Stacks Project

Version 74af77a7, compiled on Jun 27, 2023.

The following people have contributed to this work: Kian Abolfazlian, Dan Abramovich, Piotr Achinger, Juan Pablo Acosta Lopez, Nick Addington, Shishir Agrawal, Eric Ahlqvist, Jarod Alper, Kasper Andersen, Johannes Anschuetz, Benjamin Antieau, Ko Aoki (青木孔), Santiago Arango-Piñeros, Konstantin Ardakov, Dima Arinkin, Kenneth Asher, Aravind Asok, Clément Baillet, Dario Balboni, Badam Baplan, Marco Baracchini, Guillermo Barajas Ayuso, Adrian Barquero-Sanchez, Owen Barrett, Reid Barton, Tathagata Basak, Giulia Battiston, Jefferson Baudin, Hanno Becker, Mark Behrens, Dori Bejleri, Pieter Belmans, Olivier Benoist, Takagi Benseki (高城 辨積), Laurent Berger, Daniel Bergh, Fabio Bernasconi, Siddharth Bhat, Bhargav Bhatt, Wessel Bindt, Chris Birkbeck, Ingo Blechschmidt, Juhani Bonsdorff, Mark Bowron, Heiko Braun, Lucas Braune, Thomas Brazelton, Giulio Bresciani, Martin Bright, David Brown, Kong Bochao, Sebastian Bozlee, Niels Borne, Félix Baril Boudreau, Elyes Boughattas, Alexis Bouthier, Ragnar-Olaf Buchweitz, Fabio Buccoliero, Kevin Buzzard, Jakub Byszewski, Anna Cadoret, Zhaodong Cai, Nicholas Camacho, Tim Campion, Samir Canning, Adi Caplan-Bricker, Robert Cardona, Nuno Cardoso, Louis Carlin, Scott Carnahan, Kęstutis Česnavičius, Antoine Chambert-Loir, Ryan Chen, Wei Chen, Will Chen, William Chen, Raymond Cheng (程毅), Filip Chindea, Chung Ching, Nava Chitrik, Christopher Chiu, Fraser Chiu, Kat Christianson, Patrick Chu, Benjamin Voulgaris Church, Joshua Ciappara, Dustin Clausen, Jérémy Cochoy, Johan Commelin, Brian Conrad, David Corwin, Sean Cotner, Pavel Čoupek, Peadar Coyle, Moises Herradon Cueto, Marco D’Addezio, Chiara Damiolini, Lucas das Dores, Rankeya Datta, Olivier de Gaay Fortman, Aise Johan de Jong, Matt DeLand, Spencer Dembner, Ashwin Deopurkar, Maarten Derickx, Neeraj Deshmukh, Benjamin Diamond, Fred Diamond, Claus Diem, Ajneet Dhillon, Daniel Disegni, Joel Dodge, Daniel Dore, Peng Du, Taylor Dupuy, Bas Edixhoven, Jonas Ehrhard, Alexander Palen Ellis, Matthew Emerton, Aras Ergus, Dennis Eriksson, Tim Evink, Andrew Fanoe, Mitchell Faulk, Maxim Fedorchuck, Hu Fei, Peter Fleischmann, Dan Fox, Cameron Franc, Dragos Fratila, Gerard Freixas, Robert Friedman, Robert Furber, Ofer Gabber, Pierre-Yves Gaillard, Juan Sebastian Gaitan, Lennart Galinat, Martin Gallauer, Luis Garcia, Xu Gao, Toby Gee, Anton Geraschenko, Daniel Gerigk, Jérémy Gerin, Alexandru Ghitza, Harry Gindi, Alberto Gioia, Charles Godfrey, Danny A. J. Gomez-Ramirez, Julia Ramos Gonzalez, Jean-Pierre Gourdot, Tom Graber, Matt Grimes, Darij Grinberg, Rein Janssen Groesbeek, Joshua Grochow, Jonathan Gruner, Yuzhou

Gu, Zeshen Gu, Quentin Guignard, Elías Guisado Villalgordo, Albert Gunawan, Joseph Gunther, Haoyang Guo, Anton Güthge, Andrei Halanay, Yatir Halevi, Jack Hall, Daniel Halpern-Leistner, Linus Hamann, Colin Hamon, Minsik Han, Xue Hang, David Hansen, Yun Hao, Michael Harris, William Hart, Philipp Hartwig, Matthew Hase-Liu, Mohamed Hashi, Olivier Haution, Tongmu He (何通木), Hadi Hedayatzadeh, Fawzy Hegab, Lukas Heger, Florian Heiderich, Jochen Heinloth, Daniel Heiss, Jeroen Hekking, Jeremiah Heller, Reimundo Heluani, Kristen Hendricks, Aron Heleodoro, Andres Fernandez Herrero, Patrick Herter, Christian Hildebrandt, Fraser Hiu, Quoc P. Ho, Manuel Hoff, Kyle Hofmann, Amit Hogadi, David Holmes, Andreas Holmstrom, Tim Holzschuh, Ray Hoobler, John Hosack, Xiaowen Hu, Yuhao Huang, Yu-Liang Huang, Logan Hyslop, Andrés Ibáñez Núñez, Ashwin Iyengar, Kentaro Inoue, Shota Inoue, Ariyan Javanpeykar, Jia Jia (甲), Lena Min Ji, Qingyuan Jiang, Peter Johnson, Mattias Jonsson, Grayson Jorgenson, Eric Jovinelly, Christian Kappen, Hayama Kazuma (羽山籍真), Kiran Kedlaya, Timo Keller, Adeel Ahmad Khan, Derek Khu, Keenan Kidwell, Ammar Kilic, Andrew Kiluk, Dongryul Kim, Myeonhu Kim, Lars Kindler, Friedrich Knop, Lukas Kofler, Junnosuke Koizumi, János Kollár, Vladimir Kondratjew, Frank Kong, Taro Konno, Grisha Konovalov, Dmitry Korb, Thea Kosche, Praphulla Koushik, Sándor Kovács, Emmanuel Kowalski, Steve Kudla, Nick Kuhn, Girish Kulkarni, Mario Kummer, Matthias Kummerer, Manoj Kummini, Arnab Kundu, Daniel Krashen, Oleksandr Kravets, Jef Laga, Raffaele Lamagna, Thiago Landim, Matt Larson, Brian Lawrence, Davis Lazowski, Geoffrey Lee, Heejong Lee, Min Lee, Pak-Hin Lee, Wet Lee, Simon Pepin Lehalleur, Tobi Lehman, Dion Leijnse, Florian Lengyel, Brandon Levin, Daniel Levine, Paul Lessard, Mao Li, Shang Li, Shizhang Li (李璋), Wen-Wei Li, Xuanyou Li, Yixiao Li (李一笑), Carl Lian, Dun Liang, Max Lieblich, Bronson Lim, David Benjamin Lim, Joseph Lipman, Zongzhu Lin, Daniel Litt, Chunhui Liu, David Liu, Huaxin Liu, Hsing Liu, Linyuan Liu, Qing Liu, Shurui Liu, Xiaolong Liu, Xuande Liu, Yuchen Liu, Zeyu Liu, David Loeffler, François Loeser, Davide Lombardo, Dino Lorenzini, Weixiao Lu, David Lubicz, Cedric Luger, Yujie Luo, Shiji Lyu, Qixiao Ma, Zachary Maddock, Roy Magen, Svetlana Makarova, Mohammed Mammeri, Zhouhang Mao, Sonja Mapes, Christophe Marciot, Florent Martin, Yuto Masamura, Kazuki Masugi (馬杉和貴), Klaus Mattis, Akhil Mathew, Anton Mellit, Fanjun Meng, Daniel Miller, Abel Milor, Ben Moonen, Yogesh More, Laurent Moret-Bailly, Maxim Mornev, Jackson Morrow, Nicolas Müller, Alapan Mukhopadhyay, Rubén Muñoz–Bertrand, Brendan Seamas Murphy, Takumi Murayama, Yusuf Mustopa, David Mykytyn, Michael Neururer, Alejandro González Nevado, Josh Nichols-Barrer, Kien Nguyen, Thomas Nyberg, Arthur Ogus, Masahiro Ohno, Catherine O'Neil, Noah Olander, Martin Olsson, Fabrice Orgogozo, Brian Osserman, Maris Ozols, Simon Paege, Piotr Pakosz, Mike Paluch, Andrea Panontin, Thanos Papaioannou, Jinhyun Park, Roland Paulin, Rakesh Pawar, Dmitrii Pedchenko, Yang Pei, Hao Peng, Peter Percival, Alex Perry, Dat Pham, Maik Pickl, Dmitrii Pirozhkov, Gregor Pohl, Jérôme Poineau, Bjorn Poonen, Dhivya Prakash R V, Anatoly Preygel, Artem Prihodko, Thibaut Pugin, Souparna Purohit, You Qi, Haonan Qu (曲昊男), Eamon Quinlan, Patrick Rabau, Ryan Reich, Emanuel Reinecke, Charles Rezk, Gabriel Ribeiro, Alice Rizzardo, Damien Robert, David Roberts, Antonios-Alexandros Robotis, Job Rock, Herman Rohrbach, Fred Rohrer, Matthieu Romagny, Oliver Röndigs, Joe Ross, Julius Ross, Apurba Kumar Roy, Rob Roy, Cameron Ruether, Yairon Cid Ruiz, Nithi Rungtanapirom, David Rydh, Carles Sáez, Steffen Sagave, Jyoti Prakash Saha, Zeyn Sahilliogullari, Rijul Saini, Takeshi Saito (斎藤 毅), Beren Sanders, Emily de Oliveira Santos, Lukas Sauer, Will Sawin, Federico Scavia, Alex Scheffelin, Simon Schirren, Alexander Schmidt, Tobias Schmidt, Olaf Schnürer, Jakob Scholbach, Rene Schoof, Hans Schoutens, Karl Schwede, Ryan Schwiebert, Won Seong, Jaakko Seppala, Ivan Serna, Michele Serra, Emre Sertoz, Chung-chieh Shan, Yi Shan, Arpith Shanbhag, Zhenbing Shang, Liran Shaul, Che Shen, Bryan Shih, Minseon Shin, Jeroen Sijsling, Lior Silberman, Carlos Simpson, John Smith, Thomas Smith, Calle Sönne, Dylan Spence, David Speyer, Tanya Kaushal Srivastava, Axel Stäbler, Jason Starr, Stephan Snegirov, Matt Stevenson, Elie Studnia, Thierry Stulemeijer, Florian Stümpfl, Ryo Suzuki, Takashi Suzuki, Daichi Takeuchi, Lenny Taelman, Kang Taeyeoup, Tuomas Tajakka, Mattia Talpo, Longke Tang, Yiming Tang, Abolfazl Tarizadeh, John Tate, David Taylor, Titus Teodorescu, Alexios Terezakis, Samuel Tiersma, Michael Thaddeus, Stulemeijer Thierry, Shabalin Timofey, Valery Tolstov, Alex Torzewski, Dajano Tossici, Burt Totaro, Meng-Gen Tsai, Manolis Tsakiris, Minh-Tien Tran, Robin Truax, David Tweedle, Ravi Vakil, Michel Van den Bergh, Theo van den Bogaart, Matthé van der Lee, Jeroen van der Meer, Peter Bruin, Thibaud van den Hove, Remy van Dobben de Bruyn, Kevin Ventullo, Hendrik Verhoek, Antoine Vezier, Erik Visse, Angelo Vistoli, Konrad Voelkel, Fred Vu, Rishi Vyas, James Waldron, Hua Wang, Jonathan Wang, Yijin Wang, Yiyang Wang, Matthew Ward, Evan Warner, Nils Waßmuth, John Watterlond, Rachel Webb, Torsten Wedhorn, Dario Weissmann, Jakob Werner, Ian Whitehead, Jonathan Wise, Junho Won, William Wright, Zhenhua Wu, Dominic Wynter, Daxin Xu, Fei Xu, Jiachang Xu, Junyan Xu, Wei Xu, Qijun Yan, Mengxue Yang, Yuan Yang, Haodong Yao, Amnon Yekutieli, Weng Yixiang, Alex Youcis, Jize Yu, John Yu, Zhiyu Yuan, Koito Yuu, Felipe Zaldivar, Bogdan Zavyalov, Maciek Zdanowicz, Omri Zemer, Dingxin Zhang, Keke Zhang, Lei Zhang, Robin Zhang, Yuchong Zhang, Zhe Zhang, Zhiyu Zhang, Zili Zhang, Jiayu Zhao, Yifei Zhao, Yu Zhao, Zhipu Zhao, Fan Zheng, Weizhe Zheng, Shend Zhjeqi, Anfang Zhou, Yicheng Zhou, Fan Zhou, Wouter Zomervrucht, Runpu Zong, Konrad Zou, Jeroen Zuiddam, David Zureick-Brown.

Part 1

Preliminaries

CHAPTER 1

Introduction

0000 1.1. Overview

0001 Besides the book by Laumon and Moret-Bailly, see [LMB00], and the work (in progress) by Fulton et al, we think there is a place for an open source textbook on algebraic stacks and the algebraic geometry that is needed to define them. The Stacks Project attempts to do this by building the foundations starting with commutative algebra and proceeding via the theory of schemes and algebraic spaces to a comprehensive foundation for the theory of algebraic stacks.

We expect this material to be read online as a key feature are the hyperlinks giving quick access to internal references spread over many different pages. If you use an embedded pdf or dvi viewer in your browser, the cross file links should work.

This project is a collaborative effort and we encourage you to help out. Please email any typos or errors you find while reading or any suggestions, additional material, or examples you have to stacks.project@gmail.com. You can download a tarball containing all source files, extract, run make, and use a dvi or pdf viewer locally. Please feel free to edit the LaTeX files and email your improvements.

1.2. Attribution

06LB The scope of this work is such that it is a daunting task to attribute correctly and succinctly all of those mathematicians whose work has led to the development of the theory we try to explain here. We hope eventually to generate enough community interest to find contributors willing to write sections with historical remarks for each and every chapter.

Those who contributed to this work are listed on the title page of the book version of this work and online. Here we would like to name a selection of major contributions:

(1) Jarod Alper contributed a chapter discussing the literature on algebraic stacks, see Guide to Literature, Section 112.1.
(2) Bhargav Bhatt wrote the initial version of a chapter on étale morphisms of schemes, see Étale Morphisms, Section 41.1.
(3) Bhargav Bhatt wrote the initial version of More on Algebra, Section 15.89.
(4) Kiran Kedlaya contributed the initial writeup of Descent, Section 35.4.
(5) The initial versions of
    (a) Algebra, Section 10.28,
    (b) Injectives, Section 19.2, and
    (c) the chapter on fields, see Fields, Section 9.1.

    are from The CRing Project, courtesy of Akhil Mathew et al.
(6) Alex Perry wrote the material on projective modules, Mittag-Leffler modules, including the proof of Algebra, Theorem 10.95.6.

(7) Alex Perry wrote the chapter on deformation theory a la Schlessinger and Rim, see Formal Deformation Theory, Section 90.1.
(8) Thibaut Pugin, Zachary Maddock and Min Lee took notes for a course which formed the basis for a chapter on étale cohomology and a chapter on the trace formula. See Étale Cohomology, Section 59.1 and The Trace Formula, Section 64.1.
(9) David Rydh has contributed many helpful comments, pointed out several mistakes, helped out in an essential way with the material on residual gerbes, and was the originator for the material in More on Groupoids in Spaces, Sections 79.12 and 79.15.
(10) Burt Totaro contributed Examples, Sections 110.64, 110.65, and Properties of Stacks, Section 100.12.
(11) The chapter on pro-étale cohomology, see Pro-étale Cohomology, Section 61.1, is taken from a paper by Bhargav Bhatt and Peter Scholze.
(12) Bhargav Bhatt contributed Examples, Sections 110.71 and 110.75.
(13) Ofer Gabber found mistakes, contributed corrections and he contributed Varieties, Lemma 33.7.17, Formal Spaces, Lemma 87.14.7, the material in More on Groupoids, Section 40.15, the main result of Properties of Spaces, Section 66.17, and the proof of More on Flatness, Proposition 38.25.13.
(14) János Kollár contributed Algebra, Lemma 10.119.2 and Local Cohomology, Proposition 51.8.7.
(15) Kiran Kedlaya wrote the initial version of More on Algebra, Section 15.90.
(16) Matthew Emerton, Toby Gee, and Brandon Levin contributed some results on thickenings, in particular More on Morphisms of Stacks, Lemmas 106.3.7, 106.3.8, and 106.3.9.
(17) Lena Min Ji wrote the initial version of More on Algebra, Section 15.125.
(18) Matthew Emerton and Toby Gee wrote the initial versions of Geometry of Stacks, Sections 107.3 and 107.5.

1.3. Other chapters

CHAPTER 2

Conventions

0002 2.1. Comments

0003 The philosophy behind the conventions used in writing these documents is to choose those conventions that work.

2.2. Set theory

0004 We use Zermelo-Fraenkel set theory with the axiom of choice. See [Kun83]. We do not use universes (different from SGA4). We do not stress set-theoretic issues, but we make sure everything is correct (of course) and so we do not ignore them either.

2.3. Categories

0005 A category $\mathcal{C}$ consists of a set of objects and, for each pair of objects, a set of morphisms between them. In other words, it is what is called a "small" category in other texts. We will use "big" categories (categories whose objects form a proper class) as well, but only those that are listed in Categories, Remark 4.2.2.

2.4. Algebra

0006 In these notes a ring is a commutative ring with a 1. Hence the category of rings has an initial object $\mathbf{Z}$ and a final object $\{0\}$ (this is the unique ring where $1 = 0$). Modules are assumed unitary. See [Eis95].

2.5. Notation

055X The natural integers are elements of $\mathbf{N} = \{1, 2, 3, \ldots\}$. The integers are elements of $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The field of rational numbers is denoted $\mathbf{Q}$. The field of real numbers is denoted $\mathbf{R}$. The field of complex numbers is denoted $\mathbf{C}$.

2.6. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites

CHAPTER 3

Set Theory

0007 ## 3.1. Introduction

0008 We need some set theory every now and then. We use Zermelo-Fraenkel set theory with the axiom of choice (ZFC) as described in [Kun83] and [Jec02].

3.2. Everything is a set

0009 Most mathematicians think of set theory as providing the basic foundations for mathematics. So how does this really work? For example, how do we translate the sentence "X is a scheme" into set theory? Well, we just unravel the definitions: A scheme is a locally ringed space such that every point has an open neighbourhood which is an affine scheme. A locally ringed space is a ringed space such that every stalk of the structure sheaf is a local ring. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on it. A topological space is a pair (X, τ) consisting of a set X and a set of subsets $\tau \subset \mathcal{P}(X)$ satisfying the axioms of a topology. And so on and so forth.

So how, given a set S would we recognize whether it is a scheme? The first thing we look for is whether the set S is an ordered pair. This is defined (see [Jec02], page 7) as saying that S has the form $(a, b) := \{\{a\}, \{a, b\}\}$ for some sets a, b. If this is the case, then we would take a look to see whether a is an ordered pair (c, d). If so we would check whether $d \subset \mathcal{P}(c)$, and if so whether d forms the collection of sets for a topology on the set c. And so on and so forth.

So even though it would take a considerable amount of work to write a complete formula $\phi_{scheme}(x)$ with one free variable x in set theory that expresses the notion "x is a scheme", it is possible to do so. The same thing should be true for any mathematical object.

3.3. Classes

000A Informally we use the notion of a class. Given a formula $\phi(x, p_1, \ldots, p_n)$, we call

$$C = \{x : \phi(x, p_1, \ldots, p_n)\}$$

a class. A class is easier to manipulate than the formula that defines it, but it is not strictly speaking a mathematical object. For example, if R is a ring, then we may consider the class of all R-modules (since after all we may translate the sentence "M is an R-module" into a formula in set theory, which then defines a class). A proper class is a class which is not a set.

In this way we may consider the category of R-modules, which is a "big" category—in other words, it has a proper class of objects. Similarly, we may consider the "big" category of schemes, the "big" category of rings, etc.

3.4. Ordinals

05N1 A set T is transitive if $x \in T$ implies $x \subset T$. A set α is an ordinal if it is transitive and well-ordered by $\in$. In this case, we define $\alpha + 1 = \alpha \cup \{\alpha\}$, which is another ordinal called the successor of α. An ordinal α is called a successor ordinal if there exists an ordinal β such that $\alpha = \beta + 1$. The smallest ordinal is $\emptyset$ which is also denoted 0. If α is not 0, and not a successor ordinal, then α is called a limit ordinal and we have

$$\alpha = \bigcup\nolimits_{\gamma \in \alpha} \gamma.$$

The first limit ordinal is ω and it is also the first infinite ordinal. The first uncountable ordinal ω_1 is the set of all countable ordinals. The collection of all ordinals is a proper class. It is well-ordered by $\in$ in the following sense: any nonempty set (or even class) of ordinals has a least element. Given a set A of ordinals, we define the supremum of A to be $\sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha$. It is the least ordinal bigger or equal to all $\alpha \in A$. Given any well-ordered set $(S, <)$, there is a unique ordinal α such that $(S, <) \cong (\alpha, \in)$; this is called the order type of the well-ordered set.

3.5. The hierarchy of sets

000B We define by transfinite recursion $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ (power set), and for a limit ordinal α,

$$V_\alpha = \bigcup\nolimits_{\beta < \alpha} V_\beta.$$

Note that each V_α is a transitive set.

000C Lemma 3.5.1. Every set is an element of V_α for some ordinal α.

Proof. See [Jec02, Lemma 6.3]. □

In [Kun83, Chapter III] it is explained that this lemma is equivalent to the axiom of foundation. The rank of a set S is the least ordinal α such that $S \in V_{\alpha+1}$. By a partial universe we shall mean a suitably large set of the form V_α which will be clear from the context.

3.6. Cardinality

000D The cardinality of a set A is the least ordinal α such that there exists a bijection between A and α. We sometimes use the notation $\alpha = |A|$ to indicate this. We say an ordinal α is a cardinal if and only if it occurs as the cardinality of some set A—in other words, if $\alpha = |A|$. We use the greek letters κ, λ for cardinals. The first infinite cardinal is ω, and in this context it is denoted $\aleph_0$. A set is countable if its cardinality is $\leq \aleph_0$. If α is an ordinal, then we denote α^+ the least cardinal $> \alpha$. You can use this to define $\aleph_1 = \aleph_0^+$, $\aleph_2 = \aleph_1^+$, etc, and in fact you can define $\aleph_\alpha$ for any ordinal α by transfinite recursion. We note the equality $\aleph_1 = \omega_1$.

The addition of cardinals κ, λ is denoted $\kappa \oplus \lambda$; it is the cardinality of $\kappa \amalg \lambda$. The multiplication of cardinals κ, λ is denoted $\kappa \otimes \lambda$; it is the cardinality of $\kappa \times \lambda$. If κ and λ are infinite cardinals, then $\kappa \oplus \lambda = \kappa \otimes \lambda = \max(\kappa, \lambda)$. The exponentiation of cardinals κ, λ is denoted κ^λ; it is the cardinality of the set of (set) maps from λ to κ. Given any set K of cardinals, the supremum of K is $\sup_{\kappa \in K} \kappa = \bigcup_{\kappa \in K} \kappa$, which is also a cardinal.

3.7. Cofinality

000E A cofinal subset S of a well-ordered set T is a subset $S \subset T$ such that $\forall t \in T \exists s \in S(t \leq s)$. Note that a subset of a well-ordered set is a well-ordered set (with induced ordering). Given an ordinal α, the cofinality $\text{cf}(\alpha)$ of α is the least ordinal β which occurs as the order type of some cofinal subset of α. The cofinality of an ordinal is always a cardinal. Hence alternatively we can define the cofinality of α as the least cardinality of a cofinal subset of α.

05N2 Lemma 3.7.1. Suppose that $T = \text{colim}_{\alpha < \beta} T_\alpha$ is a colimit of sets indexed by ordinals less than a given ordinal β. Suppose that $\varphi : S \to T$ is a map of sets. Then φ lifts to a map into T_α for some $\alpha < \beta$ provided that β is not a limit of ordinals indexed by S, in other words, if β is an ordinal with $\text{cf}(\beta) > |S|$.

Proof. For each element $s \in S$ pick a $\alpha_s < \beta$ and an element $t_s \in T_{\alpha_s}$ which maps to $\varphi(s)$ in T. By assumption $\alpha = \sup_{s \in S} \alpha_s$ is strictly smaller than β. Hence the map $\varphi_\alpha : S \to T_\alpha$ which assigns to s the image of t_s in T_α is a solution. □

The following is essentially Grothendieck's argument for the existence of ordinals with arbitrarily large cofinality which he used to prove the existence of enough injectives in certain abelian categories, see [Gro57].

05N3 Proposition 3.7.2. Let κ be a cardinal. Then there exists an ordinal whose cofinality is bigger than κ.

Proof. If κ is finite, then $\omega = \text{cf}(\omega)$ works. Let us thus assume that κ is infinite. Consider the smallest ordinal α whose cardinality is strictly greater than κ. We claim that $\text{cf}(\alpha) > \kappa$. Note that α is a limit ordinal, since if $\alpha = \beta + 1$, then $|\alpha| = |\beta|$ (because α and β are infinite) and this contradicts the minimality of α. (Of course α is also a cardinal, but we do not need this.) To get a contradiction suppose $S \subset \alpha$ is a cofinal subset with $|S| \leq \kappa$. For $\beta \in S$, i.e., $\beta < \alpha$, we have $|\beta| \leq \kappa$ by minimality of α. As α is a limit ordinal and S cofinal in α we obtain $\alpha = \bigcup_{\beta \in S} \beta$. Hence $|\alpha| \leq |S| \otimes \kappa \leq \kappa \otimes \kappa \leq \kappa$ which is a contradiction with our choice of α. □

3.8. Reflection principle

000F Some of this material is in the chapter of [Kun83] called "Easy consistency proofs".

Let $\phi(x_1, \ldots, x_n)$ be a formula of set theory. Let us use the convention that this notation implies that all the free variables in ϕ occur among $x_1, \ldots, x_n$. Let M be a set. The formula $\phi^M(x_1, \ldots, x_n)$ is the formula obtained from $\phi(x_1, \ldots, x_n)$ by replacing all the $\forall x$ and $\exists x$ by $\forall x \in M$ and $\exists x \in M$, respectively. So the formula $\phi(x_1, x_2) = \exists x(x \in x_1 \wedge x \in x_2)$ is turned into $\phi^M(x_1, x_2) = \exists x \in M(x \in x_1 \wedge x \in x_2)$. The formula ϕ^M is called the relativization of ϕ to M.

000G Theorem 3.8.1. Suppose given $\phi_1(x_1, \ldots, x_n), \ldots, \phi_m(x_1, \ldots, x_n)$ a finite collection of formulas of set theory. Let M_0 be a set. There exists a set M such that $M_0 \subset M$ and $\forall x_1, \ldots, x_n \in M$, we have

$$\forall i = 1, \ldots, m,\ \phi_i^M(x_1, \ldots, x_n) \Leftrightarrow \forall i = 1, \ldots, m,\ \phi_i(x_1, \ldots, x_n).$$

In fact we may take $M = V_\alpha$ for some limit ordinal α.

Proof. See [Jec02, Theorem 12.14] or [Kun83, Theorem 7.4]. □

We view this theorem as saying the following: Given any $x_1, \ldots, x_n \in M$ the formulas hold with the bound variables ranging through all sets if and only if they hold for the bound variables ranging through elements of V_α. This theorem is a meta-theorem because it deals with the formulas of set theory directly. It actually says that given the finite list of formulas $\phi_1, \ldots, \phi_m$ with at most free variables $x_1, \ldots, x_n$ the sentence

$$\forall M_0 \ \exists M,\ M_0 \subset M \ \forall x_1, \ldots, x_n \in M$$
$$\phi_1(x_1, \ldots, x_n) \wedge \ldots \wedge \phi_m(x_1, \ldots, x_n) \leftrightarrow \phi_1^M(x_1, \ldots, x_n) \wedge \ldots \wedge \phi_m^M(x_1, \ldots, x_n)$$

is provable in ZFC. In other words, whenever we actually write down a finite list of formulas ϕ_i, we get a theorem.

It is somewhat hard to use this theorem in "ordinary mathematics" since the meaning of the formulas $\phi_i^M(x_1, \ldots, x_n)$ is not so clear! Instead, we will use the idea of the proof of the reflection principle to prove the existence results we need directly.

3.9. Constructing categories of schemes

000H We will discuss how to apply this to produce, given an initial set of schemes, a "small" category of schemes closed under a list of natural operations. Before we do so, we introduce the size of a scheme. Given a scheme S we define

$$\mathrm{size}(S) = \max(\aleph_0, \kappa_1, \kappa_2),$$

where we define the cardinal numbers κ_1 and κ_2 as follows:

(1) We let κ_1 be the cardinality of the set of affine opens of S.
(2) We let κ_2 be the supremum of all the cardinalities of all $\Gamma(U, \mathcal{O}_S)$ for all $U \subset S$ affine open.

000I Lemma 3.9.1. For every cardinal κ, there exists a set A such that every element of A is a scheme and such that for every scheme S with $\mathrm{size}(S) \leq \kappa$, there is an element $X \in A$ such that $X \cong S$ (isomorphism of schemes).

Proof. Omitted. Hint: think about how any scheme is isomorphic to a scheme obtained by glueing affines. $\square$

We denote *Bound* the function which to each cardinal κ associates

046U (3.9.1.1) $$Bound(\kappa) = \max\{\kappa^{\aleph_0}, \kappa^+\}.$$

We could make this function grow much more rapidly, e.g., we could set $Bound(\kappa) = \kappa^\kappa$, and the result below would still hold. For any ordinal α, we denote Sch_α the full subcategory of category of schemes whose objects are elements of V_α. Here is the result we are going to prove.

000J Lemma 3.9.2. With notations size, *Bound* and Sch_α as above. Let S_0 be a set of schemes. There exists a limit ordinal α with the following properties:

000K (1) We have $S_0 \subset V_\alpha$; in other words, $S_0 \subset \mathrm{Ob}(Sch_\alpha)$.
000L (2) For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any scheme T with $\mathrm{size}(T) \leq Bound(\mathrm{size}(S))$, there exists a scheme $S' \in \mathrm{Ob}(Sch_\alpha)$ such that $T \cong S'$.
000M (3) For any countable[1] diagram category $\mathcal{I}$ and any functor $F : \mathcal{I} \to Sch_\alpha$, the limit $\lim_\mathcal{I} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.

[1]Both the set of objects and the morphism sets are countable. In fact you can prove the lemma with $\aleph_0$ replaced by any cardinal whatsoever in (3) and (4).

000N (4) For any countable index category $\mathcal{I}$ and any functor $F : \mathcal{I} \to \mathit{Sch}_\alpha$, the colimit $\mathrm{colim}_\mathcal{I} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.

Proof. We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$, we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

(1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
(2) For any $S \in \mathrm{Ob}(\mathit{Sch}_{f(\alpha)})$ and any scheme T with $\mathrm{size}(T) \leq \mathit{Bound}(\mathrm{size}(S))$, there exists a scheme $S' \in \mathrm{Ob}(\mathit{Sch}_\beta)$ such that $T \cong S'$.
(3) For any countable index category $\mathcal{I}$ and any functor $F : \mathcal{I} \to \mathit{Sch}_{f(\alpha)}$, if the limit $\lim_\mathcal{I} F$ or the colimit $\mathrm{colim}_\mathcal{I} F$ exists in Sch, then it is isomorphic to a scheme in Sch_β.

To see β exists, we argue as follows. Since $\mathrm{Ob}(\mathit{Sch}_{f(\alpha)})$ is a set, we see that $\kappa = \sup_{S \in \mathrm{Ob}(\mathit{Sch}_{f(\alpha)})} \mathit{Bound}(\mathrm{size}(S))$ exists and is a cardinal. Let A be a set of schemes obtained starting with κ as in Lemma 3.9.1. There is a set $\mathit{CountCat}$ of countable categories such that any countable category is isomorphic to an element of $\mathit{CountCat}$. Hence in (3) above we may assume that $\mathcal{I}$ is an element in $\mathit{CountCat}$. This means that the pairs $(\mathcal{I}, F)$ in (3) range over a set. Thus, there exists a set B whose elements are schemes such that for every $(\mathcal{I}, F)$ as in (3), if the limit or colimit exists, then it is isomorphic to an element in B. Hence, if we pick any β such that $A \cup B \subset V_\beta$ and $\beta > \max\{\alpha + 1, f(\alpha)\}$, then (1)–(3) hold. Since every nonempty collection of ordinals has a least element, we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha' < \alpha} f(\alpha')$.

Pick β_0 such that $S_0 \subset V_{\beta_0}$. By construction $f(\beta) \geq \beta$ and we see that also $S_0 \subset V_{f(\beta_0)}$. Moreover, as f is nondecreasing, we see $S_0 \subset V_{f(\beta)}$ is true for any $\beta \geq \beta_0$. Next, choose any ordinal $\beta_1 > \beta_0$ with cofinality $\mathrm{cf}(\beta_1) > \omega = \aleph_0$. This is possible since the cofinality of ordinals gets arbitrarily large, see Proposition 3.7.2. We claim that $\alpha = f(\beta_1)$ is a solution to the problem posed in the lemma.

The first property of the lemma holds by our choice of $\beta_1 > \beta_0$ above.

Since β_1 is a limit ordinal (as its cofinality is infinite), we get $f(\beta_1) = \sup_{\beta < \beta_1} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_1\} \subset f(\beta_1)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_1)} = \bigcup\nolimits_{\beta < \beta_1} V_{f(\beta)}.$$

Now, let $S \in \mathrm{Ob}(\mathit{Sch}_\alpha)$. We define $\beta(S)$ to be the least ordinal β such that $S \in \mathrm{Ob}(\mathit{Sch}_{f(\beta)})$. By the above we see that always $\beta(S) < \beta_1$. Since $\mathrm{Ob}(\mathit{Sch}_{f(\beta+1)}) \subset \mathrm{Ob}(\mathit{Sch}_\alpha)$, we see by construction of f above that the second property of the lemma is satisfied.

Suppose that $\{S_1, S_2, \ldots\} \subset \mathrm{Ob}(\mathit{Sch}_\alpha)$ is a countable collection. Consider the function $\omega \to \beta_1$, $n \mapsto \beta(S_n)$. Since the cofinality of β_1 is $> \omega$, the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\{S_1, S_2, \ldots\} \subset \mathrm{Ob}(\mathit{Sch}_{f(\beta)})$. It follows that any functor $F : \mathcal{I} \to \mathit{Sch}_\alpha$ factors through one of the subcategories $\mathit{Sch}_{f(\beta)}$. Thus, if there exists a scheme X that is the colimit or limit of the diagram F, then, by construction of f, we see X is isomorphic to an object of $\mathit{Sch}_{f(\beta+1)}$ which is a subcategory of Sch_α. This proves the last two assertions of the lemma. $\square$

000O Remark 3.9.3. The lemma above can also be proved using the reflection principle. However, one has to be careful. Namely, suppose the sentence $\phi_{scheme}(X)$ expresses the property "X is a scheme", then what does the formula $\phi_{scheme}^{V_\alpha}(X)$ mean? It is true that the reflection principle says we can find α such that for all $X \in V_\alpha$ we have $\phi_{scheme}(X) \leftrightarrow \phi_{scheme}^{V_\alpha}(X)$ but this is entirely useless. It is only by combining two such statements that something interesting happens. For example suppose $\phi_{red}(X, Y)$ expresses the property "X, Y are schemes, and Y is the reduction of X" (see Schemes, Definition 26.12.5). Suppose we apply the reflection principle to the pair of formulas $\phi_1(X, Y) = \phi_{red}(X, Y)$, $\phi_2(X) = \exists Y, \phi_1(X, Y)$. Then it is easy to see that any α produced by the reflection principle has the property that given $X \in \text{Ob}(Sch_\alpha)$ the reduction of X is also an object of Sch_α (left as an exercise).

000P Lemma 3.9.4. Let S be an affine scheme. Let $R = \Gamma(S, \mathcal{O}_S)$. Then the size of S is equal to $\max\{\aleph_0, |R|\}$.

Proof. There are at most $\max\{|R|, \aleph_0\}$ affine opens of $\text{Spec}(R)$. This is clear since any affine open $U \subset \text{Spec}(R)$ is a finite union of principal opens $D(f_1) \cup \ldots \cup D(f_n)$ and hence the number of affine opens is at most $\sup_n |R|^n = \max\{|R|, \aleph_0\}$, see [Kun83, Ch. I, 10.13]. On the other hand, we see that $\Gamma(U, \mathcal{O}) \subset R_{f_1} \times \ldots \times R_{f_n}$ and hence $|\Gamma(U, \mathcal{O})| \leq \max\{\aleph_0, |R_{f_1}|, \ldots, |R_{f_n}|\}$. Thus it suffices to prove that $|R_f| \leq \max\{\aleph_0, |R|\}$ which is omitted. $\square$

000Q Lemma 3.9.5. Let S be a scheme. Let $S = \bigcup_{i \in I} S_i$ be an open covering. Then $\text{size}(S) \leq \max\{|I|, \sup_i\{\text{size}(S_i)\}\}$.

Proof. Let $U \subset S$ be any affine open. Since U is quasi-compact there exist finitely many elements $i_1, \ldots, i_n \in I$ and affine opens $U_i \subset U \cap S_i$ such that $U = U_1 \cup U_2 \cup \ldots \cup U_n$. Thus

$$|\Gamma(U, \mathcal{O}_U)| \leq |\Gamma(U_1, \mathcal{O})| \otimes \ldots \otimes |\Gamma(U_n, \mathcal{O})| \leq \sup_i\{\text{size}(S_i)\}$$

Moreover, it shows that the set of affine opens of S has cardinality less than or equal to the cardinality of the set

$$\coprod_{n \in \omega} \coprod_{i_1, \ldots, i_n \in I} \{\text{affine opens of } S_{i_1}\} \times \ldots \times \{\text{affine opens of } S_{i_n}\}.$$

Each of the sets inside the disjoint union has cardinality at most $\sup_i\{\text{size}(S_i)\}$. The index set has cardinality at most $\max\{|I|, \aleph_0\}$, see [Kun83, Ch. I, 10.13]. Hence by [Jec02, Lemma 5.8] the cardinality of the coproduct is at most $\max\{\aleph_0, |I|\} \otimes \sup_i\{\text{size}(S_i)\}$. The lemma follows. $\square$

04T6 Lemma 3.9.6. Let $f : X \to S$, $g : Y \to S$ be morphisms of schemes. Then we have $\text{size}(X \times_S Y) \leq \max\{\text{size}(X), \text{size}(Y)\}$.

Proof. Let $S = \bigcup_{k \in K} S_k$ be an affine open covering. Let $X = \bigcup_{i \in I} U_i$, $Y = \bigcup_{j \in J} V_j$ be affine open coverings with I, J of cardinality $\leq \text{size}(X), \text{size}(Y)$. For each $i \in I$ there exists a finite set K_i of $k \in K$ such that $f(U_i) \subset \bigcup_{k \in K_i} S_k$. For each $j \in J$ there exists a finite set K_j of $k \in K$ such that $g(V_j) \subset \bigcup_{k \in K_j} S_k$. Hence $f(X), g(Y)$ are contained in $S' = \bigcup_{k \in K'} S_k$ with $K' = \bigcup_{i \in I} K_i \cup \bigcup_{j \in J} K_j$. Note that the cardinality of K' is at most $\max\{\aleph_0, |I|, |J|\}$. Applying Lemma 3.9.5 we see that it suffices to prove that $\text{size}(f^{-1}(S_k) \times_{S_k} g^{-1}(S_k)) \leq \max\{\text{size}(X), \text{size}(Y))\}$ for $k \in K'$. In other words, we may assume that S is affine.

Assume S affine. Let $X = \bigcup_{i \in I} U_i$, $Y = \bigcup_{j \in J} V_j$ be affine open coverings with I, J of cardinality $\leq \text{size}(X), \text{size}(Y)$. Again by Lemma 3.9.5 it suffices to prove the lemma for the products $U_i \times_S V_j$. By Lemma 3.9.4 we see that it suffices to show that

$$|A \otimes_C B| \leq \max\{\aleph_0, |A|, |B|\}.$$

We omit the proof of this inequality. □

04T7 Lemma 3.9.7. Let S be a scheme. Let $f : X \to S$ be locally of finite type with X quasi-compact. Then $\text{size}(X) \leq \text{size}(S)$.

Proof. We can find a finite affine open covering $X = \bigcup_{i=1,\ldots n} U_i$ such that each U_i maps into an affine open S_i of S. Thus by Lemma 3.9.5 we reduce to the case where both S and X are affine. In this case by Lemma 3.9.4 we see that it suffices to show

$$|A[x_1, \ldots, x_n]| \leq \max\{\aleph_0, |A|\}.$$

We omit the proof of this inequality. □

In Algebra, Lemma 10.107.13 we will show that if $A \to B$ is an epimorphism of rings, then $|B| \leq \max(|A|, \aleph_0)$. The analogue for schemes is the following lemma.

04VA Lemma 3.9.8. Let $f : X \to Y$ be a monomorphism of schemes. If at least one of the following properties holds, then $\text{size}(X) \leq \text{size}(Y)$:

(1) f is quasi-compact,
(2) f is locally of finite presentation,
(3) add more here as needed.

But the bound does not hold for monomorphisms which are locally of finite type.

Proof. Let $Y = \bigcup_{j \in J} V_j$ be an affine open covering of Y with $|J| \leq \text{size}(Y)$. By Lemma 3.9.5 it suffices to bound the size of the inverse image of V_j in X. Hence we reduce to the case that Y is affine, say $Y = \text{Spec}(B)$. For any affine open $\text{Spec}(A) \subset X$ we have $|A| \leq \max(|B|, \aleph_0) = \text{size}(Y)$, see remark above and Lemma 3.9.4. Thus it suffices to show that X has at most $\text{size}(Y)$ affine opens. This is clear if X is quasi-compact, whence case (1) holds. In case (2) the number of isomorphism classes of B-algebras A that can occur is bounded by $\text{size}(B)$, because each A is of finite type over B, hence isomorphic to an algebra $B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some n, m, and $f_j \in B[x_1, \ldots, x_n]$. However, as $X \to Y$ is a monomorphism, there is a unique morphism $\text{Spec}(A) \to X$ over $Y = \text{Spec}(B)$ if there is one, hence the number of affine opens of X is bounded by the number of these isomorphism classes.

To prove the final statement of the lemma consider the ring $B = \prod_{n \in \mathbf{N}} \mathbf{F}_2$ and set $Y = \text{Spec}(B)$. For every ultrafilter $\mathcal{U}$ on $\mathbf{N}$ we obtain a maximal ideal $\mathfrak{m}_\mathcal{U}$ with residue field $\mathbf{F}_2$; the map $B \to \mathbf{F}_2$ sends the element (x_n) to $\lim_\mathcal{U} x_n$. Details omitted. The morphism of schemes $X = \coprod_\mathcal{U} \text{Spec}(\mathbf{F}_2) \to Y$ is a monomorphism as all the points are distinct. However the cardinality of the set of affine open subschemes of X is equal to the cardinality of the set of ultrafilters on $\mathbf{N}$ which is $2^{2^{\aleph_0}}$. We conclude as $|B| = 2^{\aleph_0} < 2^{2^{\aleph_0}}$. □

000R Lemma 3.9.9. Let α be an ordinal as in Lemma 3.9.2 above. The category Sch_α satisfies the following properties:

(1) If $X, Y, S \in \mathrm{Ob}(Sch_\alpha)$, then for any morphisms $f : X \to S$, $g : Y \to S$ the fibre product $X \times_S Y$ in Sch_α exists and is a fibre product in the category of schemes.
(2) Given any at most countable collection $S_1, S_2, \ldots$ of elements of $\mathrm{Ob}(Sch_\alpha)$, the coproduct $\coprod_i S_i$ exists in $\mathrm{Ob}(Sch_\alpha)$ and is a coproduct in the category of schemes.
(3) For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any open immersion $U \to S$, there exists a $V \in \mathrm{Ob}(Sch_\alpha)$ with $V \cong U$.
(4) For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any closed immersion $T \to S$, there exists an $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.
(5) For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any finite type morphism $T \to S$, there exists an $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.
(6) Suppose S is a scheme which has an open covering $S = \bigcup_{i \in I} S_i$ such that there exists a $T \in \mathrm{Ob}(Sch_\alpha)$ with (a) $\mathrm{size}(S_i) \leq \mathrm{size}(T)^{\aleph_0}$ for all $i \in I$, and (b) $|I| \leq \mathrm{size}(T)^{\aleph_0}$. Then S is isomorphic to an object of Sch_α.
(7) For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any morphism $f : T \to S$ locally of finite type such that T can be covered by at most $\mathrm{size}(S)^{\aleph_0}$ open affines, there exists an $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$. For example this holds if T can be covered by at most $|\mathbf{R}| = 2^{\aleph_0} = \aleph_0^{\aleph_0}$ open affines.
(8) For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any monomorphism $T \to S$ which is either locally of finite presentation or quasi-compact, there exists an $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.
(9) Suppose that $T \in \mathrm{Ob}(Sch_\alpha)$ is affine. Write $R = \Gamma(T, \mathcal{O}_T)$. Then any of the following schemes is isomorphic to a scheme in Sch_α:
    (a) For any ideal $I \subset R$ with completion $R^* = \lim_n R/I^n$, the scheme $\mathrm{Spec}(R^*)$.
    (b) For any finite type R-algebra R', the scheme $\mathrm{Spec}(R')$.
    (c) For any localization $S^{-1}R$, the scheme $\mathrm{Spec}(S^{-1}R)$.
    (d) For any prime $\mathfrak{p} \subset R$, the scheme $\mathrm{Spec}(\overline{\kappa(\mathfrak{p})})$.
    (e) For any subring $R' \subset R$, the scheme $\mathrm{Spec}(R')$.
    (f) Any scheme of finite type over a ring of cardinality at most $|R|^{\aleph_0}$.
    (g) And so on.

Proof. Statements (1) and (2) follow directly from the definitions. Statement (3) follows as the size of an open subscheme U of S is clearly smaller than or equal to the size of S. Statement (4) follows from (5). Statement (5) follows from (7). Statement (6) follows as the size of S is $\leq \max\{|I|, \sup_i \mathrm{size}(S_i)\} \leq \mathrm{size}(T)^{\aleph_0}$ by Lemma 3.9.5. Statement (7) follows from (6). Namely, for any affine open $V \subset T$ we have $\mathrm{size}(V) \leq \mathrm{size}(S)$ by Lemma 3.9.7. Thus, we see that (6) applies in the situation of (7). Part (8) follows from Lemma 3.9.8.

Statement (9) is translated, via Lemma 3.9.4, into an upper bound on the cardinality of the rings R^*, $S^{-1}R$, $\overline{\kappa(\mathfrak{p})}$, R', etc. Perhaps the most interesting one is the ring R^*. As a set, it is the image of a surjective map $R^{\mathbf{N}} \to R^*$. Since $|R^{\mathbf{N}}| = |R|^{\aleph_0}$, we see that it works by our choice of $Bound(\kappa)$ being at least $\kappa^{\aleph_0}$. Phew! (The cardinality of the algebraic closure of a field is the same as the cardinality of the field, or it is $\aleph_0$.) □

000S Remark 3.9.10. Let R be a ring. Suppose we consider the ring $\prod_{\mathfrak{p} \in \mathrm{Spec}(R)} \kappa(\mathfrak{p})$. The cardinality of this ring is bounded by $|R|^{2^{|R|}}$, but is not bounded by $|R|^{\aleph_0}$ in general. For example if $R = \mathbf{C}[x]$ it is not bounded by $|R|^{\aleph_0}$ and if $R = \prod_{n \in \mathbf{N}} \mathbf{F}_2$ it is not bounded by $|R|^{|R|}$. Thus the "And so on" of Lemma 3.9.9 above should be taken with a grain of salt. Of course, if it ever becomes necessary to consider these rings in arguments pertaining to fppf/étale cohomology, then we can change the function *Bound* above into the function $\kappa \mapsto \kappa^{2^\kappa}$.

In the following lemma we use the notion of an fpqc covering which is introduced in Topologies, Section 34.9.

0AHK Lemma 3.9.11. Let $f : X \to Y$ be a morphism of schemes. Assume there exists an fpqc covering $\{g_j : Y_j \to Y\}_{j \in J}$ such that g_j factors through f. Then $\text{size}(Y) \leq \text{size}(X)$.

Proof. Let $V \subset Y$ be an affine open. By definition there exist $n \geq 0$ and $a : \{1, \ldots, n\} \to J$ and affine opens $V_i \subset Y_{a(i)}$ such that $V = g_{a(1)}(V_1) \cup \ldots \cup g_{a(n)}(V_n)$. Denote $h_j : Y_j \to X$ a morphism such that $f \circ h_j = g_j$. Then $h_{a(1)}(V_1) \cup \ldots \cup h_{a(n)}(V_n)$ is a quasi-compact subset of $f^{-1}(V)$. Hence we can find a quasi-compact open $W \subset f^{-1}(V)$ which contains $h_{a(i)}(V_i)$ for $i = 1, \ldots, n$. In particular $V = f(W)$.

On the one hand this shows that the cardinality of the set of affine opens of Y is at most the cardinality of the set S of quasi-compact opens of X. Since every quasi-compact open of X is a finite union of affines, we see that the cardinality of this set is at most $\sup |S|^n = \max(\aleph_0, |S|)$. On the other hand, we have $\mathcal{O}_Y(V) \subset \prod_{i=1,\ldots,n} \mathcal{O}_{Y_{a(i)}}(V_i)$ because $\{V_i \to V\}$ is an fpqc covering. Hence $\mathcal{O}_Y(V) \subset \mathcal{O}_X(W)$ because $V_i \to V$ factors through W. Again since W has a finite covering by affine opens of X we conclude that $|\mathcal{O}_Y(V)|$ is bounded by the size of X. The lemma now follows from the definition of the size of a scheme. $\square$

In the following lemma we use the notion of an fppf covering which is introduced in Topologies, Section 34.7.

0AHL Lemma 3.9.12. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fppf covering of a scheme. There exists an fppf covering $\{W_j \to X\}_{j \in J}$ which is a refinement of $\{X_i \to X\}_{i \in I}$ such that $\text{size}(\coprod W_j) \leq \text{size}(X)$.

Proof. Choose an affine open covering $X = \bigcup_{a \in A} U_a$ with $|A| \leq \text{size}(X)$. For each a we can choose a finite subset $I_a \subset I$ and for $i \in I_a$ a quasi-compact open $W_{a,i} \subset X_i$ such that $U_a = \bigcup_{i \in I_a} f_i(W_{a,i})$. Then $\text{size}(W_{a,i}) \leq \text{size}(X)$ by Lemma 3.9.7. We conclude that $\text{size}(\coprod_a \coprod_{i \in I_a} W_{i,a}) \leq \text{size}(X)$ by Lemma 3.9.5. $\square$

3.10. Sets with group action

000T Let G be a group. We denote G-Sets the "big" category of G-sets. For any ordinal α, we denote $G\text{-Sets}_\alpha$ the full subcategory of G-Sets whose objects are in V_α. As a notion for size of a G-set we take $\text{size}(S) = \max\{\aleph_0, |G|, |S|\}$ (where $|G|$ and $|S|$ are the cardinality of the underlying sets). As above we use the function $Bound(\kappa) = \kappa^{\aleph_0}$.

000U Lemma 3.10.1. With notations G, $G\text{-Sets}_\alpha$, size, and *Bound* as above. Let S_0 be a set of G-sets. There exists a limit ordinal α with the following properties:

(1) We have $S_0 \cup \{_GG\} \subset \text{Ob}(G\text{-Sets}_\alpha)$.

(2) For any $S \in \mathrm{Ob}(G\text{-Sets}_\alpha)$ and any G-set T with $\text{size}(T) \leq Bound(\text{size}(S))$, there exists an $S' \in \mathrm{Ob}(G\text{-Sets}_\alpha)$ that is isomorphic to T.
(3) For any countable index category $\mathcal{I}$ and any functor $F : \mathcal{I} \to G\text{-Sets}_\alpha$, the limit $\lim_\mathcal{I} F$ and colimit $\mathrm{colim}_\mathcal{I} F$ exist in $G\text{-Sets}_\alpha$ and are the same as in G-Sets.

Proof. Omitted. Similar to but easier than the proof of Lemma 3.9.2 above. $\square$

000V Lemma 3.10.2. Let α be an ordinal as in Lemma 3.10.1 above. The category $G\text{-Sets}_\alpha$ satisfies the following properties:

(1) The G-set ${}_GG$ is an object of $G\text{-Sets}_\alpha$.
(2) (Co)Products, fibre products, and pushouts exist in $G\text{-Sets}_\alpha$ and are the same as their counterparts in G-Sets.
(3) Given an object U of $G\text{-Sets}_\alpha$, any G-stable subset $O \subset U$ is isomorphic to an object of $G\text{-Sets}_\alpha$.

Proof. Omitted. $\square$

3.11. Coverings of a site

000W Suppose that $\mathcal{C}$ is a category (as in Categories, Definition 4.2.1) and that $\mathrm{Cov}(\mathcal{C})$ is a proper class of coverings satisfying properties (1), (2), and (3) of Sites, Definition 7.6.2. We list them here:

(1) If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

For an ordinal α, we set $\mathrm{Cov}(\mathcal{C})_\alpha = \mathrm{Cov}(\mathcal{C}) \cap V_\alpha$. Given an ordinal α and a cardinal κ, we set $\mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$ equal to the set of elements $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})_\alpha$ such that $|I| \leq \kappa$.

We recall the following notion, see Sites, Definition 7.8.2. Two families of morphisms, $\{\varphi_i : U_i \to U\}_{i\in I}$ and $\{\psi_j : W_j \to U\}_{j\in J}$, with the same target of $\mathcal{C}$ are called combinatorially equivalent if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$. This defines an equivalence relation on families of morphisms having a fixed target.

000X Lemma 3.11.1. With notations as above. Let $\mathrm{Cov}_0 \subset \mathrm{Cov}(\mathcal{C})$ be a set contained in $\mathrm{Cov}(\mathcal{C})$. There exist a cardinal κ and a limit ordinal α with the following properties:

(1) We have $\mathrm{Cov}_0 \subset \mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$.
(2) The set of coverings $\mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$ satisfies (1), (2), and (3) of Sites, Definition 7.6.2 (see above). In other words $(\mathcal{C}, \mathrm{Cov}(\mathcal{C})_{\kappa,\alpha})$ is a site.
(3) Every covering in $\mathrm{Cov}(\mathcal{C})$ is combinatorially equivalent to a covering in $\mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$.

Proof. To prove this, we first consider the set $\mathcal{S}$ of all sets of morphisms of $\mathcal{C}$ with fixed target. In other words, an element of $\mathcal{S}$ is a subset T of $\mathrm{Arrows}(\mathcal{C})$ such that all elements of T have the same target. Given a family $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ of morphisms with fixed target, we define $Supp(\mathcal{U}) = \{\varphi \in \mathrm{Arrows}(\mathcal{C}) \mid \exists i \in I, \varphi = \varphi_i\}$. Note that two families $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ are combinatorially equivalent if and only if $Supp(\mathcal{U}) = Supp(\mathcal{V})$. Next, we define

$\mathcal{S}_\tau \subset \mathcal{S}$ to be the subset $\mathcal{S}_\tau = \{T \in \mathcal{S} \mid \exists\, \mathcal{U} \in \text{Cov}(\mathcal{C})\ T = Supp(\mathcal{U})\}$. For every element $T \in \mathcal{S}_\tau$, set $\beta(T)$ to equal the least ordinal β such that there exists a $\mathcal{U} \in \text{Cov}(\mathcal{C})_\beta$ such that $T = \text{Supp}(\mathcal{U})$. Finally, set $\beta_0 = \sup_{T \in S_\tau} \beta(T)$. At this point it follows that every $\mathcal{U} \in \text{Cov}(\mathcal{C})$ is combinatorially equivalent to some element of $\text{Cov}(\mathcal{C})_{\beta_0}$.

Let κ be the maximum of $\aleph_0$, the cardinality $|\text{Arrows}(\mathcal{C})|$,

$$\sup_{\{U_i \to U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\beta_0}} |I|, \quad \text{and} \quad \sup_{\{U_i \to U\}_{i \in I} \in \text{Cov}_0} |I|.$$

Since κ is an infinite cardinal, we have $\kappa \otimes \kappa = \kappa$. Note that obviously $\text{Cov}(\mathcal{C})_{\beta_0} = \text{Cov}(\mathcal{C})_{\kappa, \beta_0}$.

We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$, we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

(1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
(2) If $\{U_i \to U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$ and for each i we have $\{W_{ij} \to U_i\}_{j \in J_i} \in \text{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$, then $\{W_{ij} \to U\}_{i \in I, j \in J_i} \in \text{Cov}(\mathcal{C})_{\kappa, \beta}$.
(3) If $\{U_i \to U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, \alpha}$ and $W \to U$ is a morphism of $\mathcal{C}$, then $\{U_i \times_U W \to W\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, \beta}$.

To see β exists we note that clearly the collection of all coverings $\{W_{ij} \to U\}$ and $\{U_i \times_U W \to W\}$ that occur in (2) and (3) form a set. Hence there is some ordinal β such that V_β contains all of these coverings. Moreover, the index set of the covering $\{W_{ij} \to U\}$ has cardinality $\sum_{i \in I} |J_i| \leq \kappa \otimes \kappa = \kappa$, and hence these coverings are contained in $\text{Cov}(\mathcal{C})_{\kappa, \beta}$. Since every nonempty collection of ordinals has a least element we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha' < \alpha} f(\alpha')$.

Pick an ordinal β_1 such that $\text{Arrows}(\mathcal{C}) \subset V_{\beta_1}$, $\text{Cov}_0 \subset V_{\beta_0}$, and $\beta_1 \geq \beta_0$. By construction $f(\beta_1) \geq \beta_1$ and we see that the same properties hold for $V_{f(\beta_1)}$. Moreover, as f is nondecreasing this remains true for any $\beta \geq \beta_1$. Next, choose any ordinal $\beta_2 > \beta_1$ with cofinality $\text{cf}(\beta_2) > \kappa$. This is possible since the cofinality of ordinals gets arbitrarily large, see Proposition 3.7.2. We claim that the pair κ, $\alpha = f(\beta_2)$ is a solution to the problem posed in the lemma.

The first and third property of the lemma holds by our choices of κ, $\beta_2 > \beta_1 > \beta_0$ above.

Since β_2 is a limit ordinal (as its cofinality is infinite) we get $f(\beta_2) = \sup_{\beta < \beta_2} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_2\} \subset f(\beta_2)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_2)} = \bigcup\nolimits_{\beta < \beta_2} V_{f(\beta)}.$$

Now, let $\mathcal{U} \in \text{Cov}_{\kappa, \alpha}$. We define $\beta(\mathcal{U})$ to be the least ordinal β such that $\mathcal{U} \in \text{Cov}_{\kappa, f(\beta)}$. By the above we see that always $\beta(\mathcal{U}) < \beta_2$.

We have to show properties (1), (2), and (3) defining a site hold for the pair $(\mathcal{C}, \text{Cov}_{\kappa, \alpha})$. The first holds because by our choice of β_2 all arrows of $\mathcal{C}$ are contained in $V_{f(\beta_2)}$. For the third, we use that given a covering $\mathcal{U} = \{U_i \to U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, \alpha}$ we have $\beta(\mathcal{U}) < \beta_2$ and hence any base change of $\mathcal{U}$ is by construction of f contained in $\text{Cov}(\mathcal{C})_{\kappa, f(\beta+1)}$ and hence in $\text{Cov}(\mathcal{C})_{\kappa, \alpha}$.

Finally, for the second condition, suppose that $\{U_i \to U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$ and for each i we have $\mathcal{W}_i = \{W_{ij} \to U_i\}_{j \in J_i} \in \text{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$. Consider the function $I \to \beta_2$, $i \mapsto \beta(\mathcal{W}_i)$. Since the cofinality of β_2 is $> \kappa \geq |I|$ the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\mathcal{W}_i \in \text{Cov}_{\kappa, f(\beta)}$ for all $i \in I$. It follows that the covering $\{W_{ij} \to U\}_{i \in I, j \in J_i}$ is an element of $\text{Cov}(\mathcal{C})_{\kappa, f(\beta+1)} \subset \text{Cov}(\mathcal{C})_{\kappa, \alpha}$ as desired. □

000Y Remark 3.11.2. It is likely the case that, for some limit ordinal α, the set of coverings $\text{Cov}(\mathcal{C})_\alpha$ satisfies the conditions of the lemma. This is after all what an application of the reflection principle would appear to give (modulo caveats as described at the end of Section 3.8 and in Remark 3.9.3).

3.12. Abelian categories and injectives

000Z The following lemma applies to the category of modules over a sheaf of rings on a site.

0010 Lemma 3.12.1. Suppose given a big category $\mathcal{A}$ (see Categories, Remark 4.2.2). Assume $\mathcal{A}$ is abelian and has enough injectives. See Homology, Definitions 12.5.1 and 12.27.4. Then for any given set of objects $\{A_s\}_{s \in S}$ of $\mathcal{A}$, there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ with the following properties:

(1) $\text{Ob}(\mathcal{A}')$ is a set,
(2) $\text{Ob}(\mathcal{A}')$ contains A_s for each $s \in S$,
(3) $\mathcal{A}'$ has enough injectives, and
(4) an object of $\mathcal{A}'$ is injective if and only if it is an injective object of $\mathcal{A}$.

Proof. Omitted. □

3.13. Other chapters

CHAPTER 4

Categories

0011 ## 4.1. Introduction

0012 Categories were first introduced in [EM45]. The category of categories (which is a proper class) is a 2-category. Similarly, the category of stacks forms a 2-category. If you already know about categories, but not about 2-categories you should read Section 4.28 as an introduction to the formal definitions later on.

4.2. Definitions

0013 We recall the definitions, partly to fix notation.

0014 Definition 4.2.1. A category $\mathcal{C}$ consists of the following data:

(1) A set of objects $\text{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \text{Ob}(\mathcal{C})$ a set of morphisms $\text{Mor}_\mathcal{C}(x, y)$.
(3) For each triple $x, y, z \in \text{Ob}(\mathcal{C})$ a composition map $\text{Mor}_\mathcal{C}(y, z) \times \text{Mor}_\mathcal{C}(x, y) \to \text{Mor}_\mathcal{C}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in \text{Ob}(\mathcal{C})$ there exists a morphism $\text{id}_x \in \text{Mor}_\mathcal{C}(x, x)$ such that $\text{id}_x \circ \phi = \phi$ and $\psi \circ \text{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is associative, i.e., $(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi)$ whenever these compositions make sense.

It is customary to require all the morphism sets $\text{Mor}_\mathcal{C}(x, y)$ to be disjoint. In this way a morphism $\phi : x \to y$ has a unique source x and a unique target y. This is not strictly necessary, although care has to be taken in formulating condition (2) above if it is not the case. It is convenient and we will often assume this is the case. In this case we say that ϕ and ψ are composable if the source of ϕ is equal to the target of ψ, in which case $\phi \circ \psi$ is defined. An equivalent definition would be to define a category as a quintuple $(\text{Ob}, \text{Arrows}, s, t, \circ)$ consisting of a set of objects, a set of morphisms (arrows), source, target and composition subject to a long list of axioms. We will occasionally use this point of view.

0015 Remark 4.2.2. Big categories. In some texts a category is allowed to have a proper class of objects. We will allow this as well in these notes but only in the following list of cases (to be updated as we go along). In particular, when we say: "Let $\mathcal{C}$ be a category" then it is understood that $\text{Ob}(\mathcal{C})$ is a set.

(1) The category Sets of sets.
(2) The category Ab of abelian groups.
(3) The category Groups of groups.
(4) Given a group G the category G-Sets of sets with a left G-action.

(5) Given a ring R the category Mod_R of R-modules.
(6) Given a field k the category of vector spaces over k.
(7) The category of rings.
(8) The category of divided power rings, see Divided Power Algebra, Section 23.3.
(9) The category of schemes.
(10) The category Top of topological spaces.
(11) Given a topological space X the category $\text{PSh}(X)$ of presheaves of sets over X.
(12) Given a topological space X the category $\mathit{Sh}(X)$ of sheaves of sets over X.
(13) Given a topological space X the category $\text{PAb}(X)$ of presheaves of abelian groups over X.
(14) Given a topological space X the category $\text{Ab}(X)$ of sheaves of abelian groups over X.
(15) Given a small category $\mathcal{C}$ the category of functors from $\mathcal{C}$ to Sets.
(16) Given a category $\mathcal{C}$ the category of presheaves of sets over $\mathcal{C}$.
(17) Given a site $\mathcal{C}$ the category of sheaves of sets over $\mathcal{C}$.

One of the reason to enumerate these here is to try and avoid working with something like the "collection" of "big" categories which would be like working with the collection of all classes which I think definitively is a meta-mathematical object.

0016 Remark 4.2.3. It follows directly from the definition that any two identity morphisms of an object x of $\mathcal{A}$ are the same. Thus we may and will speak of the identity morphism id_x of x.

0017 Definition 4.2.4. A morphism $\phi : x \to y$ is an isomorphism of the category $\mathcal{C}$ if there exists a morphism $\psi : y \to x$ such that $\phi \circ \psi = \text{id}_y$ and $\psi \circ \phi = \text{id}_x$.

An isomorphism ϕ is also sometimes called an invertible morphism, and the morphism ψ of the definition is called the inverse and denoted ϕ^{-1}. It is unique if it exists. Note that given an object x of a category $\mathcal{A}$ the set of invertible elements $\text{Aut}_\mathcal{A}(x)$ of $\text{Mor}_\mathcal{A}(x, x)$ forms a group under composition. This group is called the automorphism group of x in $\mathcal{A}$.

0018 Definition 4.2.5. A groupoid is a category where every morphism is an isomorphism.

0019 Example 4.2.6. A group G gives rise to a groupoid with a single object x and morphisms $\text{Mor}(x, x) = G$, with the composition rule given by the group law in G. Every groupoid with a single object is of this form.

001A Example 4.2.7. A set C gives rise to a groupoid $\mathcal{C}$ defined as follows: As objects we take $\text{Ob}(\mathcal{C}) := C$ and for morphisms we take $\text{Mor}(x, y)$ empty if $x \neq y$ and equal to $\{\text{id}_x\}$ if $x = y$.

001B Definition 4.2.8. A functor $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : \text{Ob}(\mathcal{A}) \to \text{Ob}(\mathcal{B})$.
(2) For every $x, y \in \text{Ob}(\mathcal{A})$ a map $F : \text{Mor}_\mathcal{A}(x, y) \to \text{Mor}_\mathcal{B}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in the following manner: $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for a composable pair (ϕ, ψ) of morphisms of $\mathcal{A}$ and $F(\text{id}_x) = \text{id}_{F(x)}$.

Note that every category $\mathcal{A}$ has an identity functor $\text{id}_{\mathcal{A}}$. In addition, given a functor $G : \mathcal{B} \to \mathcal{C}$ and a functor $F : \mathcal{A} \to \mathcal{B}$ there is a composition functor $G \circ F : \mathcal{A} \to \mathcal{C}$ defined in an obvious manner.

001C Definition 4.2.9. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) We say F is faithful if for any objects $x, y \in \text{Ob}(\mathcal{A})$ the map
$$F : \text{Mor}_{\mathcal{A}}(x, y) \to \text{Mor}_{\mathcal{B}}(F(x), F(y))$$
is injective.
(2) If these maps are all bijective then F is called fully faithful.
(3) The functor F is called essentially surjective if for any object $y \in \text{Ob}(\mathcal{B})$ there exists an object $x \in \text{Ob}(\mathcal{A})$ such that $F(x)$ is isomorphic to y in $\mathcal{B}$.

001D Definition 4.2.10. A subcategory of a category $\mathcal{B}$ is a category $\mathcal{A}$ whose objects and arrows form subsets of the objects and arrows of $\mathcal{B}$ and such that source, target and composition in $\mathcal{A}$ agree with those of $\mathcal{B}$ and such that the identity morphism of an object of $\mathcal{A}$ matches the one in $\mathcal{B}$. We say $\mathcal{A}$ is a full subcategory of $\mathcal{B}$ if $\text{Mor}_{\mathcal{A}}(x, y) = \text{Mor}_{\mathcal{B}}(x, y)$ for all $x, y \in \text{Ob}(\mathcal{A})$. We say $\mathcal{A}$ is a strictly full subcategory of $\mathcal{B}$ if it is a full subcategory and given $x \in \text{Ob}(\mathcal{A})$ any object of $\mathcal{B}$ which is isomorphic to x is also in $\mathcal{A}$.

If $\mathcal{A} \subset \mathcal{B}$ is a subcategory then the identity map is a functor from $\mathcal{A}$ to $\mathcal{B}$. Furthermore a subcategory $\mathcal{A} \subset \mathcal{B}$ is full if and only if the inclusion functor is fully faithful. Note that given a category $\mathcal{B}$ the set of full subcategories of $\mathcal{B}$ is the same as the set of subsets of $\text{Ob}(\mathcal{B})$.

001E Remark 4.2.11. Suppose that $\mathcal{A}$ is a category. A functor F from $\mathcal{A}$ to Sets is a mathematical object (i.e., it is a set not a class or a formula of set theory, see Sets, Section 3.2) even though the category of sets is "big". Namely, the range of F on objects will be a set $F(\text{Ob}(\mathcal{A}))$ and then we may think of F as a functor between $\mathcal{A}$ and the full subcategory of the category of sets whose objects are elements of $F(\text{Ob}(\mathcal{A}))$.

001F Example 4.2.12. A homomorphism $p : G \to H$ of groups gives rise to a functor between the associated groupoids in Example 4.2.6. It is faithful (resp. fully faithful) if and only if p is injective (resp. an isomorphism).

001G Example 4.2.13. Given a category $\mathcal{C}$ and an object $X \in \text{Ob}(\mathcal{C})$ we define the category of objects over X, denoted $\mathcal{C}/X$ as follows. The objects of $\mathcal{C}/X$ are morphisms $Y \to X$ for some $Y \in \text{Ob}(\mathcal{C})$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : \mathcal{C}/X \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : \mathcal{C}/X' \to \mathcal{C}/X$ obtained by composition with f, and $p_X \circ F = p_{X'}$.

001H Example 4.2.14. Given a category $\mathcal{C}$ and an object $X \in \text{Ob}(\mathcal{C})$ we define the category of objects under X, denoted $X/\mathcal{C}$ as follows. The objects of $X/\mathcal{C}$ are morphisms $X \to Y$ for some $Y \in \text{Ob}(\mathcal{C})$. Morphisms between objects $X \to Y$ and $X \to Y'$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : X/\mathcal{C} \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : X/\mathcal{C} \to X'/\mathcal{C}$ obtained by composition with f, and $p_{X'} \circ F = p_X$.

001I Definition 4.2.15. Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A natural transformation, or a morphism of functors $t : F \to G$, is a collection $\{t_x\}_{x \in \mathrm{Ob}(\mathcal{A})}$ such that

(1) $t_x : F(x) \to G(x)$ is a morphism in the category $\mathcal{B}$, and
(2) for every morphism $\phi : x \to y$ of $\mathcal{A}$ the following diagram is commutative

$$\begin{tikzcd} F(x) \arrow[r, "t_x"] \arrow[d, "F(\phi)"'] & G(x) \arrow[d, "G(\phi)"] \\ F(y) \arrow[r, "t_y"] & G(y) \end{tikzcd}$$

Sometimes we use the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow t"] & \mathcal{B} \end{tikzcd}$$

to indicate that t is a morphism from F to G.

Note that every functor F comes with the identity transformation $\mathrm{id}_F : F \to F$. In addition, given a morphism of functors $t : F \to G$ and a morphism of functors $s : E \to F$ then the composition $t \circ s$ is defined by the rule

$$(t \circ s)_x = t_x \circ s_x : E(x) \to G(x)$$

for $x \in \mathrm{Ob}(\mathcal{A})$. It is easy to verify that this is indeed a morphism of functors from E to G. In this way, given categories $\mathcal{A}$ and $\mathcal{B}$ we obtain a new category, namely the category of functors between $\mathcal{A}$ and $\mathcal{B}$.

02C2 Remark 4.2.16. This is one instance where the same thing does not hold if $\mathcal{A}$ is a "big" category. For example consider functors Sets $\to$ Sets. As we have currently defined it such a functor is a class and not a set. In other words, it is given by a formula in set theory (with some variables equal to specified sets)! It is not a good idea to try to consider all possible formulae of set theory as part of the definition of a mathematical object. The same problem presents itself when considering sheaves on the category of schemes for example. We will come back to this point later.

001J Definition 4.2.17. An equivalence of categories $F : \mathcal{A} \to \mathcal{B}$ is a functor such that there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that the compositions $F \circ G$ and $G \circ F$ are isomorphic to the identity functors $\mathrm{id}_\mathcal{B}$, respectively $\mathrm{id}_\mathcal{A}$. In this case we say that G is a quasi-inverse to F.

05SG Lemma 4.2.18. Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor. Suppose for every $X \in \mathrm{Ob}(\mathcal{B})$ we are given an object $j(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then there is a unique functor $j : \mathcal{B} \to \mathcal{A}$ such that j extends the rule on objects, and the isomorphisms i_X define an isomorphism of functors $\mathrm{id}_\mathcal{B} \to F \circ j$. Moreover, j and F are quasi-inverse equivalences of categories.

Proof. To construct $j : \mathcal{B} \to \mathcal{A}$, there are two steps. Firstly, we define the map $j : \mathrm{Ob}(\mathcal{B}) \to \mathrm{Ob}(\mathcal{A})$ that associates $j(X)$ to $X \in \mathcal{B}$. Secondly, if $X, Y \in \mathrm{Ob}(\mathcal{B})$ and $\phi : X \to Y$, we consider $\phi' := i_Y \circ \phi \circ i_X^{-1}$. There is an unique φ verifying $F(\varphi) = \phi'$, using that F is fully faithful. We define $j(\phi) = \varphi$. We omit the verification that j

is a functor. By construction the diagram

$$\begin{tikzcd} X \arrow[r, "i_X"'] \arrow[d, "\phi"'] & F(j(X)) \arrow[d, "F\circ j(\phi)"] \\ Y \arrow[r, "i_Y"] & F(j(Y)) \end{tikzcd}$$

commutes. Hence, as each i_X is an isomorphism, $\{i_X\}_X$ is an isomorphism of functors $\mathrm{id}_\mathcal{B} \to F \circ j$. To conclude, we have to also prove that $j \circ F$ is isomorphic to $\mathrm{id}_\mathcal{A}$. However, since F is fully faithful, in order to do this it suffices to prove this after post-composing with F, i.e., it suffices to show that $F \circ j \circ F$ is isomorphic to $F \circ \mathrm{id}_\mathcal{A}$ (small detail omitted). Since $F \circ j \cong \mathrm{id}_\mathcal{B}$ this is clear. □

02C3 Lemma 4.2.19. A functor is an equivalence of categories if and only if it is both fully faithful and essentially surjective.

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be essentially surjective and fully faithful. As by convention all categories are small and as F is essentially surjective we can, using the axiom of choice, choose for every $X \in \mathrm{Ob}(\mathcal{B})$ an object $j(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then we apply Lemma 4.2.18 using that F is fully faithful. □

001K Definition 4.2.20. Let $\mathcal{A}$, $\mathcal{B}$ be categories. We define the product category $\mathcal{A} \times \mathcal{B}$ to be the category with objects $\mathrm{Ob}(\mathcal{A} \times \mathcal{B}) = \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ and

$$\mathrm{Mor}_{\mathcal{A}\times\mathcal{B}}((x, y), (x', y')) := \mathrm{Mor}_\mathcal{A}(x, x') \times \mathrm{Mor}_\mathcal{B}(y, y').$$

Composition is defined componentwise.

4.3. Opposite Categories and the Yoneda Lemma

001L

001M Definition 4.3.1. Given a category $\mathcal{C}$ the opposite category $\mathcal{C}^{opp}$ is the category with the same objects as $\mathcal{C}$ but all morphisms reversed.

In other words $\mathrm{Mor}_{\mathcal{C}^{opp}}(x, y) = \mathrm{Mor}_\mathcal{C}(y, x)$. Composition in $\mathcal{C}^{opp}$ is the same as in $\mathcal{C}$ except backwards: if $\phi : y \to z$ and $\psi : x \to y$ are morphisms in $\mathcal{C}^{opp}$, in other words arrows $z \to y$ and $y \to x$ in $\mathcal{C}$, then $\phi \circ^{opp} \psi$ is the morphism $x \to z$ of $\mathcal{C}^{opp}$ which corresponds to the composition $z \to y \to x$ in $\mathcal{C}$.

001N Definition 4.3.2. Let $\mathcal{C}$, $\mathcal{S}$ be categories. A contravariant functor F from $\mathcal{C}$ to $\mathcal{S}$ is a functor $\mathcal{C}^{opp} \to \mathcal{S}$.

Concretely, a contravariant functor F is given by a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{S})$ and for every morphism $\psi : x \to y$ in $\mathcal{C}$ a morphism $F(\psi) : F(y) \to F(x)$. These should satisfy the property that, given another morphism $\phi : y \to z$, we have $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$ as morphisms $F(z) \to F(x)$. (Note the reverse of order.)

02X6 Definition 4.3.3. Let $\mathcal{C}$ be a category.

1. A presheaf of sets on $\mathcal{C}$ or simply a presheaf is a contravariant functor F from $\mathcal{C}$ to Sets.
2. The category of presheaves is denoted $\mathrm{PSh}(\mathcal{C})$.

Of course the category of presheaves is a proper class.

001O Example 4.3.4. Functor of points. For any $U \in \mathrm{Ob}(\mathcal{C})$ there is a contravariant functor

$$\begin{matrix} h_U & : & \mathcal{C} & \longrightarrow & \mathrm{Sets} \\ & & X & \longmapsto & \mathrm{Mor}_\mathcal{C}(X, U) \end{matrix}$$

which takes an object X to the set $\mathrm{Mor}_\mathcal{C}(X, U)$. In other words h_U is a presheaf. Given a morphism $f : X \to Y$ the corresponding map $h_U(f) : \mathrm{Mor}_\mathcal{C}(Y, U) \to \mathrm{Mor}_\mathcal{C}(X, U)$ takes ϕ to $\phi \circ f$. We will always denote this presheaf $h_U : \mathcal{C}^{opp} \to \mathrm{Sets}$. It is called the representable presheaf associated to U. If $\mathcal{C}$ is the category of schemes this functor is sometimes referred to as the functor of points of U.

Note that given a morphism $\phi : U \to V$ in $\mathcal{C}$ we get a corresponding natural transformation of functors $h(\phi) : h_U \to h_V$ defined by composing with the morphism $U \to V$. This turns composition of morphisms in $\mathcal{C}$ into composition of transformations of functors. In other words we get a functor

$$h : \mathcal{C} \longrightarrow \mathrm{PSh}(\mathcal{C})$$

Note that the target is a "big" category, see Remark 4.2.2. On the other hand, h is an actual mathematical object (i.e. a set), compare Remark 4.2.11.

001P Lemma 4.3.5 (Yoneda lemma). Let $U, V \in \mathrm{Ob}(\mathcal{C})$. Given any morphism of functors $s : h_U \to h_V$ there is a unique morphism $\phi : U \to V$ such that $h(\phi) = s$. In other words the functor h is fully faithful. More generally, given any contravariant functor F and any object U of $\mathcal{C}$ we have a natural bijection

Appeared in some form in [Yon54]. Used by Grothendieck in a generalized form in [Gro95b].

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, F) \longrightarrow F(U), \quad s \longmapsto s_U(\mathrm{id}_U).$$

Proof. For the first statement, just take $\phi = s_U(\mathrm{id}_U) \in \mathrm{Mor}_\mathcal{C}(U, V)$. For the second statement, given $\xi \in F(U)$ define s by $s_V : h_U(V) \to F(V)$ by sending the element $f : V \to U$ of $h_U(V) = \mathrm{Mor}_\mathcal{C}(V, U)$ to $F(f)(\xi)$. $\square$

001Q Definition 4.3.6. A contravariant functor $F : \mathcal{C} \to \mathrm{Sets}$ is said to be representable if it is isomorphic to the functor of points h_U for some object U of $\mathcal{C}$.

Let $\mathcal{C}$ be a category and let $F : \mathcal{C}^{opp} \to \mathrm{Sets}$ be a representable functor. Choose an object U of $\mathcal{C}$ and an isomorphism $s : h_U \to F$. The Yoneda lemma guarantees that the pair (U, s) is unique up to unique isomorphism. The object U is called an object representing F. By the Yoneda lemma the transformation s corresponds to a unique element $\xi \in F(U)$. This element is called the universal object. It has the property that for $V \in \mathrm{Ob}(\mathcal{C})$ the map

$$\mathrm{Mor}_\mathcal{C}(V, U) \longrightarrow F(V), \quad (f : V \to U) \longmapsto F(f)(\xi)$$

is a bijection. Thus ξ is universal in the sense that every element of $F(V)$ is equal to the image of ξ via $F(f)$ for a unique morphism $f : V \to U$ in $\mathcal{C}$.

4.4. Products of pairs

001R

001S Definition 4.4.1. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A product of x and y is an object $x \times y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times y, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times y, y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms

$\alpha \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(w, y)$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times y)$ making the diagram

$$\begin{tikzcd}
w \arrow[rd, dashed, "\gamma"] \arrow[rrd, "\beta"] \arrow[rdd, "\alpha"'] & & \\
& x \times y \arrow[r, "q"'] \arrow[d, "p"] & y \\
& x &
\end{tikzcd}$$

commute.

If a product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times y$ to be an object of $\mathcal{C}$ such that

$$h_{x \times y}(w) = h_x(w) \times h_y(w)$$

functorially in w. In other words the product $x \times y$ is an object representing the functor $w \mapsto h_x(w) \times h_y(w)$.

001T Definition 4.4.2. We say the category $\mathcal{C}$ has products of pairs of objects if a product $x \times y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

We use this terminology to distinguish this notion from the notion of “having products” or “having finite products” which usually means something else (in particular it always implies there exists a final object).

4.5. Coproducts of pairs

04AN

04AO Definition 4.5.1. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A coproduct, or amalgamated sum of x and y is an object $x \amalg y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $i \in \mathrm{Mor}_\mathcal{C}(x, x \amalg y)$ and $j \in \mathrm{Mor}_\mathcal{C}(y, x \amalg y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(x, w)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(y, w)$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(x \amalg y, w)$ making the diagram

$$\begin{tikzcd}
& y \arrow[d, "j"] \arrow[rdd, "\beta"] & \\
x \arrow[r, "i"] \arrow[rrd, "\alpha"'] & x \amalg y \arrow[rd, dashed, "\gamma"] & \\
& & w
\end{tikzcd}$$

commute.

If a coproduct exists it is unique up to unique isomorphism. This follows from the Yoneda lemma (applied to the opposite category) as the definition requires $x \amalg y$ to be an object of $\mathcal{C}$ such that

$$\mathrm{Mor}_\mathcal{C}(x \amalg y, w) = \mathrm{Mor}_\mathcal{C}(x, w) \times \mathrm{Mor}_\mathcal{C}(y, w)$$

functorially in w.

04AP Definition 4.5.2. We say the category $\mathcal{C}$ has coproducts of pairs of objects if a coproduct $x \amalg y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

We use this terminology to distinguish this notion from the notion of "having coproducts" or "having finite coproducts" which usually means something else (in particular it always implies there exists an initial object in $\mathcal{C}$).

4.6. Fibre products

001U

001V Definition 4.6.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_\mathcal{C}(x, y)$ and $g \in \mathrm{Mor}_\mathcal{C}(z, y)$. A fibre product of f and g is an object $x \times_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times_y z, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times_y z, z)$ making the diagram

$$\begin{tikzcd}
x \times_y z \arrow[r, "q"'] \arrow[d, "p"'] & z \arrow[d, "g"] \\
x \arrow[r, "f"] & y
\end{tikzcd}$$

commute, and such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(w, z)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times_y z)$ making the diagram

$$\begin{tikzcd}
w \arrow[rrd, "\beta"] \arrow[rdd, "\alpha"'] \arrow[rd, dashed, "\gamma"'] & & \\
& x \times_y z \arrow[r, "q"'] \arrow[d, "p"] & z \arrow[d, "g"] \\
& x \arrow[r, "f"] & y
\end{tikzcd}$$

commute.

If a fibre product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times_y z$ to be an object of $\mathcal{C}$ such that

$$h_{x \times_y z}(w) = h_x(w) \times_{h_y(w)} h_z(w)$$

functorially in w. In other words the fibre product $x \times_y z$ is an object representing the functor $w \mapsto h_x(w) \times_{h_y(w)} h_z(w)$.

08N0 Definition 4.6.2. We say a commutative diagram

$$\begin{tikzcd}
w \arrow[r] \arrow[d] & z \arrow[d] \\
x \arrow[r] & y
\end{tikzcd}$$

in a category is cartesian if w and the morphisms $w \to x$ and $w \to z$ form a fibre product of the morphisms $x \to y$ and $z \to y$.

001W Definition 4.6.3. We say the category $\mathcal{C}$ has fibre products if the fibre product exists for any $f \in \mathrm{Mor}_\mathcal{C}(x, y)$ and $g \in \mathrm{Mor}_\mathcal{C}(z, y)$.

001X Definition 4.6.4. A morphism $f : x \to y$ of a category $\mathcal{C}$ is said to be representable if for every morphism $z \to y$ in $\mathcal{C}$ the fibre product $x \times_y z$ exists.

001Y Lemma 4.6.5. Let $\mathcal{C}$ be a category. Let $f : x \to y$, and $g : y \to z$ be representable. Then $g \circ f : x \to z$ is representable.

Proof. Let $t \in \mathrm{Ob}(\mathcal{C})$ and $\varphi \in \mathrm{Mor}_{\mathcal{C}}(t, z)$. As g and f are representable, we obtain commutative diagrams

$$\begin{tikzcd} y \times_z t \arrow[r, "q"'] \arrow[d, "p"'] & t \arrow[d, "\varphi"] \\ y \arrow[r, "g"] & z \end{tikzcd} \qquad \begin{tikzcd} x \times_y (y \times_z t) \arrow[r, "q'"'] \arrow[d, "p'"'] & y \times_z t \arrow[d, "p"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

with the universal property of Definition 4.6.1. We claim that $x \times_z t = x \times_y (y \times_z t)$ with morphisms $q \circ q' : x \times_z t \to t$ and $p' : x \times_z t \to x$ is a fibre product. First, it follows from the commutativity of the diagrams above that $\varphi \circ q \circ q' = f \circ g \circ p'$. To verify the universal property, let $w \in \mathrm{Ob}(\mathcal{C})$ and suppose $\alpha : w \to x$ and $\beta : w \to y$ are morphisms with $\varphi \circ \beta = f \circ g \circ \alpha$. By definition of the fibre product, there are unique morphisms δ and γ such that

$$\begin{tikzcd} w \arrow[drr, bend left, "\beta"] \arrow[dr, dashed, "\delta"] \arrow[ddr, bend right, "f \circ \alpha"'] & & \\ & y \times_z t \arrow[r, "q"'] \arrow[d, "p"'] & t \arrow[d, "\varphi"] \\ & y \arrow[r, "g"] & z \end{tikzcd}$$

and

$$\begin{tikzcd} w \arrow[drr, bend left, "\delta"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, bend right, "\alpha"'] & & \\ & x \times_y (y \times_z t) \arrow[r, "q'"'] \arrow[d, "p'"'] & y \times_z t \arrow[d, "p"] \\ & x \arrow[r, "f"] & y \end{tikzcd}$$

commute. Then, γ makes the diagram

$$\begin{tikzcd} w \arrow[drr, bend left, "\beta"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, bend right, "\alpha"'] & & \\ & x \times_z t \arrow[r, "q \circ q'"'] \arrow[d, "p'"'] & t \arrow[d, "\varphi"] \\ & x \arrow[r, "g \circ f"] & z \end{tikzcd}$$

commute. To show its uniqueness, let γ' verify $q \circ q' \circ \gamma' = \beta$ and $p' \circ \gamma' = \alpha$. Because γ is unique, we just need to prove that $q' \circ \gamma' = \delta$ and $p' \circ \gamma' = \alpha$ to conclude. We supposed the second equality. For the first one, we also need to use the uniqueness of delta. Notice that δ is the only morphism verifying $q \circ \delta = \beta$ and $p \circ \delta = f \circ \alpha$. We already supposed that $q \circ (q' \circ \gamma') = \beta$. Furthermore, by definition of the fibre product, we know that $f \circ p' = p \circ q'$. Therefore:

$$p \circ (q' \circ \gamma') = (p \circ q') \circ \gamma' = (f \circ p') \circ \gamma' = f \circ (p' \circ \gamma') = f \circ \alpha.$$

Then $q' \circ \gamma' = \delta$, which concludes the proof. $\square$

001Z Lemma 4.6.6. Let $\mathcal{C}$ be a category. Let $f : x \to y$ be representable. Let $y' \to y$ be a morphism of $\mathcal{C}$. Then the morphism $x' := x \times_y y' \to y'$ is representable also.

Proof. Let $z \to y'$ be a morphism. The fibre product $x' \times_{y'} z$ is supposed to represent the functor

$$\begin{array}{rcl} w & \mapsto & h_{x'}(w) \times_{h_{y'}(w)} h_z(w) \\ & = & (h_x(w) \times_{h_y(w)} h_{y'}(w)) \times_{h_{y'}(w)} h_z(w) \\ & = & h_x(w) \times_{h_y(w)} h_z(w) \end{array}$$

which is representable by assumption. □

4.7. Examples of fibre products

0020 In this section we list examples of fibre products and we describe them.

As a really trivial first example we observe that the category of sets has fibre products and hence every morphism is representable. Namely, if $f : X \to Y$ and $g : Z \to Y$ are maps of sets then we define $X \times_Y Z$ as the subset of $X \times Z$ consisting of pairs (x, z) such that $f(x) = g(z)$. The morphisms $p : X \times_Y Z \to X$ and $q : X \times_Y Z \to Z$ are the projection maps $(x, z) \mapsto x$, and $(x, z) \mapsto z$. Finally, if $\alpha : W \to X$ and $\beta : W \to Z$ are morphisms such that $f \circ \alpha = g \circ \beta$ then the map $W \to X \times Z$, $w \mapsto (\alpha(w), \beta(w))$ obviously ends up in $X \times_Y Z$ as desired.

In many categories whose objects are sets endowed with certain types of algebraic structures the fibre product of the underlying sets also provides the fibre product in the category. For example, suppose that X, Y and Z above are groups and that f, g are homomorphisms of groups. Then the set-theoretic fibre product $X \times_Y Z$ inherits the structure of a group, simply by defining the product of two pairs by the formula $(x, z) \cdot (x', z') = (xx', zz')$. Here we list those categories for which a similar reasoning works.

(1) The category Groups of groups.
(2) The category G-Sets of sets endowed with a left G-action for some fixed group G.
(3) The category of rings.
(4) The category of R-modules given a ring R.

4.8. Fibre products and representability

0021 In this section we work out fibre products in the category of contravariant functors from a category to the category of sets. This will later be superseded during the discussion of sites, presheaves, sheaves. Of some interest is the notion of a "representable morphism" between such functors.

0022 Lemma 4.8.1. Let $\mathcal{C}$ be a category. Let $F, G, H : \mathcal{C}^{opp} \to \textit{Sets}$ be functors. Let $a : F \to G$ and $b : H \to G$ be transformations of functors. Then the fibre product $F \times_{a,G,b} H$ in the category $\textit{PSh}(\mathcal{C})$ exists and is given by the formula

$$(F \times_{a,G,b} H)(X) = F(X) \times_{a_X, G(X), b_X} H(X)$$

for any object X of $\mathcal{C}$.

Proof. Omitted. □

As a special case suppose we have a morphism $a : F \to G$, an object $U \in \mathrm{Ob}(\mathcal{C})$ and an element $\xi \in G(U)$. According to the Yoneda Lemma 4.3.5 this gives a transformation $\xi : h_U \to G$. The fibre product in this case is described by the rule

$$(h_U \times_{\xi,G,a} F)(X) = \{(f, \xi') \mid f : X \to U,\ \xi' \in F(X),\ G(f)(\xi) = a_X(\xi')\}$$

If F, G are also representable, then this is the functor representing the fibre product, if it exists, see Section 4.6. The analogy with Definition 4.6.4 prompts us to define a notion of representable transformations.

0023 Definition 4.8.2. Let $\mathcal{C}$ be a category. Let $F, G : \mathcal{C}^{opp} \to \textit{Sets}$ be functors. We say a morphism $a : F \to G$ is representable, or that F is relatively representable over G, if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any $\xi \in G(U)$ the functor $h_U \times_G F$ is representable.

03KC Lemma 4.8.3. Let $\mathcal{C}$ be a category. Let $a : F \to G$ be a morphism of contravariant functors from $\mathcal{C}$ to $\textit{Sets}$. If a is representable, and G is a representable functor, then F is representable.

Proof. Omitted. $\square$

0024 Lemma 4.8.4. Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{opp} \to \textit{Sets}$ be a functor. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:

(1) the diagonal $\Delta : F \to F \times F$ is representable,
(2) for every U in $\mathcal{C}$, and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable,
(3) for every pair U, V in $\mathcal{C}$ and any $\xi \in F(U)$, $\xi' \in F(V)$ the fibre product $h_U \times_{\xi,F,\xi'} h_V$ is representable.

Proof. We will continue to use the Yoneda lemma to identify $F(U)$ with transformations $h_U \to F$ of functors.

Equivalence of (2) and (3). Let U, ξ, V, ξ' be as in (3). Both (2) and (3) tell us exactly that $h_U \times_{\xi,F,\xi'} h_V$ is representable; the only difference is that the statement (3) is symmetric in U and V whereas (2) is not.

Assume condition (1). Let U, ξ, V, ξ' be as in (3). Note that $h_U \times h_V = h_{U\times V}$ is representable. Denote $\eta : h_{U\times V} \to F \times F$ the map corresponding to the product $\xi \times \xi' : h_U \times h_V \to F \times F$. Then the fibre product $F \times_{\Delta, F\times F, \eta} h_{U\times V}$ is representable by assumption. This means there exist $W \in \mathrm{Ob}(\mathcal{C})$, morphisms $W \to U$, $W \to V$ and $h_W \to F$ such that

$$\begin{tikzcd} h_W \arrow[r] \arrow[d] & h_U \times h_V \arrow[d, "\xi\times\xi'"] \\ F \arrow[r] & F \times F \end{tikzcd}$$

is cartesian. Using the explicit description of fibre products in Lemma 4.8.1 the reader sees that this implies that $h_W = h_U \times_{\xi,F,\xi'} h_V$ as desired.

Assume the equivalent conditions (2) and (3). Let U be an object of $\mathcal{C}$ and let $(\xi, \xi') \in (F \times F)(U)$. By (3) the fibre product $h_U \times_{\xi,F,\xi'} h_U$ is representable. Choose an object W and an isomorphism $h_W \to h_U \times_{\xi,F,\xi'} h_U$. The two projections $\mathrm{pr}_i : h_U \times_{\xi,F,\xi'} h_U \to h_U$ correspond to morphisms $p_i : W \to U$ by Yoneda. Consider $W' = W \times_{(p_1,p_2), U\times U} U$. It is formal to show that W' represents $F \times_{\Delta, F\times F} h_U$ because

$$h_{W'} = h_W \times_{h_U \times h_U} h_U = (h_U \times_{\xi,F,\xi'} h_U) \times_{h_U\times h_U} h_U = F \times_{F\times F} h_U.$$

Thus Δ is representable and this finishes the proof. $\square$

4.9. Pushouts

0025 The dual notion to fibre products is that of pushouts.

0026 Definition 4.9.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_\mathcal{C}(y, x)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. A pushout of f and g is an object $x \amalg_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x, x \amalg_y z)$ and $q \in \mathrm{Mor}_\mathcal{C}(z, x \amalg_y z)$ making the diagram

$$\begin{tikzcd}
y \arrow[r, "g"'] \arrow[d, "f"'] & z \arrow[d, "q"] \\
x \arrow[r, "p"] & x \amalg_y z
\end{tikzcd}$$

commute, and such that the following universal property holds: For any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(x, w)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(z, w)$ with $\alpha \circ f = \beta \circ g$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(x \amalg_y z, w)$ making the diagram

$$\begin{tikzcd}
y \arrow[r, "g"'] \arrow[d, "f"'] & z \arrow[d, "q"] \arrow[ddrr, "\beta"] & & \\
x \arrow[r, "p"] \arrow[drrr, "\alpha"'] & x \amalg_y z \arrow[drr, dashed, "\gamma"] & & \\
& & & w
\end{tikzcd}$$

commute.

It is possible and straightforward to prove the uniqueness of the triple $(x \amalg_y z, p, q)$ up to unique isomorphism (if it exists) by direct arguments. Another possibility is to think of the pushout as the fibre product in the opposite category, thereby getting this uniqueness for free from the discussion in Section 4.6.

08N1 Definition 4.9.2. We say a commutative diagram

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & z \arrow[d] \\
x \arrow[r] & w
\end{tikzcd}$$

in a category is cocartesian if w and the morphisms $x \to w$ and $z \to w$ form a pushout of the morphisms $y \to x$ and $y \to z$.

4.10. Equalizers

0027

0028 Definition 4.10.1. Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $e : Z \to X$ is an equalizer for the pair (a, b) if $a \circ e = b \circ e$ and if (Z, e) satisfies the following universal property: For every morphism $t : W \to X$ in $\mathcal{C}$ such that $a \circ t = b \circ t$ there exists a unique morphism $s : W \to Z$ such that $t = e \circ s$.

As in the case of the fibre products above, equalizers when they exist are unique up to unique isomorphism. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

4.11. Coequalizers

0029

002A Definition 4.11.1. Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $c : Y \to Z$ is a coequalizer for the pair (a, b) if $c \circ a = c \circ b$ and if (Z, c) satisfies the following universal property: For every morphism $t : Y \to W$ in $\mathcal{C}$ such that $t \circ a = t \circ b$ there exists a unique morphism $s : Z \to W$ such that $t = s \circ c$.

As in the case of the pushouts above, coequalizers when they exist are unique up to unique isomorphism, and this follows from the uniqueness of equalizers upon considering the opposite category. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

4.12. Initial and final objects

002B

002C Definition 4.12.1. Let $\mathcal{C}$ be a category.

(1) An object x of the category $\mathcal{C}$ is called an initial object if for every object y of $\mathcal{C}$ there is exactly one morphism $x \to y$.
(2) An object x of the category $\mathcal{C}$ is called a final object if for every object y of $\mathcal{C}$ there is exactly one morphism $y \to x$.

In the category of sets the empty set $\emptyset$ is an initial object, and in fact the only initial object. Also, any singleton, i.e., a set with one element, is a final object (so it is not unique).

4.13. Monomorphisms and Epimorphisms

003A

003B Definition 4.13.1. Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathcal{C}$.

(1) We say that f is a monomorphism if for every object W and every pair of morphisms $a, b : W \to X$ such that $f \circ a = f \circ b$ we have $a = b$.
(2) We say that f is an epimorphism if for every object W and every pair of morphisms $a, b : Y \to W$ such that $a \circ f = b \circ f$ we have $a = b$.

003C Example 4.13.2. In the category of sets the monomorphisms correspond to injective maps and the epimorphisms correspond to surjective maps.

08LR Lemma 4.13.3. Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then

(1) f is a monomorphism if and only if X is the fibre product $X \times_Y X$, and
(2) f is an epimorphism if and only if Y is the pushout $Y \amalg_X Y$.

Proof. Let suppose that f is a monomorphism. Let W be an object of $\mathcal{C}$ and $\alpha, \beta \in \mathrm{Mor}_\mathcal{C}(W, X)$ such that $f \circ \alpha = f \circ \beta$. Therefore $\alpha = \beta$ as f is monic. In addition, we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

which verify the universal property with $\gamma := \alpha = \beta$. Thus X is indeed the fibre product $X \times_Y X$.

Suppose that $X \times_Y X \cong X$. The diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes and if $W \in \mathrm{Ob}(\mathcal{C})$ and $\alpha, \beta : X \to Y$ such that $f \circ \alpha = f \circ \beta$, we have a unique γ verifying

$$\gamma = \mathrm{id}_X \circ \gamma = \alpha = \beta$$

which proves that $\alpha = \beta$.

The proof is exactly the same for the second point, but with the pushout $Y \amalg_X Y = Y$. $\square$

4.14. Limits and colimits

002D Let $\mathcal{C}$ be a category. A diagram in $\mathcal{C}$ is simply a functor $M : \mathcal{I} \to \mathcal{C}$. We say that $\mathcal{I}$ is the index category or that M is an $\mathcal{I}$-diagram. We will use the notation M_i to denote the image of the object i of $\mathcal{I}$. Hence for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $M(\phi) : M_i \to M_{i'}$.

002E Definition 4.14.1. A limit of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\lim_\mathcal{I} M$ in $\mathcal{C}$ together with morphisms $p_i : \lim_\mathcal{I} M \to M_i$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $p_{i'} = M(\phi) \circ p_i$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to M_i$ (indexed by $i \in \mathrm{Ob}(\mathcal{I})$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $q_{i'} = M(\phi) \circ q_i$ there exists a unique morphism $q : W \to \lim_\mathcal{I} M$ such that $q_i = p_i \circ q$ for every object i of $\mathcal{I}$.

Limits $(\lim_\mathcal{I} M, (p_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Products of pairs, fibre products, and equalizers are examples of limits. The limit over the empty diagram is a final object of $\mathcal{C}$. In the category of sets all limits exist. The dual notion is that of colimits.

002F Definition 4.14.2. A colimit of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\mathrm{colim}_\mathcal{I} M$ in $\mathcal{C}$ together with morphisms $s_i : M_i \to \mathrm{colim}_\mathcal{I} M$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $s_i = s_{i'} \circ M(\phi)$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $t_i : M_i \to W$ (indexed by $i \in \mathrm{Ob}(\mathcal{I})$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $t_i = t_{i'} \circ M(\phi)$ there exists a unique morphism $t : \mathrm{colim}_\mathcal{I} M \to W$ such that $t_i = t \circ s_i$ for every object i of $\mathcal{I}$.

Colimits $(\mathrm{colim}_\mathcal{I} M, (s_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Coproducts of pairs, pushouts, and coequalizers are examples of colimits. The colimit over an empty diagram is an initial object of $\mathcal{C}$. In the category of sets all colimits exist.

002G Remark 4.14.3. The index category of a (co)limit will never be allowed to have a proper class of objects. In this project it means that it cannot be one of the categories listed in Remark 4.2.2

002H Remark 4.14.4. We often write $\lim_i M_i$, $\mathrm{colim}_i M_i$, $\lim_{i \in \mathcal{I}} M_i$, or $\mathrm{colim}_{i \in \mathcal{I}} M_i$ instead of the versions indexed by $\mathcal{I}$. Using this notation, and using the description of limits and colimits of sets in Section 4.15 below, we can say the following. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram.

(1) The object $\lim_i M_i$ if it exists satisfies the following property

$$\mathrm{Mor}_\mathcal{C}(W, \lim_i M_i) = \lim_i \mathrm{Mor}_\mathcal{C}(W, M_i)$$

where the limit on the right takes place in the category of sets.

(2) The object $\mathrm{colim}_i M_i$ if it exists satisfies the following property

$$\mathrm{Mor}_\mathcal{C}(\mathrm{colim}_i M_i, W) = \lim_{i \in \mathcal{I}^{opp}} \mathrm{Mor}_\mathcal{C}(M_i, W)$$

where on the right we have the limit over the opposite category with value in the category of sets.

By the Yoneda lemma (and its dual) this formula completely determines the limit, respectively the colimit.

0G2U Remark 4.14.5. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. In this setting a cone for M is given by an object W and a family of morphisms $q_i : W \to M_i$, $i \in \mathrm{Ob}(\mathcal{I})$ such that for all morphisms $\phi : i \to i'$ of $\mathcal{I}$ the diagram

$$\begin{tikzcd} & W \arrow[ld, "q_i"'] \arrow[rd, "q_{i'}"] & \\ M_i \arrow[rr, "M(\phi)"] & & M_{i'} \end{tikzcd}$$

is commutative. The collection of cones forms a category with an obvious notion of morphisms. Clearly, the limit of M, if it exists, is a final object in the category of cones. Dually, a cocone for M is given by an object W and a family of morphisms $t_i : M_i \to W$ such that for all morphisms $\phi : i \to i'$ in $\mathcal{I}$ the diagram

$$\begin{tikzcd} M_i \arrow[rr, "M(\phi)"] \arrow[rd, "t_i"'] & & M_{i'} \arrow[ld, "t_{i'}"] \\ & W & \end{tikzcd}$$

commutes. The collection of cocones forms a category with an obvious notion of morphisms. Similarly to the above the colimit of M exists if and only if the category of cocones has an initial object.

As an application of the notions of limits and colimits we define products and coproducts.

002I Definition 4.14.6. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *product* $\prod_{i \in I} M_i$ is by definition $\lim_\mathcal{I} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is a final object of the category. The morphisms $p_i : \prod M_i \to M_i$ are called the projection morphisms.

002J Definition 4.14.7. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A coproduct $\coprod_{i\in I} M_i$ is by definition $\mathrm{colim}_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the coproduct is an initial object of the category. Note that the coproduct comes equipped with morphisms $M_i \to \coprod M_i$. These are sometimes called the coprojections.

002K Lemma 4.14.8. Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose colimits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : M \to N \circ H$ is a transformation of functors. Then there is a unique morphism

$$\theta : \mathrm{colim}_{\mathcal{I}} M \longrightarrow \mathrm{colim}_{\mathcal{J}} N$$

such that all the diagrams

$$\begin{tikzcd} M_i \arrow[r] \arrow[d, "t_i"'] & \mathrm{colim}_{\mathcal{I}} M \arrow[d, "\theta"] \\ N_{H(i)} \arrow[r] & \mathrm{colim}_{\mathcal{J}} N \end{tikzcd}$$

commute.

Proof. Omitted. □

002L Lemma 4.14.9. Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose limits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : N \circ H \to M$ is a transformation of functors. Then there is a unique morphism

$$\theta : \lim_{\mathcal{J}} N \longrightarrow \lim_{\mathcal{I}} M$$

such that all the diagrams

$$\begin{tikzcd} \lim_{\mathcal{J}} N \arrow[r] \arrow[d, "\theta"] & N_{H(i)} \arrow[d, "t_i"'] \\ \lim_{\mathcal{I}} M \arrow[r] & M_i \end{tikzcd}$$

commute.

Proof. Omitted. □

002M Lemma 4.14.10. Let $\mathcal{I}$, $\mathcal{J}$ be index categories. Let $M : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor. We have

$$\mathrm{colim}_i \, \mathrm{colim}_j \, M_{i,j} = \mathrm{colim}_{i,j} \, M_{i,j} = \mathrm{colim}_j \, \mathrm{colim}_i \, M_{i,j}$$

provided all the indicated colimits exist. Similar for limits.

Proof. Omitted. □

002N Lemma 4.14.11. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = \mathrm{Ob}(\mathcal{I})$ and $A = \mathrm{Arrows}(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\prod_{i\in I} M_i$ and $\prod_{a\in A} M_{t(a)}$ exist. Suppose that the equalizer of

$$\begin{tikzcd} \prod_{i\in I} M_i \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \prod_{a\in A} M_{t(a)} \end{tikzcd}$$

exists, where the morphisms are determined by their components as follows: $p_a \circ \psi = M(a) \circ p_{s(a)}$ and $p_a \circ \phi = p_{t(a)}$. Then this equalizer is the limit of the diagram.

Proof. Omitted. $\square$

002P Lemma 4.14.12. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = \text{Ob}(\mathcal{I})$ and $A = \text{Arrows}(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\coprod_{i \in I} M_i$ and $\coprod_{a \in A} M_{s(a)}$ exist. Suppose that the coequalizer of

$$\begin{tikzcd} \coprod_{a \in A} M_{s(a)} \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \coprod_{i \in I} M_i \end{tikzcd}$$

exists, where the morphisms are determined by their components as follows: The component $M_{s(a)}$ maps via ψ to the component $M_{t(a)}$ via the morphism $M(a)$. The component $M_{s(a)}$ maps via ϕ to the component $M_{s(a)}$ by the identity morphism. Then this coequalizer is the colimit of the diagram.

Proof. Omitted. $\square$

4.15. Limits and colimits in the category of sets

002U Not only do limits and colimits exist in Sets but they are also easy to describe. Namely, let $M : \mathcal{I} \to \text{Sets}$, $i \mapsto M_i$ be a diagram of sets. Denote $I = \text{Ob}(\mathcal{I})$. The limit is described as

$$\lim_{\mathcal{I}} M = \{(m_i)_{i \in I} \in \prod\nolimits_{i \in I} M_i \mid \forall \phi : i \to i' \text{ in } \mathcal{I}, M(\phi)(m_i) = m_{i'}\}.$$

So we think of an element of the limit as a compatible system of elements of all the sets M_i.

On the other hand, the colimit is

$$\text{colim}_{\mathcal{I}} M = (\coprod\nolimits_{i \in I} M_i)/\sim$$

where the equivalence relation $\sim$ is the equivalence relation generated by setting $m_i \sim m_{i'}$ if $m_i \in M_i$, $m_{i'} \in M_{i'}$ and $M(\phi)(m_i) = m_{i'}$ for some $\phi : i \to i'$. In other words, $m_i \in M_i$ and $m_{i'} \in M_{i'}$ are equivalent if there are a chain of morphisms in $\mathcal{I}$

$$\begin{tikzcd} & i_1 \arrow[dl] \arrow[dr] & & i_3 \arrow[dl] & & i_{2n-1} \arrow[dr] & \\ i = i_0 & & i_2 & & \ldots & & i_{2n} = i' \end{tikzcd}$$

and elements $m_{i_j} \in M_{i_j}$ mapping to each other under the maps $M_{i_{2k-1}} \to M_{i_{2k-2}}$ and $M_{i_{2k-1}} \to M_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above.

This is not a very pleasant type of object to work with. But if the diagram is filtered then it is much easier to describe. We will explain this in Section 4.19.

4.16. Connected limits

04AQ A (co)limit is called connected if its index category is connected.

002S Definition 4.16.1. We say that a category $\mathcal{I}$ is connected if the equivalence relation generated by $x \sim y \Leftrightarrow \text{Mor}_{\mathcal{I}}(x, y) \neq \emptyset$ has exactly one equivalence class.

Here we follow the convention of Topology, Definition 5.7.1 that connected spaces are nonempty. The following in some vague sense characterizes connected limits.

002T Lemma 4.16.2. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to \mathcal{C}/X$ be a diagram in the category of objects over X. If the index category $\mathcal{I}$ is connected and the limit of M exists in $\mathcal{C}/X$, then the limit of the composition $\mathcal{I} \to \mathcal{C}/X \to \mathcal{C}$ exists and is the same.

Proof. Let $L \to X$ be an object representing the limit in $\mathcal{C}/X$. Consider the functor

$$W \longmapsto \lim_i \mathrm{Mor}_{\mathcal{C}}(W, M_i).$$

Let (φ_i) be an element of the set on the right. Since each M_i comes equipped with a morphism $s_i : M_i \to X$ we get morphisms $f_i = s_i \circ \varphi_i : W \to X$. But as $\mathcal{I}$ is connected we see that all f_i are equal. Since $\mathcal{I}$ is nonempty there is at least one f_i. Hence this common value $W \to X$ defines the structure of an object of W in $\mathcal{C}/X$ and (φ_i) defines an element of $\lim_i \mathrm{Mor}_{\mathcal{C}/X}(W, M_i)$. Thus we obtain a unique morphism $\phi : W \to L$ such that φ_i is the composition of ϕ with $L \to M_i$ as desired. □

04AR Lemma 4.16.3. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to X/\mathcal{C}$ be a diagram in the category of objects under X. If the index category $\mathcal{I}$ is connected and the colimit of M exists in $X/\mathcal{C}$, then the colimit of the composition $\mathcal{I} \to X/\mathcal{C} \to \mathcal{C}$ exists and is the same.

Proof. Omitted. Hint: This lemma is dual to Lemma 4.16.2. □

4.17. Cofinal and initial categories

09WN In the literature sometimes the word "final" is used instead of cofinal in the following definition.

04E6 Definition 4.17.1. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor between categories. We say $\mathcal{I}$ is cofinal in $\mathcal{J}$ or that H is cofinal if

(1) for all $y \in \mathrm{Ob}(\mathcal{J})$ there exist an $x \in \mathrm{Ob}(\mathcal{I})$ and a morphism $y \to H(x)$, and
(2) given $y \in \mathrm{Ob}(\mathcal{J})$, $x, x' \in \mathrm{Ob}(\mathcal{I})$ and morphisms $y \to H(x)$ and $y \to H(x')$ there exist a sequence of morphisms
$$x = x_0 \leftarrow x_1 \to x_2 \leftarrow x_3 \to \ldots \to x_{2n} = x'$$
in $\mathcal{I}$ and morphisms $y \to H(x_i)$ in $\mathcal{J}$ such that the diagrams
$$\begin{tikzcd} & y \arrow[dl] \arrow[d] \arrow[dr] & \\ H(x_{2k}) & H(x_{2k+1}) \arrow[l] \arrow[r] & H(x_{2k+2}) \end{tikzcd}$$
commute for $k = 0, \ldots, n-1$.

04E7 Lemma 4.17.2. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of categories. Assume $\mathcal{I}$ is cofinal in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ we have a canonical isomorphism

$$\mathrm{colim}_{\mathcal{I}}\, M \circ H = \mathrm{colim}_{\mathcal{J}}\, M$$

if either side exists.

Proof. Omitted. □

09WP Definition 4.17.3. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor between categories. We say $\mathcal{I}$ is initial in $\mathcal{J}$ or that H is initial if

(1) for all $y \in \mathrm{Ob}(\mathcal{J})$ there exist an $x \in \mathrm{Ob}(\mathcal{I})$ and a morphism $H(x) \to y$,
(2) for any $y \in \mathrm{Ob}(\mathcal{J})$, $x, x' \in \mathrm{Ob}(\mathcal{I})$ and morphisms $H(x) \to y$, $H(x') \to y$ in $\mathcal{J}$ there exist a sequence of morphisms
$$x = x_0 \leftarrow x_1 \to x_2 \leftarrow x_3 \to \ldots \to x_{2n} = x'$$
in $\mathcal{I}$ and morphisms $H(x_i) \to y$ in $\mathcal{J}$ such that the diagrams
$$\begin{tikzcd} H(x_{2k}) \arrow[dr] & H(x_{2k+1}) \arrow[l] \arrow[r] \arrow[d] & H(x_{2k+2}) \arrow[dl] \\ & y & \end{tikzcd}$$
commute for $k = 0, \ldots, n-1$.

This is just the dual notion to "cofinal" functors.

002R Lemma 4.17.4. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of categories. Assume $\mathcal{I}$ is initial in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ we have a canonical isomorphism
$$\lim_{\mathcal{I}} M \circ H = \lim_{\mathcal{J}} M$$
if either side exists.

Proof. Omitted. □

05US Lemma 4.17.5. Let $F : \mathcal{I} \to \mathcal{I}'$ be a functor. Assume

(1) the fibre categories (see Definition 4.32.2) of $\mathcal{I}$ over $\mathcal{I}'$ are all connected, and
(2) for every morphism $\alpha' : x' \to y'$ in $\mathcal{I}'$ there exists a morphism $\alpha : x \to y$ in $\mathcal{I}$ such that $F(\alpha) = \alpha'$.

Then for every diagram $M : \mathcal{I}' \to \mathcal{C}$ the colimit $\mathrm{colim}_{\mathcal{I}} M \circ F$ exists if and only if $\mathrm{colim}_{\mathcal{I}'} M$ exists and if so these colimits agree.

Proof. One can prove this by showing that $\mathcal{I}$ is cofinal in $\mathcal{I}'$ and applying Lemma 4.17.2. But we can also prove it directly as follows. It suffices to show that for any object T of $\mathcal{C}$ we have
$$\lim_{\mathcal{I}^{opp}} \mathrm{Mor}_{\mathcal{C}}(M_{F(i)}, T) = \lim_{(\mathcal{I}')^{opp}} \mathrm{Mor}_{\mathcal{C}}(M_{i'}, T)$$
If $(g_{i'})_{i' \in \mathrm{Ob}(\mathcal{I}')}$ is an element of the right hand side, then setting $f_i = g_{F(i)}$ we obtain an element $(f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ of the left hand side. Conversely, let $(f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ be an element of the left hand side. Note that on each (connected) fibre category $\mathcal{I}_{i'}$ the functor $M \circ F$ is constant with value $M_{i'}$. Hence the morphisms f_i for $i \in \mathrm{Ob}(\mathcal{I})$ with $F(i) = i'$ are all the same and determine a well defined morphism $g_{i'} : M_{i'} \to T$. By assumption (2) the collection $(g_{i'})_{i' \in \mathrm{Ob}(\mathcal{I}')}$ defines an element of the right hand side. □

0A2B Lemma 4.17.6. Let $\mathcal{I}$ and $\mathcal{J}$ be a categories and denote $p : \mathcal{I} \times \mathcal{J} \to \mathcal{J}$ the projection. If $\mathcal{I}$ is connected, then for a diagram $M : \mathcal{J} \to \mathcal{C}$ the colimit $\mathrm{colim}_{\mathcal{J}} M$ exists if and only if $\mathrm{colim}_{\mathcal{I} \times \mathcal{J}} M \circ p$ exists and if so these colimits are equal.

Proof. This is a special case of Lemma 4.17.5. □

4.18. Finite limits and colimits

04AS A finite (co)limit is a (co)limit whose index category is finite, i.e., the index category has finitely many objects and finitely many morphisms. A (co)limit is called nonempty if the index category is nonempty. A (co)limit is called connected if the index category is connected, see Definition 4.16.1. It turns out that there are "enough" finite index categories.

05XU **Lemma 4.18.1.** *Let $\mathcal{I}$ be a category with*

(1) *$\mathrm{Ob}(\mathcal{I})$ is finite, and*
(2) *there exist finitely many morphisms $f_1, \ldots, f_m \in \mathrm{Arrows}(\mathcal{I})$ such that every morphism of $\mathcal{I}$ is a composition $f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_k}$.*

Then there exists a functor $F : \mathcal{J} \to \mathcal{I}$ such that

(a) *$\mathcal{J}$ is a finite category, and*
(b) *for any diagram $M : \mathcal{I} \to \mathcal{C}$ the (co)limit of M over $\mathcal{I}$ exists if and only if the (co)limit of $M \circ F$ over $\mathcal{J}$ exists and in this case the (co)limits are canonically isomorphic.*

Moreover, $\mathcal{J}$ is connected (resp. nonempty) if and only if $\mathcal{I}$ is so.

Proof. Say $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\}$. Denote $s, t : \{1, \ldots, m\} \to \{1, \ldots, n\}$ the functions such that $f_j : x_{s(j)} \to x_{t(j)}$. We set $\mathrm{Ob}(\mathcal{J}) = \{y_1, \ldots, y_n, z_1, \ldots, z_n\}$ Besides the identity morphisms we introduce morphisms $g_j : y_{s(j)} \to z_{t(j)}$, $j = 1, \ldots, m$ and morphisms $h_i : y_i \to z_i$, $i = 1, \ldots, n$. Since all of the nonidentity morphisms in $\mathcal{J}$ go from a y to a z there are no compositions to define and no associativities to check. Set $F(y_i) = F(z_i) = x_i$. Set $F(g_j) = f_j$ and $F(h_i) = \mathrm{id}_{x_i}$. It is clear that F is a functor. It is clear that $\mathcal{J}$ is finite. It is clear that $\mathcal{J}$ is connected, resp. nonempty if and only if $\mathcal{I}$ is so.

Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Consider an object W of $\mathcal{C}$ and morphisms $q_i : W \to M(x_i)$ as in Definition 4.14.1. Then by taking $q_i : W \to M(F(y_i)) = M(F(z_i)) = M(x_i)$ we obtain a family of maps as in Definition 4.14.1 for the diagram $M \circ F$. Conversely, suppose we are given maps $qy_i : W \to M(F(y_i))$ and $qz_i : W \to M(F(z_i))$ as in Definition 4.14.1 for the diagram $M \circ F$. Since

$$M(F(h_i)) = \mathrm{id} : M(F(y_i)) = M(x_i) \longrightarrow M(x_i) = M(F(z_i))$$

we conclude that $qy_i = qz_i$ for all i. Set q_i equal to this common value. The compatibility of $q_{s(j)} = qy_{s(j)}$ and $q_{t(j)} = qz_{t(j)}$ with the morphism $M(f_j)$ guarantees that the family q_i is compatible with all morphisms in $\mathcal{I}$ as by assumption every such morphism is a composition of the morphisms f_j. Thus we have found a canonical bijection

$$\lim_{B \in \mathrm{Ob}(\mathcal{J})} \mathrm{Mor}_\mathcal{C}(W, M(F(B))) = \lim_{A \in \mathrm{Ob}(\mathcal{I})} \mathrm{Mor}_\mathcal{C}(W, M(A))$$

which implies the statement on limits in the lemma. The statement on colimits is proved in the same way (proof omitted). □

04AT **Lemma 4.18.2.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Connected finite limits exist in $\mathcal{C}$.*
(2) *Equalizers and fibre products exist in $\mathcal{C}$.*

Proof. Since equalizers and fibre products are finite connected limits we see that (1) implies (2). For the converse, let $\mathcal{I}$ be a finite connected index category. Let

$F : \mathcal{J} \to \mathcal{I}$ be the functor of index categories constructed in the proof of Lemma 4.18.1. Then we see that we may replace $\mathcal{I}$ by $\mathcal{J}$. The result is that we may assume that $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\} \amalg \{y_1, \ldots, y_m\}$ with $n, m \geq 1$ such that all nonidentity morphisms in $\mathcal{I}$ are morphisms $f : x_i \to y_j$ for some i and j.

Suppose that $n > 1$. Since $\mathcal{I}$ is connected there exist indices i_1, i_2 and j_0 and morphisms $a : x_{i_1} \to y_{j_0}$ and $b : x_{i_2} \to y_{j_0}$. Consider the category

$$\mathcal{I}' = \{x\} \amalg \{x_1, \ldots, \hat{x}_{i_1}, \ldots, \hat{x}_{i_2}, \ldots x_n\} \amalg \{y_1, \ldots, y_m\}$$

with

$$\mathrm{Mor}_{\mathcal{I}'}(x, y_j) = \mathrm{Mor}_{\mathcal{I}}(x_{i_1}, y_j) \amalg \mathrm{Mor}_{\mathcal{I}}(x_{i_2}, y_j)$$

and all other morphism sets the same as in $\mathcal{I}$. For any functor $M : \mathcal{I} \to \mathcal{C}$ we can construct a functor $M' : \mathcal{I}' \to \mathcal{C}$ by setting

$$M'(x) = M(x_{i_1}) \times_{M(a), M(y_{j_0}), M(b)} M(x_{i_2})$$

and for a morphism $f' : x \to y_j$ corresponding to, say, $f : x_{i_1} \to y_j$ we set $M'(f) = M(f) \circ \mathrm{pr}_1$. Then the functor M has a limit if and only if the functor M' has a limit (proof omitted). Hence by induction we reduce to the case $n = 1$.

If $n = 1$, then the limit of any $M : \mathcal{I} \to \mathcal{C}$ is the successive equalizer of pairs of maps $x_1 \to y_j$ hence exists by assumption. □

04AU Lemma 4.18.3. Let $\mathcal{C}$ be a category. The following are equivalent:

(1) Nonempty finite limits exist in $\mathcal{C}$.
(2) Products of pairs and equalizers exist in $\mathcal{C}$.
(3) Products of pairs and fibre products exist in $\mathcal{C}$.

Proof. Since products of pairs, fibre products, and equalizers are limits with nonempty index categories we see that (1) implies both (2) and (3). Assume (2). Then finite nonempty products and equalizers exist. Hence by Lemma 4.14.11 we see that finite nonempty limits exist, i.e., (1) holds. Assume (3). If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(\mathrm{pr}_1, \mathrm{pr}_2), A \times A, \Delta} A.$$

Thus (3) implies (2), and the lemma is proved. □

002O Lemma 4.18.4. Let $\mathcal{C}$ be a category. The following are equivalent:

(1) Finite limits exist in $\mathcal{C}$.
(2) Finite products and equalizers exist.
(3) The category has a final object and fibre products exist.

Proof. Since finite products, fibre products, equalizers, and final objects are limits over finite index categories we see that (1) implies both (2) and (3). By Lemma 4.14.11 above we see that (2) implies (1). Assume (3). Note that the product $A \times B$ is the fibre product over the final object. If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(\mathrm{pr}_1, \mathrm{pr}_2), A \times A, \Delta} A.$$

Thus (3) implies (2) and the lemma is proved. □

04AV Lemma 4.18.5. Let $\mathcal{C}$ be a category. The following are equivalent:

(1) Connected finite colimits exist in $\mathcal{C}$.
(2) Coequalizers and pushouts exist in $\mathcal{C}$.

Proof. Omitted. Hint: This is dual to Lemma 4.18.2. $\square$

04AW **Lemma 4.18.6.** Let $\mathcal{C}$ be a category. The following are equivalent:

(1) Nonempty finite colimits exist in $\mathcal{C}$.
(2) Coproducts of pairs and coequalizers exist in $\mathcal{C}$.
(3) Coproducts of pairs and pushouts exist in $\mathcal{C}$.

Proof. Omitted. Hint: This is the dual of Lemma 4.18.3. $\square$

002Q **Lemma 4.18.7.** Let $\mathcal{C}$ be a category. The following are equivalent:

(1) Finite colimits exist in $\mathcal{C}$.
(2) Finite coproducts and coequalizers exist in $\mathcal{C}$.
(3) The category has an initial object and pushouts exist.

Proof. Omitted. Hint: This is dual to Lemma 4.18.4. $\square$

4.19. Filtered colimits

04AX Colimits are easier to compute or describe when they are over a filtered diagram. Here is the definition.

002V **Definition 4.19.1.** We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *directed*, or *filtered* if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,
(2) for every pair of objects x, y of $\mathcal{I}$ there exist an object z and morphisms $x \to z$, $y \to z$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : y \to z$ of $\mathcal{I}$ such that $M(c \circ a) = M(c \circ b)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *directed*, or *filtered* if $\text{id} : \mathcal{I} \to \mathcal{I}$ is filtered (in other words you erase the M in part (3) above).

We observe that any diagram with filtered index category is filtered, and this is how filtered colimits usually come about. In fact, if $M : \mathcal{I} \to \mathcal{C}$ is a filtered diagram, then we can factor M as $\mathcal{I} \to \mathcal{I}' \to \mathcal{C}$ where $\mathcal{I}'$ is a filtered index category[1] such that $\text{colim}_\mathcal{I} M$ exists if and only if $\text{colim}_{\mathcal{I}'} M'$ exists in which case the colimits are canonically isomorphic.

Suppose that $M : \mathcal{I} \to \text{Sets}$ is a filtered diagram. In this case we may describe the equivalence relation in the formula

$$\text{colim}_\mathcal{I} M = (\textstyle\coprod_{i \in I} M_i)/\sim$$

simply as follows

$$m_i \sim m_{i'} \Leftrightarrow \exists i'', \phi : i \to i'', \phi' : i' \to i'', M(\phi)(m_i) = M(\phi')(m_{i'}).$$

In other words, two elements are equal in the colimit if and only if they "eventually become equal".

[1] Namely, let $\mathcal{I}'$ have the same objects as $\mathcal{I}$ but where $\text{Mor}_{\mathcal{I}'}(x, y)$ is the quotient of $\text{Mor}_\mathcal{I}(x, y)$ by the equivalence relation which identifies $a, b : x \to y$ if $M(a) = M(b)$.

002W Lemma 4.19.2. Let $\mathcal{I}$ and $\mathcal{J}$ be index categories. Assume that $\mathcal{I}$ is filtered and $\mathcal{J}$ is finite. Let $M : \mathcal{I} \times \mathcal{J} \to \text{Sets}$, $(i, j) \mapsto M_{i,j}$ be a diagram of diagrams of sets. In this case

$$\operatorname{colim}_i \lim_j M_{i,j} = \lim_j \operatorname{colim}_i M_{i,j}.$$

In particular, colimits over $\mathcal{I}$ commute with finite products, fibre products, and equalizers of sets.

Proof. Omitted. In fact, it is a fun exercise to prove that a category is filtered if and only if colimits over the category commute with finite limits (into the category of sets). □

We give a counter example to the lemma in the case where $\mathcal{J}$ is infinite. Namely, let $\mathcal{I}$ consist of $\mathbf{N} = \{1, 2, 3, \ldots\}$ with a unique morphism $i \to i'$ whenever $i \leq i'$. Let $\mathcal{J}$ be the discrete category $\mathbf{N} = \{1, 2, 3, \ldots\}$ (only morphisms are identities). Let $M_{i,j} = \{1, 2, \ldots, i\}$ with obvious inclusion maps $M_{i,j} \to M_{i',j}$ when $i \leq i'$. In this case $\operatorname{colim}_i M_{i,j} = \mathbf{N}$ and hence

$$\lim_j \operatorname{colim}_i M_{i,j} = \prod\nolimits_j \mathbf{N} = \mathbf{N}^{\mathbf{N}}$$

On the other hand $\lim_j M_{i,j} = \prod_j M_{i,j}$ and hence

$$\operatorname{colim}_i \lim_j M_{i,j} = \bigcup\nolimits_i \{1, 2, \ldots, i\}^{\mathbf{N}}$$

which is smaller than the other limit.

0BUC Lemma 4.19.3. Let $\mathcal{I}$ be a category. Let $\mathcal{J}$ be a full subcategory. Assume that $\mathcal{I}$ is filtered. Assume also that for any object i of $\mathcal{I}$, there exists a morphism $i \to j$ to some object j of $\mathcal{J}$. Then $\mathcal{J}$ is filtered and cofinal in $\mathcal{I}$.

Proof. Omitted. Pleasant exercise of the notions involved. □

It turns out we sometimes need a more finegrained control over the possible conditions one can impose on index categories. Thus we add some lemmas on the possible things one can require.

09WQ Lemma 4.19.4. Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every pair of objects x, y of $\mathcal{I}$ there exist an object z and morphisms $x \to z$ and $y \to z$. Then

(1) If M and N are diagrams of sets over $\mathcal{I}$, then $\operatorname{colim}(M_i \times N_i) \to \operatorname{colim} M_i \times \operatorname{colim} N_i$ is surjective,
(2) in general colimits of diagrams of sets over $\mathcal{I}$ do not commute with finite nonempty products.

Proof. Proof of (1). Let $(\overline{m}, \overline{n})$ be an element of $\operatorname{colim} M_i \times \operatorname{colim} N_i$. Then we can find $m \in M_x$ and $n \in N_y$ for some $x, y \in \operatorname{Ob}(\mathcal{I})$ such that m maps to $\overline{m}$ and n maps to $\overline{n}$. See Section 4.15. Choose $a : x \to z$ and $b : y \to z$ in $\mathcal{I}$. Then $(M(a)(m), N(b)(n))$ is an element of $(M \times N)_z$ whose image in $\operatorname{colim}(M_i \times N_i)$ maps to $(\overline{m}, \overline{n})$ as desired.

Proof of (2). Let G be a non-trivial group and let $\mathcal{I}$ be the one-object category with endomorphism monoid G. Then $\mathcal{I}$ trivially satisfies the condition stated in the lemma. Now let G act on itself by translation and view the G-set G as a set-valued $\mathcal{I}$-diagram. Then

$$\operatorname{colim}_{\mathcal{I}} G \times \operatorname{colim}_{\mathcal{I}} G \cong G/G \times G/G$$

is not isomorphic to

$$\mathrm{colim}_{\mathcal{I}}(G \times G) \cong (G \times G)/G$$

This example indicates that you cannot just drop the additional condition Lemma 4.19.2 even if you only care about finite products. □

09WR Lemma 4.19.5. Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every pair of objects x, y of $\mathcal{I}$ there exist an object z and morphisms $x \to z$ and $y \to z$. Let $M : \mathcal{I} \to \mathrm{Ab}$ be a diagram of abelian groups over $\mathcal{I}$. Then the colimit of M in the category of sets surjects onto the colimit of M in the category of abelian groups.

Proof. Recall that the colimit in the category of sets is the quotient of the disjoint union $\coprod M_i$ by relation, see Section 4.15. Similarly, the colimit in the category of abelian groups is a quotient of the direct sum $\bigoplus M_i$. The assumption of the lemma means that given $i, j \in \mathrm{Ob}(\mathcal{I})$ and $m \in M_i$ and $n \in M_j$, then we can find an object k and morphisms $a : i \to k$ and $b : j \to k$. Thus $m + n$ is represented in the colimit by the element $M(a)(m) + M(b)(n)$ of M_k. Thus the $\coprod M_i$ surjects onto the colimit. □

09WS Lemma 4.19.6. Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every solid diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d, dotted] \\ z \arrow[r, dotted] & w \end{tikzcd}$$

in $\mathcal{I}$ there exist an object w and dotted arrows making the diagram commute. Then $\mathcal{I}$ is either empty or a nonempty disjoint union of connected categories having the same property.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exist a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the assumption of the lemma. Hence $\mathrm{Ob}(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. Then $\mathcal{I} = \coprod \mathcal{I}_j$ with $\mathcal{I}_j$ nonempty as desired. □

09WT Lemma 4.19.7. Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every solid diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d, dotted] \\ z \arrow[r, dotted] & w \end{tikzcd}$$

in $\mathcal{I}$ there exist an object w and dotted arrows making the diagram commute. Then

(1) an injective morphism $M \to N$ of diagrams of sets over $\mathcal{I}$ gives rise to an injective map $\mathrm{colim}\, M_i \to \mathrm{colim}\, N_i$ of sets,
(2) in general the same is not the case for diagrams of abelian groups and their colimits.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Thus we may assume $\mathcal{I}$ is nonempty. In this case we can write $\mathcal{I} = \coprod \mathcal{I}_j$ where each $\mathcal{I}_j$ is nonempty and satisfies the same property, see Lemma 4.19.6. Since $\mathrm{colim}_\mathcal{I} M = \coprod_j \mathrm{colim}_{\mathcal{I}_j} M|_{\mathcal{I}_j}$ this reduces the proof of (1) to the connected case.

Assume $\mathcal{I}$ is connected and $M \to N$ is injective, i.e., all the maps $M_i \to N_i$ are injective. We identify M_i with the image of $M_i \to N_i$, i.e., we will think of M_i as a subset of N_i. We will use the description of the colimits given in Section 4.15 without further mention. Let $s, s' \in \mathrm{colim}\, M_i$ map to the same element of $\mathrm{colim}\, N_i$. Say s comes from an element m of M_i and s' comes from an element m' of $M_{i'}$. Then we can find a sequence $i = i_0, i_1, \ldots, i_n = i'$ of objects of $\mathcal{I}$ and morphisms

$$\begin{tikzcd} & i_1 \arrow[ld] \arrow[rd] & & i_3 \arrow[ld] & & i_{2n-1} \arrow[rd] & \\ i = i_0 & & i_2 & & \ldots & & i_{2n} = i' \end{tikzcd}$$

and elements $n_{i_j} \in N_{i_j}$ mapping to each other under the maps $N_{i_{2k-1}} \to N_{i_{2k-2}}$ and $N_{i_{2k-1}} \to N_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above with $n_{i_0} = m$ and $n_{i_{2n}} = m'$. We will prove by induction on n that this implies $s = s'$. The base case $n = 0$ is trivial. Assume $n \geq 1$. Using the assumption on $\mathcal{I}$ we find a commutative diagram

$$\begin{tikzcd} & i_1 \arrow[ld] \arrow[rd] & \\ i_0 \arrow[rd] & & i_2 \arrow[ld] \\ & w & \end{tikzcd}$$

We conclude that m and n_{i_2} map to the same element of N_w because both are the image of the element n_{i_1}. In particular, this element is an element $m'' \in M_w$ which gives rise to the same element as s in $\mathrm{colim}\, M_i$. Then we find the chain

$$\begin{tikzcd} & i_3 \arrow[ld] \arrow[rd] & & i_5 \arrow[ld] & & i_{2n-1} \arrow[rd] & \\ w & & i_4 & & \ldots & & i_{2n} = i' \end{tikzcd}$$

and the elements n_{i_j} for $j \geq 3$ which has a smaller length than the chain we started with. This proves the induction step and the proof of (1) is complete.

Let G be a group and let $\mathcal{I}$ be the one-object category with endomorphism monoid G. Then $\mathcal{I}$ satisfies the condition stated in the lemma because given $g_1, g_2 \in G$ we can find $h_1, h_2 \in G$ with $h_1 g_1 = h_2 g_2$. An diagram M over $\mathcal{I}$ in Ab is the same thing as an abelian group M with G-action and $\mathrm{colim}_\mathcal{I} M$ is the coinvariants M_G of M. Take G the group of order 2 acting trivially on $M = \mathbf{Z}/2\mathbf{Z}$ mapping into the first summand of $N = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ where the nontrivial element of G acts by $(x, y) \mapsto (x + y, y)$. Then $M_G \to N_G$ is zero. $\square$

002X Lemma 4.19.8. Let $\mathcal{I}$ be an index category, i.e., a category. Assume

(1) for every pair of morphisms $a : w \to x$ and $b : w \to y$ in $\mathcal{I}$ there exist an object z and morphisms $c : x \to z$ and $d : y \to z$ such that $c \circ a = d \circ b$, and

(2) for every pair of morphisms $a, b : x \to y$ there exists a morphism $c : y \to z$ such that $c \circ a = c \circ b$.

Then $\mathcal{I}$ is a (possibly empty) union of disjoint filtered index categories $\mathcal{I}_j$.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exist a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the first assumption of the lemma. Hence $\mathrm{Ob}(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. The rest is clear from the definitions. □

002Y Lemma 4.19.9. Let $\mathcal{I}$ be an index category satisfying the hypotheses of Lemma 4.19.8 above. Then colimits over $\mathcal{I}$ commute with fibre products and equalizers in sets (and more generally with finite connected limits).

Proof. By Lemma 4.19.8 we may write $\mathcal{I} = \coprod \mathcal{I}_j$ with each $\mathcal{I}_j$ filtered. By Lemma 4.19.2 we see that colimits of $\mathcal{I}_j$ commute with equalizers and fibre products. Thus it suffices to show that equalizers and fibre products commute with coproducts in the category of sets (including empty coproducts). In other words, given a set J and sets A_j, B_j, C_j and set maps $A_j \to B_j$, $C_j \to B_j$ for $j \in J$ we have to show that

$$(\coprod\nolimits_{j \in J} A_j) \times_{(\coprod_{j \in J} B_j)} (\coprod\nolimits_{j \in J} C_j) = \coprod\nolimits_{j \in J} A_j \times_{B_j} C_j$$

and given $a_j, a'_j : A_j \to B_j$ that

$$\mathrm{Equalizer}(\coprod\nolimits_{j \in J} a_j, \coprod\nolimits_{j \in J} a'_j) = \coprod\nolimits_{j \in J} \mathrm{Equalizer}(a_j, a'_j)$$

This is true even if $J = \emptyset$. Details omitted. □

4.20. Cofiltered limits

04AY Limits are easier to compute or describe when they are over a cofiltered diagram. Here is the definition.

04AZ Definition 4.20.1. We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is codirected or cofiltered if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,
(2) for every pair of objects x, y of $\mathcal{I}$ there exist an object z and morphisms $z \to x$, $z \to y$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : w \to x$ of $\mathcal{I}$ such that $M(a \circ c) = M(b \circ c)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is codirected, or cofiltered if $\mathrm{id} : \mathcal{I} \to \mathcal{I}$ is cofiltered (in other words you erase the M in part (3) above).

We observe that any diagram with cofiltered index category is cofiltered, and this is how this situation usually occurs.

As an example of why cofiltered limits of sets are "easier" than general ones, we mention the fact that a cofiltered diagram of finite nonempty sets has nonempty limit (Lemma 4.21.7). This result does not hold for a general limit of finite nonempty sets.

4.21. Limits and colimits over preordered sets

002Z A special case of diagrams is given by systems over preordered sets.

00D3 **Definition 4.21.1.** Let I be a set and let $\leq$ be a binary relation on I.

(1) We say $\leq$ is a preorder if it is transitive (if $i \leq j$ and $j \leq k$ then $i \leq k$) and reflexive ($i \leq i$ for all $i \in I$).
(2) A preordered set is a set endowed with a preorder.
(3) A directed set is a preordered set $(I, \leq)$ such that I is not empty and such that $\forall i, j \in I$, there exists $k \in I$ with $i \leq k, j \leq k$.
(4) We say $\leq$ is a partial order if it is a preorder which is antisymmetric (if $i \leq j$ and $j \leq i$, then $i = j$).
(5) A partially ordered set is a set endowed with a partial order.
(6) A directed partially ordered set is a directed set whose ordering is a partial order.

It is customary to drop the $\leq$ from the notation when talking about preordered sets, that is, one speaks of the preordered set I rather than of the preordered set $(I, \leq)$. Given a preordered set I the symbol $\geq$ is defined by the rule $i \geq j \Leftrightarrow j \leq i$ for all $i, j \in I$. The phrase "partially ordered set" is sometimes abbreviated to "poset".

Given a preordered set I we can construct a category: the objects are the elements of I, there is exactly one morphism $i \to i'$ if $i \leq i'$, and otherwise none. Conversely, given a category $\mathcal{C}$ with at most one arrow between any two objects, the set $\text{Ob}(\mathcal{C})$ is endowed with a preorder defined by the rule $x \leq y \Leftrightarrow \text{Mor}_\mathcal{C}(x, y) \neq \emptyset$.

0030 **Definition 4.21.2.** Let $(I, \leq)$ be a preordered set. Let $\mathcal{C}$ be a category.

(1) A system over I in $\mathcal{C}$, sometimes called a inductive system over I in $\mathcal{C}$ is given by objects M_i of $\mathcal{C}$ and for every $i \leq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \text{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \leq i' \leq i''$.
(2) An inverse system over I in $\mathcal{C}$, sometimes called a projective system over I in $\mathcal{C}$ is given by objects M_i of $\mathcal{C}$ and for every $i' \leq i$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \text{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i'' \leq i' \leq i$. (Note reversal of inequalities.)

We will say $(M_i, f_{ii'})$ is a (inverse) system over I to denote this. The maps $f_{ii'}$ are sometimes called the transition maps.

In other words a system over I is just a diagram $M : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category we associated to I above: objects are elements of I and there is a unique arrow $i \to i'$ in $\mathcal{I}$ if and only if $i \leq i'$. An inverse system is a diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$. From this point of view we could take (co)limits of any (inverse) system over I. However, it is customary to take only colimits of systems over I and only limits of inverse systems over I. More precisely: Given a system $(M_i, f_{ii'})$ over I the colimit of the system $(M_i, f_{ii'})$ is defined as

$$\text{colim}_{i \in I} M_i = \text{colim}_\mathcal{I} M,$$

i.e., as the colimit of the corresponding diagram. Given a inverse system $(M_i, f_{ii'})$ over I the limit of the inverse system $(M_i, f_{ii'})$ is defined as

$$\lim_{i \in I} M_i = \lim_{\mathcal{I}^{opp}} M,$$

i.e., as the limit of the corresponding diagram.

0CN1 Remark 4.21.3. Let I be a preordered set. From I we can construct a canonical partially ordered set $\overline{I}$ and an order preserving map $\pi : I \to \overline{I}$. Namely, we can define an equivalence relation $\sim$ on I by the rule

$$i \sim j \Leftrightarrow (i \leq j \text{ and } j \leq i).$$

We set $\overline{I} = I/\sim$ and we let $\pi : I \to \overline{I}$ be the quotient map. Finally, $\overline{I}$ comes with a unique partial ordering such that $\pi(i) \leq \pi(j) \Leftrightarrow i \leq j$. Observe that if I is a directed set, then $\overline{I}$ is a directed partially ordered set. Given an (inverse) system N over $\overline{I}$ we obtain an (inverse) system M over I by setting $M_i = N_{\pi(i)}$. This construction defines a functor between the category of inverse systems over I and $\overline{I}$. In fact, this is an equivalence. The reason is that if $i \sim j$, then for any system M over I the maps $M_i \to M_j$ and $M_j \to M_i$ are mutually inverse isomorphisms. More precisely, choosing a section $s : \overline{I} \to I$ of π a quasi-inverse of the functor above sends M to N with $N_{\bar{i}} = M_{s(\bar{i})}$. Finally, this correspondence is compatible with colimits of systems: if M and N are related as above and if either $\text{colim}_{\overline{I}} N$ or $\text{colim}_I M$ exists then so does the other and $\text{colim}_{\overline{I}} N = \text{colim}_I M$. Similar results hold for inverse systems and limits of inverse systems.

The upshot of Remark 4.21.3 is that while computing a colimit of a system or a limit of an inverse system, we may always assume the preorder is a partial order.

0031 Definition 4.21.4. Let I be a preordered set. We say a system (resp. inverse system) $(M_i, f_{ii'})$ is a directed system (resp. directed inverse system) if I is a directed set (Definition 4.21.1): I is nonempty and for all $i_1, i_2 \in I$ there exists $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$.

In this case the colimit is sometimes (unfortunately) called the "direct limit". We will not use this last terminology. It turns out that diagrams over a filtered category are no more general than directed systems in the following sense.

0032 Lemma 4.21.5. Let $\mathcal{I}$ be a filtered index category. There exist a directed set I and a system $(x_i, \varphi_{ii'})$ over I in $\mathcal{I}$ with the following properties:

(1) For every category $\mathcal{C}$ and every diagram $M : \mathcal{I} \to \mathcal{C}$ with values in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding system over I. If $\text{colim}_{i\in I} M(x_i)$ exists then so does $\text{colim}_{\mathcal{I}} M$ and the transformation
$$\theta : \text{colim}_{i\in I} M(x_i) \longrightarrow \text{colim}_{\mathcal{I}} M$$
of Lemma 4.14.8 is an isomorphism.

(2) For every category $\mathcal{C}$ and every diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$ in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding inverse system over I. If $\lim_{i\in I} M(x_i)$ exists then so does $\lim_{\mathcal{I}^{opp}} M$ and the transformation
$$\theta : \lim_{\mathcal{I}^{opp}} M \longrightarrow \lim_{i\in I} M(x_i)$$
of Lemma 4.14.9 is an isomorphism.

Proof. As explained in the text following Definition 4.21.2, we may view preordered sets as categories and systems as functors. Throughout the proof, we will freely shift between these two points of view. We prove the first statement by constructing a category $\mathcal{I}_0$, corresponding to a directed set[2], and a cofinal functor $M_0 : \mathcal{I}_0 \to \mathcal{I}$. Then, by Lemma 4.17.2, the colimit of a diagram $M : \mathcal{I} \to \mathcal{C}$ coincides with the

[2] In fact, our construction will produce a directed partially ordered set.

colimit of the diagram $M \circ M_0 : \mathcal{I}_0 \to \mathcal{C}$, from which the statement follows. The second statement is dual to the first and may be proved by interpreting a limit in $\mathcal{C}$ as a colimit in $\mathcal{C}^{opp}$. We omit the details.

A category $\mathcal{F}$ is called finitely generated if there exists a finite set F of arrows in $\mathcal{F}$, such that each arrow in $\mathcal{F}$ may be obtained by composing arrows from F. In particular, this implies that $\mathcal{F}$ has finitely many objects. We start the proof by reducing to the case when $\mathcal{I}$ has the property that every finitely generated subcategory of $\mathcal{I}$ may be extended to a finitely generated subcategory with a unique final object.

Let ω denote the directed set of finite ordinals, which we view as a filtered category. It is easy to verify that the product category $\mathcal{I} \times \omega$ is also filtered, and the projection $\Pi : \mathcal{I} \times \omega \to \mathcal{I}$ is cofinal.

Now let $\mathcal{F}$ be any finitely generated subcategory of $\mathcal{I} \times \omega$. By using the axioms of a filtered category and a simple induction argument on a finite set of generators of $\mathcal{F}$, we may construct a cocone $(\{f_i\}, i_\infty)$ in $\mathcal{I} \times \omega$ for the diagram $\mathcal{F} \to \mathcal{I} \times \omega$. That is, a morphism $f_i : i \to i_\infty$ for every object i in $\mathcal{F}$ such that for each arrow $f : i \to i'$ in $\mathcal{F}$ we have $f_i = f_{i'} \circ f$. We can also choose i_∞ such that there are no arrows from i_∞ to an object in $\mathcal{F}$. This is possible since we may always post-compose the arrows f_i with an arrow which is the identity on the $\mathcal{I}$-component and strictly increasing on the ω-component. Now let $\mathcal{F}^+$ denote the category consisting of all objects and arrows in $\mathcal{F}$ together with the object i_∞, the identity arrow id_{i_∞} and the arrows f_i. Since there are no arrows from i_∞ in $\mathcal{F}^+$ to any object of $\mathcal{F}$, the arrow set in $\mathcal{F}^+$ is closed under composition, so $\mathcal{F}^+$ is indeed a category. By construction, it is a finitely generated subcategory of $\mathcal{I}$ which has i_∞ as unique final object. Since, by Lemma 4.17.2, the colimit of any diagram $M : \mathcal{I} \to \mathcal{C}$ coincides with the colimit of $M \circ \Pi$, this gives the desired reduction.

The set of all finitely generated subcategories of $\mathcal{I}$ with a unique final object is naturally ordered by inclusion. We take $\mathcal{I}_0$ to be the category corresponding to this set. We also have a functor $M_0 : \mathcal{I}_0 \to \mathcal{I}$, which takes an arrow $\mathcal{F} \subset \mathcal{F}'$ in $\mathcal{I}_0$ to the unique map from the final object of $\mathcal{F}$ to the final object of $\mathcal{F}'$. Given any two finitely generated subcategories of $\mathcal{I}$, the category generated by these two categories is also finitely generated. By our assumption on $\mathcal{I}$, it is also contained in a finitely generated subcategory of $\mathcal{I}$ with a unique final object. This shows that $\mathcal{I}_0$ is directed.

Finally, we verify that M_0 is cofinal. Since any object of $\mathcal{I}$ is the final object in the subcategory consisting of only that object and its identity arrow, the functor M_0 is surjective on objects. In particular, Condition (1) of Definition 4.17.1 is satisfied. Given an object i of $\mathcal{I}$, objects $\mathcal{F}_1, \mathcal{F}_2$ in $\mathcal{I}_0$ and maps $\varphi_1 : i \to M_0(\mathcal{F}_1)$ and $\varphi_2 : i \to M_0(\mathcal{F}_2)$ in $\mathcal{I}$, we can take $\mathcal{F}_{12}$ to be a finitely generated category with a unique final object containing $\mathcal{F}_1$, $\mathcal{F}_2$ and the morphisms φ_1, φ_2. The resulting diagram commutes

$$\begin{tikzcd}
 & M_0(\mathcal{F}_{12}) & \\
M_0(\mathcal{F}_1) \arrow[ur] & & M_0(\mathcal{F}_2) \arrow[ul] \\
 & i \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

since it lives in the category $\mathcal{F}_{12}$ and $M_0(\mathcal{F}_{12})$ is final in this category. Hence also Condition (2) is satisfied, which concludes the proof. □

09P8 Remark 4.21.6. Note that a finite directed set $(I, \geq)$ always has a greatest object i_∞. Hence any colimit of a system $(M_i, f_{ii'})$ over such a set is trivial in the sense that the colimit equals M_{i_∞}. In contrast, a colimit indexed by a finite filtered category need not be trivial. For instance, let $\mathcal{I}$ be the category with a single object i and a single non-trivial morphism e satisfying $e = e \circ e$. The colimit of a diagram $M : \mathcal{I} \to \textit{Sets}$ is the image of the idempotent $M(e)$. This illustrates that something like the trick of passing to $\mathcal{I} \times \omega$ in the proof of Lemma 4.21.5 is essential.

086J Lemma 4.21.7. If $S : \mathcal{I} \to \text{Sets}$ is a cofiltered diagram of sets and all the S_i are finite nonempty, then $\lim_i S_i$ is nonempty. In other words, the limit of a directed inverse system of finite nonempty sets is nonempty.

Proof. The two statements are equivalent by Lemma 4.21.5. Let I be a directed set and let $(S_i)_{i \in I}$ be an inverse system of finite nonempty sets over I. Let us say that a subsystem T is a family $T = (T_i)_{i \in I}$ of nonempty subsets $T_i \subset S_i$ such that $T_{i'}$ is mapped into T_i by the transition map $S_{i'} \to S_i$ for all $i' \geq i$. Denote $\mathcal{T}$ the set of subsystems. We order $\mathcal{T}$ by inclusion. Suppose T_α, $\alpha \in A$ is a totally ordered family of elements of $\mathcal{T}$. Say $T_\alpha = (T_{\alpha,i})_{i \in I}$. Then we can find a lower bound $T = (T_i)_{i \in I}$ by setting $T_i = \bigcap_{\alpha \in A} T_{\alpha,i}$ which is manifestly a finite nonempty subset of S_i as all the $T_{\alpha,i}$ are nonempty and as the T_α form a totally ordered family. Thus we may apply Zorn's lemma to see that $\mathcal{T}$ has minimal elements.

Let's analyze what a minimal element $T \in \mathcal{T}$ looks like. First observe that the maps $T_{i'} \to T_i$ are all surjective. Namely, as I is a directed set and T_i is finite, the intersection $T'_i = \bigcap_{i' \geq i} \text{Im}(T_{i'} \to T_i)$ is nonempty. Thus $T' = (T'_i)$ is a subsystem contained in T and by minimality $T' = T$. Finally, we claim that T_i is a singleton for each i. Namely, if $x \in T_i$, then we can define $T'_{i'} = (T_{i'} \to T_i)^{-1}(\{x\})$ for $i' \geq i$ and $T'_j = T_j$ if $j \not\geq i$. This is another subsystem as we've seen above that the transition maps of the subsystem T are surjective. By minimality we see that $T = T'$ which indeed implies that T_i is a singleton. This holds for every $i \in I$, hence we see that $T_i = \{x_i\}$ for some $x_i \in S_i$ with $x_{i'} \mapsto x_i$ under the map $S_{i'} \to S_i$ for every $i' \geq i$. In other words, $(x_i) \in \lim S_i$ and the lemma is proved. □

4.22. Essentially constant systems

05PT

Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$. Assume the index category $\mathcal{I}$ is filtered. In this case there are three successively stronger notions which pick out an object X of $\mathcal{C}$. The first is just

$$X = \mathrm{colim}_{i \in \mathcal{I}} M_i.$$

Then X comes equipped with the coprojections $M_i \to X$. A stronger condition would be to require that X is the colimit and that there exist an $i \in \mathcal{I}$ and a morphism $X \to M_i$ such that the composition $X \to M_i \to X$ is id_X. An even stronger condition is the following.

05PU Definition 4.22.1. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$.

(1) Assume the index category $\mathcal{I}$ is filtered and let $(X, \{M_i \to X\}_i)$ be a cocone for M, see Remark 4.14.5. We say M is essentially constant with value X if there exist an $i \in \mathcal{I}$ and a morphism $X \to M_i$ such that
 (a) $X \to M_i \to X$ is id_X, and
 (b) for all j there exist k and morphisms $i \to k$ and $j \to k$ such that the morphism $M_j \to M_k$ equals the composition $M_j \to X \to M_i \to M_k$.
(2) Assume the index category $\mathcal{I}$ is cofiltered and let $(X, \{X \to M_i\}_i)$ be a cone for M, see Remark 4.14.5. We say M is essentially constant with value X if there exist an $i \in \mathcal{I}$ and a morphism $M_i \to X$ such that
 (a) $X \to M_i \to X$ is id_X, and
 (b) for all j there exist k and morphisms $k \to i$ and $k \to j$ such that the morphism $M_k \to M_j$ equals the composition $M_k \to M_i \to X \to M_j$.

Please keep in mind Lemma 4.22.3 when using this definition.

Which of the two versions is meant will be clear from context. If there is any confusion we will distinguish between these by saying that the first version means M is essentially constant as an ind-object, and in the second case we will say it is essentially constant as a pro-object. This terminology is further explained in Remarks 4.22.4 and 4.22.5. In fact we will often use the terminology "essentially constant system" which formally speaking is only defined for systems over directed sets.

05PV Definition 4.22.2. Let $\mathcal{C}$ be a category. A directed system $(M_i, f_{ii'})$ is an essentially constant system if M viewed as a functor $I \to \mathcal{C}$ defines an essentially constant diagram. A directed inverse system $(M_i, f_{ii'})$ is an essentially constant inverse system if M viewed as a functor $I^{opp} \to \mathcal{C}$ defines an essentially constant inverse diagram.

If $(M_i, f_{ii'})$ is an essentially constant system and the morphisms $f_{ii'}$ are monomorphisms, then for all $i \leq i'$ sufficiently large the morphisms $f_{ii'}$ are isomorphisms. On the other hand, consider the system

$$\mathbf{Z}^2 \to \mathbf{Z}^2 \to \mathbf{Z}^2 \to \ldots$$

with maps given by $(a, b) \mapsto (a + b, 0)$. This system is essentially constant with value $\mathbf{Z}$ but every transition map has a kernel.

Here is an example of a system which is not essentially constant. Let $M = \bigoplus_{n \geq 0} \mathbf{Z}$ and to let $S : M \to M$ be the shift operator $(a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots)$. In this case the system $M \to M \to M \to \ldots$ with transition maps S has colimit 0 and the composition $0 \to M \to 0$ is the identity, but the system is not essentially constant.

The following lemma is a sanity check.

0G2V Lemma 4.22.3. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. If $\mathcal{I}$ is filtered and M is essentially constant as an ind-object, then $X = \operatorname{colim} M_i$ exists and M is essentially constant with value X. If $\mathcal{I}$ is cofiltered and M is essentially constant as a pro-object, then $X = \lim M_i$ exists and M is essentially constant with value X.

Proof. Omitted. This is a good excercise in the definitions. $\square$

05PW Remark 4.22.4. Let $\mathcal{C}$ be a category. There exists a big category Ind-$\mathcal{C}$ of ind-objects of $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are filtered diagrams in $\mathcal{C}$, then we can define

$$\mathrm{Mor}_{\text{Ind-}\mathcal{C}}(F, G) = \lim_i \operatorname{colim}_j \mathrm{Mor}_\mathcal{C}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to$ Ind-$\mathcal{C}$ which maps X to the constant system on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only if F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

05PX Remark 4.22.5. Let $\mathcal{C}$ be a category. There exists a big category Pro-$\mathcal{C}$ of pro-objects of $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are cofiltered diagrams in $\mathcal{C}$, then we can define

$$\mathrm{Mor}_{\text{Pro-}\mathcal{C}}(F, G) = \lim_j \operatorname{colim}_i \mathrm{Mor}_\mathcal{C}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to$ Pro-$\mathcal{C}$ which maps X to the constant system on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only if F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

0G2W Example 4.22.6. Let $\mathcal{C}$ be a category. Let (X_n) and (Y_n) be inverse systems in $\mathcal{C}$ over $\mathbf{N}$ with the usual ordering. Picture:

$$\ldots \to X_3 \to X_2 \to X_1 \quad \text{and} \quad \ldots \to Y_3 \to Y_2 \to Y_1$$

Let $a : (X_n) \to (Y_n)$ be a morphism of pro-objects of $\mathcal{C}$. What does a amount to? Well, for each $n \in \mathbf{N}$ there should exist an $m(n)$ and a morphism $a_n : X_{m(n)} \to Y_n$. These morphisms ought to agree in the following sense: for all $n' \geq n$ there exists an $m(n', n) \geq m(n'), m(n)$ such that the diagram

$$\begin{tikzcd} X_{m(n,n')} \arrow[rr] \arrow[d] & & X_{m(n)} \arrow[d, "a_n"] \\ X_{m(n')} \arrow[r, "a_{n'}"] & Y_{n'} \arrow[r] & Y_n \end{tikzcd}$$

commutes. After replacing $m(n)$ by $\max_{k,l \leq n}\{m(n,k), m(k,l)\}$ we see that we obtain $\ldots \geq m(3) \geq m(2) \geq m(1)$ and a commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & X_{m(3)} \arrow[r] \arrow[d, "a_3"] & X_{m(2)} \arrow[r] \arrow[d, "a_2"] & X_{m(1)} \arrow[d, "a_1"] \\ \ldots \arrow[r] & Y_3 \arrow[r] & Y_2 \arrow[r] & Y_1 \end{tikzcd}$$

Given an increasing map $m' : \mathbf{N} \to \mathbf{N}$ with $m' \geq m$ and setting $a'_i : X_{m'(i)} \to X_{m(i)} \to Y_i$ the pair (m', a') defines the same morphism of pro-systems. Conversely, given two pairs (m_1, a_1) and (m_1, a_2) as above then these define the same morphism of pro-objects if and only if we can find $m' \geq m_1, m_2$ such that $a'_1 = a'_2$.

0G2X Remark 4.22.7. Let $\mathcal{C}$ be a category. Let $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ be cofiltered diagrams in $\mathcal{C}$. Consider the functors $A, B : \mathcal{C} \to \textit{Sets}$ defined by

$$A(X) = \mathrm{colim}_i \mathrm{Mor}_\mathcal{C}(F(i), X) \quad \text{and} \quad B(X) = \mathrm{colim}_j \mathrm{Mor}_\mathcal{C}(G(j), X)$$

We claim that a morphism of pro-systems from F to G is the same thing as a transformation of functors $t : B \to A$. Namely, given t we can apply t to the class of $\mathrm{id}_{G(j)}$ in $B(G(j))$ to get a compatible system of elements $\xi_j \in A(G(j)) = \mathrm{colim}_i \mathrm{Mor}_\mathcal{C}(F(i), G(j))$ which is exactly our definition of a morphism in Pro-$\mathcal{C}$ in Remark 4.22.5. We omit the construction of a transformation $B \to A$ given a morphism of pro-objects from F to G.

05SH Lemma 4.22.8. Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered (resp. cofiltered) index category $\mathcal{I}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If M is essentially constant as an ind-object (resp. pro-object), then so is $F \circ M : \mathcal{I} \to \mathcal{D}$.

Proof. If X is a value for M, then it follows immediately from the definition that $F(X)$ is a value for $F \circ M$. □

05PY Lemma 4.22.9. Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered index category $\mathcal{I}$. The following are equivalent

(1) M is an essentially constant ind-object, and
(2) $X = \mathrm{colim}_i M_i$ exists and for any W in $\mathcal{C}$ the map

$$\mathrm{colim}_i \mathrm{Mor}_\mathcal{C}(W, M_i) \longrightarrow \mathrm{Mor}_\mathcal{C}(W, X)$$

is bijective.

Proof. Assume (2) holds. Then $\mathrm{id}_X \in \mathrm{Mor}_\mathcal{C}(X, X)$ comes from a morphism $X \to M_i$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps

$$\mathrm{Mor}_\mathcal{C}(W, X) \longrightarrow \mathrm{colim}_i \mathrm{Mor}_\mathcal{C}(W, M_i) \longrightarrow \mathrm{Mor}_\mathcal{C}(W, X)$$

are bijective for all W where the first one is induced by the morphism $X \to M_i$ we found above, and the composition is the identity. This means that the composition

$$\mathrm{colim}_i \mathrm{Mor}_\mathcal{C}(W, M_i) \longrightarrow \mathrm{Mor}_\mathcal{C}(W, X) \longrightarrow \mathrm{colim}_i \mathrm{Mor}_\mathcal{C}(W, M_i)$$

is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_j \to X \to M_i \to M_k$ is equal to $M_j \to M_k$ for some k large enough. This proves (1) holds. The proof of (1) $\Rightarrow$ (2) is omitted. □

05PZ Lemma 4.22.10. Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with cofiltered index category $\mathcal{I}$. The following are equivalent

(1) M is an essentially constant pro-object, and
(2) $X = \lim_i M_i$ exists and for any W in $\mathcal{C}$ the map

$$\mathrm{colim}_{i \in \mathcal{I}^{opp}} \mathrm{Mor}_\mathcal{C}(M_i, W) \longrightarrow \mathrm{Mor}_\mathcal{C}(X, W)$$

is bijective.

Proof. Assume (2) holds. Then $\mathrm{id}_X \in \mathrm{Mor}_\mathcal{C}(X, X)$ comes from a morphism $M_i \to X$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps

$$\mathrm{Mor}_\mathcal{C}(X, W) \longrightarrow \mathrm{colim}_i \mathrm{Mor}_\mathcal{C}(M_i, W) \longrightarrow \mathrm{Mor}_\mathcal{C}(X, W)$$

are bijective for all W where the first one is induced by the morphism $M_i \to X$ we found above, and the composition is the identity. This means that the composition

$$\operatorname{colim}_i \mathrm{Mor}_{\mathcal{C}}(M_i, W) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, W) \longrightarrow \operatorname{colim}_i \mathrm{Mor}_{\mathcal{C}}(M_i, W)$$

is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_k \to M_i \to X \to M_j$ is equal to $M_k \to M_j$ for some k large enough. This proves (1) holds. The proof of (1) $\Rightarrow$ (2) is omitted. $\square$

0A1S Lemma 4.22.11. Let $\mathcal{C}$ be a category. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of filtered index categories. If H is cofinal, then any diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ H$ is essentially constant.

Proof. This follows formally from Lemmas 4.22.9 and 4.17.2. $\square$

0A2C Lemma 4.22.12. Let $\mathcal{I}$ and $\mathcal{J}$ be filtered categories and denote $p : \mathcal{I} \times \mathcal{J} \to \mathcal{J}$ the projection. Then $\mathcal{I} \times \mathcal{J}$ is filtered and a diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ p : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ is essentially constant.

Proof. We omit the verification that $\mathcal{I} \times \mathcal{J}$ is filtered. The equivalence follows from Lemma 4.22.11 because p is cofinal (verification omitted). $\square$

0A1T Lemma 4.22.13. Let $\mathcal{C}$ be a category. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of cofiltered index categories. If H is initial, then any diagram $M : \mathcal{J} \to \mathcal{C}$ is essentially constant if and only if $M \circ H$ is essentially constant.

Proof. This follows formally from Lemmas 4.22.10, 4.17.4, 4.17.2, and the fact that if $\mathcal{I}$ is initial in $\mathcal{J}$, then $\mathcal{I}^{opp}$ is cofinal in $\mathcal{J}^{opp}$. $\square$

4.23. Exact functors

0033 In this section we define exact functors.

0034 Definition 4.23.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) Suppose all finite limits exist in $\mathcal{A}$. We say F is left exact if it commutes with all finite limits.
(2) Suppose all finite colimits exist in $\mathcal{A}$. We say F is right exact if it commutes with all finite colimits.
(3) We say F is exact if it is both left and right exact.

0035 Lemma 4.23.2. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose all finite limits exist in $\mathcal{A}$, see Lemma 4.18.4. The following are equivalent:

(1) F is left exact,
(2) F commutes with finite products and equalizers, and
(3) F transforms a final object of $\mathcal{A}$ into a final object of $\mathcal{B}$, and commutes with fibre products.

Proof. Lemma 4.14.11 shows that (2) implies (1). Suppose (3) holds. The fibre product over the final object is the product. If $a, b : A \to B$ are morphisms of $\mathcal{A}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(\mathrm{pr}_1, \mathrm{pr}_2), A \times A, \Delta} A.$$

Thus (3) implies (2). Finally (1) implies (3) because the empty limit is a final object, and fibre products are limits. $\square$

0GMN Lemma 4.23.3. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose all finite colimits exist in $\mathcal{A}$, see Lemma 4.18.7. The following are equivalent:

(1) F is right exact,
(2) F commutes with finite coproducts and coequalizers, and
(3) F transforms an initial object of $\mathcal{A}$ into an initial object of $\mathcal{B}$, and commutes with pushouts.

Proof. Dual to Lemma 4.23.2. □

4.24. Adjoint functors

0036

0037 Definition 4.24.1. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors. We say that u is a left adjoint of v, or that v is a right adjoint to u if there are bijections

$$\mathrm{Mor}_{\mathcal{D}}(u(X), Y) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, v(Y))$$

functorial in $X \in \mathrm{Ob}(\mathcal{C})$, and $Y \in \mathrm{Ob}(\mathcal{D})$.

In other words, this means that there is a given isomorphism of functors $\mathcal{C}^{opp} \times \mathcal{D} \to \textit{Sets}$ from $\mathrm{Mor}_{\mathcal{D}}(u(-), -)$ to $\mathrm{Mor}_{\mathcal{C}}(-, v(-))$. For any object X of $\mathcal{C}$ we obtain a morphism $X \to v(u(X))$ corresponding to $\mathrm{id}_{u(X)}$. Similarly, for any object Y of $\mathcal{D}$ we obtain a morphism $u(v(Y)) \to Y$ corresponding to $\mathrm{id}_{v(Y)}$. These maps are called the adjunction maps. The adjunction maps are functorial in X and Y, hence we obtain morphisms of functors

$$\eta : \mathrm{id}_{\mathcal{C}} \to v \circ u \quad (\text{unit}) \quad \text{and} \quad \epsilon : u \circ v \to \mathrm{id}_{\mathcal{D}} \quad (\text{counit}).$$

Moreover, if $\alpha : u(X) \to Y$ and $\beta : X \to v(Y)$ are morphisms, then the following are equivalent

(1) α and β correspond to each other via the bijection of the definition,
(2) β is the composition $X \to v(u(X)) \xrightarrow{v(\alpha)} v(Y)$, and
(3) α is the composition $u(X) \xrightarrow{u(\beta)} u(v(Y)) \to Y$.

In this way one can reformulate the notion of adjoint functors in terms of adjunction maps.

0A8B Lemma 4.24.2. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. If for each $y \in \mathrm{Ob}(\mathcal{D})$ the functor $x \mapsto \mathrm{Mor}_{\mathcal{D}}(u(x), y)$ is representable, then u has a right adjoint.

Proof. For each y choose an object $v(y)$ and an isomorphism $\mathrm{Mor}_{\mathcal{C}}(-, v(y)) \to \mathrm{Mor}_{\mathcal{D}}(u(-), y)$ of functors. By Yoneda's lemma (Lemma 4.3.5) for any morphism $g : y \to y'$ the transformation of functors

$$\mathrm{Mor}_{\mathcal{C}}(-, v(y)) \to \mathrm{Mor}_{\mathcal{D}}(u(-), y) \to \mathrm{Mor}_{\mathcal{D}}(u(-), y') \to \mathrm{Mor}_{\mathcal{C}}(-, v(y'))$$

corresponds to a unique morphism $v(g) : v(y) \to v(y')$. We omit the verification that v is a functor and that it is right adjoint to u. □

0FWV Lemma 4.24.3. Let u be a left adjoint to v as in Definition 4.24.1. Bhargav Bhatt, private communication.

(1) If $v \circ u$ is fully faithful, then u is fully faithful.
(2) If $u \circ v$ is fully faithful, then v is fully faithful.

Proof. Proof of (2). Assume $u \circ v$ is fully faithful. Say we have X, Y in $\mathcal{D}$. Then the natural composite map

$$\mathrm{Mor}(X, Y) \to \mathrm{Mor}(v(X), v(Y)) \to \mathrm{Mor}(u(v(X)), u(v(Y)))$$

is a bijection, so v is at least faithful. To show full faithfulness, we must show that the second map above is injective. But the adjunction between u and v says that

$$\mathrm{Mor}(v(X), v(Y)) \to \mathrm{Mor}(u(v(X)), u(v(Y))) \to \mathrm{Mor}(u(v(X)), Y)$$

is a bijection, where the first map is natural one and the second map comes from the counit $u(v(Y)) \to Y$ of the adjunction. So this says that $\mathrm{Mor}(v(X), v(Y)) \to \mathrm{Mor}(u(v(X)), u(v(Y)))$ is also injective, as wanted. The proof of (1) is dual to this. □

07RB Lemma 4.24.4. Let u be a left adjoint to v as in Definition 4.24.1. Then

(1) u is fully faithful $\Leftrightarrow$ $\mathrm{id} \cong v \circ u$ $\Leftrightarrow$ $\eta : \mathrm{id} \to v \circ u$ is an isomorphism,
(2) v is fully faithful $\Leftrightarrow$ $u \circ v \cong \mathrm{id}$ $\Leftrightarrow$ $\epsilon : u \circ v \to \mathrm{id}$ is an isomorphism.

Proof. Proof of (1). Assume u is fully faithful. We will show $\eta_X : X \to v(u(X))$ is an isomorphism. Let $X' \to v(u(X))$ be any morphism. By adjointness this corresponds to a morphism $u(X') \to u(X)$. By fully faithfulness of u this corresponds to a unique morphism $X' \to X$. Thus we see that post-composing by η_X defines a bijection $\mathrm{Mor}(X', X) \to \mathrm{Mor}(X', v(u(X)))$. Hence η_X is an isomorphism. If there exists an isomorphism $\mathrm{id} \cong v \circ u$ of functors, then $v \circ u$ is fully faithful. By Lemma 4.24.3 we see that u is fully faithful. By the above this implies η is an isomorphism. Thus all 3 conditions are equivalent (and these conditions are also equivalent to $v \circ u$ being fully faithful).

Part (2) is dual to part (1). □

0038 Lemma 4.24.5. Let u be a left adjoint to v as in Definition 4.24.1.

(1) Suppose that $M : \mathcal{I} \to \mathcal{C}$ is a diagram, and suppose that $\mathrm{colim}_\mathcal{I} M$ exists in $\mathcal{C}$. Then $u(\mathrm{colim}_\mathcal{I} M) = \mathrm{colim}_\mathcal{I} u \circ M$. In other words, u commutes with (representable) colimits.
(2) Suppose that $M : \mathcal{I} \to \mathcal{D}$ is a diagram, and suppose that $\lim_\mathcal{I} M$ exists in $\mathcal{D}$. Then $v(\lim_\mathcal{I} M) = \lim_\mathcal{I} v \circ M$. In other words v commutes with representable limits.

Proof. A morphism from a colimit into an object is the same as a compatible system of morphisms from the constituents of the limit into the object, see Remark 4.14.4. So

$$\begin{aligned} \mathrm{Mor}_\mathcal{D}(u(\mathrm{colim}_{i\in\mathcal{I}} M_i), Y) &= \mathrm{Mor}_\mathcal{C}(\mathrm{colim}_{i\in\mathcal{I}} M_i, v(Y)) \\ &= \lim_{i\in\mathcal{I}^{opp}} \mathrm{Mor}_\mathcal{C}(M_i, v(Y)) \\ &= \lim_{i\in\mathcal{I}^{opp}} \mathrm{Mor}_\mathcal{D}(u(M_i), Y) \end{aligned}$$

proves that $u(\mathrm{colim}_{i\in\mathcal{I}} M_i)$ is the colimit we are looking for. A similar argument works for the other statement. □

0039 Lemma 4.24.6. Let u be a left adjoint of v as in Definition 4.24.1.

(1) If $\mathcal{C}$ has finite colimits, then u is right exact.
(2) If $\mathcal{D}$ has finite limits, then v is left exact.

Proof. Obvious from the definitions and Lemma 4.24.5. □

0GLL **Lemma 4.24.7.** Let $u : \mathcal{C} \to \mathcal{D}$ be a left adjoint to the functor $v : \mathcal{D} \to \mathcal{C}$. Let $\eta_X : X \to v(u(X))$ be the unit and $\epsilon_Y : u(v(Y)) \to Y$ be the counit. Then

$$u(X) \xrightarrow{u(\eta_X)} u(v(u(X))) \xrightarrow{\epsilon_{u(X)}} u(X) \quad \text{and} \quad v(Y) \xrightarrow{\eta_{v(Y)}} v(u(v(Y))) \xrightarrow{v(\epsilon_Y)} v(Y)$$

are the identity morphisms.

Proof. Omitted. $\square$

0B65 **Lemma 4.24.8.** Let $u_1, u_2 : \mathcal{C} \to \mathcal{D}$ be functors with right adjoints $v_1, v_2 : \mathcal{D} \to \mathcal{C}$. Let $\beta : u_2 \to u_1$ be a transformation of functors. Let $\beta^\vee : v_1 \to v_2$ be the corresponding transformation of adjoint functors. Then

$$\begin{tikzcd} u_2 \circ v_1 \arrow[r, "\beta"'] \arrow[d, "\beta^\vee"'] & u_1 \circ v_1 \arrow[d] \\ u_2 \circ v_2 \arrow[r] & \mathrm{id} \end{tikzcd}$$

is commutative where the unlabeled arrows are the counit transformations.

Proof. This is true because $\beta^\vee_D : v_1 D \to v_2 D$ is the unique morphism such that the induced maps $\mathrm{Mor}(C, v_1 D) \to \mathrm{Mor}(C, v_2 D)$ is the map $\mathrm{Mor}(u_1 C, D) \to \mathrm{Mor}(u_2 C, D)$ induced by $\beta_C : u_2 C \to u_1 C$. Namely, this means the map

$$\mathrm{Mor}(u_1 v_1 D, D') \to \mathrm{Mor}(u_2 v_1 D, D')$$

induced by $\beta_{v_1 D}$ is the same as the map

$$\mathrm{Mor}(v_1 D, v_1 D') \to \mathrm{Mor}(v_1 D, v_2 D')$$

induced by $\beta^\vee_{D'}$. Taking $D' = D$ we find that the counit $u_1 v_1 D \to D$ precomposed by $\beta_{v_1 D}$ corresponds to $\beta^\vee_D$ under adjunction. This exactly means that the diagram commutes when evaluated on D. $\square$

0DV0 **Lemma 4.24.9.** Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $v : \mathcal{A} \to \mathcal{B}$ and $v' : \mathcal{B} \to \mathcal{C}$ be functors with left adjoints u and u' respectively. Then

(1) The functor $v'' = v' \circ v$ has a left adjoint equal to $u'' = u \circ u'$.
(2) Given X in $\mathcal{A}$ we have

0DV1 (4.24.9.1) $$\epsilon^v_X \circ u(\epsilon^{v'}_{v(X)}) = \epsilon^{v''}_X : u''(v''(X)) \to X$$

Where ϵ is the counit of the adjunctions.

Proof. Let us unwind the formula in (2) because this will also immediately prove (1). First, the counit of the adjunctions for the pairs (u, v) and (u', v') are maps $\epsilon^v_X : u(v(X)) \to X$ and $\epsilon^{v'}_Y : u'(v'(Y)) \to Y$, see discussion following Definition 4.24.1. With u'' and v'' as in (1) we unwind everything

$$u''(v''(X)) = u(u'(v'(v(X)))) \xrightarrow{u(\epsilon^{v'}_{v(X)})} u(v(X)) \xrightarrow{\epsilon^v_X} X$$

to get the map on the left hand side of (4.24.9.1). Let us denote this by $\epsilon^{v''}_X$ for now. To see that this is the counit of an adjoint pair (u'', v'') we have to show that given Z in $\mathcal{C}$ the rule that sends a morphism $\beta : Z \to v''(X)$ to $\alpha = \epsilon^{v''}_X \circ u''(\beta) : u''(Z) \to X$ is a bijection on sets of morphisms. This is true because, this is the composition of the rule sending β to $\epsilon^{v'}_{v(X)} \circ u'(\beta)$ which is a bijection by assumption on (u', v') and then sending this to $\epsilon^v_X \circ u(\epsilon^{v'}_{v(X)} \circ u'(\beta))$ which is a bijection by assumption on (u, v). $\square$

4.25. A criterion for representability

0AHM The following lemma is often useful to prove the existence of universal objects in big categories, please see the discussion in Remark 4.25.2.

0AHN **Lemma 4.25.1.** Let $\mathcal{C}$ be a big[3] category which has limits. Let $F : \mathcal{C} \to \textit{Sets}$ be a functor. Assume that

(1) F commutes with limits,
(2) there exist a family $\{x_i\}_{i \in I}$ of objects of $\mathcal{C}$ and for each $i \in I$ an element $f_i \in F(x_i)$ such that for $y \in \text{Ob}(\mathcal{C})$ and $g \in F(y)$ there exist an i and a morphism $\varphi : x_i \to y$ with $F(\varphi)(f_i) = g$.

Then F is representable, i.e., there exists an object x of $\mathcal{C}$ such that

$$F(y) = \text{Mor}_\mathcal{C}(x, y)$$

functorially in y.

Proof. Let $\mathcal{I}$ be the category whose objects are the pairs (x_i, f_i) and whose morphisms $(x_i, f_i) \to (x_{i'}, f_{i'})$ are maps $\varphi : x_i \to x_{i'}$ in $\mathcal{C}$ such that $F(\varphi)(f_i) = f_{i'}$. Set

$$x = \lim_{(x_i, f_i) \in \mathcal{I}} x_i$$

(this will not be the x we are looking for, see below). The limit exists by assumption. As F commutes with limits we have

$$F(x) = \lim_{(x_i, f_i) \in \mathcal{I}} F(x_i).$$

Hence there is a universal element $f \in F(x)$ which maps to $f_i \in F(x_i)$ under F applied to the projection map $x \to x_i$. Using f we obtain a transformation of functors

$$\xi : \text{Mor}_\mathcal{C}(x, -) \longrightarrow F(-)$$

see Section 4.3. Let y be an arbitrary object of $\mathcal{C}$ and let $g \in F(y)$. Choose $x_i \to y$ such that f_i maps to g which is possible by assumption. Then F applied to the maps

$$x \longrightarrow x_i \longrightarrow y$$

(the first being the projection map of the limit defining x) sends f to g. Hence the transformation ξ is surjective.

In order to find the object representing F we let $e : x' \to x$ be the equalizer of all self maps $\varphi : x \to x$ with $F(\varphi)(f) = f$. Since F commutes with limits, it commutes with equalizers, and we see there exists an $f' \in F(x')$ mapping to f in $F(x)$. Since ξ is surjective and since f' maps to f we see that also $\xi' : \text{Mor}_\mathcal{C}(x', -) \to F(-)$ is surjective. Finally, suppose that $a, b : x' \to y$ are two maps such that $F(a)(f') = F(b)(f')$. We have to show $a = b$. Consider the equalizer $e' : x'' \to x'$. Again we find $f'' \in F(x'')$ mapping to f'. Choose a map $\psi : x \to x''$ such that $F(\psi)(f) = f''$. Then we see that $e \circ e' \circ \psi : x \to x$ is a morphism with $F(e \circ e' \circ \psi)(f) = f$. Hence $e \circ e' \circ \psi \circ e = e$. Since e is a monomorphism, this implies that e' is an epimorphism, thus $a = b$ as desired. $\square$

[3]See Remark 4.2.2.

0AHP Remark 4.25.2. The lemma above is often used to construct the free something on something. For example the free abelian group on a set, the free group on a set, etc. The idea, say in the case of the free group on a set E is to consider the functor

$$F : \text{Groups} \to \text{Sets}, \quad G \longmapsto \text{Map}(E, G)$$

This functor commutes with limits. As our family of objects we can take a family $E \to G_i$ consisting of groups G_i of cardinality at most $\max(\aleph_0, |E|)$ and set maps $E \to G_i$ such that every isomorphism class of such a structure occurs at least once. Namely, if $E \to G$ is a map from E to a group G, then the subgroup G' generated by the image has cardinality at most $\max(\aleph_0, |E|)$. The lemma tells us the functor is representable, hence there exists a group F_E such that $\text{Mor}_{\text{Groups}}(F_E, G) = \text{Map}(E, G)$. In particular, the identity morphism of F_E corresponds to a map $E \to F_E$ and one can show that F_E is generated by the image without imposing any relations.

Another typical application is that we can use the lemma to construct colimits once it is known that limits exist. We illustrate it using the category of topological spaces which has limits by Topology, Lemma 5.14.1. Namely, suppose that $\mathcal{I} \to \text{Top}$, $i \mapsto X_i$ is a functor. Then we can consider

$$F : \text{Top} \longrightarrow \text{Sets}, \quad Y \longmapsto \lim_{\mathcal{I}} \text{Mor}_{\text{Top}}(X_i, Y)$$

This functor commutes with limits. Moreover, given any topological space Y and an element $(\varphi_i : X_i \to Y)$ of $F(Y)$, there is a subspace $Y' \subset Y$ of cardinality at most $|\coprod X_i|$ such that the morphisms φ_i map into Y'. Namely, we can take the induced topology on the union of the images of the φ_i. Thus it is clear that the hypotheses of the lemma are satisfied and we find a topological space X representing the functor F, which precisely means that X is the colimit of the diagram $i \mapsto X_i$.

0AHQ Theorem 4.25.3 (Adjoint functor theorem). Let $G : \mathcal{C} \to \mathcal{D}$ be a functor of big categories. Assume $\mathcal{C}$ has limits, G commutes with them, and for every object y of $\mathcal{D}$ there exists a set of pairs $(x_i, f_i)_{i \in I}$ with $x_i \in \text{Ob}(\mathcal{C})$, $f_i \in \text{Mor}_{\mathcal{D}}(y, G(x_i))$ such that for any pair (x, f) with $x \in \text{Ob}(\mathcal{C})$, $f \in \text{Mor}_{\mathcal{D}}(y, G(x))$ there are an i and a morphism $h : x_i \to x$ such that $f = G(h) \circ f_i$. Then G has a left adjoint F.

Proof. The assumptions imply that for every object y of $\mathcal{D}$ the functor $x \mapsto \text{Mor}_{\mathcal{D}}(y, G(x))$ satisfies the assumptions of Lemma 4.25.1. Thus it is representable by an object, let's call it $F(y)$. An application of Yoneda's lemma (Lemma 4.3.5) turns the rule $y \mapsto F(y)$ into a functor which by construction is an adjoint to G. We omit the details. $\square$

4.26. Categorically compact objects

0FWW A little bit about "small" objects of a category.

0FWX Definition 4.26.1. Let $\mathcal{C}$ be a big[4] category. An object X of $\mathcal{C}$ is called a categorically compact if we have

$$\text{Mor}_{\mathcal{C}}(X, \text{colim}_i M_i) = \text{colim}_i \text{Mor}_{\mathcal{C}}(X, M_i)$$

for every filtered diagram $M : \mathcal{I} \to \mathcal{C}$ such that $\text{colim}_i M_i$ exists.

Often this definition is made only under the assumption that $\mathcal{C}$ has all filtered colimits.

[4] See Remark 4.2.2.

0FWY Lemma 4.26.2. Let $\mathcal{C}$ and $\mathcal{D}$ be big categories having filtered colimits. Let $\mathcal{C}' \subset \mathcal{C}$ be a small full subcategory consisting of categorically compact objects of $\mathcal{C}$ such that every object of $\mathcal{C}$ is a filtered colimit of objects of $\mathcal{C}'$. Then every functor $F' : \mathcal{C}' \to \mathcal{D}$ has a unique extension $F : \mathcal{C} \to \mathcal{D}$ commuting with filtered colimits.

Proof. For every object X of $\mathcal{C}$ we may write X as a filtered colimit $X = \text{colim}\, X_i$ with $X_i \in \text{Ob}(\mathcal{C}')$. Then we set

$$F(X) = \text{colim}\, F'(X_i)$$

in $\mathcal{D}$. We will show below that this construction does not depend on the choice of the colimit presentation of X.

Suppose given a morphism $\alpha : X \to Y$ of $\mathcal{C}$ and $X = \text{colim}_{i \in I} X_i$ and $Y = \text{colim}_{j \in J} Y_i$ are written as filtered colimit of objects in $\mathcal{C}'$. For each $i \in I$ since X_i is a categorically compact object of $\mathcal{C}$ we can find a $j \in J$ and a commutative diagram

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & X \arrow[d, "\alpha"] \\ Y_j \arrow[r] & Y \end{tikzcd}$$

Then we obtain a morphism $F'(X_i) \to F'(Y_j) \to F(Y)$ where the second morphism is the coprojection into $F(Y) = \text{colim}\, F'(Y_j)$. The arrow $\beta_i : F'(X_i) \to F(Y)$ does not depend on the choice of j. For $i \leq i'$ the composition

$$F'(X_i) \to F'(X_{i'}) \xrightarrow{\beta_{i'}} F(Y)$$

is equal to β_i. Thus we obtain a well defined arrow

$$F(\alpha) : F(X) = \text{colim}\, F(X_i) \to F(Y)$$

by the universal property of the colimit. If $\alpha' : Y \to Z$ is a second morphism of $\mathcal{C}$ and $Z = \text{colim}\, Z_k$ is also written as filtered colimit of objects in $\mathcal{C}'$, then it is a pleasant exercise to show that the induced morphisms $F(\alpha) : F(X) \to F(Y)$ and $F(\alpha') : F(Y) \to F(Z)$ compose to the morphism $F(\alpha' \circ \alpha)$. Details omitted.

In particular, if we are given two presentations $X = \text{colim}\, X_i$ and $X = \text{colim}\, X'_{i'}$ as filtered colimits of systems in $\mathcal{C}'$, then we get mutually inverse arrows $\text{colim}\, F'(X_i) \to \text{colim}\, F'(X'_{i'})$ and $\text{colim}\, F'(X'_{i'}) \to \text{colim}\, F'(X_i)$. In other words, the value $F(X)$ is well defined independent of the choice of the presentation of X as a filtered colimit of objects of $\mathcal{C}'$. Together with the functoriality of F discussed in the previous paragraph, we find that F is a functor. Also, it is clear that $F(X) = F'(X)$ if $X \in \text{Ob}(\mathcal{C}')$.

The uniqueness statement in the lemma is clear, provided we show that F commutes with filtered colimits (because this statement doesn't make sense otherwise). To show this, suppose that $X = \text{colim}_{\lambda \in \Lambda} X_\lambda$ is a filtered colimit of $\mathcal{C}$. Since F is a functor we certainly get a map

$$\text{colim}_\lambda F(X_\lambda) \longrightarrow F(X)$$

On the other hand, write $X = \operatorname{colim} X_i$ as a filtered colimit of objects of $\mathcal{C}'$. As above, for each $i \in I$ we can choose a $\lambda \in \Lambda$ and a commutative diagram

$$\begin{tikzcd} X_i \arrow[rr] \arrow[dr] & & X_\lambda \arrow[dl] \\ & X & \end{tikzcd}$$

As above this determines a well defined morphism $F'(X_i) \to \operatorname{colim}_\lambda F(X_\lambda)$ compatible with transition morphisms and hence a morphism

$$F(X) = \operatorname{colim}_i F'(X_i) \longrightarrow \operatorname{colim}_\lambda F(X_\lambda)$$

This morphism is inverse to the morphism above (details omitted) and proves that $F(X) = \operatorname{colim}_\lambda F(X_\lambda)$ as desired. □

4.27. Localization in categories

04VB The basic idea of this section is given a category $\mathcal{C}$ and a set of arrows S to construct a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ such that all elements of S become invertible in $S^{-1}\mathcal{C}$ and such that F is universal among all functors with this property. References for this section are [GZ67, Chapter I, Section 2] and [Ver96, Chapter II, Section 2].

04VC **Definition 4.27.1.** Let $\mathcal{C}$ be a category. A set of arrows S of $\mathcal{C}$ is called a *left multiplicative system* if it has the following properties:

- LMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.
- LMS2 Every solid diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[d, dotted, "s"] \\ Z \arrow[r, dotted, "f"] & W \end{tikzcd}$$

  with $t \in S$ can be completed to a commutative dotted square with $s \in S$.
- LMS3 For every pair of morphisms $f, g : X \to Y$ and $t \in S$ with target X such that $f \circ t = g \circ t$ there exists an $s \in S$ with source Y such that $s \circ f = s \circ g$.

A set of arrows S of $\mathcal{C}$ is called a *right multiplicative system* if it has the following properties:

- RMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.
- RMS2 Every solid diagram

$$\begin{tikzcd} X \arrow[r, dotted, "g"'] \arrow[d, dotted, "t"'] & Y \arrow[d, "s"] \\ Z \arrow[r, "f"] & W \end{tikzcd}$$

  with $s \in S$ can be completed to a commutative dotted square with $t \in S$.
- RMS3 For every pair of morphisms $f, g : X \to Y$ and $s \in S$ with source Y such that $s \circ f = s \circ g$ there exists a $t \in S$ with target X such that $f \circ t = g \circ t$.

A set of arrows S of $\mathcal{C}$ is called a *multiplicative system* if it is both a left multiplicative system and a right multiplicative system. In other words, this means that MS1, MS2, MS3 hold, where MS1 = LMS1 + RMS1, MS2 = LMS2 + RMS2, and MS3 = LMS3 + RMS3. (That said, of course LMS1 = RMS1 = MS1.)

These conditions are useful to construct the categories $S^{-1}\mathcal{C}$ as follows.

Left calculus of fractions. Let $\mathcal{C}$ be a category and let S be a left multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 4.27.2):

(1) We set $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X \to Y', s : Y \to Y')$ with $s \in S$ up to equivalence. (The equivalence is defined below. Think of the equivalence class of a pair (f, s) as $s^{-1}f : X \to Y$.)
(3) Two pairs $(f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $(f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are said to be equivalent if there exist a third pair $(f_3 : X \to Y_3, s_3 : Y \to Y_3)$ and morphisms $u : Y_1 \to Y_3$ and $v : Y_2 \to Y_3$ of $\mathcal{C}$ fitting into the commutative diagram

$$\begin{tikzcd} & Y_1 \arrow[d, "u"] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\ & Y_2 \arrow[u, "v"'] & \end{tikzcd}$$

(4) The composition of the equivalence classes of the pairs $(f : X \to Y', s : Y \to Y')$ and $(g : Y \to Z', t : Z \to Z')$ is defined as the equivalence class of a pair $(h \circ f : X \to Z'', u \circ t : Z \to Z'')$ where h and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"'] \arrow[d, "s"'] & Z' \arrow[d, "u"] \\ Y' \arrow[r, "h"] & Z'' \end{tikzcd}$$

which exists by assumption.
(5) The identity morphism $X \to X$ in $S^{-1}\mathcal{C}$ is the equivalence class of the pair $(\mathrm{id} : X \to X, \mathrm{id} : X \to X)$.

04VD Lemma 4.27.2. Let $\mathcal{C}$ be a category and let S be a left multiplicative system.

(1) The relation on pairs defined above is an equivalence relation.
(2) The composition rule given above is well defined on equivalence classes.
(3) Composition is associative (and the identity morphisms satisfy the identity axioms), and hence $S^{-1}\mathcal{C}$ is a category.

Proof. Proof of (1). Let us say two pairs $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are elementary equivalent if there exists a morphism $a : Y_1 \to Y_2$ of $\mathcal{C}$ such that $a \circ f_1 = f_2$ and $a \circ s_1 = s_2$. Diagram:

$$\begin{tikzcd} X \arrow[r, "f_1"'] \arrow[d, equal] & Y_1 \arrow[d, "a"] & Y \arrow[l, "s_1"] \arrow[d, equal] \\ X \arrow[r, "f_2"] & Y_2 & Y \arrow[l, "s_2"'] \end{tikzcd}$$

Let us denote this property by saying $p_1 E p_2$. Note that pEp and $aEb, bEc \Rightarrow aEc$. (Despite its name, E is not an equivalence relation.) Part (1) claims that the relation $p \sim p' \Leftrightarrow \exists q : pEq \wedge p'Eq$ (where q is supposed to be a pair satisfying the same conditions as p and p') is an equivalence relation. A simple formal argument, using the properties of E above, shows that it suffices to prove $p_3Ep_1, p_3Ep_2 \Rightarrow p_1 \sim p_2$. Thus suppose that we are given a commutative diagram

$$\begin{tikzcd}
 & Y_1 & \\
X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 \arrow[u, "a_{31}"] \arrow[d, "a_{32}"] & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\
 & Y_2 &
\end{tikzcd}$$

with $s_i \in S$. First we apply LMS2 to get a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "s_2"'] \arrow[d, "s_1"'] & Y_2 \arrow[d, dotted, "a_{24}"] \\
Y_1 \arrow[r, dotted, "a_{14}"] & Y_4
\end{tikzcd}$$

with $a_{24} \in S$. Then, we have

$$a_{14} \circ a_{31} \circ s_3 = a_{14} \circ s_1 = a_{24} \circ s_2 = a_{24} \circ a_{32} \circ s_3.$$

Hence, by LMS3, there exists a morphism $s_{44} : Y_4 \to Y_4'$ such that $s_{44} \in S$ and $s_{44} \circ a_{14} \circ a_{31} = s_{44} \circ a_{24} \circ a_{32}$. Hence, after replacing Y_4, a_{14} and a_{24} by Y_4', $s_{44} \circ a_{14}$ and $s_{44} \circ a_{24}$, we may assume that $a_{14} \circ a_{31} = a_{24} \circ a_{32}$ (and we still have $a_{24} \in S$ and $a_{14} \circ s_1 = a_{24} \circ s_2$). Set

$$f_4 = a_{14} \circ f_1 = a_{14} \circ a_{31} \circ f_3 = a_{24} \circ a_{32} \circ f_3 = a_{24} \circ f_2$$

and $s_4 = a_{14} \circ s_1 = a_{24} \circ s_2$. Then, the diagram

$$\begin{tikzcd}
X \arrow[r, "f_1"'] \arrow[d, equal] & Y_1 \arrow[d, "a_{14}"] & Y \arrow[l, "s_1"] \arrow[d, equal] \\
X \arrow[r, "f_4"] & Y_4 & Y \arrow[l, "s_4"']
\end{tikzcd}$$

commutes, and we have $s_4 \in S$ (by LMS1). Thus, p_1Ep_4, where $p_4 = (f_4, s_4)$. Similarly, p_2Ep_4. Combining these, we find $p_1 \sim p_2$.

Proof of (2). Let $p = (f : X \to Y', s : Y \to Y')$ and $q = (g : Y \to Z', t : Z \to Z')$ be pairs as in the definition of composition above. To compose we choose a diagram

$$\begin{tikzcd}
Y \arrow[r, "g"'] \arrow[d, "s"'] & Z' \arrow[d, "u_2"] \\
Y' \arrow[r, "h_2"] & Z_2
\end{tikzcd}$$

with $u_2 \in S$. We first show that the equivalence class of the pair $r_2 = (h_2 \circ f : X \to Z_2, u_2 \circ t : Z \to Z_2)$ is independent of the choice of (Z_2, h_2, u_2). Namely, suppose that (Z_3, h_3, u_3) is another choice with corresponding composition $r_3 = (h_3 \circ f :$

$X \to Z_3, u_3 \circ t : Z \to Z_3)$. Then by LMS2 we can choose a diagram

$$\begin{tikzcd} Z' \arrow[r, "u_3"'] \arrow[d, "u_2"'] & Z_3 \arrow[d, "u_{34}"] \\ Z_2 \arrow[r, "h_{24}"] & Z_4 \end{tikzcd}$$

with $u_{34} \in S$. We have $h_2 \circ s = u_2 \circ g$ and similarly $h_3 \circ s = u_3 \circ g$. Now,

$$u_{34} \circ h_3 \circ s = u_{34} \circ u_3 \circ g = h_{24} \circ u_2 \circ g = h_{24} \circ h_2 \circ s.$$

Hence, LMS3 shows that there exist a Z'_4 and an $s_{44} : Z_4 \to Z'_4$ such that $s_{44} \circ u_{34} \circ h_3 = s_{44} \circ h_{24} \circ h_2$. Replacing Z_4, h_{24} and u_{34} by Z'_4, $s_{44} \circ h_{24}$ and $s_{44} \circ u_{34}$, we may assume that $u_{34} \circ h_3 = h_{24} \circ h_2$. Meanwhile, the relations $u_{34} \circ u_3 = h_{24} \circ u_2$ and $u_{34} \in S$ continue to hold. We can now set $h_4 = u_{34} \circ h_3 = h_{24} \circ h_2$ and $u_4 = u_{34} \circ u_3 = h_{24} \circ u_2$. Then, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "h_2 \circ f"'] \arrow[d, equal] & Z_2 \arrow[d, "h_{24}"] & Z \arrow[l, "u_2 \circ t"] \arrow[d, equal] \\ X \arrow[r, "h_4 \circ f"] \arrow[d, equal] & Z_4 & Z \arrow[l, "u_4 \circ t"'] \arrow[d, equal] \\ X \arrow[r, "h_3 \circ f"] & Z_3 \arrow[u, "u_{34}"] & Z \arrow[l, "u_3 \circ t"'] \end{tikzcd}$$

Hence we obtain a pair $r_4 = (h_4 \circ f : X \to Z_4, u_4 \circ t : Z \to Z_4)$ and the above diagram shows that we have $r_2 E r_4$ and $r_3 E r_4$, whence $r_2 \sim r_3$, as desired. Thus it now makes sense to define $p \circ q$ as the equivalence class of all possible pairs r obtained as above.

To finish the proof of (2) we have to show that given pairs p_1, p_2, q such that $p_1 E p_2$ then $p_1 \circ q = p_2 \circ q$ and $q \circ p_1 = q \circ p_2$ whenever the compositions make sense. To do this, write $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ and let $a : Y_1 \to Y_2$ be a morphism of $\mathcal{C}$ such that $f_2 = a \circ f_1$ and $s_2 = a \circ s_1$. First assume that $q = (g : Y \to Z', t : Z \to Z')$. In this case choose a commutative diagram as the one on the left

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s_2"'] & Z' \arrow[d, "u"] \\ Y_2 \arrow[r, "h"] & Z'' \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s_1"'] & Z' \arrow[d, "u"] \\ Y_1 \arrow[r, "h \circ a"] & Z'' \end{tikzcd}$$

(with $u \in S$), which implies the diagram on the right is commutative as well. Using these diagrams we see that both compositions $q \circ p_1$ and $q \circ p_2$ are the equivalence class of $(h \circ a \circ f_1 : X \to Z'', u \circ t : Z \to Z'')$. Thus $q \circ p_1 = q \circ p_2$. The proof of the other case, in which we have to show $p_1 \circ q = p_2 \circ q$, is omitted. (It is similar to the case we did.)

Proof of (3). We have to prove associativity of composition. Consider a solid diagram

$$\begin{tikzcd}
 & & & Z \arrow[d] \\
 & & Y \arrow[r] \arrow[d] & Z' \arrow[d, dotted] \\
 & X \arrow[r] \arrow[d] & Y' \arrow[r, dotted] \arrow[d, dotted] & Z'' \arrow[d, dotted] \\
W \arrow[r] & X' \arrow[r, dotted] & Y'' \arrow[r, dotted] & Z'''
\end{tikzcd}$$

(whose vertical arrows belong to S) which gives rise to three composable pairs. Using LMS2 we can choose the dotted arrows making the squares commutative and such that the vertical arrows are in S. Then it is clear that the composition of the three pairs is the equivalence class of the pair $(W \to Z''', Z \to Z''')$ gotten by composing the horizontal arrows on the bottom row and the vertical arrows on the right column.

We leave it to the reader to check the identity axioms. □

0BM1 Remark 4.27.3. The motivation for the construction of $S^{-1}\mathcal{C}$ is to "force" the morphisms in S to be invertible by artificially creating inverses to them (at the cost of some existing morphisms possibly becoming identified with each other). This is similar to the localization of a commutative ring at a multiplicative subset, and more generally to the localization of a noncommutative ring at a right denominator set (see [Lam99, Section 10A]). This is more than just a similarity: The construction of $S^{-1}\mathcal{C}$ (or, more precisely, its version for additive categories $\mathcal{C}$) actually generalizes the latter type of localization. Namely, a noncommutative ring can be viewed as a pre-additive category with a single object (the morphisms being the elements of the ring); a multiplicative subset of this ring then becomes a set S of morphisms satisfying LMS1 (aka RMS1). Then, the conditions RMS2 and RMS3 for this category and this subset S translate into the two conditions ("right permutable" and "right reversible") of a right denominator set (and similarly for LMS and left denominator sets), and $S^{-1}\mathcal{C}$ (with a properly defined additive structure) is the one-object category corresponding to the localization of the ring.

0BM2 Definition 4.27.4. Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Given any morphism $f : X \to Y'$ in $\mathcal{C}$ and any morphism $s : Y \to Y'$ in S, we denote by $s^{-1}f$ the equivalence class of the pair $(f : X \to Y', s : Y \to Y')$. This is a morphism from X to Y in $S^{-1}\mathcal{C}$.

This notation is suggestive, and the things it suggests are true: Given any morphism $f : X \to Y'$ in $\mathcal{C}$ and any two morphisms $s : Y \to Y'$ and $t : Y' \to Y''$ in S, we have $(t \circ s)^{-1}(t \circ f) = s^{-1}f$. Also, for any $f : X \to Y'$ and $g : Y' \to Z'$ in $\mathcal{C}$ and all $s : Z \to Z'$ in S, we have $s^{-1}(g \circ f) = \left(s^{-1}g\right) \circ \left(\mathrm{id}_{Y'}^{-1} f\right)$. Finally, for any $f : X \to Y'$ in $\mathcal{C}$, all $s : Y \to Y'$ in S, and $t : Z \to Y$ in S, we have $(s \circ t)^{-1} f = \left(t^{-1}\mathrm{id}_Y\right) \circ \left(s^{-1}f\right)$. This is all clear from the definition. We can "write any finite collection of morphisms with the same target as fractions with common denominator".

04VE **Lemma 4.27.5.** Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X_i \to Y$ of morphisms of $S^{-1}\mathcal{C}$ (indexed by i), we can find an element $s : Y \to Y'$ of S and a family of morphisms $f_i : X_i \to Y'$ of $\mathcal{C}$ such that each g_i is the equivalence class of the pair $(f_i : X_i \to Y', s : Y \to Y')$.

Proof. For each i choose a representative $(X_i \to Y_i, s_i : Y \to Y_i)$ of g_i. The lemma follows if we can find a morphism $s : Y \to Y'$ in S such that for each i there is a morphism $a_i : Y_i \to Y'$ with $a_i \circ s_i = s$. If we have two indices $i = 1, 2$, then we can do this by completing the square

$$\begin{tikzcd} Y \arrow[r, "s_2"'] \arrow[d, "s_1"'] & Y_2 \arrow[d, "t_2"] \\ Y_1 \arrow[r, "a_1"] & Y' \end{tikzcd}$$

with $t_2 \in S$ as is possible by Definition 4.27.1. Then $s = t_2 \circ s_2 \in S$ works. If we have $n > 2$ morphisms, then we use the above trick to reduce to the case of $n - 1$ morphisms, and we win by induction. $\square$

There is an easy characterization of equality of morphisms if they have the same denominator.

04VF **Lemma 4.27.6.** Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X \to Y', s : Y \to Y')$ and $(g : X \to Y', s : Y \to Y')$. The following are equivalent

1. (1) $A = B$
2. (2) there exists a morphism $t : Y' \to Y''$ in S with $t \circ f = t \circ g$, and
3. (3) there exists a morphism $a : Y' \to Y''$ such that $a \circ f = a \circ g$ and $a \circ s \in S$.

Proof. We are going to use that $S^{-1}\mathcal{C}$ is a category (Lemma 4.27.2) and we will use the notation of Definition 4.27.4 as well as the discussion following that definition to identify some morphisms in $S^{-1}\mathcal{C}$. Thus we write $A = s^{-1}f$ and $B = s^{-1}g$.

If $A = B$ then $(\mathrm{id}_{Y'}^{-1}s) \circ A = (\mathrm{id}_{Y'}^{-1}s) \circ B$. We have $(\mathrm{id}_{Y'}^{-1}s) \circ A = \mathrm{id}_{Y'}^{-1}f$ and $(\mathrm{id}_{Y'}^{-1}s) \circ B = \mathrm{id}_{Y'}^{-1}g$. The equality of $\mathrm{id}_{Y'}^{-1}f$ and $\mathrm{id}_{Y'}^{-1}g$ means by definition that there exists a commutative diagram

$$\begin{tikzcd} & Y' \arrow[d, "u"] & \\ X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & Z & Y' \arrow[l, "t"'] \arrow[ul, "\mathrm{id}_{Y'}"'] \arrow[dl, "\mathrm{id}_{Y'}"] \\ & Y' \arrow[u, "v"'] & \end{tikzcd}$$

with $t \in S$. In particular $u = v = t \in S$ and $t \circ f = t \circ g$. Thus (1) implies (2).

The implication (2) $\Rightarrow$ (3) is immediate. Assume a is as in (3). Denote $s' = a \circ s \in S$. Then $\mathrm{id}_{Y''}^{-1}s'$ is an isomorphism in the category $S^{-1}\mathcal{C}$ (with inverse $(s')^{-1}\mathrm{id}_{Y''}$). Thus to check $A = B$ it suffices to check that $\mathrm{id}_{Y''}^{-1}s' \circ A = \mathrm{id}_{Y''}^{-1}s' \circ B$. We compute using the rules discussed in the text following Definition 4.27.4 that $\mathrm{id}_{Y''}^{-1}s' \circ A =$

$\mathrm{id}_{Y''}^{-1}(a \circ s) \circ s^{-1}f = \mathrm{id}_{Y''}^{-1}(a \circ f) = \mathrm{id}_{Y''}^{-1}(a \circ g) = \mathrm{id}_{Y''}^{-1}(a \circ s) \circ s^{-1}g = \mathrm{id}_{Y''}^{-1}s' \circ B$ and we see that (1) is true. □

05Q0 Remark 4.27.7. Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. Given an object Y of $\mathcal{C}$ we denote Y/S the category whose objects are $s : Y \to Y'$ with $s \in S$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} & Y \arrow[ld, "s"'] \arrow[rd, "t"] & \\ Y' \arrow[rr, "a"] & & Y'' \end{tikzcd}$$

where $a : Y' \to Y''$ is arbitrary. We claim that the category Y/S is filtered (see Definition 4.19.1). Namely, LMS1 implies that $\mathrm{id}_Y : Y \to Y$ is in Y/S; hence Y/S is nonempty. LMS2 implies that given $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ we can find a diagram

$$\begin{tikzcd} Y \arrow[r, "s_2"'] \arrow[d, "s_1"'] & Y_2 \arrow[d, "t"] \\ Y_1 \arrow[r, "a"] & Y_3 \end{tikzcd}$$

with $t \in S$. Hence $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ both have maps to $t \circ s_2 : Y \to Y_3$ in Y/S. Finally, given two morphisms a, b from $s_1 : Y \to Y_1$ to $s_2 : Y \to Y_2$ in Y/S we see that $a \circ s_1 = b \circ s_1$; hence by LMS3 there exists a $t : Y_2 \to Y_3$ in S such that $t \circ a = t \circ b$. Now the combined results of Lemmas 4.27.5 and 4.27.6 tell us that

05Q1 (4.27.7.1)
$$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{(s:Y \to Y') \in Y/S} \mathrm{Mor}_{\mathcal{C}}(X, Y')$$

This formula expressing morphism sets in $S^{-1}\mathcal{C}$ as a filtered colimit of morphism sets in $\mathcal{C}$ is occasionally useful.

04VG Lemma 4.27.8. Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$.

(1) The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, \mathrm{id}_Y : Y \to Y)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.
(2) For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.
(3) If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.

Proof. Parts (1) and (2) are clear. (In (2), the inverse of $Q(s)$ is the equivalence class of the pair (id_Y, s).) To see (3) just set $H(X) = G(X)$ and set $H((f : X \to Y', s : Y \to Y')) = G(s)^{-1} \circ G(f)$. Details omitted. □

05Q2 Lemma 4.27.9. Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite colimits.

Proof. Let $\mathcal{I}$ be a finite category and let $\mathcal{I} \to \mathcal{C}$, $i \mapsto X_i$ be a functor whose colimit exists. Then using (4.27.7.1), the fact that Y/S is filtered, and Lemma 4.19.2 we have

$$\begin{aligned}\mathrm{Mor}_{S^{-1}\mathcal{C}}(Q(\mathrm{colim}\, X_i), Q(Y)) &= \mathrm{colim}_{(s:Y\to Y')\in Y/S}\, \mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}\, X_i, Y') \\ &= \mathrm{colim}_{(s:Y\to Y')\in Y/S}\, \lim_i \mathrm{Mor}_{\mathcal{C}}(X_i, Y') \\ &= \lim_i \mathrm{colim}_{(s:Y\to Y')\in Y/S}\, \mathrm{Mor}_{\mathcal{C}}(X_i, Y') \\ &= \lim_i \mathrm{Mor}_{S^{-1}\mathcal{C}}(Q(X_i), Q(Y))\end{aligned}$$

and this isomorphism commutes with the projections from both sides to the set $\mathrm{Mor}_{S^{-1}\mathcal{C}}(Q(X_j), Q(Y))$ for each $j \in \mathrm{Ob}(\mathcal{I})$. Thus, $Q(\mathrm{colim}\, X_i)$ satisfies the universal property for the colimit of the functor $i \mapsto Q(X_i)$; hence, it is this colimit, as desired. $\square$

05Q3 Lemma 4.27.10. Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if

$$\begin{tikzcd} Q(X) \arrow[r, "a"'] \arrow[d, "Q(f)"'] & Q(X') \arrow[d, "Q(f')"] \\ Q(Y) \arrow[r, "b"] & Q(Y') \end{tikzcd}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exist a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"] \arrow[d, "f'"] \\ Y \arrow[r, "h"] & Y'' & Y' \arrow[l, "t"'] \end{tikzcd}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = s^{-1}g$, $b = t^{-1}h$.

Proof. We choose maps and objects in the following way: First write $a = s^{-1}g$ for some $s : X' \to X''$ in S and $g : X \to X''$. By LMS2 we can find $t : Y' \to Y''$ in S and $f'' : X'' \to Y''$ such that

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"] \arrow[d, "f'"] \\ Y & Y'' & Y' \arrow[l, "t"'] \end{tikzcd}$$

commutes. Now in this diagram we are going to repeatedly change our choice of

$$X'' \xrightarrow{f''} Y'' \xleftarrow{t} Y'$$

by postcomposing both t and f'' by a morphism $d : Y'' \to Y'''$ with the property that $d \circ t \in S$. According to Remark 4.27.7 we may after such a replacement assume that there exists a morphism $h : Y \to Y''$ such that $b = t^{-1}h$ holds[5]. At this point we have everything as in the lemma except that we don't know that the left square of the diagram commutes. But the definition of composition in $S^{-1}\mathcal{C}$ shows that $b \circ Q(f)$ is the equivalence class of the pair $(h \circ f : X \to Y'', t : Y' \to Y'')$ (since b is the equivalence class of the pair $(h : Y \to Y'', t : Y' \to Y'')$, while $Q(f)$

[5]Here is a more down-to-earth way to see this: Write $b = q^{-1}i$ for some $q : Y' \to Z$ in S and some $i : Y \to Z$. By LMS2 we can find $r : Y'' \to Y'''$ in S and $j : Z \to Y'''$ such that $j \circ q = r \circ t$. Now, set $d = r$ and $h = j \circ i$.

is the equivalence class of the pair $(f : X \to Y, \mathrm{id} : Y \to Y))$, while $Q(f') \circ a$ is the equivalence class of the pair $(f'' \circ g : X \to Y'', t : Y' \to Y'')$ (since a is the equivalence class of the pair $(g : X \to X'', s : X' \to X'')$, while $Q(f')$ is the equivalence class of the pair $(f' : X' \to Y', \mathrm{id} : Y' \to Y'))$. Since we know that $b \circ Q(f) = Q(f') \circ a$, we thus conclude that the equivalence classes of the pairs $(h \circ f : X \to Y'', t : Y' \to Y'')$ and $(f'' \circ g : X \to Y'', t : Y' \to Y'')$ are equal. Hence using Lemma 4.27.6 we can find a morphism $d : Y'' \to Y'''$ such that $d \circ t \in S$ and $d \circ h \circ f = d \circ f'' \circ g$. Hence we make one more replacement of the kind described above and we win. □

Right calculus of fractions. Let $\mathcal{C}$ be a category and let S be a right multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 4.27.11):

(1) We set $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X' \to Y, s : X' \to X)$ with $s \in S$ up to equivalence. (The equivalence is defined below. Think of the equivalence class of a pair (f, s) as $fs^{-1} : X \to Y$.)
(3) Two pairs $(f_1 : X_1 \to Y, s_1 : X_1 \to X)$ and $(f_2 : X_2 \to Y, s_2 : X_2 \to X)$ are said to be equivalent if there exist a third pair $(f_3 : X_3 \to Y, s_3 : X_3 \to X)$ and morphisms $u : X_3 \to X_1$ and $v : X_3 \to X_2$ of $\mathcal{C}$ fitting into the commutative diagram

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & X_3 \arrow[l, "s_3"'] \arrow[r, "f_3"] \arrow[u, "u"] \arrow[d, "v"'] & Y \\ & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

(4) The composition of the equivalence classes of the pairs $(f : X' \to Y, s : X' \to X)$ and $(g : Y' \to Z, t : Y' \to Y)$ is defined as the equivalence class of a pair $(g \circ h : X'' \to Z, s \circ u : X'' \to X)$ where h and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "h"] \arrow[d, "u"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f"] & Y \end{tikzcd}$$

which exists by assumption.
(5) The identity morphism $X \to X$ in $S^{-1}\mathcal{C}$ is the equivalence class of the pair $(\mathrm{id} : X \to X, \mathrm{id} : X \to X)$.

04VH Lemma 4.27.11. Let $\mathcal{C}$ be a category and let S be a right multiplicative system.

(1) The relation on pairs defined above is an equivalence relation.
(2) The composition rule given above is well defined on equivalence classes.
(3) Composition is associative (and the identity morphisms satisfy the identity axioms), and hence $S^{-1}\mathcal{C}$ is a category.

Proof. This lemma is dual to Lemma 4.27.2. It follows formally from that lemma by replacing $\mathcal{C}$ by its opposite category in which S is a left multiplicative system. □

0BM3 Definition 4.27.12. Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Given any morphism $f : X' \to Y$ in $\mathcal{C}$ and any morphism $s : X' \to X$ in S, we denote by fs^{-1} the equivalence class of the pair $(f : X' \to Y, s : X' \to X)$. This is a morphism from X to Y in $S^{-1}\mathcal{C}$.

Identities similar (actually, dual) to the ones in Definition 4.27.4 hold. We can "write any finite collection of morphisms with the same source as fractions with common denominator".

04VI Lemma 4.27.13. Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X \to Y_i$ of morphisms of $S^{-1}\mathcal{C}$ (indexed by i), we can find an element $s : X' \to X$ of S and a family of morphisms $f_i : X' \to Y_i$ of $\mathcal{C}$ such that g_i is the equivalence class of the pair $(f_i : X' \to Y_i, s : X' \to X)$.

Proof. This lemma is the dual of Lemma 4.27.5 and follows formally from that lemma by replacing all categories in sight by their opposites. □

There is an easy characterization of equality of morphisms if they have the same denominator.

04VJ Lemma 4.27.14. Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X' \to Y, s : X' \to X)$ and $(g : X' \to Y, s : X' \to X)$. The following are equivalent

(1) $A = B$,
(2) there exists a morphism $t : X'' \to X'$ in S with $f \circ t = g \circ t$, and
(3) there exists a morphism $a : X'' \to X'$ with $f \circ a = g \circ a$ and $s \circ a \in S$.

Proof. This is dual to Lemma 4.27.6. □

05Q4 Remark 4.27.15. Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. Given an object X of $\mathcal{C}$ we denote S/X the category whose objects are $s : X' \to X$ with $s \in S$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} X' \arrow[rr, "a"'] \arrow[dr, "s"'] & & X'' \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

where $a : X' \to X''$ is arbitrary. The category S/X is cofiltered (see Definition 4.20.1). (This is dual to the corresponding statement in Remark 4.27.7.) Now the combined results of Lemmas 4.27.13 and 4.27.14 tell us that

05Q5 (4.27.15.1) $$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{(s:X' \to X) \in (S/X)^{opp}} \mathrm{Mor}_{\mathcal{C}}(X', Y)$$

This formula expressing morphisms in $S^{-1}\mathcal{C}$ as a filtered colimit of morphisms in $\mathcal{C}$ is occasionally useful.

04VK Lemma 4.27.16. Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$.

(1) The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, \mathrm{id}_X : X \to X)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.
(2) For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.
(3) If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.

Proof. This lemma is the dual of Lemma 4.27.8 and follows formally from that lemma by replacing all categories in sight by their opposites. □

05Q6 Lemma 4.27.17. Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite limits.

Proof. This is dual to Lemma 4.27.9. □

05Q7 Lemma 4.27.18. Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if

$$\begin{tikzcd} Q(X) \arrow[r, "a"'] \arrow[d, "Q(f)"'] & Q(X') \arrow[d, "Q(f')"] \\ Q(Y) \arrow[r, "b"] & Q(Y') \end{tikzcd}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exist a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X'' \arrow[l, "s"] \arrow[r, "g"'] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\ Y & Y'' \arrow[l, "t"'] \arrow[r, "h"] & Y' \end{tikzcd}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = gs^{-1}$, $b = ht^{-1}$.

Proof. This lemma is dual to Lemma 4.27.10. □

Multiplicative systems and two sided calculus of fractions. If S is a multiplicative system then left and right calculus of fractions give canonically isomorphic categories.

04VL Lemma 4.27.19. Let $\mathcal{C}$ be a category and let S be a multiplicative system. The category of left fractions and the category of right fractions $S^{-1}\mathcal{C}$ are canonically isomorphic.

Proof. Denote $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ the two categories of fractions. By the universal properties of Lemmas 4.27.8 and 4.27.16 we obtain functors $\mathcal{C}_{left} \to \mathcal{C}_{right}$ and $\mathcal{C}_{right} \to \mathcal{C}_{left}$. By the uniqueness statement in the universal properties, these functors are each other's inverse. □

05Q8 Definition 4.27.20. Let $\mathcal{C}$ be a category and let S be a multiplicative system. We say S is saturated if, in addition to MS1, MS2, MS3, we also have

MS4 Given three composable morphisms f, g, h, if $fg, gh \in S$, then $g \in S$.

Note that a saturated multiplicative system contains all isomorphisms. Moreover, if f, g, h are composable morphisms in a category and fg, gh are isomorphisms, then g is an isomorphism (because then g has both a left and a right inverse, hence is invertible).

05Q9 **Lemma 4.27.21.** Let $\mathcal{C}$ be a category and let S be a multiplicative system. Denote $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ the localization functor. The set

$$\hat{S} = \{f \in \text{Arrows}(\mathcal{C}) \mid Q(f) \text{ is an isomorphism}\}$$

is equal to

$$S' = \{f \in \text{Arrows}(\mathcal{C}) \mid \text{there exist } g, h \text{ such that } gf, fh \in S\}$$

and is the smallest saturated multiplicative system containing S. In particular, if S is saturated, then $\hat{S} = S$.

Proof. It is clear that $S \subset S' \subset \hat{S}$ because elements of S' map to morphisms in $S^{-1}\mathcal{C}$ which have both left and right inverses. Note that S' satisfies MS4, and that $\hat{S}$ satisfies MS1. Next, we prove that $S' = \hat{S}$.

Let $f \in \hat{S}$. Let $s^{-1}g = ht^{-1}$ be the inverse morphism in $S^{-1}\mathcal{C}$. (We may use both left fractions and right fractions to describe morphisms in $S^{-1}\mathcal{C}$, see Lemma 4.27.19.) The relation $\text{id}_X = s^{-1}gf$ in $S^{-1}\mathcal{C}$ means there exists a commutative diagram

$$\begin{tikzcd} & X' \arrow[d, "u"] & \\ X \arrow[ur, "gf"] \arrow[r, "f'"] \arrow[dr, "\text{id}_X"'] & X'' & X \arrow[ul, "s"'] \arrow[l, "s'"'] \arrow[dl, "\text{id}_X"] \\ & X \arrow[u, "v"] & \end{tikzcd}$$

for some morphisms f', u, v and $s' \in S$. Hence $ugf = s' \in S$. Similarly, using that $\text{id}_Y = fht^{-1}$ one proves that $fhw \in S$ for some w. We conclude that $f \in S'$. Thus $S' = \hat{S}$. Provided we prove that $S' = \hat{S}$ is a multiplicative system it is now clear that this implies that $S' = \hat{S}$ is the smallest saturated system containing S.

Our remarks above take care of MS1 and MS4, so to finish the proof of the lemma we have to show that LMS2, RMS2, LMS3, RMS3 hold for $\hat{S}$. Let us check that LMS2 holds for $\hat{S}$. Suppose we have a solid diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[d, dotted, "s"] \\ Z \arrow[r, dotted, "f"] & W \end{tikzcd}$$

with $t \in \hat{S}$. Pick a morphism $a : Z \to Z'$ such that $at \in S$. Then we can use LMS2 for S to find a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "t"'] & Y \arrow[dd, "s"] \\ Z \arrow[d, "a"'] & \\ Z' \arrow[r, "f'"] & W \end{tikzcd}$$

and setting $f = f' \circ a$ we win. The proof of RMS2 is dual to this. Finally, suppose given a pair of morphisms $f, g : X \to Y$ and $t \in \hat{S}$ with target X such that $ft = gt$.

Then we pick a morphism b such that $tb \in S$. Then $ftb = gtb$ which implies by LMS3 for S that there exists an $s \in S$ with source Y such that $sf = sg$ as desired. The proof of RMS3 is dual to this. $\square$

4.28. Formal properties

003D In this section we discuss some formal properties of the 2-category of categories. This will lead us to the definition of a (strict) 2-category later.

Let us denote Ob(Cat) the class of all categories. For every pair of categories $\mathcal{A}, \mathcal{B} \in \text{Ob}(\text{Cat})$ we have the "small" category of functors $\text{Fun}(\mathcal{A}, \mathcal{B})$. Composition of transformation of functors such as

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F''"] \arrow[r, "F'" description] \arrow[r, bend right=50, "F"'] & \mathcal{B} \end{tikzcd} \quad (\Downarrow t',\ \Downarrow t) \quad \text{composes to} \quad \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F''"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow t\circ t'"] & \mathcal{B} \end{tikzcd}$$

is called vertical composition. We will use the usual symbol $\circ$ for this. Next, we will define horizontal composition. In order to do this we explain a bit more of the structure at hand.

Namely for every triple of categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ there is a composition law

$$\circ : \text{Ob}(\text{Fun}(\mathcal{B}, \mathcal{C})) \times \text{Ob}(\text{Fun}(\mathcal{A}, \mathcal{B})) \longrightarrow \text{Ob}(\text{Fun}(\mathcal{A}, \mathcal{C}))$$

coming from composition of functors. This composition law is associative, and identity functors act as units. In other words – forgetting about transformations of functors – we see that Cat forms a category. How does this structure interact with the morphisms between functors?

Well, given $t : F \to F'$ a transformation of functors $F, F' : \mathcal{A} \to \mathcal{B}$ and a functor $G : \mathcal{B} \to \mathcal{C}$ we can define a transformation of functors $G \circ F \to G \circ F'$. We will denote this transformation ${}_Gt$. It is given by the formula $({}_Gt)_x = G(t_x) : G(F(x)) \to G(F'(x))$ for all $x \in \mathcal{A}$. In this way composition with G becomes a functor

$$\text{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \text{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that ${}_G(\text{id}_F) = \text{id}_{G \circ F}$ and that ${}_G(t_1 \circ t_2) = {}_Gt_1 \circ {}_Gt_2$. Of course we also have that ${}_{\text{id}_\mathcal{A}}t = t$.

Similarly, given $s : G \to G'$ a transformation of functors $G, G' : \mathcal{B} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$ a functor we can define s_F to be the transformation of functors $G \circ F \to G' \circ F$ given by $(s_F)_x = s_{F(x)} : G(F(x)) \to G'(F(x))$ for all $x \in \mathcal{A}$. In this way composition with F becomes a functor

$$\text{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \text{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that $(\text{id}_G)_F = \text{id}_{G \circ F}$ and that $(s_1 \circ s_2)_F = s_{1,F} \circ s_{2,F}$. Of course we also have that $s_{\text{id}_\mathcal{B}} = s$.

These constructions satisfy the additional properties

$${}_{G_1}({}_{G_2}t) = {}_{G_1 \circ G_2}t, \ (s_{F_1})_{F_2} = s_{F_1 \circ F_2}, \text{ and } {}_H(s_F) = ({}_Hs)_F$$

whenever these make sense. Finally, given functors $F, F' : \mathcal{A} \to \mathcal{B}$, and $G, G' : \mathcal{B} \to \mathcal{C}$ and transformations $t : F \to F'$, and $s : G \to G'$ the following diagram is commutative

$$\begin{tikzcd} G \circ F \arrow[r, "{}_G t"] \arrow[d, "s_F"'] & G \circ F' \arrow[d, "s_{F'}"] \\ G' \circ F \arrow[r, "{}_{G'} t"'] & G' \circ F' \end{tikzcd}$$

in other words ${}_{G'}t \circ s_F = s_{F'} \circ {}_G t$. To prove this we just consider what happens on any object $x \in \mathrm{Ob}(\mathcal{A})$:

$$\begin{tikzcd} G(F(x)) \arrow[r, "G(t_x)"] \arrow[d, "s_{F(x)}"'] & G(F'(x)) \arrow[d, "s_{F'(x)}"] \\ G'(F(x)) \arrow[r, "G'(t_x)"'] & G'(F'(x)) \end{tikzcd}$$

which is commutative because s is a transformation of functors. This compatibility relation allows us to define horizontal composition.

003E Definition 4.28.1. Given a diagram as in the left hand side of:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow t"] & \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow s"] & \mathcal{C} \end{tikzcd} \text{ gives } \begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "G \circ F"] \arrow[rr, bend right, "G' \circ F'"'] \arrow[rr, phantom, "\Downarrow s \star t"] & & \mathcal{C} \end{tikzcd}$$

we define the horizontal composition $s \star t$ to be the transformation of functors ${}_{G'}t \circ s_F = s_{F'} \circ {}_G t$.

Now we see that we may recover our previously constructed transformations ${}_G t$ and s_F as ${}_G t = \mathrm{id}_G \star t$ and $s_F = s \star \mathrm{id}_F$. Furthermore, all of the rules we found above are consequences of the properties stated in the lemma that follows.

003F Lemma 4.28.2. The horizontal and vertical compositions have the following properties

(1) $\circ$ and $\star$ are associative,
(2) the identity transformations id_F are units for $\circ$,
(3) the identity transformations of the identity functors $\mathrm{id}_{\mathrm{id}_\mathcal{A}}$ are units for $\star$ and $\circ$, and
(4) given a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"] \arrow[r, "F'" description] \arrow[r, bend right=50, "F''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow t"] \arrow[r, phantom, shift right=3ex, "\Downarrow t'"] & \mathcal{B} \arrow[r, bend left=50, "G"] \arrow[r, "G'" description] \arrow[r, bend right=50, "G''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow s"] \arrow[r, phantom, shift right=3ex, "\Downarrow s'"] & \mathcal{C} \end{tikzcd}$$

we have $(s' \circ s) \star (t' \circ t) = (s' \star t') \circ (s \star t)$.

Proof. The last statement turns using our previous notation into the following equation

$$s'_{F''} \circ {}_{G'}t' \circ s_{F'} \circ {}_G t = (s' \circ s)_{F''} \circ {}_G(t' \circ t).$$

According to our result above applied to the middle composition we may rewrite the left hand side as $s'_{F''} \circ s_{F''} \circ_G t' \circ_G t$ which is easily shown to be equal to the right hand side. $\square$

Another way of formulating condition (4) of the lemma is that composition of functors and horizontal composition of transformation of functors gives rise to a functor

$$(\circ, \star) : \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C})$$

whose source is the product category, see Definition 4.2.20.

4.29. 2-categories

003G We will give a definition of (strict) 2-categories as they appear in the setting of stacks. Before you read this take a look at Section 4.28 and Example 4.30.2. Basically, you take this example and you write out all the rules satisfied by the objects, 1-morphisms and 2-morphisms in that example.

003H **Definition 4.29.1.** A (strict) 2-category $\mathcal{C}$ consists of the following data

1. A set of objects $\mathrm{Ob}(\mathcal{C})$.
2. For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a category $\mathrm{Mor}_\mathcal{C}(x, y)$. The objects of $\mathrm{Mor}_\mathcal{C}(x, y)$ will be called 1-morphisms and denoted $F : x \to y$. The morphisms between these 1-morphisms will be called 2-morphisms and denoted $t : F' \to F$. The composition of 2-morphisms in $\mathrm{Mor}_\mathcal{C}(x, y)$ will be called vertical composition and will be denoted $t \circ t'$ for $t : F' \to F$ and $t' : F'' \to F'$.
3. For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a functor
$$(\circ, \star) : \mathrm{Mor}_\mathcal{C}(y, z) \times \mathrm{Mor}_\mathcal{C}(x, y) \longrightarrow \mathrm{Mor}_\mathcal{C}(x, z).$$
The image of the pair of 1-morphisms (F, G) on the left hand side will be called the composition of F and G, and denoted $F \circ G$. The image of the pair of 2-morphisms (t, s) will be called the horizontal composition and denoted $t \star s$.

These data are to satisfy the following rules:

1. The set of objects together with the set of 1-morphisms endowed with composition of 1-morphisms forms a category.
2. Horizontal composition of 2-morphisms is associative.
3. The identity 2-morphism $\mathrm{id}_{\mathrm{id}_x}$ of the identity 1-morphism id_x is a unit for horizontal composition.

This is obviously not a very pleasant type of object to work with. On the other hand, there are lots of examples where it is quite clear how you work with it. The only example we have so far is that of the 2-category whose objects are a given collection of categories, 1-morphisms are functors between these categories, and 2-morphisms are natural transformations of functors, see Section 4.28. As far as this text is concerned all 2-categories will be sub 2-categories of this example. Here is what it means to be a sub 2-category.

02X7 **Definition 4.29.2.** Let $\mathcal{C}$ be a 2-category. A sub 2-category $\mathcal{C}'$ of $\mathcal{C}$, is given by a subset $\mathrm{Ob}(\mathcal{C}')$ of $\mathrm{Ob}(\mathcal{C})$ and sub categories $\mathrm{Mor}_{\mathcal{C}'}(x, y)$ of the categories $\mathrm{Mor}_\mathcal{C}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{C}')$ such that these, together with the operations $\circ$ (composition 1-morphisms), $\circ$ (vertical composition 2-morphisms), and $\star$ (horizontal composition) form a 2-category.

003J Remark 4.29.3. Big 2-categories. In many texts a 2-category is allowed to have a class of objects (but hopefully a "class of classes" is not allowed). We will allow these "big" 2-categories as well, but only in the following list of cases (to be updated as we go along):

(1) The 2-category of categories Cat.
(2) The $(2,1)$-category of categories Cat.
(3) The 2-category of groupoids Groupoids; this is a $(2,1)$-category.
(4) The 2-category of fibred categories over a fixed category.
(5) The $(2,1)$-category of fibred categories over a fixed category.

See Definition 4.30.1. Note that in each case the class of objects of the 2-category $\mathcal{C}$ is a proper class, but for all objects $x, y \in \mathrm{Ob}(C)$ the category $\mathrm{Mor}_{\mathcal{C}}(x, y)$ is "small" (according to our conventions).

The notion of equivalence of categories that we defined in Section 4.2 extends to the more general setting of 2-categories as follows.

003L Definition 4.29.4. Two objects x, y of a 2-category are equivalent if there exist 1-morphisms $F : x \to y$ and $G : y \to x$ such that $F \circ G$ is 2-isomorphic to id_y and $G \circ F$ is 2-isomorphic to id_x.

Sometimes we need to say what it means to have a functor from a category into a 2-category.

003N Definition 4.29.5. Let $\mathcal{A}$ be a category and let $\mathcal{C}$ be a 2-category.

(1) A functor from an ordinary category into a 2-category will ignore the 2-morphisms unless mentioned otherwise. In other words, it will be a "usual" functor into the category formed out of 2-category by forgetting all the 2-morphisms.
(2) A weak functor, or a pseudo functor φ from $\mathcal{A}$ into the 2-category $\mathcal{C}$ is given by the following data
   (a) a map $\varphi : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{C})$,
   (b) for every pair $x, y \in \mathrm{Ob}(\mathcal{A})$, and every morphism $f : x \to y$ a 1-morphism $\varphi(f) : \varphi(x) \to \varphi(y)$,
   (c) for every $x \in \mathrm{Ob}(A)$ a 2-morphism $\alpha_x : \mathrm{id}_{\varphi(x)} \to \varphi(\mathrm{id}_x)$, and
   (d) for every pair of composable morphisms $f : x \to y$, $g : y \to z$ of $\mathcal{A}$ a 2-morphism $\alpha_{g,f} : \varphi(g \circ f) \to \varphi(g) \circ \varphi(f)$.

   These data are subject to the following conditions:
   (a) the 2-morphisms α_x and $\alpha_{g,f}$ are all isomorphisms,
   (b) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{\mathrm{id}_y, f} = \alpha_y \star \mathrm{id}_{\varphi(f)}$:

$$\begin{tikzcd} \varphi(x) \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(f)"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\varphi(f)}"] & \varphi(y) \arrow[r, bend left, "\mathrm{id}_{\varphi(y)}"] \arrow[r, bend right, "\varphi(\mathrm{id}_y)"'] \arrow[r, phantom, "\Downarrow \alpha_y"] & \varphi(y) \end{tikzcd} = \begin{tikzcd} \varphi(x) \arrow[r, bend left, "\varphi(f)"] \arrow[r, bend right, "\varphi(\mathrm{id}_y) \circ \varphi(f)"'] \arrow[r, phantom, "\Downarrow \alpha_{\mathrm{id}_y, f}"] & \varphi(y) \end{tikzcd}$$

   (c) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{f, \mathrm{id}_x} = \mathrm{id}_{\varphi(f)} \star \alpha_x$,
   (d) for any triple of composable morphisms $f : w \to x$, $g : x \to y$, and $h : y \to z$ of $\mathcal{A}$ we have

$$(\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}) \circ \alpha_{h, g \circ f} = (\alpha_{h,g} \star \mathrm{id}_{\varphi(f)}) \circ \alpha_{h \circ g, f}$$

in other words the following diagram with objects 1-morphisms and arrows 2-morphisms commutes

$$\begin{tikzcd}
\varphi(h \circ g \circ f) \arrow[r, "\alpha_{h \circ g, f}"'] \arrow[d, "\alpha_{h, g \circ f}"'] & \varphi(h \circ g) \circ \varphi(f) \arrow[d, "\alpha_{h,g} \star \mathrm{id}_{\varphi(f)}"] \\
\varphi(h) \circ \varphi(g \circ f) \arrow[r, "\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}"] & \varphi(h) \circ \varphi(g) \circ \varphi(f)
\end{tikzcd}$$

Again this is not a very workable notion, but it does sometimes come up. There is a theorem that says that any pseudo-functor is isomorphic to a functor. Finally, there are the notions of functor between 2-categories, and pseudo functor between 2-categories. This last notion leads us into 3-category territory. We would like to avoid having to define this at almost any cost!

4.30. (2, 1)-categories

02X8 Some 2-categories have the property that all 2-morphisms are isomorphisms. These will play an important role in the following, and they are easier to work with.

003I Definition 4.30.1. A (strict) $(2, 1)$-category is a 2-category in which all 2-morphisms are isomorphisms.

003K Example 4.30.2. The 2-category Cat, see Remark 4.29.3, can be turned into a $(2, 1)$-category by only allowing isomorphisms of functors as 2-morphisms.

In fact, more generally any 2-category $\mathcal{C}$ produces a $(2, 1)$-category by considering the sub 2-category $\mathcal{C}'$ with the same objects and 1-morphisms but whose 2-morphisms are the invertible 2-morphisms of $\mathcal{C}$. In this situation we will say "let $\mathcal{C}'$ be the $(2, 1)$-category associated to $\mathcal{C}$" or similar. For example, the $(2, 1)$-category of groupoids means the 2-category whose objects are groupoids, whose 1-morphisms are functors and whose 2-morphisms are isomorphisms of functors. Except that this is a bad example as a transformation between functors between groupoids is automatically an isomorphism!

003M Remark 4.30.3. Thus there are variants of the construction of Example 4.30.2 above where we look at the 2-category of groupoids, or categories fibred in groupoids over a fixed category, or stacks. And so on.

4.31. 2-fibre products

003O In this section we introduce 2-fibre products. Suppose that $\mathcal{C}$ is a 2-category. We say that a diagram

$$\begin{tikzcd}
w \arrow[r] \arrow[d] & y \arrow[d] \\
x \arrow[r] & z
\end{tikzcd}$$

2-commutes if the two 1-morphisms $w \to y \to z$ and $w \to x \to z$ are 2-isomorphic. In a 2-category it is more natural to ask for 2-commutativity of diagrams than for actually commuting diagrams. (Indeed, some may say that we should not work with strict 2-categories at all, and in a "weak" 2-category the notion of a commutative diagram of 1-morphisms does not even make sense.) Correspondingly the notion of a fibre product has to be adjusted.

Let $\mathcal{C}$ be a 2-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_\mathcal{C}(x, z)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. In order to define the 2-fibre product of f and g we are going to look at 2-commutative diagrams

$$\begin{tikzcd} w \arrow[r, "a"'] \arrow[d, "b"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z. \end{tikzcd}$$

Now in the case of categories, the fibre product is a final object in the category of such diagrams. Correspondingly a 2-fibre product is a final object in a 2-category (see definition below). The 2-category of 2-commutative diagrams over f and g is the 2-category defined as follows:

(1) Objects are quadruples (w, a, b, ϕ) as above where ϕ is an invertible 2-morphism $\phi : f \circ a \to g \circ b$,
(2) 1-morphisms from (w', a', b', ϕ') to (w, a, b, ϕ) are given by $(k : w' \to w, \alpha : a' \to a \circ k, \beta : b' \to b \circ k)$ such that

$$\begin{tikzcd} f \circ a' \arrow[rr, "\mathrm{id}_f \star \alpha"'] \arrow[d, "\phi'"'] & & f \circ a \circ k \arrow[d, "\phi \star \mathrm{id}_k"] \\ g \circ b' \arrow[rr, "\mathrm{id}_g \star \beta"] & & g \circ b \circ k \end{tikzcd}$$

is commutative,
(3) given a second 1-morphism $(k', \alpha', \beta') : (w'', a'', b'', \phi'') \to (w', \alpha', \beta', \phi')$ the composition of 1-morphisms is given by the rule

$$(k, \alpha, \beta) \circ (k', \alpha', \beta') = (k \circ k', (\alpha \star \mathrm{id}_{k'}) \circ \alpha', (\beta \star \mathrm{id}_{k'}) \circ \beta'),$$

(4) a 2-morphism between 1-morphisms (k_i, α_i, β_i), $i = 1, 2$ with the same source and target is given by a 2-morphism $\delta : k_1 \to k_2$ such that

$$\begin{tikzcd} a' \arrow[r, "\alpha_1"'] \arrow[dr, "\alpha_2"'] & a \circ k_1 \arrow[d, "\mathrm{id}_a \star \delta"] \\ & a \circ k_2 \end{tikzcd} \qquad \begin{tikzcd} b \circ k_1 \arrow[d, "\mathrm{id}_b \star \delta"'] & b' \arrow[l, "\beta_1"] \arrow[dl, "\beta_2"] \\ b \circ k_2 & \end{tikzcd}$$

commute,
(5) vertical composition of 2-morphisms is given by vertical composition of the morphisms δ in $\mathcal{C}$, and
(6) horizontal composition of the diagram

$$\begin{tikzcd}[column sep=huge] (w'', a'', b'', \phi'') \arrow[r, bend left, "{(k'_1, \alpha'_1, \beta'_1)}"] \arrow[r, bend right, "{(k'_2, \alpha'_2, \beta'_2)}"'] \arrow[r, phantom, "\Downarrow\delta'"] & (w', a', b', \phi') \arrow[r, bend left, "{(k_1, \alpha_1, \beta_1)}"] \arrow[r, bend right, "{(k_2, \alpha_2, \beta_2)}"'] \arrow[r, phantom, "\Downarrow\delta"] & (w, a, b, \phi) \end{tikzcd}$$

is given by the diagram

$$\begin{tikzcd}[column sep=huge] (w'', a'', b'', \phi'') \arrow[r, bend left, "{(k_1 \circ k'_1, (\alpha_1 \star \mathrm{id}_{k'_1}) \circ \alpha'_1, (\beta_1 \star \mathrm{id}_{k'_1}) \circ \beta'_1)}"] \arrow[r, bend right, "{(k_2 \circ k'_2, (\alpha_2 \star \mathrm{id}_{k'_2}) \circ \alpha'_2, (\beta_2 \star \mathrm{id}_{k'_2}) \circ \beta'_2)}"'] \arrow[r, phantom, "\Downarrow\delta \star \delta'"] & (w, a, b, \phi) \end{tikzcd}$$

Note that if $\mathcal{C}$ is actually a $(2,1)$-category, the morphisms α and β in (2) above are automatically also isomorphisms[6]. In addition the 2-category of 2-commutative diagrams is also a $(2,1)$-category if $\mathcal{C}$ is a $(2,1)$-category.

003P Definition 4.31.1. A final object of a $(2,1)$-category $\mathcal{C}$ is an object x such that

(1) for every $y \in \mathrm{Ob}(\mathcal{C})$ there is a morphism $y \to x$, and
(2) every two morphisms $y \to x$ are isomorphic by a unique 2-morphism.

Likely, in the more general case of 2-categories there are different flavours of final objects. We do not want to get into this and hence we only define 2-fibre products in the $(2,1)$-case.

003Q Definition 4.31.2. Let $\mathcal{C}$ be a $(2,1)$-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_\mathcal{C}(x, z)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. A 2-fibre product of f and g is a final object in the category of 2-commutative diagrams described above. If a 2-fibre product exists we will denote it $x \times_z y \in \mathrm{Ob}(\mathcal{C})$, and denote the required morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times_z y, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times_z y, y)$ making the diagram

$$\begin{tikzcd} x \times_z y \arrow[r, "p"] \arrow[d, "q"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z \end{tikzcd}$$

2-commute and we will denote the given invertible 2-morphism exhibiting this by $\psi : f \circ p \to g \circ q$.

Thus the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $a \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $b \in \mathrm{Mor}_\mathcal{C}(w, y)$ with a given 2-isomorphism $\phi : f \circ a \to g \circ b$ there is a $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times_z y)$ making the diagram

$$\begin{tikzcd} w \arrow[rd, dashed, "\gamma"] \arrow[rrd, "a"] \arrow[rdd, "b"'] & & \\ & x \times_z y \arrow[r, "p"'] \arrow[d, "q"'] & x \arrow[d, "f"] \\ & y \arrow[r, "g"] & z \end{tikzcd}$$

2-commute such that for suitable choices of $a \to p \circ \gamma$ and $b \to q \circ \gamma$ the diagram

$$\begin{tikzcd} f \circ a \arrow[r] \arrow[d, "\phi"'] & f \circ p \circ \gamma \arrow[d, "\psi \star \mathrm{id}_\gamma"] \\ g \circ b \arrow[r] & g \circ q \circ \gamma \end{tikzcd}$$

commutes. Moreover γ is unique up to isomorphism. Of course the exact properties are finer than this. All of the cases of 2-fibre products that we will need later on come from the following example of 2-fibre products in the 2-category of categories.

003R Example 4.31.3. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. We define a category $\mathcal{A} \times_\mathcal{C} \mathcal{B}$ as follows:

[6]In fact it seems in the 2-category case that one could define another 2-category of 2-commutative diagrams where the direction of the arrows α, β is reversed, or even where the direction of only one of them is reversed. This is why we restrict to $(2,1)$-categories later on.

(1) an object of $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is a triple (A, B, f), where $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and $f : F(A) \to G(B)$ is an isomorphism in $\mathcal{C}$,
(2) a morphism $(A, B, f) \to (A', B', f')$ is given by a pair (a, b), where $a : A \to A'$ is a morphism in $\mathcal{A}$, and $b : B \to B'$ is a morphism in $\mathcal{B}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[d, "F(a)"] & G(B) \arrow[d, "G(b)"] \\ F(A') \arrow[r, "f'"] & G(B') \end{tikzcd}$$

is commutative.

Moreover, we define functors $p : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{A}$ and $q : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{B}$ by setting

$$p(A, B, f) = A, \quad q(A, B, f) = B,$$

in other words, these are the forgetful functors. We define a transformation of functors $\psi : F \circ p \to G \circ q$. On the object $\xi = (A, B, f)$ it is given by $\psi_\xi = f : F(p(\xi)) = F(A) \to G(B) = G(q(\xi))$.

02X9 Lemma 4.31.4. In the $(2, 1)$-category of categories 2-fibre products exist and are given by the construction of Example 4.31.3.

Proof. Let us check the universal property: let $\mathcal{W}$ be a category, let $a : \mathcal{W} \to \mathcal{A}$ and $b : \mathcal{W} \to \mathcal{B}$ be functors, and let $t : F \circ a \to G \circ b$ be an isomorphism of functors.

Consider the functor $\gamma : \mathcal{W} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ given by $W \mapsto (a(W), b(W), t_W)$. (Check this is a functor omitted.) Moreover, consider $\alpha : a \to p \circ \gamma$ and $\beta : b \to q \circ \gamma$ obtained from the identities $p \circ \gamma = a$ and $q \circ \gamma = b$. Then it is clear that (γ, α, β) is a morphism from (W, a, b, t) to $(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}, p, q, \psi)$.

Let $(k, \alpha', \beta') : (W, a, b, t) \to (\mathcal{A} \times_{\mathcal{C}} \mathcal{B}, p, q, \psi)$ be a second such morphism. For an object W of $\mathcal{W}$ let us write $k(W) = (a_k(W), b_k(W), t_{k,W})$. Hence $p(k(W)) = a_k(W)$ and so on. The map α' corresponds to functorial maps $\alpha' : a(W) \to a_k(W)$. Since we are working in the $(2, 1)$-category of categories, in fact each of the maps $a(W) \to a_k(W)$ is an isomorphism. We can use these (and their counterparts $b(W) \to b_k(W)$) to get isomorphisms

$$\delta_W : \gamma(W) = (a(W), b(W), t_W) \longrightarrow (a_k(W), b_k(W), t_{k,W}) = k(W).$$

It is straightforward to show that δ defines a 2-isomorphism between γ and k in the 2-category of 2-commutative diagrams as desired. $\square$

06RL Remark 4.31.5. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. Another, slightly more symmetrical, construction of a 2-fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is as follows. An object is a quintuple (A, B, C, a, b) where A, B, C are objects of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and where $a : F(A) \to C$ and $b : G(B) \to C$ are isomorphisms. A morphism $(A, B, C, a, b) \to (A', B', C', a', b')$ is given by a triple of morphisms $A \to A', B \to B', C \to C'$ compatible with the morphisms a, b, a', b'. We can prove directly that this leads to a 2-fibre product. However, it is easier to observe that the functor $(A, B, C, a, b) \mapsto (A, B, b^{-1} \circ a)$ gives an equivalence from the category of quintuples to the category constructed in Example 4.31.3.

02XA **Lemma 4.31.6.** Let

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "I"'] \arrow[dr, "K"] & \\ \mathcal{X} \arrow[r, "H"] \arrow[dr, "L"] & \mathcal{Z} \arrow[dr, "M"] & \mathcal{B} \arrow[d, "G"] \\ & \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

be a 2-commutative diagram of categories. A choice of isomorphisms $\alpha : G \circ K \to M \circ I$ and $\beta : M \circ H \to F \circ L$ determines a morphism

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$$

of 2-fibre products associated to this situation.

Proof. Just use the functor

$$(X, Y, \phi) \longmapsto (L(X), K(Y), \alpha_Y^{-1} \circ M(\phi) \circ \beta_X^{-1})$$

on objects and

$$(a, b) \longmapsto (L(a), K(b))$$

on morphisms. $\square$

02XB **Lemma 4.31.7.** Assumptions as in Lemma 4.31.6.

(1) If K and L are faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is faithful.
(2) If K and L are fully faithful and M is faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is fully faithful.
(3) If K and L are equivalences and M is fully faithful then the morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is an equivalence.

Proof. Let (X, Y, ϕ) and (X', Y', ϕ') be objects of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. Set $Z = H(X)$ and identify it with $I(Y)$ via ϕ. Also, identify $M(Z)$ with $F(L(X))$ via α_X and identify $M(Z)$ with $G(K(Y))$ via β_Y. Similarly for $Z' = H(X')$ and $M(Z')$. The map on morphisms is the map

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{X}}(X, X') \times_{\mathrm{Mor}_{\mathcal{Z}}(Z, Z')} \mathrm{Mor}_{\mathcal{Y}}(Y, Y') \arrow[d] \\ \mathrm{Mor}_{\mathcal{A}}(L(X), L(X')) \times_{\mathrm{Mor}_{\mathcal{C}}(M(Z), M(Z'))} \mathrm{Mor}_{\mathcal{B}}(K(Y), K(Y')) \end{tikzcd}$$

Hence parts (1) and (2) follow. Moreover, if K and L are equivalences and M is fully faithful, then any object (A, B, ϕ) is in the essential image for the following reasons: Pick X, Y such that $L(X) \cong A$ and $K(Y) \cong B$. Then the fully faithfulness of M guarantees that we can find an isomorphism $H(X) \cong I(Y)$. Some details omitted. $\square$

02XC **Lemma 4.31.8.** Let

$$\begin{tikzcd} \mathcal{A} \arrow[dr] & & \mathcal{C} \arrow[dl] \arrow[dr] & & \mathcal{E} \arrow[dl] \\ & \mathcal{B} & & \mathcal{D} & \end{tikzcd}$$

be a diagram of categories and functors. Then there is a canonical isomorphism

$$(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E} \cong \mathcal{A} \times_{\mathcal{B}} (\mathcal{C} \times_{\mathcal{D}} \mathcal{E})$$

of categories.

Proof. Just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, (C, E, \psi), \phi)$$

if you know what I mean. □

Henceforth we do not write the parentheses when dealing with fibre products of more than 2 categories.

04S7 **Lemma 4.31.9.** Let

$$\begin{tikzcd} \mathcal{A} \arrow[dr] & & \mathcal{C} \arrow[dl] \arrow[dr] & & \mathcal{E} \arrow[dl] \\ & \mathcal{B} \arrow[dr, "F"'] & & \mathcal{D} \arrow[dl, "G"] & \\ & & \mathcal{F} & & \end{tikzcd}$$

be a commutative diagram of categories and functors. Then there is a canonical functor

$$\mathrm{pr}_{02} : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \longrightarrow \mathcal{A} \times_{\mathcal{F}} \mathcal{E}$$

of categories.

Proof. If we write $\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ as $(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E}$ then we can just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, E, G(\psi) \circ F(\phi))$$

if you know what I mean. □

02XD **Lemma 4.31.10.** Let

$$\mathcal{A} \to \mathcal{B} \leftarrow \mathcal{C} \leftarrow \mathcal{D}$$

be a diagram of categories and functors. Then there is a canonical isomorphism

$$\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{C}} \mathcal{D} \cong \mathcal{A} \times_{\mathcal{B}} \mathcal{D}$$

of categories.

Proof. Omitted. □

We claim that this means you can work with these 2-fibre products just like with ordinary fibre products. Here are some further lemmas that actually come up later.

02XE **Lemma 4.31.11.** Let

$$\begin{tikzcd} \mathcal{C}_3 \arrow[r] \arrow[d] & \mathcal{S} \arrow[d, "\Delta"] \\ \mathcal{C}_1 \times \mathcal{C}_2 \arrow[r, "G_1 \times G_2"] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism $\mathcal{C}_3 \cong \mathcal{C}_1 \times_{G_1, \mathcal{S}, G_2} \mathcal{C}_2$.

Proof. We may assume that $\mathcal{C}_3$ is the category $(\mathcal{C}_1 \times \mathcal{C}_2) \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$ constructed in Example 4.31.3. Hence an object is a triple $((X_1, X_2), S, \phi)$ where $\phi = (\phi_1, \phi_2) : (G_1(X_1), G_2(X_2)) \to (S, S)$ is an isomorphism. Thus we can associate to this the triple $(X_1, X_2, \phi_2^{-1} \circ \phi_1)$. Conversely, if (X_1, X_2, ψ) is an object of $\mathcal{C}_1 \times_{G_1,\mathcal{S},G_2} \mathcal{C}_2$, then we can associate to this the triple $((X_1, X_2), G_2(X_2), (\psi, \mathrm{id}_{G_2(X_2)}))$. We claim these constructions given mutually inverse functors. We omit describing how to deal with morphisms and showing they are mutually inverse. $\square$

02XF **Lemma 4.31.12.** Let

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{S} \arrow[d, "\Delta"] \\ \mathcal{C} \arrow[r, "G_1\times G_2"] & \mathcal{S}\times\mathcal{S} \end{tikzcd}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism

$$\mathcal{C}' \cong (\mathcal{C} \times_{G_1,\mathcal{S},G_2} \mathcal{C}) \times_{(p,q),\mathcal{C}\times\mathcal{C},\Delta} \mathcal{C}.$$

Proof. An object of the right hand side is given by $((C_1, C_2, \phi), C_3, \psi)$ where $\phi : G_1(C_1) \to G_2(C_2)$ is an isomorphism and $\psi = (\psi_1, \psi_2) : (C_1, C_2) \to (C_3, C_3)$ is an isomorphism. Hence we can associate to this the triple $(C_3, G_1(C_1), (G_1(\psi_1^{-1}), \phi^{-1} \circ G_2(\psi_2^{-1})))$ which is an object of $\mathcal{C}'$. Details omitted. $\square$

04Z1 **Lemma 4.31.13.** Let $\mathcal{A} \to \mathcal{C}$, $\mathcal{B} \to \mathcal{C}$ and $\mathcal{C} \to \mathcal{D}$ be functors between categories. Then the diagram

$$\begin{tikzcd} \mathcal{A}\times_{\mathcal{C}}\mathcal{B} \arrow[r] \arrow[d] & \mathcal{A}\times_{\mathcal{D}}\mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r, "\Delta_{\mathcal{C}/\mathcal{D}}"] & \mathcal{C}\times_{\mathcal{D}}\mathcal{C} \end{tikzcd}$$

is a 2-fibre product diagram.

Proof. Omitted. $\square$

04YR **Lemma 4.31.14.** Let

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

be a 2-fibre product of categories. Then the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathcal{U}\times_{\mathcal{V}}\mathcal{U} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}\times_{\mathcal{Y}}\mathcal{X} \end{tikzcd}$$

is 2-cartesian.

Proof. This is a purely 2-category theoretic statement, valid in any $(2,1)$-category with 2-fibre products. Explicitly, it follows from the following chain of equivalences:

$$\begin{aligned} \mathcal{X} \times_{(\mathcal{X}\times_{\mathcal{Y}}\mathcal{X})} (\mathcal{U} \times_{\mathcal{V}} \mathcal{U}) &= \mathcal{X} \times_{(\mathcal{X}\times_{\mathcal{Y}}\mathcal{X})} ((\mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \times_{\mathcal{V}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{V})) \\ &= \mathcal{X} \times_{(\mathcal{X}\times_{\mathcal{Y}}\mathcal{X})} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \\ &= \mathcal{X} \times_{\mathcal{Y}} \mathcal{V} = \mathcal{U} \end{aligned}$$

see Lemmas 4.31.8 and 4.31.10. □

4.32. Categories over categories

02XG In this section we have a functor $p : \mathcal{S} \to \mathcal{C}$. We think of $\mathcal{S}$ as being on top and of $\mathcal{C}$ as being at the bottom. To make sure that everybody knows what we are talking about we define the 2-category of categories over $\mathcal{C}$.

003Y Definition 4.32.1. Let $\mathcal{C}$ be a category. The 2-category of categories over $\mathcal{C}$ is the 2-category defined as follows:

(1) Its objects will be functors $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$\mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

In this 2-category we define horizontal and vertical composition exactly as is done for Cat in Section 4.28. The axioms of a 2-category are satisfied for the same reason that the hold in Cat. To see this one can also use that the axioms hold in Cat and verify things such as "vertical composition of 2-morphisms over $\mathcal{C}$ gives another 2-morphism over $\mathcal{C}$". This is clear.

Analogously to the fibre of a map of spaces, we have the notion of a fibre category, and some notions of lifting associated to this situation.

02XH Definition 4.32.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.

(1) The fibre category over an object $U \in \mathrm{Ob}(\mathcal{C})$ is the category $\mathcal{S}_U$ with objects
$$\mathrm{Ob}(\mathcal{S}_U) = \{x \in \mathrm{Ob}(\mathcal{S}) : p(x) = U\}$$
and morphisms
$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) = \{\phi \in \mathrm{Mor}_{\mathcal{S}}(x, y) : p(\phi) = \mathrm{id}_U\}.$$
(2) A lift of an object $U \in \mathrm{Ob}(\mathcal{C})$ is an object $x \in \mathrm{Ob}(\mathcal{S})$ such that $p(x) = U$, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. We will also sometime say that x lies over U.
(3) Similarly, a lift of a morphism $f : V \to U$ in $\mathcal{C}$ is a morphism $\phi : y \to x$ in $\mathcal{S}$ such that $p(\phi) = f$. We sometimes say that ϕ lies over f.

There are some observations we could make here. For example if $F : (\mathcal{S}, p) \to (\mathcal{S}', p')$ is a 1-morphism of categories over $\mathcal{C}$, then F induces functors of fibre categories $F : \mathcal{S}_U \to \mathcal{S}'_U$. Similarly for 2-morphisms.

Here is the obligatory lemma describing the 2-fibre product in the $(2, 1)$-category of categories over $\mathcal{C}$.

0040 Lemma 4.32.3. Let $\mathcal{C}$ be a category. The $(2, 1)$-category of categories over $\mathcal{C}$ has 2-fibre products. Suppose that $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ are morphisms of categories over $\mathcal{C}$. An explicit 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by the following description

(1) an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a quadruple (U, x, y, f), where $U \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$, and $f : F(x) \to G(y)$ is an isomorphism in $\mathcal{S}_U$,
(2) a morphism $(U, x, y, f) \to (U', x', y', f')$ is given by a pair (a, b), where $a : x \to x'$ is a morphism in $\mathcal{X}$, and $b : y \to y'$ is a morphism in $\mathcal{Y}$ such that
  (a) a and b induce the same morphism $U \to U'$, and
  (b) the diagram

$$\begin{tikzcd} F(x) \arrow[r, "f"] \arrow[d, "F(a)"] & G(y) \arrow[d, "G(b)"] \\ F(x') \arrow[r, "f'"] & G(y') \end{tikzcd}$$

  is commutative.

The functors $p : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{Y}$ are the forgetful functors in this case. The transformation $\psi : F \circ p \to G \circ q$ is given on the object $\xi = (U, x, y, f)$ by $\psi_\xi = f : F(p(\xi)) = F(x) \to G(y) = G(q(\xi))$.

Proof. Let us check the universal property: let $p_{\mathcal{W}} : \mathcal{W} \to \mathcal{C}$ be a category over $\mathcal{C}$, let $X : \mathcal{W} \to \mathcal{X}$ and $Y : \mathcal{W} \to \mathcal{Y}$ be functors over $\mathcal{C}$, and let $t : F \circ X \to G \circ Y$ be an isomorphism of functors over $\mathcal{C}$. The desired functor $\gamma : \mathcal{W} \to \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by $W \mapsto (p_{\mathcal{W}}(W), X(W), Y(W), t_W)$. Details omitted; compare with Lemma 4.31.4. □

0H2D Example 4.32.4. The constructions of 2-fibre products of categories over categories given in Lemma 4.32.3 and of categories in Lemma 4.31.4 (as in Example 4.31.3) produce non-equivalent outputs in general. Namely, let $\mathcal{S}$ be the groupoid category with one object and two arrows, and let $\mathcal{X}$ be the discrete category with one object. Taking the 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ as categories yields the discrete category with two objects. However, if we view all of these as categories over $\mathcal{S}$, the 2-fiber product $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ as categories over $\mathcal{S}$ is the discrete category with one object. The difference is that (in the notation of Lemma 4.32.3), we were allowed to choose any comparison isomorphism f in the first situation, but could only choose the identity arrow in the second situation.

02XI Lemma 4.32.5. Let $\mathcal{C}$ be a category. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be morphisms of categories over $\mathcal{C}$. For any object U of $\mathcal{C}$ we have the following identity of fibre categories

$$(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})_U = \mathcal{X}_U \times_{\mathcal{S}_U} \mathcal{Y}_U$$

Proof. Omitted. □

4.33. Fibred categories

02XJ A very brief discussion of fibred categories is warranted.

Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Given an object $x \in \mathcal{S}$ with $p(x) = U$, and given a morphism $f : V \to U$, we can try to take some kind of "fibre product $V \times_U x$" (or a base change of x via $V \to U$). Namely, a morphism from an object $z \in \mathcal{S}$ into "$V \times_U x$" should be given by a pair (φ, g), where $\varphi : z \to x$, $g : p(z) \to V$

such that $p(\varphi) = f \circ g$. Pictorially:

$$\begin{tikzcd} z \arrow[r, dash] \arrow[d, squiggly, "p"] & ? \arrow[r] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\ p(z) \arrow[r] & V \arrow[r, "f"] & U \end{tikzcd}$$

If such a morphism $V \times_U x \to x$ exists then it is called a strongly cartesian morphism.

02XK Definition 4.33.1. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. A strongly cartesian morphism, or more precisely a strongly $\mathcal{C}$-cartesian morphism is a morphism $\varphi : y \to x$ of $\mathcal{S}$ such that for every $z \in \mathrm{Ob}(\mathcal{S})$ the map

$$\mathrm{Mor}_\mathcal{S}(z, y) \longrightarrow \mathrm{Mor}_\mathcal{S}(z, x) \times_{\mathrm{Mor}_\mathcal{C}(p(z),p(x))} \mathrm{Mor}_\mathcal{C}(p(z), p(y)),$$

given by $\psi \longmapsto (\varphi \circ \psi, p(\psi))$ is bijective.

Note that by the Yoneda Lemma 4.3.5, given $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and the morphism $f : V \to U$ of $\mathcal{C}$, if there is a strongly cartesian morphism $\varphi : y \to x$ with $p(\varphi) = f$, then (y, φ) is unique up to unique isomorphism. This is clear from the definition above, as the functor

$$z \longmapsto \mathrm{Mor}_\mathcal{S}(z, x) \times_{\mathrm{Mor}_\mathcal{C}(p(z),U)} \mathrm{Mor}_\mathcal{C}(p(z), V)$$

only depends on the data $(x, U, f : V \to U)$. Hence we will sometimes use $V \times_U x \to x$ or $f^*x \to x$ to denote a strongly cartesian morphism which is a lift of f.

02XL Lemma 4.33.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.

(1) The composition of two strongly cartesian morphisms is strongly cartesian.
(2) Any isomorphism of $\mathcal{S}$ is strongly cartesian.
(3) Any strongly cartesian morphism φ such that $p(\varphi)$ is an isomorphism, is an isomorphism.

Proof. Proof of (1). Let $\varphi : y \to x$ and $\psi : z \to y$ be strongly cartesian. Let t be an arbitrary object of $\mathcal{S}$. Then we have

$$\begin{aligned} & \mathrm{Mor}_\mathcal{S}(t, z) \\ & = \mathrm{Mor}_\mathcal{S}(t, y) \times_{\mathrm{Mor}_\mathcal{C}(p(t),p(y))} \mathrm{Mor}_\mathcal{C}(p(t), p(z)) \\ & = \mathrm{Mor}_\mathcal{S}(t, x) \times_{\mathrm{Mor}_\mathcal{C}(p(t),p(x))} \mathrm{Mor}_\mathcal{C}(p(t), p(y)) \times_{\mathrm{Mor}_\mathcal{C}(p(t),p(y))} \mathrm{Mor}_\mathcal{C}(p(t), p(z)) \\ & = \mathrm{Mor}_\mathcal{S}(t, x) \times_{\mathrm{Mor}_\mathcal{C}(p(t),p(x))} \mathrm{Mor}_\mathcal{C}(p(t), p(z)) \end{aligned}$$

hence $z \to x$ is strongly cartesian.

Proof of (2). Let $y \to x$ be an isomorphism. Then $p(y) \to p(x)$ is an isomorphism too. Hence $\mathrm{Mor}_\mathcal{C}(p(z), p(y)) \to \mathrm{Mor}_\mathcal{C}(p(z), p(x))$ is a bijection. Hence $\mathrm{Mor}_\mathcal{S}(z, x) \times_{\mathrm{Mor}_\mathcal{C}(p(z),p(x))} \mathrm{Mor}_\mathcal{C}(p(z), p(y))$ is bijective to $\mathrm{Mor}_\mathcal{S}(z, x)$. Hence the displayed map of Definition 4.33.1 is a bijection as $y \to x$ is an isomorphism, and we conclude that $y \to x$ is strongly cartesian.

Proof of (3). Assume $\varphi : y \to x$ is strongly cartesian with $p(\varphi) : p(y) \to p(x)$ an isomorphism. Applying the definition with $z = x$ shows that $(\mathrm{id}_x, p(\varphi)^{-1})$ comes from a unique morphism $\chi : x \to y$. We omit the verification that χ is the inverse of φ. $\square$

09WU Lemma 4.33.3. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be composable functors between categories. Let $x \to y$ be a morphism of $\mathcal{A}$. If $x \to y$ is strongly $\mathcal{B}$-cartesian and $F(x) \to F(y)$ is strongly $\mathcal{C}$-cartesian, then $x \to y$ is strongly $\mathcal{C}$-cartesian.

Proof. This follows directly from the definition. $\square$

06N4 Lemma 4.33.4. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. Assume

(1) $x \to y$ is strongly cartesian,
(2) $p(x) \times_{p(y)} p(z)$ exists, and
(3) there exists a strongly cartesian morphism $a : w \to z$ in $\mathcal{S}$ with $p(w) = p(x) \times_{p(y)} p(z)$ and $p(a) = \text{pr}_2 : p(x) \times_{p(y)} p(z) \to p(z)$.

Then the fibre product $x \times_y z$ exists and is isomorphic to w.

Proof. Since $x \to y$ is strongly cartesian there exists a unique morphism $b : w \to x$ such that $p(b) = \text{pr}_1$. To see that w is the fibre product we compute

$$
\begin{aligned}
&\text{Mor}_{\mathcal{S}}(t, w) \\
&= \text{Mor}_{\mathcal{S}}(t, z) \times_{\text{Mor}_{\mathcal{C}}(p(t),p(z))} \text{Mor}_{\mathcal{C}}(p(t), p(w)) \\
&= \text{Mor}_{\mathcal{S}}(t, z) \times_{\text{Mor}_{\mathcal{C}}(p(t),p(z))} (\text{Mor}_{\mathcal{C}}(p(t), p(x)) \times_{\text{Mor}_{\mathcal{C}}(p(t),p(y))} \text{Mor}_{\mathcal{C}}(p(t), p(z))) \\
&= \text{Mor}_{\mathcal{S}}(t, z) \times_{\text{Mor}_{\mathcal{C}}(p(t),p(y))} \text{Mor}_{\mathcal{C}}(p(t), p(x)) \\
&= \text{Mor}_{\mathcal{S}}(t, z) \times_{\text{Mor}_{\mathcal{S}}(t,y)} \text{Mor}_{\mathcal{S}}(t, y) \times_{\text{Mor}_{\mathcal{C}}(p(t),p(y))} \text{Mor}_{\mathcal{C}}(p(t), p(x)) \\
&= \text{Mor}_{\mathcal{S}}(t, z) \times_{\text{Mor}_{\mathcal{S}}(t,y)} \text{Mor}_{\mathcal{S}}(t, x)
\end{aligned}
$$

as desired. The first equality holds because $a : w \to z$ is strongly cartesian and the last equality holds because $x \to y$ is strongly cartesian. $\square$

02XM Definition 4.33.5. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. We say $\mathcal{S}$ is a *fibred category over $\mathcal{C}$* if given any $x \in \text{Ob}(\mathcal{S})$ lying over $U \in \text{Ob}(\mathcal{C})$ and any morphism $f : V \to U$ of $\mathcal{C}$, there exists a strongly cartesian morphism $f^*x \to x$ lying over f.

Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. For every $f : V \to U$ and $x \in \text{Ob}(\mathcal{S}_U)$ as in the definition we may choose a strongly cartesian morphism $f^*x \to x$ lying over f. By the axiom of choice we may choose $f^*x \to x$ for all $f : V \to U = p(x)$ simultaneously. We claim that for every morphism $\phi : x \to x'$ in $\mathcal{S}_U$ and $f : V \to U$ there is a unique morphism $f^*\phi : f^*x \to f^*x'$ in $\mathcal{S}_V$ such that

$$
\begin{tikzcd}
f^*x \arrow[r, "f^*\phi"'] \arrow[d] & f^*x' \arrow[d] \\
x \arrow[r, "\phi"] & x'
\end{tikzcd}
$$

commutes. Namely, the arrow exists and is unique because $f^*x' \to x'$ is strongly cartesian. The uniqueness of this arrow guarantees that f^* (now also defined on morphisms) is a functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$.

02XN Definition 4.33.6. Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category.

(1) A choice of pullbacks[7] for $p : \mathcal{S} \to \mathcal{C}$ is given by a choice of a strongly cartesian morphism $f^*x \to x$ lying over f for any morphism $f : V \to U$ of $\mathcal{C}$ and any $x \in \mathrm{Ob}(\mathcal{S}_U)$.
(2) Given a choice of pullbacks, for any morphism $f : V \to U$ of $\mathcal{C}$ the functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$ described above is called a pullback functor (associated to the choices $f^*x \to x$ made above).

Of course we may always assume our choice of pullbacks has the property that $\mathrm{id}_U^*x = x$, although in practice this is a useless property without imposing further assumptions on the pullbacks.

02XO Lemma 4.33.7. Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. Assume given a choice of pullbacks for $p : \mathcal{S} \to \mathcal{C}$.

(1) For any pair of composable morphisms $f : V \to U$, $g : W \to V$ there is a unique isomorphism
$$\alpha_{g,f} : (f \circ g)^* \longrightarrow g^* \circ f^*$$
as functors $\mathcal{S}_U \to \mathcal{S}_W$ such that for every $y \in \mathrm{Ob}(\mathcal{S}_U)$ the following diagram commutes
$$\begin{tikzcd} g^*f^*y \arrow[r] & f^*y \arrow[d] \\ (f \circ g)^*y \arrow[u, "(\alpha_{g,f})_y"] \arrow[r] & y \end{tikzcd}$$
(2) If $f = \mathrm{id}_U$, then there is a canonical isomorphism $\alpha_U : \mathrm{id} \to (\mathrm{id}_U)^*$ as functors $\mathcal{S}_U \to \mathcal{S}_U$.
(3) The quadruple $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{g,f}, \alpha_U)$ defines a pseudo functor from $\mathcal{C}^{opp}$ to the $(2,1)$-category of categories, see Definition 4.29.5.

Proof. In fact, it is clear that the commutative diagram of part (1) uniquely determines the morphism $(\alpha_{g,f})_y$ in the fibre category $\mathcal{S}_W$. It is an isomorphism since both the morphism $(f \circ g)^*y \to y$ and the composition $g^*f^*y \to f^*y \to y$ are strongly cartesian morphisms lifting $f \circ g$ (see discussion following Definition 4.33.1 and Lemma 4.33.2). In the same way, since $\mathrm{id}_x : x \to x$ is clearly strongly cartesian over id_U (with $U = p(x)$) we see that there exists an isomorphism $(\alpha_U)_x : x \to (\mathrm{id}_U)^*x$. (Of course we could have assumed beforehand that $f^*x = x$ whenever f is an identity morphism, but it is better for the sake of generality not to assume this.) We omit the verification that $\alpha_{g,f}$ and α_U so obtained are transformations of functors. We also omit the verification of (3). $\square$

042G Lemma 4.33.8. Let $\mathcal{C}$ be a category. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is fibred over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$.

Proof. Denote $p_i : \mathcal{S}_i \to \mathcal{C}$ the given functors. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. We claim that in this case F maps strongly cartesian morphisms to

[7]This is probably nonstandard terminology. In some texts this is called a "cleavage" but it conjures up the wrong image. Maybe a "cleaving" would be a better word. A related notion is that of a "splitting", but in many texts a "splitting" means a choice of pullbacks such that $g^*f^* = (f \circ g)^*$ for any composable pair of morphisms. Compare also with Definition 4.36.2.

strongly cartesian morphisms. Namely, suppose that $\varphi : y \to x$ is strongly cartesian in $\mathcal{S}_1$. Set $f : V \to U$ equal to $p_1(\varphi)$. Suppose that $z' \in \mathrm{Ob}(\mathcal{S}_2)$, with $W = p_2(z')$, and we are given $g : W \to V$ and $\psi' : z' \to F(x)$ such that $p_2(\psi') = f \circ g$. Then

$$\psi = j \circ G(\psi') : G(z') \to G(F(x)) \to x$$

is a morphism in $\mathcal{S}_1$ with $p_1(\psi) = f \circ g$. Hence by assumption there exists a unique morphism $\xi : G(z') \to y$ lying over g such that $\psi = \varphi \circ \xi$. This in turn gives a morphism

$$\xi' = F(\xi) \circ i^{-1} : z' \to F(G(z')) \to F(y)$$

lying over g with $\psi' = F(\varphi) \circ \xi'$. We omit the verification that ξ' is unique. $\square$

The conclusion from Lemma 4.33.8 is that equivalences map strongly cartesian morphisms to strongly cartesian morphisms. But this may not be the case for an arbitrary functor between fibred categories over $\mathcal{C}$. Hence we define the 2-category of fibred categories as follows.

02XP Definition 4.33.9. Let $\mathcal{C}$ be a category. The 2-category of fibred categories over $\mathcal{C}$ is the sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.32.1) defined as follows:

(1) Its objects will be fibred categories $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$\mathrm{Mor}_{\mathrm{Fib}/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Note the condition on 1-morphisms. Note also that this is a true 2-category and not a $(2,1)$-category. Hence when taking 2-fibre products we first pass to the associated $(2,1)$-category.

02XQ Lemma 4.33.10. Let $\mathcal{C}$ be a category. The $(2,1)$-category of fibred categories over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.32.3.

Proof. Basically what one has to show here is that given $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ morphisms of fibred categories over $\mathcal{C}$, then the category $\mathcal{X} \times_\mathcal{S} \mathcal{Y}$ described in Lemma 4.32.3 is fibred. Let us show that $\mathcal{X} \times_\mathcal{S} \mathcal{Y}$ has plenty of strongly cartesian morphisms. Namely, suppose we have (U, x, y, ϕ) an object of $\mathcal{X} \times_\mathcal{S} \mathcal{Y}$. And suppose $f : V \to U$ is a morphism in $\mathcal{C}$. Choose strongly cartesian morphisms $a : f^*x \to x$ in $\mathcal{X}$ lying over f and $b : f^*y \to y$ in $\mathcal{Y}$ lying over f. By assumption $F(a)$ and $G(b)$ are strongly cartesian. Since $\phi : F(x) \to G(y)$ is an isomorphism, by the uniqueness of strongly cartesian morphisms we find a unique isomorphism $f^*\phi : F(f^*x) \to G(f^*y)$ such that $G(b) \circ f^*\phi = \phi \circ F(a)$. In other words $(a, b) : (V, f^*x, f^*y, f^*\phi) \to (U, x, y, \phi)$ is a morphism in $\mathcal{X} \times_\mathcal{S} \mathcal{Y}$. We omit the verification that this is a strongly cartesian morphism (and that these are in fact the only strongly cartesian morphisms). $\square$

02XR Lemma 4.33.11. Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a fibred category and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is a fibred category.

Proof. Suppose that $\varphi : x' \to x$ is strongly cartesian with respect to p. We claim that φ is strongly cartesian with respect to p' also. Set $g = p'(\varphi)$, so that $g : V'/U \to V/U$ for some morphisms $f : V \to U$ and $f' : V' \to U$. Let $z \in \mathrm{Ob}(\mathcal{S})$. Set $p'(z) = (W \to U)$. To show that φ is strongly cartesian for p' we have to show

$$\mathrm{Mor}_{\mathcal{S}}(z, x') \longrightarrow \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}/U}(W/U, V/U)} \mathrm{Mor}_{\mathcal{C}/U}(W/U, V'/U),$$

given by $\psi' \longmapsto (\varphi \circ \psi', p'(\psi'))$ is bijective. Suppose given an element (ψ, h) of the right hand side, then in particular $g \circ h = p(\psi)$, and by the condition that φ is strongly cartesian we get a unique morphism $\psi' : z \to x'$ with $\psi = \varphi \circ \psi'$ and $p(\psi') = h$. OK, and now $p'(\psi') : W/U \to V/U$ is a morphism whose corresponding map $W \to V$ is h, hence equal to h as a morphism in $\mathcal{C}/U$. Thus ψ' is a unique morphism $z \to x'$ which maps to the given pair (ψ, h). This proves the claim.

Finally, suppose given $g : V'/U \to V/U$ and x with $p'(x) = V/U$. Since $p : \mathcal{S} \to \mathcal{C}$ is a fibred category we see there exists a strongly cartesian morphism $\varphi : x' \to x$ with $p(\varphi) = g$. By the same argument as above it follows that $p'(\varphi) = g : V'/U \to V/U$. And as seen above the morphism φ is strongly cartesian. Thus the conditions of Definition 4.33.5 are satisfied and we win. □

09WV Lemma 4.33.12. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be functors between categories. If $\mathcal{A}$ is fibred over $\mathcal{B}$ and $\mathcal{B}$ is fibred over $\mathcal{C}$, then $\mathcal{A}$ is fibred over $\mathcal{C}$.

Proof. This follows from the definitions and Lemma 4.33.3. □

06N5 Lemma 4.33.13. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$ with $x \to y$ strongly cartesian. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists, $p(x \times_y z) = p(x) \times_{p(y)} p(z)$, and $x \times_y z \to z$ is strongly cartesian.

Proof. Pick a strongly cartesian morphism $\mathrm{pr}_2^* z \to z$ lying over $\mathrm{pr}_2 : p(x) \times_{p(y)} p(z) \to p(z)$. Then $\mathrm{pr}_2^* z = x \times_y z$ by Lemma 4.33.4. □

08NF Lemma 4.33.14. Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of fibred categories over $\mathcal{C}$. There exist 1-morphisms of fibred categories over $\mathcal{C}$

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "u"] & \mathcal{X}' \arrow[l, shift left, "w"] \arrow[r, "v"] & \mathcal{Y} \end{tikzcd}$$

such that $F = v \circ u$ and such that

(1) $u : \mathcal{X} \to \mathcal{X}'$ is fully faithful,
(2) w is left adjoint to u, and
(3) $v : \mathcal{X}' \to \mathcal{Y}$ is a fibred category.

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $f : y \to F(x)$ is a morphism in $\mathcal{Y}_U$. A morphism $(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b) : U \to U'$ and such that $f' \circ b = F(a) \circ f$.

Let us make a choice of pullbacks for both p and q and let us use the same notation to indicate them. Let (U, x, y, f) be an object and let $h : V \to U$ be a morphism. Consider the morphism $c : (V, h^*x, h^*y, h^*f) \to (U, x, y, f)$ coming from the given strongly cartesian maps $h^*x \to x$ and $h^*y \to y$. We claim c is strongly cartesian in $\mathcal{X}'$ over $\mathcal{C}$. Namely, suppose we are given an object (W, x', y', f') of $\mathcal{X}'$, a morphism $(a, b) : (W, x', y', f') \to (U, x, y, f)$ lying over $W \to U$, and a factorization $W \to$

$V \to U$ of $W \to U$ through h. As $h^*x \to x$ and $h^*y \to y$ are strongly cartesian we obtain morphisms $a' : x' \to h^*x$ and $b' : y' \to h^*y$ lying over the given morphism $W \to V$. Consider the diagram

$$\begin{tikzcd} y' \arrow[r] \arrow[d, "f'"'] & h^*y \arrow[r] \arrow[d, "h^*f"'] & y \arrow[d, "f"'] \\ F(x') \arrow[r] & F(h^*x) \arrow[r] & F(x) \end{tikzcd}$$

The outer rectangle and the right square commute. Since F is a 1-morphism of fibred categories the morphism $F(h^*x) \to F(x)$ is strongly cartesian. Hence the left square commutes by the universal property of strongly cartesian morphisms. This proves that $\mathcal{X}'$ is fibred over $\mathcal{C}$.

The functor $u : \mathcal{X} \to \mathcal{X}'$ is given by $x \mapsto (p(x), x, F(x), \mathrm{id})$. This is fully faithful. The functor $\mathcal{X}' \to \mathcal{Y}$ is given by $(U, x, y, f) \mapsto y$. The functor $w : \mathcal{X}' \to \mathcal{X}$ is given by $(U, x, y, f) \mapsto x$. Each of these functors is a 1-morphism of fibred categories over $\mathcal{C}$ by our description of strongly cartesian morphisms of $\mathcal{X}'$ over $\mathcal{C}$. Adjointness of w and u means that

$$\mathrm{Mor}_{\mathcal{X}}(x, x') = \mathrm{Mor}_{\mathcal{X}'}((U, x, y, f), (p(x'), x', F(x'), \mathrm{id})),$$

which follows immediately from the definitions.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a fibred category. Let $c : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Set $V = q(y')$ and let $h = q(c) : V \to U$. Let $a : h^*x \to x$ and $b : h^*y \to y$ be the strongly cartesian morphisms covering h. Since F is a 1-morphism of fibred categories we may identify $h^*F(x) = F(h^*x)$ with strongly cartesian morphism $F(a) : F(h^*x) \to F(x)$. By the universal property of $b : h^*y \to y$ there is a morphism $c' : y' \to h^*y$ in $\mathcal{Y}_V$ such that $c = b \circ c'$. We claim that

$$(a, c) : (V, h^*x, y', h^*f \circ c') \longrightarrow (U, x, y, f)$$

is strongly cartesian in $\mathcal{X}'$ over $\mathcal{Y}$. To see this let (W, x_1, y_1, f_1) be an object of $\mathcal{X}'$, let $(a_1, b_1) : (W, x_1, y_1, f_1) \to (U, x, y, f)$ be a morphism and let $b_1 = c \circ b_1'$ for some morphism $b_1' : y_1 \to y'$. Then

$$(a_1', b_1') : (W, x_1, y_1, f_1) \longrightarrow (V, h^*x, y', h^*f \circ c')$$

(where $a_1' : x_1 \to h^*x$ is the unique morphism lying over the given morphism $q(b_1') : W \to V$ such that $a_1 = a \circ a_1'$) is the desired morphism. $\square$

4.34. Inertia

04Z2 Given fibred categories $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ over a category $\mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{S}'$ we have the diagonal morphism

$$\Delta = \Delta_{\mathcal{S}/\mathcal{S}'} : \mathcal{S} \longrightarrow \mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$$

in the $(2,1)$-category of fibred categories over $\mathcal{C}$.

034H **Lemma 4.34.1.** Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Consider the category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ over $\mathcal{C}$ whose

(1) objects are pairs (x, α) where $x \in \mathrm{Ob}(\mathcal{S})$ and $\alpha : x \to x$ is an automorphism with $F(\alpha) = \mathrm{id}$,

(2) morphisms $(x, \alpha) \to (y, \beta)$ are given by morphisms $\phi : x \to y$ such that

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "\alpha"'] & y \arrow[d, "\beta"] \\ x \arrow[r, "\phi"] & y \end{tikzcd}$$

commutes, and

(3) the functor $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ is given by $(x, \alpha) \mapsto p(x)$.

Then

(1) there is an equivalence

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$$

in the $(2,1)$-category of categories over $\mathcal{C}$, and

(2) $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ is a fibred category over $\mathcal{C}$.

Proof. Note that (2) follows from (1) by Lemmas 4.33.10 and 4.33.8. Thus it suffices to prove (1). We will use without further mention the construction of the 2-fibre product from Lemma 4.33.10. In particular an object of $\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$ is a triple $(x, y, (\iota, \kappa))$ where x and y are objects of $\mathcal{S}$, and $(\iota, \kappa) : (x, x, \text{id}_{F(x)}) \to (y, y, \text{id}_{F(y)})$ is an isomorphism in $\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$. This just means that $\iota, \kappa : x \to y$ are isomorphisms and that $F(\iota) = F(\kappa)$. Consider the functor

$$I_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$$

which to an object (x, α) of the left hand side assigns the object $(x, x, (\alpha, \text{id}_x))$ of the right hand side and to a morphism ϕ of the left hand side assigns the morphism (ϕ, ϕ) of the right hand side. We claim that a quasi-inverse to that morphism is given by the functor

$$\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S} \longrightarrow I_{\mathcal{S}/\mathcal{S}'}$$

which to an object $(x, y, (\iota, \kappa))$ of the left hand side assigns the object $(x, \kappa^{-1} \circ \iota)$ of the right hand side and to a morphism $(\phi, \phi') : (x, y, (\iota, \kappa)) \to (z, w, (\lambda, \mu))$ of the left hand side assigns the morphism ϕ. Indeed, the endo-functor of $I_{\mathcal{S}/\mathcal{S}'}$ induced by composing the two functors above is the identity on the nose, and the endo-functor induced on $\mathcal{S} \times_{\Delta, (\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}), \Delta} \mathcal{S}$ is isomorphic to the identity via the natural isomorphism

$$(\text{id}_x, \kappa) : (x, x, (\kappa^{-1} \circ \iota, \text{id}_x)) \longrightarrow (x, y, (\iota, \kappa)).$$

Some details omitted. $\square$

034I Definition 4.34.2. Let $\mathcal{C}$ be a category.

(1) Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. The relative inertia of $\mathcal{S}$ over $\mathcal{S}'$ is the fibred category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ of Lemma 4.34.1.

(2) By the inertia fibred category $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$ we mean $\mathcal{I}_{\mathcal{S}} = \mathcal{I}_{\mathcal{S}/\mathcal{C}}$.

Note that there are canonical 1-morphisms

042H (4.34.2.1)
$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \quad \text{and} \quad \mathcal{I}_{\mathcal{S}} \longrightarrow \mathcal{S}$$

of fibred categories over $\mathcal{C}$. In terms of the description of Lemma 4.34.1 these simply map the object (x, α) to the object x and the morphism $\phi : (x, \alpha) \to (y, \beta)$ to the morphism $\phi : x \to y$. There is also a neutral section

04Z3 (4.34.2.2)
$$e : \mathcal{S} \to \mathcal{I}_{\mathcal{S}/\mathcal{S}'} \quad \text{and} \quad e : \mathcal{S} \to \mathcal{I}_{\mathcal{S}}$$

defined by the rules $x \mapsto (x, \mathrm{id}_x)$ and $(\phi : x \to y) \mapsto \phi$. This is a right inverse to (4.34.2.1). Given a 2-commutative square

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, "G"'] \arrow[d, "F_1"'] & \mathcal{S}_2 \arrow[d, "F_2"] \\ \mathcal{S}'_1 \arrow[r, "G'"] & \mathcal{S}'_2 \end{tikzcd}$$

there are functoriality maps

04Z4 (4.34.2.3) $$\mathcal{I}_{\mathcal{S}_1/\mathcal{S}'_1} \longrightarrow \mathcal{I}_{\mathcal{S}_2/\mathcal{S}'_2} \quad \text{and} \quad \mathcal{I}_{\mathcal{S}_1} \longrightarrow \mathcal{I}_{\mathcal{S}_2}$$

defined by the rules $(x, \alpha) \mapsto (G(x), G(\alpha))$ and $\phi \mapsto G(\phi)$. In particular there is always a comparison map

04Z5 (4.34.2.4) $$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{I}_{\mathcal{S}}$$

and all the maps above are compatible with this.

04Z6 Lemma 4.34.3. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of categories fibred over a category $\mathcal{C}$. Then the diagram

$$\begin{tikzcd} \mathcal{I}_{\mathcal{S}/\mathcal{S}'} \arrow[r, "(4.34.2.4)"'] \arrow[d, "F\circ(4.34.2.1)"'] & \mathcal{I}_{\mathcal{S}} \arrow[d, "(4.34.2.3)"] \\ \mathcal{S}' \arrow[r, "e"] & \mathcal{I}_{\mathcal{S}'} \end{tikzcd}$$

is a 2-fibre product.

Proof. Omitted. $\square$

4.35. Categories fibred in groupoids

003S In this section we explain how to think about categories fibred in groupoids and we see how they are basically the same as functors with values in the $(2,1)$-category of groupoids.

003T Definition 4.35.1. Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. We say that $\mathcal{S}$ is fibred in groupoids over $\mathcal{C}$ if the following two conditions hold:

(1) For every morphism $f : V \to U$ in $\mathcal{C}$ and every lift x of U there is a lift $\phi : y \to x$ of f with target x.
(2) For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

Condition (2) phrased differently says that applying the functor p gives a bijection between the sets of dotted arrows in the following commutative diagram below:

$$\begin{tikzcd} y \arrow[r] & x \\ z \arrow[u, dashed] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} p(y) \arrow[r] & p(x) \\ p(z) \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$

Another way to think about the second condition is the following. Suppose that $g : W \to V$ and $f : V \to U$ are morphisms in $\mathcal{C}$. Let $x \in \mathrm{Ob}(\mathcal{S}_U)$. By the first condition we can lift f to $\phi : y \to x$ and then we can lift g to $\psi : z \to y$. Instead of doing this two step process we can directly lift $g \circ f$ to $\gamma : z' \to x$. This gives the solid arrows in the diagram

03WP (4.35.1.1)
$$\begin{tikzcd}
z' \arrow[d, dashed, leftrightarrow] \arrow[rrd, "\gamma"] & & \\
z \arrow[r, "\psi"] \arrow[d, squiggly, "p"] & y \arrow[r, "\phi"] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\
W \arrow[r, "g"] & V \arrow[r, "f"] & U
\end{tikzcd}$$

where the squiggly arrows represent not morphisms but the functor p. Applying the second condition to the arrows $\phi \circ \psi$, γ and id_W we conclude that there is a unique morphism $\chi : z \to z'$ in $\mathcal{S}_W$ such that $\gamma \circ \chi = \phi \circ \psi$. Similarly there is a unique morphism $z' \to z$. The uniqueness implies that the morphisms $z' \to z$ and $z \to z'$ are mutually inverse, in other words isomorphisms.

It should be clear from this discussion that a category fibred in groupoids is very closely related to a fibred category. Here is the result.

003V **Lemma 4.35.2.** Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent

(1) $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids, and
(2) all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Moreover, in this case every morphism of $\mathcal{S}$ is strongly cartesian. In addition, given $f^*x \to x$ lying over f for all $f : V \to U = p(x)$ the data $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{f,g}, \alpha_U)$ constructed in Lemma 4.33.7 defines a pseudo functor from $\mathcal{C}^{opp}$ in to the $(2,1)$-category of groupoids.

Proof. Assume $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. To show all fibre categories $\mathcal{S}_U$ for $U \in \mathrm{Ob}(\mathcal{C})$ are groupoids, we must exhibit for every $f : y \to x$ in $\mathcal{S}_U$ an inverse morphism. The diagram on the left (in $\mathcal{S}_U$) is mapped by p to the diagram on the right:

$$\begin{tikzcd}
y \arrow[r, "f"] & x \\
x \arrow[u, dashed] \arrow[ur, "\mathrm{id}_x"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\mathrm{id}_U"] & U \\
U \arrow[u, dashed] \arrow[ur, "\mathrm{id}_U"'] &
\end{tikzcd}$$

Since only id_U makes the diagram on the right commute, there is a unique $g : x \to y$ making the diagram on the left commute, so $fg = \mathrm{id}_x$. By a similar argument there is a unique $h : y \to x$ so that $gh = \mathrm{id}_y$. Then $fgh = f : y \to x$. We have $fg = \mathrm{id}_x$, so $h = f$. Condition (2) of Definition 4.35.1 says exactly that every morphism of $\mathcal{S}$ is strongly cartesian. Hence condition (1) of Definition 4.35.1 implies that $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Conversely, assume all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$. We have to check conditions (1) and (2) of Definition 4.35.1. The first condition follows trivially. Let $\phi : y \to x$, $\psi : z \to x$ and $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ be as in condition (2) of Definition 4.35.1. Write $U = p(x)$, $V = p(y)$, $W = p(z)$, $p(\phi) = g : V \to U$, $p(\psi) = h : W \to U$. Choose a strongly cartesian $g^*x \to x$ lying over g. Then we get a morphism $i : y \to g^*x$ in $\mathcal{S}_V$, which is therefore an isomorphism. We also get a morphism $j : z \to g^*x$ corresponding to the pair (ψ, f) as $g^*x \to x$ is strongly cartesian. Then one checks that $\chi = i^{-1} \circ j$ is a solution.

We have seen in the proof of (1) $\Rightarrow$ (2) that every morphism of $\mathcal{S}$ is strongly cartesian. The final statement follows directly from Lemma 4.33.7. □

03WQ Lemma 4.35.3. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\mathcal{S}'$ be the subcategory of $\mathcal{S}$ defined as follows

(1) $\mathrm{Ob}(\mathcal{S}') = \mathrm{Ob}(\mathcal{S})$, and
(2) for $x, y \in \mathrm{Ob}(\mathcal{S}')$ the set of morphisms between x and y in $\mathcal{S}'$ is the set of strongly cartesian morphisms between x and y in $\mathcal{S}$.

Let $p' : \mathcal{S}' \to \mathcal{C}$ be the restriction of p to $\mathcal{S}'$. Then $p' : \mathcal{S}' \to \mathcal{C}$ is fibred in groupoids.

Proof. Note that the construction makes sense since by Lemma 4.33.2 the identity morphism of any object of $\mathcal{S}$ is strongly cartesian, and the composition of strongly cartesian morphisms is strongly cartesian. The first lifting property of Definition 4.35.1 follows from the condition that in a fibred category given any morphism $f : V \to U$ and x lying over U there exists a strongly cartesian morphism $\varphi : y \to x$ lying over f. Let us check the second lifting property of Definition 4.35.1 for the category $p' : \mathcal{S}' \to \mathcal{C}$ over $\mathcal{C}$. To do this we argue as in the discussion following Definition 4.35.1. Thus in Diagram 4.35.1.1 the morphisms ϕ, ψ and γ are strongly cartesian morphisms of $\mathcal{S}$. Hence γ and $\phi \circ \psi$ are strongly cartesian morphisms of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ and having the same target in $\mathcal{S}$. By the discussion following Definition 4.33.1 this means these two arrows are isomorphic as desired (here we use also that any isomorphism in $\mathcal{S}$ is strongly cartesian, by Lemma 4.33.2 again). □

003U Example 4.35.4. A homomorphism of groups $p : G \to H$ gives rise to a functor $p : \mathcal{S} \to \mathcal{C}$ as in Example 4.2.12. This functor $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids if and only if p is surjective. The fibre category $\mathcal{S}_U$ over the (unique) object $U \in \mathrm{Ob}(\mathcal{C})$ is the category associated to the kernel of p as in Example 4.2.6.

Given $p : \mathcal{S} \to \mathcal{C}$, we can ask: if the fibre category $\mathcal{S}_U$ is a groupoid for all $U \in \mathrm{Ob}(\mathcal{C})$, must $\mathcal{S}$ be fibred in groupoids over $\mathcal{C}$? We can see the answer is no as follows. Start with a category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$. Altering the morphisms in $\mathcal{S}$ which do not map to the identity morphism on some object does not alter the categories $\mathcal{S}_U$. Hence we can violate the existence and uniqueness conditions on lifts. One example is the functor from Example 4.35.4 when $G \to H$ is not surjective. Here is another example.

02C4 Example 4.35.5. Let $\mathrm{Ob}(\mathcal{C}) = \{A, B, T\}$ and $\mathrm{Mor}_\mathcal{C}(A, B) = \{f\}$, $\mathrm{Mor}_\mathcal{C}(B, T) = \{g\}$, $\mathrm{Mor}_\mathcal{C}(A, T) = \{h\} = \{gf\}$, plus the identity morphism for each object. See the diagram below for a picture of this category. Now let $\mathrm{Ob}(\mathcal{S}) = \{A', B', T'\}$ and $\mathrm{Mor}_\mathcal{S}(A', B') = \emptyset$, $\mathrm{Mor}_\mathcal{S}(B', T') = \{g'\}$, $\mathrm{Mor}_\mathcal{S}(A', T') = \{h'\}$, plus the identity morphisms. The functor $p : \mathcal{S} \to \mathcal{C}$ is obvious. Then for every $U \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{S}_U$ is the category with one object and the identity morphism on that object, so a groupoid, but the morphism $f : A \to B$ cannot be lifted. Similarly, if we declare $\mathrm{Mor}_\mathcal{S}(A', B') = \{f'_1, f'_2\}$ and $\mathrm{Mor}_\mathcal{S}(A', T') = \{h'\} = \{g'f'_1\} = \{g'f'_2\}$, then the fibre categories are the same and $f : A \to B$ in the diagram below has two lifts.

$$\begin{tikzcd} B' \arrow[r, "g'"] & T' & & B \arrow[r, "g"] & T \\ A' \arrow[u, dashed, "??"] \arrow[ur, "h'"'] & & \textit{above} & A \arrow[u, "f"] \arrow[ur, "gf=h"'] & \end{tikzcd}$$

Later we would like to make assertions such as "any category fibred in groupoids over $\mathcal{C}$ is equivalent to a split one", or "any category fibred in groupoids whose fibre categories are setlike is equivalent to a category fibred in sets". The notion of equivalence depends on the 2-category we are working with.

02XS Definition 4.35.6. Let $\mathcal{C}$ be a category. The 2-category of categories fibred in groupoids over $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Definition 4.33.9) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in groupoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \text{id}_{p(x)}$ for all $x \in \text{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism! Hence this is actually a $(2, 1)$-category and not just a 2-category. Here is the obligatory lemma on 2-fibre products.

0041 Lemma 4.35.7. Let $\mathcal{C}$ be a category. The 2-category of categories fibred in groupoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.32.3.

Proof. By Lemma 4.33.10 the fibre product as described in Lemma 4.32.3 is a fibred category. Hence it suffices to prove that the fibre categories are groupoids, see Lemma 4.35.2. By Lemma 4.32.5 it is enough to show that the 2-fibre product of groupoids is a groupoid, which is clear (from the construction in Lemma 4.31.4 for example). $\square$

0H2E Remark 4.35.8. Let $\mathcal{C}$ be a category. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be 1-morphisms of categories fibred in groupoids over $\mathcal{C}$. Let $p : \mathcal{S} \to \mathcal{C}$ be the given functor. We claim the 2-fibre product of Lemma 4.35.7 is canonically equivalent (as a category) to the one in Example 4.31.3. Objects of the former are quadruples (U, x, y, α) where $p(\alpha) = \text{id}_U$ (see Lemma 4.32.3) and objects of the latter are triples (x, y, α) (see Example 4.31.3). The equivalence between the two categories is given by the rules $(U, x, y, \alpha) \mapsto (x, y, \alpha)$ and $(x, y, \alpha) \mapsto (p(f(x)), x, y', \alpha')$ where $\alpha' = g(\gamma)^{-1} \circ \alpha$ and $\gamma : y' \to y$ is a lift of the arrow $p(\alpha) : p(f(x)) \to p(g(y))$. Details omitted.

003Z Lemma 4.35.9. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids, and suppose that $G : \mathcal{S} \to \mathcal{S}'$ is a functor over $\mathcal{C}$.

(1) Then G is faithful (resp. fully faithful, resp. an equivalence) if and only if for each $U \in \text{Ob}(\mathcal{C})$ the induced functor $G_U : \mathcal{S}_U \to \mathcal{S}'_U$ is faithful (resp. fully faithful, resp. an equivalence).
(2) If G is an equivalence, then G is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{C}$.

Proof. Let x, y be objects of $\mathcal{S}$ lying over the same object U. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Mor}_\mathcal{S}(x,y) \arrow[rr, "G"'] \arrow[dr, "p"'] & & \mathrm{Mor}_{\mathcal{S}'}(G(x),G(y)) \arrow[dl, "p'"] \\ & \mathrm{Mor}_\mathcal{C}(U,U) & \end{tikzcd}$$

From this diagram it is clear that if G is faithful (resp. fully faithful) then so is each G_U.

Suppose G is an equivalence. For every object x' of $\mathcal{S}'$ there exists an object x of $\mathcal{S}$ such that $G(x)$ is isomorphic to x'. Suppose that x' lies over U' and x lies over U. Then there is an isomorphism $f : U' \to U$ in $\mathcal{C}$, namely, p' applied to the isomorphism $x' \to G(x)$. By the axioms of a category fibred in groupoids there exists an arrow $f^*x \to x$ of $\mathcal{S}$ lying over f. Hence there exists an isomorphism $\alpha : x' \to G(f^*x)$ such that $p'(\alpha) = \mathrm{id}_{U'}$ (this time by the axioms for $\mathcal{S}'$). All in all we conclude that for every object x' of $\mathcal{S}'$ we can choose a pair $(o_{x'}, \alpha_{x'})$ consisting of an object $o_{x'}$ of $\mathcal{S}$ and an isomorphism $\alpha_{x'} : x' \to G(o_{x'})$ with $p'(\alpha_{x'}) = \mathrm{id}_{p'(x')}$. From this point on we proceed as usual (see proof of Lemma 4.2.19) to produce an inverse functor $F : \mathcal{S}' \to \mathcal{S}$, by taking $x' \mapsto o_{x'}$ and $\varphi' : x' \to y'$ to the unique arrow $\varphi_{\varphi'} : o_{x'} \to o_{y'}$ with $\alpha_{y'}^{-1} \circ G(\varphi_{\varphi'}) \circ \alpha_{x'} = \varphi'$. With these choices F is a functor over $\mathcal{C}$. We omit the verification that $G \circ F$ and $F \circ G$ are 2-isomorphic to the respective identity functors (in the 2-category of categories fibred in groupoids over $\mathcal{C}$).

Suppose that G_U is faithful (resp. fully faithful) for all $U \in \mathrm{Ob}(\mathcal{C})$. To show that G is faithful (resp. fully faithful) we have to show for any objects $x, y \in \mathrm{Ob}(\mathcal{S})$ that G induces an injection (resp. bijection) between $\mathrm{Mor}_\mathcal{S}(x, y)$ and $\mathrm{Mor}_{\mathcal{S}'}(G(x), G(y))$. Set $U = p(x)$ and $V = p(y)$. It suffices to prove that G induces an injection (resp. bijection) between morphism $x \to y$ lying over f to morphisms $G(x) \to G(y)$ lying over f for any morphism $f : U \to V$. Now fix $f : U \to V$. Denote $f^*y \to y$ a pullback. Then also $G(f^*y) \to G(y)$ is a pullback. The set of morphisms from x to y lying over f is bijective to the set of morphisms between x and f^*y lying over id_U. (By the second axiom of a category fibred in groupoids.) Similarly the set of morphisms from $G(x)$ to $G(y)$ lying over f is bijective to the set of morphisms between $G(x)$ and $G(f^*y)$ lying over id_U. Hence the fact that G_U is faithful (resp. fully faithful) gives the desired result.

Finally suppose for all G_U is an equivalence for all U, so it is fully faithful and essentially surjective. We have seen this implies G is fully faithful, and thus to prove it is an equivalence we have to prove that it is essentially surjective. This is clear, for if $z' \in \mathrm{Ob}(\mathcal{S}')$ then $z' \in \mathrm{Ob}(\mathcal{S}'_U)$ where $U = p'(z')$. Since G_U is essentially surjective we know that z' is isomorphic, in $\mathcal{S}'_U$, to an object of the form $G_U(z)$ for some $z \in \mathrm{Ob}(\mathcal{S}_U)$. But morphisms in $\mathcal{S}'_U$ are morphisms in $\mathcal{S}'$ and hence z' is isomorphic to $G(z)$ in $\mathcal{S}'$. □

04Z7 Lemma 4.35.10. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids. Let $G : \mathcal{S} \to \mathcal{S}'$ be a functor over $\mathcal{C}$. Then G is fully faithful if and only if the diagonal

$$\Delta_G : \mathcal{S} \longrightarrow \mathcal{S} \times_{G, \mathcal{S}', G} \mathcal{S}$$

is an equivalence.

Proof. By Lemma 4.35.9 it suffices to look at fibre categories over an object U of $\mathcal{C}$. An object of the right hand side is a triple (x, x', α) where $\alpha : G(x) \to G(x')$ is a morphism in $\mathcal{S}'_U$. The functor Δ_G maps the object x of $\mathcal{S}_U$ to the triple $(x, x, \mathrm{id}_{G(x)})$. Note that (x, x', α) is in the essential image of Δ_G if and only if $\alpha = G(\beta)$ for some morphism $\beta : x \to x'$ in $\mathcal{S}_U$ (details omitted). Hence in order for Δ_G to be an equivalence, every α has to be the image of a morphism $\beta : x \to x'$, and also every two distinct morphisms $\beta, \beta' : x \to x'$ have to give distinct morphisms $G(\beta), G(\beta')$. This proves the lemma. $\square$

03YT **Lemma 4.35.11.** Let $\mathcal{C}$ be a category. Let $\mathcal{S}_i$, $i = 1, 2, 3, 4$ be categories fibred in groupoids over $\mathcal{C}$. Suppose that $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ and $\psi : \mathcal{S}_3 \to \mathcal{S}_4$ are equivalences over $\mathcal{C}$. Then

$$\mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{S}_2, \mathcal{S}_3) \longrightarrow \mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_4), \quad \alpha \longmapsto \psi \circ \alpha \circ \varphi$$

is an equivalence of categories.

Proof. This is a generality and holds in any 2-category. $\square$

042I **Lemma 4.35.12.** Let $\mathcal{C}$ be a category. If $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then so is the inertia fibred category $\mathcal{I}_\mathcal{S} \to \mathcal{C}$.

Proof. Clear from the construction in Lemma 4.34.1 or by using (from the same lemma) that $I_\mathcal{S} \to \mathcal{S} \times_{\Delta, \mathcal{S} \times_\mathcal{C} \mathcal{S}, \Delta} \mathcal{S}$ is an equivalence and appealing to Lemma 4.35.7. $\square$

02XT **Lemma 4.35.13.** Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is fibred in groupoids.

Proof. We have already seen in Lemma 4.33.11 that p' is a fibred category. Hence it suffices to prove the fibre categories are groupoids, see Lemma 4.35.2. For $V \in \mathrm{Ob}(\mathcal{C})$ we have

$$\mathcal{S}_V = \coprod\nolimits_{f : V \to U} \mathcal{S}_{(f : V \to U)}$$

where the left hand side is the fibre category of p and the right hand side is the disjoint union of the fibre categories of p'. Hence the result. $\square$

09WW **Lemma 4.35.14.** Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be functors between categories. If $\mathcal{A}$ is fibred in groupoids over $\mathcal{B}$ and $\mathcal{B}$ is fibred in groupoids over $\mathcal{C}$, then $\mathcal{A}$ is fibred in groupoids over $\mathcal{C}$.

Proof. One can prove this directly from the definition. However, we will argue using the criterion of Lemma 4.35.2. By Lemma 4.33.12 we see that $\mathcal{A}$ is fibred over $\mathcal{C}$. To finish the proof we show that the fibre category $\mathcal{A}_U$ is a groupoid for U in $\mathcal{C}$. Namely, if $x \to y$ is a morphism of $\mathcal{A}_U$, then its image in $\mathcal{B}$ is an isomorphism as $\mathcal{B}_U$ is a groupoid. But then $x \to y$ is an isomorphism, for example by Lemma 4.33.2 and the fact that every morphism of $\mathcal{A}$ is strongly $\mathcal{B}$-cartesian (see Lemma 4.35.2). $\square$

06N6 **Lemma 4.35.15.** Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists and $p(x \times_y z) = p(x) \times_{p(y)} p(z)$.

Proof. Follows from Lemma 4.33.13. $\square$

06N7 Lemma 4.35.16. Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. There exists a factorization $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ by 1-morphisms of categories fibred in groupoids over $\mathcal{C}$ such that $\mathcal{X} \to \mathcal{X}'$ is an equivalence over $\mathcal{C}$ and such that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{Y}$.

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $f : F(x) \to y$ is an isomorphism in $\mathcal{Y}_U$. A morphism $(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b)$ and such that $f' \circ F(a) = b \circ f$. In other words $\mathcal{X}' = \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ with the construction of the 2-fibre product from Lemma 4.32.3. By Lemma 4.35.7 we see that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{C}$ and that $\mathcal{X}' \to \mathcal{Y}$ is a morphism of categories over $\mathcal{C}$. As functor $\mathcal{X} \to \mathcal{X}'$ we take $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ on objects and $(a : x \to x') \mapsto (a, F(a))$ on morphisms. It is clear that the composition $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ equals F. We omit the verification that $\mathcal{X} \to \mathcal{X}'$ is an equivalence of fibred categories over $\mathcal{C}$.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a category fibred in groupoids. Let $b : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Because $\mathcal{X}$ is fibred in groupoids over $\mathcal{C}$ we can find a morphism $a : x' \to x$ lying over $U' = q(y') \to q(y) = U$. Since $\mathcal{Y}$ is fibred in groupoids over $\mathcal{C}$ and since both $F(x') \to F(x)$ and $y' \to y$ lie over the same morphism $U' \to U$ we can find $f' : F(x') \to y'$ lying over $\mathrm{id}_{U'}$ such that $f \circ F(a) = b \circ f'$. Hence we obtain $(a, b) : (U', x', y', f') \to (U, x, y, f)$. This verifies the first condition (1) of Definition 4.35.1. To see (2) let $(a, b) : (U', x', y', f') \to (U, x, y, f)$ and $(a', b') : (U'', x'', y'', f'') \to (U, x, y, f)$ be morphisms of $\mathcal{X}'$ and let $b'' : y' \to y''$ be a morphism of $\mathcal{Y}$ such that $b' \circ b'' = b$. We have to show that there exists a unique morphism $a'' : x' \to x''$ such that $f'' \circ F(a'') = b'' \circ f'$ and such that $(a', b') \circ (a'', b'') = (a, b)$. Because $\mathcal{X}$ is fibred in groupoids we know there exists a unique morphism $a'' : x' \to x''$ such that $a' \circ a'' = a$ and $p(a'') = q(b'')$. Because $\mathcal{Y}$ is fibred in groupoids we see that $F(a'')$ is the unique morphism $F(x') \to F(x'')$ such that $F(a') \circ F(a'') = F(a)$ and $q(F(a'')) = q(b'')$. The relation $f'' \circ F(a'') = b'' \circ f'$ follows from this and the given relations $f \circ F(a) = b \circ f'$ and $f \circ F(a') = b' \circ f''$. $\square$

06N8 Lemma 4.35.17. Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. Assume we have a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[dr, "f"'] & \mathcal{X} \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "F"] & \mathcal{X}'' \arrow[dl, "g"] \\ & \mathcal{Y} & \end{tikzcd}$$

where a and b are equivalences of categories over $\mathcal{C}$ and f and g are categories fibred in groupoids. Then there exists an equivalence $h : \mathcal{X}'' \to \mathcal{X}'$ of categories over $\mathcal{Y}$ such that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{C}$. If the diagram above actually commutes, then we can arrange it so that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{Y}$.

Proof. We will show that both $\mathcal{X}'$ and $\mathcal{X}''$ over $\mathcal{Y}$ are equivalent to the category fibred in groupoids $\mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ over $\mathcal{Y}$, see proof of Lemma 4.35.16. Choose a quasi-inverse $b^{-1} : \mathcal{X}'' \to \mathcal{X}$ in the 2-category of categories over $\mathcal{C}$. Since the right triangle of the diagram is 2-commutative we see that

$$\begin{tikzcd} \mathcal{X} & \mathcal{X}'' \arrow[l, "b^{-1}"] \arrow[d, "g"] \\ \mathcal{Y} \arrow[from=u, "F"'] & \mathcal{Y} \arrow[l] \end{tikzcd}$$

is 2-commutative. Hence we obtain a 1-morphism $c : \mathcal{X}'' \to \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ by the universal property of the 2-fibre product. Moreover c is a morphism of categories over $\mathcal{Y}$ (!) and an equivalence (by the assumption that b is an equivalence, see Lemma 4.31.7). Hence c is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{Y}$ by Lemma 4.35.9.

We still have to construct a 2-isomorphism between $c \circ b$ and the functor $d : \mathcal{X} \to \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$, $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ constructed in the proof of Lemma 4.35.16. Let $\alpha : F \to g \circ b$ and $\beta : b^{-1} \circ b \to \mathrm{id}$ be 2-isomorphisms between 1-morphisms of categories over $\mathcal{C}$. Note that $c \circ b$ is given by the rule

$$x \mapsto (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

on objects. Then we see that

$$(\beta_x, \alpha_x) : (p(x), x, F(x), \mathrm{id}_{F(x)}) \longrightarrow (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

is a functorial isomorphism which gives our 2-morphism $d \to b \circ c$. Finally, if the diagram commutes then α_x is the identity for all x and we see that this 2-morphism is a 2-morphism in the 2-category of categories over $\mathcal{Y}$. $\square$

4.36. Presheaves of categories

02XU In this section we compare the notion of fibred categories with the closely related notion of a "presheaf of categories". The basic construction is explained in the following example.

02XV Example 4.36.1. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Cat}$ is a functor to the 2-category of categories, see Definition 4.29.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. From this we can construct a fibred category $\mathcal{S}_F$ over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned} \mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in \mathrm{Mor}_{\mathcal{C}}(V, U), \phi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathrm{Mor}_{F(V)}(y, f^*x) \end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. Let us check that this is indeed a fibred category. Given $f : V \to U$ in $\mathcal{C}$ and (U, x) a lift of U, then we claim

$(f, \mathrm{id}_{f^*x}) : (V, f^*x) \to (U, x)$ is a strongly cartesian lift of f. We have to show a h in the diagram on the left determines (h, ν) on the right:

$$\begin{tikzcd} V \arrow[r, "f"] & U \\ W \arrow[u, dashed, "h"] \arrow[ur, "g"'] \end{tikzcd} \qquad \begin{tikzcd} (V, f^*x) \arrow[r, "{(f,\mathrm{id}_{f^*x})}"] & (U, x) \\ (W, z) \arrow[u, dashed, "{(h,\nu)}"] \arrow[ur, "{(g,\psi)}"'] \end{tikzcd}$$

Just take $\nu = \psi$ which works because $f \circ h = g$ and hence $g^*x = h^*f^*x$. Moreover, this is the only lift making the diagram (on the right) commute.

02XW Definition 4.36.2. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Cat}$ is a functor to the 2-category of categories. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the fibred category constructed in Example 4.36.1. A *split fibred category* is a fibred category isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

02XX Lemma 4.36.3. Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a fibred category over $\mathcal{C}$. Then $\mathcal{S}$ is split if and only if for some choice of pullbacks (see Definition 4.33.6) the pullback functors $(f \circ g)^*$ and $g^* \circ f^*$ are equal.

Proof. This is immediate from the definitions. $\square$

004A Lemma 4.36.4. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. There exists a contravariant functor $F : \mathcal{C} \to \textit{Cat}$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ in the 2-category of fibred categories over $\mathcal{C}$. In other words, every fibred category is equivalent to a split one.

Proof. Let us make a choice of pullbacks (see Definition 4.33.6). By Lemma 4.33.7 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^*x_1 \to f_2^*x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_U).

At this point we need to check that p' makes $\mathcal{S}'$ into a fibred category over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$ which maps strongly cartesian morphisms to strongly cartesian morphisms. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^*f_i^*x_i = (f_i \circ g)^*x_i$ from Lemma 4.33.7 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^*x_1$ to $(f_2 \circ g)^*x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. $\square$

4.37. Presheaves of groupoids

0048 In this section we compare the notion of categories fibred in groupoids with the closely related notion of a "presheaf of groupoids". The basic construction is explained in the following example.

0049 Example 4.37.1. This example is the analogue of Example 4.36.1, for "presheaves of groupoids" instead of "presheaves of categories". The output will be a category fibred in groupoids instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Groupoids}$ is a functor to the category of groupoids, see Definition 4.29.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. We construct a category $\mathcal{S}_F$ fibred in groupoids over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned} \mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in \mathrm{Mor}_\mathcal{C}(V, U), \phi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathrm{Mor}_{F(V)}(y, f^*x) \end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. The condition that $F(U)$ is a groupoid for every U guarantees that $\mathcal{S}_F$ is fibred in groupoids over $\mathcal{C}$, as we have already seen in Example 4.36.1 that $\mathcal{S}_F$ is a fibred category, see Lemma 4.35.2. But we can also prove conditions (1), (2) of Definition 4.35.1 directly as follows: (1) Lifts of morphisms exist since given $f : V \to U$ in $\mathcal{C}$ and (U, x) an object of $\mathcal{S}_F$ over U, then $(f, \mathrm{id}_{f^*x}) : (V, f^*x) \to (U, x)$ is a lift of f. (2) Suppose given solid diagrams as follows

$$\begin{tikzcd} V \arrow[r, "f"] & U \\ W \arrow[u, dashed, "h"] \arrow[ur, "g"'] \end{tikzcd} \qquad \begin{tikzcd} (V, y) \arrow[r, "{(f,\phi)}"] & (U, x) \\ (W, z) \arrow[u, dashed, "{(h,\nu)}"] \arrow[ur, "{(g,\psi)}"'] \end{tikzcd}$$

Then for the dotted arrows we have $\nu = (h^*\phi)^{-1} \circ \psi$ so given h there exists a ν which is unique by uniqueness of inverses.

04TL Definition 4.37.2. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Groupoids}$ is a functor to the 2-category of groupoids. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the category fibred in groupoids constructed in Example 4.37.1. A *split category fibred in groupoids* is a category fibred in groupoids isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

02XY Lemma 4.37.3. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. There exists a contravariant functor $F : \mathcal{C} \to \textit{Groupoids}$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ over $\mathcal{C}$. In other words, every category fibred in groupoids is equivalent to a split one.

Proof. Make a choice of pullbacks (see Definition 4.33.6). By Lemmas 4.33.7 and 4.35.2 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e.,

$x \in \mathrm{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^*x_1 \to f_2^*x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_U).

At this point we need to check that p' makes $\mathcal{S}'$ into a category fibred in groupoids over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^*f_i^*x_i = (f_i \circ g)^*x_i$ from Lemma 4.35.2 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^*x_1$ to $(f_2 \circ g)^*x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. □

We will see an alternative proof of this lemma in Section 4.42.

4.38. Categories fibred in sets

0042

02Y0 Definition 4.38.1. A category is called discrete if the only morphisms are the identity morphisms.

A discrete category has only one interesting piece of information: its set of objects. Thus we sometime confuse discrete categories with sets.

0043 Definition 4.38.2. Let $\mathcal{C}$ be a category. A category fibred in sets, or a category fibred in discrete categories is a category fibred in groupoids all of whose fibre categories are discrete.

We want to clarify the relationship between categories fibred in sets and presheaves (see Definition 4.3.3). To do this it makes sense to first make the following definition.

04S8 Definition 4.38.3. Let $\mathcal{C}$ be a category. The 2-category of categories fibred in sets over $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 4.35.6) defined as follows:

1. Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in sets.
2. Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
3. Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category. Here is the obligatory lemma on the existence of 2-fibre products.

0047 Lemma 4.38.4. Let $\mathcal{C}$ be a category. The 2-category of categories fibred in sets over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 4.32.3 returns a category fibred in sets if one starts out with such.

Proof. Omitted. □

04TM **Example 4.38.5.** This example is the analogue of Examples 4.36.1 and 4.37.1 for presheaves instead of "presheaves of categories". The output will be a category fibred in sets instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Sets}$ is a presheaf. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^* : F(U) \to F(V)$. We construct a category $\mathcal{S}_F$ fibred in sets over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) = \{f \in \mathrm{Mor}_\mathcal{C}(V, U) \mid f^*x = y\}$$

Composition is inherited from composition in $\mathcal{C}$ which works as $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. As every fibre category $\mathcal{S}_{F,U}$ is discrete with underlying set $F(U)$ and we have already see in Example 4.37.1 that $\mathcal{S}_F$ is a category fibred in groupoids, we conclude that $\mathcal{S}_F$ is fibred in sets.

02Y2 **Lemma 4.38.6.** *Let $\mathcal{C}$ be a category. The only 2-morphisms between categories fibred in sets are identities. In other words, the 2-category of categories fibred in sets is a category. Moreover, there is an equivalence of categories*

$$\left\{\begin{matrix}\text{the category of presheaves}\\ \text{of sets over } \mathcal{C}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{the category of categories}\\ \text{fibred in sets over } \mathcal{C}\end{matrix}\right\}$$

The functor from left to right is the construction $F \to \mathcal{S}_F$ discussed in Example 4.38.5. The functor from right to left assigns to $p : \mathcal{S} \to \mathcal{C}$ the presheaf of objects $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$.

Proof. The first assertion is clear, as the only morphisms in the fibre categories are identities.

Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in sets. Let $f : V \to U$ be a morphism in $\mathcal{C}$ and let $x \in \mathrm{Ob}(\mathcal{S}_U)$. Then there is exactly one choice for the object f^*x. Thus we see that $(f \circ g)^*x = g^*(f^*x)$ for f, g as in Lemma 4.35.2. It follows that we may think of the assignments $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$ and $f \mapsto f^*$ as a presheaf on $\mathcal{C}$. □

Here is an important example of a category fibred in sets.

0044 **Example 4.38.7.** Let $\mathcal{C}$ be a category. Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider the representable presheaf $h_X = \mathrm{Mor}_\mathcal{C}(-, X)$ (see Example 4.3.4). On the other hand, consider the category $p : \mathcal{C}/X \to \mathcal{C}$ from Example 4.2.13. The fibre category $(\mathcal{C}/X)_U$ has as objects morphisms $h : U \to X$, and only identities as morphisms. Hence we see that under the correspondence of Lemma 4.38.6 we have

$$h_X \longleftrightarrow \mathcal{C}/X.$$

In other words, the category $\mathcal{C}/X$ is canonically equivalent to the category $\mathcal{S}_{h_X}$ associated to h_X in Example 4.38.5.

For this reason it is tempting to define a "representable" object in the 2-category of categories fibred in groupoids to be a category fibred in sets whose associated presheaf is representable. However, this is would not be a good definition for use since we prefer to have a notion which is invariant under equivalences. To make this precise we study exactly which categories fibred in groupoids are equivalent to categories fibred in sets.

4.39. Categories fibred in setoids

04S9

02XZ Definition 4.39.1. Let us call a category a setoid[8] if it is a groupoid where every object has exactly one automorphism: the identity.

If C is a set with an equivalence relation $\sim$, then we can make a setoid $\mathcal{C}$ as follows: $\mathrm{Ob}(\mathcal{C}) = C$ and $\mathrm{Mor}_\mathcal{C}(x, y) = \emptyset$ unless $x \sim y$ in which case we set $\mathrm{Mor}_\mathcal{C}(x, y) = \{1\}$. Transitivity of $\sim$ means that we can compose morphisms. Conversely any setoid category defines an equivalence relation on its objects (isomorphism) such that you recover the category (up to unique isomorphism – not equivalence) from the procedure just described.

Discrete categories are setoids. For any setoid $\mathcal{C}$ there is a canonical procedure to make a discrete category equivalent to it, namely one replaces $\mathrm{Ob}(\mathcal{C})$ by the set of isomorphism classes (and adds identity morphisms). In terms of sets endowed with an equivalence relation this corresponds to taking the quotient by the equivalence relation.

04SA Definition 4.39.2. Let $\mathcal{C}$ be a category. A category fibred in setoids is a category fibred in groupoids all of whose fibre categories are setoids.

Below we will clarify the relationship between categories fibred in setoids and categories fibred in sets.

02Y1 Definition 4.39.3. Let $\mathcal{C}$ be a category. The 2-category of categories fibred in setoids over $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 4.35.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in setoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category.
Here is the obligatory lemma on the existence of 2-fibre products.

04SB Lemma 4.39.4. Let $\mathcal{C}$ be a category. The 2-category of categories fibred in setoids over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 4.32.3 returns a category fibred in setoids if one starts out with such.

Proof. Omitted. □

0045 Lemma 4.39.5. Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a category over $\mathcal{C}$.

(1) If $\mathcal{S} \to \mathcal{S}'$ is an equivalence over $\mathcal{C}$ with $\mathcal{S}'$ fibred in sets over $\mathcal{C}$, then
   (a) $\mathcal{S}$ is fibred in setoids over $\mathcal{C}$, and
   (b) for each $U \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Ob}(\mathcal{S}_U) \to \mathrm{Ob}(\mathcal{S}'_U)$ identifies the target as the set of isomorphism classes of the source.
(2) If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in setoids, then there exists a category fibred in sets $p' : \mathcal{S}' \to \mathcal{C}$ and an equivalence $\mathrm{can} : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$.

[8] A set on steroids!?

Proof. Let us prove (2). An object of the category $\mathcal{S}'$ will be a pair (U, ξ), where $U \in \mathrm{Ob}(\mathcal{C})$ and ξ is an isomorphism class of objects of $\mathcal{S}_U$. A morphism $(U, \xi) \to (V, \psi)$ is given by a morphism $x \to y$, where $x \in \xi$ and $y \in \psi$. Here we identify two morphisms $x \to y$ and $x' \to y'$ if they induce the same morphism $U \to V$, and if for some choices of isomorphisms $x \to x'$ in $\mathcal{S}_U$ and $y \to y'$ in $\mathcal{S}_V$ the compositions $x \to x' \to y'$ and $x \to y \to y'$ agree. By construction there are surjective maps on objects and morphisms from $\mathcal{S} \to \mathcal{S}'$. We define composition of morphisms in $\mathcal{S}'$ to be the unique law that turns $\mathcal{S} \to \mathcal{S}'$ into a functor. Some details omitted. $\square$

Thus categories fibred in setoids are exactly the categories fibred in groupoids which are equivalent to categories fibred in sets. Moreover, an equivalence of categories fibred in sets is an isomorphism by Lemma 4.38.6.

04SC Lemma 4.39.6. Let $\mathcal{C}$ be a category. The construction of Lemma 4.39.5 part (2) gives a functor

$$F : \left\{ \begin{matrix} \text{the 2-category of categories} \\ \text{fibred in setoids over } \mathcal{C} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{the category of categories} \\ \text{fibred in sets over } \mathcal{C} \end{matrix} \right\}$$

(see Definition 4.29.5). This functor is an equivalence in the following sense:

(1) for any two 1-morphisms $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ there exists a unique 2-isomorphism $f \to g$,
(2) for any morphism $h : F(\mathcal{S}_1) \to F(\mathcal{S}_2)$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = h$, and
(3) any category fibred in sets $\mathcal{S}$ is equal to $F(\mathcal{S})$.

In particular, defining $F_i \in \mathrm{PSh}(\mathcal{C})$ by the rule $F_i(U) = \mathrm{Ob}(\mathcal{S}_{i,U})/\cong$, we have

$$\mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_2)\Big/\text{2-isomorphism} = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(F_1, F_2)$$

More precisely, given any map $\phi : F_1 \to F_2$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Lemma 4.38.6 the target of F is a category hence the assertion makes sense. The construction of Lemma 4.39.5 part (2) assigns to $\mathcal{S}$ the category fibred in sets whose value over U is the set of isomorphism classes in $\mathcal{S}_U$. Hence it is clear that it defines a functor as indicated. Let $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ be as in (1). For each object U of $\mathcal{C}$ and each object x of $\mathcal{S}_{1,U}$ we see that $f(x) \cong g(x)$ by assumption. As $\mathcal{S}_2$ is fibred in setoids there exists a unique isomorphism $t_x : f(x) \to g(x)$ in $\mathcal{S}_{2,U}$. Clearly the rule $x \mapsto t_x$ gives the desired 2-isomorphism $f \to g$. We omit the proofs of (2) and (3). To see the final assertion use Lemma 4.38.6 to see that the right hand side is equal to $\mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(F(\mathcal{S}_1), F(\mathcal{S}_2))$ and apply (1) and (2) above. $\square$

Here is another characterization of categories fibred in setoids among all categories fibred in groupoids.

042J Lemma 4.39.7. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. The following are equivalent:

(1) $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in setoids, and
(2) the canonical 1-morphism $\mathcal{I}_\mathcal{S} \to \mathcal{S}$, see (4.34.2.1), is an equivalence (of categories over $\mathcal{C}$).

Proof. Assume (2). The category $\mathcal{I}_\mathcal{S}$ has objects (x, α) where $x \in \mathcal{S}$, say with $p(x) = U$, and $\alpha : x \to x$ is a morphism in $\mathcal{S}_U$. Hence if $\mathcal{I}_\mathcal{S} \to \mathcal{S}$ is an equivalence over $\mathcal{C}$ then every pair of objects (x, α), (x, α') are isomorphic in the fibre category of $\mathcal{I}_\mathcal{S}$ over U. Looking at the definition of morphisms in $\mathcal{I}_\mathcal{S}$ we conclude that α, α' are conjugate in the group of automorphisms of x. Hence taking $\alpha' = \mathrm{id}_x$ we conclude that every automorphism of x is equal to the identity. Since $\mathcal{S} \to \mathcal{C}$ is fibred in groupoids this implies that $\mathcal{S} \to \mathcal{C}$ is fibred in setoids. We omit the proof of (1) $\Rightarrow$ (2). □

04SD Lemma 4.39.8. Let $\mathcal{C}$ be a category. The construction of Lemma 4.39.6 which associates to a category fibred in setoids a presheaf is compatible with products, in the sense that the presheaf associated to a 2-fibre product $\mathcal{X} \times_\mathcal{Y} \mathcal{Z}$ is the fibre product of the presheaves associated to $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$.

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. The lemma just says that

$$\mathrm{Ob}((\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_U)/\cong \quad \text{equals} \quad \mathrm{Ob}(\mathcal{X}_U)/\cong \ \times_{\mathrm{Ob}(\mathcal{Y}_U)/\cong} \mathrm{Ob}(\mathcal{Z}_U)/\cong$$

the proof of which we omit. (But note that this would not be true in general if the category $\mathcal{Y}_U$ is not a setoid.) □

4.40. Representable categories fibred in groupoids

04SE Here is our definition of a representable category fibred in groupoids. As promised this is invariant under equivalences.

0046 Definition 4.40.1. Let $\mathcal{C}$ be a category. A category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$ is called representable if there exist an object X of $\mathcal{C}$ and an equivalence $j : \mathcal{S} \to \mathcal{C}/X$ (in the 2-category of groupoids over $\mathcal{C}$).

The usual abuse of notation is to say that X represents $\mathcal{S}$ and not mention the equivalence j. We spell out what this entails.

02Y3 Lemma 4.40.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids.

1. $\mathcal{S}$ is representable if and only if the following conditions are satisfied:
   (a) $\mathcal{S}$ is fibred in setoids, and
   (b) the presheaf $U \mapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$ is representable.
2. If $\mathcal{S}$ is representable the pair (X, j), where j is the equivalence $j : \mathcal{S} \to \mathcal{C}/X$, is uniquely determined up to isomorphism.

Proof. The first assertion follows immediately from Lemma 4.39.5. For the second, suppose that $j' : \mathcal{S} \to \mathcal{C}/X'$ is a second such pair. Choose a 1-morphism $t' : \mathcal{C}/X' \to \mathcal{S}$ such that $j' \circ t' \cong \mathrm{id}_{\mathcal{C}/X'}$ and $t' \circ j' \cong \mathrm{id}_\mathcal{S}$. Then $j \circ t' : \mathcal{C}/X' \to \mathcal{C}/X$ is an equivalence. Hence it is an isomorphism, see Lemma 4.38.6. Hence by the Yoneda Lemma 4.3.5 (via Example 4.38.7 for example) it is given by an isomorphism $X' \to X$. □

04SF Lemma 4.40.3. Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Assume that $\mathcal{X}$, $\mathcal{Y}$ are representable by objects X, Y of $\mathcal{C}$. Then

$$\mathrm{Mor}_{\mathit{Cat}/\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \Big/ \text{2-isomorphism} = \mathrm{Mor}_\mathcal{C}(X, Y)$$

More precisely, given $\phi : X \to Y$ there exists a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Example 4.38.7 we have $\mathcal{C}/X = \mathcal{S}_{h_X}$ and $\mathcal{C}/Y = \mathcal{S}_{h_Y}$. By Lemma 4.39.6 we have

$$\mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{X}, \mathcal{Y})\Big/\text{2-isomorphism} = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_X, h_Y)$$

By the Yoneda Lemma 4.3.5 we have $\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_X, h_Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$. $\square$

4.41. The 2-Yoneda lemma

0GWH Let $\mathcal{C}$ be a category. The 2-category of fibred categories over $\mathcal{C}$ was constructed/defined in Definition 4.33.9. If $\mathcal{S}$, $\mathcal{S}'$ are fibred categories over $\mathcal{C}$ then

$$\mathrm{Mor}_{\mathrm{Fib}/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

denotes the category of 1-morphisms in this 2-category. Here is the 2-category analogue of the Yoneda lemma in the setting of fibred categories.

0GWI **Lemma 4.41.1** (2-Yoneda lemma for fibred categories). *Let $\mathcal{C}$ be a category. Let $\mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor*

$$\mathrm{Mor}_{\mathrm{Fib}/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \longrightarrow \mathcal{S}_U$$

given by $G \mapsto G(\mathrm{id}_U)$ is an equivalence.

Proof. Make a choice of pullbacks for $\mathcal{S}$ (see Definition 4.33.6). We define a functor

$$\mathcal{S}_U \longrightarrow \mathrm{Mor}_{\mathrm{Fib}/\mathcal{C}}(\mathcal{C}/U, \mathcal{S})$$

as follows. Given $x \in \mathrm{Ob}(\mathcal{S}_U)$ the associated functor is

(1) on objects: $(f : V \to U) \mapsto f^*x$, and
(2) on morphisms: the arrow $(g : V'/U \to V/U)$ maps to the composition

$$(f \circ g)^*x \xrightarrow{(\alpha_{g,f})_x} g^*f^*x \to f^*x$$

where $\alpha_{g,f}$ is as in Lemma 4.33.7.

We omit the verification that this is an inverse to the functor of the lemma. $\square$

Let $\mathcal{C}$ be a category. The 2-category of categories fibred in groupoids over $\mathcal{C}$ is a "full" sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.35.6). Hence if $\mathcal{S}$, $\mathcal{S}'$ are fibred in groupoids over $\mathcal{C}$ then

$$\mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

denotes the category of 1-morphisms in this 2-category (see Definition 4.32.1). These are all groupoids, see remarks following Definition 4.35.6. Here is the 2-category analogue of the Yoneda lemma.

004B **Lemma 4.41.2** (2-Yoneda lemma). *Let $\mathcal{S} \to \mathcal{C}$ be fibred in groupoids. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor*

$$\mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \longrightarrow \mathcal{S}_U$$

given by $G \mapsto G(\mathrm{id}_U)$ is an equivalence.

Proof. Make a choice of pullbacks for $\mathcal{S}$ (see Definition 4.33.6). We define a functor

$$\mathcal{S}_U \longrightarrow \mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{C}/U, \mathcal{S})$$

as follows. Given $x \in \mathrm{Ob}(\mathcal{S}_U)$ the associated functor is

(1) on objects: $(f : V \to U) \mapsto f^*x$, and

(2) on morphisms: the arrow $(g : V'/U \to V/U)$ maps to the composition

$$(f \circ g)^*x \xrightarrow{(\alpha_{g,f})_x} g^*f^*x \to f^*x$$

where $\alpha_{g,f}$ is as in Lemma 4.35.2.

We omit the verification that this is an inverse to the functor of the lemma. $\square$

076J Remark 4.41.3. We can use the 2-Yoneda lemma to give an alternative proof of Lemma 4.37.3. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. We define a contravariant functor F from $\mathcal{C}$ to the category of groupoids as follows: for $U \in \mathrm{Ob}(\mathcal{C})$ let

$$F(U) = \mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}).$$

If $f : U \to V$ the induced functor $\mathcal{C}/U \to \mathcal{C}/V$ induces the morphism $F(f) : F(V) \to F(U)$. Clearly F is a functor. Let $\mathcal{S}'$ be the associated category fibred in groupoids from Example 4.37.1. There is an obvious functor $G : \mathcal{S}' \to \mathcal{S}$ over $\mathcal{C}$ given by taking the pair (U, x), where $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(U)$, to $x(\mathrm{id}_U) \in \mathcal{S}$. Now Lemma 4.41.2 implies that for each U,

$$G_U : \mathcal{S}'_U = F(U) = \mathrm{Mor}_{\mathrm{Cat}/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \to \mathcal{S}_U$$

is an equivalence, and thus G is an equivalence between $\mathcal{S}$ and $\mathcal{S}'$ by Lemma 4.35.9.

4.42. Representable 1-morphisms

02Y4 Let $\mathcal{C}$ be a category. In this section we explain what it means for a 1-morphism between categories fibred in groupoids over $\mathcal{C}$ to be representable.

Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{C}/U \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $\mathcal{C}$. We want to describe the 2-fibre product

$$\begin{tikzcd}
(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "F"] \\
\mathcal{C}/U \arrow[r, "G"] & \mathcal{Y}
\end{tikzcd}$$

Let $y = G(\mathrm{id}_U) \in \mathcal{Y}_U$. Make a choice of pullbacks for $\mathcal{Y}$ (see Definition 4.33.6). Then G is isomorphic to the functor $(f : V \to U) \mapsto f^*y$, see Lemma 4.41.2 and its proof. We may think of an object of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ as a quadruple $(V, f : V \to U, x, \phi)$, see Lemma 4.32.3. Using the description of G above we may think of ϕ as an isomorphism $\phi : f^*y \to F(x)$ in $\mathcal{Y}_V$.

02Y5 Lemma 4.42.1. In the situation above the fibre category of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ over an object $f : V \to U$ of $\mathcal{C}/U$ is the category described as follows:

(1) objects are pairs (x, ϕ), where $x \in \mathrm{Ob}(\mathcal{X}_V)$, and $\phi : f^*y \to F(x)$ is a morphism in $\mathcal{Y}_V$,
(2) the set of morphisms between (x, ϕ) and (x', ϕ') is the set of morphisms $\psi : x \to x'$ in $\mathcal{X}_V$ such that $F(\psi) = \phi' \circ \phi^{-1}$.

Proof. See discussion above. $\square$

02Y6 Lemma 4.42.2. Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Let $G : \mathcal{C}/U \to \mathcal{Y}$ be a 1-morphism. Then

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is a category fibred in groupoids.

Proof. We have already seen in Lemma 4.35.7 that the composition

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U \longrightarrow \mathcal{C}$$

is a category fibred in groupoids. Then the lemma follows from Lemma 4.35.13. □

02Y7 Definition 4.42.3. Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. We say F is representable, or that $\mathcal{X}$ is relatively representable over $\mathcal{Y}$, if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any $G : \mathcal{C}/U \to \mathcal{Y}$ the category fibred in groupoids

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is representable.

02Y8 Lemma 4.42.4. Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. If F is representable then every one of the functors

$$F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$$

between fibre categories is faithful.

Proof. Clear from the description of fibre categories in Lemma 4.42.1 and the characterization of representable fibred categories in Lemma 4.40.2. □

02Y9 Lemma 4.42.5. Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Make a choice of pullbacks for $\mathcal{Y}$. Assume

(1) each functor $F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and
(2) for each U and each $y \in \mathcal{Y}_U$ the presheaf

$$(f : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathcal{X}_V, \phi : f^*y \to F(x)\}/\cong$$

is a representable presheaf on $\mathcal{C}/U$.

Then F is representable.

Proof. Clear from the description of fibre categories in Lemma 4.42.1 and the characterization of representable fibred categories in Lemma 4.40.2. □

Before we state the next lemma we point out that the 2-category of categories fibred in groupoids is a $(2,1)$-category, and hence we know what it means to say that it has a final object (see Definition 4.31.1). And it has a final object namely $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. Thus we define 2-products of categories fibred in groupoids over $\mathcal{C}$ as the 2-fibre products

$$\mathcal{X} \times \mathcal{Y} := \mathcal{X} \times_{\mathcal{C}} \mathcal{Y}.$$

With this definition in place the following lemma makes sense.

02YA Lemma 4.42.6. Let $\mathcal{C}$ be a category. Let $\mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:

(1) The diagonal $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable.

(2) For every U in $\mathcal{C}$, any $G : \mathcal{C}/U \to \mathcal{S}$ is representable.

Proof. Suppose the diagonal is representable, and let U, G be given. Consider any $V \in \text{Ob}(\mathcal{C})$ and any $G' : \mathcal{C}/V \to \mathcal{S}$. Note that $\mathcal{C}/U \times \mathcal{C}/V = \mathcal{C}/U \times V$ is representable. Hence the fibre product

$$\begin{tikzcd} (\mathcal{C}/U \times V) \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \mathcal{C}/U \times V \arrow[r, "{(G,G')}"] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

is representable by assumption. This means there exists $W \to U \times V$ in $\mathcal{C}$, such that

$$\begin{tikzcd} \mathcal{C}/W \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \mathcal{C}/U \times \mathcal{C}/V \arrow[r] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

is cartesian. This implies that $\mathcal{C}/W \cong \mathcal{C}/U \times_{\mathcal{S}} \mathcal{C}/V$ (see Lemma 4.31.11 and Remark 4.35.8) as desired.

Assume (2) holds. Consider any $V \in \text{Ob}(\mathcal{C})$ and any $(G, G') : \mathcal{C}/V \to \mathcal{S} \times \mathcal{S}$. We have to show that $\mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$ is representable. What we know is that $\mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$ is representable, say by $a : W \to V$ in $\mathcal{C}/V$. The equivalence

$$\mathcal{C}/W \to \mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$$

followed by the second projection to $\mathcal{C}/V$ gives a second morphism $a' : W \to V$. Consider $W' = W \times_{(a,a'),V\times V} V$. There exists an equivalence

$$\mathcal{C}/W' \cong \mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$$

namely

$$\begin{aligned} \mathcal{C}/W' &\cong \mathcal{C}/W \times_{(\mathcal{C}/V\times\mathcal{C}/V)} \mathcal{C}/V \\ &\cong \big(\mathcal{C}/V \times_{(G,\mathcal{S},G')} \mathcal{C}/V\big) \times_{(\mathcal{C}/V\times\mathcal{C}/V)} \mathcal{C}/V \\ &\cong \mathcal{C}/V \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} \end{aligned}$$

(for the last isomorphism see Lemma 4.31.12 and Remark 4.35.8) which proves the lemma. □

Bibliographic notes: Parts of this have been taken from Vistoli's notes [Vis04].

4.43. Monoidal categories

0FFJ Let $\mathcal{C}$ be a category. Suppose we are given a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

We often want to know whether $\otimes$ satisfies an associative rule and whether there is a unit for $\otimes$.

An associativity constraint for $(\mathcal{C}, \otimes)$ is a functorial isomorphism

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

such that for all objects X, Y, Z, W the diagram

$$\begin{tikzcd} X \otimes (Y \otimes (Z \otimes W)) \arrow[r] \arrow[d] & (X \otimes Y) \otimes (Z \otimes W) \arrow[r] & ((X \otimes Y) \otimes Z) \otimes W \\ X \otimes ((Y \otimes Z) \otimes W) \arrow[rr] & & (X \otimes (Y \otimes Z)) \otimes W \arrow[u] \end{tikzcd}$$

is commutative where every arrow is determined by a suitable application of ϕ and functoriality of $\otimes$. Given an associativity constraint there are well defined functors

$$\mathcal{C} \times \ldots \times \mathcal{C} \longrightarrow \mathcal{C}, \quad (X_1, \ldots, X_n) \longmapsto X_1 \otimes \ldots \otimes X_n$$

for all $n \geq 1$.

Let ϕ be an associativity constraint. A unit for $(\mathcal{C}, \otimes, \phi)$ is an object $\mathbf{1}$ of $\mathcal{C}$ together with functorial isomorphisms

$$\mathbf{1} \otimes X \to X \quad \text{and} \quad X \otimes \mathbf{1} \to X$$

such that for all objects X, Y the diagram

$$\begin{tikzcd} X \otimes (\mathbf{1} \otimes Y) \arrow[rr, "\phi"'] \arrow[dr] & & (X \otimes \mathbf{1}) \otimes Y \arrow[dl] \\ & X \otimes Y & \end{tikzcd}$$

is commutative where the diagonal arrows are given by the isomorphisms introduced above.

An equivalent definition would be that a unit is a pair $(\mathbf{1}, 1)$ where $\mathbf{1}$ is an object of $\mathcal{C}$ and $1 : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ is an isomorphism such that the functors $L : X \mapsto \mathbf{1} \otimes X$ and $R : X \mapsto X \otimes \mathbf{1}$ are equivalences. Certainly, given a unit as above we get the isomorphism $1 : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ for free and L and R are equivalences as they are isomorphic to the identity functor. Conversely, given $(\mathbf{1}, 1)$ such that L and R are equivalences, we obtain functorial isomorphisms $l : \mathbf{1} \otimes X \to X$ and $r : X \otimes \mathbf{1} \to X$ characterized by $L(l) = 1 \otimes \text{id}_X$ and $R(r) = \text{id}_X \otimes 1$. Then we can use r and l in the notion of unit as above.

A unit is unique up to unique isomorphism if it exists (exercise).

0FFK Definition 4.43.1. A triple $(\mathcal{C}, \otimes, \phi)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, and ϕ is an associativity constraint is called a monoidal category if there exists a unit $\mathbf{1}$.

We always write $\mathbf{1}$ to denote a unit of a monoidal category; as it is determined up to unique isomorphism there is no harm in choosing one. From now on we no longer write the brackets when taking tensor products in monoidal categories and we always identify $X \otimes \mathbf{1}$ and $\mathbf{1} \otimes X$ with X. Moreover, we will say "let $\mathcal{C}$ be a monoidal category" with $\otimes, \phi, \mathbf{1}$ understood.

0FFL Definition 4.43.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be monoidal categories. A functor of monoidal categories $F : \mathcal{C} \to \mathcal{C}'$ is given by a functor F as indicated and a isomorphism

$$F(X) \otimes F(Y) \to F(X \otimes Y)$$

functorial in X and Y such that for all objects X, Y, and Z the diagram

$$\begin{tikzcd} F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r] \arrow[d] & F(X) \otimes F(Y \otimes Z) \arrow[r] & F(X \otimes (Y \otimes Z)) \arrow[d] \\ (F(X) \otimes F(Y)) \otimes F(Z) \arrow[r] & F(X \otimes Y) \otimes F(Z) \arrow[r] & F((X \otimes Y) \otimes Z) \end{tikzcd}$$

commutes and such that $F(\mathbf{1})$ is a unit in $\mathcal{C}'$.

By our conventions about units, we may always assume $F(\mathbf{1}) = \mathbf{1}$ if F is a functor of monoidal categories. As an example, if $A \to B$ is a ring homomorphism, then the functor $M \mapsto M \otimes_A B$ is functor of monoidal categories from Mod_A to Mod_B.

0FFM Lemma 4.43.3. Let $\mathcal{C}$ be a monoidal category. Let X be an object of $\mathcal{C}$. The following are equivalent

(1) the functor $L : Y \mapsto X \otimes Y$ is an equivalence,
(2) the functor $R : Y \mapsto Y \otimes X$ is an equivalence,
(3) there exists an object X' such that $X \otimes X' \cong X' \otimes X \cong \mathbf{1}$.

Proof. Assume (1). Choose X' such that $L(X') = \mathbf{1}$, i.e., $X \otimes X' \cong \mathbf{1}$. Denote L' and R' the functors corresponding to X'. The equation $X \otimes X' \cong \mathbf{1}$ implies $L \circ L' \cong \text{id}$. Thus L' must be the quasi-inverse to L (which exists by assumption). Hence $L' \circ L \cong \text{id}$. Hence $X' \otimes X \cong \mathbf{1}$. Thus (3) holds.

The proof of (2) $\Rightarrow$ (3) is dual to what we just said.

Assume (3). Then it is clear that L' and L are quasi-inverse to each other and it is clear that R' and R are quasi-inverse to each other. Thus (1) and (2) hold. $\square$

0FFN Definition 4.43.4. Let $\mathcal{C}$ be a monoidal category. An object X of $\mathcal{C}$ is called invertible if any (or all) of the equivalent conditions of Lemma 4.43.3 hold.

Observe that if $F : \mathcal{C} \to \mathcal{C}'$ is a functor of monoidal categories, then F sends invertible objects to invertible objects.

0FFP Definition 4.43.5. Given a monoidal category $(\mathcal{C}, \otimes, \phi)$ and an object X a left dual is an object Y together with morphisms $\eta : \mathbf{1} \to X \otimes Y$ and $\epsilon : Y \otimes X \to \mathbf{1}$ such that the diagrams

$$\begin{tikzcd} X \arrow[r, "\eta\otimes 1"'] \arrow[dr, "1"'] & X \otimes Y \otimes X \arrow[d, "1\otimes\epsilon"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r, "1\otimes\eta"'] \arrow[dr, "1"'] & Y \otimes X \otimes Y \arrow[d, "\epsilon\otimes 1"] \\ & Y \end{tikzcd}$$

commute. In this situation we say that X is a right dual of Y.

Observe that if $F : \mathcal{C} \to \mathcal{C}'$ is a functor of monoidal categories, then $F(Y)$ is a left dual of $F(X)$ if Y is a left dual of X.

0FFQ Lemma 4.43.6. Let $\mathcal{C}$ be a monoidal category. If Y is a left dual to X, then

$$\text{Mor}(Z' \otimes X, Z) = \text{Mor}(Z', Z \otimes Y) \quad \text{and} \quad \text{Mor}(Y \otimes Z', Z) = \text{Mor}(Z', X \otimes Z)$$

functorially in Z and Z'.

Proof. Consider the maps

$$\mathrm{Mor}(Z' \otimes X, Z) \to \mathrm{Mor}(Z' \otimes X \otimes Y, Z \otimes Y) \to \mathrm{Mor}(Z', Z \otimes Y)$$

where we use η in the second arrow and the sequence of maps

$$\mathrm{Mor}(Z', Z \otimes Y) \to \mathrm{Mor}(Z' \otimes X, Z \otimes Y \otimes X) \to \mathrm{Mor}(Z' \otimes X, Z)$$

where we use ϵ in the second arrow. A straightforward calculation using the properties of η and ϵ shows that the compositions of these are mutually inverse. Similarly for the other equality. $\square$

0FFR Remark 4.43.7. Lemma 4.43.6 says in particular that $Z \mapsto Z \otimes Y$ is the right adjoint of $Z' \mapsto Z' \otimes X$. In particular, uniqueness of adjoint functors guarantees that a left dual of X, if it exists, is unique up to unique isomorphism. Conversely, assume the functor $Z \mapsto Z \otimes Y$ is a right adjoint of the functor $Z' \mapsto Z' \otimes X$, i.e., we're given a bijection

$$\mathrm{Mor}(Z' \otimes X, Z) \longrightarrow \mathrm{Mor}(Z', Z \otimes Y)$$

functorial in both Z and Z'. The unit of the adjunction produces maps

$$\eta_Z : Z \to Z \otimes X \otimes Y$$

functorial in Z and the counit of the adjoint produces maps

$$\epsilon_{Z'} : Z' \otimes Y \otimes X \to Z'$$

functorial in Z'. In particular, we find $\eta = \eta_{\mathbf{1}} : \mathbf{1} \to X \otimes Y$ and $\epsilon = \epsilon_{\mathbf{1}} : Y \otimes X \to \mathbf{1}$. As an exercise in the relationship between units, counits, and the adjunction isomorphism, the reader can show that we have

$$(\epsilon \otimes \mathrm{id}_Y) \circ \eta_Y = \mathrm{id}_Y \quad \text{and} \quad \epsilon_X \circ (\eta \otimes \mathrm{id}_X) = \mathrm{id}_X$$

However, this isn't enough to show that $(\epsilon \otimes \mathrm{id}_Y) \circ (\mathrm{id}_Y \otimes \eta) = \mathrm{id}_Y$ and $(\mathrm{id}_X \otimes \epsilon) \circ (\eta \otimes \mathrm{id}_X) = \mathrm{id}_X$, because we don't know in general that $\eta_Y = \mathrm{id}_Y \otimes \eta$ and we don't know that $\epsilon_X = \epsilon \otimes \mathrm{id}_X$. For this it would suffice to know that our adjunction isomorphism has the following property: for every W, Z, Z' the diagram

$$\begin{tikzcd}
\mathrm{Mor}(Z' \otimes X, Z) \arrow[r] \arrow[d, "\mathrm{id}_W \otimes -"'] & \mathrm{Mor}(Z', Z \otimes Y) \arrow[d, "\mathrm{id}_W \otimes -"] \\
\mathrm{Mor}(W \otimes Z' \otimes X, W \otimes Z) \arrow[r] & \mathrm{Mor}(W \otimes Z', W \otimes Z \otimes Y)
\end{tikzcd}$$

If this holds, we will say the adjunction is compatible with the given tensor structure. Thus the requirement that $Z \mapsto Z \otimes Y$ be the right adjoint of $Z' \mapsto Z' \otimes X$ compatible with the given tensor structure is an equivalent formulation of the property of being a left dual.

0FFS Lemma 4.43.8. Let $\mathcal{C}$ be a monoidal category. If Y_i, $i = 1, 2$ are left duals of X_i, $i = 1, 2$, then $Y_2 \otimes Y_1$ is a left dual of $X_1 \otimes X_2$.

Proof. Follows from uniqueness of adjoints and Remark 4.43.7. $\square$

A commutativity constraint for $(\mathcal{C}, \otimes)$ is a functorial isomorphism

$$\psi : X \otimes Y \longrightarrow Y \otimes X$$

such that the composition

$$X \otimes Y \xrightarrow{\psi} Y \otimes X \xrightarrow{\psi} X \otimes Y$$

is the identity. We say ψ is compatible with a given associativity constraint ϕ if for all objects X, Y, Z the diagram

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\phi"'] \arrow[d, "\psi"] & (X \otimes Y) \otimes Z \arrow[r, "\psi"'] & Z \otimes (X \otimes Y) \arrow[d, "\phi"] \\ X \otimes (Z \otimes Y) \arrow[r, "\phi"] & (X \otimes Z) \otimes Y \arrow[r, "\psi"] & (Z \otimes X) \otimes Y \end{tikzcd}$$

commutes.

0FFW Definition 4.43.9. A quadruple $(\mathcal{C}, \otimes, \phi, \psi)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is a functor, ϕ is an associativity constraint, and ψ is a commutativity constraint compatible with ϕ is called a symmetric monoidal category if there exists a unit.

To be sure, if $(\mathcal{C}, \otimes, \phi, \psi)$ is a symmetric monoidal category, then $(\mathcal{C}, \otimes, \phi)$ is a monoidal category.

0FN8 Lemma 4.43.10. Let $(\mathcal{C}, \otimes, \phi, \psi)$ be a symmetric monoidal category. Let X be an object of $\mathcal{C}$ and let Y, $\eta : \mathbf{1} \to X \otimes Y$, and $\epsilon : Y \otimes X \to \mathbf{1}$ be a left dual of X as in Definition 4.43.5. Then $\eta' = \psi \circ \eta : \mathbf{1} \to Y \otimes X$ and $\epsilon' = \epsilon \circ \psi : X \otimes Y \to \mathbf{1}$ makes X into a left dual of Y.

Proof. Omitted. Hint: pleasant exercise in the definitions. $\square$

0FFY Definition 4.43.11. Let $\mathcal{C}$ and $\mathcal{C}'$ be symmetric monoidal categories. A functor of symmetric monoidal categories $F : \mathcal{C} \to \mathcal{C}'$ is given by a functor F as indicated and an isomorphism

$$F(X) \otimes F(Y) \to F(X \otimes Y)$$

functorial in X and Y such that F is a functor of monoidal categories and such that for all objects X and Y the diagram

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r] \arrow[d] & F(X \otimes Y) \arrow[d] \\ F(Y) \otimes F(X) \arrow[r] & F(Y \otimes X) \end{tikzcd}$$

commutes.

0GWJ Remark 4.43.12. Let $\mathcal{C}$ be a monoidal category. We say $\mathcal{C}$ has an internal hom if for every pair of objects X, Y of $\mathcal{C}$ there is an object $hom(X, Y)$ of $\mathcal{C}$ such that we have

$$\text{Mor}(X, hom(Y, Z)) = \text{Mor}(X \otimes Y, Z)$$

functorially in X, Y, Z. By the Yoneda lemma the bifunctor $(X, Y) \mapsto hom(X, Y)$ is determined up to unique isomorphism if it exists. Given an internal hom we obtain canonical maps

(1) $hom(X, Y) \otimes X \to Y$,
(2) $hom(Y, Z) \otimes hom(X, Y) \to hom(X, Z)$,
(3) $Z \otimes hom(X, Y) \to hom(X, Z \otimes Y)$,
(4) $Y \to hom(X, Y \otimes X)$, and
(5) $hom(Y, Z) \otimes X \to hom(hom(X, Y), Z)$ in case $\mathcal{C}$ is symmetric monoidal.

Namely, the map in (1) is the image of $\mathrm{id}_{hom(X,Y)}$ by $\mathrm{Mor}(hom(X,Y), hom(X,Y)) \to \mathrm{Mor}(hom(X,Y) \otimes X, Y)$. To construct the map in (2) by the defining property of $hom(X,Z)$ we need to construct a map

$$hom(Y,Z) \otimes hom(X,Y) \otimes X \longrightarrow Z$$

and such a map exists since by (1) we have maps $hom(X,Y) \otimes X \to Y$ and $hom(Y,Z) \otimes Y \to Z$. To construct the map in (3) by the defining property of $hom(X, Z \otimes Y)$ we need to construct a map

$$Z \otimes hom(X,Y) \otimes X \to Z \otimes Y$$

for which we use $\mathrm{id}_Z \otimes a$ where a is the map in (1). To construct the map in (4) we note that we already have the map $Y \otimes hom(X,X) \to hom(X, Y \otimes X)$ by (3). Thus it suffices to construct a map $\mathbf{1} \to hom(X,X)$ and for this we take the element in $\mathrm{Mor}(\mathbf{1}, hom(X,X))$ corresponding to the canonical isomorphism $\mathbf{1} \otimes X \to X$ in $\mathrm{Mor}(\mathbf{1} \otimes X, X)$. Finally, we come to (5). By the universal property of $hom(hom(X,Y), Z)$ it suffices to construct a map

$$hom(Y,Z) \otimes X \otimes hom(X,Y) \longrightarrow Z$$

We do this by swapping the last two tensor products using the commutativity constraint and then using the maps $hom(X,Y) \otimes X \to Y$ and $hom(Y,Z) \otimes Y \to Z$.

4.44. Categories of dotted arrows

0H17 We discuss certain "categories of dotted arrows" in $(2,1)$-categories. These will appear when formulating various lifting criteria for algebraic stacks, see for example Morphisms of Stacks, Section 101.39 and More on Morphisms of Stacks, Section 106.8.

0H18 **Definition 4.44.1.** Let $\mathcal{C}$ be a $(2,1)$-category. Consider a 2-commutative solid diagram

0H19 (4.44.1.1)
$$\begin{tikzcd} S \arrow[r, "x"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ T \arrow[r, "y"] \arrow[ru, dotted] & Y \end{tikzcd}$$

in $\mathcal{C}$. Fix a 2-isomorphism

$$\gamma : y \circ j \to f \circ x$$

witnessing the 2-commutativity of the diagram. Given (4.44.1.1) and γ, a dotted arrow is a triple (a, α, β) consisting of a morphism $a \colon T \to X$ and and 2-isomorphisms $\alpha : a \circ j \to x$, $\beta : y \to f \circ a$ such that $\gamma = (\mathrm{id}_f \star \alpha) \circ (\beta \star \mathrm{id}_j)$, in other words such that

$$\begin{tikzcd} & f \circ a \circ j \arrow[rd, "\mathrm{id}_f \star \alpha"] & \\ y \circ j \arrow[ru, "\beta \star \mathrm{id}_j"] \arrow[rr, "\gamma"] & & f \circ x \end{tikzcd}$$

is commutative. A morphism of dotted arrows $(a, \alpha, \beta) \to (a', \alpha', \beta')$ is a 2-arrow $\theta : a \to a'$ such that $\alpha = \alpha' \circ (\theta \star \mathrm{id}_j)$ and $\beta' = (\mathrm{id}_f \star \theta) \circ \beta$.

In the situation of Definition 4.44.1, there is an associated category of dotted arrows. This category is a groupoid. It may depend on γ in general. The next two lemmas say that categories of dotted arrows are well-behaved with respect to base change and composition for f.

0H1A Lemma 4.44.2. Let $\mathcal{C}$ be a $(2,1)$-category. Assume given a 2-commutative diagram

$$\begin{tikzcd} S \arrow[r, "x'"'] \arrow[d, "j"'] & X' \arrow[r, "q"'] \arrow[d, "p"] & X \arrow[d, "f"] \\ T \arrow[r, "y'"] & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in $\mathcal{C}$, where the right square is 2-cartesian with respect to a 2-isomorphism $\phi\colon g \circ p \to f \circ q$. Choose a 2-arrow $\gamma' : y' \circ j \to p \circ x'$. Set $x = q \circ x'$, $y = g \circ y'$ and let $\gamma : y \circ j \to f \circ x$ be the 2-isomorphism $\gamma = (\phi \star \mathrm{id}_{x'}) \circ (\mathrm{id}_g \star \gamma')$. Then the category $\mathcal{D}'$ of dotted arrows for the left square and γ' is equivalent to the category $\mathcal{D}$ of dotted arrows for the outer rectangle and γ.

Proof. There is a functor $\mathcal{D}' \to \mathcal{D}$ which is $(a, \alpha, \beta) \mapsto (q \circ a, \mathrm{id}_q \star \alpha, (\phi \star \mathrm{id}_a) \circ (\mathrm{id}_g \star \beta))$ on objects and $\theta \mapsto \mathrm{id}_q \star \theta$ on arrows. Checking that this functor $\mathcal{D}' \to \mathcal{D}$ is an equivalence follows formally from the universal property for 2-fibre products as in Section 4.31. Details omitted. $\square$

0H1B Lemma 4.44.3. Let $\mathcal{C}$ be a $(2,1)$-category. Assume given a solid 2-commutative diagram

$$\begin{tikzcd} S \arrow[r, "x"'] \arrow[dd, "j"'] & X \arrow[d, "f"] \\ & Y \arrow[d, "g"] \\ T \arrow[r, "z"] \arrow[uur, dotted] & Z \end{tikzcd}$$

in $\mathcal{C}$. Choose a 2-isomorphism $\gamma\colon z \circ j \to g \circ f \circ x$. Let $\mathcal{D}$ be the category of dotted arrows for the outer rectangle and γ. Let $\mathcal{D}'$ be the category of dotted arrows for the solid square

$$\begin{tikzcd} S \arrow[r, "f \circ x"'] \arrow[d, "j"'] & Y \arrow[d, "g"] \\ T \arrow[r, "z"] \arrow[ur, dotted] & Z \end{tikzcd}$$

and γ. Then $\mathcal{D}$ is equivalent to a category $\mathcal{D}''$ which has the following property: there is a functor $\mathcal{D}'' \to \mathcal{D}'$ which turns $\mathcal{D}''$ into a category fibred in groupoids over $\mathcal{D}'$ and whose fibre categories are isomorphic to categories of dotted arrows for certain solid squares of the form

$$\begin{tikzcd} S \arrow[r, "x"'] \arrow[d, "j"'] & X \arrow[d, "f"] \\ T \arrow[r, "y"] \arrow[ur, dotted] & Y \end{tikzcd}$$

and some choices of 2-isomorphism $y \circ j \to f \circ x$.

Proof. Construct the category $\mathcal{D}''$ whose objects are tuples $(a, \alpha, \beta, b, \eta)$ where (a, α, β) is an object of $\mathcal{D}$ and $b : T \to Y$ is a 1-morphism and $\eta : b \to f \circ a$ is a 2-isomorphism. Morphisms $(a, \alpha, \beta, b, \eta) \to (a', \alpha', \beta', b', \eta')$ in $\mathcal{D}''$ are pairs (θ_1, θ_2), where $\theta_1 : a \to a'$ defines an arrow $(a, \alpha, \beta) \to (a', \alpha', \beta')$ in $\mathcal{D}$ and $\theta_2 : b \to b'$ is a 2-isomorphism with the compatibility condition $\eta' \circ \theta_2 = (\mathrm{id}_f \star \theta_1) \circ \eta$.

There is a functor $\mathcal{D}'' \to \mathcal{D}'$ which is $(a, \alpha, \beta, b, \eta) \mapsto (b, (\mathrm{id}_f \star \alpha) \circ (\eta \star \mathrm{id}_j), (\mathrm{id}_g \star \eta^{-1}) \circ \beta)$ on objects and $(\theta_1, \theta_2) \mapsto \theta_2$ on arrows. Then $\mathcal{D}'' \to \mathcal{D}'$ is fibred in groupoids.

If (y, δ, ϵ) is an object of $\mathcal{D}'$, write $\mathcal{D}_{y,\delta}$ for the category of dotted arrows for the last displayed diagram with $y \circ j \to f \circ x$ given by δ. There is a functor $\mathcal{D}_{y,\delta} \to \mathcal{D}''$ given by $(a, \alpha, \eta) \mapsto (a, \alpha, (\mathrm{id}_g \star \eta) \circ \epsilon, y, \eta)$ on objects and $\theta \mapsto (\theta, \mathrm{id}_y)$ on arrows. This exhibits an isomorphism from $\mathcal{D}_{y,\delta}$ to the fibre category of $\mathcal{D}'' \to \mathcal{D}'$ over (y, δ, ϵ).

There is also a functor $\mathcal{D} \to \mathcal{D}''$ which is $(a, \alpha, \beta) \mapsto (a, \alpha, \beta, f \circ a, \mathrm{id}_{f \circ a})$ on objects and $\theta \mapsto (\theta, \mathrm{id}_f \star \theta)$ on arrows. This functor is fully faithful and essentially surjective, hence an equivalence. Details omitted. $\square$

4.45. Other chapters

CHAPTER 5

Topology

004C 5.1. Introduction

004D Basic topology will be explained in this document. A reference is [Eng77].

5.2. Basic notions

004E The following is a list of basic notions in topology. Some of these notions are discussed in more detail in the text that follows and some are defined in the list, but others are considered basic and will not be defined. If you are not familiar with most of the italicized concepts, then we suggest looking at an introductory text on topology before continuing.

004F (1) X is a topological space,
004G (2) $x \in X$ is a point,
0B12 (3) $E \subset X$ is a locally closed subset,
004H (4) $x \in X$ is a closed point,
08ZA (5) $E \subset X$ is a dense subset,
004I (6) $f : X_1 \to X_2$ is continuous,
0BBW (7) an extended real function $f : X \to \mathbf{R} \cup \{\infty, -\infty\}$ is upper semi-continuous if $\{x \in X \mid f(x) < a\}$ is open for all $a \in \mathbf{R}$,
0BBX (8) an extended real function $f : X \to \mathbf{R} \cup \{\infty, -\infty\}$ is lower semi-continuous if $\{x \in X \mid f(x) > a\}$ is open for all $a \in \mathbf{R}$,
(9) a continuous map of spaces $f : X \to Y$ is open if $f(U)$ is open in Y for $U \subset X$ open,
(10) a continuous map of spaces $f : X \to Y$ is closed if $f(Z)$ is closed in Y for $Z \subset X$ closed,
004J (11) a neighbourhood of $x \in X$ is any subset $E \subset X$ which contains an open subset that contains x,
09R7 (12) the induced topology on a subset $E \subset X$,
004K (13) $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an open covering of U (note: we allow any U_i to be empty and we even allow, in case U is empty, the empty set for I),
0GM1 (14) a subcover of a covering as in (13) is an open covering $\mathcal{U}' : U = \bigcup_{i \in I'} U_i$ where $I' \subset I$,
004L (15) the open covering $\mathcal{V}$ is a refinement of the open covering $\mathcal{U}$ (if $\mathcal{V} : U = \bigcup_{j \in J} V_j$ and $\mathcal{U} : U = \bigcup_{i \in I} U_i$ this means each V_j is completely contained in one of the U_i),
004M (16) $\{E_i\}_{i \in I}$ is a fundamental system of neighbourhoods of x in X,
004N (17) a topological space X is called Hausdorff or separated if and only if for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$,

08ZB (18) the product of two topological spaces,
08ZC (19) the fibre product $X \times_Y Z$ of a pair of continuous maps $f : X \to Y$ and $g : Z \to Y$,
0B30 (20) the discrete topology and the indiscrete topology on a set,
(21) etc.

5.3. Hausdorff spaces

08ZD The category of topological spaces has finite products.

08ZE **Lemma 5.3.1.** *Let X be a topological space. The following are equivalent:*

1. *X is Hausdorff,*
2. *the diagonal $\Delta(X) \subset X \times X$ is closed.*

Proof. We suppose that X is Hausdorff. Let $(x, y) \notin \Delta(X)$, i.e., $x \neq y$. There are U and V disjoint open sets of X such that $x \in U$ and $y \in V$. This implies that $U \times V \subset (X \times X) \setminus \Delta(X)$. This shows that $(X \times X) \setminus \Delta(X)$ is an open set of $X \times X$ which is equivalent to say that the diagonal $\Delta(X) \subset X \times X$ is closed in $X \times X$. The converse is similar: The complement $(X \times X) \setminus \Delta(X)$ consist precisely of $(x, y) \in X \times X$ with $x \neq y$. Thus, if $\Delta(X) \subset X \times X$ is closed, then, by the definition of the product topology, for every such (x, y), there are opens $U, V \subset X$ with $(x, y) \in U \times V$ and $(U \times V) \cap \Delta(X) = \emptyset$. In other words, with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$. □

08ZF **Lemma 5.3.2.** *Let $f : X \to Y$ be a continuous map of topological spaces. If Y is Hausdorff, then the graph of f is closed in $X \times Y$.*

Proof. The graph is the inverse image of the diagonal under the map $X \times Y \to Y \times Y$. Thus the lemma follows from Lemma 5.3.1. □

08ZG **Lemma 5.3.3.** *Let $f : X \to Y$ be a continuous map of topological spaces. Let $s : Y \to X$ be a continuous map such that $f \circ s = \text{id}_Y$. If X is Hausdorff, then $s(Y)$ is closed.*

Proof. This follows from Lemma 5.3.1 as $s(Y) = \{x \in X \mid x = s(f(x))\}$. □

08ZH **Lemma 5.3.4.** *Let $X \to Z$ and $Y \to Z$ be continuous maps of topological spaces. If Z is Hausdorff, then $X \times_Z Y$ is closed in $X \times Y$.*

Proof. This follows from Lemma 5.3.1 as $X \times_Z Y$ is the inverse image of $\Delta(Z)$ under $X \times Y \to Z \times Z$. □

5.4. Separated maps

0CY0 Just the definition and some simple lemmas.

0CY1 **Definition 5.4.1.** A continuous map $f : X \to Y$ of topological spaces is called *separated* if and only if the diagonal $\Delta : X \to X \times_Y X$ is a closed map.

0CY2 **Lemma 5.4.2.** *Let $f : X \to Y$ be continuous map of topological spaces. The following are equivalent:*

1. *f is separated,*
2. *$\Delta(X) \subset X \times_Y X$ is a closed subset,*
3. *given distinct points $x, x' \in X$ mapping to the same point of Y, there exist disjoint open neighbourhoods of x and x'.*

Proof. If f is separated, by Definition 5.4.1, Δ is a closed map. The fact that X is closed in X gives us that $\Delta(X)$ is closed in $X \times_Y X$. Thus (1) implies (2).

Assune $\Delta(X) \subset X \times_Y X$ is a closed subset and denote U the complementary open. This means we have an open set $W \subset X \times X$ such that $W \cap (X \times_Y X) = U$. However, by definition of the product topology, if $(x, x') \in W \cap (X \times_Y X)$, we have V and V' open sets of X such that $x \in V$, $x' \in V'$ and $V \times V' \subset W$. If we had $V \cap V' \neq \emptyset$, we would have $z \in V \cap V'$. However, $(z, z) \in X \times_Y X$, so $(z, z) \in (V \times V') \cap (X \times_Y X) \subset U$, which is absurd. Therefore $V \cap V' = \emptyset$, and we have two disjoint open neighborhoods for x and x'. It proves that (2) implies (3).

Finally, we suppose that given distinct points $x, x' \in X$ mapping to the same point of Y, there exist disjoint open neighbourhoods of x and x'. Let F be a closed set of X. We will show that $\Delta(F)$ is a closed subset of $X \times_Y X$. Let $(x, x') \in X \times_Y X$ be a point not contained in $\Delta(F)$. Then either $x \neq x'$ or $x \notin F$. In the first case, we choose disjoint open neighbourhoods $V, V' \subset X$ of x, x' and we see that $(V \times V') \cap X \times_Y X$ is an open neighbourood of (x, x') not meeting $\Delta(F)$. In the second case, we see that $((X \setminus F) \times X) \cap X \times_Y X$ is an open neighbourood of (x, x') not meeting $\Delta(F)$. We have shown that (3) implies (1). $\square$

0CY3 **Lemma 5.4.3.** Let $f : X \to Y$ be continuous map of topological spaces. If X is Hausdorff, then f is separated.

Proof. Clear from Lemmas 5.4.2 and 5.3.1 as $\Delta(X)$ closed in $X \times X$ implies $\Delta(X)$ closed in $X \times_Y X$. $\square$

0CY4 **Lemma 5.4.4.** Let $f : X \to Y$ and $Y' \to Y$ be continuous maps of topological spaces. If f is separated, then $f' : Y' \times_Y X \to Z$ is separated.

Proof. Follows from characterization (3) of Lemma 5.4.2. Namely, with $X' = Y' \times_Y X$ the image $\Delta(X')$ of the diagonal in the fibre product $X' \times_{Y'} X'$ is the inverse image of $\Delta(X)$ in $X \times_Y X$. $\square$

5.5. Bases

004O Basic material on bases for topological spaces.

004P **Definition 5.5.1.** Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a *base for the topology on X* or a *basis for the topology on X* if the following conditions hold:

(1) Every element $B \in \mathcal{B}$ is open in X.
(2) For every open $U \subset X$ and every $x \in U$, there exists an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

The following lemma is sometimes used to define a topology.

0D5P **Lemma 5.5.2.** Let X be a set and let $\mathcal{B}$ be a collection of subsets. Assume that $X = \bigcup_{B \in \mathcal{B}} B$ and that given $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$ there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$. Then there is a unique topology on X such that $\mathcal{B}$ is a basis for this topology.

Proof. Let $\sigma(\mathcal{B})$ be the set of subsets of X which can be written as unions of elements of $\mathcal{B}$. We claim $\sigma(\mathcal{B})$ is a topology. Namely, the empty set is an element of $\sigma(\mathcal{B})$ (as an empty union) and X is an element of $\sigma(\mathcal{B})$ (as the union of all elements of $\mathcal{B}$). It is clear that $\sigma(\mathcal{B})$ is preserved under unions. Finally, if $U, V \in \sigma(\mathcal{B})$ then write $U = \bigcup_{i \in I} U_i$ and $V = \bigcup_{j \in J} V_j$ with $U_i, V_j \in \mathcal{B}$ for all $i \in I$ and $j \in J$. Then

$$U \cap V = \bigcup\nolimits_{i \in I, j \in J} U_i \cap V_j$$

The assumption in the lemma tells us that $U_i \cap V_j \in \sigma(\mathcal{B})$ hence we see that $U \cap V$ is too. Thus $\sigma(\mathcal{B}$ is a topology. Properties (1) and (2) of Definition 5.5.1 are immediate for this topology. To prove the uniqueness of this topology let $\mathcal{T}$ be a topology on X such that $\mathcal{B}$ is a base for it. Then of course every element of $\mathcal{B}$ is in $\mathcal{T}$ by (1) of Definition 5.5.1 and hence $\sigma(\mathcal{B} \subset \mathcal{T}$. Conversely, part (2) of Definition 5.5.1 tells us that every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$, i.e., $\mathcal{T} \subset \sigma(\mathcal{B})$. This finishes the proof. □

004Q Lemma 5.5.3. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{U} : U = \bigcup_i U_i$ be an open covering of $U \subset X$. There exists an open covering $U = \bigcup V_j$ which is a refinement of $\mathcal{U}$ such that each V_j is an element of the basis $\mathcal{B}$.

Proof. If $x \in U = \bigcup_{i \in I} U_i$, there is an $i_x \in I$ such that $x \in U_{i_x}$. Thus we have a $B_{i_x} \in \mathcal{B}$ verifying $x \in B_{i_x} \subset U_{i_x}$. Set $J = \{i_x | x \in U\}$ and for $j = i_x \in J$ set $V_j = B_{i_x}$. This gives the desired open covering of U by $\{V_j\}_{j \in J}$. □

08ZI Definition 5.5.4. Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a subbase for the topology on X or a subbasis for the topology on X if the finite intersections of elements of $\mathcal{B}$ form a basis for the topology on X.

In particular every element of $\mathcal{B}$ is open.

08ZJ Lemma 5.5.5. Let X be a set. Given any collection $\mathcal{B}$ of subsets of X there is a unique topology on X such that $\mathcal{B}$ is a subbase for this topology.

Proof. By convention $\bigcap_{\emptyset} B = X$. Thus we can apply Lemma 5.5.2 to the set of finite intersections of elements from $\mathcal{B}$. □

0D5Q Lemma 5.5.6. Let X be a topological space. Let $\mathcal{B}$ be a collection of opens of X. Assume $X = \bigcup_{U \in \mathcal{B}} U$ and for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$. Then there is a continuous map $f : X \to Y$ of topological spaces such that

(1) for $U \in \mathcal{B}$ the image $f(U)$ is open,
(2) for $U \in \mathcal{B}$ we have $f^{-1}(f(U)) = U$, and
(3) the opens $f(U)$, $U \in \mathcal{B}$ form a basis for the topology on Y.

Proof. Define an equivalence relation $\sim$ on points of X by the rule

$$x \sim y \Leftrightarrow (\forall U \in \mathcal{B} : x \in U \Leftrightarrow y \in U)$$

Let Y be the set of equivalence classes and $f : X \to Y$ the natural map. Part (2) holds by construction. The assumptions on $\mathcal{B}$ exactly mirror the assumptions in Lemma 5.5.2 on the set of subsets $f(U)$, $U \in \mathcal{B}$. Hence there is a unique topology on Y such that (3) holds. Then (1) is clear as well. □

5.6. Submersive maps

0405 If X is a topological space and $E \subset X$ is a subset, then we usually endow E with the induced topology.

09R8 **Lemma 5.6.1.** Let X be a topological space. Let Y be a set and let $f : Y \to X$ be an injective map of sets. The induced topology on Y is the topology characterized by each of the following statements:

(1) it is the weakest topology on Y such that f is continuous,
(2) the open subsets of Y are $f^{-1}(U)$ for $U \subset X$ open,
(3) the closed subsets of Y are the sets $f^{-1}(Z)$ for $Z \subset X$ closed.

Proof. The set $\mathcal{T} = \{f^{-1}(U) | U \subset X \text{ open}\}$ is a topology on Y. Firstly, $\emptyset = f^{-1}(\emptyset)$ and $f^{-1}(X) = Y$. So $\mathcal{T}$ contains $\emptyset$ and Y.

Now let $\{V_i\}_{i \in I}$ be a collection of open subsets where $V_i \in \mathcal{T}$ and write $V_i = f^{-1}(U_i)$ where U_i is an open subset of X, then

$$\bigcup\nolimits_{i \in I} V_i = \bigcup\nolimits_{i \in I} f^{-1}(U_i) = f^{-1}\left(\bigcup\nolimits_{i \in I} U_i\right)$$

So $\bigcup_{i \in I} V_i \in \mathcal{T}$ as $\bigcup_{i \in I} U_i$ is open in X. Now let $V_1, V_2 \in \mathcal{T}$. We have U_1, U_2 open in X such that $V_1 = f^{-1}(U_1)$ and $V_2 = f^{-1}(U_2)$. Then

$$V_1 \cap V_2 = f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2)$$

So $V_1 \cap V_2 \in \mathcal{T}$ because $U_1 \cap U_2$ is open in X.

Any topology on Y such that f is continuous contains $\mathcal{T}$ according to the definition of a continuous map. Thus $\mathcal{T}$ is indeed the weakest topology on Y such that f is continuous. This proves that (1) and (2) are equivalent.

The equivalence of (2) and (3) follows from the equality $f^{-1}(X \setminus E) = Y \setminus f^{-1}(E)$ for all subsets $E \subset X$. $\square$

Dually, if X is a topological space and $X \to Y$ is a surjection of sets, then Y can be endowed with the quotient topology.

08ZK **Lemma 5.6.2.** Let X be a topological space. Let Y be a set and let $f : X \to Y$ be a surjective map of sets. The quotient topology on Y is the topology characterized by each of the following statements:

(1) it is the strongest topology on Y such that f is continuous,
(2) a subset V of Y is open if and only if $f^{-1}(V)$ is open,
(3) a subset Z of Y is closed if and only if $f^{-1}(Z)$ is closed.

Proof. The set $\mathcal{T} = \{V \subset Y | f^{-1}(V) \text{ is open}\}$ is a topology on Y. Firstly $\emptyset = f^{-1}(\emptyset)$ and $f^{-1}(Y) = X$. So $\mathcal{T}$ contains $\emptyset$ and Y.

Let $(V_i)_{i \in I}$ be a family of elements $V_i \in \mathcal{T}$. Then

$$\bigcup\nolimits_{i \in I} f^{-1}(V_i) = f^{-1}\left(\bigcup\nolimits_{i \in I} V_i\right)$$

Thus $\bigcup_{i \in I} V_i \in \mathcal{T}$ as $\bigcup_{i \in I} f^{-1}(V_i)$ is open in X. Furthermore if $V_1, V_2 \in \mathcal{T}$ then

$$f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2)$$

So $V_1 \cap V_2 \in \mathcal{T}$ because $f^{-1}(V_1) \cap f^{-1}(V_2)$ is open in X.

Finally a topology on Y such that f is continuous is included in $\mathcal{T}$ according to the definition of a continuous function, so $\mathcal{T}$ is the strongest topology on Y such that f is continuous. It proves that (1) and (2) are equivalent.

Finally, (2) and (3) equivalence follows from $f^{-1}(X \setminus E) = Y \setminus f^{-1}(E)$ for all subsets $E \subset X$. $\square$

Let $f : X \to Y$ be a continuous map of topological spaces. In this case we obtain a factorization $X \to f(X) \to Y$ of maps of sets. We can endow $f(X)$ with the quotient topology coming from the surjection $X \to f(X)$ or with the induced topology coming from the injection $f(X) \to Y$. The map

$$(f(X), \text{quotient topology}) \longrightarrow (f(X), \text{induced topology})$$

is continuous.

0406 Definition 5.6.3. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say f is a strict map of topological spaces if the induced topology and the quotient topology on $f(X)$ agree (see discussion above).
(2) We say f is submersive[1] if f is surjective and strict.

Thus a continuous map $f : X \to Y$ is submersive if f is a surjection and for any $T \subset Y$ we have T is open or closed if and only if $f^{-1}(T)$ is so. In other words, Y has the quotient topology relative to the surjection $X \to Y$.

02YB Lemma 5.6.4. Let $f : X \to Y$ be surjective, open, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then

(1) $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$,
(2) $T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed,
(3) $T \subset Y$ is open if and only if $f^{-1}(T)$ is open, and
(4) $T \subset Y$ is locally closed if and only if $f^{-1}(T)$ is locally closed.

In particular we see that f is submersive.

Proof. It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. If $x \in X$, and $x \not\in \overline{f^{-1}(T)}$, then there exists an open neighbourhood $x \in U \subset X$ with $U \cap f^{-1}(T) = \emptyset$. Since f is open we see that $f(U)$ is an open neighbourhood of $f(x)$ not meeting T. Hence $x \not\in f^{-1}(\overline{T})$. This proves (1). Part (2) is an easy consequence of (1). Part (3) is obvious from the fact that f is open and surjective. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)} = f^{-1}(\overline{T})$ is open, and hence by (3) applied to the map $f^{-1}(\overline{T}) \to \overline{T}$ we see that T is open in $\overline{T}$, i.e., T is locally closed. $\square$

0AAU Lemma 5.6.5. Let $f : X \to Y$ be surjective, closed, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then

(1) $\overline{T} = f(\overline{f^{-1}(T)})$,
(2) $T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed,
(3) $T \subset Y$ is open if and only if $f^{-1}(T)$ is open, and
(4) $T \subset Y$ is locally closed if and only if $f^{-1}(T)$ is locally closed.

In particular we see that f is submersive.

Proof. It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. Then $T \subset f(\overline{f^{-1}(T)}) \subset \overline{T}$ is a closed subset, hence we get (1). Part (2) is obvious from the fact that f is closed and surjective. Part (3) follows from (2) applied to the complement of T. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)}$ is open. Since the map $\overline{f^{-1}(T)} \to \overline{T}$ is surjective by (1) we can apply part (3) to the map $\overline{f^{-1}(T)} \to \overline{T}$ induced by f to conclude that T is open in $\overline{T}$, i.e., T is locally closed. $\square$

[1] This is very different from the notion of a submersion between differential manifolds! It is probably a good idea to use "strict and surjective" in stead of "submersive".

5.7. Connected components

004R

004S Definition 5.7.1. Let X be a topological space.

(1) We say X is connected if X is not empty and whenever $X = T_1 \amalg T_2$ with $T_i \subset X$ open and closed, then either $T_1 = \emptyset$ or $T_2 = \emptyset$.
(2) We say $T \subset X$ is a connected component of X if T is a maximal connected subset of X.

The empty space is not connected.

0376 Lemma 5.7.2. Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is a connected subset, then $f(E) \subset Y$ is connected as well.

Proof. Let $A \subset f(E)$ an open and closed subset of $f(E)$. Because f is continuous, $f^{-1}(A)$ is an open and closed subset of E. As E is connected, $f^{-1}(A) = \emptyset$ or $f^{-1}(A) = E$. However, $A \subset f(E)$ implies that $A = f(f^{-1}(A))$. Indeed, if $x \in f(f^{-1}(A))$ then there is $y \in f^{-1}(A)$ such that $f(y) = x$ and because $y \in f^{-1}(A)$, we have $f(y) \in A$ i.e. $x \in A$. Reciprocally, if $x \in A$, $A \subset f(E)$ implies that there is $y \in E$ such that $f(y) = x$. Therefore $y \in f^{-1}(A)$, and then $x \in f(f^{-1}(A))$. Thus $A = \emptyset$ or $A = f(E)$ proving that $f(E)$ is connected. $\square$

004T Lemma 5.7.3. Let X be a topological space.

(1) If $T \subset X$ is connected, then so is its closure.
(2) Any connected component of X is closed (but not necessarily open).
(3) Every connected subset of X is contained in a unique connected component of X.
(4) Every point of X is contained in a unique connected component, in other words, X is the disjoint union of its connected components.

Proof. Let $\overline{T}$ be the closure of the connected subset T. Suppose $\overline{T} = T_1 \amalg T_2$ with $T_i \subset \overline{T}$ open and closed. Then $T = (T \cap T_1) \amalg (T \cap T_2)$. Hence T equals one of the two, say $T = T_1 \cap T$. Thus $\overline{T} \subset T_1$. This implies (1) and (2).

Let A be a nonempty set of connected subsets of X such that $\Omega = \bigcap_{T \in A} T$ is nonempty. We claim $E = \bigcup_{T \in A} T$ is connected. Namely, E is nonempty as it contains Ω. Say $E = E_1 \amalg E_2$ with E_i closed in E. We may assume E_1 meets Ω (after renumbering). Then each $T \in A$ meets E_1 and hence must be contained in E_1 as T is connected. Hence $E \subset E_1$ which proves the claim.

Let $W \subset X$ be a nonempty connected subset. If we apply the result of the previous paragraph to the set of all connected subsets of X containing W, then we see that E is a connected component of X. This implies existence and uniqueness in (3).

Let $x \in X$. Taking $W = \{x\}$ in the previous paragraph we see that x is contained in a unique connected component of X. Any two distinct connected components must be disjoint (by the result of the second paragraph).

To get an example where connected components are not open, just take an infinite product $\prod_{n \in \mathbf{N}} \{0, 1\}$ with the product topology. Its connected components are singletons, which are not open. $\square$

0FIY Remark 5.7.4. Let X be a topological space and $x \in X$. Let $Z \subset X$ be the connected component of X passing through x. Consider the intersection E of all [Eng77, Example 6.1.24] open and closed subsets of X containing x. It is clear that $Z \subset E$. In general $Z \neq E$. For example, let $X = \{x, y, z_1, z_2, \ldots\}$ with the topology with the following basis of opens, $\{z_n\}$, $\{x, z_n, z_{n+1}, \ldots\}$, and $\{y, z_n, z_{n+1}, \ldots\}$ for all n. Then $Z = \{x\}$ and $E = \{x, y\}$. We omit the details.

0377 Lemma 5.7.5. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that

(1) all fibres of f are connected, and
(2) a set $T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed.

Then f induces a bijection between the sets of connected components of X and Y.

Proof. Let $T \subset Y$ be a connected component. Note that T is closed, see Lemma 5.7.3. The lemma follows if we show that $f^{-1}(T)$ is connected because any connected subset of X maps into a connected component of Y by Lemma 5.7.2. Suppose that $f^{-1}(T) = Z_1 \amalg Z_2$ with Z_1, Z_2 closed. For any $t \in T$ we see that $f^{-1}(\{t\}) = Z_1 \cap f^{-1}(\{t\}) \amalg Z_2 \cap f^{-1}(\{t\})$. By (1) we see $f^{-1}(\{t\})$ is connected we conclude that either $f^{-1}(\{t\}) \subset Z_1$ or $f^{-1}(\{t\}) \subset Z_2$. In other words $T = T_1 \amalg T_2$ with $f^{-1}(T_i) = Z_i$. By (2) we conclude that T_i is closed in Y. Hence either $T_1 = \emptyset$ or $T_2 = \emptyset$ as desired. $\square$

0378 Lemma 5.7.6. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, (b) all fibres of f are connected. Then f induces a bijection between the sets of connected components of X and Y.

Proof. This is a special case of Lemma 5.7.5. $\square$

07VB Lemma 5.7.7. Let $f : X \to Y$ be a continuous map of nonempty topological spaces. Assume that (a) Y is connected, (b) f is open and closed, and (c) there is a point $y \in Y$ such that the fiber $f^{-1}(y)$ is a finite set. Then X has at most $|f^{-1}(y)|$ connected components. Hence any connected component T of X is open and closed, and $f(T)$ is a nonempty open and closed subset of Y, which is therefore equal to Y.

Proof. If the topological space X has at least N connected components for some $N \in \mathbf{N}$, we find by induction a decomposition $X = X_1 \amalg \ldots \amalg X_N$ of X as a disjoint union of N nonempty open and closed subsets $X_1, \ldots, X_N$ of X. As f is open and closed, each $f(X_i)$ is a nonempty open and closed subset of Y and is hence equal to Y. In particular the intersection $X_i \cap f^{-1}(y)$ is nonempty for each $1 \leq i \leq N$. Hence $f^{-1}(y)$ has at least N elements. $\square$

04MC Definition 5.7.8. A topological space is totally disconnected if the connected components are all singletons.

A discrete space is totally disconnected. A totally disconnected space need not be discrete, for example $\mathbf{Q} \subset \mathbf{R}$ is totally disconnected but not discrete.

08ZL Lemma 5.7.9. Let X be a topological space. Let $\pi_0(X)$ be the set of connected components of X. Let $X \to \pi_0(X)$ be the map which sends $x \in X$ to the connected component of X passing through x. Endow $\pi_0(X)$ with the quotient topology. Then $\pi_0(X)$ is a totally disconnected space and any continuous map $X \to Y$ from X to a totally disconnected space Y factors through $\pi_0(X)$.

Proof. By Lemma 5.7.5 the connected components of $\pi_0(X)$ are the singletons. We omit the proof of the second statement. $\square$

04MD Definition 5.7.10. A topological space X is called locally connected if every point $x \in X$ has a fundamental system of connected neighbourhoods.

04ME Lemma 5.7.11. Let X be a topological space. If X is locally connected, then

(1) any open subset of X is locally connected, and
(2) the connected components of X are open.

So also the connected components of open subsets of X are open. In particular, every point has a fundamental system of open connected neighbourhoods.

Proof. For all $x \in X$ let write $\mathcal{N}(x)$ the fundamental system of connected neighbourhoods of x and let $U \subset X$ be an open subset of X. Then for all $x \in U$, U is a neighbourhood of x, so the set $\{V \in \mathcal{N}(x) | V \subset U\}$ is not empty and is a fundamental system of connected neighbourhoods of x in U. Thus U is locally connected and it proves (1).

Let $x \in \mathcal{C} \subset X$ where $\mathcal{C}$ is the connected component of x. Because X is locally connected, there exists $\mathcal{N}$ a connected neighbourhood of x. Therefore by the definition of a connected component, we have $\mathcal{N} \subset \mathcal{C}$ and then $\mathcal{C}$ is a neighbourhood of x. It implies that $\mathcal{C}$ is a neighbourhood of each of his point, in other words $\mathcal{C}$ is open and (2) is proven. □

5.8. Irreducible components

004U

004V Definition 5.8.1. Let X be a topological space.

(1) We say X is irreducible, if X is not empty, and whenever $X = Z_1 \cup Z_2$ with Z_i closed, we have $X = Z_1$ or $X = Z_2$.
(2) We say $Z \subset X$ is an irreducible component of X if Z is a maximal irreducible subset of X.

An irreducible space is obviously connected.

0379 Lemma 5.8.2. Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is an irreducible subset, then $f(E) \subset Y$ is irreducible as well.

Proof. Clearly we may assume $E = X$ (i.e., X irreducible) and $f(E) = Y$ (i.e., f surjective). First, $Y \neq \emptyset$ since $X \neq \emptyset$. Next, assume $Y = Y_1 \cup Y_2$ with Y_1, Y_2 closed. Then $X = X_1 \cup X_2$ with $X_i = f^{-1}(Y_i)$ closed in X. By assumption on X, we must have $X = X_1$ or $X = X_2$, hence $Y = Y_1$ or $Y = Y_2$ since f is surjective. □

004W Lemma 5.8.3. Let X be a topological space.

(1) If $T \subset X$ is irreducible so is its closure in X.
(2) Any irreducible component of X is closed.
(3) Any irreducible subset of X is contained in an irreducible component of X.
(4) Every point of X is contained in some irreducible component of X, in other words, X is the union of its irreducible components.

Proof. Let $\overline{T}$ be the closure of the irreducible subset T. If $\overline{T} = Z_1 \cup Z_2$ with $Z_i \subset \overline{T}$ closed, then $T = (T \cap Z_1) \cup (T \cap Z_2)$ and hence T equals one of the two, say $T = Z_1 \cap T$. Thus clearly $\overline{T} \subset Z_1$. This proves (1). Part (2) follows immediately from (1) and the definition of irreducible components.

Let $T \subset X$ be irreducible. Consider the set A of irreducible subsets $T \subset T_\alpha \subset X$. Note that A is nonempty since $T \in A$. There is a partial ordering on A coming from inclusion: $\alpha \leq \alpha' \Leftrightarrow T_\alpha \subset T_{\alpha'}$. Choose a maximal totally ordered subset $A' \subset A$, and let $T' = \bigcup_{\alpha \in A'} T_\alpha$. We claim that T' is irreducible. Namely, suppose that $T' = Z_1 \cup Z_2$ is a union of two closed subsets of T'. For each $\alpha \in A'$ we have either $T_\alpha \subset Z_1$ or $T_\alpha \subset Z_2$, by irreducibility of T_α. Suppose that for some $\alpha_0 \in A'$ we have $T_{\alpha_0} \not\subset Z_1$ (say, if not we're done anyway). Then, since A' is totally ordered we see immediately that $T_\alpha \subset Z_2$ for all $\alpha \in A'$. Hence $T' = Z_2$. This proves (3). Part (4) is an immediate consequence of (3) as a singleton space is irreducible. □

0G2Y Lemma 5.8.4. Let X be a topological space and suppose $X = \bigcup_{i=1,\ldots,n} X_i$ where each X_i is an irreducible closed subset of X and no X_i is contained in the union of the other members. Then each X_i is an irreducible component of X and each irreducible component of X is one of the X_i.

Proof. Let Y be an irreducible component of X. Write $Y = \bigcup_{i=1,\ldots,n}(Y \cap X_i)$ and note that each $Y \cap X_i$ is closed in Y since X_i is closed in X. By irreducibility of Y we see that Y is equal to one of the $Y \cap X_i$, i.e., $Y \subset X_i$. By maximality of irreducible components we get $Y = X_i$.

Conversely, take one of the X_i and expand it to an irreducible component Y, which we have already shown is one of the X_j. So $X_i \subset X_j$ and since the original union does not have redundant members, $X_i = X_j$, which is an irreducible component. □

0GM2 Lemma 5.8.5. Let $f : X \to Y$ be a surjective, continuous map of topological spaces. If X has a finite number, say n, of irreducible components, then Y has $\leq n$ irreducible components.

Proof. Say $X_1, \ldots, X_n$ are the irreducible components of X. By Lemmas 5.8.2 and 5.8.3 the closure $Y_i \subset Y$ of $f(X_i)$ is irreducible. Since f is surjective, we see that Y is the union of the Y_i. We may choose a minimal subset $I \subset \{1, \ldots, n\}$ such that $Y = \bigcup_{i \in I} Y_i$. Then we may apply Lemma 5.8.4 to see that the Y_i for $i \in I$ are the irreducible components of Y. □

A singleton is irreducible. Thus if $x \in X$ is a point then the closure $\overline{\{x\}}$ is an irreducible closed subset of X.

004X Definition 5.8.6. Let X be a topological space.

(1) Let $Z \subset X$ be an irreducible closed subset. A generic point of Z is a point $\xi \in Z$ such that $Z = \overline{\{\xi\}}$.
(2) The space X is called Kolmogorov, if for every $x, x' \in X$, $x \neq x'$ there exists a closed subset of X which contains exactly one of the two points.
(3) The space X is called quasi-sober if every irreducible closed subset has a generic point.
(4) The space X is called sober if every irreducible closed subset has a unique generic point.

A topological space X is Kolmogorov, quasi-sober, resp. sober if and only if the map $x \mapsto \overline{\{x\}}$ from X to the set of irreducible closed subsets of X is injective, surjective, resp. bijective. Hence we see that a topological space is sober if and only if it is quasi-sober and Kolmogorov.

0B31 Lemma 5.8.7. Let X be a topological space and let $Y \subset X$.

(1) If X is Kolmogorov then so is Y.
(2) Suppose Y is locally closed in X. If X is quasi-sober then so is Y.
(3) Suppose Y is locally closed in X. If X is sober then so is Y.

Proof. Proof of (1). Suppose X is Kolmogorov. Let $x, y \in Y$ with $x \neq y$. Then $\overline{\overline{\{x\}} \cap Y} = \overline{\{x\}} \neq \overline{\{y\}} = \overline{\overline{\{y\}} \cap Y}$. Hence $\overline{\{x\}} \cap Y \neq \overline{\{y\}} \cap Y$. This shows that Y is Kolmogorov.

Proof of (2). Suppose X is quasi-sober. It suffices to consider the cases Y is closed and Y is open. First, suppose Y is closed. Let Z be an irreducible closed subset of Y. Then Z is an irreducible closed subset of X. Hence there exists $x \in Z$ with $\overline{\{x\}} = Z$. It follows $\overline{\{x\}} \cap Y = Z$. This shows Y is quasi-sober. Second, suppose Y is open. Let Z be an irreducible closed subset of Y. Then $\overline{Z}$ is an irreducible closed subset of X. Hence there exists $x \in \overline{Z}$ with $\overline{\{x\}} = \overline{Z}$. If $x \notin Y$ we get the contradiction $Z = Z \cap Y \subset \overline{Z} \cap Y = \overline{\{x\}} \cap Y = \emptyset$. Therefore $x \in Y$. It follows $Z = \overline{Z} \cap Y = \overline{\{x\}} \cap Y$. This shows Y is quasi-sober.

Proof of (3). Immediately from (1) and (2). □

06N9 Lemma 5.8.8. Let X be a topological space and let $(X_i)_{i \in I}$ be a covering of X.

(1) Suppose X_i is locally closed in X for every $i \in I$. Then, X is Kolmogorov if and only if X_i is Kolmogorov for every $i \in I$.
(2) Suppose X_i is open in X for every $i \in I$. Then, X is quasi-sober if and only if X_i is quasi-sober for every $i \in I$.
(3) Suppose X_i is open in X for every $i \in I$. Then, X is sober if and only if X_i is sober for every $i \in I$.

Proof. Proof of (1). If X is Kolmogorov then so is X_i for every $i \in I$ by Lemma 5.8.7. Suppose X_i is Kolmogorov for every $i \in I$. Let $x, y \in X$ with $\overline{\{x\}} = \overline{\{y\}}$. There exists $i \in I$ with $x \in X_i$. There exists an open subset $U \subset X$ such that X_i is a closed subset of U. If $y \notin U$ we get the contradiction $x \in \overline{\{x\}} \cap U = \overline{\{y\}} \cap U = \emptyset$. Hence $y \in U$. It follows $y \in \overline{\{y\}} \cap U = \overline{\{x\}} \cap U \subset X_i$. This shows $y \in X_i$. It follows $\overline{\{x\}} \cap X_i = \overline{\{y\}} \cap X_i$. Since X_i is Kolmogorov we get $x = y$. This shows X is Kolmogorov.

Proof of (2). If X is quasi-sober then so is X_i for every $i \in I$ by Lemma 5.8.7. Suppose X_i is quasi-sober for every $i \in I$. Let Y be an irreducible closed subset of X. As $Y \neq \emptyset$ there exists $i \in I$ with $X_i \cap Y \neq \emptyset$. As X_i is open in X it follows $X_i \cap Y$ is non-empty and open in Y, hence irreducible and dense in Y. Thus $X_i \cap Y$ is an irreducible closed subset of X_i. As X_i is quasi-sober there exists $x \in X_i \cap Y$ with $X_i \cap Y = \overline{\{x\}} \cap X_i \subset \overline{\{x\}}$. Since $X_i \cap Y$ is dense in Y and Y is closed in X it follows $Y = \overline{X_i \cap Y} \cap Y \subset \overline{X_i \cap Y} \subset \overline{\{x\}} \subset Y$. Therefore $Y = \overline{\{x\}}$. This shows X is quasi-sober.

Proof of (3). Immediately from (1) and (2). □

0B32 Example 5.8.9. Let X be an indiscrete space of cardinality at least 2. Then X is quasi-sober but not Kolmogorov. Moreover, the family of its singletons is a covering of X by discrete and hence Kolmogorov spaces.

0B33 Example 5.8.10. Let Y be an infinite set, furnished with the topology whose closed sets are Y and the finite subsets of Y. Then Y is Kolmogorov but not quasi-sober.

However, the family of its singletons (which are its irreducible components) is a covering by discrete and hence sober spaces.

0B34 Example 5.8.11. Let X and Y be as in Example 5.8.9 and Example 5.8.10. Then, $X \amalg Y$ is neither Kolmogorov nor quasi-sober.

0B35 Example 5.8.12. Let Z be an infinite set and let $z \in Z$. We furnish Z with the topology whose closed sets are Z and the finite subsets of $Z \setminus \{z\}$. Then Z is sober but its subspace $Z \setminus \{z\}$ is not quasi-sober.

004Y Example 5.8.13. Recall that a topological space X is Hausdorff iff for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$. In this case X is irreducible if and only if X is a singleton. Similarly, any subset of X is irreducible if and only if it is a singleton. Hence a Hausdorff space is sober.

004Z Lemma 5.8.14. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) Y is irreducible, (b) f is open, and (c) there exists a dense collection of points $y \in Y$ such that $f^{-1}(y)$ is irreducible. Then X is irreducible.

Proof. Suppose $X = Z_1 \cup Z_2$ with Z_i closed. Consider the open sets $U_1 = Z_1 \setminus Z_2 = X \setminus Z_2$ and $U_2 = Z_2 \setminus Z_1 = X \setminus Z_1$. To get a contradiction assume that U_1 and U_2 are both nonempty. By (b) we see that $f(U_i)$ is open. By (a) we have Y irreducible and hence $f(U_1) \cap f(U_2) \neq \emptyset$. By (c) there is a point y which corresponds to a point of this intersection such that the fibre $X_y = f^{-1}(y)$ is irreducible. Then $X_y \cap U_1$ and $X_y \cap U_2$ are nonempty disjoint open subsets of X_y which is a contradiction. □

037A Lemma 5.8.15. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, and (b) for every $y \in Y$ the fibre $f^{-1}(y)$ is irreducible. Then f induces a bijection between irreducible components.

Proof. We point out that assumption (b) implies that f is surjective (see Definition 5.8.1). Let $T \subset Y$ be an irreducible component. Note that T is closed, see Lemma 5.8.3. The lemma follows if we show that $f^{-1}(T)$ is irreducible because any irreducible subset of X maps into an irreducible component of Y by Lemma 5.8.2. Note that $f^{-1}(T) \to T$ satisfies the assumptions of Lemma 5.8.14. Hence we win. □

The construction of the following lemma is sometimes called the "soberification".

0A2N Lemma 5.8.16. Let X be a topological space. There is a canonical continuous map

$$c : X \longrightarrow X'$$

from X to a sober topological space X' which is universal among continuous maps from X to sober topological spaces. Moreover, the assignment $U' \mapsto c^{-1}(U')$ is a bijection between opens of X' and X which commutes with finite intersections and arbitrary unions. The image $c(X)$ is a Kolmogorov topological space and the map $c : X \to c(X)$ is universal for maps of X into Kolmogorov spaces.

Proof. Let X' be the set of irreducible closed subsets of X and let

$$c : X \to X', \quad x \mapsto \overline{\{x\}}.$$

For $U \subset X$ open, let $U' \subset X'$ denote the set of irreducible closed subsets of X which meet U. Then $c^{-1}(U') = U$. In particular, if $U_1 \neq U_2$ are open in X, then

$U'_1 \neq U'_2$. Hence c induces a bijection between the subsets of X' of the form U' and the opens of X.

Let U_1, U_2 be open in X. Suppose that $Z \in U'_1$ and $Z \in U'_2$. Then $Z \cap U_1$ and $Z \cap U_2$ are nonempty open subsets of the irreducible space Z and hence $Z \cap U_1 \cap U_2$ is nonempty. Thus $(U_1 \cap U_2)' = U'_1 \cap U'_2$. The rule $U \mapsto U'$ is also compatible with arbitrary unions (details omitted). Thus it is clear that the collection of U' form a topology on X' and that we have a bijection as stated in the lemma.

Next we show that X' is sober. Let $T \subset X'$ be an irreducible closed subset. Let $U \subset X$ be the open such that $X' \setminus T = U'$. Then $Z = X \setminus U$ is irreducible because of the properties of the bijection of the lemma. We claim that $Z \in T$ is the unique generic point. Namely, any open of the form $V' \subset X'$ which does not contain Z must come from an open $V \subset X$ which misses Z, i.e., is contained in U.

Finally, we check the universal property. Let $f : X \to Y$ be a continuous map to a sober topological space. Then we let $f' : X' \to Y$ be the map which sends the irreducible closed $Z \subset X$ to the unique generic point of $\overline{f(Z)}$. It follows immediately that $f' \circ c = f$ as maps of sets, and the properties of c imply that f' is continuous. We omit the verification that the continuous map f' is unique. We also omit the proof of the statements on Kolmogorov spaces. □

0GM3 Lemma 5.8.17. Let X be a connected topological space with a finite number of irreducible components $X_1, \ldots, X_n$. If $n > 1$ there is an $1 \leq j \leq n$ such that $X' = \bigcup_{i \neq j} X_i$ is connected.

Proof. This is a graph theory problem. Let Γ be the graph with vertices $V = \{1, \ldots, n\}$ and an edge between i and j if and only if $X_i \cap X_j$ is nonempty. Connectedness of X means that Γ is connected. Our problem is to find $1 \leq j \leq n$ such that $\Gamma \setminus \{j\}$ is still connected. You can do this by choosing $j, j' \in E$ with maximal distance and then j works (choose a leaf!). Details omitted. □

5.9. Noetherian topological spaces

0050

0051 Definition 5.9.1. A topological space is called Noetherian if the descending chain condition holds for closed subsets of X. A topological space is called locally Noetherian if every point has a neighbourhood which is Noetherian.

0052 Lemma 5.9.2. Let X be a Noetherian topological space.

(1) Any subset of X with the induced topology is Noetherian.
(2) The space X has finitely many irreducible components.
(3) Each irreducible component of X contains a nonempty open of X.

Proof. Let $T \subset X$ be a subset of X. Let $T_1 \supset T_2 \supset \ldots$ be a descending chain of closed subsets of T. Write $T_i = T \cap Z_i$ with $Z_i \subset X$ closed. Consider the descending chain of closed subsets $Z_1 \supset Z_1 \cap Z_2 \supset Z_1 \cap Z_2 \cap Z_3 \ldots$ This stabilizes by assumption and hence the original sequence of T_i stabilizes. Thus T is Noetherian.

Let A be the set of closed subsets of X which do not have finitely many irreducible components. Assume that A is not empty to arrive at a contradiction. The set A is partially ordered by inclusion: $\alpha \leq \alpha' \Leftrightarrow Z_\alpha \subset Z_{\alpha'}$. By the descending chain condition we may find a smallest element of A, say Z. As Z is not a finite union of irreducible components, it is not irreducible. Hence we can write $Z = Z' \cup Z''$ and both are strictly smaller closed subsets. By construction $Z' = \bigcup Z'_i$ and $Z'' = \bigcup Z''_j$ are finite unions of their irreducible components. Hence $Z = \bigcup Z'_i \cup \bigcup Z''_j$ is a finite union of irreducible closed subsets. After removing redundant members of this expression, this will be the decomposition of Z into its irreducible components (Lemma 5.8.4), a contradiction.

Let $Z \subset X$ be an irreducible component of X. Let $Z_1, \ldots, Z_n$ be the other irreducible components of X. Consider $U = Z \setminus (Z_1 \cup \ldots \cup Z_n)$. This is not empty since otherwise the irreducible space Z would be contained in one of the other Z_i. Because $X = Z \cup Z_1 \cup \ldots Z_n$ (see Lemma 5.8.3), also $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ and hence open in X. Thus Z contains a nonempty open of X. □

04Z8 Lemma 5.9.3. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) If X is Noetherian, then $f(X)$ is Noetherian.
(2) If X is locally Noetherian and f open, then $f(X)$ is locally Noetherian.

Proof. In case (1), suppose that $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ is a descending chain of closed subsets of $f(X)$ (as usual with the induced topology as a subset of Y). Then $f^{-1}(Z_1) \supset f^{-1}(Z_2) \supset f^{-1}(Z_3) \supset \ldots$ is a descending chain of closed subsets of X. Hence this chain stabilizes. Since $f(f^{-1}(Z_i)) = Z_i$ we conclude that $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ stabilizes also. In case (2), let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. By assumption there exists a neighbourhood $E \subset X$ of x which is Noetherian. Then $f(E) \subset f(X)$ is a neighbourhood which is Noetherian by part (1). □

0053 Lemma 5.9.4. Let X be a topological space. Let $X_i \subset X$, $i = 1, \ldots, n$ be a finite collection of subsets. If each X_i is Noetherian (with the induced topology), then $\bigcup_{i=1,\ldots,n} X_i$ is Noetherian (with the induced topology).

Proof. Let $\{F_m\}_{m \in \mathbf{N}}$ a decreasing sequence of closed subsets of $X' = \bigcup_{i=1,\ldots,n} X_i$ with the induced topology. Then we can find a decreasing sequence $\{G_m\}_{m \in \mathbf{N}}$ of closed subsets of X verifying $F_m = G_m \cap X'$ for all m (small detail omitted). As X_i is noetherian and $\{G_m \cap X_i\}_{m \in \mathbf{N}}$ a decreasing sequence of closed subsets of X_i, there exists $m_i \in \mathbf{N}$ such that for all $m \geq m_i$ we have $G_m \cap X_i = G_{m_i} \cap X_i$. Let $m_0 = \max_{i=1,\ldots,n} m_i$. Then clearly

$$F_m = G_m \cap X' = G_m \cap (X_1 \cup \ldots \cup X_n) = (G_m \cap X_1) \cup \ldots (G_m \cap X_n)$$

stabilizes for $m \geq m_0$ and the proof is complete. □

02HZ Example 5.9.5. Any nonempty, Kolmogorov Noetherian topological space has a closed point (combine Lemmas 5.12.8 and 5.12.13). Let $X = \{1, 2, 3, \ldots\}$. Define a topology on X with opens $\emptyset$, $\{1, 2, \ldots, n\}$, $n \geq 1$ and X. Thus X is a locally Noetherian topological space, without any closed points. This space cannot be the underlying topological space of a locally Noetherian scheme, see Properties, Lemma 28.5.9.

04MF Lemma 5.9.6. Let X be a locally Noetherian topological space. Then X is locally connected.

Proof. Let $x \in X$. Let E be a neighbourhood of x. We have to find a connected neighbourhood of x contained in E. By assumption there exists a neighbourhood E' of x which is Noetherian. Then $E \cap E'$ is Noetherian, see Lemma 5.9.2. Let $E \cap E' = Y_1 \cup \ldots \cup Y_n$ be the decomposition into irreducible components, see Lemma 5.9.2. Let $E'' = \bigcup_{x \in Y_i} Y_i$. This is a connected subset of $E \cap E'$ containing x. It contains the open $E \cap E' \setminus (\bigcup_{x \notin Y_i} Y_i)$ of $E \cap E'$ and hence it is a neighbourhood of x in X. This proves the lemma. $\square$

5.10. Krull dimension

0054

0055 **Definition 5.10.1.** Let X be a topological space.

(1) A chain of irreducible closed subsets of X is a sequence $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ with Z_i closed irreducible and $Z_i \neq Z_{i+1}$ for $i = 0, \ldots, n-1$.
(2) The length of a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ of irreducible closed subsets of X is the integer n.
(3) The dimension or more precisely the Krull dimension $\dim(X)$ of X is the element of $\{-\infty, 0, 1, 2, 3, \ldots, \infty\}$ defined by the formula:

$$\dim(X) = \sup\{\text{lengths of chains of irreducible closed subsets}\}$$

Thus $\dim(X) = -\infty$ if and only if X is the empty space.
(4) Let $x \in X$. The Krull dimension of X at x is defined as

$$\dim_x(X) = \min\{\dim(U), x \in U \subset X \text{ open}\}$$

the minimum of $\dim(U)$ where U runs over the open neighbourhoods of x in X.

Note that if $U' \subset U \subset X$ are open then $\dim(U') \leq \dim(U)$. Hence if $\dim_x(X) = d$ then x has a fundamental system of open neighbourhoods U with $\dim(U) = \dim_x(X)$.

0B7I **Lemma 5.10.2.** Let X be a topological space. Then $\dim(X) = \sup \dim_x(X)$ where the supremum runs over the points x of X.

Proof. It is clear that $\dim(X) \geq \dim_x(X)$ for all $x \in X$ (see discussion following Definition 5.10.1). Thus an inequality in one direction. For the converse, let $n \geq 0$ and suppose that $\dim(X) \geq n$. Then we can find a chain of irreducible closed subsets $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$. Pick $x \in Z_0$. For every open neighbourhood U of x we get a chain of irreducible closed subsets

$$Z_0 \cap U \subset Z_1 \cap U \subset \ldots \subset Z_n \cap U$$

in U. Namely, the sets $U \cap Z_i$ are irreducible closed in U and the inclusions are strict (details omitted; hint: the closure of $U \cap Z_i$ is Z_i). In this way we see that $\dim_x(X) \geq n$ which proves the other inequality. $\square$

0056 **Example 5.10.3.** The Krull dimension of the usual Euclidean space $\mathbf{R}^n$ is 0.

0057 **Example 5.10.4.** Let $X = \{s, \eta\}$ with open sets given by $\{\emptyset, \{\eta\}, \{s, \eta\}\}$. In this case a maximal chain of irreducible closed subsets is $\{s\} \subset \{s, \eta\}$. Hence $\dim(X) = 1$. It is easy to generalize this example to get a $(n+1)$-element topological space of Krull dimension n.

0058 **Definition 5.10.5.** Let X be a topological space. We say that X is equidimensional if every irreducible component of X has the same dimension.

5.11. Codimension and catenary spaces

02I0 We only define the codimension of irreducible closed subsets.

02I3 Definition 5.11.1. Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. The codimension of Y in X is the supremum of the lengths e of chains

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e \subset X$$

of irreducible closed subsets in X starting with Y. We will denote this $\text{codim}(Y, X)$.

The codimension is an element of $\{0, 1, 2, \ldots\} \cup \{\infty\}$. If $\text{codim}(Y, X) < \infty$, then every chain can be extended to a maximal chain (but these do not all have to have the same length).

02I4 Lemma 5.11.2. Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. Let $U \subset X$ be an open subset such that $Y \cap U$ is nonempty. Then

$$\text{codim}(Y, X) = \text{codim}(Y \cap U, U)$$

Proof. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. □

02I5 Example 5.11.3. Let $X = [0, 1]$ be the unit interval with the following topology: The sets $[0, 1]$, $(1 - 1/n, 1]$ for $n \in \mathbf{N}$, and $\emptyset$ are open. So the closed sets are $\emptyset$, $\{0\}$, $[0, 1 - 1/n]$ for $n > 1$ and $[0, 1]$. This is clearly a Noetherian topological space. But the irreducible closed subset $Y = \{0\}$ has infinite codimension $\text{codim}(Y, X) = \infty$. To see this we just remark that all the closed sets $[0, 1 - 1/n]$ are irreducible.

02I1 Definition 5.11.4. Let X be a topological space. We say X is catenary if for every pair of irreducible closed subsets $T \subset T'$ we have $\text{codim}(T, T') < \infty$ and every maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

has the same length (equal to the codimension).

02I2 Lemma 5.11.5. Let X be a topological space. The following are equivalent:

1. X is catenary,
2. X has an open covering by catenary spaces.

Moreover, in this case any locally closed subspace of X is catenary.

Proof. Suppose that X is catenary and that $U \subset X$ is an open subset. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. □

02I6 Lemma 5.11.6. Let X be a topological space. The following are equivalent:

1. X is catenary, and
2. for every pair of irreducible closed subsets $Y \subset Y'$ we have $\text{codim}(Y, Y') < \infty$ and for every triple $Y \subset Y' \subset Y''$ of irreducible closed subsets we have
$$\text{codim}(Y, Y'') = \text{codim}(Y, Y') + \text{codim}(Y', Y'').$$

Proof. Let suppose that X is catenary. According to Definition 5.11.4, for every pair of irreducible closed subsets $Y \subset Y'$ we have $\operatorname{codim}(Y, Y') < \infty$. Let $Y \subset Y' \subset Y''$ be a triple of irreducible closed subsets of X. Let

$$Y = Y_0 \subset Y_1 \subset ... \subset Y_{e_1} = Y'$$

be a maximal chain of irreducible closed subsets between Y and Y' where $e_1 = \operatorname{codim}(Y, Y')$. Let also

$$Y' = Y_{e_1} \subset Y_{e_1+1} \subset ... \subset Y_{e_1+e_2} = Y''$$

be a maximal chain of irreducible closed subsets between Y' and Y'' where $e_2 = \operatorname{codim}(Y', Y'')$. As the two chains are maximal, the concatenation

$$Y = Y_0 \subset Y_1 \subset ... \subset Y_{e_1} = Y' = Y_{e_1} \subset Y_{e_1+1} \subset ... \subset Y_{e_1+e_2} = Y''$$

is maximal too (between Y and Y'') and its length equals to $e_1 + e_2$. As X is catenary, each maximal chain has the same length equals to the codimension. Thus the point (2) that $\operatorname{codim}(Y, Y'') = e_1+e_2 = \operatorname{codim}(Y, Y')+\operatorname{codim}(Y', Y'')$ is verified.

For the reciprocal, we show by induction that : if $Y = Y_1 \subset ... \subset Y_n = Y'$, then $\operatorname{codim}(Y, Y') = \operatorname{codim}(Y_1, Y_2) + ... + \operatorname{codim}(Y_{n-1}, Y_n)$. Therefore, it forces maximal chains to have the same length. □

5.12. Quasi-compact spaces and maps

0059 The phrase "compact" will be reserved for Hausdorff topological spaces. And many spaces occurring in algebraic geometry are not Hausdorff.

005A Definition 5.12.1. Quasi-compactness.

(1) We say that a topological space X is quasi-compact if every open covering of X has a finite subcover.
(2) We say that a continuous map $f : X \to Y$ is quasi-compact if the inverse image $f^{-1}(V)$ of every quasi-compact open $V \subset Y$ is quasi-compact.
(3) We say a subset $Z \subset X$ is retrocompact if the inclusion map $Z \to X$ is quasi-compact.

In many texts on topology a space is called compact if it is quasi-compact and Hausdorff; and in other texts the Hausdorff condition is omitted. To avoid confusion in algebraic geometry we use the term quasi-compact. The notion of quasi-compactness of a map is very different from the notion of a "proper map", since there we require (besides closedness and separatedness) the inverse image of any quasi-compact subset of the target to be quasi-compact, whereas in the definition above we only consider quasi-compact open sets.

005B Lemma 5.12.2. A composition of quasi-compact maps is quasi-compact.

Proof. This is immediate from the definition. □

005C Lemma 5.12.3. A closed subset of a quasi-compact topological space is quasi-compact.

Proof. Let $E \subset X$ be a closed subset of the quasi-compact space X. Let $E = \bigcup V_j$ be an open covering. Choose $U_j \subset X$ open such that $V_j = E \cap U_j$. Then $X = (X \setminus E) \cup \bigcup U_j$ is an open covering of X. Hence $X = (X \setminus E) \cup U_{j_1} \cup \ldots \cup U_{j_n}$ for some n and indices j_i. Thus $E = V_{j_1} \cup \ldots \cup V_{j_n}$ as desired. □

08YB **Lemma 5.12.4.** Let X be a Hausdorff topological space.

(1) If $E \subset X$ is quasi-compact, then it is closed.
(2) If $E_1, E_2 \subset X$ are disjoint quasi-compact subsets then there exists opens $E_i \subset U_i$ with $U_1 \cap U_2 = \emptyset$.

Proof. Proof of (1). Let $x \in X$, $x \notin E$. For every $e \in E$ we can find disjoint opens V_e and U_e with $e \in V_e$ and $x \in U_e$. Since $E \subset \bigcup V_e$ we can find finitely many $e_1, \ldots, e_n$ such that $E \subset V_{e_1} \cup \ldots \cup V_{e_n}$. Then $U = U_{e_1} \cap \ldots \cap U_{e_n}$ is an open neighbourhood of x which avoids $V_{e_1} \cup \ldots \cup V_{e_n}$. In particular it avoids E. Thus E is closed.

Proof of (2). In the proof of (1) we have seen that given $x \in E_1$ we can find an open neighbourhood $x \in U_x$ and an open $E_2 \subset V_x$ such that $U_x \cap V_x = \emptyset$. Because E_1 is quasi-compact we can find a finite number $x_i \in E_1$ such that $E_1 \subset U = U_{x_1} \cup \ldots \cup U_{x_n}$. We take $V = V_{x_1} \cap \ldots \cap V_{x_n}$ to finish the proof. $\square$

08YC **Lemma 5.12.5.** Let X be a quasi-compact Hausdorff space. Let $E \subset X$. The following are equivalent: (a) E is closed in X, (b) E is quasi-compact.

Proof. The implication (a) $\Rightarrow$ (b) is Lemma 5.12.3. The implication (b) $\Rightarrow$ (a) is Lemma 5.12.4. $\square$

The following is really a reformulation of the quasi-compact property.

005D **Lemma 5.12.6.** Let X be a quasi-compact topological space. If $\{Z_\alpha\}_{\alpha \in A}$ is a collection of closed subsets such that the intersection of each finite subcollection is nonempty, then $\bigcap_{\alpha \in A} Z_\alpha$ is nonempty.

Proof. We suppose that $\bigcap_{\alpha \in A} Z_\alpha = \emptyset$. So we have $\bigcup_{\alpha \in A}(X \setminus Z_\alpha) = X$ by complementation. As the subsets Z_α are closed, $\bigcup_{\alpha \in A}(X \setminus Z_\alpha)$ is an open covering of the quasi-compact space X. Thus there exists a finite subset $J \subset A$ such that $X = \bigcup_{\alpha \in J}(X \setminus Z_\alpha)$. The complementary is then empty, which means that $\bigcap_{\alpha \in J} Z_\alpha = \emptyset$. It proves there exists a finite subcollection of $\{Z_\alpha\}_{\alpha \in J}$ verifying $\bigcap_{\alpha \in J} Z_\alpha = \emptyset$, which concludes by contraposition. $\square$

04Z9 **Lemma 5.12.7.** Let $f : X \to Y$ be a continuous map of topological spaces.

(1) If X is quasi-compact, then $f(X)$ is quasi-compact.
(2) If f is quasi-compact, then $f(X)$ is retrocompact.

Proof. If $f(X) = \bigcup V_i$ is an open covering, then $X = \bigcup f^{-1}(V_i)$ is an open covering. Hence if X is quasi-compact then $X = f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_n})$ for some $i_1, \ldots, i_n \in I$ and hence $f(X) = V_{i_1} \cup \ldots \cup V_{i_n}$. This proves (1). Assume f is quasi-compact, and let $V \subset Y$ be quasi-compact open. Then $f^{-1}(V)$ is quasi-compact, hence by (1) we see that $f(f^{-1}(V)) = f(X) \cap V$ is quasi-compact. Hence $f(X)$ is retrocompact. $\square$

005E **Lemma 5.12.8.** Let X be a topological space. Assume that

(1) X is nonempty,
(2) X is quasi-compact, and
(3) X is Kolmogorov.

Then X has a closed point.

Proof. Consider the set

$$\mathcal{T} = \{Z \subset X \mid Z = \overline{\{x\}} \text{ for some } x \in X\}$$

of all closures of singletons in X. It is nonempty since X is nonempty. Make $\mathcal{T}$ into a partially ordered set using the relation of inclusion. Suppose Z_α, $\alpha \in A$ is a totally ordered subset of $\mathcal{T}$. By Lemma 5.12.6 we see that $\bigcap_{\alpha \in A} Z_\alpha \neq \emptyset$. Hence there exists some $x \in \bigcap_{\alpha \in A} Z_\alpha$ and we see that $Z = \overline{\{x\}} \in \mathcal{T}$ is a lower bound for the family. By Zorn's lemma there exists a minimal element $Z \in \mathcal{T}$. As X is Kolmogorov we conclude that $Z = \{x\}$ for some x and $x \in X$ is a closed point. $\square$

08ZM Lemma 5.12.9. Let X be a quasi-compact Kolmogorov space. Then the set X_0 of closed points of X is quasi-compact.

Proof. Let $X_0 = \bigcup U_{i,0}$ be an open covering. Write $U_{i,0} = X_0 \cap U_i$ for some open $U_i \subset X$. Consider the complement Z of $\bigcup U_i$. This is a closed subset of X, hence quasi-compact (Lemma 5.12.3) and Kolmogorov. By Lemma 5.12.8 if Z is nonempty it would have a closed point which contradicts the fact that $X_0 \subset \bigcup U_i$. Hence $Z = \emptyset$ and $X = \bigcup U_i$. Since X is quasi-compact this covering has a finite subcover and we conclude. $\square$

005F Lemma 5.12.10. Let X be a topological space. Assume

(1) X is quasi-compact,
(2) X has a basis for the topology consisting of quasi-compact opens, and
(3) the intersection of two quasi-compact opens is quasi-compact.

For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha \in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not, then there exists a disjoint union decomposition $S = B \amalg C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 5.12.3 and assumption (1). By assumption (2) there exist quasi-compact opens $U, V \subset X$ with $B = S \cap U$ and $C = S \cap V$ (details omitted). Then $U \cap V \cap S = \emptyset$. Hence $\bigcap_\alpha U \cap V \cap Z_\alpha = \emptyset$. By assumption (3) the intersection $U \cap V$ is quasi-compact. By Lemma 5.12.6 for some $\alpha' \in A$ we have $U \cap V \cap Z_{\alpha'} = \emptyset$. Since $X \setminus (U \cup V)$ is disjoint from S and closed in X hence quasi-compact, we can use the same lemma to see that $Z_{\alpha''} \subset U \cup V$ for some $\alpha'' \in A$. Then $Z_\alpha = Z_{\alpha'} \cap Z_{\alpha''}$ is contained in $U \cup V$ and disjoint from $U \cap V$. Hence $Z_\alpha = U \cap Z_\alpha \amalg V \cap Z_\alpha$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. $\square$

08ZN Lemma 5.12.11. Let X be a topological space. Assume X is quasi-compact and Hausdorff. For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha \in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not, then there exists a disjoint union decomposition $S = B \amalg C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 5.12.3. By Lemma 5.12.4 there exist disjoint opens $U, V \subset X$ with $B \subset U$ and $C \subset V$. Then $X \setminus U \cup V$ is closed in X hence quasi-compact (Lemma 5.12.3). It follows that $(X \setminus U \cup V) \cap Z_\alpha = \emptyset$ for some α by Lemma 5.12.6. In other words, $Z_\alpha \subset U \cup V$. Thus $Z_\alpha = Z_\alpha \cap V \amalg Z_\alpha \cap U$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. □

04PL Lemma 5.12.12. Let X be a topological space. Assume

(1) X is quasi-compact,
(2) X has a basis for the topology consisting of quasi-compact opens, and
(3) the intersection of two quasi-compact opens is quasi-compact.

For a subset $T \subset X$ the following are equivalent:

(a) T is an intersection of open and closed subsets of X, and
(b) T is closed in X and is a union of connected components of X.

Proof. It is clear that (a) implies (b). Assume (b). Let $x \in X$, $x \notin T$. Let $x \in C \subset X$ be the connected component of X containing x. By Lemma 5.12.10 we see that $C = \bigcap V_\alpha$ is the intersection of all open and closed subsets V_α of X which contain C. In particular, any pairwise intersection $V_\alpha \cap V_\beta$ occurs as a V_α. As T is a union of connected components of X we see that $C \cap T = \emptyset$. Hence $T \cap \bigcap V_\alpha = \emptyset$. Since T is quasi-compact as a closed subset of a quasi-compact space (see Lemma 5.12.3) we deduce that $T \cap V_\alpha = \emptyset$ for some α, see Lemma 5.12.6. For this α we see that $U_\alpha = X \setminus V_\alpha$ is an open and closed subset of X which contains T and not x. The lemma follows. □

04ZA Lemma 5.12.13. Let X be a Noetherian topological space.

(1) The space X is quasi-compact.
(2) Any subset of X is retrocompact.

Proof. Suppose $X = \bigcup U_i$ is an open covering of X indexed by the set I which does not have a refinement by a finite open covering. Choose $i_1, i_2, \ldots$ elements of I inductively in the following way: Choose i_{n+1} such that $U_{i_{n+1}}$ is not contained in $U_{i_1} \cup \ldots \cup U_{i_n}$. Thus we see that $X \supset (X \setminus U_{i_1}) \supset (X \setminus U_{i_1} \cup U_{i_2}) \supset \ldots$ is a strictly decreasing infinite sequence of closed subsets. This contradicts the fact that X is Noetherian. This proves the first assertion. The second assertion is now clear since every subset of X is Noetherian by Lemma 5.9.2. □

04ZB Lemma 5.12.14. A quasi-compact locally Noetherian space is Noetherian.

Proof. The conditions imply immediately that X has a finite covering by Noetherian subsets, and hence is Noetherian by Lemma 5.9.4. □

08ZP Lemma 5.12.15 (Alexander subbase theorem). Let X be a topological space. Let $\mathcal{B}$ be a subbase for X. If every covering of X by elements of $\mathcal{B}$ has a finite refinement, then X is quasi-compact.

Proof. Assume there is an open covering of X which does not have a finite refinement. Using Zorn's lemma we can choose a maximal open covering $X = \bigcup_{i \in I} U_i$ which does not have a finite refinement (details omitted). In other words, if $U \subset X$ is any open which does not occur as one of the U_i, then the covering $X = U \cup \bigcup_{i \in I} U_i$

does have a finite refinement. Let $I' \subset I$ be the set of indices such that $U_i \in \mathcal{B}$. Then $\bigcup_{i \in I'} U_i \neq X$, since otherwise we would get a finite refinement covering X by our assumption on $\mathcal{B}$. Pick $x \in X$, $x \notin \bigcup_{i \in I'} U_i$. Pick $i \in I$ with $x \in U_i$. Pick $V_1, \ldots, V_n \in \mathcal{B}$ such that $x \in V_1 \cap \ldots \cap V_n \subset U_i$. This is possible as $\mathcal{B}$ is a subbasis for X. Note that V_j does not occur as a U_i. By maximality of the chosen covering we see that for each j there exist $i_{j,1}, \ldots, i_{j,n_j} \in I$ such that $X = V_j \cup U_{i_{j,1}} \cup \ldots \cup U_{i_{j,n_j}}$. Since $V_1 \cap \ldots \cap V_n \subset U_i$ we conclude that $X = U_i \cup \bigcup U_{i_{j,l}}$ a contradiction. $\square$

5.13. Locally quasi-compact spaces

08ZQ Recall that a neighbourhood of a point need not be open.

0068 **Definition 5.13.1.** A topological space X is called *locally quasi-compact*[2] if every point has a fundamental system of quasi-compact neighbourhoods.

The term *locally compact space* in the literature often refers to a space as in the following lemma.

08ZR **Lemma 5.13.2.** *A Hausdorff space is locally quasi-compact if and only if every point has a quasi-compact neighbourhood.*

Proof. Let X be a Hausdorff space. Let $x \in X$ and let $x \in E \subset X$ be a quasi-compact neighbourhood. Then E is closed by Lemma 5.12.4. Suppose that $x \in U \subset X$ is an open neighbourhood of x. Then $Z = E \setminus U$ is a closed subset of E not containing x. Hence we can find a pair of disjoint open subsets $W, V \subset E$ of E such that $x \in V$ and $Z \subset W$, see Lemma 5.12.4. It follows that $\overline{V} \subset E$ is a closed neighbourhood of x contained in $E \cap U$. Also $\overline{V}$ is quasi-compact as a closed subset of E (Lemma 5.12.3). In this way we obtain a fundamental system of quasi-compact neighbourhoods of x. $\square$

0CQN **Lemma 5.13.3** (Baire category theorem)**.** *Let X be a locally quasi-compact Hausdorff space. Let $U_n \subset X$, $n \geq 1$ be dense open subsets. Then $\bigcap_{n \geq 1} U_n$ is dense in X.*

Proof. After replacing U_n by $\bigcap_{i=1,\ldots,n} U_i$ we may assume that $U_1 \supset U_2 \supset \ldots$. Let $x \in X$. We will show that x is in the closure of $\bigcap_{n \geq 1} U_n$. Thus let E be a neighbourhood of x. To show that $E \cap \bigcap_{n \geq 1} U_n$ is nonempty we may replace E by a smaller neighbourhood. After replacing E by a smaller neighbourhood, we may assume that E is quasi-compact.

Set $x_0 = x$ and $E_0 = E$. Below, we will inductively choose a point $x_i \in E_{i-1} \cap U_i$ and a quasi-compact neighbourhood E_i of x_i with $E_i \subset E_{i-1} \cap U_i$. Because X is Hausdorff, the subsets $E_i \subset X$ are closed (Lemma 5.12.4). Since the E_i are also nonempty we conclude that $\bigcap_{i \geq 1} E_i$ is nonempty (Lemma 5.12.6). Since $\bigcap_{i \geq 1} E_i \subset E \cap \bigcap_{n \geq 1} U_n$ this proves the lemma.

The base case $i = 0$ we have done above. Induction step. Since E_{i-1} is a neighbourhood of x_{i-1} we can find an open $x_{i-1} \in W \subset E_{i-1}$. Since U_i is dense in X we

[2]This may not be standard notation. Alternative notions used in the literature are: (1) Every point has some quasi-compact neighbourhood, and (2) Every point has a closed quasi-compact neighbourhood. A scheme has the property that every point has a fundamental system of open quasi-compact neighbourhoods.

see that $W \cap U_i$ is nonempty. Pick any $x_i \in W \cap U_i$. By definition of locally quasi-compact spaces we can find a quasi-compact neighbourhood E_i of x_i contained in $W \cap U_i$. Then $E_i \subset E_{i-1} \cap U_i$ as desired. $\square$

09UV Lemma 5.13.4. Let X be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then there exists an open covering $X = \bigcup_{i \in I} V_i$ such that $\overline{V_i} \subset U_i$ for all i.

Proof. Let $x \in X$. Choose an $i(x) \in I$ such that $x \in U_{i(x)}$. Since $X \setminus U_{i(x)}$ and $\{x\}$ are disjoint closed subsets of X, by Lemmas 5.12.3 and 5.12.4 there exists an open neighbourhood U_x of x whose closure is disjoint from $X \setminus U_{i(x)}$. Thus $\overline{U_x} \subset U_{i(x)}$. Since X is quasi-compact, there is a finite list of points $x_1, \ldots, x_m$ such that $X = U_{x_1} \cup \ldots \cup U_{x_m}$. Setting $V_i = \bigcup_{i = i(x_j)} U_{x_j}$ the proof is finished. $\square$

09UW Lemma 5.13.5. Let X be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Suppose given an integer $p \geq 0$ and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open covering $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$. Then there exists an open covering $X = \bigcup_{j \in J} V_j$ and a map $\alpha : J \to I$ such that $\overline{V_j} \subset U_{\alpha(j)}$ and such that each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some k.

Proof. Since X is quasi-compact, there is a reduction to the case where I is finite (details omitted). We prove the result for I finite by induction on p. The base case $p = 0$ is immediate by taking a covering as in Lemma 5.13.4 refining the open covering $X = \bigcup W_{i_0, k}$.

Induction step. Assume the lemma proven for $p - 1$. For all p-tuples $i'_0, \ldots, i'_{p-1}$ of I let $U_{i'_0} \cap \ldots \cap U_{i'_{p-1}} = \bigcup W_{i'_0 \ldots i'_{p-1}, k}$ be a common refinement of the coverings $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$ for those $(p+1)$-tuples such that $\{i'_0, \ldots, i'_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). (There are finitely many of these as I is finite.) By induction there exists a solution for these opens, say $X = \bigcup V_j$ and $\alpha : J \to I$. At this point the covering $X = \bigcup_{j \in J} V_j$ and α satisfy $\overline{V_j} \subset U_{\alpha(j)}$ and each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some k if there is a repetition in $\alpha(j_0), \ldots, \alpha(j_p)$. Of course, we may and do assume that J is finite.

Fix $i_0, \ldots, i_p \in I$ pairwise distinct. Consider $(p+1)$-tuples $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is not contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for any k. Let N be the number of such $(p+1)$-tuples. We will show how to decrease N. Since

$$\overline{V_{j_0}} \cap \ldots \cap \overline{V_{j_p}} \subset U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$$

we find a finite set $K = \{k_1, \ldots, k_t\}$ such that the LHS is contained in $\bigcup_{k \in K} W_{i_0 \ldots i_p, k}$. Then we consider the open covering

$$V_{j_0} = (V_{j_0} \setminus (\overline{V_{j_1}} \cap \ldots \cap \overline{V_{j_p}})) \cup (\bigcup\nolimits_{k \in K} V_{j_0} \cap W_{i_0 \ldots i_p, k})$$

The first open on the RHS intersects $V_{j_1} \cap \ldots \cap V_{j_p}$ in the empty set and the other opens $V_{j_0, k}$ of the RHS satisfy $V_{j_0, k} \cap V_{j_1} \ldots \cap V_{j_p} \subset W_{\alpha(j_0) \ldots \alpha(j_p), k}$. Set $J' = J \amalg K$. For $j \in J$ set $V'_j = V_j$ if $j \neq j_0$ and set $V'_{j_0} = V_{j_0} \setminus (\overline{V_{j_1}} \cap \ldots \cap \overline{V_{j_p}})$. For $k \in K$ set $V'_k = V_{j_0, k}$. Finally, the map $\alpha' : J' \to I$ is given by α on J and maps every element of K to i_0. A simple check shows that N has decreased by one under this replacement. Repeating this procedure N times we arrive at the situation where $N = 0$.

To finish the proof we argue by induction on the number M of $(p+1)$-tuples $i_0, \ldots, i_p \in I$ with pairwise distinct entries for which there exists a $(p+1)$-tuple $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is not contained in $W_{\alpha(j_0)\ldots\alpha(j_p),k}$ for any k. To do this, we claim that the operation performed in the previous paragraph does not increase M. This follows formally from the fact that the map $\alpha' : J' \to I$ factors through a map $\beta : J' \to J$ such that $V'_{j'} \subset V_{\beta(j')}$. □

09UX Lemma 5.13.6. Let X be a Hausdorff and locally quasi-compact space. Let $Z \subset X$ be a quasi-compact (hence closed) subset. Suppose given an integer $p \geq 0$, a set I, for every $i \in I$ an open $U_i \subset X$, and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open $W_{i_0\ldots i_p} \subset U_{i_0} \cap \ldots \cap U_{i_p}$ such that

(1) $Z \subset \bigcup U_i$, and
(2) for every $i_0, \ldots, i_p$ we have $W_{i_0\ldots i_p} \cap Z = U_{i_0} \cap \ldots \cap U_{i_p} \cap Z$.

Then there exist opens V_i of X such that we have $Z \subset \bigcup V_i$, for all i we have $\overline{V_i} \subset U_i$, and we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0\ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$.

Proof. Since Z is quasi-compact, there is a reduction to the case where I is finite (details omitted). Because X is locally quasi-compact and Z is quasi-compact, we can find a neighbourhood $Z \subset E$ which is quasi-compact, i.e., E is quasi-compact and contains an open neighbourhood of Z in X. If we prove the result after replacing X by E, then the result follows. Hence we may assume X is quasi-compact.

We prove the result in case I is finite and X is quasi-compact by induction on p. The base case is $p = 0$. In this case we have $X = (X \setminus Z) \cup \bigcup W_i$. By Lemma 5.13.4 we can find a covering $X = V \cup \bigcup V_i$ by opens $V_i \subset W_i$ and $V \subset X \setminus Z$ with $\overline{V_i} \subset W_i$ for all i. Then we see that we obtain a solution of the problem posed by the lemma.

Induction step. Assume the lemma proven for $p-1$. Set $W_{j_0\ldots j_{p-1}}$ equal to the intersection of all $W_{i_0\ldots i_p}$ with $\{j_0, \ldots, j_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). By induction there exists a solution for these opens, say $V_i \subset U_i$. It follows from our choice of $W_{j_0\ldots j_{p-1}}$ that we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0\ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$ where $i_a = i_b$ for some $0 \leq a < b \leq p$. Thus we only need to modify our choice of V_i if $V_{i_0} \cap \ldots \cap V_{i_p} \not\subset W_{i_0\ldots i_p}$ for some $(p+1)$-tuple $i_0, \ldots, i_p$ with pairwise distinct elements. In this case we have

$$T = \overline{V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0\ldots i_p}} \subset \overline{V_{i_0}} \cap \ldots \cap \overline{V_{i_p}} \setminus W_{i_0\ldots i_p}$$

is a closed subset of X contained in $U_{i_0} \cap \ldots \cap U_{i_p}$ not meeting Z. Hence we can replace V_{i_0} by $V_{i_0} \setminus T$ to "fix" the problem. After repeating this finitely many times for each of the problem tuples, the lemma is proven. □

0CY5 Lemma 5.13.7. Let X be a topological space. Let $Z \subset X$ be a quasi-compact subset such that any two points of Z have disjoint open neighbourhoods in X. Suppose given an integer $p \geq 0$, a set I, for every $i \in I$ an open $U_i \subset X$, and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open $W_{i_0\ldots i_p} \subset U_{i_0} \cap \ldots \cap U_{i_p}$ such that

(1) $Z \subset \bigcup U_i$, and
(2) for every $i_0, \ldots, i_p$ we have $W_{i_0\ldots i_p} \cap Z = U_{i_0} \cap \ldots \cap U_{i_p} \cap Z$.

Then there exist opens V_i of X such that

(1) $Z \subset \bigcup V_i$,

(2) $V_i \subset U_i$ for all i,
(3) $\overline{V_i} \cap Z \subset U_i$ for all i, and
(4) $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0 \ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$.

Proof. Since Z is quasi-compact, there is a reduction to the case where I is finite (details omitted). We prove the result in case I is finite by induction on p.

The base case is $p = 0$. For $z \in Z \cap U_i$ and $z' \in Z \setminus U_i$ there exist disjoint opens $z \in V_{z,z'}$ and $z' \in W_{z,z'}$ of X. Since $Z \setminus U_i$ is quasi-compact (Lemma 5.12.3), we can choose a finite nunber $z'_1, \ldots, z'_r$ such that $Z \setminus U_i \subset W_{z,z'_1} \cup \ldots \cup W_{z,z'_r}$. Then we see that $V_z = V_{z,z'_1} \cap \ldots \cap V_{z,z'_r} \cap U_i$ is an open neighbourhood of z contained in U_i with the property that $\overline{V_z} \cap Z \subset U_i$. Since z and i where arbitrary and since Z is quasi-compact we can find a finite list $z_1, i_1, \ldots, z_t, i_t$ and opens $V_{z_j} \subset U_{i_j}$ with $\overline{V_{z_j}} \cap Z \subset U_{i_j}$ and $Z \subset \bigcup V_{z_j}$. Then we can set $V_i = W_i \cap (\bigcup_{j : i = i_j} V_{z_j})$ to solve the problem for $p = 0$.

Induction step. Assume the lemma proven for $p - 1$. Set $W_{j_0 \ldots j_{p-1}}$ equal to the intersection of all $W_{i_0 \ldots i_p}$ with $\{j_0, \ldots, j_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). By induction there exists a solution for these opens, say $V_i \subset U_i$. It follows from our choice of $W_{j_0 \ldots j_{p-1}}$ that we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0 \ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$ where $i_a = i_b$ for some $0 \leq a < b \leq p$. Thus we only need to modify our choice of V_i if $V_{i_0} \cap \ldots \cap V_{i_p} \not\subset W_{i_0 \ldots i_p}$ for some $(p+1)$-tuple $i_0, \ldots, i_p$ with pairwise distinct elements. In this case we have

$$T = \overline{V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0 \ldots i_p}} \subset \overline{V_{i_0}} \cap \ldots \cap \overline{V_{i_p}} \setminus W_{i_0 \ldots i_p}$$

is a closed subset of X not meeting Z by our property (3) of the opens V_i. Hence we can replace V_{i_0} by $V_{i_0} \setminus T$ to "fix" the problem. After repeating this finitely many times for each of the problem tuples, the lemma is proven. □

5.14. Limits of spaces

08ZS The category of topological spaces has products. Namely, if I is a set and for $i \in I$ we are given a topological space X_i then we endow $\prod_{i \in I} X_i$ with the product topology. As a basis for the topology we use sets of the form $\prod U_i$ where $U_i \subset X_i$ is open and $U_i = X_i$ for almost all i.

The category of topological spaces has equalizers. Namely, if $a, b : X \to Y$ are morphisms of topological spaces, then the equalizer of a and b is the subset $\{x \in X \mid a(x) = b(x)\} \subset X$ endowed with the induced topology.

08ZT Lemma 5.14.1. The category of topological spaces has limits and the forgetful functor to sets commutes with them.

Proof. This follows from the discussion above and Categories, Lemma 4.14.11. It follows from the description above that the forgetful functor commutes with limits. Another way to see this is to use Categories, Lemma 4.24.5 and use that the forgetful functor has a left adjoint, namely the functor which assigns to a set the corresponding discrete topological space. □

0A2P Lemma 5.14.2. Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of topological spaces over $\mathcal{I}$. Let $X = \lim X_i$ be the limit with projection maps $f_i : X \to X_i$.

(1) Any open of X is of the form $\bigcup_{j \in J} f_j^{-1}(U_j)$ for some subset $J \subset I$ and opens $U_j \subset X_j$.
(2) Any quasi-compact open of X is of the form $f_i^{-1}(U_i)$ for some i and some $U_i \subset X_i$ open.

Proof. The construction of the limit given above shows that $X \subset \prod X_i$ with the induced topology. A basis for the topology of $\prod X_i$ are the opens $\prod U_i$ where $U_i \subset X_i$ is open and $U_i = X_i$ for almost all i. Say $i_1, \ldots, i_n \in \text{Ob}(\mathcal{I})$ are the objects such that $U_{i_j} \neq X_{i_j}$. Then

$$X \cap \prod U_i = f_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap f_{i_n}^{-1}(U_{i_n})$$

For a general limit of topological spaces these form a basis for the topology on X. However, if $\mathcal{I}$ is cofiltered as in the statement of the lemma, then we can pick a $j \in \text{Ob}(\mathcal{I})$ and morphisms $j \to i_l$, $l = 1, \ldots, n$. Let

$$U_j = (X_j \to X_{i_1})^{-1}(U_{i_1}) \cap \ldots \cap (X_j \to X_{i_n})^{-1}(U_{i_n})$$

Then it is clear that $X \cap \prod U_i = f_j^{-1}(U_j)$. Thus for any open W of X there is a set A and a map $\alpha : A \to \text{Ob}(\mathcal{I})$ and opens $U_a \subset X_{\alpha(a)}$ such that $W = \bigcup f_{\alpha(a)}^{-1}(U_a)$. Set $J = \text{Im}(\alpha)$ and for $j \in J$ set $U_j = \bigcup_{\alpha(a)=j} U_a$ to see that $W = \bigcup_{j \in J} f_j^{-1}(U_j)$. This proves (1).

To see (2) suppose that $\bigcup_{j \in J} f_j^{-1}(U_j)$ is quasi-compact. Then it is equal to $f_{j_1}^{-1}(U_{j_1}) \cup \ldots \cup f_{j_m}^{-1}(U_{j_m})$ for some $j_1, \ldots, j_m \in J$. Since $\mathcal{I}$ is cofiltered, we can pick a $i \in \text{Ob}(\mathcal{I})$ and morphisms $i \to j_l$, $l = 1, \ldots, m$. Let

$$U_i = (X_i \to X_{j_1})^{-1}(U_{j_1}) \cup \ldots \cup (X_i \to X_{j_m})^{-1}(U_{j_m})$$

Then our open equals $f_i^{-1}(U_i)$ as desired. □

0A2Q Lemma 5.14.3. Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of topological spaces over $\mathcal{I}$. Let X be a topological space such that

(1) $X = \lim X_i$ as a set (denote f_i the projection maps),
(2) the sets $f_i^{-1}(U_i)$ for $i \in \text{Ob}(\mathcal{I})$ and $U_i \subset X_i$ open form a basis for the topology of X.

Then X is the limit of the X_i as a topological space.

Proof. Follows from the description of the limit topology in Lemma 5.14.2. □

08ZU Theorem 5.14.4 (Tychonov). A product of quasi-compact spaces is quasi-compact.

Proof. Let I be a set and for $i \in I$ let X_i be a quasi-compact topological space. Set $X = \prod X_i$. Let $\mathcal{B}$ be the set of subsets of X of the form $U_i \times \prod_{i' \in I, i' \neq i} X_{i'}$ where $U_i \subset X_i$ is open. By construction this family is a subbasis for the topology on X. By Lemma 5.12.15 it suffices to show that any covering $X = \bigcup_{j \in J} B_j$ by elements B_j of $\mathcal{B}$ has a finite refinement. We can decompose $J = \coprod J_i$ so that if $j \in J_i$, then $B_j = U_j \times \prod_{i' \neq i} X_{i'}$ with $U_j \subset X_i$ open. If $X_i = \bigcup_{j \in J_i} U_j$, then there is a finite refinement and we conclude that $X = \bigcup_{j \in J} B_j$ has a finite refinement. If this is not the case, then for every i we can choose an point $x_i \in X_i$ which is not in $\bigcup_{j \in J_i} U_j$. But then the point $x = (x_i)_{i \in I}$ is an element of X not contained in $\bigcup_{j \in J} B_j$, a contradiction. □

The following lemma does not hold if one drops the assumption that the spaces X_i are Hausdorff, see Examples, Section 110.4.

08ZV **Lemma 5.14.5.** Let $\mathcal{I}$ be a category and let $i \mapsto X_i$ be a diagram over $\mathcal{I}$ in the category of topological spaces. If each X_i is quasi-compact and Hausdorff, then $\lim X_i$ is quasi-compact.

Proof. Recall that $\lim X_i$ is a subspace of $\prod X_i$. By Theorem 5.14.4 this product is quasi-compact. Hence it suffices to show that $\lim X_i$ is a closed subspace of $\prod X_i$ (Lemma 5.12.3). If $\varphi : j \to k$ is a morphism of $\mathcal{I}$, then let $\Gamma_\varphi \subset X_j \times X_k$ denote the graph of the corresponding continuous map $X_j \to X_k$. By Lemma 5.3.2 this graph is closed. It is clear that $\lim X_i$ is the intersection of the closed subsets

$$\Gamma_\varphi \times \prod\nolimits_{l \neq j,k} X_l \subset \prod X_i$$

Thus the result follows. □

The following lemma generalizes Categories, Lemma 4.21.7 and partially generalizes Lemma 5.12.6.

0A2R **Lemma 5.14.6.** Let $\mathcal{I}$ be a cofiltered category and let $i \mapsto X_i$ be a diagram over $\mathcal{I}$ in the category of topological spaces. If each X_i is quasi-compact, Hausdorff, and nonempty, then $\lim X_i$ is nonempty.

Proof. In the proof of Lemma 5.14.5 we have seen that $X = \lim X_i$ is the intersection of the closed subsets

$$Z_\varphi = \Gamma_\varphi \times \prod\nolimits_{l \neq j,k} X_l$$

inside the quasi-compact space $\prod X_i$ where $\varphi : j \to k$ is a morphism of $\mathcal{I}$ and $\Gamma_\varphi \subset X_j \times X_k$ is the graph of the corresponding morphism $X_j \to X_k$. Hence by Lemma 5.12.6 it suffices to show any finite intersection of these subsets is nonempty. Assume $\varphi_t : j_t \to k_t$, $t = 1, \ldots, n$ is a finite collection of morphisms of $\mathcal{I}$. Since $\mathcal{I}$ is cofiltered, we can pick an object j and a morphism $\psi_t : j \to j_t$ for each t. For each pair t, t' such that either (a) $j_t = j_{t'}$, or (b) $j_t = k_{t'}$, or (c) $k_t = k_{t'}$ we obtain two morphisms $j \to l$ with $l = j_t$ in case (a), (b) or $l = k_t$ in case (c). Because $\mathcal{I}$ is cofiltered and since there are finitely many pairs (t, t') we may choose a map $j' \to j$ which equalizes these two morphisms for all such pairs (t, t'). Pick an element $x \in X_{j'}$ and for each t let x_{j_t}, resp. x_{k_t} be the image of x under the morphism $X_{j'} \to X_j \to X_{j_t}$, resp. $X_{j'} \to X_j \to X_{j_t} \to X_{k_t}$. For any index $l \in \mathrm{Ob}(\mathcal{I})$ which is not equal to j_t or k_t for some t we pick an arbitrary element $x_l \in X_l$ (using the axiom of choice). Then $(x_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ is in the intersection

$$Z_{\varphi_1} \cap \ldots \cap Z_{\varphi_n}$$

by construction and the proof is complete. □

5.15. Constructible sets

04ZC
005G **Definition 5.15.1.** Let X be a topological space. Let $E \subset X$ be a subset of X.

(1) We say E is constructible[3] in X if E is a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.

[3]In the second edition of EGA I [GD71] this was called a "globally constructible" set and a the terminology "constructible" was used for what we call a locally constructible set.

(2) We say E is locally constructible in X if there exists an open covering $X = \bigcup V_i$ such that each $E \cap V_i$ is constructible in V_i.

005H Lemma 5.15.2. The collection of constructible sets is closed under finite intersections, finite unions and complements.

Proof. Note that if U_1, U_2 are open and retrocompact in X then so is $U_1 \cup U_2$ because the union of two quasi-compact subsets of X is quasi-compact. It is also true that $U_1 \cap U_2$ is retrocompact. Namely, suppose $U \subset X$ is quasi-compact open, then $U_2 \cap U$ is quasi-compact because U_2 is retrocompact in X, and then we conclude $U_1 \cap (U_2 \cap U)$ is quasi-compact because U_1 is retrocompact in X. From this it is formal to show that the complement of a constructible set is constructible, that finite unions of constructibles are constructible, and that finite intersections of constructibles are constructible. $\square$

005I Lemma 5.15.3. Let $f : X \to Y$ be a continuous map of topological spaces. If the inverse image of every retrocompact open subset of Y is retrocompact in X, then inverse images of constructible sets are constructible.

Proof. This is true because $f^{-1}(U \cap V^c) = f^{-1}(U) \cap f^{-1}(V)^c$, combined with the definition of constructible sets. $\square$

005J Lemma 5.15.4. Let $U \subset X$ be open. For a constructible set $E \subset X$ the intersection $E \cap U$ is constructible in U.

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap U$ is retrocompact in U by Lemma 5.15.3. To show this let $W \subset U$ be open and quasi-compact. Then W is open and quasi-compact in X. Hence $V \cap W = V \cap U \cap W$ is quasi-compact as V is retrocompact in X. $\square$

09YD Lemma 5.15.5. Let $U \subset X$ be a retrocompact open. Let $E \subset U$. If E is constructible in U, then E is constructible in X.

Proof. Suppose that $V, W \subset U$ are retrocompact open in U. Then V, W are retrocompact open in X (Lemma 5.12.2). Hence $V \cap (U \setminus W) = V \cap (X \setminus W)$ is constructible in X. We conclude since every constructible subset of U is a finite union of subsets of the form $V \cap (U \setminus W)$. $\square$

053W Lemma 5.15.6. Let X be a topological space. Let $E \subset X$ be a subset. Let $X = V_1 \cup \ldots \cup V_m$ be a finite covering by retrocompact opens. Then E is constructible in X if and only if $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$.

Proof. If E is constructible in X, then by Lemma 5.15.4 we see that $E \cap V_j$ is constructible in V_j for all j. Conversely, suppose that $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$. Then $E = \bigcup E \cap V_j$ is a finite union of constructible sets by Lemma 5.15.5 and hence constructible. $\square$

09YE Lemma 5.15.7. Let X be a topological space. Let $Z \subset X$ be a closed subset such that $X \setminus Z$ is quasi-compact. Then for a constructible set $E \subset X$ the intersection $E \cap Z$ is constructible in Z.

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap Z$ is retrocompact in Z by Lemma 5.15.3. To show this let $W \subset Z$ be open and quasi-compact. The subset $W' = W \cup (X \setminus Z)$ is quasi-compact, open, and $W = Z \cap W'$. Hence $V \cap Z \cap W = V \cap Z \cap W'$ is a closed subset of the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap Z \cap W$ is quasi-compact by Lemma 5.12.3. $\square$

09YF Lemma 5.15.8. Let X be a topological space. Let $T \subset X$ be a subset. Suppose

(1) T is retrocompact in X,
(2) quasi-compact opens form a basis for the topology on X.

Then for a constructible set $E \subset X$ the intersection $E \cap T$ is constructible in T.

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap T$ is retrocompact in T by Lemma 5.15.3. To show this let $W \subset T$ be open and quasi-compact. By assumption (2) we can find a quasi-compact open $W' \subset X$ such that $W = T \cap W'$ (details omitted). Hence $V \cap T \cap W = V \cap T \cap W'$ is the intersection of T with the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap T \cap W$ is quasi-compact. $\square$

09YG Lemma 5.15.9. Let $Z \subset X$ be a closed subset whose complement is retrocompact open. Let $E \subset Z$. If E is constructible in Z, then E is constructible in X.

Proof. Suppose that $V \subset Z$ is retrocompact open in Z. Consider the open subset $\tilde{V} = V \cup (X \setminus Z)$ of X. Let $W \subset X$ be quasi-compact open. Then

$$W \cap \tilde{V} = (V \cap W) \cup ((X \setminus Z) \cap W).$$

The first part is quasi-compact as $V \cap W = V \cap (Z \cap W)$ and $(Z \cap W)$ is quasi-compact open in Z (Lemma 5.12.3) and V is retrocompact in Z. The second part is quasi-compact as $(X \setminus Z)$ is retrocompact in X. In this way we see that $\tilde{V}$ is retrocompact in X. Thus if $V_1, V_2 \subset Z$ are retrocompact open, then

$$V_1 \cap (Z \setminus V_2) = \tilde{V}_1 \cap (X \setminus \tilde{V}_2)$$

is constructible in X. We conclude since every constructible subset of Z is a finite union of subsets of the form $V_1 \cap (Z \setminus V_2)$. $\square$

09YH Lemma 5.15.10. Let X be a topological space. Every constructible subset of X is retrocompact.

Proof. Let $E = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c$ with U_i, V_i retrocompact open in X. Let $W \subset X$ be quasi-compact open. Then $E \cap W = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c \cap W$. Thus it suffices to show that $U \cap V^c \cap W$ is quasi-compact if U, V are retrocompact open and W is quasi-compact open. This is true because $U \cap V^c \cap W$ is a closed subset of the quasi-compact $U \cap W$ so Lemma 5.12.3 applies. $\square$

Question: Does the following lemma also hold if we assume X is a quasi-compact topological space? Compare with Lemma 5.15.7.

09YI Lemma 5.15.11. Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. For E, E' constructible in X, the intersection $E \cap E'$ is constructible in E.

Proof. Combine Lemmas 5.15.8 and 5.15.10. $\square$

09YJ Lemma 5.15.12. Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. Let E be constructible in X and $F \subset E$ constructible in E. Then F is constructible in X.

Proof. Observe that any retrocompact subset T of X has a basis for the induced topology consisting of quasi-compact opens. In particular this holds for any constructible subset (Lemma 5.15.10). Write $E = E_1 \cup \ldots \cup E_n$ with $E_i = U_i \cap V_i^c$ where $U_i, V_i \subset X$ are retrocompact open. Note that $E_i = E \cap E_i$ is constructible in E by Lemma 5.15.11. Hence $F \cap E_i$ is constructible in E_i by Lemma 5.15.11. Thus it suffices to prove the lemma in case $E = U \cap V^c$ where $U, V \subset X$ are retrocompact open. In this case the inclusion $E \subset X$ is a composition

$$E = U \cap V^c \to U \to X$$

Then we can apply Lemma 5.15.9 to the first inclusion and Lemma 5.15.5 to the second. □

0F2K Lemma 5.15.13. Let X be a quasi-compact topological space having a basis consisting of quasi-compact opens such that the intersection of any two quasi-compact opens is quasi-compact. Let $T \subset X$ be a locally closed subset such that T is quasi-compact and T^c is retrocompact in X. Then T is constructible in X.

Proof. Note that T is quasi-compact and open in $\overline{T}$. Using our basis of quasi-compact opens we can write $T = U \cap \overline{T}$ where U is quasi-compact open in X. Then $V = U \setminus T = U \cap T^c$ is retrocompact in U as T^c is retrocompact in X. Hence V is quasi-compact. Since the intersection of any two quasi-compact opens is quasi-compact any quasi-compact open of X is retrocompact. Thus $T = U \cap V^c$ with U and $V = U \setminus T$ retrocompact opens of X. A fortiori, T is constructible in X. □

09YK Lemma 5.15.14. Let X be a topological space which has a basis for the topology consisting of quasi-compact opens. Let $E \subset X$ be a subset. Let $X = E_1 \cup \ldots \cup E_m$ be a finite covering by constructible subsets. Then E is constructible in X if and only if $E \cap E_j$ is constructible in E_j for each $j = 1, \ldots, m$.

Proof. Combine Lemmas 5.15.11 and 5.15.12. □

005K Lemma 5.15.15. Let X be a topological space. Suppose that $Z \subset X$ is irreducible. Let $E \subset X$ be a finite union of locally closed subsets (e.g. E is constructible). The following are equivalent

(1) The intersection $E \cap Z$ contains an open dense subset of Z.
(2) The intersection $E \cap Z$ is dense in Z.

If Z has a generic point ξ, then this is also equivalent to

(3) We have $\xi \in E$.

Proof. The implication (1) $\Rightarrow$ (2) is clear. Assume (2). Note that $E \cap Z$ is a finite union of locally closed subsets Z_i of Z. Since Z is irreducible, one of the Z_i must be dense in Z. Then this Z_i is dense open in Z as it is open in its closure. Hence (1) holds.

Suppose that $\xi \in Z$ is a generic point. If the equivalent conditions (1) and (2) hold, then $\xi \in E$. Conversely, if $\xi \in E$ then $\xi \in E \cap Z$ and hence $E \cap Z$ is dense in Z. □

5.16. Constructible sets and Noetherian spaces

053X

005L Lemma 5.16.1. Let X be a Noetherian topological space. The constructible sets in X are precisely the finite unions of locally closed subsets of X.

Proof. This follows immediately from Lemma 5.12.13. ☐

053Y Lemma 5.16.2. Let $f : X \to Y$ be a continuous map of Noetherian topological spaces. If $E \subset Y$ is constructible in Y, then $f^{-1}(E)$ is constructible in X.

Proof. Follows immediately from Lemma 5.16.1 and the definition of a continuous map. ☐

053Z Lemma 5.16.3. Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent:

(1) E is constructible in X, and
(2) for every irreducible closed $Z \subset X$ the intersection $E \cap Z$ either contains a nonempty open of Z or is not dense in Z.

Proof. Assume E is constructible and $Z \subset X$ irreducible closed. Then $E \cap Z$ is constructible in Z by Lemma 5.16.2. Hence $E \cap Z$ is a finite union of nonempty locally closed subsets T_i of Z. Clearly if none of the T_i is open in Z, then $E \cap Z$ is not dense in Z. In this way we see that (1) implies (2).

Conversely, assume (2) holds. Consider the set $\mathcal{S}$ of closed subsets Y of X such that $E \cap Y$ is not constructible in Y. If $\mathcal{S} \neq \emptyset$, then it has a smallest element Y as X is Noetherian. Let $Y = Y_1 \cup \ldots \cup Y_r$ be the decomposition of Y into its irreducible components, see Lemma 5.9.2. If $r > 1$, then each $Y_i \cap E$ is constructible in Y_i and hence a finite union of locally closed subsets of Y_i. Thus $E \cap Y$ is a finite union of locally closed subsets of Y too and we conclude that $E \cap Y$ is constructible in Y by Lemma 5.16.1. This is a contradiction and so $r = 1$. If $r = 1$, then Y is irreducible, and by assumption (2) we see that $E \cap Y$ either (a) contains an open V of Y or (b) is not dense in Y. In case (a) we see, by minimality of Y, that $E \cap (Y \setminus V)$ is a finite union of locally closed subsets of $Y \setminus V$. Thus $E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 5.16.1. This is a contradiction and so we must be in case (b). In case (b) we see that $E \cap Y = E \cap Y'$ for some proper closed subset $Y' \subset Y$. By minimality of Y we see that $E \cap Y'$ is a finite union of locally closed subsets of Y' and we see that $E \cap Y' = E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 5.16.1. This contradiction finishes the proof of the lemma. ☐

0540 Lemma 5.16.4. Let X be a Noetherian topological space. Let $x \in X$. Let $E \subset X$ be constructible in X. The following are equivalent:

(1) E is a neighbourhood of x, and
(2) for every irreducible closed subset Y of X which contains x the intersection $E \cap Y$ is dense in Y.

Proof. It is clear that (1) implies (2). Assume (2). Consider the set $\mathcal{S}$ of closed subsets Y of X containing x such that $E \cap Y$ is not a neighbourhood of x in Y. If $\mathcal{S} \neq \emptyset$, then it has a minimal element Y as X is Noetherian. Suppose $Y = Y_1 \cup Y_2$ with two smaller nonempty closed subsets Y_1, Y_2. If $x \in Y_i$ for $i = 1, 2$, then $Y_i \cap E$ is a neighbourhood of x in Y_i and we conclude $Y \cap E$ is a neighbourhood of x in Y which is a contradiction. If $x \in Y_1$ but $x \notin Y_2$ (say), then $Y_1 \cap E$ is a neighbourhood of x in Y_1 and hence also in Y, which is a contradiction as well. We conclude that Y is irreducible closed. By assumption (2) we see that $E \cap Y$ is dense in Y. Thus $E \cap Y$ contains an open V of Y, see Lemma 5.16.3. If $x \in V$ then $E \cap Y$ is a neighbourhood of x in Y which is a contradiction. If $x \notin V$, then

$Y' = Y \setminus V$ is a proper closed subset of Y containing x. By minimality of Y we see that $E \cap Y'$ contains an open neighbourhood $V' \subset Y'$ of x in Y'. But then $V' \cup V$ is an open neighbourhood of x in Y contained in E, a contradiction. This contradiction finishes the proof of the lemma. ☐

0541 Lemma 5.16.5. Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent:

(1) E is open in X, and
(2) for every irreducible closed subset Y of X the intersection $E \cap Y$ is either empty or contains a nonempty open of Y.

Proof. This follows formally from Lemmas 5.16.3 and 5.16.4. ☐

5.17. Characterizing proper maps

005M We include a section discussing the notion of a proper map in usual topology. We define a continuous map of topological spaces to be proper if it is universally closed and separated. Although this matches well with the definition of a proper morphism in algebraic geometry, this is different from the definition in Bourbaki. With our definition of a proper map of topological spaces, the proper base change theorem (Cohomology, Theorem 20.18.2) holds without any further assumptions. Furthermore, given a morphism $f : X \to Y$ of finite type schemes over $\mathbf{C}$ one has: f is proper as a morphism of schemes if and only if the continuous map $f : X(\mathbf{C}) \to Y(\mathbf{C})$ on $\mathbf{C}$-points with the classical topology is proper. This is explained in [Gro71, Exp. XII, Prop. 3.2(v)] which also has a footnote pointing out that they take properness in topology to be Bourbaki's notion with separatedness added on.

We find it useful to have names for three distinct concepts: separated, universally closed, and both of those together (i.e., properness). For a continuous map $f : X \to Y$ of locally compact Hausdorff spaces the word "proper" has long been used for the notion "f^{-1}(compact) = compact" and this is equivalent to universal closedness for such nice spaces. In fact, we will see the preimage condition formulated for clarity using the word "quasi-compact" is equivalent to universal closedness in general, if one includes the assumption of the map being closed. See also [Lan93, Exercises 22-26 in Chapter II] but beware that Lang uses "proper" as a synonym for "universally closed", like Bourbaki does.

005N Lemma 5.17.1 (Tube lemma). Let X and Y be topological spaces. Let $A \subset X$ and $B \subset Y$ be quasi-compact subsets. Let $A \times B \subset W \subset X \times Y$ with W open in $X \times Y$. Then there exists opens $A \subset U \subset X$ and $B \subset V \subset Y$ such that $U \times V \subset W$.

Proof. For every $a \in A$ and $b \in B$ there exist opens $U_{(a,b)}$ of X and $V_{(a,b)}$ of Y such that $(a, b) \in U_{(a,b)} \times V_{(a,b)} \subset W$. Fix b and we see there exist a finite number $a_1, \ldots, a_n$ such that $A \subset U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}$. Hence

$$A \times \{b\} \subset (U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}) \times (V_{(a_1,b)} \cap \ldots \cap V_{(a_n,b)}) \subset W.$$

Thus for every $b \in B$ there exists opens $U_b \subset X$ and $V_b \subset Y$ such that $A \times \{b\} \subset U_b \times V_b \subset W$. As above there exist a finite number $b_1, \ldots, b_m$ such that $B \subset V_{b_1} \cup \ldots \cup V_{b_m}$. Then we win because $A \times B \subset (U_{b_1} \cap \ldots \cap U_{b_m}) \times (V_{b_1} \cup \ldots \cup V_{b_m})$. ☐

The notation in the following definition may be slightly different from what you are used to.

005O Definition 5.17.2. Let $f : X \to Y$ be a continuous map between topological spaces.

(1) We say that the map f is closed if the image of every closed subset is closed.
(2) We say that the map f is Bourbaki-proper[4] if the map $Z \times X \to Z \times Y$ is closed for any topological space Z.
(3) We say that the map f is quasi-proper if the inverse image $f^{-1}(V)$ of every quasi-compact subset $V \subset Y$ is quasi-compact.
(4) We say that f is universally closed if the map $f' : Z \times_Y X \to Z$ is closed for any continuous map $g : Z \to Y$.
(5) We say that f is proper if f is separated and universally closed.

The following lemma is useful later.

005P Lemma 5.17.3. A topological space X is quasi-compact if and only if the projection map $Z \times X \to Z$ is closed for any topological space Z.

Combination of [Bou71, I, p. 75, Lemme 1] and [Bou71, I, p. 76, Corrolaire 1].

Proof. (See also remark below.) If X is not quasi-compact, there exists an open covering $X = \bigcup_{i \in I} U_i$ such that no finite number of U_i cover X. Let Z be the subset of the power set $\mathcal{P}(I)$ of I consisting of I and all nonempty finite subsets of I. Define a topology on Z with as a basis for the topology the following sets:

(1) All subsets of $Z \setminus \{I\}$.
(2) For every finite subset K of I the set $U_K := \{J \subset I \mid J \in Z,\ K \subset J\})$.

It is left to the reader to verify this is the basis for a topology. Consider the subset of $Z \times X$ defined by the formula

$$M = \{(J, x) \mid J \in Z,\ x \in \bigcap\nolimits_{i \in J} U_i^c)\}$$

If $(J, x) \notin M$, then $x \in U_i$ for some $i \in J$. Hence $U_{\{i\}} \times U_i \subset Z \times X$ is an open subset containing (J, x) and not intersecting M. Hence M is closed. The projection of M to Z is $Z - \{I\}$ which is not closed. Hence $Z \times X \to Z$ is not closed.

Assume X is quasi-compact. Let Z be a topological space. Let $M \subset Z \times X$ be closed. Let $z \in Z$ be a point which is not in $\text{pr}_1(M)$. By the Tube Lemma 5.17.1 there exists an open $U \subset Z$ such that $U \times X$ is contained in the complement of M. Hence $\text{pr}_1(M)$ is closed. □

005Q Remark 5.17.4. Lemma 5.17.3 is a combination of [Bou71, I, p. 75, Lemme 1] and [Bou71, I, p. 76, Corollaire 1].

005R Theorem 5.17.5. Let $f : X \to Y$ be a continuous map between topological spaces. The following conditions are equivalent:

(1) The map f is quasi-proper and closed.
(2) The map f is Bourbaki-proper.
(3) The map f is universally closed.
(4) The map f is closed and $f^{-1}(y)$ is quasi-compact for any $y \in Y$.

In [Bou71, I, p. 75, Theorem 1] you can find: (2) ⇔ (4). In [Bou71, I, p. 77, Proposition 6] you can find: (2) ⇒ (1).

Proof. (See also the remark below.) If the map f satisfies (1), it automatically satisfies (4) because any single point is quasi-compact.

[4]This is the terminology used in [Bou71]. Sometimes this property may be called "universally closed" in the literature.

Assume map f satisfies (4). We will prove it is universally closed, i.e., (3) holds. Let $g : Z \to Y$ be a continuous map of topological spaces and consider the diagram

$$\begin{tikzcd} Z \times_Y X \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

During the proof we will use that $Z \times_Y X \to Z \times X$ is a homeomorphism onto its image, i.e., that we may identify $Z \times_Y X$ with the corresponding subset of $Z \times X$ with the induced topology. The image of $f' : Z \times_Y X \to Z$ is $\mathrm{Im}(f') = \{z : g(z) \in f(X)\}$. Because $f(X)$ is closed, we see that $\mathrm{Im}(f')$ is a closed subspace of Z. Consider a closed subset $P \subset Z \times_Y X$. Let $z \in Z$, $z \notin f'(P)$. If $z \notin \mathrm{Im}(f')$, then $Z \setminus \mathrm{Im}(f')$ is an open neighbourhood which avoids $f'(P)$. If z is in $\mathrm{Im}(f')$ then $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\}$ and $f^{-1}\{g(z)\}$ is quasi-compact by assumption. Because P is a closed subset of $Z \times_Y X$, we have a closed P' of $Z \times X$ such that $P = P' \cap Z \times_Y X$. Since $(f')^{-1}\{z\}$ is a subset of $P^c = P'^c \cup (Z \times_Y X)^c$, and since $(f')^{-1}\{z\}$ is disjoint from $(Z \times_Y X)^c$ we see that $(f')^{-1}\{z\}$ is contained in P'^c. We may apply the Tube Lemma 5.17.1 to $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\} \subset (P')^c \subset Z \times X$. This gives $V \times U$ containing $(f')^{-1}\{z\}$ where U and V are open sets in X and Z respectively and $V \times U$ has empty intersection with P'. Then the set $V \cap g^{-1}(Y - f(U^c))$ is open in Z since f is closed, contains z, and has empty intersection with the image of P. Thus $f'(P)$ is closed. In other words, the map f is universally closed.

The implication (3) $\Rightarrow$ (2) is trivial. Namely, given any topological space Z consider the projection morphism $g : Z \times Y \to Y$. Then it is easy to see that f' is the map $Z \times X \to Z \times Y$, in other words that $(Z \times Y) \times_Y X = Z \times X$. (This identification is a purely categorical property having nothing to do with topological spaces per se.)

Assume f satisfies (2). We will prove it satisfies (1). Note that f is closed as f can be identified with the map $\{pt\} \times X \to \{pt\} \times Y$ which is assumed closed. Choose any quasi-compact subset $K \subset Y$. Let Z be any topological space. Because $Z \times X \to Z \times Y$ is closed we see the map $Z \times f^{-1}(K) \to Z \times K$ is closed (if T is closed in $Z \times f^{-1}(K)$, write $T = Z \times f^{-1}(K) \cap T'$ for some closed $T' \subset Z \times X$). Because K is quasi-compact, $K \times Z \to Z$ is closed by Lemma 5.17.3. Hence the composition $Z \times f^{-1}(K) \to Z \times K \to Z$ is closed and therefore $f^{-1}(K)$ must be quasi-compact by Lemma 5.17.3 again. $\square$

005S Remark 5.17.6. Here are some references to the literature. In [Bou71, I, p. 75, Theorem 1] you can find: (2) $\Leftrightarrow$ (4). In [Bou71, I, p. 77, Proposition 6] you can find: (2) $\Rightarrow$ (1). Of course, trivially we have (1) $\Rightarrow$ (4). Thus (1), (2) and (4) are equivalent. The equivalence of (3) and (4) is [Lan93, Chapter II, Exercise 25].

08YD Lemma 5.17.7. Let $f : X \to Y$ be a continuous map of topological spaces. If X is quasi-compact and Y is Hausdorff, then f is universally closed.

Proof. Since every point of Y is closed, we see from Lemma 5.12.3 that the closed subset $f^{-1}(y)$ of X is quasi-compact for all $y \in Y$. Thus, by Theorem 5.17.5 it suffices to show that f is closed. If $E \subset X$ is closed, then it is quasi-compact (Lemma 5.12.3), hence $f(E) \subset Y$ is quasi-compact (Lemma 5.12.7), hence $f(E)$ is closed in Y (Lemma 5.12.4). $\square$

08YE Lemma 5.17.8. Let $f : X \to Y$ be a continuous map of topological spaces. If f is bijective, X is quasi-compact, and Y is Hausdorff, then f is a homeomorphism.

Proof. It suffices to prove f is closed, because this implies that f^{-1} is continuous. If $T \subset X$ is closed, then T is quasi-compact by Lemma 5.12.3, hence $f(T)$ is quasi-compact by Lemma 5.12.7, hence $f(T)$ is closed by Lemma 5.12.4. $\square$

5.18. Jacobson spaces

005T

005U Definition 5.18.1. Let X be a topological space. Let X_0 be the set of closed points of X. We say that X is Jacobson if every closed subset $Z \subset X$ is the closure of $Z \cap X_0$.

Note that a topological space X is Jacobson if and only if every nonempty locally closed subset of X has a point closed in X.

Let X be a Jacobson space and let X_0 be the set of closed points of X with the induced topology. Clearly, the definition implies that the morphism $X_0 \to X$ induces a bijection between the closed subsets of X_0 and the closed subsets of X. Thus many properties of X are inherited by X_0. For example, the Krull dimensions of X and X_0 are the same.

005V Lemma 5.18.2. Let X be a topological space. Let X_0 be the set of closed points of X. Suppose that for every point $x \in X$ the intersection $X_0 \cap \overline{\{x\}}$ is dense in $\overline{\{x\}}$. Then X is Jacobson.

Proof. Let Z be closed subset of X and U be and open subset of X such that $U \cap Z$ is nonempty. Then for $x \in U \cap Z$ we have that $\overline{\{x\}} \cap U$ is a nonempty subset of $Z \cap U$, and by hypothesis it contains a point closed in X as required. $\square$

02I7 Lemma 5.18.3. Let X be a Kolmogorov topological space with a basis of quasi-compact open sets. If X is not Jacobson, then there exists a non-closed point $x \in X$ such that $\{x\}$ is locally closed.

Proof. As X is not Jacobson there exists a closed set Z and an open set U in X such that $Z \cap U$ is nonempty and does not contain points closed in X. As X has a basis of quasi-compact open sets we may replace U by an open quasi-compact neighborhood of a point in $Z \cap U$ and so we may assume that U is quasi-compact open. By Lemma 5.12.8, there exists a point $x \in Z \cap U$ closed in $Z \cap U$, and so $\{x\}$ is locally closed but not closed in X. $\square$

005W Lemma 5.18.4. Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Then X is Jacobson if and only if each U_i is Jacobson. Moreover, in this case $X_0 = \bigcup U_{i,0}$.

Proof. Let X be a topological space. Let X_0 be the set of closed points of X. Let $U_{i,0}$ be the set of closed points of U_i. Then $X_0 \cap U_i \subset U_{i,0}$ but equality may not hold in general.

First, assume that each U_i is Jacobson. We claim that in this case $X_0 \cap U_i = U_{i,0}$. Namely, suppose that $x \in U_{i,0}$, i.e., x is closed in U_i. Let $\overline{\{x\}}$ be the closure in X. Consider $\overline{\{x\}} \cap U_j$. If $x \notin U_j$, then $\overline{\{x\}} \cap U_j = \emptyset$. If $x \in U_j$, then $U_i \cap U_j \subset U_j$ is an open subset of U_j containing x. Let $T' = U_j \setminus U_i \cap U_j$ and $T = \{x\} \amalg T'$. Then

T, T' are closed subsets of U_j and T contains x. As U_j is Jacobson we see that the closed points of U_j are dense in T. Because $T = \{x\} \amalg T'$ this can only be the case if x is closed in U_j. Hence $\overline{\{x\}} \cap U_j = \{x\}$. We conclude that $\overline{\{x\}} = \{x\}$ as desired.

Let $Z \subset X$ be a closed subset (still assuming each U_i is Jacobson). Since now we know that $X_0 \cap Z \cap U_i = U_{i,0} \cap Z$ are dense in $Z \cap U_i$ it follows immediately that $X_0 \cap Z$ is dense in Z.

Conversely, assume that X is Jacobson. Let $Z \subset U_i$ be closed. Then $X_0 \cap \overline{Z}$ is dense in $\overline{Z}$. Hence also $X_0 \cap Z$ is dense in Z, because $\overline{Z} \setminus Z$ is closed. As $X_0 \cap U_i \subset U_{i,0}$ we see that $U_{i,0} \cap Z$ is dense in Z. Thus U_i is Jacobson as desired. □

005X Lemma 5.18.5. Let X be Jacobson. The following types of subsets $T \subset X$ are Jacobson:

(1) Open subspaces.
(2) Closed subspaces.
(3) Locally closed subspaces.
(4) Unions of locally closed subspaces.
(5) Constructible sets.
(6) Any subset $T \subset X$ which locally on X is a union of locally closed subsets.

In each of these cases closed points of T are closed in X.

Proof. Let X_0 be the set of closed points of X. For any subset $T \subset X$ we let $(*)$ denote the property:

(*) Every nonempty locally closed subset of T has a point closed in X.

Note that always $X_0 \cap T \subset T_0$. Hence property $(*)$ implies that T is Jacobson. In addition it clearly implies that every closed point of T is closed in X.

Suppose that $T = \bigcup_i T_i$ with T_i locally closed in X. Take $A \subset T$ a locally closed nonempty subset in T, then there exists a T_i such that $A \cap T_i$ is nonempty, it is locally closed in T_i and so in X. As X is Jacobson A has a point closed in X. □

07JU Lemma 5.18.6. A finite Jacobson space is discrete.

Proof. If X is finite Jacobson, $X_0 \subset X$ the subset of closed points, then, on the one hand, $\overline{X_0} = X$. On the other hand, X, and hence X_0 is finite, so $X_0 = \{x_1, \ldots, x_n\} = \bigcup_{i=1,\ldots,n}\{x_i\}$ is a finite union of closed sets, hence closed, so $X = \overline{X_0} = X_0$. Every point is closed, and by finiteness, every point is open. □

005Z Lemma 5.18.7. Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence

$$\{\text{finite unions loc. closed subsets of } X\} \leftrightarrow \{\text{finite unions loc. closed subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subsets of locally closed, of open and of closed subsets.

Proof. We just prove that the correspondence $E \mapsto E \cap X_0$ is injective. Indeed if $E \neq E'$ then without loss of generality $E \setminus E'$ is nonempty, and it is a finite union of locally closed sets (details omitted). As X is Jacobson, we see that $(E \setminus E') \cap X_0 = E \cap X_0 \setminus E' \cap X_0$ is not empty. □

005Y Lemma 5.18.8. Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence

$$\{\text{constructible subsets of } X\} \leftrightarrow \{\text{constructible subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subset of retrocompact open subsets, as well as complements of these.

Proof. From Lemma 5.18.7 above, we just have to see that if U is open in X then $U \cap X_0$ is retrocompact in X_0 if and only if U is retrocompact in X. This follows if we prove that for U open in X then $U \cap X_0$ is quasi-compact if and only if U is quasi-compact. From Lemma 5.18.5 it follows that we may replace X by U and assume that $U = X$. Finally notice that any collection of opens $\mathcal{U}$ of X cover X if and only if they cover X_0, using the Jacobson property of X in the closed $X \setminus \bigcup \mathcal{U}$ to find a point in X_0 if it were nonempty. $\square$

5.19. Specialization

0060

0061 Definition 5.19.1. Let X be a topological space.

(1) If $x, x' \in X$ then we say x is a specialization of x', or x' is a generalization of x if $x \in \overline{\{x'\}}$. Notation: $x' \rightsquigarrow x$.
(2) A subset $T \subset X$ is stable under specialization if for all $x' \in T$ and every specialization $x' \rightsquigarrow x$ we have $x \in T$.
(3) A subset $T \subset X$ is stable under generalization if for all $x \in T$ and every generalization $x' \rightsquigarrow x$ we have $x' \in T$.

0062 Lemma 5.19.2. Let X be a topological space.

(1) Any closed subset of X is stable under specialization.
(2) Any open subset of X is stable under generalization.
(3) A subset $T \subset X$ is stable under specialization if and only if the complement T^c is stable under generalization.

Proof. Let F be a closed subset of X, if $y \in F$ then $\{y\} \subset F$, so $\overline{\{y\}} \subset \overline{F} = F$ as F is closed. Thus for all specialization x of y, we have $x \in F$.

Let $x, y \in X$ such that $x \in \overline{\{y\}}$ and let T be a subset of X. Saying that T is stable under specialization means that $y \in T$ implies $x \in T$ and reciprocally saying that T is stable under generalization means that $x \in T$ implies $y \in T$. Therefore (3) is proven using contraposition.

The second property follows from (1) and (3) by considering the complement. $\square$

0EES Lemma 5.19.3. Let $T \subset X$ be a subset of a topological space X. The following are equivalent

(1) T is stable under specialization, and
(2) T is a (directed) union of closed subsets of X.

Proof. Suppose that T is stable under specialization, then for all $y \in T$ we have $\overline{\{y\}} \subset T$. Thus $T = \bigcup_{y \in T} \overline{\{y\}}$ which is an union of closed subsets of X. Reciprocally, suppose that $T = \bigcup_{i \in I} F_i$ where F_i are closed subsets of X. If $y \in T$ then there exists $i \in I$ such that $y \in F_i$. As F_i is closed, we have $\overline{\{y\}} \subset F_i \subset T$, which proves that T is stable under specialization. $\square$

0063 Definition 5.19.4. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say that specializations lift along f or that f is specializing if given $y' \leadsto y$ in Y and any $x' \in X$ with $f(x') = y'$ there exists a specialization $x' \leadsto x$ of x' in X such that $f(x) = y$.
(2) We say that generalizations lift along f or that f is generalizing if given $y' \leadsto y$ in Y and any $x \in X$ with $f(x) = y$ there exists a generalization $x' \leadsto x$ of x in X such that $f(x') = y'$.

0064 Lemma 5.19.5. Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. If specializations lift along both f and g then specializations lift along $g \circ f$. Similarly for "generalizations lift along".

Proof. Let $z' \leadsto z$ be a specialization in Z and let $x' \in X$ such as $g \circ f(x') = z'$. Then because specializations lift along g, there exists a specialization $f(x') \leadsto y$ of $f(x')$ in Y such that $g(y) = z$. Likewise, because specializations lift along f, there exists a specialization $x' \leadsto x$ of x' in X such that $f(x) = y$. It provides a specialization $x' \leadsto x$ of x' in X such that $g \circ f(x) = z$. In other words, specialization lift along $g \circ f$. □

0065 Lemma 5.19.6. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) If specializations lift along f, and if $T \subset X$ is stable under specialization, then $f(T) \subset Y$ is stable under specialization.
(2) If generalizations lift along f, and if $T \subset X$ is stable under generalization, then $f(T) \subset Y$ is stable under generalization.

Proof. Let $y' \leadsto y$ be a specialization in Y where $y' \in f(T)$ and let $x' \in T$ such that $f(x') = y'$. Because specialization lift along f, there exists a specialization $x' \leadsto x$ of x' in X such that $f(x) = y$. But T is stable under specialization so $x \in T$ and then $y \in f(T)$. Therefore $f(T)$ is stable under specialization.

The proof of (2) is identical, using that generalizations lift along f. □

0066 Lemma 5.19.7. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) If f is closed then specializations lift along f.
(2) If f is open, X is a Noetherian topological space, each irreducible closed subset of X has a generic point, and Y is Kolmogorov then generalizations lift along f.

Proof. Assume f is closed. Let $y' \leadsto y$ in Y and any $x' \in X$ with $f(x') = y'$ be given. Consider the closed subset $T = \overline{\{x'\}}$ of X. Then $f(T) \subset Y$ is a closed subset, and $y' \in f(T)$. Hence also $y \in f(T)$. Hence $y = f(x)$ with $x \in T$, i.e., $x' \leadsto x$.

Assume f is open, X Noetherian, every irreducible closed subset of X has a generic point, and Y is Kolmogorov. Let $y' \leadsto y$ in Y and any $x \in X$ with $f(x) = y$ be given. Consider $T = f^{-1}(\{y'\}) \subset X$. Take an open neighbourhood $x \in U \subset X$ of x. Then $f(U) \subset Y$ is open and $y \in f(U)$. Hence also $y' \in f(U)$. In other words, $T \cap U \neq \emptyset$. This proves that $x \in \overline{T}$. Since X is Noetherian, T is Noetherian (Lemma 5.9.2). Hence it has a decomposition $T = T_1 \cup \ldots \cup T_n$ into irreducible components. Then correspondingly $\overline{T} = \overline{T_1} \cup \ldots \cup \overline{T_n}$. By the above $x \in \overline{T_i}$ for some i. By assumption there exists a generic point $x' \in \overline{T_i}$, and we see that $x' \leadsto x$. As $x' \in \overline{T}$ we see that $f(x') \in \overline{\{y'\}}$. Note that $f(\overline{T_i}) = f(\overline{\{x'\}}) \subset \overline{\{f(x')\}}$. If $f(x') \neq y'$, then since Y is Kolmogorov $f(x')$ is not a generic point of the irreducible closed subset $\overline{\{y'\}}$ and the inclusion $\overline{\{f(x')\}} \subset \overline{\{y'\}}$ is strict, i.e., $y' \not\in f(\overline{T_i})$. This contradicts the fact that $f(T_i) = \{y'\}$. Hence $f(x') = y'$ and we win. $\square$

06NA Lemma 5.19.8. Suppose that $s, t : R \to U$ and $\pi : U \to X$ are continuous maps of topological spaces such that

(1) π is open,
(2) U is sober,
(3) s, t have finite fibres,
(4) generalizations lift along s, t,
(5) $(t, s)(R) \subset U \times U$ is an equivalence relation on U and X is the quotient of U by this equivalence relation (as a set).

Then X is Kolmogorov.

Proof. Properties (3) and (5) imply that a point x corresponds to an finite equivalence class $\{u_1, \ldots, u_n\} \subset U$ of the equivalence relation. Suppose that $x' \in X$ is a second point corresponding to the equivalence class $\{u'_1, \ldots, u'_m\} \subset U$. Suppose that $u_i \rightsquigarrow u'_j$ for some i, j. Then for any $r' \in R$ with $s(r') = u'_j$ by (4) we can find $r \rightsquigarrow r'$ with $s(r) = u_i$. Hence $t(r) \rightsquigarrow t(r')$. Since $\{u'_1, \ldots, u'_m\} = t(s^{-1}(\{u'_j\}))$ we conclude that every element of $\{u'_1, \ldots, u'_m\}$ is the specialization of an element of $\{u_1, \ldots, u_n\}$. Thus $\overline{\{u_1\}} \cup \ldots \cup \overline{\{u_n\}}$ is a union of equivalence classes, hence of the form $\pi^{-1}(Z)$ for some subset $Z \subset X$. By (1) we see that Z is closed in X and in fact $Z = \overline{\{x\}}$ because $\pi(\overline{\{u_i\}}) \subset \overline{\{x\}}$ for each i. In other words, $x \rightsquigarrow x'$ if and only if some lift of x in U specializes to some lift of x' in U, if and only if every lift of x' in U is a specialization of some lift of x in U.

Suppose that both $x \rightsquigarrow x'$ and $x' \rightsquigarrow x$. Say x corresponds to $\{u_1, \ldots, u_n\}$ and x' corresponds to $\{u'_1, \ldots, u'_m\}$ as above. Then, by the results of the preceding paragraph, we can find a sequence

$$\ldots \rightsquigarrow u'_{j_3} \rightsquigarrow u_{i_3} \rightsquigarrow u'_{j_2} \rightsquigarrow u_{i_2} \rightsquigarrow u'_{j_1} \rightsquigarrow u_{i_1}$$

which must repeat, hence by (2) we conclude that $\{u_1, \ldots, u_n\} = \{u'_1, \ldots, u'_m\}$, i.e., $x = x'$. Thus X is Kolmogorov. $\square$

02JF Lemma 5.19.9. Let $f : X \to Y$ be a morphism of topological spaces. Suppose that Y is a sober topological space, and f is surjective. If either specializations or generalizations lift along f, then $\dim(X) \geq \dim(Y)$.

Proof. Assume specializations lift along f. Let $Z_0 \subset Z_1 \subset \ldots Z_e \subset Y$ be a chain of irreducible closed subsets of X. Let $\xi_e \in X$ be a point mapping to the generic point of Z_e. By assumption there exists a specialization $\xi_e \rightsquigarrow \xi_{e-1}$ in X such that ξ_{e-1} maps to the generic point of Z_{e-1}. Continuing in this manner we find a sequence of specializations

$$\xi_e \rightsquigarrow \xi_{e-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

with ξ_i mapping to the generic point of Z_i. This clearly implies the sequence of irreducible closed subsets

$$\overline{\{\xi_0\}} \subset \overline{\{\xi_1\}} \subset \ldots \overline{\{\xi_e\}}$$

is a chain of length e in X. The case when generalizations lift along f is similar. $\square$

0542 Lemma 5.19.10. Let X be a Noetherian sober topological space. Let $E \subset X$ be a subset of X.

(1) If E is constructible and stable under specialization, then E is closed.
(2) If E is constructible and stable under generalization, then E is open.

Proof. Let E be constructible and stable under generalization. Let $Y \subset X$ be an irreducible closed subset with generic point $\xi \in Y$. If $E \cap Y$ is nonempty, then it contains ξ (by stability under generalization) and hence is dense in Y, hence it contains a nonempty open of Y, see Lemma 5.16.3. Thus E is open by Lemma 5.16.5. This proves (2). To prove (1) apply (2) to the complement of E in X. $\square$

5.20. Dimension functions

02I8 It scarcely makes sense to consider dimension functions unless the space considered is sober (Definition 5.8.6). Thus the definition below can be improved by considering the sober topological space associated to X. Since the underlying topological space of a scheme is sober we do not bother with this improvement.

02I9 **Definition 5.20.1.** Let X be a topological space.

(1) Let $x, y \in X$, $x \neq y$. Suppose $x \rightsquigarrow y$, that is y is a specialization of x. We say y is an *immediate specialization* of x if there is no $z \in X \setminus \{x, y\}$ with $x \rightsquigarrow z$ and $z \rightsquigarrow y$.
(2) A map $\delta : X \to \mathbf{Z}$ is called a *dimension function*[5] if
   (a) whenever $x \rightsquigarrow y$ and $x \neq y$ we have $\delta(x) > \delta(y)$, and
   (b) for every immediate specialization $x \rightsquigarrow y$ in X we have $\delta(x) = \delta(y) + 1$.

It is clear that if δ is a dimension function, then so is $\delta + t$ for any $t \in \mathbf{Z}$. Here is a fun lemma.

02IA **Lemma 5.20.2.** Let X be a topological space. If X is sober and has a dimension function, then X is catenary. Moreover, for any $x \rightsquigarrow y$ we have

$$\delta(x) - \delta(y) = \operatorname{codim}\left(\overline{\{y\}},\ \overline{\{x\}}\right).$$

Proof. Suppose $Y \subset Y' \subset X$ are irreducible closed subsets. Let $\xi \in Y$, $\xi' \in Y'$ be their generic points. Then we see immediately from the definitions that $\operatorname{codim}(Y, Y') \leq \delta(\xi) - \delta(\xi') < \infty$. In fact the first inequality is an equality. Namely, suppose

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e = Y'$$

is any maximal chain of irreducible closed subsets. Let $\xi_i \in Y_i$ denote the generic point. Then we see that $\xi_i \rightsquigarrow \xi_{i+1}$ is an immediate specialization. Hence we see that $e = \delta(\xi) - \delta(\xi')$ as desired. This also proves the last statement of the lemma. $\square$

02IB **Lemma 5.20.3.** Let X be a topological space. Let δ, δ' be two dimension functions on X. If X is locally Noetherian and sober then $\delta - \delta'$ is locally constant on X.

Proof. Let $x \in X$ be a point. We will show that $\delta - \delta'$ is constant in a neighbourhood of x. We may replace X by an open neighbourhood of x in X which is Noetherian. Hence we may assume X is Noetherian and sober. Let $Z_1, \ldots, Z_r$ be the irreducible components of X passing through x. (There are finitely many as X is Noetherian, see Lemma 5.9.2.) Let $\xi_i \in Z_i$ be the generic point. Note $Z_1 \cup \ldots \cup Z_r$ is a

[5] This is likely nonstandard notation. This notion is usually introduced only for (locally) Noetherian schemes, in which case condition (a) is implied by (b).

neighbourhood of x in X (not necessarily closed). We claim that $\delta - \delta'$ is constant on $Z_1 \cup \ldots \cup Z_r$. Namely, if $y \in Z_i$, then

$$\delta(x) - \delta(y) = \delta(x) - \delta(\xi_i) + \delta(\xi_i) - \delta(y) = -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i)$$

by Lemma 5.20.2. Similarly for δ'. Whence the result. □

02IC **Lemma 5.20.4.** Let X be locally Noetherian, sober and catenary. Then any point has an open neighbourhood $U \subset X$ which has a dimension function.

Proof. We will use repeatedly that an open subspace of a catenary space is catenary, see Lemma 5.11.5 and that a Noetherian topological space has finitely many irreducible components, see Lemma 5.9.2. In the proof of Lemma 5.20.3 we saw how to construct such a function. Namely, we first replace X by a Noetherian open neighbourhood of x. Next, we let $Z_1, \ldots, Z_r \subset X$ be the irreducible components of X. Let

$$Z_i \cap Z_j = \bigcup Z_{ijk}$$

be the decomposition into irreducible components. We replace X by

$$X \setminus \left(\bigcup\nolimits_{x \not\in Z_i} Z_i \cup \bigcup\nolimits_{x \not\in Z_{ijk}} Z_{ijk}\right)$$

so that we may assume $x \in Z_i$ for all i and $x \in Z_{ijk}$ for all i, j, k. For $y \in X$ choose any i such that $y \in Z_i$ and set

$$\delta(y) = -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i).$$

We claim this is a dimension function. First we show that it is well defined, i.e., independent of the choice of i. Namely, suppose that $y \in Z_{ijk}$ for some i, j, k. Then we have (using Lemma 5.11.6)

$$\begin{aligned}\delta(y) &= -\text{codim}(\overline{\{x\}}, Z_i) + \text{codim}(\overline{\{y\}}, Z_i) \\ &= -\text{codim}(\overline{\{x\}}, Z_{ijk}) - \text{codim}(Z_{ijk}, Z_i) + \text{codim}(\overline{\{y\}}, Z_{ijk}) + \text{codim}(Z_{ijk}, Z_i) \\ &= -\text{codim}(\overline{\{x\}}, Z_{ijk}) + \text{codim}(\overline{\{y\}}, Z_{ijk})\end{aligned}$$

which is symmetric in i and j. We omit the proof that it is a dimension function. □

02ID **Remark 5.20.5.** Combining Lemmas 5.20.3 and 5.20.4 we see that on a catenary, locally Noetherian, sober topological space the obstruction to having a dimension function is an element of $H^1(X, \mathbf{Z})$.

5.21. Nowhere dense sets

03HM

03HN **Definition 5.21.1.** Let X be a topological space.

1. Given a subset $T \subset X$ the *interior* of T is the largest open subset of X contained in T.
2. A subset $T \subset X$ is called *nowhere dense* if the closure of T has empty interior.

03HO **Lemma 5.21.2.** Let X be a topological space. The union of a finite number of nowhere dense sets is a nowhere dense set.

Proof. Omitted. □

03J0 Lemma 5.21.3. Let X be a topological space. Let $U \subset X$ be an open. Let $T \subset U$ be a subset. If T is nowhere dense in U, then T is nowhere dense in X.

Proof. Assume T is nowhere dense in U. Suppose that $x \in X$ is an interior point of the closure $\overline{T}$ of T in X. Say $x \in V \subset \overline{T}$ with $V \subset X$ open in X. Note that $\overline{T} \cap U$ is the closure of T in U. Hence the interior of $\overline{T} \cap U$ being empty implies $V \cap U = \emptyset$. Thus x cannot be in the closure of U, a fortiori cannot be in the closure of T, a contradiction. $\square$

03HP Lemma 5.21.4. Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $T \subset X$ be a subset. If $T \cap U_i$ is nowhere dense in U_i for all i, then T is nowhere dense in X.

Proof. Denote $\overline{T}_i$ the closure of $T \cap U_i$ in U_i. We have $\overline{T} \cap U_i = \overline{T}_i$. Taking the interior commutes with intersection with opens, thus

$$(\text{interior of } \overline{T}) \cap U_i = \text{interior of } (\overline{T} \cap U_i) = \text{interior in } U_i \text{ of } \overline{T}_i$$

By assumption the last of these is empty. Hence T is nowhere dense in X. $\square$

03HQ Lemma 5.21.5. Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset X$ be a subset. If f is a homeomorphism of X onto a closed subset of Y and T is nowhere dense in X, then also $f(T)$ is nowhere dense in Y.

Proof. Omitted. $\square$

03HR Lemma 5.21.6. Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset Y$ be a subset. If f is open and T is a closed nowhere dense subset of Y, then also $f^{-1}(T)$ is a closed nowhere dense subset of X. If f is surjective and open, then T is closed nowhere dense if and only if $f^{-1}(T)$ is closed nowhere dense.

Proof. Omitted. (Hint: In the first case the interior of $f^{-1}(T)$ maps into the interior of T, and in the second case the interior of $f^{-1}(T)$ maps onto the interior of T.) $\square$

5.22. Profinite spaces

08ZW Here is the definition.

08ZX Definition 5.22.1. A topological space is profinite if it is homeomorphic to a limit of a diagram of finite discrete spaces.

This is not the most convenient characterization of a profinite space.

08ZY Lemma 5.22.2. Let X be a topological space. The following are equivalent

(1) X is a profinite space, and
(2) X is Hausdorff, quasi-compact, and totally disconnected.

If this is true, then X is a cofiltered limit of finite discrete spaces.

Proof. Assume (1). Choose a diagram $i \mapsto X_i$ of finite discrete spaces such that $X = \lim X_i$. As each X_i is Hausdorff and quasi-compact we find that X is quasi-compact by Lemma 5.14.5. If $x, x' \in X$ are distinct points, then x and x' map to distinct points in some X_i. Hence x and x' have disjoint open neighbourhoods, i.e., X is Hausdorff. In exactly the same way we see that X is totally disconnected.

Assume (2). Let $\mathcal{I}$ be the set of finite disjoint union decompositions $X = \coprod_{i \in I} U_i$ with U_i nonempty open (and closed) for all $i \in I$. For each $I \in \mathcal{I}$ there is a continuous map $X \to I$ sending a point of U_i to i. We define a partial ordering: $I \leq I'$ for $I, I' \in \mathcal{I}$ if and only if the covering corresponding to I' refines the covering corresponding to I. In this case we obtain a canonical map $I' \to I$. In other words we obtain an inverse system of finite discrete spaces over $\mathcal{I}$. The maps $X \to I$ fit together and we obtain a continuous map

$$X \longrightarrow \lim_{I \in \mathcal{I}} I$$

We claim this map is a homeomorphism, which finishes the proof. (The final assertion follows too as the partially ordered set $\mathcal{I}$ is directed: given two disjoint union decompositions of X we can find a third refining both.) Namely, the map is injective as X is totally disconnected and hence $\{x\}$ is the intersection of all open and closed subsets of X containing x (Lemma 5.12.11) and the map is surjective by Lemma 5.12.6. By Lemma 5.17.8 the map is a homeomorphism. $\square$

0ET8 Lemma 5.22.3. A limit of profinite spaces is profinite.

Proof. Let $i \mapsto X_i$ be a diagram of profinite spaces over the index category $\mathcal{I}$. Let us use the characterization of profinite spaces in Lemma 5.22.2. In particular each X_i is Hausdorff, quasi-compact, and totally disconnected. By Lemma 5.14.1 the limit $X = \lim X_i$ exists. By Lemma 5.14.5 the limit X is quasi-compact. Let $x, x' \in X$ be distinct points. Then there exists an i such that x and x' have distinct images x_i and x'_i in X_i under the projection $X \to X_i$. Then x_i and x'_i have disjoint open neighbourhoods in X_i. Taking the inverse images of these opens we conclude that X is Hausdorff. Similarly, x_i and x'_i are in distinct connected components of X_i whence necessarily x and x' must be in distinct connected components of X. Hence X is totally disconnected. This finishes the proof. $\square$

08ZZ Lemma 5.22.4. Let X be a profinite space. Every open covering of X has a refinement by a finite covering $X = \coprod U_i$ with U_i open and closed.

Proof. Write $X = \lim X_i$ as a limit of an inverse system of finite discrete spaces over a directed set I (Lemma 5.22.2). Denote $f_i : X \to X_i$ the projection. For every point $x = (x_i) \in X$ a fundamental system of open neighbourhoods is the collection $f_i^{-1}(\{x_i\})$. Thus, as X is quasi-compact, we may assume we have an open covering

$$X = f_{i_1}^{-1}(\{x_{i_1}\}) \cup \ldots \cup f_{i_n}^{-1}(\{x_{i_n}\})$$

Choose $i \in I$ with $i \geq i_j$ for $j = 1, \ldots, n$ (this is possible as I is a directed set). Then we see that the covering

$$X = \coprod_{t \in X_i} f_i^{-1}(\{t\})$$

refines the given covering and is of the desired form. $\square$

0900 Lemma 5.22.5. Let X be a topological space. If X is quasi-compact and every connected component of X is the intersection of the open and closed subsets containing it, then $\pi_0(X)$ is a profinite space.

Proof. We will use Lemma 5.22.2 to prove this. Since $\pi_0(X)$ is the image of a quasi-compact space it is quasi-compact (Lemma 5.12.7). It is totally disconnected by construction (Lemma 5.7.9). Let $C, D \subset X$ be distinct connected components of X. Write $C = \bigcap U_\alpha$ as the intersection of the open and closed subsets of X containing C. Any finite intersection of U_α's is another. Since $\bigcap U_\alpha \cap D = \emptyset$ we conclude that $U_\alpha \cap D = \emptyset$ for some α (use Lemmas 5.7.3, 5.12.3 and 5.12.6) Since U_α is open and closed, it is the union of the connected components it contains, i.e., U_α is the inverse image of some open and closed subset $V_\alpha \subset \pi_0(X)$. This proves that the points corresponding to C and D are contained in disjoint open subsets, i.e., $\pi_0(X)$ is Hausdorff. □

5.23. Spectral spaces

08YF The material in this section is taken from [Hoc69] and [Hoc67]. In his thesis Hochster proves (among other things) that the spectral spaces are exactly the topological spaces that occur as the spectrum of a ring.

08YG Definition 5.23.1. A topological space X is called spectral if it is sober, quasi-compact, the intersection of two quasi-compact opens is quasi-compact, and the collection of quasi-compact opens forms a basis for the topology. A continuous map $f : X \to Y$ of spectral spaces is called spectral if the inverse image of a quasi-compact open is quasi-compact.

In other words a continuous map of spectral spaces is spectral if and only if it is quasi-compact (Definition 5.12.1).

Let X be a spectral space. The constructible topology on X is the topology which has as a subbase of opens the sets U and U^c where U is a quasi-compact open of X. Note that since X is spectral an open $U \subset X$ is retrocompact if and only if U is quasi-compact. Hence the constructible topology can also be characterized as the coarsest topology such that every constructible subset of X is both open and closed (see Section 5.15 for definitions and properties of constructible sets). It follows that a subset of X is open, resp. closed in the constructible topology if and only if it is a union, resp. intersection of constructible subsets. Since the collection of quasi-compact opens is a basis for the topology on X we see that the constructible topology is stronger than the given topology on X.

0901 Lemma 5.23.2. Let X be a spectral space. The constructible topology is Hausdorff, totally disconnected, and quasi-compact.

Proof. Let $x, y \in X$ with $x \neq y$. Since X is sober, there is an open subset U containing exactly one of the two points x, y. Say $x \in U$. We may replace U by a quasi-compact open neighbourhood of x contained in U. Then U and U^c are open and closed in the constructible topology. Hence X is Hausdorff in the constructible topology because $x \in U$ and $y \in U^c$ are disjoint opens in the constructible topology. The existence of U also implies x and y are in distinct connected components in the constructible topology, whence X is totally disconnected in the constructible topology.

Let $\mathcal{B}$ be the collection of subsets $B \subset X$ with B either quasi-compact open or closed with quasi-compact complement. If $B \in \mathcal{B}$ then $B^c \in \mathcal{B}$. It suffices to show every covering $X = \bigcup_{i \in I} B_i$ with $B_i \in \mathcal{B}$ has a finite refinement, see Lemma 5.12.15. Taking complements we see that we have to show that any family $\{B_i\}_{i \in I}$ of elements of $\mathcal{B}$ such that $B_{i_1} \cap \ldots \cap B_{i_n} \neq \emptyset$ for all n and all $i_1, \ldots, i_n \in I$ has a common point of intersection. We may and do assume $B_i \neq B_{i'}$ for $i \neq i'$.

To get a contradiction assume $\{B_i\}_{i \in I}$ is a family of elements of $\mathcal{B}$ having the finite intersection property but empty intersection. An application of Zorn's lemma shows that we may assume our family is maximal (details omitted). Let $I' \subset I$ be those indices such that B_i is closed and set $Z = \bigcap_{i \in I'} B_i$. This is a closed subset of X which is nonempty by Lemma 5.12.6. If Z is reducible, then we can write $Z = Z' \cup Z''$ as a union of two closed subsets, neither equal to Z. This means in particular that we can find a quasi-compact open $U' \subset X$ meeting Z' but not Z''. Similarly, we can find a quasi-compact open $U'' \subset X$ meeting Z'' but not Z'. Set $B' = X \setminus U'$ and $B'' = X \setminus U''$. Note that $Z'' \subset B'$ and $Z' \subset B''$. If there exist a finite number of indices $i_1, \ldots, i_n \in I$ such that $B' \cap B_{i_1} \cap \ldots \cap B_{i_n} = \emptyset$ as well as a finite number of indices $j_1, \ldots, j_m \in I$ such that $B'' \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$ then we find that $Z \cap B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$. However, the set $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m}$ is quasi-compact hence we would find a finite number of indices $i'_1, \ldots, i'_l \in I'$ with $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} \cap B_{i'_1} \cap \ldots \cap B_{i'_l} = \emptyset$, a contradiction. Thus we see that we may add either B' or B'' to the given family contradicting maximality. We conclude that Z is irreducible. However, this leads to a contradiction as well, as now every nonempty (by the same argument as above) open $Z \cap B_i$ for $i \in I \setminus I'$ contains the unique generic point of Z. This contradiction proves the lemma. □

0A2S Lemma 5.23.3. Let $f : X \to Y$ be a spectral map of spectral spaces. Then

(1) f is continuous in the constructible topology,
(2) the fibres of f are quasi-compact, and
(3) the image is closed in the constructible topology.

Proof. Let X' and Y' denote X and Y endowed with the constructible topology which are quasi-compact Hausdorff spaces by Lemma 5.23.2. Part (1) says $X' \to Y'$ is continuous and follows immediately from the definitions. Part (3) follows as $f(X')$ is a quasi-compact subset of the Hausdorff space Y', see Lemma 5.12.4. We have a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of continuous maps of topological spaces. Since Y' is Hausdorff we see that the fibres X'_y are closed in X'. As X' is quasi-compact we see that X'_y is quasi-compact (Lemma 5.12.3). As $X'_y \to X_y$ is a surjective continuous map we conclude that X_y is quasi-compact (Lemma 5.12.7). □

0G1J Lemma 5.23.4. Let X and Y be spectral spaces. Let $f : X \to Y$ be a continuous map. Then f is spectral if and only if f is continuous in the constructible topology.

Proof. The only if part of this is Lemma 5.23.3. Assume f is continuous in the constructible topology. Let $V \subset Y$ be quasi-compact open. Then V is open and closed in the constructible topology. Hence $f^{-1}(V)$ is open and closed in the constructible topology. Hence $f^{-1}(V)$ is quasi-compact in the constructible topology as X is quasi-compact in the constructible topology by Lemma 5.23.2. Since the identity $f^{-1}(V) \to f^{-1}(V)$ is surjective and continuous from the constructible topology to the usual topology, we conclude that $f^{-1}(V)$ is quasi-compact in the topology of X by Lemma 5.12.7. This finishes the proof. □

0902 **Lemma 5.23.5.** *Let X be a spectral space. Let $E \subset X$ be closed in the constructible topology (for example constructible or closed). Then E with the induced topology is a spectral space.*

Proof. Let $Z \subset E$ be a closed irreducible subset. Let η be the generic point of the closure $\overline{Z}$ of Z in X. To prove that E is sober, we show that $\eta \in E$. If not, then since E is closed in the constructible topology, there exists a constructible subset $F \subset X$ such that $\eta \in F$ and $F \cap E = \emptyset$. By Lemma 5.15.15 this implies $F \cap \overline{Z}$ contains a nonempty open subset of $\overline{Z}$. But this is impossible as $\overline{Z}$ is the closure of Z and $Z \cap F = \emptyset$.

Since E is closed in the constructible topology, it is quasi-compact in the constructible topology (Lemmas 5.12.3 and 5.23.2). Hence a fortiori it is quasi-compact in the topology coming from X. If $U \subset X$ is a quasi-compact open, then $E \cap U$ is closed in the constructible topology, hence quasi-compact (as seen above). It follows that the quasi-compact open subsets of E are the intersections $E \cap U$ with U quasi-compact open in X. These form a basis for the topology. Finally, given two $U, U' \subset X$ quasi-compact opens, the intersection $(E \cap U) \cap (E \cap U') = E \cap (U \cap U')$ and $U \cap U'$ is quasi-compact as X is spectral. This finishes the proof. $\square$

0903 **Lemma 5.23.6.** *Let X be a spectral space. Let $E \subset X$ be a subset closed in the constructible topology (for example constructible).*

1. *If $x \in \overline{E}$, then x is the specialization of a point of E.*
2. *If E is stable under specialization, then E is closed.*
3. *If $E' \subset X$ is open in the constructible topology (for example constructible) and stable under generalization, then E' is open.*

Proof. Proof of (1). Let $x \in \overline{E}$. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of x. A finite intersection of the U_i is another one. The intersection $U_i \cap E$ is nonempty for all i. Since the subsets $U_i \cap E$ are closed in the constructible topology we see that $\bigcap (U_i \cap E)$ is nonempty by Lemma 5.23.2 and Lemma 5.12.6. Since $\{U_i\}$ is a fundamental system of open neighbourhoods of x, we see that $\bigcap U_i$ is the set of generalizations of x. Thus x is a specialization of a point of E.

Part (2) is immediate from (1).

Proof of (3). Assume E' is as in (3). The complement of E' is closed in the constructible topology (Lemma 5.15.2) and closed under specialization (Lemma 5.19.2). Hence the complement is closed by (2), i.e., E' is open. $\square$

0904 **Lemma 5.23.7.** *Let X be a spectral space. Let $x, y \in X$. Then either there exists a third point specializing to both x and y, or there exist disjoint open neighbourhoods containing x and y.*

Proof. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of x. A finite intersection of the U_i is another one. Let $\{V_j\}$ be the set of quasi-compact open neighbourhoods of y. A finite intersection of the V_j is another one. If $U_i \cap V_j$ is empty for some i, j we are done. If not, then the intersection $U_i \cap V_j$ is nonempty for all i and j. The sets $U_i \cap V_j$ are closed in the constructible topology on X. By Lemma 5.23.2 we see that $\bigcap (U_i \cap V_j)$ is nonempty (Lemma 5.12.6). Since X is a sober space and $\{U_i\}$ is a fundamental system of open neighbourhoods of x, we see that $\bigcap U_i$ is the set of generalizations of x. Similarly, $\bigcap V_j$ is the set of generalizations of y. Thus any element of $\bigcap(U_i \cap V_j)$ specializes to both x and y. $\square$

0905 **Lemma 5.23.8.** Let X be a spectral space. The following are equivalent:

(1) X is profinite,
(2) X is Hausdorff,
(3) X is totally disconnected,
(4) every quasi-compact open is closed,
(5) there are no nontrivial specializations between points,
(6) every point of X is closed,
(7) every point of X is the generic point of an irreducible component of X,
(8) the constructible topology equals the given topology on X, and
(9) add more here.

Proof. Lemma 5.22.2 shows the implication (1) $\Rightarrow$ (3). Irreducible components are closed, so if X is totally disconnected, then every point is closed. So (3) implies (6). The equivalence of (6) and (5) is immediate, and (6) $\Leftrightarrow$ (7) holds because X is sober. Assume (5). Then all constructible subsets of X are closed (Lemma 5.23.6), in particular all quasi-compact opens are closed. So (5) implies (4). Since X is sober, for any two points there is a quasi-compact open containing exactly one of them, hence (4) implies (2). Parts (4) and (8) are equivalent by the definition of the constructible topology. It remains to prove (2) implies (1). Suppose X is Hausdorff. Every quasi-compact open is also closed (Lemma 5.12.4). This implies X is totally disconnected. Hence it is profinite, by Lemma 5.22.2. $\square$

0906 **Lemma 5.23.9.** If X is a spectral space, then $\pi_0(X)$ is a profinite space.

Proof. Combine Lemmas 5.12.10 and 5.22.5. $\square$

0907 **Lemma 5.23.10.** The product of two spectral spaces is spectral.

Proof. Let X, Y be spectral spaces. Denote $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections. Let $Z \subset X \times Y$ be a closed irreducible subset. Then $p(Z) \subset X$ is irreducible and $q(Z) \subset Y$ is irreducible. Let $x \in X$ be the generic point of the closure of $p(X)$ and let $y \in Y$ be the generic point of the closure of $q(Y)$. If $(x, y) \notin Z$, then there exist opens $x \in U \subset X$, $y \in V \subset Y$ such that $Z \cap U \times V = \emptyset$. Hence Z is contained in $(X \setminus U) \times Y \cup X \times (Y \setminus V)$. Since Z is irreducible, we see that either $Z \subset (X \setminus U) \times Y$ or $Z \subset X \times (Y \setminus V)$. In the first case $p(Z) \subset (X \setminus U)$ and in the second case $q(Z) \subset (Y \setminus V)$. Both cases are absurd as x is in the closure of $p(Z)$ and y is in the closure of $q(Z)$. Thus we conclude that $(x, y) \in Z$, which means that (x, y) is the generic point for Z.

A basis of the topology of $X \times Y$ are the opens of the form $U \times V$ with $U \subset X$ and $V \subset Y$ quasi-compact open (here we use that X and Y are spectral). Then $U \times V$ is quasi-compact as the product of quasi-compact spaces is quasi-compact. Moreover, any quasi-compact open of $X \times Y$ is a finite union of such quasi-compact rectangles $U \times V$. It follows that the intersection of two such is again quasi-compact (since X and Y are spectral). This concludes the proof. $\square$

09XU **Lemma 5.23.11.** Let $f : X \to Y$ be a continuous map of topological spaces. If

(1) X and Y are spectral,
(2) f is spectral and bijective, and

(3) generalizations (resp. specializations) lift along f.

Then f is a homeomorphism.

Proof. Since f is spectral it defines a continuous map between X and Y in the constructible topology. By Lemmas 5.23.2 and 5.17.8 it follows that $X \to Y$ is a homeomorphism in the constructible topology. Let $U \subset X$ be quasi-compact open. Then $f(U)$ is constructible in Y. Let $y \in Y$ specialize to a point in $f(U)$. By the last assumption we see that $f^{-1}(y)$ specializes to a point of U. Hence $f^{-1}(y) \in U$. Thus $y \in f(U)$. It follows that $f(U)$ is open, see Lemma 5.23.6. Whence f is a homeomorphism. To prove the lemma in case specializations lift along f one shows instead that $f(Z)$ is closed if $X \setminus Z$ is a quasi-compact open of X. $\square$

09XV Lemma 5.23.12. The inverse limit of a directed inverse system of finite sober topological spaces is a spectral topological space.

Proof. Let I be a directed set. Let X_i be an inverse system of finite sober spaces over I. Let $X = \lim X_i$ which exists by Lemma 5.14.1. As a set $X = \lim X_i$. Denote $p_i : X \to X_i$ the projection. Because I is directed we may apply Lemma 5.14.2. A basis for the topology is given by the opens $p_i^{-1}(U_i)$ for $U_i \subset X_i$ open. Since an open covering of $p_i^{-1}(U_i)$ is in particular an open covering in the profinite topology, we conclude that $p_i^{-1}(U_i)$ is quasi-compact. Given $U_i \subset X_i$ and $U_j \subset X_j$, then $p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k)$ for some $k \geq i, j$ and open $U_k \subset X_k$. Finally, if $Z \subset X$ is irreducible and closed, then $p_i(Z) \subset X_i$ is irreducible and therefore has a unique generic point ξ_i (because X_i is a finite sober topological space). Then $\xi = \lim \xi_i$ is a generic point of Z (it is a point of Z as Z is closed). This finishes the proof. $\square$

09XW Lemma 5.23.13. Let W be the topological space with two points, one closed, the other not. A topological space is spectral if and only if it is homeomorphic to a subspace of a product of copies of W which is closed in the constructible topology.

Proof. Write $W = \{0, 1\}$ where 0 is a specialization of 1 but not vice versa. Let I be a set. The space $\prod_{i \in I} W$ is spectral by Lemma 5.23.12. Thus we see that a subspace of $\prod_{i \in I} W$ closed in the constructible topology is a spectral space by Lemma 5.23.5.

For the converse, let X be a spectral space. Let $U \subset X$ be a quasi-compact open. Consider the continuous map

$$f_U : X \longrightarrow W$$

which maps every point in U to 1 and every point in $X \setminus U$ to 0. Taking the product of these maps we obtain a continuous map

$$f = \prod f_U : X \longrightarrow \prod\nolimits_U W$$

By construction the map $f : X \to Y$ is spectral. By Lemma 5.23.3 the image of f is closed in the constructible topology. If $x', x \in X$ are distinct, then since X is sober either x' is not a specialization of x or conversely. In either case (as the quasi-compact opens form a basis for the topology of X) there exists a quasi-compact open $U \subset X$ such that $f_U(x') \neq f_U(x)$. Thus f is injective. Let $Y = f(X)$ endowed with the induced topology. Let $y' \rightsquigarrow y$ be a specialization in Y and say $f(x') = y'$ and $f(x) = y$. Arguing as above we see that $x' \rightsquigarrow x$, since otherwise there is a U

such that $x \in U$ and $x' \notin U$, which would imply $f_U(x') \not\rightsquigarrow f_U(x)$. We conclude that $f : X \to Y$ is a homeomorphism by Lemma 5.23.11. □

09XX Lemma 5.23.14. A topological space is spectral if and only if it is a directed inverse limit of finite sober topological spaces.

Proof. One direction is given by Lemma 5.23.12. For the converse, assume X is spectral. Then we may assume $X \subset \prod_{i \in I} W$ is a subset closed in the constructible topology where $W = \{0, 1\}$ as in Lemma 5.23.13. We can write

$$\prod\nolimits_{i \in I} W = \lim\nolimits_{J \subset I \text{ finite}} \prod\nolimits_{j \in J} W$$

as a cofiltered limit. For each J, let $X_J \subset \prod_{j \in J} W$ be the image of X. Then we see that $X = \lim X_J$ as sets because X is closed in the product with the constructible topology (detail omitted). A formal argument (omitted) on limits shows that $X = \lim X_J$ as topological spaces. □

0A2T Lemma 5.23.15. Let X be a topological space and let $c : X \to X'$ be the universal map from X to a sober topological space, see Lemma 5.8.16.

(1) If X is quasi-compact, so is X'.
(2) If X is quasi-compact, has a basis of quasi-compact opens, and the intersection of two quasi-compact opens is quasi-compact, then X' is spectral.
(3) If X is Noetherian, then X' is a Noetherian spectral space.

Proof. Let $U \subset X$ be open and let $U' \subset X'$ be the corresponding open, i.e., the open such that $c^{-1}(U') = U$. Then U is quasi-compact if and only if U' is quasi-compact, as pulling back by c is a bijection between the opens of X and X' which commutes with unions. This in particular proves (1).

Proof of (2). It follows from the above that X' has a basis of quasi-compact opens. Since c^{-1} also commutes with intersections of pairs of opens, we see that the intersection of two quasi-compact opens X' is quasi-compact. Finally, X' is quasi-compact by (1) and sober by construction. Hence X' is spectral.

Proof of (3). It is immediate that X' is Noetherian as this is defined in terms of the acc for open subsets which holds for X. We have already seen in (2) that X' is spectral. □

5.24. Limits of spectral spaces

0A2U Lemma 5.23.14 tells us that every spectral space is a cofiltered limit of finite sober spaces. Every finite sober space is a spectral space and every continuous map of finite sober spaces is a spectral map of spectral spaces. In this section we prove some lemmas concerning limits of systems of spectral topological spaces along spectral maps.

0A2V Lemma 5.24.1. Let $\mathcal{I}$ be a category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral.

(1) Given subsets $Z_i \subset X_i$ closed in the constructible topology with $f_a(Z_j) \subset Z_i$ for all $a : j \to i$ in $\mathcal{I}$, then $\lim Z_i$ is quasi-compact.
(2) The space $X = \lim X_i$ is quasi-compact.

Proof. The limit $Z = \lim Z_i$ exists by Lemma 5.14.1. Denote X'_i the space X_i endowed with the constructible topology and Z'_i the corresponding subspace of X'_i. Let $a : j \to i$ in $\mathcal{I}$ be a morphism. As f_a is spectral it defines a continuous map $f_a : X'_j \to X'_i$. Thus $f_a|_{Z_j} : Z'_j \to Z'_i$ is a continuous map of quasi-compact Hausdorff spaces (by Lemmas 5.23.2 and 5.12.3). Thus $Z' = \lim Z_i$ is quasi-compact by Lemma 5.14.5. The maps $Z'_i \to Z_i$ are continuous, hence $Z' \to Z$ is continuous and a bijection on underlying sets. Hence Z is quasi-compact as the image of the surjective continuous map $Z' \to Z$ (Lemma 5.12.7). □

0A2W Lemma 5.24.2. Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral.

(1) Given nonempty subsets $Z_i \subset X_i$ closed in the constructible topology with $f_a(Z_j) \subset Z_i$ for all $a : j \to i$ in $\mathcal{I}$, then $\lim Z_i$ is nonempty.
(2) If each X_i is nonempty, then $X = \lim X_i$ is nonempty.

Proof. Denote X'_i the space X_i endowed with the constructible topology and Z'_i the corresponding subspace of X'_i. Let $a : j \to i$ in $\mathcal{I}$ be a morphism. As f_a is spectral it defines a continuous map $f_a : X'_j \to X'_i$. Thus $f_a|_{Z_j} : Z'_j \to Z'_i$ is a continuous map of quasi-compact Hausdorff spaces (by Lemmas 5.23.2 and 5.12.3). By Lemma 5.14.6 the space $\lim Z'_i$ is nonempty. Since $\lim Z'_i = \lim Z_i$ as sets we conclude. □

0A2X Lemma 5.24.3. Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Let $X = \lim X_i$ with projections $p_i : X \to X_i$. Let $i \in \mathrm{Ob}(\mathcal{I})$ and let $E, F \subset X_i$ be subsets with E closed in the constructible topology and F open in the constructible topology. Then $p_i^{-1}(E) \subset p_i^{-1}(F)$ if and only if there is a morphism $a : j \to i$ in $\mathcal{I}$ such that $f_a^{-1}(E) \subset f_a^{-1}(F)$.

Proof. Observe that

$$p_i^{-1}(E) \setminus p_i^{-1}(F) = \lim_{a:j\to i} f_a^{-1}(E) \setminus f_a^{-1}(F)$$

Since f_a is a spectral map, it is continuous in the constructible topology hence the set $f_a^{-1}(E) \setminus f_a^{-1}(F)$ is closed in the constructible topology. Hence Lemma 5.24.2 applies to show that the LHS is nonempty if and only if each of the spaces of the RHS is nonempty. □

0A2Y Lemma 5.24.4. Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Let $X = \lim X_i$ with projections $p_i : X \to X_i$. Let $E \subset X$ be a constructible subset. Then there exists an $i \in \mathrm{Ob}(\mathcal{I})$ and a constructible subset $E_i \subset X_i$ such that $p_i^{-1}(E_i) = E$. If E is open, resp. closed, we may choose E_i open, resp. closed.

Proof. Assume E is a quasi-compact open of X. By Lemma 5.14.2 we can write $E = p_i^{-1}(U_i)$ for some i and some open $U_i \subset X_i$. Write $U_i = \bigcup U_{i,\alpha}$ as a union of quasi-compact opens. As E is quasi-compact we can find $\alpha_1, \ldots, \alpha_n$ such that $E = p_i^{-1}(U_{i,\alpha_1} \cup \ldots \cup U_{i,\alpha_n})$. Hence $E_i = U_{i,\alpha_1} \cup \ldots \cup U_{i,\alpha_n}$ works.

Assume E is a constructible closed subset. Then E^c is quasi-compact open. So $E^c = p_i^{-1}(F_i)$ for some i and quasi-compact open $F_i \subset X_i$ by the result of the previous paragraph. Then $E = p_i^{-1}(F_i^c)$ as desired.

If E is general we can write $E = \bigcup_{l=1,\ldots,n} U_l \cap Z_l$ with U_l constructible open and Z_l constructible closed. By the result of the previous paragraphs we may write $U_l = p_{i_l}^{-1}(U_{l,i_l})$ and $Z_l = p_{j_l}^{-1}(Z_{l,j_l})$ with $U_{l,i_l} \subset X_{i_l}$ constructible open and $Z_{l,j_l} \subset X_{j_l}$ constructible closed. As $\mathcal{I}$ is cofiltered we may choose an object k of $\mathcal{I}$ and morphism $a_l : k \to i_l$ and $b_l : k \to j_l$. Then taking $E_k = \bigcup_{l=1,\ldots,n} f_{a_l}^{-1}(U_{l,i_l}) \cap f_{b_l}^{-1}(Z_{l,j_l})$ we obtain a constructible subset of X_k whose inverse image in X is E. □

0A2Z Lemma 5.24.5. Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Then the inverse limit $X = \lim X_i$ is a spectral topological space and the projection maps $p_i : X \to X_i$ are spectral.

Proof. The limit $X = \lim X_i$ exists (Lemma 5.14.1) and is quasi-compact by Lemma 5.24.1.

Denote $p_i : X \to X_i$ the projection. Because $\mathcal{I}$ is cofiltered we can apply Lemma 5.14.2. Hence a basis for the topology on X is given by the opens $p_i^{-1}(U_i)$ for $U_i \subset X_i$ open. Since a basis for the topology of X_i is given by the quasi-compact open, we conclude that a basis for the topology on X is given by $p_i^{-1}(U_i)$ with $U_i \subset X_i$ quasi-compact open. A formal argument shows that

$$p_i^{-1}(U_i) = \lim\nolimits_{a:j \to i} f_a^{-1}(U_i)$$

as topological spaces. Since each f_a is spectral the sets $f_a^{-1}(U_i)$ are closed in the constructible topology of X_j and hence $p_i^{-1}(U_i)$ is quasi-compact by Lemma 5.24.1. Thus X has a basis for the topology consisting of quasi-compact opens.

Any quasi-compact open U of X is of the form $U = p_i^{-1}(U_i)$ for some i and some quasi-compact open $U_i \subset X_i$ (see Lemma 5.24.4). Given $U_i \subset X_i$ and $U_j \subset X_j$ quasi-compact open, then $p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k)$ for some k and quasi-compact open $U_k \subset X_k$. Namely, choose k and morphisms $k \to i$ and $k \to j$ and let U_k be the intersection of the pullbacks of U_i and U_j to X_k. Thus we see that the intersection of two quasi-compact opens of X is quasi-compact open.

Finally, let $Z \subset X$ be irreducible and closed. Then $p_i(Z) \subset X_i$ is irreducible and therefore $Z_i = \overline{p_i(Z)}$ has a unique generic point ξ_i (because X_i is a spectral space). Then $f_a(\xi_j) = \xi_i$ for $a : j \to i$ in $\mathcal{I}$ because $\overline{f_a(Z_j)} = Z_i$. Hence $\xi = \lim \xi_i$ is a point of X. Claim: $\xi \in Z$. Namely, if not we can find a quasi-compact open containing ξ disjoint from Z. This would be of the form $p_i^{-1}(U_i)$ for some i and quasi-compact open $U_i \subset X_i$. Then $\xi_i \in U_i$ but $p_i(Z) \cap U_i = \emptyset$ which contradicts $\xi_i \in \overline{p_i(Z)}$. So $\xi \in Z$ and hence $\overline{\{\xi\}} \subset Z$. Conversely, every $z \in Z$ is in the closure of ξ. Namely, given a quasi-compact open neighbourhood U of z we write $U = p_i^{-1}(U_i)$ for some i and quasi-compact open $U_i \subset X_i$. We see that $p_i(z) \in U_i$ hence $\xi_i \in U_i$ hence $\xi \in U$. Thus ξ is a generic point of Z. We omit the proof that ξ is the unique generic point of Z (hint: show that a second generic point has to be equal to ξ by showing that it has to map to ξ_i in X_i since by spectrality of X_i the irreducible Z_i has a unique generic point). This finishes the proof. □

0A30 Lemma 5.24.6. Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Set $X = \lim X_i$ and denote $p_i : X \to X_i$ the projection.

(1) Given any quasi-compact open $U \subset X$ there exists an $i \in \mathrm{Ob}(\mathcal{I})$ and a quasi-compact open $U_i \subset X_i$ such that $p_i^{-1}(U_i) = U$.

(2) Given $U_i \subset X_i$ and $U_j \subset X_j$ quasi-compact opens such that $p_i^{-1}(U_i) \subset p_j^{-1}(U_j)$ there exist $k \in \mathrm{Ob}(\mathcal{I})$ and morphisms $a : k \to i$ and $b : k \to j$ such that $f_a^{-1}(U_i) \subset f_b^{-1}(U_j)$.
(3) If $U_i, U_{1,i}, \ldots, U_{n,i} \subset X_i$ are quasi-compact opens and $p_i^{-1}(U_i) = p_i^{-1}(U_{1,i}) \cup \ldots \cup p_i^{-1}(U_{n,i})$ then $f_a^{-1}(U_i) = f_a^{-1}(U_{1,i}) \cup \ldots \cup f_a^{-1}(U_{n,i})$ for some morphism $a : j \to i$ in $\mathcal{I}$.
(4) Same statement as in (3) but for intersections.

Proof. Part (1) is a special case of Lemma 5.24.4. Part (2) is a special case of Lemma 5.24.3 as quasi-compact opens are both open and closed in the constructible topology. Parts (3) and (4) follow formally from (1) and (2) and the fact that taking inverse images of subsets commutes with taking unions and intersections. □

0A31 Lemma 5.24.7. Let W be a subset of a spectral space X. The following are equivalent:

(1) W is an intersection of constructible sets and closed under generalizations,
(2) W is quasi-compact and closed under generalizations,
(3) there exists a quasi-compact subset $E \subset X$ such that W is the set of points specializing to E,
(4) W is an intersection of quasi-compact open subsets,
0ANZ (5) there exists a nonempty set I and quasi-compact opens $U_i \subset X$, $i \in I$ such that $W = \bigcap U_i$ and for all $i, j \in I$ there exists a $k \in I$ with $U_k \subset U_i \cap U_j$.

In this case we have (a) W is a spectral space, (b) $W = \lim U_i$ as topological spaces, and (c) for any open U containing W there exists an i with $U_i \subset U$.

Proof. Let $W \subset X$ satisfy (1). Then W is closed in the constructible topology, hence quasi-compact in the constructible topology (by Lemmas 5.23.2 and 5.12.3), hence quasi-compact in the topology of X (because opens in X are open in the constructible topology). Thus (2) holds.

It is clear that (2) implies (3) by taking $E = W$.

Let X be a spectral space and let $E \subset W$ be as in (3). Since every point of W specializes to a point of E we see that an open of W which contains E is equal to W. Hence since E is quasi-compact, so is W. If $x \in X$, $x \notin W$, then $Z = \overline{\{x\}}$ is disjoint from W. Since W is quasi-compact we can find a quasi-compact open U with $W \subset U$ and $U \cap Z = \emptyset$. We conclude that (4) holds.

If $W = \bigcap_{j \in J} U_j$ then setting I equal to the set of finite subsets of J and $U_i = U_{j_1} \cap \ldots \cap U_{j_r}$ for $i = \{j_1, \ldots, j_r\}$ shows that (4) implies (5). It is immediate that (5) implies (1).

Let I and U_i be as in (5). Since $W = \bigcap U_i$ we have $W = \lim U_i$ by the universal property of limits. Then W is a spectral space by Lemma 5.24.5. Let $U \subset X$ be an open neighbourhood of W. Then $E_i = U_i \cap (X \setminus U)$ is a family of constructible subsets of the spectral space $Z = X \setminus U$ with empty intersection. Using that the spectral topology on Z is quasi-compact (Lemma 5.23.2) we conclude from Lemma 5.12.6 that $E_i = \emptyset$ for some i. □

0AP0 Lemma 5.24.8. Let X be a spectral space. Let $E \subset X$ be a constructible subset. Let $W \subset X$ be the set of points of X which specialize to a point of E. Then

$W \setminus E$ is a spectral space. If $W = \bigcap U_i$ with U_i as in Lemma 5.24.7 (5) then $W \setminus E = \lim(U_i \setminus E)$.

Proof. Since E is constructible, it is quasi-compact and hence Lemma 5.24.7 applies to W. If E is constructible, then E is constructible in U_i for all $i \in I$. Hence $U_i \setminus E$ is spectral by Lemma 5.23.5. Since $W \setminus E = \bigcap(U_i \setminus E)$ we have $W \setminus E = \lim U_i \setminus E$ by the universal property of limits. Then $W \setminus E$ is a spectral space by Lemma 5.24.5. □

5.25. Stone-Čech compactification

0908 The Stone-Čech compactification of a topological space X is a map $X \to \beta(X)$ from X to a Hausdorff quasi-compact space $\beta(X)$ which is universal for such maps. We prove this exists by a standard argument using the following simple lemma.

0909 Lemma 5.25.1. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that $f(X)$ is dense in Y and that Y is Hausdorff. Then the cardinality of Y is at most the cardinality of $P(P(X))$ where P is the power set operation.

Proof. Let $S = f(X) \subset Y$. Let $\mathcal{D}$ be the set of all closed domains of Y, i.e., subsets $D \subset Y$ which equal the closure of its interior. Note that the closure of an open subset of Y is a closed domain. For $y \in Y$ consider the set

$$I_y = \{T \subset S \mid \text{ there exists } D \in \mathcal{D} \text{ with } T = S \cap D \text{ and } y \in D\}.$$

Since S is dense in Y for every closed domain D we see that $S \cap D$ is dense in D. Hence, if $D \cap S = D' \cap S$ for $D, D' \in \mathcal{D}$, then $D = D'$. Thus $I_y = I_{y'}$ implies that $y = y'$ because the Hausdorff condition assures us that we can find a closed domain containing y but not y'. The result follows. □

Let X be a topological space. By Lemma 5.25.1, there is a set I of isomorphism classes of continuous maps $f : X \to Y$ which have dense image and where Y is Hausdorff and quasi-compact. For $i \in I$ choose a representative $f_i : X \to Y_i$. Consider the map

$$\prod f_i : X \longrightarrow \prod\nolimits_{i \in I} Y_i$$

and denote $\beta(X)$ the closure of the image. Since each Y_i is Hausdorff, so is $\beta(X)$. Since each Y_i is quasi-compact, so is $\beta(X)$ (use Theorem 5.14.4 and Lemma 5.12.3).

Let us show the canonical map $X \to \beta(X)$ satisfies the universal property with respect to maps to Hausdorff, quasi-compact spaces. Namely, let $f : X \to Y$ be such a morphism. Let $Z \subset Y$ be the closure of $f(X)$. Then $X \to Z$ is isomorphic to one of the maps $f_i : X \to Y_i$, say $f_{i_0} : X \to Y_{i_0}$. Thus f factors as $X \to \beta(X) \to \prod Y_i \to Y_{i_0} \cong Z \to Y$ as desired.

090A Lemma 5.25.2. Let X be a Hausdorff, locally quasi-compact space. There exists a map $X \to X^*$ which identifies X as an open subspace of a quasi-compact Hausdorff space X^* such that $X^* \setminus X$ is a singleton (one point compactification). In particular, the map $X \to \beta(X)$ identifies X with an open subspace of $\beta(X)$.

Proof. Set $X^* = X \amalg \{\infty\}$. We declare a subset V of X^* to be open if either $V \subset X$ is open in X, or $\infty \in V$ and $U = V \cap X$ is an open of X such that $X \setminus U$ is quasi-compact. We omit the verification that this defines a topology. It is clear that $X \to X^*$ identifies X with an open subspace of X.

Since X is locally quasi-compact, every point $x \in X$ has a quasi-compact neighbourhood $x \in E \subset X$. Then E is closed (Lemma 5.12.4 part (1)) and $V = (X \setminus E) \amalg \{\infty\}$ is an open neighbourhood of ∞ disjoint from the interior of E. Thus X^* is Hausdorff.

Let $X^* = \bigcup V_i$ be an open covering. Then for some i, say i_0, we have $\infty \in V_{i_0}$. By construction $Z = X^* \setminus V_{i_0}$ is quasi-compact. Hence the covering $Z \subset \bigcup_{i \neq i_0} Z \cap V_i$ has a finite refinement which implies that the given covering of X^* has a finite refinement. Thus X^* is quasi-compact.

The map $X \to X^*$ factors as $X \to \beta(X) \to X^*$ by the universal property of the Stone-Čech compactification. Let $\varphi : \beta(X) \to X^*$ be this factorization. Then $X \to \varphi^{-1}(X)$ is a section to $\varphi^{-1}(X) \to X$ hence has closed image (Lemma 5.3.3). Since the image of $X \to \beta(X)$ is dense we conclude that $X = \varphi^{-1}(X)$. □

5.26. Extremally disconnected spaces

08YH The material in this section is taken from [Gle58] (with a slight modification as in [Rai59]). In Gleason's paper it is shown that in the category of quasi-compact Hausdorff spaces, the "projective objects" are exactly the extremally disconnected spaces.

08YI Definition 5.26.1. A topological space X is called extremally disconnected if the closure of every open subset of X is open.

If X is Hausdorff and extremally disconnected, then X is totally disconnected (this isn't true in general). If X is quasi-compact, Hausdorff, and extremally disconnected, then X is profinite by Lemma 5.22.2, but the converse does not hold in general. For example the p-adic integers $\mathbf{Z}_p = \lim \mathbf{Z}/p^n\mathbf{Z}$ is a profinite space which is not extremally disconnected. Namely, if $U \subset \mathbf{Z}_p$ is the set of nonzero elements whose valuation is even, then U is open but its closure is $U \cup \{0\}$ which is not open.

08YJ Lemma 5.26.2. Let $f : X \to Y$ be a continuous map of topological spaces. Assume f is surjective and $f(E) \neq Y$ for all proper closed subsets $E \subset X$. Then for $U \subset X$ open the subset $f(U)$ is contained in the closure of $Y \setminus f(X \setminus U)$.

Proof. Pick $y \in f(U)$ and let $V \subset Y$ be any open neighbourhood of y. We will show that V intersects $Y \setminus f(X \setminus U)$. Note that $W = U \cap f^{-1}(V)$ is a nonempty open subset of X, hence $f(X \setminus W) \neq Y$. Take $y' \in Y$, $y' \notin f(X \setminus W)$. It is elementary to show that $y' \in V$ and $y' \in Y \setminus f(X \setminus U)$. □

08YK Lemma 5.26.3. Let X be an extremally disconnected space. If $U, V \subset X$ are disjoint open subsets, then $\overline{U}$ and $\overline{V}$ are disjoint too.

Proof. By assumption $\overline{U}$ is open, hence $V \cap \overline{U}$ is open and disjoint from U, hence empty because $\overline{U}$ is the intersection of all the closed subsets of X containing U. This means the open $\overline{V} \cap \overline{U}$ avoids V hence is empty by the same argument. □

08YL Lemma 5.26.4. Let $f : X \to Y$ be a continuous map of Hausdorff quasi-compact topological spaces. If Y is extremally disconnected, f is surjective, and $f(Z) \neq Y$ for every proper closed subset Z of X, then f is a homeomorphism.

Proof. By Lemma 5.17.8 it suffices to show that f is injective. Suppose that $x, x' \in X$ are distinct points with $y = f(x) = f(x')$. Choose disjoint open neighbourhoods $U, U' \subset X$ of x, x'. Observe that f is closed (Lemma 5.17.7) hence $T = f(X \setminus U)$

and $T' = f(X \setminus U')$ are closed in Y. Since X is the union of $X \setminus U$ and $X \setminus U'$ we see that $Y = T \cup T'$. By Lemma 5.26.2 we see that y is contained in the closure of $Y \setminus T$ and the closure of $Y \setminus T'$. On the other hand, by Lemma 5.26.3, this intersection is empty. In this way we obtain the desired contradiction. □

08YM Lemma 5.26.5. Let $f : X \to Y$ be a continuous surjective map of Hausdorff quasi-compact topological spaces. There exists a quasi-compact subset $E \subset X$ such that $f(E) = Y$ but $f(E') \neq Y$ for all proper closed subsets $E' \subset E$.

Proof. We will use without further mention that the quasi-compact subsets of X are exactly the closed subsets (Lemma 5.12.5). Consider the collection $\mathcal{E}$ of all quasi-compact subsets $E \subset X$ with $f(E) = Y$ ordered by inclusion. We will use Zorn's lemma to show that $\mathcal{E}$ has a minimal element. To do this it suffices to show that given a totally ordered family E_λ of elements of $\mathcal{E}$ the intersection $\bigcap E_\lambda$ is an element of $\mathcal{E}$. It is quasi-compact as it is closed. For every $y \in Y$ the sets $E_\lambda \cap f^{-1}(\{y\})$ are nonempty and closed, hence the intersection $\bigcap E_\lambda \cap f^{-1}(\{y\}) = \bigcap(E_\lambda \cap f^{-1}(\{y\}))$ is nonempty by Lemma 5.12.6. This finishes the proof. □

08YN Proposition 5.26.6. Let X be a Hausdorff, quasi-compact topological space. The following are equivalent

(1) X is extremally disconnected,
(2) for any surjective continuous map $f : Y \to X$ with Y Hausdorff quasi-compact there exists a continuous section, and
(3) for any solid commutative diagram

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & Z \end{tikzcd}$$

of continuous maps of quasi-compact Hausdorff spaces with $Y \to Z$ surjective, there is a dotted arrow in the category of topological spaces making the diagram commute.

Proof. It is clear that (3) implies (2). On the other hand, if (2) holds and $X \to Z$ and $Y \to Z$ are as in (3), then (2) assures there is a section to the projection $X \times_Z Y \to X$ which implies a suitable dotted arrow exists (details omitted). Thus (3) is equivalent to (2).

Assume X is extremally disconnected and let $f : Y \to X$ be as in (2). By Lemma 5.26.5 there exists a quasi-compact subset $E \subset Y$ such that $f(E) = X$ but $f(E') \neq X$ for all proper closed subsets $E' \subset E$. By Lemma 5.26.4 we find that $f|_E : E \to X$ is a homeomorphism, the inverse of which gives the desired section.

Assume (2). Let $U \subset X$ be open with complement Z. Consider the continuous surjection $f : \overline{U} \amalg Z \to X$. Let σ be a section. Then $\overline{U} = \sigma^{-1}(\overline{U})$ is open. Thus X is extremally disconnected. □

090B Lemma 5.26.7. Let $f : X \to X$ be a surjective continuous selfmap of a Hausdorff topological space. If f is not id_X, then there exists a proper closed subset $E \subset X$ such that $X = E \cup f(E)$.

Proof. Pick $p \in X$ with $f(p) \neq p$. Choose disjoint open neighbourhoods $p \in U$, $f(p) \in V$ and set $E = X \setminus U \cap f^{-1}(V)$. Then $p \notin E$ hence E is a proper closed subset. If $x \in X$, then either $x \in E$, or if not, then $x \in U \cap f^{-1}(V)$ and writing $x = f(y)$ (possible as f is surjective) we find $y \in V \subset E$ and $x \in f(E)$. $\square$

090C **Example 5.26.8.** We can use Proposition 5.26.6 to see that the Stone-Čech compactification $\beta(X)$ of a discrete space X is extremally disconnected. Namely, let $f : Y \to \beta(X)$ be a continuous surjection where Y is quasi-compact and Hausdorff. Then we can lift the map $X \to \beta(X)$ to a continuous (!) map $X \to Y$ as X is discrete. By the universal property of the Stone-Čech compactification we see that we obtain a factorization $X \to \beta(X) \to Y$. Since $\beta(X) \to Y \to \beta(X)$ equals the identity on the dense subset X we conclude that we get a section. In particular, we conclude that the Stone-Čech compactification of a discrete space is totally disconnected, whence profinite (see discussion following Definition 5.26.1 and Lemma 5.22.2).

Using the supply of extremally disconnected spaces given by Example 5.26.8 we can prove that every quasi-compact Hausdorff space has a "projective cover" in the category of quasi-compact Hausdorff spaces.

090D **Lemma 5.26.9.** *Let X be a quasi-compact Hausdorff space. There exists a continuous surjection $X' \to X$ with X' quasi-compact, Hausdorff, and extremally disconnected. If we require that every proper closed subset of X' does not map onto X, then X' is unique up to isomorphism.*

Proof. Let $Y = X$ but endowed with the discrete topology. Let $X' = \beta(Y)$. The continuous map $Y \to X$ factors as $Y \to X' \to X$. This proves the first statement of the lemma by Example 5.26.8.

By Lemma 5.26.5 we can find a quasi-compact subset $E \subset X'$ surjecting onto X such that no proper closed subset of E surjects onto X. Because X' is extremally disconnected there exists a continuous map $f : X' \to E$ over X (Proposition 5.26.6). Composing f with the map $E \to X'$ gives a continuous selfmap $f|_E : E \to E$. Observe that $f|_E$ has to be surjective as otherwise the image would be a proper closed subset surjecting onto X. Hence $f|_E$ has to be id_E as otherwise Lemma 5.26.7 shows that E isn't minimal. Thus the id_E factors through the extremally disconnected space X'. A formal, categorical argument (using the characterization of Proposition 5.26.6) shows that E is extremally disconnected.

To prove uniqueness, suppose we have a second $X'' \to X$ minimal cover. By the lifting property proven in Proposition 5.26.6 we can find a continuous map $g : X' \to X''$ over X. Observe that g is a closed map (Lemma 5.17.7). Hence $g(X') \subset X''$ is a closed subset surjecting onto X and we conclude $g(X') = X''$ by minimality of X''. On the other hand, if $E \subset X'$ is a proper closed subset, then $g(E) \neq X''$ as E does not map onto X by minimality of X'. By Lemma 5.26.4 we see that g is an isomorphism. $\square$

090E **Remark 5.26.10.** Let X be a quasi-compact Hausdorff space. Let κ be an infinite cardinal bigger or equal than the cardinality of X. Then the cardinality of the minimal quasi-compact, Hausdorff, extremally disconnected cover $X' \to X$ (Lemma 5.26.9) is at most 2^{2^κ}. Namely, choose a subset $S \subset X'$ mapping bijectively to X. By minimality of X' the set S is dense in X'. Thus $|X'| \leq 2^{2^\kappa}$ by Lemma 5.25.1.

5.27. Miscellany

0067 The following lemma applies to the underlying topological space associated to a quasi-separated scheme.

0069 Lemma 5.27.1. Let X be a topological space which

1. has a basis of the topology consisting of quasi-compact opens, and
2. has the property that the intersection of any two quasi-compact opens is quasi-compact.

Then

1. X is locally quasi-compact,
2. a quasi-compact open $U \subset X$ is retrocompact,
3. any quasi-compact open $U \subset X$ has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j \in J} U_j$ with J finite and all U_j and $U_j \cap U_{j'}$ quasi-compact,
4. add more here.

Proof. Omitted. $\square$

06RM Definition 5.27.2. Let X be a topological space. We say $x \in X$ is an isolated point of X if $\{x\}$ is open in X.

5.28. Partitions and stratifications

09XY Stratifications can be defined in many different ways. We welcome comments on the choice of definitions in this section.

09XZ Definition 5.28.1. Let X be a topological space. A partition of X is a decomposition $X = \coprod X_i$ into locally closed subsets X_i. The X_i are called the parts of the partition. Given two partitions of X we say one refines the other if the parts of one are unions of parts of the other.

Any topological space X has a partition into connected components. If X has finitely many irreducible components $Z_1, \ldots, Z_r$, then there is a partition with parts $X_I = \bigcap_{i \in I} Z_i \setminus (\bigcup_{i \not\in I} Z_i)$ whose indices are subsets $I \subset \{1, \ldots, r\}$ which refines the partition into connected components.

09Y0 Definition 5.28.2. Let X be a topological space. A good stratification of X is a partition $X = \coprod X_i$ such that for all $i, j \in I$ we have

$$X_i \cap \overline{X_j} \neq \emptyset \Rightarrow X_i \subset \overline{X_j}.$$

Given a good stratification $X = \coprod_{i \in I} X_i$ we obtain a partial ordering on I by setting $i \leq j$ if and only if $X_i \subset \overline{X_j}$. Then we see that

$$\overline{X_j} = \bigcup\nolimits_{i \leq j} X_i$$

However, what often happens in algebraic geometry is that one just has that the left hand side is a subset of the right hand side in the last displayed formula. This leads to the following definition.

09Y1 Definition 5.28.3. Let X be a topological space. A stratification of X is given by a partition $X = \coprod_{i \in I} X_i$ and a partial ordering on I such that for each $j \in I$ we have

$$\overline{X_j} \subset \bigcup\nolimits_{i \leq j} X_i$$

The parts X_i are called the strata of the stratification.

We often impose additional conditions on the stratification. For example, stratifications are particularly nice if they are locally finite, which means that every point has a neighbourhood which meets only finitely many strata. More generally we introduce the following definition.

0BDS Definition 5.28.4. Let X be a topological space. Let I be a set and for $i \in I$ let $E_i \subset X$ be a subset. We say the collection $\{E_i\}_{i \in I}$ is locally finite if for all $x \in X$ there exists an open neighbourhood U of x such that $\{i \in I | E_i \cap U \neq \emptyset\}$ is finite.

09Y2 Remark 5.28.5. Given a locally finite stratification $X = \coprod X_i$ of a topological space X, we obtain a family of closed subsets $Z_i = \bigcup_{j \leq i} X_j$ of X indexed by I such that

$$Z_i \cap Z_j = \bigcup\nolimits_{k \leq i, j} Z_k$$

Conversely, given closed subsets $Z_i \subset X$ indexed by a partially ordered set I such that $X = \bigcup Z_i$, such that every point has a neighbourhood meeting only finitely many Z_i, and such that the displayed formula holds, then we obtain a locally finite stratification of X by setting $X_i = Z_i \setminus \bigcup_{j < i} Z_j$.

09Y3 Lemma 5.28.6. Let X be a topological space. Let $X = \coprod X_i$ be a finite partition of X. Then there exists a finite stratification of X refining it.

Proof. Let $T_i = \overline{X_i}$ and $\Delta_i = T_i \setminus X_i$. Let S be the set of all intersections of T_i and Δ_i. (For example $T_1 \cap T_2 \cap \Delta_4$ is an element of S.) Then $S = \{Z_s\}$ is a finite collection of closed subsets of X such that $Z_s \cap Z_{s'} \in S$ for all $s, s' \in S$. Define a partial ordering on S by inclusion. Then set $Y_s = Z_s \setminus \bigcup_{s' < s} Z_{s'}$ to get the desired stratification. $\square$

09Y4 Lemma 5.28.7. Let X be a topological space. Suppose $X = T_1 \cup \ldots \cup T_n$ is written as a union of constructible subsets. There exists a finite stratification $X = \coprod X_i$ with each X_i constructible such that each T_k is a union of strata.

Proof. By definition of constructible subsets, we can write each T_i as a finite union of $U \cap V^c$ with $U, V \subset X$ retrocompact open. Hence we may assume that $T_i = U_i \cap V_i^c$ with $U_i, V_i \subset X$ retrocompact open. Let S be the finite set of closed subsets of X consisting of $\emptyset, X, U_i^c, V_i^c$ and finite intersections of these. If $Z \in S$, then Z is constructible in X (Lemma 5.15.2). Moreover, $Z \cap Z' \in S$ for all $Z, Z' \in S$. Define a partial ordering on S by inclusion. For $Z \in S$ set $X_Z = Z \setminus \bigcup_{Z' < Z,\ Z' \in S} Z'$ to get a stratification $X = \coprod_{Z \in S} X_Z$ satisfying the properties stated in the lemma. $\square$

09Y5 Lemma 5.28.8. Let X be a Noetherian topological space. Any finite partition of X can be refined by a finite good stratification.

Proof. Let $X = \coprod X_i$ be a finite partition of X. Let Z be an irreducible component of X. Since $X = \bigcup \overline{X_i}$ with finite index set, there is an i such that $Z \subset \overline{X_i}$. Since X_i is locally closed this implies that $Z \cap X_i$ contains an open of Z. Thus $Z \cap X_i$ contains an open U of X (Lemma 5.9.2). Write $X_i = U \amalg X_i^1 \amalg X_i^2$ with $X_i^1 = (X_i \setminus U) \cap \overline{U}$ and $X_i^2 = (X_i \setminus U) \cap \overline{U}^c$. For $i' \neq i$ we set $X_{i'}^1 = X_{i'} \cap \overline{U}$ and $X_{i'}^2 = X_{i'} \cap \overline{U}^c$. Then

$$X \setminus U = \coprod X_l^k$$

is a partition such that $\overline{U} \setminus U = \bigcup X_l^1$. Note that $X \setminus U$ is closed and strictly smaller than X. By Noetherian induction we can refine this partition by a finite good stratification $X \setminus U = \coprod_{\alpha \in A} T_\alpha$. Then $X = U \amalg \coprod_{\alpha \in A} T_\alpha$ is a finite good stratification of X refining the partition we started with. $\square$

5.29. Colimits of spaces

0B1W The category of topological spaces has coproducts. Namely, if I is a set and for $i \in I$ we are given a topological space X_i then we endow the set $\coprod_{i \in I} X_i$ with the coproduct topology. As a basis for this topology we use sets of the form U_i where $U_i \subset X_i$ is open.

The category of topological spaces has coequalizers. Namely, if $a, b : X \to Y$ are morphisms of topological spaces, then the coequalizer of a and b is the coequalizer $Y/\sim$ in the category of sets endowed with the quotient topology (Section 5.6).

0B1X Lemma 5.29.1. The category of topological spaces has colimits and the forgetful functor to sets commutes with them.

Proof. This follows from the discussion above and Categories, Lemma 4.14.12. Another proof of existence of colimits is sketched in Categories, Remark 4.25.2. It follows from the above that the forgetful functor commutes with colimits. Another way to see this is to use Categories, Lemma 4.24.5 and use that the forgetful functor has a right adjoint, namely the functor which assigns to a set the corresponding chaotic (or indiscrete) topological space. $\square$

5.30. Topological groups, rings, modules

0B1Y This is just a short section with definitions and elementary properties.

0B1Z Definition 5.30.1. A topological group is a group G endowed with a topology such that multiplication $G \times G \to G$, $(x, y) \mapsto xy$ and inverse $G \to G$, $x \mapsto x^{-1}$ are continuous. A homomorphism of topological groups is a homomorphism of groups which is continuous.

If G is a topological group and $H \subset G$ is a subgroup, then H with the induced topology is a topological group. If G is a topological group and $G \to H$ is a surjection of groups, then H endowed with the quotient topology is a topological group.

0BMC Example 5.30.2. Let E be a set. We can endow the set of self maps $\text{Map}(E, E)$ with the compact open topology, i.e., the topology such that given $f : E \to E$ a fundamental system of neighbourhoods of f is given by the sets $U_S(f) = \{f' : E \to E \mid f'|_S = f|_S\}$ where $S \subset E$ is finite. With this topology the action

$$\text{Map}(E, E) \times E \longrightarrow E, \quad (f, e) \longmapsto f(e)$$

is continuous when E is given the discrete topology. If X is a topological space and $X \times E \to E$ is a continuous map, then the map $X \to \text{Map}(E, E)$ is continuous. In other words, the compact open topology is the coarsest topology such that the "action" map displayed above is continuous. The composition

$$\text{Map}(E, E) \times \text{Map}(E, E) \to \text{Map}(E, E)$$

is continuous as well (as is easily verified using the description of neighbourhoods above). Finally, if $\text{Aut}(E) \subset \text{Map}(E, E)$ is the subset of invertible maps, then the inverse $i : \text{Aut}(E) \to \text{Aut}(E)$, $f \mapsto f^{-1}$ is continuous too. Namely, say $S \subset E$ is finite, then $i^{-1}(U_S(f^{-1})) = U_{f^{-1}(S)}(f)$. Hence $\mathrm{Aut}(E)$ is a topological group as in Definition 5.30.1.

0B20 Lemma 5.30.3. The category of topological groups has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of groups.

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 4.14.11. Let G_i, $i \in I$ be a collection of topological groups. Take the usual product $G = \prod G_i$ with the product topology. Since $G \times G = \prod (G_i \times G_i)$ as a topological space (because products commutes with products in any category), we see that multiplication on G is continuous. Similarly for the inverse map. Let $a, b : G \to H$ be two homomorphisms of topological groups. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of groups endowed with the induced topology. □

0BR1 Lemma 5.30.4. Let G be a topological group. The following are equivalent

(1) G as a topological space is profinite,
(2) G is a limit of a diagram of finite discrete topological groups,
(3) G is a cofiltered limit of finite discrete topological groups.

Proof. We have the corresponding result for topological spaces, see Lemma 5.22.2. Combined with Lemma 5.30.3 we see that it suffices to prove that (1) implies (3).

We first prove that every neighbourhood E of the neutral element e contains an open subgroup. Namely, since G is the cofiltered limit of finite discrete topological spaces (Lemma 5.22.2), we can choose a continuous map $f : G \to T$ to a finite discrete space T such that $f^{-1}(f(\{e\})) \subset E$. Consider

$$H = \{g \in G \mid f(gg') = f(g') \text{ for all } g' \in G\}$$

This is a subgroup of G and contained in E. Thus it suffices to show that H is open. Pick $t \in T$ and set $W = f^{-1}(\{t\})$. Observe that $W \subset G$ is open and closed, in particular quasi-compact. For each $w \in W$ there exist open neighbourhoods $e \in U_w \subset G$ and $w \in U'_w \subset W$ such that $U_w U'_w \subset W$. By quasi-compactness we can find $w_1, \ldots, w_n$ such that $W = \bigcup U'_{w_i}$. Then $U_t = U_{w_1} \cap \ldots \cap U_{w_n}$ is an open neighbourhood of e such that $f(gw) = t$ for all $w \in W$. Since T is finite we see that $\bigcap_{t \in T} U_t \subset H$ is an open neighbourhood of e. Since $H \subset G$ is a subgroup it follows that H is open.

Suppose that $H \subset G$ is an open subgroup. Since G is quasi-compact we see that the index of H in G is finite. Say $G = Hg_1 \cup \ldots \cup Hg_n$. Then $N = \bigcap_{i=1,\ldots,n} g_i H g_i^{-1}$ is an open normal subgroup contained in H. Since N also has finite index we see that $G \to G/N$ is a surjection to a finite discrete topological group.

Consider the map

$$G \longrightarrow \lim_{N \subset G \text{ open and normal}} G/N$$

We claim that this map is an isomorphism of topological groups. This finishes the proof of the lemma as the limit on the right is cofiltered (the intersection of two open normal subgroups is open and normal). The map is continuous as each $G \to G/N$ is continuous. The map is injective as G is Hausdorff and every neighbourhood of e contains an N by the arguments above. The map is surjective by Lemma 5.12.6. By Lemma 5.17.8 the map is a homeomorphism. □

0BR2 Definition 5.30.5. A topological group is called a profinite group if it satisfies the equivalent conditions of Lemma 5.30.4.

If $G_1 \to G_2 \to G_3 \to \ldots$ is a system of topological groups then the colimit $G = \text{colim}\, G_n$ as a topological group (Lemma 5.30.6) is in general different from the colimit as a topological space (Lemma 5.29.1) even though these have the same underlying set. See Examples, Section 110.77.

0B21 Lemma 5.30.6. The category of topological groups has colimits and colimits commute with the forgetful functor to the category of groups.

Proof. We will use the argument of Categories, Remark 4.25.2 to prove existence of colimits. Namely, suppose that $\mathcal{I} \to \text{Top}$, $i \mapsto G_i$ is a functor into the category TopGroup of topological groups. Then we can consider

$$F : \text{TopGroup} \longrightarrow \text{Sets}, \quad H \longmapsto \lim_{\mathcal{I}} \text{Mor}_{\text{TopGroup}}(G_i, H)$$

This functor commutes with limits. Moreover, given any topological group H and an element $(\varphi_i : G_i \to H)$ of $F(H)$, there is a subgroup $H' \subset H$ of cardinality at most $|\coprod G_i|$ (coproduct in the category of groups, i.e., the free product on the G_i) such that the morphisms φ_i map into H'. Namely, we can take the induced topology on the subgroup generated by the images of the φ_i. Thus it is clear that the hypotheses of Categories, Lemma 4.25.1 are satisfied and we find a topological group G representing the functor F, which precisely means that G is the colimit of the diagram $i \mapsto G_i$.

To see the statement on commutation with the forgetful functor to groups we will use Categories, Lemma 4.24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a group the corresponding chaotic (or indiscrete) topological group. $\square$

0B22 Definition 5.30.7. A topological ring is a ring R endowed with a topology such that addition $R \times R \to R$, $(x, y) \mapsto x + y$ and multiplication $R \times R \to R$, $(x, y) \mapsto xy$ are continuous. A homomorphism of topological rings is a homomorphism of rings which is continuous.

In the Stacks project rings are commutative with 1. If R is a topological ring, then $(R, +)$ is a topological group since $x \mapsto -x$ is continuous. If R is a topological ring and $R' \subset R$ is a subring, then R' with the induced topology is a topological ring. If R is a topological ring and $R \to R'$ is a surjection of rings, then R' endowed with the quotient topology is a topological ring.

0B23 Lemma 5.30.8. The category of topological rings has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of rings.

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 4.14.11. Let R_i, $i \in I$ be a collection of topological rings. Take the usual product $R = \prod R_i$ with the product topology. Since $R \times R = \prod (R_i \times R_i)$ as a topological space (because products commutes with products in any category), we see that addition and multiplication on R are continuous. Let $a, b : R \to R'$ be two homomorphisms of topological rings. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of rings endowed with the induced topology. $\square$

0B24 Lemma 5.30.9. The category of topological rings has colimits and colimits commute with the forgetful functor to the category of rings.

Proof. The exact same argument as used in the proof of Lemma 5.30.6 shows existence of colimits. To see the statement on commutation with the forgetful functor to rings we will use Categories, Lemma 4.24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a ring the corresponding chaotic (or indiscrete) topological ring. □

0B25 Definition 5.30.10. Let R be a topological ring. A topological module is an R-module M endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous. A homomorphism of topological modules is a homomorphism of modules which is continuous.

If R is a topological ring and M is a topological module, then $(M, +)$ is a topological group since $x \mapsto -x$ is continuous. If R is a topological ring, M is a topological module and $M' \subset M$ is a submodule, then M' with the induced topology is a topological module. If R is a topological ring, M is a topological module, and $M \to M'$ is a surjection of modules, then M' endowed with the quotient topology is a topological module.

0B26 Lemma 5.30.11. Let R be a topological ring. The category of topological modules over R has limits and limits commute with the forgetful functors to (a) the category of topological spaces and (b) the category of R-modules.

Proof. It is enough to prove the existence and commutation for products and equalizers, see Categories, Lemma 4.14.11. Let M_i, $i \in I$ be a collection of topological modules over R. Take the usual product $M = \prod M_i$ with the product topology. Since $M \times M = \prod(M_i \times M_i)$ as a topological space (because products commutes with products in any category), we see that addition on M is continuous. Similarly for multiplication $R \times M \to M$. Let $a, b : M \to M'$ be two homomorphisms of topological modules over R. Then as the equalizer we can simply take the equalizer of a and b as maps of topological spaces, which is the same thing as the equalizer as maps of modules endowed with the induced topology. □

0B27 Lemma 5.30.12. Let R be a topological ring. The category of topological modules over R has colimits and colimits commute with the forgetful functor to the category of modules over R.

Proof. The exact same argument as used in the proof of Lemma 5.30.6 shows existence of colimits. To see the statement on commutation with the forgetful functor to R-modules we will use Categories, Lemma 4.24.5. Indeed, the forgetful functor has a right adjoint, namely the functor which assigns to a module the corresponding chaotic (or indiscrete) topological module. □

5.31. Other chapters

Preliminaries

CHAPTER 6

Sheaves on Spaces

006A 6.1. Introduction

006B Basic properties of sheaves on topological spaces will be explained in this document. A reference is [God73].

This will be superseded by the discussion of sheaves over sites later in the documents. But perhaps it makes sense to briefly define some of the notions here.

6.2. Basic notions

006C The following is a list of basic notions in topology.

(1) Let X be a topological space. The phrase: "Let $U = \bigcup_{i \in I} U_i$ be an open covering" means the following: I is a set and for each $i \in I$ we are given an open subset $U_i \subset X$ such that U is the union of the U_i. It is allowed to have $I = \emptyset$ in which case there are no U_i and $U = \emptyset$. It is also allowed, in case $I \neq \emptyset$ to have any or all of the U_i be empty.
(2) etc, etc.

6.3. Presheaves

006D

006E Definition 6.3.1. Let X be a topological space.

(1) A presheaf $\mathcal{F}$ of sets on X is a rule which assigns to each open $U \subset X$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ a map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho^U_U = \text{id}_{\mathcal{F}(U)}$ and whenever $W \subset V \subset U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on X is a rule which assigns to each open $U \subset X$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps, i.e., whenever $V \subset U \subset X$ are open the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi"] \arrow[d, "\rho^U_V"] & \mathcal{G}(U) \arrow[d, "\rho^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi"] & \mathcal{G}(V) \end{tikzcd}$$

commutes.
(3) The category of presheaves of sets on X will be denoted $\text{PSh}(X)$.

The elements of the set $\mathcal{F}(U)$ are called the sections of $\mathcal{F}$ over U. For every $V \subset U$ the map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the restriction map. We will use the notation $s|_V := \rho^U_V(s)$ if $s \in \mathcal{F}(U)$. This notation is consistent with the notion of restriction of functions from topology because if $W \subset V \subset U$ and s is a section of $\mathcal{F}$ over

U then $s|_W = (s|_V)|_W$ by the property of the restriction maps expressed in the definition above.

Another notation that is often used is to indicate sections over an open U by the symbol $\Gamma(U, -)$ or by $H^0(U, -)$. In other words, the following equalities are tautological

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U) = H^0(U, \mathcal{F}).$$

In this chapter we will not use this notation, but in others we will.

006F Definition 6.3.2. Let X be a topological space. Let A be a set. The constant presheaf with value A is the presheaf that assigns the set A to every open $U \subset X$, and such that all restriction mappings are id_A.

6.4. Abelian presheaves

006G In this section we briefly point out some features of the category of presheaves that allow one to define presheaves of abelian groups.

006H Example 6.4.1. Let X be a topological space. Consider a rule $\mathcal{F}$ that associates to every open subset of X a singleton set. Since every set has a unique map into a singleton set, there exist unique restriction maps ρ^U_V. The resulting structure is a presheaf of sets on X. It is a final object in the category of presheaves of sets on X, by the property of singleton sets mentioned above. Hence it is also unique up to unique isomorphism. We will sometimes write $*$ for this presheaf.

006I Lemma 6.4.2. Let X be a topological space. The category of presheaves of sets on X has products (see Categories, Definition 4.14.6). Moreover, the set of sections of the product $\mathcal{F} \times \mathcal{G}$ over an open U is the product of the sets of sections of $\mathcal{F}$ and $\mathcal{G}$ over U.

Proof. Namely, suppose $\mathcal{F}$ and $\mathcal{G}$ are presheaves of sets on the topological space X. Consider the rule $U \mapsto \mathcal{F}(U) \times \mathcal{G}(U)$, denoted $\mathcal{F} \times \mathcal{G}$. If $V \subset U \subset X$ are open then define the restriction mapping

$$(\mathcal{F} \times \mathcal{G})(U) \longrightarrow (\mathcal{F} \times \mathcal{G})(V)$$

by mapping $(s, t) \mapsto (s|_V, t|_V)$. Then it is immediately clear that $\mathcal{F} \times \mathcal{G}$ is a presheaf. Also, there are projection maps $p : \mathcal{F} \times \mathcal{G} \to \mathcal{F}$ and $q : \mathcal{F} \times \mathcal{G} \to \mathcal{G}$. We leave it to the reader to show that for any third presheaf $\mathcal{H}$ we have $\text{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G}) = \text{Mor}(\mathcal{H}, \mathcal{F}) \times \text{Mor}(\mathcal{H}, \mathcal{G})$. $\square$

Recall that if $(A, + : A \times A \to A, - : A \to A, 0 \in A)$ is an abelian group, then the zero and the negation maps are uniquely determined by the addition law. In other words, it makes sense to say "let $(A, +)$ be an abelian group".

006J Lemma 6.4.3. Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets. Consider the following types of structure on $\mathcal{F}$:

(1) For every open U the structure of an abelian group on $\mathcal{F}(U)$ such that all restriction maps are abelian group homomorphisms.
(2) A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a map of presheaves $- : \mathcal{F} \to \mathcal{F}$ and a map $0 : * \to \mathcal{F}$ (see Example 6.4.1) satisfying all the axioms of $+, -, 0$ in a usual abelian group.

(3) A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a map of presheaves $- : \mathcal{F} \to \mathcal{F}$ and a map $0 : * \to \mathcal{F}$ such that for each open $U \subset X$ the quadruple $(\mathcal{F}(U), +, -, 0)$ is an abelian group,
(4) A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ such that for every open $U \subset X$ the map $+ : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an abelian group.

There are natural bijections between the collections of types of data (1) - (4) above.

Proof. Omitted. □

The lemma says that to give an abelian group object $\mathcal{F}$ in the category of presheaves is the same as giving a presheaf of sets $\mathcal{F}$ such that all the sets $\mathcal{F}(U)$ are endowed with the structure of an abelian group and such that all the restriction mappings are group homomorphisms. For most algebra structures we will take this approach to (pre)sheaves of such objects, i.e., we will define a (pre)sheaf of such objects to be a (pre)sheaf $\mathcal{F}$ of sets all of whose sets of sections $\mathcal{F}(U)$ are endowed with this structure compatibly with the restriction mappings.

006K Definition 6.4.4. Let X be a topological space.

(1) A presheaf of abelian groups on X or an abelian presheaf over X is a presheaf of sets $\mathcal{F}$ such that for each open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an abelian group, and such that all restriction maps ρ^U_V are homomorphisms of abelian groups, see Lemma 6.4.3 above.
(2) A morphism of abelian presheaves over X $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets which induces a homomorphism of abelian groups $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$.
(3) The category of presheaves of abelian groups on X is denoted $\text{PAb}(X)$.

006L Example 6.4.5. Let X be a topological space. For each $x \in X$ suppose given an abelian group M_x. For $U \subset X$ open we set

$$\mathcal{F}(U) = \bigoplus\nolimits_{x \in U} M_x.$$

We denote a typical element in this abelian group by $\sum_{i=1}^n m_{x_i}$, where $x_i \in U$ and $m_{x_i} \in M_{x_i}$. (Of course we may always choose our representation such that $x_1, \ldots, x_n$ are pairwise distinct.) We define for $V \subset U \subset X$ open a restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ by mapping an element $s = \sum_{i=1}^n m_{x_i}$ to the element $s|_V = \sum_{x_i \in V} m_{x_i}$. We leave it to the reader to verify that this is a presheaf of abelian groups.

6.5. Presheaves of algebraic structures

006M Let us clarify the definition of presheaves of algebraic structures. Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to \text{Sets}$ is a faithful functor. Typically F is a "forgetful" functor. For an object $M \in \text{Ob}(\mathcal{C})$ we often call $F(M)$ the underlying set of the object M. If $M \to M'$ is a morphism in $\mathcal{C}$ we call $F(M) \to F(M')$ the underlying map of sets. In fact, we will often not distinguish between an object and its underlying set, and similarly for morphisms. So we will say a map of sets $F(M) \to F(M')$ is a morphism of algebraic structures, if it is equal to $F(f)$ for some morphism $f : M \to M'$ in $\mathcal{C}$.

In analogy with Definition 6.4.4 above a "presheaf of objects of $\mathcal{C}$" could be defined by the following data:

(1) a presheaf of sets $\mathcal{F}$, and
(2) for every open $U \subset X$ a choice of an object $A(U) \in \mathrm{Ob}(\mathcal{C})$

subject to the following conditions (using the phraseology above)

(1) for every open $U \subset X$ the set $\mathcal{F}(U)$ is the underlying set of $A(U)$, and
(2) for every $V \subset U \subset X$ open the map of sets $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of algebraic structures.

In other words, for every $V \subset U$ open in X the restriction mappings ρ^U_V is the image $F(\alpha^U_V)$ for some unique morphism $\alpha^U_V : A(U) \to A(V)$ in the category $\mathcal{C}$. The uniqueness is forced by the condition that F is faithful; it also implies that $\alpha^U_W = \alpha^V_W \circ \alpha^U_V$ whenever $W \subset V \subset U$ are open in X. The system $(A(-), \alpha^U_V)$ is what we will define as a presheaf with values in $\mathcal{C}$ on X, compare Sites, Definition 7.2.2. We recover our presheaf of sets $(\mathcal{F}, \rho^U_V)$ via the rules $\mathcal{F}(U) = F(A(U))$ and $\rho^U_V = F(\alpha^U_V)$.

006N Definition 6.5.1. Let X be a topological space. Let $\mathcal{C}$ be a category.

(1) A presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$ is given by a rule which assigns to every open $U \subset X$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ a morphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that whenever $W \subset V \subset U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$ is given by a morphism $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ compatible with restriction morphisms.

006O Definition 6.5.2. Let X be a topological space. Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to \textit{Sets}$ be a faithful functor. Let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. The presheaf of sets $U \mapsto F(\mathcal{F}(U))$ is called the underlying presheaf of sets of $\mathcal{F}$.

It is customary to use the same letter $\mathcal{F}$ to denote the underlying presheaf of sets, and this makes sense according to our discussion preceding Definition 6.5.1. In particular, the phrase "let $s \in \mathcal{F}(U)$" or "let s be a section of $\mathcal{F}$ over U" signifies that $s \in F(\mathcal{F}(U))$.

This notation and these definitions apply in particular to: Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field, etc and morphisms between these.

6.6. Presheaves of modules

006P Suppose that $\mathcal{O}$ is a presheaf of rings on X. We would like to define the notion of a presheaf of $\mathcal{O}$-modules over X. In analogy with Definition 6.4.4 we are tempted to define this as a presheaf of sets $\mathcal{F}$ such that for every open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an $\mathcal{O}(U)$-module compatible with restriction mappings (of $\mathcal{F}$ and $\mathcal{O}$). However, it is customary (and equivalent) to define it as in the following definition.

006Q Definition 6.6.1. Let X be a topological space, and let $\mathcal{O}$ be a presheaf of rings on X.

(1) A presheaf of $\mathcal{O}$-modules is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets

$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$

such that for every open $U \subset X$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.

(2) A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id}\times\varphi"'] & \mathcal{F} \arrow[d, "\varphi"] \\ \mathcal{O} \times \mathcal{G} \arrow[r] & \mathcal{G} \end{tikzcd}$$

commutes.

(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

(4) The category of presheaves of $\mathcal{O}$-modules is denoted $\mathrm{PMod}(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the restriction of $\mathcal{F}$. We obtain the restriction functor

$$\mathrm{PMod}(\mathcal{O}_2) \longrightarrow \mathrm{PMod}(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$\left(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}\right)(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the change of rings functor

$$\mathrm{PMod}(\mathcal{O}_1) \longrightarrow \mathrm{PMod}(\mathcal{O}_2)$$

006R Lemma 6.6.2. With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. ☐

6.7. Sheaves

006S In this section we explain the sheaf condition.

006T Definition 6.7.1. Let X be a topological space.

(1) A sheaf $\mathcal{F}$ of sets on X is a presheaf of sets which satisfies the following additional property: Given any open covering $U = \bigcup_{i \in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

(2) A morphism of sheaves of sets is simply a morphism of presheaves of sets.

(3) The category of sheaves of sets on X is denoted $Sh(X)$.

006U Remark 6.7.2. There is always a bit of confusion as to whether it is necessary to say something about the set of sections of a sheaf over the empty set $\emptyset \subset X$. It is necessary, and we already did if you read the definition right. Namely, note that the empty set is covered by the empty open covering, and hence the "collection of sections s_i" from the definition above actually form an element of the empty product which is the final object of the category the sheaf has values in. In other words, if you read the definition right you automatically deduce that $\mathcal{F}(\emptyset)$ = a final object, which in the case of a sheaf of sets is a singleton. If you do not like this argument, then you can just require that $\mathcal{F}(\emptyset) = \{*\}$.

In particular, this condition will then ensure that if $U, V \subset X$ are open and disjoint then

$$\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V).$$

(Because the fibre product over a final object is a product.)

006V Example 6.7.3. Let X, Y be topological spaces. Consider the rule $\mathcal{F}$ which associates to the open $U \subset X$ the set

$$\mathcal{F}(U) = \{f : U \to Y \mid f \text{ is continuous}\}$$

with the obvious restriction mappings. We claim that $\mathcal{F}$ is a sheaf. To see this suppose that $U = \bigcup_{i \in I} U_i$ is an open covering, and $f_i \in \mathcal{F}(U_i)$, $i \in I$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. In this case define $f : U \to Y$ by setting $f(u)$ equal to the value of $f_i(u)$ for any $i \in I$ such that $u \in U_i$. This is well defined by assumption. Moreover, $f : U \to Y$ is a map such that its restriction to U_i agrees with the continuous map f_i. Hence clearly f is continuous!

We can use the result of the example to define constant sheaves. Namely, suppose that A is a set. Endow A with the discrete topology. Let $U \subset X$ be an open subset. Then we have

$$\{f : U \to A \mid f \text{ continuous}\} = \{f : U \to A \mid f \text{ locally constant}\}.$$

Thus the rule which assigns to an open all locally constant maps into A is a sheaf.

006W Definition 6.7.4. Let X be a topological space. Let A be a set. The constant sheaf with value A denoted $\underline{A}$, or $\underline{A}_X$ is the sheaf that assigns to an open $U \subset X$ the set of all locally constant maps $U \to A$ with restriction mappings given by restrictions of functions.

006X Example 6.7.5. Let X be a topological space. Let $(A_x)_{x \in X}$ be a family of sets A_x indexed by points $x \in X$. We are going to construct a sheaf of sets Π from this data. For $U \subset X$ open set

$$\Pi(U) = \prod\nolimits_{x \in U} A_x.$$

For $V \subset U \subset X$ open define a restriction mapping by the following rule: An element $s = (a_x)_{x \in U} \in \Pi(U)$ restricts to $s|_V = (a_x)_{x \in V}$. It is obvious that this defines a presheaf of sets. We claim this is a sheaf. Namely, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i \in \Pi(U_i)$ are such that s_i and s_j agree over $U_i \cap U_j$. Write $s_i = (a_{i,x})_{x \in U_i}$. The compatibility condition implies that $a_{i,x} = a_{j,x}$ in the set A_x whenever $x \in U_i \cap U_j$. Hence there exists a unique element $s = (a_x)_{x \in U}$ in $\Pi(U) = \prod_{x \in U} A_x$ with the property that $a_x = a_{i,x}$ whenever $x \in U_i$ for some i. Of course this element s has the property that $s|_{U_i} = s_i$ for all i.

006Y Example 6.7.6. Let X be a topological space. Suppose for each $x \in X$ we are given an abelian group M_x. Consider the presheaf $\mathcal{F} : U \mapsto \bigoplus_{x \in U} M_x$ defined in Example 6.4.5. This is not a sheaf in general. For example, if X is an infinite set with the discrete topology, then the sheaf condition would imply that $\mathcal{F}(X) = \prod_{x \in X} \mathcal{F}(\{x\})$ but by definition we have $\mathcal{F}(X) = \bigoplus_{x \in X} M_x = \bigoplus_{x \in X} \mathcal{F}(\{x\})$. And an infinite direct sum is in general different from an infinite direct product.

However, if X is a topological space such that every open of X is quasi-compact, then $\mathcal{F}$ is a sheaf. This is left as an exercise to the reader.

6.8. Abelian sheaves

006Z

0070 Definition 6.8.1. Let X be a topological space.

(1) An abelian sheaf on X or sheaf of abelian groups on X is an abelian presheaf on X such that the underlying presheaf of sets is a sheaf.
(2) The category of sheaves of abelian groups is denoted $\textit{Ab}(X)$.

Let X be a topological space. In the case of an abelian presheaf $\mathcal{F}$ the sheaf condition with regards to an open covering $U = \bigcup U_i$ is often expressed by saying that the complex of abelian groups

$$0 \to \mathcal{F}(U) \to \prod\nolimits_i \mathcal{F}(U_i) \to \prod\nolimits_{(i_0, i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is exact. The first map is the usual one, whereas the second maps the element $(s_i)_{i \in I}$ to the element

$$(s_{i_0}|_{U_{i_0} \cap U_{i_1}} - s_{i_1}|_{U_{i_0} \cap U_{i_1}})_{(i_0, i_1)} \in \prod\nolimits_{(i_0, i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

6.9. Sheaves of algebraic structures

0071 Let us clarify the definition of sheaves of certain types of structures. First, let us reformulate the sheaf condition. Namely, suppose that $\mathcal{F}$ is a presheaf of sets on the topological space X. The sheaf condition can be reformulated as follows. Let $U = \bigcup_{i \in I} U_i$ be an open covering. Consider the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

Here the left map is defined by the rule $s \mapsto \prod_{i \in I} s|_{U_i}$. The two maps on the right are the maps

$$\prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0, i_1)} s_{i_0}|_{U_{i_0} \cap U_{i_1}} \text{ resp. } \prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0, i_1)} s_{i_1}|_{U_{i_0} \cap U_{i_1}}.$$

The sheaf condition exactly says that the left arrow is the equalizer of the right two. This generalizes immediately to the case of presheaves with values in a category as long as the category has products.

0072 Definition 6.9.1. Let X be a topological space. Let $\mathcal{C}$ be a category with products. A presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X is a sheaf if for every open covering the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is an equalizer diagram in the category $\mathcal{C}$.

Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to \text{Sets}$ is a faithful functor. A good example to keep in mind is the case where $\mathcal{C}$ is the category of abelian groups and F is the forgetful functor. Consider a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X. We would like to reformulate the condition above in terms of the underlying presheaf of sets (Definition 6.5.2). Note that the underlying presheaf of sets is a sheaf of sets if and only if all the diagrams

$$F(\mathcal{F}(U)) \longrightarrow \prod_{i\in I} F(\mathcal{F}(U_i)) \rightrightarrows \prod_{(i_0,i_1)\in I\times I} F(\mathcal{F}(U_{i_0} \cap U_{i_1}))$$

of sets – after applying the forgetful functor F – are equalizer diagrams! Thus we would like $\mathcal{C}$ to have products and equalizers and we would like F to commute with them. This is equivalent to the condition that $\mathcal{C}$ has limits and that F commutes with them, see Categories, Lemma 4.14.11. But this is not yet good enough (see Example 6.9.4); we also need F to reflect isomorphisms. This property means that given a morphism $f : A \to A'$ in $\mathcal{C}$, then f is an isomorphism if (and only if) $F(f)$ is a bijection.

0073 Lemma 6.9.2. Suppose the category $\mathcal{C}$ and the functor $F : \mathcal{C} \to \text{Sets}$ have the following properties:

(1) F is faithful,
(2) $\mathcal{C}$ has limits and F commutes with them, and
(3) the functor F reflects isomorphisms.

Let X be a topological space. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then $\mathcal{F}$ is a sheaf if and only if the underlying presheaf of sets is a sheaf.

Proof. Assume that $\mathcal{F}$ is a sheaf. Then $\mathcal{F}(U)$ is the equalizer of the diagram above and by assumption we see $F(\mathcal{F}(U))$ is the equalizer of the corresponding diagram of sets. Hence $F(\mathcal{F})$ is a sheaf of sets.

Assume that $F(\mathcal{F})$ is a sheaf. Let $E \in \text{Ob}(\mathcal{C})$ be the equalizer of the two parallel arrows in Definition 6.9.1. We get a canonical morphism $\mathcal{F}(U) \to E$, simply because $\mathcal{F}$ is a presheaf. By assumption, the induced map $F(\mathcal{F}(U)) \to F(E)$ is an isomorphism, because $F(E)$ is the equalizer of the corresponding diagram of sets. Hence we see $\mathcal{F}(U) \to E$ is an isomorphism by condition (3) of the lemma. □

The lemma in particular applies to sheaves of groups, rings, algebras over a fixed ring, modules over a fixed ring, vector spaces over a fixed field, etc. In other words, these are presheaves of groups, rings, modules over a fixed ring, vector spaces over a fixed field, etc such that the underlying presheaf of sets is a sheaf.

0074 Example 6.9.3. Let X be a topological space. For each open $U \subset X$ consider the $\mathbf{R}$-algebra $\mathcal{C}^0(U) = \{f : U \to \mathbf{R} \mid f \text{ is continuous}\}$. There are obvious restriction mappings that turn this into a presheaf of $\mathbf{R}$-algebras over X. By Example 6.7.3 it is a sheaf of sets. Hence by the Lemma 6.9.2 it is a sheaf of $\mathbf{R}$-algebras over X.

0075 Example 6.9.4. Consider the category of topological spaces Top. There is a natural faithful functor Top $\to$ Sets which commutes with products and equalizers. But it does not reflect isomorphisms. And, in fact it turns out that the analogue of Lemma 6.9.2 is wrong. Namely, suppose $X = \mathbf{N}$ with the discrete topology. Let A_i, for $i \in \mathbf{N}$ be a discrete topological space. For any subset $U \subset \mathbf{N}$ define $\mathcal{F}(U) = \prod_{i\in U} A_i$ with the discrete topology. Then this is a presheaf of topological spaces whose underlying presheaf of sets is a sheaf, see Example 6.7.5. However, if each A_i has at least two elements, then this is not a sheaf of topological spaces according to Definition 6.9.1. The reader may check that putting the product topology on each $\mathcal{F}(U) = \prod_{i \in U} A_i$ does lead to a sheaf of topological spaces over X.

6.10. Sheaves of modules

0076

0077 Definition 6.10.1. Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X.

(1) A sheaf of $\mathcal{O}$-modules is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 6.6.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
(2) A morphism of sheaves of $\mathcal{O}$-modules is a morphism of presheaves of $\mathcal{O}$-modules.
(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\text{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
(4) The category of sheaves of $\mathcal{O}$-modules is denoted $\text{Mod}(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology "sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

6.11. Stalks

0078 Let X be a topological space. Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The stalk of $\mathcal{F}$ at x is the set

$$\mathcal{F}_x = \text{colim}_{x \in U} \mathcal{F}(U)$$

where the colimit is over the set of open neighbourhoods U of x in X. The set of open neighbourhoods is partially ordered by (reverse) inclusion: We say $U \geq U' \Leftrightarrow U \subset U'$. The transition maps in the system are given by the restriction maps of $\mathcal{F}$. See Categories, Section 4.21 for notation and terminology regarding (co)limits over systems. Note that the colimit is a directed colimit. Thus it is easy to describe $\mathcal{F}_x$. Namely,

$$\mathcal{F}_x = \{(U, s) \mid x \in U, s \in \mathcal{F}(U)\}/\sim$$

with equivalence relation given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ with $x \in U''$ and $s|_{U''} = s'|_{U''}$. By abuse of notation we will often denote (U, s), s_x, or even s the corresponding element in $\mathcal{F}_x$. Also we will say $s = s'$ in $\mathcal{F}_x$ for two local sections of $\mathcal{F}$ defined in an open neighbourhood of x to denote that they have the same image in $\mathcal{F}_x$.

An obvious consequence of this definition is that for any open $U \subset X$ there is a canonical map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

defined by $s \mapsto \prod_{x \in U}(U, s)$. Think about it!

0079 Lemma 6.11.1. Let $\mathcal{F}$ be a sheaf of sets on the topological space X. For every open $U \subset X$ the map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

is injective.

Proof. Suppose that $s, s' \in \mathcal{F}(U)$ map to the same element in every stalk $\mathcal{F}_x$ for all $x \in U$. This means that for every $x \in U$, there exists an open $V^x \subset U$, $x \in V^x$ such that $s|_{V^x} = s'|_{V^x}$. But then $U = \bigcup_{x \in U} V^x$ is an open covering. Thus by the uniqueness in the sheaf condition we see that $s = s'$. $\square$

007A Definition 6.11.2. Let X be a topological space. A presheaf of sets $\mathcal{F}$ on X is separated if for every open $U \subset X$ the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective.

Another observation is that the construction of the stalk $\mathcal{F}_x$ is functorial in the presheaf $\mathcal{F}$. In other words, it gives a functor

$$\mathrm{PSh}(X) \longrightarrow \mathrm{Sets}, \ \mathcal{F} \longmapsto \mathcal{F}_x.$$

This functor is called the stalk functor. Namely, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then we define $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ by the rule $(U, s) \mapsto (U, \varphi(s))$. To see that this works we have to check that if $(U, s) = (U', s')$ in $\mathcal{F}_x$ then also $(U, \varphi(s)) = (U', \varphi(s'))$ in $\mathcal{G}_x$. This is clear since φ is compatible with the restriction mappings.

007B Example 6.11.3. Let X be a topological space. Let A be a set. Denote temporarily A_p the constant presheaf with value A (p for presheaf – not for point). There is a canonical map of presheaves $A_p \to \underline{A}$ into the constant sheaf with value A. For every point we have canonical bijections $A = (A_p)_x = \underline{A}_x$, where the second map is induced by functoriality from the map $A_p \to \underline{A}$.

007C Example 6.11.4. Suppose $X = \mathbf{R}^n$ with the Euclidean topology. Consider the presheaf of $\mathcal{C}^\infty$ functions on X, denoted $\mathcal{C}^\infty_{\mathbf{R}^n}$. In other words, $\mathcal{C}^\infty_{\mathbf{R}^n}(U)$ is the set of $\mathcal{C}^\infty$-functions $f : U \to \mathbf{R}$. As in Example 6.7.3 it is easy to show that this is a sheaf. In fact it is a sheaf of $\mathbf{R}$-vector spaces.

Next, let $x \in X = \mathbf{R}^n$ be a point. How do we think of an element in the stalk $\mathcal{C}^\infty_{\mathbf{R}^n,x}$? Such an element is given by a $\mathcal{C}^\infty$-function f whose domain contains x. And a pair of such functions f, g determine the same element of the stalk if they agree in a neighbourhood of x. In other words, an element if $\mathcal{C}^\infty_{\mathbf{R}^n,x}$ is the same thing as what is sometimes called a germ of a $\mathcal{C}^\infty$-function at x.

007D Example 6.11.5. Let X be a topological space. Let A_x be a set for each $x \in X$. Consider the sheaf $\mathcal{F} : U \mapsto \prod_{x \in U} A_x$ of Example 6.7.5. We would just like to point out here that the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is in general not equal to the set A_x. Of course there is a map $\mathcal{F}_x \to A_x$, but that is in general the best you can say. For example, suppose $x = \lim x_n$ with $x_n \neq x_m$ for all $n \neq m$ and suppose that $A_y = \{0, 1\}$ for all $y \in X$. Then $\mathcal{F}_x$ maps onto the (infinite) set of tails of sequences of 0s and 1s. Namely, every open neighbourhood of x contains almost all of the x_n. On the other hand, if every neighbourhood of x contains a point y such that $A_y = \emptyset$, then $\mathcal{F}_x = \emptyset$.

6.12. Stalks of abelian presheaves

007E We first deal with the case of abelian groups as a model for the general case.

007F Lemma 6.12.1. Let X be a topological space. Let $\mathcal{F}$ be a presheaf of abelian groups on X. There exists a unique structure of an abelian group on $\mathcal{F}_x$ such that for every $U \subset X$ open, $x \in U$ the map $\mathcal{F}(U) \to \mathcal{F}_x$ is a group homomorphism. Moreover,

$$\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$$

holds in the category of abelian groups.

Proof. We define addition of a pair of elements (U, s) and (V, t) as the pair $(U \cap V, s|_{U\cap V} + t|_{U\cap V})$. The rest is easy to check. $\square$

What is crucial in the proof above is that the partially ordered set of open neighbourhoods is a directed set (Categories, Definition 4.21.1). Namely, the coproduct of two abelian groups A, B is the direct sum $A \oplus B$, whereas the coproduct in the category of sets is the disjoint union $A \amalg B$, showing that colimits in the category of abelian groups do not agree with colimits in the category of sets in general.

6.13. Stalks of presheaves of algebraic structures

007G The proof of Lemma 6.12.1 will work for any type of algebraic structure such that directed colimits commute with the forgetful functor.

007H Lemma 6.13.1. Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to \text{Sets}$ be a functor. Assume that

(1) F is faithful, and
(2) directed colimits exist in $\mathcal{C}$ and F commutes with them.

Let X be a topological space. Let $x \in X$. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then

$$\mathcal{F}_x = \text{colim}_{x\in U} \mathcal{F}(U)$$

exists in $\mathcal{C}$. Its underlying set is equal to the stalk of the underlying presheaf of sets of $\mathcal{F}$. Furthermore, the construction $\mathcal{F} \mapsto \mathcal{F}_x$ is a functor from the category of presheaves with values in $\mathcal{C}$ to $\mathcal{C}$.

Proof. Omitted. $\square$

By the very definition, all the morphisms $\mathcal{F}(U) \to \mathcal{F}_x$ are morphisms in the category $\mathcal{C}$ which (after applying the forgetful functor F) turn into the corresponding maps for the underlying sheaf of sets. As usual we will not distinguish between the morphism in $\mathcal{C}$ and the underlying map of sets, which is permitted since F is faithful.

This lemma applies in particular to: Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field.

6.14. Stalks of presheaves of modules

007I

007J Lemma 6.14.1. Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $x \in X$. The canonical map $\mathcal{O}_x \times \mathcal{F}_x \to \mathcal{F}_x$ coming from the multiplication map $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ defines a $\mathcal{O}_x$-module structure on the abelian group $\mathcal{F}_x$.

Proof. Omitted. $\square$

007K Lemma 6.14.2. Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of presheaves of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $x \in X$. We have

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Omitted. $\square$

6.15. Algebraic structures

007L In this section we mildly formalize the notions we have encountered in the sections above.

007M Definition 6.15.1. A type of algebraic structure is given by a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \textit{Sets}$ with the following properties

(1) F is faithful,
(2) $\mathcal{C}$ has limits and F commutes with limits,
(3) $\mathcal{C}$ has filtered colimits and F commutes with them, and
(4) F reflects isomorphisms.

We make this definition to point out the properties we will use in a number of arguments below. But we will not actually study this notion in any great detail, since we are prohibited from studying "big" categories by convention, except for those listed in Categories, Remark 4.2.2. Among those the following have the required properties.

007N Lemma 6.15.2. The following categories, endowed with the obvious forgetful functor, define types of algebraic structures:

(1) The category of pointed sets.
(2) The category of abelian groups.
(3) The category of groups.
(4) The category of monoids.
(5) The category of rings.
(6) The category of R-modules for a fixed ring R.
(7) The category of Lie algebras over a fixed field.

Proof. Omitted. $\square$

From now on we will think of a (pre)sheaf of algebraic structures and their stalks, in terms of the underlying (pre)sheaf of sets. This is allowable by Lemmas 6.9.2 and 6.13.1.

In the rest of this section we point out some results on algebraic structures that will be useful in the future.

007O Lemma 6.15.3. Let $(\mathcal{C}, F)$ be a type of algebraic structure.

(1) $\mathcal{C}$ has a final object 0 and $F(0) = \{*\}$.
(2) $\mathcal{C}$ has products and $F(\prod A_i) = \prod F(A_i)$.
(3) $\mathcal{C}$ has fibre products and $F(A \times_B C) = F(A) \times_{F(B)} F(C)$.
(4) $\mathcal{C}$ has equalizers, and if $E \to A$ is the equalizer of $a, b : A \to B$, then $F(E) \to F(A)$ is the equalizer of $F(a), F(b) : F(A) \to F(B)$.
(5) $A \to B$ is a monomorphism if and only if $F(A) \to F(B)$ is injective.
(6) if $F(a) : F(A) \to F(B)$ is surjective, then a is an epimorphism.
(7) given $A_1 \to A_2 \to A_3 \to \ldots$, then $\text{colim}\, A_i$ exists and $F(\text{colim}\, A_i) = \text{colim}\, F(A_i)$, and more generally for any filtered colimit.

Proof. Omitted. The only interesting statement is (5) which follows because $A \to B$ is a monomorphism if and only if $A \to A \times_B A$ is an isomorphism, and then applying the fact that F reflects isomorphisms. $\square$

007P Lemma 6.15.4. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $A, B, C \in \mathrm{Ob}(\mathcal{C})$. Let $f : A \to B$ and $g : C \to B$ be morphisms of $\mathcal{C}$. If $F(g)$ is injective, and $\mathrm{Im}(F(f)) \subset \mathrm{Im}(F(g))$, then f factors as $f = g \circ t$ for some morphism $t : A \to C$.

Proof. Consider $A \times_B C$. The assumptions imply that $F(A \times_B C) = F(A) \times_{F(B)} F(C) = F(A)$. Hence $A = A \times_B C$ because F reflects isomorphisms. The result follows. □

007Q Example 6.15.5. The lemma will be applied often to the following situation. Suppose that we have a diagram

$$\begin{tikzcd} A \arrow[r] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$. Suppose $C \to D$ is injective on underlying sets, and suppose that the composition $A \to B \to D$ has image on underlying sets in the image of $C \to D$. Then we get a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$.

007R Example 6.15.6. Let $F : \mathcal{C} \to \textit{Sets}$ be a type of algebraic structures. Let X be a topological space. Suppose that for every $x \in X$ we are given an object $A_x \in \mathrm{Ob}(\mathcal{C})$. Consider the presheaf Π with values in $\mathcal{C}$ on X defined by the rule $\Pi(U) = \prod_{x \in U} A_x$ (with obvious restriction mappings). Note that the associated presheaf of sets $U \mapsto F(\Pi(U)) = \prod_{x \in U} F(A_x)$ is a sheaf by Example 6.7.5. Hence Π is a sheaf of algebraic structures of type $(\mathcal{C}, F)$. This gives many examples of sheaves of abelian groups, groups, rings, etc.

6.16. Exactness and points

007S In any category we have the notion of epimorphism, monomorphism, isomorphism, etc.

007T Lemma 6.16.1. Let X be a topological space. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on X.

(1) The map φ is a monomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective.
(2) The map φ is an epimorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.
(3) The map φ is an isomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective.

Proof. Omitted. □

It follows that in the category of sheaves of sets the notions epimorphism and monomorphism can be described as follows.

007U Definition 6.16.2. Let X be a topological space.

(1) A presheaf $\mathcal{F}$ is called a subpresheaf of a presheaf $\mathcal{G}$ if $\mathcal{F}(U) \subset \mathcal{G}(U)$ for all open $U \subset X$ such that the restriction maps of $\mathcal{G}$ induce the restriction maps of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\mathcal{F}$ is called a subsheaf of $\mathcal{G}$. We sometimes indicate this by the notation $\mathcal{F} \subset \mathcal{G}$.
(2) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called injective if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(3) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called surjective if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all U open in X.
(4) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called injective if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(5) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called surjective if and only if for every open U of X and every section s of $\mathcal{G}(U)$ there exists an open covering $U = \bigcup U_i$ such that $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$ for all i.

007V Lemma 6.16.3. Let X be a topological space.

(1) Epimorphisms (resp. monomorphisms) in the category of presheaves are exactly the surjective (resp. injective) maps of presheaves.
(2) Epimorphisms (resp. monomorphisms) in the category of sheaves are exactly the surjective (resp. injective) maps of sheaves, and are exactly those maps which are surjective (resp. injective) on all the stalks.
(3) The sheafification of a surjective (resp. injective) morphism of presheaves of sets is surjective (resp. injective).

Proof. Omitted. □

007W Lemma 6.16.4. let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $\mathcal{F}, \mathcal{G}$ are sheaves on X with values in $\mathcal{C}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of the underlying sheaves of sets. If for all points $x \in X$ the map $\mathcal{F}_x \to \mathcal{G}_x$ is a morphism of algebraic structures, then φ is a morphism of sheaves of algebraic structures.

Proof. Let U be an open subset of X. Consider the diagram of (underlying) sets

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \prod_{x \in U} \mathcal{F}_x \arrow[d] \\ \mathcal{G}(U) \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd}$$

By assumption, and previous results, all but the left vertical arrow are morphisms of algebraic structures. In addition the bottom horizontal arrow is injective, see Lemma 6.11.1. Hence we conclude by Lemma 6.15.4, see also Example 6.15.5 □

Short exact sequences of abelian sheaves, etc will be discussed in the chapter on sheaves of modules. See Modules, Section 17.3.

6.17. Sheafification

007X In this section we explain how to get the sheafification of a presheaf on a topological space. We will use stalks to describe the sheafification in this case. This is different from the general procedure described in Sites, Section 7.10, and perhaps somewhat easier to understand.

The basic construction is the following. Let $\mathcal{F}$ be a presheaf of sets on a topological space X. For every open $U \subset X$ we define

$$\mathcal{F}^\#(U) = \{(s_u) \in \prod\nolimits_{u \in U} \mathcal{F}_u \text{ such that } (*)\}$$

where $(*)$ is the property:

(*) For every $u \in U$, there exists an open neighbourhood $u \in V \subset U$, and a section $\sigma \in \mathcal{F}(V)$ such that for all $v \in V$ we have $s_v = (V, \sigma)$ in $\mathcal{F}_v$.

Note that $(*)$ is a condition for each $u \in U$, and that given $u \in U$ the truth of this condition depends only on the values s_v for v in any open neighbourhood of u. Thus it is clear that, if $V \subset U \subset X$ are open, the projection maps

$$\prod\nolimits_{u \in U} \mathcal{F}_u \longrightarrow \prod\nolimits_{v \in V} \mathcal{F}_v$$

maps elements of $\mathcal{F}^\#(U)$ into $\mathcal{F}^\#(V)$. Using these maps as the restriction mappings, we turn $\mathcal{F}^\#$ into a presheaf of sets on X.

Furthermore, the map $\mathcal{F}(U) \to \prod_{u \in U} \mathcal{F}_u$ described in Section 6.11 clearly has image in $\mathcal{F}^\#(U)$. In addition, if $V \subset U \subset X$ are open then we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}^\#(U) \arrow[r] \arrow[d] & \prod_{u \in U} \mathcal{F}_u \arrow[d] \\
\mathcal{F}(V) \arrow[r] & \mathcal{F}^\#(V) \arrow[r] & \prod_{v \in V} \mathcal{F}_v
\end{tikzcd}$$

where the vertical maps are induced from the restriction mappings. Thus we see that there is a canonical morphism of presheaves $\mathcal{F} \to \mathcal{F}^\#$.

In Example 6.7.5 we saw that the rule $\Pi(\mathcal{F}) : U \mapsto \prod_{u \in U} \mathcal{F}_u$ is a sheaf, with obvious restriction mappings. And by construction $\mathcal{F}^\#$ is a subpresheaf of this. In other words, we have morphisms of presheaves

$$\mathcal{F} \to \mathcal{F}^\# \to \Pi(\mathcal{F}).$$

In addition the rule that associates to $\mathcal{F}$ the sequence above is clearly functorial in the presheaf $\mathcal{F}$. This notation will be used in the proofs of the lemmas below.

007Y **Lemma 6.17.1.** The presheaf $\mathcal{F}^\#$ is a sheaf.

Proof. It is probably better for the reader to find their own explanation of this than to read the proof here. In fact the lemma is true for the same reason as why the presheaf of continuous function is a sheaf, see Example 6.7.3 (and this analogy can be made precise using the "espace étalé").

Anyway, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i = (s_{i,u})_{u \in U_i} \in \mathcal{F}^\#(U_i)$ such that s_i and s_j agree over $U_i \cap U_j$. Because $\Pi(\mathcal{F})$ is a sheaf, we find an element $s = (s_u)_{u \in U}$ in $\prod_{u \in U} \mathcal{F}_u$ restricting to s_i on U_i. We have to check property $(*)$. Pick $u \in U$. Then $u \in U_i$ for some i. Hence by $(*)$ for s_i, there exists a V open, $u \in V \subset U_i$ and a $\sigma \in \mathcal{F}(V)$ such that $s_{i,v} = (V, \sigma)$ in $\mathcal{F}_v$ for all $v \in V$. Since $s_{i,v} = s_v$ we get $(*)$ for s. □

007Z **Lemma 6.17.2.** Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $x \in X$. Then $\mathcal{F}_x = \mathcal{F}^\#_x$.

Proof. The map $\mathcal{F}_x \to \mathcal{F}_x^\#$ is injective, since already the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ is injective. Namely, there is a canonical map $\Pi(\mathcal{F})_x \to \mathcal{F}_x$ which is a left inverse to the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$, see Example 6.11.5. To show that it is surjective, suppose that $\overline{s} \in \mathcal{F}_x^\#$. We can find an open neighbourhood U of x such that $\overline{s}$ is the equivalence class of (U, s) with $s \in \mathcal{F}^\#(U)$. By definition, this means there exists an open neighbourhood $V \subset U$ of x and a section $\sigma \in \mathcal{F}(V)$ such that $s|_V$ is the image of σ in $\Pi(\mathcal{F})(V)$. Clearly the class of (V, σ) defines an element of $\mathcal{F}_x$ mapping to $\overline{s}$. $\square$

0080 Lemma 6.17.3. Let $\mathcal{F}$ be a presheaf of sets on X. Any map $\mathcal{F} \to \mathcal{G}$ into a sheaf of sets factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.

Proof. Clearly, there is a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^\# \arrow[r] \arrow[d] & \Pi(\mathcal{F}) \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{G}^\# \arrow[r] & \Pi(\mathcal{G}) \end{tikzcd}$$

So it suffices to prove that $\mathcal{G} = \mathcal{G}^\#$. To see this it suffices to prove, for every point $x \in X$ the map $\mathcal{G}_x \to \mathcal{G}_x^\#$ is bijective, by Lemma 6.16.1. And this is Lemma 6.17.2 above. $\square$

This lemma really says that there is an adjoint pair of functors: $i : \mathit{Sh}(X) \to \mathrm{PSh}(X)$ (inclusion) and $\# : \mathrm{PSh}(X) \to \mathit{Sh}(X)$ (sheafification). The formula is that

$$\mathrm{Mor}_{\mathrm{PSh}(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{\mathit{Sh}(X)}(\mathcal{F}^\#, \mathcal{G})$$

which says that sheafification is a left adjoint of the inclusion functor. See Categories, Section 4.24.

0081 Example 6.17.4. See Example 6.11.3 for notation. The map $A_p \to \underline{A}$ induces a map $A_p^\# \to \underline{A}$. It is easy to see that this is an isomorphism. In words: The sheafification of the constant presheaf with value A is the constant sheaf with value A.

0082 Lemma 6.17.5. Let X be a topological space. A presheaf $\mathcal{F}$ is separated (see Definition 6.11.2) if and only if the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is injective.

Proof. This is clear from the construction of $\mathcal{F}^\#$ in this section. $\square$

6.18. Sheafification of abelian presheaves

0083 The following strange looking lemma is likely unnecessary, but very convenient to deal with sheafification of presheaves of algebraic structures.

0084 Lemma 6.18.1. Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $U \subset X$ be open. There is a canonical fibre product diagram

$$\begin{tikzcd} \mathcal{F}^\#(U) \arrow[r] \arrow[d] & \Pi(\mathcal{F})(U) \arrow[d] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \Pi(\mathcal{F})_x \end{tikzcd}$$

where the maps are the following:

(1) The left vertical map has components $\mathcal{F}^\#(U) \to \mathcal{F}^\#_x = \mathcal{F}_x$ where the equality is Lemma 6.17.2.
(2) The top horizontal map comes from the map of presheaves $\mathcal{F} \to \Pi(\mathcal{F})$ described in Section 6.17.
(3) The right vertical map has obvious component maps $\Pi(\mathcal{F})(U) \to \Pi(\mathcal{F})_x$.
(4) The bottom horizontal map has components $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ which come from the map of presheaves $\mathcal{F} \to \Pi(\mathcal{F})$ described in Section 6.17.

Proof. It is clear that the diagram commutes. We have to show it is a fibre product diagram. The bottom horizontal arrow is injective since all the maps $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ are injective (see beginning proof of Lemma 6.17.2). A section $s \in \Pi(\mathcal{F})(U)$ is in $\mathcal{F}^\#$ if and only if $(*)$ holds. But $(*)$ says that around every point the section s comes from a section of $\mathcal{F}$. By definition of the stalk functors, this is equivalent to saying that the value of s in every stalk $\Pi(\mathcal{F})_x$ comes from an element of the stalk $\mathcal{F}_x$. Hence the lemma. □

0085 Lemma 6.18.2. Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Then there exists a unique structure of abelian sheaf on $\mathcal{F}^\#$ such that $\mathcal{F} \to \mathcal{F}^\#$ is a morphism of abelian presheaves. Moreover, the following adjointness property holds

$$\mathrm{Mor}_{\mathrm{PAb}(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{\mathrm{Ab}(X)}(\mathcal{F}^\#, \mathcal{G}).$$

Proof. Recall the sheaf of sets $\Pi(\mathcal{F})$ defined in Section 6.17. All the stalks $\mathcal{F}_x$ are abelian groups, see Lemma 6.12.1. Hence $\Pi(\mathcal{F})$ is a sheaf of abelian groups by Example 6.15.6. Also, it is clear that the map $\mathcal{F} \to \Pi(\mathcal{F})$ is a morphism of abelian presheaves. If we show that condition $(*)$ of Section 6.17 defines a subgroup of $\Pi(\mathcal{F})(U)$ for all open subsets $U \subset X$, then $\mathcal{F}^\#$ canonically inherits the structure of abelian sheaf. This is quite easy to do by hand, and we leave it to the reader to find a good simple argument. The argument we use here, which generalizes to presheaves of algebraic structures is the following: Lemma 6.18.1 show that $\mathcal{F}^\#(U)$ is the fibre product of a diagram of abelian groups. Thus $\mathcal{F}^\#$ is an abelian subgroup as desired.

Note that at this point $\mathcal{F}^\#_x$ is an abelian group by Lemma 6.12.1 and that $\mathcal{F}_x \to \mathcal{F}^\#_x$ is a bijection (Lemma 6.17.2) and a homomorphism of abelian groups. Hence $\mathcal{F}_x \to \mathcal{F}^\#_x$ is an isomorphism of abelian groups. This will be used below without further mention.

To prove the adjointness property we use the adjointness property of sheafification of presheaves of sets. For example if $\psi : \mathcal{F} \to i(\mathcal{G})$ is morphism of presheaves then we obtain a morphism of sheaves $\psi' : \mathcal{F}^\# \to \mathcal{G}$. What we have to do is to check that this is a morphism of abelian sheaves. We may do this for example by noting that it is true on stalks, by Lemma 6.17.2, and then using Lemma 6.16.4 above. □

6.19. Sheafification of presheaves of algebraic structures

0086

0087 Lemma 6.19.1. Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$ on X. Then there exists a sheaf $\mathcal{F}^\#$ with values in $\mathcal{C}$ and a morphism $\mathcal{F} \to \mathcal{F}^\#$ of presheaves with values in $\mathcal{C}$ with the following properties:

(1) The map $\mathcal{F} \to \mathcal{F}^\#$ identifies the underlying sheaf of sets of $\mathcal{F}^\#$ with the sheafification of the underlying presheaf of sets of $\mathcal{F}$.
(2) For any morphism $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf with values in $\mathcal{C}$ there exists a unique factorization $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.

Proof. The proof is the same as the proof of Lemma 6.18.2, with repeated application of Lemma 6.15.4 (see also Example 6.15.5). The main idea however, is to define $\mathcal{F}^\#(U)$ as the fibre product in $\mathcal{C}$ of the diagram

$$\begin{tikzcd} & \Pi(\mathcal{F})(U) \arrow[d] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \Pi(\mathcal{F})_x \end{tikzcd}$$

compare Lemma 6.18.1. □

6.20. Sheafification of presheaves of modules

0088

0089 Lemma 6.20.1. Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O}^\# \times \mathcal{F}^\# \arrow[r] & \mathcal{F}^\# \end{tikzcd}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. □

This actually means that the functor $i : \text{Mod}(\mathcal{O}^\#) \to \text{PMod}(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# : \text{PMod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}^\#)$ are adjoint. In a formula

$$\text{Mor}_{\text{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \text{Mor}_{\text{Mod}(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on X. In Section 6.6 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$\text{Mod}(\mathcal{O}_2) \longrightarrow \text{Mod}(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the tensor product sheaf $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^{\#}$$

as the sheafification of our construction for presheaves. We obtain the change of rings functor

$$\mathrm{Mod}(\mathcal{O}_1) \longrightarrow \mathrm{Mod}(\mathcal{O}_2)$$

008A Lemma 6.20.2. With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 6.6.2 and the fact that $\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. $\square$

008B Lemma 6.20.3. Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf $\mathcal{O}$-modules. Let $x \in X$. We have

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Follows directly from Lemma 6.14.2 and the fact that taking stalks commutes with sheafification. $\square$

6.21. Continuous maps and sheaves

008C Let $f : X \to Y$ be a continuous map of topological spaces. We will define the pushforward and pullback functors for presheaves and sheaves.

Let $\mathcal{F}$ be a presheaf of sets on X. We define the pushforward of $\mathcal{F}$ by the rule

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for any open $V \subset Y$. Given $V_1 \subset V_2 \subset Y$ open the restriction map is given by the commutativity of the diagram

$$\begin{tikzcd} f_*\mathcal{F}(V_2) \arrow[r, equal] \arrow[d] & \mathcal{F}(f^{-1}(V_2)) \arrow[d, "\text{restriction for } \mathcal{F}"] \\ f_*\mathcal{F}(V_1) \arrow[r, equal] & \mathcal{F}(f^{-1}(V_1)) \end{tikzcd}$$

It is clear that this defines a presheaf of sets. The construction is clearly functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* : \mathrm{PSh}(X) \longrightarrow \mathrm{PSh}(Y).$$

008D Lemma 6.21.1. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X. Then $f_*\mathcal{F}$ is a sheaf on Y.

Proof. This immediately follows from the fact that if $V = \bigcup V_j$ is an open covering in Y, then $f^{-1}(V) = \bigcup f^{-1}(V_j)$ is an open covering in X. $\square$

As a consequence we obtain a functor

$$f_* : Sh(X) \longrightarrow Sh(Y).$$

This is compatible with composition in the following strong sense.

008E Lemma 6.21.2. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal (on both presheaves and sheaves of sets).

Proof. This is because $(g \circ f)_*\mathcal{F}(W) = \mathcal{F}((g \circ f)^{-1}W)$ and $(g_* \circ f_*)\mathcal{F}(W) = \mathcal{F}(f^{-1}g^{-1}W)$ and $(g \circ f)^{-1}W = f^{-1}g^{-1}W$. $\square$

Let $\mathcal{G}$ be a presheaf of sets on Y. The pullback presheaf $f_p\mathcal{G}$ of a given presheaf $\mathcal{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words it should be a presheaf $f_p\mathcal{G}$ on X such that

$$\mathrm{Mor}_{\mathrm{PSh}(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PSh}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

By the Yoneda lemma this determines the pullback uniquely. It turns out that it actually exists.

008F Lemma 6.21.3. Let $f : X \to Y$ be a continuous map. There exists a functor $f_p : \mathrm{PSh}(Y) \to \mathrm{PSh}(X)$ which is left adjoint to f_*. For a presheaf $\mathcal{G}$ it is determined by the rule

$$f_p\mathcal{G}(U) = \mathrm{colim}_{f(U) \subset V}\, \mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods V of $f(U)$ in Y. The colimits are over directed partially ordered sets. (The restriction mappings of $f_p\mathcal{G}$ are explained in the proof.)

Proof. The colimit is over the partially ordered set consisting of open subsets $V \subset Y$ which contain $f(U)$ with ordering by reverse inclusion. This is a directed partially ordered set, since if V, V' are in it then so is $V \cap V'$. Furthermore, if $U_1 \subset U_2$, then every open neighbourhood of $f(U_2)$ is an open neighbourhood of $f(U_1)$. Hence the system defining $f_p\mathcal{G}(U_2)$ is a subsystem of the one defining $f_p\mathcal{G}(U_1)$ and we obtain a restriction map (for example by applying the generalities in Categories, Lemma 4.14.8).

Note that the construction of the colimit is clearly functorial in $\mathcal{G}$, and similarly for the restriction mappings. Hence we have defined f_p as a functor.

A small useful remark is that there exists a canonical map $\mathcal{G}(U) \to f_p\mathcal{G}(f^{-1}(U))$, because the system of open neighbourhoods of $f(f^{-1}(U))$ contains the element U. This is compatible with restriction mappings. In other words, there is a canonical map $i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G}$.

Let $\mathcal{F}$ be a presheaf of sets on X. Suppose that $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of presheaves of sets. The corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$.

Another small useful remark is that there exists a canonical map $c_\mathcal{F} : f_pf_*\mathcal{F} \to \mathcal{F}$. Namely, let $U \subset X$ open. For every open neighbourhood $V \supset f(U)$ in Y there exists a map $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$, namely the restriction map on $\mathcal{F}$. And this is compatible with the restriction mappings between values of $\mathcal{F}$ on f^{-1} of varying opens containing $f(U)$. Thus we obtain a canonical map $f_pf_*\mathcal{F}(U) \to \mathcal{F}(U)$. Another trivial verification shows that these maps are compatible with restriction maps and define a map $c_\mathcal{F}$ of presheaves of sets.

Suppose that $\varphi : \mathcal{G} \to f_*\mathcal{F}$ is a map of presheaves of sets. Consider $f_p\varphi : f_p\mathcal{G} \to f_pf_*\mathcal{F}$. Postcomposing with $c_\mathcal{F}$ gives the desired map $c_\mathcal{F} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$. We omit the verification that this construction is inverse to the construction in the other direction given above. □

008G Lemma 6.21.4. Let $f : X \to Y$ be a continuous map. Let $x \in X$. Let $\mathcal{G}$ be a presheaf of sets on Y. There is a canonical bijection of stalks $(f_p\mathcal{G})_x = \mathcal{G}_{f(x)}$.

Proof. This you can see as follows

$$\begin{aligned} (f_p\mathcal{G})_x &= \operatorname{colim}_{x \in U} f_p\mathcal{G}(U) \\ &= \operatorname{colim}_{x \in U} \operatorname{colim}_{f(U) \subset V} \mathcal{G}(V) \\ &= \operatorname{colim}_{f(x) \in V} \mathcal{G}(V) \\ &= \mathcal{G}_{f(x)} \end{aligned}$$

Here we have used Categories, Lemma 4.14.10, and the fact that any V open in Y containing $f(x)$ occurs in the third description above. Details omitted. □

Let $\mathcal{G}$ be a sheaf of sets on Y. The pullback sheaf $f^{-1}\mathcal{G}$ is defined by the formula

$$f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#.$$

The pullback f^{-1} is a left adjoint of pushforward on sheaves. In other words,

$$\operatorname{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Mor}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Namely, we have

$$\begin{aligned} \operatorname{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &= \operatorname{Mor}_{\mathrm{PSh}(X)}(f_p\mathcal{G}, \mathcal{F}) \\ &= \operatorname{Mor}_{\mathrm{PSh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ &= \operatorname{Mor}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}) \end{aligned}$$

For the first equality we use that sheafification is a left adjoint to the inclusion of sheaves in presheaves. For the second equality we use that f_p is a left adjoint to f_* on presheaves. We will return to this statement in the proof of Lemma 6.21.8.

008H Lemma 6.21.5. Let $x \in X$. Let $\mathcal{G}$ be a sheaf of sets on Y. There is a canonical bijection of stalks $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.

Proof. This is a combination of Lemmas 6.17.2 and 6.21.4. □

008I Lemma 6.21.6. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly $(g \circ f)_p \cong f_p \circ g_p$ on presheaves.

Proof. To see this use that adjoint functors are unique up to unique isomorphism, and Lemma 6.21.2. □

008J Definition 6.21.7. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X and let $\mathcal{G}$ be a sheaf of sets on Y. An f-map $\xi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ indexed by open subsets $V \subset Y$ such that

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\xi_V"'] \arrow[d, "\text{restriction of } \mathcal{G}"'] & \mathcal{F}(f^{-1}V) \arrow[d, "\text{restriction of } \mathcal{F}"] \\ \mathcal{G}(V') \arrow[r, "\xi_{V'}"] & \mathcal{F}(f^{-1}V') \end{tikzcd}$$

commutes for all $V' \subset V \subset Y$ open.

In the literature we sometimes find this defined alternatively as in part (4) of Lemma 6.21.8 but as the lemma shows there is really no difference.

008K Lemma 6.21.8. Let $f : X \to Y$ be a continuous map. There are bijections between the following four sets

(1) the set of maps $\mathcal{G} \to f_*\mathcal{F}$,
(2) the set of maps $f^{-1}\mathcal{G} \to \mathcal{F}$,
(3) the set of f-maps $\xi : \mathcal{G} \to \mathcal{F}$, and
(4) the set of all collections of maps $\xi_{U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ for all $U \subset X$ and $V \subset Y$ open such that $f(U) \subset V$ compatible with all restriction maps,

functorially in $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$.

Proof. A map of sheaves $a : \mathcal{G} \to f_*\mathcal{F}$ is by definition a rule which to each open V of Y assigns a map $a_V : \mathcal{G}(V) \to f_*\mathcal{F}(V)$ and we have $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. Thus at least the "data" corresponds exactly to what you need for an f-map ξ from $\mathcal{G}$ to $\mathcal{F}$. To show that the sets (1) and (3) are in bijection we observe that a is a map of sheaves if and only if corresponding family of maps a_V satisfy the condition in Definition 6.21.7.

Recall that $f^{-1}\mathcal{G}$ is the sheafification of $f_p\mathcal{G}$. By the universal property of sheafification a map of sheaves $b : f^{-1}\mathcal{G} \to \mathcal{F}$ is the same thing as a map of presheaves $b_p : f_p\mathcal{G} \to \mathcal{F}$ where f_p is the functor defined earlier in the section. To give such a map b_p you need to specify for each open U of X a map

$$b_{p,U} : \mathrm{colim}_{f(U) \subset V}\, \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction mappings. We may and do view $b_{p,U}$ as a collection of maps $b_{p,U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ for all V open in Y with $f(U) \subset V$. These maps have to be compatible with all possible restriction mappings you can think of. In other words, we see that b_p corresponds to a collection of maps as in (4). Of course, conversely such a collection defines a map b_p and in turn a map $b : f^{-1}\mathcal{G} \to \mathcal{F}$.

To finish the proof of the lemma you have to show that by "forgetting structure" the rule that to a collection $\xi_{U,V}$ as in (4) associates the f-map ξ with $\xi_V = \xi_{f^{-1}(V),V}$ is bijective. To do this, if ξ is a usual f-map then we just define $\tilde{\xi}_{U,V}$ to be the composition of $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ by the restruction map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ which makes sense exactly because $f(U) \subset V$, i.e., $U \subset f^{-1}(V)$. This finishes the proof. □

It is sometimes convenient to think about f-maps instead of maps between sheaves either on X or on Y. We define composition of f-maps as follows.

008L Definition 6.21.9. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. The composition of φ and ψ is the $(g \circ f)$-map $\varphi \circ \psi$ defined by the commutativity of the diagrams

$$\begin{tikzcd}
\mathcal{H}(W) \arrow[rr, "(\varphi\circ\psi)_W"'] \arrow[dr, "\psi_W"'] & & \mathcal{F}(f^{-1}g^{-1}W) \\
& \mathcal{G}(g^{-1}W) \arrow[ur, "\varphi_{g^{-1}W}"'] &
\end{tikzcd}$$

We leave it to the reader to verify that this works. Another way to think about this is to think of $\varphi \circ \psi$ as the composition

$$\mathcal{H} \xrightarrow{\psi} g_*\mathcal{G} \xrightarrow{g_*\varphi} g_*f_*\mathcal{F} = (g \circ f)_*\mathcal{F}$$

Now, doesn't it seem that thinking about f-maps is somehow easier?

Finally, given a continuous map $f : X \to Y$, and an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ there is a natural map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

for all $x \in X$. The image of a representative (V, s) of an element in $\mathcal{G}_{f(x)}$ is mapped to the element in $\mathcal{F}_x$ with representative $(f^{-1}V, \varphi_V(s))$. We leave it to the reader to see that this is well defined. Another way to state it is that it is the unique map such that all diagrams

$$\begin{tikzcd}
\mathcal{F}(f^{-1}V) \arrow[r] & \mathcal{F}_x \\
\mathcal{G}(V) \arrow[r] \arrow[u, "\varphi_V"] & \mathcal{G}_{f(x)} \arrow[u, "\varphi_x"]
\end{tikzcd}$$

(for $f(x) \in V \subset Y$ open) commute.

008M Lemma 6.21.10. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. Let $x \in X$ be a point. The map on stalks $(\varphi \circ \psi)_x : \mathcal{H}_{g(f(x))} \to \mathcal{F}_x$ is the composition

$$\mathcal{H}_{g(f(x))} \xrightarrow{\psi_{f(x)}} \mathcal{G}_{f(x)} \xrightarrow{\varphi_x} \mathcal{F}_x$$

Proof. Immediate from Definition 6.21.9 and the definition of the map on stalks above. □

6.22. Continuous maps and abelian sheaves

008N Let $f : X \to Y$ be a continuous map. We claim there are functors

$$\begin{aligned}
f_* : \mathrm{PAb}(X) &\longrightarrow \mathrm{PAb}(Y) \\
f_* : \mathrm{Ab}(X) &\longrightarrow \mathrm{Ab}(Y) \\
f_p : \mathrm{PAb}(Y) &\longrightarrow \mathrm{PAb}(X) \\
f^{-1} : \mathrm{Ab}(Y) &\longrightarrow \mathrm{Ab}(X)
\end{aligned}$$

with similar properties to their counterparts in Section 6.21. To see this we argue in the following way.

Each of the functors will be constructed in the same way as the corresponding functor in Section 6.21. This works because all the colimits in that section are directed colimits (but we will work through it below).

First off, given an abelian presheaf $\mathcal{F}$ on X and an abelian presheaf $\mathcal{G}$ on Y we define

$$\begin{aligned}
f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}(V)) \\
f_p\mathcal{G}(U) &= \mathrm{colim}_{f(U)\subset V}\, \mathcal{G}(V)
\end{aligned}$$

as abelian groups. The restriction mappings are the same as the restriction mappings for presheaves of sets (and they are all homomorphisms of abelian groups).

The assignments $\mathcal{F} \mapsto f_*\mathcal{F}$ and $\mathcal{G} \to f_p\mathcal{G}$ are functors on the categories of presheaves of abelian groups. This is clear, as (for example) a map of abelian presheaves $\mathcal{G}_1 \to \mathcal{G}_2$ gives rise to a map of directed systems $\{\mathcal{G}_1(V)\}_{f(U)\subset V} \to \{\mathcal{G}_2(V)\}_{f(U)\subset V}$ all of whose maps are homomorphisms and hence gives rise to a homomorphism of abelian groups $f_p\mathcal{G}_1(U) \to f_p\mathcal{G}_2(U)$.

The functors f_* and f_p are adjoint on the category of presheaves of abelian groups, i.e., we have

$$\mathrm{Mor}_{\mathrm{PAb}(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PAb}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

To prove this, note that the map $i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G}$ from the proof of Lemma 6.21.3 is a map of abelian presheaves. Hence if $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves, then the corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$ is also a map of abelian presheaves. For the other direction we point out that the map $c_\mathcal{F} : f_pf_*\mathcal{F} \to \mathcal{F}$ from the proof of Lemma 6.21.3 is a map of abelian presheaves as well (since it is made out of restriction mappings of $\mathcal{F}$ which are all homomorphisms). Hence given a map of abelian presheaves $\varphi : \mathcal{G} \to f_*\mathcal{F}$ the map $c_\mathcal{F} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves as well. Since these constructions $\psi \mapsto f_*\psi$ and $\varphi \mapsto c_\mathcal{F} \circ f_p\varphi$ are inverse to each other as constructions on maps of presheaves of sets we see they are also inverse to each other on maps of abelian presheaves.

If $\mathcal{F}$ is an abelian sheaf on Y, then $f_*\mathcal{F}$ is an abelian sheaf on X. This is true because of the definition of an abelian sheaf and because this is true for sheaves of sets, see Lemma 6.21.1. This defines the functor f_* on the category of abelian sheaves.

We define $f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#$ as before. Adjointness of f_* and f^{-1} follows formally as in the case of presheaves of sets. Here is the argument:

$$\begin{aligned}
\mathrm{Mor}_{\mathrm{Ab}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{\mathrm{PAb}(X)}(f_p\mathcal{G}, \mathcal{F}) \\
&= \mathrm{Mor}_{\mathrm{PAb}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\
&= \mathrm{Mor}_{\mathrm{Ab}(Y)}(\mathcal{G}, f_*\mathcal{F})
\end{aligned}$$

008O **Lemma 6.22.1.** Let $f : X \to Y$ be a continuous map.

1. Let $\mathcal{G}$ be an abelian presheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f_p\mathcal{G})_x$ of Lemma 6.21.4 is an isomorphism of abelian groups.
2. Let $\mathcal{G}$ be an abelian sheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x$ of Lemma 6.21.5 is an isomorphism of abelian groups.

Proof. Omitted. □

Given a continuous map $f : X \to Y$ and sheaves of abelian groups $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of abelian groups makes sense. We can just define it exactly as in Definition 6.21.7 (replacing maps of sets with homomorphisms of abelian groups) or we can simply say that it is the same as a map of abelian sheaves $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The group of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the groups $\mathrm{Mor}_{\mathrm{Ab}(X)}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{\mathrm{Ab}(Y)}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are abelian group homomorphisms.

6.23. Continuous maps and sheaves of algebraic structures

008P Let $(\mathcal{C}, F)$ be a type of algebraic structure. For a topological space X let us introduce the notation:

(1) $\text{PSh}(X, \mathcal{C})$ will be the category of presheaves with values in $\mathcal{C}$.
(2) $\textit{Sh}(X, \mathcal{C})$ will be the category of sheaves with values in $\mathcal{C}$.

Let $f : X \to Y$ be a continuous map of topological spaces. The same arguments as in the previous section show there are functors

$$\begin{aligned} f_* : \text{PSh}(X, \mathcal{C}) &\longrightarrow \text{PSh}(Y, \mathcal{C}) \\ f_* : \textit{Sh}(X, \mathcal{C}) &\longrightarrow \textit{Sh}(Y, \mathcal{C}) \\ f_p : \text{PSh}(Y, \mathcal{C}) &\longrightarrow \text{PSh}(X, \mathcal{C}) \\ f^{-1} : \textit{Sh}(Y, \mathcal{C}) &\longrightarrow \textit{Sh}(X, \mathcal{C}) \end{aligned}$$

constructed in the same manner and with the same properties as the functors constructed for abelian (pre)sheaves. In particular there are commutative diagrams

$$\begin{tikzcd} \text{PSh}(X, \mathcal{C}) \arrow[r, "f_*"] \arrow[d, "F"] & \text{PSh}(Y, \mathcal{C}) \arrow[d, "F"] \\ \text{PSh}(X) \arrow[r, "f_*"] & \text{PSh}(Y) \end{tikzcd} \qquad \begin{tikzcd} \textit{Sh}(X, \mathcal{C}) \arrow[r, "f_*"] \arrow[d, "F"] & \textit{Sh}(Y, \mathcal{C}) \arrow[d, "F"] \\ \textit{Sh}(X) \arrow[r, "f_*"] & \textit{Sh}(Y) \end{tikzcd}$$

$$\begin{tikzcd} \text{PSh}(Y, \mathcal{C}) \arrow[r, "f_p"] \arrow[d, "F"] & \text{PSh}(X, \mathcal{C}) \arrow[d, "F"] \\ \text{PSh}(Y) \arrow[r, "f_p"] & \text{PSh}(X) \end{tikzcd} \qquad \begin{tikzcd} \textit{Sh}(Y, \mathcal{C}) \arrow[r, "f^{-1}"] \arrow[d, "F"] & \textit{Sh}(X, \mathcal{C}) \arrow[d, "F"] \\ \textit{Sh}(Y) \arrow[r, "f^{-1}"] & \textit{Sh}(X) \end{tikzcd}$$

The main formulas to keep in mind are the following

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}(V)) \\ f_p\mathcal{G}(U) &= \text{colim}_{f(U) \subset V}\, \mathcal{G}(V) \\ f^{-1}\mathcal{G} &= (f_p\mathcal{G})^\# \\ (f_p\mathcal{G})_x &= \mathcal{G}_{f(x)} \\ (f^{-1}\mathcal{G})_x &= \mathcal{G}_{f(x)} \end{aligned}$$

Each of these formulas has the property that they hold in the category $\mathcal{C}$ and that upon taking underlying sets we get the corresponding formula for presheaves of sets. In addition we have the adjointness properties

$$\begin{aligned} \text{Mor}_{\text{PSh}(X, \mathcal{C})}(f_p\mathcal{G}, \mathcal{F}) &= \text{Mor}_{\text{PSh}(Y, \mathcal{C})}(\mathcal{G}, f_*\mathcal{F}) \\ \text{Mor}_{\textit{Sh}(X, \mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) &= \text{Mor}_{\textit{Sh}(Y, \mathcal{C})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

To prove these, the main step is to construct the maps

$$i_\mathcal{G} : \mathcal{G} \longrightarrow f_*f_p\mathcal{G}$$

and

$$c_{\mathcal{F}} : f_p f_* \mathcal{F} \longrightarrow \mathcal{F}$$

which occur in the proof of Lemma 6.21.3 as morphisms of presheaves with values in $\mathcal{C}$. This may be safely left to the reader since the constructions are exactly the same as in the case of presheaves of sets.

Given a continuous map $f : X \to Y$ and sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of algebraic structures makes sense. We can just define it exactly as in Definition 6.21.7 (replacing maps of sets with morphisms in $\mathcal{C}$) or we can simply say that it is the same as a map of sheaves of algebraic structures $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $\mathrm{Mor}_{Sh(X,\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{Sh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are homomorphisms of algebraic structures.

008Q Lemma 6.23.1. Let $f : X \to Y$ be a continuous map of topological spaces. Suppose given sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map of underlying sheaves of sets. If for every $V \subset Y$ open the map of sets $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ is the effect of a morphism in $\mathcal{C}$ on underlying sets, then φ comes from a unique f-morphism between sheaves of algebraic structures.

Proof. Omitted. □

6.24. Continuous maps and sheaves of modules

008R The case of sheaves of modules is more complicated. The reason is that the natural setting for defining the pullback and pushforward functors, is the setting of ringed spaces, which we will define below. First we state a few obvious lemmas.

008S Lemma 6.24.1. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a presheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Let $V \subset Y$ is open. We define the map of the lemma to be the map

$$f_*\mathcal{O}(V) \times f_*\mathcal{F}(V) = \mathcal{O}(f^{-1}V) \times \mathcal{F}(f^{-1}V) \to \mathcal{F}(f^{-1}V) = f_*\mathcal{F}(V).$$

Here the arrow in the middle is the multiplication map on X. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_*\mathcal{O}$-module on $f_*\mathcal{F}$. □

008T Lemma 6.24.2. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets

$$f_p\mathcal{O} \times f_p\mathcal{G} \longrightarrow f_p\mathcal{G}$$

which turns $f_p\mathcal{G}$ into a presheaf of $f_p\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Let $U \subset X$ is open. We define the map of the lemma to be the map

$$\begin{aligned} f_p\mathcal{O}(U) \times f_p\mathcal{G}(U) &= \operatorname{colim}_{f(U)\subset V} \mathcal{O}(V) \times \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= \operatorname{colim}_{f(U)\subset V} (\mathcal{O}(V) \times \mathcal{G}(V)) \\ &\to \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= f_p\mathcal{G}(U). \end{aligned}$$

Here the arrow in the middle is the multiplication map on Y. The second equality holds because directed colimits commute with finite limits, see Categories, Lemma 4.19.2. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_p\mathcal{O}$-module on $f_p\mathcal{G}$. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a presheaf of rings on X and let $\mathcal{O}_Y$ be a presheaf of rings on Y. So at the moment we have defined functors

$$\begin{aligned} f_* : \mathrm{PMod}(\mathcal{O}_X) &\longrightarrow \mathrm{PMod}(f_*\mathcal{O}_X) \\ f_p : \mathrm{PMod}(\mathcal{O}_Y) &\longrightarrow \mathrm{PMod}(f_p\mathcal{O}_Y) \end{aligned}$$

These satisfy some compatibilities as follows.

008U Lemma 6.24.3. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a presheaf of $f_p\mathcal{O}$-modules. Then

$$\mathrm{Mor}_{\mathrm{PMod}(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PMod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.2 and 6.24.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module via the map $i_\mathcal{O} : \mathcal{O} \to f_*f_p\mathcal{O}$ (defined first in the proof of Lemma 6.21.3).

Proof. Note that we have

$$\mathrm{Mor}_{\mathrm{PAb}(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PAb}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

according to Section 6.22. So what we have to prove is that under this correspondence, the subsets of module maps correspond. In addition, the correspondence is determined by the rule

$$(\psi : f_p\mathcal{G} \to \mathcal{F}) \longmapsto (f_*\psi \circ i_\mathcal{G} : \mathcal{G} \to f_*\mathcal{F})$$

and in the other direction by the rule

$$(\varphi : \mathcal{G} \to f_*\mathcal{F}) \longmapsto (c_\mathcal{F} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F})$$

where $i_\mathcal{G}$ and $c_\mathcal{F}$ are as in Section 6.22. Hence, using the functoriality of f_* and f_p we see that it suffices to check that the maps $i_\mathcal{G} : \mathcal{G} \to f_*f_p\mathcal{G}$ and $c_\mathcal{F} : f_pf_*\mathcal{F} \to \mathcal{F}$ are compatible with module structures, which we leave to the reader. □

008V Lemma 6.24.4. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $f_*\mathcal{O}$-modules. Then

$$\mathrm{Mor}_{\mathrm{PMod}(\mathcal{O})}(\mathcal{O} \otimes_{p, f_pf_*\mathcal{O}} f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PMod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.2 and 6.24.1, and we use the map $c_\mathcal{O} : f_pf_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned}
\mathrm{Mor}_{\mathrm{PMod}(\mathcal{O})}(\mathcal{O} \otimes_{p, f_p f_* \mathcal{O}} f_p \mathcal{G}, \mathcal{F}) & = \mathrm{Mor}_{\mathrm{PMod}(f_p f_* \mathcal{O})}(f_p \mathcal{G}, \mathcal{F}_{f_p f_* \mathcal{O}}) \\
& = \mathrm{Mor}_{\mathrm{PMod}(f_* \mathcal{O})}(\mathcal{G}, f_*(\mathcal{F}_{f_p f_* \mathcal{O}})) \\
& = \mathrm{Mor}_{\mathrm{PMod}(f_* \mathcal{O})}(\mathcal{G}, f_* \mathcal{F}).
\end{aligned}$$

The first equality is Lemma 6.6.2. The second equality is Lemma 6.24.3. The third equality is given by the equality $f_*(\mathcal{F}_{f_p f_* \mathcal{O}}) = f_* \mathcal{F}$ of abelian sheaves which is $f_*\mathcal{O}$-linear. Namely, $\mathrm{id}_{f_*\mathcal{O}}$ corresponds to $c_\mathcal{O}$ under the adjunction described in the proof of Lemma 6.21.3 and thus $\mathrm{id}_{f_*\mathcal{O}} = f_* c_\mathcal{O} \circ i_{f_*\mathcal{O}}$. ☐

008W Lemma 6.24.5. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. The pushforward $f_*\mathcal{F}$, as defined in Lemma 6.24.1 is a sheaf of $f_*\mathcal{O}$-modules.

Proof. Obvious from the definition and Lemma 6.21.1. ☐

008X Lemma 6.24.6. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules.

Proof. Recall that f^{-1} is defined as the composition of the functor f_p and sheafification. Thus the lemma is a combination of Lemma 6.24.2 and Lemma 6.20.1. ☐

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a sheaf of rings on X and let $\mathcal{O}_Y$ be a sheaf of rings on Y. So now we have defined functors

$$\begin{aligned}
f_* : \mathrm{Mod}(\mathcal{O}_X) & \longrightarrow \mathrm{Mod}(f_*\mathcal{O}_X) \\
f^{-1} : \mathrm{Mod}(\mathcal{O}_Y) & \longrightarrow \mathrm{Mod}(f^{-1}\mathcal{O}_Y)
\end{aligned}$$

These satisfy some compatibilities as follows.

008Y Lemma 6.24.7. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then

$$\mathrm{Mor}_{\mathrm{Mod}(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.6 and 6.24.5, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_* f^{-1}\mathcal{O}$.

Proof. Argue by the equalities

$$\begin{aligned}
\mathrm{Mor}_{\mathrm{Mod}(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) & = \mathrm{Mor}_{\mathrm{Mod}(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) \\
& = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).
\end{aligned}$$

where the second is Lemmas 6.24.3 and the first is by Lemma 6.20.1. ☐

008Z Lemma 6.24.8. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then

$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.6 and 6.24.5, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned}\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{\mathrm{Mod}(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}}) \\ &= \mathrm{Mor}_{\mathrm{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).\end{aligned}$$

which are a combination of Lemma 6.20.2 and 6.24.7. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a (pre)sheaf of rings on X and let $\mathcal{O}_Y$ be a (pre)sheaf of rings on Y. So at the moment we have defined functors

$$\begin{aligned} f_* : \mathrm{PMod}(\mathcal{O}_X) &\longrightarrow \mathrm{PMod}(f_*\mathcal{O}_X) \\ f_* : \mathrm{Mod}(\mathcal{O}_X) &\longrightarrow \mathrm{Mod}(f_*\mathcal{O}_X) \\ f_p : \mathrm{PMod}(\mathcal{O}_Y) &\longrightarrow \mathrm{PMod}(f_p\mathcal{O}_Y) \\ f^{-1} : \mathrm{Mod}(\mathcal{O}_Y) &\longrightarrow \mathrm{Mod}(f^{-1}\mathcal{O}_Y)\end{aligned}$$

Clearly, usually the pair of functors (f_*, f^{-1}) on sheaves of modules are not adjoint, because their target categories do not match. Namely, as we saw above, it works only if by some miracle the sheaves of rings $\mathcal{O}_X, \mathcal{O}_Y$ satisfy the relations $\mathcal{O}_X = f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_Y = f_*\mathcal{O}_X$. This is almost never true in practice. We interrupt the discussion to define the correct notion of morphism for which a suitable adjoint pair of functors on sheaves of modules exists.

6.25. Ringed spaces

0090 Let X be a topological space and let $\mathcal{O}_X$ be a sheaf of rings on X. We are supposed to think of the sheaf of rings $\mathcal{O}_X$ as a sheaf of functions on X. And if $f : X \to Y$ is a "suitable" map, then by composition a function on Y turns into a function on X. Thus there should be a natural f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$, see Definition 6.21.7 and Lemma 6.21.8. For a precise example, see Example 6.25.2 below. Here is the relevant abstract definition.

0091 Definition 6.25.1. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair consisting of a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$.

0092 Example 6.25.2. Let $f : X \to Y$ be a continuous map of topological spaces. Consider the sheaves of continuous real valued functions $\mathcal{C}^0_X$ on X and $\mathcal{C}^0_Y$ on Y, see Example 6.9.3. We claim that there is a natural f-map $f^\sharp : \mathcal{C}^0_Y \to \mathcal{C}^0_X$ associated to f. Namely, we simply define it by the rule

$$\begin{aligned}\mathcal{C}^0_Y(V) &\longrightarrow \mathcal{C}^0_X(f^{-1}V) \\ h &\longmapsto h \circ f\end{aligned}$$

Strictly speaking we should write $f^\sharp(h) = h \circ f|_{f^{-1}(V)}$. It is clear that this is a family of maps as in Definition 6.21.7 and compatible with the $\mathbf{R}$-algebra structures. Hence it is an f-map of sheaves of $\mathbf{R}$-algebras, see Lemma 6.23.1.

Of course there are lots of other situations where there is a canonical morphism of ringed spaces associated to a geometrical type of morphism. For example, if M, N are $\mathcal{C}^\infty$-manifolds and $f : M \to N$ is a infinitely differentiable map, then f induces a canonical morphism of ringed spaces $(M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$. The construction (which is identical to the above) is left to the reader.

It may not be completely obvious how to compose morphisms of ringed spaces hence we spell it out here.

0093 Definition 6.25.3. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then we define the composition of morphisms of ringed spaces by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of f-maps defined in Definition 6.21.9.

6.26. Morphisms of ringed spaces and modules

0094 We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed spaces.

0095 Definition 6.26.1. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the pushforward of $\mathcal{F}$ as the sheaf of $\mathcal{O}_Y$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of the module structure given in Lemma 6.24.5.

(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. We define the pullback $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_X$-modules defined by the formula

$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

where the ring map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is the map corresponding to $f^\sharp$, and where the module structure is given by Lemma 6.24.6.

Thus we have defined functors

$$\begin{array}{rcl} f_* : \mathrm{Mod}(\mathcal{O}_X) & \longrightarrow & \mathrm{Mod}(\mathcal{O}_Y) \\ f^* : \mathrm{Mod}(\mathcal{O}_Y) & \longrightarrow & \mathrm{Mod}(\mathcal{O}_X) \end{array}$$

The final result on these functors is that they are indeed adjoint as expected.

0096 Lemma 6.26.2. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}, \mathcal{F}) \\ & = & \mathrm{Mor}_{\mathrm{Mod}(f^{-1}\mathcal{O}_Y)}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_Y}) \\ & = & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

Here we use Lemmas 6.20.2 and 6.24.7. $\square$

0097 Lemma 6.26.3. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal. There is a canonical isomorphism of functors $(g \circ f)^* \cong f^* \circ g^*$.

Proof. The result on pushforwards is a consequence of Lemma 6.21.2 and our definitions. The result on pullbacks follows from this by the same argument as in the proof of Lemma 6.21.6. $\square$

Given a morphism of ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, a sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$ on Y, the notion of an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of sheaves of modules makes sense. We can just define it as an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of abelian sheaves (see Definition 6.21.7 and Lemma 6.21.8) such that for all open $V \subset Y$ the map

$$\mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

is an $\mathcal{O}_Y(V)$-module map. Here we think of $\mathcal{F}(f^{-1}V)$ as an $\mathcal{O}_Y(V)$-module via the map $f^\sharp_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O}_X)}(f^*\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O}_Y)}(\mathcal{G}, f_*\mathcal{F})$. See above.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, and $x \in X$ the induced map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is an $\mathcal{O}_{Y,f(x)}$-module map where the $\mathcal{O}_{Y,f(x)}$-module structure on $\mathcal{F}_x$ comes from the $\mathcal{O}_{X,x}$-module structure via the map $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Here is a related lemma.

0098 Lemma 6.26.4. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $x \in X$. Then

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$

as $\mathcal{O}_{X,x}$-modules where the tensor product on the right uses $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

Proof. This follows from Lemma 6.20.3 and the identification of the stalks of pullback sheaves at x with the corresponding stalks at $f(x)$. See the formulae in Section 6.23 for example. $\square$

6.27. Skyscraper sheaves and stalks

0099

009A Definition 6.27.1. Let X be a topological space.

(1) Let $x \in X$ be a point. Denote $i_x : \{x\} \to X$ the inclusion map. Let A be a set and think of A as a sheaf on the one point space $\{x\}$. We call $i_{x,*}A$ the skyscraper sheaf at x with value A.
(2) If in (1) above A is an abelian group then we think of $i_{x,*}A$ as a sheaf of abelian groups on X.
(3) If in (1) above A is an algebraic structure then we think of $i_{x,*}A$ as a sheaf of algebraic structures.
(4) If $(X, \mathcal{O}_X)$ is a ringed space, then we think of $i_x : \{x\} \to X$ as a morphism of ringed spaces $(\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$ and if A is a $\mathcal{O}_{X,x}$-module, then we think of $i_{x,*}A$ as a sheaf of $\mathcal{O}_X$-modules.
(5) We say a sheaf of sets $\mathcal{F}$ is a skyscraper sheaf if there exists a point x of X and a set A such that $\mathcal{F} \cong i_{x,*}A$.
(6) We say a sheaf of abelian groups $\mathcal{F}$ is a skyscraper sheaf if there exists a point x of X and an abelian group A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of abelian groups.
(7) We say a sheaf of algebraic structures $\mathcal{F}$ is a skyscraper sheaf if there exists a point x of X and an algebraic structure A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of algebraic structures.

(8) If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we say $\mathcal{F}$ is a skyscraper sheaf if there exists a point $x \in X$ and a $\mathcal{O}_{X,x}$-module A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of $\mathcal{O}_X$-modules.

009B Lemma 6.27.2. Let X be a topological space, $x \in X$ a point, and A a set. For any point $x' \in X$ the stalk of the skyscraper sheaf at x with value A at x' is

$$(i_{x,*}A)_{x'} = \begin{cases} A & \text{if} \quad x' \in \overline{\{x\}} \\ \{*\} & \text{if} \quad x' \notin \overline{\{x\}} \end{cases}$$

A similar description holds for the case of abelian groups, algebraic structures and sheaves of modules.

Proof. Omitted. □

009C Lemma 6.27.3. Let X be a topological space, and let $x \in X$ a point. The functors $\mathcal{F} \mapsto \mathcal{F}_x$ and $A \mapsto i_{x,*}A$ are adjoint. In a formula

$$\mathrm{Mor}_{\mathrm{Sets}}(\mathcal{F}_x, A) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}, i_{x,*}A).$$

A similar statement holds for the case of abelian groups, algebraic structures. In the case of sheaves of modules we have

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, A) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*}A).$$

Proof. Omitted. Hint: The stalk functor can be seen as the pullback functor for the morphism $i_x : \{x\} \to X$. Then the adjointness follows from adjointness of i_x^{-1} and $i_{x,*}$ (resp. i_x^* and $i_{x,*}$ in the case of sheaves of modules). □

6.28. Limits and colimits of presheaves

009D Let X be a topological space. Let $\mathcal{I} \to \mathrm{PSh}(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $\lim_i \mathcal{F}_i$ and $\mathrm{colim}_i \mathcal{F}_i$ exist.
(2) For any open $U \subset X$ we have
$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U)$$
and
$$(\mathrm{colim}_i \mathcal{F}_i)(U) = \mathrm{colim}_i \mathcal{F}_i(U).$$
(3) Let $x \in X$ be a point. In general the stalk of $\lim_i \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the index category is finite then it is the case. In other words, the stalk functor is left exact (see Categories, Definition 4.23.1).
(4) Let $x \in X$. We always have
$$(\mathrm{colim}_i \mathcal{F}_i)_x = \mathrm{colim}_i \mathcal{F}_{i,x}.$$

The proofs are all easy.

6.29. Limits and colimits of sheaves

009E Let X be a topological space. Let $\mathcal{I} \to Sh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $\lim_i \mathcal{F}_i$ and $\mathrm{colim}_i \mathcal{F}_i$ exist.
(2) The inclusion functor $i : Sh(X) \to \mathrm{PSh}(X)$ commutes with limits. In other words, we may compute the limit in the category of sheaves as the limit in the category of presheaves. In particular, for any open $U \subset X$ we have
$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U).$$

(3) The inclusion functor $i : Sh(X) \to \mathrm{PSh}(X)$ does not commute with colimits in general (not even with finite colimits – think surjections). The colimit is computed as the sheafification of the colimit in the category of presheaves:

$$\mathrm{colim}_i \mathcal{F}_i = \Big(U \mapsto \mathrm{colim}_i \mathcal{F}_i(U)\Big)^{\#}.$$

(4) Let $x \in X$ be a point. In general the stalk of $\lim_i \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the index category is finite then it is the case. In other words, the stalk functor is left exact.

(5) Let $x \in X$. We always have

$$(\mathrm{colim}_i \mathcal{F}_i)_x = \mathrm{colim}_i \mathcal{F}_{i,x}.$$

(6) The sheafification functor $^{\#} : \mathrm{PSh}(X) \to Sh(X)$ commutes with all colimits, and with finite limits. But it does not commute with all limits.

The proofs are all easy. Here is an example of what is true for directed colimits of sheaves.

009F Lemma 6.29.1. Let X be a topological space. Let I be a directed set. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of sheaves of sets over I, see Categories, Section 4.21. Let $U \subset X$ be an open subset. Consider the canonical map

$$\Psi : \mathrm{colim}_i \mathcal{F}_i(U) \longrightarrow (\mathrm{colim}_i \mathcal{F}_i)(U)$$

(1) If all the transition maps are injective then Ψ is injective for any open U.
(2) If U is quasi-compact, then Ψ is injective.
(3) If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.
(4) If U has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j\in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \mathrm{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 6.11.2). By the discussion above we have $(\mathcal{F}')^{\#} = \mathrm{colim}_i \mathcal{F}_i$. By Lemma 6.17.5 we see that $\mathcal{F}' \to (\mathcal{F}')^{\#}$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. Since U is quasi-compact this means there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$ and for each j an index $i_j \in I$, $i_j \geq i$, $i_j \geq i'$ such that $\varphi_{ii_j}(s) = \varphi_{i'i_j}(s')$. Let $i'' \in I$ be $\geq$ than all of the i_j. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on the opens U_j for all j and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. Since U is quasi-compact there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ comes from s_j for all j. Pick $i \in I$ which is $\geq$ than all of the i_j. By (1) the sections $\varphi_{i_j i}(s_j)$ agree over the overlaps $U_j \cap U_{j'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). In particular we see that U is quasi-compact and hence by (2) we have injectivity of Ψ. Let s be an element of the target of Ψ. By assumption there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, with $U_j \cap U_{j'}$

quasi-compact for all $j, j' \in J$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \cap U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_j i_{jj'}}(s_j)$ and $\varphi_{i_{j'} i_{jj'}}(s_{j'})$ agree over $U_j \cap U_{j'}$. Choose an index $i \in I$ wich is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_j i}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. □

009G Example 6.29.2. Let $X = \{s_1, s_2, \xi_1, \xi_2, \xi_3, \ldots\}$ as a set. Declare a subset $U \subset X$ to be open if $s_1 \in U$ or $s_2 \in U$ implies U contains all of the ξ_i. Let $U_n = \{\xi_n, \xi_{n+1}, \ldots\}$, and let $j_n : U_n \to X$ be the inclusion map. Set $\mathcal{F}_n = j_{n,*}\underline{\mathbf{Z}}$. There are transition maps $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Let $\mathcal{F} = \text{colim}\, \mathcal{F}_n$. Note that $\mathcal{F}_{n,\xi_m} = 0$ if $m < n$ because $\{\xi_m\}$ is an open subset of X which misses U_n. Hence we see that $\mathcal{F}_{\xi_n} = 0$ for all n. On the other hand the stalk $\mathcal{F}_{s_i}$, $i = 1, 2$ is the colimit

$$M = \text{colim}_n \prod\nolimits_{m \geq n} \mathbf{Z}$$

which is not zero. We conclude that the sheaf $\mathcal{F}$ is the direct sum of the skyscraper sheaves with value M at the closed points s_1 and s_2. Hence $\Gamma(X, \mathcal{F}) = M \oplus M$. On the other hand, the reader can verify that $\text{colim}_n \Gamma(X, \mathcal{F}_n) = M$. Hence some condition is necessary in part (4) of Lemma 6.29.1 above.

There is a version of the previous lemma dealing with sheaves on a diagram of spectral spaces. To state it we introduce some notation. Let $\mathcal{I}$ be a cofiltered index category. Let $i \mapsto X_i$ be a diagram of spectral spaces over $\mathcal{I}$ such that for $a : j \to i$ in $\mathcal{I}$ the corresponding map $f_a : X_j \to X_i$ is spectral. Set $X = \lim X_i$ and denote $p_i : X \to X_i$ the projection.

0A32 Lemma 6.29.3. In the situation described above, let $i \in \text{Ob}(\mathcal{I})$ and let $\mathcal{G}$ be a sheaf on X_i. For $U_i \subset X_i$ quasi-compact open we have

$$p_i^{-1}\mathcal{G}(p_i^{-1}(U_i)) = \text{colim}_{a:j\to i}\, f_a^{-1}\mathcal{G}(f_a^{-1}(U_i))$$

Proof. Let us prove the canonical map $\text{colim}_{a:j\to i}\, f_a^{-1}\mathcal{G}(f_a^{-1}(U_i)) \to p_i^{-1}\mathcal{G}(p_i^{-1}(U_i))$ is injective. Let s, s' be sections of $f_a^{-1}\mathcal{G}$ over $f_a^{-1}(U_i)$ for some $a : j \to i$. For $b : k \to j$ let $Z_k \subset f_{a\circ b}^{-1}(U_i)$ be the closed subset of points x such that the image of s and s' in the stalk $(f_{a\circ b}^{-1}\mathcal{G})_x$ are different. If Z_k is nonempty for all $b : k \to j$, then by Topology, Lemma 5.24.2 we see that $\lim_{b:k\to j} Z_k$ is nonempty too. Then for $x \in \lim_{b:k\to j} Z_k \subset X$ (observe that $\mathcal{I}/j \to \mathcal{I}$ is initial) we see that the image of s and s' in the stalk of $p_i^{-1}\mathcal{G}$ at x are different too since $(p_i^{-1}\mathcal{G})_x = (f_{b\circ a}^{-1}\mathcal{G})_{p_k(x)}$ for all $b : k \to j$ as above. Thus if the images of s and s' in $p_i^{-1}\mathcal{G}(p_i^{-1}(U_i))$ are the same, then Z_k is empty for some $b : k \to j$. This proves injectivity.

Surjectivity. Let s be a section of $p_i^{-1}\mathcal{G}$ over $p_i^{-1}(U_i)$. By Topology, Lemma 5.24.5 the set $p_i^{-1}(U_i)$ is a quasi-compact open of the spectral space X. By construction of the pullback sheaf, we can find an open covering $p_i^{-1}(U_i) = \bigcup_{l\in L} W_l$, opens $V_{l,i} \subset X_i$, sections $s_{l,i} \in \mathcal{G}(V_{l,i})$ such that $p_i(W_l) \subset V_{l,i}$ and $p_i^{-1}s_{l,i}|_{W_l} = s|_{W_l}$. Because X and X_i are spectral and $p_i^{-1}(U_i)$ is quasi-compact open, we may assume L is finite and W_l and $V_{l,i}$ quasi-compact open for all l. Then we can apply Topology, Lemma 5.24.6 to find $a : j \to i$ and open covering $f_a^{-1}(U_i) = \bigcup_{l\in L} W_{l,j}$ by quasi-compact opens whose pullback to X is the covering $p_i^{-1}(U_i) = \bigcup_{l\in L} W_l$ and such that moreover $W_{l,j} \subset f_a^{-1}(V_{l,i})$. Write $s_{l,j}$ the restriction of the pullback of $s_{l,i}$ by f_a to $W_{l,j}$. Then we see that $s_{l,j}$ and $s_{l',j}$ restrict to elements of $(f_a^{-1}\mathcal{G})(W_{l,j} \cap W_{l',j})$

which pullback to the same element $(p_i^{-1}\mathcal{G})(W_l \cap W_{l'})$, namely, the restriction of s. Hence by injectivity, we can find $b : k \to j$ such that the sections $f_b^{-1}s_{l,j}$ glue to a section over $f_{a\circ b}^{-1}(U_i)$ as desired. $\square$

Next, in addition to the cofiltered system X_i of spectral spaces, assume given

(1) a sheaf $\mathcal{F}_i$ on X_i for all $i \in \mathrm{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ an f_a-map $\varphi_a : \mathcal{F}_i \to \mathcal{F}_j$

such that $\varphi_c = \varphi_b \circ \varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \mathrm{colim}\, p_i^{-1}\mathcal{F}_i$ on X.

0A33 Lemma 6.29.4. In the situation described above, let $i \in \mathrm{Ob}(\mathcal{I})$ and let $U_i \subset X_i$ be a quasi-compact open. Then

$$\mathrm{colim}_{a:j\to i}\, \mathcal{F}_j(f_a^{-1}(U_i)) = \mathcal{F}(p_i^{-1}(U_i))$$

Proof. Recall that $p_i^{-1}(U_i)$ is a quasi-compact open of the spectral space X, see Topology, Lemma 5.24.5. Hence Lemma 6.29.1 applies and we have

$$\mathcal{F}(p_i^{-1}(U_i)) = \mathrm{colim}_{a:j\to i}\, p_j^{-1}\mathcal{F}_j(p_i^{-1}(U_i)).$$

A formal argument shows that

$$\mathrm{colim}_{a:j\to i}\, \mathcal{F}_j(f_a^{-1}(U_i)) = \mathrm{colim}_{a:j\to i}\, \mathrm{colim}_{b:k\to j}\, f_b^{-1}\mathcal{F}_j(f_{a\circ b}^{-1}(U_i))$$

Thus it suffices to show that

$$p_j^{-1}\mathcal{F}_j(p_i^{-1}(U_i)) = \mathrm{colim}_{b:k\to j}\, f_b^{-1}\mathcal{F}_j(f_{a\circ b}^{-1}(U_i))$$

This is Lemma 6.29.3 applied to $\mathcal{F}_j$ and the quasi-compact open $f_a^{-1}(U_i)$. $\square$

6.30. Bases and sheaves

009H Sometimes there exists a basis for the topology consisting of opens that are easier to work with than general opens. For convenience we give here some definitions and simple lemmas in order to facilitate working with (pre)sheaves in such a situation.

009I Definition 6.30.1. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

(1) A presheaf $\mathcal{F}$ of sets on $\mathcal{B}$ is a rule which assigns to each $U \in \mathcal{B}$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$ for all $U \in \mathcal{B}$ whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.
(2) A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$ is a rule which assigns to each element $U \in \mathcal{B}$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.

As in the case of usual presheaves we use the terminology of sections, restrictions of sections, etc. In particular, we may define the stalk of $\mathcal{F}$ at a point $x \in X$ by the colimit

$$\mathcal{F}_x = \mathrm{colim}_{U\in\mathcal{B}, x\in U}\, \mathcal{F}(U).$$

As in the case of the stalk of a presheaf on X this limit is directed. The reason is that the collection of $U \in \mathcal{B}$, $x \in U$ is a fundamental system of open neighbourhoods of x.

It is easy to make examples to show that the notion of a presheaf on X is very different from the notion of a presheaf on a basis for the topology on X. This does not happen in the case of sheaves. A much more useful notion therefore, is the following.

009J Definition 6.30.2. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

(1) A sheaf $\mathcal{F}$ of sets on $\mathcal{B}$ is a presheaf of sets on $\mathcal{B}$ which satisfies the following additional property: Given any $U \in \mathcal{B}$, and any covering $U = \bigcup_{i \in I} U_i$ with $U_i \in \mathcal{B}$, and any coverings $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$ with $U_{ijk} \in \mathcal{B}$ the sheaf condition holds:
   (**) For any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$, $\forall k \in I_{ij}$
   $$s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$$
   there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.
(2) A morphism of sheaves of sets on $\mathcal{B}$ is simply a morphism of presheaves of sets.

First we explain that it suffices to check the sheaf condition $(**)$ on a cofinal system of coverings. In the situation of the definition, suppose $U \in \mathcal{B}$. Let us temporarily denote $\text{Cov}_{\mathcal{B}}(U)$ the set of all coverings of U by elements of $\mathcal{B}$. Note that $\text{Cov}_{\mathcal{B}}(U)$ is preordered by refinement. A subset $C \subset \text{Cov}_{\mathcal{B}}(U)$ is a cofinal system, if for every $\mathcal{U} \in \text{Cov}_{\mathcal{B}}(U)$ there exists a covering $\mathcal{V} \in C$ which refines $\mathcal{U}$.

009K Lemma 6.30.3. With notation as above. For each $U \in \mathcal{B}$, let $C(U) \subset \text{Cov}_{\mathcal{B}}(U)$ be a cofinal system. For each $U \in \mathcal{B}$, and each $\mathcal{U} : U = \bigcup U_i$ in $C(U)$, let coverings $\mathcal{U}_{ij} : U_i \cap U_j = \bigcup U_{ijk}$, $U_{ijk} \in \mathcal{B}$ be given. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent

(1) The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.
(2) For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ the sheaf condition $(**)$ holds (for the given coverings $\mathcal{U}_{ij}$).

Proof. We have to show that (2) implies (1). Suppose that $U \in \mathcal{B}$, and that $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an arbitrary covering by elements of $\mathcal{B}$. Because the system $C(U)$ is cofinal we can find an element $\mathcal{V} : U = \bigcup_{j \in J} V_j$ in $C(U)$ which refines $\mathcal{U}$. This means there exists a map $\alpha : J \to I$ such that $V_j \subset U_{\alpha(j)}$.

Note that if $s, s' \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = s'|_{U_i}$, then

$$s|_{V_j} = (s|_{U_{\alpha(j)}})|_{V_j} = (s'|_{U_{\alpha(j)}})|_{V_j} = s'|_{V_j}$$

for all j. Hence by the uniqueness in $(**)$ for the covering $\mathcal{V}$ we conclude that $s = s'$. Thus we have proved the uniqueness part of $(**)$ for our arbitrary covering $\mathcal{U}$.

Suppose furthermore that $U_i \cap U_{i'} = \bigcup_{k \in I_{ii'}} U_{ii'k}$ are arbitrary coverings by $U_{ii'k} \in \mathcal{B}$. Let us try to prove the existence part of $(**)$ for the system $(\mathcal{U}, \mathcal{U}_{ij})$. Thus let $s_i \in \mathcal{F}(U_i)$ and suppose we have

$$s_i|_{U_{ii'k}} = s_{i'}|_{U_{ii'k}}$$

for all i, i', k. Set $t_j = s_{\alpha(j)}|_{V_j}$, where $\mathcal{V}$ and α are as above.

There is one small kink in the argument here. Namely, let $\mathcal{V}_{jj'} : V_j \cap V_{j'} = \bigcup_{l \in J_{jj'}} V_{jj'l}$ be the covering given to us by the statement of the lemma. It is not a priori clear that

$$t_j|_{V_{jj'l}} = t_{j'}|_{V_{jj'l}}$$

for all j, j', l. To see this, note that we do have

$$t_j|_W = t_{j'}|_W \text{ for all } W \in \mathcal{B}, W \subset V_{jj'l} \cap U_{\alpha(j)\alpha(j')k}$$

for all $k \in I_{\alpha(j)\alpha(j')}$, by our assumption on the family of elements s_i. And since $V_j \cap V_{j'} \subset U_{\alpha(j)} \cap U_{\alpha(j')}$ we see that $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$ agree on the members of a covering of $V_{jj'l}$ by elements of $\mathcal{B}$. Hence by the uniqueness part proved above we finally deduce the desired equality of $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$. Then we get the existence of an element $t \in \mathcal{F}(U)$ by property $(**)$ for $(\mathcal{V}, \mathcal{V}_{jj'})$.

Again there is a small snag. We know that t restricts to t_j on V_j but we do not yet know that t restricts to s_i on U_i. To conclude this note that the sets $U_i \cap V_j$, $j \in J$ cover U_i. Hence also the sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$ cover U_i. We leave it to the reader to see that t and s_i restrict to the same section of $\mathcal{F}$ on any $W \in \mathcal{B}$ which is contained in one of the open sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$. Hence by the uniqueness part seen above we win. □

009L Lemma 6.30.4. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Assume that for every triple $U, U', U'' \in \mathcal{B}$ with $U' \subset U$ and $U'' \subset U$ we have $U' \cap U'' \in \mathcal{B}$. For each $U \in \mathcal{B}$, let $C(U) \subset \text{Cov}_{\mathcal{B}}(U)$ be a cofinal system. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent

(1) The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.
(2) For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ and for every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ there exists a unique section $s \in \mathcal{F}(U)$ which restricts to s_i on U_i.

Proof. This is a reformulation of Lemma 6.30.3 above in the special case where the coverings $\mathcal{U}_{ij}$ each consist of a single element. But also this case is much easier and is an easy exercise to do directly. □

009M Lemma 6.30.5. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $U \in \mathcal{B}$. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. The map

$$\mathcal{F}(U) \to \prod\nolimits_{x \in U} \mathcal{F}_x$$

identifies $\mathcal{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property

(*) For any $x \in U$ there exists a $V \in \mathcal{B}$, with $x \in V \subset U$ and a section $\sigma \in \mathcal{F}(V)$ such that for all $y \in V$ we have $s_y = (V, \sigma)$ in $\mathcal{F}_y$.

Proof. First note that the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective by the uniqueness in the sheaf condition of Definition 6.30.2. Let (s_x) be any element on the right hand side which satisfies $(*)$. Clearly this means we can find a covering $U = \bigcup U_i$, $U_i \in \mathcal{B}$ such that $(s_x)_{x \in U_i}$ comes from certain $\sigma_i \in \mathcal{F}(U_i)$. For every $y \in U_i \cap U_j$ the sections σ_i and σ_j agree in the stalk $\mathcal{F}_y$. Hence there exists an element $V_{ijy} \in \mathcal{B}$, $y \in V_{ijy}$ such that $\sigma_i|_{V_{ijy}} = \sigma_j|_{V_{ijy}}$. Thus the sheaf condition $(**)$ of Definition 6.30.2 applies to the system of σ_i and we obtain a section $s \in \mathcal{F}(U)$ with the desired property. □

Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. There is a natural restriction functor from the category of sheaves of sets on X to the category of sheaves of sets on $\mathcal{B}$. It turns out that this is an equivalence of categories. In down to earth terms this means the following.

009N Lemma 6.30.6. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. There exists a unique sheaf of sets $\mathcal{F}^{ext}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.

Proof. We first construct a presheaf $\mathcal{F}^{ext}$ with the desired property. Namely, for an arbitrary open $U \subset X$ we define $\mathcal{F}^{ext}(U)$ as the set of elements $(s_x)_{x \in U}$ such that $(*)$ of Lemma 6.30.5 holds. It is clear that there are restriction mappings that turn $\mathcal{F}^{ext}$ into a presheaf of sets. Also, by Lemma 6.30.5 we see that $\mathcal{F}(U) = \mathcal{F}^{ext}(U)$ whenever U is an element of the basis $\mathcal{B}$. To see $\mathcal{F}^{ext}$ is a sheaf one may argue as in the proof of Lemma 6.17.1. $\square$

Note that we have
$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$
in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

009O Lemma 6.30.7. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Denote $Sh(\mathcal{B})$ the category of sheaves on $\mathcal{B}$. There is an equivalence of categories
$$Sh(X) \longrightarrow Sh(\mathcal{B})$$
which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.6 above. Checking the obvious functorialities is left to the reader. $\square$

This ends the discussion of sheaves of sets on a basis $\mathcal{B}$. Let $(\mathcal{C}, F)$ be a type of algebraic structure. At the end of this section we would like to point out that the constructions above work for sheaves with values in $\mathcal{C}$. Let us briefly define the relevant notions.

009P Definition 6.30.8. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure.

(1) A presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$ is a rule which assigns to each $U \in \mathcal{B}$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a morphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that $\rho^U_U = \text{id}_{\mathcal{F}(U)}$ for all $U \in \mathcal{B}$ and whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$ is a rule which assigns to each element $U \in \mathcal{B}$ a morphism of algebraic structures $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.
(3) Given a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$ we say that $U \mapsto F(\mathcal{F}(U))$ is the underlying presheaf of sets.
(4) A sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$ is a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ whose underlying presheaf of sets is a sheaf.

At this point we can define the stalk at $x \in X$ of a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ as the directed colimit
$$\mathcal{F}_x = \text{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It exists as an object of $\mathcal{C}$ because of our assumptions on $\mathcal{C}$. Also, we see that the underlying set of $\mathcal{F}_x$ is the stalk of the underlying presheaf of sets on $\mathcal{B}$.

Note that Lemmas 6.30.3, 6.30.4 and 6.30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf with values in $\mathcal{C}$. The analogue of Lemma 6.30.6 need some care. Here it is.

009Q Lemma 6.30.9. Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on $\mathcal{B}$. There exists a unique sheaf $\mathcal{F}^{ext}$ with values in $\mathcal{C}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.

Proof. By the conditions imposed on the pair $(\mathcal{C}, F)$ it suffices to come up with a presheaf $\mathcal{F}^{ext}$ which does the correct thing on the level of underlying presheaves of sets. Thus our first task is to construct a suitable object $\mathcal{F}^{ext}(U)$ for all open $U \subset X$. We could do this by imitating Lemma 6.18.1 in the setting of presheaves on $\mathcal{B}$. However, a slightly different method (but basically equivalent) is the following: Define it as the directed colimit

$$\mathcal{F}^{ext}(U) := \operatorname{colim}_{\mathcal{U}} FIB(\mathcal{U})$$

over all coverings $\mathcal{U} : U = \bigcup_{i \in I} U_i$ by $U_i \in \mathcal{B}$ of the fibre product

$$\begin{tikzcd} FIB(\mathcal{U}) \arrow[r] \arrow[d] & \prod_{x \in U} \mathcal{F}_x \arrow[d] \\ \prod_{i \in I} \mathcal{F}(U_i) \arrow[r] & \prod_{i \in I} \prod_{x \in U_i} \mathcal{F}_x \end{tikzcd}$$

By the usual arguments, see Lemma 6.15.4 and Example 6.15.5 it suffices to show that this construction on underlying sets is the same as the definition using (∗∗) above. Details left to the reader. $\square$

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

as objects in $\mathcal{C}$ in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

009R Lemma 6.30.10. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Denote $Sh(\mathcal{B}, \mathcal{C})$ the category of sheaves with values in $\mathcal{C}$ on $\mathcal{B}$. There is an equivalence of categories

$$Sh(X, \mathcal{C}) \longrightarrow Sh(\mathcal{B}, \mathcal{C})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.9 above. Checking the obvious functorialities is left to the reader. $\square$

Finally we come to the case of (pre)sheaves of modules on a basis. We will use the easy fact that the category of presheaves of sets on a basis has products and that they are described by taking products of values on elements of the bases.

009S Definition 6.30.11. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{B}$.

(1) A presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$ is a presheaf of abelian groups on $\mathcal{B}$ together with a morphism of presheaves of sets $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ such that for all $U \in \mathcal{B}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ turns the group $\mathcal{F}(U)$ into an $\mathcal{O}(U)$-module.
(2) A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$ is a morphism of abelian presheaves on $\mathcal{B}$ which induces an $\mathcal{O}(U)$-module homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ for every $U \in \mathcal{B}$.
(3) Suppose that $\mathcal{O}$ is a sheaf of rings on $\mathcal{B}$. A sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$ is a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ whose underlying presheaf of abelian groups is a sheaf.

We can define the stalk at $x \in X$ of a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = \mathrm{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It is a $\mathcal{O}_x$-module.

Note that Lemmas 6.30.3, 6.30.4 and 6.30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf of $\mathcal{O}$-modules. The analogue of Lemma 6.30.6 is as follows.

009T Lemma 6.30.12. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{B}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{B}$. Let $\mathcal{O}^{ext}$ be the sheaf of rings on X extending $\mathcal{O}$ and let $\mathcal{F}^{ext}$ be the abelian sheaf on X extending $\mathcal{F}$, see Lemma 6.30.9. There exists a canonical map

$$\mathcal{O}^{ext} \times \mathcal{F}^{ext} \longrightarrow \mathcal{F}^{ext}$$

which agrees with the given map over elements of $\mathcal{B}$ and which endows $\mathcal{F}^{ext}$ with the structure of an $\mathcal{O}^{ext}$-module.

Proof. It suffices to construct the multiplication map on the level of presheaves of sets. Perhaps the easiest way to see this is to prove directly that if $(f_x)_{x \in U}$, $f_x \in \mathcal{O}_x$ and $(m_x)_{x \in U}$, $m_x \in \mathcal{F}_x$ satisfy $(*)$, then the element $(f_x m_x)_{x \in U}$ also satisfies $(*)$. Then we get the desired result, because in the proof of Lemma 6.30.6 we construct the extension in terms of families of elements of stalks satisfying $(*)$. $\square$

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

as $\mathcal{O}_x$-modules in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x, or simply because it is true on the underlying sets.

009U Lemma 6.30.13. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a sheaf of rings on X. Denote $\mathrm{Mod}(\mathcal{O}|_{\mathcal{B}})$ the category of sheaves of $\mathcal{O}|_{\mathcal{B}}$-modules on $\mathcal{B}$. There is an equivalence of categories

$$\mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O}|_{\mathcal{B}})$$

which assigns to a sheaf of $\mathcal{O}$-modules on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.12 above. Checking the obvious functorialities is left to the reader. $\square$

Finally, we address the question of the relationship of this with continuous maps. This is now very easy thanks to the work above. First we do the case where there is a basis on the target given.

009V Lemma 6.30.14. Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a morphism

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 6.21.7 and discussion of f-maps in Section 6.23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. This is trivial because the collection of maps amounts to a morphism between the restrictions of $\mathcal{G}$ and $f_*\mathcal{F}$ to $\mathcal{B}$. By Lemma 6.30.10 this is the same as giving a morphism from $\mathcal{G}$ to $f_*\mathcal{F}$, which by Lemma 6.21.8 is the same as an f-map. See also Lemma 6.23.1 and the discussion preceding it for how to deal with the case of sheaves of algebraic structures. $\square$

Here is the analogue for ringed spaces.

009W Lemma 6.30.15. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a $\mathcal{O}_Y(V)$-module map

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

(where $\mathcal{F}(f^{-1}V)$ has a module structure using $f^\sharp_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$) compatible with restriction mappings. Then there is a unique f-map (see discussion of f-maps in Section 6.26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. Same as the proof of the corresponding lemma for sheaves of algebraic structures above. $\square$

009X Lemma 6.30.16. Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a morphism

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 6.21.7 and the discussion of f-maps in Section 6.23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{\text{restr.}} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Let us first proves this for sheaves of sets. Fix $V \subset Y$ open. Pick $s \in \mathcal{G}(V)$. We are going to construct an element $\varphi_V(s) \in \mathcal{F}(f^{-1}V)$. We can define a value $\varphi(s)_x$ in the stalk $\mathcal{F}_x$ for every $x \in f^{-1}V$ by picking a $U \in \mathcal{B}_X$ with $x \in U \subset f^{-1}V$ and setting $\varphi(s)_x$ equal to the equivalence class of $(U, \varphi_V^U(s))$ in the stalk. Clearly, the family $(\varphi(s)_x)_{x \in f^{-1}V}$ satisfies condition $(*)$ because the maps φ_V^U for varying U

are compatible with restrictions in the sheaf $\mathcal{F}$. Thus, by the proof of Lemma 6.30.6 we see that $(\varphi(s)_x)_{x\in f^{-1}V}$ corresponds to a unique element $\varphi_V(s)$ of $\mathcal{F}(f^{-1}V)$. Thus we have defined a set map $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$. The compatibility between φ_V and φ_V^U follows from Lemma 6.30.5.

We leave it to the reader to show that the construction of φ_V is compatible with restriction mappings as we vary $V \in \mathcal{B}_Y$. Thus we may apply Lemma 6.30.14 above to "glue" them to the desired f-map.

Finally, we note that the map of sheaves of sets so constructed satisfies the property that the map on stalks

$$\mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is the colimit of the system of maps φ_V^U as $V \in \mathcal{B}_Y$ varies over those elements that contain $f(x)$ and $U \in \mathcal{B}_X$ varies over those elements that contain x. In particular, if $\mathcal{G}$ and $\mathcal{F}$ are the underlying sheaves of sets of sheaves of algebraic structures, then we see that the maps on stalks is a morphism of algebraic structures. Hence we conclude that the associated map of sheaves of underlying sets $f^{-1}\mathcal{G} \to \mathcal{F}$ satisfies the assumptions of Lemma 6.23.1. We conclude that $f^{-1}\mathcal{G} \to \mathcal{F}$ is a morphism of sheaves with values in $\mathcal{C}$. And by adjointness this means that φ is an f-map of sheaves of algebraic structures. □

009Y Lemma 6.30.17. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a $\mathcal{O}_Y(V)$-module map

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction mappings. Here the $\mathcal{O}_Y(V)$-module structure on $\mathcal{F}(U)$ comes from the $\mathcal{O}_X(U)$-module structure via the map $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V) \to \mathcal{O}_X(U)$. Then there is a unique f-map of sheaves of modules (see Definition 6.21.7 and the discussion of f-maps in Section 6.26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{\text{restr.}} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Similar to the above and omitted. □

6.31. Open immersions and (pre)sheaves

009Z Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X. In Section 6.21 we have defined functors j_* and j^{-1} such that j_* is right adjoint to j^{-1}. It turns out that for an open immersion there is a left adjoint for j^{-1}, which we will denote $j_!$. First we point out that j^{-1} has a particularly simple description in the case of an open immersion.

00A0 Lemma 6.31.1. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X.

(1) Let $\mathcal{G}$ be a presheaf of sets on X. The presheaf $j_p\mathcal{G}$ (see Section 6.21) is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.
(2) Let $\mathcal{G}$ be a sheaf of sets on X. The sheaf $j^{-1}\mathcal{G}$ is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.

(3) For any point $u \in U$ and any sheaf $\mathcal{G}$ on X we have a canonical identification of stalks
$$j^{-1}\mathcal{G}_u = (\mathcal{G}|_U)_u = \mathcal{G}_u.$$
(4) On the category of presheaves of U we have $j_p j_* = \text{id}$.
(5) On the category of sheaves of U we have $j^{-1}j_* = \text{id}$.

The same description holds for (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures, and (pre)sheaves of modules.

Proof. The colimit in the definition of $j_p\mathcal{G}(V)$ is over collection of all $W \subset X$ open such that $V \subset W$ ordered by reverse inclusion. Hence this has a largest element, namely V. This proves (1). And (2) follows because the assignment $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open is clearly a sheaf if $\mathcal{G}$ is a sheaf. Assertion (3) follows from (2) since the collection of open neighbourhoods of u which are contained in U is cofinal in the collection of all open neighbourhoods of u in X. Parts (4) and (5) follow by computing $j^{-1}j_*\mathcal{F}(V) = j_*\mathcal{F}(V) = \mathcal{F}(V)$.

The exact same arguments work for (pre)sheaves of abelian groups and (pre)sheaves of algebraic structures. $\square$

00A1 Definition 6.31.2. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{G}$ be a presheaf of sets, abelian groups or algebraic structures on X. The presheaf $j_p\mathcal{G}$ described in Lemma 6.31.1 is called the restriction of $\mathcal{G}$ to U and denoted $\mathcal{G}|_U$.
(2) Let $\mathcal{G}$ be a sheaf of sets on X, abelian groups or algebraic structures on X. The sheaf $j^{-1}\mathcal{G}$ is called the restriction of $\mathcal{G}$ to U and denoted $\mathcal{G}|_U$.
(3) If $(X, \mathcal{O})$ is a ringed space, then the pair $(U, \mathcal{O}|_U)$ is called the open subspace of $(X, \mathcal{O})$ associated to U.
(4) If $\mathcal{G}$ is a presheaf of $\mathcal{O}$-modules then $\mathcal{G}|_U$ together with the multiplication map $\mathcal{O}|_U \times \mathcal{G}|_U \to \mathcal{G}|_U$ (see Lemma 6.24.6) is called the restriction of $\mathcal{G}$ to U.

We leave a definition of the restriction of presheaves of modules to the reader. Ok, so in this section we will discuss a left adjoint to the restriction functor. Here is the definition in the case of (pre)sheaves of sets.

00A2 Definition 6.31.3. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be a presheaf of sets on U. We define the extension of $\mathcal{F}$ by the empty set $j_{p!}\mathcal{F}$ to be the presheaf of sets on X defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} \emptyset & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.
(2) Let $\mathcal{F}$ be a sheaf of sets on U. We define the extension of $\mathcal{F}$ by the empty set $j_!\mathcal{F}$ to be the sheafification of the presheaf $j_{p!}\mathcal{F}$.

00A3 Lemma 6.31.4. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).

(2) The functor $j_!$ is a left adjoint to restriction, in a formula
$$\mathrm{Mor}_{Sh(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(3) Let $\mathcal{F}$ be a sheaf of sets on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows
$$j_!\mathcal{F}_x = \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) On the category of presheaves of U we have $j_p j_{p!} = \mathrm{id}$.

(5) On the category of sheaves of U we have $j^{-1}j_! = \mathrm{id}$.

Proof. To map $j_{p!}\mathcal{F}$ into $\mathcal{G}$ it is enough to map $\mathcal{F}(V) \to \mathcal{G}(V)$ whenever $V \subset U$ compatibly with restriction mappings. And by Lemma 6.31.1 the same description holds for maps $\mathcal{F} \to \mathcal{G}|_U$. The adjointness of $j_!$ and restriction follows from this and the properties of sheafification. The identification of stalks is obvious from the definition of the extension by the empty set and the definition of a stalk. Statements (4) and (5) follow by computing the value of the sheaf on any open of U. □

Note that if $\mathcal{F}$ is a sheaf of abelian groups on U, then in general $j_!\mathcal{F}$ as defined above, is not a sheaf of abelian groups, for example because some of its stalks are empty (hence not abelian groups for sure). Thus we need to modify the definition of $j_!$ depending on the type of sheaves we consider. The reason for choosing the empty set in the definition of the extension by the empty set, is that it is the initial object in the category of sets. Thus in the case of abelian groups we use 0 (and more generally for sheaves with values in any abelian category).

00A4 Definition 6.31.5. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be an abelian presheaf on U. We define the extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by 0 to be the abelian presheaf on X defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} 0 & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.

(2) Let $\mathcal{F}$ be an abelian sheaf on U. We define the extension $j_!\mathcal{F}$ of $\mathcal{F}$ by 0 to be the sheafification of the abelian presheaf $j_{p!}\mathcal{F}$.

(3) Let $\mathcal{C}$ be a category having an initial object e. Let $\mathcal{F}$ be a presheaf on U with values in $\mathcal{C}$. We define the extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by e to be the presheaf on X with values in $\mathcal{C}$ defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} e & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.

(4) Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Let $\mathcal{F}$ be a sheaf of algebraic structures on U (of the give type). We define the extension $j_!\mathcal{F}$ of $\mathcal{F}$ by e to be the sheafification of the presheaf $j_{p!}\mathcal{F}$ defined above.

(5) Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}|_U$-modules. In this case we define the extension by 0 to be the presheaf of $\mathcal{O}$-modules which is equal to $j_{p!}\mathcal{F}$ as an abelian presheaf endowed with the multiplication map $\mathcal{O} \times j_{p!}\mathcal{F} \to j_{p!}\mathcal{F}$.

(6) Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}|_U$-modules. In this case we define the extension by 0 to be the $\mathcal{O}$-module which is equal to $j_!\mathcal{F}$ as an abelian sheaf endowed with the multiplication map $\mathcal{O} \times j_!\mathcal{F} \to j_!\mathcal{F}$.

It is true that one can define $j_!$ in the setting of sheaves of algebraic structures (see below). However, it depends on the type of algebraic structures involved what the resulting object is. For example, if $\mathcal{O}$ is a sheaf of rings on U, then $j_{!,rings}\mathcal{O} \neq j_{!,abelian}\mathcal{O}$ since the initial object in the category of rings is $\mathbf{Z}$ and the initial object in the category of abelian groups is 0. In particular the functor $j_!$ does not commute with taking underlying sheaves of sets, in contrast to what we have seen so far! We separate out the case of (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures and (pre)sheaves of modules as usual.

00A5 **Lemma 6.31.6.** Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Consider the functors of restriction and extension by 0 for abelian (pre)sheaves.

(1) The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).
(2) The functor $j_!$ is a left adjoint to restriction, in a formula
$$\mathrm{Mor}_{\mathrm{Ab}(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{\mathrm{Ab}(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{\mathrm{Ab}(U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) Let $\mathcal{F}$ be an abelian sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows
$$j_!\mathcal{F}_x = \begin{cases} 0 & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) On the category of abelian presheaves of U we have $j_p j_{p!} = \mathrm{id}$.
(5) On the category of abelian sheaves of U we have $j^{-1}j_! = \mathrm{id}$.

Proof. Omitted. □

00A6 **Lemma 6.31.7.** Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Consider the functors of restriction and extension by e for (pre)sheaves of algebraic structure defined above.

(1) The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).
(2) The functor $j_!$ is a left adjoint to restriction, in a formula
$$\mathrm{Mor}_{Sh(X,\mathcal{C})}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(U,\mathcal{C})}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Sh(U,\mathcal{C})}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) Let $\mathcal{F}$ be a sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows
$$j_!\mathcal{F}_x = \begin{cases} e & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) On the category of presheaves of algebraic structures on U we have $j_p j_{p!} = \mathrm{id}$.
(5) On the category of sheaves of algebraic structures on U we have $j^{-1}j_! = \mathrm{id}$.

Proof. Omitted. □

00A7 Lemma 6.31.8. Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. Consider the functors of restriction and extension by 0 for (pre)sheaves of modules defined above.

(1) The functor $j_{p!}$ is a left adjoint to restriction, in a formula
$$\mathrm{Mor}_{\mathrm{PMod}(\mathcal{O})}(j_{p!}\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{\mathrm{PMod}(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(2) The functor $j_!$ is a left adjoint to restriction, in a formula
$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(3) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows
$$j_!\mathcal{F}_x = \begin{cases} 0 & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) On the category of sheaves of $\mathcal{O}|_U$-modules on U we have $j^{-1}j_! = \mathrm{id}$.

Proof. Omitted. □

Note that by the lemmas above, both the functors j_* and $j_!$ are fully faithful embeddings of the category of sheaves on U into the category of sheaves on X. It is only true for the functor $j_!$ that one can easily describe the essential image of this functor.

00A8 Lemma 6.31.9. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor
$$j_! : Sh(U) \longrightarrow Sh(X)$$
is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \emptyset$ for all $x \in X \setminus U$.

Proof. Fully faithfulness follows formally from $j^{-1}j_! = \mathrm{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that
$$j_!j^{-1}\mathcal{G} \to \mathcal{G}$$
is an isomorphism on all stalks and hence an isomorphism. □

00A9 Lemma 6.31.10. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor
$$j_! : \mathrm{Ab}(U) \longrightarrow \mathrm{Ab}(X)$$
is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. □

00AA Lemma 6.31.11. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. The functor
$$j_! : Sh(U, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = e$ for all $x \in X \setminus U$.

Proof. Omitted. ☐

00AB Lemma 6.31.12. Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. The functor

$$j_! : \text{Mod}(\mathcal{O}|_U) \longrightarrow \text{Mod}(\mathcal{O})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. ☐

00AC Remark 6.31.13. Let $j : U \to X$ be an open immersion of topological spaces as above. Let $x \in X$, $x \notin U$. Let $\mathcal{F}$ be a sheaf of sets on U. Then $j_!\mathcal{F}_x = \emptyset$ by Lemma 6.31.4. Hence $j_!$ does not transform a final object of $Sh(U)$ into a final object of $Sh(X)$ unless $U = X$. According to our conventions in Categories, Section 4.23 this means that the functor $j_!$ is not left exact as a functor between the categories of sheaves of sets. It will be shown later that $j_!$ on abelian sheaves is exact, see Modules, Lemma 17.3.4.

6.32. Closed immersions and (pre)sheaves

00AD Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. In Section 6.21 we have defined functors i_* and i^{-1} such that i_* is right adjoint to i^{-1}.

00AE Lemma 6.32.1. Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. Let $\mathcal{F}$ be a sheaf of sets on Z. The stalks of $i_*\mathcal{F}$ are described as follows

$$i_*\mathcal{F}_x = \begin{cases} \{*\} & \text{if} \quad x \notin Z \\ \mathcal{F}_x & \text{if} \quad x \in Z \end{cases}$$

where $\{*\}$ denotes a singleton set. Moreover, $i^{-1}i_* = \text{id}$ on the category of sheaves of sets on Z. Moreover, the same holds for abelian sheaves on Z, resp. sheaves of algebraic structures on Z where $\{*\}$ has to be replaced by 0, resp. a final object of the category of algebraic structures.

Proof. If $x \notin Z$, then there exist arbitrarily small open neighbourhoods U of x which do not meet Z. Because $\mathcal{F}$ is a sheaf we have $\mathcal{F}(i^{-1}(U)) = \{*\}$ for any such U, see Remark 6.7.2. This proves the first case. The second case comes from the fact that for $z \in Z$ any open neighbourhood of z is of the form $Z \cap U$ for some open U of X. For the statement that $i^{-1}i_* = \text{id}$ consider the canonical map $i^{-1}i_*\mathcal{F} \to \mathcal{F}$. This is an isomorphism on stalks (see above) and hence an isomorphism.

For sheaves of abelian groups, and sheaves of algebraic structures you argue in the same manner. ☐

00AF Lemma 6.32.2. Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor

$$i_* : Sh(Z) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \{*\}$ for all $x \in X \setminus Z$.

Proof. Fully faithfulness follows formally from $i^{-1}i_* = \text{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$\mathcal{G} \to i_* i^{-1}\mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. □

00AG Lemma 6.32.3. Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor

$$i_* : \text{Ab}(Z) \longrightarrow \text{Ab}(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. □

00AH Lemma 6.32.4. Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure with final object 0. The functor

$$i_* : Sh(Z, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. □

00AI Remark 6.32.5. Let $i : Z \to X$ be a closed immersion of topological spaces as above. Let $x \in X$, $x \notin Z$. Let $\mathcal{F}$ be a sheaf of sets on Z. Then $(i_*\mathcal{F})_x = \{*\}$ by Lemma 6.32.1. Hence if $\mathcal{F} = * \amalg *$, where $*$ is the singleton sheaf, then $i_*\mathcal{F}_x = \{*\} \neq i_*(*)_x \amalg i_*(*)_x$ because the latter is a two point set. According to our conventions in Categories, Section 4.23 this means that the functor i_* is not right exact as a functor between the categories of sheaves of sets. In particular, it cannot have a right adjoint, see Categories, Lemma 4.24.6.

On the other hand, we will see later (see Modules, Lemma 17.6.3) that i_* on abelian sheaves is exact, and does have a right adjoint, namely the functor that associates to an abelian sheaf on X the sheaf of sections supported in Z.

00AJ Remark 6.32.6. We have not discussed the relationship between closed immersions and ringed spaces. This is because the notion of a closed immersion of ringed spaces is best discussed in the setting of quasi-coherent sheaves, see Modules, Section 17.13.

6.33. Glueing sheaves

00AK In this section we glue sheaves defined on the members of a covering of X. We first deal with maps.

04TN Lemma 6.33.1. Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of sets on X. Given a collection

$$\varphi_i : \mathcal{F}|_{U_i} \longrightarrow \mathcal{G}|_{U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathcal{F}|_{U_i \cap U_j} \to \mathcal{G}|_{U_i \cap U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$$

whose restriction to each U_i agrees with φ_i.

Proof. For each open subset $U \subset X$ define

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U), \quad s \mapsto \varphi_U(s)$$

where $\varphi_U(s)$ is the unique section verifying

$$(\varphi_U(s))|_{U \cap U_i} = (\varphi_i)_{U \cap U_i}(s|_{U \cap U_i}).$$

Existence and uniqueness of such a section follows from the sheaf axioms due to the fact that

$$\begin{aligned} ((\varphi_i)_{U \cap U_i}(s|_{U \cap U_i}))|_{U \cap U_i \cap U_j} &= (\varphi_i)_{U \cap U_i \cap U_j}(s|_{U \cap U_i \cap U_j}) \\ &= (\varphi_j)_{U \cap U_i \cap U_j}(s|_{U \cap U_i \cap U_j}) \\ &= ((\varphi_j)_{U \cap U_j}(s|_{U \cap U_j}))|_{U \cap U_i \cap U_j}. \end{aligned}$$

This family of maps gives us indeed a map of sheaves: Let $V \subset U \subset X$ be open subsets then

$$(\varphi_U(s))|_V = \varphi_V(s|_V)$$

since for each $i \in I$ the following holds

$$\begin{aligned} (\varphi_U(s))|_{V \cap U_i} &= ((\varphi_U(s))|_{U \cap U_i})|_{V \cap U_i} \\ &= ((\varphi_i)_{U \cap U_i}(s|_{U \cap U_i}))|_{V \cap U_i} \\ &= (\varphi_i)_{V \cap U_i}(s|_{V \cap U_i}) \\ &= \varphi_V(s_V)|_{V \cap U_i}. \end{aligned}$$

Furthermore, its restriction to each U_i agrees with φ_i since given $U \subset X$ open subset and $s \in \mathcal{F}(U \cap U_i)$ then

$$\begin{aligned} \varphi_{U \cap U_i}(s) &= \varphi_{U \cap U_i}(s)|_{U \cap U_i} \\ &= (\varphi_i)_{U \cap U_i}(s|_{U \cap U_i}) \\ &= (\varphi_i)_{U \cap U_i}(s). \end{aligned}$$

□

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on the topological space X the rule

$$U \longmapsto \mathrm{Mor}_{Sh(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

defines a sheaf. This is a kind of internal hom sheaf. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules, Section 17.22.

Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on U_i. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \longrightarrow \mathcal{F}_j|_{U_i \cap U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{tikzcd} \mathcal{F}_i|_{U_i \cap U_j \cap U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathcal{F}_k|_{U_i \cap U_j \cap U_k} \\ & \mathcal{F}_j|_{U_i \cap U_j \cap U_k} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a glueing data for sheaves of sets with respect to the covering $X = \bigcup U_i$.

00AL Lemma 6.33.2. Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $X = \bigcup U_i$ there exists a sheaf of sets $\mathcal{F}$ on X together with isomorphisms

$$\varphi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{tikzcd} \mathcal{F}|_{U_i \cap U_j} \arrow[r, "\varphi_i"'] \arrow[d, "\mathrm{id}"'] & \mathcal{F}_i|_{U_i \cap U_j} \arrow[d, "\varphi_{ij}"] \\ \mathcal{F}|_{U_i \cap U_j} \arrow[r, "\varphi_j"] & \mathcal{F}_j|_{U_i \cap U_j} \end{tikzcd}$$

are commutative.

Proof. First proof. In this proof we give a formula for the set of sections of $\mathcal{F}$ over an open $W \subset X$. Namely, we define

$$\mathcal{F}(W) = \{(s_i)_{i \in I} \mid s_i \in \mathcal{F}_i(W \cap U_i), \varphi_{ij}(s_i|_{W \cap U_i \cap U_j}) = s_j|_{W \cap U_i \cap U_j}\}.$$

Restriction mappings for $W' \subset W$ are defined by the restricting each of the s_i to $W' \cap U_i$. The sheaf condition for $\mathcal{F}$ follows immediately from the sheaf condition for each of the $\mathcal{F}_i$.

We still have to prove that $\mathcal{F}|_{U_i}$ maps isomorphically to $\mathcal{F}_i$. Let $W \subset U_i$. In this case the condition in the definition of $\mathcal{F}(W)$ implies that $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. And the commutativity of the diagrams in the definition of a glueing data assures that we may start with any section $s \in \mathcal{F}_i(W)$ and obtain a compatible collection by setting $s_i = s$ and $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$.

Second proof (sketch). Let $\mathcal{B}$ be the set of opens $U \subset X$ such that $U \subset U_i$ for some $i \in I$. Then $\mathcal{B}$ is a base for the topology on X. For $U \in \mathcal{B}$ we pick $i \in I$ with $U \subset U_i$ and we set $\mathcal{F}(U) = \mathcal{F}_i(U)$. Using the isomorphisms φ_{ij} we see that this prescription is "independent of the choice of i". Using the restriction mappings of $\mathcal{F}_i$ we find that $\mathcal{F}$ is a sheaf on $\mathcal{B}$. Finally, use Lemma 6.30.6 to extend $\mathcal{F}$ to a unique sheaf $\mathcal{F}$ on X. □

00AM Lemma 6.33.3. Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $(\mathcal{F}_i, \varphi_{ij})$ be a glueing data of sheaves of abelian groups, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules for some sheaf of rings $\mathcal{O}$ on X. Then the construction in the proof of Lemma 6.33.2 above leads to a sheaf of abelian groups, resp. sheaf of algebraic structures, resp. sheaf of $\mathcal{O}$-modules.

Proof. This is true because in the construction the set of sections $\mathcal{F}(W)$ over an open W is given as the equalizer of the maps

$$\prod_{i \in I} \mathcal{F}_i(W \cap U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}_i(W \cap U_i \cap U_j)$$

And in each of the cases envisioned this equalizer gives an object in the relevant category whose underlying set is the object considered in the cited lemma. □

00AN **Lemma 6.33.4.** Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. The functor which associates to a sheaf of sets $\mathcal{F}$ the following collection of glueing data

$$(\mathcal{F}|_{U_i}, (\mathcal{F}|_{U_i})|_{U_i \cap U_j} \to (\mathcal{F}|_{U_j})|_{U_i \cap U_j})$$

with respect to the covering $X = \bigcup U_i$ defines an equivalence of categories between $Sh(X)$ and the category of glueing data. A similar statement holds for abelian sheaves, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules.

Proof. The functor is fully faithful by Lemma 6.33.1 and essentially surjective (via an explicitly given quasi-inverse functor) by Lemma 6.33.2. $\square$

This lemma means that if the sheaf $\mathcal{F}$ was constructed from the glueing data $(\mathcal{F}_i, \varphi_{ij})$ and if $\mathcal{G}$ is a sheaf on X, then a morphism $f : \mathcal{F} \to \mathcal{G}$ is given by a collection of morphisms of sheaves

$$f_i : \mathcal{F}_i \longrightarrow \mathcal{G}|_{U_i}$$

compatible with the glueing maps φ_{ij}. Similarly, to give a morphism of sheaves $g : \mathcal{G} \to \mathcal{F}$ is the same as giving a collection of morphisms of sheaves

$$g_i : \mathcal{G}|_{U_i} \longrightarrow \mathcal{F}_i$$

compatible with the glueing maps φ_{ij}.

6.34. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

Topics in Scheme Theory

(42) Chow Homology
(43) Intersection Theory
(44) Picard Schemes of Curves
(45) Weil Cohomology Theories
(46) Adequate Modules
(47) Dualizing Complexes
(48) Duality for Schemes

CHAPTER 7

Sites and Sheaves

00UZ 7.1. Introduction

00V0 The notion of a site was introduced by Grothendieck to be able to study sheaves in the étale topology of schemes. The basic reference for this notion is perhaps [AGV71]. Our notion of a site differs from that in [AGV71]; what we call a site is called a category endowed with a pretopology in [AGV71, Exposé II, Définition 1.3]. The reason we do this is that in algebraic geometry it is often convenient to work with a given class of coverings, for example when defining when a property of schemes is local in a given topology, see Descent, Section 35.15. Our exposition will closely follow [Art62]. We will not use universes.

7.2. Presheaves

00V1 Let $\mathcal{C}$ be a category. A presheaf of sets is a contravariant functor $\mathcal{F}$ from $\mathcal{C}$ to Sets (see Categories, Remark 4.2.11). So for every object U of $\mathcal{C}$ we have a set $\mathcal{F}(U)$. The elements of this set are called the sections of $\mathcal{F}$ over U. For every morphism $f : V \to U$ the map $\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the restriction map and is often denoted $f^* : \mathcal{F}(U) \to \mathcal{F}(V)$. Another way of expressing this is to say that $f^*(s)$ is the pullback of s via f. Functoriality means that $g^* f^*(s) = (f \circ g)^*(s)$. Sometimes we use the notation $s|_V := f^*(s)$. This notation is consistent with the notion of restriction of functions from topology because if $W \to V \to U$ are morphisms in $\mathcal{C}$ and s is a section of $\mathcal{F}$ over U then $s|_W = (s|_V)|_W$ by the functorial nature of $\mathcal{F}$. Of course we have to be careful since it may very well happen that there is more than one morphism $V \to U$ and it is certainly not going to be the case that the corresponding pullback maps are equal.

00V2 Definition 7.2.1. A presheaf of sets on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to Sets. Morphisms of presheaves are transformations of functors. The category of presheaves of sets is denoted $\textit{PSh}(\mathcal{C})$.

Note that for any object U of $\mathcal{C}$ the functor of points h_U, see Categories, Example 4.3.4 is a presheaf. These are called the representable presheaves. These presheaves have the pleasing property that for any presheaf $\mathcal{F}$ we have

090F (7.2.1.1)
$$\mathrm{Mor}_{\textit{PSh}(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

This is the Yoneda lemma (Categories, Lemma 4.3.5).

Similarly, we can define the notion of a presheaf of abelian groups, rings, etc. More generally we may define a presheaf with values in a category.

00V3 Definition 7.2.2. Let $\mathcal{C}$, $\mathcal{A}$ be categories. A presheaf $\mathcal{F}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}$, i.e., $\mathcal{F} : \mathcal{C}^{opp} \to \mathcal{A}$. A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a transformation of functors from $\mathcal{F}$ to $\mathcal{G}$.

These form the objects and morphisms of the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$.

00V4 Remark 7.2.3. As already pointed out we may consider the category of presheaves with values in any of the "big" categories listed in Categories, Remark 4.2.2. These will be "big" categories as well and they will be listed in the above mentioned remark as we go along.

7.3. Injective and surjective maps of presheaves

00V5

00V6 Definition 7.3.1. Let $\mathcal{C}$ be a category, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets.

(1) We say that φ is injective if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is surjective if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

00V7 Lemma 7.3.2. The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of $\mathrm{PSh}(\mathcal{C})$. A map is an isomorphism if and only if it is both injective and surjective.

Proof. We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if it is a monomorphism of $\mathrm{PSh}(\mathcal{C})$. Indeed, the "only if" direction is straightforward, so let us show the "if" direction. Assume that φ is a monomorphism. Let $U \in \mathrm{Ob}(\mathcal{C})$; we need to show that φ_U is injective. So let $a, b \in \mathcal{F}(U)$ be such that $\varphi_U(a) = \varphi_U(b)$; we need to check that $a = b$. Under the isomorphism (7.2.1.1), the elements a and b of $\mathcal{F}(U)$ correspond to two natural transformations $a', b' \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, \mathcal{F})$. Similarly, under the analogous isomorphism $\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, \mathcal{G}) = \mathcal{G}(U)$, the two equal elements $\varphi_U(a)$ and $\varphi_U(b)$ of $\mathcal{G}(U)$ correspond to the two natural transformations $\varphi \circ a', \varphi \circ b' \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, \mathcal{G})$, which therefore must also be equal. So $\varphi \circ a' = \varphi \circ b'$, and thus $a' = b'$ (since φ is monic), whence $a = b$. This finishes (1).

We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if it is an epimorphism of $\mathrm{PSh}(\mathcal{C})$. Indeed, the "only if" direction is straightforward, so let us show the "if" direction. Assume that φ is an epimorphism.

For any two morphisms $f : A \to B$ and $g : A \to C$ in the category Sets, we let $\mathrm{inl}_{f,g}$ and $\mathrm{inr}_{f,g}$ denote the two canonical maps from B and C to $B \coprod_A C$. (Here, the pushout is evaluated in Sets.)

Now, we define a presheaf $\mathcal{H}$ of sets on $\mathcal{C}$ by setting $\mathcal{H}(U) = \mathcal{G}(U) \coprod_{\mathcal{F}(U)} \mathcal{G}(U)$ (where the pushout is evaluated in Sets and induced by the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$) for every $U \in \mathrm{Ob}(\mathcal{C})$; its action on morphisms is defined in the obvious way (by the functoriality of pushout). Then, there are two natural transformations $i_1 : \mathcal{G} \to \mathcal{H}$ and $i_2 : \mathcal{G} \to \mathcal{H}$ whose components at an object $U \in \mathrm{Ob}(\mathcal{C})$ are given by the maps $\mathrm{inl}_{\varphi_U, \varphi_U}$ and $\mathrm{inr}_{\varphi_U, \varphi_U}$, respectively. The definition of a pushout shows that $i_1 \circ \varphi = i_2 \circ \varphi$, whence $i_1 = i_2$ (since φ is an epimorphism). Thus, for every $U \in \mathrm{Ob}(\mathcal{C})$, we have $\mathrm{inl}_{\varphi_U, \varphi_U} = \mathrm{inr}_{\varphi_U, \varphi_U}$. Thus, φ_U must be surjective (since a simple combinatorial argument shows that if $f : A \to B$ is a morphism in Sets, then $\mathrm{inl}_{f,f} = \mathrm{inr}_{f,f}$ if and only if f is surjective). In other words, φ is surjective, and (2) is proven.

We shall show that $\varphi : \mathcal{F} \to \mathcal{G}$ is both injective and surjective if and only if it is an isomorphism of $\text{PSh}(\mathcal{C})$. This time, the "if" direction is straightforward. To prove the "only if" direction, it suffices to observe that if φ is both injective and surjective, then φ_U is an invertible map for every $U \in \text{Ob}(\mathcal{C})$, and the inverses of these maps for all U can be combined to a natural transformation $\mathcal{G} \to \mathcal{F}$ which is an inverse to φ. □

00V8 Definition 7.3.3. We say $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$ if for every object $U \in \text{Ob}(\mathcal{C})$ the set $\mathcal{F}(U)$ is a subset of $\mathcal{G}(U)$, compatibly with the restriction mappings.

In other words, the inclusion maps $\mathcal{F}(U) \to \mathcal{G}(U)$ glue together to give an (injective) morphism of presheaves $\mathcal{F} \to \mathcal{G}$.

00V9 Lemma 7.3.4. Let $\mathcal{C}$ be a category. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{C}$. There exists a unique subpresheaf $\mathcal{G}' \subset \mathcal{G}$ such that φ factors as $\mathcal{F} \to \mathcal{G}' \to \mathcal{G}$ and such that the first map is surjective.

Proof. To prove existence, just set $\mathcal{G}'(U) = \varphi_U(\mathcal{F}(U))$ for every $U \in \text{Ob}(C)$ (and inherit the action on morphisms from $\mathcal{G}$), and prove that this defines a subpresheaf of $\mathcal{G}$ and that φ factors as $\mathcal{F} \to \mathcal{G}' \to \mathcal{G}$ with the first map being surjective. Uniqueness is straightforward. □

00VA Definition 7.3.5. Notation as in Lemma 7.3.4. We say that $\mathcal{G}'$ is the image of φ.

7.4. Limits and colimits of presheaves

00VB Let $\mathcal{C}$ be a category. Limits and colimits exist in the category $\text{PSh}(\mathcal{C})$. In addition, for any $U \in \text{Ob}(\mathcal{C})$ the functor

$$\text{PSh}(\mathcal{C}) \longrightarrow \text{Sets}, \quad \mathcal{F} \longmapsto \mathcal{F}(U)$$

commutes with limits and colimits. Perhaps the easiest way to prove these statements is the following. Given a diagram $\mathcal{F} : \mathcal{I} \to \text{PSh}(\mathcal{C})$ define presheaves

$$\mathcal{F}_{\lim} : U \longmapsto \lim_{i \in \mathcal{I}} \mathcal{F}_i(U) \text{ and } \mathcal{F}_{\text{colim}} : U \longmapsto \text{colim}_{i \in \mathcal{I}} \mathcal{F}_i(U)$$

There are clearly projection maps $\mathcal{F}_{\lim} \to \mathcal{F}_i$ and canonical maps $\mathcal{F}_i \to \mathcal{F}_{\text{colim}}$. These maps satisfy the requirements of the maps of a limit (resp. colimit) of Categories, Definition 4.14.1 (resp. Categories, Definition 4.14.2). Indeed, they clearly form a cone, resp. a cocone, over $\mathcal{F}$. Furthermore, if $(\mathcal{G}, q_i : \mathcal{G} \to \mathcal{F}_i)$ is another system (as in the definition of a limit), then we get for every U a system of maps $\mathcal{G}(U) \to \mathcal{F}_i(U)$ with suitable functoriality requirements. And thus a unique map $\mathcal{G}(U) \to \mathcal{F}_{\lim}(U)$. It is easy to verify these are compatible as we vary U and arise from the desired map $\mathcal{G} \to \mathcal{F}_{\lim}$. A similar argument works in the case of the colimit.

7.5. Functoriality of categories of presheaves

00VC Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. In this case we denote

$$u^p : \text{PSh}(\mathcal{D}) \longrightarrow \text{PSh}(\mathcal{C})$$

the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. Note that by the previous section this functor commutes with all limits.

For $V \in \mathrm{Ob}(\mathcal{D})$ let $\mathcal{I}_V^u$ denote the category with

053L (7.5.0.1)
$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{I}_V^u) & = & \{(U, \phi) \mid U \in \mathrm{Ob}(\mathcal{C}), \phi : V \to u(U)\} \\ \mathrm{Mor}_{\mathcal{I}_V^u}((U, \phi), (U', \phi')) & = & \{f : U \to U' \text{ in } \mathcal{C} \mid u(f) \circ \phi = \phi'\} \end{array}$$

We sometimes drop the subscript ^{u} from the notation and we simply write $\mathcal{I}_V$. We will use these categories to define a left adjoint to the functor u^p. Before we do so we prove a few technical lemmas.

00X4 **Lemma 7.5.1.** Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Suppose that $\mathcal{C}$ has fibre products and equalizers, and that u commutes with them. Then the categories $(\mathcal{I}_V)^{opp}$ satisfy the hypotheses of Categories, Lemma 4.19.8.

Proof. There are two conditions to check.

First, suppose we are given three objects $\phi : V \to u(U)$, $\phi' : V \to u(U')$, and $\phi'' : V \to u(U'')$ and morphisms $a : U' \to U$, $b : U'' \to U$ such that $u(a) \circ \phi' = \phi$ and $u(b) \circ \phi'' = \phi$. We have to show there exists another object $\phi''' : V \to u(U''')$ and morphisms $c : U''' \to U'$ and $d : U''' \to U''$ such that $u(c) \circ \phi''' = \phi'$, $u(d) \circ \phi''' = \phi''$ and $a \circ c = b \circ d$. We take $U''' = U' \times_U U''$ with c and d the projection morphisms. This works as u commutes with fibre products; we omit the verification.

Second, suppose we are given two objects $\phi : V \to u(U)$ and $\phi' : V \to u(U')$ and morphisms $a, b : (U, \phi) \to (U', \phi')$. We have to find a morphism $c : (U'', \phi'') \to (U, \phi)$ which equalizes a and b. Let $c : U'' \to U$ be the equalizer of a and b in the category $\mathcal{C}$. As u commutes with equalizers and since $u(a) \circ \phi = u(b) \circ \phi = \phi'$ we obtain a morphism $\phi'' : V \to u(U'')$. $\square$

00X3 **Lemma 7.5.2.** Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Assume

(1) the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and
(2) the category $\mathcal{C}$ has fibre products and u commutes with them.

Then the index categories $(\mathcal{I}_V^u)^{opp}$ are filtered (see Categories, Definition 4.19.1).

Proof. The assumptions imply that the assumptions of Lemma 7.5.1 are satisfied (see the discussion in Categories, Section 4.18). By Categories, Lemma 4.19.8 we see that $\mathcal{I}_V$ is a (possibly empty) disjoint union of directed categories. Hence it suffices to show that $\mathcal{I}_V$ is connected.

First, we show that $\mathcal{I}_V$ is nonempty. Namely, let X be the final object of $\mathcal{C}$, which exists by assumption. Let $V \to u(X)$ be the morphism coming from the fact that $u(X)$ is final in $\mathcal{D}$ by assumption. This gives an object of $\mathcal{I}_V$.

Second, we show that $\mathcal{I}_V$ is connected. Let $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$ be in $\mathrm{Ob}(\mathcal{I}_V)$. By assumption $U_1 \times U_2$ exists and $u(U_1 \times U_2) = u(U_1) \times u(U_2)$. Consider the morphism $\phi : V \to u(U_1 \times U_2)$ corresponding to (ϕ_1, ϕ_2) by the universal property of products. Clearly the object $\phi : V \to u(U_1 \times U_2)$ maps to both $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$. $\square$

Given $g : V' \to V$ in $\mathcal{D}$ we get a functor $\overline{g} : \mathcal{I}_V \to \mathcal{I}_{V'}$ by setting $\overline{g}(U, \phi) = (U, \phi \circ g)$ on objects. Given a presheaf $\mathcal{F}$ on $\mathcal{C}$ we obtain a functor

$$\mathcal{F}_V : \mathcal{I}_V^{opp} \longrightarrow \textit{Sets}, \quad (U, \phi) \longmapsto \mathcal{F}(U).$$

In other words, $\mathcal{F}_V$ is a presheaf of sets on $\mathcal{I}_V$. Note that we have $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$. We define

$$u_p\mathcal{F}(V) = \mathrm{colim}_{\mathcal{I}_V^{opp}} \mathcal{F}_V$$

As a colimit we obtain for each $(U, \phi) \in \mathrm{Ob}(\mathcal{I}_V)$ a canonical map $\mathcal{F}(U) \xrightarrow{c(\phi)} u_p\mathcal{F}(V)$. For $g : V' \to V$ as above there is a canonical restriction map $g^* : u_p\mathcal{F}(V) \to u_p\mathcal{F}(V')$ compatible with $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$ by Categories, Lemma 4.14.8. It is the unique map so that for all $(U, \phi) \in \mathrm{Ob}(\mathcal{I}_V)$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "c(\phi)"] \arrow[d, "\mathrm{id}"'] & u_p\mathcal{F}(V) \arrow[d, "g^*"] \\ \mathcal{F}(U) \arrow[r, "c(\phi \circ g)"] & u_p\mathcal{F}(V') \end{tikzcd}$$

commutes. The uniqueness of these maps implies that we obtain a presheaf. This presheaf will be denoted $u_p\mathcal{F}$.

00VD Lemma 7.5.3. There is a canonical map $\mathcal{F}(U) \to u_p\mathcal{F}(u(U))$, which is compatible with restriction maps (on $\mathcal{F}$ and on $u_p\mathcal{F}$).

Proof. This is just the map $c(\mathrm{id}_{u(U)})$ introduced above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors $\mathcal{F}_V \to \mathcal{F}'_V$, and hence to a map of presheaves $u_p\mathcal{F} \to u_p\mathcal{F}'$. In other words, we have defined a functor

$$u_p : \mathrm{PSh}(\mathcal{C}) \longrightarrow \mathrm{PSh}(\mathcal{D})$$

00VE Lemma 7.5.4. The functor u_p is a left adjoint to the functor u^p. In other words the formula

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G})$$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We will show that the displayed formula holds by constructing maps either way. We will leave it to the reader to verify they are each others inverse.

Given a map $\alpha : u_p\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^p u_p\mathcal{F} \to u^p\mathcal{G}$. Lemma 7.5.3 says that there is a map $\mathcal{F} \to u^p u_p\mathcal{F}$. The composition of the two gives the desired map. (The good thing about this construction is that it is clearly functorial in everything in sight.)

Conversely, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $u_p\beta : u_p\mathcal{F} \to u_p u^p\mathcal{G}$. We claim that the functor $u^p\mathcal{G}_Y$ on $\mathcal{I}_Y$ has a canonical map to the constant functor with value $\mathcal{G}(Y)$. Namely, for every object (X, ϕ) of $\mathcal{I}_Y$, the value of $u^p\mathcal{G}_Y$ on this object is $\mathcal{G}(u(X))$ which maps to $\mathcal{G}(Y)$ by $\mathcal{G}(\phi) = \phi^*$. This is a transformation of functors because $\mathcal{G}$ is a functor itself. This leads to a map $u_p u^p\mathcal{G}(Y) \to \mathcal{G}(Y)$. Another trivial verification shows that this is functorial in Y leading to a map of presheaves $u_p u^p\mathcal{G} \to \mathcal{G}$. The composition $u_p\mathcal{F} \to u_p u^p\mathcal{G} \to \mathcal{G}$ is the desired map. □

00VF Remark 7.5.5. Suppose that $\mathcal{A}$ is a category such that any diagram $\mathcal{I}_Y \to \mathcal{A}$ has a colimit in $\mathcal{A}$. In this case it is clear that there are functors u^p and u_p, defined in exactly the same way as above, on the categories of presheaves with values in $\mathcal{A}$. Moreover, the adjointness of the pair u^p and u_p continues to hold in this setting.

04D2 Lemma 7.5.6. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. For any object U of $\mathcal{C}$ we have $u_p h_U = h_{u(U)}$.

Proof. By adjointness of u_p and u^p we have

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(u_p h_U, \mathcal{G}) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, u^p\mathcal{G}) = u^p\mathcal{G}(U) = \mathcal{G}(u(U))$$

and hence by Yoneda's lemma we see that $u_p h_U = h_{u(U)}$ as presheaves. $\square$

7.6. Sites

00VG Our notion of a site uses the following type of structures.

0396 **Definition 7.6.1.** Let $\mathcal{C}$ be a category, see Conventions, Section 2.3. A *family of morphisms with fixed target* in $\mathcal{C}$ is given by an object $U \in \mathrm{Ob}(\mathcal{C})$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this.

It can happen that the set I is empty! This notation is meant to suggest an open covering as in topology.

00VH **Definition 7.6.2.** A *site*[1] is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, called *coverings of* $\mathcal{C}$, satisfying the following axioms

(1) If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Clarifications. In axiom (1) we require there should be a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that $I = \{i\}$ is a singleton set and such that the morphism $U_i \to U$ is equal to the morphism $V \to U$ given in (1). In the following we often denote $\{V \to U\}$ a family of morphisms with fixed target whose index set is a singleton. In axiom (3) we require the existence of the covering for some choice of the fibre products $U_i \times_U V$ for $i \in I$.

00VI **Remark 7.6.3.** (On set theoretic issues – skip on a first reading.) The main reason for introducing sites is to study the category of sheaves on a site, because it is the generalization of the category of sheaves on a topological space that has been so important in algebraic geometry. In order to avoid thinking about things like "classes of classes" and so on, we will not allow sites to be "big" categories, in contrast to what we do for categories and 2-categories.

Suppose that $\mathcal{C}$ is a category and that $\mathrm{Cov}(\mathcal{C})$ is a proper class of coverings satisfying (1), (2) and (3) above. We will not allow this as a site either, mainly because we are going to take limits over coverings. However, there are several natural ways to replace $\mathrm{Cov}(\mathcal{C})$ by a set of coverings or a slightly different structure that give rise to the same category of sheaves. For example:

(1) In Sets, Section 3.11 we show how to pick a suitable set of coverings that gives the same category of sheaves.
(2) Another thing we can do is to take the associated topology (see Definition 7.48.2). The resulting topology on $\mathcal{C}$ has the same category of sheaves. Two topologies have the same categories of sheaves if and only if they are equal, see Theorem 7.50.2. A topology on a category is given by a

[1]This notation differs from that of [AGV71], as explained in the introduction.

choice of sieves on objects. The collection of all possible sieves and even all possible topologies on $\mathcal{C}$ is a set.

(3) We could also slightly modify the notion of a site, see Remark 7.48.4 below, and end up with a canonical set of coverings.

Each of these solutions has some minor drawback. For the first, one has to check that constructions later on do not depend on the choice of the set of coverings. For the second, one has to learn about topologies and redo many of the arguments for sites. For the third, see the last sentence of Remark 7.48.4.

Our approach will be to work with sites as in Definition 7.6.2 above. Given a category $\mathcal{C}$ with a proper class of coverings as above, we will replace this by a set of coverings producing a site using Sets, Lemma 3.11.1. It is shown in Lemma 7.8.8 below that the resulting category of sheaves (the topos) is independent of this choice. We leave it to the reader to use one of the other two strategies to deal with these issues if he/she so desires.

00VJ Example 7.6.4. Let X be a topological space. Let X_{Zar} be the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in X_{Zar}. Now define $\{U_i \to U\}_{i\in I} \in \text{Cov}(X_{Zar})$ if and only if $\bigcup U_i = U$. Conditions (1) and (2) above are clear, and (3) is also clear once we realize that in X_{Zar} we have $U \times V = U \cap V$. Note that in particular the empty set has to be an element of X_{Zar} since otherwise this would not work in general. Furthermore, it is equally important, as we will see later, to allow the empty covering of the empty set as a covering! We turn X_{Zar} into a site by choosing a suitable set of coverings $\text{Cov}(X_{Zar})_{\kappa,\alpha}$ as in Sets, Lemma 3.11.1. Presheaves and sheaves (as defined below) on the site X_{Zar} agree exactly with the usual notion of a presheaves and sheaves on a topological space, as defined in Sheaves, Section 6.1.

00VK Example 7.6.5. Let G be a group. Consider the category G-Sets whose objects are sets X with a left G-action, with G-equivariant maps as the morphisms. An important example is ${}_GG$ which is the G-set whose underlying set is G and action given by left multiplication. This category has fiber products, see Categories, Section 4.7. We declare $\{\varphi_i : U_i \to U\}_{i\in I}$ to be a covering if $\bigcup_{i\in I} \varphi_i(U_i) = U$. This gives a class of coverings on G-Sets which is easily seen to satisfy conditions (1), (2), and (3) of Definition 7.6.2. The result is not a site since both the collection of objects of the underlying category and the collection of coverings form a proper class. We first replace by G-Sets by a full subcategory G-Sets_α as in Sets, Lemma 3.10.1. After this the site $(G\text{-Sets}_\alpha, \text{Cov}_{\kappa,\alpha'}(G\text{-Sets}_\alpha))$ gotten by suitably restricting the collection of coverings as in Sets, Lemma 3.11.1 will be denoted $\mathcal{T}_G$.

As a special case, if the group G is countable, then we can let $\mathcal{T}_G$ be the category of countable G-sets and coverings those jointly surjective families of morphisms $\{\varphi_i : U_i \to U\}_{i\in I}$ such that I is countable.

07GE Example 7.6.6. Let $\mathcal{C}$ be a category. There is a canonical way to turn this into a site where $\{f : V \to U \mid f \text{ is an isomorphism}\}$ are the coverings of U. Sheaves on this site are the presheaves on $\mathcal{C}$. This corresponding topology is called the chaotic or indiscrete topology.

7.7. Sheaves

00VL Let $\mathcal{C}$ be a site. Before we introduce the notion of a sheaf with values in a category we explain what it means for a presheaf of sets to be a sheaf. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$ and let $\{U_i \to U\}_{i \in I}$ be an element of $\text{Cov}(\mathcal{C})$. By assumption all the fibre products $U_i \times_U U_j$ exist in $\mathcal{C}$. There are two natural maps

$$\prod\nolimits_{i \in I} \mathcal{F}(U_i) \underset{\text{pr}_1^*}{\overset{\text{pr}_0^*}{\rightrightarrows}} \prod\nolimits_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

which we will denote pr_i^*, $i = 0, 1$ as indicated in the displayed equation. Namely, an element of the left hand side corresponds to a family $(s_i)_{i \in I}$, where each s_i is a section of $\mathcal{F}$ over U_i. For each pair $(i_0, i_1) \in I \times I$ we have the projection morphisms

$$\text{pr}^{(i_0,i_1)}_{i_0} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_0} \text{ and } \text{pr}^{(i_0,i_1)}_{i_1} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_1}.$$

Thus we may pull back either the section s_{i_0} via the first of these maps or the section s_{i_1} via the second. Explicitly the maps we referred to above are

$$\text{pr}_0^* : (s_i)_{i \in I} \longmapsto \left(\text{pr}^{(i_0,i_1),*}_{i_0}(s_{i_0})\right)_{(i_0,i_1) \in I \times I}$$

and

$$\text{pr}_1^* : (s_i)_{i \in I} \longmapsto \left(\text{pr}^{(i_0,i_1),*}_{i_1}(s_{i_1})\right)_{(i_0,i_1) \in I \times I}.$$

Finally consider the natural map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i), \quad s \longmapsto (s|_{U_i})_{i \in I}$$

where we have used the notation $s|_{U_i}$ to indicate the pullback of s via the map $U_i \to U$. It is clear from the functorial nature of $\mathcal{F}$ and the commutativity of the fibre product diagrams that $\text{pr}_0^*((s|_{U_i})_{i \in I}) = \text{pr}_1^*((s|_{U_i})_{i \in I})$.

00VM **Definition 7.7.1.** Let $\mathcal{C}$ be a site, and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say $\mathcal{F}$ is a *sheaf* if for every covering $\{U_i \to U\}_{i \in I} \in \text{Cov}(\mathcal{C})$ the diagram

00VN (7.7.1.1)
$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \underset{\text{pr}_1^*}{\overset{\text{pr}_0^*}{\rightrightarrows}} \prod\nolimits_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*.

Loosely speaking this means that given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$$

in $\mathcal{F}(U_i \times_U U_j)$ for all pairs $(i, j) \in I \times I$ then there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

04B3 **Remark 7.7.2.** If the covering $\{U_i \to U\}_{i \in I}$ is the empty family (this means that $I = \emptyset$), then the sheaf condition signifies that $\mathcal{F}(U) = \{*\}$ is a singleton set. This is because in (7.7.1.1) the second and third sets are empty products in the category of sets, which are final objects in the category of sets, hence singletons.

00VO **Example 7.7.3.** Let X be a topological space. Let X_{Zar} be the site constructed in Example 7.6.4. The notion of a sheaf on X_{Zar} coincides with the notion of a sheaf on X introduced in Sheaves, Definition 6.7.1.

00VP Example 7.7.4. Let X be a topological space. Let us consider the site X'_{Zar} which is the same as the site X_{Zar} of Example 7.6.4 except that we disallow the empty covering of the empty set. In other words, we do allow the covering $\{\emptyset \to \emptyset\}$ but we do not allow the covering whose index set is empty. It is easy to show that this still defines a site. However, we claim that the sheaves on X'_{Zar} are different from the sheaves on X_{Zar}. For example, as an extreme case consider the situation where $X = \{p\}$ is a singleton. Then the objects of X'_{Zar} are $\emptyset, X$ and every covering of $\emptyset$ can be refined by $\{\emptyset \to \emptyset\}$ and every covering of X by $\{X \to X\}$. Clearly, a sheaf on this is given by any choice of a set $\mathcal{F}(\emptyset)$ and any choice of a set $\mathcal{F}(X)$, together with any restriction map $\mathcal{F}(X) \to \mathcal{F}(\emptyset)$. Thus sheaves on X'_{Zar} are the same as usual sheaves on the two point space $\{\eta, p\}$ with open sets $\{\emptyset, \{\eta\}, \{p, \eta\}\}$. In general sheaves on X'_{Zar} are the same as sheaves on the space $X \amalg \{\eta\}$, with opens given by the empty set and any set of the form $U \cup \{\eta\}$ for $U \subset X$ open.

00VQ Definition 7.7.5. The category $Sh(\mathcal{C})$ of sheaves of sets is the full subcategory of the category $\text{PSh}(\mathcal{C})$ whose objects are the sheaves of sets.

Let $\mathcal{A}$ be a category. If products indexed by I, and $I \times I$ exist in $\mathcal{A}$ for any I that occurs as an index set for covering families then Definition 7.7.1 above makes sense, and defines a notion of a sheaf on $\mathcal{C}$ with values in $\mathcal{A}$. Note that the diagram in $\mathcal{A}$

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \overset{\text{pr}_0^*}{\underset{\text{pr}_1^*}{\rightrightarrows}} \prod\nolimits_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

is an equalizer diagram if and only if for every object X of $\mathcal{A}$ the diagram of sets

$$\text{Mor}_{\mathcal{A}}(X, \mathcal{F}(U)) \longrightarrow \prod \text{Mor}_{\mathcal{A}}(X, \mathcal{F}(U_i)) \overset{\text{pr}_0^*}{\underset{\text{pr}_1^*}{\rightrightarrows}} \prod \text{Mor}_{\mathcal{A}}(X, \mathcal{F}(U_{i_0} \times_U U_{i_1}))$$

is an equalizer diagram.

Suppose $\mathcal{A}$ is arbitrary. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{A}$. Choose any object $X \in \text{Ob}(\mathcal{A})$. Then we get a presheaf of sets $\mathcal{F}_X$ defined by the rule

$$\mathcal{F}_X(U) = \text{Mor}_{\mathcal{A}}(X, \mathcal{F}(U)).$$

From the above it follows that a good definition is obtained by requiring all the presheaves $\mathcal{F}_X$ to be sheaves of sets.

00VR Definition 7.7.6. Let $\mathcal{C}$ be a site, let $\mathcal{A}$ be a category and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$ with values in $\mathcal{A}$. We say that $\mathcal{F}$ is a sheaf if for all objects X of $\mathcal{A}$ the presheaf of sets $\mathcal{F}_X$ (defined above) is a sheaf.

7.8. Families of morphisms with fixed target

00VS This section is meant to introduce some notions regarding families of morphisms with the same target.

00VT Definition 7.8.1. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another.

(1) A morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$, or simply a morphism from $\mathcal{U}$ to $\mathcal{V}$ is given by a morphism $U \to V$, a map

of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $U_i \to V_{\alpha(i)}$ such that the diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

is commutative.

(2) In the special case that $U = V$ and $U \to V$ is the identity we call $\mathcal{U}$ a refinement of the family $\mathcal{V}$.

A trivial but important remark is that if $\mathcal{V} = \{V_j \to V\}_{j \in J}$ is the empty family of maps, i.e., if $J = \emptyset$, then no family $\mathcal{U} = \{U_i \to V\}_{i \in I}$ with $I \neq \emptyset$ can refine $\mathcal{V}$!

00VU Definition 7.8.2. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with fixed target.

1. We say $\mathcal{U}$ and $\mathcal{V}$ are combinatorially equivalent if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$.
2. We say $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ and for all $i \in I$ and $j \in J$ commutative diagrams

$$\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & V_{\alpha(i)} \arrow[dl] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} V_j \arrow[rr] \arrow[dr] & & U_{\beta(j)} \arrow[dl] \\ & U & \end{tikzcd}$$

with isomorphisms as horizontal arrows.

00VV Lemma 7.8.3. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target.

1. If $\mathcal{U}$ and $\mathcal{V}$ are combinatorially equivalent then they are tautologically equivalent.
2. If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent then $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$.
3. The relation "being combinatorially equivalent" is an equivalence relation on all families of morphisms with fixed target.
4. The relation "being tautologically equivalent" is an equivalence relation on all families of morphisms with fixed target.
5. The relation "$\mathcal{U}$ refines $\mathcal{V}$ and $\mathcal{V}$ refines $\mathcal{U}$" is an equivalence relation on all families of morphisms with fixed target.

Proof. Omitted. □

In the following lemma, given a category $\mathcal{C}$, a presheaf $\mathcal{F}$ on $\mathcal{C}$, a family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that all fibre products $U_i \times_U U_{i'}$ exist, we say that the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$ holds if the diagram (7.7.1.1) is an equalizer diagram.

00VW Lemma 7.8.4. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target. Assume that the fibre products $U_i \times_U U_{i'}$ and $V_j \times_U V_{j'}$ exist. If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent, then for any presheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$ is equivalent to the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{V}$.

Proof. First, note that if $\varphi : A \to B$ is an isomorphism in the category $\mathcal{C}$, then $\varphi^* : \mathcal{F}(B) \to \mathcal{F}(A)$ is an isomorphism. Let $\beta : J \to I$ be a map and let $\chi_j : V_j \to U_{\beta(j)}$ be isomorphisms over U which are assumed to exist by hypothesis. Let us show that the sheaf condition for $\mathcal{V}$ implies the sheaf condition for $\mathcal{U}$. Suppose given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$$

in $\mathcal{F}(U_i \times_U U_{i'})$ for all pairs $(i, i') \in I \times I$. Then we can define $s_j = \chi_j^* s_{\beta(j)}$. For any pair $(j, j') \in J \times J$ the morphism $\chi_j \times_{\mathrm{id}_U} \chi_{j'} : V_j \times_U V_{j'} \to U_{\beta(j)} \times_U U_{\beta(j')}$ is an isomorphism as well. Hence by transport of structure we see that

$$s_j|_{V_j \times_U V_{j'}} = s_{j'}|_{V_j \times_U V_{j'}}$$

as well. The sheaf condition w.r.t. $\mathcal{V}$ implies there exists a unique s such that $s|_{V_j} = s_j$ for all $j \in J$. By the first remark of the proof this implies that $s|_{U_i} = s_i$ for all $i \in \mathrm{Im}(\beta)$ as well. Suppose that $i \in I$, $i \notin \mathrm{Im}(\beta)$. For such an i we have isomorphisms $U_i \to V_{\alpha(i)} \to U_{\beta(\alpha(i))}$ over U. This gives a morphism $U_i \to U_i \times_U U_{\beta(\alpha(i))}$ which is a section of the projection. Because s_i and $s_{\beta(\alpha(i))}$ restrict to the same element on the fibre product we conclude that $s_{\beta(\alpha(i))}$ pulls back to s_i via $U_i \to U_{\beta(\alpha(i))}$. Thus we see that also $s_i = s|_{U_i}$ as desired. $\square$

0G1K Lemma 7.8.5. Let $\mathcal{C}$ be a category. Let $\mathcal{V} = \{V_j \to U\}_{j \in J} \to \mathcal{U} = \{U_i \to U\}_{i \in I}$ be a morphism of families of maps with fixed target of $\mathcal{C}$ given by $\mathrm{id} : U \to U$, $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. If $\mathcal{F}(U) \to \prod_{j \in J} \mathcal{F}(V_j)$ is injective then $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ is injective.

Proof. Omitted. $\square$

0G1L Lemma 7.8.6. Let $\mathcal{C}$ be a category. Let $\mathcal{V} = \{V_j \to U\}_{j \in J} \to \mathcal{U} = \{U_i \to U\}_{i \in I}$ be a morphism of families of maps with fixed target of $\mathcal{C}$ given by $\mathrm{id} : U \to U$, $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. If

(1) the fibre products $U_i \times_U U_{i'}$, $U_i \times_U V_j$, $V_j \times_U V_{j'}$ exist,
(2) $\mathcal{F}$ satisfies the sheaf condition with respect to $\mathcal{V}$, and
(3) for every $i \in I$ the map $\mathcal{F}(U_i) \to \prod_{j \in J} \mathcal{F}(V_j \times_U U_i)$ is injective.

Then $\mathcal{F}$ satisfies the sheaf condition with respect to $\mathcal{U}$.

Proof. By Lemma 7.8.5 the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective. Suppose given $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Set $s_j = f_j^*(s_{\alpha(j)}) \in \mathcal{F}(V_j)$. Since the morphisms f_j are morphisms over U we obtain induced morphisms $f_{jj'} : V_j \times_U V_{j'} \to U_{\alpha(i)} \times_U U_{\alpha(i')}$ compatible with the $f_j, f_{j'}$ via the projection maps. It follows that

$$s_j|_{V_j \times_U V_{j'}} = f_{jj'}^*(s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = f_{jj'}^*(s_{\alpha(j')}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = s_{j'}|_{V_j \times_U V_{j'}}$$

for all $j, j' \in J$. Hence, by the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{V}$, we get a section $s \in \mathcal{F}(U)$ which restricts to s_j on each V_j. We are done if we show s restricts to s_i on U_i for any $i \in I$. Since $\mathcal{F}$ satisfies (3) it suffices to show that s and s_i restrict to the same element over $U_i \times_U V_j$ for all $j \in J$. To see this we use

$$s|_{U_i \times_U V_j} = s_j|_{U_i \times_U V_j} = (\mathrm{id} \times f_j)^* s_{\alpha(j)}|_{U_i \times_U U_{\alpha(j)}} = (\mathrm{id} \times f_j)^* s_i|_{U_i \times_U U_{\alpha(j)}} = s_i|_{U_i \times_U V_j}$$

as desired. $\square$

00VX Lemma 7.8.7. Let $\mathcal{C}$ be a category. Let Cov_i, $i = 1, 2$ be two sets of families of morphisms with fixed target which each define the structure of a site on $\mathcal{C}$.

(1) If every $\mathcal{U} \in \text{Cov}_1$ is tautologically equivalent to some $\mathcal{V} \in \text{Cov}_2$, then $Sh(\mathcal{C}, \text{Cov}_2) \subset Sh(\mathcal{C}, \text{Cov}_1)$. If also, every $\mathcal{U} \in \text{Cov}_2$ is tautologically equivalent to some $\mathcal{V} \in \text{Cov}_1$ then the category of sheaves are equal.
(2) Suppose that for each $\mathcal{U} \in \text{Cov}_1$ there exists a $\mathcal{V} \in \text{Cov}_2$ such that $\mathcal{V}$ refines $\mathcal{U}$. In this case $Sh(\mathcal{C}, \text{Cov}_2) \subset Sh(\mathcal{C}, \text{Cov}_1)$. If also for every $\mathcal{U} \in \text{Cov}_2$ there exists a $\mathcal{V} \in \text{Cov}_1$ such that $\mathcal{V}$ refines $\mathcal{U}$, then the categories of sheaves are equal.

Proof. Part (1) follows directly from Lemma 7.8.4 and the definitions.

Proof of (2). Let $\mathcal{F}$ be a sheaf of sets for the site $(\mathcal{C}, \text{Cov}_2)$. Let $\mathcal{U} \in \text{Cov}_1$, say $\mathcal{U} = \{U_i \to U\}_{i \in I}$. By assumption we may choose a refinement $\mathcal{V} \in \text{Cov}_2$ of $\mathcal{U}$, say $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and refinement given by $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Observe that $\mathcal{F}$ satisfies the sheaf condition for $\mathcal{V}$ and for the coverings $\{V_j \times_U U_i \to U_i\}_{j \in J}$ as these are in Cov_2. Hence $\mathcal{F}$ satisfies the sheaf condition for $\mathcal{U}$ by Lemma 7.8.6. □

00VY Lemma 7.8.8. Let $\mathcal{C}$ be a category. Let $\text{Cov}(\mathcal{C})$ be a proper class of coverings satisfying conditions (1), (2) and (3) of Definition 7.6.2. Let $\text{Cov}_1, \text{Cov}_2 \subset \text{Cov}(\mathcal{C})$ be two subsets of $\text{Cov}(\mathcal{C})$ which endow $\mathcal{C}$ with the structure of a site. If every covering $\mathcal{U} \in \text{Cov}(\mathcal{C})$ is combinatorially equivalent to a covering in Cov_1 and combinatorially equivalent to a covering in Cov_2, then $Sh(\mathcal{C}, \text{Cov}_1) = Sh(\mathcal{C}, \text{Cov}_2)$.

Proof. This is clear from Lemmas 7.8.7 and 7.8.3 above as the hypothesis implies that every covering $\mathcal{U} \in \text{Cov}_1 \subset \text{Cov}(\mathcal{C})$ is combinatorially equivalent to an element of Cov_2, and similarly with the roles of Cov_1 and Cov_2 reversed. □

7.9. The example of G-sets

00VZ As an example, consider the site $\mathcal{T}_G$ of Example 7.6.5. We will describe the category of sheaves on $\mathcal{T}_G$. The answer will turn out to be independent of the choices made in defining $\mathcal{T}_G$. In fact, during the proof we will need only the following properties of the site $\mathcal{T}_G$:

(a) $\mathcal{T}_G$ is a full subcategory of G-Sets,
(b) $\mathcal{T}_G$ contains the G-set ${}_GG$,
(c) $\mathcal{T}_G$ has fibre products and they are the same as in G-Sets,
(d) given $U \in \text{Ob}(\mathcal{T}_G)$ and a G-invariant subset $O \subset U$, there exists an object of $\mathcal{T}_G$ isomorphic to O, and
(e) any surjective family of maps $\{U_i \to U\}_{i \in I}$, with $U, U_i \in \text{Ob}(\mathcal{T}_G)$ is combinatorially equivalent to a covering of $\mathcal{T}_G$.

These properties hold by Sets, Lemmas 3.10.2 and 3.11.1.

Remark that the map

$$\text{Hom}_G({}_GG, {}_GG) \longrightarrow G^{opp}, \varphi \longmapsto \varphi(1)$$

is an isomorphism of groups. The inverse map sends $g \in G$ to the map $R_g : s \mapsto sg$ (i.e. right multiplication). Note that $R_{g_1 g_2} = R_{g_2} \circ R_{g_1}$ so the opposite is necessary.

This implies that for every presheaf $\mathcal{F}$ on $\mathcal{T}_G$ the value $\mathcal{F}({}_GG)$ inherits the structure of a G-set as follows: $g \cdot s$ for $g \in G$ and $s \in \mathcal{F}({}_GG)$ defined by $\mathcal{F}(R_g)(s)$. This is a left action because

$$(g_1 g_2) \cdot s = \mathcal{F}(R_{g_1 g_2})(s) = \mathcal{F}(R_{g_2} \circ R_{g_1})(s) = \mathcal{F}(R_{g_1})(\mathcal{F}(R_{g_2})(s)) = g_1 \cdot (g_2 \cdot s).$$

Here we've used that $\mathcal{F}$ is contravariant. Note that if $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{T}_G$ then we get a map $\mathcal{F}({}_GG) \to \mathcal{G}({}_GG)$ which is compatible with the G-actions we have just defined. All in all we have constructed a functor

$$\mathrm{PSh}(\mathcal{T}_G) \longrightarrow G\text{-Sets}, \quad \mathcal{F} \longmapsto \mathcal{F}({}_GG).$$

We leave it to the reader to verify that this construction has the pleasing property that the representable presheaf h_U is mapped to something canonically isomorphic to U. In a formula $h_U({}_GG) = \mathrm{Hom}_G({}_GG, U) \cong U$.

Suppose that S is a G-set. We define a presheaf $\mathcal{F}_S$ by the formula[2]

$$\mathcal{F}_S(U) = \mathrm{Mor}_{G\text{-Sets}}(U, S).$$

This is clearly a presheaf. On the other hand, suppose that $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{T}_G$. This implies that $\coprod_i U_i \to U$ is surjective. Thus it is clear that the map

$$\mathcal{F}_S(U) = \mathrm{Mor}_{G\text{-Sets}}(U, S) \longrightarrow \prod \mathcal{F}_S(U_i) = \prod \mathrm{Mor}_{G\text{-Sets}}(U_i, S)$$

is injective. And, given a family of G-equivariant maps $s_i : U_i \to S$, such that all the diagrams

$$\begin{tikzcd} U_i \times_U U_j \arrow[r] \arrow[d] & U_j \arrow[d, "s_j"] \\ U_i \arrow[r, "s_i"] & S \end{tikzcd}$$

commute, there is a unique G-equivariant map $s : U \to S$ such that s_i is the composition $U_i \to U \to S$. Namely, we just define $s(u) = s_i(u_i)$ where $i \in I$ is any index such that there exists some $u_i \in U_i$ mapping to u under the map $U_i \to U$. The commutativity of the diagrams above implies exactly that this construction is well defined. All in all we have constructed a functor

$$G\text{-Sets} \longrightarrow \mathit{Sh}(\mathcal{T}_G), \quad S \longmapsto \mathcal{F}_S.$$

We now have the following diagram of categories and functors

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{T}_G) \arrow[rr, "\mathcal{F}\mapsto\mathcal{F}({}_GG)"] & & G\text{-Sets} \arrow[dl, "S\mapsto\mathcal{F}_S"] \\ & \mathit{Sh}(\mathcal{T}_G) \arrow[ul] & \end{tikzcd}$$

It is immediate from the definitions that $\mathcal{F}_S({}_GG) = \mathrm{Mor}_G({}_GG, S) = S$, the last equality by evaluation at 1. This almost proves the following.

00W0 Proposition 7.9.1. The functors $\mathcal{F} \mapsto \mathcal{F}({}_GG)$ and $S \mapsto \mathcal{F}_S$ define quasi-inverse equivalences between $\mathit{Sh}(\mathcal{T}_G)$ and G-Sets.

Proof. We have already seen that composing the functors one way around is isomorphic to the identity functor. In the other direction, for any sheaf $\mathcal{H}$ there is a natural map of sheaves

$$can : \mathcal{H} \longrightarrow \mathcal{F}_{\mathcal{H}({}_GG)}.$$

[2] It may appear this is the representable presheaf defined by S. This may not be the case because S may not be an object of $\mathcal{T}_G$ which was chosen to be a sufficiently large set of G-sets.

Namely, for any object U of $\mathcal{T}_G$ we let can_U be the map

$$\begin{array}{ccc} \mathcal{H}(U) & \longrightarrow & \mathcal{F}_{\mathcal{H}({}_GG)}(U) = \mathrm{Mor}_G(U, \mathcal{H}({}_GG)) \\ s & \longmapsto & (u \mapsto \alpha_u^* s). \end{array}$$

Here $\alpha_u : {}_GG \to U$ is the map $\alpha_u(g) = gu$ and $\alpha_u^* : \mathcal{H}(U) \to \mathcal{H}({}_GG)$ is the pullback map. A trivial but confusing verification shows that this is indeed a map of presheaves. We have to show that can is an isomorphism. We do this by showing can_U is an isomorphism for all $U \in \mathrm{Ob}(\mathcal{T}_G)$. We leave the (important but easy) case that $U = {}_GG$ to the reader. A general object U of $\mathcal{T}_G$ is a disjoint union of G-orbits: $U = \coprod_{i \in I} O_i$. The family of maps $\{O_i \to U\}_{i \in I}$ is tautologically equivalent to a covering in $\mathcal{T}_G$ (by the properties of $\mathcal{T}_G$ listed at the beginning of this section). Hence by Lemma 7.8.4 the sheaf $\mathcal{H}$ satisfies the sheaf property with respect to $\{O_i \to U\}_{i \in I}$. The sheaf property for this covering implies $\mathcal{H}(U) = \prod_i \mathcal{H}(O_i)$. Hence it suffices to show that can_U is an isomorphism when U consists of a single G-orbit. Let $u \in U$ and let $H \subset G$ be its stabilizer. Clearly, $\mathrm{Mor}_G(U, \mathcal{H}({}_GG)) = \mathcal{H}({}_GG)^H$ equals the subset of H-invariant elements. On the other hand consider the covering $\{{}_GG \to U\}$ given by $g \mapsto gu$ (again it is just combinatorially equivalent to some covering of $\mathcal{T}_G$, and again this doesn't matter). Note that the fibre product $({}_GG) \times_U ({}_GG)$ is equal to $\{(g, gh), g \in G, h \in H\} \cong \coprod_{h \in H} {}_GG$. Hence the sheaf property for this covering reads as

$$\begin{tikzcd} \mathcal{H}(U) \arrow[r] & \mathcal{H}({}_GG) \arrow[r, shift left, "\mathrm{pr}_0^*"] \arrow[r, shift right, "\mathrm{pr}_1^*"'] & \prod_{h \in H} \mathcal{H}({}_GG). \end{tikzcd}$$

The two maps pr_i^* into the factor $\mathcal{H}({}_GG)$ differ by multiplication by h. Now the result follows from this and the fact that can is an isomorphism for $U = {}_GG$. $\square$

7.10. Sheafification

00W1 In order to define the sheafification we study the zeroth Čech cohomology group of a covering and its functoriality properties.

Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$, and let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let us use the notation $\mathcal{F}(\mathcal{U})$ to indicate the equalizer

$$H^0(\mathcal{U}, \mathcal{F}) = \{(s_i)_{i \in I} \in \prod\nolimits_i \mathcal{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \ \forall i, j \in I\}.$$

As we will see later, this is the zeroth Čech cohomology of $\mathcal{F}$ over U with respect to the covering $\mathcal{U}$. A small remark is that we can define $H^0(\mathcal{U}, \mathcal{F})$ as soon as all the morphisms $U_i \to U$ are representable, i.e., $\mathcal{U}$ need not be a covering of the site. There is a canonical map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$. It is clear that a morphism of coverings $\mathcal{U} \to \mathcal{V}$ induces commutative diagrams

$$\begin{tikzcd} & U_i \arrow[rr] & & V_{\alpha(i)} \\ U_i \times_U U_j \arrow[ur] \arrow[r] \arrow[dr] & & V_{\alpha(i)} \times_V V_{\alpha(j)} \arrow[ur] \arrow[dr] & \\ & U_j \arrow[rr] & & V_{\alpha(j)} \end{tikzcd}$$

This in turn produces a map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$, compatible with the map $\mathcal{F}(V) \to \mathcal{F}(U)$.

By construction, a presheaf $\mathcal{F}$ is a sheaf if and only if for every covering $\mathcal{U}$ of $\mathcal{C}$ the natural map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ is bijective. We will use this notion to prove the following simple lemma about limits of sheaves.

00W2 Lemma 7.10.1. Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $\lim_\mathcal{I} \mathcal{F}$ exists and is equal to the limit in the category of presheaves.

Proof. Let $\lim_i \mathcal{F}_i$ be the limit as a presheaf. We will show that this is a sheaf and then it will trivially follow that it is a limit in the category of sheaves. To prove the sheaf property, let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be a covering. Let $(s_j)_{j\in J}$ be an element of $H^0(\mathcal{V}, \lim_i \mathcal{F}_i)$. Using the projection maps we get elements $(s_{j,i})_{j\in J}$ in $H^0(\mathcal{V}, \mathcal{F}_i)$. By the sheaf property for $\mathcal{F}_i$ we see that there is a unique $s_i \in \mathcal{F}_i(V)$ such that $s_{j,i} = s_i|_{V_j}$. Let $\phi : i \to i'$ be a morphism of the index category. We would like to show that $\mathcal{F}(\phi) : \mathcal{F}_i \to \mathcal{F}_{i'}$ maps s_i to $s_{i'}$. We know this is true for the sections $s_{i,j}$ and $s_{i',j}$ for all j and hence by the sheaf property for $\mathcal{F}_{i'}$ this is true. At this point we have an element $s = (s_i)_{i\in \mathrm{Ob}(\mathcal{I})}$ of $(\lim_i \mathcal{F}_i)(V)$. We leave it to the reader to see this element has the required property that $s_j = s|_{V_j}$. □

00W3 Example 7.10.2. A particular example is the limit over the empty diagram. This gives the final object in the category of (pre)sheaves. It is the presheaf that associates to each object U of $\mathcal{C}$ a singleton set, with unique restriction mappings and moreover this presheaf is a sheaf. We often denote this sheaf by $*$.

Let $\mathcal{J}_U$ be the category of all coverings of U. In other words, the objects of $\mathcal{J}_U$ are the coverings of U in $\mathcal{C}$, and the morphisms are the refinements. By our conventions on sites this is indeed a category, i.e., the collection of objects and morphisms forms a set. Note that $\mathrm{Ob}(\mathcal{J}_U)$ is not empty since $\{\mathrm{id}_U\}$ is an object of it. According to the remarks above the construction $\mathcal{U} \mapsto H^0(\mathcal{U}, \mathcal{F})$ is a contravariant functor on $\mathcal{J}_U$. We define

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{J}_U^{opp}} H^0(\mathcal{U}, \mathcal{F})$$

See Categories, Section 4.14 for a discussion of limits and colimits. We point out that later we will see that $\mathcal{F}^+(U)$ is the zeroth Čech cohomology of $\mathcal{F}$ over U.

Before we say more about the structure of the colimit, we turn the collection of sets $\mathcal{F}^+(U)$, $U \in \mathrm{Ob}(\mathcal{C})$ into a presheaf. Namely, let $V \to U$ be a morphism of $\mathcal{C}$. By the axioms of a site there is a functor[3]

$$\mathcal{J}_U \longrightarrow \mathcal{J}_V, \quad \{U_i \to U\} \longmapsto \{U_i \times_U V \to V\}.$$

Note that the projection maps furnish a functorial morphism of coverings $\{U_i \times_U V \to V\} \to \{U_i \to U\}$ and hence, by the construction above, a functorial map of sets $H^0(\{U_i \to U\}, \mathcal{F}) \to H^0(\{U_i \times_U V \to V\}, \mathcal{F})$. In other words, there is a transformation of functors from $H^0(-, \mathcal{F}) : \mathcal{J}_U^{opp} \to \textit{Sets}$ to the composition $\mathcal{J}_U^{opp} \to \mathcal{J}_V^{opp} \xrightarrow{H^0(-,\mathcal{F})} \textit{Sets}$. Hence by generalities of colimits we obtain a canonical map $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$. In terms of the description of the set $\mathcal{F}^+(U)$ above, it just takes the element associated with $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$ to the element associated with $(s_i|_{V\times_U U_i}) \in H^0(\{U_i \times_U V \to V\}, \mathcal{F})$.

[3]This construction actually involves a choice of the fibre products $U_i \times_U V$ and hence the axiom of choice. The resulting map does not depend on the choices made, see below.

00W4 **Lemma 7.10.3.** The constructions above define a presheaf $\mathcal{F}^+$ together with a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.

Proof. All we have to do is to show that given morphisms $W \to V \to U$ the composition $\mathcal{F}^+(U) \to \mathcal{F}^+(V) \to \mathcal{F}^+(W)$ equals the map $\mathcal{F}^+(U) \to \mathcal{F}^+(W)$. This can be shown directly by verifying that, given a covering $\{U_i \to U\}$ and $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$, we have canonically $W \times_U U_i \cong W \times_V (V \times_U U_i)$, and $s_i|_{W \times_U U_i}$ corresponds to $(s_i|_{V \times_U U_i})|_{W \times_V (V \times_U U_i)}$ via this isomorphism. $\square$

More indirectly, the result of Lemma 7.10.6 shows that we may pullback an element s as above via any morphism from any covering of W to $\{U_i \to U\}$ and we will always end up with the same element in $\mathcal{F}^+(W)$.

00W5 **Lemma 7.10.4.** The association $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^+)$ is a functor.

Proof. Instead of proving this we state exactly what needs to be proven. Let $\mathcal{F} \to \mathcal{G}$ be a map of presheaves. Prove the commutativity of:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^+ \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{G}^+ \end{tikzcd}$$

$\square$

The next two lemmas imply that the colimits above are colimits over a directed set.

00W6 **Lemma 7.10.5.** Given a pair of coverings $\{U_i \to U\}$ and $\{V_j \to U\}$ of a given object U of the site $\mathcal{C}$, there exists a covering which is a common refinement.

Proof. Since $\mathcal{C}$ is a site we have that for every i the family $\{V_j \times_U U_i \to U_i\}_j$ is a covering. And, then another axiom implies that $\{V_j \times_U U_i \to U\}_{i,j}$ is a covering of U. Clearly this covering refines both given coverings. $\square$

00W7 **Lemma 7.10.6.** Any two morphisms $f, g : \mathcal{U} \to \mathcal{V}$ of coverings inducing the same morphism $U \to V$ induce the same map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$.

Proof. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$. The morphism f consists of a map $U \to V$, a map $\alpha : I \to J$ and maps $f_i : U_i \to V_{\alpha(i)}$. Likewise, g determines a map $\beta : I \to J$ and maps $g_i : U_i \to V_{\beta(i)}$. As f and g induce the same map $U \to V$, the diagram

$$\begin{tikzcd} & V_{\alpha(i)} \arrow[dr] & \\ U_i \arrow[ur, "f_i"] \arrow[dr, "g_i"'] & & V \\ & V_{\beta(i)} \arrow[ur] & \end{tikzcd}$$

is commutative for every $i \in I$. Hence f and g factor through the fibre product

$$\begin{tikzcd} & V_{\alpha(i)} \\ U_i \arrow[ur, "f_i"] \arrow[r, "\varphi"] \arrow[dr, "g_i"'] & V_{\alpha(i)} \times_V V_{\beta(i)} \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\ & V_{\beta(i)}. \end{tikzcd}$$

Now let $s = (s_j)_j \in H^0(\mathcal{V}, \mathcal{F})$. Then for all $i \in I$:

$$(f^*s)_i = f_i^*(s_{\alpha(i)}) = \varphi^*\mathrm{pr}_1^*(s_{\alpha(i)}) = \varphi^*\mathrm{pr}_2^*(s_{\beta(i)}) = g_i^*(s_{\beta(i)}) = (g^*s)_i,$$

where the middle equality is given by the definition of $H^0(\mathcal{V}, \mathcal{F})$. This shows that the maps $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ induced by f and g are equal. □

00W8 Remark 7.10.7. In particular this lemma shows that if $\mathcal{U}$ is a refinement of $\mathcal{V}$, and if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a canonical identification $H^0(\mathcal{U}, \mathcal{F}) = H^0(\mathcal{V}, \mathcal{F})$.

From these two lemmas, and the fact that $\mathcal{J}_U$ is nonempty, it follows that the diagram $H^0(-, \mathcal{F}) : \mathcal{J}_U^{opp} \to \textit{Sets}$ is filtered, see Categories, Definition 4.19.1. Hence, by Categories, Section 4.19 the colimit $\mathcal{F}^+(U)$ may be described in the following straightforward manner. Namely, every element in the set $\mathcal{F}^+(U)$ arises from an element $s \in H^0(\mathcal{U}, \mathcal{F})$ for some covering $\mathcal{U}$ of U. Given a second element $s' \in H^0(\mathcal{U}', \mathcal{F})$ then s and s' determine the same element of the colimit if and only if there exists a covering $\mathcal{V}$ of U and refinements $f : \mathcal{V} \to \mathcal{U}$ and $f' : \mathcal{V} \to \mathcal{U}'$ such that $f^*s = (f')^*s'$ in $H^0(\mathcal{V}, \mathcal{F})$. Since the trivial covering $\{\mathrm{id}_U\}$ is an object of $\mathcal{J}_U$ we get a canonical map $\mathcal{F}(U) \to \mathcal{F}^+(U)$.

00W9 Lemma 7.10.8. The map $\theta : \mathcal{F} \to \mathcal{F}^+$ has the following property: For every object U of $\mathcal{C}$ and every section $s \in \mathcal{F}^+(U)$ there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ is in the image of $\theta : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.

Proof. Namely, let $\{U_i \to U\}$ be a covering such that s arises from the element $(s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$. According to Lemma 7.10.6 we may consider the covering $\{U_i \to U_i\}$ and the (obvious) morphism of coverings $\{U_i \to U_i\} \to \{U_i \to U\}$ to compute the pullback of s to an element of $\mathcal{F}^+(U_i)$. And indeed, using this covering we get exactly $\theta(s_i)$ for the restriction of s to U_i. □

00WA Definition 7.10.9. We say that a presheaf of sets $\mathcal{F}$ on a site $\mathcal{C}$ is separated if, for all coverings of $\{U_i \to U\}$, the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective.

00WB Theorem 7.10.10. With $\mathcal{F}$ as above

00WC (1) The presheaf $\mathcal{F}^+$ is separated.
00WD (2) If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and the map of presheaves $\mathcal{F} \to \mathcal{F}^+$ is injective.
00WE (3) If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.
00WF (4) The presheaf $\mathcal{F}^{++}$ is always a sheaf.

Proof. Proof of (1). Suppose that $s, s' \in \mathcal{F}^+(U)$ and suppose that there exists some covering $\{U_i \to U\}$ such that $s|_{U_i} = s'|_{U_i}$ for all i. We now have three coverings of U: the covering $\{U_i \to U\}$ above, a covering $\mathcal{U}$ for s as in Lemma 7.10.8, and a similar covering $\mathcal{U}'$ for s'. By Lemma 7.10.5, we can find a common refinement, say $\{W_j \to U\}$. This means we have $s_j, s'_j \in \mathcal{F}(W_j)$ such that $s|_{W_j} = \theta(s_j)$, similarly for $s'|_{W_j}$, and such that $\theta(s_j) = \theta(s'_j)$. This last equality means that there exists some covering $\{W_{jk} \to W_j\}$ such that $s_j|_{W_{jk}} = s'_j|_{W_{jk}}$. Then since $\{W_{jk} \to U\}$ is a covering we see that s, s' map to the same element of $H^0(\{W_{jk} \to U\}, \mathcal{F})$ as desired.

Proof of (2). It is clear that $\mathcal{F} \to \mathcal{F}^+$ is injective because all the maps $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ are injective. It is also clear that, if $\mathcal{U} \to \mathcal{U}'$ is a refinement, then $H^0(\mathcal{U}', \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ is injective. Now, suppose that $\{U_i \to U\}$ is a covering, and let (s_i) be a family of elements of $\mathcal{F}^+(U_i)$ satisfying the sheaf condition $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Choose coverings (as in Lemma 7.10.8) $\{U_{ij} \to U_i\}$ such that $s_i|_{U_{ij}}$ is the image of the (unique) element $s_{ij} \in \mathcal{F}(U_{ij})$. The sheaf condition implies that s_{ij} and $s_{i'j'}$ agree over $U_{ij} \times_U U_{i'j'}$ because it maps to $U_i \times_U U_{i'}$ and we have the equality there. Hence $(s_{ij}) \in H^0(\{U_{ij} \to U\}, \mathcal{F})$ gives rise to an element $s \in \mathcal{F}^+(U)$. We leave it to the reader to verify that $s|_{U_i} = s_i$.

Proof of (3). This is immediate from the definitions because the sheaf property says exactly that every map $\mathcal{F} \to H^0(\mathcal{U}, \mathcal{F})$ is bijective (for every covering $\mathcal{U}$ of U).

Statement (4) is now obvious. □

00WG Definition 7.10.11. Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := \mathcal{F}^{++}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the sheaf associated to $\mathcal{F}$.

00WH Proposition 7.10.12. The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.

Proof. By Lemma 7.10.4 we get a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^+ \arrow[r] \arrow[d] & \mathcal{F}^{++} \arrow[d] \\
\mathcal{G} \arrow[r] & \mathcal{G}^+ \arrow[r] & \mathcal{G}^{++}
\end{tikzcd}$$

and by Theorem 7.10.10 the lower horizontal maps are isomorphisms. The uniqueness follows from Lemma 7.10.8 which says that every section of $\mathcal{F}^\#$ locally comes from sections of $\mathcal{F}$. □

It is clear from this result that the functor $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^\#)$ is unique up to unique isomorphism of functors. Actually, let us temporarily denote $i : Sh(\mathcal{C}) \to \text{PSh}(\mathcal{C})$ the functor of inclusion. The result above actually says that

$$\text{Mor}_{\text{PSh}(\mathcal{C})}(\mathcal{F}, i(\mathcal{G})) = \text{Mor}_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G}).$$

In other words, the functor of sheafification is the left adjoint to the inclusion functor i. We finish this section with a couple of lemmas.

00WI Lemma 7.10.13. Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $\text{colim}_\mathcal{I} \mathcal{F}$ exists and is the sheafification of the colimit in the category of presheaves.

Proof. Since the sheafification functor is a left adjoint it commutes with all colimits, see Categories, Lemma 4.24.5. Hence, since $\text{PSh}(\mathcal{C})$ has colimits, we deduce that $Sh(\mathcal{C})$ has colimits (which are the sheafifications of the colimits in presheaves). □

00WJ **Lemma 7.10.14.** The functor $\text{PSh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}^\#$ is exact.

Proof. Since it is a left adjoint it is right exact, see Categories, Lemma 4.24.6. On the other hand, by Lemmas 7.10.5 and Lemma 7.10.6 the colimits in the construction of $\mathcal{F}^+$ are really over the directed set $\text{Ob}(\mathcal{J}_U)$ where $\mathcal{U} \geq \mathcal{U}'$ if and only if $\mathcal{U}$ is a refinement of $\mathcal{U}'$. Hence by Categories, Lemma 4.19.2 we see that $\mathcal{F} \to \mathcal{F}^+$ commutes with finite limits (as a functor from presheaves to presheaves). Then we conclude using Lemma 7.10.1. $\square$

00WK **Lemma 7.10.15.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. Denote $\theta^2 : \mathcal{F} \to \mathcal{F}^\#$ the canonical map of $\mathcal{F}$ into its sheafification. Let U be an object of $\mathcal{C}$. Let $s \in \mathcal{F}^\#(U)$. There exists a covering $\{U_i \to U\}$ and sections $s_i \in \mathcal{F}(U_i)$ such that

(1) $s|_{U_i} = \theta^2(s_i)$, and
(2) for every i, j there exists a covering $\{U_{ijk} \to U_i \times_U U_j\}$ of $\mathcal{C}$ such that the pullback of s_i and s_j to each U_{ijk} agree.

Conversely, given any covering $\{U_i \to U\}$, elements $s_i \in \mathcal{F}(U_i)$ such that (2) holds, then there exists a unique section $s \in \mathcal{F}^\#(U)$ such that (1) holds.

Proof. Omitted. $\square$

7.11. Injective and surjective maps of sheaves

00WL

00WM **Definition 7.11.1.** Let $\mathcal{C}$ be a site, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ and every section $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ such that for all i the restriction $s|_{U_i}$ is in the image of $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

00WN **Lemma 7.11.2.** The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of the category $\mathit{Sh}(\mathcal{C})$. A map of sheaves is an isomorphism if and only if it is both injective and surjective.

Proof. Omitted. $\square$

086K **Lemma 7.11.3.** Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of sheaves of sets. Then the diagram

$$\mathcal{F} \times_\mathcal{G} \mathcal{F} \rightrightarrows \mathcal{F} \longrightarrow \mathcal{G}$$

represents $\mathcal{G}$ as a coequalizer.

Proof. Let $\mathcal{H}$ be a sheaf of sets and let $\varphi : \mathcal{F} \to \mathcal{H}$ be a map of sheaves equalizing the two maps $\mathcal{F} \times_\mathcal{G} \mathcal{F} \to \mathcal{F}$. Let $\mathcal{G}' \subset \mathcal{G}$ be the presheaf image of the map $\mathcal{F} \to \mathcal{G}$. As the product $\mathcal{F} \times_\mathcal{G} \mathcal{F}$ may be computed in the category of presheaves we see that it is equal to the presheaf product $\mathcal{F} \times_{\mathcal{G}'} \mathcal{F}$. Hence φ induces a unique map of presheaves $\psi' : \mathcal{G}' \to \mathcal{H}$. Since $\mathcal{G}$ is the sheafification of $\mathcal{G}'$ by Lemma 7.11.2 we conclude that ψ' extends uniquely to a map of sheaves $\psi : \mathcal{G} \to \mathcal{H}$. We omit the verification that φ is equal to the composition of ψ and the given map. $\square$

7.12. Representable sheaves

00WO Let $\mathcal{C}$ be a category. The canonical topology is the finest topology such that all representable presheaves are sheaves (it is formally defined in Definition 7.47.12 but we will not need this). This topology is not always the topology associated to the structure of a site on $\mathcal{C}$. We will give a collection of coverings that generates this topology in case $\mathcal{C}$ has fibered products. First we give the following general definition.

00WP Definition 7.12.1. Let $\mathcal{C}$ be a category. We say that a family $\{U_i \to U\}_{i \in I}$ is an effective epimorphism if all the morphisms $U_i \to U$ are representable (see Categories, Definition 4.6.4), and for any $X \in \mathrm{Ob}(\mathcal{C})$ the sequence

$$\mathrm{Mor}_\mathcal{C}(U, X) \longrightarrow \prod\nolimits_{i \in I} \mathrm{Mor}_\mathcal{C}(U_i, X) \rightrightarrows \prod\nolimits_{(i,j) \in I^2} \mathrm{Mor}_\mathcal{C}(U_i \times_U U_j, X)$$

is an equalizer diagram. We say that a family $\{U_i \to U\}$ is a universal effective epimorphism if for any morphism $V \to U$ the base change $\{U_i \times_U V \to V\}$ is an effective epimorphism.

The class of families which are universal effective epimorphisms satisfies the axioms of Definition 7.6.2. If $\mathcal{C}$ has fibre products, then the associated topology is the canonical topology. (In this case, to get a site argue as in Sets, Lemma 3.11.1.)

Conversely, suppose that $\mathcal{C}$ is a site such that all representable presheaves are sheaves. Then clearly, all coverings are universal effective epimorphisms. Thus the following definition is the "correct" one in the setting of sites.

00WQ Definition 7.12.2. We say that the topology on a site $\mathcal{C}$ is weaker than the canonical topology, or that the topology is subcanonical if all the coverings of $\mathcal{C}$ are universal effective epimorphisms.

A representable sheaf is a representable presheaf which is also a sheaf. Since it is perhaps better to avoid this terminology when the topology is not subcanonical, we only define it formally in that case.

00WR Definition 7.12.3. Let $\mathcal{C}$ be a site whose topology is subcanonical. The Yoneda embedding h (see Categories, Section 4.3) presents $\mathcal{C}$ as a full subcategory of the category of sheaves of $\mathcal{C}$. In this case we call sheaves of the form h_U with $U \in \mathrm{Ob}(\mathcal{C})$ representable sheaves on $\mathcal{C}$. Notation: Sometimes, the representable sheaf h_U associated to U is denoted $\underline{U}$.

Note that we have in the situation of the definition

$$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

for every sheaf $\mathcal{F}$, since it holds for presheaves, see (7.2.1.1). In general the presheaves h_U are not sheaves and to get a sheaf you have to sheafify them. In this case we still have

090I (7.12.3.1) $$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

for every sheaf $\mathcal{F}$. Namely, the first equality holds by the adjointness property of # and the second is (7.2.1.1).

00WT Lemma 7.12.4. Let $\mathcal{C}$ be a site. If $\{U_i \to U\}_{i \in I}$ is a covering of the site $\mathcal{C}$, then the morphism of presheaves of sets

$$\coprod\nolimits_{i \in I} h_{U_i} \to h_U$$

becomes surjective after sheafification.

Proof. By Lemma 7.11.2 above we have to show that $\coprod_{i \in I} h_{U_i}^\# \to h_U^\#$ is an epimorphism. Let $\mathcal{F}$ be a sheaf of sets. A morphism $h_U^\# \to \mathcal{F}$ corresponds to a section $s \in \mathcal{F}(U)$. Hence the injectivity of $\mathrm{Mor}(h_U^\#, \mathcal{F}) \to \prod_i \mathrm{Mor}(h_{U_i}^\#, \mathcal{F})$ follows directly from the sheaf property of $\mathcal{F}$. □

The next lemma says, in the case the topology is weaker than the canonical topology, that every sheaf is made up out of representable sheaves in a way.

00WS Lemma 7.12.5. Let $\mathcal{C}$ be a site. Let $E \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every object of $\mathcal{C}$ has a covering by elements of E. Let $\mathcal{F}$ be a sheaf of sets. There exists a diagram of sheaves of sets

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

which represents $\mathcal{F}$ as a coequalizer, such that $\mathcal{F}_i$, $i = 0, 1$ are coproducts of sheaves of the form $h_U^\#$ with $U \in E$.

Proof. First we show there is an epimorphism $\mathcal{F}_0 \to \mathcal{F}$ of the desired type. Namely, just take

$$\mathcal{F}_0 = \coprod\nolimits_{U \in E, s \in \mathcal{F}(U)} (h_U)^\# \longrightarrow \mathcal{F}$$

Here the arrow restricted to the component corresponding to (U, s) maps the element $\mathrm{id}_U \in h_U^\#(U)$ to the section $s \in \mathcal{F}(U)$. This is an epimorphism according to Lemma 7.11.2 and our condition on E. To construct $\mathcal{F}_1$ first set $\mathcal{G} = \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ and then construct an epimorphism $\mathcal{F}_1 \to \mathcal{G}$ as above. See Lemma 7.11.3. □

0GLW Lemma 7.12.6. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{C}$. Then there exists a diagram $\mathcal{I} \to \mathcal{C}$, $i \mapsto U_i$ such that

$$\mathcal{F} = \mathrm{colim}_{i \in \mathcal{I}} h_{U_i}^\#$$

Moreover, if $E \subset \mathrm{Ob}(\mathcal{C})$ is a subset such that every object of $\mathcal{C}$ has a covering by elements of E, then we may assume U_i is an element of E for all $i \in \mathrm{Ob}(\mathcal{I})$.

Proof. Let $\mathcal{I}$ be the category whose objects are pairs (U, s) with $U \in \mathrm{Ob}(\mathcal{C})$ and $s \in \mathcal{F}(U)$ and whose morphisms $(U, s) \to (U', s')$ are morphisms $f : U \to U'$ in $\mathcal{C}$ with $f^*s' = s$. For each object (U, s) of $\mathcal{I}$ the element s defines by the Yoneda lemma a map $s : h_U \to \mathcal{F}$ of presheaves. Hence by the universal property of sheafification a map $h_U^\# \to \mathcal{F}$. These maps are immediately seen to be compatible with the morphisms in the category $\mathcal{I}$. Hence we obtain a map $\mathrm{colim}_{(U,s)} h_U \to \mathcal{F}$ of presheaves (where the colimit is taken in the category of presheaves) and a map $\mathrm{colim}_{(U,s)} (h_U)^\# \to \mathcal{F}$ of sheaves (where the colimit is taken in the category of sheaves). Since sheafification is the left adjoint to the inclusion functor $\mathit{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ (Proposition 7.10.12) we have $\mathrm{colim}(h_U)^\# = (\mathrm{colim}\, h_U)^\#$ by Categories, Lemma 4.24.5. Thus it suffices to show that $\mathrm{colim}_{(U,s)} h_U \to \mathcal{F}$ is an isomorphism of presheaves. To see this we show that for every object X of $\mathcal{C}$ the map $\mathrm{colim}_{(U,s)} h_U(X) \to \mathcal{F}(X)$ is bijective. Namely, it has an inverse sending the element $t \in \mathcal{F}(X)$ to the element of the set $\mathrm{colim}_{(U,s)} h_U(X)$ corresponding to (X, t) and $\mathrm{id}_X \in h_X(X)$.

We omit the proof of the final statement. □

7.13. Continuous functors

00WU

00WV Definition 7.13.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A functor $u : \mathcal{C} \to \mathcal{D}$ is called continuous if for every $\{V_i \to V\}_{i\in I} \in \text{Cov}(\mathcal{C})$ we have the following

(1) $\{u(V_i) \to u(V)\}_{i\in I}$ is in $\text{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Recall that given a functor u as above, and a presheaf of sets $\mathcal{F}$ on $\mathcal{D}$ we have defined $u^p\mathcal{F}$ to be simply the presheaf $\mathcal{F} \circ u$, in other words

$$u^p\mathcal{F}(V) = \mathcal{F}(u(V))$$

for every object V of $\mathcal{C}$.

00WW Lemma 7.13.2. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor. If $\mathcal{F}$ is a sheaf on $\mathcal{D}$ then $u^p\mathcal{F}$ is a sheaf as well.

Proof. Let $\{V_i \to V\}$ be a covering. By assumption $\{u(V_i) \to u(V)\}$ is a covering in $\mathcal{D}$ and $u(V_i \times_V V_j) = u(V_i) \times_{u(V)} u(V_j)$. Hence the sheaf condition for $u^p\mathcal{F}$ and the covering $\{V_i \to V\}$ is precisely the same as the sheaf condition for $\mathcal{F}$ and the covering $\{u(V_i) \to u(V)\}$. $\square$

In order to avoid confusion we sometimes denote

$$u^s : Sh(\mathcal{D}) \longrightarrow Sh(\mathcal{C})$$

the functor u^p restricted to the subcategory of sheaves of sets. Recall that u^p has a left adjoint $u_p : \text{PSh}(\mathcal{C}) \to \text{PSh}(\mathcal{D})$, see Section 7.5.

00WX Lemma 7.13.3. In the situation of Lemma 7.13.2. The functor $u_s : \mathcal{G} \mapsto (u_p\mathcal{G})^\#$ is a left adjoint to u^s.

Proof. Follows directly from Lemma 7.5.4 and Proposition 7.10.12. $\square$

Here is a technical lemma.

00WY Lemma 7.13.4. In the situation of Lemma 7.13.2. For any presheaf $\mathcal{G}$ on $\mathcal{C}$ we have $(u_p\mathcal{G})^\# = (u_p(\mathcal{G}^\#))^\#$.

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{D}$ we have

$$\begin{aligned}
\text{Mor}_{Sh(\mathcal{D})}(u_s(\mathcal{G}^\#), \mathcal{F}) &= \text{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^\#, u^s\mathcal{F}) \\
&= \text{Mor}_{\text{PSh}(\mathcal{C})}(\mathcal{G}^\#, u^p\mathcal{F}) \\
&= \text{Mor}_{\text{PSh}(\mathcal{C})}(\mathcal{G}, u^p\mathcal{F}) \\
&= \text{Mor}_{\text{PSh}(\mathcal{D})}(u_p\mathcal{G}, \mathcal{F}) \\
&= \text{Mor}_{Sh(\mathcal{D})}((u_p\mathcal{G})^\#, \mathcal{F})
\end{aligned}$$

and the result follows from the Yoneda lemma. $\square$

04D3 Lemma 7.13.5. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor between sites. For any object U of $\mathcal{C}$ we have $u_s h_U^\# = h_{u(U)}^\#$.

Proof. Follows from Lemmas 7.5.6 and 7.13.4. $\square$

00WZ Remark 7.13.6. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let us use the definition of tautologically equivalent families of maps, see Definition 7.8.2 to (slightly) weaken the conditions defining continuity. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let us call u quasi-continuous if for every $\mathcal{V} = \{V_i \to V\}_{i \in I} \in \text{Cov}(\mathcal{C})$ we have the following

(1') the family of maps $\{u(V_i) \to u(V)\}_{i \in I}$ is tautologically equivalent to an element of $\text{Cov}(\mathcal{D})$, and

(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

We are going to see that Lemmas 7.13.2 and 7.13.3 hold in case u is quasi-continuous as well.

We first remark that the morphisms $u(V_i) \to u(V)$ are representable, since they are isomorphic to representable morphisms (by the first condition). In particular, the family $u(\mathcal{V}) = \{u(V_i) \to u(V)\}_{i \in I}$ gives rise to a zeroth Čech cohomology group $H^0(u(\mathcal{V}), \mathcal{F})$ for any presheaf $\mathcal{F}$ on $\mathcal{D}$. Let $\mathcal{U} = \{U_j \to u(V)\}_{j \in J}$ be an element of $\text{Cov}(\mathcal{D})$ tautologically equivalent to $\{u(V_i) \to u(V)\}_{i \in I}$. Note that $u(\mathcal{V})$ is a refinement of $\mathcal{U}$ and vice versa. Hence by Remark 7.10.7 we see that $H^0(u(\mathcal{V}), \mathcal{F}) = H^0(\mathcal{U}, \mathcal{F})$. In particular, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}(u(V)) = H^0(u(\mathcal{V}), \mathcal{F})$ because of the sheaf property expressed in terms of zeroth Čech cohomology groups. We conclude that $u^p\mathcal{F}$ is a sheaf if $\mathcal{F}$ is a sheaf, since $H^0(\mathcal{V}, u^p\mathcal{F}) = H^0(u(\mathcal{V}), \mathcal{F})$ which we just observed is equal to $\mathcal{F}(u(V)) = u^p\mathcal{F}(V)$. Thus Lemma 7.13.2 holds. Lemma 7.13.3 follows immediately.

7.14. Morphisms of sites

00X0

00X1 Definition 7.14.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A morphism of sites $f : \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ such that the functor u_s is exact.

Notice how the functor u goes in the direction opposite the morphism f. If $f \leftrightarrow u$ is a morphism of sites then we use the notation $f^{-1} = u_s$ and $f_* = u^s$. The functor f^{-1} is called the pullback functor and the functor f_* is called the pushforward functor. As in topology we have the following adjointness property

$$\text{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$

The motivation for this definition comes from the following example.

00X2 Example 7.14.2. Let $f : X \to Y$ be a continuous map of topological spaces. Recall that we have sites X_{Zar} and Y_{Zar}, see Example 7.6.4. Consider the functor $u : Y_{Zar} \to X_{Zar}$, $V \mapsto f^{-1}(V)$. This functor is clearly continuous because inverse images of open coverings are open coverings. (Actually, this depends on how you chose sets of coverings for X_{Zar} and Y_{Zar}. But in any case the functor is quasi-continuous, see Remark 7.13.6.) It is easy to verify that the functor u^s equals the usual pushforward functor f_* from topology. Hence, since u_s is an adjoint and since the usual topological pullback functor f^{-1} is an adjoint as well, we get a canonical isomorphism $f^{-1} \cong u_s$. Since f^{-1} is exact we deduce that u_s is exact. Hence u defines a morphism of sites $f : X_{Zar} \to Y_{Zar}$, which we may denote f as well since we've already seen the functors u_s, u^s agree with their usual notions anyway.

0EWI Example 7.14.3. Let $\mathcal{C}$ be a category. Let

$$\text{Cov}(\mathcal{C}) \supset \text{Cov}'(\mathcal{C})$$

be two sets of families of morphisms with fixed target which turn $\mathcal{C}$ into a site. Denote $\mathcal{C}_\tau$ the site corresponding to $\text{Cov}(\mathcal{C})$ and $\mathcal{C}_{\tau'}$ the site corresponding to $\text{Cov}'(\mathcal{C})$. We claim the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

Namely, observe that $\text{id} : \mathcal{C}_{\tau'} \to \mathcal{C}_\tau$ is continuous as every τ'-covering is a τ-covering. Thus the functor $\epsilon_* = \text{id}^s$ is the identity functor on underlying presheaves. Hence the left adjoint ϵ^{-1} of ϵ_* sends a τ'-sheaf $\mathcal{F}$ to the τ-sheafification of $\mathcal{F}$ (by the universal property of sheafification). Finite limits of τ'-sheaves agree with finite limits of presheaves (Lemma 7.10.1) and τ-sheafification commutes with finite limits (Lemma 7.10.14). Thus ϵ^{-1} is left exact. Since ϵ^{-1} is a left adjoint it is also right exact (Categories, Lemma 4.24.6). Thus ϵ^{-1} is exact and we have checked all the conditions of Definition 7.14.1.

03CB Lemma 7.14.4. Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$ be continuous functors which induce morphisms of sites. Then the functor $u \circ v : \mathcal{C}_3 \to \mathcal{C}_1$ is continuous and defines a morphism of sites $\mathcal{C}_1 \to \mathcal{C}_3$.

Proof. It is immediate from the definitions that $u \circ v$ is a continuous functor. In addition, we clearly have $(u \circ v)^p = v^p \circ u^p$, and hence $(u \circ v)^s = v^s \circ u^s$. Hence functors $(u \circ v)_s$ and $u_s \circ v_s$ are both left adjoints of $(u \circ v)^s$. Therefore $(u \circ v)_s \cong u_s \circ v_s$ and we conclude that $(u \circ v)_s$ is exact as a composition of exact functors. □

03CC Definition 7.14.5. Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ and $g : \mathcal{C}_2 \to \mathcal{C}_3$ be morphisms of sites given by continuous functors $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$. The composition $g \circ f$ is the morphism of sites corresponding to the functor $u \circ v$.

In this situation we have $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ (see proof of Lemma 7.14.4).

00X5 Lemma 7.14.6. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume all the categories $(\mathcal{I}_V^u)^{opp}$ of Section 7.5 are filtered. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.

Proof. Since u_s is the left adjoint of u^s we see that u_s is right exact, see Categories, Lemma 4.24.6. Hence it suffices to show that u_s is left exact. In other words we have to show that u_s commutes with finite limits. Because the categories $\mathcal{I}_Y^{opp}$ are filtered we see that u_p commutes with finite limits, see Categories, Lemma 4.19.2 (this also uses the description of limits in PSh, see Section 7.4). And since sheafification commutes with finite limits as well (Lemma 7.10.14) we conclude because $u_s = \# \circ u_p$. □

00X6 Proposition 7.14.7. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume furthermore the following:

(1) the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and
(2) the category $\mathcal{C}$ has fibre products and u commutes with them.

Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.

Proof. This follows from Lemmas 7.5.2 and 7.14.6. □

00X7 Remark 7.14.8. The conditions of Proposition 7.14.7 above are equivalent to saying that u is left exact, i.e., commutes with finite limits. See Categories, Lemmas 4.18.4 and 4.23.2. It seems more natural to phrase it in terms of final objects and fibre products since this seems to have more geometric meaning in the examples.

Lemma 7.19.4 will provide another way to prove a continuous functor gives rise to a morphism of sites.

00X8 Remark 7.14.9. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Analogously to Definition 7.14.1 we say that a quasi-morphism of sites $f : \mathcal{D} \to \mathcal{C}$ is given by a quasi-continuous functor $u : \mathcal{C} \to \mathcal{D}$ (see Remark 7.13.6) such that u_s is exact. The analogue of Proposition 7.14.7 in this setting is obtained by replacing the word "continuous" by the word "quasi-continuous", and replacing the word "morphism" by "quasi-morphism". The proof is literally the same.

In Definition 7.14.1 the condition that u_s be exact cannot be omitted. For example, the conclusion of the following lemma need not hold if one only assumes that u is continuous.

08H2 Lemma 7.14.10. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$. Given any object V of $\mathcal{D}$ there exists a covering $\{V_j \to V\}$ such that for every j there exists a morphism $V_j \to u(U_j)$ for some object U_j of $\mathcal{C}$.

Proof. Since $f^{-1} = u_s$ is exact we have $f^{-1}* = *$ where $*$ denotes the final object of the category of sheaves (Example 7.10.2). Since $f^{-1}* = u_s*$ is the sheafification of u_p* we see there exists a covering $\{V_j \to V\}$ such that $(u_p*)(V_j)$ is nonempty. Since $(u_p*)(V_j)$ is a colimit over the category $\mathcal{I}^u_{V_j}$ whose objects are morphisms $V_j \to u(U)$ the lemma follows. □

7.15. Topoi

00X9 Here is a definition of a topos which is suitable for our purposes. Namely, a topos is the category of sheaves on a site. In order to specify a topos you just specify the site. The real difference between a topos and a site lies in the definition of morphisms. Namely, it turns out that there are lots of morphisms of topoi which do not come from morphisms of the underlying sites.

00XA Definition 7.15.1 (Topoi). A topos is the category $Sh(\mathcal{C})$ of sheaves on a site $\mathcal{C}$.

(1) Let $\mathcal{C}$, $\mathcal{D}$ be sites. A morphism of topoi f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ is given by a pair of functors $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $f^{-1} : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ such that
   (a) we have
   $$\mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$
   bifunctorially, and
   (b) the functor f^{-1} commutes with finite limits, i.e., is left exact.
(2) Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be sites. Given morphisms of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $g : Sh(\mathcal{E}) \to Sh(\mathcal{D})$ the composition $f \circ g$ is the morphism of topoi defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Suppose that $\alpha : \mathcal{S}_1 \to \mathcal{S}_2$ is an equivalence of (possibly "big") categories. If $\mathcal{S}_1$, $\mathcal{S}_2$ are topoi, then setting $f_* = \alpha$ and f^{-1} equal to a quasi-inverse of α gives a morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ of topoi. Moreover this morphism is an equivalence in the 2-category of topoi (see Section 7.36). Thus it makes sense to say "$\mathcal{S}$ is a topos" if $\mathcal{S}$ is equivalent to the category of sheaves on a site (and not necessarily equal to the category of sheaves on a site). We will occasionally use this abuse of notation.

The empty topos is topos of sheaves on the site $\mathcal{C}$, where $\mathcal{C}$ is the empty category. We will sometimes write $\emptyset$ for this site. This is a site which has a unique sheaf (since $\emptyset$ has no objects). Thus $Sh(\emptyset)$ is equivalent to the category having a single object and a single morphism.

The punctual topos is the topos of sheaves on the site $\mathcal{C}$ which has a single object pt and one morphism id_{pt} and whose only covering is the covering $\{\text{id}_{pt}\}$. We will simply write pt for this site. It is clear that the category of sheaves = the category of presheaves = the category of sets. In a formula $Sh(pt) = \text{Sets}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Note that f_* commutes with all limits and that f^{-1} commutes with all colimits, see Categories, Lemma 4.24.5. In particular, the condition on f^{-1} in the definition above guarantees that f^{-1} is exact. Morphisms of topoi are often constructed using either Lemma 7.21.1 or the following lemma.

00XC Lemma 7.15.2. Given a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ corresponding to the functor $u : \mathcal{C} \to \mathcal{D}$ the pair of functors $(f^{-1} = u_s, f_* = u^s)$ is a morphism of topoi.

Proof. This is obvious from Definition 7.14.1. $\square$

00XD Remark 7.15.3. There are many sites that give rise to the topos $Sh(pt)$. A useful example is the following. Suppose that S is a set (of sets) which contains at least one nonempty element. Let $\mathcal{S}$ be the category whose objects are elements of S and whose morphisms are arbitrary set maps. Assume that $\mathcal{S}$ has fibre products. For example this will be the case if $S = \mathcal{P}(\text{infinite set})$ is the power set of any infinite set (exercise in set theory). Make $\mathcal{S}$ into a site by declaring surjective families of maps to be coverings (and choose a suitable sufficiently large set of covering families as in Sets, Section 3.11). We claim that $Sh(\mathcal{S})$ is equivalent to the category of sets.

We first prove this in case S contains $e \in S$ which is a singleton. In this case, there is an equivalence of topoi $i : Sh(pt) \to Sh(\mathcal{S})$ given by the functors

05UW (7.15.3.1) $$i^{-1}\mathcal{F} = \mathcal{F}(e), \quad i_*E = (U \mapsto \text{Mor}_{\text{Sets}}(U, E))$$

Namely, suppose that $\mathcal{F}$ is a sheaf on $\mathcal{S}$. For any $U \in \text{Ob}(\mathcal{S}) = S$ we can find a covering $\{\varphi_u : e \to U\}_{u \in U}$, where φ_u maps the unique element of e to $u \in U$. The sheaf condition implies in this case that $\mathcal{F}(U) = \prod_{u \in U} \mathcal{F}(e)$. In other words $\mathcal{F}(U) = \text{Mor}_{\text{Sets}}(U, \mathcal{F}(e))$. Moreover, this rule is compatible with restriction mappings. Hence the functor

$$i_* : \text{Sets} = Sh(pt) \longrightarrow Sh(\mathcal{S}), \quad E \longmapsto (U \mapsto \text{Mor}_{\text{Sets}}(U, E))$$

is an equivalence of categories, and its inverse is the functor i^{-1} given above.

If $\mathcal{S}$ does not contain a singleton, then the functor i_* as defined above still makes sense. To show that it is still an equivalence in this case, choose any nonempty $\tilde{e} \in S$ and a map $\varphi : \tilde{e} \to \tilde{e}$ whose image is a singleton. For any sheaf $\mathcal{F}$ set

$$\mathcal{F}(e) := \text{Im}(\mathcal{F}(\varphi) : \mathcal{F}(\tilde{e}) \longrightarrow \mathcal{F}(\tilde{e}))$$

and show that this is a quasi-inverse to i_*. Details omitted.

00XB Remark 7.15.4. (Set theoretical issues related to morphisms of topoi. Skip on a first reading.) A morphism of topoi as defined above is not a set but a class. In other words it is given by a mathematical formula rather than a mathematical object. Although we may contemplate the collection of all morphisms between two given topoi, it is not a good idea to introduce it as a mathematical object. On the other hand, suppose $\mathcal{C}$ and $\mathcal{D}$ are given sites. Consider a functor $\Phi : \mathcal{C} \to \mathit{Sh}(\mathcal{D})$. Such a thing is a set, in other words, it is a mathematical object. We may, in succession, ask the following questions on Φ.

(1) Is it true, given a sheaf $\mathcal{F}$ on $\mathcal{D}$, that the rule $U \mapsto \mathrm{Mor}_{\mathit{Sh}(\mathcal{D})}(\Phi(U), \mathcal{F})$ defines a sheaf on $\mathcal{C}$? If so, this defines a functor $\Phi_* : \mathit{Sh}(\mathcal{D}) \to \mathit{Sh}(\mathcal{C})$.
(2) Is it true that Φ_* has a left adjoint? If so, write Φ^{-1} for this left adjoint.
(3) Is it true that Φ^{-1} is exact?

If the last question still has the answer "yes", then we obtain a morphism of topoi (Φ_*, Φ^{-1}). Moreover, given any morphism of topoi (f_*, f^{-1}) we may set $\Phi(U) = f^{-1}(h_U^\#)$ and obtain a functor Φ as above with $f_* \cong \Phi_*$ and $f^{-1} \cong \Phi^{-1}$ (compatible with adjoint property). The upshot is that by working with the collection of Φ instead of morphisms of topoi, we (a) replaced the notion of a morphism of topoi by a mathematical object, and (b) the collection of Φ forms a class (and not a collection of classes). Of course, more can be said, for example one can work out more precisely the significance of conditions (2) and (3) above; we do this in the case of points of topoi in Section 7.32.

00XE Remark 7.15.5. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A quasi-morphism of sites $f : \mathcal{D} \to \mathcal{C}$ (see Remark 7.14.9) gives rise to a morphism of topoi f from $\mathit{Sh}(\mathcal{D})$ to $\mathit{Sh}(\mathcal{C})$ exactly as in Lemma 7.15.2.

7.16. G-sets and morphisms

04D4 Let $\varphi : G \to H$ be a homomorphism of groups. Choose (suitable) sites $\mathcal{T}_G$ and $\mathcal{T}_H$ as in Example 7.6.5 and Section 7.9. Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor which assigns to a H-set U the G-set U_φ which has the same underlying set but G action defined by $g \cdot \xi = \varphi(g)\xi$ for $g \in G$ and $\xi \in U$. It is clear that u commutes with finite limits and is continuous[4]. Applying Proposition 7.14.7 and Lemma 7.15.2 we obtain a morphism of topoi

$$f : \mathit{Sh}(\mathcal{T}_G) \longrightarrow \mathit{Sh}(\mathcal{T}_H)$$

associated with φ. Using Proposition 7.9.1 we see that we get a pair of adjoint functors

$$f_* : G\text{-Sets} \longrightarrow H\text{-Sets}, \quad f^{-1} : H\text{-Sets} \longrightarrow G\text{-Sets}.$$

Let's work out what are these functors in this case.

We first work out a formula for f_*. Recall that given a G-set S the corresponding sheaf $\mathcal{F}_S$ on $\mathcal{T}_G$ is given by the rule $\mathcal{F}_S(U) = \mathrm{Mor}_G(U, S)$. And on the other hand, given a sheaf $\mathcal{G}$ on $\mathcal{T}_H$ the corresponding H-set is given by the rule $\mathcal{G}({}_HH)$. Hence we see that

$$f_*S = \mathrm{Mor}_{G\text{-Sets}}(({}_HH)_\varphi, S).$$

If we work this out a little bit more then we get

$$f_*S = \{a : H \to S \mid a(gh) = ga(h)\}$$

with left H-action given by $(h \cdot a)(h') = a(h'h)$ for any element $a \in f_*S$.

[4]Set theoretical remark: First choose $\mathcal{T}_H$. Then choose $\mathcal{T}_G$ to contain $u(\mathcal{T}_H)$ and such that every covering in $\mathcal{T}_H$ corresponds to a covering in $\mathcal{T}_G$. This is possible by Sets, Lemmas 3.10.1, 3.10.2 and 3.11.1.

Next, we explicitly compute f^{-1}. Note that since the topology on $\mathcal{T}_G$ and $\mathcal{T}_H$ is subcanonical, all representable presheaves are sheaves. Moreover, given an object V of $\mathcal{T}_H$ we see that $f^{-1}h_V$ is equal to $h_{u(V)}$ (see Lemma 7.13.5). Hence we see that $f^{-1}S = S_\varphi$ for representable sheaves. Since every sheaf on $\mathcal{T}_H$ is a coproduct of representable sheaves we conclude that this is true in general. Hence we see that for any H-set T we have

$$f^{-1}T = T_\varphi.$$

The adjunction between f^{-1} and f_* is evidenced by the formula

$$\mathrm{Mor}_{G\text{-Sets}}(T_\varphi, S) = \mathrm{Mor}_{H\text{-Sets}}(T, f_*S)$$

with f_*S as above. This can be proved directly. Moreover, it is then clear that (f^{-1}, f_*) form an adjoint pair and that f^{-1} is exact. So alternatively to the above the morphism of topoi $f : G\text{-Sets} \to H\text{-Sets}$ can be defined directly in this manner.

7.17. Quasi-compact objects and colimits

090G To be able to use the same language as in the case of topological spaces we introduce the following terminology.

090H **Definition 7.17.1.** Let $\mathcal{C}$ be a site. An object U of $\mathcal{C}$ is *quasi-compact* if given a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ there exists another covering $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and a morphism $\mathcal{V} \to \mathcal{U}$ of families of maps with fixed target given by $\mathrm{id} : U \to U$, $\alpha : J \to I$, and $V_j \to U_{\alpha(j)}$ (see Definition 7.8.1) such that the image of α is a finite subset of I.

Of course the usual notion is sufficient to conclude that U is quasi-compact.

0D05 **Lemma 7.17.2.** Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. Consider the following conditions

(1) U is quasi-compact,
(2) for every covering $\{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ there exists a finite covering $\{V_j \to U\}_{j=1,\ldots,m}$ of $\mathcal{C}$ refining $\mathcal{U}$, and
(3) for every covering $\{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ there exists a finite subset $I' \subset I$ such that $\{U_i \to U\}_{i \in I'}$ is a covering in $\mathcal{C}$.

Then we always have (3) $\Rightarrow$ (2) $\Rightarrow$ (1) but the reverse implications do not hold in general.

Proof. The implications are immediate from the definitions. Let $X = [0,1] \subset \mathbf{R}$ as a topological space (with the usual ϵ-δ topology). Let $\mathcal{C}$ be the category of open subspaces of X with inclusions as morphisms and usual open coverings (compare with Example 7.6.4). However, then we change the notion of covering in $\mathcal{C}$ to exclude all finite coverings, except for the coverings of the form $\{U \to U\}$. It is easy to see that this will be a site as in Definition 7.6.2. In this site the object $X = U = [0,1]$ is quasi-compact in the sense of Definition 7.17.1 but U does not satisfy (2). We leave it to the reader to make an example where (2) holds but not (3). $\square$

Here is the topos theoretic meaning of a quasi-compact object.

0D06 **Lemma 7.17.3.** Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following are equivalent

(1) U is quasi-compact, and

(2) for every surjection of sheaves $\coprod_{i \in I} \mathcal{F}_i \to h_U^\#$ there is a finite subset $J \subset I$ such that $\coprod_{i \in J} \mathcal{F}_i \to h_U^\#$ is surjective.

Proof. Assume (1) and let $\coprod_{i \in I} \mathcal{F}_i \to h_U^\#$ be a surjection. Then id_U is a section of $h_U^\#$ over U. Hence there exists a covering $\{U_a \to U\}_{a \in A}$ and for each $a \in A$ a section s_a of $\coprod_{i \in I} \mathcal{F}_i$ over U_a mapping to id_U. By the construction of coproducts as sheafification of coproducts of presheaves (Lemma 7.10.13), for each a there exists a covering $\{U_{ab} \to U_a\}_{b \in B_a}$ and for all $b \in B_a$ an $\iota(b) \in I$ and a section s_b of $\mathcal{F}_{\iota(b)}$ over U_{ab} mapping to $\mathrm{id}_U|_{U_{ab}}$. Thus after replacing the covering $\{U_a \to U\}_{a \in A}$ by $\{U_{ab} \to U\}_{a \in A, b \in B_a}$ we may assume we have a map $\iota : A \to I$ and for each $a \in A$ a section s_a of $\mathcal{F}_{\iota(a)}$ over U_a mapping to id_U. Since U is quasi-compact, there is a covering $\{V_c \to U\}_{c \in C}$, a map $\alpha : C \to A$ with finite image, and $V_c \to U_{\alpha(c)}$ over U. Then we see that $J = \mathrm{Im}(\iota \circ \alpha) \subset I$ works because $\coprod_{c \in C} h_{V_c}^\# \to h_U^\#$ is surjective (Lemma 7.12.4) and factors through $\coprod_{i \in J} \mathcal{F}_i \to h_U^\#$. (Here we use that the composition $h_{V_c}^\# \to h_{U_{\alpha(c)}} \xrightarrow{s_{\alpha(c)}} \mathcal{F}_{\iota(\alpha(c))} \to h_U^\#$ is the map $h_{V_c}^\# \to h_U^\#$ coming from the morphism $V_c \to U$ because $s_{\alpha(c)}$ maps to $\mathrm{id}_U|_{U_{\alpha(c)}}$.)

Assume (2). Let $\{U_i \to U\}_{i \in I}$ be a covering. By Lemma 7.12.4 we see that $\coprod_{i \in I} h_{U_i}^\# \to h_U^\#$ is surjective. Thus we find a finite subset $J \subset I$ such that $\coprod_{j \in J} h_{U_j}^\# \to h_U^\#$ is surjective. Then arguing as above we find a covering $\{V_c \to U\}_{c \in C}$ of U in $\mathcal{C}$ and a map $\iota : C \to J$ such that id_U lifts to a section of s_c of $h_{U_{\iota(c)}}^\#$ over V_c. Refining the covering even further we may assume $s_c \in h_{U_{\iota(c)}}(V_c)$ mapping to id_U. Then $s_c : V_c \to U_{\iota(c)}$ is a morphism over U and we conclude. □

The lemma above motivates the following definition.

0D07 Definition 7.17.4. An object $\mathcal{F}$ of a topos $Sh(\mathcal{C})$ is quasi-compact if for any surjective map $\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$ of $Sh(\mathcal{C})$ there exists a finite subset $J \subset I$ such that $\coprod_{i \in J} \mathcal{F}_i \to \mathcal{F}$ is surjective. A topos $Sh(\mathcal{C})$ is said to be quasi-compact if its final object $*$ is a quasi-compact object.

By Lemma 7.17.3 if the site $\mathcal{C}$ has a final object X, then $Sh(\mathcal{C})$ is quasi-compact if and only if X is quasi-compact.

0GMP Lemma 7.17.5. Let $\mathcal{C}$ be a site.

(1) If $U \to V$ is a morphism of $\mathcal{C}$ such that $h_U^\# \to h_V^\#$ is surjective and U is quasi-compact, then V is quasi-compact.
(2) If $\mathcal{F} \to \mathcal{G}$ is a surjection of sheaves of sets and $\mathcal{F}$ is quasi-compact, then $\mathcal{G}$ is quasi-compact.

Proof. Omitted. □

0GMQ Lemma 7.17.6. Let $\mathcal{C}$ be a site. If $n \geq 1$ and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are quasi-compact sheaves on $\mathcal{C}$, then $\coprod_{i=1,\ldots,n} \mathcal{F}_i$ is quasi-compact.

Proof. Omitted. □

The following two lemmas form the analogue of Sheaves, Lemma 6.29.1 for sites.

0738 Lemma 7.17.7. Let $\mathcal{C}$ be a site. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a filtered diagram of sheaves of sets. Let $U \in \mathrm{Ob}(\mathcal{C})$. Consider the canonical map

$$\Psi : \mathrm{colim}_i\, \mathcal{F}_i(U) \longrightarrow \left(\mathrm{colim}_i\, \mathcal{F}_i\right)(U)$$

With the terminology introduced above:

(1) If all the transition maps are injective then Ψ is injective for any U.
(2) If U is quasi-compact, then Ψ is injective.
(3) If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.
(4) If U has a cofinal system of coverings $\{U_j \to U\}_{j \in J}$ with J finite and $U_j \times_U U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \operatorname{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 7.10.9). By Lemma 7.10.13 we have $(\mathcal{F}')^\# = \operatorname{colim}_i \mathcal{F}_i$. By Theorem 7.10.10 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. This means we can choose a covering $\{U_a \to U\}_{a \in A}$ and for each $a \in A$ an index $i_a \in I$, $i_a \geq i$, $i_a \geq i'$ such that $\varphi_{ii_a}(s) = \varphi_{i'i_a}(s')$. Because U is quasi-compact we can choose a covering $\{V_b \to U\}_{b \in B}$, a map $\alpha : B \to A$ with finite image, and morphisms $V_b \to U_{\alpha(b)}$ over U. Pick $i'' \in I$ to be $\geq$ than all of the $i_{\alpha(b)}$ which is possible because the image of α is finite. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on V_b for all $b \in B$ and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. There exists a covering $\{U_a \to U\}_{a \in A}$ and for each $a \in A$ an index $i_a \in I$ and a section $s_a \in \mathcal{F}_{i_a}(U_a)$ such that $s|_{U_a}$ comes from s_a for all $a \in A$. Because U is quasi-compact we can choose a covering $\{V_b \to U\}_{b \in B}$, a map $\alpha : B \to A$ with finite image, and morphisms $V_b \to U_{\alpha(b)}$ over U. Pick $i \in I$ to be $\geq$ than all of the $i_{\alpha(b)}$ which is possible because the image of α is finite. By (1) the sections $s_b = \varphi_{i_{\alpha(b)}i}(s_{\alpha(b)})|_{V_b}$ agree over $V_b \times_U V_{b'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). By Lemma 7.17.2 the object U is quasi-compact, hence Ψ is injective by (2). To prove surjectivity, let s be an element of the target of Ψ. By assumption there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$, with $U_j \times_U U_{j'}$ quasi-compact for all $1 \leq j, j' \leq m$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \times_U U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_ji_{jj'}}(s_j)$ and $\varphi_{i_{j'}i_{jj'}}(s_{j'})$ agree over $U_j \times_U U_{j'}$. Choose an index $i \in I$ wich is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_ji}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. $\square$

0GMR Lemma 7.17.8. Let $\mathcal{C}$ be a site. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a filtered diagram of sheaves of sets. Consider the canonical map

$$\Psi : \operatorname{colim}_i \Gamma(\mathcal{C}, \mathcal{F}_i) \longrightarrow \Gamma(\mathcal{C}, \operatorname{colim}_i \mathcal{F}_i)$$

We have the following:

(1) If all the transition maps are injective then Ψ is injective.
(2) If $Sh(\mathcal{C})$ is quasi-compact, then Ψ is injective.
(3) If $Sh(\mathcal{C})$ is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.
(4) Assume there exists a set $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ with the following properties:

(a) for every surjection $\mathcal{F} \to *$ there exists a $\mathcal{K} \in S$ and a map $\mathcal{K} \to \mathcal{F}$ such that $\mathcal{K} \to *$ is surjective,
(b) for $\mathcal{K} \in S$ the product $\mathcal{K} \times \mathcal{K}$ is quasi-compact.

Then Ψ is bijective.

Proof. Proof of (1). Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \mathrm{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 7.10.9). By Lemma 7.10.13 we have $(\mathcal{F}')^\# = \mathrm{colim}_i \mathcal{F}_i$. By Theorem 7.10.10 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Proof of (2). Assume $Sh(\mathcal{C})$ is quasi-compact. Recall that $\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}(*, \mathcal{F})$ for all $\mathcal{F}$ in $Sh(\mathcal{C})$. Let $a_i, b_i : * \to \mathcal{F}_i$ and for $i' \geq i$ denote $a_{i'}, b_{i'} : * \to \mathcal{F}_{i'}$ the composition with the transition maps of the system. Set $a = \mathrm{colim}_{i' \geq i} a_{i'}$ and similary for b. For $i' \geq i$ denote

$$E_{i'} = \mathrm{Equalizer}(a_{i'}, b_{i'}) \subset * \quad \text{and} \quad E = \mathrm{Equalizer}(a, b) \subset *$$

By Categories, Lemma 4.19.2 we have $E = \mathrm{colim}_{i' \geq i} E_{i'}$. It follows that $\coprod_{i' \geq i} E_{i'} \to E$ is a surjective map of sheaves. Hence, if $E = *$, i.e., if $a = b$, then because $*$ is quasi-compact, we see that $E_{i'} = *$ for some $i' \geq i$, and we conclude $a_{i'} = b_{i'}$ for some $i' \geq i$. This proves (2).

Proof of (3). Assume $Sh(\mathcal{C})$ is quasi-compact and all transition maps are injective. Let $a : * \to \mathrm{colim}\, \mathcal{F}_i$ be a map. Then $E_i = a^{-1}(\mathcal{F}_i) \subset *$ is a subsheaf and we have $\mathrm{colim}\, E_i = *$ (by the reference above). Hence for some i we have $E_i = *$ and we see that the image of a is contained in $\mathcal{F}_i$ as desired.

Proof of (4). Let $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ satisfy (4)(a), (b). Applying (4)(a) to $\mathrm{id} : * \to *$ we find there exists a $\mathcal{K} \in S$ such that $\mathcal{K} \to *$ is surjective. The maps $\mathcal{K} \times \mathcal{K} \to \mathcal{K} \to *$ are surjective. By (4)(b) and Lemma 7.17.5 we conclude that $\mathcal{K}$ and $Sh(\mathcal{C})$ are quasi-compact. Thus Ψ is injective by (2). Set $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$. Let $s : * \to \mathcal{F}$ be a global section of the colimit. Since $\coprod \mathcal{F}_i \to \mathcal{F}$ is surjective, we see that the projection

$$\coprod\nolimits_{i \in I} * \times_{s, \mathcal{F}} \mathcal{F}_i \to *$$

is surjective. By (4)(a) we obtain $\mathcal{K} \in S$ and a map $\mathcal{K} \to \coprod_{i \in I} * \times_{s,\mathcal{F}} \mathcal{F}_i$ with $\mathcal{K} \to *$ surjective. Since $\mathcal{K}$ is quasi-compact we obtain a factorization $\mathcal{K} \to \coprod_{i' \in I'} * \times_{s,\mathcal{F}} \mathcal{F}_{i'}$ for some finite subset $I' \subset I$. Let $i \in I$ be an upper bound for the finite subset I'. The transition maps define a map $\coprod_{i' \in I'} \mathcal{F}_{i'} \to \mathcal{F}_i$. This in turn produces a map $\mathcal{K} \to * \times_{s,\mathcal{F}} \mathcal{F}_i$. In other words, we obtain $\mathcal{K} \in S$ with $\mathcal{K} \to *$ surjective and a commutative diagram

$$\begin{tikzcd} \mathcal{K} \times \mathcal{K} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{K} \arrow[r] \arrow[d] & * \arrow[d, "s"] & \\ & \mathcal{F}_i \arrow[r] & \mathcal{F} \arrow[r, equal] & \mathrm{colim}\, \mathcal{F}_i \end{tikzcd}$$

Observe that the top row of this diagram is a coequalizer. Hence it suffices to show that after increasing i the two induced maps $a_i, b_i : \mathcal{K} \times \mathcal{K} \to \mathcal{F}_i$ are equal. This is done shown in the next paragraph using the exact same argument as in the proof of (2) and we urge the reader to skip the rest of the proof.

For $i' \geq i$ denote $a_{i'}, b_{i'} : \mathcal{K} \times \mathcal{K} \to \mathcal{F}_{i'}$ the composition of a_i, b_i with the transition maps of the system. Set $a = \mathrm{colim}_{i' \geq i} a_{i'} : \mathcal{K} \times \mathcal{K} \to \mathcal{F}$ and similary for b. We have

$a = b$ by the commutativity of the diagram above. For $i' \geq i$ denote

$$E_{i'} = \text{Equalizer}(a_{i'}, b_{i'}) \subset \mathcal{K} \times \mathcal{K} \quad \text{and} \quad E = \text{Equalizer}(a, b) \subset \mathcal{K} \times \mathcal{K}$$

By Categories, Lemma 4.19.2 we have $E = \text{colim}_{i' \geq i} E_{i'}$. It follows that $\coprod_{i' \geq i} E_{i'} \to E$ is a surjective map of sheaves. Since $a = b$ we have $E = \mathcal{K} \times \mathcal{K}$. As $\mathcal{K} \times \mathcal{K}$ is quasi-compact by (4)(b), we see that $E_{i'} = \mathcal{K} \times \mathcal{K}$ for some $i' \geq i$, and we conclude $a_{i'} = b_{i'}$ for some $i' \geq i$. $\square$

0GMS Remark 7.17.9. Let $\mathcal{C}$ be a site. There are several ways to ensure that the hypotheses of part (4) of Lemma 7.17.8 are satisfied. Here are a few.

(1) Assume there exists a set $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ with the following properties:
   (a) for every surjection $\mathcal{F} \to *$ there exist $m \geq 0$ and $U_1, \ldots, U_m \in \mathcal{B}$ with $\mathcal{F}(U_j)$ nonempty and $\coprod h_{U_j}^\# \to *$ surjective,
   (b) for $U, U' \in \mathcal{B}$ the sheaf $h_U^\# \times h_{U'}^\#$ is quasi-compact.
(2) Assume there exists a set $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ with the following properties:
   (a) there exist $m \geq 0$ and $U_1, \ldots, U_m \in \mathcal{B}$ with $\coprod h_{U_j}^\# \to *$ surjective,
   (b) for $U \in \mathcal{B}$ any covering of U can be refined by a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$ with $U_j \in \mathcal{B}$, and
   (c) for $U, U' \in \mathcal{B}$ there exist $m \geq 0$, $U_1, \ldots, U_m \in \mathcal{B}$, and morphisms $U_j \to U$ and $U_j \to U'$ such that $\coprod h_{U_j}^\# \to h_U^\# \times h_{U'}^\#$ is surjective.
(3) Suppose that
   (a) $Sh(\mathcal{C})$ is quasi-compact,
   (b) every object of $\mathcal{C}$ has a covering whose members are quasi-compact objects,
   (c) if U and U' are quasi-compact, then the sheaf $h_U^\# \times h_{U'}^\#$ is quasi-compact.

In cases (1) and (2) we set $S \subset \text{Ob}(Sh(\mathcal{C}))$ equal to the set of finite coproducts of the sheaves $h_U^\#$ for $U \in \mathcal{B}$. In case (3) we set $S \subset \text{Ob}(Sh(\mathcal{C}))$ equal to the set of finite coproducts of the sheaves $h_U^\#$ for $U \in \text{Ob}(\mathcal{C})$ quasi-compact.

Later we will need a bound on what can happen with colimits as follows.

0GS0 Lemma 7.17.10. Let $\mathcal{C}$ be a site. Let β be an ordinal. Let $\beta \to Sh(\mathcal{C})$, $\alpha \mapsto \mathcal{F}_\alpha$ be a system of sheaves over β. For $U \in \text{Ob}(\mathcal{C})$ consider the canonical map

$$\text{colim}_{\alpha < \beta} \mathcal{F}_\alpha(U) \longrightarrow (\text{colim}_{\alpha < \beta} \mathcal{F}_\alpha)(U)$$

If the cofinality of β is large enough, then this map is bijective for all U.

Proof. The left hand side is the value on U of the colimit $\mathcal{F}_{\text{colim}}$ taken in the category of presheaves, see Section 7.4. Recall that $\text{colim}_{\alpha < \beta} \mathcal{F}_\alpha$ is the sheafification $\mathcal{F}_{\text{colim}}^\#$ of $\mathcal{F}_{\text{colim}}$, see Lemma 7.10.13. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be an element of the set $\text{Cov}(\mathcal{C})$ of coverings of $\mathcal{C}$. If the cofinality of β is larger than the cardinality of I, then we claim

$$H^0(\mathcal{U}, \mathcal{F}_{\text{colim}}) = \text{colim}\, H^0(\mathcal{U}, \mathcal{F}_\alpha) = \text{colim}\, \mathcal{F}_\alpha(U) = \mathcal{F}_{\text{colim}}(U)$$

The second and third equality signs are clear. For the first, say $s = (s_i) \in H^0(\mathcal{U}, \mathcal{F}_{\text{colim}})$. Then for each i the element s_i comes from an element $s_{i,\alpha_i} \in \mathcal{F}_{\alpha_i}(U_i)$ for some $\alpha_i < \beta$. By the assumption on cofinality, we can choose $\alpha_i = \alpha$ independent of i. Then s_i and s_j map to the same element of $\mathcal{F}_{\alpha_{i,j}}(U_i \times_U U_j)$ for some $\alpha_{i,j} < \beta$. Since the cardinality if $I \times I$ is also less than the cofinality of β, we see that we may after increasing α assume $\alpha_{i,j} = \alpha$ for all i, j. This proves that the natural map $\text{colim}\, H^0(\mathcal{U}, \mathcal{F}_\alpha) \to H^0(\mathcal{U}, \mathcal{F}_{\text{colim}})$ is surjective. A very similar argument shows that it is injective. In particular, we see that $\mathcal{F}_{\text{colim}}$ satisfies the sheaf condition for $\mathcal{U}$. Thus if the cofinality of β is larger than the supremum of the cardinalities of the set of index sets I of coverings, then we conclude. □

7.18. Colimits of sites

0EXI We need an analogue of Lemma 7.17.7 in the case that the site is the limit of an inverse system of sites. For simplicity we only explain the construction in case the index sets of coverings are finite.

0A34 Situation 7.18.1. Here we are given

(1) a cofiltered index category $\mathcal{I}$,
(2) for $i \in \text{Ob}(\mathcal{I})$ a site $\mathcal{C}_i$ such that every covering in $\mathcal{C}_i$ has a finite index set,
(3) for a morphism $a : i \to j$ in $\mathcal{I}$ a morphism of sites $f_a : \mathcal{C}_i \to \mathcal{C}_j$ given by a continuous functor $u_a : \mathcal{C}_j \to \mathcal{C}_i$,

such that $f_a \circ f_b = f_c$ whenever $c = a \circ b$ in $\mathcal{I}$.

09YL Lemma 7.18.2. In Situation 7.18.1 we can construct a site $(\mathcal{C}, \text{Cov}(\mathcal{C}))$ as follows

(1) as a category $\mathcal{C} = \text{colim}\, \mathcal{C}_i$, and
(2) $\text{Cov}(\mathcal{C})$ is the union of the images of $\text{Cov}(\mathcal{C}_i)$ by $u_i : \mathcal{C}_i \to \mathcal{C}$.

Proof. Our definition of composition of morphisms of sites implies that $u_b \circ u_a = u_c$ whenever $c = a \circ b$ in $\mathcal{I}$. The formula $\mathcal{C} = \text{colim}\, \mathcal{C}_i$ means that $\text{Ob}(\mathcal{C}) = \text{colim}\, \text{Ob}(\mathcal{C}_i)$ and $\text{Arrows}(\mathcal{C}) = \text{colim}\, \text{Arrows}(\mathcal{C}_i)$. Then source, target, and composition are inherited from the source, target, and composition on $\text{Arrows}(\mathcal{C}_i)$. In this way we obtain a category. Denote $u_i : \mathcal{C}_i \to \mathcal{C}$ the obvious functor. Remark that given any finite diagram in $\mathcal{C}$ there exists an i such that this diagram is the image of a diagram in $\mathcal{C}_i$.

Let $\{U^t \to U\}$ be a covering of $\mathcal{C}$. We first prove that if $V \to U$ is a morphism of $\mathcal{C}$, then $U^t \times_U V$ exists. By our remark above and our definition of coverings, we can find an i, a covering $\{U_i^t \to U_i\}$ of $\mathcal{C}_i$ and a morphism $V_i \to U_i$ whose image by u_i is the given data. We claim that $U^t \times_U V$ is the image of $U_i^t \times_{U_i} V_i$ by u_i. Namely, for every $a : j \to i$ in $\mathcal{I}$ the functor u_a is continuous, hence $u_a(U_i^t \times_{U_i} V_i) = u_a(U_i^t) \times_{u_a(U_i)} u_a(V_i)$. In particular we can replace i by j, if we so desire. Thus, if W is another object of $\mathcal{C}$, then we may assume $W = u_i(W_i)$ and we see that

$$
\begin{aligned}
&\text{Mor}_\mathcal{C}(W, u_i(U_i^t \times_{U_i} V_i)) \\
&= \text{colim}_{a:j\to i} \text{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i^t \times_{U_i} V_i)) \\
&= \text{colim}_{a:j\to i} \text{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i^t)) \times_{\text{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(U_i))} \text{Mor}_{\mathcal{C}_j}(u_a(W_i), u_a(V_i)) \\
&= \text{Mor}_\mathcal{C}(W, U^t) \times_{\text{Mor}_\mathcal{C}(W,U)} \text{Mor}_\mathcal{C}(W, V)
\end{aligned}
$$

as filtered colimits commute with finite limits (Categories, Lemma 4.19.2). It also follows that $\{U^t \times_U V \to V\}$ is a covering in $\mathcal{C}$. In this way we see that axiom (3) of Definition 7.6.2 holds.

To verify axiom (2) of Definition 7.6.2 let $\{U^t \to U\}_{t\in T}$ be a covering of $\mathcal{C}$ and for each t let $\{U^{ts} \to U^t\}$ be a covering of $\mathcal{C}$. Then we can find an i and a covering

$\{U_i^t \to U_i\}_{t \in T}$ of $\mathcal{C}_i$ whose image by u_i is $\{U^t \to U\}$. Since T is finite we may choose an $a : j \to i$ in $\mathcal{I}$ and coverings $\{U_j^{ts} \to u_a(U_i^t)\}$ of $\mathcal{C}_j$ whose image by u_j gives $\{U^{ts} \to U^t\}$. Then we conclude that $\{U^{ts} \to U\}$ is a covering of $\mathcal{C}$ by an application of axiom (2) to the site $\mathcal{C}_j$.

We omit the proof of axiom (1) of Definition 7.6.2. □

0A35 Lemma 7.18.3. In Situation 7.18.1 let $u_i : \mathcal{C}_i \to \mathcal{C}$ be as constructed in Lemma 7.18.2. Then u_i defines a morphism of sites $f_i : \mathcal{C} \to \mathcal{C}_i$. For $U_i \in \mathrm{Ob}(\mathcal{C}_i)$ and sheaf $\mathcal{F}$ on $\mathcal{C}_i$ we have

09YM (7.18.3.1) $$f_i^{-1}\mathcal{F}(u_i(U_i)) = \mathrm{colim}_{a:j\to i}\, f_a^{-1}\mathcal{F}(u_a(U_i))$$

Proof. It is immediate from the arguments in the proof of Lemma 7.18.2 that the functors u_i are continuous. To finish the proof we have to show that $f_i^{-1} := u_{i,s}$ is an exact functor $\mathit{Sh}(\mathcal{C}_i) \to \mathit{Sh}(\mathcal{C})$. In fact it suffices to show that f_i^{-1} is left exact, because it is right exact as a left adjoint (Categories, Lemma 4.24.6). We first prove (7.18.3.1) and then we deduce exactness.

For an arbitrary object V of $\mathcal{C}$ we can pick a $a : j \to i$ and an object $V_j \in \mathrm{Ob}(\mathcal{C})$ with $V = u_j(V_j)$. Then we can set

$$\mathcal{G}(V) = \mathrm{colim}_{b:k\to j}\, f_{a\circ b}^{-1}\mathcal{F}(u_b(V_j))$$

The value $\mathcal{G}(V)$ of the colimit is independent of the choice of $b : j \to i$ and of the object V_j with $u_j(V_j) = V$; we omit the verification. Moreover, if $\alpha : V \to V'$ is a morphism of $\mathcal{C}$, then we can choose $b : j \to i$ and a morphism $\alpha_j : V_j \to V_j'$ with $u_j(\alpha_j) = \alpha$. This induces a map $\mathcal{G}(V') \to \mathcal{G}(V)$ by using the restrictions along the morphisms $u_b(\alpha_j) : u_b(V_j) \to u_b(V_j')$. A check shows that $\mathcal{G}$ is a presheaf (omitted). In fact, $\mathcal{G}$ satisfies the sheaf condition. Namely, any covering $\mathcal{U} = \{U^t \to U\}$ in $\mathcal{C}$ comes from a finite level. Say $\mathcal{U}_j = \{U_j^t \to U_j\}$ is mapped to $\mathcal{U}$ by u_j for some $a : j \to i$ in $\mathcal{I}$. Then we have

$$H^0(\mathcal{U}, \mathcal{G}) = \mathrm{colim}_{b:k\to j}\, H^0(u_b(\mathcal{U}_j), f_{b\circ a}^{-1}\mathcal{F}) = \mathrm{colim}_{b:k\to j}\, f_{b\circ a}^{-1}\mathcal{F}(u_b(U_j)) = \mathcal{G}(U)$$

as desired. The first equality holds because filtered colimits commute with finite limits (Categories, Lemma 4.19.2). By construction $\mathcal{G}(U)$ is given by the right hand side of (7.18.3.1). Hence (7.18.3.1) is true if we can show that $\mathcal{G}$ is equal to $f_i^{-1}\mathcal{F}$.

In this paragraph we check that $\mathcal{G}$ is canonically isomorphic to $f_i^{-1}\mathcal{F}$. We strongly encourage the reader to skip this paragraph. To check this we have to show there is a bijection $\mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(\mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{\mathit{Sh}(\mathcal{C}_i)}(\mathcal{F}, f_{i,*}\mathcal{H})$ functorial in the sheaf $\mathcal{H}$ on $\mathcal{C}$ where $f_{i,*} = u_i^p$. A map $\mathcal{G} \to \mathcal{H}$ is the same thing as a compatible system of maps

$$\varphi_{a,b,V_j} : f_{a\circ b}^{-1}\mathcal{F}(u_b(V_j)) \longrightarrow \mathcal{H}(u_j(V_j))$$

for all $a : j \to i$, $b : k \to j$ and $V_j \in \mathrm{Ob}(\mathcal{C}_j)$. The compatibilities force the maps φ_{a,b,V_j} to be equal to $\varphi_{a\circ b,\mathrm{id},u_b(V_j)}$. Given $a : j \to i$, the family of maps $\varphi_{a,\mathrm{id},V_j}$ corresponds to a map of sheaves $\varphi_a : f_a^{-1}\mathcal{F} \to f_{j,*}\mathcal{H}$. The compatibilities between the $\varphi_{a,\mathrm{id},u_a(V_i)}$ and the $\varphi_{\mathrm{id},\mathrm{id},V_i}$ implies that φ_a is the adjoint of the map φ_{id} via

$$\mathrm{Mor}_{\mathit{Sh}(\mathcal{C}_j)}(f_a^{-1}\mathcal{F}, f_{j,*}\mathcal{H}) = \mathrm{Mor}_{\mathit{Sh}(\mathcal{C}_i)}(\mathcal{F}, f_{a,*}f_{j,*}\mathcal{H}) = \mathrm{Mor}_{\mathit{Sh}(\mathcal{C}_i)}(\mathcal{F}, f_{i,*}\mathcal{H})$$

Thus finally we see that the whole system of maps φ_{a,b,V_j} is determined by the map $\varphi_{id} : \mathcal{F} \to f_{i,*}\mathcal{H}$. Conversely, given such a map $\psi : \mathcal{F} \to f_{i,*}\mathcal{H}$ we can read the argument just given backwards to construct the family of maps φ_{a,b,V_j}. This finishes the proof that $\mathcal{G} = f_i^{-1}\mathcal{F}$.

Assume (7.18.3.1) holds. Then the functor $\mathcal{F} \mapsto f_i^{-1}\mathcal{F}(U)$ commutes with finite limits because finite limits of sheaves are computed in the category of presheaves (Lemma 7.10.1), the functors f_a^{-1} commutes with finite limits, and filtered colimits commute with finite limits. To see that $\mathcal{F} \mapsto f_i^{-1}\mathcal{F}(V)$ commutes with finite limits for a general object V of $\mathcal{C}$, we can use the same argument using the formula for $f_i^{-1}\mathcal{F}(V) = \mathcal{G}(V)$ given above. Thus f_i^{-1} is left exact and the proof of the lemma is complete. $\square$

09YN Lemma 7.18.4. In Situation 7.18.1 assume given

(1) a sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in \text{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ a map $\varphi_a : f_a^{-1}\mathcal{F}_i \to \mathcal{F}_j$ of sheaves on $\mathcal{C}_j$

such that $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \text{colim}\, f_i^{-1}\mathcal{F}_i$ on the site $\mathcal{C}$ of Lemma 7.18.2. Let $i \in \text{Ob}(\mathcal{I})$ and $X_i \in \text{Ob}(\mathcal{C}_i)$. Then

$$\text{colim}_{a:j\to i}\, \mathcal{F}_j(u_a(X_i)) = \mathcal{F}(u_i(X_i))$$

Proof. A formal argument shows that

$$\text{colim}_{a:j\to i}\, \mathcal{F}_i(u_a(X_i)) = \text{colim}_{a:j\to i}\, \text{colim}_{b:k\to j}\, f_b^{-1}\mathcal{F}_j(u_{a\circ b}(X_i))$$

By (7.18.3.1) we see that the inner colimit is equal to $f_j^{-1}\mathcal{F}_j(u_i(X_i))$ hence we conclude by Lemma 7.17.7. $\square$

0EXJ Lemma 7.18.5. In Situation 7.18.1 assume we have a sheaf $\mathcal{F}$ on $\mathcal{C}$. Then

$$\mathcal{F} = \text{colim}\, f_i^{-1}f_{i,*}\mathcal{F}$$

where the transition maps are $f_j^{-1}\varphi_a$ for $a : j \to i$ where $\varphi_a : f_a^{-1}f_{i,*}\mathcal{F} \to f_{j,*}\mathcal{F}$ is a canonical map satisfying a cocycle condition as in Lemma 7.18.4.

Proof. For the morphism

$$\varphi_a : f_a^{-1}f_{i,*}\mathcal{F} \to f_{j,*}\mathcal{F}$$

we choose the adjoint to the identity map

$$f_{i,*}\mathcal{F} \to f_{a,*}f_{j,*}\mathcal{F}$$

Hence φ_a is the counit for the adjunction given by $(f_a^{-1}, f_{a,*})$. We must prove that for all $a : j \to i$ and $b : k \to i$ with composition $c = a \circ b$ we have $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$. This follows from Categories, Lemma 4.24.9. Lastly, we must prove that the map given by adjunction

$$\text{colim}_{i\in I}\, f_i^{-1}f_{i,*}\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. For an object U of $\mathcal{C}$ we need to show the map

$$(\text{colim}_{i\in I}\, f_i^{-1}\mathcal{F}_i)(U) \to \mathcal{F}(U)$$

is bijective. Choose an i and an object U_i of $\mathcal{C}_i$ with $u_i(U_i) = U$. Then the left hand side is equal to

$$(\text{colim}_{i\in I}\, f_i^{-1}\mathcal{F}_i)(U) = \text{colim}_{a:j\to i}\, f_{j,*}\mathcal{F}(u_a(U_i))$$

by Lemma 7.18.4. Since $u_j(u_a(U_i)) = U$ we have $f_{j,*}\mathcal{F}(u_a(U_i)) = \mathcal{F}(U)$ for all $a : j \to i$ by definition. Hence the value of the colimit is $\mathcal{F}(U)$ and the proof is complete. $\square$

7.19. More functoriality of presheaves

00XF In this section we revisit the material of Section 7.5. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Recall that

$$u^p : \mathrm{PSh}(\mathcal{D}) \longrightarrow \mathrm{PSh}(\mathcal{C})$$

is the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. It turns out that this functor not only has a left adjoint (namely u_p) but also a right adjoint.

Namely, for any $V \in \mathrm{Ob}(\mathcal{D})$ we define a category ${}_V\mathcal{I} = {}_V^u\mathcal{I}$. Its objects are pairs $(U, \psi : u(U) \to V)$. Note that the arrow is in the opposite direction from the arrow we used in defining the category $\mathcal{I}_V^u$ in Section 7.5. A morphism $(U, \psi) \to (U', \psi')$ is given by a morphism $\alpha : U \to U'$ such that $\psi = \psi' \circ u(\alpha)$. In addition, given any presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ we introduce the functor ${}_V\mathcal{F} : {}_V\mathcal{I}^{opp} \to \textit{Sets}$, which is defined by the rule ${}_V\mathcal{F}(U, \psi) = \mathcal{F}(U)$. We define

$${}_pu(\mathcal{F})(V) := \lim\nolimits_{{}_V\mathcal{I}^{opp}} {}_V\mathcal{F}$$

As a limit there are projection maps $c(\psi) : {}_pu(\mathcal{F})(V) \to \mathcal{F}(U)$ for every object (U, ψ) of ${}_V\mathcal{I}$. In fact,

$${}_pu(\mathcal{F})(V) = \left\{ \begin{matrix} \text{collections } s_{(U,\psi)} \in \mathcal{F}(U) \\ \forall \beta : (U_1, \psi_1) \to (U_2, \psi_2) \text{ in } {}_V\mathcal{I} \\ \text{we have } \beta^* s_{(U_2,\psi_2)} = s_{(U_1,\psi_1)} \end{matrix} \right\}$$

where the correspondence is given by $s \mapsto s_{(U,\psi)} = c(\psi)(s)$. We leave it to the reader to define the restriction mappings ${}_pu(\mathcal{F})(V) \to {}_pu(\mathcal{F})(V')$ associated to any morphism $V' \to V$ of $\mathcal{D}$. The resulting presheaf will be denoted ${}_pu\mathcal{F}$.

00XG **Lemma 7.19.1.** There is a canonical map ${}_pu\mathcal{F}(u(U)) \to \mathcal{F}(U)$, which is compatible with restriction maps.

Proof. This is just the projection map $c(\mathrm{id}_{u(U)})$ above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors ${}_V\mathcal{F} \to {}_V\mathcal{F}'$, and hence to a map of presheaves ${}_pu\mathcal{F} \to {}_pu\mathcal{F}'$. In other words, we have defined a functor

$${}_pu : \mathrm{PSh}(\mathcal{C}) \longrightarrow \mathrm{PSh}(\mathcal{D})$$

00XH **Lemma 7.19.2.** The functor ${}_pu$ is a right adjoint to the functor u^p. In other words the formula

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F})$$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. This is proved in exactly the same way as the proof of Lemma 7.5.4. We note that the map $u^p{}_pu\mathcal{F} \to \mathcal{F}$ from Lemma 7.19.1 is the map that is used to go from the right to the left.

Alternately, think of a presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ as a presheaf $\mathcal{F}'$ on $\mathcal{C}^{opp}$ with values in $\textit{Sets}^{opp}$, and similarly on $\mathcal{D}$. Check that $({}_pu\mathcal{F})' = u_p(\mathcal{F}')$, and that $(u^p\mathcal{G})' = u^p(\mathcal{G}')$. By Remark 7.5.5 we have the adjointness of u_p and u^p for presheaves with values in $\textit{Sets}^{opp}$. The result then follows formally from this. □

Thus given a functor $u : \mathcal{C} \to \mathcal{D}$ of categories we obtain a sequence of functors

$$u_p, u^p, {}_pu$$

between categories of presheaves where in each consecutive pair the first is left adjoint to the second.

09VQ **Lemma 7.19.3.** Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors of categories. Assume that v is right adjoint to u. Then we have

(1) $u^p h_V = h_{v(V)}$ for any V in $\mathcal{D}$,
(2) the category $\mathcal{I}_U^v$ has an initial object,
(3) the category ${}_V^u\mathcal{I}$ has a final object,
(4) ${}_pu = v^p$, and
(5) $u^p = v_p$.

Proof. Proof of (1). Let V be an object of $\mathcal{D}$. We have $u^p h_V = h_{v(V)}$ because $u^p h_V(U) = \mathrm{Mor}_\mathcal{D}(u(U), V) = \mathrm{Mor}_\mathcal{C}(U, v(V))$ by assumption.

Proof of (2). Let U be an object of $\mathcal{C}$. Let $\eta : U \to v(u(U))$ be the map adjoint to the map $\mathrm{id} : u(U) \to u(U)$. Then we claim $(u(U), \eta)$ is an initial object of $\mathcal{I}_U^v$. Namely, given an object $(V, \phi : U \to v(V))$ of $\mathcal{I}_U^v$ the morphism ϕ is adjoint to a map $\psi : u(U) \to V$ which then defines a morphism $(u(U), \eta) \to (V, \phi)$.

Proof of (3). Let V be an object of $\mathcal{D}$. Let $\xi : u(v(V)) \to V$ be the map adjoint to the map $\mathrm{id} : v(V) \to v(V)$. Then we claim $(v(V), \xi)$ is a final object of ${}_V^u\mathcal{I}$. Namely, given an object $(U, \psi : u(U) \to V)$ of ${}_V^u\mathcal{I}$ the morphism ψ is adjoint to a map $\phi : U \to v(V)$ which then defines a morphism $(U, \psi) \to (v(V), \xi)$.

Hence for any presheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} v^p\mathcal{F}(V) &= \mathcal{F}(v(V)) \\ &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_{v(V)}, \mathcal{F}) \\ &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(u^p h_V, \mathcal{F}) \\ &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(h_V, {}_pu\mathcal{F}) \\ &= {}_pu\mathcal{F}(V) \end{aligned}$$

which proves part (4). Part (5) follows by the uniqueness of adjoint functors. ☐

09VR **Lemma 7.19.4.** A continuous functor of sites which has a continuous left adjoint defines a morphism of sites.

Proof. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $w : \mathcal{D} \to \mathcal{C}$ be a continuous left adjoint. Then $u_p = w^p$ by Lemma 7.19.3. Hence $u_s = w^s$ has a left adjoint, namely w_s (Lemma 7.13.3). Thus u_s has both a right and a left adjoint, whence is exact (Categories, Lemma 4.24.6). ☐

7.20. Cocontinuous functors

00XI There is another way to construct morphisms of topoi. This involves using cocontinuous functors between sites defined as follows.

00XJ **Definition 7.20.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. The functor u is called *cocontinuous* if for every $U \in \mathrm{Ob}(\mathcal{C})$ and every covering $\{V_j \to u(U)\}_{j\in J}$ of $\mathcal{D}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that the family of maps $\{u(U_i) \to u(U)\}_{i\in I}$ refines the covering $\{V_j \to u(U)\}_{j\in J}$.

Note that $\{u(U_i) \to u(U)\}_{i \in I}$ is in general not a covering of the site $\mathcal{D}$.

00XK Lemma 7.20.2. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then ${}_pu\mathcal{F}$ is a sheaf on $\mathcal{D}$, which we will denote ${}_su\mathcal{F}$.

Proof. Let $\{V_j \to V\}_{j \in J}$ be a covering of the site $\mathcal{D}$. We have to show that

$$\begin{tikzcd} {}_pu\mathcal{F}(V) \arrow[r] & \prod {}_pu\mathcal{F}(V_j) \arrow[r, shift left] \arrow[r, shift right] & \prod {}_pu\mathcal{F}(V_j \times_V V_{j'}) \end{tikzcd}$$

is an equalizer diagram. Since ${}_pu$ is right adjoint to u^p we have

$${}_pu\mathcal{F}(V) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(h_V, {}_pu\mathcal{F}) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(u^p h_V, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}((u^p h_V)^\#, \mathcal{F})$$

Hence it suffices to show that

07GF (7.20.2.1)
$$\begin{tikzcd} \coprod u^p h_{V_j \times_V V_{j'}} \arrow[r, shift left] \arrow[r, shift right] & \coprod u^p h_{V_j} \arrow[r] & u^p h_V \end{tikzcd}$$

becomes a coequalizer diagram after sheafification. (Recall that a coproduct in the category of sheaves is the sheafification of the coproduct in the category of presheaves, see Lemma 7.10.13.)

We first show that the second arrow of (7.20.2.1) becomes surjective after sheafification. To do this we use Lemma 7.11.2. Thus it suffices to show a section s of $u^p h_V$ over U lifts to a section of $\coprod u^p h_{V_j}$ on the members of a covering of U. Note that s is a morphism $s : u(U) \to V$. Then $\{V_j \times_{V,s} u(U) \to u(U)\}$ is a covering of $\mathcal{D}$. Hence, as u is cocontinuous, there is a covering $\{U_i \to U\}$ such that $\{u(U_i) \to u(U)\}$ refines $\{V_j \times_{V,s} u(U) \to u(U)\}$. This means that each restriction $s|_{U_i} : u(U_i) \to V$ factors through a morphism $s_i : u(U_i) \to V_j$ for some j, i.e., $s|_{U_i}$ is in the image of $u^p h_{V_j}(U_i) \to u^p h_V(U_i)$ as desired.

Let $s, s' \in (\coprod u^p h_{V_j})^\#(U)$ map to the same element of $(u^p h_V)^\#(U)$. To finish the proof of the lemma we show that after replacing U by the members of a covering that s, s' are the image of the same section of $\coprod u^p h_{V_j \times_V V_{j'}}$ by the two maps of (7.20.2.1). We may first replace U by the members of a covering and assume that $s \in u^p h_{V_j}(U)$ and $s' \in u^p h_{V_{j'}}(U)$. A second such replacement guarantees that s and s' have the same image in $u^p h_V(U)$ instead of in the sheafification. Hence $s : u(U) \to V_j$ and $s' : u(U) \to V_{j'}$ are morphisms of $\mathcal{D}$ such that

$$\begin{tikzcd} u(U) \arrow[r, "s'"'] \arrow[d, "s"'] & V_{j'} \arrow[d] \\ V_j \arrow[r] & V \end{tikzcd}$$

is commutative. Thus we obtain $t = (s, s') : u(U) \to V_j \times_V V_{j'}$, i.e., a section $t \in u^p h_{V_j \times_V V_{j'}}(U)$ which maps to s, s' as desired. $\square$

00XL Lemma 7.20.3. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functor $Sh(\mathcal{D}) \to Sh(\mathcal{C})$, $\mathcal{G} \mapsto (u^p\mathcal{G})^\#$ is a left adjoint to the functor ${}_su$ introduced in Lemma 7.20.2 above. Moreover, it is exact.

Proof. Let us prove the adjointness property as follows

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F}) &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, {}_su\mathcal{F}). \end{aligned}$$

Thus it is a left adjoint and hence right exact, see Categories, Lemma 4.24.6. We have seen that sheafification is left exact, see Lemma 7.10.14. Moreover, the inclusion $i : Sh(\mathcal{D}) \to \mathrm{PSh}(\mathcal{D})$ is left exact by Lemma 7.10.1. Finally, the functor u^p is left exact because it is a right adjoint (namely to u_p). Thus the functor is the composition ${}^\# \circ u^p \circ i$ of left exact functors, hence left exact. □

We finish this section with a technical lemma.

00XM Lemma 7.20.4. In the situation of Lemma 7.20.3. For any presheaf $\mathcal{G}$ on $\mathcal{D}$ we have $(u^p\mathcal{G})^\# = (u^p(\mathcal{G}^\#))^\#$.

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}((u^p(\mathcal{G}^\#))^\#, \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_s u\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_p u\mathcal{F}) \\ &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{G}, {}_p u\mathcal{F}) \\ &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F}) \end{aligned}$$

and the result follows from the Yoneda lemma. □

09W7 Remark 7.20.5. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Given morphisms $g : u(U) \to V$ and $f : W \to V$ in $\mathcal{D}$ we can consider the functor

$$\mathcal{C}^{opp} \longrightarrow \mathrm{Sets}, \quad T \longmapsto \mathrm{Mor}_\mathcal{C}(T, U) \times_{\mathrm{Mor}_\mathcal{D}(u(T), V)} \mathrm{Mor}_\mathcal{D}(u(T), W)$$

If this functor is representable, denote $U \times_{g,V,f} W$ the corresponding object of $\mathcal{C}$. Assume that $\mathcal{C}$ and $\mathcal{D}$ are sites. Consider the property P: for every covering $\{f_j : V_j \to V\}$ of $\mathcal{D}$ and any morphism $g : u(U) \to V$ we have

(1) $U \times_{g,V,f_i} V_i$ exists for all i, and
(2) $\{U \times_{g,V,f_i} V_i \to U\}$ is a covering of $\mathcal{C}$.

Please note the similarity with the definition of continuous functors. If u has P then u is cocontinuous (details omitted). Many of the cocontinuous functors we will encounter satisfy P.

7.21. Cocontinuous functors and morphisms of topoi

00XN It is clear from the above that a cocontinuous functor u gives a morphism of topoi in the same direction as u. Thus this is in the opposite direction from the morphism of topoi associated (under certain conditions) to a continuous u as in Definition 7.14.1, Proposition 7.14.7, and Lemma 7.15.2.

00XO Lemma 7.21.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functors $g_* = {}_s u$ and $g^{-1} = (u^p\ )^\#$ define a morphism of topoi g from $Sh(\mathcal{C})$ to $Sh(\mathcal{D})$.

Proof. This is exactly the content of Lemma 7.20.3. □

03L5 Lemma 7.21.2. Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{E}$ be cocontinuous functors. Then $v \circ u$ is cocontinuous and we have $h = g \circ f$ where $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, resp. $g : Sh(\mathcal{D}) \to Sh(\mathcal{E})$, resp. $h : Sh(\mathcal{C}) \to Sh(\mathcal{E})$ is the morphism of topoi associated to u, resp. v, resp. $v \circ u$.

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $\{E_i \to v(u(U))\}$ be a covering of U in $\mathcal{E}$. By assumption there exists a covering $\{D_j \to u(U)\}$ in $\mathcal{D}$ such that $\{v(D_j) \to v(u(U))\}$ refines $\{E_i \to v(u(U))\}$. Also by assumption there exists a covering $\{C_l \to U\}$ in $\mathcal{C}$ such that $\{u(C_l) \to u(U)\}$ refines $\{D_j \to u(U)\}$. Then it is true that $\{v(u(C_l)) \to v(u(U))\}$ refines the covering $\{E_i \to v(u(U))\}$. This proves that $v \circ u$ is cocontinuous. To prove the last assertion it suffices to show that ${}_sv \circ {}_su = {}_s(v \circ u)$. It suffices to prove that ${}_pv \circ {}_pu = {}_p(v \circ u)$, see Lemma 7.20.2. Since ${}_pu$, resp. ${}_pv$, resp. ${}_p(v \circ u)$ is right adjoint to u^p, resp. v^p, resp. $(v \circ u)^p$ it suffices to prove that $u^p \circ v^p = (v \circ u)^p$. And this is direct from the definitions. □

00XP Example 7.21.3. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subspace. Recall that we have sites X_{Zar} and U_{Zar}, see Example 7.6.4. Recall that we have the functor $u : X_{Zar} \to U_{Zar}$ associated to j which is continuous and gives rise to a morphism of sites $U_{Zar} \to X_{Zar}$, see Example 7.14.2. This also gives a morphism of topoi (j_*, j^{-1}). Next, consider the functor $v : U_{Zar} \to X_{Zar}$, $V \mapsto v(V) = V$ (just the same open but now thought of as an object of X_{Zar}). This functor is cocontinuous. Namely, if $v(V) = \bigcup_{j\in J} W_j$ is an open covering in X, then each W_j must be a subset of U and hence is of the form $v(V_j)$, and trivially $V = \bigcup_{j \in J} V_j$ is an open covering in U. We conclude by Lemma 7.21.1 above that there is a morphism of topoi associated to v

$$Sh(U) \longrightarrow Sh(X)$$

given by ${}_sv$ and $(v^p\ )^\#$. We claim that actually $(v^p\ )^\# = j^{-1}$ and that ${}_sv = j_*$, in other words, that this is the same morphism of topoi as the one given above. Perhaps the easiest way to see this is to realize that for any sheaf $\mathcal{G}$ on X we have $v^p\mathcal{G}(V) = \mathcal{G}(V)$ which according to Sheaves, Lemma 6.31.1 is a description of $j^{-1}\mathcal{G}$ (and hence sheafification is superfluous in this case). The equality of ${}_sv$ and j_* follows by uniqueness of adjoint functors (but may also be computed directly).

00XQ Example 7.21.4. This example is a slight generalization of Example 7.21.3. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that f is open. Recall that we have sites X_{Zar} and Y_{Zar}, see Example 7.6.4. Recall that we have the functor $u : Y_{Zar} \to X_{Zar}$ associated to f which is continuous and gives rise to a morphism of sites $X_{Zar} \to Y_{Zar}$, see Example 7.14.2. This also gives a morphism of topoi (f_*, f^{-1}). Next, consider the functor $v : X_{Zar} \to Y_{Zar}$, $U \mapsto v(U) = f(U)$. This functor is cocontinuous. Namely, if $f(U) = \bigcup_{j\in J} V_j$ is an open covering in Y, then setting $U_j = f^{-1}(V_j) \cap U$ we get an open covering $U = \bigcup U_j$ such that $f(U) = \bigcup f(U_j)$ is a refinement of $f(U) = \bigcup V_j$. We conclude by Lemma 7.21.1 above that there is a morphism of topoi associated to v

$$Sh(X) \longrightarrow Sh(Y)$$

given by ${}_sv$ and $(v^p\ )^\#$. We claim that actually $(v^p\ )^\# = f^{-1}$ and that ${}_sv = f_*$, in other words, that this is the same morphism of topoi as the one given above. For any sheaf $\mathcal{G}$ on Y we have $v^p\mathcal{G}(U) = \mathcal{G}(f(U))$. On the other hand, we may compute $u_p\mathcal{G}(U) = \mathrm{colim}_{f(U)\subset V}\, \mathcal{G}(V) = \mathcal{G}(f(U))$ because clearly $(f(U), U \subset f^{-1}(f(U)))$ is an initial object of the category $\mathcal{I}^u_U$ of Section 7.5. Hence $u_p = v^p$ and we conclude $f^{-1} = u_s = (v^p\ )^\#$. The equality of ${}_sv$ and f_* follows by uniqueness of adjoint functors (but may also be computed directly).

In the first Example 7.21.3 the functor v is also continuous. But in the second Example 7.21.4 it is generally not continuous because condition (2) of Definition

7.13.1 may fail. Hence the following lemma applies to the first example, but not to the second.

00XR Lemma 7.21.5. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(a) u is cocontinuous, and
(b) u is continuous.

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. Then

(1) sheafification in the formula $g^{-1} = (u^p\ )^\#$ is unnecessary, in other words $g^{-1}(\mathcal{G})(U) = \mathcal{G}(u(U))$,
(2) g^{-1} has a left adjoint $g_! = (u_p\ )^\#$, and
(3) g^{-1} commutes with arbitrary limits and colimits.

Proof. By Lemma 7.13.2 for any sheaf $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G}$ is a sheaf on $\mathcal{C}$. And then we see the adjointness by the following string of equalities

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) \\ &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

The statement on limits and colimits follows from the discussion in Categories, Section 4.24. □

In the situation of Lemma 7.21.5 above we see that we have a sequence of adjoint functors

$$g_!,\ g^{-1},\ g_*.$$

The functor $g_!$ is not exact in general, because it does not transform a final object of $Sh(\mathcal{C})$ into a final object of $Sh(\mathcal{D})$ in general. See Sheaves, Remark 6.31.13. On the other hand, in the topological setting of Example 7.21.3 the functor $j_!$ is exact on abelian sheaves, see Modules, Lemma 17.3.4. The following lemma gives the generalization to the case of sites.

00XS Lemma 7.21.6. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(a) u is cocontinuous,
(b) u is continuous, and
(c) fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.

In this case the functor $g_!$ above commutes with fibre products and equalizers (and more generally with finite connected limits).

Proof. Assume (a), (b), and (c). We have $g_! = (u_p\ )^\#$. Recall (Lemma 7.10.1) that limits of sheaves are equal to the corresponding limits as presheaves. And sheafification commutes with finite limits (Lemma 7.10.14). Thus it suffices to show that u_p commutes with fibre products and equalizers. To do this it suffices that colimits over the categories $(\mathcal{I}_V^u)^{opp}$ of Section 7.5 commute with fibre products and equalizers. This follows from Lemma 7.5.1 and Categories, Lemma 4.19.9. □

The following lemma deals with a case that is even more like the morphism associated to an open immersion of topological spaces.

00XT Lemma 7.21.7. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(a) u is cocontinuous,
(b) u is continuous, and
(c) u is fully faithful.

For $g_!, g^{-1}, g_*$ as above the canonical maps $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ and $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ are isomorphisms for all sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. Let X be an object of $\mathcal{C}$. In Lemmas 7.20.2 and 7.21.5 we have seen that sheafification is not necessary for the functors $g^{-1} = (u^p\ )^\#$ and $g_* = ({}_pu\ )^\#$. We may compute $(g^{-1}g_*\mathcal{F})(X) = g_*\mathcal{F}(u(X)) = \lim \mathcal{F}(Y)$. Here the limit is over the category of pairs $(Y, u(Y) \to u(X))$ where the morphisms $u(Y) \to u(X)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the limit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. This proves that $g^{-1}g_*\mathcal{F} = \mathcal{F}$.

On the other hand, $(g^{-1}g_!\mathcal{F})(X) = g_!\mathcal{F}(u(X)) = (u_p\mathcal{F})^\#(u(X))$, and $u_p\mathcal{F}(u(X)) = \mathrm{colim}\, \mathcal{F}(Y)$. Here the colimit is over the category of pairs $(Y, u(X) \to u(Y))$ where the morphisms $u(X) \to u(Y)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the colimit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. Thus for every $X \in \mathrm{Ob}(\mathcal{C})$ we have $u_p\mathcal{F}(u(X)) = \mathcal{F}(X)$. Since u is cocontinuous and continuous any covering of $u(X)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(X_i) \to u(X)\}$ of $\mathcal{D}$ where $\{X_i \to X\}$ is a covering in $\mathcal{C}$. This implies that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ also, since in the colimit defining the value of $(u_p\mathcal{F})^+$ on $u(X)$ we may restrict to the cofinal system of coverings $\{u(X_i) \to u(X)\}$ as above. Hence we see that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(u_p\mathcal{F})^\#(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

Finally, here is a case that does not have any corresponding topological example. We will use this lemma to see what happens when we enlarge a "partial universe" of schemes keeping the same topology. In the situation of the lemma, the morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ identifies $Sh(\mathcal{C})$ as a subtopos of $Sh(\mathcal{D})$ (Section 7.43) and moreover, the given embedding has a retraction.

00XU Lemma 7.21.8. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(a) u is cocontinuous,
(b) u is continuous,
(c) u is fully faithful,
(d) fibre products exist in $\mathcal{C}$ and u commutes with them, and
(e) there exist final objects $e_\mathcal{C} \in \mathrm{Ob}(\mathcal{C})$, $e_\mathcal{D} \in \mathrm{Ob}(\mathcal{D})$ such that $u(e_\mathcal{C}) = e_\mathcal{D}$.

Let $g_!, g^{-1}, g_*$ be as above. Then, u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ with $f_* = g^{-1}$, $f^{-1} = g_!$. The composition

$$Sh(\mathcal{C}) \xrightarrow{g} Sh(\mathcal{D}) \xrightarrow{f} Sh(\mathcal{C})$$

is isomorphic to the identity morphism of the topos $Sh(\mathcal{C})$. Moreover, the functor f^{-1} is fully faithful.

Proof. By assumption the functor u satisfies the hypotheses of Proposition 7.14.7. Hence u defines a morphism of sites and hence a morphism of topoi f as in Lemma 7.15.2. The formulas $f_* = g^{-1}$ and $f^{-1} = g_!$ are clear from the lemma cited and Lemma 7.21.5. We have $f_* \circ g_* = g^{-1} \circ g_* \cong \mathrm{id}$, and $g^{-1} \circ f^{-1} = g^{-1} \circ g_! \cong \mathrm{id}$ by Lemma 7.21.7.

We still have to show that f^{-1} is fully faithful. Let $\mathcal{F}, \mathcal{G} \in \text{Ob}(Sh(\mathcal{C}))$. We have to show that the map

$$\text{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G})$$

is bijective. But the right hand side is equal to

$$\begin{aligned}\text{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G}) &= \text{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, f_*f^{-1}\mathcal{G}) \\ &= \text{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, g^{-1}f^{-1}\mathcal{G}) \\ &= \text{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G})\end{aligned}$$

(the first equality by adjunction) which proves what we want. $\square$

00XV Example 7.21.9. Let X be a topological space. Let $i : Z \to X$ be the inclusion of a subset (with induced topology). Consider the functor $u : X_{Zar} \to Z_{Zar}$, $U \mapsto u(U) = Z \cap U$. At first glance it may appear that this functor is cocontinuous as well. After all, since Z has the induced topology, shouldn't any covering of $U \cap Z$ it come from a covering of U in X? Not so! Namely, what if $U \cap Z = \emptyset$? In that case, the empty covering is a covering of $U \cap Z$, and the empty covering can only be refined by the empty covering. Thus we conclude that u cocontinuous $\Rightarrow$ every nonempty open U of X has nonempty intersection with Z. But this is not sufficient. For example, if $X = \mathbf{R}$ the real number line with the usual topology, and $Z = \mathbf{R} \setminus \{0\}$, then there is an open covering of Z, namely $Z = \{x < 0\} \cup \bigcup_n \{1/n < x\}$ which cannot be refined by the restriction of any open covering of X.

7.22. Cocontinuous functors which have a right adjoint

00XW It may happen that a cocontinuous functor u has a right adjoint v. In this case it is often the case that v is continuous, and if so, then it defines a morphism of topoi (which is the same as the one defined by u).

00XX Lemma 7.22.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{C}$ be functors. Assume that u is cocontinuous, and that v is a right adjoint to u. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to u, see Lemma 7.21.1. Then $g_*\mathcal{F}$ is equal to the presheaf $v^p\mathcal{F}$, in other words, $(g_*\mathcal{F})(V) = \mathcal{F}(v(V))$.

Proof. We have $u^p h_V = h_{v(V)}$ by Lemma 7.19.3. By Lemma 7.20.4 this implies that $g^{-1}(h_V^\#) = (u^p h_V^\#)^\# = (u^p h_V)^\# = h_{v(V)}^\#$. Hence for any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned}(g_*\mathcal{F})(V) &= \text{Mor}_{Sh(\mathcal{D})}(h_V^\#, g_*\mathcal{F}) \\ &= \text{Mor}_{Sh(\mathcal{C})}(g^{-1}(h_V^\#), \mathcal{F}) \\ &= \text{Mor}_{Sh(\mathcal{C})}(h_{v(V)}^\#, \mathcal{F}) \\ &= \mathcal{F}(v(V))\end{aligned}$$

which proves the lemma. $\square$

In the situation of Lemma 7.22.1 we see that v^p transforms sheaves into sheaves. Hence we can define $v^s = v^p$ restricted to sheaves. Just as in Lemma 7.13.3 we see that $v_s : \mathcal{G} \mapsto (v_p\mathcal{G})^\#$ is a left adjoint to v^s. On the other hand, we have $v^s = g_*$ and g^{-1} is a left adjoint of g_* as well. We conclude that $g^{-1} = v_s$ is exact.

00XY Lemma 7.22.2. In the situation of Lemma 7.22.1. We have $g_* = v^s = v^p$ and $g^{-1} = v_s = (v_p\ )^\#$. If v is continuous then v defines a morphism of sites f from $\mathcal{C}$ to $\mathcal{D}$ whose associated morphism of topoi is equal to the morphism g associated to the cocontinuous functor u. In other words, a continuous functor which has a cocontinuous left adjoint defines a morphism of sites.

Proof. Clear from the discussion above the lemma and Definitions 7.14.1 and Lemma 7.15.2. $\square$

0EWJ Example 7.22.3. This example continues the discussion of Example 7.14.3 from which we borrow the notation $\mathcal{C}, \tau, \tau', \epsilon$. Observe that the identity functor $v : \mathcal{C}_{\tau'} \to \mathcal{C}_\tau$ is a continuous functor and the identity functor $u : \mathcal{C}_\tau \to \mathcal{C}_{\tau'}$ is a cocontinuous functor. Moreover u is left adjoint to v. Hence the results of Lemmas 7.22.1 and 7.22.2 apply and we conclude v defines a morphism of sites, namely

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

whose corresponding morphism of topoi is the same as the morphism of topoi associated to the cocontinuous functor u.

7.23. Cocontinuous functors which have a left adjoint

08NG It may happen that a cocontinuous functor u has a left adjoint w.

08NH Lemma 7.23.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to a continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$, see Lemmas 7.21.1 and 7.21.5.

(1) If $w : \mathcal{D} \to \mathcal{C}$ is a left adjoint to u, then
   (a) $g_!\mathcal{F}$ is the sheaf associated to the presheaf $w^p\mathcal{F}$, and
   (b) $g_!$ is exact.
(2) if w is a continuous left adjoint, then $g_!$ has a left adjoint.
(3) If w is a cocontinuous left adjoint, then $g_! = h^{-1}$ and $g^{-1} = h_*$ where $h : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to w.

Proof. Recall that $g_!\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$. Hence (1)(a) follows from the fact that $u_p = w^p$ by Lemma 7.19.3.

To see (1)(b) note that $g_!$ commutes with all colimits as $g_!$ is a left adjoint (Categories, Lemma 4.24.5). Let $i \mapsto \mathcal{F}_i$ be a finite diagram in $Sh(\mathcal{C})$. Then $\lim \mathcal{F}_i$ is computed in the category of presheaves (Lemma 7.10.1). Since w^p is a right adjoint (Lemma 7.5.4) we see that $w^p \lim \mathcal{F}_i = \lim w^p\mathcal{F}_i$. Since sheafification is exact (Lemma 7.10.14) we conclude by (1)(a).

Assume w is continuous. Then $g_! = (w^p\ )^\# = w^s$ but sheafification isn't necessary and one has the left adjoint w_s, see Lemmas 7.13.2 and 7.13.3.

Assume w is cocontinuous. The equality $g_! = h^{-1}$ follows from (1)(a) and the definitions. The equality $g^{-1} = h_*$ follows from the equality $g_! = h^{-1}$ and uniqueness of adjoint functor. Alternatively one can deduce it from Lemma 7.22.1. $\square$

7.24. Existence of lower shriek

09YW In this section we discuss some cases of morphisms of topoi f for which f^{-1} has a left adjoint $f_!$.

09YX **Lemma 7.24.1.** Let $\mathcal{C}$, $\mathcal{D}$ be two sites. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $E \subset \mathrm{Ob}(\mathcal{D})$ be a subset such that

(1) for $V \in E$ there exists a sheaf $\mathcal{G}$ on $\mathcal{C}$ such that $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{F})$ functorially for $\mathcal{F}$ in $Sh(\mathcal{C})$,
(2) every object of $\mathcal{D}$ has a covering by objects of E.

Then f^{-1} has a left adjoint $f_!$.

Proof. By the Yoneda lemma (Categories, Lemma 4.3.5) the sheaf $\mathcal{G}_V$ corresponding to $V \in E$ is defined up to unique isomorphism by the formula $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}_V, \mathcal{F})$. Recall that $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f^{-1}\mathcal{F})$. Denote $i_V : h_V^\# \to f^{-1}\mathcal{G}_V$ the map corresponding to id in $\mathrm{Mor}(\mathcal{G}_V, \mathcal{G}_V)$. Functoriality in (1) implies that the bijection is given by

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}_V, \mathcal{F}) \to \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f^{-1}\mathcal{F}), \quad \varphi \mapsto f^{-1}\varphi \circ i_V$$

For any $V_1, V_2 \in E$ there is a canonical map

$$\mathrm{Mor}_{Sh(\mathcal{D})}(h_{V_2}^\#, h_{V_1}^\#) \to \mathrm{Hom}_{Sh(\mathcal{C})}(\mathcal{G}_{V_2}, \mathcal{G}_{V_1}), \quad \varphi \mapsto f_!(\varphi)$$

which is characterized by $f^{-1}(f_!(\varphi)) \circ i_{V_2} = i_{V_1} \circ \varphi$. Note that $\varphi \mapsto f_!(\varphi)$ is compatible with composition; this can be seen directly from the characterization. Hence $h_V^\# \mapsto \mathcal{G}_V$ and $\varphi \mapsto f_!\varphi$ is a functor from the full subcategory of $Sh(\mathcal{D})$ whose objects are the $h_V^\#$.

Let J be a set and let $J \to E$, $j \mapsto V_j$ be a map. Then we have a functorial bijection

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\coprod \mathcal{G}_{V_j}, \mathcal{F}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{D})}(\coprod h_{V_j}^\#, f^{-1}\mathcal{F})$$

using the product of the bijections above. Hence we can extend the functor $f_!$ to the full subcategory of $Sh(\mathcal{D})$ whose objects are coproducts of $h_V^\#$ with $V \in E$.

Given an arbitrary sheaf $\mathcal{H}$ on $\mathcal{D}$ we choose a coequalizer diagram

$$\mathcal{H}_1 \rightrightarrows \mathcal{H}_0 \longrightarrow \mathcal{H}$$

where $\mathcal{H}_i = \coprod h_{V_{i,j}}^\#$ is a coproduct with $V_{i,j} \in E$. This is possible by assumption (2), see Lemma 7.12.5 (for those worried about set theoretical issues, note that the construction given in Lemma 7.12.5 is canonical). Define $f_!(\mathcal{H})$ to be the sheaf on $\mathcal{C}$ which makes

$$f_!\mathcal{H}_1 \rightrightarrows f_!\mathcal{H}_0 \longrightarrow f_!\mathcal{H}$$

a coequalizer diagram. Then

$$\begin{aligned}
\mathrm{Mor}(f_!\mathcal{H}, \mathcal{F}) &= \mathrm{Equalizer}(\ \mathrm{Mor}(f_!\mathcal{H}_0, \mathcal{F}) \rightrightarrows \mathrm{Mor}(f_!\mathcal{H}_1, \mathcal{F})\ ) \\
&= \mathrm{Equalizer}(\ \mathrm{Mor}(\mathcal{H}_0, f^{-1}\mathcal{F}) \rightrightarrows \mathrm{Mor}(\mathcal{H}_1, f^{-1}\mathcal{F})\ ) \\
&= \mathrm{Hom}(\mathcal{H}, f^{-1}\mathcal{F})
\end{aligned}$$

Hence we see that we can extend $f_!$ to the whole category of sheaves on $\mathcal{D}$. $\square$

7.25. Localization

00XZ Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. See Categories, Example 4.2.13 for the definition of the category $\mathcal{C}/U$ of objects over U. We turn $\mathcal{C}/U$ into a site by declaring a family of morphisms $\{V_j \to V\}$ of objects over U to be a covering of $\mathcal{C}/U$ if and only if it is a covering in $\mathcal{C}$. Consider the forgetful functor

$$j_U : \mathcal{C}/U \longrightarrow \mathcal{C}.$$

This is clearly cocontinuous and continuous. Hence by the results of the previous sections we obtain a morphism of topoi

$$j_U : \mathit{Sh}(\mathcal{C}/U) \longrightarrow \mathit{Sh}(\mathcal{C})$$

given by j_U^{-1} and j_{U*}, as well as a functor $j_{U!}$.

00Y0 Definition 7.25.1. Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$.

(1) The site $\mathcal{C}/U$ is called the localization of the site $\mathcal{C}$ at the object U.
(2) The morphism of topoi $j_U : \mathit{Sh}(\mathcal{C}/U) \to \mathit{Sh}(\mathcal{C})$ is called the localization morphism.
(3) The functor j_{U*} is called the direct image functor.
(4) For a sheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^{-1}\mathcal{F}$ is called the restriction of $\mathcal{F}$ to $\mathcal{C}/U$.
(5) For a sheaf $\mathcal{G}$ on $\mathcal{C}/U$ the sheaf $j_{U!}\mathcal{G}$ is called the extension of $\mathcal{G}$ by the empty set.

The restriction $j_U^{-1}\mathcal{F}$ is the sheaf defined by the rule $j_U^{-1}\mathcal{F}(X/U) = \mathcal{F}(X)$ as expected. The extension by the empty set also has a very easy description in this case; here it is.

03CD Lemma 7.25.2. Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}/U$. Then $j_{U!}(\mathcal{G}^\#)$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings.

Proof. By Lemma 7.21.5 we have $j_{U!}(\mathcal{G}^\#) = ((j_U)_p\mathcal{G}^\#)^\#$. By Lemma 7.13.4 this is equal to $((j_U)_p\mathcal{G})^\#$. Hence it suffices to prove that $(j_U)_p$ is given by the formula above for any presheaf $\mathcal{G}$ on $\mathcal{C}/U$. OK, and by the definition in Section 7.5 we have

$$(j_U)_p\mathcal{G}(V) = \mathrm{colim}_{(W/U, V \to W)}\, \mathcal{G}(W)$$

Now it is clear that the category of pairs $(W/U, V \to W)$ has an object $O_\varphi = (\varphi : V \to U, \mathrm{id} : V \to V)$ for every $\varphi : V \to U$, and moreover for any object there is a unique morphism from one of the O_φ into it. The result follows. □

03HU Lemma 7.25.3. Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let X/U be an object of $\mathcal{C}/U$. Then we have $j_{U!}(h_{X/U}^\#) = h_X^\#$.

Proof. Denote $p : X \to U$ the structure morphism of X. By Lemma 7.25.2 we see $j_{U!}(h_{X/U}^\#)$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \{\psi : V \to X \mid p \circ \psi = \varphi\}$$

This is clearly the same thing as $\mathrm{Mor}_\mathcal{C}(V, X)$. Hence the lemma follows. □

We have $j_{U!}(*) = h_U^\#$ by either of the two lemmas above. Hence for every sheaf $\mathcal{G}$ over $\mathcal{C}/U$ there is a canonical map of sheaves $j_{U!}\mathcal{G} \to h_U^\#$. This characterizes sheaves in the essential image of $j_{U!}$.

00Y1 Lemma 7.25.4. Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ gives an equivalence of categories

$$Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})/h_U^\#$$

Proof. Let us denote objects of $\mathcal{C}/U$ as pairs (X, a) where X is an object of $\mathcal{C}$ and $a : X \to U$ is a morphism of $\mathcal{C}$. Similarly, objects of $Sh(\mathcal{C})/h_U^\#$ are pairs $(\mathcal{F}, \varphi)$. The functor $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$ sends $\mathcal{G}$ to the pair $(j_{U!}\mathcal{G}, \gamma)$ where γ is the composition of $j_{U!}\mathcal{G} \to j_{U!}*$ with the identification $j_{U!}* = h_U^\#$.

Let us construct a functor from $Sh(\mathcal{C})/h_U^\#$ to $Sh(\mathcal{C}/U)$. Suppose that $(\mathcal{F}, \varphi)$ is given. For an object (X, a) of $\mathcal{C}/U$ we consider the set $\mathcal{F}_\varphi(X, a)$ of elements $s \in \mathcal{F}(X)$ which under φ map to the image of $a \in \mathrm{Mor}_\mathcal{C}(X, U) = h_U(X)$ in $h_U^\#(X)$. It is easy to see that $(X, a) \mapsto \mathcal{F}_\varphi(X, a)$ is a sheaf on $\mathcal{C}/U$. Clearly, the rule $(\mathcal{F}, \varphi) \mapsto \mathcal{F}_\varphi$ defines a functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C}/U)$.

Consider also the functor $\mathrm{PSh}(\mathcal{C})/h_U \to \mathrm{PSh}(\mathcal{C}/U)$, $(\mathcal{F}, \varphi) \mapsto \mathcal{F}_\varphi$ where $\mathcal{F}_\varphi(X, a)$ is defined as the set of elements of $\mathcal{F}(X)$ mapping to $a \in h_U(X)$. We claim that the diagram

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{C})/h_U \arrow[r] \arrow[d] & \mathrm{PSh}(\mathcal{C}/U) \arrow[d] \\ Sh(\mathcal{C})/h_U^\# \arrow[r] & Sh(\mathcal{C}/U) \end{tikzcd}$$

commutes, where the vertical arrows are given by sheafification. To see this[5], it suffices to prove that the construction commutes with the functor $\mathcal{F} \mapsto \mathcal{F}^+$ of Lemmas 7.10.3 and 7.10.4 and Theorem 7.10.10. Commutation with $\mathcal{F} \mapsto \mathcal{F}^+$ follows from the fact that given (X, a) the categories of coverings of (X, a) in $\mathcal{C}/U$ and coverings of X in $\mathcal{C}$ are canonically identified.

Next, let $\mathrm{PSh}(\mathcal{C}/U) \to \mathrm{PSh}(\mathcal{C})/h_U$ send $\mathcal{G}$ to the pair $(j_{U!}^{PSh}\mathcal{G}, \gamma)$ where $j_{U!}^{PSh}\mathcal{G}$ the presheaf defined by the formula in Lemma 7.25.2 and γ is the composition of $j_{U!}^{PSh}\mathcal{G} \to j_{U!}*$ with the identification $j_{U!}^{PSh}* = h_U$ (obvious from the formula). Then it is immediately clear that the diagram

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{C}/U) \arrow[r] \arrow[d] & \mathrm{PSh}(\mathcal{C})/h_U \arrow[d] \\ Sh(\mathcal{C}/U) \arrow[r] & Sh(\mathcal{C})/h_U^\# \end{tikzcd}$$

[5]An alternative is to describe $\mathcal{F}_\varphi$ by the cartesian diagram

$$\begin{tikzcd} \mathcal{F}_\varphi \arrow[r] \arrow[d] & * \arrow[d] \\ \mathcal{F}|_{\mathcal{C}/U} \arrow[r] & h_U|_{\mathcal{C}/U} \end{tikzcd} \quad \text{for presheaves and} \quad \begin{tikzcd} \mathcal{F}_\varphi \arrow[r] \arrow[d] & * \arrow[d] \\ \mathcal{F}|_{\mathcal{C}/U} \arrow[r] & h_U^\#|_{\mathcal{C}/U} \end{tikzcd}$$

for sheaves and use that restriction to $\mathcal{C}/U$ commutes with sheafification.

commutes, where the vertical arrows are sheafification. Putting everything together it suffices to show there are functorial isomorphisms $(j_{U!}^{PSh}\mathcal{G})_\gamma = \mathcal{G}$ for $\mathcal{G}$ in $\text{PSh}(\mathcal{C}/U)$ and $j_{U!}^{PSh}\mathcal{F}_\varphi = \mathcal{F}$ for $(\mathcal{F}, \varphi)$ in $\text{PSh}(\mathcal{C})/h_U$. The value of the presheaf $(j_{U!}^{PSh}\mathcal{G})_\gamma$ on (X, a) is the fibre of the map

$$\coprod\nolimits_{a' : X \to U} \mathcal{G}(X, a') \to \text{Mor}_\mathcal{C}(X, U)$$

over a which is $\mathcal{G}(X, a)$. This proves the first equality. The value of the presheaf $j_{U!}^{PSh}\mathcal{F}_\varphi$ is on X is

$$\coprod\nolimits_{a : X \to U} \mathcal{F}_\varphi(X, a) = \mathcal{F}(X)$$

because given a set map $S \to S'$ the set S is the disjoint union of its fibres. $\square$

Lemma 7.25.4 says the functor $j_{U!}$ is the composition

$$\mathit{Sh}(\mathcal{C}/U) \to \mathit{Sh}(\mathcal{C})/h_U^\# \to \mathit{Sh}(\mathcal{C})$$

where the first arrow is an equivalence.

04BB Lemma 7.25.5. Let $\mathcal{C}$ be a site. Let $U \in \text{Ob}(\mathcal{C})$. The functor $j_{U!}$ commutes with fibre products and equalizers (and more generally finite connected limits). In particular, if $\mathcal{F} \subset \mathcal{F}'$ in $\mathit{Sh}(\mathcal{C}/U)$, then $j_{U!}\mathcal{F} \subset j_{U!}\mathcal{F}'$.

Proof. Via Lemma 7.25.4 and the fact that an equivalence of categories commutes with all limits, this reduces to the fact that the functor $\mathit{Sh}(\mathcal{C})/h_U^\# \to \mathit{Sh}(\mathcal{C})$ commutes with fibre products and equalizers. Alternatively, one can prove this directly using the description of $j_{U!}$ in Lemma 7.25.2 using that sheafification is exact. (Also, in case $\mathcal{C}$ has fibre products and equalizers, the result follows from Lemma 7.21.6.) $\square$

0E8E Lemma 7.25.6. Let $\mathcal{C}$ be a site. Let $U \in \text{Ob}(\mathcal{C})$. The functor $j_{U!}$ reflects injections and surjections.

Proof. We have to show $j_{U!}$ reflects monomorphisms and epimorphisms, see Lemma 7.11.2. Via Lemma 7.25.4 this reduces to the fact that the functor $\mathit{Sh}(\mathcal{C})/h_U^\# \to \mathit{Sh}(\mathcal{C})$ reflects monomorphisms and epimorphisms. $\square$

03EE Lemma 7.25.7. Let $\mathcal{C}$ be a site. Let $U \in \text{Ob}(\mathcal{C})$. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $j_{U!}j_U^{-1}\mathcal{F} = \mathcal{F} \times h_U^\#$.

Proof. This is clear from the description of $j_{U!}$ in Lemma 7.25.2. $\square$

03EH Lemma 7.25.8. Let $\mathcal{C}$ be a site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{C}/V \arrow[rr, "j"'] \arrow[dr, "j_V"'] & & \mathcal{C}/U \arrow[dl, "j_U"] \\
& \mathcal{C} &
\end{tikzcd}$$

of continuous and cocontinuous functors. The functor $j : \mathcal{C}/V \to \mathcal{C}/U$, $(a : W \to V) \mapsto (f \circ a : W \to U)$ is identified with the functor $j_{V/U} : (\mathcal{C}/U)/(V/U) \to \mathcal{C}/U$ via the identification $(\mathcal{C}/U)/(V/U) = \mathcal{C}/V$. Moreover we have $j_{V!} = j_{U!} \circ j_!$, $j_V^{-1} = j^{-1} \circ j_U^{-1}$, and $j_{V*} = j_{U*} \circ j_*$.

Proof. The commutativity of the diagram is immediate. The agreement of j with $j_{V/U}$ follows from the definitions. By Lemma 7.21.2 we see that the following diagram of morphisms of topoi

04IK (7.25.8.1)
$$\begin{tikzcd} Sh(\mathcal{C}/V) \arrow[rr, "j"] \arrow[dr, "j_V"'] & & Sh(\mathcal{C}/U) \arrow[dl, "j_U"] \\ & Sh(\mathcal{C}) & \end{tikzcd}$$

is commutative. This proves that $j_V^{-1} = j^{-1} \circ j_U^{-1}$ and $j_{V*} = j_{U*} \circ j_*$. The equality $j_{V!} = j_{U!} \circ j_!$ follows formally from adjointness properties. $\square$

04IL **Lemma 7.25.9.** Notation $\mathcal{C}$, $f : V \to U$, j_U, j_V, and j as in Lemma 7.25.8. Via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 7.25.4 we have

(1) the functor j^{-1} has the following description
$$j^{-1}(\mathcal{H} \xrightarrow{\varphi} h_U^\#) = (\mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \to h_V^\#).$$

(2) the functor $j_!$ has the following description
$$j_!(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (\mathcal{H} \xrightarrow{h_f \circ \varphi} h_U^\#)$$

Proof. Proof of (2). Recall that the identification $Sh(\mathcal{C}/V) \to Sh(\mathcal{C})/h_V^\#$ sends $\mathcal{G}$ to $j_{V!}\mathcal{G} \to j_{V!}(*) = h_V^\#$ and similarly for $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$. Thus $j_!\mathcal{G}$ is mapped to $j_{U!}(j_!\mathcal{G}) \to j_{U!}(*) = h_U^\#$ and (2) follows because $j_{U!}j_! = j_{V!}$ by Lemma 7.25.8.

The reader can now prove (1) by using that j^{-1} is the right adjoint to $j_!$ and using that the rule in (1) is the right adjoint to the rule in (2). Here is a direct proof. Suppose that $\varphi : \mathcal{H} \to h_U^\#$ is an object of $Sh(\mathcal{C})/h_U^\#$. By the proof of Lemma 7.25.4 this corresponds to the sheaf $\mathcal{H}_\varphi$ on $\mathcal{C}/U$ defined by the rule
$$(a : W \to U) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = a\}$$
on $\mathcal{C}/U$. The pullback $j^{-1}\mathcal{H}_\varphi$ to $\mathcal{C}/V$ is given by the rule
$$(a : W \to V) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = f \circ a\}$$
by the description of $j^{-1} = j_{U/V}^{-1}$ as the restriction of $\mathcal{H}_\varphi$ to $\mathcal{C}/V$. On the other hand, applying the rule to the object
$$\mathcal{H}' = \mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \xrightarrow{\varphi'} h_V^\#$$
of $Sh(\mathcal{C})/h_V^\#$ we get $\mathcal{H}'_{\varphi'}$ given by
$$\begin{aligned}(a : W \to V) \longmapsto & \{s' \in \mathcal{H}'(W) \mid \varphi'(s') = a\} \\ = & \{(s, a') \in \mathcal{H}(W) \times h_V^\#(W) \mid a' = a \text{ and } \varphi(s) = f \circ a'\}\end{aligned}$$
which is exactly the same rule as the one describing $j^{-1}\mathcal{H}_\varphi$ above. $\square$

0494 **Remark 7.25.10.** Localization and presheaves. Let $\mathcal{C}$ be a category. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^{-1}, j_{U*} and $j_{U!}$ have not been defined for presheaves. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Example 7.6.6). Hence we also obtain a functor

$$j_U^{-1} : \text{PSh}(\mathcal{C}) \longrightarrow \text{PSh}(\mathcal{C}/U)$$

and functors

$$j_{U*}, j_{U!} : \text{PSh}(\mathcal{C}/U) \longrightarrow \text{PSh}(\mathcal{C})$$

which are right, left adjoint to j_U^{-1}. By Lemma 7.25.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \text{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ commutes with fibre products and equalizers.

09W8 Remark 7.25.11. Let $\mathcal{C}$ be a site. Let $U \to V$ be a morphism of $\mathcal{C}$. The cocontinuous functors $\mathcal{C}/U \to \mathcal{C}$ and $j : \mathcal{C}/U \to \mathcal{C}/V$ (Lemma 7.25.8) satisfy property P of Remark 7.20.5. For example, if we have objects (X/U), (W/V), a morphism $g : j(X/U) \to (W/V)$, and a covering $\{f_i : (W_i/V) \to (W/V)\}$ then $(X \times_W W_i/U)$ is an avatar of $(X/U) \times_{g,(W/V),f_i} (W_i/V)$ and the family $\{(X \times_W W_i/U) \to (X/U)\}$ is a covering of $\mathcal{C}/U$.

7.26. Glueing sheaves

04TP This section is the analogue of Sheaves, Section 6.33.

04TQ Lemma 7.26.1. Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves on $\mathcal{C}$. Given a collection

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \longrightarrow \mathcal{G}|_{\mathcal{C}/U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\varphi_{ij} : \mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} \to \mathcal{G}|_{\mathcal{C}/U_i \times_U U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F}|_{\mathcal{C}/U} \longrightarrow \mathcal{G}|_{\mathcal{C}/U}$$

whose restriction to each $\mathcal{C}/U_i$ agrees with φ_i.

Proof. The restrictions used in the lemma are those of Lemma 7.25.8. Let V/U be an object of $\mathcal{C}/U$. Set $V_i = U_i \times_U V$ and denote $\mathcal{V} = \{V_i \to V\}$. Observe that $(U_i \times_U U_j) \times_U V = V_i \times_V V_j$. Then we have $\mathcal{F}|_{\mathcal{C}/U_i}(V_i/U_i) = \mathcal{F}(V_i)$ and $\mathcal{F}|_{\mathcal{C}/U_i \times_U U_j}(V_i \times_V V_j/U_i \times_U U_j) = \mathcal{F}(V_i \times_V V_j)$ and similarly for $\mathcal{G}$. Thus we can define φ on sections over V as the dotted arrows in the diagram

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, equal] & H^0(\mathcal{V}, \mathcal{F}) \arrow[r] \arrow[d, dotted] & \prod \mathcal{F}(V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\prod \varphi_i"'] & \prod \mathcal{F}(V_i \times_V V_j) \arrow[d, "\prod \varphi_{ij}"'] \\
\mathcal{G}(V) \arrow[r, equal] & H^0(\mathcal{V}, \mathcal{G}) \arrow[r] & \prod \mathcal{G}(V_i) \arrow[r, shift left] \arrow[r, shift right] & \prod \mathcal{G}(V_i \times_V V_j)
\end{tikzcd}$$

The equality signs come from the sheaf condition; see Section 7.10 for the notation $H^0(\mathcal{V}, -)$. We omit the verification that these maps are compatible with the restriction maps. $\square$

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on a site $\mathcal{C}$ the rule

$$U \longmapsto \text{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U})$$

defines a sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$. This is a kind of internal hom sheaf. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules on Sites, Section 18.27.

0BWQ **Lemma 7.26.2.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be sheaves on $\mathcal{C}$. There is a canonical bijection

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries.

Proof. The lemma says that the functors $- \times \mathcal{G}$ and $\mathcal{H}om(\mathcal{G}, -)$ are adjoint to each other. To show this, we use the notion of unit and counit, see Categories, Section 4.24. The unit

$$\eta_\mathcal{F} : \mathcal{F} \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{F} \times \mathcal{G})$$

sends $s \in \mathcal{F}(U)$ to the map $\mathcal{G}|_{\mathcal{C}/U} \to \mathcal{F}|_{\mathcal{C}/U} \times \mathcal{G}|_{\mathcal{C}/U}$ which over V/U is given by

$$\mathcal{G}(V) \longrightarrow \mathcal{F}(V) \times \mathcal{G}(V), \quad t \longmapsto (s|_V, t).$$

The counit

$$\epsilon_\mathcal{H} : \mathcal{H}om(\mathcal{G}, \mathcal{H}) \times \mathcal{G} \longrightarrow \mathcal{H}$$

is the evaluation map. It is given by the rule

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{G}|_{\mathcal{C}/U}, \mathcal{H}|_{\mathcal{C}/U}) \times \mathcal{G}(U) \longrightarrow \mathcal{H}(U), \quad (\varphi, s) \longmapsto \varphi(s).$$

Then for each $\varphi : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$, the corresponding morphism $\mathcal{F} \to \mathcal{H}om(\mathcal{G}, \mathcal{H})$ is given by mapping each section $s \in \mathcal{F}(U)$ to the morphism of sheaves on $\mathcal{C}/U$ which on sections over V/U is given by

$$\mathcal{G}(V) \longrightarrow \mathcal{H}(V), \quad t \longmapsto \varphi(s|_V, t).$$

Conversely, for each $\psi : \mathcal{F} \to \mathcal{H}om(\mathcal{G}, \mathcal{H})$, the corresponding morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is given by

$$\mathcal{F}(U) \times \mathcal{G}(U) \longrightarrow \mathcal{H}(U), \quad (s, t) \longmapsto \psi(s)(t)$$

on sections over an object U. We omit the details of the proof showing that these constructions are mutually inverse. □

0D7X **Lemma 7.26.3.** Let $\mathcal{C}$ be a site and $U \in \mathrm{Ob}(\mathcal{C})$. Then $\mathcal{H}om(h_U^\#, \mathcal{F}) = j_*(\mathcal{F}|_{\mathcal{C}/U})$ for $\mathcal{F}$ in $Sh(\mathcal{C})$.

Proof. This can be shown by directly constructing an isomorphism of sheaves. Instead we argue as follows. Let $\mathcal{G}$ be a sheaf on $\mathcal{C}$. Then

$$\begin{aligned}
\mathrm{Mor}(\mathcal{G}, j_*(\mathcal{F}|_{\mathcal{C}/U})) &= \mathrm{Mor}(\mathcal{G}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}) \\
&= \mathrm{Mor}(j_!(\mathcal{G}|_{\mathcal{C}/U}), \mathcal{F}) \\
&= \mathrm{Mor}(\mathcal{G} \times h_U^\#, \mathcal{F}) \\
&= \mathrm{Mor}(\mathcal{G}, \mathcal{H}om(h_U^\#, \mathcal{F}))
\end{aligned}$$

and we conclude by the Yoneda lemma. Here we used Lemmas 7.26.2 and 7.25.7. □

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on $\mathcal{C}/U_i$. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{\mathcal{C}/U_i \times_U U_j} \longrightarrow \mathcal{F}_j|_{\mathcal{C}/U_i \times_U U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{tikzcd} \mathcal{F}_i|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathcal{F}_k|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} \\ & \mathcal{F}_j|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a glueing data for sheaves of sets with respect to the covering $\{U_i \to U\}_{i \in I}$.

04TR Lemma 7.26.4. Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $\{U_i \to U\}_{i \in I}$ there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}/U$ together with isomorphisms

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{tikzcd} \mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} \arrow[r, "\varphi_i"] \arrow[d, "\mathrm{id}"'] & \mathcal{F}_i|_{\mathcal{C}/U_i \times_U U_j} \arrow[d, "\varphi_{ij}"] \\ \mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} \arrow[r, "\varphi_j"] & \mathcal{F}_j|_{\mathcal{C}/U_i \times_U U_j} \end{tikzcd}$$

are commutative.

Proof. Let us describe how to construct the sheaf $\mathcal{F}$ on $\mathcal{C}/U$. Let $a : V \to U$ be an object of $\mathcal{C}/U$. Then

$$\mathcal{F}(V/U) = \{(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}_i(U_i \times_U V/U_i) \mid \varphi_{ij}(s_i|_{U_i \times_U U_j \times_U V}) = s_j|_{U_i \times_U U_j \times_U V}\}$$

We omit the construction of the restriction mappings. We omit the verification that this is a sheaf. We omit the construction of the isomorphisms φ_i, and we omit proving the commutativity of the diagrams of the lemma. $\square$

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. Associated to $\mathcal{F}$ we have its canonical glueing data given by the restrictions $\mathcal{F}|_{\mathcal{C}/U_i}$ and the canonical isomorphisms

$$\left(\mathcal{F}|_{\mathcal{C}/U_i}\right)|_{\mathcal{C}/U_i \times_U U_j} = \left(\mathcal{F}|_{\mathcal{C}/U_j}\right)|_{\mathcal{C}/U_i \times_U U_j}$$

coming from the fact that the composition of the functors $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_i \to \mathcal{C}/U$ and $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_j \to \mathcal{C}/U$ are equal.

04TS Lemma 7.26.5. Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. The category $Sh(\mathcal{C}/U)$ is equivalent to the category of glueing data via the functor that associates to $\mathcal{F}$ on $\mathcal{C}/U$ the canonical glueing data.

Proof. In Lemma 7.26.1 we saw that the functor is fully faithful, and in Lemma 7.26.4 we proved that it is essentially surjective (by explicitly constructing a quasi-inverse functor). $\square$

Let $\mathcal{C}$ be a site. We are going to discuss a version of glueing sheaves on the entire site $\mathcal{C}$. For each object U in $\mathcal{C}$, let $\mathcal{F}_U$ be a sheaf on $\mathcal{C}/U$. Recall that there is a functor $j_f : \mathcal{C}/V \to \mathcal{C}/U$ associated to each morphism $f : V \to U$ in $\mathcal{C}$, given by $(a : W \to V) \mapsto (f \circ a : W \to U)$. For each such f, let

$$c_f : j_f^{-1}\mathcal{F}_U \to \mathcal{F}_V$$

be an isomorphism of sheaves. Assume that given any two arrows $f : V \to U$ and $g : W \to V$ in $\mathcal{C}$, the composition $c_g \circ j_g^{-1}c_f$ is equal to $c_{f\circ g}$. We will call such a collection of data $(\mathcal{F}_U, c_f)$ an absolute glueing data for sheaves of sets on $\mathcal{C}$. A morphism of absolute glueing data $(\mathcal{F}_U, c_f) \to (\mathcal{G}_U, c'_f)$ is given by a collection (φ_U) of morphisms of sheaves $\varphi_U : \mathcal{F}_U \to \mathcal{G}_U$, such that

$$\begin{tikzcd} j_f^{-1}\mathcal{F}_U \arrow[r, "c_f"'] \arrow[d, "j_f^{-1}\varphi_U"'] & \mathcal{F}_V \arrow[d, "\varphi_V"] \\ j_f^{-1}\mathcal{G}_U \arrow[r, "c'_f"] & \mathcal{G}_V \end{tikzcd}$$

commutes for every morphism $f : V \to U$ in $\mathcal{C}$.

Associated to any sheaf $\mathcal{F}$ on $\mathcal{C}$ is its canonical absolute glueing data $(\mathcal{F}|_{\mathcal{C}/U}, c_f)$, where the canonical isomorphisms $c_f : j_f^{-1}\mathcal{F}|_{\mathcal{C}/U} \to \mathcal{F}|_{\mathcal{C}/V}$ for $f : V \to U$ come from the relation $j_V = j_U \circ j_f$ as in Lemma 7.25.8. Any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{C}$ induces a morphism $(\varphi|_{\mathcal{C}/U})$ of canonical absolute glueing data.

0GWK **Lemma 7.26.6.** Let $\mathcal{C}$ be a site. The category $Sh(\mathcal{C})$ is equivalent to the category of absolute glueing data via the functor that associates to $\mathcal{F}$ on $\mathcal{C}$ the canonical absolute glueing data.

Proof. Given an absolute glueing data $(\mathcal{F}_U, c_f)$, we construct a sheaf $\mathcal{F}$ on $\mathcal{C}$ by setting $\mathcal{F}(U) = \mathcal{F}_U(U)$, where restriction along $f : V \to U$ given by the commutative diagram

$$\begin{tikzcd} \mathcal{F}_U(U) \arrow[r] \arrow[d, equal] & \mathcal{F}_U(V) \arrow[r, "c_f"] & \mathcal{F}_V(V) \arrow[d, equal] \\ \mathcal{F}(U) \arrow[rr] & & \mathcal{F}(V) \end{tikzcd}$$

The compatibility condition $c_g \circ j_g^{-1}c_f = c_{f\circ g}$ ensures that $\mathcal{F}$ is a presheaf, and also ensures that the maps $c_f : \mathcal{F}_U(V) \to \mathcal{F}(V)$ define an isomorphism $\mathcal{F}_U \to \mathcal{F}|_{\mathcal{C}/U}$ for each U. Since each $\mathcal{F}_U$ is a sheaf, this implies that $\mathcal{F}$ is a sheaf as well. The functor $(\mathcal{F}_U, c_f) \mapsto \mathcal{F}$ just constructed is quasi-inverse to the functor which takes a sheaf on $\mathcal{C}$ to its canonical glueing data. Further details omitted. $\square$

0GWL **Remark 7.26.7.** There is a variant of Lemma 7.26.6 which comes up in algebraic geometry. Namely, suppose that $\mathcal{C}$ is a site with all fibre products and for each $U \in \mathrm{Ob}(\mathcal{C})$ we are given a full subcategory $U_\tau \subset \mathcal{C}/U$ with the following properties

1. U/U is in U_τ,
2. for V/U in U_τ and covering $\{V_j \to V\}$ of $\mathcal{C}$ we have V_j/U in U_τ and
3. for a morphism $U' \to U$ of $\mathcal{C}$ and V/U in U_τ the base change $V' = V \times_U U'$ is in U'_τ.

In this setting U_τ is a site for all U in $\mathcal{C}$ and the base change functor $U_\tau \to U'_\tau$ defines a morphism $f_\tau : U_\tau \to U'_\tau$ of sites for all morphisms $f : U' \to U$ of $\mathcal{C}$. The glueing statement we obtain then reads as follows: A sheaf $\mathcal{F}$ on $\mathcal{C}$ is given by the following data:

(1) for every $U \in \mathrm{Ob}(\mathcal{C})$ a sheaf $\mathcal{F}_U$ on U_τ,
(2) for every $f : U' \to U$ in $\mathcal{C}$ a map $c_f : f_\tau^{-1}\mathcal{F}_U \to \mathcal{F}_{U'}$.

These data are subject to the following conditions:

(a) given $f : U' \to U$ and $g : U'' \to U'$ in $\mathcal{C}$ the composition $c_g \circ g_\tau^{-1}c_f$ is equal to $c_{f\circ g}$, and
(b) if $f : U' \to U$ is in U_τ then c_f is an isomorphism.

If we ever need this we will precisely state and prove this here. (Note that this result is slightly different from the statements above as we are not requiring all the maps c_f to be isomorphisms!)

7.27. More localization

04IM In this section we prove a few lemmas on localization where we impose some additional hypotheses on the site on or the object we are localizing at.

03HT **Lemma 7.27.1.** Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. If the topology on $\mathcal{C}$ is subcanonical, see Definition 7.12.2, and if $\mathcal{G}$ is a sheaf on $\mathcal{C}/U$, then

$$j_{U!}(\mathcal{G})(V) = \coprod\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

in other words sheafification is not necessary in Lemma 7.25.2.

Proof. Let $\mathcal{V} = \{V_i \to V\}_{i \in I}$ be a covering of V in the site $\mathcal{C}$. We are going to check the sheaf condition for the presheaf $\mathcal{H}$ of Lemma 7.25.2 directly. Let $(s_i, \varphi_i)_{i \in I} \in \prod_i \mathcal{H}(V_i)$, This means $\varphi_i : V_i \to U$ is a morphism in $\mathcal{C}$, and $s_i \in \mathcal{G}(V_i \xrightarrow{\varphi_i} U)$. The restriction of the pair (s_i, φ_i) to $V_i \times_V V_j$ is the pair $(s_i|_{V_i \times_V V_j/U}, \mathrm{pr}_1 \circ \varphi_i)$, and likewise the restriction of the pair (s_j, φ_j) to $V_i \times_V V_j$ is the pair $(s_j|_{V_i \times_V V_j/U}, \mathrm{pr}_2 \circ \varphi_j)$. Hence, if the family (s_i, φ_i) lies in $\check{H}^0(\mathcal{V}, \mathcal{H})$, then we see that $\mathrm{pr}_1 \circ \varphi_i = \mathrm{pr}_2 \circ \varphi_j$. The condition that the topology on $\mathcal{C}$ is weaker than the canonical topology then implies that there exists a unique morphism $\varphi : V \to U$ such that φ_i is the composition of $V_i \to V$ with φ. At this point the sheaf condition for $\mathcal{G}$ guarantees that the sections s_i glue to a unique section $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$. Hence $(s, \varphi) \in \mathcal{H}(V)$ as desired. $\square$

03CE **Lemma 7.27.2.** Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume $\mathcal{C}$ has products of pairs of objects. Then

(1) the functor j_U has a continuous right adjoint, namely the functor $v(X) = X \times U/U$,
(2) the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}$ whose associated morphism of topoi equals $j_U : \mathit{Sh}(\mathcal{C}/U) \to \mathit{Sh}(\mathcal{C})$, and
(3) we have $j_{U*}\mathcal{F}(X) = \mathcal{F}(X \times U/U)$.

Proof. The functor v being right adjoint to j_U means that given Y/U and X we have

$$\mathrm{Mor}_\mathcal{C}(Y, X) = \mathrm{Mor}_{\mathcal{C}/U}(Y/U, X \times U/U)$$

which is clear. To check that v is continuous let $\{X_i \to X\}$ be a covering of $\mathcal{C}$. By the third axiom of a site (Definition 7.6.2) we see that

$$\{X_i \times_X (X \times U) \to X \times_X (X \times U)\} = \{X_i \times U \to X \times U\}$$

is a covering of $\mathcal{C}$ also. Hence v is continuous. The other statements of the lemma follow from Lemmas 7.22.1 and 7.22.2. $\square$

09W9 **Lemma 7.27.3.** Let $\mathcal{C}$ be a site. Let $U \to V$ be a morphism of $\mathcal{C}$. Assume $\mathcal{C}$ has fibre products. Let j be as in Lemma 7.25.8. Then

(1) the functor $j : \mathcal{C}/U \to \mathcal{C}/V$ has a continuous right adjoint, namely the functor $v : (X/V) \mapsto (X \times_V U/U)$,
(2) the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}/V$ whose associated morphism of topoi equals j, and
(3) we have $j_*\mathcal{F}(X/V) = \mathcal{F}(X \times_V U/U)$.

Proof. Follows from Lemma 7.27.2 since j may be viewed as a localization functor by Lemma 7.25.8. □

A fundamental property of an open immersion is that the restriction of the pushforward and the restriction of the extension by the empty set produces back the original sheaf. This is not always true for the functors associated to j_U above. It is true when U is a "subobject of the final object".

00Y2 **Lemma 7.27.4.** Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume that every X in $\mathcal{C}$ has at most one morphism to U. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. The canonical maps $\mathcal{F} \to j_U^{-1} j_{U!}\mathcal{F}$ and $j_U^{-1} j_{U*}\mathcal{F} \to \mathcal{F}$ are isomorphisms.

Proof. This is a special case of Lemma 7.21.7 because the assumption on U is equivalent to the fully faithfulness of the localization functor $\mathcal{C}/U \to \mathcal{C}$. □

0EYV **Lemma 7.27.5.** Let $\mathcal{C}$ be a site. Let

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

be a commutative diagram of $\mathcal{C}$. The morphisms of Lemma 7.25.8 produce commutative diagrams

$$\begin{tikzcd} \mathcal{C}/U' \arrow[r, "j_{U'/U}"'] \arrow[d, "j_{U'/V'}"'] & \mathcal{C}/U \arrow[d, "j_{U/V}"] \\ \mathcal{C}/V' \arrow[r, "j_{V'/V}"] & \mathcal{C}/V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Sh(\mathcal{C}/U') \arrow[r, "j_{U'/U}"'] \arrow[d, "j_{U'/V'}"'] & Sh(\mathcal{C}/U) \arrow[d, "j_{U/V}"] \\ Sh(\mathcal{C}/V') \arrow[r, "j_{V'/V}"] & Sh(\mathcal{C}/V) \end{tikzcd}$$

of continuous and cocontinuous functors and of topoi. Moreover, if the initial diagram of $\mathcal{C}$ is cartesian, then we have $j_{V'/V}^{-1} \circ j_{U/V,*} = j_{U'/V',*} \circ j_{U'/U}^{-1}$.

Proof. The commutativity of the left square in the first statement of the lemma is immediate from the definitions. It implies the commutativity of the diagram of topoi by Lemma 7.21.2. Assume the diagram is cartesian. By the uniqueness of adjoint functors, to show $j_{V'/V}^{-1} \circ j_{U/V,*} = j_{U'/V',*} \circ j_{U'/U}^{-1}$ is equivalent to showing $j_{U/V}^{-1} \circ j_{V'/V!} = j_{U'/U!} \circ j_{U'/V'}^{-1}$. Via the identifications of Lemma 7.25.4 we may think of our diagram of topoi as

$$\begin{tikzcd} Sh(\mathcal{C})/h_{U'}^\# \arrow[r] \arrow[d] & Sh(\mathcal{C})/h_U^\# \arrow[d] \\ Sh(\mathcal{C})/h_{V'}^\# \arrow[r] & Sh(\mathcal{C})/h_V^\# \end{tikzcd}$$

and we know how to interpret the functors j^{-1} and $j_!$ by Lemma 7.25.9. Thus we have to show given $\mathcal{F} \to h_{V'}^\#$ that

$$\mathcal{F} \times_{h_{V'}^\#} h_{U'}^\# = \mathcal{F} \times_{h_V^\#} h_U^\#$$

as sheaves with map to $h_U^\#$. This is true because $h_{U'} = h_{V'} \times_{h_V} h_U$ and hence also

$$h_{U'}^\# = h_{V'}^\# \times_{h_V^\#} h_U^\#$$

as sheafification is exact. □

7.28. Localization and morphisms

04I8 The following lemma is important in order to understand relation between localization and morphisms of sites and topoi.

03CF **Lemma 7.28.1.** Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$ and set $U = u(V)$. Then the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/U$ determines a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The morphism f' fits into a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d, "f'"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}/V) \arrow[r, "j_V"] & Sh(\mathcal{D}). \end{tikzcd}$$

Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 7.25.4 the functor $(f')^{-1}$ is described by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} h_U^\#).$$

Finally, we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. It is clear that u' is continuous, and hence we get functors $f'_* = (u')^s = (u')^p$ (see Sections 7.5 and 7.13) and an adjoint $(f')^{-1} = (u')_s = ((u')_p\ )^\#$. The assertion $f'_* j_U^{-1} = j_V^{-1} f_*$ follows as

$$(j_V^{-1} f_* \mathcal{F})(V'/V) = f_*\mathcal{F}(V') = \mathcal{F}(u(V')) = (j_U^{-1}\mathcal{F})(u(V')/U) = (f'_* j_U^{-1}\mathcal{F})(V'/V)$$

which holds even for presheaves. What isn't clear a priori is that $(f')^{-1}$ is exact, that the diagram commutes, and that the description of $(f')^{-1}$ holds.

Let $\mathcal{H}$ be a sheaf on $\mathcal{D}/V$. Let us compute $j_{U!}(f')^{-1}\mathcal{H}$. We have

$$\begin{aligned} j_{U!}(f')^{-1}\mathcal{H} &= ((j_U)_p(u'_p\mathcal{H})^\#)^\# \\ &= ((j_U)_p u'_p \mathcal{H})^\# \\ &= (u_p (j_V)_p \mathcal{H})^\# \\ &= f^{-1} j_{V!}\mathcal{H} \end{aligned}$$

The first equality by unwinding the definitions. The second equality by Lemma 7.13.4. The third equality because $u \circ j_V = j_U \circ u'$. The fourth equality by Lemma 7.13.4 again. All of the equalities above are isomorphisms of functors, and hence we may interpret this as saying that the following diagram of categories and functors is commutative

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_{U!}"'] & Sh(\mathcal{C})/h_U^\# \arrow[r] & Sh(\mathcal{C}) \\ Sh(\mathcal{D}/V) \arrow[r, "j_{V!}"] \arrow[u, "(f')^{-1}"] & Sh(\mathcal{D})/h_V^\# \arrow[r] \arrow[u, "f^{-1}"] & Sh(\mathcal{D}) \arrow[u, "f^{-1}"] \end{tikzcd}$$

The middle arrow makes sense as $f^{-1}h_V^\# = (h_{u(V)})^\# = h_U^\#$, see Lemma 7.13.5. In particular this proves the description of $(f')^{-1}$ given in the statement of the lemma. Since by Lemma 7.25.4 the left horizontal arrows are equivalences and since f^{-1} is exact by assumption we conclude that $(f')^{-1} = u'_s$ is exact. Namely, because it is a left adjoint it is already right exact (Categories, Lemma 4.24.5). Hence we only need to show that it transforms a final object into a final object and commutes with fibre products (Categories, Lemma 4.23.2). Both are clear for the induced functor $f^{-1} : Sh(\mathcal{D})/h_V^\# \to Sh(\mathcal{C})/h_U^\#$. This proves that f' is a morphism of sites.

We still have to verify that $(f')^{-1}j_V^{-1} = j_U^{-1}f^{-1}$. To see this use the formula above and the description in Lemma 7.25.7. Namely, combined these give, for any sheaf $\mathcal{G}$ on $\mathcal{D}$, that

$$j_{U!}(f')^{-1}j_V^{-1}\mathcal{G} = f^{-1}j_{V!}j_V^{-1}\mathcal{G} = f^{-1}(\mathcal{G} \times h_V^\#) = f^{-1}\mathcal{G} \times h_U^\# = j_{U!}j_U^{-1}f^{-1}\mathcal{G}.$$

Since the functor $j_{U!}$ induces an equivalence $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$ we conclude. □

The following lemma is a special case of the more general Lemma 7.28.1 above.

03EF Lemma 7.28.2. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a functor. Let $V \in \mathrm{Ob}(\mathcal{D})$. Set $U = u(V)$. Assume that

(1) $\mathcal{C}$ and $\mathcal{D}$ have all finite limits,
(2) u is continuous, and
(3) u commutes with finite limits.

There exists a commutative diagram of morphisms of sites

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "j_U"'] \arrow[d, "f'"'] & \mathcal{C} \arrow[d, "f"] \\ \mathcal{D}/V \arrow[r, "j_V"] & \mathcal{D} \end{tikzcd}$$

where the right vertical arrow corresponds to u, the left vertical arrow corresponds to the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/u(V)$ and the horizontal arrows correspond to the functors $\mathcal{C} \to \mathcal{C}/U$, $X \mapsto X \times U$ and $\mathcal{D} \to \mathcal{D}/V$, $Y \mapsto Y \times V$ as in Lemma 7.27.2. Moreover, the associated diagram of morphisms of topoi is equal to the diagram of Lemma 7.28.1. In particular we have $f'_*j_U^{-1} = j_V^{-1}f_*$.

Proof. Note that u satisfies the assumptions of Proposition 7.14.7 and hence induces a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ by that proposition. It is clear that u induces a functor u' as indicated. It is clear that this functor also satisfies the assumptions of Proposition 7.14.7. Hence we get a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The diagram commutes by our definition of composition of morphisms of sites (see Definition 7.14.5) and because

$$u(Y \times V) = u(Y) \times u(V) = u(Y) \times U$$

which shows that the diagram of categories and functors opposite to the diagram of the lemma commutes. □

At this point we can localize a site, we know how to relocalize, and we can localize a morphism of sites at an object of the site downstairs. If we combine these then we get the following kind of diagram.

04IN Lemma 7.28.3. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d, "f_c"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}/V) \arrow[r, "j_V"] & Sh(\mathcal{D}). \end{tikzcd}$$

We have $f_c = f' \circ j_{U/u(V)}$ where $f' : Sh(\mathcal{C}/u(V)) \to Sh(\mathcal{D}/V)$ is as in Lemma 7.28.1 and $j_{U/u(V)} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C}/u(V))$ is as in Lemma 7.25.8. Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 7.25.4 the functor $(f_c)^{-1}$ is described by the rule

$$(f_c)^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, h^\#_{u(V)}, c} h_U^\# \to h_U^\#).$$

Finally, given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{tikzcd} U' \arrow[r, "c'"'] \arrow[d, "a"'] & u(V') \arrow[d, "u(b)"] \\ U \arrow[r, "c"] & u(V) \end{tikzcd}$$

commutes, then the diagram

$$\begin{tikzcd} Sh(\mathcal{C}/U') \arrow[r, "j_{U'/U}"'] \arrow[d, "f_{c'}"'] & Sh(\mathcal{C}/U) \arrow[d, "f_c"] \\ Sh(\mathcal{D}/V') \arrow[r, "j_{V'/V}"] & Sh(\mathcal{D}/V). \end{tikzcd}$$

commutes.

Proof. This lemma proves itself, and is more a collection of things we know at this stage of the development of theory. For example the commutativity of the first square follows from the commutativity of Diagram (7.25.8.1) and the commutativity of the diagram in Lemma 7.28.1. The description of f_c^{-1} follows on combining Lemma 7.25.9 with Lemma 7.28.1. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{h^\#_{u(V)}, c} h_U^\# \times_{h_U^\#} h_{U'}^\# = f^{-1}(\mathcal{H} \times_{h_V^\#} h_{V'}^\#) \times_{h^\#_{u(V'), c'}} h_{U'}^\#$$

which is formal using that $f^{-1}h_V^\# = h_{u(V)}^\#$ and $f^{-1}h_{V'}^\# = h_{u(V')}^\#$, see Lemma 7.13.5. □

In the following lemma we find another kind of functoriality of localization, in case the morphism of topoi comes from a cocontinuous functor. This is a kind of diagram which is different from the diagram in Lemma 7.28.1, and in particular, in general the equality $f'_*j_U^{-1} = j_V^{-1}f_*$ seen in Lemma 7.28.1 does not hold in the situation of the following lemma.

03EG Lemma 7.28.4. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let U be an object of $\mathcal{C}$, and set $V = u(U)$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "j_U"'] \arrow[d, "u'"'] & \mathcal{C} \arrow[d, "u"] \\ \mathcal{D}/V \arrow[r, "j_V"] & \mathcal{D} \end{tikzcd}$$

where the left vertical arrow is $u' : \mathcal{C}/U \to \mathcal{D}/V$, $U'/U \mapsto V'/V$. Then u' is cocontinuous also and we get a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d, "f'"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}/V) \arrow[r, "j_V"] & Sh(\mathcal{D}) \end{tikzcd}$$

where f (resp. f') corresponds to u (resp. u').

Proof. The commutativity of the first diagram is clear. It implies the commutativity of the second diagram provided we show that u' is cocontinuous.

Let U'/U be an object of $\mathcal{C}/U$. Let $\{V_j/V \to u(U')/V\}_{j\in J}$ be a covering of $u(U')/V$ in $\mathcal{D}/V$. Since u is cocontinuous there exists a covering $\{U'_i \to U'\}_{i\in I}$ such that the family $\{u(U'_i) \to u(U')\}$ refines the covering $\{V_j \to u(U')\}$ in $\mathcal{D}$. In other words, there exists a map of index sets $\alpha : I \to J$ and morphisms $\phi_i : u(U'_i) \to V_{\alpha(i)}$ over U'. Think of U'_i as an object over U via the composition $U'_i \to U' \to U$. Then $\{U'_i/U \to U'/U\}$ is a covering of $\mathcal{C}/U$ such that $\{u(U'_i)/V \to u(U')/V\}$ refines $\{V_j/V \to u(U')/V\}$ (use the same α and the same maps ϕ_i). Hence $u' : \mathcal{C}/U \to \mathcal{D}/V$ is cocontinuous. $\square$

0D5R Lemma 7.28.5. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let V be an object of $\mathcal{D}$. Let ${}^u_V\mathcal{I}$ be the category introduced in Section 7.19. We have a commutative diagram

$$\begin{tikzcd} {}^u_V\mathcal{I} \arrow[r, "j"'] \arrow[d, "u'"'] & \mathcal{C} \arrow[d, "u"] \\ \mathcal{D}/V \arrow[r, "j_V"] & \mathcal{D} \end{tikzcd} \quad \text{where} \quad \begin{matrix} j : (U, \psi) \mapsto U \\ u' : (U, \psi) \mapsto (\psi : u(U) \to V) \end{matrix}$$

Declare a family of morphisms $\{(U_i, \psi_i) \to (U, \psi)\}$ of ${}^u_V\mathcal{I}$ to be a covering if and only if $\{U_i \to U\}$ is a covering in $\mathcal{C}$. Then

(1) ${}^u_V\mathcal{I}$ is a site,
(2) j is continuous and cocontinuous,
(3) u' is cocontinuous,

(4) we get a commutative diagram of topoi

$$\begin{tikzcd} Sh({}^u_V\mathcal{I}) \arrow[r, "j"'] \arrow[d, "f'"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}/V) \arrow[r, "j_V"] & Sh(\mathcal{D}) \end{tikzcd}$$

where f (resp. f') corresponds to u (resp. u'), and

(5) we have $f'_*j^{-1} = j_V^{-1}f_*$.

Proof. Parts (1), (2), (3), and (4) are straightforward consequences of the definitions and the fact that the functor j commutes with fibre products. We omit the details. To see (5) recall that f_* is given by ${}_su = {}_pu$. Hence the value of $j_V^{-1}f_*\mathcal{F}$ on V'/V is the value of ${}_pu\mathcal{F}$ on V' which is the limit of the values of $\mathcal{F}$ on the category ${}^u_{V'}\mathcal{I}$. Clearly, there is an equivalence of categories

$$ {}^u_{V'}\mathcal{I} \to {}^{u'}_{V'/V}\mathcal{I} $$

Since the value of $f'_*j^{-1}\mathcal{F}$ on V'/V is given by the limit of the values of $j^{-1}\mathcal{F}$ on the category ${}^{u'}_{V'/V}\mathcal{I}$ and since the values of $j^{-1}\mathcal{F}$ on objects of ${}^u_V\mathcal{I}$ are just the values of $\mathcal{F}$ (by Lemma 7.21.5 as j is continuous and cocontinuous) we see that (5) is true. □

The following two results are of a slightly different nature.

0FN1 Lemma 7.28.6. Assume given sites $\mathcal{C}', \mathcal{C}, \mathcal{D}', \mathcal{D}$ and functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "v'"'] \arrow[d, "u'"'] & \mathcal{C} \arrow[d, "u"] \\ \mathcal{D}' \arrow[r, "v"] & \mathcal{D} \end{tikzcd}$$

Assume

(1) u, u', v, and v' are cocontinuous giving rise to morphisms of topoi f, f', g, and g' by Lemma 7.21.1,
(2) $v \circ u' = u \circ v'$,
(3) v and v' are continuous as well as cocontinuous, and
(4) for any object V' of $\mathcal{D}'$ the functor ${}^{u'}_{V'}\mathcal{I} \to {}^u_{v(V')}\mathcal{I}$ given by v is cofinal.

Then $f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$.

Proof. The categories ${}^{u'}_{V'}\mathcal{I}$ and ${}^u_{v(V')}\mathcal{I}$ are defined in Section 7.19. The functor in condition (4) sends the object $\psi : u'(U') \to V'$ of ${}^{u'}_{V'}\mathcal{I}$ to the object $v(\psi) : uv'(U') = vu'(U') \to v(V')$ of ${}^u_{v(V')}\mathcal{I}$. Recall that g^{-1} is given by v^p (Lemma 7.21.5) and f_* is given by ${}_su = {}_pu$. Hence the value of $g^{-1}f_*\mathcal{F}$ on V' is the value of ${}_pu\mathcal{F}$ on $v(V')$ which is the limit

$$ \lim_{u(U) \to v(V') \in \mathrm{Ob}({}^u_{v(V')}\mathcal{I}^{opp})} \mathcal{F}(U) $$

By the same reasoning, the value of $f'_*(g')^{-1}\mathcal{F}$ on V' is given by the limit

$$ \lim_{u'(U') \to V' \in \mathrm{Ob}({}^{u'}_{V'}\mathcal{I}^{opp})} \mathcal{F}(v'(U')) $$

Thus assumption (4) and Categories, Lemma 4.17.4 show that these agree and the first equality of the lemma is proved. The second equality follows from the first by uniqueness of adjoints. □

0FN2 Lemma 7.28.7. Assume given sites $\mathcal{C}', \mathcal{C}, \mathcal{D}', \mathcal{D}$ and functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "v'"'] & \mathcal{C} \\ \mathcal{D}' \arrow[u, "u'"] \arrow[r, "v"] & \mathcal{D} \arrow[u, "u"'] \end{tikzcd}$$

With notation as in Sections 7.14 and 7.21 assume

(1) u and u' are continuous giving rise to morphisms of sites f and f',
(2) v and v' are cocontinuous giving rise to morphisms of topoi g and g',
(3) $u \circ v = v' \circ u'$, and
(4) v and v' are continuous as well as cocontinuous.

Then[6] $f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$.

Proof. Namely, we have

$$f'_*(g')^{-1}\mathcal{F} = (u')^p((v')^p\mathcal{F})^\# = (u')^p(v')^p\mathcal{F}$$

The first equality by definition and the second by Lemma 7.21.5. We have

$$g^{-1}f_*\mathcal{F} = (v^pu^p\mathcal{F})^\# = ((u')^p(v')^p\mathcal{F})^\# = (u')^p(v')^p\mathcal{F}$$

The first equality by definition, the second because $u \circ v = v' \circ u'$, the third because we already saw that $(u')^p(v')^p\mathcal{F}$ is a sheaf. This proves $f'_* \circ (g')^{-1} = g^{-1} \circ f_*$ and the equality $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ follows by uniqueness of left adjoints. □

7.29. Morphisms of topoi

039Z In this section we show that any morphism of topoi is equivalent to a morphism of topoi which comes from a morphism of sites. Please compare with [AGV71, Exposé IV, Proposition 4.9.4].

03A0 Lemma 7.29.1. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(1) u is cocontinuous,
(2) u is continuous,
(3) given $a, b : U' \to U$ in $\mathcal{C}$ such that $u(a) = u(b)$, then there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$,
(4) given $U', U \in \text{Ob}(\mathcal{C})$ and a morphism $c : u(U') \to u(U)$ in $\mathcal{D}$ there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ and morphisms $c_i : U'_i \to U$ such that $u(c_i) = c \circ u(f_i)$, and
(5) given $V \in \text{Ob}(\mathcal{D})$ there exists a covering of V in $\mathcal{D}$ of the form $\{u(U_i) \to V\}_{i \in I}$.

Then the morphism of topoi

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{D})$$

associated to the cocontinuous functor u by Lemma 7.21.1 is an equivalence.

Proof. Assume u satisfies properties (1) – (5). We will show that the adjunction mappings

$$\mathcal{G} \longrightarrow g_*g^{-1}\mathcal{G} \quad \text{and} \quad g^{-1}g_*\mathcal{F} \longrightarrow \mathcal{F}$$

are isomorphisms.

[6]In this generality we don't know $f \circ g'$ is equal to $g \circ f'$ as morphisms of topoi (there is a canonical 2-arrow from the first to the second which may not be an isomorphism).

Note that Lemma 7.21.5 applies and we have $g^{-1}\mathcal{G}(U) = \mathcal{G}(u(U))$ for any sheaf $\mathcal{G}$ on $\mathcal{D}$. Next, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let V be an object of $\mathcal{D}$. By definition we have $g_*\mathcal{F}(V) = \lim_{u(U)\to V} \mathcal{F}(U)$. Hence

$$g^{-1}g_*\mathcal{F}(U) = \lim_{U', u(U')\to u(U)} \mathcal{F}(U')$$

where the morphisms $\psi : u(U') \to u(U)$ need not be of the form $u(\alpha)$. The category of such pairs (U', ψ) has a final object, namely (U, id), which gives rise to the map from the limit into $\mathcal{F}(U)$. Let $(s_{(U',\psi)})$ be an element of the limit. We want to show that $s_{(U',\psi)}$ is uniquely determined by the value $s_{(U,\text{id})} \in \mathcal{F}(U)$. By property (4) given any (U', ψ) there exists a covering $\{U'_i \to U'\}$ such that the compositions $u(U'_i) \to u(U') \to u(U)$ are of the form $u(c_i)$ for some $c_i : U'_i \to U$ in $\mathcal{C}$. Hence

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s_{(U,\text{id})}).$$

Since $\mathcal{F}$ is a sheaf it follows that indeed $s_{(U',\psi)}$ is determined by $s_{(U,\text{id})}$. This proves uniqueness. For existence, assume given any $s \in \mathcal{F}(U)$, $\psi : u(U') \to u(U)$, $\{f_i : U'_i \to U'\}$ and $c_i : U'_i \to U$ such that $\psi \circ u(f_i) = u(c_i)$ as above. We claim there exists a (unique) element $s_{(U',\psi)} \in \mathcal{F}(U')$ such that

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s).$$

Namely, a priori it is not clear the elements $c_i^*(s)|_{U'_i\times_{U'}U'_j}$ and $c_j^*(s)|_{U'_i\times_{U'}U'_j}$ agree, since the diagram

$$\begin{tikzcd} U'_i \times_{U'} U'_j \arrow[r, "\text{pr}_2"'] \arrow[d, "\text{pr}_1"'] & U'_j \arrow[d, "c_j"] \\ U'_i \arrow[r, "c_i"] & U \end{tikzcd}$$

need not commute. But condition (3) of the lemma guarantees that there exist coverings $\{f_{ijk} : U'_{ijk} \to U'_i \times_{U'} U'_j\}_{k\in K_{ij}}$ such that $c_i \circ \text{pr}_1 \circ f_{ijk} = c_j \circ \text{pr}_2 \circ f_{ijk}$. Hence

$$f_{ijk}^*\left(c_i^*s|_{U'_i\times_{U'}U'_j}\right) = f_{ijk}^*\left(c_j^*s|_{U'_i\times_{U'}U'_j}\right)$$

Hence $c_i^*(s)|_{U'_i\times_{U'}U'_j} = c_j^*(s)|_{U'_i\times_{U'}U'_j}$ by the sheaf condition for $\mathcal{F}$ and hence the existence of $s_{(U',\psi)}$ also by the sheaf condition for $\mathcal{F}$. The uniqueness guarantees that the collection $(s_{(U',\psi)})$ so obtained is an element of the limit with $s_{(U,\psi)} = s$. This proves that $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Let V be an object of $\mathcal{D}$. Then we see that

$$g_*g^{-1}\mathcal{G}(V) = \lim_{U,\psi:u(U)\to V} \mathcal{G}(u(U))$$

By the preceding paragraph we see that the value of the sheaf $g_*g^{-1}\mathcal{G}$ on an object V of the form $V = u(U)$ is equal to $\mathcal{G}(u(U))$. (Formally, this holds because we have $g^{-1}g_*g^{-1} \cong g^{-1}$, and the description of g^{-1} given at the beginning of the proof; informally just by comparing limits here and above.) Hence the adjunction mapping $\mathcal{G} \to g_*g^{-1}\mathcal{G}$ has the property that it is a bijection on sections over any object of the form $u(U)$. Since by axiom (5) there exists a covering of V by objects of the form $u(U)$ we see easily that the adjunction map is an isomorphism. $\square$

It will be convenient to give cocontinuous functors as in Lemma 7.29.1 a name.

03CG Definition 7.29.2. Let $\mathcal{C}$, $\mathcal{D}$ be sites. A special cocontinuous functor u from $\mathcal{C}$ to $\mathcal{D}$ is a cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the assumptions and conclusions of Lemma 7.29.1.

03CH Lemma 7.29.3. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. For every object U of $\mathcal{C}$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "j_U"'] \arrow[d] & \mathcal{C} \arrow[d, "u"] \\ \mathcal{D}/u(U) \arrow[r, "j_{u(U)}"] & \mathcal{D} \end{tikzcd}$$

as in Lemma 7.28.4. The left vertical arrow is a special cocontinuous functor. Hence in the commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d] & Sh(\mathcal{C}) \arrow[d, "u"] \\ Sh(\mathcal{D}/u(U)) \arrow[r, "j_{u(U)}"] & Sh(\mathcal{D}) \end{tikzcd}$$

the vertical arrows are equivalences.

Proof. We have seen the existence and commutativity of the diagrams in Lemma 7.28.4. We have to check hypotheses (1) – (5) of Lemma 7.29.1 for the induced functor $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. This is completely mechanical.

Property (1). This is Lemma 7.28.4.

Property (2). Let $\{U'_i/U \to U'/U\}_{i\in I}$ be a covering of U'/U in $\mathcal{C}/U$. Because u is continuous we see that $\{u(U'_i)/u(U) \to u(U')/u(U)\}_{i\in I}$ is a covering of $u(U')/u(U)$ in $\mathcal{D}/u(U)$. Hence (2) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (3). Let $a, b : U''/U \to U'/U$ in $\mathcal{C}/U$ be morphisms such that $u(a) = u(b)$ in $\mathcal{D}/u(U)$. Because u satisfies (3) we see there exists a covering $\{f_i : U''_i \to U''\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$. This gives a covering $\{f_i : U''_i/U \to U''/U\}$ in $\mathcal{C}/U$ such that $a \circ f_i = b \circ f_i$. Hence (3) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (4). Let $U''/U, U'/U \in \text{Ob}(\mathcal{C}/U)$ and a morphism $c : u(U'')/u(U) \to u(U')/u(U)$ in $\mathcal{D}/u(U)$ be given. Because u satisfies property (4) there exists a covering $\{f_i : U''_i \to U''\}$ in $\mathcal{C}$ and morphisms $c_i : U''_i \to U'$ such that $u(c_i) = c \circ u(f_i)$. We think of U''_i as an object over U via the composition $U''_i \to U'' \to U$. It may not be true that c_i is a morphism over U! But since $u(c_i)$ is a morphism over $u(U)$ we may apply property (3) for u and find coverings $\{f_{ik} : U''_{ik} \to U''_i\}$ such that $c_{ik} = c_i \circ f_{ik} : U''_{ik} \to U'$ are morphisms over U. Hence $\{f_i \circ f_{ik} : U''_{ik}/U \to U''/U\}$ is a covering in $\mathcal{C}/U$ such that $u(c_{ik}) = c \circ u(f_{ik})$. Hence (4) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (5). Let $h : V \to u(U)$ be an object of $\mathcal{D}/u(U)$. Because u satisfies property (5) there exists a covering $\{c_i : u(U_i) \to V\}$ in $\mathcal{D}$. By property (4) we can find coverings $\{f_{ij} : U_{ij} \to U_i\}$ and morphisms $c_{ij} : U_{ij} \to U$ such that $u(c_{ij}) = h \circ c_i \circ u(f_{ij})$. Hence $\{u(U_{ij})/u(U) \to V/u(U)\}$ is a covering in $\mathcal{D}/u(U)$ of the desired shape and we conclude that (5) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. □

03A1 Lemma 7.29.4. Let $\mathcal{C}$ be a site. Let $\mathcal{C}' \subset Sh(\mathcal{C})$ be a full subcategory (with a set of objects) such that

(1) $h_U^\# \in \mathrm{Ob}(\mathcal{C}')$ for all $U \in \mathrm{Ob}(\mathcal{C})$, and
(2) $\mathcal{C}'$ is preserved under fibre products in $\mathit{Sh}(\mathcal{C})$.

Declare a covering of $\mathcal{C}'$ to be any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ of maps such that $\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$ is a surjective map of sheaves. Then

(1) $\mathcal{C}'$ is a site (after choosing a set of coverings, see Sets, Lemma 3.11.1),
(2) representable presheaves on $\mathcal{C}'$ are sheaves (i.e., the topology on $\mathcal{C}'$ is subcanonical, see Definition 7.12.2),
(3) the functor $v : \mathcal{C} \to \mathcal{C}'$, $U \mapsto h_U^\#$ is a special cocontinuous functor, hence induces an equivalence $g : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{C}')$,
(4) for any $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ we have $g^{-1}h_\mathcal{F} = \mathcal{F}$, and
(5) for any $U \in \mathrm{Ob}(\mathcal{C})$ we have $g_*h_U^\# = h_{v(U)} = h_{h_U^\#}$.

Proof. Warning: Some of the statements above may look be a bit confusing at first; this is because objects of $\mathcal{C}'$ can also be viewed as sheaves on $\mathcal{C}$! We omit the proof that the coverings of $\mathcal{C}'$ as described in the lemma satisfy the conditions of Definition 7.6.2.

Suppose that $\{\mathcal{F}_i \to \mathcal{F}\}$ is a surjective family of morphisms of sheaves. Let $\mathcal{G}$ be another sheaf. Part (2) of the lemma says that the equalizer of

$$\mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(\textstyle\coprod_{i\in I} \mathcal{F}_i, \mathcal{G}) \rightrightarrows \mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(\textstyle\coprod_{(i_0,i_1)\in I\times I} \mathcal{F}_{i_0} \times_\mathcal{F} \mathcal{F}_{i_1}, \mathcal{G})$$

is $\mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(\mathcal{F}, \mathcal{G})$. This is clear (for example use Lemma 7.11.3).

To prove (3) we have to check conditions (1) – (5) of Lemma 7.29.1. The fact that v is cocontinuous is equivalent to the description of surjective maps of sheaves in Lemma 7.11.2. The functor v is continuous because $U \mapsto h_U^\#$ commutes with fibre products, and transforms coverings into coverings (see Lemma 7.10.14, and Lemma 7.12.4). Properties (3), (4) of Lemma 7.29.1 are statements about morphisms $f : h_{U'}^\# \to h_U^\#$. Such a morphism is the same thing as an element of $h_U^\#(U')$. Hence (3) and (4) are immediate from the construction of the sheafification. Property (5) of Lemma 7.29.1 is Lemma 7.12.5. Denote $g : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{C}')$ the equivalence of topoi associated with v by Lemma 7.29.1.

Let $\mathcal{F}$ be as in part (4) of the lemma. For any $U \in \mathrm{Ob}(\mathcal{C})$ we have

$$g^{-1}h_\mathcal{F}(U) = h_\mathcal{F}(v(U)) = \mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$$

The first equality by Lemma 7.21.5. Thus part (4) holds.

Let $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then

$$\begin{aligned} g_*h_U^\#(\mathcal{F}) &= \mathrm{Mor}_{\mathit{Sh}(\mathcal{C}')}(h_\mathcal{F}, g_*h_U^\#) \\ &= \mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(g^{-1}h_\mathcal{F}, h_U^\#) \\ &= \mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(\mathcal{F}, h_U^\#) \\ &= \mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}, h_U^\#) \end{aligned}$$

as desired (where the third equality was shown above). $\square$

Using this we can massage any topos to live over a site having all finite limits.

03CI Lemma 7.29.5. Let $\mathit{Sh}(\mathcal{C})$ be a topos. Let $\{\mathcal{F}_i\}_{i\in I}$ be a set of sheaves on $\mathcal{C}$. There exists an equivalence of topoi $g : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{C}')$ induced by a special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$ of sites such that

(1) $\mathcal{C}'$ has a subcanonical topology,
(2) a family $\{V_j \to V\}$ of morphisms of $\mathcal{C}'$ is (combinatorially equivalent to) a covering of $\mathcal{C}'$ if and only if $\coprod h_{V_j} \to h_V$ is surjective,
(3) $\mathcal{C}'$ has fibre products and a final object (i.e., $\mathcal{C}'$ has all finite limits),
(4) every subsheaf of a representable sheaf on $\mathcal{C}'$ is representable, and
(5) each $g_*\mathcal{F}_i$ is a representable sheaf.

Proof. Consider the full subcategory $\mathcal{C}_1 \subset \mathit{Sh}(\mathcal{C})$ consisting of all $h_U^\#$ for all $U \in \mathrm{Ob}(\mathcal{C})$, the given sheaves $\mathcal{F}_i$ and the final sheaf $*$ (see Example 7.10.2). We are going to inductively define full subcategories

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_2 \subset \ldots \subset \mathit{Sh}(\mathcal{C})$$

Namely, given $\mathcal{C}_n$ let $\mathcal{C}_{n+1}$ be the full subcategory consisting of all fibre products and subsheaves of objects of $\mathcal{C}_n$. (Note that $\mathcal{C}_{n+1}$ has a set of objects.) Set $\mathcal{C}' = \bigcup_{n \geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any family $\{\mathcal{G}_j \to \mathcal{G}\}_{j \in J}$ of morphisms of objects of $\mathcal{C}'$ such that $\coprod \mathcal{G}_j \to \mathcal{G}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. Apply Lemma 7.29.4. $\square$

Here is the goal of the current section.

03A2 Lemma 7.29.6. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{D})$ be a morphism of topoi. Then there exists a site $\mathcal{C}'$ and a diagram of functors

$$\mathcal{C} \xrightarrow{v} \mathcal{C}' \xleftarrow{u} \mathcal{D}$$

such that

(1) the functor v is a special cocontinuous functor,
(2) the functor u commutes with fibre products, is continuous and defines a morphism of sites $\mathcal{C}' \to \mathcal{D}$, and
(3) the morphism of topoi f agrees with the composition of morphisms of topoi

$$\mathit{Sh}(\mathcal{C}) \longrightarrow \mathit{Sh}(\mathcal{C}') \longrightarrow \mathit{Sh}(\mathcal{D})$$

where the first arrow comes from v via Lemma 7.29.1 and the second arrow from u via Lemma 7.15.2.

This statement is closely related to [AGV71, Proposition 4.9.4. Exposé IV]. In order to get the whole result, one should also use [AGV71, Remarque 4.7.4, Exposé IV].

Proof. Consider the full subcategory $\mathcal{C}_1 \subset \mathit{Sh}(\mathcal{C})$ consisting of all $h_U^\#$ and all $f^{-1}h_V^\#$ for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $V \in \mathrm{Ob}(\mathcal{D})$. Let $\mathcal{C}_{n+1}$ be a full subcategory consisting of all fibre products of objects of $\mathcal{C}_n$. Set $\mathcal{C}' = \bigcup_{n \geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i \in I}$ such that $\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. The functor $u : \mathcal{D} \to \mathcal{C}'$ is given by $V \mapsto f^{-1}h_V^\#$.

Part (1) follows from Lemma 7.29.4.

Proof of (2) and (3) of the lemma. The functor u commutes with fibre products as both $V \mapsto h_V^\#$ and f^{-1} do. Moreover, since f^{-1} is exact and commutes with arbitrary colimits we see that it transforms a covering into a surjective family of morphisms of sheaves. Hence u is continuous. To see that it defines a morphism of sites we still have to see that u_s is exact. In order to do this we will show that $g^{-1} \circ u_s = f^{-1}$. Namely, then since g^{-1} is an equivalence and f^{-1} is exact we will conclude. Because g^{-1} is adjoint to g_*, and u_s is adjoint to u^s, and f^{-1} is adjoint to f_* it also suffices to prove that $u^s \circ g_* = f_*$. Let U be an object of $\mathcal{C}$ and let V be an object of $\mathcal{D}$. Then

$$\begin{aligned}(u^s g_* h_U^\#)(V) &= g_* h_U^\#(f^{-1}h_V^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(f^{-1}h_V^\#, h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f_*h_U^\#) \\ &= f_*h_U^\#(V)\end{aligned}$$

The first equality because $u^s = u^p$. The second equality by Lemma 7.29.4 (5). The third equality by adjointness of f_* and f^{-1} and the final equality by properties of sheafification and the Yoneda lemma. We omit the verification that these identities are functorial in U and V. Hence we see that we have $u^s \circ g_* = f_*$ for sheaves of the form $h_U^\#$. This implies that $u^s \circ g_* = f_*$ and we win (some details omitted). $\square$

03CJ Remark 7.29.7. Notation and assumptions as in Lemma 7.29.6. If the site $\mathcal{D}$ has a final object and fibre products then the functor $u : \mathcal{D} \to \mathcal{C}'$ satisfies all the assumptions of Proposition 7.14.7. Namely, in addition to the properties mentioned in the lemma u also transforms the final object of $\mathcal{D}$ into the final object of $\mathcal{C}'$. This is clear from the construction of u. Hence, if we first apply Lemmas 7.29.5 to $\mathcal{D}$ and then Lemma 7.29.6 to the resulting morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D}')$ we obtain the following statement: Any morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ fits into a commutative diagram

$$\begin{tikzcd} Sh(\mathcal{C}) \arrow[r, "f"'] \arrow[d, "g"'] & Sh(\mathcal{D}) \arrow[d, "e"] \\ Sh(\mathcal{C}') \arrow[r, "f'"] & Sh(\mathcal{D}') \end{tikzcd}$$

where the following properties hold:

(1) the morphisms e and g are equivalences given by special cocontinuous functors $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{D} \to \mathcal{D}'$,
(2) the sites $\mathcal{C}'$ and $\mathcal{D}'$ have fibre products, final objects and have subcanonical topologies,
(3) the morphism $f' : \mathcal{C}' \to \mathcal{D}'$ comes from a morphism of sites corresponding to a functor $u : \mathcal{D}' \to \mathcal{C}'$ to which Proposition 7.14.7 applies, and
(4) given any set of sheaves $\mathcal{F}_i$ (resp. $\mathcal{G}_j$) on $\mathcal{C}$ (resp. $\mathcal{D}$) we may assume each of these is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).

It is often useful to replace $\mathcal{C}$ and $\mathcal{D}$ by $\mathcal{C}'$ and $\mathcal{D}'$.

03CK Remark 7.29.8. Notation and assumptions as in Lemma 7.29.6. Suppose that in addition the original morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D})$ is an equivalence. Then the construction in the proof of Lemma 7.29.6 gives two functors

$$\mathcal{C} \to \mathcal{C}' \leftarrow \mathcal{D}$$

which are both special cocontinuous functors. Hence in this case we can actually factor the morphism of topoi as a composition

$$Sh(\mathcal{C}) \to Sh(\mathcal{C}') = Sh(\mathcal{D}') \leftarrow Sh(\mathcal{D})$$

as in Remark 7.29.7, but with the middle morphism an identity.

7.30. Localization of topoi

04GY We repeat some of the material on localization to the apparently more general case of topoi. In reality this is not more general since we may always enlarge the underlying sites to assume that we are localizing at objects of the site.

04GZ **Lemma 7.30.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the category $Sh(\mathcal{C})/\mathcal{F}$ is a topos. There is a canonical morphism of topoi

$$j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \longrightarrow Sh(\mathcal{C})$$

which is a localization as in Section 7.25 such that

(1) the functor $j_{\mathcal{F}}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H} \times \mathcal{F}/\mathcal{F}$, and
(2) the functor $j_{\mathcal{F}!}$ is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$.

Proof. Apply Lemma 7.29.5. This means we may assume $\mathcal{C}$ is a site with subcanonical topology, and $\mathcal{F} = h_U = h_U^\#$ for some $U \in \mathrm{Ob}(\mathcal{C})$. Hence the material of Section 7.25 applies. In particular, there is an equivalence $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ such that the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$$

is equal to $j_{U!}$, see Lemma 7.25.4. Denote $a : Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C}/U)$ the inverse functor, so $j_{\mathcal{F}!} = j_{U!} \circ a$, $j_{\mathcal{F}}^{-1} = a^{-1} \circ j_U^{-1}$, and $j_{\mathcal{F},*} = j_{U,*} \circ a$. The description of $j_{\mathcal{F}!}$ follows from the above. The description of $j_{\mathcal{F}}^{-1}$ follows from Lemma 7.25.7. $\square$

04H0 **Lemma 7.30.2.** In the situation of Lemma 7.30.1, the functor $j_{\mathcal{F},*}$ is the one associates to $\varphi : \mathcal{G} \to \mathcal{F}$ the sheaf

$$U \longmapsto \{\alpha : \mathcal{F}|_U \to \mathcal{G}|_U \text{ such that } \alpha \text{ is a right inverse to } \varphi|_U\}.$$

Proof. For any $\varphi : \mathcal{G} \to \mathcal{F}$, let us use the notation $\mathcal{G}/\mathcal{F}$ to denote the corresponding object of $Sh(\mathcal{C})/\mathcal{F}$. We have

$$(j_{\mathcal{F},*}(\mathcal{G}/\mathcal{F}))(U) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, j_{\mathcal{F},*}(\mathcal{G}/\mathcal{F})) = \mathrm{Mor}_{Sh(\mathcal{C})/\mathcal{F}}(j_{\mathcal{F}}^{-1} h_U^\#, (\mathcal{G}/\mathcal{F})).$$

By Lemma 7.30.1 this set is the fiber over the element $h_U^\# \times \mathcal{F} \to \mathcal{F}$ under the map of sets

$$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{G}) \xrightarrow{\varphi\circ} \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{F}).$$

By the adjunction in Lemma 7.26.2, we have

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{G}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \\ &= \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}), \\ \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\# \times \mathcal{F}, \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{H}om(\mathcal{F}, \mathcal{F})) \\ &= \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}), \end{aligned}$$

and under the adjunction, the map $\varphi\circ$ becomes the map

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}), \quad \psi \longmapsto \varphi|_{\mathcal{C}/U} \circ \psi,$$

the element $h_U^\# \times \mathcal{F} \to \mathcal{F}$ becomes $\mathrm{id}_{\mathcal{F}|_{\mathcal{C}/U}}$. Therefore $(j_{\mathcal{F},*}\mathcal{G}/\mathcal{F})(U)$ is isomorphic to the fiber of $\mathrm{id}_{\mathcal{F}|_{\mathcal{C}/U}}$ under the map

$$\mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U}) \xrightarrow{\varphi|_{\mathcal{C}/U}\circ} \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{F}|_{\mathcal{C}/U}),$$

which is $\{\alpha : \mathcal{F}|_U \to \mathcal{G}|_U \text{ such that } \alpha \text{ is a right inverse to } \varphi|_U\}$ as desired. $\square$

0791 **Lemma 7.30.3.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $\mathcal{C}/\mathcal{F}$ be the category of pairs (U, s) where $U \in \mathrm{Ob}(\mathcal{C})$ and $s \in \mathcal{F}(U)$. Let a covering in $\mathcal{C}/\mathcal{F}$ be a family $\{(U_i, s_i) \to (U, s)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then $j : \mathcal{C}/\mathcal{F} \to \mathcal{C}$ is a continuous and cocontinuous functor of sites which induces a morphism of topoi $j : \mathit{Sh}(\mathcal{C}/\mathcal{F}) \to \mathit{Sh}(\mathcal{C})$. In fact, there is an equivalence $\mathit{Sh}(\mathcal{C}/\mathcal{F}) = \mathit{Sh}(\mathcal{C})/\mathcal{F}$ which turns j into $j_\mathcal{F}$.

Proof. We omit the verification that $\mathcal{C}/\mathcal{F}$ is a site and that j is continuous and cocontinuous. By Lemma 7.21.5 there exists a morphism of topoi j as indicated, with $j^{-1}\mathcal{G}(U, s) = \mathcal{G}(U)$, and there is a left adjoint $j_!$ to j^{-1}. A morphism $\varphi : * \to j^{-1}\mathcal{G}$ on $\mathcal{C}/\mathcal{F}$ is the same thing as a rule which assigns to every pair (U, s) a section $\varphi(s) \in \mathcal{G}(U)$ compatible with restriction maps. Hence this is the same thing as a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$. We conclude that $j_!* = \mathcal{F}$. In particular, for every $\mathcal{H} \in \mathit{Sh}(\mathcal{C}/\mathcal{F})$ there is a canonical map

$$j_!\mathcal{H} \to j_!* = \mathcal{F}$$

i.e., we obtain a functor $j'_! : \mathit{Sh}(\mathcal{C}/\mathcal{F}) \to \mathit{Sh}(\mathcal{C})/\mathcal{F}$. An inverse to this functor is the rule which assigns to an object $\varphi : \mathcal{G} \to \mathcal{F}$ of $\mathit{Sh}(\mathcal{C})/\mathcal{F}$ the sheaf

$$a(\mathcal{G}/\mathcal{F}) : (U, s) \longmapsto \{t \in \mathcal{G}(U) \mid \varphi(t) = s\}$$

We omit the verification that $a(\mathcal{G}/\mathcal{F})$ is a sheaf and that a is inverse to $j'_!$. $\square$

04IP **Definition 7.30.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$.

(1) The topos $\mathit{Sh}(\mathcal{C})/\mathcal{F}$ is called the *localization of the topos $\mathit{Sh}(\mathcal{C})$ at $\mathcal{F}$*.
(2) The morphism of topoi $j_\mathcal{F} : \mathit{Sh}(\mathcal{C})/\mathcal{F} \to \mathit{Sh}(\mathcal{C})$ of Lemma 7.30.1 is called the *localization morphism*.

We are going to show that whenever the sheaf $\mathcal{F}$ is equal to $h_U^\#$ for some object U of the site, then the localization of the topos is equal to the category of sheaves on the localization of the site at U. Moreover, we are going to check that any functorialities are compatible with this identification.

04IQ **Lemma 7.30.5.** Let $\mathcal{C}$ be a site. Let $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$. Then $j_\mathcal{F} : \mathit{Sh}(\mathcal{C})/\mathcal{F} \to \mathit{Sh}(\mathcal{C})$ constructed in Lemma 7.30.1 agrees with the morphism of topoi $j_U : \mathit{Sh}(\mathcal{C}/U) \to \mathit{Sh}(\mathcal{C})$ constructed in Section 7.25 via the identification $\mathit{Sh}(\mathcal{C}/U) = \mathit{Sh}(\mathcal{C})/h_U^\#$ of Lemma 7.25.4.

Proof. We have seen in Lemma 7.25.4 that the composition $\mathit{Sh}(\mathcal{C}/U) \to \mathit{Sh}(\mathcal{C})/h_U^\# \to \mathit{Sh}(\mathcal{C})$ is $j_{U!}$. The functor $\mathit{Sh}(\mathcal{C})/h_U^\# \to \mathit{Sh}(\mathcal{C})$ is $j_{\mathcal{F}!}$ by Lemma 7.30.1. Hence $j_{\mathcal{F}!} = j_{U!}$ via the identification. So $j_\mathcal{F}^{-1} = j_U^{-1}$ (by adjointness) and so $j_{\mathcal{F},*} = j_{U,*}$ (by adjointness again). $\square$

04IR **Lemma 7.30.6.** Let $\mathcal{C}$ be a site. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of topoi

$$\begin{tikzcd} \mathit{Sh}(\mathcal{C})/\mathcal{G} \arrow[rr, "j"'] \arrow[dr, "j_\mathcal{G}"'] & & \mathit{Sh}(\mathcal{C})/\mathcal{F} \arrow[dl, "j_\mathcal{F}"] \\ & \mathit{Sh}(\mathcal{C}) & \end{tikzcd}$$

where $j = j_{\mathcal{G}/\mathcal{F}}$ is the localization of the topos $Sh(\mathcal{C})/\mathcal{F}$ at the object $\mathcal{G}/\mathcal{F}$. In particular we have

$$j^{-1}(\mathcal{H} \to \mathcal{F}) = (\mathcal{H} \times_\mathcal{F} \mathcal{G} \to \mathcal{G})$$

and

$$j_!(\mathcal{E} \xrightarrow{e} \mathcal{F}) = (\mathcal{E} \xrightarrow{s \circ e} \mathcal{G}).$$

Proof. The description of j^{-1} and $j_!$ comes from the description of those functors in Lemma 7.30.1. The equality of functors $j_{\mathcal{G}!} = j_{\mathcal{F}!} \circ j_!$ is clear from the description of these functors (as forgetful functors). By adjointness we also obtain the equalities $j_\mathcal{G}^{-1} = j^{-1} \circ j_\mathcal{F}^{-1}$, and $j_{\mathcal{G},*} = j_{\mathcal{F},*} \circ j_*$. □

04IS Lemma 7.30.7. Assume $\mathcal{C}$ and $s : \mathcal{G} \to \mathcal{F}$ are as in Lemma 7.30.6. If $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and $s : \mathcal{G} \to \mathcal{F}$ comes from a morphism $V \to U$ of $\mathcal{C}$ then the diagram in Lemma 7.30.6 is identified with diagram (7.25.8.1) via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 7.25.4.

Proof. This is true because the descriptions of j^{-1} agree. See Lemma 7.25.9 and Lemma 7.30.6. □

7.31. Localization and morphisms of topoi

04IT This section is the analogue of Section 7.28 for morphisms of topoi.

04H1 Lemma 7.31.1. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C})/\mathcal{F} \arrow[r, "j_\mathcal{F}"'] \arrow[d, "f'"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D})/\mathcal{G} \arrow[r, "j_\mathcal{G}"] & Sh(\mathcal{D}). \end{tikzcd}$$

The morphism f' is characterized by the property that

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F})$$

and we have $f'_* j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1} f_*$.

Proof. Since the statement is about topoi and does not refer to the underlying sites we may change sites at will. Hence by the discussion in Remark 7.29.7 we may assume that f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Proposition 7.14.7 between sites having all finite limits and subcanonical topologies, and such that $\mathcal{G} = h_V$ for some object V of $\mathcal{D}$. Then $\mathcal{F} = f^{-1}h_V = h_{u(V)}$ by Lemma 7.13.5. By Lemma 7.28.1 we obtain a commutative diagram of morphisms of topoi

$$\begin{tikzcd} Sh(\mathcal{C}/U) \arrow[r, "j_U"'] \arrow[d, "f'"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}/V) \arrow[r, "j_V"] & Sh(\mathcal{D}), \end{tikzcd}$$

and we have $f'_* j_U^{-1} = j_V^{-1} f_*$. By Lemma 7.30.5 we may identify $j_\mathcal{F}$ and j_U and $j_\mathcal{G}$ and j_V. The description of $(f')^{-1}$ is given in Lemma 7.28.1. □

04IU Lemma 7.31.2. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Set $U = u(V)$. Set $\mathcal{G} = h_V^\#$, and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$ (see Lemma 7.13.5). Then the diagram of morphisms of topoi of Lemma 7.31.1 agrees with the diagram of morphisms of topoi of Lemma 7.28.1 via the identifications $j_\mathcal{F} = j_U$ and $j_\mathcal{G} = j_V$ of Lemma 7.30.5.

Proof. This is not a complete triviality as the choice of morphism of sites giving rise to f made in the proof of Lemma 7.31.1 may be different from the morphisms of sites given to us in the lemma. But in both cases the functor $(f')^{-1}$ is described by the same rule. Hence they agree and the associated morphism of topoi is the same. Some details omitted. □

04IV Lemma 7.31.3. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G} \in Sh(\mathcal{D})$, $\mathcal{F} \in Sh(\mathcal{C})$ and $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{C})/\mathcal{F} \arrow[r, "j_\mathcal{F}"'] \arrow[d, "f_s"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D})/\mathcal{G} \arrow[r, "j_\mathcal{G}"] & Sh(\mathcal{D}). \end{tikzcd}$$

We have $f_s = f' \circ j_{\mathcal{F}/f^{-1}\mathcal{G}}$ where $f' : Sh(\mathcal{C})/f^{-1}\mathcal{G} \to Sh(\mathcal{D})/\mathcal{F}$ is as in Lemma 7.31.1 and $j_{\mathcal{F}/f^{-1}\mathcal{G}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})/f^{-1}\mathcal{G}$ is as in Lemma 7.30.6. The functor $(f_s)^{-1}$ is described by the rule

$$(f_s)^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, f^{-1}\mathcal{G}, s} \mathcal{F} \to \mathcal{F}).$$

Finally, given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$ and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "s'"'] \arrow[d, "a"'] & f^{-1}\mathcal{G}' \arrow[d, "f^{-1}b"] \\ \mathcal{F} \arrow[r, "s"] & f^{-1}\mathcal{G} \end{tikzcd}$$

commutes, then the diagram

$$\begin{tikzcd} Sh(\mathcal{C})/\mathcal{F}' \arrow[r, "j_{\mathcal{F}'/\mathcal{F}}"'] \arrow[d, "f_{s'}"'] & Sh(\mathcal{C})/\mathcal{F} \arrow[d, "f_s"] \\ Sh(\mathcal{D})/\mathcal{G}' \arrow[r, "j_{\mathcal{G}'/\mathcal{G}}"] & Sh(\mathcal{D})/\mathcal{G}. \end{tikzcd}$$

commutes.

Proof. The commutativity of the first square follows from the commutativity of the diagram in Lemma 7.30.6 and the commutativity of the diagram in Lemma 7.31.1. The description of f_s^{-1} follows on combining the descriptions of $(f')^{-1}$ in Lemma 7.31.1 with the description of $(j_{\mathcal{F}/f^{-1}\mathcal{G}})^{-1}$ in Lemma 7.30.6. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{f^{-1}\mathcal{G}, s} \mathcal{F} \times_\mathcal{F} \mathcal{F}' = f^{-1}(\mathcal{H} \times_\mathcal{G} \mathcal{G}') \times_{f^{-1}\mathcal{G}', s'} \mathcal{F}'$$

which is formal. □

04IW Lemma 7.31.4. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Let $c : U \to u(V)$ be a morphism. Set $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$. Let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be the map induced by c. Then the diagram of morphisms of topoi of Lemma 7.28.3 agrees with the diagram of morphisms of topoi of Lemma 7.31.3 via the identifications $j_\mathcal{F} = j_U$ and $j_\mathcal{G} = j_V$ of Lemma 7.30.5.

Proof. This follows on combining Lemmas 7.30.7 and 7.31.2. □

7.32. Points

00Y3

00Y4 Definition 7.32.1. Let $\mathcal{C}$ be a site. A point of the topos $Sh(\mathcal{C})$ is a morphism of topoi p from $Sh(pt)$ to $Sh(\mathcal{C})$.

We will define a point of a site in terms of a functor $u : \mathcal{C} \to \text{Sets}$. It will turn out later that u will define a morphism of sites which gives rise to a point of the topos associated to $\mathcal{C}$, see Lemma 7.32.8.

Let $\mathcal{C}$ be a site. Let $p = u$ be a functor $u : \mathcal{C} \to \text{Sets}$. This curious language is introduced because it seems funny to talk about neighbourhoods of functors; so we think of a "point" p as a geometric thing which is given by a categorical datum, namely the functor u. The fact that p is actually equal to u does not matter. A neighbourhood of p is a pair (U, x) with $U \in \text{Ob}(\mathcal{C})$ and $x \in u(U)$. A morphism of neighbourhoods $(V, y) \to (U, x)$ is given by a morphism $\alpha : V \to U$ of $\mathcal{C}$ such that $u(\alpha)(y) = x$. Note that the category of neighbourhoods isn't a "big" category.

We define the stalk of a presheaf $\mathcal{F}$ at p as

04EH (7.32.1.1)
$$\mathcal{F}_p = \text{colim}_{\{(U,x)\}^{opp}} \mathcal{F}(U).$$

The colimit is over the opposite of the category of neighbourhoods of p. In other words, an element of $\mathcal{F}_p$ is given by a triple (U, x, s), where (U, x) is a neighbourhood of p and $s \in \mathcal{F}(U)$. Equality of triples is the equivalence relation generated by $(U, x, s) \sim (V, y, \alpha^*s)$ when α is as above.

Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets, then we get a canonical map of stalks $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$. Thus we obtain a stalk functor

$$\text{PSh}(\mathcal{C}) \longrightarrow \text{Sets}, \quad \mathcal{F} \longmapsto \mathcal{F}_p.$$

We have defined the stalk functor using any functor $p = u : \mathcal{C} \to \text{Sets}$. No conditions are necessary for the definition to work[7]. On the other hand, it is probably better not to use this notion unless p actually is a point (see definition below), since in general the stalk functor does not have good properties.

00Y5 Definition 7.32.2. Let $\mathcal{C}$ be a site. A point p of the site $\mathcal{C}$ is given by a functor $u : \mathcal{C} \to \text{Sets}$ such that

(1) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ the map $\coprod u(U_i) \to u(U)$ is surjective.
(2) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ and every morphism $V \to U$ the maps $u(U_i \times_U V) \to u(U_i) \times_{u(U)} u(V)$ are bijective.
(3) The stalk functor $Sh(\mathcal{C}) \to \text{Sets}$, $\mathcal{F} \mapsto \mathcal{F}_p$ is left exact.

[7]One should try to avoid the case where $u(U) = \emptyset$ for all U.

The conditions should be familiar since they are modeled after those of Definitions 7.13.1 and 7.14.1. Note that (3) implies that $*_p = \{*\}$, see Example 7.10.2. Hence $u(U) \neq \emptyset$ for at least some U (because the empty colimit produces the empty set). We will show below (Lemma 7.32.7) that this does give rise to a point of the topos $Sh(\mathcal{C})$. Before we do so, we prove some lemmas for general functors u.

00Y6 Lemma 7.32.3. Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to \textit{Sets}$ be a functor. There are functorial isomorphisms $(h_U)_p = u(U)$ for $U \in \text{Ob}(\mathcal{C})$.

Proof. An element of $(h_U)_p$ is given by a triple (V, y, f), where $V \in \text{Ob}(\mathcal{C})$, $y \in u(V)$ and $f \in h_U(V) = \text{Mor}_\mathcal{C}(V, U)$. Two such (V, y, f), (V', y', f') determine the same object if there exists a morphism $\phi : V \to V'$ such that $u(\phi)(y) = y'$ and $f' \circ \phi = f$, and in general you have to take chains of identities like this to get the correct equivalence relation. In any case, every (V, y, f) is equivalent to the element $(U, u(f)(y), \text{id}_U)$. If ϕ exists as above, then the triples (V, y, f), (V', y', f') determine the same triple $(U, u(f)(y), \text{id}_U) = (U, u(f')(y'), \text{id}_U)$. This proves that the map $u(U) \to (h_U)_p$, $x \mapsto$ class of (U, x, id_U) is bijective. $\square$

Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to \textit{Sets}$ be a functor. In analogy with the constructions in Section 7.5 given a set E we define a presheaf u^pE by the rule

04EI (7.32.3.1) $$U \longmapsto u^pE(U) = \text{Mor}_{\textit{Sets}}(u(U), E) = \text{Map}(u(U), E).$$

This defines a functor $u^p : \textit{Sets} \to \textit{PSh}(\mathcal{C})$, $E \mapsto u^pE$.

00Y7 Lemma 7.32.4. For any functor $u : \mathcal{C} \to \textit{Sets}$. The functor u^p is a right adjoint to the stalk functor on presheaves.

Proof. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. Let E be a set. A morphism $\mathcal{F} \to u^pE$ is given by a compatible system of maps $\mathcal{F}(U) \to \text{Map}(u(U), E)$, i.e., a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. And by definition of $\mathcal{F}_p$ a map $\mathcal{F}_p \to E$ is given by a rule associating with each triple (U, x, σ) an element in E such that equivalent triples map to the same element, see discussion surrounding Equation (7.32.1.1). This also means a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. $\square$

In analogy with Section 7.13 we have the following lemma.

00Y8 Lemma 7.32.5. Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to \textit{Sets}$ be a functor. Suppose that for every covering $\{U_i \to U\}$ of $\mathcal{C}$

(1) the map $\coprod u(U_i) \to u(U)$ is surjective, and
(2) the maps $u(U_i \times_U U_j) \to u(U_i) \times_{u(U)} u(U_j)$ are surjective.

Then we have

(1) the presheaf u^pE is a sheaf for all sets E, denote it u^sE,
(2) the stalk functor $Sh(\mathcal{C}) \to \textit{Sets}$ and the functor $u^s : \textit{Sets} \to Sh(\mathcal{C})$ are adjoint, and
(3) we have $\mathcal{F}_p = \mathcal{F}_p^\#$ for every presheaf of sets $\mathcal{F}$.

Proof. The first assertion is immediate from the definition of a sheaf, assumptions (1) and (2), and the definition of u^pE. The second is a restatement of the adjointness of u^p and the stalk functor (Lemma 7.32.4) restricted to sheaves. The third assertion follows as, for any set E, we have

$$\text{Map}(\mathcal{F}_p, E) = \text{Mor}_{\textit{PSh}(\mathcal{C})}(\mathcal{F}, u^pE) = \text{Mor}_{Sh(\mathcal{C})}(\mathcal{F}^\#, u^sE) = \text{Map}(\mathcal{F}^\#_p, E)$$

by the adjointness property of sheafification. $\square$

In particular Lemma 7.32.5 holds when $p = u$ is a point. In this case we think of the sheaf u^sE as the "skyscraper" sheaf with value E at p.

00Y9 Definition 7.32.6. Let p be a point of the site $\mathcal{C}$ given by the functor u. For a set E we define $p_*E = u^sE$ the sheaf described in Lemma 7.32.5 above. We sometimes call this a skyscraper sheaf.

In particular we have the following adjointness property of skyscraper sheaves and stalks:

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, p_*E) = \mathrm{Map}(\mathcal{F}_p, E)$$

This motivates the notation $p^{-1}\mathcal{F} = \mathcal{F}_p$ which we will sometimes use.

00YA Lemma 7.32.7. Let $\mathcal{C}$ be a site.

(1) Let p be a point of the site $\mathcal{C}$. Then the pair of functors (p_*, p^{-1}) introduced above define a morphism of topoi $Sh(pt) \to Sh(\mathcal{C})$.
(2) Let $p = (p_*, p^{-1})$ be a point of the topos $Sh(\mathcal{C})$. Then the functor $u : U \mapsto p^{-1}(h_U^\#)$ gives rise to a point p' of the site $\mathcal{C}$ whose associated morphism of topoi $(p'_*, (p')^{-1})$ is equal to p.

Proof. Proof of (1). By the above the functors p_* and p^{-1} are adjoint. The functor p^{-1} is required to be exact by Definition 7.32.2. Hence the conditions imposed in Definition 7.15.1 are all satisfied and we see that (1) holds.

Proof of (2). Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then $\coprod (h_{U_i})^\# \to h_U^\#$ is surjective, see Lemma 7.12.4. Since p^{-1} is exact (by definition of a morphism of topoi) we conclude that $\coprod u(U_i) \to u(U)$ is surjective. This proves part (1) of Definition 7.32.2. Sheafification is exact, see Lemma 7.10.14. Hence if $U \times_V W$ exists in $\mathcal{C}$, then

$$h_{U \times_V W}^\# = h_U^\# \times_{h_V^\#} h_W^\#$$

and we see that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ since p^{-1} is exact. This proves part (2) of Definition 7.32.2. Let $p' = u$, and let $\mathcal{F}_{p'}$ be the stalk functor defined by Equation (7.32.1.1) using u. There is a canonical comparison map $c : \mathcal{F}_{p'} \to \mathcal{F}_p = p^{-1}\mathcal{F}$. Namely, given a triple (U, x, σ) representing an element ξ of $\mathcal{F}_{p'}$ we think of σ as a map $\sigma : h_U^\# \to \mathcal{F}$ and we can set $c(\xi) = p^{-1}(\sigma)(x)$ since $x \in u(U) = p^{-1}(h_U^\#)$. By Lemma 7.32.3 we see that $(h_U)_{p'} = u(U)$. Since conditions (1) and (2) of Definition 7.32.2 hold for p' we also have $(h_U^\#)_{p'} = (h_U)_{p'}$ by Lemma 7.32.5. Hence we have

$$(h_U^\#)_{p'} = (h_U)_{p'} = u(U) = p^{-1}(h_U^\#)$$

We claim this bijection equals the comparison map $c : (h_U^\#)_{p'} \to p^{-1}(h_U^\#)$ (verification omitted). Any sheaf on $\mathcal{C}$ is a coequalizer of maps of coproducts of sheaves of the form $h_U^\#$, see Lemma 7.12.5. The stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ and the functor p^{-1} commute with arbitrary colimits (as they are both left adjoints). We conclude c is an isomorphism for every sheaf $\mathcal{F}$. Thus the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ is isomorphic to p^{-1} and we in particular see that it is exact. This proves condition (3) of Definition 7.32.2 holds and p' is a point. The final assertion has already been shown above, since we saw that $p^{-1} = (p')^{-1}$. □

Actually a point always corresponds to a morphism of sites as we show in the following lemma.

04EL **Lemma 7.32.8.** Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to \text{Sets}$. Let S_0 be an infinite set such that $u(U) \subset S_0$ for all $U \in \text{Ob}(\mathcal{C})$. Let $\mathcal{S}$ be the site constructed out of the powerset $S = \mathcal{P}(S_0)$ in Remark 7.15.3. Then

(1) there is an equivalence $i : Sh(pt) \to Sh(\mathcal{S})$,
(2) the functor $u : \mathcal{C} \to \mathcal{S}$ induces a morphism of sites $f : \mathcal{S} \to \mathcal{C}$, and
(3) the composition
$$Sh(pt) \to Sh(\mathcal{S}) \to Sh(\mathcal{C})$$
is the morphism of topoi (p_*, p^{-1}) of Lemma 7.32.7.

Proof. Part (1) we saw in Remark 7.15.3. Moreover, recall that the equivalence associates to the set E the sheaf i_*E on $\mathcal{S}$ defined by the rule $V \mapsto \text{Mor}_{\text{Sets}}(V, E)$. Part (2) is clear from the definition of a point of $\mathcal{C}$ (Definition 7.32.2) and the definition of a morphism of sites (Definition 7.14.1). Finally, consider f_*i_*E. By construction we have
$$f_*i_*E(U) = i_*E(u(U)) = \text{Mor}_{\text{Sets}}(u(U), E)$$
which is equal to $p_*E(U)$, see Equation (7.32.3.1). This proves (3). $\square$

Contrary to what happens in the topological case it is not always true that the stalk of the skyscraper sheaf with value E is E. Here is what is true in general.

05UX **Lemma 7.32.9.** Let $\mathcal{C}$ be a site. Let $p : Sh(pt) \to Sh(\mathcal{C})$ be a point of the topos associated to $\mathcal{C}$. For any set E there are canonical maps
$$E \longrightarrow (p_*E)_p \longrightarrow E$$
whose composition is id_E.

Proof. There is always an adjunction map $(p_*E)_p = p^{-1}p_*E \to E$. This map is an isomorphism when $E = \{*\}$ because p_* and p^{-1} are both left exact, hence transform the final object into the final object. Hence given $e \in E$ we can consider the map $i_e : \{*\} \to E$ which gives
$$\begin{tikzcd} p^{-1}p_*\{*\} \arrow[r, "p^{-1}p_*i_e"'] \arrow[d, "\cong"'] & p^{-1}p_*E \arrow[d] \\ \{*\} \arrow[r, "i_e"] & E \end{tikzcd}$$
whence the map $E \to (p_*E)_p = p^{-1}p_*E$ as desired. $\square$

05UY **Lemma 7.32.10.** Let $\mathcal{C}$ be a site. Let $p : Sh(pt) \to Sh(\mathcal{C})$ be a point of the topos associated to $\mathcal{C}$. The functor $p_* : \text{Sets} \to Sh(\mathcal{C})$ has the following properties: It commutes with arbitrary limits, it is left exact, it is faithful, it transforms surjections into surjections, it commutes with coequalizers, it reflects injections, it reflects surjections, and it reflects isomorphisms.

Proof. Because p_* is a right adjoint it commutes with arbitrary limits and it is left exact. The fact that $p^{-1}p_*E \to E$ is a canonically split surjection implies that p_* is faithful, reflects injections, reflects surjections, and reflects isomorphisms. By Lemma 7.32.7 we may assume that p comes from a point $u : \mathcal{C} \to \text{Sets}$ of the underlying site $\mathcal{C}$. In this case the sheaf p_*E is given by
$$p_*E(U) = \text{Mor}_{\text{Sets}}(u(U), E)$$

see Equation (7.32.3.1) and Definition 7.32.6. It follows immediately from this formula that p_* transforms surjections into surjections and coequalizers into coequalizers. $\square$

7.33. Constructing points

05UZ In this section we give criteria for when a functor from a site to the category of sets defines a point of that site.

0F4E **Lemma 7.33.1.** Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to \textit{Sets}$ be a functor. If the category of neighbourhoods of p is cofiltered, then the stalk functor (7.32.1.1) is left exact.

Proof. Let $\mathcal{I} \to \textit{Sh}(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a finite diagram of sheaves. We have to show that the stalk of the limit of this system agrees with the limit of the stalks. Let $\mathcal{F}$ be the limit of the system as a presheaf. According to Lemma 7.10.1 this is a sheaf and it is the limit in the category of sheaves. Hence we have to show that $\mathcal{F}_p = \lim_\mathcal{I} \mathcal{F}_{i,p}$. Recall also that $\mathcal{F}$ has a simple description, see Section 7.4. Thus we have to show that

$$\lim_i \operatorname{colim}_{\{(U,x)\}^{opp}} \mathcal{F}_i(U) = \operatorname{colim}_{\{(U,x)\}^{opp}} \lim_i \mathcal{F}_i(U).$$

This holds, by Categories, Lemma 4.19.2, because the opposite of the category of neighbourhoods is filtered by assumption. $\square$

00YB **Lemma 7.33.2.** Let $\mathcal{C}$ be a site. Assume that $\mathcal{C}$ has a final object X and fibred products. Let $p = u : \mathcal{C} \to \textit{Sets}$ be a functor such that

(1) $u(X)$ is a singleton set, and
(2) for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is bijective.

Then the the category of neighbourhoods of p is cofiltered and consequently the stalk functor $\textit{Sh}(\mathcal{C}) \to \textit{Sets}$, $\mathcal{F} \to \mathcal{F}_p$ commutes with finite limits.

Proof. Please note the analogy with Lemma 7.5.2. The assumptions on $\mathcal{C}$ imply that $\mathcal{C}$ has finite limits. See Categories, Lemma 4.18.4. Assumption (1) implies that the category of neighbourhoods is nonempty. Suppose (U, x) and (V, y) are neighbourhoods. Then $u(U \times V) = u(U \times_X V) = u(U) \times_{u(X)} u(V) = u(U) \times u(V)$ by (2). Hence there exists a neighbourhood $(U \times_X V, z)$ mapping to both (U, x) and (V, y). Let $a, b : (V, y) \to (U, x)$ be two morphisms in the category of neighbourhoods. Let W be the equalizer of $a, b : V \to U$. As in the proof of Categories, Lemma 4.18.4 we may write W in terms of fibre products:

$$W = (V \times_{a,U,b} V) \times_{(pr_1, pr_2), V \times V, \Delta} V$$

The bijectivity in (2) guarantees there exists an element $z \in u(W)$ which maps to $((y, y), y)$. Then $(W, z) \to (V, y)$ equalizes a, b as desired. The "consequently" clause is Lemma 7.33.1. $\square$

00YC **Proposition 7.33.3.** Let $\mathcal{C}$ be a site. Assume that finite limits exist in $\mathcal{C}$. (I.e., $\mathcal{C}$ has fibre products, and a final object.) A point p of such a site $\mathcal{C}$ is given by a functor $u : \mathcal{C} \to \textit{Sets}$ such that

(1) u commutes with finite limits, and
(2) if $\{U_i \to U\}$ is a covering, then $\coprod_i u(U_i) \to u(U)$ is surjective.

Proof. Suppose first that p is a point (Definition 7.32.2) given by a functor u. Condition (2) is satisfied directly from the definition of a point. By Lemma 7.32.3 we have $(h_U)_p = u(U)$. By Lemma 7.32.5 we have $(h_U^\#)_p = (h_U)_p$. Thus we see that u is equal to the composition of functors

$$\mathcal{C} \xrightarrow{h} \textit{PSh}(\mathcal{C}) \xrightarrow{\#} \textit{Sh}(\mathcal{C}) \xrightarrow{()_p} \textit{Sets}$$

Each of these functors is left exact, and hence we see u satisfies (1).

Conversely, suppose that u satisfies (1) and (2). In this case we immediately see that u satisfies the first two conditions of Definition 7.32.2. And its stalk functor is exact, because it is a left adjoint by Lemma 7.32.5 and it commutes with finite limits by Lemma 7.33.2. $\square$

00YD Remark 7.33.4. In fact, let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has a final object X and fibre products. Let $p = u : \mathcal{C} \to \textit{Sets}$ be a functor such that

(1) $u(X) = \{*\}$ a singleton, and
(2) for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is surjective.
(3) for every covering $\{U_i \to U\}$ the map $\coprod u(U_i) \to u(U)$ is surjective.

Then, in general, p is not a point of $\mathcal{C}$. An example is the category $\mathcal{C}$ with two objects $\{U, X\}$ and exactly one non-identity arrow, namely $U \to X$. We endow $\mathcal{C}$ with the trivial topology, i.e., the only coverings are $\{U \to U\}$ and $\{X \to X\}$. A sheaf $\mathcal{F}$ is the same thing as a presheaf and consists of a triple $(A, B, A \to B)$: namely $A = \mathcal{F}(X)$, $B = \mathcal{F}(U)$ and $A \to B$ is the restriction mapping corresponding to $U \to X$. Note that $U \times_X U = U$ so fibre products exist. Consider the functor $u = p$ with $u(X) = \{*\}$ and $u(U) = \{*_1, *_2\}$. This satisfies (1), (2), and (3), but the corresponding stalk functor (7.32.1.1) is the functor

$$(A, B, A \to B) \longmapsto B \amalg_A B$$

which isn't exact. Namely, consider $(\emptyset, \{1\}, \emptyset \to \{1\}) \to (\{1\}, \{1\}, \{1\} \to \{1\})$ which is an injective map of sheaves, but is transformed into the noninjective map of sets

$$\{1\} \amalg \{1\} \longrightarrow \{1\} \amalg_{\{1\}} \{1\}$$

by the stalk functor.

00YE Example 7.33.5. Let X be a topological space. Let X_{Zar} be the site of Example 7.6.4. Let $x \in X$ be a point. Consider the functor

$$u : X_{Zar} \longrightarrow \textit{Sets}, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \{*\} & \text{if} \quad x \in U \end{cases}$$

This functor commutes with product and fibred products, and turns coverings into surjective families of maps. Hence we obtain a point p of the site X_{Zar}. It is immediately verified that the stalk functor agrees with the stalk at x defined in Sheaves, Section 6.11.

04EJ Example 7.33.6. Let X be a topological space. What are the points of the topos $\textit{Sh}(X)$? To see this, let X_{Zar} be the site of Example 7.6.4. By Lemma 7.32.7 a point of $\textit{Sh}(X)$ corresponds to a point of this site. Let p be a point of the site X_{Zar} given by the functor $u : X_{Zar} \to \textit{Sets}$. We are going to use the characterization of such a u in Proposition 7.33.3. This implies immediately that $u(\emptyset) = \emptyset$ and $u(U \cap V) = u(U) \times u(V)$. In particular we have $u(U) = u(U) \times u(U)$ via the diagonal map which implies that $u(U)$ is either a singleton or empty. Moreover, if $U = \bigcup U_i$ is an open covering then

$$u(U) = \emptyset \Rightarrow \forall i,\ u(U_i) = \emptyset \quad \text{and} \quad u(U) \neq \emptyset \Rightarrow \exists i,\ u(U_i) \neq \emptyset.$$

We conclude that there is a unique largest open $W \subset X$ with $u(W) = \emptyset$, namely the union of all the opens U with $u(U) = \emptyset$. Let $Z = X \setminus W$. If $Z = Z_1 \cup Z_2$ with $Z_i \subset Z$ closed, then $W = (X \setminus Z_1) \cap (X \setminus Z_2)$ so $\emptyset = u(W) = u(X \setminus Z_1) \times u(X \setminus Z_2)$ and we conclude that $u(X \setminus Z_1) = \emptyset$ or that $u(X \setminus Z_2) = \emptyset$. This means that $X \setminus Z_1 = W$ or that $X \setminus Z_2 = W$. In other words, Z is irreducible. Now we see that u is described by the rule

$$u : X_{Zar} \longrightarrow \text{Sets}, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad Z \cap U = \emptyset \\ \{*\} & \text{if} \quad Z \cap U \neq \emptyset \end{cases}$$

Note that for any irreducible closed $Z \subset X$ this functor satisfies assumptions (1), (2) of Proposition 7.33.3 and hence defines a point. In other words we see that points of the site X_{Zar} are in one-to-one correspondence with irreducible closed subsets of X. In particular, if X is a sober topological space, then points of X_{Zar} and points of X are in one to one correspondence, see Example 7.33.5.

00YF Example 7.33.7. Consider the site $\mathcal{T}_G$ described in Example 7.6.5 and Section 7.9. The forgetful functor $u : \mathcal{T}_G \to \text{Sets}$ commutes with products and fibred products and turns coverings into surjective families. Hence it defines a point of $\mathcal{T}_G$. We identify $Sh(\mathcal{T}_G)$ and G-Sets. The stalk functor

$$p^{-1} : Sh(\mathcal{T}_G) = G\text{-Sets} \longrightarrow \text{Sets}$$

is the forgetful functor. The pushforward p_* is the functor

$$\text{Sets} \longrightarrow Sh(\mathcal{T}_G) = G\text{-Sets}$$

which maps a set S to the G-set $\text{Map}(G, S)$ with action $g \cdot \psi = \psi \circ R_g$ where R_g is right multiplication. In particular we have $p^{-1}p_*S = \text{Map}(G, S)$ as a set and the maps $S \to \text{Map}(G, S) \to S$ of Lemma 7.32.9 are the obvious ones.

08RH Example 7.33.8. Let $\mathcal{C}$ be a category endowed with the chaotic topology (Example 7.6.6). For every object U_0 of $\mathcal{C}$ the functor $u : U \mapsto \text{Mor}_\mathcal{C}(U_0, U)$ defines a point p of $\mathcal{C}$. Namely, conditions (1) and (2) of Definition 7.32.2 are immediate as the only coverings are given by identity maps. Condition (2) holds because $\mathcal{F}_p = \mathcal{F}(U_0)$ and since the topology is discrete taking sections over U_0 is an exact functor.

7.34. Points and morphisms of topoi

05V0 In this section we make a few remarks about points and morphisms of topoi.

0F4F Lemma 7.34.1. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let $v : \mathcal{D} \to \text{Sets}$ be a functor and set $w = v \circ u$. Denote q, resp., p the stalk functor (7.32.1.1) associated to v, resp. w. Then $(u_p\mathcal{F})_q = \mathcal{F}_p$ functorially in the presheaf $\mathcal{F}$ on $\mathcal{C}$.

Proof. This is a simple categorical fact. We have

$$\begin{aligned} (u_p\mathcal{F})_q &= \text{colim}_{(V,y)} \text{colim}_{U,\phi:V \to u(U)} \mathcal{F}(U) \\ &= \text{colim}_{(V,y,U,\phi:V \to u(U))} \mathcal{F}(U) \\ &= \text{colim}_{(U,x)} \mathcal{F}(U) \\ &= \mathcal{F}_p \end{aligned}$$

The first equality holds by the definition of u_p and the definition of the stalk functor. Observe that $y \in v(V)$. In the second equality we simply combine colimits. To see the third equality we apply Categories, Lemma 4.17.5 to the functor F of diagram categories defined by the rule

$$F((V, y, U, \phi : V \to u(U))) = (U, v(\phi)(y)).$$

This makes sense because $w(U) = v(u(U))$. Let us check the hypotheses of Categories, Lemma 4.17.5. Observe that F has a right inverse, namely $(U, x) \mapsto (u(U), x, U, \mathrm{id} : u(U) \to u(U))$. Again this makes sense because $x \in w(U) = v(u(U))$. On the other hand, there is always a morphism

$$(V, y, U, \phi : V \to u(U)) \longrightarrow (u(U), v(\phi)(y), U, \mathrm{id} : u(U) \to u(U))$$

in the fibre category over (U, x) which shows the fibre categories are connected. The fourth and final equality is clear. □

05V1 Lemma 7.34.2. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let q be a point of $\mathcal{D}$ given by the functor $v : \mathcal{D} \to \textit{Sets}$, see Definition 7.32.2. Then the functor $v \circ u : \mathcal{C} \to \textit{Sets}$ defines a point p of $\mathcal{C}$ and moreover there is a canonical identification

$$(f^{-1}\mathcal{F})_q = \mathcal{F}_p$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

First proof Lemma 7.34.2. Note that since u is continuous and since v defines a point, it is immediate that $v \circ u$ satisfies conditions (1) and (2) of Definition 7.32.2. Let us prove the displayed equality. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then

$$(f^{-1}\mathcal{F})_q = (u_s\mathcal{F})_q = (u_p\mathcal{F})_q = \mathcal{F}_p$$

The first equality since $f^{-1} = u_s$, the second equality by Lemma 7.32.5, and the third by Lemma 7.34.1. Hence now we see that p also satisfies condition (3) of Definition 7.32.2 because it is a composition of exact functors. This finishes the proof. □

Second proof Lemma 7.34.2. By Lemma 7.32.8 we may factor (q_*, q^{-1}) as

$$Sh(pt) \xrightarrow{i} Sh(\mathcal{S}) \xrightarrow{h} Sh(\mathcal{D})$$

where the second morphism of topoi comes from a morphism of sites $h : \mathcal{S} \to \mathcal{D}$ induced by the functor $v : \mathcal{D} \to \mathcal{S}$ (which makes sense as $\mathcal{S} \subset \textit{Sets}$ is a full subcategory containing every object in the image of v). By Lemma 7.14.4 the composition $v \circ u : \mathcal{C} \to \mathcal{S}$ defines a morphism of sites $g : \mathcal{S} \to \mathcal{C}$. In particular, the functor $v \circ u : \mathcal{C} \to \mathcal{S}$ is continuous which by the definition of the coverings in $\mathcal{S}$, see Remark 7.15.3, means that $v \circ u$ satisfies conditions (1) and (2) of Definition 7.32.2. On the other hand, we see that

$$g_* i_* E(U) = i_* E(v(u(U)) = \mathrm{Mor}_{\textit{Sets}}(v(u(U)), E)$$

by the construction of i in Remark 7.15.3. Note that this is the same as the formula for which is equal to $(v \circ u)^p E$, see Equation (7.32.3.1). By Lemma 7.32.5 the functor $g_* i_* = (v \circ u)^p = (v \circ u)^s$ is right adjoint to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_q$. Hence we see that the stalk functor p^{-1} is canonically isomorphic to $i^{-1} \circ g^{-1}$. Hence it is exact and we conclude that p is a point. Finally, as we have $g = f \circ h$ by construction we see that $p^{-1} = i^{-1} \circ h^{-1} \circ f^{-1} = q^{-1} \circ f^{-1}$, i.e., we have the displayed formula of the lemma. □

05V2 **Lemma 7.34.3.** Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $q : Sh(pt) \to Sh(\mathcal{D})$ be a point. Then $p = f \circ q$ is a point of the topos $Sh(\mathcal{C})$ and we have a canonical identification

$$(f^{-1}\mathcal{F})_q = \mathcal{F}_p$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

Proof. This is immediate from the definitions and the fact that we can compose morphisms of topoi. $\square$

7.35. Localization and points

04EK In this section we show that points of a localization $\mathcal{C}/U$ are constructed in a simple manner from the points of $\mathcal{C}$.

04H2 **Lemma 7.35.1.** Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to \text{Sets}$. Let U be an object of $\mathcal{C}$ and let $x \in u(U)$. The functor

$$v : \mathcal{C}/U \longrightarrow \text{Sets}, \quad (\varphi : V \to U) \longmapsto \{y \in u(V) \mid u(\varphi)(y) = x\}$$

defines a point q of the site $\mathcal{C}/U$ such that the diagram

$$\begin{tikzcd} & Sh(pt) \arrow[dl, "q"'] \arrow[d, "p"] \\ Sh(\mathcal{C}/U) \arrow[r, "j_U"] & Sh(\mathcal{C}) \end{tikzcd}$$

commutes. In other words $\mathcal{F}_p = (j_U^{-1}\mathcal{F})_q$ for any sheaf on $\mathcal{C}$.

Proof. Choose S and $\mathcal{S}$ as in Lemma 7.32.8. We may identify $Sh(pt) = Sh(\mathcal{S})$ as in that lemma, and we may write $p = f : Sh(\mathcal{S}) \to Sh(\mathcal{C})$ for the morphism of topoi induced by u. By Lemma 7.28.1 we get a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{S}/u(U)) \arrow[r, "j_{u(U)}"'] \arrow[d, "p'"'] & Sh(\mathcal{S}) \arrow[d, "p"] \\ Sh(\mathcal{C}/U) \arrow[r, "j_U"] & Sh(\mathcal{C}), \end{tikzcd}$$

where p' is given by the functor $u' : \mathcal{C}/U \to \mathcal{S}/u(U)$, $V/U \mapsto u(V)/u(U)$. Consider the functor $j_x : \mathcal{S} \cong \mathcal{S}/x$ obtained by assigning to a set E the set E endowed with the constant map $E \to u(U)$ with value x. Then j_x is a fully faithful cocontinuous functor which has a continuous right adjoint $v_x : (\psi : E \to u(U)) \mapsto \psi^{-1}(\{x\})$. Note that $j_{u(U)} \circ j_x = \text{id}_{\mathcal{S}}$, and $v_x \circ u' = v$. These observations imply that we have the following commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{S}) \arrow[dr, "a"] \arrow[ddr, "q"'] \arrow[rrdd, bend left, "p"] & & \\ & Sh(\mathcal{S}/u(U)) \arrow[r, "j_{u(U)}"'] \arrow[d, "p'"] & Sh(\mathcal{S}) \arrow[d, "p"] \\ & Sh(\mathcal{C}/U) \arrow[r, "j_U"] & Sh(\mathcal{C}) \end{tikzcd}$$

Namely:

(1) The morphism $a : Sh(\mathcal{S}) \to Sh(\mathcal{S}/u(U))$ is the morphism of topoi associated to the cocontinuous functor j_x, which equals the morphism associated to the continuous functor v_x, see Lemma 7.21.1 and Section 7.22.
(2) The composition $p \circ j_{u(U)} \circ a = p$ since $j_{u(U)} \circ j_x = \mathrm{id}_{\mathcal{S}}$.
(3) The composition $p' \circ a$ gives a morphism of topoi. Moreover, it is the morphism of topoi associated to the continuous functor $v_x \circ u' = v$. Hence v does indeed define a point q of $\mathcal{C}/U$ which fits into the diagram above by construction.

This ends the proof of the lemma. □

04H3 Lemma 7.35.2. Let $\mathcal{C}$, p, u, U be as in Lemma 7.35.1. The construction of Lemma 7.35.1 gives a one to one correspondence between points q of $\mathcal{C}/U$ lying over p and elements x of $u(U)$.

Proof. Let q be a point of $\mathcal{C}/U$ given by the functor $v : \mathcal{C}/U \to \textit{Sets}$ such that $j_U \circ q = p$ as morphisms of topoi. Recall that $u(V) = p^{-1}(h_V^\#)$ for any object V of $\mathcal{C}$, see Lemma 7.32.7. Similarly $v(V/U) = q^{-1}(h_{V/U}^\#)$ for any object V/U of $\mathcal{C}/U$. Consider the following two diagrams

$$
\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}/U}(W/U, V/U) \arrow[r] \arrow[d] & \mathrm{Mor}_{\mathcal{C}}(W, V) \arrow[d] \\
\mathrm{Mor}_{\mathcal{C}/U}(W/U, U/U) \arrow[r] & \mathrm{Mor}_{\mathcal{C}}(W, U)
\end{tikzcd}
\qquad
\begin{tikzcd}
h_{V/U}^\# \arrow[r] \arrow[d] & j_U^{-1}(h_V^\#) \arrow[d] \\
h_{U/U}^\# \arrow[r] & j_U^{-1}(h_U^\#)
\end{tikzcd}
$$

The right hand diagram is the sheafification of the diagram of presheaves on $\mathcal{C}/U$ which maps W/U to the left hand diagram of sets. (There is a small technical point to make here, namely, that we have $(j_U^{-1}h_V)^\# = j_U^{-1}(h_V^\#)$ and similarly for h_U, see Lemma 7.20.4.) Note that the left hand diagram of sets is cartesian. Since sheafification is exact (Lemma 7.10.14) we conclude that the right hand diagram is cartesian.

Apply the exact functor q^{-1} to the right hand diagram to get a cartesian diagram

$$
\begin{tikzcd}
v(V/U) \arrow[r] \arrow[d] & u(V) \arrow[d] \\
v(U/U) \arrow[r] & u(U)
\end{tikzcd}
$$

of sets. Here we have used that $q^{-1} \circ j^{-1} = p^{-1}$. Since U/U is a final object of $\mathcal{C}/U$ we see that $v(U/U)$ is a singleton. Hence the image of $v(U/U)$ in $u(U)$ is an element x, and the top horizontal map gives a bijection $v(V/U) \to \{y \in u(V) \mid y \mapsto x \text{ in } u(U)\}$ as desired. □

04H4 Lemma 7.35.3. Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to \textit{Sets}$. Let U be an object of $\mathcal{C}$. For any sheaf $\mathcal{G}$ on $\mathcal{C}/U$ we have

$$(j_{U!}\mathcal{G})_p = \coprod\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ associated to elements $x \in u(U)$ as in Lemma 7.35.1.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \coprod_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 7.25.2. Also, the stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 7.32.5. Hence we see that

$$(j_{U!}\mathcal{G})_p = \mathrm{colim}_{(V,y)} \coprod\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

To each element (V, y, φ, s) of this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \coprod\nolimits_{x \in u(U)} \mathrm{colim}_{(\varphi : V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by our construction of the points q. $\square$

04H5 Remark 7.35.4. Warning: The result of Lemma 7.35.3 has no analogue for $j_{U,*}$.

7.36. 2-morphisms of topoi

04I9 This is a brief section concerning the notion of a 2-morphism of topoi.

04IA Definition 7.36.1. Let $f, g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be two morphisms of topoi. A 2-morphism from f to g is given by a transformation of functors $t : f_* \to g_*$.

Pictorially we sometimes represent t as follows:

$$\begin{tikzcd} Sh(\mathcal{C}) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow t"] & Sh(\mathcal{D}) \end{tikzcd}$$

Note that since f^{-1} is adjoint to f_* and g^{-1} is adjoint to g_* we see that t induces also a transformation of functors $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) uniquely characterized by the condition that the diagram

$$\begin{tikzcd} \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}) \arrow[r, equal] \arrow[d, "t\circ -"'] & \mathrm{Mor}_{Sh(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) \arrow[d, "-\circ t"] \\ \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, g_*\mathcal{F}) \arrow[r, equal] & \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1}\mathcal{G}, \mathcal{F}) \end{tikzcd}$$

commutes. Because of set theoretic difficulties (see Remark 7.15.4) we do not obtain a 2-category of topoi. But we can still define horizontal and vertical composition and show that the axioms of a strict 2-category listed in Categories, Section 4.29 hold. Namely, vertical composition of 2-morphisms is clear (just compose transformations of functors), composition of 1-morphisms has been defined in Definition 7.15.1, and horizontal composition of

$$\begin{tikzcd} Sh(\mathcal{C}) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow t"] & Sh(\mathcal{D}) \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow s"] & Sh(\mathcal{E}) \end{tikzcd}$$

is defined by the transformation of functors $s \star t$ introduced in Categories, Definition 4.28.1. Explicitly, $s \star t$ is given by

$$f'_* f_* \mathcal{F} \xrightarrow{f'_* t} f'_* g_* \mathcal{F} \xrightarrow{s} g'_* g_* \mathcal{F} \quad \text{or} \quad f'_* f_* \mathcal{F} \xrightarrow{s} g'_* f_* \mathcal{F} \xrightarrow{g'_* t} g'_* g_* \mathcal{F}$$

(these maps are equal). Since these definitions agree with the ones in Categories, Section 4.28 it follows from Categories, Lemma 4.28.2 that the axioms of a strict 2-category hold with these definitions.

7.37. Morphisms between points

00YG

00YH Lemma 7.37.1. Let $\mathcal{C}$ be a site. Let $u, u' : \mathcal{C} \to \text{Sets}$ be two functors, and let $t : u' \to u$ be a transformation of functors. Then we obtain a canonical transformation of stalk functors $t_{stalk} : \mathcal{F}_{p'} \to \mathcal{F}_p$ which agrees with t via the identifications of Lemma 7.32.3.

Proof. Omitted. □

00YI Definition 7.37.2. Let $\mathcal{C}$ be a site. Let p, p' be points of $\mathcal{C}$ given by functors $u, u' : \mathcal{C} \to \text{Sets}$. A morphism $f : p \to p'$ is given by a transformation of functors

$$f_u : u' \to u.$$

Note how the transformation of functors goes the other way. This makes sense, as we will see later, by thinking of the morphism f as a kind of 2-arrow pictorially as follows:

$$\begin{tikzcd} \text{Sets} = Sh(pt) \arrow[r, bend left, "p"] \arrow[r, bend right, "p'"'] \arrow[r, phantom, "\Downarrow f"] & Sh(\mathcal{C}) \end{tikzcd}$$

Namely, we will see later that f_u induces a canonical transformation of functors $p_* \to p'_*$ between the skyscraper sheaf constructions.

This is a fairly important notion, and deserves a more complete treatment here. List of desiderata

(1) Describe the automorphisms of the point of $\mathcal{T}_G$ described in Example 7.33.7.
(2) Describe $\text{Mor}(p, p')$ in terms of $\text{Mor}(p_*, p'_*)$.
(3) Specialization of points in topological spaces. Show that if $x' \in \overline{\{x\}}$ in the topological space X, then there is a morphism $p \to p'$, where p (resp. p') is the point of X_{Zar} associated to x (resp. x').

7.38. Sites with enough points

00YJ

00YK Definition 7.38.1. Let $\mathcal{C}$ be a site.

(1) A family of points $\{p_i\}_{i \in I}$ is called conservative if every map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ which is an isomorphism on all the fibres $\mathcal{F}_{p_i} \to \mathcal{G}_{p_i}$ is an isomorphism.
(2) We say that $\mathcal{C}$ has enough points if there exists a conservative family of points.

It turns out that you can then check "exactness" at the stalks.

00YL Lemma 7.38.2. Let $\mathcal{C}$ be a site and let $\{p_i\}_{i \in I}$ be a conservative family of points. Then

(1) Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ injective implies φ injective.
(2) Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ surjective implies φ surjective.
(3) Given any pair of maps of sheaves $\varphi_1, \varphi_2 : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{1,p_i} = \varphi_{2,p_i}$ implies $\varphi_1 = \varphi_2$.

(4) Given a finite diagram $\mathcal{G} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{F}$ and morphisms $q_j : \mathcal{F} \to \mathcal{G}_j$ then $(\mathcal{F}, q_j)$ is a limit of the diagram if and only if for each i the stalk $(\mathcal{F}_{p_i}, (q_j)_{p_i})$ is one.
(5) Given a finite diagram $\mathcal{F} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{G}$ and morphisms $e_j : \mathcal{F}_j \to \mathcal{G}$ then $(\mathcal{G}, e_j)$ is a colimit of the diagram if and only if for each i the stalk $(\mathcal{G}_{p_i}, (e_j)_{p_i})$ is one.

Proof. We will use over and over again that all the stalk functors commute with any finite limits and colimits and hence with products, fibred products, etc. We will also use that injective maps are the monomorphisms and the surjective maps are the epimorphisms. A map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ is an isomorphism. Hence (1). Similarly, $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism. Hence (2). The maps $a, b : \mathcal{F} \to \mathcal{G}$ are equal if and only if $\mathcal{F} \times_{a,\mathcal{G},b} \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is an isomorphism. Hence (3). The assertions (4) and (5) follow immediately from the definitions and the remarks at the start of this proof. □

00YM Lemma 7.38.3. Let $\mathcal{C}$ be a site and let $\{(p_i, u_i)\}_{i \in I}$ be a family of points. The family is conservative if and only if for every sheaf $\mathcal{F}$ and every $U \in \mathrm{Ob}(\mathcal{C})$ and every pair of distinct sections $s, s' \in \mathcal{F}(U)$, $s \neq s'$ there exists an i and $x \in u_i(U)$ such that the triples (U, x, s) and (U, x, s') define distinct elements of $\mathcal{F}_{p_i}$.

Proof. Suppose that the family is conservative and that $\mathcal{F}$, U, and s, s' are as in the lemma. The sections s, s' define maps $a, a' : (h_U)^\# \to \mathcal{F}$ which are distinct. Hence, by Lemma 7.38.2 there is an i such that $a_{p_i} \neq a'_{p_i}$. Recall that $(h_U)^\#_{p_i} = u_i(U)$, by Lemmas 7.32.3 and 7.32.5. Hence there exists an $x \in u_i(U)$ such that $a_{p_i}(x) \neq a'_{p_i}(x)$ in $\mathcal{F}_{p_i}$. Unwinding the definitions you see that (U, x, s) and (U, x, s') are as in the statement of the lemma.

To prove the converse, assume the condition on the existence of points of the lemma. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves which is an isomorphism at all the stalks. We have to show that ϕ is both injective and surjective, see Lemma 7.11.2. Injectivity is an immediate consequence of the assumption. To show surjectivity we have to show that $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism (Categories, Lemma 4.13.3). Since this map is clearly surjective, it suffices to check injectivity which follows as $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is injective on all stalks by assumption. □

In the following lemma the points $q_{i,x}$ are exactly all the points of $\mathcal{C}/U$ lying over the point p_i according to Lemma 7.35.2.

04H6 Lemma 7.38.4. Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. let $\{(p_i, u_i)\}_{i \in I}$ be a family of points of $\mathcal{C}$. For $x \in u_i(U)$ let $q_{i,x}$ be the point of $\mathcal{C}/U$ constructed in Lemma 7.35.1. If $\{p_i\}$ is a conservative family of points, then $\{q_{i,x}\}_{i \in I, x \in u_i(U)}$ is a conservative family of points of $\mathcal{C}/U$. In particular, if $\mathcal{C}$ has enough points, then so does every localization $\mathcal{C}/U$.

Proof. We know that $j_{U!}$ induces an equivalence $j_{U!} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$, see Lemma 7.25.4. Moreover, we know that $(j_{U!}\mathcal{G})_{p_i} = \coprod_x \mathcal{G}_{q_{i,x}}$, see Lemma 7.35.3. Hence the result follows formally. □

The following lemma tells us we can check the existence of points locally on the site.

06UL Lemma 7.38.5. Let $\mathcal{C}$ be a site. Let $\{U_i\}_{i \in I}$ be a family of objects of $\mathcal{C}$. Assume

(1) $\coprod h_{U_i}^\# \to *$ is a surjective map of sheaves, and
(2) each localization $\mathcal{C}/U_i$ has enough points.

Then $\mathcal{C}$ has enough points.

Proof. For each $i \in I$ let $\{p_j\}_{j \in J_i}$ be a conservative family of points of $\mathcal{C}/U_i$. For $j \in J_i$ denote $q_j : Sh(pt) \to Sh(\mathcal{C})$ the composition of p_j with the localization morphism $Sh(\mathcal{C}/U_i) \to Sh(\mathcal{C})$. Then q_j is a point, see Lemma 7.34.3. We claim that the family of points $\{q_j\}_{j \in \coprod J_i}$ is conservative. Namely, let $\mathcal{F} \to \mathcal{G}$ be a map of sheaves on $\mathcal{C}$ such that $\mathcal{F}_{q_j} \to \mathcal{G}_{q_j}$ is an isomorphism for all $j \in \coprod J_i$. Let W be an object of $\mathcal{C}$. By assumption (1) there exists a covering $\{W_a \to W\}$ and morphisms $W_a \to U_{i(a)}$. Since $(\mathcal{F}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{F}_{q_j}$ and $(\mathcal{G}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{G}_{q_j}$ by Lemma 7.34.3 we see that $\mathcal{F}|_{U_{i(a)}} \to \mathcal{G}|_{U_{i(a)}}$ is an isomorphism since the family of points $\{p_j\}_{j \in J_{i(a)}}$ is conservative. Hence $\mathcal{F}(W_a) \to \mathcal{G}(W_a)$ is bijective for each a. Similarly $\mathcal{F}(W_a \times_W W_b) \to \mathcal{G}(W_a \times_W W_b)$ is bijective for each a, b. By the sheaf condition this shows that $\mathcal{F}(W) \to \mathcal{G}(W)$ is bijective, i.e., $\mathcal{F} \to \mathcal{G}$ is an isomorphism. $\square$

0F4G Lemma 7.38.6. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $\{(q_i, v_i)\}_{i \in I}$ be a conservative family of points of $\mathcal{D}$. If each functor $u_i = v_i \circ u$ defines a point of $\mathcal{C}$, then u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$.

Proof. Denote p_i the stalk functor (7.32.1.1) on $\mathrm{PSh}(\mathcal{C})$ corresponding to the functor u_i. We have

$$(f^{-1}\mathcal{F})_{q_i} = (u_s\mathcal{F})_{q_i} = (u_p\mathcal{F})_{q_i} = \mathcal{F}_{p_i}$$

The first equality since $f^{-1} = u_s$, the second equality by Lemma 7.32.5, and the third by Lemma 7.34.1. Hence if p_i is a point, then pulling back by f and then taking stalks at q_i is an exact functor. Since the family of points $\{q_i\}$ is conservative, this implies that f^{-1} is an exact functor and we see that f is a morphism of sites by Definition 7.14.1. $\square$

7.39. Criterion for existence of points

00YN This section corresponds to Deligne's appendix to [AGV71, Exposé VI]. In fact it is almost literally the same.

Let $\mathcal{C}$ be a site. Suppose that $(I, \geq)$ is a directed set, and that $(U_i, f_{ii'})$ is an inverse system over I, see Categories, Definition 4.21.2. Given the data $(I, \geq, U_i, f_{ii'})$ we define

$$u : \mathcal{C} \longrightarrow \text{Sets}, \quad u(V) = \mathrm{colim}_i \, \mathrm{Mor}_\mathcal{C}(U_i, V)$$

Let $\mathcal{F} \mapsto \mathcal{F}_p$ be the stalk functor associated to u as in Section 7.32. It is direct from the definition that actually

$$\mathcal{F}_p = \mathrm{colim}_i \, \mathcal{F}(U_i)$$

in this special case. Note that u commutes with all finite limits (I mean those that are representable in $\mathcal{C}$) because each of the functors $V \mapsto \mathrm{Mor}_\mathcal{C}(U_i, V)$ do, see Categories, Lemma 4.19.2.

We say that a system $(I, \geq, U_i, f_{ii'})$ is a refinement of $(J, \geq, V_j, g_{jj'})$ if $J \subset I$, the ordering on J induced from that of I and $V_j = U_j$, $g_{jj'} = f_{jj'}$ (in words, the inverse system over J is induced by that over I). Let u be the functor associated to

$(I, \geq, U_i, f_{ii'})$ and let u' be the functor associated to $(J, \geq, V_j, g_{jj'})$. This induces a transformation of functors

$$u' \longrightarrow u$$

simply because the colimits for u' are over a subsystem of the systems in the colimits for u. In particular we get an associated transformation of stalk functors $\mathcal{F}_{p'} \to \mathcal{F}_p$, see Lemma 7.37.1.

00YO Lemma 7.39.1. Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. Let $\{W_k \to W\}$ be a finite covering of $\mathcal{C}$. Let $f \in u'(W)$. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and that the image of f in $u(W)$ is in the image of one of the $u(W_k)$.

Proof. There exists a $j_0 \in J$ such that f is defined by $f' : V_{j_0} \to W$. For $j \geq j_0$ we set $V_{j,k} = V_j \times_{f' \circ f_{jj_0}, W} W_k$. Then $\{V_{j,k} \to V_j\}$ is a finite covering in the site $\mathcal{C}$. Hence $\mathcal{F}(V_j) \subset \prod_k \mathcal{F}(V_{j,k})$. By Categories, Lemma 4.19.2 once again we see that

$$\mathcal{F}_{p'} = \mathrm{colim}_j\, \mathcal{F}(V_j) \longrightarrow \prod\nolimits_k \mathrm{colim}_j\, \mathcal{F}(V_{j,k})$$

is injective. Hence there exists a k such that s and s' have distinct image in $\mathrm{colim}_j\, \mathcal{F}(V_{j,k})$. Let $J_0 = \{j \in J, j \geq j_0\}$ and $I = J \amalg J_0$. We order I so that no element of the second summand is smaller than any element of the first, but otherwise using the ordering on J. If $j \in I$ is in the first summand then we use V_j and if $j \in I$ is in the second summand then we use $V_{j,k}$. We omit the definition of the transition maps of the inverse system. By the above it follows that s, s' have distinct image in $\mathcal{F}_p$. Moreover, the restriction of f' to $V_{j,k}$ factors through W_k by construction. $\square$

00YP Lemma 7.39.2. Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and such that for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$, and any $f \in u'(W)$ the image of f in $u(W)$ is in the image of one of the $u(W_k)$.

Proof. Let E be the set of pairs $(\{W_k \to W\}, f \in u'(W))$. Consider pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ such that

1. $(I, \geq, U_i, g_{ii'})$ is a refinement of $(J, \geq, V_j, g_{jj'})$,
2. s, s' map to distinct elements of $\mathcal{F}_p$, and
3. for every pair $(\{W_k \to W\}, f \in u'(W)) \in E'$ we have that the image of f in $u(W)$ is in the image of one of the $u(W_k)$.

We order such pairs by inclusion in the first factor and by refinement in the second. Denote $\mathcal{S}$ the class of all pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ as above. We claim that the hypothesis of Zorn's lemma holds for $\mathcal{S}$. Namely, suppose that $(E'_a, (I_a, \geq, U_i, f_{ii'}))_{a \in A}$ is a totally ordered subset of $\mathcal{S}$. Then we can define $E' = \bigcup_{a \in A} E'_a$ and we can set $I = \bigcup_{a \in A} I_a$. We claim that the corresponding pair $(E', (I, \geq, U_i, f_{ii'}))$ is an element of $\mathcal{S}$. Conditions (1) and (3) are clear. For condition (2) you note that

$$u = \mathrm{colim}_{a \in A}\, u_a \text{ and correspondingly } \mathcal{F}_p = \mathrm{colim}_{a \in A}\, \mathcal{F}_{p_a}$$

The distinctness of the images of s, s' in this stalk follows from the description of a directed colimit of sets, see Categories, Section 4.19. We will simply write $(E', (I, \ldots)) = \bigcup_{a \in A}(E'_a, (I_a, \ldots))$ in this situation.

OK, so Zorn's Lemma would apply if $\mathcal{S}$ was a set, and this would, combined with Lemma 7.39.1 above easily prove the lemma. It doesn't since $\mathcal{S}$ is a class. In order to circumvent this we choose a well ordering on E. For $e \in E$ set $E'_e = \{e' \in E \mid e' \leq e\}$. Using transfinite recursion we construct pairs $(E'_e, (I_e, \ldots)) \in \mathcal{S}$ such that $e_1 \leq e_2 \Rightarrow (E'_{e_1}, (I_{e_1}, \ldots)) \leq (E'_{e_2}, (I_{e_2}, \ldots))$. Let $e \in E$, say $e = (\{W_k \to W\}, f \in u'(W))$. If e has a predecessor $e-1$, then we let $(I_e, \ldots)$ be a refinement of $(I_{e-1}, \ldots)$ as in Lemma 7.39.1 with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. If e does not have a predecessor, then we let $(I_e, \ldots)$ be a refinement of $\bigcup_{e' < e}(I_{e'}, \ldots)$ with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. Finally, the union $\bigcup_{e \in E} I_e$ will be a solution to the problem posed in the lemma. $\square$

00YQ Proposition 7.39.3. Let $\mathcal{C}$ be a site. Assume that [AGV71, Exposé VI, Appendix by Deligne, Proposition 9.0]

(1) finite limits exist in $\mathcal{C}$, and
(2) every covering $\{U_i \to U\}_{i \in I}$ has a refinement by a finite covering of $\mathcal{C}$.

Then $\mathcal{C}$ has enough points.

Proof. We have to show that given any sheaf $\mathcal{F}$ on $\mathcal{C}$, any $U \in \text{Ob}(\mathcal{C})$, and any distinct sections $s, s' \in \mathcal{F}(U)$, there exists a point p such that s, s' have distinct image in $\mathcal{F}_p$. See Lemma 7.38.3. Consider the system $(J, \geq, V_j, g_{jj'})$ with $J = \{1\}$, $V_1 = U$, $g_{11} = \text{id}_U$. Apply Lemma 7.39.2. By the result of that lemma we get a system $(I, \geq, U_i, f_{ii'})$ refining our system such that $s_p \neq s'_p$ and such that moreover for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$ the map $\coprod_k u(W_k) \to u(W)$ is surjective. Since every covering of $\mathcal{C}$ can be refined by a finite covering we conclude that $\coprod_k u(W_k) \to u(W)$ is surjective for any covering $\{W_k \to W\}$ of the site $\mathcal{C}$. This implies that $u = p$ is a point, see Proposition 7.33.3 (and the discussion at the beginning of this section which guarantees that u commutes with finite limits). $\square$

0DW0 Lemma 7.39.4. Let $\mathcal{C}$ be a site. Let I be a set and for $i \in I$ let U_i be an object of $\mathcal{C}$ such that

(1) $\coprod h_{U_i}$ surjects onto the final object of $Sh(\mathcal{C})$, and
(2) $\mathcal{C}/U_i$ satisfies the hypotheses of Proposition 7.39.3.

Then $\mathcal{C}$ has enough points.

Proof. By assumption (2) and the proposition $\mathcal{C}/U_i$ has enough points. The points of $\mathcal{C}/U_i$ give points of $\mathcal{C}$ via the procedure of Lemma 7.34.2. Thus it suffices to show: if $\phi : \mathcal{F} \to \mathcal{G}$ is a map of sheaves on $\mathcal{C}$ such that $\phi|_{\mathcal{C}/U_i}$ is an isomorphism for all i, then ϕ is an isomorphism. By assumption (1) for every object W of $\mathcal{C}$ there is a covering $\{W_j \to W\}_{j \in J}$ such that for $j \in J$ there is an $i \in I$ and a morphism $f_j : W_j \to U_i$. Then the maps $\mathcal{F}(W_j) \to \mathcal{G}(W_j)$ are bijective and similarly for $\mathcal{F}(W_j \times_W W_{j'}) \to \mathcal{G}(W_j \times_W W_{j'})$. The sheaf condition tells us that $\mathcal{F}(W) \to \mathcal{G}(W)$ is bijective as desired. $\square$

7.40. Weakly contractible objects

090J A weakly contractible object of a site is one that satisfies the equivalent conditions of the following lemma.

090K Lemma 7.40.1. Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following conditions are equivalent

(1) For every covering $\{U_i \to U\}$ there exists a map of sheaves $h_U^\# \to \coprod h_{U_i}^\#$ right inverse to the sheafification of $\coprod h_{U_i} \to h_U$.
(2) For every surjection of sheaves of sets $\mathcal{F} \to \mathcal{G}$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

Proof. Assume (1) and let $\mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves of sets. For $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ and $t_i \in \mathcal{F}(U_i)$ mapping to $s|_{U_i}$, see Definition 7.11.1. Think of t_i as a map $t_i : h_{U_i}^\# \to \mathcal{F}$ via (7.12.3.1). Then precomposing $\coprod t_i : \coprod h_{U_i}^\# \to \mathcal{F}$ with the map $h_U^\# \to \coprod h_{U_i}^\#$ we get from (1) we obtain a section $t \in \mathcal{F}(U)$ mapping to s. Thus (2) holds.

Assume (2) holds. Let $\{U_i \to U\}$ be a covering. Then $\coprod h_{U_i}^\# \to h_U^\#$ is surjective (Lemma 7.12.4). Hence by (2) there exists a section s of $\coprod h_{U_i}^\#$ mapping to the section id_U of $h_U^\#$. This section corresponds to a map $h_U^\# \to \coprod h_{U_i}^\#$ which is right inverse to the sheafification of $\coprod h_{U_i} \to h_U$ which proves (1). $\square$

090L Definition 7.40.2. Let $\mathcal{C}$ be a site.

(1) We say an object U of $\mathcal{C}$ is weakly contractible if the equivalent conditions of Lemma 7.40.1 hold.
(2) We say a site has enough weakly contractible objects if every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ with U_i weakly contractible for all i.
(3) More generally, if P is a property of objects of $\mathcal{C}$ we say that $\mathcal{C}$ has enough P objects if every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ such that U_i has P for all i.

The small étale site of $\mathbf{A}^1_{\mathbf{C}}$ does not have any weakly contractible objects. On the other hand, the small pro-étale site of any scheme has enough contractible objects.

7.41. Exactness properties of pushforward

04D5 Let f be a morphism of topoi. The functor f_* in general is only left exact. There are many additional conditions one can impose on this functor to single out particular classes of morphisms of topoi. We collect them here and note some of the logical dependencies. Some parts of the following lemma are purely category theoretical (i.e., they do not depend on having a morphism of topoi, just having a pair of adjoint functors is enough).

04D6 Lemma 7.41.1. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Consider the following properties (on sheaves of sets):

(1) f_* is faithful,
(2) f_* is fully faithful,
(3) $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Sh(\mathcal{C})$,
(4) f_* transforms surjections into surjections,
(5) f_* commutes with coequalizers,
(6) f_* commutes with pushouts,
(7) $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is an isomorphism for all $\mathcal{F}$ in $Sh(\mathcal{C})$,
(8) f_* reflects injections,
(9) f_* reflects surjections,

(10) f_* reflects bijections, and
(11) for any surjection $\mathcal{F} \to f^{-1}\mathcal{G}$ there exists a surjection $\mathcal{G}' \to \mathcal{G}$ such that $f^{-1}\mathcal{G}' \to f^{-1}\mathcal{G}$ factors through $\mathcal{F} \to f^{-1}\mathcal{G}$.

Then we have the following implications

(a) (2) $\Rightarrow$ (1),
(b) (3) $\Rightarrow$ (1),
(c) (7) $\Rightarrow$ (1), (2), (3), (8), (9), (10).
(d) (3) $\Leftrightarrow$ (9),
(e) (6) $\Rightarrow$ (4) and (5) $\Rightarrow$ (4),
(f) (4) $\Leftrightarrow$ (11),
(g) (9) $\Rightarrow$ (8), (10), and
(h) (2) $\Leftrightarrow$ (7).

Picture

$$
\begin{tikzcd}
(6) \arrow[dr, Rightarrow] & & & & & (9) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (8) \\
& (4) \arrow[r, Leftrightarrow] & (11) & (2) \arrow[r, Leftrightarrow] & (7) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (10) \\
(5) \arrow[uur, Rightarrow, start anchor=north east, end anchor=south west, shift left=0] & & & & & (3) \arrow[r, Rightarrow] & (1)
\end{tikzcd}
$$

Proof. Proof of (a): This is immediate from the definitions.

Proof of (b). Suppose that $a, b : \mathcal{F} \to \mathcal{F}'$ are maps of sheaves on $\mathcal{C}$. If $f_*a = f_*b$, then $f^{-1}f_*a = f^{-1}f_*b$. Consider the commutative diagram

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{F}' \\
f^{-1}f_*\mathcal{F} \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & f^{-1}f_*\mathcal{F}' \arrow[u]
\end{tikzcd}
$$

If the bottom two arrows are equal and the vertical arrows are surjective then the top two arrows are equal. Hence (b) follows.

Proof of (c). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$. Consider the commutative diagram

$$
\begin{tikzcd}
\mathcal{F} \arrow[r] & \mathcal{F}' \\
f^{-1}f_*\mathcal{F} \arrow[u] \arrow[r] & f^{-1}f_*\mathcal{F}' \arrow[u]
\end{tikzcd}
$$

If (7) holds, then the vertical arrows are isomorphisms. Hence if f_*a is injective (resp. surjective, resp. bijective) then the bottom arrow is injective (resp. surjective, resp. bijective) and hence the top arrow is injective (resp. surjective, resp. bijective). Thus we see that (7) implies (8), (9), (10). It is clear that (7) implies (3). The implications (7) $\Rightarrow$ (2), (1) follow from (a) and (h) which we will see below.

Proof of (d). Assume (3). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (3) this implies that a is surjective. This means that (9) holds. Assume (9). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. It suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

Proof of (e). We use Categories, Lemma 4.13.3 without further mention. If $\mathcal{F} \to \mathcal{F}'$ is surjective then $\mathcal{F}' \amalg_\mathcal{F} \mathcal{F}' \to \mathcal{F}'$ is an isomorphism. Hence (6) implies that

$$f_*\mathcal{F}' \amalg_{f_*\mathcal{F}} f_*\mathcal{F}' = f_*(\mathcal{F}' \amalg_\mathcal{F} \mathcal{F}') \longrightarrow f_*\mathcal{F}'$$

is an isomorphism also. And this in turn implies that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence we see that (6) implies (4). If $\mathcal{F} \to \mathcal{F}'$ is surjective then $\mathcal{F}'$ is the coequalizer of the two projections $\mathcal{F} \times_{\mathcal{F}'} \mathcal{F} \to \mathcal{F}$ by Lemma 7.11.3. Hence if (5) holds, then $f_*\mathcal{F}'$ is the coequalizer of the two projections

$$f_*(\mathcal{F} \times_{\mathcal{F}'} \mathcal{F}) = f_*\mathcal{F} \times_{f_*\mathcal{F}'} f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which clearly means that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence (5) implies (4) as well.

Proof of (f). Assume (4). Let $\mathcal{F} \to f^{-1}\mathcal{G}$ be a surjective map of sheaves on $\mathcal{C}$. By (4) we see that $f_*\mathcal{F} \to f_*f^{-1}\mathcal{G}$ is surjective. Let $\mathcal{G}'$ be the fibre product

$$\begin{tikzcd} f_*\mathcal{F} \arrow[r] & f_*f^{-1}\mathcal{G} \\ \mathcal{G}' \arrow[u] \arrow[r] & \mathcal{G} \arrow[u] \end{tikzcd}$$

so that $\mathcal{G}' \to \mathcal{G}$ is surjective also. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & f^{-1}\mathcal{G} \\ f^{-1}f_*\mathcal{F} \arrow[u] \arrow[r] & f^{-1}f_*f^{-1}\mathcal{G} \arrow[u] \\ f^{-1}\mathcal{G}' \arrow[u] \arrow[r] & f^{-1}\mathcal{G} \arrow[u] \end{tikzcd}$$

and we see the required result. Conversely, assume (11). Let $a : \mathcal{F} \to \mathcal{F}'$ be surjective map of sheaves on $\mathcal{C}$. Consider the fibre product diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] & \mathcal{F}' \\ \mathcal{F}'' \arrow[u] \arrow[r] & f^{-1}f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

Because the lower horizontal arrow is surjective and by (11) we can find a surjection $\gamma : \mathcal{G}' \to f_*\mathcal{F}'$ such that $f^{-1}\gamma$ factors through $\mathcal{F}'' \to f^{-1}f_*\mathcal{F}'$:

$$\begin{tikzcd} & \mathcal{F} \arrow[r] & \mathcal{F}' \\ f^{-1}\mathcal{G}' \arrow[r] & \mathcal{F}'' \arrow[u] \arrow[r] & f^{-1}f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

Pushing this down using f_* we get a commutative diagram

$$\begin{tikzcd} & f_*\mathcal{F} \arrow[r] & f_*\mathcal{F}' \\ f_*f^{-1}\mathcal{G}' \arrow[r] & f_*\mathcal{F}'' \arrow[r] \arrow[u] & f_*f^{-1}f_*\mathcal{F}' \arrow[u] \\ \mathcal{G}' \arrow[u] \arrow[rr] & & f_*\mathcal{F}' \arrow[u] \end{tikzcd}$$

which proves that (4) holds.

Proof of (g). Assume (9). We use Categories, Lemma 4.13.3 without further mention. Let $a : \mathcal{F} \to \mathcal{F}'$ be a map of sheaves on $\mathcal{C}$ such that f_*a is injective. This means that $f_*\mathcal{F} \to f_*\mathcal{F} \times_{f_*\mathcal{F}'} f_*\mathcal{F} = f_*(\mathcal{F} \times_{\mathcal{F}'} \mathcal{F})$ is an isomorphism. Thus by (9) we see that $\mathcal{F} \to \mathcal{F} \times_{\mathcal{F}'} \mathcal{F}$ is surjective, i.e., an isomorphism. Thus a is injective, i.e., (8) holds. Since (10) is trivially equivalent to (8) + (9) we are done with (g).

Proof of (h). This is Categories, Lemma 4.24.4. $\square$

Here is a condition on a morphism of sites which guarantees that the functor f_* transforms surjective maps into surjective maps.

04D7 Lemma 7.41.2. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for any object U of $\mathcal{C}$ and any covering $\{V_j \to u(U)\}$ in $\mathcal{D}$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that the map of sheaves

$$\coprod h^\#_{u(U_i)} \to h^\#_{u(U)}$$

factors through the map of sheaves

$$\coprod h^\#_{V_j} \to h^\#_{u(U)}.$$

Then f_* transforms surjective maps of sheaves into surjective maps of sheaves.

Proof. Let $a : \mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves on $\mathcal{D}$. Let U be an object of $\mathcal{C}$ and let $s \in f_*\mathcal{G}(U) = \mathcal{G}(u(U))$. By assumption there exists a covering $\{V_j \to u(U)\}$ and sections $s_j \in \mathcal{F}(V_j)$ with $a(s_j) = s|_{V_j}$. Now we may think of the sections s, s_j and a as giving a commutative diagram of maps of sheaves

$$\begin{tikzcd} \coprod h^\#_{V_j} \arrow[r, "\coprod s_j"'] \arrow[d] & \mathcal{F} \arrow[d, "a"] \\ h^\#_{u(U)} \arrow[r, "s"] & \mathcal{G} \end{tikzcd}$$

By assumption there exists a covering $\{U_i \to U\}$ such that we can enlarge the commutative diagram above as follows

$$\begin{tikzcd} & \coprod h^\#_{V_j} \arrow[r, "\coprod s_j"'] \arrow[d] & \mathcal{F} \arrow[d, "a"] \\ \coprod h^\#_{u(U_i)} \arrow[r] \arrow[ur] & h^\#_{u(U)} \arrow[r, "s"] & \mathcal{G} \end{tikzcd}$$

Because $\mathcal{F}$ is a sheaf the map from the left lower corner to the right upper corner corresponds to a family of sections $s_i \in \mathcal{F}(u(U_i))$, i.e., sections $s_i \in f_*\mathcal{F}(U_i)$. The commutativity of the diagram implies that $a(s_i)$ is equal to the restriction of s to U_i. In other words we have shown that f_*a is a surjective map of sheaves. □

04D8 Example 7.41.3. Assume $f : \mathcal{D} \to \mathcal{C}$ satisfies the assumptions of Lemma 7.41.2. Then it is in general not the case that f_* commutes with coequalizers or pushouts. Namely, suppose that f is the morphism of sites associated to the morphism of topological spaces $X = \{1, 2\} \to Y = \{*\}$ (see Example 7.14.2), where Y is a singleton space, and $X = \{1, 2\}$ is a discrete space with two points. A sheaf $\mathcal{F}$ on X is given by a pair (A_1, A_2) of sets. Then $f_*\mathcal{F}$ corresponds to the set $A_1 \times A_2$. Hence if $a = (a_1, a_2), b = (b_1, b_2) : (A_1, A_2) \to (B_1, B_2)$ are maps of sheaves on X, then the coequalizer of a, b is (C_1, C_2) where C_i is the coequalizer of a_i, b_i, and the coequalizer of f_*a, f_*b is the coequalizer of

$$a_1 \times a_2, b_1 \times b_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$$

which is in general different from $C_1 \times C_2$. Namely, if $A_2 = \emptyset$ then $A_1 \times A_2 = \emptyset$, and hence the coequalizer of the displayed arrows is $B_1 \times B_2$, but in general $C_1 \neq B_1$. A similar example works for pushouts.

The following lemma gives a criterion for when a morphism of sites has a functor f_* which reflects injections and surjections. Note that this also implies that f_* is faithful and that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is always surjective.

04D9 Lemma 7.41.4. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for every object V of $\mathcal{D}$ there exist objects U_i of $\mathcal{C}$ and morphisms $u(U_i) \to V$ such that $\{u(U_i) \to V\}$ is a covering of $\mathcal{D}$. In this case the functor $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ reflects injections and surjections.

Proof. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be maps of sheaves on $\mathcal{D}$. By assumption for every object V of $\mathcal{D}$ we get $\mathcal{F}(V) \subset \prod \mathcal{F}(u(U_i)) = \prod f_*\mathcal{F}(U_i)$ by the sheaf condition for some $U_i \in \text{Ob}(\mathcal{C})$ and similarly for $\mathcal{G}$. Hence it is clear that if $f_*\alpha$ is injective, then α is injective. In other words f_* reflects injections.

Suppose that $f_*\alpha$ is surjective. Then for $V, U_i, u(U_i) \to V$ as above and a section $s \in \mathcal{G}(V)$, there exist coverings $\{U_{ij} \to U_i\}$ such that $s|_{u(U_{ij})}$ is in the image of $\mathcal{F}(u(U_{ij}))$. Since $\{u(U_{ij}) \to V\}$ is a covering (as u is continuous and by the axioms of a site) we conclude that s is locally in the image. Thus α is surjective. In other words f_* reflects surjections. □

08LS Example 7.41.5. We construct a morphism $f : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Lemma 7.41.4. Namely, let $\varphi : G \to H$ be a morphism of finite groups. Consider the sites $\mathcal{D} = \mathcal{T}_G$ and $\mathcal{C} = \mathcal{T}_H$ of countable G-sets and H-sets and coverings countable families of jointly surjective maps (Example 7.6.5). Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor described in Section 7.16 and $f : \mathcal{T}_G \to \mathcal{T}_H$ the corresponding morphism of sites. If φ is injective, then every countable G-set is, as a G-set, the quotient of a countable H-set (this fails if φ isn't injective). Thus f satisfies the hypothesis of Lemma 7.41.4. If the sheaf $\mathcal{F}$ on $\mathcal{T}_G$ corresponds to the G-set S, then the canonical map

$$f^{-1}f_*\mathcal{F} \longrightarrow \mathcal{F}$$

corresponds to the map

$$\text{Map}_G(H, S) \longrightarrow S, \quad a \longmapsto a(1_H)$$

If φ is injective but not surjective, then this map is surjective (as it should according to Lemma 7.41.4) but not injective in general (for example take $G = \{1\}$, $H = \{1, \sigma\}$, and $S = \{1, 2\}$). Moreover, the functor f_* does not commute with coequalizers or pushouts (for $G = \{1\}$ and $H = \{1, \sigma\}$).

7.42. Almost cocontinuous functors

04B4 Let $\mathcal{C}$ be a site. The category $\text{PSh}(\mathcal{C})$ has an initial object, namely the presheaf which assigns the empty set to each object of $\mathcal{C}$. Let us denote this presheaf by $\emptyset$. It follows from the properties of sheafification that the sheafification $\emptyset^\#$ of $\emptyset$ is an initial object of the category $Sh(\mathcal{C})$ of sheaves on $\mathcal{C}$.

04B5 Definition 7.42.1. Let $\mathcal{C}$ be a site. We say an object U of $\mathcal{C}$ is sheaf theoretically empty if $\emptyset^\# \to h_U^\#$ is an isomorphism of sheaves.

The following lemma makes this notion more explicit.

04B6 Lemma 7.42.2. Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following are equivalent:

(1) U is sheaf theoretically empty,
(2) $\mathcal{F}(U)$ is a singleton for each sheaf $\mathcal{F}$,
(3) $\emptyset^\#(U)$ is a singleton,
(4) $\emptyset^\#(U)$ is nonempty, and
(5) the empty family is a covering of U in $\mathcal{C}$.

Moreover, if U is sheaf theoretically empty, then for any morphism $U' \to U$ of $\mathcal{C}$ the object U' is sheaf theoretically empty.

Proof. For any sheaf $\mathcal{F}$ we have $\mathcal{F}(U) = \text{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F})$. Hence, we see that (1) and (2) are equivalent. It is clear that (2) implies (3) implies (4). If every covering of U is given by a nonempty family, then $\emptyset^+(U)$ is empty by definition of the plus construction. Note that $\emptyset^+ = \emptyset^\#$ as $\emptyset$ is a separated presheaf, see Theorem 7.10.10. Thus we see that (4) implies (5). If (5) holds, then $\mathcal{F}(U)$ is a singleton for every sheaf $\mathcal{F}$ by the sheaf condition for $\mathcal{F}$, see Remark 7.7.2. Thus (5) implies (2) and (1) – (5) are equivalent. The final assertion of the lemma follows from Axiom (3) of Definition 7.6.2 applied the empty covering of U. $\square$

04B7 Definition 7.42.3. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. We say u is almost cocontinuous if for every object U of $\mathcal{C}$ and every covering $\{V_j \to u(U)\}_{j \in J}$ there exists a covering $\{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ such that for each i in I we have at least one of the following two conditions

(1) $u(U_i)$ is sheaf theoretically empty, or
(2) the morphism $u(U_i) \to u(U)$ factors through V_j for some $j \in J$.

The motivation for this definition comes from a closed immersion $i : Z \to X$ of topological spaces. As discussed in Example 7.21.9 the continuous functor $X_{Zar} \to Z_{Zar}$, $U \mapsto Z \cap U$ is not cocontinuous. But it is almost cocontinuous in the sense defined above. We know that i_* while not exact on sheaves of sets, is exact on sheaves of abelian groups, see Sheaves, Remark 6.32.5. And this holds in general for continuous and almost cocontinuous functors.

04B8 Lemma 7.42.4. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ such that $\mathcal{G}(V)$ is a singleton whenever V is sheaf theoretically empty. Then $(u^p\mathcal{G})^\# = u^p(\mathcal{G}^\#)$.

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. We have to show that $(u^p\mathcal{G})^\#(U) = u^p(\mathcal{G}^\#)(U)$. It suffices to show that $(u^p\mathcal{G})^+(U) = u^p(\mathcal{G}^+)(U)$ since $\mathcal{G}^+$ is another presheaf for which the assumption of the lemma holds. We have

$$u^p(\mathcal{G}^+)(U) = \mathcal{G}^+(u(U)) = \mathrm{colim}_\mathcal{V}\, \check{H}^0(\mathcal{V}, \mathcal{G})$$

where the colimit is over the coverings $\mathcal{V}$ of $u(U)$ in $\mathcal{D}$. On the other hand, we see that

$$u^p(\mathcal{G})^+(U) = \mathrm{colim}_\mathcal{U}\, \check{H}^0(u(\mathcal{U}), \mathcal{G})$$

where the colimit is over the category of coverings $\mathcal{U} = \{U_i \to U\}_{i\in I}$ of U in $\mathcal{C}$ and $u(\mathcal{U}) = \{u(U_i) \to u(U)\}_{i\in I}$. The condition that u is continuous means that each $u(\mathcal{U})$ is a covering. Write $I = I_1 \amalg I_2$, where

$$I_2 = \{i \in I \mid u(U_i) \text{ is sheaf theoretically empty}\}$$

Then $u(\mathcal{U})' = \{u(U_i) \to u(U)\}_{i\in I_1}$ is still a covering of because each of the other pieces can be covered by the empty family and hence can be dropped by Axiom (2) of Definition 7.6.2. Moreover, $\check{H}^0(u(\mathcal{U}), \mathcal{G}) = \check{H}^0(u(\mathcal{U})', \mathcal{G})$ by our assumption on $\mathcal{G}$. Finally, the condition that u is almost cocontinuous implies that for every covering $\mathcal{V}$ of $u(U)$ there exists a covering $\mathcal{U}$ of U such that $u(\mathcal{U})'$ refines $\mathcal{V}$. It follows that the two colimits displayed above have the same value as desired. □

04B9 Lemma 7.42.5. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Then $u^s = u^p : \mathit{Sh}(\mathcal{D}) \to \mathit{Sh}(\mathcal{C})$ commutes with pushouts and coequalizers (and more generally finite connected colimits).

Proof. Let $\mathcal{I}$ be a finite connected index category. Let $\mathcal{I} \to \mathit{Sh}(\mathcal{D})$, $i \mapsto \mathcal{G}_i$ by a diagram. We know that the colimit of this diagram is the sheafification of the colimit in the category of presheaves, see Lemma 7.10.13. Denote colim^{Psh} the colimit in the category of presheaves. Since $\mathcal{I}$ is finite and connected we see that $\mathrm{colim}^{Psh}_i \mathcal{G}_i$ is a presheaf satisfying the assumptions of Lemma 7.42.4 (because a finite connected colimit of singleton sets is a singleton). Hence that lemma gives

$$\begin{aligned} u^s(\mathrm{colim}_i \mathcal{G}_i) &= u^s((\mathrm{colim}^{Psh}_i \mathcal{G}_i)^\#) \\ &= (u^p(\mathrm{colim}^{Psh}_i \mathcal{G}_i))^\# \\ &= (\mathrm{colim}^{PSh}_i u^p(\mathcal{G}_i))^\# \\ &= \mathrm{colim}_i u^s(\mathcal{G}_i) \end{aligned}$$

as desired. □

04BA Lemma 7.42.6. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. If u is almost cocontinuous then f_* commutes with pushouts and coequalizers (and more generally finite connected colimits).

Proof. This is a special case of Lemma 7.42.5. □

7.43. Subtopoi

08LT Here is the definition.

08LU Definition 7.43.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A morphism of topoi $f : \mathit{Sh}(\mathcal{D}) \to \mathit{Sh}(\mathcal{C})$ is called an embedding if f_* is fully faithful.

According to Lemma 7.41.1 this is equivalent to asking the adjunction map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ to be an isomorphism for every sheaf $\mathcal{F}$ on $\mathcal{D}$.

08LV Definition 7.43.2. Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is a subtopos if there exists an embedding of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ such that E is equal to the essential image of the functor f_*.

The subtopoi constructed in the following lemma will be dubbed "open" in the definition later on.

08LW Lemma 7.43.3. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. The following are equivalent

(1) $\mathcal{F}$ is a subobject of the final object of $Sh(\mathcal{C})$, and
(2) the topos $Sh(\mathcal{C})/\mathcal{F}$ is a subtopos of $Sh(\mathcal{C})$.

Proof. We have seen in Lemma 7.30.1 that $Sh(\mathcal{C})/\mathcal{F}$ is a topos. In fact, we recall the proof. First we apply Lemma 7.29.5 to see that we may assume $\mathcal{C}$ is a site with a subcanonical topology, fibre products, a final object X, and an object U with $\mathcal{F} = h_U$. The proof of Lemma 7.30.1 shows that the morphism of topoi $j_\mathcal{F} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ is equal (modulo certain identifications) to the localization morphism $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$.

Assume (2). This means that $j_U^{-1}j_{U,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism for all sheaves $\mathcal{G}$ on $\mathcal{C}/U$. For any object Z/U of $\mathcal{C}/U$ we have

$$(j_{U,*}h_{Z/U})(U) = \mathrm{Mor}_{\mathcal{C}/U}(U \times_X U/U, Z/U)$$

by Lemma 7.27.2. Setting $\mathcal{G} = h_{Z/U}$ in the equality above we obtain

$$\mathrm{Mor}_{\mathcal{C}/U}(U \times_X U/U, Z/U) = \mathrm{Mor}_{\mathcal{C}/U}(U, Z/U)$$

for all Z/U. By Yoneda's lemma (Categories, Lemma 4.3.5) this implies $U \times_X U = U$. By Categories, Lemma 4.13.3 $U \to X$ is a monomorphism, in other words (1) holds.

Assume (1). Then $j_U^{-1}j_{U,*} = \mathrm{id}$ by Lemma 7.27.4. $\square$

08LX Definition 7.43.4. Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is an open subtopos if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(\mathcal{C})$ such that E is the subtopos $Sh(\mathcal{C})/\mathcal{F}$ described in Lemma 7.43.3.

This means there is a bijection between the collection of open subtopoi of $Sh(\mathcal{C})$ and the set of subobjects of the final object of $Sh(\mathcal{C})$. Given an open subtopos there is a "closed" complement.

08LY Lemma 7.43.5. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a subsheaf of the final object $*$ of $Sh(\mathcal{C})$. The full subcategory of sheaves $\mathcal{G}$ such that $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is an isomorphism is a subtopos of $Sh(\mathcal{C})$.

Proof. We apply Lemma 7.29.5 to see that we may assume $\mathcal{C}$ is a site with the properties listed in that lemma. In particular $\mathcal{C}$ has a final object X (so that $* = h_X$) and an object U with $\mathcal{F} = h_U$.

Let $\mathcal{D} = \mathcal{C}$ as a category but a covering is a family $\{V_j \to V\}$ of morphisms such that $\{V_i \to V\} \cup \{U \times_X V \to V\}$ is a covering. By our choice of $\mathcal{C}$ this means exactly that

$$h_{U \times_X V} \amalg \coprod h_{V_i} \longrightarrow h_V$$

is surjective. We claim that $\mathcal{D}$ is a site, i.e., the coverings satisfy the conditions (1), (2), (3) of Definition 7.6.2. Condition (1) holds. For condition (2) suppose that $\{V_i \to V\}$ and $\{V_{ij} \to V_i\}$ are coverings of $\mathcal{D}$. Then the composition

$$\coprod \left(h_{U \times_X V_i} \amalg \coprod h_{V_{ij}}\right) \longrightarrow h_{U \times_X V} \amalg \coprod h_{V_i} \longrightarrow h_V$$

is surjective. Since each of the morphisms $U \times_X V_i \to V$ factors through $U \times_X V$ we see that

$$h_{U \times_X V} \amalg \coprod h_{V_{ij}} \longrightarrow h_V$$

is surjective, i.e., $\{V_{ij} \to V\}$ is a covering of V in $\mathcal{D}$. Condition (3) follows similarly as a base change of a surjective map of sheaves is surjective.

Note that the (identity) functor $u : \mathcal{C} \to \mathcal{D}$ is continuous and commutes with fibre products and final objects. Hence we obtain a morphism $f : \mathcal{D} \to \mathcal{C}$ of sites (Proposition 7.14.7). Observe that f_* is the identity functor on underlying presheaves, hence fully faithful. To finish the proof we have to show that the essential image of f_* is the full subcategory $E \subset Sh(\mathcal{C})$ singled out in the lemma. To do this, note that $\mathcal{G} \in \mathrm{Ob}(Sh(\mathcal{C}))$ is in E if and only if $\mathcal{G}(U \times_X V)$ is a singleton for all objects V of $\mathcal{C}$. Thus such a sheaf satisfies the sheaf property for all coverings of $\mathcal{D}$ (argument omitted). Conversely, if $\mathcal{G}$ satisfies the sheaf property for all coverings of $\mathcal{D}$, then $\mathcal{G}(U \times_X V)$ is a singleton, as in $\mathcal{D}$ the object $U \times_X V$ is covered by the empty covering. □

08LZ Definition 7.43.6. Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is an closed subtopos if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(\mathcal{C})$ such that E is the subtopos described in Lemma 7.43.5.

All right, and now we can define what it means to have a closed immersion and an open immersion of topoi.

08M0 Definition 7.43.7. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi.

(1) We say f is an open immersion if f is an embedding and the essential image of f_* is an open subtopos.
(2) We say f is a closed immersion if f is an embedding and the essential image of f_* is a closed subtopos.

08M1 Lemma 7.43.8. Let $i : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a closed immersion of topoi. Then i_* is fully faithful, transforms surjections into surjections, commutes with coequalizers, commutes with pushouts, reflects injections, reflects surjections, and reflects bijections.

Proof. Let $\mathcal{F}$ be a subsheaf of the final object $*$ of $Sh(\mathcal{C})$ and let $E \subset Sh(\mathcal{C})$ be the full subcategory consisting of those $\mathcal{G}$ such that $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is an isomorphism. By Lemma 7.43.5 the functor i_* is isomorphic to the inclusion functor $\iota : E \to Sh(\mathcal{C})$.

Let $j_\mathcal{F} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ be the localization functor (Lemma 7.30.1). Note that E can also be described as the collection of sheaves $\mathcal{G}$ such that $j_\mathcal{F}^{-1}\mathcal{G} = *$.

Let $a, b : \mathcal{G}_1 \to \mathcal{G}_2$ be two morphism of E. To prove ι commutes with coequalizers it suffices to show that the coequalizer of a, b in $Sh(\mathcal{C})$ lies in E. This is clear because the coequalizer of two morphisms $* \to *$ is $*$ and because $j_\mathcal{F}^{-1}$ is exact. Similarly for pushouts.

Thus i_* satisfies properties (5), (6), and (7) of Lemma 7.41.1 and hence the morphism i satisfies all properties mentioned in that lemma, in particular the ones mentioned in this lemma. $\square$

7.44. Sheaves of algebraic structures

00YR In Sheaves, Section 6.15 we introduced a type of algebraic structure to be a pair $(\mathcal{A}, s)$, where $\mathcal{A}$ is a category, and $s : \mathcal{A} \to \text{Sets}$ is a functor such that

(1) s is faithful,
(2) $\mathcal{A}$ has limits and s commutes with limits,
(3) $\mathcal{A}$ has filtered colimits and s commutes with them, and
(4) s reflects isomorphisms.

For such a type of algebraic structure we saw that a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ on a space X is a sheaf if and only if the associated presheaf of sets is a sheaf. Moreover, we worked out the notion of stalk, and given a continuous map $f : X \to Y$ we defined adjoint functors pushforward and pullback on sheaves of algebraic structures which agrees with pushforward and pullback on the underlying sheaves of sets. In addition extending a sheaf of algebraic structures from a basis to all opens of a space, works as expected.

Part of this material still works in the setting of sites and sheaves. Let $(\mathcal{A}, s)$ be a type of algebraic structure. Let $\mathcal{C}$ be a site. Let us denote $\text{PSh}(\mathcal{C}, \mathcal{A})$, resp. $Sh(\mathcal{C}, \mathcal{A})$ the category of presheaves, resp. sheaves with values in $\mathcal{A}$ on $\mathcal{C}$.

(α) A presheaf with values in $\mathcal{A}$ is a sheaf if and only if its underlying presheaf of sets is a sheaf. See the proof of Sheaves, Lemma 6.9.2.
(β) Given a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ the presheaf $\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf. This is true since the colimits in the sheafification process are filtered, and even colimits over directed sets (see Section 7.10, especially the proof of Lemma 7.10.14) and since s commutes with filtered colimits.
(γ) We get the following commutative diagram

$$\begin{tikzcd} Sh(\mathcal{C}, \mathcal{A}) \arrow[r, shift left] \arrow[d, "s"'] & \text{PSh}(\mathcal{C}, \mathcal{A}) \arrow[l, shift left, "\#"] \arrow[d, "s"] \\ Sh(\mathcal{C}) \arrow[r, shift left] & \text{PSh}(\mathcal{C}) \arrow[l, shift left] \end{tikzcd}$$

(δ) We have $\mathcal{F} = \mathcal{F}^\#$ if and only if $\mathcal{F}$ is a sheaf of algebraic structures.
(ϵ) The functor $^\#$ is adjoint to the inclusion functor:

$$\text{Mor}_{\text{PSh}(\mathcal{C},\mathcal{A})}(\mathcal{G}, \mathcal{F}) = \text{Mor}_{Sh(\mathcal{C},\mathcal{A})}(\mathcal{G}^\#, \mathcal{F})$$

The proof is the same as the proof of Proposition 7.10.12.
(ζ) The functor $\mathcal{F} \mapsto \mathcal{F}^\#$ is left exact. The proof is the same as the proof of Lemma 7.10.14.

00YS Definition 7.44.1. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a functor $u : \mathcal{C} \to \mathcal{D}$. We define the pushforward functor for presheaves of algebraic structures by the rule $u^p\mathcal{F}(U) = \mathcal{F}(uU)$, and for sheaves of algebraic structures by the same rule, namely $f_*\mathcal{F}(U) = \mathcal{F}(uU)$.

The problem comes with trying the define the pullback. The reason is that the colimits defining the functor u_p in Section 7.5 may not be filtered. Thus the axioms above are not enough in general to define the pullback of a (pre)sheaf of algebraic structures. Nonetheless, in almost all cases the following lemma is sufficient to define pushforward, and pullback of (pre)sheaves of algebraic structures.

00YT Lemma 7.44.2. Suppose the functor $u : \mathcal{C} \to \mathcal{D}$ satisfies the hypotheses of Proposition 7.14.7, and hence gives rise to a morphism of sites $f : \mathcal{D} \to \mathcal{C}$. In this case the pullback functor f^{-1} (resp. u_p) and the pushforward functor f_* (resp. u^p) extend to an adjoint pair of functors on the categories of sheaves (resp. presheaves) of algebraic structures. Moreover, these functors commute with taking the underlying sheaf (resp. presheaf) of sets.

Proof. We have defined $f_* = u^p$ above. In the course of the proof of Proposition 7.14.7 we saw that all the colimits used to define u_p are filtered under the assumptions of the proposition. Hence we conclude from the definition of a type of algebraic structure that we may define u_p by exactly the same colimits as a functor on presheaves of algebraic structures. Adjointness of u_p and u^p is proved in exactly the same way as the proof of Lemma 7.5.4. The discussion of sheafification of presheaves of algebraic structures above then implies that we may define $f^{-1}(\mathcal{F}) = (u_p\mathcal{F})^\#$. $\square$

We briefly discuss a method for dealing with pullback and pushforward for a general morphism of sites, and more generally for any morphism of topoi.

Let $\mathcal{C}$ be a site. In the case $\mathcal{A} = \text{Ab}$, we may think of an abelian (pre)sheaf on $\mathcal{C}$ as a quadruple $(\mathcal{F}, +, 0, i)$. Here the data are

(D1) $\mathcal{F}$ is a sheaf of sets,
(D2) $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets,
(D3) $0 : * \to \mathcal{F}$ is a morphism from the singleton sheaf (see Example 7.10.2) to $\mathcal{F}$, and
(D4) $i : \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets.

These data have to satisfy the following axioms

(A1) $+$ is associative and commutative,
(A2) 0 is a unit for $+$, and
(A3) $+ \circ (1, i) = 0 \circ (\mathcal{F} \to *)$.

Compare Sheaves, Lemma 6.4.3. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites. Note that since f^{-1} is exact we have $f^{-1}* = *$ and $f^{-1}(\mathcal{F} \times \mathcal{F}) = f^{-1}\mathcal{F} \times f^{-1}\mathcal{F}$. Thus we can define $f^{-1}\mathcal{F}$ simply as the quadruple $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. The axioms are going to be preserved because f^{-1} is a functor which commutes with finite limits. Finally it is not hard to check that f_* and f^{-1} are adjoint as usual.

In [AGV71] this method is used. They introduce something called an "espèce the structure algébrique ≪définie par limites projectives finie≫". For such an espèce you can use the method described above to define a pair of adjoint functors f^{-1} and f_* as above. This clearly works for most algebraic structures that one encounters in practice. Instead of formalizing this construction we simply list those algebraic structures for which this method works (to be verified case by case). In fact, this method works for any morphism of topoi.

00YV Proposition 7.44.3. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f = (f^{-1}, f_*)$ be a morphism of topoi from $Sh(\mathcal{D}) \to Sh(\mathcal{C})$. The method introduced above gives rise to an adjoint pair of functors (f^{-1}, f_*) on sheaves of algebraic structures compatible with taking the underlying sheaves of sets for the following types of algebraic structures:

(1) pointed sets,
(2) abelian groups,
(3) groups,
(4) monoids,
(5) rings,
(6) modules over a fixed ring, and
(7) lie algebras over a fixed field.

Moreover, in each of these cases the results above labeled (α), (β), (γ), (δ), (ϵ), and (ζ) hold.

Proof. The final statement of the proposition holds simply since each of the listed categories, endowed with the obvious forgetful functor, is indeed a type of algebraic structure in the sense explained at the beginning of this section. See Sheaves, Lemma 6.15.2.

Proof of (2). We think of a sheaf of abelian groups as a quadruple $(\mathcal{F}, +, 0, i)$ as explained in the discussion preceding the proposition. If $(\mathcal{F}, +, 0, i)$ lives on $\mathcal{C}$, then its pullback is defined as $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. If $(\mathcal{G}, +, 0, i)$ lives on $\mathcal{D}$, then its pushforward is defined as $(f_*\mathcal{G}, f_*+, f_*0, f_*i)$. This works because $f_*(\mathcal{G} \times \mathcal{G}) = f_*\mathcal{G} \times f_*\mathcal{G}$. Adjointness follows from adjointness of the set based functors, since

$$\mathrm{Mor}_{\mathrm{Ab}(\mathcal{C})}((\mathcal{F}_1, +, 0, i), (\mathcal{F}_2, +, 0, i)) = \left\{ \begin{matrix} \varphi \in \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}_1, \mathcal{F}_2) \\ \varphi \text{ is compatible with } +, 0, i \end{matrix} \right\}$$

Details left to the reader.

This method also works for sheaves of rings by thinking of a sheaf of rings (with unit) as a sextuple $(\mathcal{O}, +, 0, i, \cdot, 1)$ satisfying a list of axioms that you can find in any elementary algebra book.

A sheaf of pointed sets is a pair $(\mathcal{F}, p)$, where $\mathcal{F}$ is a sheaf of sets, and $p : * \to \mathcal{F}$ is a map of sheaves of sets.

A sheaf of groups is given by a quadruple $(\mathcal{F}, \cdot, 1, i)$ with suitable axioms.

A sheaf of monoids is given by a pair $(\mathcal{F}, \cdot)$ with suitable axiom.

Let R be a ring. An sheaf of R-modules is given by a quintuple $(\mathcal{F}, +, 0, i, \{\lambda_r\}_{r \in R})$, where the quadruple $(\mathcal{F}, +, 0, i)$ is a sheaf of abelian groups as above, and $\lambda_r : \mathcal{F} \to \mathcal{F}$ is a family of morphisms of sheaves of sets such that $\lambda_r \circ 0 = 0$, $\lambda_r \circ + = + \circ (\lambda_r, \lambda_r)$, $\lambda_{r+r'} = + \circ \lambda_r \times \lambda_{r'} \circ (\mathrm{id}, \mathrm{id})$, $\lambda_{rr'} = \lambda_r \circ \lambda_{r'}$, $\lambda_1 = \mathrm{id}$, $\lambda_0 = 0 \circ (\mathcal{F} \to *)$. □

We will discuss the category of sheaves of modules over a sheaf of rings in Modules on Sites, Section 18.10.

00YU Remark 7.44.4. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a continuous functor which gives rise to a morphism of sites $\mathcal{C} \to \mathcal{D}$. Note that even in the case of abelian groups we have not defined a pullback functor for presheaves of abelian groups. Since all colimits are representable in the category of abelian groups, we certainly may define a functor u_p^{ab} on abelian presheaves by the same colimits as we have used to define u_p on presheaves of sets. It will also be the case that u_p^{ab} is adjoint to u^p on the categories of abelian presheaves. However, it will not always be the case that u_p^{ab} agrees with u_p on the underlying presheaves of sets.

7.45. Pullback maps

06UM It sometimes happens that a site $\mathcal{C}$ does not have a final object. In this case we define the global section functor as follows.

06UN Definition 7.45.1. The global sections of a presheaf of sets $\mathcal{F}$ over a site $\mathcal{C}$ is the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(*, \mathcal{F})$$

where $*$ is the final object in the category of presheaves on $\mathcal{C}$, i.e., the presheaf which associates to every object a singleton.

Of course the same definition applies to sheaves as well. Here is one way to compute global sections.

0792 Lemma 7.45.2. Let $\mathcal{C}$ be a site. Let $a, b : V \to U$ be objects of $\mathcal{C}$ such that

$$\begin{tikzcd} h_V^\# \arrow[r, shift left] \arrow[r, shift right] & h_U^\# \arrow[r] & * \end{tikzcd}$$

is a coequalizer in $\mathit{Sh}(\mathcal{C})$. Then $\Gamma(\mathcal{C}, \mathcal{F})$ is the equalizer of $a^*, b^* : \mathcal{F}(U) \to \mathcal{F}(V)$.

Proof. Since $\mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$ this is clear from the definitions. $\square$

Now, let $f : \mathit{Sh}(\mathcal{D}) \to \mathit{Sh}(\mathcal{C})$ be a morphism of topoi. Then for any sheaf $\mathcal{F}$ on $\mathcal{C}$ there is a pullback map

$$f^{-1} : \Gamma(\mathcal{C}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{D}, f^{-1}\mathcal{F})$$

Namely, as f^{-1} is exact it transforms $*$ into $*$. Hence a global section s of $\mathcal{F}$ over $\mathcal{C}$, which is a map of sheaves $s : * \to \mathcal{F}$, can be pulled back to $f^{-1}s : * = f^{-1}* \to f^{-1}\mathcal{F}$.

We can generalize this a bit by considering a pair of sheaves $\mathcal{F}$, $\mathcal{G}$ on $\mathcal{C}$, $\mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$. Then we compose the construction above with the obvious map $\Gamma(\mathcal{D}, f^{-1}\mathcal{F}) \to \Gamma(\mathcal{D}, \mathcal{G})$ to get a map

$$\Gamma(\mathcal{C}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{D}, \mathcal{G})$$

This map is sometimes also called a pullback map.

A slightly more general construction which occurs frequently in nature is the following. Suppose that we have a commutative diagram of morphisms of topoi

$$\begin{tikzcd} \mathit{Sh}(\mathcal{D}) \arrow[rr, "f"'] \arrow[dr, "h"'] & & \mathit{Sh}(\mathcal{C}) \arrow[dl, "g"] \\ & \mathit{Sh}(\mathcal{B}) & \end{tikzcd}$$

Next, suppose that we have a sheaf $\mathcal{F}$ on $\mathcal{C}$. Then there is a pullback map

$$f^{-1} : g_*\mathcal{F} \longrightarrow h_*f^{-1}\mathcal{F}$$

Namely, it is just the map coming from the identification $g_*f_*f^{-1}\mathcal{F} = h_*f^{-1}\mathcal{F}$ together with g_* applied to the canonical map $\mathcal{F} \to f_*f^{-1}\mathcal{F}$. If g is the identity, then this map on global sections agrees with the pullback map above.

In the situation of the previous paragraph, suppose we have a pair of sheaves $\mathcal{F}$, $\mathcal{G}$ on $\mathcal{C}$, $\mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$, then we compose the pullback map above with h_* applied to $f^{-1}\mathcal{F} \to \mathcal{G}$ to get a map

$$g_*\mathcal{F} \longrightarrow h_*\mathcal{G}$$

Restricting to sections over an object of $\mathcal{B}$ one recovers the "pullback map" on global sections discussed above (with suitable choices of sites).

An even more general situation is where we have a commutative diagram of topoi

$$\begin{tikzcd} Sh(\mathcal{D}) \arrow[r, "f"'] \arrow[d, "h"'] & Sh(\mathcal{C}) \arrow[d, "g"] \\ Sh(\mathcal{B}) \arrow[r, "e"] & Sh(\mathcal{A}) \end{tikzcd}$$

and a sheaf $\mathcal{G}$ on $\mathcal{C}$. Then there is a base change map

$$e^{-1}g_*\mathcal{G} \longrightarrow h_*f^{-1}\mathcal{G}.$$

Namely, this map is adjoint to a map $g_*\mathcal{G} \to e_*h_*f^{-1}\mathcal{G} = (e \circ h)_*f^{-1}\mathcal{G}$ which is the pullback map just described.

0F6X Remark 7.45.3. Consider a commutative diagram

$$\begin{tikzcd} Sh(\mathcal{B}') \arrow[r, "k"'] \arrow[d, "f'"'] & Sh(\mathcal{B}) \arrow[d, "f"] \\ Sh(\mathcal{C}') \arrow[r, "l"] \arrow[d, "g'"'] & Sh(\mathcal{C}) \arrow[d, "g"] \\ Sh(\mathcal{D}') \arrow[r, "m"] & Sh(\mathcal{D}) \end{tikzcd}$$

of topoi. Then the base change maps for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} m^{-1} \circ (g \circ f)_* &= m^{-1} \circ g_* \circ f_* \\ &\to g'_* \circ l^{-1} \circ f_* \\ &\to g'_* \circ f'_* \circ k^{-1} \\ &= (g' \circ f')_* \circ k^{-1} \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0F6Y Remark 7.45.4. Consider a commutative diagram

$$\begin{tikzcd} Sh(\mathcal{C}'') \arrow[r, "g'"'] \arrow[d, "f''"'] & Sh(\mathcal{C}') \arrow[r, "g"'] \arrow[d, "f'"'] & Sh(\mathcal{C}) \arrow[d, "f"] \\ Sh(\mathcal{D}'') \arrow[r, "h'"] & Sh(\mathcal{D}') \arrow[r, "h"] & Sh(\mathcal{D}) \end{tikzcd}$$

of ringed topoi. Then the base change maps for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned}(h \circ h')^{-1} \circ f_* &= (h')^{-1} \circ h^{-1} \circ f_* \\ &\to (h')^{-1} \circ f'_* \circ g^{-1} \\ &\to f''_* \circ (g')^{-1} \circ g^{-1} \\ &= f''_* \circ (g \circ g')^{-1}\end{aligned}$$

is the base change map for the rectangle. We omit the verification.

7.46. Comparison with SGA4

0CMZ Our notation for the functors u^p and u_p from Section 7.5 and u^s and u_s from Section 7.13 is taken from [Art62, pages 14 and 42]. Having made these choices, the notation for the functor ${}_pu$ in Section 7.19 and ${}_su$ in Section 7.20 seems reasonable. In this section we compare our notation with that of SGA4.

Presheaves: Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. The functor u^p is denoted u^* in [AGV71, Exposee I, Section 5]. The functor u_p is denoted $u_!$ in [AGV71, Exposee I, Proposition 5.1]. The functor ${}_pu$ is denoted u_* in [AGV71, Exposee I, Proposition 5.1]. In other words, we have

$$u_p, u^p, {}_pu \quad (SP) \quad \text{versus} \quad u_!, u^*, u_* \quad (SGA4)$$

The reader should be cautioned that different notation is used for these functors in different parts of SGA4.

Sheaves and continuous functors: Suppose that $\mathcal{C}$ and $\mathcal{D}$ are sites and that $u : \mathcal{C} \to \mathcal{D}$ is a continuous functor (Definition 7.13.1). The functor u^s is denoted u_s in [AGV71, Exposee III, 1.11]. The functor u_s is denoted u^s in [AGV71, Exposee III, Proposition 1.2]. In other words, we have

$$u_s, u^s \quad (SP) \quad \text{versus} \quad u^s, u_s \quad (SGA4)$$

When u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ (Definition 7.14.1) we see that the associated morphism of topoi (Lemma 7.15.2) is the same as that in [AGV71, Exposee IV, (4.9.1.1)].

Sheaves and cocontinuous functors: Suppose that $\mathcal{C}$ and $\mathcal{D}$ are sites and that $u : \mathcal{C} \to \mathcal{D}$ is a cocontinuous functor (Definition 7.20.1). The functor ${}_su$ (Lemma 7.20.2) is denoted u_* in [AGV71, Exposee III, Proposition 2.3]. The functor $(u^p\ )^\#$ is denoted u^* in [AGV71, Exposee III, Proposition 2.3]. In other words, we have

$$(u^p\ )^\#, {}_su \quad (SP) \quad \text{versus} \quad u^*, u_* \quad (SGA4)$$

Thus the morphism of topoi associated to u in Lemma 7.21.1 is the same as that in [AGV71, Exposee IV, 4.7].

Morphisms of Topoi: If f is a morphism of topoi given by the functors (f^{-1}, f_*) then the functor f^{-1} is denoted f^* in [AGV71, Exposee IV, Definition 3.1]. We will use f^{-1} to denote pullback of sheaves of sets or more generally sheaves of algebraic structure (Section 7.44). We will use f^* to denote pullback of sheaves of modules for a morphism of ringed topoi (Modules on Sites, Definition 18.13.1).

7.47. Topologies

00YW In this section we define what a topology on a category is as defined in [AGV71]. One can develop all of the machinery of sheaves and topoi in this language. A modern exposition of this material can be found in [KS06]. However, the case of most interest for algebraic geometry is the topology defined by a site on its underlying category. Thus we strongly suggest the first time reader skip this section and all other sections of this chapter!

00YX Definition 7.47.1. Let $\mathcal{C}$ be a category. Let $U \in \text{Ob}(\mathcal{C})$. A sieve S on U is a subpresheaf $S \subset h_U$.

In other words, a sieve on U picks out for each object $T \in \text{Ob}(\mathcal{C})$ a subset $S(T)$ of the set of all morphisms $T \to U$. In fact, the only condition on the collection of subsets $S(T) \subset h_U(T) = \text{Mor}_\mathcal{C}(T, U)$ is the following rule

00YY (7.47.1.1)
$$\left.\begin{matrix}(\alpha : T \to U) \in S(T) \\ g : T' \to T\end{matrix}\right\} \Rightarrow (\alpha \circ g : T' \to U) \in S(T')$$

A good mental picture to keep in mind is to think of the map $S \to h_U$ as a "morphism from S to U".

00YZ Lemma 7.47.2. Let $\mathcal{C}$ be a category. Let $U \in \text{Ob}(\mathcal{C})$.

(1) The collection of sieves on U is a set.
(2) Inclusion defines a partial ordering on this set.
(3) Unions and intersections of sieves are sieves.
00Z0 (4) Given a family of morphisms $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ with target U there exists a unique smallest sieve S on U such that each $U_i \to U$ belongs to $S(U_i)$.
(5) The sieve $S = h_U$ is the maximal sieve.
(6) The empty subpresheaf is the minimal sieve.

Proof. By our definition of subpresheaf, the collection of all subpresheaves of a presheaf $\mathcal{F}$ is a subset of $\prod_{U \in \text{Ob}(\mathcal{C})} \mathcal{P}(\mathcal{F}(U))$. And this is a set. (Here $\mathcal{P}(A)$ denotes the powerset of A.) Hence the collection of sieves on U is a set.

The partial ordering is defined by: $S \leq S'$ if and only if $S(T) \subset S'(T)$ for all $T \to U$. Notation: $S \subset S'$.

Given a collection of sieves S_i, $i \in I$ on U we can define $\bigcup S_i$ as the sieve with values $(\bigcup S_i)(T) = \bigcup S_i(T)$ for all $T \in \text{Ob}(\mathcal{C})$. We define the intersection $\bigcap S_i$ in the same way.

Given $\{U_i \to U\}_{i \in I}$ as in the statement, consider the morphisms of presheaves $h_{U_i} \to h_U$. We simply define S as the union of the images (Definition 7.3.5) of these maps of presheaves.

The last two statements of the lemma are obvious. □

00Z1 Definition 7.47.3. Let $\mathcal{C}$ be a category. Given a family of morphisms $\{f_i : U_i \to U\}_{i \in I}$ of $\mathcal{C}$ with target U we say the sieve S on U described in Lemma 7.47.2 part (4) is the sieve on U generated by the morphisms f_i.

00Z2 Definition 7.47.4. Let $\mathcal{C}$ be a category. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Let $S \subset h_U$ be a sieve. We define the pullback of S by f to be the sieve $S \times_U V$ of V defined by the rule

$$(\alpha : T \to V) \in (S \times_U V)(T) \Leftrightarrow (f \circ \alpha : T \to U) \in S(T)$$

We leave it to the reader to see that this is indeed a sieve (hint: use Equation 7.47.1.1). We also sometimes call $S \times_U V$ the base change of S by $f : V \to U$.

00Z3 **Lemma 7.47.5.** Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let S be a sieve on U. If $f : V \to U$ is in S, then $S \times_U V = h_V$ is maximal.

Proof. Trivial from the definitions. $\square$

00Z4 **Definition 7.47.6.** Let $\mathcal{C}$ be a category. A *topology* on $\mathcal{C}$ is given by a rule which assigns to every $U \in \mathrm{Ob}(\mathcal{C})$ a subset $J(U)$ of the set of all sieves on U satisfying the following conditions

(1) For every morphism $f : V \to U$ in $\mathcal{C}$, and every element $S \in J(U)$ the pullback $S \times_U V$ is an element of $J(V)$.
(2) If S and S' are sieves on $U \in \mathrm{Ob}(\mathcal{C})$, if $S \in J(U)$, and if for all $f \in S(V)$ the pullback $S' \times_U V$ belongs to $J(V)$, then S' belongs to $J(U)$.
(3) For every $U \in \mathrm{Ob}(\mathcal{C})$ the maximal sieve $S = h_U$ belongs to $J(U)$.

In this case, the sieves belonging to $J(U)$ are called the *covering sieves*.

00Z5 **Lemma 7.47.7.** Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$.

(1) Finite intersections of elements of $J(U)$ are in $J(U)$.
(2) If $S \in J(U)$ and $S' \supset S$, then $S' \in J(U)$.

Proof. Let $S, S' \in J(U)$. Consider $S'' = S \cap S'$. For every $V \to U$ in $S(U)$ we have

$$S' \times_U V = S'' \times_U V$$

simply because $V \to U$ already is in S. Hence by the second axiom of the definition we see that $S'' \in J(U)$.

Let $S \in J(U)$ and $S' \supset S$. For every $V \to U$ in $S(U)$ we have $S' \times_U V = h_V$ by Lemma 7.47.5. Thus $S' \times_U V \in J(V)$ by the third axiom. Hence $S' \in J(U)$ by the second axiom. $\square$

00Z6 **Definition 7.47.8.** Let $\mathcal{C}$ be a category. Let J, J' be two topologies on $\mathcal{C}$. We say that J is *finer* or *stronger* than J' if and only if for every object U of $\mathcal{C}$ we have $J'(U) \subset J(U)$. In this case we also say that J' is *coarser* or *weaker* than J.

In other words, any covering sieve of J' is a covering sieve of J. There exists a finest topology on $\mathcal{C}$, namely that topology where any sieve is a covering sieve. This is called the *discrete topology* of $\mathcal{C}$. There also exists a coarsest topology. Namely, the topology where $J(U) = \{h_U\}$ for all objects U. This is called the *chaotic* or *indiscrete topology*.

00Z7 **Lemma 7.47.9.** Let $\mathcal{C}$ be a category. Let $\{J_i\}_{i \in I}$ be a set of topologies.

(1) The rule $J(U) = \bigcap J_i(U)$ defines a topology on $\mathcal{C}$.
(2) There is a coarsest topology finer than all of the topologies J_i.

Proof. The first part is direct from the definitions. The second follows by taking the intersection of all topologies finer than all of the J_i. $\square$

At this point we can define without any motivation what a sheaf is.

00Z8 Definition 7.47.10. Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say that $\mathcal{F}$ is a sheaf on $\mathcal{C}$ if for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every covering sieve S of U the canonical map

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, \mathcal{F}) \longrightarrow \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F})$$

is bijective.

Recall that the left hand side of the displayed formula equals $\mathcal{F}(U)$. In other words, $\mathcal{F}$ is a sheaf if and only if a section of $\mathcal{F}$ over U is the same thing as a compatible collection of sections $s_{T,\alpha} \in \mathcal{F}(T)$ parametrized by $(\alpha : T \to U) \in S(T)$, and this for every covering sieve S on U.

00Z9 Lemma 7.47.11. Let $\mathcal{C}$ be a category. Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of presheaves of sets on $\mathcal{C}$. For each $U \in \mathrm{Ob}(\mathcal{C})$ denote $J(U)$ the set of sieves S with the following property: For every morphism $V \to U$, the maps

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_V, \mathcal{F}_i) \longrightarrow \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S \times_U V, \mathcal{F}_i)$$

are bijective for all $i \in I$. Then J defines a topology on $\mathcal{C}$. This topology is the finest topology in which all of the $\mathcal{F}_i$ are sheaves.

Proof. If we show that J is a topology, then the last statement of the lemma immediately follows. The first and third axioms of a topology are immediately verified. Thus, assume that we have an object U, and sieves S, S' of U such that $S \in J(U)$, and for all $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. We have to show that $S' \in J(U)$. In other words, we have to show that for any $f : W \to U$, the maps

$$\mathcal{F}_i(W) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_W, \mathcal{F}_i) \longrightarrow \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$$

are bijective for all $i \in I$. Pick an element $i \in I$ and pick an element $\varphi \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$. We will construct a section $s \in \mathcal{F}_i(W)$ mapping to φ.

Suppose $\alpha : V \to W$ is an element of $S \times_U W$. According to the definition of pullbacks we see that the composition $f \circ \alpha : V \to W \to U$ is in S. Hence $S' \times_U V$ is in $J(W)$ by assumption on the pair of sieves S, S'. Now we have a commutative diagram of presheaves

$$\begin{tikzcd} S' \times_U V \arrow[r] \arrow[d] & h_V \arrow[d] \\ S' \times_U W \arrow[r] & h_W \end{tikzcd}$$

The restriction of φ to $S' \times_U V$ corresponds to an element $s_{V,\alpha} \in \mathcal{F}_i(V)$. This we see from the definition of J, and because $S' \times_U V$ is in $J(W)$. We leave it to the reader to check that the rule $(V, \alpha) \mapsto s_{V,\alpha}$ defines an element $\psi \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S \times_U W, \mathcal{F}_i)$. Since $S \in J(U)$ we see immediately from the definition of J that ψ corresponds to an element s of $\mathcal{F}_i(W)$.

We leave it to the reader to verify that the construction $\varphi \mapsto s$ is inverse to the natural map displayed above. $\square$

00ZA Definition 7.47.12. Let $\mathcal{C}$ be a category. The finest topology on $\mathcal{C}$ such that all representable presheaves are sheaves, see Lemma 7.47.11, is called the canonical topology of $\mathcal{C}$.

7.48. The topology defined by a site

00ZB Suppose that $\mathcal{C}$ is a category, and suppose that $\text{Cov}_1(\mathcal{C})$ and $\text{Cov}_2(\mathcal{C})$ are sets of coverings that define the structure of a site on $\mathcal{C}$. In this situation it can happen that the categories of sheaves (of sets) for $\text{Cov}_1(\mathcal{C})$ and $\text{Cov}_2(\mathcal{C})$ are the same, see for example Lemma 7.8.7.

It turns out that the category of sheaves on $\mathcal{C}$ with respect to some topology J determines and is determined by the topology J. This is a nontrivial statement which we will address later, see Theorem 7.50.2.

Accepting this for the moment it makes sense to study the topology determined by a site.

00ZC **Lemma 7.48.1.** Let $\mathcal{C}$ be a site with coverings $\text{Cov}(\mathcal{C})$. For every object U of $\mathcal{C}$, let $J(U)$ denote the set of sieves S on U with the following property: there exists a covering $\{f_i : U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$ so that the sieve S' generated by the f_i (see Definition 7.47.3) is contained in S.

(1) This J is a topology on $\mathcal{C}$.
(2) A presheaf $\mathcal{F}$ is a sheaf for this topology (see Definition 7.47.10) if and only if it is a sheaf on the site (see Definition 7.7.1).

Proof. To prove the first assertion we just note that axioms (1), (2) and (3) of the definition of a site (Definition 7.6.2) directly imply the axioms (3), (2) and (1) of the definition of a topology (Definition 7.47.6). As an example we prove J has property (2). Namely, let U be an object of $\mathcal{C}$, let S, S' be sieves on U such that $S \in J(U)$, and such that for every $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. By definition of $J(U)$ we can find a covering $\{f_i : U_i \to U\}$ of the site such that S the image of $h_{U_i} \to h_U$ is contained in S. Since each $S' \times_U U_i$ is in $J(U_i)$ we see that there are coverings $\{U_{ij} \to U_i\}$ of the site such that $h_{U_{ij}} \to h_{U_i}$ is contained in $S' \times_U U_i$. By definition of the base change this means that $h_{U_{ij}} \to h_U$ is contained in the subpresheaf $S' \subset h_U$. By axiom (2) for sites we see that $\{U_{ij} \to U\}$ is a covering of U and we conclude that $S' \in J(U)$ by definition of J.

Let $\mathcal{F}$ be a presheaf. Suppose that $\mathcal{F}$ is a sheaf in the topology J. We will show that $\mathcal{F}$ is a sheaf on the site as well. Let $\{f_i : U_i \to U\}_{i\in I}$ be a covering of the site. Let $s_i \in \mathcal{F}(U_i)$ be a family of sections such that $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for all i, j. We have to show that there exists a unique section $s \in \mathcal{F}(U)$ restricting back to the s_i on the U_i. Let $S \subset h_U$ be the sieve generated by the f_i. Note that $S \in J(U)$ by definition. In stead of constructing s, by the sheaf condition in the topology, it suffices to construct an element

$$\varphi \in \text{Mor}_{\text{PSh}(\mathcal{C})}(S, \mathcal{F}).$$

Take $\alpha \in S(T)$ for some object $T \in \mathcal{U}$. This means exactly that $\alpha : T \to U$ is a morphism which factors through f_i for some $i \in I$ (and maybe more than 1). Pick such an index i and a factorization $\alpha = f_i \circ \alpha_i$. Define $\varphi(\alpha) = \alpha_i^* s_i$. If i', $\alpha = f_i \circ \alpha'_{i'}$ is a second choice, then $\alpha_i^* s_i = (\alpha'_{i'})^* s_{i'}$ exactly because of our condition $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for all i, j. Thus $\varphi(\alpha)$ is well defined. We leave it to the reader to verify that φ, which in turn determines s is correct in the sense that s restricts back to s_i.

Let $\mathcal{F}$ be a presheaf. Suppose that $\mathcal{F}$ is a sheaf on the site $(\mathcal{C}, \text{Cov}(\mathcal{C}))$. We will show that $\mathcal{F}$ is a sheaf for the topology J as well. Let U be an object of $\mathcal{C}$. Let S

be a covering sieve on U with respect to the topology J. Let

$$\varphi \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F}).$$

We have to show there is a unique element in $\mathcal{F}(U) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, \mathcal{F})$ which restricts back to φ. By definition there exists a covering $\{f_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ such that $f_i : U_i \in U$ belongs to $S(U_i)$. Hence we can set $s_i = \varphi(f_i) \in \mathcal{F}(U_i)$. Then it is a pleasant exercise to see that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. Thus we obtain the desired section s by the sheaf condition for $\mathcal{F}$ on the site $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$. Details left to the reader. □

00ZD Definition 7.48.2. Let $\mathcal{C}$ be a site with coverings $\mathrm{Cov}(\mathcal{C})$. The topology associated to $\mathcal{C}$ is the topology J constructed in Lemma 7.48.1 above.

Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ be two coverings defining the structure of a site on $\mathcal{C}$. It may very well happen that the topologies defined by these are the same. If this happens then we say $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ define the same topology on $\mathcal{C}$. And if this happens then the categories of sheaves are the same, by Lemma 7.48.1.

It is usually the case that we only care about the topology defined by a collection of coverings, and we view the possibility of choosing different sets of coverings as a tool to study the topology.

00ZE Remark 7.48.3. Enlarging the class of coverings. Clearly, if $\mathrm{Cov}(\mathcal{C})$ defines the structure of a site on $\mathcal{C}$ then we may add to $\mathcal{C}$ any set of families of morphisms with fixed target tautologically equivalent (see Definition 7.8.2) to elements of $\mathrm{Cov}(\mathcal{C})$ without changing the topology.

00ZF Remark 7.48.4. Shrinking the class of coverings. Let $\mathcal{C}$ be a site. Consider the set

$$\mathcal{S} = P(\mathrm{Arrows}(\mathcal{C})) \times \mathrm{Ob}(\mathcal{C})$$

where $P(\mathrm{Arrows}(\mathcal{C}))$ is the power set of the set of morphisms, i.e., the set of all sets of morphisms. Let $\mathcal{S}_\tau \subset \mathcal{S}$ be the subset consisting of those $(T, U) \in \mathcal{S}$ such that (a) all $\varphi \in T$ have target U, (b) the collection $\{\varphi\}_{\varphi \in T}$ is tautologically equivalent (see Definition 7.8.2) to some covering in $\mathrm{Cov}(\mathcal{C})$. Clearly, considering the elements of $\mathcal{S}_\tau$ as the coverings, we do not get exactly the notion of a site as defined in Definition 7.6.2. The structure $(\mathcal{C}, \mathcal{S}_\tau)$ we get satisfies slightly modified conditions. The modified conditions are:

- (0') $\mathrm{Cov}(\mathcal{C}) \subset P(\mathrm{Arrows}(\mathcal{C})) \times \mathrm{Ob}(\mathcal{C})$,
- (1') If $V \to U$ is an isomorphism then $(\{V \to U\}, U) \in \mathrm{Cov}(\mathcal{C})$.
- (2') If $(T, U) \in \mathrm{Cov}(\mathcal{C})$ and for $f : U' \to U$ in T we are given $(T_f, U') \in \mathrm{Cov}(\mathcal{C})$, then setting $T' = \{f \circ f' \mid f \in T,\ f' \in T_f\}$, we get $(T', U) \in \mathrm{Cov}(\mathcal{C})$.
- (3') If $(T, U) \in \mathrm{Cov}(\mathcal{C})$ and $g : V \to U$ is a morphism of $\mathcal{C}$ then
  - (a) $U' \times_{f,U,g} V$ exists for $f : U' \to U$ in T, and
  - (b) setting $T' = \{\mathrm{pr}_2 : U' \times_{f,U,g} V \to V \mid f : U' \to U \in T\}$ for some choice of fibre products we get $(T', V) \in \mathrm{Cov}(\mathcal{C})$.

And it is easy to verify that, given a structure satisfying (0') – (3') above, then after suitably enlarging $\mathrm{Cov}(\mathcal{C})$ (compare Sets, Section 3.11) we get a site. Obviously there is little difference between this notion and the actual notion of a site, at least from the point of view of the topology. There are two benefits: because of condition (0') above the coverings automatically form a set, and because of (0') the totality of all structures of this type forms a set as well. The price you pay for this is that you have to keep writing "tautologically equivalent" everywhere.

7.49. Sheafification in a topology

00ZG In this section we explain the analogue of the sheafification construction in a topology.

Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets. For every $U \in \mathrm{Ob}(\mathcal{C})$ we define

$$L\mathcal{F}(U) = \mathrm{colim}_{S \in J(U)^{opp}} \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F})$$

as a colimit. Here we think of $J(U)$ as a partially ordered set, ordered by inclusion, see Lemma 7.47.2. The transition maps in the system are defined as follows. If $S \subset S'$ are in $J(U)$, then $S \to S'$ is a morphism of presheaves. Hence there is a natural restriction mapping

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S', \mathcal{F}) \longrightarrow \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F}).$$

Thus we see that $S \mapsto \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F})$ is a directed system as in Categories, Definition 4.21.2 provided we reverse the ordering on $J(U)$ (which is what the superscript ^{opp} is supposed to indicate). In particular, since $h_U \in J(U)$ there is a canonical map

$$\ell : \mathcal{F}(U) \longrightarrow L\mathcal{F}(U)$$

coming from the identification $\mathcal{F}(U) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_U, \mathcal{F})$. In addition, the colimit defining $L\mathcal{F}(U)$ is directed since for any pair of covering sieves S, S' on U the sieve $S \cap S'$ is a covering sieve too, see Lemma 7.47.2.

Let $f : V \to U$ be a morphism in $\mathcal{C}$. Let $S \in J(U)$. There is a commutative diagram

$$\begin{tikzcd} S \times_U V \arrow[r] \arrow[d] & h_V \arrow[d] \\ S \arrow[r] & h_U \end{tikzcd}$$

We can use the left vertical map to get canonical restriction maps

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F}) \to \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S \times_U V, \mathcal{F}).$$

Base change $S \mapsto S \times_U V$ induces an order preserving map $J(U) \to J(V)$. And the restriction maps define a transformation of functors as in Categories, Lemma categories-lemma-functorial-colimit. Hence we get a natural restriction map

$$L\mathcal{F}(U) \longrightarrow L\mathcal{F}(V).$$

00ZH Lemma 7.49.1. In the situation above.

(1) The assignment $U \mapsto L\mathcal{F}(U)$ combined with the restriction mappings defined above is a presheaf.
(2) The maps ℓ glue to give a morphism of presheaves $\ell : \mathcal{F} \to L\mathcal{F}$.
(3) The rule $\mathcal{F} \mapsto (\mathcal{F} \xrightarrow{\ell} L\mathcal{F})$ is a functor.
(4) If $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$, then $L\mathcal{F}$ is a subpresheaf of $L\mathcal{G}$.
(5) The map $\ell : \mathcal{F} \to L\mathcal{F}$ has the following property: For every section $s \in L\mathcal{F}(U)$ there exists a covering sieve S on U and an element $\varphi \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F})$ such that $\ell(\varphi)$ equals the restriction of s to S.

Proof. Omitted. $\square$

00ZI Definition 7.49.2. Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. We say that a presheaf of sets $\mathcal{F}$ is separated if for every object U and every covering sieve S on U the canonical map $\mathcal{F}(U) \to \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F})$ is injective.

00ZJ Theorem 7.49.3. Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets.

(1) The presheaf $L\mathcal{F}$ is separated.
(2) If $\mathcal{F}$ is separated, then $L\mathcal{F}$ is a sheaf and the map of presheaves $\mathcal{F} \to L\mathcal{F}$ is injective.
(3) If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to L\mathcal{F}$ is an isomorphism.
(4) The presheaf $LL\mathcal{F}$ is always a sheaf.

Proof. Part (3) is trivial from the definition of L and the definition of a sheaf (Definition 7.47.10). Part (4) follows formally from the others.

We sketch the proof of (1). Suppose S is a covering sieve of the object U. Suppose that $\varphi_i \in L\mathcal{F}(U)$, $i = 1, 2$ map to the same element in $\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, L\mathcal{F})$. We may find a single covering sieve S' on U such that both φ_i are represented by elements $\varphi_i \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S', \mathcal{F})$. We may assume that $S' = S$ by replacing both S and S' by $S' \cap S$ which is also a covering sieve, see Lemma 7.47.2. Suppose $V \in \mathrm{Ob}(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. Then we have $S \times_U V = h_V$, see Lemma 7.47.5. Thus the restrictions of φ_i via $V \to U$ correspond to sections $s_{i,V,\alpha}$ of $\mathcal{F}$ over V. The assumption is that there exist a covering sieve $S_{V,\alpha}$ of V such that $s_{i,V,\alpha}$ restrict to the same element of $\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S_{V,\alpha}, \mathcal{F})$. Consider the sieve S'' on U defined by the rule

00ZK
$$(f : T \to U) \in S''(T) \quad \Leftrightarrow \quad \begin{aligned} &\exists\ V,\ \alpha : V \to U,\ \alpha \in S(V), \\ &\exists\ g : T \to V,\ g \in S_{V,\alpha}(T), \\ &f = \alpha \circ g \end{aligned} \tag{7.49.3.1}$$

By axiom (2) of a topology we see that S'' is a covering sieve on U. By construction we see that φ_1 and φ_2 restrict to the same element of $\mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S'', L\mathcal{F})$ as desired.

We sketch the proof of (2). Assume that $\mathcal{F}$ is a separated presheaf of sets on $\mathcal{C}$ with respect to the topology J. Let S be a covering sieve of the object U of $\mathcal{C}$. Suppose that $\varphi \in \mathrm{Mor}_{\mathcal{C}}(S, L\mathcal{F})$. We have to find an element $s \in L\mathcal{F}(U)$ restricting to φ. Suppose $V \in \mathrm{Ob}(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. The value $\varphi(\alpha) \in L\mathcal{F}(V)$ is given by a covering sieve $S_{V,\alpha}$ of V and a morphism of presheaves $\varphi_{V,\alpha} : S_{V,\alpha} \to \mathcal{F}$. As in the proof above, define a covering sieve S'' on U by Equation (7.49.3.1). We define

$$\varphi'' : S'' \longrightarrow \mathcal{F}$$

by the following simple rule: For every $f : T \to U$, $f \in S''(T)$ choose V, α, g as in Equation (7.49.3.1). Then set

$$\varphi''(f) = \varphi_{V,\alpha}(g).$$

We claim this is independent of the choice of V, α, g. Consider a second such choiceV', α', g'. The restrictions of $\varphi_{V,\alpha}$ and $\varphi_{V',\alpha'}$ to the intersection of the following covering sieves on T

$$(S_{V,\alpha} \times_{V,g} T) \cap (S_{V',\alpha'} \times_{V',g'} T)$$

agree. Namely, these restrictions both correspond to the restriction of φ to T (via f) and the desired equality follows because $\mathcal{F}$ is separated. Denote the common restriction ψ. The independence of choice follows because $\varphi_{V,\alpha}(g) = \psi(\mathrm{id}_T) = \varphi_{V',\alpha'}(g')$. OK, so now φ'' gives an element $s \in L\mathcal{F}(U)$. We leave it to the reader to check that s restricts to φ. $\square$

00ZL Definition 7.49.4. Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := LL\mathcal{F}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the sheaf associated to $\mathcal{F}$.

00ZM Proposition 7.49.5. Let $\mathcal{C}$ be a category endowed with a topology. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.

Proof. Same as the proof of Proposition 7.10.12. $\square$

7.50. Topologies and sheaves

00ZN

00ZO Lemma 7.50.1. Let $\mathcal{C}$ be a category endowed with a topology J. Let U be an object of $\mathcal{C}$. Let S be a sieve on U. The following are equivalent

(1) The sieve S is a covering sieve.
(2) The sheafification $S^\# \to h_U^\#$ of the map $S \to h_U$ is an isomorphism.

Proof. First we make a couple of general remarks. We will use that $S^\# = LLS$, and $h_U^\# = LLh_U$. In particular, by Lemma 7.49.1, we see that $S^\# \to h_U^\#$ is injective. Note that $\mathrm{id}_U \in h_U(U)$. Hence it gives rise to sections of Lh_U and $h_U^\# = LLh_U$ over U which we will also denote id_U.

Suppose S is a covering sieve. It clearly suffices to find a morphism $h_U \to S^\#$ such that the composition $h_U \to h_U^\#$ is the canonical map. To find such a map it suffices to find a section $s \in S^\#(U)$ wich restricts to id_U. But since S is a covering sieve, the element $\mathrm{id}_S \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S, S)$ gives rise to a section of LS over U which restricts to id_U in Lh_U. Hence we win.

Suppose that $S^\# \to h_U^\#$ is an isomorphism. Let $1 \in S^\#(U)$ be the element corresponding to id_U in $h_U^\#(U)$. Because $S^\# = LLS$ there exists a covering sieve S' on U such that 1 comes from a

$$\varphi \in \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(S', LS).$$

This in turn means that for every $\alpha : V \to U$, $\alpha \in S'(V)$ there exists a covering sieve $S_{V,\alpha}$ on V such that $\varphi(\alpha)$ corresponds to a morphism of presheaves $S_{V,\alpha} \to S$. In other words $S_{V,\alpha}$ is contained in $S \times_U V$. By the second axiom of a topology we see that S is a covering sieve. $\square$

00ZP Theorem 7.50.2. Let $\mathcal{C}$ be a category. Let J, J' be topologies on $\mathcal{C}$. The following are equivalent

(1) $J = J'$,
(2) sheaves for the topology J are the same as sheaves for the topology J'.

Proof. It is a tautology that if $J = J'$ then the notions of sheaves are the same. Conversely, Lemma 7.50.1 characterizes covering sieves in terms of the sheafification functor. But the sheafification functor $\text{PSh}(\mathcal{C}) \to Sh(\mathcal{C}, J)$ is the left adjoint of the inclusion functor $Sh(\mathcal{C}, J) \to \text{PSh}(\mathcal{C})$. Hence if the subcategories $Sh(\mathcal{C}, J)$ and $Sh(\mathcal{C}, J')$ are the same, then the sheafification functors are the same and hence the collections of covering sieves are the same. □

00ZQ **Lemma 7.50.3.** *Assumption and notation as in Theorem 7.50.2. Then $J \subset J'$ if and only if every sheaf for the topology J' is a sheaf for the topology J.*

Proof. One direction is clear. For the other direction suppose that $Sh(\mathcal{C}, J') \subset Sh(\mathcal{C}, J)$. By formal nonsense this implies that if $\mathcal{F}$ is a presheaf of sets, and $\mathcal{F} \to \mathcal{F}^\#$, resp. $\mathcal{F} \to \mathcal{F}^{\#,\prime}$ is the sheafification wrt J, resp. J' then there is a canonical map $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ equals the canonical map $\mathcal{F} \to \mathcal{F}^{\#,\prime}$. Of course, $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ identifies the second sheaf as the sheafification of the first with respect to the topology J'. Apply this to the map $S \to h_U$ of Lemma 7.50.1. We get a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S^\# \arrow[r] \arrow[d] & S^{\#,\prime} \arrow[d] \\ h_U \arrow[r] & h_U^\# \arrow[r] & h_U^{\#,\prime} \end{tikzcd}$$

And clearly, if S is a covering sieve for the topology J then the middle vertical map is an isomorphism (by the lemma) and we conclude that the right vertical map is an isomorphism as it is the sheafification of the one in the middle wrt J'. By the lemma again we conclude that S is a covering sieve for J' as well. □

7.51. Topologies and continuous functors

00ZR Explain how a continuous functor gives an adjoint pair of functors on sheaves.

7.52. Points and topologies

00ZS Recall from Section 7.32 that given a functor $p = u : \mathcal{C} \to \text{Sets}$ we can define a stalk functor

$$\text{PSh}(\mathcal{C}) \longrightarrow \text{Sets}, \mathcal{F} \longmapsto \mathcal{F}_p.$$

00ZT **Definition 7.52.1.** Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. A *point p of the topology* is given by a functor $u : \mathcal{C} \to \text{Sets}$ such that

(1) For every covering sieve S on U the map $S_p \to (h_U)_p$ is surjective.
(2) The stalk functor $Sh(\mathcal{C}) \to \text{Sets}$, $\mathcal{F} \to \mathcal{F}_p$ is exact.

7.53. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories

(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

Topics in Scheme Theory

(42) Chow Homology
(43) Intersection Theory
(44) Picard Schemes of Curves
(45) Weil Cohomology Theories
(46) Adequate Modules
(47) Dualizing Complexes
(48) Duality for Schemes
(49) Discriminants and Differents
(50) de Rham Cohomology
(51) Local Cohomology
(52) Algebraic and Formal Geometry
(53) Algebraic Curves
(54) Resolution of Surfaces
(55) Semistable Reduction
(56) Functors and Morphisms
(57) Derived Categories of Varieties
(58) Fundamental Groups of Schemes
(59) Étale Cohomology
(60) Crystalline Cohomology
(61) Pro-étale Cohomology
(62) Relative Cycles
(63) More Étale Cohomology
(64) The Trace Formula

Algebraic Spaces

(65) Algebraic Spaces
(66) Properties of Algebraic Spaces
(67) Morphisms of Algebraic Spaces
(68) Decent Algebraic Spaces
(69) Cohomology of Algebraic Spaces
(70) Limits of Algebraic Spaces
(71) Divisors on Algebraic Spaces
(72) Algebraic Spaces over Fields
(73) Topologies on Algebraic Spaces
(74) Descent and Algebraic Spaces
(75) Derived Categories of Spaces
(76) More on Morphisms of Spaces
(77) Flatness on Algebraic Spaces
(78) Groupoids in Algebraic Spaces
(79) More on Groupoids in Spaces
(80) Bootstrap
(81) Pushouts of Algebraic Spaces

Topics in Geometry

(82) Chow Groups of Spaces
(83) Quotients of Groupoids
(84) More on Cohomology of Spaces
(85) Simplicial Spaces
(86) Duality for Spaces
(87) Formal Algebraic Spaces
(88) Algebraization of Formal Spaces
(89) Resolution of Surfaces Revisited

Deformation Theory

(90) Formal Deformation Theory
(91) Deformation Theory
(92) The Cotangent Complex
(93) Deformation Problems

Algebraic Stacks

(94) Algebraic Stacks
(95) Examples of Stacks
(96) Sheaves on Algebraic Stacks

CHAPTER 8

Stacks

0266

8.1. Introduction

0267 In this very short chapter we introduce stacks, and stacks in groupoids. See [DM69], and [Vis04].

8.2. Presheaves of morphisms associated to fibred categories

02Z9 Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.33. Suppose that $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ are objects in the fibre category over U. We are going to define a functor

$$Mor(x, y) : (\mathcal{C}/U)^{opp} \longrightarrow \mathrm{Sets}.$$

In other words this will be a presheaf on $\mathcal{C}/U$, see Sites, Definition 7.2.2. Make a choice of pullbacks as in Categories, Definition 4.33.6. Then, for $f : V \to U$ we set

$$Mor(x, y)(f : V \to U) = \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y).$$

Let $f' : V' \to U$ be a second object of $\mathcal{C}/U$. We also have to define the restriction map corresponding to a morphism $g : V'/U \to V/U$ in $\mathcal{C}/U$, in other words $g : V' \to V$ and $f' = f \circ g$. This will be a map

$$\mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow \mathrm{Mor}_{\mathcal{S}_{V'}}({f'}^*x, {f'}^*y), \quad \phi \longmapsto \phi|_{V'}$$

This map will basically be g^*, except that this transforms an element ϕ of the left hand side into an element $g^*\phi$ of $\mathrm{Mor}_{\mathcal{S}_{V'}}(g^*f^*x, g^*f^*y)$. At this point we use the transformation $\alpha_{g,f}$ of Categories, Lemma 4.33.7. In a formula, the restriction map is described by

$$\phi|_{V'} = (\alpha_{g,f})_y^{-1} \circ g^*\phi \circ (\alpha_{g,f})_x.$$

Of course, nobody thinks of this restriction map in this way. We will only do this once in order to verify the following lemma.

026A **Lemma 8.2.1.** This actually does give a presheaf.

Proof. Let $g : V'/U \to V/U$ be as above and similarly $g' : V''/U \to V'/U$ be morphisms in $\mathcal{C}/U$. So $f' = f \circ g$ and $f'' = f' \circ g' = f \circ g \circ g'$. Let $\phi \in \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$. Then we have

$$\begin{aligned}
& (\alpha_{g\circ g',f})_y^{-1} \circ (g \circ g')^*\phi \circ (\alpha_{g\circ g',f})_x \\
= \quad & (\alpha_{g\circ g',f})_y^{-1} \circ (\alpha_{g',g})_{f^*y}^{-1} \circ (g')^*g^*\phi \circ (\alpha_{g',g})_{f^*x} \circ (\alpha_{g\circ g',f})_x \\
= \quad & (\alpha_{g',f'})_y^{-1} \circ (g')^*(\alpha_{g,f})_y^{-1} \circ (g')^*g^*\phi \circ (g')^*(\alpha_{g,f})_x \circ (\alpha_{g',f'})_x \\
= \quad & (\alpha_{g',f'})_y^{-1} \circ (g')^*\Big((\alpha_{g,f})_y^{-1} \circ g^*\phi \circ (\alpha_{g,f})_x\Big) \circ (\alpha_{g',f'})_x
\end{aligned}$$

which is what we want, namely $\phi|_{V''} = (\phi|_{V'})|_{V''}$. The first equality holds because $\alpha_{g',g}$ is a transformation of functors, and hence

$$\begin{tikzcd} (g \circ g')^* f^* x \arrow[r, "(g\circ g')^*\phi"'] \arrow[d, "(\alpha_{g',g})_{f^*x}"'] & (g \circ g')^* f^* y \arrow[d, "(\alpha_{g',g})_{f^*y}"] \\ (g')^* g^* f^* x \arrow[r, "(g')^* g^* \phi"] & (g')^* g^* f^* y \end{tikzcd}$$

commutes. The second equality holds because of property (d) of a pseudo functor since $f' = f \circ g$ (see Categories, Definition 4.29.5). The last equality follows from the fact that $(g')^*$ is a functor. $\square$

From now on we often omit mentioning the transformations $\alpha_{g,f}$ and we simply identify the functors $g^* \circ f^*$ and $(f \circ g)^*$. In particular, given $g : V'/U \to V/U$ the restriction mappings for the presheaf $Mor(x, y)$ will sometimes be denoted $\phi \mapsto g^*\phi$. We formalize the construction in a definition.

02ZB Definition 8.2.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.33. Given an object U of $\mathcal{C}$ and objects x, y of the fibre category, the presheaf of morphisms from x to y is the presheaf

$$(f : V \to U) \longmapsto \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$$

described above. It is denoted $Mor(x, y)$. The subpresheaf $Isom(x, y)$ whose values over V is the set of isomorphisms $f^*x \to f^*y$ in the fibre category $\mathcal{S}_V$ is called the presheaf of isomorphisms from x to y.

If $\mathcal{S}$ is fibred in groupoids then of course $Isom(x, y) = Mor(x, y)$, and it is customary to use the $Isom$ notation.

042V Lemma 8.2.3. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism of fibred categories over the category $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}((\mathcal{S}_1)_U)$. Then F defines a canonical morphism of presheaves

$$Mor_{\mathcal{S}_1}(x, y) \longrightarrow Mor_{\mathcal{S}_2}(F(x), F(y))$$

on $\mathcal{C}/U$.

Proof. By Categories, Definition 4.33.9 the functor F maps strongly cartesian morphisms to strongly cartesian morphisms. Hence if $f : V \to U$ is a morphism in $\mathcal{C}$, then there are canonical isomorphisms $\alpha_V : f^*F(x) \to F(f^*x)$, $\beta_V : f^*F(y) \to F(f^*y)$ such that $f^*F(x) \to F(f^*x) \to F(x)$ is the canonical morphism $f^*F(x) \to F(x)$, and similarly for β_V. Thus we may define

$$\begin{tikzcd} Mor_{\mathcal{S}_1}(x, y)(f : V \to U) \arrow[r, equal] & \mathrm{Mor}_{\mathcal{S}_{1,V}}(f^*x, f^*y) \arrow[d] \\ Mor_{\mathcal{S}_2}(F(x), F(y))(f : V \to U) \arrow[r, equal] & \mathrm{Mor}_{\mathcal{S}_{2,V}}(f^*F(x), f^*F(y)) \end{tikzcd}$$

by $\phi \mapsto \beta_V^{-1} \circ F(\phi) \circ \alpha_V$. We omit the verification that this is compatible with the restriction mappings. $\square$

02ZA Remark 8.2.4. Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. In this case we can prove Lemma 8.2.1 using Categories, Lemma 4.36.4 which says that $\mathcal{S} \to \mathcal{C}$ is equivalent to the category associated to a contravariant functor $F : \mathcal{C} \to \text{Groupoids}$.

In the case of the fibred category associated to F we have $g^* \circ f^* = (f \circ g)^*$ on the nose and there is no need to use the maps $\alpha_{g,f}$. In this case the lemma is (even more) trivial. Of course then one uses that the $Mor(x, y)$ presheaf is unchanged when passing to an equivalent fibred category which follows from Lemma 8.2.3.

04SI Lemma 8.2.5. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.33. Let $U \in \mathrm{Ob}(\mathcal{C})$ and let $x, y \in \mathrm{Ob}(\mathcal{S}_U)$. Denote $x, y : \mathcal{C}/U \to \mathcal{S}$ also the corresponding 1-morphisms, see Categories, Lemma 4.41.2. Then

(1) the 2-fibre product $\mathcal{S} \times_{\mathcal{S} \times \mathcal{S}, (x,y)} \mathcal{C}/U$ is fibred in setoids over $\mathcal{C}/U$, and
(2) $Isom(x, y)$ is the presheaf of sets corresponding to this category fibred in setoids, see Categories, Lemma 4.39.6.

Proof. Omitted. Hint: Objects of the 2-fibre product are $(a : V \to U, z, (\alpha, \beta))$ where $\alpha : z \to a^*x$ and $\beta : z \to a^*y$ are isomorphisms in $\mathcal{S}_V$. Thus the relationship with $Isom(x, y)$ comes by assigning to such an object the isomorphism $\beta \circ \alpha^{-1}$. $\square$

8.3. Descent data in fibred categories

02ZC In this section we define the notion of a descent datum in the abstract setting of a fibred category. Before we do so we point out that this is completely analogous to descent data for quasi-coherent sheaves (Descent, Section 35.2) and descent data for schemes over schemes (Descent, Section 35.34).

We will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

026B Definition 8.3.1. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 4.33.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$. Assume all the fibre products $U_i \times_U U_j$, and $U_i \times_U U_j \times_U U_k$ exist.

(1) A descent datum (X_i, φ_{ij}) in $\mathcal{S}$ relative to the family $\{f_i : U_i \to U\}$ is given by an object X_i of $\mathcal{S}_{U_i}$ for each $i \in I$, an isomorphism $\varphi_{ij} : \mathrm{pr}_0^* X_i \to \mathrm{pr}_1^* X_j$ in $\mathcal{S}_{U_i \times_U U_j}$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram
$$\begin{tikzcd} \mathrm{pr}_0^* X_i \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & \mathrm{pr}_2^* X_k \\ & \mathrm{pr}_1^* X_j \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$
in the category $\mathcal{S}_{U_i \times_U U_j \times_U U_k}$ commutes. This is called the cocycle condition.
(2) A morphism $\psi : (X_i, \varphi_{ij}) \to (X'_i, \varphi'_{ij})$ of descent data is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms $\psi_i : X_i \to X'_i$ in $\mathcal{S}_{U_i}$ such that all the diagrams
$$\begin{tikzcd} \mathrm{pr}_0^* X_i \arrow[r, "\varphi_{ij}"'] \arrow[d, "\mathrm{pr}_0^*\psi_i"'] & \mathrm{pr}_1^* X_j \arrow[d, "\mathrm{pr}_1^*\psi_j"] \\ \mathrm{pr}_0^* X'_i \arrow[r, "\varphi'_{ij}"] & \mathrm{pr}_1^* X'_j \end{tikzcd}$$
in the categories $\mathcal{S}_{U_i \times_U U_j}$ commute.

(3) The category of descent data relative to $\mathcal{U}$ is denoted $DD(\mathcal{U})$.

The fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ will exist if each of the morphisms $f_i : U_i \to U$ is representable, see Categories, Definition 4.6.4. Recall that in a site one of the conditions for a covering $\{U_i \to U\}$ is that each of the morphisms is representable, see Sites, Definition 7.6.2 part (3). In fact the main interest in the definition above is where $\mathcal{C}$ is a site and $\{U_i \to U\}$ is a covering of $\mathcal{C}$. However, a descent datum is just an abstract gadget that can be defined as above. This is useful: for example, given a fibred category over $\mathcal{C}$ one can look at the collection of families with respect to which descent data are effective, and try to use these as the family of coverings for a site.

026C Remarks 8.3.2. Two remarks on Definition 8.3.1 are in order. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\{f_i : U_i \to U\}_{i\in I}$, and (X_i, φ_{ij}) be as in Definition 8.3.1.

(1) There is a diagonal morphism $\Delta : U_i \to U_i \times_U U_i$. We can pull back φ_{ii} via this morphism to get an automorphism $\Delta^*\varphi_{ii} \in \mathrm{Aut}_{U_i}(X_i)$. On pulling back the cocycle condition for the triple (i, i, i) by $\Delta_{123} : U_i \to U_i \times_U U_i \times_U U_i$ we deduce that $\Delta^*\varphi_{ii} \circ \Delta^*\varphi_{ii} = \Delta^*\varphi_{ii}$; thus $\Delta^*\varphi_{ii} = \mathrm{id}_{X_i}$.
(2) There is a morphism $\Delta_{13} : U_i \times_U U_j \to U_i \times_U U_j \times_U U_i$ and we can pull back the cocycle condition for the triple (i, j, i) to get the identity $(\sigma^*\varphi_{ji}) \circ \varphi_{ij} = \mathrm{id}_{\mathrm{pr}_0^* X_i}$, where $\sigma : U_i \times_U U_j \to U_j \times_U U_i$ is the switching morphism.

02ZD Lemma 8.3.3. (Pullback of descent data.) Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice pullbacks as in Categories, Definition 4.33.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$, and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be a families of morphisms of $\mathcal{C}$ with fixed target. Assume all the fibre products $U_i \times_U U_{i'}$, $U_i \times_U U_{i'} \times_U U_{i''}$, $V_j \times_V V_{j'}$, and $V_j \times_V V_{j'} \times_V V_{j''}$ exist. Let $\alpha : I \to J$, $h : U \to V$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1.

(1) Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \to V\}$. The system
$$\left(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum relative to $\mathcal{U}$.
(2) This construction defines a functor between descent data relative to $\mathcal{V}$ and descent data relative to $\mathcal{U}$.
(3) Given a second $\alpha' : I \to J$, $h' : U \to V$ and $g'_i : U_i \to V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.

Proof. Omitted. $\square$

02ZE Definition 8.3.4. With $\mathcal{U} = \{U_i \to U\}_{i\in I}$, $\mathcal{V} = \{V_j \to V\}_{j\in J}$, $\alpha : I \to J$, $h : U \to V$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 8.3.3 the functor
$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')})$$
constructed in that lemma is called the pullback functor on descent data.

Given $h : U \to V$, if there exists a morphism $\tilde{h} : \mathcal{U} \to \mathcal{V}$ covering h then $\tilde{h}^*$ is independent of the choice of $\tilde{h}$ as we saw in Lemma 8.3.3. Hence we will sometimes simply write h^* to indicate the pullback functor.

026E Definition 8.3.5. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 4.33.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with target U. Assume all the fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exist.

(1) Given an object X of $\mathcal{S}_U$ the trivial descent datum is the descent datum (X, id_X) with respect to the family $\{\text{id}_U : U \to U\}$.
(2) Given an object X of $\mathcal{S}_U$ we have a canonical descent datum on the family of objects f_i^*X by pulling back the trivial descent datum (X, id_X) via the obvious map $\{f_i : U_i \to U\} \to \{\text{id}_U : U \to U\}$. We denote this descent datum (f_i^*X, can).
(3) A descent datum (X_i, φ_{ij}) relative to $\{f_i : U_i \to U\}$ is called effective if there exists an object X of $\mathcal{S}_U$ such that (X_i, φ_{ij}) is isomorphic to (f_i^*X, can).

Note that the rule that associates to $X \in \mathcal{S}_U$ its canonical descent datum relative to $\mathcal{U}$ defines a functor

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U}).$$

A descent datum is effective if and only if it is in the essential image of this functor. Let us make explicit the canonical descent datum as follows.

026D Lemma 8.3.6. In the situation of Definition 8.3.5 part (2) the maps $can_{ij} : \text{pr}_0^* f_i^* X \to \text{pr}_1^* f_j^* X$ are equal to $(\alpha_{\text{pr}_1, f_j})_X \circ (\alpha_{\text{pr}_0, f_i})_X^{-1}$ where $\alpha_{\cdot,\cdot}$ is as in Categories, Lemma 4.33.7 and where we use the equality $f_i \circ \text{pr}_0 = f_j \circ \text{pr}_1$ as maps $U_i \times_U U_j \to U$.

Proof. Omitted. $\square$

0GEA Lemma 8.3.7. Let $\mathcal{C}$ be a category. Let $\mathcal{V} = \{V_j \to U\}_{j \in J} \to \mathcal{U} = \{U_i \to U\}_{i \in I}$ be a morphism of families of maps with fixed target of $\mathcal{C}$ given by $\text{id} : U \to U$, $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. If

(1) for $0 \leq p \leq 3$ and $0 \leq q \leq 3$ with $p + q \geq 2$ and $i_1, \ldots, i_p \in I$ and $j_1, \ldots, j_q \in J$ the fibre products $U_{i_1} \times_U \ldots \times_U U_{i_p} \times_U V_{j_1} \times_U \ldots \times_U V_{j_q}$ exist,
(2) the functor $\mathcal{S}_U \to DD(\mathcal{V})$ is an equivalence,
(3) for every $i \in I$ the functor $\mathcal{S}_{U_i} \to DD(\mathcal{V}_i)$ is fully faithful, and
(4) for every $i, i' \in I$ the functor $\mathcal{S}_{U_i \times_U U_{i'}} \to DD(\mathcal{V}_{ii'})$ is faithful.

Here $\mathcal{V}_i = \{U_i \times_U V_j \to U_i\}_{j \in J}$ and $\mathcal{V}_{ii'} = \{U_i \times_U U_{i'} \times_U V_j \to U_i \times_U U_{i'}\}_{j \in J}$. Then $\mathcal{S}_U \to DD(\mathcal{U})$ is an equivalence.

Proof. Condition (1) guarantees we have enough fibre products so that the statement makes sense. We will show that the functor $\mathcal{S}_U \to DD(\mathcal{U})$ is essentially surjective. Suppose given a descent datum $(X_i, \varphi_{ii'})$ relative to $\mathcal{U}$. By Lemma 8.3.3 we can pull this back to a descent datum $(X_j, \varphi_{jj'})$ for $\mathcal{V}$. By assumption (2) this descent datum is effective, hence we get an object X of $\mathcal{S}_U$ such that the pullback of the trivial descent datum (X, id_X) by the morphism $\mathcal{V} \to \{U \to U\}$ is isomorphic to $(X_j, \varphi_{jj'})$. Next, observe that we have a diagram

$$\begin{tikzcd} \mathcal{V}_i \arrow[r] \arrow[d] & \mathcal{V} \arrow[r] & \mathcal{U} \arrow[d] \\ \{U_i \to U_i\} \arrow[rru] \arrow[rr] & & \{U \to U\} \end{tikzcd}$$

of morphisms of families of maps with fixed target of $\mathcal{C}$. This diagram does not commute, but by Lemma 8.3.3 the pullback functors on descent data one gets are canonically isomorphic. Hence (X, id_X) and (X_i, id_{X_i}) pull back to isomorphic objects in $DD(\mathcal{V}_i)$. Hence by assumption (3) we obtain an isomorphism $(U_i \to U)^*X \to X_i$ in the category $\mathcal{S}_{U_i}$. We omit the verification that these arrows are compatible with the morphisms $\varphi_{ii'}$; hint: use the faithfulness of the functors in condition (4). We also omit the verification that the functor $\mathcal{S}_U \to DD(\mathcal{U})$ is fully faithful. □

8.4. Stacks

0268 Here is the definition of a stack. It mixes the notion of a fibred category with the notion of descent.

026F Definition 8.4.1. Let $\mathcal{C}$ be a site. A stack over $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ which satisfies the following conditions:

(1) $p : \mathcal{S} \to \mathcal{C}$ is a fibred category, see Categories, Definition 4.33.5,
(2) for any $U \in \mathrm{Ob}(\mathcal{C})$ and any $x, y \in \mathcal{S}_U$ the presheaf $Mor(x, y)$ (see Definition 8.2.2) is a sheaf on the site $\mathcal{C}/U$, and
(3) for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ of the site $\mathcal{C}$, any descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.

We find the formulation above the most convenient way to think about a stack. Namely, given a category over $\mathcal{C}$ in order to verify that it is a stack you proceed to check properties (1), (2) and (3) in that order. Certainly properties (2) and (3) do not make sense if the category isn't fibred. Without (2) we cannot prove that the descent in (3) is unique up to unique isomorphism and functorial.

The following lemma provides an alternative definition.

02ZF Lemma 8.4.2. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. The following are equivalent

(1) $\mathcal{S}$ is a stack over $\mathcal{C}$, and
(2) for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ of the site $\mathcal{C}$ the functor
$$\mathcal{S}_U \longrightarrow DD(\mathcal{U})$$
which associates to an object its canonical descent datum is an equivalence.

Proof. Omitted. □

04TU Lemma 8.4.3. Let $p : \mathcal{S} \to \mathcal{C}$ be a stack over the site $\mathcal{C}$. Let $\mathcal{S}'$ be a subcategory of $\mathcal{S}$. Assume

(1) if $\varphi : y \to x$ is a strongly cartesian morphism of $\mathcal{S}$ and x is an object of $\mathcal{S}'$, then y is isomorphic to an object of $\mathcal{S}'$,
(2) $\mathcal{S}'$ is a full subcategory of $\mathcal{S}$, and
(3) if $\{f_i : U_i \to U\}$ is a covering of $\mathcal{C}$, and x an object of $\mathcal{S}$ over U such that f_i^*x is isomorphic to an object of $\mathcal{S}'$ for each i, then x is isomorphic to an object of $\mathcal{S}'$.

Then $\mathcal{S}' \to \mathcal{C}$ is a stack.

Proof. Omitted. Hints: The first condition guarantees that $\mathcal{S}'$ is a fibred category. The second condition guarantees that the *Isom*-presheaves of $\mathcal{S}'$ are sheaves (as they are identical to their counter parts in $\mathcal{S}$). The third condition guarantees that the descent condition holds in $\mathcal{S}'$ as we can first descend in $\mathcal{S}$ and then (3) implies the resulting object is isomorphic to an object of $\mathcal{S}'$. $\square$

042W Lemma 8.4.4. Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack over $\mathcal{C}$.

Proof. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. By Categories, Lemma 4.33.8 we see that $\mathcal{S}_1$ is fibred if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$. Hence we may assume that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are fibred. Moreover, the proof of Categories, Lemma 4.33.8 shows that F and G map strongly cartesian morphisms to strongly cartesian morphisms, i.e., F and G are 1-morphisms of fibred categories over $\mathcal{C}$. This means that given $U \in \mathrm{Ob}(\mathcal{C})$, and $x, y \in \mathcal{S}_{1,U}$ then the presheaves

$$Mor_{\mathcal{S}_1}(x, y), Mor_{\mathcal{S}_1}(F(x), F(y)) : (\mathcal{C}/U)^{opp} \longrightarrow \mathrm{Sets}.$$

are identified, see Lemma 8.2.3. Hence the first is a sheaf if and only if the second is a sheaf. Finally, we have to show that if every descent datum in $\mathcal{S}_1$ is effective, then so is every descent datum in $\mathcal{S}_2$. To do this, let $(X_i, \varphi_{ii'})$ be a descent datum in $\mathcal{S}_2$ relative the covering $\{U_i \to U\}$ of the site $\mathcal{C}$. Then $(G(X_i), G(\varphi_{ii'}))$ is a descent datum in $\mathcal{S}_1$ relative the covering $\{U_i \to U\}$. Let X be an object of $\mathcal{S}_{1,U}$ such that the descent datum (f_i^*X, can) is isomorphic to $(G(X_i), G(\varphi_{ii'}))$. Then $F(X)$ is an object of $\mathcal{S}_{2,U}$ such that the descent datum $(f_i^*F(X), can)$ is isomorphic to $(F(G(X_i)), F(G(\varphi_{ii'})))$ which in turn is isomorphic to the original descent datum $(X_i, \varphi_{ii'})$ using i. $\square$

The 2-category of stacks over $\mathcal{C}$ is defined as follows.

02ZG Definition 8.4.5. Let $\mathcal{C}$ be a site. The 2-category of stacks over $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Categories, Definition 4.33.9) defined as follows:

(1) Its objects will be stacks $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

026G Lemma 8.4.6. Let $\mathcal{C}$ be a site. The $(2, 1)$-category of stacks over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.32.3.

Proof. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be 1-morphisms of stacks over $\mathcal{C}$ as defined above. The category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Categories, Lemma 4.32.3 is a fibred category according to Categories, Lemma 4.33.10. (This is where we use that f and g preserve strongly cartesian morphisms.) It remains to show that the morphism presheaves are sheaves and that descent relative to coverings of $\mathcal{C}$ is effective.

Recall that an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by a quadruple (U, x, y, ϕ). It lies over the object U of $\mathcal{C}$. Next, let (U, x', y', ϕ') be second object lying over U. Recall that $\phi : f(x) \to g(y)$, and $\phi' : f(x') \to g(y')$ are isomorphisms in the category $\mathcal{S}_U$. Let us use these isomorphisms to identify $z = f(x) = g(y)$ and $z' = f(x') = g(y')$. With this identifications it is clear that

$$Mor((U, x, y, \phi), (U, x', y', \phi')) = Mor(x, x') \times_{Mor(z,z')} Mor(y, y')$$

as presheaves. However, as the fibred product in the category of presheaves preserves sheaves (Sites, Lemma 7.10.1) we see that this is a sheaf.

Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a covering of the site $\mathcal{C}$. Let (X_i, χ_{ij}) be a descent datum in $\mathcal{X} \times_\mathcal{S} \mathcal{Y}$ relative to $\mathcal{U}$. Write $X_i = (U_i, x_i, y_i, \phi_i)$ as above. Write $\chi_{ij} = (\varphi_{ij}, \psi_{ij})$ as in the definition of the category $\mathcal{X} \times_\mathcal{S} \mathcal{Y}$ (see Categories, Lemma 4.32.3). It is clear that (x_i, φ_{ij}) is a descent datum in $\mathcal{X}$ and that (y_i, ψ_{ij}) is a descent datum in $\mathcal{Y}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are stacks these descent data are effective. Thus we get $x \in \mathrm{Ob}(\mathcal{X}_U)$, and $y \in \mathrm{Ob}(\mathcal{Y}_U)$ with $x_i = x|_{U_i}$, and $y_i = y|_{U_i}$ compatibly with descent data. Set $z = f(x)$ and $z' = g(y)$ which are both objects of $\mathcal{S}_U$. The morphisms ϕ_i are elements of $Isom(z, z')(U_i)$ with the property that $\phi_i|_{U_i \times_U U_j} = \phi_j|_{U_i \times_U U_j}$. Hence by the sheaf property of $Isom(z, z')$ we obtain an isomorphism $\phi : z = f(x) \to z' = g(y)$. We omit the verification that the canonical descent datum associated to the object (U, x, y, ϕ) of $(\mathcal{X} \times_\mathcal{S} \mathcal{Y})_U$ is isomorphic to the descent datum we started with. $\square$

04WQ Lemma 8.4.7. Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be stacks over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism. Then the following are equivalent

(1) F is fully faithful,
(2) for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every $x, y \in \mathrm{Ob}(\mathcal{S}_{1,U})$ the map
$$F : Mor_{\mathcal{S}_1}(x, y) \longrightarrow Mor_{\mathcal{S}_2}(F(x), F(y))$$
is an isomorphism of sheaves on $\mathcal{C}/U$.

Proof. Assume (1). For U, x, y as in (2) the displayed map F evaluates to the map $F : \mathrm{Mor}_{\mathcal{S}_{1,V}}(x|_V, y|_V) \to \mathrm{Mor}_{\mathcal{S}_{2,V}}(F(x|_V), F(y|_V))$ on an object V of $\mathcal{C}$ lying over U. Now, since F is fully faithful, the corresponding map $\mathrm{Mor}_{\mathcal{S}_1}(x|_V, y|_V) \to \mathrm{Mor}_{\mathcal{S}_2}(F(x|_V), F(y|_V))$ is a bijection. Morphisms in the fibre category $\mathcal{S}_{1,V}$ are exactly those morphisms between $x|_V$ and $y|_V$ in $\mathcal{S}_1$ lying over id_V. Similarly, morphisms in the fibre category $\mathcal{S}_{2,V}$ are exactly those morphisms between $F(x|_V)$ and $F(y|_V)$ in $\mathcal{S}_2$ lying over id_V. Thus we find that F induces a bijection between these also. Hence (2) holds.

Assume (2). Suppose given objects U, V of $\mathcal{C}$ and $x \in \mathrm{Ob}(\mathcal{S}_{1,U})$ and $y \in \mathrm{Ob}(\mathcal{S}_{1,V})$. To show that F is fully faithful, it suffices to prove it induces a bijection on morphisms lying over a fixed $f : U \to V$. Choose a strongly Cartesian $f^*y \to y$ in $\mathcal{S}_1$ lying above f. This results in a bijection between the set of morphisms $x \to y$ in $\mathcal{S}_1$ lying over f and $\mathrm{Mor}_{\mathcal{S}_{1,U}}(x, f^*y)$. Since F preserves strongly Cartesian morphisms as a 1-morphism in the 2-category of stacks over $\mathcal{C}$, we also get a bijection between the set of morphisms $F(x) \to F(y)$ in $\mathcal{S}_2$ lying over f and $\mathrm{Mor}_{\mathcal{S}_{2,U}}(F(x), F(f^*y))$. Since F induces a bijection $\mathrm{Mor}_{\mathcal{S}_{1,U}}(x, f^*y) \to \mathrm{Mor}_{\mathcal{S}_{2,U}}(F(x), F(f^*y))$ we conclude (1) holds. $\square$

046N Lemma 8.4.8. Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be stacks over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism which is fully faithful. Then the following are equivalent

(1) F is an equivalence,

(2) for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every $x \in \mathrm{Ob}(\mathcal{S}_{2,U})$ there exists a covering $\{f_i : U_i \to U\}$ such that f_i^*x is in the essential image of the functor $F : \mathcal{S}_{1,U_i} \to \mathcal{S}_{2,U_i}$.

Proof. The implication (1) $\Rightarrow$ (2) is immediate. To see that (2) implies (1) we have to show that every x as in (2) is in the essential image of the functor F. To do this choose a covering as in (2), $x_i \in \mathrm{Ob}(\mathcal{S}_{1,U_i})$, and isomorphisms $\varphi_i : F(x_i) \to f_i^*x$. Then we get a descent datum for $\mathcal{S}_1$ relative to $\{f_i : U_i \to U\}$ by taking

$$\varphi_{ij} : x_i|_{U_i \times_U U_j} \longrightarrow x_j|_{U_i \times_U U_j}$$

the arrow such that $F(\varphi_{ij}) = \varphi_j^{-1} \circ \varphi_i$. This descent datum is effective by the axioms of a stack, and hence we obtain an object x_1 of $\mathcal{S}_1$ over U. We omit the verification that $F(x_1)$ is isomorphic to x over U. □

03ZZ Remark 8.4.9. (Cutting down a “big” stack to get a stack.) Let $\mathcal{C}$ be a site. Suppose that $p : \mathcal{S} \to \mathcal{C}$ is functor from a “big” category to $\mathcal{C}$, i.e., suppose that the collection of objects of $\mathcal{S}$ forms a proper class. Finally, suppose that $p : \mathcal{S} \to \mathcal{C}$ satisfies conditions (1), (2), (3) of Definition 8.4.1. In general there is no way to replace $p : \mathcal{S} \to \mathcal{C}$ by a equivalent category such that we obtain a stack. The reason is that it can happen that a fibre categories $\mathcal{S}_U$ may have a proper class of isomorphism classes of objects. On the other hand, suppose that

(4) for every $U \in \mathrm{Ob}(\mathcal{C})$ there exists a set $S_U \subset \mathrm{Ob}(\mathcal{S}_U)$ such that every object of $\mathcal{S}_U$ is isomorphic in $\mathcal{S}_U$ to an element of S_U.

In this case we can find a full subcategory $\mathcal{S}_{small}$ of $\mathcal{S}$ such that, setting $p_{small} = p|_{\mathcal{S}_{small}}$, we have

(a) the functor $p_{small} : \mathcal{S}_{small} \to \mathcal{C}$ defines a stack, and
(b) the inclusion $\mathcal{S}_{small} \to \mathcal{S}$ is fully faithful and essentially surjective.

(Hint: For every $U \in \mathrm{Ob}(\mathcal{C})$ let $\alpha(U)$ denote the smallest ordinal such that $\mathrm{Ob}(\mathcal{S}_U) \cap V_{\alpha(U)}$ surjects onto the set of isomorphism classes of $\mathcal{S}_U$, and set $\alpha = \sup_{U \in \mathrm{Ob}(\mathcal{C})} \alpha(U)$. Then take $\mathrm{Ob}(\mathcal{S}_{small}) = \mathrm{Ob}(\mathcal{S}) \cap V_\alpha$. For notation used see Sets, Section 3.5.)

8.5. Stacks in groupoids

02ZH Among stacks those which are fibred in groupoids are somewhat easier to comprehend. We redefine them as follows.

02ZI Definition 8.5.1. A stack in groupoids over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids over $\mathcal{C}$ (see Categories, Definition 4.35.1),
(2) for all $U \in \mathrm{Ob}(\mathcal{C})$, for all $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $\mathcal{C}/U$, and
(3) for all coverings $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

Usually the hardest part to check is the third condition. Here is the lemma comparing this with the notion of a stack.

02ZJ Lemma 8.5.2. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. The following are equivalent

(1) $\mathcal{S}$ is a stack in groupoids over $\mathcal{C}$,

(2) $\mathcal{S}$ is a stack over $\mathcal{C}$ and all fibre categories are groupoids, and
(3) $\mathcal{S}$ is fibred in groupoids over $\mathcal{C}$ and is a stack over $\mathcal{C}$.

Proof. Omitted, but see Categories, Lemma 4.35.2. □

03YI Lemma 8.5.3. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a stack. Let $p' : \mathcal{S}' \to \mathcal{C}$ be the category fibred in groupoids associated to $\mathcal{S}$ constructed in Categories, Lemma 4.35.3. Then $p' : \mathcal{S}' \to \mathcal{C}$ is a stack in groupoids.

Proof. Recall that the morphisms in $\mathcal{S}'$ are exactly the strongly cartesian morphisms of $\mathcal{S}$, and that any isomorphism of $\mathcal{S}$ is such a morphism. Hence descent data in $\mathcal{S}'$ are exactly the same thing as descent data in $\mathcal{S}$. Now apply Lemma 8.4.2. Some details omitted. □

042X Lemma 8.5.4. Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack in groupoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack in groupoids over $\mathcal{C}$.

Proof. Follows by combining Lemmas 8.5.2 and 8.4.4. □

The 2-category of stacks in groupoids over $\mathcal{C}$ is defined as follows.

02ZK Definition 8.5.5. Let $\mathcal{C}$ be a site. The 2-category of stacks in groupoids over $\mathcal{C}$ is the sub 2-category of the 2-category of stacks over $\mathcal{C}$ (see Definition 8.4.5) defined as follows:

(1) Its objects will be stacks in groupoids $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in groupoids over $\mathcal{C}$ is a (strict) $(2,1)$-category.

02ZL Lemma 8.5.6. Let $\mathcal{C}$ be a category. The 2-category of stacks in groupoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.32.3.

Proof. This is clear from Categories, Lemma 4.35.7 and Lemmas 8.5.2 and 8.4.6. □

8.6. Stacks in setoids

042Y This is just a brief section saying that a stack in sets is the same thing as a sheaf of sets. Please consult Categories, Section 4.39 for notation.

042Z Definition 8.6.1. Let $\mathcal{C}$ be a site.

(1) A stack in setoids over $\mathcal{C}$ is a stack over $\mathcal{C}$ all of whose fibre categories are setoids.
(2) A stack in sets, or a stack in discrete categories is a stack over $\mathcal{C}$ all of whose fibre categories are discrete.

From the discussion in Section 8.5 this is the same thing as a stack in groupoids whose fibre categories are setoids (resp. discrete). Moreover, it is also the same thing as a category fibred in setoids (resp. sets) which is a stack.

0430 **Lemma 8.6.2.** Let $\mathcal{C}$ be a site. Under the equivalence

$$\left\{\begin{matrix}\text{the category of presheaves}\\ \text{of sets over } \mathcal{C}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{the category of categories}\\ \text{fibred in sets over } \mathcal{C}\end{matrix}\right\}$$

of Categories, Lemma 4.38.6 the stacks in sets correspond precisely to the sheaves.

Proof. Omitted. Hint: Show that effectivity of descent corresponds exactly to the sheaf condition. □

0432 **Lemma 8.6.3.** Let $\mathcal{C}$ be a site. Let $\mathcal{S}$ be a category fibred in setoids over $\mathcal{C}$. Then $\mathcal{S}$ is a stack in setoids if and only if the unique equivalent category $\mathcal{S}'$ fibred in sets (see Categories, Lemma 4.39.5) is a stack in sets. In other words, if and only if the presheaf

$$U \longmapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$$

is a sheaf.

Proof. Omitted. □

0431 **Lemma 8.6.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack in setoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack in setoids over $\mathcal{C}$.

Proof. By Categories, Lemma 4.39.5 we see that a category $\mathcal{S}$ over $\mathcal{C}$ is fibred in setoids over $\mathcal{C}$ if and only if it is equivalent over $\mathcal{C}$ to a category fibred in sets. Hence we see that $\mathcal{S}_1$ is fibred in setoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred in setoids over $\mathcal{C}$. Hence now the lemma follows from Lemma 8.6.3. □

The 2-category of stacks in setoids over $\mathcal{C}$ is defined as follows.

0433 **Definition 8.6.5.** Let $\mathcal{C}$ be a site. The *2-category of stacks in setoids over $\mathcal{C}$* is the sub 2-category of the 2-category of stacks over $\mathcal{C}$ (see Definition 8.4.5) defined as follows:

(1) Its objects will be stacks in setoids $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in setoids over $\mathcal{C}$ is a (strict) $(2,1)$-category.

0434 **Lemma 8.6.6.** Let $\mathcal{C}$ be a site. The 2-category of stacks in setoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.32.3.

Proof. This is clear from Categories, Lemmas 4.35.7 and 4.39.4 and Lemmas 8.5.2 and 8.4.6. □

05UI **Lemma 8.6.7.** Let $\mathcal{C}$ be a site. Let $\mathcal{S}, \mathcal{T}$ be stacks in groupoids over $\mathcal{C}$ and let $\mathcal{R}$ be a stack in setoids over $\mathcal{C}$. Let $f : \mathcal{T} \to \mathcal{S}$ and $g : \mathcal{R} \to \mathcal{S}$ be 1-morphisms. If f is faithful, then the 2-fibre product

$$\mathcal{T} \times_{f, \mathcal{S}, g} \mathcal{R}$$

is a stack in setoids over $\mathcal{C}$.

Proof. Immediate from the explicit description of the 2-fibre product in Categories, Lemma 4.32.3. $\square$

05UJ **Lemma 8.6.8.** Let $\mathcal{C}$ be a site. Let $\mathcal{S}$ be a stack in groupoids over $\mathcal{C}$ and let $\mathcal{S}_i$, $i = 1, 2$ be stacks in setoids over $\mathcal{C}$. Let $f_i : \mathcal{S}_i \to \mathcal{S}$ be 1-morphisms. Then the 2-fibre product

$$\mathcal{S}_1 \times_{f_1, \mathcal{S}, f_2} \mathcal{S}_2$$

is a stack in setoids over $\mathcal{C}$.

Proof. This is a special case of Lemma 8.6.7 as f_2 is faithful. $\square$

06DV **Lemma 8.6.9.** Let $\mathcal{C}$ be a site. Let

$$\begin{tikzcd} \mathcal{T}_2 \arrow[r] \arrow[d, "G'"'] & \mathcal{T}_1 \arrow[d, "G"] \\ \mathcal{S}_2 \arrow[r, "F"] & \mathcal{S}_1 \end{tikzcd}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. Assume

(1) for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$, and
(2) G' is faithful,

then G is faithful.

Proof. We may assume that $\mathcal{T}_2$ is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 4.32.3. By Categories, Lemma 4.35.9 the faithfulness of G, G' can be checked on fibre categories. Suppose that y, y' are objects of $\mathcal{T}_1$ over the object U of $\mathcal{C}$. Let $\alpha, \beta : y \to y'$ be morphisms of $(\mathcal{T}_1)_U$ such that $G(\alpha) = G(\beta)$. Our object is to show that $\alpha = \beta$. Considering instead $\gamma = \alpha^{-1} \circ \beta$ we see that $G(\gamma) = \mathrm{id}_{G(y)}$ and we have to show that $\gamma = \mathrm{id}_y$. By assumption we can find a covering $\{U_i \to U\}$ such that $G(y)|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$. Since it suffices to show that $\gamma|_{U_i} = \mathrm{id}$ for each i, we may therefore assume that we have $f : F(x) \to G(y)$ for some object x of $\mathcal{S}_2$ over U and morphisms f of $(\mathcal{S}_1)_U$. In this case we get a morphism

$$(1, \gamma) : (U, x, y, f) \longrightarrow (U, x, y, f)$$

in the fibre category of $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ over U whose image under G' in $\mathcal{S}_1$ is id_x. As G' is faithful we conclude that $\gamma = \mathrm{id}_y$ and we win. $\square$

05W9 **Lemma 8.6.10.** Let $\mathcal{C}$ be a site. Let

$$\begin{tikzcd} \mathcal{T}_2 \arrow[r] \arrow[d] & \mathcal{T}_1 \arrow[d, "G"] \\ \mathcal{S}_2 \arrow[r, "F"] & \mathcal{S}_1 \end{tikzcd}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. If

(1) $F : \mathcal{S}_2 \to \mathcal{S}_1$ is fully faithful,
(2) for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$, and
(3) $\mathcal{T}_2$ is a stack in setoids.

then $\mathcal{T}_1$ is a stack in setoids.

Proof. We may assume that $\mathcal{T}_2$ is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 4.32.3. Pick $U \in \mathrm{Ob}(\mathcal{C})$ and $y \in \mathrm{Ob}((\mathcal{T}_1)_U)$. We have to show that the sheaf $Aut(y)$ on $\mathcal{C}/U$ is trivial. To to this we may replace U by the members of a covering of U. Hence by assumption (2) we may assume that there exists an object $x \in \mathrm{Ob}((\mathcal{S}_2)_U)$ and an isomorphism $f : F(x) \to G(y)$. Then $y' = (U, x, y, f)$ is an object of $\mathcal{T}_2$ over U which is mapped to y under the projection $\mathcal{T}_2 \to \mathcal{T}_1$. Because F is fully faithful by (1) the map $Aut(y') \to Aut(y)$ is surjective, use the explicit description of morphisms in $\mathcal{T}_2$ in Categories, Lemma 4.32.3. Since by (3) the sheaf $Aut(y')$ is trivial we get the result of the lemma. $\square$

0CKJ Lemma 8.6.11. Let $\mathcal{C}$ be a site. Let $F : \mathcal{S} \to \mathcal{T}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. Assume that

(1) $\mathcal{T}$ is a stack in groupoids over $\mathcal{C}$,
(2) for every $U \in \mathrm{Ob}(\mathcal{C})$ the functor $\mathcal{S}_U \to \mathcal{T}_U$ of fibre categories is faithful,
(3) for each U and each $y \in \mathrm{Ob}(\mathcal{T}_U)$ the presheaf

$$(h : V \to U) \longmapsto \{(x, f) \mid x \in \mathrm{Ob}(\mathcal{S}_V), f : F(x) \to f^*y \text{ over } V\}/\cong$$

is a sheaf on $\mathcal{C}/U$.

Then $\mathcal{S}$ is a stack in groupoids over $\mathcal{C}$.

Proof. We have to prove descent for morphisms and descent for objects.

Descent for morphisms. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let x, x' be objects of $\mathcal{S}$ over U. For each i let $\alpha_i : x|_{U_i} \to x'|_{U_i}$ be a morphism over U_i such that α_i and α_j restrict to the same morphism $x|_{U_i \times_U U_j} \to x'|_{U_i \times_U U_j}$. Because $\mathcal{T}$ is a stack in groupoids, there is a morphism $\beta : F(x) \to F(x')$ over U whose restriction to U_i is $F(\alpha_i)$. Then we can think of $\xi = (x, \beta)$ and $\xi' = (x', \mathrm{id}_{F(x')})$ as sections of the presheaf associated to $y = F(x')$ over U in assumption (3). On the other hand, the restrictions of ξ and ξ' to U_i are $(x|_{U_i}, F(\alpha_i))$ and $(x'|_{U_i}, \mathrm{id}_{F(x'|_{U_i})})$. These are isomorphic to each other by the morphism α_i. Thus ξ and ξ' are isomorphic by assumption (3). This means there is a morphism $\alpha : x \to x'$ over U with $F(\alpha) = \beta$. Since F is faithful on fibre categories we obtain $\alpha|_{U_i} = \alpha_i$.

Descent of objects. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let (x_i, φ_{ij}) be a descent datum for $\mathcal{S}$ with respect to the given covering. Because $\mathcal{T}$ is a stack in groupoids, there is an object y in $\mathcal{T}_U$ and isomorphisms $\beta_i : F(x_i) \to y|_{U_i}$ such that $F(\varphi_{ij}) = \beta_j|_{U_i \times_U U_j} \circ (\beta_i|_{U_i \times_U U_j})^{-1}$. Then (x_i, β_i) are sections of the presheaf associated to y over U defined in assumption (3). Moreover, φ_{ij} defines an isomorphism from the pair $(x_i, \beta_i)|_{U_i \times_U U_j}$ to the pair $(x_j, \beta_j)|_{U_i \times_U U_j}$. Hence by assumption (3) there exists a pair (x, β) over U whose restriction to U_i is isomorphic to (x_i, β_i). This means there are morphisms $\alpha_i : x_i \to x|_{U_i}$ with $\beta_i = \beta|_{U_i} \circ F(\alpha_i)$. Since F is faithful on fibre categories a calculation shows that $\varphi_{ij} = \alpha_j|_{U_i \times_U U_j} \circ (\alpha_i|_{U_i \times_U U_j})^{-1}$. This finishes the proof. $\square$

8.7. The inertia stack

036X Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories over the category $\mathcal{C}$. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Recall that we have defined in Categories, Definition 4.34.2 a relative inertia fibred category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ as the category whose objects are pairs (x, α) where $x \in \mathrm{Ob}(\mathcal{S})$ and $\alpha : x \to x$ with

$F(\alpha) = \mathrm{id}_{F(x)}$. There is also an absolute version, namely the inertia $\mathcal{I}_\mathcal{S}$ of $\mathcal{S}$. These inertia categories are actually stacks over $\mathcal{C}$ provided that $\mathcal{S}$ and $\mathcal{S}'$ are stacks.

036Y Lemma 8.7.1. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be stacks over the site $\mathcal{C}$. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of stacks over $\mathcal{C}$.

(1) The inertia $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_\mathcal{S}$ are stacks over $\mathcal{C}$.
(2) If $\mathcal{S}, \mathcal{S}'$ are stacks in groupoids over $\mathcal{C}$, then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_\mathcal{S}$.
(3) If $\mathcal{S}, \mathcal{S}'$ are stacks in setoids over $\mathcal{C}$, then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_\mathcal{S}$.

Proof. The first three assertions follow from Lemmas 8.4.6, 8.5.6, and 8.6.6 and the equivalence in Categories, Lemma 4.34.1 part (1). □

04ZM Lemma 8.7.2. Let $\mathcal{C}$ be a site. If $\mathcal{S}$ is a stack in groupoids, then the canonical 1-morphism $\mathcal{I}_\mathcal{S} \to \mathcal{S}$ is an equivalence if and only if $\mathcal{S}$ is a stack in setoids.

Proof. Follows directly from Categories, Lemma 4.39.7. □

8.8. Stackification of fibred categories

02ZM Here is the result.

02ZN Lemma 8.8.1. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. There exists a stack $p' : \mathcal{S}' \to \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \to \mathcal{S}'$ of fibred categories over $\mathcal{C}$ (see Categories, Definition 4.33.9) such that

(1) for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the map
$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$
induced by G identifies the right hand side with the sheafification of the left hand side, and
(2) for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x' \in \mathrm{Ob}(\mathcal{S}'_U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_{U_i} \to \mathcal{S}'_{U_i}$.

Moreover the stack $\mathcal{S}'$ is determined up to unique 2-isomorphism by these conditions.

Proof by naive method. In this proof method we proceed in stages:

First, given x lying over U and any object y of $\mathcal{S}$, we say that two morphisms $a, b : x \to y$ of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ are locally equal if there exists a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}$ such that the compositions
$$f_i^*x \to x \xrightarrow{a} y, \quad f_i^*x \to x \xrightarrow{b} y$$
are equal. This gives an equivalence relation $\sim$ on arrows of $\mathcal{S}$. If $b \sim b'$ then $a \circ b \circ c \sim a \circ b' \circ c$ (verification omitted). Hence we can quotient out by this equivalence relation to obtain a new category $\mathcal{S}^1$ over $\mathcal{C}$ together with a morphism $G^1 : \mathcal{S} \to \mathcal{S}^1$.

One checks that G^1 preserves strongly cartesian morphisms and that $\mathcal{S}^1$ is a fibred category over $\mathcal{C}$. Checks omitted. Thus we reduce to the case where locally equal morphisms are equal.

Next, we add morphisms as follows. Given x lying over U and any object y of lying over V a locally defined morphism from x to y is given by

(1) a morphism $f : U \to V$,

(2) a covering $\{f_i : U_i \to U\}$ of U, and
(3) morphisms $a_i : f_i^*x \to y$ with $p(a_i) = f \circ f_i$

with the property that the compositions

$$(f_i \times f_j)^*x \to f_i^*x \xrightarrow{a_i} y, \quad (f_i \times f_j)^*x \to f_j^*x \xrightarrow{a_j} y$$

are equal. Note that a usual morphism $a : x \to y$ gives a locally defined morphism $(p(a) : U \to V, \{\mathrm{id}_U\}, a)$. We say two locally defined morphisms $(f, \{f_i : U_i \to U\}, a_i)$ and $(g, \{g_j : U'_j \to U\}, b_j)$ are equal if $f = g$ and the compositions

$$(f_i \times g_j)^*x \to f_i^*x \xrightarrow{a_i} y, \quad (f_i \times g_j)^*x \to g_j^*x \xrightarrow{b_j} y$$

are equal (this is the right condition since we are in the situation where locally equal morphisms are equal). To compose locally defined morphisms $(f, \{f_i : U_i \to U\}, a_i)$ from x to y and $(g, \{g_j : V_j \to V\}, b_j)$ from y to z lying over W, just take $g \circ f : U \to W$, the covering $\{U_i \times_V V_j \to U\}$, and as maps the compositions

$$x|_{U_i \times_V V_j} \xrightarrow{\mathrm{pr}_0^* a_i} y|_{V_j} \xrightarrow{b_j} z$$

We omit the verification that this is a locally defined morphism.

One checks that $\mathcal{S}^2$ with the same objects as $\mathcal{S}$ and with locally defined morphisms as morphisms is a category over $\mathcal{C}$, that there is a functor $G^2 : \mathcal{S} \to \mathcal{S}^2$ over $\mathcal{C}$, that this functor preserves strongly cartesian objects, and that $\mathcal{S}^2$ is a fibred category over $\mathcal{C}$. Checks omitted. This reduces one to the case where the morphism presheaves of $\mathcal{S}$ are all sheaves, by checking that the effect of using locally defined morphisms is to take the sheafification of the (separated) morphisms presheaves.

Finally, in the case where the morphism presheaves are all sheaves we have to add objects in order to make sure descent conditions are effective in the end result. The simplest way to do this is to consider the category $\mathcal{S}'$ whose objects are pairs $(\mathcal{U}, \xi)$ where $\mathcal{U} = \{U_i \to U\}$ is a covering of $\mathcal{C}$ and $\xi = (X_i, \varphi_{ii'})$ is a descent datum relative $\mathcal{U}$. Suppose given two such data $(\mathcal{U}, \xi) = (\{f_i : U_i \to U\}, x_i, \varphi_{ii'})$ and $(\mathcal{V}, \eta) = (\{g_j : V_j \to V\}, y_j, \psi_{jj'})$. We define

$$\mathrm{Mor}_{\mathcal{S}'}((\mathcal{U}, \xi), (\mathcal{V}, \eta))$$

as the set of (f, a_{ij}), where $f : U \to V$ and

$$a_{ij} : x_i|_{U_i \times_V V_j} \longrightarrow y_j$$

are morphisms of $\mathcal{S}$ lying over $U_i \times_V V_j \to V_j$. These have to satisfy the following condition: for any $i, i' \in I$ and $j, j' \in J$ set $W = (U_i \times_U U_{i'}) \times_V (V_j \times_V V_{j'})$. Then

$$\begin{tikzcd} x_i|_W \arrow[r, "a_{ij}|_W"'] \arrow[d, "\varphi_{ii'}|_W"'] & y_j|_W \arrow[d, "\psi_{jj'}|_W"] \\ x_{i'}|_W \arrow[r, "a_{i'j'}|_W"] & y_{j'}|_W \end{tikzcd}$$

commutes. At this point you have to verify the following things:

(1) there is a well defined composition on morphisms as above,
(2) this turns $\mathcal{S}'$ into a category over $\mathcal{C}$,
(3) there is a functor $G : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$,
(4) for x, y objects of $\mathcal{S}$ we have $\mathrm{Mor}_{\mathcal{S}}(x, y) = \mathrm{Mor}_{\mathcal{S}'}(G(x), G(y))$,

(5) any object of $\mathcal{S}'$ locally comes from an object of $\mathcal{S}$, i.e., part (2) of the lemma holds,
(6) G preserves strongly cartesian morphisms,
(7) $\mathcal{S}'$ is a fibred category over $\mathcal{C}$, and
(8) $\mathcal{S}'$ is a stack over $\mathcal{C}$.

This is all not hard but there is a lot of it. Details omitted. $\square$

Less naive proof. Here is a less naive proof. By Categories, Lemma 4.36.4 there exists an equivalence of fibred categories $\mathcal{S} \to \mathcal{S}'$ where $\mathcal{S}'$ is a split fibred category, i.e., one in which the pullback functors compose on the nose. Obviously the lemma for $\mathcal{S}'$ implies the lemma for $\mathcal{S}$. Hence we may think of $\mathcal{S}$ as a presheaf in categories.

Consider the 2-category Cat temporarily as a category by forgetting about 2-morphisms. Let us think of a category as a quintuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, \circ)$ as in Categories, Section 4.2. Consider the forgetful functor

$$forget : \mathrm{Cat} \to \mathrm{Sets} \times \mathrm{Sets}, \quad (\mathrm{Ob}, \mathrm{Arrows}, s, t, \circ) \mapsto (\mathrm{Ob}, \mathrm{Arrows}).$$

Then $forget$ is faithful, Cat has limits and $forget$ commutes with them, Cat has directed colimits and $forget$ commutes with them, and $forget$ reflects isomorphisms. We can sheafify presheaves with values in Cat, and by an argument similar to the one in the first part of Sites, Section 7.44 the result commutes with $forget$. Applying this to $\mathcal{S}$ we obtain a sheafification $\mathcal{S}^\#$ which has a sheaf of objects and a sheaf of morphisms both of which are the sheafifications of the corresponding presheaves for $\mathcal{S}$. In this case it is quite easy to see that the map $\mathcal{S} \to \mathcal{S}^\#$ has the properties (1) and (2) of the lemma.

However, the category $\mathcal{S}^\#$ may not yet be a stack since, although the presheaf of objects is a sheaf, the descent condition may not yet be satisfied. To remedy this we have to add more objects. But the argument above does reduce us to the case where $\mathcal{S} = \mathcal{S}_F$ for some sheaf(!) $F : \mathcal{C}^{opp} \to \mathrm{Cat}$ of categories. In this case consider the functor $F' : \mathcal{C}^{opp} \to \mathrm{Cat}$ defined by

(1) The set $\mathrm{Ob}(F'(U))$ is the set of pairs $(\mathcal{U}, \xi)$ where $\mathcal{U} = \{U_i \to U\}$ is a covering of U and $\xi = (x_i, \varphi_{ii'})$ is a descent datum relative to $\mathcal{U}$.
(2) A morphism in $F'(U)$ from $(\mathcal{U}, \xi)$ to $(\mathcal{V}, \eta)$ is an element of

$$\mathrm{colim}\, \mathrm{Mor}_{DD(\mathcal{W})}(a^*\xi, b^*\eta)$$

where the colimit is over all common refinements $a : \mathcal{W} \to \mathcal{U}$, $b : \mathcal{W} \to \mathcal{V}$. This colimit is filtered (verification omitted). Hence composition of morphisms in $F(U)$ is defined by finding a common refinement and composing in $DD(\mathcal{W})$.
(3) Given $h : V \to U$ and an object $(\mathcal{U}, \xi)$ of $F'(U)$ we set $F'(h)(\mathcal{U}, \xi)$ equal to $(V \times_U \mathcal{U}, \mathrm{pr}_1^*\xi)$. More precisely, if $\mathcal{U} = \{U_i \to U\}$ and $\xi = (x_i, \varphi_{ii'})$, then $V \times_U \mathcal{U} = \{V \times_U U_i \to V\}$ which comes with a canonical morphism $\mathrm{pr}_1 : V \times_U \mathcal{U} \to \mathcal{U}$ and $\mathrm{pr}_1^*\xi$ is the pullback of ξ with respect to this morphism (see Definition 8.3.4).
(4) Given $h : V \to U$, objects $(\mathcal{U}, \xi)$ and $(\mathcal{V}, \eta)$ and a morphism between them, represented by $a : \mathcal{W} \to \mathcal{U}$, $b : \mathcal{W} \to \mathcal{V}$, and $\alpha : a^*\xi \to b^*\eta$, then $F'(h)(\alpha)$ is represented by $a' : V \times_U \mathcal{W} \to V \times_U \mathcal{U}$, $b' : V \times_U \mathcal{W} \to V \times_U \mathcal{V}$, and the pullback α' of the morphism α via the map $V \times_U \mathcal{W} \to \mathcal{W}$. This works since pullbacks in $\mathcal{S}_F$ commute on the nose.

There is a map $F \to F'$ given by associating to an object x of $F(U)$ the object $(\{U \to U\}, (x, triv))$ of $F'(U)$. At this point you have to check that the corresponding functor $\mathcal{S}_F \to \mathcal{S}_{F'}$ has properties (1) and (2) of the lemma, and finally that $\mathcal{S}_{F'}$ is a stack. Details omitted. □

0435 Lemma 8.8.2. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. Let $p' : \mathcal{S}' \to \mathcal{C}$ and $G : \mathcal{S} \to \mathcal{S}'$ the stack and 1-morphism constructed in Lemma 8.8.1. This construction has the following universal property: Given a stack $q : \mathcal{X} \to \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{X}$ of fibred categories over $\mathcal{C}$ there exists a 1-morphism $H : \mathcal{S}' \to \mathcal{X}$ such that the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "F"'] \arrow[dr, "G"'] & & \mathcal{X} \\ & \mathcal{S}' \arrow[ur, "H"'] & \end{tikzcd}$$

is 2-commutative.

Proof. Omitted. Hint: Suppose that $x' \in \mathrm{Ob}(\mathcal{S}'_U)$. By the result of Lemma 8.8.1 there exists a covering $\{U_i \to U\}_{i \in I}$ such that $x'|_{U_i} = G(x_i)$ for some $x_i \in \mathrm{Ob}(\mathcal{S}_{U_i})$. Moreover, there exist coverings $\{U_{ijk} \to U_i \times_U U_j\}$ and isomorphisms $\alpha_{ijk} : x_i|_{U_{ijk}} \to x_j|_{U_{ijk}}$ with $G(\alpha_{ijk}) = \mathrm{id}_{x'|_{U_{ijk}}}$. Set $y_i = F(x_i)$. Then you can check that

$$F(\alpha_{ijk}) : y_i|_{U_{ijk}} \to y_j|_{U_{ijk}}$$

agree on overlaps and therefore (as $\mathcal{X}$ is a stack) define a morphism $\beta_{ij} : y_i|_{U_i \times_U U_j} \to y_j|_{U_i \times_U U_j}$. Next, you check that the β_{ij} define a descent datum. Since $\mathcal{X}$ is a stack these descent data are effective and we find an object y of $\mathcal{X}_U$ agreeing with $G(x_i)$ over U_i. The hint is to set $H(x') = y$. □

04W9 Lemma 8.8.3. Notation and assumptions as in Lemma 8.8.2. There is a canonical equivalence of categories

$$\mathrm{Mor}_{\mathrm{Fib}/\mathcal{C}}(\mathcal{S}, \mathcal{X}) = \mathrm{Mor}_{\mathrm{Stacks}/\mathcal{C}}(\mathcal{S}', \mathcal{X})$$

given by the constructions in the proof of the aforementioned lemma.

Proof. Omitted. □

04Y1 Lemma 8.8.4. Let $\mathcal{C}$ be a site. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ be morphisms of fibred categories over $\mathcal{C}$. In this case the stackification of the 2-fibre product is the 2-fibre product of the stackifications.

Proof. Let us denote $\mathcal{X}', \mathcal{Y}', \mathcal{Z}'$ the stackifications and $\mathcal{W}$ the stackification of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$. By construction of 2-fibre products there is a canonical 1-morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$. As the second 2-fibre product is a stack (see Lemma 8.4.6) this 1-morphism induces a 1-morphism $h : \mathcal{W} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ by the universal property of stackification, see Lemma 8.8.2. Now h is a morphism of stacks, and we may check that it is an equivalence using Lemmas 8.4.7 and 8.4.8.

Thus we first prove that h induces isomorphisms of Mor-sheaves. Let ξ, ξ' be objects of $\mathcal{W}$ over $U \in \mathrm{Ob}(\mathcal{C})$. We want to show that

$$h : \mathit{Mor}(\xi, \xi') \longrightarrow \mathit{Mor}(h(\xi), h(\xi'))$$

is an isomorphism. To do this we may work locally on U (see Sites, Section 7.26). Hence by construction of $\mathcal{W}$ (see Lemma 8.8.1) we may assume that ξ, ξ' actually come from objects (x, z, α) and (x', z', α') of $\mathcal{X} \times_\mathcal{Y} \mathcal{Z}$ over U. By the same lemma once more we see that in this case $Mor(\xi, \xi')$ is the sheafification of

$$V/U \longmapsto \mathrm{Mor}_{\mathcal{X}_V}(x|_V, x'|_V) \times_{\mathrm{Mor}_{\mathcal{Y}_V}(f(x)|_V, f(x')|_V)} \mathrm{Mor}_{\mathcal{Z}_V}(z|_V, z'|_V)$$

and that $Mor(h(\xi), h(\xi'))$ is equal to the fibre product

$$Mor(i(x), i(x')) \times_{Mor(j(f(x)), j(f(x'))} Mor(k(z), k(z'))$$

where $i : \mathcal{X} \to \mathcal{X}'$, $j : \mathcal{Y} \to \mathcal{Y}'$, and $k : \mathcal{Z} \to \mathcal{Z}'$ are the canonical functors. Thus the first displayed map of this paragraph is an isomorphism as sheafification is exact (and hence the sheafification of a fibre product of presheaves is the fibre product of the sheafifications).

Finally, we have to check that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of h. Write such an object as a triple (x', z', α). Then x' locally comes from an object of $\mathcal{X}$, z' locally comes from an object of $\mathcal{Z}$, and having made suitable replacements for x', z' the morphism α of $\mathcal{Y}'_U$ locally comes from a morphism of $\mathcal{Y}$. In other words, we have shown that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of $\mathcal{X} \times_\mathcal{Y} \mathcal{Z} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$, hence a fortiori it is locally in the essential image of h. □

06NS Lemma 8.8.5. Let $\mathcal{C}$ be a site. Let $\mathcal{X}$ be a fibred category over $\mathcal{C}$. The stackification of the inertia fibred category $\mathcal{I}_\mathcal{X}$ is inertia of the stackification of $\mathcal{X}$.

Proof. This follows from the fact that stackification is compatible with 2-fibre products by Lemma 8.8.4 and the fact that there is a formula for the inertia in terms of 2-fibre products of categories over $\mathcal{C}$, see Categories, Lemma 4.34.1. □

8.9. Stackification of categories fibred in groupoids

02ZO Here is the result.

02ZP Lemma 8.9.1. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. There exists a stack in groupoids $p' : \mathcal{S}' \to \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \to \mathcal{S}'$ of categories fibred in groupoids over $\mathcal{C}$ (see Categories, Definition 4.35.6) such that

(1) for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the map
$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$
induced by G identifies the right hand side with the sheafification of the left hand side, and
(2) for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x' \in \mathrm{Ob}(\mathcal{S}'_U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_{U_i} \to \mathcal{S}'_{U_i}$.

Moreover the stack in groupoids $\mathcal{S}'$ is determined up to unique 2-isomorphism by these conditions.

Proof. Apply Lemma 8.8.1. The result will be a stack in groupoids by applying Lemma 8.5.2. □

0436 Lemma 8.9.2. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. Let $p' : \mathcal{S}' \to \mathcal{C}$ and $G : \mathcal{S} \to \mathcal{S}'$ the stack in groupoids and 1-morphism constructed in Lemma 8.9.1. This construction has the following universal property:

Given a stack in groupoids $q : \mathcal{X} \to \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{X}$ of categories over $\mathcal{C}$ there exists a 1-morphism $H : \mathcal{S}' \to \mathcal{X}$ such that the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "F"'] \arrow[rd, "G"'] & & \mathcal{X} \\ & \mathcal{S}' \arrow[ru, "H"'] & \end{tikzcd}$$

is 2-commutative.

Proof. This is a special case of Lemma 8.8.2. □

04Y2 Lemma 8.9.3. Let $\mathcal{C}$ be a site. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ be morphisms of categories fibred in groupoids over $\mathcal{C}$. In this case the stackification of the 2-fibre product is the 2-fibre product of the stackifications.

Proof. This is a special case of Lemma 8.8.4. □

8.10. Inherited topologies

06NT It turns out that a fibred category over a site inherits a canonical topology from the underlying site.

06NU Lemma 8.10.1. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\text{Cov}(\mathcal{S})$ be the set of families $\{x_i \to x\}_{i \in I}$ of morphisms in $\mathcal{S}$ with fixed target such that (a) each $x_i \to x$ is strongly cartesian, and (b) $\{p(x_i) \to p(x)\}_{i \in I}$ is a covering of $\mathcal{C}$. Then $(\mathcal{S}, \text{Cov}(\mathcal{S}))$ is a site.

Proof. We have to check the three conditions of Sites, Definition 7.6.2.

(1) If $x \to y$ is an isomorphism of $\mathcal{S}$, then it is strongly cartesian by Categories, Lemma 4.33.2 and $p(x) \to p(y)$ is an isomorphism of $\mathcal{C}$. Thus $\{p(x) \to p(y)\}$ is a covering of $\mathcal{C}$ whence $\{x \to y\} \in \text{Cov}(\mathcal{S})$.
(2) If $\{x_i \to x\}_{i \in I} \in \text{Cov}(\mathcal{S})$ and for each i we have $\{y_{ij} \to x_i\}_{j \in J_i} \in \text{Cov}(\mathcal{S})$, then each composition $p(y_{ij}) \to p(x)$ is strongly cartesian by Categories, Lemma 4.33.2 and $\{p(y_{ij}) \to p(x)\}_{i \in I, j \in J_i} \in \text{Cov}(\mathcal{C})$. Hence also $\{y_{ij} \to x\}_{i \in I, j \in J_i} \in \text{Cov}(\mathcal{S})$.
(3) Suppose $\{x_i \to x\}_{i \in I} \in \text{Cov}(\mathcal{S})$ and $y \to x$ is a morphism of $\mathcal{S}$. As $\{p(x_i) \to p(x)\}$ is a covering of $\mathcal{C}$ we see that $p(x_i) \times_{p(x)} p(y)$ exists. Hence Categories, Lemma 4.33.13 implies that $x_i \times_x y$ exists, that $p(x_i \times_x y) = p(x_i) \times_{p(x)} p(y)$, and that $x_i \times_x y \to y$ is strongly cartesian. Since also $\{p(x_i) \times_{p(x)} p(y) \to p(y)\}_{i \in I} \in \text{Cov}(\mathcal{C})$ we conclude that $\{x_i \times_x y \to y\}_{i \in I} \in \text{Cov}(\mathcal{S})$

This finishes the proof. □

Note that if $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then the coverings of the site $\mathcal{S}$ in Lemma 8.10.1 are characterized by

$$\{x_i \to x\} \in \text{Cov}(\mathcal{S}) \Leftrightarrow \{p(x_i) \to p(x)\} \in \text{Cov}(\mathcal{C})$$

because every morphism of $\mathcal{S}$ is strongly cartesian.

06NV Definition 8.10.2. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. We say $(\mathcal{S}, \text{Cov}(\mathcal{S}))$ as in Lemma 8.10.1 is the structure of site on $\mathcal{S}$ inherited from $\mathcal{C}$. We sometimes indicate this by saying that $\mathcal{S}$ is endowed with the topology inherited from $\mathcal{C}$.

In particular we obtain a topos of sheaves $Sh(\mathcal{S})$ in this situation. It turns out that this topos is functorial with respect to 1-morphisms of fibred categories.

06NW Lemma 8.10.3. Let $\mathcal{C}$ be a site. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of fibred categories over $\mathcal{C}$. Then F is a continuous and cocontinuous functor between the structure of sites inherited from $\mathcal{C}$. Hence F induces a morphism of topoi $f : Sh(\mathcal{X}) \to Sh(\mathcal{Y})$ with $f_* = {}_sF = {}_pF$ and $f^{-1} = F^s = F^p$. In particular $f^{-1}(\mathcal{G})(x) = \mathcal{G}(F(x))$ for a sheaf $\mathcal{G}$ on $\mathcal{Y}$ and object x of $\mathcal{X}$.

Proof. We first prove that F is continuous. Let $\{x_i \to x\}_{i \in I}$ be a covering of $\mathcal{X}$. By Categories, Definition 4.33.9 the functor F transforms strongly cartesian morphisms into strongly cartesian morphisms, hence $\{F(x_i) \to F(x)\}_{i \in I}$ is a covering of $\mathcal{Y}$. This proves part (1) of Sites, Definition 7.13.1. Moreover, let $x' \to x$ be a morphism of $\mathcal{X}$. By Categories, Lemma 4.33.13 the fibre product $x_i \times_x x'$ exists and $x_i \times_x x' \to x'$ is strongly cartesian. Hence $F(x_i \times_x x') \to F(x')$ is strongly cartesian. By Categories, Lemma 4.33.13 applied to $\mathcal{Y}$ this means that $F(x_i \times_x x') = F(x_i) \times_{F(x)} F(x')$. This proves part (2) of Sites, Definition 7.13.1 and we conclude that F is continuous.

Next we prove that F is cocontinuous. Let $x \in \mathrm{Ob}(\mathcal{X})$ and let $\{y_i \to F(x)\}_{i \in I}$ be a covering in $\mathcal{Y}$. Denote $\{U_i \to U\}_{i \in I}$ the corresponding covering of $\mathcal{C}$. For each i choose a strongly cartesian morphism $x_i \to x$ in $\mathcal{X}$ lying over $U_i \to U$. Then $F(x_i) \to F(x)$ and $y_i \to F(x)$ are both a strongly cartesian morphisms in $\mathcal{Y}$ lying over $U_i \to U$. Hence there exists a unique isomorphism $F(x_i) \to y_i$ in $\mathcal{Y}_{U_i}$ compatible with the maps to $F(x)$. Thus $\{x_i \to x\}_{i \in I}$ is a covering of $\mathcal{X}$ such that $\{F(x_i) \to F(x)\}_{i \in I}$ is isomorphic to $\{y_i \to F(x)\}_{i \in I}$. Hence F is cocontinuous, see Sites, Definition 7.20.1.

The final assertion follows from the first two, see Sites, Lemmas 7.21.1, 7.20.2, and 7.21.5. $\square$

0CN0 Lemma 8.10.4. Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ be a category fibred in groupoids. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The functor p induces an equivalence of sites $\mathcal{X}/x \to \mathcal{C}/U$ where $\mathcal{X}$ is endowed with the topology inherited from $\mathcal{C}$.

Proof. Here $\mathcal{C}/U$ is the localization of the site $\mathcal{C}$ at the object U and similarly for $\mathcal{X}/x$. It follows from Categories, Definition 4.35.1 that the rule $x'/x \mapsto p(x')/p(x)$ defines an equivalence of categories $\mathcal{X}/x \to \mathcal{C}/U$. Whereupon it follows from Definition 8.10.2 that coverings of x' in $\mathcal{X}/x$ are in bijective correspondence with coverings of $p(x')$ in $\mathcal{C}/U$. $\square$

06NX Lemma 8.10.5. Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. If F turns $\mathcal{X}$ into a category fibred in groupoids over $\mathcal{Y}$, then $\mathcal{X}$ is a stack in groupoids over $\mathcal{Y}$ (with topology inherited from $\mathcal{C}$).

Proof. Let us prove descent for objects. Let $\{y_i \to y\}$ be a covering of $\mathcal{Y}$. Let (x_i, φ_{ij}) be a descent datum in $\mathcal{X}$ with respect to this covering. Then (x_i, φ_{ij}) is also a descent datum with respect to the covering $\{q(y_i) \to q(y)\}$ of $\mathcal{C}$. As $\mathcal{X}$ is a stack in groupoids we obtain an object x over $q(y)$ and isomorphisms $\psi_i : x|_{q(y_i)} \to x_i$ over $q(y_i)$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \psi_j|_{q(y_i) \times_{q(y)} q(y_j)} \circ \psi_i^{-1}|_{q(y_i) \times_{q(y)} q(y_j)}.$$

Consider the sheaf $I = Isom_{\mathcal{Y}}(F(x), y)$ on $\mathcal{C}/p(x)$. Note that $s_i = F(\psi_i) \in I(q(x_i))$ because $F(x_i) = y_i$. Because $F(\varphi_{ij}) = \text{id}$ (as we started with a descent datum over $\{y_i \to y\}$) the displayed formula shows that $s_i|_{q(y_i) \times_{q(y)} q(y_j)} = s_j|_{q(y_i) \times_{q(y)} q(y_j)}$. Hence the local sections s_i glue to $s : F(x) \to y$. As F is fibred in groupoids we see that x is isomorphic to an object x' with $F(x') = y$. We omit the verification that x' in the fibre category of $\mathcal{X}$ over y is a solution to the problem of descent posed by the descent datum (x_i, φ_{ij}). We also omit the proof of the sheaf property of the *Isom*-presheaves of $\mathcal{X}/\mathcal{Y}$. □

09WX Lemma 8.10.6. Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ be a stack. Endow $\mathcal{X}$ with the topology inherited from $\mathcal{C}$ and let $q : \mathcal{Y} \to \mathcal{X}$ be a stack. Then $\mathcal{Y}$ is a stack over $\mathcal{C}$. If p and q define stacks in groupoids, then $\mathcal{Y}$ is a stack in groupoids over $\mathcal{C}$.

Proof. We check the three conditions in Definition 8.4.1 to prove that $\mathcal{Y}$ is a stack over $\mathcal{C}$. By Categories, Lemma 4.33.12 we find that $\mathcal{Y}$ is a fibred category over $\mathcal{C}$. Thus condition (1) holds.

Let U be an object of $\mathcal{C}$ and let y_1, y_2 be objects of $\mathcal{Y}$ over U. Denote $x_i = q(y_i)$ in $\mathcal{X}$. Consider the map of presheaves

$$q : Mor_{\mathcal{Y}/\mathcal{C}}(y_1, y_2) \longrightarrow Mor_{\mathcal{X}/\mathcal{C}}(x_1, x_2)$$

on $\mathcal{C}/U$, see Lemma 8.2.3. Let $\{U_i \to U\}$ be a covering and let φ_i be a section of the presheaf on the left over U_i such that φ_i and φ_j restrict to the same section over $U_i \times_U U_j$. We have to find a morphism $\varphi : x_1 \to x_2$ restricting to φ_i. Note that $q(\varphi_i) = \psi|_{U_i}$ for some morphism $\psi : x_1 \to x_2$ over U because the second presheaf is a sheaf (by assumption). Let $y_{12} \to y_2$ be the stronly $\mathcal{X}$-cartesian morphism of $\mathcal{Y}$ lying over ψ. Then φ_i corresponds to a morphism $\varphi'_i : y_1|_{U_i} \to y_{12}|_{U_i}$ over $x_1|_{U_i}$. In other words, φ'_i now define local sections of the presheaf

$$Mor_{\mathcal{Y}/\mathcal{X}}(y_1, y_{12})$$

over the members of the covering $\{x_1|_{U_i} \to x_1\}$. By assumption these glue to a unique morphism $y_1 \to y_{12}$ which composed with the given morphism $y_{12} \to y_2$ produces the desired morphism $y_1 \to y_2$.

Finally, we show that descent data are effective. Let $\{f_i : U_i \to U\}$ be a covering of $\mathcal{C}$ and let (y_i, φ_{ij}) be a descent datum relative to this covering (Definition 8.3.1). Setting $x_i = q(y_i)$ and $\psi_{ij} = q(\varphi_{ij})$ we obtain a descent datum (x_i, ψ_{ij}) for the covering in $\mathcal{X}$. By assumption on $\mathcal{X}$ we may assume $x_i = x|_{U_i}$ and the ψ_{ij} equal to the canonical descent datum (Definition 8.3.5). In this case $\{x|_{U_i} \to x\}$ is a covering and we can view (y_i, φ_{ij}) as a descent datum relative to this covering. By our assumption that $\mathcal{Y}$ is a stack over $\mathcal{C}$ we see that it is effective which finishes the proof of condition (3).

The final assertion follows because $\mathcal{Y}$ is a stack over $\mathcal{C}$ and is fibred in groupoids by Categories, Lemma 4.35.14. □

8.11. Gerbes

06NY Gerbes are a special kind of stacks in groupoids.

06NZ Definition 8.11.1. A gerbe over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is a stack in groupoids over $\mathcal{C}$ (see Definition 8.5.1),

(2) for $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $\mathcal{S}_{U_i}$ is nonempty, and
(3) for $U \in \text{Ob}(\mathcal{C})$ and $x, y \in \text{Ob}(\mathcal{S}_U)$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $x|_{U_i} \cong y|_{U_i}$ in $\mathcal{S}_{U_i}$.

In other words, a gerbe is a stack in groupoids such that any two objects are locally isomorphic and such that objects exist locally.

06P0 Lemma 8.11.2. Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a gerbe over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a gerbe over $\mathcal{C}$.

Proof. Assume $\mathcal{S}_1$ is a gerbe over $\mathcal{C}$. By Lemma 8.5.4 we see $\mathcal{S}_2$ is a stack in groupoids over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be equivalences of categories over $\mathcal{C}$. Given $U \in \text{Ob}(\mathcal{C})$ we see that there exists a covering $\{U_i \to U\}$ such that $(\mathcal{S}_1)_{U_i}$ is nonempty. Applying F we see that $(\mathcal{S}_2)_{U_i}$ is nonempty. Given $U \in \text{Ob}(\mathcal{C})$ and $x, y \in \text{Ob}((\mathcal{S}_2)_U)$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $G(x)|_{U_i} \cong G(y)|_{U_i}$ in $(\mathcal{S}_1)_{U_i}$. By Categories, Lemma 4.35.9 this implies $x|_{U_i} \cong y|_{U_i}$ in $(\mathcal{S}_2)_{U_i}$. □

We want to generalize the definition of gerbes a bit. Namely, let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over a site $\mathcal{C}$. We want to say what it means for $\mathcal{X}$ to be a gerbe over $\mathcal{Y}$. By Section 8.10 the category $\mathcal{Y}$ inherits the structure of a site from $\mathcal{C}$. A naive guess is: Just require that $\mathcal{X} \to \mathcal{Y}$ is a gerbe in the sense above. Except the notion so obtained is not invariant under replacing $\mathcal{X}$ by an equivalent stack in groupoids over $\mathcal{C}$; this is even the case for the property of being fibred in groupoids over $\mathcal{Y}$. However, it turns out that we can replace $\mathcal{X}$ by an equivalent stack in groupoids over $\mathcal{C}$ which is fibred in groupoids over $\mathcal{Y}$, and then the property of being a gerbe over $\mathcal{Y}$ is independent of this choice. Here is the precise formulation.

06P1 Lemma 8.11.3. Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. The following are equivalent

(1) For some (equivalently any) factorization $F = F' \circ a$ where $a : \mathcal{X} \to \mathcal{X}'$ is an equivalence of categories over $\mathcal{C}$ and F' is fibred in groupoids, the map $F' : \mathcal{X}' \to \mathcal{Y}$ is a gerbe (with the topology on $\mathcal{Y}$ inherited from $\mathcal{C}$).
(2) The following two conditions are satisfied
   (a) for $y \in \text{Ob}(\mathcal{Y})$ lying over $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and objects x_i of $\mathcal{X}$ over U_i such that $F(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$, and
   (b) for $U \in \text{Ob}(\mathcal{C})$, $x, x' \in \text{Ob}(\mathcal{X}_U)$, and $b : F(x) \to F(x')$ in $\mathcal{Y}_U$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and morphisms $a_i : x|_{U_i} \to x'|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = b|_{U_i}$.

Proof. By Categories, Lemma 4.35.16 there exists a factorization $F = F' \circ a$ where $a : \mathcal{X} \to \mathcal{X}'$ is an equivalence of categories over $\mathcal{C}$ and F' is fibred in groupoids. By Categories, Lemma 4.35.17 given any two such factorizations $F = F' \circ a = F'' \circ b$ we have that $\mathcal{X}'$ is equivalent to $\mathcal{X}''$ as categories over $\mathcal{Y}$. Hence Lemma 8.11.2 guarantees that the condition (1) is independent of the choice of the factorization. Moreover, this means that we may assume $\mathcal{X}' = \mathcal{X} \times_{F,\mathcal{Y},\text{id}} \mathcal{Y}$ as in the proof of Categories, Lemma 4.35.16

Let us prove that (a) and (b) imply that $\mathcal{X}' \to \mathcal{Y}$ is a gerbe. First of all, by Lemma 8.10.5 we see that $\mathcal{X}' \to \mathcal{Y}$ is a stack in groupoids. Next, let y be an object of $\mathcal{Y}$ lying over $U \in \mathrm{Ob}(\mathcal{C})$. By (a) we can find a covering $\{U_i \to U\}$ in $\mathcal{C}$ and objects x_i of $\mathcal{X}$ over U_i and isomorphisms $f_i : F(x_i) \to y|_{U_i}$ in $\mathcal{Y}_{U_i}$. Then $(U_i, x_i, y|_{U_i}, f_i)$ are objects of $\mathcal{X}'_{U_i}$, i.e., the second condition of Definition 8.11.1 holds. Finally, let (U, x, y, f) and (U, x', y, f') be objects of $\mathcal{X}'$ lying over the same object $y \in \mathrm{Ob}(\mathcal{Y})$. Set $b = (f')^{-1} \circ f$. By condition (b) we can find a covering $\{U_i \to U\}$ and isomorphisms $a_i : x|_{U_i} \to x'|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = b|_{U_i}$. Then

$$(a_i, \mathrm{id}) : (U, x, y, f)|_{U_i} \to (U, x', y, f')|_{U_i}$$

is a morphism in $\mathcal{X}'_{U_i}$ as desired. This proves that (2) implies (1).

To prove that (1) implies (2) one reads the arguments in the preceding paragraph backwards. Details omitted. □

06P2 Definition 8.11.4. Let $\mathcal{C}$ be a site. Let $\mathcal{X}$ and $\mathcal{Y}$ be stacks in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. We say $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ if the equivalent conditions of Lemma 8.11.3 are satisfied.

This definition does not conflict with Definition 8.11.1 when $\mathcal{Y} = \mathcal{C}$ because in this case we may take $\mathcal{X}' = \mathcal{X}$ in part (1) of Lemma 8.11.3. Note that conditions (2)(a) and (2)(b) of Lemma 8.11.3 are quite close in spirit to conditions (2) and (3) of Definition 8.11.1. Namely, (2)(a) says that the map of presheaves of isomorphism classes of objects becomes a surjection after sheafification. Moreover, (2)(b) says that

$$\mathit{Isom}_{\mathcal{X}}(x, x') \longrightarrow \mathit{Isom}_{\mathcal{Y}}(F(x), F(x'))$$

is a surjection of sheaves on $\mathcal{C}/U$ for any U and $x, x' \in \mathrm{Ob}(\mathcal{X}_U)$.

06P3 Lemma 8.11.5. Let $\mathcal{C}$ be a site. Let

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G'"'] \arrow[d, "F'"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}' \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

be a 2-fibre product of stacks in groupoids over $\mathcal{C}$. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 4.32.3. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$.

Let y' be an object of $\mathcal{Y}'$ lying over the object U of $\mathcal{C}$. By assumption there exists a covering $\{U_i \to U\}$ of U and objects $x_i \in \mathcal{X}_{U_i}$ with isomorphisms $\alpha_i : G(y')|_{U_i} \to F(x_i)$. Then $(U_i, y'|_{U_i}, x_i, \alpha_i)$ is an object of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U_i whose image in $\mathcal{Y}'$ is $y'|_{U_i}$. Thus (2)(a) holds.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x'_1, x'_2 be objects of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U, and let $b' : F'(x'_1) \to F'(x'_2)$ be a morphism in $\mathcal{Y}'_U$. Write $x'_i = (U, y'_i, x_i, \alpha_i)$. Note that $F'(x'_i) = x_i$ and $G'(x'_i) = y'_i$. By assumption there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and morphisms $a_i : x_1|_{U_i} \to x_2|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = G(b')|_{U_i}$. Then $(b'|_{U_i}, a_i)$ is a morphism $x'_1|_{U_i} \to x'_2|_{U_i}$ as required in (2)(b). □

06R3 Lemma 8.11.6. Let $\mathcal{C}$ be a site. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids over $\mathcal{C}$. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.

Proof. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{X} \to \mathcal{Z}$.

Let z be an object of $\mathcal{Z}$ lying over the object U of $\mathcal{C}$. By assumption on G there exists a covering $\{U_i \to U\}$ of U and objects $y_i \in \mathcal{Y}_{U_i}$ such that $G(y_i) \cong z|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \to U_i\}$ and objects $x_{ij} \in \mathcal{X}_{U_{ij}}$ such that $F(x_{ij}) \cong y_i|_{U_{ij}}$. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $(G \circ F)(x_{ij}) \cong z|_{U_{ij}}$. Thus (2)(a) holds.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x_1, x_2 be objects of $\mathcal{X}$ over U, and let $c : (G \circ F)(x_1) \to (G \circ F)(x_2)$ be a morphism in $\mathcal{Z}_U$. By assumption on G there exists a covering $\{U_i \to U\}$ of U and morphisms $b_i : F(x_1)|_{U_i} \to F(x_2)|_{U_i}$ in $\mathcal{Y}_{U_i}$ such that $G(b_i) = c|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \to U_i\}$ and morphisms $a_{ij} : x_1|_{U_{ij}} \to x_2|_{U_{ij}}$ in $\mathcal{X}_{U_{ij}}$ such that $F(a_{ij}) = b_i|_{U_{ij}}$. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $(G \circ F)(a_{ij}) = c|_{U_{ij}}$ as required in (2)(b). □

06P4 **Lemma 8.11.7.** Let $\mathcal{C}$ be a site. Let

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G'"] \arrow[d, "F'"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Y}' \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. If for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(\mathcal{Y}_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $G : \mathcal{Y}'_{U_i} \to \mathcal{Y}_{U_i}$ and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 4.32.3. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{X} \to \mathcal{Y}$.

Let y be an object of $\mathcal{Y}$ lying over the object U of $\mathcal{C}$. By assumption there exists a covering $\{U_i \to U\}$ of U and objects $y'_i \in \mathcal{Y}'_{U_i}$ with $G(y'_i) \cong y|_{U_i}$. By (2)(a) for $\mathcal{X}' \to \mathcal{Y}'$ there exist coverings $\{U_{ij} \to U_i\}$ and objects x'_{ij} of $\mathcal{X}'$ over U_{ij} with $F'(x'_{ij})$ isomorphic to the restriction of y'_i to U_{ij}. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $G'(x'_{ij})$ are objects of $\mathcal{X}$ over U_{ij} whose images in $\mathcal{Y}$ are isomorphic to the restrictions $y|_{U_{ij}}$. This proves (2)(a) for $\mathcal{X} \to \mathcal{Y}$.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x_1, x_2 be objects of $\mathcal{X}$ over U, and let $b : F(x_1) \to F(x_2)$ be a morphism in $\mathcal{Y}_U$. By assumption we may choose a covering $\{U_i \to U\}$ and objects y'_i of $\mathcal{Y}'$ over U_i such that there exist isomorphisms $\alpha_i : G(y'_i) \to F(x_1)|_{U_i}$. Then we get objects

$$x'_{1i} = (U_i, y'_i, x_1|_{U_i}, \alpha_i) \quad \text{and} \quad x'_{2i} = (U_i, y'_i, x_2|_{U_i}, b|_{U_i} \circ \alpha_i)$$

of $\mathcal{X}'$ over U_i. The identity morphism on y'_i is a morphism $F'(x'_{1i}) \to F'(x'_{2i})$. By (2)(b) for $\mathcal{X}' \to \mathcal{Y}'$ there exist coverings $\{U_{ij} \to U_i\}$ and morphisms $a'_{ij} : x'_{1i}|_{U_{ij}} \to x'_{2i}|_{U_{ij}}$ such that $F'(a'_{ij}) = \mathrm{id}_{y'_i}|_{U_{ij}}$. Unwinding the definition of morphisms in $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ we see that $G'(a'_{ij}) : x_1|_{U_{ij}} \to x_2|_{U_{ij}}$ are the morphisms we're looking for, i.e., (2)(b) holds for $\mathcal{X} \to \mathcal{Y}$. □

Gerbes all of whose automorphism sheaves are abelian play an important role in algebraic geometry.

0CJY **Lemma 8.11.8.** Let $p : \mathcal{S} \to \mathcal{C}$ be a gerbe over a site $\mathcal{C}$. Assume that for all $U \in \text{Ob}(\mathcal{C})$ and $x \in \text{Ob}(\mathcal{S}_U)$ the sheaf of groups $\mathit{Aut}(x) = \mathit{Isom}(x, x)$ on $\mathcal{C}/U$ is abelian. Then there exist

(1) a sheaf $\mathcal{G}$ of abelian groups on $\mathcal{C}$,
(2) for every $U \in \text{Ob}(\mathcal{C})$ and every $x \in \text{Ob}(\mathcal{S}_U)$ an isomorphism $\mathcal{G}|_U \to \mathit{Aut}(x)$

such that for every U and every morphism $\varphi : x \to y$ in $\mathcal{S}_U$ the diagram

$$\begin{tikzcd} \mathcal{G}|_U \arrow[r, equal] \arrow[d] & \mathcal{G}|_U \arrow[d] \\ \mathit{Aut}(x) \arrow[r, "\alpha \mapsto \varphi \circ \alpha \circ \varphi^{-1}"] & \mathit{Aut}(y) \end{tikzcd}$$

is commutative.

Proof. Let x, y be two objects of $\mathcal{S}$ with $U = p(x) = p(y)$.

If there is a morphism $\varphi : x \to y$ over U, then it is an isomorphism and then we indeed get an isomorphism $\mathit{Aut}(x) \to \mathit{Aut}(y)$ sending α to $\varphi \circ \alpha \circ \varphi^{-1}$. Moreover, since we are assuming $\mathit{Aut}(x)$ is commutative, this isomorphism is independent of the choice of φ by a simple computation: namely, if ψ is a second such map, then

$$\varphi \circ \alpha \circ \varphi^{-1} = \psi \circ \psi^{-1} \circ \varphi \circ \alpha \circ \varphi^{-1} = \psi \circ \alpha \circ \psi^{-1} \circ \varphi \circ \varphi^{-1} = \psi \circ \alpha \circ \psi^{-1}$$

The upshot is a canonical isomorphism of sheaves $\mathit{Aut}(x) \to \mathit{Aut}(y)$. Furthermore, if there is a third object z and a morphism $y \to z$ (and hence also a morphism $x \to z$), then the canonical isomorphisms $\mathit{Aut}(x) \to \mathit{Aut}(y)$, $\mathit{Aut}(y) \to \mathit{Aut}(z)$, and $\mathit{Aut}(x) \to \mathit{Aut}(z)$ are compatible in the sense that

$$\begin{tikzcd} \mathit{Aut}(x) \arrow[rr] \arrow[dr] & & \mathit{Aut}(z) \\ & \mathit{Aut}(y) \arrow[ur] & \end{tikzcd}$$

commutes.

If there is no morphism from x to y over U, then we can choose a covering $\{U_i \to U\}$ such that there exist morphisms $x|_{U_i} \to y|_{U_i}$. This gives canonical isomorphisms

$$\mathit{Aut}(x)|_{U_i} \longrightarrow \mathit{Aut}(y)|_{U_i}$$

which agree over $U_i \times_U U_j$ (by canonicity). By glueing of sheaves (Sites, Lemma 7.26.1) we get a unique isomorphism $\mathit{Aut}(x) \to \mathit{Aut}(y)$ whose restriction to any U_i is the canonical isomorphism of the previous paragraph. Similarly to the above these canonical isomorphisms satisfy a compatibility if we have a third object over U.

What if the fibre category of $\mathcal{S}$ over U is empty? Well, in this case we can find a covering $\{U_i \to U\}$ and objects x_i of $\mathcal{S}$ over U_i. Then we set $\mathcal{G}_i = \mathit{Aut}(x_i)$. By the above we obtain canonical isomorphisms

$$\varphi_{ij} : \mathcal{G}_i|_{U_i \times_U U_j} \longrightarrow \mathcal{G}_j|_{U_i \times_U U_j}$$

whose restrictions to $U_i \times_U U_j \times_U U_k$ satisfy the cocycle condition explained in Sites, Section 7.26. By Sites, Lemma 7.26.4 we obtain a sheaf $\mathcal{G}$ over U whose restriction to U_i gives $\mathcal{G}_i$ in a manner compatible with the glueing maps φ_{ij}.

If $\mathcal{C}$ has a final object U, then this finishes the proof as we can take $\mathcal{G}$ equal to the sheaf we just constructed. In the general case we need to verify that the sheaves $\mathcal{G}$ constructed over varying U are compatible in a canonical manner. This is omitted. $\square$

8.12. Functoriality for stacks

04WA In this section we study what happens if we want to change the base site of a stack. This section can be skipped on a first reading.

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \to \mathcal{D}$ be a category over $\mathcal{D}$. In this situation we denote $u^p\mathcal{S}$ the category over $\mathcal{C}$ defined as follows

(1) An object of $u^p\mathcal{S}$ is a pair (U, y) consisting of an object U of $\mathcal{C}$ and an object y of $\mathcal{S}_{u(U)}$.
(2) A morphism $(a, \beta) : (U, y) \to (U', y')$ is given by a morphism $a : U \to U'$ of $\mathcal{C}$ and a morphism $\beta : y \to y'$ of $\mathcal{S}$ such that $p(\beta) = u(a)$.

Note that with these definitions the fibre category of $u^p\mathcal{S}$ over U is equal to the fibre category of $\mathcal{S}$ over $u(U)$.

04WB **Lemma 8.12.1.** In the situation above, if $\mathcal{S}$ is a fibred category over $\mathcal{D}$ then $u^p\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Proof. Please take a look at the discussion surrounding Categories, Definitions 4.33.1 and 4.33.5 before reading this proof. Let $(a, \beta) : (U, y) \to (U', y')$ be a morphism of $u^p\mathcal{S}$. We claim that (a, β) is strongly cartesian if and only if β is strongly cartesian. First, assume β is strongly cartesian. Consider any second morphism $(a_1, \beta_1) : (U_1, y_1) \to (U', y')$ of $u^p\mathcal{S}$. Then

$$
\begin{aligned}
&\mathrm{Mor}_{u^p\mathcal{S}}((U_1, y_1), (U, y)) \\
&= \mathrm{Mor}_\mathcal{C}(U_1, U) \times_{\mathrm{Mor}_\mathcal{D}(u(U_1), u(U))} \mathrm{Mor}_\mathcal{S}(y_1, y) \\
&= \mathrm{Mor}_\mathcal{C}(U_1, U) \times_{\mathrm{Mor}_\mathcal{D}(u(U_1), u(U))} \mathrm{Mor}_\mathcal{S}(y_1, y') \times_{\mathrm{Mor}_\mathcal{D}(u(U_1), u(U'))} \mathrm{Mor}_\mathcal{D}(u(U_1), u(U)) \\
&= \mathrm{Mor}_\mathcal{S}(y_1, y') \times_{\mathrm{Mor}_\mathcal{D}(u(U_1), u(U'))} \mathrm{Mor}_\mathcal{C}(U_1, U) \\
&= \mathrm{Mor}_{u^p\mathcal{S}}((U_1, y_1), (U', y')) \times_{\mathrm{Mor}_\mathcal{C}(U_1, U')} \mathrm{Mor}_\mathcal{C}(U_1, U)
\end{aligned}
$$

the second equality as β is strongly cartesian. Hence we see that indeed (a, β) is strongly cartesian. Conversely, suppose that (a, β) is strongly cartesian. Choose a strongly cartesian morphism $\beta' : y'' \to y'$ in $\mathcal{S}$ with $p(\beta') = u(a)$. Then bot $(a, \beta) : (U, y) \to (U, y')$ and $(a, \beta') : (U, y'') \to (U, y)$ are strongly cartesian and lift a. Hence, by the uniqueness of strongly cartesian morphisms (see discussion in Categories, Section 4.33) there exists an isomorphism $\iota : y \to y''$ in $\mathcal{S}_{u(U)}$ such that $\beta = \beta' \circ \iota$, which implies that β is strongly cartesian in $\mathcal{S}$ by Categories, Lemma 4.33.2.

Finally, we have to show that given (U', y') and $U \to U'$ we can find a strongly cartesian morphism $(U, y) \to (U', y')$ in $u^p\mathcal{S}$ lifting the morphism $U \to U'$. This follows from the above as by assumption we can find a strongly cartesian morphism $y \to y'$ lifting the morphism $u(U) \to u(U')$. $\square$

04WC **Lemma 8.12.2.** Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \to \mathcal{D}$ be a stack over $\mathcal{D}$. Then $u^p\mathcal{S}$ is a stack over $\mathcal{C}$.

Proof. We have seen in Lemma 8.12.1 that $u^p\mathcal{S}$ is a fibred category over $\mathcal{C}$. Moreover, in the proof of that lemma we have seen that a morphism (a, β) of $u^p\mathcal{S}$ is strongly cartesian if and only β is strongly cartesian in $\mathcal{S}$. Hence, given a morphism $a : U \to U'$ of $\mathcal{C}$, not only do we have the equalities $(u^p\mathcal{S})_U = \mathcal{S}_U$ and $(u^p\mathcal{S})_{U'} = \mathcal{S}_{U'}$, but via these equalities the pullback functors agree; in a formula $a^*(U', y') = (U, u(a)^*y')$.

Having said this, let $\mathcal{U} = \{U_i \to U\}$ be a covering of $\mathcal{C}$. As u is continuous we see that $\mathcal{V} = \{u(U_i) \to u(U)\}$ is a covering of $\mathcal{D}$, and that $u(U_i \times_U U_j) = u(U_i) \times_{u(U)} u(U_j)$ and similarly for the triple fibre products $U_i \times_U U_j \times_U U_k$. As we have the identifications of fibre categories and pullbacks we see that descend data relative to $\mathcal{U}$ are identical to descend data relative to $\mathcal{V}$. Since by assumption we have effective descent in $\mathcal{S}$ we conclude the same holds for $u^p\mathcal{S}$. □

04WD Lemma 8.12.3. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \to \mathcal{D}$ be a stack in groupoids over $\mathcal{D}$. Then $u^p\mathcal{S}$ is a stack in groupoids over $\mathcal{C}$.

Proof. This follows immediately from Lemma 8.12.2 and the fact that all fibre categories are groupoids. □

04WE Definition 8.12.4. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{S}$ be a fibred category over $\mathcal{D}$. In this setting we write $f_*\mathcal{S}$ for the fibred category $u^p\mathcal{S}$ defined above. We say that $f_*\mathcal{S}$ is the pushforward of $\mathcal{S}$ along f.

By the results above we know that $f_*\mathcal{S}$ is a stack (in groupoids) if $\mathcal{S}$ is a stack (in groupoids). It is harder to define the pullback of a stack (and we'll need additional assumptions for our particular construction – feel free to write up and submit a more general construction). We do this in several steps.

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. In this setting we define a category $u_{pp}\mathcal{S}$ as follows:

(1) An object of $u_{pp}\mathcal{S}$ is a triple $(U, \phi : V \to u(U), x)$ where $U \in \mathrm{Ob}(\mathcal{C})$, the map $\phi : V \to u(U)$ is a morphism in $\mathcal{D}$, and $x \in \mathrm{Ob}(\mathcal{S}_U)$.
(2) A morphism
$$(U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \to u(U_2), x_2)$$
of $u_{pp}\mathcal{S}$ is given by a (a, b, α) where $a : U_1 \to U_2$ is a morphism of $\mathcal{C}$, $b : V_1 \to V_2$ is a morphism of $\mathcal{D}$, and $\alpha : x_1 \to x_2$ is morphism of $\mathcal{S}$, such that $p(\alpha) = a$ and the diagram
$$\begin{tikzcd} V_1 \arrow[r, "b"'] \arrow[d, "\phi_1"'] & V_2 \arrow[d, "\phi_2"] \\ u(U_1) \arrow[r, "u(a)"] & u(U_2) \end{tikzcd}$$
commutes in $\mathcal{D}$.

We think of $u_{pp}\mathcal{S}$ as a category over $\mathcal{D}$ via
$$p_{pp} : u_{pp}\mathcal{S} \longrightarrow \mathcal{D}, \quad (U, \phi : V \to u(U), x) \longmapsto V.$$
The fibre category of $u_{pp}\mathcal{S}$ over an object V of $\mathcal{D}$ does not have a simple description.

04WF Lemma 8.12.5. In the situation above assume

(1) $p : \mathcal{S} \to \mathcal{C}$ is a fibred category,
(2) $\mathcal{C}$ has nonempty finite limits, and
(3) $u : \mathcal{C} \to \mathcal{D}$ commutes with nonempty finite limits.

Consider the set $R \subset \text{Arrows}(u_{pp}\mathcal{S})$ of morphisms of the form

$$(a, \text{id}_V, \alpha) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with α strongly cartesian. Then R is a right multiplicative system.

Proof. According to Categories, Definition 4.27.1 we have to check RMS1, RMS2, RMS3. Condition RMS1 holds as a composition of strongly cartesian morphisms is strongly cartesian, see Categories, Lemma 4.33.2.

To check RMS2 suppose we have a morphism

$$(a, b, \alpha) : (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U, \phi : V \to u(U), x)$$

of $u_{pp}\mathcal{S}$ and a morphism

$$(c, \text{id}_V, \gamma) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with γ strongly cartesian from R. In this situation set $U'_1 = U_1 \times_U U'$, and denote $a' : U'_1 \to U'$ and $c' : U'_1 \to U_1$ the projections. As $u(U'_1) = u(U_1) \times_{u(U)} u(U')$ we see that $\phi'_1 = (\phi_1, \phi') : V_1 \to u(U'_1)$ is a morphism in $\mathcal{D}$. Let $\gamma_1 : x'_1 \to x_1$ be a strongly cartesian morphism of $\mathcal{S}$ with $p(\gamma_1) = \phi'_1$ (which exists because $\mathcal{S}$ is a fibred category over $\mathcal{C}$). Then as $\gamma : x' \to x$ is strongly cartesian there exists a unique morphism $\alpha' : x'_1 \to x'$ with $p(\alpha') = a'$. At this point we see that

$$(a', b, \alpha') : (U_1, \phi_1 : V_1 \to u(U'_1), x'_1) \longrightarrow (U, \phi : V \to u(U'), x')$$

is a morphism and that

$$(c', \text{id}_{V_1}, \gamma_1) : (U'_1, \phi'_1 : V_1 \to u(U'_1), x'_1) \longrightarrow (U_1, \phi : V_1 \to u(U_1), x_1)$$

is an element of R which form a solution of the existence problem posed by RMS2.

Finally, suppose that

$$(a, b, \alpha), (a', b', \alpha') : (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U, \phi : V \to u(U), x)$$

are two morphisms of $u_{pp}\mathcal{S}$ and suppose that

$$(c, \text{id}_V, \gamma) : (U, \phi : V \to u(U), x) \longrightarrow (U', \phi : V \to u(U'), x')$$

is an element of R which equalizes the morphisms (a, b, α) and (a', b', α'). This implies in particular that $b = b'$. Let $d : U_2 \to U_1$ be the equalizer of a, a' which exists (see Categories, Lemma 4.18.3). Moreover, $u(d) : u(U_2) \to u(U_1)$ is the equalizer of $u(a), u(a')$ hence (as $b = b'$) there is a morphism $\phi_2 : V_1 \to u(U_2)$ such that $\phi_1 = u(d) \circ \phi_1$. Let $\delta : x_2 \to x_1$ be a strongly cartesian morphism of $\mathcal{S}$ with $p(\delta) = u(d)$. Now we claim that $\alpha \circ \delta = \alpha' \circ \delta$. This is true because γ is strongly cartesian, $\gamma \circ \alpha \circ \delta = \gamma \circ \alpha' \circ \delta$, and $p(\alpha \circ \delta) = p(\alpha' \circ \delta)$. Hence the arrow

$$(d, \text{id}_{V_1}, \delta) : (U_2, \phi_2 : V_1 \to u(U_2), x_2) \longrightarrow (U_1, \phi_1 : V_1 \to u(U_1), x_1)$$

is an element of R and equalizes (a, b, α) and (a', b', α'). Hence R satisfies RMS3 as well. ☐

04WG Lemma 8.12.6. With notation and assumptions as in Lemma 8.12.5. Set $u_p\mathcal{S} = R^{-1}u_{pp}\mathcal{S}$, see Categories, Section 4.27. Then $u_p\mathcal{S}$ is a fibred category over $\mathcal{D}$.

Proof. We use the description of $u_p\mathcal{S}$ given just above Categories, Lemma 4.27.11. Note that the functor $p_{pp} : u_{pp}\mathcal{S} \to \mathcal{D}$ transforms every element of R to an identity morphism. Hence by Categories, Lemma 4.27.16 we obtain a canonical functor $p_p : u_p\mathcal{S} \to \mathcal{D}$ extending the given functor. This is how we think of $u_p\mathcal{S}$ as a category over $\mathcal{D}$.

First we want to characterize the $\mathcal{D}$-strongly cartesian morphisms in $u_p\mathcal{S}$. A morphism $f : X \to Y$ of $u_p\mathcal{S}$ is the equivalence class of a pair $(f' : X' \to Y, r : X' \to X)$ with $r \in R$. In fact, in $u_p\mathcal{S}$ we have $f = (f', 1) \circ (r, 1)^{-1}$ with obvious notation. Note that an isomorphism is always strongly cartesian, as are compositions of strongly cartesian morphisms, see Categories, Lemma 4.33.2. Hence f is strongly cartesian if and only if $(f', 1)$ is so. Thus the following claim completely characterizes strongly cartesian morphisms. Claim: A morphism

$$(a, b, \alpha) : X_1 = (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \to u(U_2), x_2) = X_2$$

of $u_{pp}\mathcal{S}$ has image $f = ((a, b, \alpha), 1)$ strongly cartesian in $u_p\mathcal{S}$ if and only if α is a strongly cartesian morphism of $\mathcal{S}$.

Assume α strongly cartesian. Let $X = (U, \phi : V \to u(U), x)$ be another object, and let $f_2 : X \to X_2$ be a morphism of $u_p\mathcal{S}$ such that $p_p(f_2) = b \circ b_1$ for some $b_1 : U \to U_1$. To show that f is strongly cartesian we have to show that there exists a unique morphism $f_1 : X \to X_1$ in $u_p\mathcal{S}$ such that $p_p(f_1) = b_1$ and $f_2 = f \circ f_1$ in $u_p\mathcal{S}$. Write $f_2 = (f_2' : X' \to X_2, r : X' \to X)$. Again we can write $f_2 = (f_2', 1) \circ (r, 1)^{-1}$ in $u_p\mathcal{S}$. Since $(r, 1)$ is an isomorphism whose image in $\mathcal{D}$ is an identity we see that finding a morphism $f_1 : X \to X_1$ with the required properties is the same thing as finding a morphism $f_1' : X' \to X_1$ in $u_p\mathcal{S}$ with $p(f_1') = b_1$ and $f_2' = f \circ f_1'$. Hence we may assume that f_2 is of the form $f_2 = ((a_2, b_2, \alpha_2), 1)$ with $b_2 = b \circ b_1$. Here is a picture

$$\begin{tikzcd}
 & (U_1, V_1 \to u(U_1), x_1) \arrow[d, "{(a,b,\alpha)}"] \\
(U, V \to u(U), x) \arrow[r, "{(a_2,b_2,\alpha_2)}"] & (U_2, V_2 \to u(U_2), x_2)
\end{tikzcd}$$

Now it is clear how to construct the morphism f_1. Namely, set $U' = U \times_{U_2} U_1$ with projections $c : U' \to U$ and $a_1 : U' \to U_1$. Pick a strongly cartesian morphism $\gamma : x' \to x$ lifting the morphism c. Since $b_2 = b \circ b_1$, and since $u(U') = u(U) \times_{u(U_2)} u(U_1)$ we see that $\phi' = (\phi, \phi_1 \circ b_1) : V \to u(U')$. Since α is strongly cartesian, and $a \circ a_1 = a_2 \circ c = p(\alpha_2 \circ \gamma)$ there exists a morphism $\alpha_1 : x' \to x_1$ lifting a_1 such that $\alpha \circ \alpha_1 = \alpha_2 \circ \gamma$. Set $X' = (U', \phi' : V \to u(U'), x')$. Thus we see that

$$f_1 = ((a_1, b_1, \alpha_1) : X' \to X_1, (c, \mathrm{id}_V, \gamma) : X' \to X) : X \longrightarrow X_1$$

works, in fact the diagram

$$\begin{tikzcd}
(U', \phi' : V \to u(U'), x') \arrow[r, "{(a_1,b_1,\alpha_1)}"'] \arrow[d, "{(c,\mathrm{id}_V,\gamma)}"'] & (U_1, V_1 \to u(U_1), x_1) \arrow[d, "{(a,b,\alpha)}"] \\
(U, V \to u(U), x) \arrow[r, "{(a_2,b_2,\alpha_2)}"] & (U_2, V_2 \to u(U_2), x_2)
\end{tikzcd}$$

is commutative by construction. This proves existence.

Next we prove uniqueness, still in the special case $f = ((a, b, \alpha), 1)$ and $f_2 = ((a_2, b_2, \alpha_2), 1)$. We strongly advise the reader to skip this part. Suppose that

$g_1, g_1' : X \to X_1$ are two morphisms of $u_p\mathcal{S}$ such that $p_p(g_1) = p_p(g_1') = b_1$ and $f_2 = f \circ g_1 = f \circ g_1'$. Our goal is to show that $g_1 = g_1'$. By Categories, Lemma 4.27.13 we may represent g_1 and g_1' as the equivalence classes of $(f_1 : X' \to X_1, r : X' \to X)$ and $(f_1' : X' \to X_1, r : X' \to X)$ for some $r \in R$. By Categories, Lemma 4.27.14 we see that $f_2 = f \circ g_1 = f \circ g_1'$ means that there exists a morphism $r' : X'' \to X'$ in $u_{pp}\mathcal{S}$ such that $r' \circ r \in R$ and

$$(a, b, \alpha) \circ f_1 \circ r' = (a, b, \alpha) \circ f_1' \circ r' = (a_2, b_2, \alpha_2) \circ r'$$

in $u_{pp}\mathcal{S}$. Note that now g_1 is represented by $(f_1 \circ r', r \circ r')$ and similarly for g_1'. Hence we may assume that

$$(a, b, \alpha) \circ f_1 = (a, b, \alpha) \circ f_1' = (a_2, b_2, \alpha_2).$$

Write $r = (c, \mathrm{id}_V, \gamma) : (U', \phi' : V \to u(U'), x')$, $f_1 = (a_1, b_1, \alpha_1)$, and $f_1' = (a_1', b_1, \alpha_1')$. Here we have used the condition that $p_p(g_1) = p_p(g_1')$. The equalities above are now equivalent to $a \circ a_1 = a \circ a_1' = a_2 \circ c$ and $\alpha \circ \alpha_1 = \alpha \circ \alpha_1' = \alpha_2 \circ \gamma$. It need not be the case that $a_1 = a_1'$ in this situation. Thus we have to precompose by one more morphism from R. Namely, let $U'' = \mathrm{Eq}(a_1, a_1')$ be the equalizer of a_1 and a_1' which is a subobject of U'. Denote $c' : U'' \to U'$ the canonical monomorphism. Because of the relations among the morphisms above we see that $V \to u(U')$ maps into $u(U'') = u(\mathrm{Eq}(a_1, a_1')) = \mathrm{Eq}(u(a_1), u(a_1'))$. Hence we get a new object $(U'', \phi'' : V \to u(U''), x'')$, where $\gamma' : x'' \to x'$ is a strongly cartesian morphism lifting γ. Then we see that we may precompose f_1 and f_1' with the element $(c', \mathrm{id}_V, \gamma')$ of R. After doing this, i.e., replacing $(U', \phi' : V \to u(U'), x')$ with $(U'', \phi'' : V \to u(U''), x'')$, we get back to the previous situation where in addition we now have that $a_1 = a_1'$. In this case it follows formally from the fact that α is strongly cartesian (!) that $\alpha_1 = \alpha_1'$. This shows that $g_1 = g_1'$ as desired.

We omit the proof of the fact that for any strongly cartesian morphism of $u_p\mathcal{S}$ of the form $((a, b, \alpha), 1)$ the morphism α is strongly cartesian in $\mathcal{S}$. (We do not need the characterization of strongly cartesian morphisms in the rest of the proof, although we do use it later in this section.)

Let $(U, \phi : V \to u(U), x)$ be an object of $u_p\mathcal{S}$. Let $b : V' \to V$ be a morphism of $\mathcal{D}$. Then the morphism

$$(\mathrm{id}_U, b, \mathrm{id}_x) : (U, \phi \circ b : V' \to u(U), x) \longrightarrow (U, \phi : V \to u(U), x)$$

is strongly cartesian by the result of the preceding paragraphs and we win. □

04WH Lemma 8.12.7. With notation and assumptions as in Lemma 8.12.6. If $\mathcal{S}$ is fibred in groupoids, then $u_p\mathcal{S}$ is fibred in groupoids.

Proof. By Lemma 8.12.6 we know that $u_p\mathcal{S}$ is a fibred category. Let $f : X \to Y$ be a morphism of $u_p\mathcal{S}$ with $p_p(f) = \mathrm{id}_V$. We are done if we can show that f is invertible, see Categories, Lemma 4.35.2. Write f as the equivalence class of a pair $((a, b, \alpha), r)$ with $r \in R$. Then $p_p(r) = \mathrm{id}_V$, hence $p_{pp}((a, b, \alpha)) = \mathrm{id}_V$. Hence $b = \mathrm{id}_V$. But any morphism of $\mathcal{S}$ is strongly cartesian, see Categories, Lemma 4.35.2 hence we see that $(a, b, \alpha) \in R$ is invertible in $u_p\mathcal{S}$ as desired. □

04WI Lemma 8.12.8. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let $p : \mathcal{S} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}$ be categories over $\mathcal{C}$ and $\mathcal{D}$. Assume that

(1) $p : \mathcal{S} \to \mathcal{C}$ is a fibred category,
(2) $q : \mathcal{T} \to \mathcal{D}$ is a fibred category,

(3) $\mathcal{C}$ has nonempty finite limits, and
(4) $u : \mathcal{C} \to \mathcal{D}$ commutes with nonempty finite limits.

Then we have a canonical equivalence of categories

$$\mathrm{Mor}_{\mathrm{Fib}/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = \mathrm{Mor}_{\mathrm{Fib}/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. In this proof we use the notation x/U to denote an object x of $\mathcal{S}$ which lies over U in $\mathcal{C}$. Similarly y/V denotes an object y of $\mathcal{T}$ which lies over V in $\mathcal{D}$. In the same vein $\alpha/a : x/U \to x'/U'$ denotes the morphism $\alpha : x \to x'$ with image $a : U \to U'$ in $\mathcal{C}$.

Let $G : u_p\mathcal{S} \to \mathcal{T}$ be a 1-morphism of fibred categories over $\mathcal{D}$. Denote $G' : u_{pp}\mathcal{S} \to \mathcal{T}$ the composition of G with the canonical (localization) functor $u_{pp}\mathcal{S} \to u_p\mathcal{S}$. Then consider the functor $H : \mathcal{S} \to u^p\mathcal{T}$ given by

$$H(x/U) = (U, G'(U, \mathrm{id}_{u(U)} : u(U) \to u(U), x))$$

on objects and by

$$H((\alpha, a) : x/U \to x'/U') = G'(a, u(a), \alpha)$$

on morphisms. Since G transforms strongly cartesian morphisms into strongly cartesian morphisms, we see that if α is strongly cartesian, then $H(\alpha)$ is strongly cartesian. Namely, we've seen in the proof of Lemma 8.12.6 that in this case the map $(a, u(a), \alpha)$ becomes strongly cartesian in $u_p\mathcal{S}$. Clearly this construction is functorial in G and we obtain a functor

$$A : \mathrm{Mor}_{\mathrm{Fib}/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T}) \longrightarrow \mathrm{Mor}_{\mathrm{Fib}/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T})$$

Conversely, let $H : \mathcal{S} \to u^p\mathcal{T}$ be a 1-morphism of fibred categories. Recall that an object of $u^p\mathcal{T}$ is a pair (U, y) with $y \in \mathrm{Ob}(\mathcal{T}_{u(U)})$. We denote $\mathrm{pr} : u^p\mathcal{T} \to \mathcal{T}$ the functor $(U, y) \mapsto y$. In this case we define a functor $G' : u_{pp}\mathcal{S} \to \mathcal{T}$ by the rules

$$G'(U, \phi : V \to u(U), x) = \phi^*\mathrm{pr}(H(x))$$

on objects and we let

$$G'((a, b, \alpha) : (U, \phi : V \to u(U), x) \to (U', \phi' : V' \to u(U'), x')) = \beta$$

be the unique morphism $\beta : \phi^*\mathrm{pr}(H(x)) \to (\phi')^*\mathrm{pr}(H(x'))$ such that $q(\beta) = b$ and the diagram

$$\begin{tikzcd} \phi^*\mathrm{pr}(H(x)) \arrow[r, "\beta"'] \arrow[d] & (\phi')^*\mathrm{pr}(H(x')) \arrow[d] \\ \mathrm{pr}(H(x)) \arrow[r, "\mathrm{pr}(H(a,\alpha))"] & \mathrm{pr}(H(x')) \end{tikzcd}$$

Such a morphism exists and is unique because $\mathcal{T}$ is a fibred category.

We check that $G'(r)$ is an isomorphism if $r \in R$. Namely, if

$$(a, \mathrm{id}_V, \alpha) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with α strongly cartesian is an element of the right multiplicative system R of Lemma 8.12.5 then $H(\alpha)$ is strongly cartesian, and $\mathrm{pr}(H(\alpha))$ is strongly cartesian, see proof of Lemma 8.12.1. Hence in this case the morphism β has $q(\beta) = \mathrm{id}_V$ and is strongly cartesian. Hence β is an isomorphism by Categories, Lemma 4.33.2. Thus by Categories, Lemma 4.27.16 we obtain a canonical extension $G : u_p\mathcal{S} \to \mathcal{T}$.

Next, let us prove that G transforms strongly cartesian morphisms into strongly cartesian morphisms. Suppose that $f : X \to Y$ is a strongly cartesian. By the characterization of strongly cartesian morphisms in $u_p\mathcal{S}$ we can write f as $((a, b, \alpha) : X' \to Y, r : X' \to Y)$ where $r \in R$ and α strongly cartesian in $\mathcal{S}$. By the above it suffices to show that $G'(a, b\alpha)$ is strongly cartesian. As before the condition that α is strongly cartesian implies that $\text{pr}(H(a, \alpha)) : \text{pr}(H(x)) \to \text{pr}(H(x'))$ is strongly cartesian in $\mathcal{T}$. Since in the commutative square above now all arrows except possibly β is strongly cartesian it follows that also β is strongly cartesian as desired. Clearly the construction $H \mapsto G$ is functorial in H and we obtain a functor

$$B : \text{Mor}_{\text{Fib}/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) \longrightarrow \text{Mor}_{\text{Fib}/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

To finish the proof of the lemma we have to show that the functors A and B are mutually quasi-inverse. We omit the verifications. □

04WJ Definition 8.12.9. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.14.7. Let $\mathcal{S}$ be a stack over $\mathcal{C}$. In this setting we write $f^{-1}\mathcal{S}$ for the stackification of the fibred category $u_p\mathcal{S}$ over $\mathcal{D}$ constructed above. We say that $f^{-1}\mathcal{S}$ is the pullback of $\mathcal{S}$ along f.

Of course, if $\mathcal{S}$ is a stack in groupoids, then $f^{-1}\mathcal{S}$ is a stack in groupoids by Lemmas 8.9.1 and 8.12.7.

04WK Lemma 8.12.10. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.14.7. Let $p : \mathcal{S} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}$ be stacks. Then we have a canonical equivalence of categories

$$\text{Mor}_{\text{Stacks}/\mathcal{C}}(\mathcal{S}, f_*\mathcal{T}) = \text{Mor}_{\text{Stacks}/\mathcal{D}}(f^{-1}\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. For $i = 1, 2$ an i-morphism of stacks is the same thing as a i-morphism of fibred categories, see Definition 8.4.5. By Lemma 8.12.8 we have already

$$\text{Mor}_{\text{Fib}/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = \text{Mor}_{\text{Fib}/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

Hence the result follows from Lemma 8.8.3 as $u^p\mathcal{T} = f_*\mathcal{T}$ and $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$. □

04WR Lemma 8.12.11. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.14.7. Let $\mathcal{S} \to \mathcal{C}$ be a fibred category, and let $\mathcal{S} \to \mathcal{S}'$ be the stackification of $\mathcal{S}$. Then $f^{-1}\mathcal{S}'$ is the stackification of $u_p\mathcal{S}$.

Proof. Omitted. Hint: This is the analogue of Sites, Lemma 7.13.4. □

The following lemma tells us that the 2-category of stacks over Sch_{fppf} is a "full 2-sub category" of the 2-category of stacks over Sch'_{fppf} provided that Sch'_{fppf} contains Sch_{fppf} (see Topologies, Section 34.12).

04WS Lemma 8.12.12. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor satisfying the assumptions of Sites, Lemma 7.21.8. Let $f : \mathcal{D} \to \mathcal{C}$ be the corresponding morphism of sites. Then

(1) for every stack $p : \mathcal{S} \to \mathcal{C}$ the canonical functor $\mathcal{S} \to f_* f^{-1}\mathcal{S}$ is an equivalence of stacks,
(2) given stacks $\mathcal{S}, \mathcal{S}'$ over $\mathcal{C}$ the construction f^{-1} induces an equivalence
$$\mathrm{Mor}_{\textit{Stacks}/\mathcal{C}}(\mathcal{S}, \mathcal{S}') \longrightarrow \mathrm{Mor}_{\textit{Stacks}/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}')$$
of morphism categories.

Proof. Note that by Lemma 8.12.10 we have an equivalence of categories
$$\mathrm{Mor}_{\textit{Stacks}/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}') = \mathrm{Mor}_{\textit{Stacks}/\mathcal{C}}(\mathcal{S}, f_* f^{-1}\mathcal{S}')$$
Hence (2) follows from (1).

To prove (1) we are going to use Lemma 8.4.8. This lemma tells us that we have to show that $can : \mathcal{S} \to f_* f^{-1}\mathcal{S}$ is fully faithful and that all objects of $f_* f^{-1}\mathcal{S}$ are locally in the essential image.

We quickly describe the functor can, see proof of Lemma 8.12.8. To do this we introduce the functor $c'' : \mathcal{S} \to u_{pp}\mathcal{S}$ defined by $c''(x/U) = (U, \mathrm{id} : u(U) \to u(U), x)$, and $c''(\alpha/a) = (a, u(a), \alpha)$. We set $c' : \mathcal{S} \to u_p\mathcal{S}$ equal to the composition of c'' and the canonical functor $u_{pp}\mathcal{S} \to u_p\mathcal{S}$. We set $c : \mathcal{S} \to f^{-1}\mathcal{S}$ equal to the composition of c' and the canonical functor $u_p\mathcal{S} \to f^{-1}\mathcal{S}$. Then $can : \mathcal{S} \to f_* f^{-1}\mathcal{S}$ is the functor which to x/U associates the pair $(U, c(x))$ and to α/a the morphism $(a, c(\alpha))$.

Fully faithfulness. To prove this we are going to use Lemma 8.4.7. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $x, y \in \mathcal{S}_U$. First off, as u is fully faithful, we have
$$\mathrm{Mor}_{(f_* f^{-1}\mathcal{S})_U}(can(x), can(y)) = \mathrm{Mor}_{(f^{-1}\mathcal{S})_{u(U)}}(c(x), c(y))$$
directly from the definition of f_*. Similar holds after pulling back to any U'/U. Because $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$, and since u is continuous and cocontinuous the presheaf
$$U'/U \longmapsto \mathrm{Mor}_{(f^{-1}\mathcal{S})_{u(U')}}(c(x|_{U'}), c(y|_{U'}))$$
is the sheafification of the presheaf
$$U'/U \longmapsto \mathrm{Mor}_{(u_p\mathcal{S})_{u(U')}}(c'(x|_{U'}), c'(y|_{U'}))$$
Hence to finish the proof of fully faithfulness it suffices to show that for any U and x, y the map
$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) \longrightarrow \mathrm{Mor}_{(u_p\mathcal{S})_U}(c'(x), c'(y))$$
is bijective. A morphism $f : x \to y$ in $u_p\mathcal{S}$ over $u(U)$ is given by an equivalence class of diagrams
$$\begin{tikzcd}
(U', \phi : u(U) \to u(U'), x') \arrow[r, "(a,b,\alpha)"'] \arrow[d, "(c, \mathrm{id}_{u(U)}, \gamma)"'] & (U, \mathrm{id} : u(U) \to u(U), y) \\
(U, \mathrm{id} : u(U) \to u(U), x)
\end{tikzcd}$$
with γ strongly cartesian and $b = \mathrm{id}_{u(U)}$. But since u is fully faithful we can write $\phi = u(c')$ for some morphism $c' : U \to U'$ and then we see that $a \circ c' = \mathrm{id}_U$ and $c \circ c' = \mathrm{id}_{U'}$. Because γ is strongly cartesian we can find a morphism $\gamma' : x \to x'$ lifting c' such that $\gamma \circ \gamma' = \mathrm{id}_x$. By definition of the equivalence classes defining morphisms in $u_p\mathcal{S}$ it follows that the morphism
$$(U, \mathrm{id} : u(U) \to u(U), x) \xrightarrow[(\mathrm{id}, \mathrm{id}, \alpha \circ \gamma')]{} (U, \mathrm{id} : u(U) \to u(U), y)$$

of $u_{pp}\mathcal{S}$ induces the morphism f in $u_p\mathcal{S}$. This proves that the map is surjective. We omit the proof that it is injective.

Finally, we have to show that any object of $f_*f^{-1}\mathcal{S}$ locally comes from an object of $\mathcal{S}$. This is clear from the constructions (details omitted). □

8.13. Stacks and localization

04WT Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. We want to understand stacks over $\mathcal{C}/U$ as stacks over $\mathcal{C}$ together with a morphism towards U. The following lemma is the reason why this is easier to do when the presheaf h_U is a sheaf.

04WU **Lemma 8.13.1.** Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then $j_U : \mathcal{C}/U \to \mathcal{C}$ is a stack over $\mathcal{C}$ if and only if h_U is a sheaf.

Proof. Combine Lemma 8.6.3 with Categories, Example 4.38.7. □

Assume that $\mathcal{C}$ is a site, and U is an object of $\mathcal{C}$ whose associated representable presheaf is a sheaf. We denote $j : \mathcal{C}/U \to \mathcal{C}$ the localization functor.

Construction A. Let $p : \mathcal{S} \to \mathcal{C}/U$ be a stack over the site $\mathcal{C}/U$. We define a stack $j_!p : j_!\mathcal{S} \to \mathcal{C}$ as follows:

(1) As a category $j_!\mathcal{S} = \mathcal{S}$, and
(2) the functor $j_!p : j_!\mathcal{S} \to \mathcal{C}$ is just the composition $j \circ p$.

We omit the verification that this is a stack (hint: Use that h_U is a sheaf to glue morphisms to U). There is a canonical functor

$$j_!\mathcal{S} \longrightarrow \mathcal{C}/U$$

namely the functor p which is a 1-morphism of stacks over $\mathcal{C}$.

Construction B. Let $q : \mathcal{T} \to \mathcal{C}$ be a stack over $\mathcal{C}$ which is endowed with a morphism of stacks $p : \mathcal{T} \to \mathcal{C}/U$ over $\mathcal{C}$. In this case it is automatically the case that $p : \mathcal{T} \to \mathcal{C}/U$ is a stack over $\mathcal{C}/U$.

04WV **Lemma 8.13.2.** Assume that $\mathcal{C}$ is a site, and U is an object of $\mathcal{C}$ whose associated representable presheaf is a sheaf. Constructions A and B above define mutually inverse (!) functors of 2-categories

$$\left\{\begin{matrix}\text{2-category of}\\ \text{stacks over } \mathcal{C}/U\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{2-category of pairs } (\mathcal{T}, p) \text{ consisting}\\ \text{of a stack } \mathcal{T} \text{ over } \mathcal{C} \text{ and a morphism}\\ p : \mathcal{T} \to \mathcal{C}/U \text{ of stacks over } \mathcal{C}\end{matrix}\right\}$$

Proof. This is clear. □

8.14. Other chapters

CHAPTER 9

Fields

09FA 9.1. Introduction

09FB In this chapter, we shall discuss the theory of fields. Recall that a field is a ring in which all nonzero elements are invertible. Equivalently, the only two ideals of a field are (0) and (1) since any nonzero element is a unit. Consequently fields will be the simplest cases of much of the theory developed later.

The theory of field extensions has a different feel from standard commutative algebra since, for instance, any morphism of fields is injective. Nonetheless, it turns out that questions involving rings can often be reduced to questions about fields. For instance, any domain can be embedded in a field (its quotient field), and any local ring (that is, a ring with a unique maximal ideal; we have not defined this term yet) has associated to it its residue field (that is, its quotient by the maximal ideal). A knowledge of field extensions will thus be useful.

9.2. Basic definitions

09FC Because we have placed this chapter before the chapter discussing commutative algebra we need to introduce some of the basic definitions here before we discuss these in greater detail in the algebra chapters.

09FD Definition 9.2.1. A field is a nonzero ring where every nonzero element is invertible. Given a field a subfield is a subring that is itself a field.

For a field k, we write k^* for the subset $k \setminus \{0\}$. This generalizes the usual notation R^* that refers to the group of invertible elements in a ring R.

09FE Definition 9.2.2. A domain or an integral domain is a nonzero ring where 0 is the only zerodivisor.

9.3. Examples of fields

09FF To get started, let us begin by providing several examples of fields. The reader should recall that if R is a ring and $I \subset R$ an ideal, then R/I is a field precisely when I is a maximal ideal.

09FG Example 9.3.1 (Rational numbers). The rational numbers form a field. It is called the field of rational numbers and denoted $\mathbf{Q}$.

09FH Example 9.3.2 (Prime fields). If p is a prime number, then $\mathbf{Z}/(p)$ is a field, denoted $\mathbf{F}_p$. Indeed, (p) is a maximal ideal in $\mathbf{Z}$. Thus, fields may be finite: $\mathbf{F}_p$ contains p elements.

09FI Example 9.3.3. In a principal ideal domain, an ideal generated by an irreducible element is maximal. Now, if k is a field, then the polynomial ring $k[x]$ is a principal ideal domain. It follows that if $P \in k[x]$ is an irreducible polynomial (that is, a nonconstant polynomial that does not admit a factorization into terms of smaller degrees), then $k[x]/(P)$ is a field. It contains a copy of k in a natural way. This is a very general way of constructing fields. For instance, the complex numbers $\mathbf{C}$ can be constructed as $\mathbf{R}[x]/(x^2+1)$.

09FJ Example 9.3.4 (Quotient fields). Recall that, given a domain A, there is an imbedding $A \to F$ into a field F constructed from A in exactly the same manner that $\mathbf{Q}$ is constructed from $\mathbf{Z}$. Formally the elements of F are (equivalence classes of) fractions a/b, $a, b \in A$, $b \neq 0$. As usual $a/b = a'/b'$ if and only if $ab' = ba'$. The field F is called the quotient field, or field of fractions, or fraction field of A. The quotient field has the following universal property: given an injective ring map $\varphi : A \to K$ to a field K, there is a unique map $\psi : F \to K$ making

$$\begin{tikzcd} F \arrow[r, "\psi"] & K \\ A \arrow[u] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commute. Indeed, it is clear how to define such a map: we set $\psi(a/b) = \varphi(a)\varphi(b)^{-1}$ where injectivity of φ assures that $\varphi(b) \neq 0$ if $b \neq 0$.

09FK Example 9.3.5 (Field of rational functions). If k is a field, then we can consider the field $k(x)$ of rational functions over k. This is the quotient field of the polynomial ring $k[x]$. In other words, it is the set of quotients F/G for $F, G \in k[x]$, $G \neq 0$ with the obvious equivalence relation.

09FL Example 9.3.6. Let X be a Riemann surface. Let $\mathbf{C}(X)$ denote the set of meromorphic functions on X. Then $\mathbf{C}(X)$ is a ring under multiplication and addition of functions. It turns out that in fact $\mathbf{C}(X)$ is a field. Namely, if a nonzero function $f(z)$ is meromorphic, so is $1/f(z)$. For example, let S^2 be the Riemann sphere; then we know from complex analysis that the ring of meromorphic functions $\mathbf{C}(S^2)$ is the field of rational functions $\mathbf{C}(z)$.

9.4. Vector spaces

09FM One reason fields are so nice is that the theory of modules over fields (i.e. vector spaces), is very simple.

09FN Lemma 9.4.1. If k is a field, then every k-module is free.

Proof. Indeed, by linear algebra we know that a k-module (i.e. vector space) V has a basis $\mathcal{B} \subset V$, which defines an isomorphism from the free vector space on $\mathcal{B}$ to V. □

09FP Lemma 9.4.2. Every exact sequence of modules over a field splits.

Proof. This follows from Lemma 9.4.1 as every vector space is a projective module. □

This is another reason why much of the theory in future chapters will not say very much about fields, since modules behave in such a simple manner. Note that Lemma 9.4.2 is a statement about the category of k-modules (for k a field), because the notion of exactness is inherently arrow-theoretic, i.e., makes use of purely categorical notions, and can in fact be phrased within a so-called abelian category.

Henceforth, since the study of modules over a field is linear algebra, and since the ideal theory of fields is not very interesting, we shall study what this chapter is really about: extensions of fields.

9.5. The characteristic of a field

09FQ In the category of rings, there is an initial object $\mathbf{Z}$: any ring R has a map from $\mathbf{Z}$ into it in precisely one way. For fields, there is no such initial object. Nonetheless, there is a family of objects such that every field can be mapped into in exactly one way by exactly one of them, and in no way by the others.

Let F be a field. Think of F as a ring to get a ring map $f : \mathbf{Z} \to F$. The image of this ring map is a domain (as a subring of a field) hence the kernel of f is a prime ideal in $\mathbf{Z}$. Hence the kernel of f is either (0) or (p) for some prime number p.

In the first case we see that f is injective, and in this case we think of $\mathbf{Z}$ as a subring of F. Moreover, since every nonzero element of F is invertible we see that it makes sense to talk about $p/q \in F$ for $p, q \in \mathbf{Z}$ with $q \neq 0$. Hence in this case we may and we do think of $\mathbf{Q}$ as a subring of F. One can easily see that this is the smallest subfield of F in this case.

In the second case, i.e., when $\mathrm{Ker}(f) = (p)$ we see that $\mathbf{Z}/(p) = \mathbf{F}_p$ is a subring of F. Clearly it is the smallest subfield of F.

Arguing in this way we see that every field contains a smallest subfield which is either $\mathbf{Q}$ or finite equal to $\mathbf{F}_p$ for some prime number p.

09FR Definition 9.5.1. The characteristic of a field F is 0 if $\mathbf{Z} \subset F$, or is a prime p if $p = 0$ in F. The prime subfield of F is the smallest subfield of F which is either $\mathbf{Q} \subset F$ if the characteristic is zero, or $\mathbf{F}_p \subset F$ if the characteristic is $p > 0$.

It is easy to see that if $E \subset F$ is a subfield, then the characteristic of E is the same as the characteristic of F.

09FS Example 9.5.2. The characteristic of $\mathbf{F}_p$ is p, and that of $\mathbf{Q}$ is 0.

9.6. Field extensions

09FT In general, though, we are interested not so much in fields by themselves but in field extensions. This is perhaps analogous to studying not rings but algebras over a fixed ring. The nice thing for fields is that the notion of a "field over another field" just recovers the notion of a field extension, by the next result.

09FU Lemma 9.6.1. If F is a field and R is a nonzero ring, then any ring homomorphism $\varphi : F \to R$ is injective.

Proof. Indeed, let $a \in \mathrm{Ker}(\varphi)$ be a nonzero element. Then we have $\varphi(1) = \varphi(a^{-1}a) = \varphi(a^{-1})\varphi(a) = 0$. Thus $1 = \varphi(1) = 0$ and R is the zero ring. □

09FV Definition 9.6.2. If F is a field contained in a field E, then E is said to be a field extension of F. We shall write E/F to indicate that E is an extension of F.

So if F, F' are fields, and $F \to F'$ is any ring-homomorphism, we see by Lemma 9.6.1 that it is injective, and F' can be regarded as an extension of F, by a slight abuse of language. Alternatively, a field extension of F is just an F-algebra that happens to be a field. This is completely different than the situation for general rings, since a ring homomorphism is not necessarily injective.

Let k be a field. There is a category of field extensions of k. An object of this category is an extension E/k, that is a (necessarily injective) morphism of fields

$$k \to E,$$

while a morphism between extensions E/k and E'/k is a k-algebra morphism $E \to E'$; alternatively, it is a commutative diagram

$$\begin{tikzcd} E \arrow[rr] & & E' \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The set of morphisms from $E \to E'$ in the category of extensions of k will be denoted by $\mathrm{Mor}_k(E, E')$.

09FW Definition 9.6.3. A tower of fields $E_n/E_{n-1}/\ldots/E_0$ consists of a sequence of extensions of fields E_n/E_{n-1}, E_{n-1}/E_{n-2}, $\ldots$, E_1/E_0.

Let us give a few examples of field extensions.

09FX Example 9.6.4. Let k be a field, and $P \in k[x]$ an irreducible polynomial. We have seen that $k[x]/(P)$ is a field (Example 9.3.3). Since it is also a k-algebra in the obvious way, it is an extension of k.

09FY Example 9.6.5. If X is a Riemann surface, then the field of meromorphic functions $\mathbf{C}(X)$ (Example 9.3.6) is an extension field of $\mathbf{C}$, because any element of $\mathbf{C}$ induces a meromorphic — indeed, holomorphic — constant function on X.

Let F/k be a field extension. Let $S \subset F$ be any subset. Then there is a smallest subextension of F (that is, a subfield of F containing k) that contains S. To see this, consider the family of subfields of F containing S and k, and take their intersection; one checks that this is a field. By a standard argument one shows, in fact, that this is the set of elements of F that can be obtained via a finite number of elementary algebraic operations (addition, multiplication, subtraction, and division) involving elements of k and S.

09FZ Definition 9.6.6. Let k be a field. If F/k is an extension of fields and $S \subset F$, we write $k(S)$ for the smallest subfield of F containing k and S. We will say that S generates the field extension $k(S)/k$. If $S = \{\alpha\}$ is a singleton, then we write $k(\alpha)$ instead of $k(\{\alpha\})$. We say F/k is a finitely generated field extension if there exists a finite subset $S \subset F$ with $F = k(S)$.

For instance, $\mathbf{C}$ is generated by i over $\mathbf{R}$.

09G0 Exercise 9.6.7. Show that $\mathbf{C}$ does not have a countable set of generators over $\mathbf{Q}$.

Let us now classify extensions generated by one element.

09G1 Lemma 9.6.8 (Classification of simple extensions). If a field extension F/k is generated by one element, then it is k-isomorphic either to the rational function field $k(t)/k$ or to one of the extensions $k[t]/(P)$ for $P \in k[t]$ irreducible.

We will see that many of the most important cases of field extensions are generated by one element, so this is actually useful.

Proof. Let $\alpha \in F$ be such that $F = k(\alpha)$; by assumption, such an α exists. There is a morphism of rings

$$k[t] \to F$$

sending the indeterminate t to α. The image is a domain, so the kernel is a prime ideal. Thus, it is either (0) or (P) for $P \in k[t]$ irreducible.

If the kernel is (P) for $P \in k[t]$ irreducible, then the map factors through $k[t]/(P)$, and induces a morphism of fields $k[t]/(P) \to F$. Since the image contains α, we see easily that the map is surjective, hence an isomorphism. In this case, $k[t]/(P) \simeq F$.

If the kernel is trivial, then we have an injection $k[t] \to F$. One may thus define a morphism of the quotient field $k(t)$ into F; given a quotient $R(t)/Q(t)$ with $R(t), Q(t) \in k[t]$, we map this to $R(\alpha)/Q(\alpha)$. The hypothesis that $k[t] \to F$ is injective implies that $Q(\alpha) \neq 0$ unless Q is the zero polynomial. The quotient field of $k[t]$ is the rational function field $k(t)$, so we get a morphism $k(t) \to F$ whose image contains α. It is thus surjective, hence an isomorphism. $\square$

9.7. Finite extensions

09G2 If F/E is a field extension, then evidently F is also a vector space over E (the scalar action is just multiplication in F).

09G3 **Definition 9.7.1.** Let F/E be an extension of fields. The dimension of F considered as an E-vector space is called the *degree* of the extension and is denoted $[F : E]$. If $[F : E] < \infty$ then F is said to be a *finite* extension of E.

09G4 **Example 9.7.2.** The field $\mathbf{C}$ is a two dimensional vector space over $\mathbf{R}$ with basis $1, i$. Thus $\mathbf{C}$ is a finite extension of $\mathbf{R}$ of degree 2.

09G5 **Lemma 9.7.3.** Let $K/E/F$ be a tower of algebraic field extensions. If K is finite over F, then K is finite over E.

Proof. Direct from the definition. $\square$

Let us now consider the degree in the most important special example, that given by Lemma 9.6.8, in the next two examples.

09G6 **Example 9.7.4** (Degree of a rational function field)**.** If k is any field, then the rational function field $k(t)$ is not a finite extension. For example the elements $\{t^n, n \in \mathbf{Z}\}$ are linearly independent over k.

In fact, if k is uncountable, then $k(t)$ is uncountably dimensional as a k-vector space. To show this, we claim that the family of elements $\{1/(t-\alpha), \alpha \in k\} \subset k(t)$ is linearly independent over k. A nontrivial relation between them would lead to a contradiction: for instance, if one works over $\mathbf{C}$, then this follows because $\frac{1}{t-\alpha}$, when considered as a meromorphic function on $\mathbf{C}$, has a pole at α and nowhere else. Consequently any sum $\sum c_i \frac{1}{t-\alpha_i}$ for the $c_i \in k^*$, and $\alpha_i \in k$ distinct, would have poles at each of the α_i. In particular, it could not be zero.

Amusingly, this leads to a quick proof of the Hilbert Nullstellensatz over the complex numbers. For a slightly more general result, see Algebra, Theorem 10.35.11.

0BU1 Lemma 9.7.5. A finite extension of fields is a finitely generated field extension. The converse is not true.

Proof. Let F/E be a finite extension of fields. Let $\alpha_1, \ldots, \alpha_n$ be a basis of F as a vector space over E. Then $F = E(\alpha_1, \ldots, \alpha_n)$ hence F/E is a finitely generated field extension. The converse is not true as follows from Example 9.7.4. □

09G7 Example 9.7.6 (Degree of a simple algebraic extension). Consider a monogenic field extension E/k of the form discussed in Example 9.6.4. In other words, $E = k[t]/(P)$ for $P \in k[t]$ an irreducible polynomial. Then the degree $[E : k]$ is just the degree $d = \deg(P)$ of the polynomial P. Indeed, say

09G8 (9.7.6.1)
$$P = a_d t^d + a_{d-1} t^{d-1} + \ldots + a_0.$$

with $a_d \neq 0$. Then the images of $1, t, \ldots, t^{d-1}$ in $k[t]/(P)$ are linearly independent over k, because any relation involving them would have degree strictly smaller than that of P, and P is the element of smallest degree in the ideal (P).

Conversely, the set $S = \{1, t, \ldots, t^{d-1}\}$ (or more properly their images) spans $k[t]/(P)$ as a vector space. Indeed, we have by (9.7.6.1) that $a_d t^d$ lies in the span of S. Since a_d is invertible, we see that t^d is in the span of S. Similarly, the relation $tP(t) = 0$ shows that the image of t^{d+1} lies in the span of $\{1, t, \ldots, t^d\}$ — by what was just shown, thus in the span of S. Working upward inductively, we find that the image of t^n for $n \geq d$ lies in the span of S.

This confirms the observation that $[\mathbf{C} : \mathbf{R}] = 2$, for instance. More generally, if k is a field, and $\alpha \in k$ is not a square, then the irreducible polynomial $x^2 - \alpha \in k[x]$ allows one to construct an extension $k[x]/(x^2 - \alpha)$ of degree two. We shall write this as $k(\sqrt{\alpha})$. Such extensions will be called quadratic, for obvious reasons.

The basic fact about the degree is that it is multiplicative in towers.

09G9 Lemma 9.7.7 (Multiplicativity). Suppose given a tower of fields $F/E/k$. Then

$$[F : k] = [F : E][E : k]$$

Proof. Let $\alpha_1, \ldots, \alpha_n \in F$ be an E-basis for F. Let $\beta_1, \ldots, \beta_m \in E$ be a k-basis for E. Then the claim is that the set of products $\{\alpha_i \beta_j, 1 \leq i \leq n, 1 \leq j \leq m\}$ is a k-basis for F. Indeed, let us check first that they span F over k.

By assumption, the $\{\alpha_i\}$ span F over E. So if $f \in F$, there are $a_i \in E$ with

$$f = \sum\nolimits_i a_i \alpha_i,$$

and, for each i, we can write $a_i = \sum b_{ij} \beta_j$ for some $b_{ij} \in k$. Putting these together, we find

$$f = \sum\nolimits_{i,j} b_{ij} \alpha_i \beta_j,$$

proving that the $\{\alpha_i \beta_j\}$ span F over k.

Suppose now that there existed a nontrivial relation

$$\sum\nolimits_{i,j} c_{ij} \alpha_i \beta_j = 0$$

for the $c_{ij} \in k$. In that case, we would have

$$\sum\nolimits_i \alpha_i \left(\sum\nolimits_j c_{ij} \beta_j \right) = 0,$$

and the inner terms lie in E as the β_j do. Now E-linear independence of the $\{\alpha_i\}$ shows that the inner sums are all zero. Then k-linear independence of the $\{\beta_j\}$ shows that the c_{ij} all vanish. □

We sidetrack to a slightly tangential definition.

09GA Definition 9.7.8. A field K is said to be a number field if it has characteristic 0 and the extension $K/\mathbf{Q}$ is finite.

Number fields are the basic objects in algebraic number theory. We shall see later that, for the analog of the integers $\mathbf{Z}$ in a number field, something kind of like unique factorization still holds (though strict unique factorization generally does not!).

9.8. Algebraic extensions

09GB An important class of extensions are those where every element generates a finite extension.

09GC Definition 9.8.1. Consider a field extension F/E. An element $\alpha \in F$ is said to be algebraic over E if α is the root of some nonzero polynomial with coefficients in E. If all elements of F are algebraic then F is said to be an algebraic extension of E.

By Lemma 9.6.8, the subextension $E(\alpha)$ is isomorphic either to the rational function field $E(t)$ or to a quotient ring $E[t]/(P)$ for $P \in E[t]$ an irreducible polynomial. In the latter case, α is algebraic over E (in fact, the proof of Lemma 9.6.8 shows that we can pick P such that α is a root of P); in the former case, it is not.

09GD Example 9.8.2. The field $\mathbf{C}$ is algebraic over $\mathbf{R}$. Namely, if $\alpha = a + ib$ in $\mathbf{C}$, then $\alpha^2 - 2a\alpha + a^2 + b^2 = 0$ is a polynomial equation for α over $\mathbf{R}$.

09GE Example 9.8.3. Let X be a compact Riemann surface, and let $f \in \mathbf{C}(X) - \mathbf{C}$ any nonconstant meromorphic function on X (see Example 9.3.6). Then it is known that $\mathbf{C}(X)$ is algebraic over the subextension $\mathbf{C}(f)$ generated by f. We shall not prove this.

09GF Lemma 9.8.4. Let $K/E/F$ be a tower of field extensions.

(1) If $\alpha \in K$ is algebraic over F, then α is algebraic over E.
(2) If K is algebraic over F, then K is algebraic over E.

Proof. This is immediate from the definitions. □

We now show that there is a deep connection between finiteness and being algebraic.

09GG Lemma 9.8.5. A finite extension is algebraic. In fact, an extension E/k is algebraic if and only if every subextension $k(\alpha)/k$ generated by some $\alpha \in E$ is finite.

In general, it is very false that an algebraic extension is finite.

Proof. Let E/k be finite, say of degree n. Choose $\alpha \in E$. Then the elements $\{1, \alpha, \ldots, \alpha^n\}$ are linearly dependent over E, or we would necessarily have $[E : k] > n$. A relation of linear dependence now gives the desired polynomial that α must satisfy.

For the last assertion, note that a monogenic extension $k(\alpha)/k$ is finite if and only if α is algebraic over k, by Examples 9.7.4 and 9.7.6. So if E/k is algebraic, then each $k(\alpha)/k$, $\alpha \in E$, is a finite extension, and conversely. □

We can extract a lemma of the last proof (really of Examples 9.7.4 and 9.7.6): a monogenic extension is finite if and only if it is algebraic. We shall use this observation in the next result.

09GH Lemma 9.8.6. Let k be a field, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be elements of some extension field such that each α_i is algebraic over k. Then the extension $k(\alpha_1, \ldots, \alpha_n)/k$ is finite. That is, a finitely generated algebraic extension is finite.

Proof. Indeed, each extension $k(\alpha_1, \ldots, \alpha_{i+1})/k(\alpha_1, \ldots, \alpha_i)$ is generated by one element and algebraic, hence finite. By multiplicativity of degree (Lemma 9.7.7) we obtain the result. □

The set of complex numbers that are algebraic over $\mathbf{Q}$ are simply called the algebraic numbers. For instance, $\sqrt{2}$ is algebraic, i is algebraic, but π is not. It is a basic fact that the algebraic numbers form a field, although it is not obvious how to prove this from the definition that a number is algebraic precisely when it satisfies a nonzero polynomial equation with rational coefficients (e.g. by polynomial equations).

09GI Lemma 9.8.7. Let E/k be a field extension. Then the elements of E algebraic over k form a subextension of E/k.

Proof. Let $\alpha, \beta \in E$ be algebraic over k. Then $k(\alpha, \beta)/k$ is a finite extension by Lemma 9.8.6. It follows that $k(\alpha + \beta) \subset k(\alpha, \beta)$ is a finite extension, which implies that $\alpha + \beta$ is algebraic by Lemma 9.8.5. Similarly for the difference, product and quotient of α and β. □

Many nice properties of field extensions, like those of rings, will have the property that they will be preserved by towers and composita.

09GJ Lemma 9.8.8. Let E/k and F/E be algebraic extensions of fields. Then F/k is an algebraic extension of fields.

Proof. Choose $\alpha \in F$. Then α is algebraic over E. The key observation is that α is algebraic over a finitely generated subextension of k. That is, there is a finite set $S \subset E$ such that α is algebraic over $k(S)$: this is clear because being algebraic means that a certain polynomial in $E[x]$ that α satisfies exists, and as S we can take the coefficients of this polynomial. It follows that α is algebraic over $k(S)$. In particular, the extension $k(S, \alpha)/k(S)$ is finite. Since S is a finite set, and $k(S)/k$ is algebraic, Lemma 9.8.6 shows that $k(S)/k$ is finite. Using multiplicativity (Lemma 9.7.7) we find that $k(S, \alpha)/k$ is finite, so α is algebraic over k. □

The method of proof in the previous argument — that being algebraic over E was a property that descended to a finitely generated subextension of E — is an idea that recurs throughout algebra. It often allows one to reduce general commutative algebra questions to the Noetherian case for example.

09GK Lemma 9.8.9. Let E/F be an algebraic extension of fields. Then the cardinality $|E|$ of E is at most $\max(\aleph_0, |F|)$.

Proof. Let S be the set of nonconstant polynomials with coefficients in F. For every $P \in S$ the set of roots $r(P, E) = \{\alpha \in E \mid P(\alpha) = 0\}$ is finite (details omitted). Moreover, the fact that E is algebraic over F implies that $E = \bigcup_{P \in S} r(P, E)$. It is clear that S has cardinality bounded by $\max(\aleph_0, |F|)$ because the cardinality of a countable product of copies of F has cardinality at most $\max(\aleph_0, |F|)$. Thus so does E. □

0BID Lemma 9.8.10. Let E/F be a finite or more generally an algebraic extension of fields. Any subring $F \subset R \subset E$ is a field.

Proof. Let $\alpha \in R$ be nonzero. Then $1, \alpha, \alpha^2, \ldots$ are contained in R. By Lemma 9.8.5 we find a nontrivial relation $a_0 + a_1\alpha + \ldots + a_d\alpha^d = 0$. We may assume $a_0 \neq 0$ because if not we can divide the relation by α to decrease d. Then we see that

$$a_0 = \alpha(-a_1 - \ldots - a_d\alpha^{d-1})$$

which proves that the inverse of α is the element $a_0^{-1}(-a_1 - \ldots - a_d\alpha^{d-1})$ of R. $\square$

0BMD Lemma 9.8.11. Let E/F an algebraic extension of fields. Any F-algebra map $f : E \to E$ is an automorphism.

Proof. If E/F is finite, then $f : E \to E$ is an F-linear injective map (Lemma 9.6.1) of finite dimensional vector spaces, and hence bijective. In general we still see that f is injective. Let $\alpha \in E$ and let $P \in F[x]$ be a polynomial such that $P(\alpha) = 0$. Let $E' \subset E$ be the subfield of E generated by the roots $\alpha = \alpha_1, \ldots, \alpha_n$ of P in E. Then E' is finite over F by Lemma 9.8.6. Since f preserves the set of roots, we find that $f|_{E'} : E' \to E'$. Hence $f|_{E'}$ is an isomorphism by the first part of the proof and we conclude that α is in the image of f. $\square$

9.9. Minimal polynomials

09GL Let E/k be a field extension, and let $\alpha \in E$ be algebraic over k. Then α satisfies a (nontrivial) polynomial equation in $k[x]$. Consider the set of polynomials $P \in k[x]$ such that $P(\alpha) = 0$; by hypothesis, this set does not just contain the zero polynomial. It is easy to see that this set is an ideal. Indeed, it is the kernel of the map

$$k[x] \to E, \quad x \mapsto \alpha$$

Since $k[x]$ is a PID, there is a generator $P \in k[x]$ of this ideal. If we assume P monic, without loss of generality, then P is uniquely determined.

09GM Definition 9.9.1. The polynomial P above is called the minimal polynomial of α over k.

The minimal polynomial has the following characterization: it is the monic polynomial, of smallest degree, that annihilates α. Any nonconstant multiple of P will have larger degree, and only multiples of P can annihilate α. This explains the name minimal.

Clearly the minimal polynomial is irreducible. This is equivalent to the assertion that the ideal in $k[x]$ consisting of polynomials annihilating α is prime. This follows from the fact that the map $k[x] \to E, x \mapsto \alpha$ is a map into a domain (even a field), so the kernel is a prime ideal.

09GN Lemma 9.9.2. The degree of the minimal polynomial is $[k(\alpha) : k]$.

Proof. This is just a restatement of the argument in Lemma 9.6.8: the observation is that if P is the minimal polynomial of α, then the map

$$k[x]/(P) \to k(\alpha), \quad x \mapsto \alpha$$

is an isomorphism as in the aforementioned proof, and we have counted the degree of such an extension (see Example 9.7.6). $\square$

So the observation of the above proof is that if $\alpha \in E$ is algebraic, then $k(\alpha) \subset E$ is isomorphic to $k[x]/(P)$.

9.10. Algebraic closure

09GP The "fundamental theorem of algebra" states that $\mathbf{C}$ is algebraically closed. A beautiful proof of this result uses Liouville's theorem in complex analysis, we shall give another proof (see Lemma 9.23.1).

09GQ Definition 9.10.1. A field F is said to be algebraically closed if every algebraic extension E/F is trivial, i.e., $E = F$.

This may not be the definition in every text. Here is the lemma comparing it with the other one.

09GR Lemma 9.10.2. Let F be a field. The following are equivalent

(1) F is algebraically closed,
(2) every irreducible polynomial over F is linear,
(3) every nonconstant polynomial over F has a root,
(4) every nonconstant polynomial over F is a product of linear factors.

Proof. If F is algebraically closed, then every irreducible polynomial is linear. Namely, if there exists an irreducible polynomial of degree > 1, then this generates a nontrivial finite (hence algebraic) field extension, see Example 9.7.6. Thus (1) implies (2). If every irreducible polynomial is linear, then every irreducible polynomial has a root, whence every nonconstant polynomial has a root. Thus (2) implies (3).

Assume every nonconstant polynomial has a root. Let $P \in F[x]$ be nonconstant. If $P(\alpha) = 0$ with $\alpha \in F$, then we see that $P = (x - \alpha)Q$ for some $Q \in F[x]$ (by division with remainder). Thus we can argue by induction on the degree that any nonconstant polynomial can be written as a product $c \prod (x - \alpha_i)$.

Finally, suppose that every nonconstant polynomial over F is a product of linear factors. Let E/F be an algebraic extension. Then all the simple subextensions $F(\alpha)/F$ of E are necessarily trivial (because the only irreducible polynomials are linear by assumption). Thus $E = F$. We see that (4) implies (1) and we are done. $\square$

Now we want to define a "universal" algebraic extension of a field. Actually, we should be careful: the algebraic closure is not a universal object. That is, the algebraic closure is not unique up to unique isomorphism: it is only unique up to isomorphism. But still, it will be very handy, if not functorial.

09GS Definition 9.10.3. Let F be a field. An algebraic closure of F is a field $\overline{F}$ containing F such that:

(1) $\overline{F}$ is algebraic over F.
(2) $\overline{F}$ is algebraically closed.

If F is algebraically closed, then F is its own algebraic closure. We now prove the basic existence result.

09GT Theorem 9.10.4. Every field has an algebraic closure.

The proof will mostly be a red herring to the rest of the chapter. However, we will want to know that it is possible to embed a field inside an algebraically closed field, and we will often assume it done.

Proof. Let F be a field. By Lemma 9.8.9 the cardinality of an algebraic extension of F is bounded by $\max(\aleph_0, |F|)$. Choose a set S containing F with $|S| > \max(\aleph_0, |F|)$. Let's consider triples (E, σ_E, μ_E) where

(1) E is a set with $F \subset E \subset S$, and
(2) $\sigma_E : E \times E \to E$ and $\mu_E : E \times E \to E$ are maps of sets such that (E, σ_E, μ_E) defines the structure of a field extension of F (in particular $\sigma_E(a, b) = a +_F b$ for $a, b \in F$ and similarly for μ_E), and
(3) E/F is an algebraic field extension.

The collection of all triples (E, σ_E, μ_E) forms a set I. For $i \in I$ we will denote $E_i = (E_i, \sigma_i, \mu_i)$ the corresponding field extension to F. We define a partial ordering on I by declaring $i \leq i'$ if and only if $E_i \subset E_{i'}$ (this makes sense as E_i and $E_{i'}$ are subsets of the same set S) and we have $\sigma_i = \sigma_{i'}|_{E_i \times E_i}$ and $\mu_i = \mu_{i'}|_{E_i \times E_i}$, in other words, $E_{i'}$ is a field extension of E_i.

Let $T \subset I$ be a totally ordered subset. Then it is clear that $E_T = \bigcup_{i \in T} E_i$ with induced maps $\sigma_T = \bigcup \sigma_i$ and $\mu_T = \bigcup \mu_i$ is another element of I. In other words every totally order subset of I has a upper bound in I. By Zorn's lemma there exists a maximal element (E, σ_E, μ_E) in I. We claim that E is an algebraic closure. Since by definition of I the extension E/F is algebraic, it suffices to show that E is algebraically closed.

To see this we argue by contradiction. Namely, suppose that E is not algebraically closed. Then there exists an irreducible polynomial P over E of degree > 1, see Lemma 9.10.2. By Lemma 9.8.5 we obtain a nontrivial finite extension $E' = E[x]/(P)$. Observe that E'/F is algebraic by Lemma 9.8.8. Thus the cardinality of E' is $\leq \max(\aleph_0, |F|)$. By elementary set theory we can extend the given injection $E \subset S$ to an injection $E' \to S$. In other words, we may think of E' as an element of our set I contradicting the maximality of E. This contradiction completes the proof. $\square$

09GU Lemma 9.10.5. Let F be a field. Let $\overline{F}$ be an algebraic closure of F. Let M/F be an algebraic extension. Then there is a morphism of F-extensions $M \to \overline{F}$.

Proof. Consider the set I of pairs (E, φ) where $F \subset E \subset M$ is a subextension and $\varphi : E \to \overline{F}$ is a morphism of F-extensions. We partially order the set I by declaring $(E, \varphi) \leq (E', \varphi')$ if and only if $E \subset E'$ and $\varphi'|_E = \varphi$. If $T = \{(E_t, \varphi_t)\} \subset I$ is a totally ordered subset, then $\bigcup \varphi_t : \bigcup E_t \to \overline{F}$ is an element of I. Thus every totally ordered subset of I has an upper bound. By Zorn's lemma there exists a maximal element (E, φ) in I. We claim that $E = M$, which will finish the proof. If not, then pick $\alpha \in M$, $\alpha \notin E$. The α is algebraic over E, see Lemma 9.8.4. Let P be the minimal polynomial of α over E. Let P^φ be the image of P by φ in $\overline{F}[x]$. Since $\overline{F}$ is algebraically closed there is a root β of P^φ in $\overline{F}$. Then we can extend φ to $\varphi' : E(\alpha) = E[x]/(P) \to \overline{F}$ by mapping x to β. This contradicts the maximality of (E, φ) as desired. $\square$

09GV Lemma 9.10.6. Any two algebraic closures of a field are isomorphic.

Proof. Let F be a field. If M and $\overline{F}$ are algebraic closures of F, then there exists a morphism of F-extensions $\varphi : M \to \overline{F}$ by Lemma 9.10.5. Now the image $\varphi(M)$ is algebraically closed. On the other hand, the extension $\varphi(M) \subset \overline{F}$ is algebraic by Lemma 9.8.4. Thus $\varphi(M) = \overline{F}$. □

9.11. Relatively prime polynomials

09GW Let K be an algebraically closed field. Then the ring $K[x]$ has a very simple ideal structure as we saw in Lemma 9.10.2. In particular, every polynomial $P \in K[x]$ can be written as

$$P = c(x - \alpha_1) \ldots (x - \alpha_n),$$

where c is the constant term and the $\alpha_1, \ldots, \alpha_n \in k$ are the roots of P (counted with multiplicity). Clearly, the only irreducible polynomials in $K[x]$ are the linear polynomials $c(x - \alpha)$, $c, \alpha \in K$ (and $c \neq 0$).

09GX **Definition 9.11.1.** If k is any field, we say that two polynomials in $k[x]$ are *relatively prime* if they generate the unit ideal in $k[x]$.

Continuing the discussion above, if K is an algebraically closed field, two polynomials in $K[x]$ are relatively prime if and only if they have no common roots. This follows because the maximal ideals of $K[x]$ are of the form $(x - \alpha)$, $\alpha \in K$. So if $F, G \in K[x]$ have no common root, then (F, G) cannot be contained in any $(x - \alpha)$ (as then they would have a common root at α).

If k is not algebraically closed, then this still gives information about when two polynomials in $k[x]$ generate the unit ideal.

09GY **Lemma 9.11.2.** *Two polynomials in $k[x]$ are relatively prime precisely when they have no common roots in an algebraic closure $\overline{k}$ of k.*

Proof. The claim is that any two polynomials P, Q generate (1) in $k[x]$ if and only if they generate (1) in $\overline{k}[x]$. This is a piece of linear algebra: a system of linear equations with coefficients in k has a solution if and only if it has a solution in any extension of k. Consequently, we can reduce to the case of an algebraically closed field, in which case the result is clear from what we have already proved. □

9.12. Separable extensions

09GZ In characteristic p something funny happens with irreducible polynomials over fields. We explain this in the following lemma.

09H0 **Lemma 9.12.1.** *Let F be a field. Let $P \in F[x]$ be an irreducible polynomial over F. Let $P' = \mathrm{d}P/\mathrm{d}x$ be the derivative of P with respect to x. Then one of the following two cases happens*

1. *P and P' are relatively prime, or*
2. *P' is the zero polynomial.*

The second case can only happen if F has characteristic $p > 0$. In this case $P(x) = Q(x^q)$ where $q = p^f$ is a power of p and $Q \in F[x]$ is an irreducible polynomial such that Q and Q' are relatively prime.

Proof. Note that P' has degree $< \deg(P)$. Hence if P and P' are not relatively prime, then $(P, P') = (R)$ where R is a polynomial of degree $< \deg(P)$ contradicting the irreducibility of P. This proves we have the dichotomy between (1) and (2).

Assume we are in case (2) and $P = a_d x^d + \ldots + a_0$. Then $P' = da_d x^{d-1} + \ldots + a_1$. In characteristic 0 we see that this forces $a_d, \ldots, a_1 = 0$ which would mean P is constant a contradiction. Thus we conclude that the characteristic p is positive. In this case the condition $P' = 0$ forces $a_i = 0$ whenever p does not divide i. In other words, $P(x) = P_1(x^p)$ for some nonconstant polynomial P_1. Clearly, P_1 is irreducible as well. By induction on the degree we see that $P_1(x) = Q(x^q)$ as in the statement of the lemma, hence $P(x) = Q(x^{pq})$ and the lemma is proved. □

09H1 Definition 9.12.2. Let F be a field. Let K/F be an extension of fields.

(1) We say an irreducible polynomial P over F is separable if it is relatively prime to its derivative.
(2) Given $\alpha \in K$ algebraic over F we say α is separable over F if its minimal polynomial is separable over F.
(3) If K is an algebraic extension of F, we say K is separable[1] over F if every element of K is separable over F.

By Lemma 9.12.1 in characteristic 0 every irreducible polynomial is separable, every algebraic element in an extension is separable, and every algebraic extension is separable.

09H2 Lemma 9.12.3. Let $K/E/F$ be a tower of algebraic field extensions.

(1) If $\alpha \in K$ is separable over F, then α is separable over E.
(2) if K is separable over F, then K is separable over E.

Proof. We will use Lemma 9.12.1 without further mention. Let P be the minimal polynomial of α over F. Let Q be the minimal polynomial of α over E. Then Q divides P in the polynomial ring $E[x]$, say $P = QR$. Then $P' = Q'R + QR'$. Thus if $Q' = 0$, then Q divides P and P' hence $P' = 0$ by the lemma. This proves (1). Part (2) follows immediately from (1) and the definitions. □

09H3 Lemma 9.12.4. Let F be a field. An irreducible polynomial P over F is separable if and only if P has pairwise distinct roots in an algebraic closure of F.

Proof. Suppose that $\alpha \in \overline{F}$ is a root of both P and P'. Then $P = (x - \alpha)Q$ for some polynomial Q. Taking derivatives we obtain $P' = Q + (x - \alpha)Q'$. Thus α is a root of Q. Hence we see that if P and P' have a common root, then P does not have pairwise distinct roots. Conversely, if P has a repeated root, i.e., $(x-\alpha)^2$ divides P, then α is a root of both P and P'. Combined with Lemma 9.11.2 this proves the lemma. □

09H4 Lemma 9.12.5. Let F be a field and let $\overline{F}$ be an algebraic closure of F. Let $p > 0$ be the characteristic of F. Let P be a polynomial over F. Then the set of roots of P and $P(x^p)$ in $\overline{F}$ have the same cardinality (not counting multiplicity).

Proof. Clearly, α is a root of $P(x^p)$ if and only if α^p is a root of P. In other words, the roots of $P(x^p)$ are the roots of $x^p - \beta$, where β is a root of P. Thus it suffices to show that the map $\overline{F} \to \overline{F}$, $\alpha \mapsto \alpha^p$ is bijective. It is surjective, as $\overline{F}$ is algebraically closed which means that every element has a pth root. It is injective because $\alpha^p = \beta^p$ implies $(\alpha - \beta)^p = 0$ because the characteristic is p. And of course in a field $x^p = 0$ implies $x = 0$. □

[1]For nonalgebraic extensions this definition does not make sense and is not the correct one. We refer the reader to Algebra, Sections 10.42 and 10.44.

Let F be a field and let P be an irreducible polynomial over F. Then we know that $P = Q(x^q)$ for some separable irreducible polynomial Q (Lemma 9.12.1) where q is a power of the characteristic p (and if the characteristic is zero, then $q = 1$[2] and $Q = P$). By Lemma 9.12.5 the number of roots of P and Q in any algebraic closure of F is the same. By Lemma 9.12.4 this number is equal to the degree of Q.

09H5 Definition 9.12.6. Let F be a field. Let P be an irreducible polynomial over F. The separable degree of P is the cardinality of the set of roots of P in any algebraic closure of F (see discussion above). Notation $\deg_s(P)$.

The separable degree of P always divides the degree and the quotient is a power of the characteristic. If the characteristic is zero, then $\deg_s(P) = \deg(P)$.

09H6 Situation 9.12.7. Here F be a field and K/F is a finite extension generated by elements $\alpha_1, \ldots, \alpha_n \in K$. We set $K_0 = F$ and

$$K_i = F(\alpha_1, \ldots, \alpha_i)$$

to obtain a tower of finite extensions $K = K_n/K_{n-1}/\ldots/K_0 = F$. Denote P_i the minimal polynomial of α_i over K_{i-1}. Finally, we fix an algebraic closure $\overline{F}$ of F.

Let F, K, α_i, and $\overline{F}$ be as in Situation 9.12.7. Suppose that $\varphi : K \to \overline{F}$ is a morphism of extensions of F. Then we obtain maps $\varphi_i : K_i \to \overline{F}$. In particular, we can take the image of $P_i \in K_{i-1}[x]$ by φ_{i-1} to get a polynomial $P_i^\varphi \in \overline{F}[x]$.

09H7 Lemma 9.12.8. In Situation 9.12.7 the correspondence

$$\mathrm{Mor}_F(K, \overline{F}) \longrightarrow \{(\beta_1, \ldots, \beta_n) \text{ as below}\}, \quad \varphi \longmapsto (\varphi(\alpha_1), \ldots, \varphi(\alpha_n))$$

is a bijection. Here the right hand side is the set of n-tuples $(\beta_1, \ldots, \beta_n)$ of elements of $\overline{F}$ such that β_i is a root of P_i^φ.

Proof. Let $(\beta_1, \ldots, \beta_n)$ be an element of the right hand side. We construct a map of fields corresponding to it by induction. Namely, we set $\varphi_0 : K_0 \to \overline{F}$ equal to the given map $K_0 = F \subset \overline{F}$. Having constructed $\varphi_{i-1} : K_{i-1} \to \overline{F}$ we observe that $K_i = K_{i-1}[x]/(P_i)$. Hence we can set φ_i equal to the unique map $K_i \to \overline{F}$ inducing φ_{i-1} on K_{i-1} and mapping x to β_i. This works precisely as β_i is a root of P_i^φ. Uniqueness implies that the two constructions are mutually inverse. □

09H8 Lemma 9.12.9. In Situation 9.12.7 we have $|\mathrm{Mor}_F(K, \overline{F})| = \prod_{i=1}^n \deg_s(P_i)$.

Proof. This follows immediately from Lemma 9.12.8. Observe that a key ingredient we are tacitly using here is the well-definedness of the separable degree of an irreducible polynomial which was observed just prior to Definition 9.12.6. □

We now use the result above to characterize separable field extensions.

09H9 Lemma 9.12.10. Assumptions and notation as in Situation 9.12.7. If each P_i is separable, i.e., each α_i is separable over K_{i-1}, then

$$|\mathrm{Mor}_F(K, \overline{F})| = [K : F]$$

and the field extension K/F is separable. If one of the α_i is not separable over K_{i-1}, then $|\mathrm{Mor}_F(K, \overline{F})| < [K : F]$.

[2] A good convention for this chapter is to set $0^0 = 1$.

Proof. If α_i is separable over K_{i-1} then $\deg_s(P_i) = \deg(P_i) = [K_i : K_{i-1}]$ (last equality by Lemma 9.9.2). By multiplicativity (Lemma 9.7.7) we have

$$[K : F] = \prod [K_i : K_{i-1}] = \prod \deg(P_i) = \prod \deg_s(P_i) = |\operatorname{Mor}_F(K, \overline{F})|$$

where the last equality is Lemma 9.12.9. By the exact same argument we get the strict inequality $|\operatorname{Mor}_F(K, \overline{F})| < [K : F]$ if one of the α_i is not separable over K_{i-1}.

Finally, assume again that each α_i is separable over K_{i-1}. We will show K/F is separable. Let $\gamma = \gamma_1 \in K$ be arbitrary. Then we can find additional elements $\gamma_2, \ldots, \gamma_m$ such that $K = F(\gamma_1, \ldots, \gamma_m)$ (for example we could take $\gamma_2 = \alpha_1, \ldots, \gamma_{n+1} = \alpha_n$). Then we see by the last part of the lemma (already proven above) that if γ is not separable over F we would have the strict inequality $|\operatorname{Mor}_F(K, \overline{F})| < [K : F]$ contradicting the very first part of the lemma (already prove above as well). $\square$

09HA Lemma 9.12.11. Let K/F be a finite extension of fields. Let $\overline{F}$ be an algebraic closure of F. Then we have

$$|\operatorname{Mor}_F(K, \overline{F})| \leq [K : F]$$

with equality if and only if K is separable over F.

Proof. This is a corollary of Lemma 9.12.10. Namely, since K/F is finite we can find finitely many elements $\alpha_1, \ldots, \alpha_n \in K$ generating K over F (for example we can choose the α_i to be a basis of K over F). If K/F is separable, then each α_i is separable over $F(\alpha_1, \ldots, \alpha_{i-1})$ by Lemma 9.12.3 and we get equality by Lemma 9.12.10. On the other hand, if we have equality, then no matter how we choose $\alpha_1, \ldots, \alpha_n$ we get that α_1 is separable over F by Lemma 9.12.10. Since we can start the sequence with an arbitrary element of K it follows that K is separable over F. $\square$

09HB Lemma 9.12.12. Let E/k and F/E be separable algebraic extensions of fields. Then F/k is a separable extension of fields.

Proof. Choose $\alpha \in F$. Then α is separable algebraic over E. Let $P = x^d + \sum_{i<d} a_i x^i$ be the minimal polynomial of α over E. Each a_i is separable algebraic over k. Consider the tower of fields

$$k \subset k(a_0) \subset k(a_0, a_1) \subset \ldots \subset k(a_0, \ldots, a_{d-1}) \subset k(a_0, \ldots, a_{d-1}, \alpha)$$

Because a_i is separable algebraic over k it is separable algebraic over $k(a_0, \ldots, a_{i-1})$ by Lemma 9.12.3. Finally, α is separable algebraic over $k(a_0, \ldots, a_{d-1})$ because it is a root of P which is irreducible (as it is irreducible over the possibly bigger field E) and separable (as it is separable over E). Thus $k(a_0, \ldots, a_{d-1}, \alpha)$ is separable over k by Lemma 9.12.10 and we conclude that α is separable over k as desired. $\square$

09HC Lemma 9.12.13. Let E/k be a field extension. Then the elements of E separable over k form a subextension of E/k.

Proof. Let $\alpha, \beta \in E$ be separable over k. Then β is separable over $k(\alpha)$ by Lemma 9.12.3. Thus we can apply Lemma 9.12.12 to $k(\alpha, \beta)$ to see that $k(\alpha, \beta)$ is separable over k. $\square$

9.13. Linear independence of characters

0CKK Here is the statement.

0CKL **Lemma 9.13.1.** Let L be a field. Let G be a monoid, for example a group. Let $\chi_1, \ldots, \chi_n : G \to L$ be pairwise distinct homomorphisms of monoids where L is regarded as a monoid by multiplication. Then $\chi_1, \ldots, \chi_n$ are L-linearly independent: if $\lambda_1, \ldots, \lambda_n \in L$ not all zero, then $\sum \lambda_i \chi_i(g) \neq 0$ for some $g \in G$.

Proof. If $n = 1$ this is true because $\chi_1(e) = 1$ if $e \in G$ is the neutral (identity) element. We prove the result by induction for $n > 1$. Suppose that $\lambda_1, \ldots, \lambda_n \in L$ not all zero. If $\lambda_i = 0$ for some, then we win by induction on n. Since we want to show that $\sum \lambda_i \chi_i(g) \neq 0$ for some $g \in G$ we may after dividing by $-\lambda_n$ assume that $\lambda_n = -1$. Then the only way we get in trouble is if

$$\chi_n(g) = \sum\nolimits_{i=1,\ldots,n-1} \lambda_i \chi_i(g)$$

for all $g \in G$. Fix $h \in G$. Then we would also get

$$\begin{aligned} \chi_n(h)\chi_n(g) &= \chi_n(hg) \\ &= \sum\nolimits_{i=1,\ldots,n-1} \lambda_i \chi_i(hg) \\ &= \sum\nolimits_{i=1,\ldots,n-1} \lambda_i \chi_i(h)\chi_i(g) \end{aligned}$$

Multiplying the previous relation by $\chi_n(h)$ and substracting we obtain

$$0 = \sum\nolimits_{i=1,\ldots,n-1} \lambda_i(\chi_n(h) - \chi_i(h))\chi_i(g)$$

for all $g \in G$. Since $\lambda_i \neq 0$ we conclude that $\chi_n(h) = \chi_i(h)$ for all i by induction. The choice of h above was arbitrary, so we conclude that $\chi_i = \chi_n$ for $i \leq n-1$ which contradicts the assumption that our characters χ_i are pairwise distinct. $\square$

0EM9 **Lemma 9.13.2.** Let L be a field. Let $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in L$ pairwise distinct elements of L. Then there exists an $e \geq 0$ such that $\sum_{i=1,\ldots,n} \alpha_i^e \neq 0$.

Proof. Apply linear independence of characters (Lemma 9.13.1) to the monoid homomorphisms $\mathbf{Z}_{\geq 0} \to L$, $e \mapsto \alpha_i^e$. $\square$

0CKM **Lemma 9.13.3.** Let K/F and L/F be field extensions. Let $\sigma_1, \ldots, \sigma_n : K \to L$ be pairwise distinct morphisms of F-extensions. Then $\sigma_1, \ldots, \sigma_n$ are L-linearly independent: if $\lambda_1, \ldots, \lambda_n \in L$ not all zero, then $\sum \lambda_i \sigma_i(\alpha) \neq 0$ for some $\alpha \in K$.

Proof. Apply Lemma 9.13.1 to the restrictions of σ_i to the groups of units. $\square$

0CKN **Lemma 9.13.4.** Let K/F and L/F be field extensions with K/F finite separable and L algebraically closed. Then the map

$$K \otimes_F L \longrightarrow \prod\nolimits_{\sigma \in \mathrm{Hom}_F(K,L)} L, \quad \alpha \otimes \beta \mapsto (\sigma(\alpha)\beta)_\sigma$$

is an isomorphism of L-algebras.

Proof. Choose a basis $\alpha_1, \ldots, \alpha_n$ of K as a vector space over F. By Lemma 9.12.11 (and a tiny omitted argument) the set $\mathrm{Hom}_F(K, L)$ has n elements, say $\sigma_1, \ldots, \sigma_n$. In particular, the two sides have the same dimension n as vector spaces over L. Thus if the map is not an isomorphism, then it has a kernel. In other words, there would exist $\mu_j \in L$, $j = 1, \ldots, n$ not all zero, with $\sum \alpha_j \otimes \mu_j$ in the kernel. In other words, $\sum \sigma_i(\alpha_j)\mu_j = 0$ for all i. This would mean the $n \times n$ matrix with entries $\sigma_i(\alpha_j)$ is not invertible. Thus we can find $\lambda_1, \ldots, \lambda_n \in L$ not all zero, such that $\sum \lambda_i \sigma_i(\alpha_j) = 0$ for all j. Now any element $\alpha \in K$ can be written as $\alpha = \sum \beta_j \alpha_j$ with $\beta_j \in F$ and we would get

$$\sum \lambda_i \sigma_i(\alpha) = \sum \lambda_i \sigma_i(\sum \beta_j \alpha_j) = \sum \beta_j \sum \lambda_i \sigma_i(\alpha_j) = 0$$

which contradicts Lemma 9.13.3. □

9.14. Purely inseparable extensions

09HD Purely inseparable extensions are the opposite of the separable extensions defined in the previous section. These extensions only show up in positive characteristic.

09HE Definition 9.14.1. Let F be a field of characteristic $p > 0$. Let K/F be an extension.

(1) An element $\alpha \in K$ is purely inseparable over F if there exists a power q of p such that $\alpha^q \in F$.
(2) The extension K/F is said to be purely inseparable if and only if every element of K is purely inseparable over F.

Observe that a purely inseparable extension is necessarily algebraic. Let F be a field of characteristic $p > 0$. An example of a purely inseparable extension is gotten by adjoining the pth root of an element $t \in F$ which does not yet have one. Namely, the lemma below shows that $P = x^p - t$ is irreducible, and hence

$$K = F[x]/(P) = F[t^{1/p}]$$

is a field. And K is purely inseparable over F because every element

$$a_0 + a_1 t^{1/p} + \ldots + a_{p-1} t^{(p-1)/p}, \quad a_i \in F$$

of K has pth power equal to

$$(a_0 + a_1 t^{1/p} + \ldots + a_{p-1} t^{(p-1)/p})^p = a_0^p + a_1^p t + \ldots + a_{p-1}^p t^{p-1} \in F$$

This situation occurs for the field $\mathbf{F}_p(t)$ of rational functions over $\mathbf{F}_p$.

09HF Lemma 9.14.2. Let p be a prime number. Let F be a field of characteristic p. Let $t \in F$ be an element which does not have a pth root in F. Then the polynomial $x^p - t$ is irreducible over F.

Proof. To see this, suppose that we have a factorization $x^p - t = fg$. Taking derivatives we get $f'g + fg' = 0$. Note that neither $f' = 0$ nor $g' = 0$ as the degrees of f and g are smaller than p. Moreover, $\deg(f') < \deg(f)$ and $\deg(g') < \deg(g)$. We conclude that f and g have a factor in common. Thus if $x^p - t$ is reducible, then it is of the form $x^p - t = cf^n$ for some irreducible f, $c \in F^*$, and $n > 1$. Since p is a prime number this implies $n = p$ and f linear, which would imply $x^p - t$ has a root in F. Contradiction. □

We will see that taking pth roots is a very important operation in characteristic p.

09HG Lemma 9.14.3. Let E/k and F/E be purely inseparable extensions of fields. Then F/k is a purely inseparable extension of fields.

Proof. Say the characteristic of k is p. Choose $\alpha \in F$. Then $\alpha^q \in E$ for some p-power q. Whereupon $(\alpha^q)^{q'} \in k$ for some p-power q'. Hence $\alpha^{qq'} \in k$. □

09HH Lemma 9.14.4. Let E/k be a field extension. Then the elements of E purely-inseparable over k form a subextension of E/k.

Proof. Let p be the characteristic of k. Let $\alpha, \beta \in E$ be purely inseparable over k. Say $\alpha^q \in k$ and $\beta^{q'} \in k$ for some p-powers q, q'. If q'' is a p-power, then $(\alpha + \beta)^{q''} = \alpha^{q''} + \beta^{q''}$. Hence if $q'' \geq q, q'$, then we conclude that $\alpha + \beta$ is purely inseparable over k. Similarly for the difference, product and quotient of α and β. □

09HI Lemma 9.14.5. Let E/F be a finite purely inseparable field extension of characteristic $p > 0$. Then there exists a sequence of elements $\alpha_1, \ldots, \alpha_n \in E$ such that we obtain a tower of fields

$$E = F(\alpha_1, \ldots, \alpha_n) \supset F(\alpha_1, \ldots, \alpha_{n-1}) \supset \ldots \supset F(\alpha_1) \supset F$$

such that each intermediate extension is of degree p and comes from adjoining a pth root. Namely, $\alpha_i^p \in F(\alpha_1, \ldots, \alpha_{i-1})$ is an element which does not have a pth root in $F(\alpha_1, \ldots, \alpha_{i-1})$ for $i = 1, \ldots, n$.

Proof. By induction on the degree of E/F. If the degree of the extension is 1 then the result is clear (with $n = 0$). If not, then choose $\alpha \in E$, $\alpha \notin F$. Say $\alpha^{p^r} \in F$ for some $r > 0$. Pick r minimal and replace α by $\alpha^{p^{r-1}}$. Then $\alpha \notin F$, but $\alpha^p \in F$. Then $t = \alpha^p$ is not a pth power in F (because that would imply $\alpha \in F$, see Lemma 9.12.5 or its proof). Thus $F \subset F(\alpha)$ is a subextension of degree p (Lemma 9.14.2). By induction we find $\alpha_1, \ldots, \alpha_n \in E$ generating $E/F(\alpha)$ satisfying the conclusions of the lemma. The sequence $\alpha, \alpha_1, \ldots, \alpha_n$ does the job for the extension E/F. □

030K Lemma 9.14.6. Let E/F be an algebraic field extension. There exists a unique subextension $E/E_{sep}/F$ such that E_{sep}/F is separable and E/E_{sep} is purely inseparable.

Proof. If the characteristic is zero we set $E_{sep} = E$. Assume the characteristic is $p > 0$. Let E_{sep} be the set of elements of E which are separable over F. This is a subextension by Lemma 9.12.13 and of course E_{sep} is separable over F. Given an α in E there exists a p-power q such that α^q is separable over F. Namely, q is that power of p such that the minimal polynomial of α is of the form $P(x^q)$ with P separable algebraic, see Lemma 9.12.1. Hence E/E_{sep} is purely inseparable. Uniqueness is clear. □

030L Definition 9.14.7. Let E/F be an algebraic field extension. Let E_{sep} be the subextension found in Lemma 9.14.6.

(1) The integer $[E_{sep} : F]$ is called the separable degree of the extension. Notation $[E : F]_s$.
(2) The integer $[E : E_{sep}]$ is called the inseparable degree, or the degree of inseparability of the extension. Notation $[E : F]_i$.

Of course in characteristic 0 we have $[E : F] = [E : F]_s$ and $[E : F]_i = 1$. By multiplicativity (Lemma 9.7.7) we have

$$[E : F] = [E : F]_s [E : F]_i$$

even in case some of these degrees are infinite. In fact, the separable degree and the inseparable degree are multiplicative too (see Lemma 9.14.9).

09HJ **Lemma 9.14.8.** Let K/F be a finite extension. Let $\overline{F}$ be an algebraic closure of F. Then $[K : F]_s = |\mathrm{Mor}_F(K, \overline{F})|$.

Proof. We first prove this when K/F is purely inseparable. Namely, we claim that in this case there is a unique map $K \to \overline{F}$. This can be seen by choosing a sequence of elements $\alpha_1, \ldots, \alpha_n \in K$ as in Lemma 9.14.5. The irreducible polynomial of α_i over $F(\alpha_1, \ldots, \alpha_{i-1})$ is $x^p - \alpha_i^p$. Applying Lemma 9.12.9 we see that $|\mathrm{Mor}_F(K, \overline{F})| = 1$. On the other hand, $[K : F]_s = 1$ in this case hence the equality holds.

Let's return to a general finite extension K/F. In this case choose $F \subset K_s \subset K$ as in Lemma 9.14.6. By Lemma 9.12.11 we have $|\mathrm{Mor}_F(K_s, \overline{F})| = [K_s : F] = [K : F]_s$. On the other hand, every field map $\sigma' : K_s \to \overline{F}$ extends to a unique field map $\sigma : K \to \overline{F}$ by the result of the previous paragraph. In other words $|\mathrm{Mor}_F(K, \overline{F})| = |\mathrm{Mor}_F(K_s, \overline{F})|$ and the proof is done. $\square$

09HK **Lemma 9.14.9** (Multiplicativity). Suppose given a tower of algebraic field extensions $K/E/F$. Then

$$[K : F]_s = [K : E]_s[E : F]_s \quad \text{and} \quad [K : F]_i = [K : E]_i[E : F]_i$$

Proof. We first prove this in case K is finite over F. Since we have multiplicativity for the usual degree (by Lemma 9.7.7) it suffices to prove one of the two formulas. By Lemma 9.14.8 we have $[K : F]_s = |\mathrm{Mor}_F(K, \overline{F})|$. By the same lemma, given any $\sigma \in \mathrm{Mor}_F(E, \overline{F})$ the number of extensions of σ to a map $\tau : K \to \overline{F}$ is $[K : E]_s$. Namely, via $E \cong \sigma(E) \subset \overline{F}$ we can view $\overline{F}$ as an algebraic closure of E. Combined with the fact that there are $[E : F]_s = |\mathrm{Mor}_F(E, \overline{F})|$ choices for σ we obtain the result.

We omit the proof if the extensions are infinite. $\square$

9.15. Normal extensions

09HL Let $P \in F[x]$ be a nonconstant polynomial over a field F. We say P *splits completely into linear factors over F* or *splits completely over F* if there exist $c \in F^*$, $n \geq 1$, $\alpha_1, \ldots, \alpha_n \in F$ such that

$$P = c(x - \alpha_1) \ldots (x - \alpha_n)$$

in $F[x]$. Normal extensions are defined as follows.

09HM **Definition 9.15.1.** Let E/F be an algebraic field extension. We say E is *normal* over F if for all $\alpha \in E$ the minimal polynomial P of α over F splits completely into linear factors over E.

As in the case of separable extensions, it takes a bit of work to establish the basic properties of this notion.

09HN **Lemma 9.15.2.** Let $K/E/F$ be a tower of algebraic field extensions. If K is normal over F, then K is normal over E.

Proof. Let $\alpha \in K$. Let P be the minimal polynomial of α over F. Let Q be the minimal polynomial of α over E. Then Q divides P in the polynomial ring $E[x]$, say $P = QR$. Hence, if P splits completely over K, then so does Q. $\square$

09HP **Lemma 9.15.3.** Let F be a field. Let M/F be an algebraic extension. Let $M/E_i/F$, $i \in I$ be subextensions with E_i/F normal. Then $\bigcap E_i$ is normal over F.

Proof. Direct from the definitions. □

0EXK Lemma 9.15.4. Let E/F be a normal algebraic field extension. Then the subextension $E/E_{sep}/F$ of Lemma 9.14.6 is normal.

Proof. If the characteristic is zero, then $E_{sep} = E$, and the result is clear. If the characteristic is $p > 0$, then E_{sep} is the set of elements of E which are separable over F. Then if $\alpha \in E_{sep}$ has minimal polynomial P write $P = c(x-\alpha)(x-\alpha_2)\ldots(x-\alpha_d)$ with $\alpha_2, \ldots, \alpha_d \in E$. Since P is a separable polynomial and since α_i is a root of P, we conclude $\alpha_i \in E_{sep}$ as desired. □

09HQ Lemma 9.15.5. Let E/F be an algebraic extension of fields. Let $\overline{F}$ be an algebraic closure of F. The following are equivalent

(1) E is normal over F, and
(2) for every pair $\sigma, \sigma' \in \mathrm{Mor}_F(E, \overline{F})$ we have $\sigma(E) = \sigma'(E)$.

Proof. Let $\mathcal{P}$ be the set of all minimal polynomials over F of all elements of E. Set

$$T = \{\beta \in \overline{F} \mid P(\beta) = 0 \text{ for some } P \in \mathcal{P}\}$$

It is clear that if E is normal over F, then $\sigma(E) = T$ for all $\sigma \in \mathrm{Mor}_F(E, \overline{F})$. Thus we see that (1) implies (2).

Conversely, assume (2). Pick $\beta \in T$. We can find a corresponding $\alpha \in E$ whose minimal polynomial $P \in \mathcal{P}$ annihilates β. Because $F(\alpha) = F[x]/(P)$ we can find an element $\sigma_0 \in \mathrm{Mor}_F(F(\alpha), \overline{F})$ mapping α to β. By Lemma 9.10.5 we can extend σ_0 to a $\sigma \in \mathrm{Mor}_F(E, \overline{F})$. Whence we see that β is in the common image of all embeddings $\sigma : E \to \overline{F}$. It follows that $\sigma(E) = T$ for any σ. Fix a σ. Now let $P \in \mathcal{P}$. Then we can write

$$P = (x - \beta_1)\ldots(x - \beta_n)$$

for some n and $\beta_i \in \overline{F}$ by Lemma 9.10.2. Observe that $\beta_i \in T$. Thus $\beta_i = \sigma(\alpha_i)$ for some $\alpha_i \in E$. Thus $P = (x-\alpha_1)\ldots(x-\alpha_n)$ splits completely over E. This finishes the proof. □

0BR3 Lemma 9.15.6. Let E/F be an algebraic extension of fields. If E is generated by $\alpha_i \in E$, $i \in I$ over F and if for each i the minimal polynomial of α_i over F splits completely in E, then E/F is normal.

Proof. Let P_i be the minimal polynomial of α_i over F. Let $\alpha_i = \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,d_i}$ be the roots of P_i over E. Given two embeddings $\sigma, \sigma' : E \to \overline{F}$ over F we see that

$$\{\sigma(\alpha_{i,1}), \ldots, \sigma(\alpha_{i,d_i})\} = \{\sigma'(\alpha_{i,1}), \ldots, \sigma'(\alpha_{i,d_i})\}$$

because both sides are equal to the set of roots of P_i in $\overline{F}$. The elements $\alpha_{i,j}$ generate E over F and we find that $\sigma(E) = \sigma'(E)$. Hence E/F is normal by Lemma 9.15.5. □

0BME Lemma 9.15.7. Let $L/M/K$ be a tower of algebraic extensions.

(1) If M/K is normal, then any automorphism τ of L/K induces an automorphism $\tau|_M : M \to M$.
(2) If L/K is normal, then any K-algebra map $\sigma : M \to L$ extends to an automorphism of L.

Proof. Choose an algebraic closure $\overline{L}$ of L (Theorem 9.10.4).

Let τ be as in (1). Then $\tau(M) = M$ as subfields of $\overline{L}$ by Lemma 9.15.5 and hence $\tau|_M : M \to M$ is an automorphism.

Let $\sigma : M \to L$ be as in (2). By Lemma 9.10.5 we can extend σ to a map $\tau : L \to \overline{L}$, i.e., such that

$$\begin{tikzcd}
L \arrow[r, "\tau"] & \overline{L} \\
M \arrow[u] \arrow[ur, "\sigma"] & K \arrow[l] \arrow[u]
\end{tikzcd}$$

is commutative. By Lemma 9.15.5 we see that $\tau(L) = L$. Hence $\tau : L \to L$ is an automorphism which extends σ. □

09HR **Definition 9.15.8.** Let E/F be an extension of fields. Then $\text{Aut}(E/F)$ or $\text{Aut}_F(E)$ denotes the automorphism group of E as an object of the category of F-extensions. Elements of $\text{Aut}(E/F)$ are called *automorphisms of E over F* or *automorphisms of E/F*.

Here is a characterization of normal extensions in terms of automorphisms.

09HS **Lemma 9.15.9.** *Let E/F be a finite extension. We have*

$$|\text{Aut}(E/F)| \leq [E : F]_s$$

with equality if and only if E is normal over F.

Proof. Choose an algebraic closure $\overline{F}$ of F. Recall that $[E : F]_s = |\,\text{Mor}_F(E, \overline{F})|$. Pick an element $\sigma_0 \in \text{Mor}_F(E, \overline{F})$. Then the map

$$\text{Aut}(E/F) \longrightarrow \text{Mor}_F(E, \overline{F}), \quad \tau \longmapsto \sigma_0 \circ \tau$$

is injective. Thus the inequality. If equality holds, then every $\sigma \in \text{Mor}_F(E, \overline{F})$ is gotten by precomposing σ_0 by an automorphism. Hence $\sigma(E) = \sigma_0(E)$. Thus E is normal over F by Lemma 9.15.5.

Conversely, assume that E/F is normal. Then by Lemma 9.15.5 we have $\sigma(E) = \sigma_0(E)$ for all $\sigma \in \text{Mor}_F(E, \overline{F})$. Thus we get an automorphism of E over F by setting $\tau = \sigma_0^{-1} \circ \sigma$. Whence the map displayed above is surjective. □

0BR4 **Lemma 9.15.10.** *Let L/K be an algebraic normal extension of fields. Let E/K be an extension of fields. Then either there is no K-embedding from L to E or there is one $\tau : L \to E$ and every other one is of the form $\tau \circ \sigma$ where $\sigma \in \text{Aut}(L/K)$.*

Proof. Given τ replace L by $\tau(L) \subset E$ and apply Lemma 9.15.7. □

9.16. Splitting fields

09HT The following lemma is a useful tool for constructing normal field extensions.

09HU **Lemma 9.16.1.** *Let F be a field. Let $P \in F[x]$ be a nonconstant polynomial. There exists a smallest field extension E/F such that P splits completely over E. Moreover, the field extension E/F is normal and unique up to (nonunique) isomorphism.*

Proof. Choose an algebraic closure $\overline{F}$. Then we can write $P = c(x-\beta_1)\ldots(x-\beta_n)$ in $\overline{F}[x]$, see Lemma 9.10.2. Note that $c \in F^*$. Set $E = F(\beta_1, \ldots, \beta_n)$. Then it is clear that E is minimal with the requirement that P splits completely over E.

Next, let E' be another minimal field extension of F such that P splits completely over E'. Write $P = c(x-\alpha_1)\ldots(x-\alpha_n)$ with $c \in F$ and $\alpha_i \in E'$. Again it follows from minimality that $E' = F(\alpha_1, \ldots, \alpha_n)$. Moreover, if we pick any $\sigma : E' \to \overline{F}$ (Lemma 9.10.5) then we immediately see that $\sigma(\alpha_i) = \beta_{\tau(i)}$ for some permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Thus $\sigma(E') = E$. This implies that E' is a normal extension of F by Lemma 9.15.5 and that $E \cong E'$ as extensions of F thereby finishing the proof. □

09HV **Definition 9.16.2.** Let F be a field. Let $P \in F[x]$ be a nonconstant polynomial. The field extension E/F constructed in Lemma 9.16.1 is called the *splitting field of P over F*.

09DT **Lemma 9.16.3.** Let E/F be a finite extension of fields. There exists a unique smallest finite extension K/E such that K is normal over F.

Proof. Choose generators $\alpha_1, \ldots, \alpha_n$ of E over F. Let $P_1, \ldots, P_n$ be the minimal polynomials of $\alpha_1, \ldots, \alpha_n$ over F. Set $P = P_1 \ldots P_n$. Observe that $(x-\alpha_1)\ldots(x-\alpha_n)$ divides P, since each $(x-\alpha_i)$ divides P_i. Say $P = (x-\alpha_1)\ldots(x-\alpha_n)Q$. Let K/E be the splitting field of P over E. We claim that K is the splitting field of P over F as well (which implies that K is normal over F). This is clear because K/E is generated by the roots of Q over E and E is generated by the roots of $(x-\alpha_1)\ldots(x-\alpha_n)$ over F, hence K is generated by the roots of P over F.

Uniqueness. Suppose that K'/E is a second smallest extension such that K'/F is normal. Choose an algebraic closure $\overline{F}$ and an embedding $\sigma_0 : E \to \overline{F}$. By Lemma 9.10.5 we can extend σ_0 to $\sigma : K \to \overline{F}$ and $\sigma' : K' \to \overline{F}$. By Lemma 9.15.3 we see that $\sigma(K) \cap \sigma'(K')$ is normal over F. By minimality we conclude that $\sigma(K) = \sigma(K')$. Thus $\sigma \circ (\sigma')^{-1} : K' \to K$ gives an isomorphism of extensions of E. □

0BMF **Definition 9.16.4.** Let E/F be a finite extension of fields. The field extension K/E constructed in Lemma 9.16.3 is called the *normal closure E over F*.

One can construct the normal closure inside any given normal extension.

0BMG **Lemma 9.16.5.** Let L/K be an algebraic normal extension.

(1) If $L/M/K$ is a subextension with M/K finite, then there exists a tower $L/M'/M/K$ with M'/K finite and normal.
(2) If $L/M'/M/K$ is a tower with M/K normal and M'/M finite, then there exists a tower $L/M''/M'/M/K$ with M''/M finite and M''/K normal.

Proof. Proof of (1). Let M' be the smallest subextension of L/K containing M which is normal over K. By Lemma 9.16.3 this is the normal closure of M/K and is finite over K.

Proof of (2). Let $\alpha_1, \ldots, \alpha_n \in M'$ generate M' over M. Let $P_1, \ldots, P_n$ be the minimal polynomials of $\alpha_1, \ldots, \alpha_n$ over K. Let $\alpha_{i,j}$ be the roots of P_i in L. Let $M'' = M(\alpha_{i,j})$. It follows from Lemma 9.15.6 (applied with the set of generators $M \cup \{\alpha_{i,j}\}$) that M'' is normal over K. □

The following lemma can sometimes be used to prove properties of the normal closure.

0EXL Lemma 9.16.6. Let L/K be a finite extension. Let M/L be the normal closure of L over K. Then there is a surjective map

$$L \otimes_K L \otimes_K \ldots \otimes_K L \longrightarrow M$$

of K-algebras where the number of tensors can be taken $[L : K]_s \leq [L : K]$.

Proof. Choose an algebraic closure $\overline{K}$ of K. Set $n = [L : K]_s = |\mathrm{Mor}_K(L, \overline{K})|$ with equality by Lemma 9.14.8. Say $\mathrm{Mor}_K(L, \overline{K}) = \{\sigma_1, \ldots, \sigma_n\}$. Let $M' \subset \overline{K}$ be the K-subalgebra generated by $\sigma_i(L)$, $i = 1, \ldots, n$. It follows from Lemma 9.15.5 that M' is normal over K and that it is the smallest normal subextension of $\overline{K}$ containing $\sigma_1(L)$. By uniqueness of normal closure we have $M \cong M'$. Finally, there is a surjective map

$$L \otimes_K L \otimes_K \ldots \otimes_K L \longrightarrow M', \quad \lambda_1 \otimes \ldots \otimes \lambda_n \longmapsto \sigma_1(\lambda_1) \ldots \sigma_n(\lambda_n)$$

and note that $n \leq [L : K]$ by definition. □

9.17. Roots of unity

09HW Let F be a field. For an integer $n \geq 1$ we set

$$\mu_n(F) = \{\zeta \in F \mid \zeta^n = 1\}$$

This is called the group of nth roots of unity or nth roots of 1. It is an abelian group under multiplication with neutral element given by 1. Observe that in a field the number of roots of a polynomial of degree d is always at most d. Hence we see that $|\mu_n(F)| \leq n$ as it is defined by a polynomial equation of degree n. Of course every element of $\mu_n(F)$ has order dividing n. Moreover, the subgroups

$$\mu_d(F) \subset \mu_n(F), \quad d|n$$

each have at most d elements. This implies that $\mu_n(F)$ is cyclic.

09HX Lemma 9.17.1. Let A be an abelian group of exponent dividing n such that $\{x \in A \mid dx = 0\}$ has cardinality at most d for all $d|n$. Then A is cyclic of order dividing n.

Proof. The conditions imply that $|A| \leq n$, in particular A is finite. The structure of finite abelian groups shows that $A = \mathbf{Z}/e_1\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}/e_r\mathbf{Z}$ for some integers $1 < e_1|e_2|\ldots|e_r$. This would imply that $\{x \in A \mid e_1 x = 0\}$ has cardinality e_1^r. Hence $r = 1$. □

Applying this to the field $\mathbf{F}_p$ we obtain the celebrated result that the group $(\mathbf{Z}/p\mathbf{Z})^*$ is a cyclic group. More about this in the section on finite fields.

One more observation is often useful: If F has characteristic $p > 0$, then $\mu_{p^n}(F) = \{1\}$. This is true because raising to the pth power is an injective map on fields of characteristic p as we have seen in the proof of Lemma 9.12.5. (Of course, it also follows from the statement of that lemma itself.)

9.18. Finite fields

09HY Let F be a finite field. It is clear that F has positive characteristic as we cannot have an injection $\mathbf{Q} \to F$. Say the characteristic of F is p. The extension $\mathbf{F}_p \subset F$ is finite. Hence we see that F has $q = p^f$ elements for some $f \geq 1$.

Let us think about the group of units F^*. This is a finite abelian group, so it has some exponent e. Then $F^* = \mu_e(F)$ and we see from the discussion in Section 9.17 that F^* is a cyclic group of order $q - 1$. (A posteriori it follows that $e = q - 1$ as well.) In particular, if $\alpha \in F^*$ is a generator then it clearly is true that

$$F = \mathbf{F}_p(\alpha)$$

In other words, the extension $F/\mathbf{F}_p$ is generated by a single element. Of course, the same thing is true for any extension of finite fields E/F (because E is already generated by a single element over the prime field).

9.19. Primitive elements

09HZ Let E/F be a finite extension of fields. An element $\alpha \in E$ is called a primitive element of E over F if $E = F(\alpha)$.

030N Lemma 9.19.1 (Primitive element). Let E/F be a finite extension of fields. The following are equivalent

(1) there exists a primitive element for E over F, and
(2) there are finitely many subextensions $E/K/F$.

Moreover, (1) and (2) hold if E/F is separable.

Proof. Let $\alpha \in E$ be a primitive element. Let P be the minimal polynomial of α over F. Let $E \subset M$ be a splitting field for P over E, so that $P(x) = (x - \alpha)(x - \alpha_2) \ldots (x - \alpha_n)$ over M. For ease of notation we set $\alpha_1 = \alpha$. Next, let $E/K/F$ be a subextension. Let Q be the minimal polynomial of α over K. Observe that $\deg(Q) = [E : K]$. Writing $Q = x^d + \sum_{i<d} a_i x^i$ we claim that K is equal to $L = F(a_0, \ldots, a_{d-1})$. Indeed α has degree d over L and $L \subset K$. Hence $[E : L] = [E : K]$ and it follows that $[K : L] = 1$, i.e., $K = L$. Thus it suffices to show there are at most finitely many possibilities for the polynomial Q. This is clear because we have a factorization $P = QR$ in $K[x]$ in particular in $E[x]$. Since we have unique factorization in $E[x]$ there are at most finitely many monic factors of P in $E[x]$.

If F is a finite field (equivalently E is a finite field), then E/F has a primitive element by the discussion in Section 9.18. Next, assume F is infinite and there are at most finitely many proper subfields $E/K/F$. List them, say $K_1, \ldots, K_N$. Then each $K_i \subset E$ is a proper sub F-vector space. As F is infinite we can find a vector $\alpha \in E$ with $\alpha \notin K_i$ for all i (a vector space can never be equal to a finite union of proper subvector spaces; details omitted). Then α is a primitive element for E over F.

Having established the equivalence of (1) and (2) we now turn to the final statement of the lemma. Choose an algebraic closure $\overline{F}$ of F. Enumerate the elements $\sigma_1, \ldots, \sigma_n \in \mathrm{Mor}_F(E, \overline{F})$. Since E/F is separable we have $n = [E : F]$ by Lemma 9.12.11. Note that if $i \neq j$, then

$$V_{ij} = \mathrm{Ker}(\sigma_i - \sigma_j : E \longrightarrow \overline{F})$$

is not equal to E. Hence arguing as in the preceding paragraph we can find $\alpha \in E$ with $\alpha \notin V_{ij}$ for all $i \neq j$. It follows that $|\operatorname{Mor}_F(F(\alpha), \overline{F})| \geq n$. On the other hand $[F(\alpha) : F] \leq [E : F]$. Hence equality by Lemma 9.12.11 and we conclude that $E = F(\alpha)$. □

9.20. Trace and norm

0BIE Let L/K be a finite extension of fields. By Lemma 9.4.1 we can choose an isomorphism $L \cong K^{\oplus n}$ of K-modules. Of course $n = [L : K]$ is the degree of the field extension. Using this isomorphism we get for a K-algebra map

$$L \longrightarrow \operatorname{Mat}(n \times n, K), \quad \alpha \longmapsto \text{matrix of multiplication by } \alpha$$

Thus given $\alpha \in L$ we can take the trace and the determinant of the corresponding matrix. Of course these quantities are independent of the choice of the basis chosen above. More canonically, simply thinking of L as a finite dimensional K-vector space we have $\operatorname{Trace}_K(\alpha : L \to L)$ and the determinant $\det_K(\alpha : L \to L)$.

0BIF Definition 9.20.1. Let L/K be a finite extension of fields. For $\alpha \in L$ we define the trace $\operatorname{Trace}_{L/K}(\alpha) = \operatorname{Trace}_K(\alpha : L \to L)$ and the norm $\operatorname{Norm}_{L/K}(\alpha) = \det_K(\alpha : L \to L)$.

It is clear from the definition that $\operatorname{Trace}_{L/K}$ is K-linear and satisfies $\operatorname{Trace}_{L/K}(\alpha) = [L : K]\alpha$ for $\alpha \in K$. Similarly $\operatorname{Norm}_{L/K}$ is multiplicative and $\operatorname{Norm}_{L/K}(\alpha) = \alpha^{[L:K]}$ for $\alpha \in K$. This is a special case of the more general construction discussed in Exercises, Exercises 111.22.6 and 111.22.7.

0BIG Lemma 9.20.2. Let L/K be a finite extension of fields. Let $\alpha \in L$ and let P be the minimal polynomial of α over K. Then the characteristic polynomial of the K-linear map $\alpha : L \to L$ is equal to P^e with $e \deg(P) = [L : K]$.

Proof. Choose a basis $\beta_1, \ldots, \beta_e$ of L over $K(\alpha)$. Then e satisfies $e \deg(P) = [L : K]$ by Lemmas 9.9.2 and 9.7.7. Then we see that $L = \bigoplus K(\alpha)\beta_i$ is a direct sum decomposition into α-invariant subspaces hence the characteristic polynomial of $\alpha : L \to L$ is equal to the characteristic polynomial of $\alpha : K(\alpha) \to K(\alpha)$ to the power e.

To finish the proof we may assume that $L = K(\alpha)$. In this case by Cayley-Hamilton we see that α is a root of the characteristic polynomial. And since the characteristic polynomial has the same degree as the minimal polynomial, we find that equality holds. □

0BIH Lemma 9.20.3. Let L/K be a finite extension of fields. Let $\alpha \in L$ and let $P = x^d + a_1 x^{d-1} + \ldots + a_d$ be the minimal polynomial of α over K. Then

$$\operatorname{Norm}_{L/K}(\alpha) = (-1)^{[L:K]} a_d^e \quad \text{and} \quad \operatorname{Trace}_{L/K}(\alpha) = -e a_1$$

where $ed = [L : K]$.

Proof. Follows immediately from Lemma 9.20.2 and the definitions. □

0BII Lemma 9.20.4. Let L/K be a finite extension of fields. Let V be a finite dimensional vector space over L. Let $\varphi : V \to V$ be an L-linear map. Then

$$\operatorname{Trace}_K(\varphi : V \to V) = \operatorname{Trace}_{L/K}(\operatorname{Trace}_L(\varphi : V \to V))$$

and

$$\det_K(\varphi : V \to V) = \operatorname{Norm}_{L/K}(\det_L(\varphi : V \to V))$$

Proof. Choose an isomorphism $V = L^{\oplus n}$ so that φ corresponds to an $n \times n$ matrix. In the case of traces, both sides of the formula are additive in φ. Hence we can assume that φ corresponds to the matrix with exactly one nonzero entry in the (i, j) spot. In this case a direct computation shows both sides are equal.

In the case of norms both sides are zero if φ has a nonzero kernel. Hence we may assume φ corresponds to an element of $\mathrm{GL}_n(L)$. Both sides of the formula are multiplicative in φ. Since every element of $\mathrm{GL}_n(L)$ is a product of elementary matrices we may assume that φ either looks like

$$E_{12}(\lambda) = \begin{pmatrix} 1 & \lambda & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \quad \text{or} \quad E_1(a) = \begin{pmatrix} a & 0 & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

(because we may also permute the basis elements if we like). In both cases the formula is easy to verify by direct computation. $\square$

0BIJ **Lemma 9.20.5.** Let $M/L/K$ be a tower of finite extensions of fields. Then

$$\mathrm{Trace}_{M/K} = \mathrm{Trace}_{L/K} \circ \mathrm{Trace}_{M/L} \quad \text{and} \quad \mathrm{Norm}_{M/K} = \mathrm{Norm}_{L/K} \circ \mathrm{Norm}_{M/L}$$

Proof. Think of M as a vector space over L and apply Lemma 9.20.4. $\square$

The trace pairing is defined using the trace.

0BIK **Definition 9.20.6.** Let L/K be a finite extension of fields. The *trace pairing* for L/K is the symmetric K-bilinear form

$$Q_{L/K} : L \times L \longrightarrow K, \quad (\alpha, \beta) \longmapsto \mathrm{Trace}_{L/K}(\alpha\beta)$$

It turns out that a finite extension of fields is separable if and only if the trace pairing is nondegenerate.

0BIL **Lemma 9.20.7.** Let L/K be a finite extension of fields. The following are equivalent:

(1) L/K is separable,
(2) $\mathrm{Trace}_{L/K}$ is not identically zero, and
(3) the trace pairing $Q_{L/K}$ is nondegenerate.

Proof. It is clear that (3) implies (2). If (2) holds, then pick $\gamma \in L$ with $\mathrm{Trace}_{L/K}(\gamma) \neq 0$. Then if $\alpha \in L$ is nonzero, we see that $Q_{L/K}(\alpha, \gamma/\alpha) \neq 0$. Hence $Q_{L/K}$ is nondegenerate. This proves the equivalence of (2) and (3).

Suppose that K has characteristic p and $L = K(\alpha)$ with $\alpha \notin K$ and $\alpha^p \in K$. Then $\mathrm{Trace}_{L/K}(1) = p = 0$. For $i = 1, \ldots, p-1$ we see that $x^p - \alpha^{pi}$ is the minimal polynomial for α^i over K and we find $\mathrm{Trace}_{L/K}(\alpha^i) = 0$ by Lemma 9.20.3. Hence for this kind of purely inseparable degree p extension we see that $\mathrm{Trace}_{L/K}$ is identically zero.

Assume that L/K is not separable. Then there exists a subfield $L/K'/K$ such that L/K' is a purely inseparable degree p extension as in the previous paragraph, see Lemmas 9.14.6 and 9.14.5. Hence by Lemma 9.20.5 we see that $\mathrm{Trace}_{L/K}$ is identically zero.

Assume on the other hand that L/K is separable. By induction on the degree we will show that $\mathrm{Trace}_{L/K}$ is not identically zero. Thus by Lemma 9.20.5 we may assume that L/K is generated by a single element α (use that if the trace is nonzero then it is surjective). We have to show that $\mathrm{Trace}_{L/K}(\alpha^e)$ is nonzero for some $e \geq 0$.

Let $P = x^d + a_1 x^{d-1} + \ldots + a_d$ be the minimal polynomial of α over K. Then P is also the characteristic polynomial of the linear maps $\alpha : L \to L$, see Lemma 9.20.2. Since L/k is separable we see from Lemma 9.12.4 that P has d pairwise distinct roots $\alpha_1, \ldots, \alpha_d$ in an algebraic closure $\overline{K}$ of K. Thus these are the eigenvalues of $\alpha : L \to L$. By linear algebra, the trace of α^e is equal to $\alpha_1^e + \ldots + \alpha_d^e$. Thus we conclude by Lemma 9.13.2. □

Let K be a field and let $Q : V \times V \to K$ be a bilinear form on a finite dimensional vector space over K. Say $\dim_K(V) = n$. Then Q defines a linear map $Q : V \to V^*$, $v \mapsto Q(v, -)$ where $V^* = \text{Hom}_K(V, K)$ is the dual vector space. Hence a linear map

$$\det(Q) : \wedge^n(V) \longrightarrow \wedge^n(V)^*$$

If we pick a basis element $\omega \in \wedge^n(V)$, then we can write $\det(Q)(\omega) = \lambda\omega^*$, where ω^* is the dual basis element in $\wedge^n(V)^*$. If we change our choice of ω into $c\omega$ for some $c \in K^*$, then ω^* changes into $c^{-1}\omega^*$ and therefore λ changes into $c^2\lambda$. Thus the class of λ in $K/(K^*)^2$ is well defined and is called the discriminant of Q. Unwinding the definitions we see that

$$\lambda = \det(Q(v_i, v_j)_{1 \leq i,j \leq n})$$

if $\{v_1, \ldots, v_n\}$ is a basis for V over K. Observe that the discriminant is nonzero if and only if Q is nondegenerate.

0BIM Definition 9.20.8. Let L/K be a finite extension of fields. The discriminant of L/K is the discriminant of the trace pairing $Q_{L/K}$.

By the discussion above and Lemma 9.20.7 we see that the discriminant is nonzero if and only if L/K is separable. For $a \in K$ we often say "the discriminant is a" when it would be more correct to say the discriminant is the class of a in $K/(K^*)^2$.

0BIN Exercise 9.20.9. Let L/K be an extension of degree 2. Show that exactly one of the following happens

1. the discriminant is 0, the characteristic of K is 2, and L/K is purely inseparable obtained by taking a square root of an element of K,
2. the discriminant is 1, the characteristic of K is 2, and L/K is separable of degree 2,
3. the discriminant is not a square, the characteristic of K is not 2, and L is obtained from K by taking the square root of the discriminant.

9.21. Galois theory

09DU Here is the definition.

09I0 Definition 9.21.1. A field extension E/F is called Galois if it is algebraic, separable, and normal.

It turns out that a finite extension is Galois if and only if it has the "correct" number of automorphisms.

09I1 Lemma 9.21.2. Let E/F be a finite extension of fields. Then E is Galois over F if and only if $|\text{Aut}(E/F)| = [E : F]$.

Proof. Assume $|\text{Aut}(E/F)| = [E : F]$. By Lemma 9.15.9 this implies that E/F is separable and normal, hence Galois. Conversely, if E/F is separable then $[E : F] = [E : F]_s$ and if E/F is in addition normal, then Lemma 9.15.9 implies that $|\text{Aut}(E/F)| = [E : F]$. □

Motivated by the lemma above we introduce the Galois group as follows.

09DV **Definition 9.21.3.** If E/F is a Galois extension, then the group $\text{Aut}(E/F)$ is called the *Galois group* and it is denoted $\text{Gal}(E/F)$.

If L/K is an infinite Galois extension, then one should think of the Galois group as a topological group. We will return to this in Section 9.22.

09I2 **Lemma 9.21.4.** Let $K/E/F$ be a tower of algebraic field extensions. If K is Galois over F, then K is Galois over E.

Proof. Combine Lemmas 9.15.2 and 9.12.3. □

0EXM **Lemma 9.21.5.** Let L/K be a finite separable extension of fields. Let M be the normal closure of L over K (Definition 9.16.4). Then M/K is Galois.

Proof. The subextension $M/M_{sep}/K$ of Lemma 9.14.6 is normal by Lemma 9.15.4. Since L/K is separable we have $L \subset M_{sep}$. By minimality $M = M_{sep}$ and the proof is done. □

Let G be a group acting on a field K (by field automorphisms). We will often use the notation

$$K^G = \{x \in K \mid \sigma(x) = x \ \forall \sigma \in G\}$$

and we will call this the *fixed field* for the action of G on K.

09I3 **Lemma 9.21.6.** Let K be a field. Let G be a finite group acting faithfully on K. Then the extension K/K^G is Galois, we have $[K : K^G] = |G|$, and the Galois group of the extension is G.

Proof. Given $\alpha \in K$ consider the orbit $G \cdot \alpha \subset K$ of α under the group action. Consider the polynomial

$$P = \prod\nolimits_{\beta \in G \cdot \alpha} (x - \beta) \in K[x]$$

The key to the whole lemma is that this polynomial is invariant under the action of G and hence has coefficients in K^G. Namely, for $\tau \in G$ we have

$$P^\tau = \prod\nolimits_{\beta \in G \cdot \alpha} (x - \tau(\beta)) = \prod\nolimits_{\beta \in G \cdot \alpha} (x - \beta) = P$$

because the map $\beta \mapsto \tau(\beta)$ is a permutation of the orbit $G \cdot \alpha$. Thus $P \in K^G[x]$. Since also $P(\alpha) = 0$ as α is an element of its orbit we conclude that the extension K/K^G is algebraic. Moreover, the minimal polynomial Q of α over K^G divides the polynomial P just constructed. Hence Q is separable (by Lemma 9.12.4 for example) and we conclude that K/K^G is separable. Thus K/K^G is Galois. To finish the proof it suffices to show that $[K : K^G] = |G|$ since then G will be the Galois group by Lemma 9.21.2.

Pick finitely many elements $\alpha_i \in K$, $i = 1, \ldots, n$ such that $\sigma(\alpha_i) = \alpha_i$ for $i = 1, \ldots, n$ implies σ is the neutral element of G. Set

$$L = K^G(\{\sigma(\alpha_i); 1 \leq i \leq n, \sigma \in G\}) \subset K$$

and observe that the action of G on K induces an action of G on L. We will show that L has degree $|G|$ over K^G. This will finish the proof, since if $L \subset K$ is proper, then we can add an element $\alpha \in K$, $\alpha \notin L$ to our list of elements $\alpha_1, \ldots, \alpha_n$ without increasing L which is absurd. This reduces us to the case that K/K^G is finite which is treated in the next paragraph.

Assume K/K^G is finite. By Lemma 9.19.1 we can find $\alpha \in K$ such that $K = K^G(\alpha)$. By the construction in the first paragraph of this proof we see that α has degree at most $|G|$ over K. However, the degree cannot be less than $|G|$ as G acts faithfully on $K^G(\alpha) = L$ by construction and the inequality of Lemma 9.15.9. □

09DW Theorem 9.21.7 (Fundamental theorem of Galois theory). Let L/K be a finite Galois extension with Galois group G. Then we have $K = L^G$ and the map

$$\{\text{subgroups of } G\} \longrightarrow \{\text{subextensions } L/M/K\}, \quad H \longmapsto L^H$$

is a bijection whose inverse maps M to $\text{Gal}(L/M)$. The normal subgroups H of G correspond exactly to those subextensions M with M/K Galois.

Proof. By Lemma 9.21.4 given a subextension $L/M/K$ the extension L/M is Galois. Of course L/M is also finite (Lemma 9.7.3). Thus $|\text{Gal}(L/M)| = [L : M]$ by Lemma 9.21.2. Conversely, if $H \subset G$ is a finite subgroup, then $[L : L^H] = |H|$ by Lemma 9.21.6. It follows formally from these two observations that we obtain a bijective correspondence as in the theorem.

If $H \subset G$ is normal, then L^H is fixed by the action of G and we obtain a canonical map $G/H \to \text{Aut}(L^H/K)$. This map has to be injective as $\text{Gal}(L/L^H) = H$. Hence $|G/H| = [L^H : K]$ and L^H is Galois by Lemma 9.21.2.

Conversely, assume that $K \subset M \subset L$ with M/K Galois. By Lemma 9.15.7 we see that every element $\tau \in \text{Gal}(L/K)$ induces an element $\tau|_M \in \text{Gal}(M/K)$. This induces a homomorphism of Galois groups $\text{Gal}(L/K) \to \text{Gal}(M/K)$ whose kernel is H. Thus H is a normal subgroup. □

0BMH Lemma 9.21.8. Let $L/M/K$ be a tower of fields. Assume L/K and M/K are finite Galois. Then we obtain a short exact sequence

$$1 \to \text{Gal}(L/M) \to \text{Gal}(L/K) \to \text{Gal}(M/K) \to 1$$

of finite groups.

Proof. Namely, by Lemma 9.15.7 we see that every element $\tau \in \text{Gal}(L/K)$ induces an element $\tau|_M \in \text{Gal}(M/K)$ which gives us the homomorphism on the right. The map on the left identifies the left group with the kernel of the right arrow. The sequence is exact because the sizes of the groups work out correctly by multiplicativity of degrees in towers of finite extensions (Lemma 9.7.7). One can also use Lemma 9.15.7 directly to see that the map on the right is surjective. □

9.22. Infinite Galois theory

0BMI The Galois group comes with a canonical topology.

0BMJ Lemma 9.22.1. Let E/F be a Galois extension. Endow $\text{Gal}(E/F)$ with the coarsest topology such that

$$\text{Gal}(E/F) \times E \longrightarrow E$$

is continuous when E is given the discrete topology. Then

(1) for any topological space X and map $X \to \mathrm{Aut}(E/F)$ such that the action $X \times E \to E$ is continuous the induced map $X \to \mathrm{Gal}(E/F)$ is continuous,
(2) this topology turns $\mathrm{Gal}(E/F)$ into a profinite topological group.

Proof. Throughout this proof we think of E as a discrete topological space. Recall that the compact open topology on the set of self maps $\mathrm{Map}(E, E)$ is the universal topology such that the action $\mathrm{Map}(E, E) \times E \to E$ is continuous. See Topology, Example 5.30.2 for a precise statement. The topology of the lemma on $\mathrm{Gal}(E/F)$ is the induced topology coming from the injective map $\mathrm{Gal}(E/F) \to \mathrm{Map}(E, E)$. Hence the universal property (1) follows from the corresponding universal property of the compact open topology. Since the set of invertible self maps $\mathrm{Aut}(E)$ endowed with the compact open topology forms a topological group, see Topology, Example 5.30.2, and since $\mathrm{Gal}(E/F) = \mathrm{Aut}(E/F) \to \mathrm{Map}(E, E)$ factors through $\mathrm{Aut}(E)$ we obtain a topological group. In other words, we are using the injection

$$\mathrm{Gal}(E/F) \subset \mathrm{Aut}(E)$$

to endow $\mathrm{Gal}(E/F)$ with the induced structure of a topological group (see Topology, Section 5.30) and by construction this is the coarsest structure of a topological group such that the action $\mathrm{Gal}(E/F) \times E \to E$ is continuous.

To show that $\mathrm{Gal}(E/F)$ is profinite we argue as follows (our argument is necessarily nonstandard because we have defined the topology before showing that the Galois group is an inverse limit of finite groups). By Topology, Lemma 5.30.4 it suffices to show that the underlying topological space of $\mathrm{Gal}(E/F)$ is profinite. For any subset $S \subset E$ consider the set

$$G(S) = \{f : S \to E \mid \begin{matrix} f(\alpha) \text{ is a root of the minimal polynomial} \\ \text{of } \alpha \text{ over } F \text{ for all } \alpha \in S \end{matrix}\}$$

Since a polynomial has only a finite number of roots we see that $G(S)$ is finite for all $S \subset E$ finite. If $S \subset S'$ then restriction gives a map $G(S') \to G(S)$. Also, observe that if $\alpha \in S \cap F$ and $f \in G(S)$, then $f(\alpha) = \alpha$ because the minimal polynomial is linear in this case. Consider the profinite topological space

$$G = \lim\nolimits_{S \subset E \text{ finite}} G(S)$$

Consider the canonical map

$$c : \mathrm{Gal}(E/F) \longrightarrow G, \quad \sigma \longmapsto (\sigma|_S : S \to E)_S$$

This is injective and unwinding the definitions the reader sees the topology on $\mathrm{Gal}(E/F)$ as defined above is the induced topology from G. An element $(f_S) \in G$ is in the image of c exactly if (A) $f_S(\alpha) + f_S(\beta) = f_S(\alpha + \beta)$ and (M) $f_S(\alpha) f_S(\beta) = f_S(\alpha\beta)$ whenever this makes sense (i.e., $\alpha, \beta, \alpha + \beta, \alpha\beta \in S$). Namely, this means $\lim f_S : E \to E$ will be an F-algebra map and hence an automorphism by Lemma 9.8.11. The conditions (A) and (M) for a given triple (S, α, β) define a closed subset of G and hence $\mathrm{Gal}(E/F)$ is homeomorphic to a closed subset of a profinite space and therefore profinite itself. □

0BMK Lemma 9.22.2. Let $L/M/K$ be a tower of fields. Assume both L/K and M/K are Galois. Then there is a canonical surjective continuous homomorphism $c : \mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K)$.

Proof. By Lemma 9.15.7 given $\tau : L \to L$ in $\text{Gal}(L/K)$ the restriction $\tau|_M : M \to M$ is an element of $\text{Gal}(M/K)$. This defines the homomorphism c. Continuity follows from the universal property of the topology: the action

$$\text{Gal}(L/K) \times M \longrightarrow M, \quad (\tau, x) \longmapsto \tau(x) = c(\tau)(x)$$

is continuous as $M \subset L$ and the action $\text{Gal}(L/K) \times L \to L$ is continuous. Hence continuity of c by part (1) of Lemma 9.22.1. Lemma 9.15.7 also shows that the map is surjective. □

Here is a more standard way to think about the Galois group of an infinite Galois extension.

0BU2 **Lemma 9.22.3.** Let L/K be a Galois extension with Galois group G. Let Λ be the set of finite Galois subextensions, i.e., $\lambda \in \Lambda$ corresponds to $L/L_\lambda/K$ with L_λ/K finite Galois with Galois group G_λ. Define a partial ordering on Λ by the rule $\lambda \geq \lambda'$ if and only if $L_\lambda \supset L_{\lambda'}$. Then

(1) Λ is a directed partially ordered set,
(2) L_λ is a system of K-extensions over Λ and $L = \text{colim}\, L_\lambda$,
(3) G_λ is an inverse system of finite groups over Λ, the transition maps are surjective, and
$$G = \lim_{\lambda \in \Lambda} G_\lambda$$
as a profinite group, and
(4) each of the projections $G \to G_\lambda$ is continuous and surjective.

Proof. Every subfield of L containing K is separable over K (follows immediately from the definition). Let $S \subset L$ be a finite subset. Then $K(S)/K$ is finite and there exists a tower $L/E/K(S)/K$ such that E/K is finite Galois, see Lemma 9.16.5. Hence $E = L_\lambda$ for some $\lambda \in \Lambda$. This certainly implies the set Λ is not empty. Also, given $\lambda_1, \lambda_2 \in \Lambda$ we can write $L_{\lambda_i} = K(S_i)$ for finite sets $S_1, S_2 \subset L$ (Lemma 9.7.5). Then there exists a $\lambda \in \Lambda$ such that $K(S_1 \cup S_2) \subset L_\lambda$. Hence $\lambda \geq \lambda_1, \lambda_2$ and Λ is directed (Categories, Definition 4.21.4). Finally, since every element in L is contained in L_λ for some $\lambda \in \Lambda$, it follows from the description of filtered colimits in Categories, Section 4.19 that $\text{colim}\, L_\lambda = L$.

If $\lambda \geq \lambda'$ in Λ, then we obtain a canonical surjective map $G_\lambda \to G_{\lambda'}$, $\sigma \mapsto \sigma|_{L_{\lambda'}}$ by Lemma 9.21.8. Thus we get an inverse system of finite groups with surjective transition maps.

Recall that $G = \text{Aut}(L/K)$. By Lemma 9.22.2 the restriction $\sigma|_{L_\lambda}$ of a $\sigma \in G$ to L_λ is an element of G_λ. Moreover, this procedure gives a continuous surjection $G \to G_\lambda$. Since the transition mappings in the inverse system of G_λ are given by restriction also, it is clear that we obtain a canonical continuous map

$$G \longrightarrow \lim_{\lambda \in \Lambda} G_\lambda$$

Continuity by definition of limits in the category of topological groups; recall that these limits commute with the forgetful functor to the categories of sets and topological spaces by Topology, Lemma 5.30.3. On the other hand, since $L = \text{colim}\, L_\lambda$ it is clear that any element of the inverse limit (viewed as a set) defines an automorphism of L. Thus the map is bijective. Since the topology on both sides is profinite, and since a bijective continuous map of profinite spaces is a homeomorphism (Topology, Lemma 5.17.8), the proof is complete. □

0BML **Theorem 9.22.4** (Fundamental theorem of infinite Galois theory). Let L/K be a Galois extension. Let $G = \text{Gal}(L/K)$ be the Galois group viewed as a profinite topological group (Lemma 9.22.1). Then we have $K = L^G$ and the map

$$\{\text{closed subgroups of } G\} \longrightarrow \{\text{subextensions } L/M/K\}, \quad H \longmapsto L^H$$

is a bijection whose inverse maps M to $\text{Gal}(L/M)$. The finite subextensions M correspond exactly to the open subgroups $H \subset G$. The normal closed subgroups H of G correspond exactly to subextensions M Galois over K.

Proof. We will use the result of finite Galois theory (Theorem 9.21.7) without further mention. Let $S \subset L$ be a finite subset. There exists a tower $L/E/K$ such that $K(S) \subset E$ and such that E/K is finite Galois, see Lemma 9.16.5. In other words, we see that L/K is the union of its finite Galois subextensions. For such an E, by Lemma 9.22.2 the map $\text{Gal}(L/K) \to \text{Gal}(E/K)$ is surjective and continuous, i.e., the kernel is open because the topology on $\text{Gal}(E/K)$ is discrete. In particular we see that no element of $L \setminus K$ is fixed by $\text{Gal}(L/K)$ as $E^{\text{Gal}(E/K)} = K$. This proves that $L^G = K$.

By Lemma 9.21.4 given a subextension $L/M/K$ the extension L/M is Galois. It is immediate from the definition of the topology on G that the subgroup $\text{Gal}(L/M)$ is closed. By the above applied to L/M we see that $L^{\text{Gal}(L/M)} = M$.

Conversely, let $H \subset G$ be a closed subgroup. We claim that $H = \text{Gal}(L/L^H)$. The inclusion $H \subset \text{Gal}(L/L^H)$ is clear. Suppose that $g \in \text{Gal}(L/L^H)$. Let $S \subset L$ be a finite subset. We will show that the open neighbourhood $U_S(g) = \{g' \in G \mid g'(s) = g(s)\}$ of g meets H. This implies that $g \in H$ because H is closed. Let $L/E/K$ be a finite Galois subextension containing $K(S)$ as in the first paragraph of the proof and consider the homomorphism $c : \text{Gal}(L/K) \to \text{Gal}(E/K)$. Then $L^H \cap E = E^{c(H)}$. Since g fixes L^H it fixes $E^{c(H)}$ and hence $c(g) \in c(H)$ by finite Galois theory. Pick $h \in H$ with $c(h) = c(g)$. Then $h \in U_S(g)$ as desired.

At this point we have established the correspondence between closed subgroups and subextensions.

Assume $H \subset G$ is open. Arguing as above we find that H containes $\text{Gal}(L/E)$ for some large enough finite Galois subextension E and we find that L^H is contained in E whence finite over K. Conversely, if M is a finite subextension, then M is generated by a finite subset S and the corresponding subgroup is the open subset $U_S(e)$ where $e \in G$ is the neutral element.

Assume that $K \subset M \subset L$ with M/K Galois. By Lemma 9.22.2 there is a surjective continuous homomorphism of Galois groups $\text{Gal}(L/K) \to \text{Gal}(M/K)$ whose kernel is $\text{Gal}(L/M)$. Thus $\text{Gal}(L/M)$ is a normal closed subgroup.

Finally, assume $N \subset G$ is normal and closed. For any $L/E/K$ as in the first paragraph of the proof, the image $c(N) \subset \text{Gal}(E/K)$ is a normal subgroup. Hence $L^N = \bigcup E^{c(N)}$ is a union of Galois extensions of K (by finite Galois theory) whence Galois over K. □

0BMM **Lemma 9.22.5.** Let $L/M/K$ be a tower of fields. Assume L/K and M/K are Galois. Then we obtain a short exact sequence

$$1 \to \text{Gal}(L/M) \to \text{Gal}(L/K) \to \text{Gal}(M/K) \to 1$$

of profinite topological groups.

Proof. This is a reformulation of Lemma 9.22.2. □

9.23. The complex numbers

09I4 The fundamental theorem of algebra states that the field of complex numbers is an algebraically closed field. In this section we discuss this briefly.

The first remark we'd like to make is that you need to use a little bit of input from calculus in order to prove this. We will use the intuitively clear fact that every odd degree polynomial over the reals has a real root. Namely, let $P(x) = a_{2k+1}x^{2k+1} + \ldots + a_0 \in \mathbf{R}[x]$ for some $k \geq 0$ and $a_{2k+1} \neq 0$. We may and do assume $a_{2k+1} > 0$. Then for $x \in \mathbf{R}$ very large (positive) we see that $P(x) > 0$ as the term $a_{2k+1}x^{2k+1}$ dominates all the other terms. Similarly, if $x \ll 0$, then $P(x) < 0$ by the same reason (and this is where we use that the degree is odd). Hence by the intermediate value theorem there is an $x \in \mathbf{R}$ with $P(x) = 0$.

A conclusion we can draw from the above is that $\mathbf{R}$ has no nontrivial odd degree field extensions, as elements of such extensions would have odd degree minimal polynomials.

Next, let $K/\mathbf{R}$ be a finite Galois extension with Galois group G. Let $P \subset G$ be a 2-sylow subgroup. Then $K^P/\mathbf{R}$ is an odd degree extension, hence by the above $K^P = \mathbf{R}$, which in turn implies $G = P$. (All of these arguments rely on Galois theory of course.) Thus G is a 2-group. If G is nontrivial, then we see that $\mathbf{C} \subset K$ as $\mathbf{C}$ is (up to isomorphism) the only degree 2 extension of $\mathbf{R}$. If G has more than 2 elements we would obtain a quadratic extension of $\mathbf{C}$. This is absurd as every complex number has a square root.

The conclusion: $\mathbf{C}$ is algebraically closed. Namely, if not then we'd get a nontrivial finite extension $K/\mathbf{C}$ which we could assume normal (hence Galois) over $\mathbf{R}$ by Lemma 9.16.3. But we've seen above that then $K = \mathbf{C}$.

09I5 Lemma 9.23.1 (Fundamental theorem of algebra). The field $\mathbf{C}$ is algebraically closed.

Proof. See discussion above. □

9.24. Kummer extensions

09I6 Let K be a field. Let $n \geq 2$ be an integer such that K contains a primitive nth root of 1. Let $a \in K$. Let L be an extension of K obtained by adjoining a root b of the equation $x^n = a$. Then L/K is Galois. If $G = \text{Gal}(L/K)$ is the Galois group, then the map

$$G \longrightarrow \mu_n(K), \quad \sigma \longmapsto \sigma(b)/b$$

is an injective homomorphism of groups. In particular, G is cyclic of order dividing n as a subgroup of the cyclic group $\mu_n(K)$. Kummer theory gives a converse.

09DX Lemma 9.24.1 (Kummer extensions). Let L/K be a Galois extension of fields whose Galois group is $\mathbf{Z}/n\mathbf{Z}$. Assume moreover that the characteristic of K is prime to n and that K contains a primitive nth root of 1. Then $L = K[z]$ with $z^n \in K$.

Proof. Let $\zeta \in K$ be a primitive nth root of 1. Let σ be a generator of $\text{Gal}(L/K)$. Consider $\sigma : L \to L$ as a K-linear operator. Note that $\sigma^n - 1 = 0$ as a linear operator. Applying linear independence of characters (Lemma 9.13.1), we see that there cannot be a polynomial over K of degree $< n$ annihilating σ. Hence the minimal polynomial of σ as a linear operator is $x^n - 1$. Since ζ is a root of $x^n - 1$ by linear algebra there is a $0 \neq z \in L$ such that $\sigma(z) = \zeta z$. This z satisfies $z^n \in K$ because $\sigma(z^n) = (\zeta z)^n = z^n$. Moreover, we see that $z, \sigma(z), \ldots, \sigma^{n-1}(z) = z, \zeta z, \ldots \zeta^{n-1} z$ are pairwise distinct which guarantees that z generates L over K. Hence $L = K[z]$ as required. □

0EXN Lemma 9.24.2. Let K be a field with algebraic closure $\overline{K}$. Let p be a prime different from the characteristic of K. Let $\zeta \in \overline{K}$ be a primitive pth root of 1. Then $K(\zeta)/K$ is a Galois extension of degree dividing $p - 1$.

Proof. The polynomial $x^p - 1$ splits completely over $K(\zeta)$ as its roots are $1, \zeta, \zeta^2, \ldots, \zeta^{p-1}$. Hence $K(\zeta)/K$ is a splitting field and hence normal. The extension is separable as $x^p - 1$ is a separable polynomial. Thus the extension is Galois. Any automorphism of $K(\zeta)$ over K sends ζ to ζ^i for some $1 \leq i \leq p - 1$. Thus the Galois group is a subgroup of $(\mathbf{Z}/p\mathbf{Z})^*$. □

0EXP Lemma 9.24.3. Let K be a field. Let L/K be a finite extension of degree e which is generated by an element α with $a = \alpha^e \in K$. Then any sub extension $L/L'/K$ is generated by α^d for some $d|e$.

Proof. Observe that for $d|e$ the subfield $K(\alpha^d)$ has $[K(\alpha^d) : K] = e/d$ and $[L : K(\alpha^d)] = d$ and that both extensions $K(\alpha^d)/K$ and $L/K(\alpha^d)$ are extensions as in the lemma.

We will use induction on the pair of integers $([L : L'], [L' : K])$ ordered lexicographically. Let p be a prime number dividing e and set $d = e/p$. If $K(\alpha^d)$ is contained in L', then we win by induction, because then it suffices to prove the lemma for $L/L'/K(\alpha^d)$. If not, then $[L'(\alpha^d) : L'] = p$ and by induction hypothesis we have $L'(\alpha^d) = K(\alpha^i)$ for some $i|d$. If $i \neq 1$ we are done by induction. Thus we may assume that $[L : L'] = p$.

If e is not a power of p, then we can do this trick again with a second prime number and we win. Thus we may assume e is a power of p.

If the characteristic of K is p and e is a pth power, then L/K is purely inseparable. Hence L/L' is purely inseparable of degree p and hence $\alpha^p \in L'$. Thus $L' = K(\alpha^p)$ and this case is done.

The final case is where e is a power of p, the characteristic of K is not p, L/L' is a degree p extension, and $L = L'(\alpha^{e/p})$. Claim: this can only happen if $e = p$ and $L' = K$. The claim finishes the proof.

First, we prove the claim when K contains a primitive pth root of unity ζ. In this case the degree p extension $K(\alpha^{e/p})/K$ is Galois with Galois group generated by the automorphism $\alpha^{e/p} \mapsto \zeta\alpha^{e/p}$. On the other hand, since L is generated by $\alpha^{e/p}$ and L' we see that the map

$$K(\alpha^{e/p}) \otimes_K L' \longrightarrow L$$

is an isomorphism of K-algebras (look at dimensions). Thus L has an automorphism σ of order p over K sending $\alpha^{e/p}$ to $\zeta\alpha^{e/p}$. Then $\sigma(\alpha) = \zeta'\alpha$ for some eth root of unity ζ' (as α^e is in K). Then on the one hand $(\zeta')^{e/p} = \zeta$ and on the other hand ζ' has to be a pth root of 1 as σ has order p. Thus $e/p = 1$ and the claim has been shown.

Finally, suppose that K does not contain a primitive pth root of 1. Choose a primitive pth root ζ in some algebraic closure $\overline{L}$ of L. Consider the diagram

$$\begin{tikzcd} K(\zeta) \arrow[r] & L(\zeta) \\ K \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

By Lemma 9.24.2 the vertical extensions have degree prime to p. Hence $[L(\zeta) : K(\zeta)]$ is divisible by e. On the other hand, $L(\zeta)$ is generated by α over $K(\zeta)$ and hence $[L(\zeta) : K(\zeta)] \leq e$. Thus $[L(\zeta) : K(\zeta)] = e$. Similarly we have $[K(\alpha^{e/p}, \zeta) : K(\zeta)] = p$ and $[L(\zeta) : L'(\zeta)] = p$. Thus the fields $K(\zeta), L'(\zeta), L(\zeta)$ and the element α fall into the case discussed in the previous paragraph we conclude $e = p$ as desired. $\square$

9.25. Artin-Schreier extensions

09I7 Let K be a field of characteristic $p > 0$. Let $a \in K$. Let L be an extension of K obtained by adjoining a root b of the equation $x^p - x = a$. Then L/K is Galois. If $G = \text{Gal}(L/K)$ is the Galois group, then the map

$$G \longrightarrow \mathbf{Z}/p\mathbf{Z}, \quad \sigma \longmapsto \sigma(b) - b$$

is an injective homomorphism of groups. In particular, G is cyclic of order dividing p as a subgroup of $\mathbf{Z}/p\mathbf{Z}$. The theory of Artin-Schreier extensions gives a converse.

09DY **Lemma 9.25.1** (Artin-Schreier extensions). *Let L/K be a Galois extension of fields of characteristic $p > 0$ with Galois group $\mathbf{Z}/p\mathbf{Z}$. Then $L = K[z]$ with $z^p - z \in K$.*

Proof. Let σ be a generator of $\text{Gal}(L/K)$. Consider $\sigma : L \to L$ as a K-linear operator. Observe that $\sigma^p - 1 = 0$ as a linear operator. Applying linear independence of characters (Lemma 9.13.1), there cannot be a polynomial of degree $< p$ annihilating σ. We conclude that the minimal polynomial of σ is $x^p - 1 = (x - 1)^p$. This implies that there exists $w \in L$ such that $(\sigma - 1)^{p-1}(w) = y$ is nonzero. Then $\sigma(y) = y$, i.e., $y \in K$. Thus $z = y^{-1}(\sigma - 1)^{p-2}(w)$ satisfies $\sigma(z) = z + 1$. Since $z \notin K$ we have $L = K[z]$. Moreover since $\sigma(z^p - z) = (z + 1)^p - (z + 1) = z^p - z$ we see that $z^p - z \in K$ and the proof is complete. $\square$

9.26. Transcendence

030D We recall the standard definitions.

030E **Definition 9.26.1.** Let K/k be a field extension.

(1) A collection of elements $\{x_i\}_{i \in I}$ of K is called *algebraically independent* over k if the map
$$k[X_i; i \in I] \longrightarrow K$$
which maps X_i to x_i is injective.
(2) The field of fractions of a polynomial ring $k[x_i; i \in I]$ is denoted $k(x_i; i \in I)$.
(3) A *purely transcendental extension* of k is any field extension K/k isomorphic to the field of fractions of a polynomial ring over k.
(4) A *transcendence basis* of K/k is a collection of elements $\{x_i\}_{i \in I}$ which are algebraically independent over k and such that the extension $K/k(x_i; i \in I)$ is algebraic.

09I8 Example 9.26.2. The field $\mathbf{Q}(\pi)$ is purely transcendental because π isn't the root of a nonzero polynomial with rational coefficients. In particular, $\mathbf{Q}(\pi) \cong \mathbf{Q}(x)$.

030F Lemma 9.26.3. Let E/F be a field extension. A transcendence basis of E over F exists. Any two transcendence bases have the same cardinality.

Proof. Let A be an algebraically independent subset of E. Let G be a subset of E containing A that generates E/F. We claim we can find a transcendence basis B such that $A \subset B \subset G$. To prove this, consider the collection $\mathcal{B}$ of algebraically independent subsets whose members are subsets of G that contain A. Define a partial ordering on $\mathcal{B}$ using inclusion. Then $\mathcal{B}$ contains at least one element A. The union of the elements of a totally ordered subset T of $\mathcal{B}$ is an algebraically independent subset of E over F since any algebraic dependence relation would have occurred in one of the elements of T (since polynomials only involve finitely many variables). The union also contains A and is contained in G. By Zorn's lemma, there is a maximal element $B \in \mathcal{B}$. Now we claim E is algebraic over $F(B)$. This is because if it wasn't then there would be an element $f \in G$ transcendental over $F(B)$ since $F(G) = E$. Then $B \cup \{f\}$ would be algebraically independent contradicting the maximality of B. Thus B is our transcendence basis.

Let B and B' be two transcendence bases. Without loss of generality, we can assume that $|B'| \leq |B|$. Now we divide the proof into two cases: the first case is that B is an infinite set. Then for each $\alpha \in B'$, there is a finite set $B_\alpha \subset B$ such that α is algebraic over $F(B_\alpha)$ since any algebraic dependence relation only uses finitely many indeterminates. Then we define $B^* = \bigcup_{\alpha \in B'} B_\alpha$. By construction, $B^* \subset B$, but we claim that in fact the two sets are equal. To see this, suppose that they are not equal, say there is an element $\beta \in B \setminus B^*$. We know β is algebraic over $F(B')$ which is algebraic over $F(B^*)$. Therefore β is algebraic over $F(B^*)$, a contradiction. So $|B| \leq |\bigcup_{\alpha \in B'} B_\alpha|$. Now if B' is finite, then so is B so we can assume B' is infinite; this means

$$|B| \leq |\bigcup\nolimits_{\alpha \in B'} B_\alpha| = |B'|$$

because each B_α is finite and B' is infinite. Therefore in the infinite case, $|B| = |B'|$.

Now we need to look at the case where B is finite. In this case, B' is also finite, so suppose $B = \{\alpha_1, \ldots, \alpha_n\}$ and $B' = \{\beta_1, \ldots, \beta_m\}$ with $m \leq n$. We perform induction on m: if $m = 0$ then E/F is algebraic so $B = \emptyset$ so $n = 0$. If $m > 0$, there is an irreducible polynomial $f \in F[x, y_1, \ldots, y_n]$ such that $f(\beta_1, \alpha_1, \ldots, \alpha_n) = 0$ and such that x occurs in f. Since β_1 is not algebraic over F, f must involve some y_i so without loss of generality, assume f uses y_1. Let $B^* = \{\beta_1, \alpha_2, \ldots, \alpha_n\}$. We claim that B^* is a basis for E/F. To prove this claim, we see that we have a tower of algebraic extensions

$$E/F(B^*, \alpha_1)/F(B^*)$$

since α_1 is algebraic over $F(B^*)$. Now we claim that B^* (counting multiplicity of elements) is algebraically independent over F because if it weren't, then there would be an irreducible $g \in F[x, y_2, \ldots, y_n]$ such that $g(\beta_1, \alpha_2, \ldots, \alpha_n) = 0$ which must involve x making β_1 algebraic over $F(\alpha_2, \ldots, \alpha_n)$ which would make α_1 algebraic over $F(\alpha_2, \ldots, \alpha_n)$ which is impossible. So this means that $\{\alpha_2, \ldots, \alpha_n\}$ and $\{\beta_2, \ldots, \beta_m\}$ are bases for E over $F(\beta_1)$ which means by induction, $m = n$. $\square$

030G Definition 9.26.4. Let K/k be a field extension. The transcendence degree of K over k is the cardinality of a transcendence basis of K over k. It is denoted $\text{trdeg}_k(K)$.

030H Lemma 9.26.5. Let $L/K/k$ be field extensions. Then

$$\text{trdeg}_k(L) = \text{trdeg}_K(L) + \text{trdeg}_k(K).$$

Proof. Choose a transcendence basis $A \subset K$ of K over k. Choose a transcendence basis $B \subset L$ of L over K. Then it is straightforward to see that $A \cup B$ is a transcendence basis of L over k. $\square$

09I9 Example 9.26.6. Consider the field extension $\mathbf{Q}(e, \pi)$ formed by adjoining the numbers e and π. This field extension has transcendence degree at least 1 since both e and π are transcendental over the rationals. However, this field extension might have transcendence degree 2 if e and π are algebraically independent. Whether or not this is true is unknown and whence the problem of determining $\text{trdeg}(\mathbf{Q}(e, \pi))$ is open.

09IA Example 9.26.7. Let F be a field and $E = F(t)$. Then $\{t\}$ is a transcendence basis since $E = F(t)$. However, $\{t^2\}$ is also a transcendence basis since $F(t)/F(t^2)$ is algebraic. This illustrates that while we can always decompose an extension E/F into an algebraic extension E/F' and a purely transcendental extension F'/F, this decomposition is not unique and depends on choice of transcendence basis.

09IB Example 9.26.8. Let X be a compact Riemann surface. Then the function field $\mathbf{C}(X)$ (see Example 9.3.6) has transcendence degree one over $\mathbf{C}$. In fact, any finitely generated extension of $\mathbf{C}$ of transcendence degree one arises from a Riemann surface. There is even an equivalence of categories between the category of compact Riemann surfaces and (non-constant) holomorphic maps and the opposite of the category of finitely generated extensions of $\mathbf{C}$ of transcendence degree 1 and morphisms of $\mathbf{C}$-algebras. See [For91].

There is an algebraic version of the above statement as well. Given an (irreducible) algebraic curve in projective space over an algebraically closed field k (e.g. the complex numbers), one can consider its "field of rational functions": basically, functions that look like quotients of polynomials, where the denominator does not identically vanish on the curve. There is a similar anti-equivalence of categories (Algebraic Curves, Theorem 53.2.6) between smooth projective curves and non-constant morphisms of curves and finitely generated extensions of k of transcendence degree one. See [Har77].

037I Definition 9.26.9. Let K/k be a field extension.

(1) The algebraic closure of k in K is the subfield k' of K consisting of elements of K which are algebraic over k.
(2) We say k is algebraically closed in K if every element of K which is algebraic over k is contained in k.

0G1M Lemma 9.26.10. Let k'/k be a finite extension of fields. Let $k'(x_1, \ldots, x_r)/k(x_1, \ldots, x_r)$ be the induced extension of purely transcendental extensions. Then $[k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] = [k' : k] < \infty$.

Proof. By multiplicativity of degrees of extensions (Lemma 9.7.7) it suffices to prove this when k' is generated by a single element $\alpha \in k'$ over k. Let $f \in k[T]$ be the minimal polynomial of α over k. Then $k'(x_1, \ldots, x_r)$ is generated by $\alpha, x_1, \ldots, x_r$

over k and hence $k'(x_1, \ldots, x_r)$ is generated by α over $k(x_1, \ldots, x_r)$. Thus it suffices to show that f is still irreducible as an element of $k(x_1, \ldots, x_r)[T]$. We only sketch the proof. It is clear that f is irreducible as an element of $k[x_1, \ldots, x_r, T]$ for example because f is monic as a polynomial in T and any putative factorization in $k[x_1, \ldots, x_r, T]$ would lead to a factorization in $k[T]$ by setting x_i equal to 0. By Gauss' lemma we conclude. $\square$

037J Lemma 9.26.11. Let K/k be a finitely generated field extension. The algebraic closure of k in K is finite over k.

Proof. Let $x_1, \ldots, x_r \in K$ be a transcendence basis for K over k. Then $n = [K : k(x_1, \ldots, x_r)] < \infty$. Suppose that $k \subset k' \subset K$ with k'/k finite. In this case $[k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] = [k' : k] < \infty$, see Lemma 9.26.10. Hence

$$[k' : k] = [k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] \leq [K : k(x_1, \ldots, x_r)] = n.$$

In other words, the degrees of finite subextensions are bounded and the lemma follows. $\square$

9.27. Linearly disjoint extensions

09IC Let k be a field, K and L field extensions of k. Suppose also that K and L are embedded in some larger field Ω.

09ID Definition 9.27.1. Consider a diagram

09IE (9.27.1.1)
$$\begin{tikzcd} L \arrow[r] & \Omega \\ k \arrow[u] \arrow[r] & K \arrow[u] \end{tikzcd}$$

of field extensions. The compositum of K and L in Ω written KL is the smallest subfield of Ω containing both L and K.

It is clear that KL is generated by the set $K \cup L$ over k, generated by the set K over L, and generated by the set L over K.

Warning: The (isomorphism class of the) composition depends on the choice of the embeddings of K and L into Ω. For example consider the number fields $K = \mathbf{Q}(2^{1/8}) \subset \mathbf{R}$ and $L = \mathbf{Q}(2^{1/12}) \subset \mathbf{R}$. The compositum inside $\mathbf{R}$ is the field $\mathbf{Q}(2^{1/24})$ of degree 24 over $\mathbf{Q}$. However, if we embed $K = \mathbf{Q}[x]/(x^8 - 2)$ into $\mathbf{C}$ by mapping x to $2^{1/8}e^{2\pi i/8}$, then the compositum $\mathbf{Q}(2^{1/12}, 2^{1/8}e^{2\pi i/8})$ contains $i = e^{2\pi i/4}$ and has degree 48 over $\mathbf{Q}$ (we omit showing the degree is 48, but the existence of i certainly proves the two composita are not isomorphic).

09IF Definition 9.27.2. Consider a diagram of fields as in (9.27.1.1). We say that K and L are linearly disjoint over k in Ω if the map

$$K \otimes_k L \longrightarrow KL, \quad \sum x_i \otimes y_i \longmapsto \sum x_i y_i$$

is injective.

The following lemma does not seem to fit anywhere else.

030M Lemma 9.27.3. Let E/F be a normal algebraic field extension. There exist subextensions $E/E_{sep}/F$ and $E/E_{insep}/F$ such that

(1) $F \subset E_{sep}$ is Galois and $E_{sep} \subset E$ is purely inseparable,

(2) $F \subset E_{insep}$ is purely inseparable and $E_{insep} \subset E$ is Galois,
(3) $E = E_{sep} \otimes_F E_{insep}$.

Proof. We found the subfield E_{sep} in Lemma 9.14.6. We set $E_{insep} = E^{\mathrm{Aut}(E/F)}$. Details omitted. □

9.28. Review

037H In this section we give a quick review of what has transpired above.

Let K/k be a field extension. Let $\alpha \in K$. Then we have the following possibilities:

(1) The element α is transcendental over k.
(2) The element α is algebraic over k. Denote $P(T) \in k[T]$ its minimal polynomial. This is a monic polynomial $P(T) = T^d + a_1 T^{d-1} + \ldots + a_d$ with coefficients in k. It is irreducible and $P(\alpha) = 0$. These properties uniquely determine P, and the integer d is called the degree of α over k. There are two subcases:
    (a) The polynomial $\mathrm{d}P/\mathrm{d}T$ is not identically zero. This is equivalent to the condition that $P(T) = \prod_{i=1,\ldots,d}(T - \alpha_i)$ for pairwise distinct elements $\alpha_1, \ldots, \alpha_d$ in the algebraic closure of k. In this case we say that α is separable over k.
    (b) The $\mathrm{d}P/\mathrm{d}T$ is identically zero. In this case the characteristic p of k is > 0, and P is actually a polynomial in T^p. Clearly there exists a largest power $q = p^e$ such that P is a polynomial in T^q. Then the element α^q is separable over k.

030J Definition 9.28.1. Algebraic field extensions.

(1) A field extension K/k is called algebraic if every element of K is algebraic over k.
(2) An algebraic extension k'/k is called separable if every $\alpha \in k'$ is separable over k.
(3) An algebraic extension k'/k is called purely inseparable if the characteristic of k is $p > 0$ and for every element $\alpha \in k'$ there exists a power q of p such that $\alpha^q \in k$.
(4) An algebraic extension k'/k is called normal if for every $\alpha \in k'$ the minimal polynomial $P(T) \in k[T]$ of α over k splits completely into linear factors over k'.
(5) An algebraic extension k'/k is called Galois if it is separable and normal.

The following lemma does not seem to fit anywhere else.

031V Lemma 9.28.2. Let K be a field of characteristic $p > 0$. Let L/K be a separable algebraic extension. Let $\alpha \in L$.

(1) If the coefficients of the minimal polynomial of α over K are pth powers in K then α is a pth power in L.
(2) More generally, if $P \in K[T]$ is a polynomial such that (a) α is a root of P, (b) P has pairwise distinct roots in an algebraic closure, and (c) all coefficients of P are pth powers, then α is a pth power in L.

Proof. It follows from the definitions that (2) implies (1). Assume P is as in (2). Write $P(T) = \sum_{i=0}^{d} a_i T^{d-i}$ and $a_i = b_i^p$. The polynomial $Q(T) = \sum_{i=0}^{d} b_i T^{d-i}$ has distinct roots in an algebraic closure as well, because the roots of Q are the

pth roots of the roots of P. If α is not a pth power, then $T^p - \alpha$ is an irreducible polynomial over L (Lemma 9.14.2). Moreover Q and $T^p - \alpha$ have a root in common in an algebraic closure $\overline{L}$. Thus Q and $T^p - \alpha$ are not relatively prime, which implies $T^p - \alpha | Q$ in $L[T]$. This contradicts the fact that the roots of Q are pairwise distinct. $\square$

9.29. Other chapters

CHAPTER 10

Commutative Algebra

00AO 10.1. Introduction

00AP Basic commutative algebra will be explained in this document. A reference is [Mat70a].

10.2. Conventions

00AQ A ring is commutative with 1. The zero ring is a ring. In fact it is the only ring that does not have a prime ideal. The Kronecker symbol δ_{ij} will be used. If $R \to S$ is a ring map and $\mathfrak q$ a prime of S, then we use the notation "$\mathfrak p = R \cap \mathfrak q$" to indicate the prime which is the inverse image of $\mathfrak q$ under $R \to S$ even if R is not a subring of S and even if $R \to S$ is not injective.

10.3. Basic notions

00AR The following is a list of basic notions in commutative algebra. Some of these notions are discussed in more detail in the text that follows and some are defined in the list, but others are considered basic and will not be defined. If you are not familiar with most of the italicized concepts, then we suggest looking at an introductory text on algebra before continuing.

00AS (1) R is a ring,
00AT (2) $x \in R$ is nilpotent,
00AU (3) $x \in R$ is a zerodivisor,
00AV (4) $x \in R$ is a unit,
00AW (5) $e \in R$ is an idempotent,
00AX (6) an idempotent $e \in R$ is called trivial if $e = 1$ or $e = 0$,
00AY (7) $\varphi : R_1 \to R_2$ is a ring homomorphism,
00AZ (8) $\varphi : R_1 \to R_2$ is of finite presentation, or R_2 is a finitely presented R_1-algebra, see Definition 10.6.1,
00B0 (9) $\varphi : R_1 \to R_2$ is of finite type, or R_2 is a finite type R_1-algebra, see Definition 10.6.1,
00B1 (10) $\varphi : R_1 \to R_2$ is finite, or R_2 is a finite R_1-algebra,
00B2 (11) R is a (integral) domain,
00B3 (12) R is reduced,
00B4 (13) R is Noetherian,
00B5 (14) R is a principal ideal domain or a PID,
00B6 (15) R is a Euclidean domain,
00B7 (16) R is a unique factorization domain or a UFD,
00B8 (17) R is a discrete valuation ring or a dvr,

00B9 (18) K is a field,
00BA (19) L/K is a field extension,
00BB (20) L/K is an algebraic field extension,
00BC (21) $\{t_i\}_{i \in I}$ is a transcendence basis for L over K,
00BD (22) the transcendence degree $\text{trdeg}(L/K)$ of L over K,
00BE (23) the field k is algebraically closed,
00BF (24) if L/K is algebraic, and Ω/K an extension with Ω algebraically closed, then there exists a ring map $L \to \Omega$ extending the map on K,
00BG (25) $I \subset R$ is an ideal,
00BH (26) $I \subset R$ is radical,
00BI (27) if I is an ideal then we have its radical $\sqrt{I}$,
00BJ (28) $I \subset R$ is nilpotent means that $I^n = 0$ for some $n \in \mathbf{N}$,
0543 (29) $I \subset R$ is locally nilpotent means that every element of I is nilpotent,
00BK (30) $\mathfrak p \subset R$ is a prime ideal,
00BL (31) if $\mathfrak p \subset R$ is prime and if $I, J \subset R$ are ideal, and if $IJ \subset \mathfrak p$, then $I \subset \mathfrak p$ or $J \subset \mathfrak p$.
00BM (32) $\mathfrak m \subset R$ is a maximal ideal,
00BN (33) any nonzero ring has a maximal ideal,
00BO (34) the Jacobson radical of R is $\text{rad}(R) = \bigcap_{\mathfrak m \subset R} \mathfrak m$ the intersection of all the maximal ideals of R,
00BP (35) the ideal (T) generated by a subset $T \subset R$,
00BQ (36) the quotient ring R/I,
00BR (37) an ideal I in the ring R is prime if and only if R/I is a domain,
00BS (38) an ideal I in the ring R is maximal if and only if the ring R/I is a field,
00BT (39) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_2$ is an ideal, then $\varphi^{-1}(I)$ is an ideal of R_1,
00BU (40) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_1$ is an ideal, then $\varphi(I) \cdot R_2$ (sometimes denoted $I \cdot R_2$, or IR_2) is the ideal of R_2 generated by $\varphi(I)$,
00BV (41) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $\mathfrak p \subset R_2$ is a prime ideal, then $\varphi^{-1}(\mathfrak p)$ is a prime ideal of R_1,
00BW (42) M is an R-module,
055Y (43) for $m \in M$ the annihilator $I = \{f \in R \mid fm = 0\}$ of m in R,
00BX (44) $N \subset M$ is an R-submodule,
00BY (45) M is an Noetherian R-module,
00BZ (46) M is a finite R-module,
00C0 (47) M is a finitely generated R-module,
00C1 (48) M is a finitely presented R-module,
00C2 (49) M is a free R-module,
0516 (50) if $0 \to K \to L \to M \to 0$ is a short exact sequence of R-modules and K, M are free, then L is free,
00C3 (51) if $N \subset M \subset L$ are R-modules, then $L/M = (L/N)/(M/N)$,
00C4 (52) S is a multiplicative subset of R,
00C5 (53) the localization $R \to S^{-1}R$ of R,
00C6 (54) if R is a ring and S is a multiplicative subset of R then $S^{-1}R$ is the zero ring if and only if S contains 0,

00C7 (55) if R is a ring and if the multiplicative subset S consists completely of nonzerodivisors, then $R \to S^{-1}R$ is injective,
(56) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and S is a multiplicative subsets of R_1, then $\varphi(S)$ is a multiplicative subset of R_2,
00C8 (57) if S, S' are multiplicative subsets of R, and if SS' denotes the set of products $SS' = \{r \in R \mid \exists s \in S, \exists s' \in S', r = ss'\}$ then SS' is a multiplicative subset of R,
00C9 (58) if S, S' are multiplicative subsets of R, and if $\overline{S}$ denotes the image of S in $(S')^{-1}R$, then $(SS')^{-1}R = \overline{S}^{-1}((S')^{-1}R)$,
00CA (59) the localization $S^{-1}M$ of the R-module M,
00CB (60) the functor $M \mapsto S^{-1}M$ preserves injective maps, surjective maps, and exactness,
00CC (61) if S, S' are multiplicative subsets of R, and if M is an R-module, then $(SS')^{-1}M = S^{-1}((S')^{-1}M)$,
00CD (62) if R is a ring, I an ideal of R, and S a multiplicative subset of R, then $S^{-1}I$ is an ideal of $S^{-1}R$, and we have $S^{-1}R/S^{-1}I = \overline{S}^{-1}(R/I)$, where $\overline{S}$ is the image of S in R/I,
00CE (63) if R is a ring, and S a multiplicative subset of R, then any ideal I' of $S^{-1}R$ is of the form $S^{-1}I$, where one can take I to be the inverse image of I' in R,
00CF (64) if R is a ring, M an R-module, and S a multiplicative subset of R, then any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule $N \subset M$, where one can take N to be the inverse image of N' in M,
00CG (65) if $S = \{1, f, f^2, \ldots\}$ then $R_f = S^{-1}R$ and $M_f = S^{-1}M$,
00CH (66) if $S = R \setminus \mathfrak{p} = \{x \in R \mid x \notin \mathfrak{p}\}$ for some prime ideal $\mathfrak{p}$, then it is customary to denote $R_\mathfrak{p} = S^{-1}R$ and $M_\mathfrak{p} = S^{-1}M$,
00CI (67) a local ring is a ring with exactly one maximal ideal,
03C0 (68) a semi-local ring is a ring with finitely many maximal ideals,
00CJ (69) if $\mathfrak{p}$ is a prime in R, then $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$,
00CK (70) the residue field, denoted $\kappa(\mathfrak{p})$, of the prime $\mathfrak{p}$ in the ring R is the field of fractions of the domain $R/\mathfrak{p}$; it is equal to $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R/\mathfrak{p}$,
00CL (71) given R and M_1, M_2 the tensor product $M_1 \otimes_R M_2$,
0F0K (72) given matrices A and B in a ring R of sizes $m \times n$ and $n \times m$ we have $\det(AB) = \sum \det(A_S)\det({}_SB)$ in R where the sum is over subsets $S \subset \{1, \ldots, n\}$ of size m and A_S is the $m \times m$ submatrix of A with columns corresponding to S and ${}_SB$ is the $m \times m$ submatrix of B with rows corresponding to S,
(73) etc.

10.4. Snake lemma

07JV The snake lemma and its variants are discussed in the setting of abelian categories in Homology, Section 12.5.

07JW Lemma 10.4.1. Given a commutative diagram [CE56, III, Lemma 3.3]

$$\begin{tikzcd} & X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & U \arrow[r] & V \arrow[r] & W & \end{tikzcd}$$

of abelian groups with exact rows, there is a canonical exact sequence

$$\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma)$$

Moreover: if $X \to Y$ is injective, then the first map is injective; if $V \to W$ is surjective, then the last map is surjective.

Proof. The map $\partial : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ is defined as follows. Take $z \in \mathrm{Ker}(\gamma)$. Choose $y \in Y$ mapping to z. Then $\beta(y) \in V$ maps to zero in W. Hence $\beta(y)$ is the image of some $u \in U$. Set $\partial z = \overline{u}$, the class of u in the cokernel of α. Proof of exactness is omitted. □

10.5. Finite modules and finitely presented modules

0517 Just some basic notation and lemmas.

0518 Definition 10.5.1. Let R be a ring. Let M be an R-module.

(1) We say M is a finite R-module, or a finitely generated R-module if there exist $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in M$ such that every element of M is an R-linear combination of the x_i. Equivalently, this means there exists a surjection $R^{\oplus n} \to M$ for some $n \in \mathbf{N}$.
(2) We say M is a finitely presented R-module or an R-module of finite presentation if there exist integers $n, m \in \mathbf{N}$ and an exact sequence

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

Informally, M is a finitely presented R-module if and only if it is finitely generated and the module of relations among these generators is finitely generated as well. A choice of an exact sequence as in the definition is called a presentation of M.

07JX Lemma 10.5.2. Let R be a ring. Let $\alpha : R^{\oplus n} \to M$ and $\beta : N \to M$ be module maps. If $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$, then there exists an R-module map $\gamma : R^{\oplus n} \to N$ such that $\alpha = \beta \circ \gamma$.

Proof. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith basis vector of $R^{\oplus n}$. Let $x_i \in N$ be an element with $\alpha(e_i) = \beta(x_i)$ which exists by assumption. Set $\gamma(a_1, \ldots, a_n) = \sum a_i x_i$. By construction $\alpha = \beta \circ \gamma$. □

0519 Lemma 10.5.3. Let R be a ring. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of R-modules.

(1) If M_1 and M_3 are finite R-modules, then M_2 is a finite R-module.
(2) If M_1 and M_3 are finitely presented R-modules, then M_2 is a finitely presented R-module.
(3) If M_2 is a finite R-module, then M_3 is a finite R-module.
(4) If M_2 is a finitely presented R-module and M_1 is a finite R-module, then M_3 is a finitely presented R-module.

(5) If M_3 is a finitely presented R-module and M_2 is a finite R-module, then M_1 is a finite R-module.

Proof. Proof of (1). If $x_1, \ldots, x_n$ are generators of M_1 and $y_1, \ldots, y_m \in M_2$ are elements whose images in M_3 are generators of M_3, then $x_1, \ldots, x_n, y_1, \ldots, y_m$ generate M_2.

Part (3) is immediate from the definition.

Proof of (5). Assume M_3 is finitely presented and M_2 finite. Choose a presentation

$$R^{\oplus m} \to R^{\oplus n} \to M_3 \to 0$$

By Lemma 10.5.2 there exists a map $R^{\oplus n} \to M_2$ such that the solid diagram

$$\begin{tikzcd} & R^{\oplus m} \arrow[r] \arrow[d, dotted] & R^{\oplus n} \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

commutes. This produces the dotted arrow. By the snake lemma (Lemma 10.4.1) we see that we get an isomorphism

$$\mathrm{Coker}(R^{\oplus m} \to M_1) \cong \mathrm{Coker}(R^{\oplus n} \to M_2)$$

In particular we conclude that $\mathrm{Coker}(R^{\oplus m} \to M_1)$ is a finite R-module. Since $\mathrm{Im}(R^{\oplus m} \to M_1)$ is finite by (3), we see that M_1 is finite by part (1).

Proof of (4). Assume M_2 is finitely presented and M_1 is finite. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M_2 \to 0$. Choose a surjection $R^{\oplus k} \to M_1$. By Lemma 10.5.2 there exists a factorization $R^{\oplus k} \to R^{\oplus n} \to M_2$ of the composition $R^{\oplus k} \to M_1 \to M_2$. Then $R^{\oplus k+m} \to R^{\oplus n} \to M_3 \to 0$ is a presentation.

Proof of (2). Assume that M_1 and M_3 are finitely presented. The argument in the proof of part (1) produces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R^{\oplus n} \arrow[r] \arrow[d] & R^{\oplus n+m} \arrow[r] \arrow[d] & R^{\oplus m} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

with surjective vertical arrows. By the snake lemma we obtain a short exact sequence

$$0 \to \mathrm{Ker}(R^{\oplus n} \to M_1) \to \mathrm{Ker}(R^{\oplus n+m} \to M_2) \to \mathrm{Ker}(R^{\oplus m} \to M_3) \to 0$$

By part (5) we see that the outer two modules are finite. Hence the middle one is finite too. By (4) we see that M_2 is of finite presentation. $\square$

00KZ Lemma 10.5.4. Let R be a ring, and let M be a finite R-module. There exists a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to R/I_i for some ideal I_i of R.

Proof. By induction on the number of generators of M. Let $x_1, \ldots, x_r \in M$ be a minimal number of generators. Let $M' = Rx_1 \subset M$. Then M/M' has $r-1$ generators and the induction hypothesis applies. And clearly $M' \cong R/I_1$ with $I_1 = \{f \in R \mid fx_1 = 0\}$. $\square$

0560 Lemma 10.5.5. Let $R \to S$ be a ring map. Let M be an S-module. If M is finite as an R-module, then M is finite as an S-module.

Proof. In fact, any R-generating set of M is also an S-generating set of M, since the R-module structure is induced by the image of R in S. □

10.6. Ring maps of finite type and of finite presentation

00F2

00F3 Definition 10.6.1. Let $R \to S$ be a ring map.

(1) We say $R \to S$ is of finite type, or that S is a finite type R-algebra if there exist an $n \in \mathbf{N}$ and an surjection of R-algebras $R[x_1, \ldots, x_n] \to S$.
(2) We say $R \to S$ is of finite presentation if there exist integers $n, m \in \mathbf{N}$ and polynomials $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ and an isomorphism of R-algebras $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \cong S$.

Informally, $R \to S$ is of finite presentation if and only if S is finitely generated as an R-algebra and the ideal of relations among the generators is finitely generated. A choice of a surjection $R[x_1, \ldots, x_n] \to S$ as in the definition is sometimes called a presentation of S.

00F4 Lemma 10.6.2. The notions finite type and finite presentation have the following permanence properties.

(1) A composition of ring maps of finite type is of finite type.
(2) A composition of ring maps of finite presentation is of finite presentation.
(3) Given $R \to S' \to S$ with $R \to S$ of finite type, then $S' \to S$ is of finite type.
(4) Given $R \to S' \to S$, with $R \to S$ of finite presentation, and $R \to S'$ of finite type, then $S' \to S$ is of finite presentation.

Proof. We only prove the last assertion. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $S' = R[y_1, \ldots, y_a]/I$. Say that the class $\bar{y}_i$ of y_i maps to $h_i \bmod (f_1, \ldots, f_m)$ in S. Then it is clear that $S = S'[x_1, \ldots, x_n]/(f_1, \ldots, f_m, h_1 - \bar{y}_1, \ldots, h_a - \bar{y}_a)$. □

00R2 Lemma 10.6.3. Let $R \to S$ be a ring map of finite presentation. For any surjection $\alpha : R[x_1, \ldots, x_n] \to S$ the kernel of α is a finitely generated ideal in $R[x_1, \ldots, x_n]$.

Proof. Write $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_k)$. Choose $g_i \in R[y_1, \ldots, y_m]$ which are lifts of $\alpha(x_i)$. Then we see that $S = R[x_i, y_j]/(f_l, x_i - g_i)$. Choose $h_j \in R[x_1, \ldots, x_n]$ such that $\alpha(h_j)$ corresponds to $y_j \bmod (f_1, \ldots, f_k)$. Consider the map $\psi : R[x_i, y_j] \to R[x_i]$, $x_i \mapsto x_i$, $y_j \mapsto h_j$. Then the kernel of α is the image of $(f_l, x_i - g_i)$ under ψ and we win. □

0561 Lemma 10.6.4. Let $R \to S$ be a ring map. Let M be an S-module. Assume $R \to S$ is of finite type and M is finitely presented as an R-module. Then M is finitely presented as an S-module.

Proof. This is similar to the proof of part (4) of Lemma 10.6.2. We may assume $S = R[x_1, \ldots, x_n]/J$. Choose $y_1, \ldots, y_m \in M$ which generate M as an R-module and choose relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ which generate the kernel of $R^{\oplus m} \to M$. For any $i = 1, \ldots, n$ and $j = 1, \ldots, m$ write

$$x_i y_j = \sum a_{ijk} y_k$$

for some $a_{ijk} \in R$. Consider the S-module N generated by $y_1, \ldots, y_m$ subject to the relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ and $x_i y_j = \sum a_{ijk} y_k$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then N has a presentation

$$S^{\oplus nm+t} \longrightarrow S^{\oplus m} \longrightarrow N \longrightarrow 0$$

By construction there is a surjective map $\varphi : N \to M$. To finish the proof we show φ is injective. Suppose $z = \sum b_j y_j \in N$ for some $b_j \in S$. We may think of b_j as a polynomial in $x_1, \ldots, x_n$ with coefficients in R. By applying the relations of the form $x_i y_j = \sum a_{ijk} y_k$ we can inductively lower the degree of the polynomials. Hence we see that $z = \sum c_j y_j$ for some $c_j \in R$. Hence if $\varphi(z) = 0$ then the vector $(c_1, \ldots, c_m)$ is an R-linear combination of the vectors $(a_{i1}, \ldots, a_{im})$ and we conclude that $z = 0$ as desired. □

10.7. Finite ring maps

0562 Here is the definition.

0563 Definition 10.7.1. Let $\varphi : R \to S$ be a ring map. We say $\varphi : R \to S$ is finite if S is finite as an R-module.

00GJ Lemma 10.7.2. Let $R \to S$ be a finite ring map. Let M be an S-module. Then M is finite as an R-module if and only if M is finite as an S-module.

Proof. One of the implications follows from Lemma 10.5.5. To see the other assume that M is finite as an S-module. Pick $x_1, \ldots, x_n \in S$ which generate S as an R-module. Pick $y_1, \ldots, y_m \in M$ which generate M as an S-module. Then $x_i y_j$ generate M as an R-module. □

00GL Lemma 10.7.3. Suppose that $R \to S$ and $S \to T$ are finite ring maps. Then $R \to T$ is finite.

Proof. If t_i generate T as an S-module and s_j generate S as an R-module, then $t_i s_j$ generate T as an R-module. (Also follows from Lemma 10.7.2.) □

0D46 Lemma 10.7.4. Let $\varphi : R \to S$ be a ring map.

(1) If φ is finite, then φ is of finite type.
(2) If S is of finite presentation as an R-module, then φ is of finite presentation.

Proof. For (1) if $x_1, \ldots, x_n \in S$ generate S as an R-module, then $x_1, \ldots, x_n$ generate S as an R-algebra. For (2), suppose that $\sum r^i_j x_i = 0$, $j = 1, \ldots, m$ is a set of generators of the relations among the x_i when viewed as R-module generators of S. Furthermore, write $1 = \sum r_i x_i$ for some $r_i \in R$ and $x_i x_j = \sum r^k_{ij} x_k$ for some $r^k_{ij} \in R$. Then

$$S = R[t_1, \ldots, t_n]/(\sum r^i_j t_i,\ 1 - \sum r_i t_i,\ t_i t_j - \sum r^k_{ij} t_k)$$

as an R-algebra which proves (2). □

For more information on finite ring maps, please see Section 10.36.

10.8. Colimits

07N7 Some of the material in this section overlaps with the general discussion on colimits in Categories, Sections 4.14 – 4.21. The notion of a preordered set is defined in Categories, Definition 4.21.1. It is a slightly weaker notion than a partially ordered set.

00D4 Definition 10.8.1. Let $(I, \leq)$ be a preordered set. A system (M_i, μ_{ij}) of R-modules over I consists of a family of R-modules $\{M_i\}_{i \in I}$ indexed by I and a family of R-module maps $\{\mu_{ij} : M_i \to M_j\}_{i \leq j}$ such that for all $i \leq j \leq k$

$$\mu_{ii} = \mathrm{id}_{M_i} \quad \mu_{ik} = \mu_{jk} \circ \mu_{ij}$$

We say (M_i, μ_{ij}) is a directed system if I is a directed set.

This is the same as the notion defined in Categories, Definition 4.21.2 and Section 4.21. We refer to Categories, Definition 4.14.2 for the definition of a colimit of a diagram/system in any category.

00D5 Lemma 10.8.2. Let (M_i, μ_{ij}) be a system of R-modules over the preordered set I. The colimit of the system (M_i, μ_{ij}) is the quotient R-module $(\bigoplus_{i \in I} M_i)/Q$ where Q is the R-submodule generated by all elements

$$\iota_i(x_i) - \iota_j(\mu_{ij}(x_i))$$

where $\iota_i : M_i \to \bigoplus_{i \in I} M_i$ is the natural inclusion. We denote the colimit $M = \mathrm{colim}_i M_i$. We denote $\pi : \bigoplus_{i \in I} M_i \to M$ the projection map and $\phi_i = \pi \circ \iota_i : M_i \to M$.

Proof. This lemma is a special case of Categories, Lemma 4.14.12 but we will also prove it directly in this case. Namely, note that $\phi_i = \phi_j \circ \mu_{ij}$ in the above construction. To show the pair (M, ϕ_i) is the colimit we have to show it satisfies the universal property: for any other such pair (Y, ψ_i) with $\psi_i : M_i \to Y$, $\psi_i = \psi_j \circ \mu_{ij}$, there is a unique R-module homomorphism $g : M \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} M_i \arrow[rr, "\mu_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & M_j \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\ & M \arrow[d, "g"] & \\ & Y & \end{tikzcd}$$

And this is clear because we can define g by taking the map ψ_i on the summand M_i in the direct sum $\bigoplus M_i$. $\square$

00D6 Lemma 10.8.3. Let (M_i, μ_{ij}) be a system of R-modules over the preordered set I. Assume that I is directed. The colimit of the system (M_i, μ_{ij}) is canonically isomorphic to the module M defined as follows:

(1) as a set let
$$M = \left(\coprod\nolimits_{i \in I} M_i\right) / \sim$$
where for $m \in M_i$ and $m' \in M_{i'}$ we have
$$m \sim m' \Leftrightarrow \mu_{ij}(m) = \mu_{i'j}(m') \text{ for some } j \geq i, i'$$

(2) as an abelian group for $m \in M_i$ and $m' \in M_{i'}$ we define the sum of the classes of m and m' in M to be the class of $\mu_{ij}(m) + \mu_{i'j}(m')$ where $j \in I$ is any index with $i \leq j$ and $i' \leq j$, and
(3) as an R-module define for $m \in M_i$ and $x \in R$ the product of x and the class of m in M to be the class of xm in M.

The canonical maps $\phi_i : M_i \to M$ are induced by the canonical maps $M_i \to \coprod_{i \in I} M_i$.

Proof. Omitted. Compare with Categories, Section 4.19. □

00D7 Lemma 10.8.4. Let (M_i, μ_{ij}) be a directed system. Let $M = \operatorname{colim} M_i$ with $\mu_i : M_i \to M$. Then, $\mu_i(x_i) = 0$ for $x_i \in M_i$ if and only if there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.

Proof. This is clear from the description of the directed colimit in Lemma 10.8.3. □

00D8 Example 10.8.5. Consider the partially ordered set $I = \{a, b, c\}$ with $a < b$ and $a < c$ and no other strict inequalities. A system $(M_a, M_b, M_c, \mu_{ab}, \mu_{ac})$ over I consists of three R-modules M_a, M_b, M_c and two R-module homomorphisms $\mu_{ab} : M_a \to M_b$ and $\mu_{ac} : M_a \to M_c$. The colimit of the system is just

$$M := \operatorname{colim}_{i \in I} M_i = \operatorname{Coker}(M_a \to M_b \oplus M_c)$$

where the map is $\mu_{ab} \oplus -\mu_{ac}$. Thus the kernel of the canonical map $M_a \to M$ is $\operatorname{Ker}(\mu_{ab}) + \operatorname{Ker}(\mu_{ac})$. And the kernel of the canonical map $M_b \to M$ is the image of $\operatorname{Ker}(\mu_{ac})$ under the map μ_{ab}. Hence clearly the result of Lemma 10.8.4 is false for general systems.

00D9 Definition 10.8.6. Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same preordered set I. A homomorphism of systems Φ from (M_i, μ_{ij}) to (N_i, ν_{ij}) is by definition a family of R-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ for all $i \leq j$.

This is the same notion as a transformation of functors between the associated diagrams $M : I \to \text{Mod}_R$ and $N : I \to \text{Mod}_R$, in the language of categories. The following lemma is a special case of Categories, Lemma 4.14.8.

00DA Lemma 10.8.7. Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same preordered set. A morphism of systems $\Phi = (\phi_i)$ from (M_i, μ_{ij}) to (N_i, ν_{ij}) induces a unique homomorphism

$$\operatorname{colim} \phi_i : \operatorname{colim} M_i \longrightarrow \operatorname{colim} N_i$$

such that

$$\begin{tikzcd} M_i \arrow[r] \arrow[d, "\phi_i"'] & \operatorname{colim} M_i \arrow[d, "\operatorname{colim} \phi_i"] \\ N_i \arrow[r] & \operatorname{colim} N_i \end{tikzcd}$$

commutes for all $i \in I$.

Proof. Write $M = \operatorname{colim} M_i$ and $N = \operatorname{colim} N_i$ and $\phi = \operatorname{colim} \phi_i$ (as yet to be constructed). We will use the explicit description of M and N in Lemma 10.8.2

without further mention. The condition of the lemma is equivalent to the condition that

$$\begin{tikzcd} \bigoplus_{i\in I} M_i \arrow[r] \arrow[d, "\bigoplus \phi_i"'] & M \arrow[d, "\phi"] \\ \bigoplus_{i\in I} N_i \arrow[r] & N \end{tikzcd}$$

commutes. Hence it is clear that if ϕ exists, then it is unique. To see that ϕ exists, it suffices to show that the kernel of the upper horizontal arrow is mapped by $\bigoplus \phi_i$ to the kernel of the lower horizontal arrow. To see this, let $j \leq k$ and $x_j \in M_j$. Then

$$(\bigoplus \phi_i)(x_j - \mu_{jk}(x_j)) = \phi_j(x_j) - \phi_k(\mu_{jk}(x_j)) = \phi_j(x_j) - \nu_{jk}(\phi_j(x_j))$$

which is in the kernel of the lower horizontal arrow as required. □

00DB Lemma 10.8.8. Let I be a directed set. Let (L_i, λ_{ij}), (M_i, μ_{ij}), and (N_i, ν_{ij}) be systems of R-modules over I. Let $\varphi_i : L_i \to M_i$ and $\psi_i : M_i \to N_i$ be morphisms of systems over I. Assume that for all $i \in I$ the sequence of R-modules

$$L_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} N_i$$

is a complex with homology H_i. Then the R-modules H_i form a system over I, the sequence of R-modules

$$\mathrm{colim}_i\, L_i \xrightarrow{\varphi} \mathrm{colim}_i\, M_i \xrightarrow{\psi} \mathrm{colim}_i\, N_i$$

is a complex as well, and denoting H its homology we have

$$H = \mathrm{colim}_i\, H_i.$$

Proof. It is clear that $\mathrm{colim}_i\, L_i \xrightarrow{\varphi} \mathrm{colim}_i\, M_i \xrightarrow{\psi} \mathrm{colim}_i\, N_i$ is a complex. For each $i \in I$, there is a canonical R-module morphism $H_i \to H$ (sending each $[m] \in H_i = \mathrm{Ker}(\psi_i)/\mathrm{Im}(\varphi_i)$ to the residue class in $H = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi)$ of the image of m in $\mathrm{colim}_i\, M_i$). These give rise to a morphism $\mathrm{colim}_i\, H_i \to H$. It remains to show that this morphism is surjective and injective.

We are going to repeatedly use the description of colimits over I as in Lemma 10.8.3 without further mention. Let $h \in H$. Since $H = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi)$ we see that h is the class mod $\mathrm{Im}(\varphi)$ of an element $[m]$ in $\mathrm{Ker}(\psi) \subset \mathrm{colim}_i\, M_i$. Choose an i such that $[m]$ comes from an element $m \in M_i$. Choose a $j \geq i$ such that $\nu_{ij}(\psi_i(m)) = 0$ which is possible since $[m] \in \mathrm{Ker}(\psi)$. After replacing i by j and m by $\mu_{ij}(m)$ we see that we may assume $m \in \mathrm{Ker}(\psi_i)$. This shows that the map $\mathrm{colim}_i\, H_i \to H$ is surjective.

Suppose that $h_i \in H_i$ has image zero on H. Since $H_i = \mathrm{Ker}(\psi_i)/\mathrm{Im}(\varphi_i)$ we may represent h_i by an element $m \in \mathrm{Ker}(\psi_i) \subset M_i$. The assumption on the vanishing of h_i in H means that the class of m in $\mathrm{colim}_i\, M_i$ lies in the image of φ. Hence there exists a $j \geq i$ and an $l \in L_j$ such that $\varphi_j(l) = \mu_{ij}(m)$. Clearly this shows that the image of h_i in H_j is zero. This proves the injectivity of $\mathrm{colim}_i\, H_i \to H$. □

00DC Example 10.8.9. Taking colimits is not exact in general. Consider the partially ordered set $I = \{a, b, c\}$ with $a < b$ and $a < c$ and no other strict inequalities, as in Example 10.8.5. Consider the map of systems $(0, \mathbf{Z}, \mathbf{Z}, 0, 0) \to (\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 1, 1)$. From the description of the colimit in Example 10.8.5 we see that the associated map of colimits is not injective, even though the map of systems is injective on each object. Hence the result of Lemma 10.8.8 is false for general systems.

04B0 Lemma 10.8.10. Let $\mathcal{I}$ be an index category satisfying the assumptions of Categories, Lemma 4.19.8. Then taking colimits of diagrams of abelian groups over $\mathcal{I}$ is exact (i.e., the analogue of Lemma 10.8.8 holds in this situation).

Proof. By Categories, Lemma 4.19.8 we may write $\mathcal{I} = \coprod_{j \in J} \mathcal{I}_j$ with each $\mathcal{I}_j$ a filtered category, and J possibly empty. By Categories, Lemma 4.21.5 taking colimits over the index categories $\mathcal{I}_j$ is the same as taking the colimit over some directed set. Hence Lemma 10.8.8 applies to these colimits. This reduces the problem to showing that coproducts in the category of R-modules over the set J are exact. In other words, exact sequences $L_j \to M_j \to N_j$ of R modules we have to show that

$$\bigoplus\nolimits_{j \in J} L_j \longrightarrow \bigoplus\nolimits_{j \in J} M_j \longrightarrow \bigoplus\nolimits_{j \in J} N_j$$

is exact. This can be verified by hand, and holds even if J is empty. ☐

10.9. Localization

00CM

00CN Definition 10.9.1. Let R be a ring, S a subset of R. We say S is a multiplicative subset of R if $1 \in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.

Given a ring A and a multiplicative subset S, we define a relation on $A \times S$ as follows:

$$(x, s) \sim (y, t) \Leftrightarrow \exists u \in S \text{ such that } (xt - ys)u = 0$$

It is easily checked that this is an equivalence relation. Let x/s (or $\frac{x}{s}$) be the equivalence class of (x, s) and $S^{-1}A$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}A$ as follows:

$$x/s + y/t = (xt + ys)/st, \quad x/s \cdot y/t = xy/st$$

One can check that $S^{-1}A$ becomes a ring under these operations.

00CO Definition 10.9.2. This ring is called the localization of A with respect to S.

We have a natural ring map from A to its localization $S^{-1}A$,

$$A \longrightarrow S^{-1}A, \quad x \longmapsto x/1$$

which is sometimes called the localization map. In general the localization map is not injective, unless S contains no zerodivisors. For, if $x/1 = 0$, then there is a $u \in S$ such that $xu = 0$ in A and hence $x = 0$ since there are no zerodivisors in S. The localization of a ring has the following universal property.

00CP Proposition 10.9.3. Let $f : A \to B$ be a ring map that sends every element in S to a unit of B. Then there is a unique homomorphism $g : S^{-1}A \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & S^{-1}A \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Existence. We define a map g as follows. For $x/s \in S^{-1}A$, let $g(x/s) = f(x)f(s)^{-1} \in B$. It is easily checked from the definition that this is a well-defined ring map. And it is also clear that this makes the diagram commutative.

Uniqueness. We now show that if $g' : S^{-1}A \to B$ satisfies $g'(x/1) = f(x)$, then $g = g'$. Hence $f(s) = g'(s/1)$ for $s \in S$ by the commutativity of the diagram. But then $g'(1/s)f(s) = 1$ in B, which implies that $g'(1/s) = f(s)^{-1}$ and hence $g'(x/s) = g'(x/1)g'(1/s) = f(x)f(s)^{-1} = g(x/s)$. □

00CQ Lemma 10.9.4. The localization $S^{-1}A$ is the zero ring if and only if $0 \in S$.

Proof. If $0 \in S$, any pair $(a, s) \sim (0, 1)$ by definition. If $0 \notin S$, then clearly $1/1 \neq 0/1$ in $S^{-1}A$. □

07JY Lemma 10.9.5. Let R be a ring. Let $S \subset R$ be a multiplicative subset. The category of $S^{-1}R$-modules is equivalent to the category of R-modules N with the property that every $s \in S$ acts as an automorphism on N.

Proof. The functor which defines the equivalence associates to an $S^{-1}R$-module M the same module but now viewed as an R-module via the localization map $R \to S^{-1}R$. Conversely, if N is an R-module, such that every $s \in S$ acts via an automorphism s_N, then we can think of N as an $S^{-1}R$-module by letting x/s act via $x_N \circ s_N^{-1}$. We omit the verification that these two functors are quasi-inverse to each other. □

The notion of localization of a ring can be generalized to the localization of a module. Let A be a ring, S a multiplicative subset of A and M an A-module. We define a relation on $M \times S$ as follows

$$(m, s) \sim (n, t) \Leftrightarrow \exists u \in S \text{ such that } (mt - ns)u = 0$$

This is clearly an equivalence relation. Denote by m/s (or $\frac{m}{s}$) be the equivalence class of (m, s) and $S^{-1}M$ be the set of all equivalence classes. Define the addition and scalar multiplication as follows

$$m/s + n/t = (mt + ns)/st, \quad m/s \cdot n/t = mn/st$$

It is clear that this makes $S^{-1}M$ an $S^{-1}A$-module.

07JZ Definition 10.9.6. The $S^{-1}A$-module $S^{-1}M$ is called the localization of M at S.

Note that there is an A-module map $M \to S^{-1}M$, $m \mapsto m/1$ which is sometimes called the localization map. It satisfies the following universal property.

07K0 Lemma 10.9.7. Let R be a ring. Let $S \subset R$ a multiplicative subset. Let M, N be R-modules. Assume all the elements of S act as automorphisms on N. Then the canonical map

$$\mathrm{Hom}_R(S^{-1}M, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

induced by the localization map, is an isomorphism.

Proof. It is clear that the map is well-defined and R-linear. Injectivity: Let $\alpha \in \mathrm{Hom}_R(S^{-1}M, N)$ and take an arbitrary element $m/s \in S^{-1}M$. Then, since $s \cdot \alpha(m/s) = \alpha(m/1)$, we have $\alpha(m/s) = s^{-1}(\alpha(m/1))$, so α is completely determined by what it does on the image of M in $S^{-1}M$. Surjectivity: Let $\beta : M \to N$ be a given R-linear map. We need to show that it can be "extended" to $S^{-1}M$. Define a map of sets

$$M \times S \to N, \quad (m, s) \mapsto s^{-1}\beta(m)$$

Clearly, this map respects the equivalence relation from above, so it descends to a well-defined map $\alpha : S^{-1}M \to N$. It remains to show that this map is R-linear, so take $r, r' \in R$ as well as $s, s' \in S$ and $m, m' \in M$. Then

$$\begin{aligned} \alpha(r \cdot m/s + r' \cdot m'/s') &= \alpha((r \cdot s' \cdot m + r' \cdot s \cdot m')/(ss')) \\ &= (ss')^{-1}\beta(r \cdot s' \cdot m + r' \cdot s \cdot m') \\ &= (ss')^{-1}(r \cdot s'\beta(m) + r' \cdot s\beta(m')) \\ &= r\alpha(m/s) + r'\alpha(m'/s') \end{aligned}$$

and we win. □

02C5 Example 10.9.8. Let A be a ring and let M be an A-module. Here are some important examples of localizations.

(1) Given $\mathfrak{p}$ a prime ideal of A consider $S = A \setminus \mathfrak{p}$. It is immediately checked that S is a multiplicative set. In this case we denote $A_\mathfrak{p}$ and $M_\mathfrak{p}$ the localization of A and M with respect to S respectively. These are called the localization of A, resp. M at $\mathfrak{p}$.
(2) Let $f \in A$. Consider $S = \{1, f, f^2, \ldots\}$. This is clearly a multiplicative subset of A. In this case we denote A_f (resp. M_f) the localization $S^{-1}A$ (resp. $S^{-1}M$). This is called the localization of A, resp. M with respect to f. Note that $A_f = 0$ if and only if f is nilpotent in A.
(3) Let $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called either the total quotient ring, or the total ring of fractions of A.
(4) If A is a domain, then the total quotient ring $Q(A)$ is the field of fractions of A. Please see Fields, Example 9.3.4.

00CR Lemma 10.9.9. Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module. Then

$$S^{-1}M = \mathrm{colim}_{f \in S} M_f$$

where the preorder on S is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f'' \in R$ in which case the map $M_{f'} \to M_f$ is given by $m/(f')^e \mapsto m(f'')^e/f^e$.

Proof. Omitted. Hint: Use the universal property of Lemma 10.9.7. □

In the following paragraph, let A denote a ring, and M, N denote modules over A.

If S and S' are multiplicative sets of A, then it is clear that

$$SS' = \{ss' : s \in S,\ s' \in S'\}$$

is also a multiplicative set of A. Then the following holds.

02C6 Proposition 10.9.10. Let $\overline{S}$ be the image of S in $S'^{-1}A$, then $(SS')^{-1}A$ is isomorphic to $\overline{S}^{-1}(S'^{-1}A)$.

Proof. The map sending $x \in A$ to $x/1 \in (SS')^{-1}A$ induces a map sending $x/s \in S'^{-1}A$ to $x/s \in (SS')^{-1}A$, by universal property. The image of the elements in $\overline{S}$ are invertible in $(SS')^{-1}A$. By the universal property we get a map $f : \overline{S}^{-1}(S'^{-1}A) \to (SS')^{-1}A$ which maps $(x/t')/(s/s')$ to $(x/t') \cdot (s/s')^{-1}$.

On the other hand, the map from A to $\overline{S}^{-1}(S'^{-1}A)$ sending $x \in A$ to $(x/1)/(1/1)$ also induces a map $g : (SS')^{-1}A \to \overline{S}^{-1}(S'^{-1}A)$ which sends x/ss' to $(x/s')/(s/1)$, by the universal property again. It is immediately checked that f and g are inverse to each other, hence they are both isomorphisms. □

For the module M we have

02C7 Proposition 10.9.11. View $S'^{-1}M$ as an A-module, then $S^{-1}(S'^{-1}M)$ is isomorphic to $(SS')^{-1}M$.

Proof. Note that given a A-module M, we have not proved any universal property for $S^{-1}M$. Hence we cannot reason as in the preceding proof; we have to construct the isomorphism explicitly.

We define the maps as follows

$$f : S^{-1}(S'^{-1}M) \longrightarrow (SS')^{-1}M, \quad \frac{x/s'}{s} \mapsto x/ss'$$

$$g : (SS')^{-1}M \longrightarrow S^{-1}(S'^{-1}M), \quad x/t \mapsto \frac{x/s'}{s} \text{ for some } s \in S, s' \in S', \text{ and } t = ss'$$

We have to check that these homomorphisms are well-defined, that is, independent the choice of the fraction. This is easily checked and it is also straightforward to show that they are inverse to each other. □

If $u : M \to N$ is an A homomorphism, then the localization indeed induces a well-defined $S^{-1}A$ homomorphism $S^{-1}u : S^{-1}M \to S^{-1}N$ which sends x/s to $u(x)/s$. It is immediately checked that this construction is functorial, so that S^{-1} is actually a functor from the category of A-modules to the category of $S^{-1}A$-modules. Moreover this functor is exact, as we show in the following proposition.

00CS Proposition 10.9.12. Let $L \xrightarrow{u} M \xrightarrow{v} N$ be an exact sequence of R-modules. Then $S^{-1}L \to S^{-1}M \to S^{-1}N$ is also exact.

Proof. First it is clear that $S^{-1}L \to S^{-1}M \to S^{-1}N$ is a complex since localization is a functor. Next suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in M, which means $xt \in \mathrm{Ker}(v)$. By the exactness of $L \to M \to N$ we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness. □

02C8 Lemma 10.9.13. Localization respects quotients, i.e. if N is a submodule of M, then $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.

Proof. From the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$$

The corollary then follows. □

If, in the preceding Corollary, we take $N = I$ and $M = A$ for an ideal I of A, we see that $S^{-1}A/S^{-1}I \simeq S^{-1}(A/I)$ as A-modules. The next proposition shows that they are isomorphic as rings.

00CT Proposition 10.9.14. Let I be an ideal of A, S a multiplicative set of A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and $\overline{S}^{-1}(A/I)$ is isomorphic to $S^{-1}A/S^{-1}I$, where $\overline{S}$ is the image of S in A/I.

Proof. The fact that $S^{-1}I$ is an ideal is clear since I itself is an ideal. Define

$$f : S^{-1}A \longrightarrow \overline{S}^{-1}(A/I), \quad x/s \mapsto \overline{x}/\overline{s}$$

where $\overline{x}$ and $\overline{s}$ are the images of x and s in A/I. We shall keep similar notations in this proof. This map is well-defined by the universal property of $S^{-1}A$, and $S^{-1}I$ is contained in the kernel of it, therefore it induces a map

$$\overline{f} : S^{-1}A/S^{-1}I \longrightarrow \overline{S}^{-1}(A/I), \quad \overline{x/s} \mapsto \overline{x}/\overline{s}$$

On the other hand, the map $A \to S^{-1}A/S^{-1}I$ sending x to $\overline{x/1}$ induces a map $A/I \to S^{-1}A/S^{-1}I$ sending $\overline{x}$ to $\overline{x/1}$. The image of $\overline{S}$ is invertible in $S^{-1}A/S^{-1}I$, thus induces a map

$$g : \overline{S}^{-1}(A/I) \longrightarrow S^{-1}A/S^{-1}I, \quad \frac{\overline{x}}{\overline{s}} \mapsto \overline{x/s}$$

by the universal property. It is then clear that $\overline{f}$ and g are inverse to each other, hence are both isomorphisms. □

We now consider how submodules behave in localization.

00CU Lemma 10.9.15. Any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some $N \subset M$. Indeed one can take N to be the inverse image of N' in M.

Proof. Let N be the inverse image of N' in M. Then one can see that $S^{-1}N \supset N'$. To show they are equal, take x/s in $S^{-1}N$, where $s \in S$ and $x \in N$. This yields that $x/1 \in N'$. Since N' is an $S^{-1}R$-submodule we have $x/s = x/1 \cdot 1/s \in N'$. This finishes the proof. □

Taking $M = A$ and $N = I$ an ideal of A, we have the following corollary, which can be viewed as a converse of the first part of Proposition 10.9.14.

02C9 Lemma 10.9.16. Each ideal I' of $S^{-1}A$ takes the form $S^{-1}I$, where one can take I to be the inverse image of I' in A.

Proof. Immediate from Lemma 10.9.15. □

10.10. Internal Hom

0581 If R is a ring, and M, N are R-modules, then

$$\mathrm{Hom}_R(M, N) = \{\varphi : M \to N\}$$

is the set of R-linear maps from M to N. This set comes with the structure of an abelian group by setting $(\varphi+\psi)(m) = \varphi(m)+\psi(m)$, as usual. In fact, $\mathrm{Hom}_R(M, N)$ is also an R-module via the rule $(x\varphi)(m) = x\varphi(m) = \varphi(xm)$.

Given maps $a : M \to M'$ and $b : N \to N'$ of R-modules, we can pre-compose and post-compose homomorphisms by a and b. This leads to the following commutative

diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(M', N) \arrow[r, "b\circ -"'] \arrow[d, "-\circ a"'] & \mathrm{Hom}_R(M', N') \arrow[d, "-\circ a"] \\
\mathrm{Hom}_R(M, N) \arrow[r, "b\circ -"] & \mathrm{Hom}_R(M, N')
\end{tikzcd}$$

In fact, the maps in this diagram are R-module maps. Thus Hom_R defines an additive functor

$$\mathrm{Mod}_R^{opp} \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R, \quad (M, N) \longmapsto \mathrm{Hom}_R(M, N)$$

0582 Lemma 10.10.1. Exactness and Hom_R. Let R be a ring.

(1) Let $M_1 \to M_2 \to M_3 \to 0$ be a complex of R-modules. Then $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if $0 \to \mathrm{Hom}_R(M_3, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$ is exact for all R-modules N.
(2) Let $0 \to M_1 \to M_2 \to M_3$ be a complex of R-modules. Then $0 \to M_1 \to M_2 \to M_3$ is exact if and only if $0 \to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3)$ is exact for all R-modules N.

Proof. Omitted. □

0583 Lemma 10.10.2. Let R be a ring. Let M be a finitely presented R-module. Let N be an R-module.

(1) For $f \in R$ we have $\mathrm{Hom}_R(M, N)_f = \mathrm{Hom}_{R_f}(M_f, N_f) = \mathrm{Hom}_R(M_f, N_f)$,
(2) for a multiplicative subset S of R we have

$$S^{-1}\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) = \mathrm{Hom}_R(S^{-1}M, S^{-1}N).$$

Proof. Part (1) is a special case of part (2). The second equality in (2) follows from Lemma 10.9.7. Choose a presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} R \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} R \to M \to 0.$$

By Lemma 10.10.1 this gives an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \bigoplus\nolimits_{i=1,\ldots,n} N \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} N.$$

Inverting S and using Proposition 10.9.12 we get an exact sequence

$$0 \to S^{-1}\mathrm{Hom}_R(M, N) \to \bigoplus\nolimits_{i=1,\ldots,n} S^{-1}N \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} S^{-1}N$$

and the result follows since $S^{-1}M$ sits in an exact sequence

$$\bigoplus\nolimits_{j=1,\ldots,m} S^{-1}R \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} S^{-1}R \to S^{-1}M \to 0$$

which induces (by Lemma 10.10.1) the exact sequence

$$0 \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \to \bigoplus\nolimits_{i=1,\ldots,n} S^{-1}N \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} S^{-1}N$$

which is the same as the one above. □

10.11. Characterizing finite and finitely presented modules

0G8M Given a module N over a ring R, you can characterize whether or not N is a finite module or a finitely presented module in terms of the functor $\mathrm{Hom}_R(N, -)$.

0G8N **Lemma 10.11.1.** Let R be a ring. Let N be an R-module. The following are equivalent

1. N is a finite R-module,
2. for any filtered colimit $M = \mathrm{colim}\, M_i$ of R-modules the map $\mathrm{colim}\, \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M)$ is injective.

Proof. Assume (1) and choose generators $x_1, \ldots, x_m$ for N. If $N \to M_i$ is a module map and the composition $N \to M_i \to M$ is zero, then because $M = \mathrm{colim}_{i' \geq i} M_{i'}$ for each $j \in \{1, \ldots, m\}$ we can find a $i' \geq i$ such that x_j maps to zero in $M_{i'}$. Since there are finitely many x_j we can find a single i' which works for all of them. Then the composition $N \to M_i \to M_{i'}$ is zero and we conclude the map is injective, i.e., part (2) holds.

Assume (2). For a finite subset $E \subset N$ denote $N_E \subset N$ the R-submodule generated by the elements of E. Then $0 = \mathrm{colim}\, N/N_E$ is a filtered colimit. Hence we see that $\mathrm{id} : N \to N$ maps into N_E for some E, i.e., N is finitely generated. $\square$

For purposes of reference, we define what it means to have a relation between elements of a module.

07N8 **Definition 10.11.2.** Let R be a ring. Let M be an R-module. Let $n \geq 0$ and $x_i \in M$ for $i = 1, \ldots, n$. A *relation* between $x_1, \ldots, x_n$ in M is a sequence of elements $f_1, \ldots, f_n \in R$ such that $\sum_{i=1,\ldots,n} f_i x_i = 0$.

00HA **Lemma 10.11.3.** Let R be a ring and let M be an R-module. Then M is the colimit of a directed system (M_i, μ_{ij}) of R-modules with all M_i finitely presented R-modules.

Proof. Consider any finite subset $S \subset M$ and any finite collection of relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in M$ and each $e \in E$ consists of a vector of elements $f_{e,s} \in R$ such that $\sum f_{e,s} x_s = 0$. Let $M_{S,E}$ be the cokernel of the map

$$R^{\#E} \longrightarrow R^{\#S}, \quad (g_e)_{e \in E} \longmapsto (\sum g_e f_{e,s})_{s \in S}.$$

There are canonical maps $M_{S,E} \to M$. If $S \subset S'$ and if the elements of E correspond, via this map, to relations in E', then there is an obvious map $M_{S,E} \to M_{S',E'}$ commuting with the maps to M. Let I be the set of pairs (S, E) with ordering by inclusion as above. It is clear that the colimit of this directed system is M. $\square$

0G8P **Lemma 10.11.4.** Let R be a ring. Let N be an R-module. The following are equivalent

1. N is a finitely presented R-module,
2. for any filtered colimit $M = \mathrm{colim}\, M_i$ of R-modules the map $\mathrm{colim}\, \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M)$ is bijective.

Proof. Assume (1) and choose an exact sequence $F_{-1} \to F_0 \to N \to 0$ with F_i finite free. Then we have an exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(F_0, M) \to \mathrm{Hom}_R(F_{-1}, M)$$

functorial in the R-module M. The functors $\mathrm{Hom}_R(F_i, M)$ commute with filtered colimits as $\mathrm{Hom}_R(R^{\oplus n}, M) = M^{\oplus n}$. Since filtered colimits are exact (Lemma 10.8.8) we see that (2) holds.

Assume (2). By Lemma 10.11.3 we can write $N = \mathrm{colim}\, N_i$ as a filtered colimit such that N_i is of finite presentation for all i. Thus id_N factors through N_i for some i. This means that N is a direct summand of a finitely presented R-module (namely N_i) and hence finitely presented. □

10.12. Tensor products

00CV

00CW Definition 10.12.1. Let R be a ring, M, N, P be three R-modules. A mapping $f : M \times N \to P$ (where $M \times N$ is viewed only as Cartesian product of two R-modules) is said to be R-bilinear if for each $x \in M$ the mapping $y \mapsto f(x, y)$ of N into P is R-linear, and for each $y \in N$ the mapping $x \mapsto f(x, y)$ is also R-linear.

00CX Lemma 10.12.2. Let M, N be R-modules. Then there exists a pair (T, g) where T is an R-module, and $g : M \times N \to T$ an R-bilinear mapping, with the following universal property: For any R-module P and any R-bilinear mapping $f : M \times N \to P$, there exists a unique R-linear mapping $\tilde{f} : T \to P$ such that $f = \tilde{f} \circ g$. In other words, the following diagram commutes:

$$
\begin{tikzcd}
M \times N \arrow[rr, "f"] \arrow[dr, "g"'] & & P \\
& T \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}
$$

Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

The R-module T which satisfies the above universal property is called the tensor product of R-modules M and N, denoted as $M \otimes_R N$.

Proof. We first prove the existence of such R-module T. Let M, N be R-modules. Let T be the quotient module P/Q, where P is the free R-module $R^{(M \times N)}$ and Q is the R-module generated by all elements of the following types: $(x \in M, y \in N)$

$$
\begin{aligned}
(x + x', y) - (x, y) - (x', y),\\
(x, y + y') - (x, y) - (x, y'),\\
(ax, y) - a(x, y),\\
(x, ay) - a(x, y)
\end{aligned}
$$

Let $\pi : M \times N \to T$ denote the natural map. This map is R-bilinear, as implied by the above relations when we check the bilinearity conditions. Denote the image $\pi(x, y) = x \otimes y$, then these elements generate T. Now let $f : M \times N \to P$ be an R-bilinear map, then we can define $f' : T \to P$ by extending the mapping $f'(x \otimes y) = f(x, y)$. Clearly $f = f' \circ \pi$. Moreover, f' is uniquely determined by the value on the generating sets $\{x \otimes y : x \in M, y \in N\}$. Suppose there is another pair (T', g') satisfying the same properties. Then there is a unique $j : T \to T'$ and also $j' : T' \to T$ such that $g' = j \circ g$, $g = j' \circ g'$. But then both the maps $(j \circ j') \circ g$ and g satisfies the universal properties, so by uniqueness they are equal, and hence

$j' \circ j$ is identity on T. Similarly $(j' \circ j) \circ g' = g'$ and $j \circ j'$ is identity on T'. So j is an isomorphism. $\square$

00CY **Lemma 10.12.3.** Let M, N, P be R-modules, then the bilinear maps

$$\begin{aligned}(x, y) &\mapsto y \otimes x \\ (x + y, z) &\mapsto x \otimes z + y \otimes z \\ (r, x) &\mapsto rx\end{aligned}$$

induce unique isomorphisms

$$\begin{aligned}M \otimes_R N &\to N \otimes_R M, \\ (M \oplus N) \otimes_R P &\to (M \otimes_R P) \oplus (N \otimes_R P), \\ R \otimes_R M &\to M\end{aligned}$$

Proof. Omitted. $\square$

We may generalize the tensor product of two R-modules to finitely many R-modules, and set up a correspondence between the multi-tensor product with multilinear mappings. Using almost the same construction one can prove that:

00CZ **Lemma 10.12.4.** Let $M_1, \ldots, M_r$ be R-modules. Then there exists a pair (T, g) consisting of an R-module T and an R-multilinear mapping $g : M_1 \times \ldots \times M_r \to T$ with the universal property: For any R-multilinear mapping $f : M_1 \times \ldots \times M_r \to P$ there exists a unique R-module homomorphism $f' : T \to P$ such that $f' \circ g = f$. Such a module T is unique up to unique isomorphism. We denote it $M_1 \otimes_R \ldots \otimes_R M_r$ and we denote the universal multilinear map $(m_1, \ldots, m_r) \mapsto m_1 \otimes \ldots \otimes m_r$.

Proof. Omitted. $\square$

00D0 **Lemma 10.12.5.** The homomorphisms

$$(M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P \to M \otimes_R (N \otimes_R P)$$

such that $f((x \otimes y) \otimes z) = x \otimes y \otimes z$ and $g(x \otimes y \otimes z) = x \otimes (y \otimes z)$, $x \in M, y \in N, z \in P$ are well-defined and are isomorphisms.

Proof. We shall prove f is well-defined and is an isomorphism, and this proof carries analogously to g. Fix any $z \in P$, then the mapping $(x, y) \mapsto x \otimes y \otimes z$, $x \in M, y \in N$, is R-bilinear in x and y, and hence induces homomorphism $f_z : M \otimes N \to M \otimes N \otimes P$ which sends $f_z(x \otimes y) = x \otimes y \otimes z$. Then consider $(M \otimes N) \times P \to M \otimes N \otimes P$ given by $(w, z) \mapsto f_z(w)$. The map is R-bilinear and thus induces $f : (M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P$ and $f((x \otimes y) \otimes z) = x \otimes y \otimes z$. To construct the inverse, we note that the map $\pi : M \times N \times P \to (M \otimes N) \otimes P$ is R-trilinear. Therefore, it induces an R-linear map $h : M \otimes N \otimes P \to (M \otimes N) \otimes P$ which agrees with the universal property. Here we see that $h(x \otimes y \otimes z) = (x \otimes y) \otimes z$. From the explicit expression of f and h, $f \circ h$ and $h \circ f$ are identity maps of $M \otimes N \otimes P$ and $(M \otimes N) \otimes P$ respectively, hence f is our desired isomorphism. $\square$

Doing induction we see that this extends to multi-tensor products. Combined with Lemma 10.12.3 we see that the tensor product operation on the category of R-modules is associative, commutative and distributive.

00D1 Definition 10.12.6. An abelian group N is called an (A,B)-bimodule if it is both an A-module and a B-module, and the actions $A \to End(M)$ and $B \to End(M)$ are compatible in the sense that $(ax)b = a(xb)$ for all $a \in A, b \in B, x \in N$. Usually we denote it as ${}_AN_B$.

00D2 Lemma 10.12.7. For A-module M, B-module P and (A,B)-bimodule N, the modules $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$ can both be given (A,B)-bimodule structure, and moreover

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

Proof. A priori $M \otimes_A N$ is an A-module, but we can give it a B-module structure by letting

$$(x \otimes y)b = x \otimes yb, \quad x \in M, y \in N, b \in B$$

Thus $M \otimes_A N$ becomes an (A,B)-bimodule. Similarly for $N \otimes_B P$, and thus for $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$. By Lemma 10.12.5, these two modules are isomorphic as both as A-module and B-module via the same mapping. □

00DE Lemma 10.12.8. For any three R-modules M, N, P,

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

Proof. An R-linear map $\hat{f} \in \mathrm{Hom}_R(M \otimes_R N, P)$ corresponds to an R-bilinear map $f : M \times N \to P$. For each $x \in M$ the mapping $y \mapsto f(x,y)$ is R-linear by the universal property. Thus f corresponds to a map $\phi_f : M \to \mathrm{Hom}_R(N,P)$. This map is R-linear since

$$\phi_f(ax+y)(z) = f(ax+y,z) = af(x,z) + f(y,z) = (a\phi_f(x) + \phi_f(y))(z),$$

for all $a \in R$, $x \in M$, $y \in M$ and $z \in N$. Conversely, any $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$ defines an R-bilinear map $M \times N \to P$, namely $(x,y) \mapsto f(x)(y)$. So this is a natural one-to-one correspondence between the two modules $\mathrm{Hom}_R(M \otimes_R N, P)$ and $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$. □

00DD Lemma 10.12.9 (Tensor products commute with colimits). Let (M_i, μ_{ij}) be a system over the preordered set I. Let N be an R-module. Then

$$\mathrm{colim}(M_i \otimes N) \cong (\mathrm{colim}\, M_i) \otimes N.$$

Moreover, the isomorphism is induced by the homomorphisms $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ where $M = \mathrm{colim}_i M_i$ with natural maps $\mu_i : M_i \to M$.

Proof. First proof. The functor $M' \mapsto M' \otimes_R N$ is left adjoint to the functor $N' \mapsto \mathrm{Hom}_R(N, N')$ by Lemma 10.12.8. Thus $M' \mapsto M' \otimes_R N$ commutes with all colimits, see Categories, Lemma 4.24.5.

Second direct proof. Let $P = \mathrm{colim}(M_i \otimes N)$ with coprojections $\lambda_i : M_i \otimes N \to P$. Let $M = \mathrm{colim}\, M_i$ with coprojections $\mu_i : M_i \to M$. Then for all $i \leq j$, the following diagram commutes:

$$\begin{tikzcd} M_i \otimes N \arrow[r, "\mu_i \otimes 1"'] \arrow[d, "\mu_{ij} \otimes 1"'] & M \otimes N \arrow[d, "\mathrm{id}"] \\ M_j \otimes N \arrow[r, "\mu_j \otimes 1"] & M \otimes N \end{tikzcd}$$

By Lemma 10.8.7 these maps induce a unique homomorphism $\psi : P \to M \otimes N$ such that $\mu_i \otimes 1 = \psi \circ \lambda_i$.

To construct the inverse map, for each $i \in I$, there is the canonical R-bilinear mapping $g_i : M_i \times N \to M_i \otimes N$. This induces a unique mapping $\widehat{\phi} : M \times N \to P$ such that $\widehat{\phi} \circ (\mu_i \times 1) = \lambda_i \circ g_i$. It is R-bilinear. Thus it induces an R-linear mapping $\phi : M \otimes N \to P$. From the commutative diagram below:

$$\begin{tikzcd}
M_i \times N \arrow[r, "g_i"] \arrow[d, "\mu_i \times \mathrm{id}"] & M_i \otimes N \arrow[r, "\mathrm{id}"'] \arrow[d, "\lambda_i"'] & M_i \otimes N \arrow[d, "\mu_i \otimes \mathrm{id}"'] \arrow[dr, "\lambda_i"] & \\
M \times N \arrow[r, "\widehat{\phi}"] & P \arrow[r, "\psi"] & M \otimes N \arrow[r, "\phi"] & P
\end{tikzcd}$$

we see that $\psi \circ \widehat{\phi} = g$, the canonical R-bilinear mapping $g : M \times N \to M \otimes N$. So $\psi \circ \phi$ is identity on $M \otimes N$. From the right-hand square and triangle, $\phi \circ \psi$ is also identity on P. $\square$

00DF Lemma 10.12.10. Let

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules and homomorphisms, and let N be any R-module. Then the sequence

00DG (10.12.10.1)
$$M_1 \otimes N \xrightarrow{f \otimes 1} M_2 \otimes N \xrightarrow{g \otimes 1} M_3 \otimes N \to 0$$

is exact. In other words, the functor $- \otimes_R N$ is right exact, in the sense that tensoring each term in the original right exact sequence preserves the exactness.

Proof. We apply the functor $\mathrm{Hom}(-, \mathrm{Hom}(N, P))$ to the first exact sequence. We obtain

$$0 \to \mathrm{Hom}(M_3, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_2, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_1, \mathrm{Hom}(N, P))$$

By Lemma 10.12.8, we have

$$0 \to \mathrm{Hom}(M_3 \otimes N, P) \to \mathrm{Hom}(M_2 \otimes N, P) \to \mathrm{Hom}(M_1 \otimes N, P)$$

Using the pullback property again, we arrive at the desired exact sequence. $\square$

00DH Remark 10.12.11. However, tensor product does NOT preserve exact sequences in general. In other words, if $M_1 \to M_2 \to M_3$ is exact, then it is not necessarily true that $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for arbitrary R-module N.

00DI Example 10.12.12. Consider the injective map $2 : \mathbf{Z} \to \mathbf{Z}$ viewed as a map of $\mathbf{Z}$-modules. Let $N = \mathbf{Z}/2$. Then the induced map $\mathbf{Z} \otimes \mathbf{Z}/2 \to \mathbf{Z} \otimes \mathbf{Z}/2$ is NOT injective. This is because for $x \otimes y \in \mathbf{Z} \otimes \mathbf{Z}/2$,

$$(2 \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

Therefore the induced map is the zero map while $\mathbf{Z} \otimes N \neq 0$.

00DJ Remark 10.12.13. For R-modules N, if the functor $- \otimes_R N$ is exact, i.e. tensoring with N preserves all exact sequences, then N is said to be flat R-module. We will discuss this later in Section 10.39.

05BS Lemma 10.12.14. Let R be a ring. Let M and N be R-modules.

(1) If N and M are finite, then so is $M \otimes_R N$.
(2) If N and M are finitely presented, then so is $M \otimes_R N$.

Proof. Suppose M is finite. Then choose a presentation $0 \to K \to R^{\oplus n} \to M \to 0$. This gives an exact sequence $K \otimes_R N \to N^{\oplus n} \to M \otimes_R N \to 0$ by Lemma 10.12.10. We conclude that if N is finite too then $M \otimes_R N$ is a quotient of a finite module, hence finite, see Lemma 10.5.3. Similarly, if both N and M are finitely presented, then we see that K is finite and that $M \otimes_R N$ is a quotient of the finitely presented module $N^{\oplus n}$ by a finite module, namely $K \otimes_R N$, and hence finitely presented, see Lemma 10.5.3. $\square$

00DK Lemma 10.12.15. Let M be an R-module. Then the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are canonically isomorphic, and the canonical isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ is given by

$$f((a/s) \otimes m) = am/s, \forall a \in R, m \in M, s \in S$$

Proof. Obviously, the map $f' : S^{-1}R \times M \to S^{-1}M$ given by $f'(a/s, m) = am/s$ is bilinear, and thus by the universal property, this map induces a unique $S^{-1}R$-module homomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ as in the statement of the lemma. Actually every element in $S^{-1}M$ is of the form m/s, $m \in M, s \in S$ and every element in $S^{-1}R \otimes_R M$ is of the form $1/s \otimes m$. To see the latter fact, write an element in $S^{-1}R \otimes_R M$ as

$$\sum_k \frac{a_k}{s_k} \otimes m_k = \sum_k \frac{a_k t_k}{s} \otimes m_k = \frac{1}{s} \otimes \sum_k a_k t_k m_k = \frac{1}{s} \otimes m$$

Where $m = \sum_k a_k t_k m_k$. Then it is obvious that f is surjective, and if $f(\frac{1}{s} \otimes m) = m/s = 0$ then there exists $t' \in S$ with $tm = 0$ in M. Then we have

$$\frac{1}{s} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0$$

Therefore f is injective. $\square$

00DL Lemma 10.12.16. Let M, N be R-modules, then there is a canonical $S^{-1}R$-module isomorphism $f : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N)$, given by

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st$$

Proof. We may use Lemma 10.12.7 and Lemma 10.12.15 repeatedly to see that these two $S^{-1}R$-modules are isomorphic, noting that $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule:

$$\begin{aligned} S^{-1}(M \otimes_R N) &\cong S^{-1}R \otimes_R (M \otimes_R N) \\ &\cong S^{-1}M \otimes_R N \\ &\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \\ &\cong S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \\ &\cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N \end{aligned}$$

This isomorphism is easily seen to be the one stated in the lemma. $\square$

10.13. Tensor algebra

00DM Let R be a ring. Let M be an R-module. We define the tensor algebra of M over R to be the noncommutative R-algebra

$$\mathrm{T}(M) = \mathrm{T}_R(M) = \bigoplus\nolimits_{n \geq 0} \mathrm{T}^n(M)$$

with $T^0(M) = R$, $T^1(M) = M$, $T^2(M) = M \otimes_R M$, $T^3(M) = M \otimes_R M \otimes_R M$, and so on. Multiplication is defined by the rule that on pure tensors we have

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_n) \cdot (y_1 \otimes y_2 \otimes \ldots \otimes y_m) = x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes y_1 \otimes y_2 \otimes \ldots \otimes y_m$$

and we extend this by linearity.

We define the exterior algebra $\wedge(M)$ of M over R to be the quotient of $T(M)$ by the two sided ideal generated by the elements $x \otimes x \in T^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\wedge^n(M)$ is denoted $x_1 \wedge \ldots \wedge x_n$. These elements generate $\wedge^n(M)$, they are R-linear in each x_i and they are zero when two of the x_i are equal (i.e., they are alternating as functions of $x_1, x_2, \ldots, x_n$). The multiplication on $\wedge(M)$ is graded commutative, i.e., every $x \in M$ and $y \in M$ satisfy $x \wedge y = -y \wedge x$.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\wedge(M)$ is free over R with basis the elements

$$x_{i_1} \wedge \ldots \wedge x_{i_r}$$

with $0 \leq r \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_r \leq n$.

We define the symmetric algebra $\mathrm{Sym}(M)$ of M over R to be the quotient of $T(M)$ by the two sided ideal generated by the elements $x \otimes y - y \otimes x \in T^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\mathrm{Sym}^n(M)$ is denoted just $x_1 \ldots x_n$. These elements generate $\mathrm{Sym}^n(M)$, these are R-linear in each x_i and $x_1 \ldots x_n = x_1' \ldots x_n'$ if the sequence of elements $x_1, \ldots, x_n$ is a permutation of the sequence $x_1', \ldots, x_n'$. Thus we see that $\mathrm{Sym}(M)$ is commutative.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\mathrm{Sym}(M) = R[x_1, \ldots, x_n]$ is a polynomial algebra.

00DN Lemma 10.13.1. Let R be a ring. Let M be an R-module. If M is a free R-module, so is each symmetric and exterior power.

Proof. Omitted, but see above for the finite free case. $\square$

00DO Lemma 10.13.2. Let R be a ring. Let $M_2 \to M_1 \to M \to 0$ be an exact sequence of R-modules. There are exact sequences

$$M_2 \otimes_R \mathrm{Sym}^{n-1}(M_1) \to \mathrm{Sym}^n(M_1) \to \mathrm{Sym}^n(M) \to 0$$

and similarly

$$M_2 \otimes_R \wedge^{n-1}(M_1) \to \wedge^n(M_1) \to \wedge^n(M) \to 0$$

Proof. Omitted. $\square$

00DP Lemma 10.13.3. Let R be a ring. Let M be an R-module. Let x_i, $i \in I$ be a given system of generators of M as an R-module. Let $n \geq 2$. There exists a canonical exact sequence

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} T^{n-2}(M) \oplus \bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i \in I} T^{n-2}(M) \to T^n(M) \to \wedge^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ in the first summand maps to

$$\underbrace{m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_1} \text{ and } x_{i_2} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$
$$+ \underbrace{m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_2} \text{ and } x_{i_1} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$

and $m_1 \otimes \ldots \otimes m_{n-2}$ in the second summand maps to

$$\underbrace{m_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes x_i \otimes \ldots \otimes m_{n-2}}_{\text{with } x_i \text{ and } x_i \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$

There is also a canonical exact sequence

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} \mathrm{T}^{n-2}(M) \to \mathrm{T}^n(M) \to \mathrm{Sym}^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ maps to

$$\underbrace{m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_1} \text{ and } x_{i_2} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}} - \underbrace{m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_2} \text{ and } x_{i_1} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$

Proof. Omitted. □

0H1C Lemma 10.13.4. Let $A \to B$ be a ring map. Let M be a B-module. Let $n > 1$. The kernel of the A-linear map $M \otimes_A \ldots \otimes_A M \to \wedge^n_B(M)$ is generated as an A-module by the elements $m_1 \otimes \ldots \otimes m_n$ with $m_i = m_j$ for $i \neq j$, $m_1, \ldots, m_n \in M$ and the elements $m_1 \otimes \ldots \otimes bm_i \otimes \ldots \otimes m_n - m_1 \otimes \ldots \otimes bm_j \otimes \ldots \otimes m_n$ for $i \neq j$, $m_1, \ldots, m_n \in M$, and $b \in B$.

Proof. Omitted. □

00DQ Lemma 10.13.5. Let R be a ring. Let M_i be a directed system of R-modules. Then $\mathrm{colim}_i \mathrm{T}(M_i) = \mathrm{T}(\mathrm{colim}_i M_i)$ and similarly for the symmetric and exterior algebras.

Proof. Omitted. Hint: Apply Lemma 10.12.9. □

0C6F Lemma 10.13.6. Let R be a ring and let $S \subset R$ be a multiplicative subset. Then $S^{-1}T_R(M) = T_{S^{-1}R}(S^{-1}M)$ for any R-module M. Similar for symmetric and exterior algebras.

Proof. Omitted. Hint: Apply Lemma 10.12.16. □

10.14. Base change

05G3 We formally introduce base change in algebra as follows.

05G4 Definition 10.14.1. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be any ring map. The base change of φ by $R \to R'$ is the ring map $R' \to S \otimes_R R'$. In this situation we often write $S' = S \otimes_R R'$. The base change of the S-module M is the S'-module $M \otimes_R R'$.

If $S = R[x_i]/(f_j)$ for some collection of variables x_i, $i \in I$ and some collection of polynomials $f_j \in R[x_i]$, $j \in J$, then $S \otimes_R R' = R'[x_i]/(f'_j)$, where $f'_j \in R'[x_i]$ is the image of f_j under the map $R[x_i] \to R'[x_i]$ induced by $R \to R'$. This simple remark is the key to understanding base change.

05G5 Lemma 10.14.2. Let $R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be a ring map and let $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$ be the base changes.

(1) If M is a finite S-module, then the base change M' is a finite S'-module.

(2) If M is an S-module of finite presentation, then the base change M' is an S'-module of finite presentation.
(3) If $R \to S$ is of finite type, then the base change $R' \to S'$ is of finite type.
(4) If $R \to S$ is of finite presentation, then the base change $R' \to S'$ is of finite presentation.

Proof. Proof of (1). Take a surjective, S-linear map $S^{\oplus n} \to M \to 0$. By Lemma 10.12.3 and 10.12.10 the result after tensoring with R' is a surjection $S'^{\oplus n} \to M' \to 0$, so M' is a finitely generated S'-module. Proof of (2). Take a presentation $S^{\oplus m} \to S^{\oplus n} \to M \to 0$. By Lemma 10.12.3 and 10.12.10 the result after tensoring with R' gives a finite presentation $S'^{\oplus m} \to S'^{\oplus n} \to M' \to 0$, of the S'-module M'. Proof of (3). This follows by the remark preceding the lemma as we can take I to be finite by assumption. Proof of (4). This follows by the remark preceding the lemma as we can take I and J to be finite by assumption. $\square$

Let $\varphi : R \to S$ be a ring map. Given an S-module N we obtain an R-module N_R by the rule $r \cdot n = \varphi(r)n$. This is sometimes called the restriction of N to R.

05DQ **Lemma 10.14.3.** Let $R \to S$ be a ring map. The functors $\text{Mod}_S \to \text{Mod}_R$, $N \mapsto N_R$ (restriction) and $\text{Mod}_R \to \text{Mod}_S$, $M \mapsto M \otimes_R S$ (base change) are adjoint functors. In a formula

$$\text{Hom}_R(M, N_R) = \text{Hom}_S(M \otimes_R S, N)$$

Proof. If $\alpha : M \to N_R$ is an R-module map, then we define $\alpha' : M \otimes_R S \to N$ by the rule $\alpha'(m \otimes s) = s\alpha(m)$. If $\beta : M \otimes_R S \to N$ is an S-module map, we define $\beta' : M \to N_R$ by the rule $\beta'(m) = \beta(m \otimes 1)$. We omit the verification that these constructions are mutually inverse. $\square$

The lemma above tells us that restriction has a left adjoint, namely base change. It also has a right adjoint.

08YP **Lemma 10.14.4.** Let $R \to S$ be a ring map. The functors $\text{Mod}_S \to \text{Mod}_R$, $N \mapsto N_R$ (restriction) and $\text{Mod}_R \to \text{Mod}_S$, $M \mapsto \text{Hom}_R(S, M)$ are adjoint functors. In a formula

$$\text{Hom}_R(N_R, M) = \text{Hom}_S(N, \text{Hom}_R(S, M))$$

Proof. If $\alpha : N_R \to M$ is an R-module map, then we define $\alpha' : N \to \text{Hom}_R(S, M)$ by the rule $\alpha'(n) = (s \mapsto \alpha(sn))$. If $\beta : N \to \text{Hom}_R(S, M)$ is an S-module map, we define $\beta' : N_R \to M$ by the rule $\beta'(n) = \beta(n)(1)$. We omit the verification that these constructions are mutually inverse. $\square$

08YQ **Lemma 10.14.5.** Let $R \to S$ be a ring map. Given S-modules M, N and an R-module P we have

$$\text{Hom}_R(M \otimes_S N, P) = \text{Hom}_S(M, \text{Hom}_R(N, P))$$

Proof. This can be proved directly, but it is also a consequence of Lemmas 10.14.4 and 10.12.8. Namely, we have

$$\begin{aligned}
\text{Hom}_R(M \otimes_S N, P) &= \text{Hom}_S(M \otimes_S N, \text{Hom}_R(S, P)) \\
&= \text{Hom}_S(M, \text{Hom}_S(N, \text{Hom}_R(S, P))) \\
&= \text{Hom}_S(M, \text{Hom}_R(N, P))
\end{aligned}$$

as desired. $\square$

10.15. Miscellany

00DR The proofs in this section should not refer to any results except those from the section on basic notions, Section 10.3.

07K1 **Lemma 10.15.1.** Let R be a ring, I and J two ideals and $\mathfrak{p}$ a prime ideal containing the product IJ. Then $\mathfrak{p}$ contains I or J.

Proof. Assume the contrary and take $x \in I \setminus \mathfrak{p}$ and $y \in J \setminus \mathfrak{p}$. Their product is an element of $IJ \subset \mathfrak{p}$, which contradicts the assumption that $\mathfrak{p}$ was prime. □

00DS **Lemma 10.15.2** (Prime avoidance)**.** Let R be a ring. Let $I_i \subset R$, $i = 1, \ldots, r$, and $J \subset R$ be ideals. Assume

(1) $J \not\subset I_i$ for $i = 1, \ldots, r$, and
(2) all but two of I_i are prime ideals.

Then there exists an $x \in J$, $x \notin I_i$ for all i.

Proof. The result is true for $r = 1$. If $r = 2$, then let $x, y \in J$ with $x \notin I_1$ and $y \notin I_2$. We are done unless $x \in I_2$ and $y \in I_1$. Then the element $x + y$ cannot be in I_1 (since that would mean $x + y - y \in I_1$) and it also cannot be in I_2.

For $r \geq 3$, assume the result holds for $r - 1$. After renumbering we may assume that I_r is prime. We may also assume there are no inclusions among the I_i. Pick $x \in J$, $x \notin I_i$ for all $i = 1, \ldots, r - 1$. If $x \notin I_r$ we are done. So assume $x \in I_r$. If $JI_1 \ldots I_{r-1} \subset I_r$ then $J \subset I_r$ (by Lemma 10.15.1) a contradiction. Pick $y \in JI_1 \ldots I_{r-1}$, $y \notin I_r$. Then $x + y$ works. □

0EHL **Lemma 10.15.3.** Let R be a ring. Let $x \in R$, $I \subset R$ an ideal, and $\mathfrak{p}_i$, $i = 1, \ldots, r$ be prime ideals. Suppose that $x + I \not\subset \mathfrak{p}_i$ for $i = 1, \ldots, r$. Then there exists a $y \in I$ such that $x + y \notin \mathfrak{p}_i$ for all i.

Proof. We may assume there are no inclusions among the $\mathfrak{p}_i$. After reordering we may assume $x \notin \mathfrak{p}_i$ for $i < s$ and $x \in \mathfrak{p}_i$ for $i \geq s$. If $s = r + 1$ then we are done. If not, then we can find $y \in I$ with $y \notin \mathfrak{p}_s$. Choose $f \in \bigcap_{i<s} \mathfrak{p}_i$ with $f \notin \mathfrak{p}_s$. Then $x + fy$ is not contained in $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Thus we win by induction on s. □

00DT **Lemma 10.15.4** (Chinese remainder)**.** Let R be a ring.

(1) If $I_1, \ldots, I_r$ are ideals such that $I_a + I_b = R$ when $a \neq b$, then $I_1 \cap \ldots \cap I_r = I_1 I_2 \ldots I_r$ and $R/(I_1 I_2 \ldots I_r) \cong R/I_1 \times \ldots \times R/I_r$.
(2) If $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are pairwise distinct maximal ideals then $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and the above applies.

Proof. Let us first prove $I_1 \cap \ldots \cap I_r = I_1 \ldots I_r$ as this will also imply the injectivity of the induced ring homomorphism $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$. The inclusion $I_1 \cap \ldots \cap I_r \supset I_1 \ldots I_r$ is always fulfilled since ideals are closed under multiplication with arbitrary ring elements. To prove the other inclusion, we claim that the ideals

$$I_1 \ldots \hat{I}_i \ldots I_r, \quad i = 1, \ldots, r$$

generate the ring R. We prove this by induction on r. It holds when $r = 2$. If $r > 2$, then we see that R is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_{r-1}$, $i = 1, \ldots, r-1$. Hence I_r is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 1, \ldots, r - 1$. Applying the same argument with the reverse ordering on the ideals we see that I_1 is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 2, \ldots, r$. Since $R = I_1 + I_r$ by assumption we see that R is the sum of the ideals displayed above. Therefore we can find elements $a_i \in I_1 \ldots \hat{I_i} \ldots I_r$ such that their sum is one. Multiplying this equation by an element of $I_1 \cap \ldots \cap I_r$ gives the other inclusion. It remains to show that the canonical map $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$ is surjective. For this, consider its action on the equation $1 = \sum_{i=1}^r a_i$ we derived above. On the one hand, a ring morphism sends 1 to 1 and on the other hand, the image of any a_i is zero in R/I_j for $j \neq i$. Therefore, the image of a_i in R/I_i is the identity. So given any element $(\bar{b_1}, \ldots, \bar{b_r}) \in R/I_1 \times \ldots \times R/I_r$, the element $\sum_{i=1}^r a_i \cdot b_i$ is an inverse image in R.

To see (2), by the very definition of being distinct maximal ideals, we have $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and so the above applies. $\square$

07DQ Lemma 10.15.5. Let R be a ring. Let $n \geq m$. Let A be an $n \times m$ matrix with coefficients in R. Let $J \subset R$ be the ideal generated by the $m \times m$ minors of A.

1. For any $f \in J$ there exists a $m \times n$ matrix B such that $BA = f 1_{m \times m}$.
2. If $f \in R$ and $BA = f 1_{m \times m}$ for some $m \times n$ matrix B, then $f^m \in J$.

Proof. For $I \subset \{1, \ldots, n\}$ with $|I| = m$, we denote by E_I the $m \times n$ matrix of the projection

$$R^{\oplus n} = \bigoplus\nolimits_{i \in \{1, \ldots, n\}} R \longrightarrow \bigoplus\nolimits_{i \in I} R$$

and set $A_I = E_I A$, i.e., A_I is the $m \times m$ matrix whose rows are the rows of A with indices in I. Let B_I be the adjugate (transpose of cofactor) matrix to A_I, i.e., such that $A_I B_I = B_I A_I = \det(A_I) 1_{m \times m}$. The $m \times m$ minors of A are the determinants $\det A_I$ for all the $I \subset \{1, \ldots, n\}$ with $|I| = m$. If $f \in J$ then we can write $f = \sum c_I \det(A_I)$ for some $c_I \in R$. Set $B = \sum c_I B_I E_I$ to see that (1) holds.

If $f 1_{m \times m} = BA$ then by the Cauchy-Binet formula (72) we have $f^m = \sum b_I \det(A_I)$ where b_I is the determinant of the $m \times m$ matrix whose columns are the columns of B with indices in I. $\square$

080R Lemma 10.15.6. Let R be a ring. Let $n \geq m$. Let $A = (a_{ij})$ be an $n \times m$ matrix with coefficients in R, written in block form as

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 has size $m \times m$. Let B be the adjugate (transpose of cofactor) matrix to A_1. Then

$$AB = \begin{pmatrix} f 1_{m \times m} \\ C \end{pmatrix}$$

where $f = \det(A_1)$ and c_{ij} is (up to sign) the determinant of the $m \times m$ minor of A corresponding to the rows $1, \ldots, \hat{j}, \ldots, m, i$.

Proof. Since the adjugate has the property $A_1 B = B A_1 = f$ the first block of the expression for AB is correct. Note that

$$c_{ij} = \sum\nolimits_k a_{ik} b_{kj} = \sum (-1)^{j+k} a_{ik} \det(A_1^{jk})$$

where A_1^{ij} means A_1 with the jth row and kth column removed. This last expression is the row expansion of the determinant of the matrix in the statement of the lemma. $\square$

05WI Lemma 10.15.7. Let R be a nonzero ring. Let $n \geq 1$. Let M be an R-module generated by $< n$ elements. Then any R-module map $f : R^{\oplus n} \to M$ has a nonzero kernel.

Proof. Choose a surjection $R^{\oplus n-1} \to M$. We may lift the map f to a map $f' : R^{\oplus n} \to R^{\oplus n-1}$ (Lemma 10.5.2). It suffices to prove f' has a nonzero kernel. The map $f' : R^{\oplus n} \to R^{\oplus n-1}$ is given by a matrix $A = (a_{ij})$. If one of the a_{ij} is not nilpotent, say $a = a_{ij}$ is not, then we can replace R by the localization R_a and we may assume a_{ij} is a unit. Since if we find a nonzero kernel after localization then there was a nonzero kernel to start with as localization is exact, see Proposition 10.9.12. In this case we can do a base change on both $R^{\oplus n}$ and $R^{\oplus n-1}$ and reduce to the case where

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & a_{22} & a_{23} & \ldots \\ 0 & a_{32} & \ldots & \\ \ldots & \ldots & & \end{pmatrix}$$

Hence in this case we win by induction on n. If not then each a_{ij} is nilpotent. Set $I = (a_{ij}) \subset R$. Note that $I^{m+1} = 0$ for some $m \geq 0$. Let m be the largest integer such that $I^m \neq 0$. Then we see that $(I^m)^{\oplus n}$ is contained in the kernel of the map and we win. □

0FJ7 Lemma 10.15.8. Let R be a nonzero ring. Let $n, m \geq 0$ be integers. If $R^{\oplus n}$ is isomorphic to $R^{\oplus m}$ as R-modules, then $n = m$.

Proof. Immediate from Lemma 10.15.7. □

10.16. Cayley-Hamilton

05G6

00DX Lemma 10.16.1. Let R be a ring. Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R. Let $P(x) \in R[x]$ be the characteristic polynomial of A (defined as $\det(x\mathrm{id}_{n\times n} - A)$). Then $P(A) = 0$ in $\mathrm{Mat}(n \times n, R)$.

Proof. We reduce the question to the well-known Cayley-Hamilton theorem from linear algebra in several steps:

(1) If $\phi : S \to R$ is a ring morphism and b_{ij} are inverse images of the a_{ij} under this map, then it suffices to show the statement for S and (b_{ij}) since ϕ is a ring morphism.
(2) If $\psi : R \hookrightarrow S$ is an injective ring morphism, it clearly suffices to show the result for S and the a_{ij} considered as elements of S.
(3) Thus we may first reduce to the case $R = \mathbf{Z}[X_{ij}]$, $a_{ij} = X_{ij}$ of a polynomial ring and then further to the case $R = \mathbf{Q}(X_{ij})$ where we may finally apply Cayley-Hamilton.

□

05BT Lemma 10.16.2. Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism. Then there exists a monic polynomial $P \in R[T]$ such that $P(\varphi) = 0$ as an endomorphism of M.

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in R^{\oplus n}$ such that $\varphi(x_i) =$

$\sum a_{ij}x_j$. In other words the diagram

$$\begin{tikzcd} R^{\oplus n} \arrow[r] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ R^{\oplus n} \arrow[r] & M \end{tikzcd}$$

is commutative where $A = (a_{ij})$. By Lemma 10.16.1 there exists a monic polynomial P such that $P(A) = 0$. Then it follows that $P(\varphi) = 0$. □

05G7 Lemma 10.16.3. Let R be a ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism such that $\varphi(M) \subset IM$. Then there exists a monic polynomial $P = t^n + a_1 t^{n-1} + \ldots + a_n \in R[T]$ such that $a_j \in I^j$ and $P(\varphi) = 0$ as an endomorphism of M.

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in I^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij}x_j$. In other words the diagram

$$\begin{tikzcd} R^{\oplus n} \arrow[r] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ I^{\oplus n} \arrow[r] & M \end{tikzcd}$$

is commutative where $A = (a_{ij})$. By Lemma 10.16.1 the polynomial $P(t) = \det(t\mathrm{id}_{n\times n} - A)$ has all the desired properties. □

As a fun example application we prove the following surprising lemma.

05G8 Lemma 10.16.4. Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.

First proof. Write $R' = R[x]$ and think of M as a finite R'-module with x acting via φ. Set $I = (x) \subset R'$. By our assumption that φ is surjective we have $IM = M$. Hence we may apply Lemma 10.16.3 to M as an R'-module, the ideal I and the endomorphism id_M. We conclude that $(1 + a_1 + \ldots + a_n)\mathrm{id}_M = 0$ with $a_j \in I$. Write $a_j = b_j(x)x$ for some $b_j(x) \in R[x]$. Translating back into φ we see that $\mathrm{id}_M = -(\sum_{j=1,\ldots,n} b_j(\varphi))\varphi$, and hence φ is invertible. □

Second proof. We perform induction on the number of generators of M over R. If M is generated by one element, then $M \cong R/I$ for some ideal $I \subset R$. In this case we may replace R by R/I so that $M = R$. In this case $\varphi : R \to R$ is given by multiplication on M by an element $r \in R$. The surjectivity of φ forces r invertible, since φ must hit 1, which implies that φ is invertible.

Now assume that we have proven the lemma in the case of modules generated by $n - 1$ elements, and are examining a module M generated by n elements. Let A mean the ring $R[t]$, and regard the module M as an A-module by letting t act via φ; since M is finite over R, it is finite over $R[t]$ as well, and since we're trying to prove φ injective, a set-theoretic property, we might as well prove the endomorphism $t : M \to M$ over A injective. We have reduced our problem to the case our endomorphism is multiplication by an element of the ground ring. Let

$M' \subset M$ denote the sub-A-module generated by the first $n-1$ of the generators of M, and consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "\varphi|_{M'}"] & M \arrow[r] \arrow[d, "\varphi"] & M/M' \arrow[r] \arrow[d, "\varphi \bmod M'"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M/M' \arrow[r] & 0, \end{tikzcd}$$

where the restriction of φ to M' and the map induced by φ on the quotient M/M' are well-defined since φ is multiplication by an element in the base, and M' and M/M' are A-modules in their own right. By the case $n = 1$ the map $M/M' \to M/M'$ is an isomorphism. A diagram chase implies that $\varphi|_{M'}$ is surjective hence by induction $\varphi|_{M'}$ is an isomorphism. This forces the middle column to be an isomorphism by the snake lemma. □

10.17. The spectrum of a ring

00DY We arbitrarily decide that the spectrum of a ring as a topological space is part of the algebra chapter, whereas an affine scheme is part of the chapter on schemes.

00DZ Definition 10.17.1. Let R be a ring.

(1) The spectrum of R is the set of prime ideals of R. It is usually denoted $\mathrm{Spec}(R)$.
(2) Given a subset $T \subset R$ we let $V(T) \subset \mathrm{Spec}(R)$ be the set of primes containing T, i.e., $V(T) = \{\mathfrak p \in \mathrm{Spec}(R) \mid \forall f \in T, f \in \mathfrak p\}$.
(3) Given an element $f \in R$ we let $D(f) \subset \mathrm{Spec}(R)$ be the set of primes not containing f.

00E0 Lemma 10.17.2. Let R be a ring.

(1) The spectrum of a ring R is empty if and only if R is the zero ring.
(2) Every nonzero ring has a maximal ideal.
(3) Every nonzero ring has a minimal prime ideal.
(4) Given an ideal $I \subset R$ and a prime ideal $I \subset \mathfrak p$ there exists a prime $I \subset \mathfrak q \subset \mathfrak p$ such that $\mathfrak q$ is minimal over I.
(5) If $T \subset R$, and if (T) is the ideal generated by T in R, then $V((T)) = V(T)$.
(6) If I is an ideal and $\sqrt{I}$ is its radical, see basic notion (27), then $V(I) = V(\sqrt{I})$.
(7) Given an ideal I of R we have $\sqrt{I} = \bigcap_{I \subset \mathfrak p} \mathfrak p$.
(8) If I is an ideal then $V(I) = \emptyset$ if and only if I is the unit ideal.
(9) If I, J are ideals of R then $V(I) \cup V(J) = V(I \cap J)$.
(10) If $(I_a)_{a\in A}$ is a set of ideals of R then $\bigcap_{a\in A} V(I_a) = V(\bigcup_{a\in A} I_a)$.
(11) If $f \in R$, then $D(f) \amalg V(f) = \mathrm{Spec}(R)$.
(12) If $f \in R$ then $D(f) = \emptyset$ if and only if f is nilpotent.
(13) If $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$.
(14) If $I \subset R$ is an ideal, and $\mathfrak p$ is a prime of R with $\mathfrak p \notin V(I)$, then there exists an $f \in R$ such that $\mathfrak p \in D(f)$, and $D(f) \cap V(I) = \emptyset$.
(15) If $f, g \in R$, then $D(fg) = D(f) \cap D(g)$.
(16) If $f_i \in R$ for $i \in I$, then $\bigcup_{i\in I} D(f_i)$ is the complement of $V(\{f_i\}_{i\in I})$ in $\mathrm{Spec}(R)$.
(17) If $f \in R$ and $D(f) = \mathrm{Spec}(R)$, then f is a unit.

Proof. We address each part in the corresponding item below.

(1) This is a direct consequence of (2) or (3).
(2) Let $\mathfrak{A}$ be the set of all proper ideals of R. This set is ordered by inclusion and is non-empty, since $(0) \in \mathfrak{A}$ is a proper ideal. Let A be a totally ordered subset of $\mathfrak{A}$. Then $\bigcup_{I \in A} I$ is in fact an ideal. Since $1 \notin I$ for all $I \in A$, the union does not contain 1 and thus is proper. Hence $\bigcup_{I \in A} I$ is in $\mathfrak{A}$ and is an upper bound for the set A. Thus by Zorn's lemma $\mathfrak{A}$ has a maximal element, which is the sought-after maximal ideal.
(3) Since R is nonzero, it contains a maximal ideal which is a prime ideal. Thus the set $\mathfrak{A}$ of all prime ideals of R is nonempty. $\mathfrak{A}$ is ordered by reverse-inclusion. Let A be a totally ordered subset of $\mathfrak{A}$. It's pretty clear that $J = \bigcap_{I \in A} I$ is in fact an ideal. Not so clear, however, is that it is prime. Let $xy \in J$. Then $xy \in I$ for all $I \in A$. Now let $B = \{I \in A | y \in I\}$. Let $K = \bigcap_{I \in B} I$. Since A is totally ordered, either $K = J$ (and we're done, since then $y \in J$) or $K \supset J$ and for all $I \in A$ such that I is properly contained in K, we have $y \notin I$. But that means that for all those $I, x \in I$, since they are prime. Hence $x \in J$. In either case, J is prime as desired. Hence by Zorn's lemma we get a maximal element which in this case is a minimal prime ideal.
(4) This is the same exact argument as (3) except you only consider prime ideals contained in $\mathfrak{p}$ and containing I.
(5) (T) is the smallest ideal containing T. Hence if $T \subset I$, some ideal, then $(T) \subset I$ as well. Hence if $I \in V(T)$, then $I \in V((T))$ as well. The other inclusion is obvious.
(6) Since $I \subset \sqrt{I}, V(\sqrt{I}) \subset V(I)$. Now let $\mathfrak{p} \in V(I)$. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some n. Hence $x^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, a boring induction argument gets you that $x \in \mathfrak{p}$. Hence $\sqrt{I} \subset \mathfrak{p}$ and $\mathfrak{p} \in V(\sqrt{I})$.
(7) Let $f \in R \setminus \sqrt{I}$. Then $f^n \notin I$ for all n. Hence $S = \{1, f, f^2, \ldots\}$ is a multiplicative subset, not containing 0. Take a prime ideal $\bar{\mathfrak{p}} \subset S^{-1}R$ containing $S^{-1}I$. Then the pull-back $\mathfrak{p}$ in R of $\bar{\mathfrak{p}}$ is a prime ideal containing I that does not intersect S. This shows that $\bigcap_{I \subset \mathfrak{p}} \mathfrak{p} \subset \sqrt{I}$. Now if $a \in \sqrt{I}$, then $a^n \in I$ for some n. Hence if $I \subset \mathfrak{p}$, then $a^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$. Thus the equality is shown.
(8) I is not the unit ideal if and only if I is contained in some maximal ideal (to see this, apply (2) to the ring R/I) which is therefore prime.
(9) If $\mathfrak{p} \in V(I) \cup V(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ which means that $I \cap J \subset \mathfrak{p}$. Now if $I \cap J \subset \mathfrak{p}$, then $IJ \subset \mathfrak{p}$ and hence either I of J is in $\mathfrak{p}$, since $\mathfrak{p}$ is prime.
(10) $\mathfrak{p} \in \bigcap_{a \in A} V(I_a) \Leftrightarrow I_a \subset \mathfrak{p}, \forall a \in A \Leftrightarrow \mathfrak{p} \in V(\bigcup_{a \in A} I_a)$
(11) If $\mathfrak{p}$ is a prime ideal and $f \in R$, then either $f \in \mathfrak{p}$ or $f \notin \mathfrak{p}$ (strictly) which is what the disjoint union says.
(12) If $a \in R$ is nilpotent, then $a^n = 0$ for some n. Hence $a^n \in \mathfrak{p}$ for any prime ideal. Thus $a \in \mathfrak{p}$ as can be shown by induction and $D(a) = \emptyset$. Now, as shown in (7), if $a \in R$ is not nilpotent, then there is a prime ideal that does not contain it.
(13) $f \in \mathfrak{p} \Leftrightarrow uf \in \mathfrak{p}$, since u is invertible.
(14) If $\mathfrak{p} \notin V(I)$, then $\exists f \in I \setminus \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so $\mathfrak{p} \in D(f)$. Also if $\mathfrak{q} \in D(f)$, then $f \notin \mathfrak{q}$ and thus I is not contained in $\mathfrak{q}$. Thus $D(f) \cap V(I) = \emptyset$.

(15) If $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Hence if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, if $fg \notin \mathfrak{p}$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.
(16) $\mathfrak{p} \in \bigcup_{i \in I} D(f_i) \Leftrightarrow \exists i \in I, f_i \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in \operatorname{Spec}(R) \setminus V(\{f_i\}_{i \in I})$
(17) If $D(f) = \operatorname{Spec}(R)$, then $V(f) = \emptyset$ and hence $fR = R$, so f is a unit.

□

The lemma implies that the subsets $V(T)$ from Definition 10.17.1 form the closed subsets of a topology on $\operatorname{Spec}(R)$. And it also shows that the sets $D(f)$ are open and form a basis for this topology.

00E1 Definition 10.17.3. Let R be a ring. The topology on $\operatorname{Spec}(R)$ whose closed sets are the sets $V(T)$ is called the Zariski topology. The open subsets $D(f)$ are called the standard opens of $\operatorname{Spec}(R)$.

It should be clear from context whether we consider $\operatorname{Spec}(R)$ just as a set or as a topological space.

00E2 Lemma 10.17.4. Suppose that $\varphi : R \to R'$ is a ring homomorphism. The induced map

$$\operatorname{Spec}(\varphi) : \operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R), \quad \mathfrak{p}' \longmapsto \varphi^{-1}(\mathfrak{p}')$$

is continuous for the Zariski topologies. In fact, for any element $f \in R$ we have $\operatorname{Spec}(\varphi)^{-1}(D(f)) = D(\varphi(f))$.

Proof. It is basic notion (41) that $\mathfrak{p} := \varphi^{-1}(\mathfrak{p}')$ is indeed a prime ideal of R. The last assertion of the lemma follows directly from the definitions, and implies the first. □

If $\varphi' : R' \to R''$ is a second ring homomorphism then the composition

$$\operatorname{Spec}(R'') \longrightarrow \operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R)$$

equals $\operatorname{Spec}(\varphi' \circ \varphi)$. In other words, Spec is a contravariant functor from the category of rings to the category of topological spaces.

00E3 Lemma 10.17.5. Let R be a ring. Let $S \subset R$ be a multiplicative subset. The map $R \to S^{-1}R$ induces via the functoriality of Spec a homeomorphism

$$\operatorname{Spec}(S^{-1}R) \longrightarrow \{\mathfrak{p} \in \operatorname{Spec}(R) \mid S \cap \mathfrak{p} = \emptyset\}$$

where the topology on the right hand side is that induced from the Zariski topology on $\operatorname{Spec}(R)$. The inverse map is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

Proof. Denote the right hand side of the arrow of the lemma by D. Choose a prime $\mathfrak{p}' \subset S^{-1}R$ and let $\mathfrak{p}$ the inverse image of $\mathfrak{p}'$ in R. Since $\mathfrak{p}'$ does not contain 1 we see that $\mathfrak{p}$ does not contain any element of S. Hence $\mathfrak{p} \in D$ and we see that the image is contained in D. Let $\mathfrak{p} \in D$. By assumption the image $\overline{S}$ does not contain 0. By basic notion (54) $\overline{S}^{-1}(R/\mathfrak{p})$ is not the zero ring. By basic notion (62) we see $S^{-1}R/S^{-1}\mathfrak{p} = \overline{S}^{-1}(R/\mathfrak{p})$ is a domain, and hence $S^{-1}\mathfrak{p}$ is a prime. The equality of rings also shows that the inverse image of $S^{-1}\mathfrak{p}$ in R is equal to $\mathfrak{p}$, because $R/\mathfrak{p} \to \overline{S}^{-1}(R/\mathfrak{p})$ is injective by basic notion (55). This proves that the map $\operatorname{Spec}(S^{-1}R) \to \operatorname{Spec}(R)$ is bijective onto D with inverse as given. It is continuous by Lemma 10.17.4. Finally, let $D(g) \subset \operatorname{Spec}(S^{-1}R)$ be a standard open. Write $g = h/s$ for some $h \in R$ and $s \in S$. Since g and $h/1$ differ by a unit we have $D(g) = D(h/1)$ in $\operatorname{Spec}(S^{-1}R)$. Hence by Lemma 10.17.4 and the bijectivity above the image of $D(g) = D(h/1)$ is $D \cap D(h)$. This proves the map is open as well. □

00E4 **Lemma 10.17.6.** Let R be a ring. Let $f \in R$. The map $R \to R_f$ induces via the functoriality of Spec a homeomorphism

$$\mathrm{Spec}(R_f) \longrightarrow D(f) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p} \cdot R_f$.

Proof. This is a special case of Lemma 10.17.5. □

It is not the case that every "affine open" of a spectrum is a standard open. See Example 10.27.4.

00E5 **Lemma 10.17.7.** Let R be a ring. Let $I \subset R$ be an ideal. The map $R \to R/I$ induces via the functoriality of Spec a homeomorphism

$$\mathrm{Spec}(R/I) \longrightarrow V(I) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p}/I$.

Proof. It is immediate that the image is contained in $V(I)$. On the other hand, if $\mathfrak{p} \in V(I)$ then $\mathfrak{p} \supset I$ and we may consider the ideal $\mathfrak{p}/I \subset R/I$. Using basic notion (51) we see that $(R/I)/(\mathfrak{p}/I) = R/\mathfrak{p}$ is a domain and hence $\mathfrak{p}/I$ is a prime ideal. From this it is immediately clear that the image of $D(f + I)$ is $D(f) \cap V(I)$, and hence the map is a homeomorphism. □

00E6 **Remark 10.17.8.** A fundamental commutative diagram associated to a ring map $\varphi : R \to S$, a prime $\mathfrak{q} \subset S$ and the corresponding prime $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ of R is the following

$$\begin{tikzcd}
\kappa(\mathfrak{q}) = S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} & S_{\mathfrak{q}} \arrow[l] & S \arrow[l] \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) \\
\kappa(\mathfrak{p}) \otimes_R S = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \arrow[u] & S_{\mathfrak{p}} \arrow[l] \arrow[u] & S \arrow[l] \arrow[r] \arrow[u] & S/\mathfrak{p}S \arrow[r] \arrow[u] & (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S \arrow[u] \\
\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \arrow[u] & R_{\mathfrak{p}} \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u] & R/\mathfrak{p} \arrow[r] \arrow[u] & \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

In this diagram the arrows in the outer left and outer right columns are identical. The horizontal maps induce on the associated spectra always a homeomorphism onto the image. The lower two rows of the diagram make sense without assuming $\mathfrak{q}$ exists. The lower squares induce fibre squares of topological spaces. This diagram shows that $\mathfrak{p}$ is in the image of the map on Spec if and only if $S \otimes_R \kappa(\mathfrak{p})$ is not the zero ring.

00E7 **Lemma 10.17.9.** Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. The following are equivalent

(1) $\mathfrak{p}$ is in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$,
(2) $S \otimes_R \kappa(\mathfrak{p}) \neq 0$,
(3) $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \neq 0$,
(4) $(S/\mathfrak{p}S)_{\mathfrak{p}} \neq 0$, and
(5) $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$.

Proof. We have already seen the equivalence of the first two in Remark 10.17.8. The others are just reformulations of this. □

00E8 **Lemma 10.17.10.** Let R be a ring. The space $\mathrm{Spec}(R)$ is quasi-compact.

Proof. It suffices to prove that any covering of $\mathrm{Spec}(R)$ by standard opens can be refined by a finite covering. Thus suppose that $\mathrm{Spec}(R) = \cup D(f_i)$ for a set of elements $\{f_i\}_{i\in I}$ of R. This means that $\cap V(f_i) = \emptyset$. According to Lemma 10.17.2 this means that $V(\{f_i\}) = \emptyset$. According to the same lemma this means that the ideal generated by the f_i is the unit ideal of R. This means that we can write 1 as a finite sum: $1 = \sum_{i\in J} r_i f_i$ with $J \subset I$ finite. And then it follows that $\mathrm{Spec}(R) = \cup_{i\in J} D(f_i)$. $\square$

04PM **Lemma 10.17.11.** Let R be a ring. The topology on $X = \mathrm{Spec}(R)$ has the following properties:

(1) X is quasi-compact,
(2) X has a basis for the topology consisting of quasi-compact opens, and
(3) the intersection of any two quasi-compact opens is quasi-compact.

Proof. The spectrum of a ring is quasi-compact, see Lemma 10.17.10. It has a basis for the topology consisting of the standard opens $D(f) = \mathrm{Spec}(R_f)$ (Lemma 10.17.6) which are quasi-compact by the first remark. The intersection of two standard opens is quasi-compact as $D(f) \cap D(g) = D(fg)$. Given any two quasi-compact opens $U, V \subset X$ we may write $U = D(f_1) \cup \ldots \cup D(f_n)$ and $V = D(g_1) \cup \ldots \cup D(g_m)$. Then $U \cap V = \bigcup D(f_i g_j)$ which is quasi-compact. $\square$

10.18. Local rings

07BH Local rings are the bread and butter of algebraic geometry.

07BI **Definition 10.18.1.** A *local ring* is a ring with exactly one maximal ideal. The maximal ideal is often denoted $\mathfrak{m}_R$ in this case. We often say "let $(R, \mathfrak{m}, \kappa)$ be a local ring" to indicate that R is local, $\mathfrak{m}$ is its unique maximal ideal and $\kappa = R/\mathfrak{m}$ is its residue field. A *local homomorphism of local rings* is a ring map $\varphi : R \to S$ such that R and S are local rings and such that $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$. If it is given that R and S are local rings, then the phrase "local ring map $\varphi : R \to S$" means that φ is a local homomorphism of local rings.

A field is a local ring. Any ring map between fields is a local homomorphism of local rings.

00E9 **Lemma 10.18.2.** Let R be a ring. The following are equivalent:

(1) R is a local ring,
(2) $\mathrm{Spec}(R)$ has exactly one closed point,
(3) R has a maximal ideal $\mathfrak{m}$ and every element of $R \setminus \mathfrak{m}$ is a unit, and
(4) R is not the zero ring and for every $x \in R$ either x or $1 - x$ is invertible or both.

Proof. Let R be a ring, and $\mathfrak{m}$ a maximal ideal. If $x \in R \setminus \mathfrak{m}$, and x is not a unit then there is a maximal ideal $\mathfrak{m}'$ containing x. Hence R has at least two maximal ideals. Conversely, if $\mathfrak{m}'$ is another maximal ideal, then choose $x \in \mathfrak{m}'$, $x \notin \mathfrak{m}$. Clearly x is not a unit. This proves the equivalence of (1) and (3). The equivalence (1) and (2) is tautological. If R is local then (4) holds since x is either in $\mathfrak{m}$ or not. If (4) holds, and $\mathfrak{m}$, $\mathfrak{m}'$ are distinct maximal ideals then we may choose $x \in R$ such that $x \bmod \mathfrak{m}' = 0$ and $x \bmod \mathfrak{m} = 1$ by the Chinese remainder theorem

(Lemma 10.15.4). This element x is not invertible and neither is $1-x$ which is a contradiction. Thus (4) and (1) are equivalent. $\square$

The localization $R_\mathfrak{p}$ of a ring R at a prime $\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Namely, the quotient $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the fraction field of the domain $R/\mathfrak{p}$ and every element of $R_\mathfrak{p}$ which is not contained in $\mathfrak{p}R_\mathfrak{p}$ is invertible.

07BJ Lemma 10.18.3. Let $\varphi : R \to S$ be a ring map. Assume R and S are local rings. The following are equivalent:

(1) φ is a local ring map,
(2) $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$, and
(3) $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.
(4) For any $x \in R$, if $\varphi(x)$ is invertible in S, then x is invertible in R.

Proof. Conditions (1) and (2) are equivalent by definition. If (3) holds then (2) holds. Conversely, if (2) holds, then $\varphi^{-1}(\mathfrak{m}_S)$ is a prime ideal containing the maximal ideal $\mathfrak{m}_R$, hence $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$. Finally, (4) is the contrapositive of (2) by Lemma 10.18.2. $\square$

Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime and set $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Then the induced ring map $R_\mathfrak{p} \to S_\mathfrak{q}$ is a local ring map.

10.19. The Jacobson radical of a ring

0AMD We recall that the Jacobson radical $\text{rad}(R)$ of a ring R is the intersection of all maximal ideals of R. If R is local then $\text{rad}(R)$ is the maximal ideal of R.

0AME Lemma 10.19.1. Let R be a ring with Jacobson radical $\text{rad}(R)$. Let $I \subset R$ be an ideal. The following are equivalent

(1) $I \subset \text{rad}(R)$, and
(2) every element of $1 + I$ is a unit in R.

In this case every element of R which maps to a unit of R/I is a unit.

Proof. If $f \in \text{rad}(R)$, then $f \in \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Hence $1 + f \not\in \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Thus the closed subset $V(1+f)$ of $\text{Spec}(R)$ is empty. This implies that $1 + f$ is a unit, see Lemma 10.17.2.

Conversely, assume that $1 + f$ is a unit for all $f \in I$. If $\mathfrak{m}$ is a maximal ideal and $I \not\subset \mathfrak{m}$, then $I + \mathfrak{m} = R$. Hence $1 = f + g$ for some $g \in \mathfrak{m}$ and $f \in I$. Then $g = 1 + (-f)$ is not a unit, contradiction.

For the final statement let $f \in R$ map to a unit in R/I. Then we can find $g \in R$ mapping to the multiplicative inverse of $f \bmod I$. Then $fg = 1 \bmod I$. Hence fg is a unit of R by (2) which implies that f is a unit. $\square$

0B7C Lemma 10.19.2. Let $\varphi : R \to S$ be a ring map such that the induced map $\text{Spec}(S) \to \text{Spec}(R)$ is surjective. Then an element $x \in R$ is a unit if and only if $\varphi(x) \in S$ is a unit.

Proof. If x is a unit, then so is $\varphi(x)$. Conversely, if $\varphi(x)$ is a unit, then $\varphi(x) \not\in \mathfrak{q}$ for all $\mathfrak{q} \in \text{Spec}(S)$. Hence $x \not\in \varphi^{-1}(\mathfrak{q}) = \text{Spec}(\varphi)(\mathfrak{q})$ for all $\mathfrak{q} \in \text{Spec}(S)$. Since $\text{Spec}(\varphi)$ is surjective we conclude that x is a unit by part (17) of Lemma 10.17.2. $\square$

10.20. Nakayama's lemma

07RC We quote from [Mat70a]: "This simple but important lemma is due to T. Nakayama, G. Azumaya and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name."

00DV **Lemma 10.20.1** (Nakayama's lemma). Let R be a ring with Jacobson radical $\text{rad}(R)$. Let M be an R-module. Let $I \subset R$ be an ideal. [Mat70a, 1.M Lemma (NAK) page 11]

00DW
(1) If $IM = M$ and M is finite, then there exists an $f \in 1 + I$ such that $fM = 0$.
(2) If $IM = M$, M is finite, and $I \subset \text{rad}(R)$, then $M = 0$.
(3) If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists an $f \in 1 + I$ such that $fM \subset N$ and $M_f = N_f$.
(4) If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset \text{rad}(R)$, then $M = N$.
(5) If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists an $f \in 1 + I$ such that $N_f \to M_f$ is surjective.
(6) If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset \text{rad}(R)$, then $N \to M$ is surjective.
(7) If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.
(8) If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset \text{rad}(R)$, then M is generated by $x_1, \ldots, x_n$.
(9) If $IM = M$, I is nilpotent, then $M = 0$.
(10) If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.
(11) If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.
(12) If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$ (since $M = IM$). In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$ (since the matrix A is entrywise congruent to the $m \times m$ identity matrix modulo I). By Lemma 10.15.5 (1), there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $\sum_i b_{hi} a_{ij} = f \delta_{hj}$ for all h and j; hence, $\sum_{i,j} b_{hi} a_{ij} y_j = \sum_j f \delta_{hj} y_j = f y_h$ for every h. In other words, $0 = f y_h$ for every h (since each i satisfies $\sum_j a_{ij} y_j = 0$). This implies that f annihilates M.

By Lemma 10.19.1 an element of $1 + \text{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$.

Part (9) holds because if $M = IM$ then $M = I^n M$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. $\square$

0GLX Lemma 10.20.2. Let R be a ring, let $S \subset R$ be a multiplicative subset, let $I \subset R$ be an ideal, and let M be a finite R-module. If $x_1, \ldots, x_r \in M$ generate $S^{-1}(M/IM)$ as an $S^{-1}(R/I)$-module, then there exists an $f \in S+I$ such that $x_1, \ldots, x_r$ generate M_f as an R_f-module.[1]

Proof. Special case $I = 0$. Let $y_1, \ldots, y_s$ be generators for M over R. Since $S^{-1}M$ is generated by $x_1, \ldots, x_r$, for each i we can write $y_i = \sum (a_{ij}/s_{ij})x_j$ for some $a_{ij} \in R$ and $s_{ij} \in S$. Let $s \in S$ be the product of all of the s_{ij}. Then we see that y_i is contained in the R_s-submodule of M_s generated by $x_1, \ldots, x_r$. Hence $x_1, \ldots, x_r$ generates M_s.

General case. By the special case, we can find an $s \in S$ such that $x_1, \ldots, x_r$ generate $(M/IM)_s$ over $(R/I)_s$. By Lemma 10.20.1 we can find a $g \in 1 + I_s \subset R_s$ such that $x_1, \ldots, x_r$ generate $(M_s)_g$ over $(R_s)_g$. Write $g = 1 + i/s'$. Then $f = ss' + is$ works; details omitted. □

0E8M Lemma 10.20.3. Let $A \to B$ be a local homomorphism of local rings. Assume

(1) B is finite as an A-module,
(2) $\mathfrak{m}_B$ is a finitely generated ideal,
(3) $A \to B$ induces an isomorphism on residue fields, and
(4) $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.

Then $A \to B$ is surjective.

Proof. To show that $A \to B$ is surjective, we view it as a map of A-modules and apply Lemma 10.20.1 (6). We conclude it suffices to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B$ is surjective. As $A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_A B \to \mathfrak{m}_B$ is surjective. View $\mathfrak{m}_A B \to \mathfrak{m}_B$ as a map of B-modules and apply Lemma 10.20.1 (6). We conclude it suffices to see that $\mathfrak{m}_A B/\mathfrak{m}_A\mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. This follows from assumption (4). □

10.21. Open and closed subsets of spectra

04PN It turns out that open and closed subsets of a spectrum correspond to idempotents of the ring.

00EC Lemma 10.21.1. Let R be a ring. Let $e \in R$ be an idempotent. In this case

$$\mathrm{Spec}(R) = D(e) \amalg D(1-e).$$

Proof. Note that an idempotent e of a domain is either 1 or 0. Hence we see that

$$\begin{aligned} D(e) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \not\in \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \neq 0 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e = 1 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$

Similarly we have

$$\begin{aligned} D(1-e) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid 1-e \not\in \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \neq 1 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e = 0 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$

[1]Special cases: (I) $I = 0$. The lemma says if $x_1, \ldots, x_r$ generate $S^{-1}M$, then $x_1, \ldots, x_r$ generate M_f for some $f \in S$. (II) $I = \mathfrak{p}$ is a prime ideal and $S = R \setminus \mathfrak{p}$. The lemma says if $x_1, \ldots, x_r$ generate $M \otimes_R \kappa(\mathfrak{p})$ then $x_1, \ldots, x_r$ generate M_f for some $f \in R$, $f \not\in \mathfrak{p}$.

Since the image of e in any residue field is either 1 or 0 we deduce that $D(e)$ and $D(1-e)$ cover all of $\mathrm{Spec}(R)$. □

00ED Lemma 10.21.2. Let R_1 and R_2 be rings. Let $R = R_1 \times R_2$. The maps $R \to R_1$, $(x, y) \mapsto x$ and $R \to R_2$, $(x, y) \mapsto y$ induce continuous maps $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(R_2) \to \mathrm{Spec}(R)$. The induced map

$$\mathrm{Spec}(R_1) \amalg \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R)$$

is a homeomorphism. In other words, the spectrum of $R = R_1 \times R_2$ is the disjoint union of the spectrum of R_1 and the spectrum of R_2.

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. We leave it to the reader to show that $R_1 = R_{e_1}$ is the localization of R at e_1. Similarly for e_2. Thus the statement of the lemma follows from Lemma 10.21.1 combined with Lemma 10.17.6. □

We reprove the following lemma later after introducing a glueing lemma for functions. See Section 10.24.

00EE Lemma 10.21.3. Let R be a ring. For each $U \subset \mathrm{Spec}(R)$ which is open and closed there exists a unique idempotent $e \in R$ such that $U = D(e)$. This induces a 1-1 correspondence between open and closed subsets $U \subset \mathrm{Spec}(R)$ and idempotents $e \in R$.

Proof. Let $U \subset \mathrm{Spec}(R)$ be open and closed. Since U is closed it is quasi-compact by Lemma 10.17.10, and similarly for its complement. Write $U = \bigcup_{i=1}^n D(f_i)$ as a finite union of standard opens. Similarly, write $\mathrm{Spec}(R) \setminus U = \bigcup_{j=1}^m D(g_j)$ as a finite union of standard opens. Since $\emptyset = D(f_i) \cap D(g_j) = D(f_i g_j)$ we see that $f_i g_j$ is nilpotent by Lemma 10.17.2. Let $I = (f_1, \ldots, f_n) \subset R$ and let $J = (g_1, \ldots, g_m) \subset R$. Note that $V(J)$ equals U, that $V(I)$ equals the complement of U, so $\mathrm{Spec}(R) = V(I) \amalg V(J)$. By the remark on nilpotency above, we see that $(IJ)^N = (0)$ for some sufficiently large integer N. Since $\bigcup D(f_i) \cup \bigcup D(g_j) = \mathrm{Spec}(R)$ we see that $I + J = R$, see Lemma 10.17.2. By raising this equation to the $2N$th power we conclude that $I^N + J^N = R$. Write $1 = x + y$ with $x \in I^N$ and $y \in J^N$. Then $0 = xy = x(1-x)$ as $I^N J^N = (0)$. Thus $x = x^2$ is idempotent and contained in $I^N \subset I$. The idempotent $y = 1 - x$ is contained in $J^N \subset J$. This shows that the idempotent x maps to 1 in every residue field $\kappa(\mathfrak p)$ for $\mathfrak p \in V(J)$ and that x maps to 0 in $\kappa(\mathfrak p)$ for every $\mathfrak p \in V(I)$.

To see uniqueness suppose that e_1, e_2 are distinct idempotents in R. We have to show there exists a prime $\mathfrak p$ such that $e_1 \in \mathfrak p$ and $e_2 \notin \mathfrak p$, or conversely. Write $e_i' = 1 - e_i$. If $e_1 \neq e_2$, then $0 \neq e_1 - e_2 = e_1(e_2 + e_2') - (e_1 + e_1')e_2 = e_1 e_2' - e_1' e_2$. Hence either the idempotent $e_1 e_2' \neq 0$ or $e_1' e_2 \neq 0$. An idempotent is not nilpotent, and hence we find a prime $\mathfrak p$ such that either $e_1 e_2' \notin \mathfrak p$ or $e_1' e_2 \notin \mathfrak p$, by Lemma 10.17.2. It is easy to see this gives the desired prime. □

00EF Lemma 10.21.4. Let R be a nonzero ring. Then $\mathrm{Spec}(R)$ is connected if and only if R has no nontrivial idempotents.

Proof. Obvious from Lemma 10.21.3 and the definition of a connected topological space. □

00EH Lemma 10.21.5. Let $I \subset R$ be a finitely generated ideal of a ring R such that $I = I^2$. Then

(1) there exists an idempotent $e \in R$ such that $I = (e)$,
(2) $R/I \cong R_{e'}$ for the idempotent $e' = 1 - e \in R$, and
(3) $V(I)$ is open and closed in $\text{Spec}(R)$.

Proof. By Nakayama's Lemma 10.20.1 there exists an element $f = 1+i$, $i \in I$ such that $fI = 0$. Then $f^2 = f + fi = f$ is an idempotent. Consider the idempotent $e = 1 - f = -i \in I$. For $j \in I$ we have $ej = j - fj = j$ hence $I = (e)$. This proves (1).

Parts (2) and (3) follow from (1). Namely, we have $V(I) = V(e) = \text{Spec}(R) \setminus D(e)$ which is open and closed by either Lemma 10.21.1 or Lemma 10.21.3. This proves (3). For (2) observe that the map $R \to R_{e'}$ is surjective since $x/(e')^n = x/e' = xe'/(e')^2 = xe'/e' = x/1$ in $R_{e'}$. The kernel of the map $R \to R_{e'}$ is the set of elements of R annihilated by a positive power of e'. Since e' is idempotent this is the ideal of elements annihilated by e' which is the ideal $I = (e)$ as $e + e' = 1$ is a pair of orthognal idempotents. This proves (2). □

10.22. Connected components of spectra

00EB Connected components of spectra are not as easy to understand as one may think at first. This is because we are used to the topology of locally connected spaces, but the spectrum of a ring is in general not locally connected.

04PP Lemma 10.22.1. Let R be a ring. Let $T \subset \text{Spec}(R)$ be a subset of the spectrum. The following are equivalent

(1) T is closed and is a union of connected components of $\text{Spec}(R)$,
(2) T is an intersection of open and closed subsets of $\text{Spec}(R)$, and
(3) $T = V(I)$ where $I \subset R$ is an ideal generated by idempotents.

Moreover, the ideal in (3) if it exists is unique.

Proof. By Lemma 10.17.11 and Topology, Lemma 5.12.12 we see that (1) and (2) are equivalent. Assume (2) and write $T = \bigcap U_\alpha$ with $U_\alpha \subset \text{Spec}(R)$ open and closed. Then $U_\alpha = D(e_\alpha)$ for some idempotent $e_\alpha \in R$ by Lemma 10.21.3. Then setting $I = (1 - e_\alpha)$ we see that $T = V(I)$, i.e., (3) holds. Finally, assume (3). Write $T = V(I)$ and $I = (e_\alpha)$ for some collection of idempotents e_α. Then it is clear that $T = \bigcap V(e_\alpha) = \bigcap D(1 - e_\alpha)$.

Suppose that I is an ideal generated by idempotents. Let $e \in R$ be an idempotent such that $V(I) \subset V(e)$. Then by Lemma 10.17.2 we see that $e^n \in I$ for some $n \geq 1$. As e is an idempotent this means that $e \in I$. Hence we see that I is generated by exactly those idempotents e such that $T \subset V(e)$. In other words, the ideal I is completely determined by the closed subset T which proves uniqueness. □

00EG Lemma 10.22.2. Let R be a ring. A connected component of $\text{Spec}(R)$ is of the form $V(I)$, where I is an ideal generated by idempotents such that every idempotent of R either maps to 0 or 1 in R/I.

Proof. Let $\mathfrak{p}$ be a prime of R. By Lemma 10.17.11 we have see that the hypotheses of Topology, Lemma 5.12.10 are satisfied for the topological space $\text{Spec}(R)$. Hence the connected component of $\mathfrak{p}$ in $\text{Spec}(R)$ is the intersection of open and closed subsets of $\text{Spec}(R)$ containing $\mathfrak{p}$. Hence it equals $V(I)$ where I is generated by the idempotents $e \in R$ such that e maps to 0 in $\kappa(\mathfrak{p})$, see Lemma 10.21.3. Any idempotent e which is not in this collection clearly maps to 1 in R/I. □

10.23. Glueing properties

00EN In this section we put a number of standard results of the form: if something is true for all members of a standard open covering then it is true. In fact, it often suffices to check things on the level of local rings as in the following lemma.

00HN **Lemma 10.23.1.** Let R be a ring.

(1) For an element x of an R-module M the following are equivalent
   (a) $x = 0$,
   (b) x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \in \text{Spec}(R)$,
   (c) x maps to zero in $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R.
   In other words, the map $M \to \prod_\mathfrak{m} M_\mathfrak{m}$ is injective.
(2) Given an R-module M the following are equivalent
   (a) M is zero,
   (b) $M_\mathfrak{p}$ is zero for all $\mathfrak{p} \in \text{Spec}(R)$,
   (c) $M_\mathfrak{m}$ is zero for all maximal ideals $\mathfrak{m}$ of R.
(3) Given a complex $M_1 \to M_2 \to M_3$ of R-modules the following are equivalent
   (a) $M_1 \to M_2 \to M_3$ is exact,
   (b) for every prime $\mathfrak{p}$ of R the localization $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$ is exact,
   (c) for every maximal ideal $\mathfrak{m}$ of R the localization $M_{1,\mathfrak{m}} \to M_{2,\mathfrak{m}} \to M_{3,\mathfrak{m}}$ is exact.
(4) Given a map $f : M \to M'$ of R-modules the following are equivalent
   (a) f is injective,
   (b) $f_\mathfrak{p} : M_\mathfrak{p} \to M'_\mathfrak{p}$ is injective for all primes $\mathfrak{p}$ of R,
   (c) $f_\mathfrak{m} : M_\mathfrak{m} \to M'_\mathfrak{m}$ is injective for all maximal ideals $\mathfrak{m}$ of R.
(5) Given a map $f : M \to M'$ of R-modules the following are equivalent
   (a) f is surjective,
   (b) $f_\mathfrak{p} : M_\mathfrak{p} \to M'_\mathfrak{p}$ is surjective for all primes $\mathfrak{p}$ of R,
   (c) $f_\mathfrak{m} : M_\mathfrak{m} \to M'_\mathfrak{m}$ is surjective for all maximal ideals $\mathfrak{m}$ of R.
(6) Given a map $f : M \to M'$ of R-modules the following are equivalent
   (a) f is bijective,
   (b) $f_\mathfrak{p} : M_\mathfrak{p} \to M'_\mathfrak{p}$ is bijective for all primes $\mathfrak{p}$ of R,
   (c) $f_\mathfrak{m} : M_\mathfrak{m} \to M'_\mathfrak{m}$ is bijective for all maximal ideals $\mathfrak{m}$ of R.

Proof. Let $x \in M$ as in (1). Let $I = \{f \in R \mid fx = 0\}$. It is easy to see that I is an ideal (it is the annihilator of x). Condition (1)(c) means that for all maximal ideals $\mathfrak{m}$ there exists an $f \in R \setminus \mathfrak{m}$ such that $fx = 0$. In other words, $V(I)$ does not contain a closed point. By Lemma 10.17.2 we see I is the unit ideal. Hence x is zero, i.e., (1)(a) holds. This proves (1).

Part (2) follows by applying (1) to all elements of M simultaneously.

Proof of (3). Let H be the homology of the sequence, i.e., $H = \text{Ker}(M_2 \to M_3)/\text{Im}(M_1 \to M_2)$. By Proposition 10.9.12 we have that $H_\mathfrak{p}$ is the homology of the sequence $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$. Hence (3) is a consequence of (2).

Parts (4) and (5) are special cases of (3). Part (6) follows formally on combining (4) and (5). $\square$

00EO Lemma 10.23.2. Let R be a ring. Let M be an R-module. Let S be an R-algebra. Suppose that $f_1, \ldots, f_n$ is a finite list of elements of R such that $\bigcup D(f_i) = \mathrm{Spec}(R)$, in other words $(f_1, \ldots, f_n) = R$.

(1) If each $M_{f_i} = 0$ then $M = 0$.
(2) If each M_{f_i} is a finite R_{f_i}-module, then M is a finite R-module.
(3) If each M_{f_i} is a finitely presented R_{f_i}-module, then M is a finitely presented R-module.
(4) Let $M \to N$ be a map of R-modules. If $M_{f_i} \to N_{f_i}$ is an isomorphism for each i then $M \to N$ is an isomorphism.
(5) Let $0 \to M'' \to M \to M' \to 0$ be a complex of R-modules. If $0 \to M''_{f_i} \to M_{f_i} \to M'_{f_i} \to 0$ is exact for each i, then $0 \to M'' \to M \to M' \to 0$ is exact.
(6) If each R_{f_i} is Noetherian, then R is Noetherian.
(7) If each S_{f_i} is a finite type R-algebra, so is S.
(8) If each S_{f_i} is of finite presentation over R, so is S.

Proof. We prove each of the parts in turn.

(1) By Proposition 10.9.10 this implies $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, so we conclude by Lemma 10.23.1.
(2) For each i take a finite generating set X_i of M_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $M \to M_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in M. Let Y be the union of these sets. This is still a finite set. Consider the obvious R-linear map $R^Y \to M$ sending the basis element e_y to y. By assumption this map is surjective after localizing at an arbitrary prime ideal $\mathfrak{p}$ of R, so it is surjective by Lemma 10.23.1 and M is finitely generated.
(3) By (2) we have a short exact sequence
$$0 \to K \to R^n \to M \to 0$$
Since localization is an exact functor and M_{f_i} is finitely presented we see that K_{f_i} is finitely generated for all $1 \leq i \leq n$ by Lemma 10.5.3. By (2) this implies that K is a finite R-module and therefore M is finitely presented.
(4) By Proposition 10.9.10 the assumption implies that the induced morphism on localizations at all prime ideals is an isomorphism, so we conclude by Lemma 10.23.1.
(5) By Proposition 10.9.10 the assumption implies that the induced sequence of localizations at all prime ideals is short exact, so we conclude by Lemma 10.23.1.
(6) We will show that every ideal of R has a finite generating set: For this, let $I \subset R$ be an arbitrary ideal. By Proposition 10.9.12 each $I_{f_i} \subset R_{f_i}$ is an ideal. These are all finitely generated by assumption, so we conclude by (2).
(7) For each i take a finite generating set X_i of S_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $S \to S_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in S. Let Y be the union of these sets. This is still a finite set. Consider the algebra homomorphism $R[X_y]_{y \in Y} \to S$ induced by Y. Since it is an

algebra homomorphism, the image T is an R-submodule of the R-module S, so we can consider the quotient module S/T. By assumption, this is zero if we localize at the f_i, so it is zero by (1) and therefore S is an R-algebra of finite type.

(8) By the previous item, there exists a surjective R-algebra homomorphism $R[X_1, \ldots, X_n] \to S$. Let K be the kernel of this map. This is an ideal in $R[X_1, \ldots, X_n]$, finitely generated in each localization at f_i. Since the f_i generate the unit ideal in R, they also generate the unit ideal in $R[X_1, \ldots, X_n]$, so an application of (2) finishes the proof.

□

00EP Lemma 10.23.3. Let $R \to S$ be a ring map. Suppose that $g_1, \ldots, g_n$ is a finite list of elements of S such that $\bigcup D(g_i) = \mathrm{Spec}(S)$ in other words $(g_1, \ldots, g_n) = S$.

(1) If each S_{g_i} is of finite type over R, then S is of finite type over R.
(2) If each S_{g_i} is of finite presentation over R, then S is of finite presentation over R.

Proof. Choose $h_1, \ldots, h_n \in S$ such that $\sum h_i g_i = 1$.

Proof of (1). For each i choose a finite list of elements $x_{i,j} \in S_{g_i}$, $j = 1, \ldots, m_i$ which generate S_{g_i} as an R-algebra. Write $x_{i,j} = y_{i,j}/g_i^{n_{i,j}}$ for some $y_{i,j} \in S$ and some $n_{i,j} \geq 0$. Consider the R-subalgebra $S' \subset S$ generated by $g_1, \ldots, g_n$, $h_1, \ldots, h_n$ and $y_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$. Since localization is exact (Proposition 10.9.12), we see that $S'_{g_i} \to S_{g_i}$ is injective. On the other hand, it is surjective by our choice of $y_{i,j}$. The elements $g_1, \ldots, g_n$ generate the unit ideal in S' as $h_1, \ldots, h_n \in S'$. Thus $S' \to S$ viewed as an S'-module map is an isomorphism by Lemma 10.23.2.

Proof of (2). We already know that S is of finite type. Write $S = R[x_1, \ldots, x_m]/J$ for some ideal J. For each i choose a lift $g'_i \in R[x_1, \ldots, x_m]$ of g_i and we choose a lift $h'_i \in R[x_1, \ldots, x_m]$ of h_i. Then we see that

$$S_{g_i} = R[x_1, \ldots, x_m, y_i]/(J_i + (1 - y_i g'_i))$$

where J_i is the ideal of $R[x_1, \ldots, x_m, y_i]$ generated by J. Small detail omitted. By Lemma 10.6.3 we may choose a finite list of elements $f_{i,j} \in J$, $j = 1, \ldots, m_i$ such that the images of $f_{i,j}$ in J_i and $1 - y_i g'_i$ generate the ideal $J_i + (1 - y_i g'_i)$. Set

$$S' = R[x_1, \ldots, x_m]/\left(\sum h'_i g'_i - 1, f_{i,j}; i = 1, \ldots, n, j = 1, \ldots, m_i\right)$$

There is a surjective R-algebra map $S' \to S$. The classes of the elements $g'_1, \ldots, g'_n$ in S' generate the unit ideal and by construction the maps $S'_{g'_i} \to S_{g_i}$ are injective. Thus we conclude as in part (1). □

10.24. Glueing functions

00EI In this section we show that given an open covering

$$\mathrm{Spec}(R) = \bigcup\nolimits_{i=1}^{n} D(f_i)$$

by standard opens, and given an element $h_i \in R_{f_i}$ for each i such that $h_i = h_j$ as elements of $R_{f_i f_j}$ then there exists a unique $h \in R$ such that the image of h in R_{f_i} is h_i. This result can be interpreted in two ways:

(1) The rule $D(f) \mapsto R_f$ is a sheaf of rings on the standard opens, see Sheaves, Section 6.30.

(2) If we think of elements of R_f as the "algebraic" or "regular" functions on $D(f)$, then these glue as would continuous, resp. differentiable functions on a topological, resp. differentiable manifold.

00EK Lemma 10.24.1. Let R be a ring. Let $f_1, \ldots, f_n$ be elements of R generating the unit ideal. Let M be an R-module. The sequence

$$0 \to M \xrightarrow{\alpha} \bigoplus\nolimits_{i=1}^{n} M_{f_i} \xrightarrow{\beta} \bigoplus\nolimits_{i,j=1}^{n} M_{f_i f_j}$$

is exact, where $\alpha(m) = (m/1, \ldots, m/1)$ and $\beta(m_1/f_1^{e_1}, \ldots, m_n/f_n^{e_n}) = (m_i/f_i^{e_i} - m_j/f_j^{e_j})_{(i,j)}$.

Proof. It suffices to show that the localization of the sequence at any maximal ideal $\mathfrak{m}$ is exact, see Lemma 10.23.1. Since $f_1, \ldots, f_n$ generate the unit ideal, there is an i such that $f_i \not\in \mathfrak{m}$. After renumbering we may assume $i = 1$. Note that $(M_{f_i})_\mathfrak{m} = (M_\mathfrak{m})_{f_i}$ and $(M_{f_i f_j})_\mathfrak{m} = (M_\mathfrak{m})_{f_i f_j}$, see Proposition 10.9.11. In particular $(M_{f_1})_\mathfrak{m} = M_\mathfrak{m}$ and $(M_{f_1 f_i})_\mathfrak{m} = (M_\mathfrak{m})_{f_i}$, because f_1 is a unit. Note that the maps in the sequence are the canonical ones coming from Lemma 10.9.7 and the identity map on M. Having said all of this, after replacing R by $R_\mathfrak{m}$, M by $M_\mathfrak{m}$, and f_i by their image in $R_\mathfrak{m}$, and f_1 by $1 \in R_\mathfrak{m}$, we reduce to the case where $f_1 = 1$.

Assume $f_1 = 1$. Injectivity of α is now trivial. Let $m = (m_i) \in \bigoplus_{i=1}^{n} M_{f_i}$ be in the kernel of β. Then $m_1 \in M_{f_1} = M$. Moreover, $\beta(m) = 0$ implies that m_1 and m_i map to the same element of $M_{f_1 f_i} = M_{f_i}$. Thus $\alpha(m_1) = m$ and the proof is complete. □

00EJ Lemma 10.24.2. Let R be a ring, and let $f_1, f_2, \ldots f_n \in R$ generate the unit ideal in R. Then the following sequence is exact:

$$0 \longrightarrow R \longrightarrow \bigoplus\nolimits_i R_{f_i} \longrightarrow \bigoplus\nolimits_{i,j} R_{f_i f_j}$$

where the maps $\alpha : R \longrightarrow \bigoplus_i R_{f_i}$ and $\beta : \bigoplus_i R_{f_i} \longrightarrow \bigoplus_{i,j} R_{f_i f_j}$ are defined as

$$\alpha(x) = \left(\frac{x}{1}, \ldots, \frac{x}{1}\right) \text{ and } \beta\left(\frac{x_1}{f_1^{r_1}}, \ldots, \frac{x_n}{f_n^{r_n}}\right) = \left(\frac{x_i}{f_i^{r_i}} - \frac{x_j}{f_j^{r_j}} \text{ in } R_{f_i f_j}\right).$$

Proof. Special case of Lemma 10.24.1. □

The following we have already seen above, but we state it explicitly here for convenience.

00EM Lemma 10.24.3. Let R be a ring. If $\text{Spec}(R) = U \amalg V$ with both U and V open then $R \cong R_1 \times R_2$ with $U \cong \text{Spec}(R_1)$ and $V \cong \text{Spec}(R_2)$ via the maps in Lemma 10.21.2. Moreover, both R_1 and R_2 are localizations as well as quotients of the ring R.

Proof. By Lemma 10.21.3 we have $U = D(e)$ and $V = D(1-e)$ for some idempotent e. By Lemma 10.24.2 we see that $R \cong R_e \times R_{1-e}$ (since clearly $R_{e(1-e)} = 0$ so the glueing condition is trivial; of course it is trivial to prove the product decomposition directly in this case). The lemma follows. □

0565 Lemma 10.24.4. Let R be a ring. Let $f_1, \ldots, f_n \in R$. Let M be an R-module. Then $M \to \bigoplus M_{f_i}$ is injective if and only if

$$M \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} M, \quad m \longmapsto (f_1 m, \ldots, f_n m)$$

is injective.

Proof. The map $M \to \bigoplus M_{f_i}$ is injective if and only if for all $m \in M$ and $e_1, \ldots, e_n \geq 1$ such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$ we have $m = 0$. This clearly implies the displayed map is injective. Conversely, suppose the displayed map is injective and $m \in M$ and $e_1, \ldots, e_n \geq 1$ are such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$. If $e_i = 1$ for all i, then we immediately conclude that $m = 0$ from the injectivity of the displayed map. Next, we prove this holds for any such data by induction on $e = \sum e_i$. The base case is $e = n$, and we have just dealt with this. If some $e_i > 1$, then set $m' = f_i m$. By induction we see that $m' = 0$. Hence we see that $f_i m = 0$, i.e., we may take $e_i = 1$ which decreases e and we win. $\square$

The following lemma is better stated and proved in the more general context of flat descent. However, it makes sense to state it here since it fits well with the above.

00EQ **Lemma 10.24.5.** Let R be a ring. Let $f_1, \ldots, f_n \in R$. Suppose we are given the following data:

(1) For each i an R_{f_i}-module M_i.
(2) For each pair i, j an $R_{f_i f_j}$-module isomorphism $\psi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$.

which satisfy the "cocycle condition" that all the diagrams

$$\begin{tikzcd} (M_i)_{f_j f_k} \arrow[rr, "\psi_{ik}"] \arrow[dr, "\psi_{ij}"'] & & (M_k)_{f_i f_j} \\ & (M_j)_{f_i f_k} \arrow[ur, "\psi_{jk}"'] & \end{tikzcd}$$

commute (for all triples i, j, k). Given this data define

$$M = \mathrm{Ker}\left(\bigoplus\nolimits_{1 \leq i \leq n} M_i \longrightarrow \bigoplus\nolimits_{1 \leq i, j \leq n} (M_i)_{f_j}\right)$$

where $(m_1, \ldots, m_n)$ maps to the element whose (i, j)th entry is $m_i/1 - \psi_{ji}(m_j/1)$. Then the natural map $M \to M_i$ induces an isomorphism $M_{f_i} \to M_i$. Moreover $\psi_{ij}(m/1) = m/1$ for all $m \in M$ (with obvious notation).

Proof. To show that $M_{f_1} \to M_1$ is an isomorphism, it suffices to show that its localization at every prime $\mathfrak{p}'$ of R_{f_1} is an isomorphism, see Lemma 10.23.1. Write $\mathfrak{p}' = \mathfrak{p} R_{f_1}$ for some prime $\mathfrak{p} \subset R$, $f_1 \notin \mathfrak{p}$, see Lemma 10.17.6. Since localization is exact (Proposition 10.9.12), we see that

$$\begin{aligned} (M_{f_1})_{\mathfrak{p}'} &= M_{\mathfrak{p}} \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_{1 \leq i \leq n} M_{i,\mathfrak{p}} \longrightarrow \bigoplus\nolimits_{1 \leq i, j \leq n} ((M_i)_{f_j})_{\mathfrak{p}}\right) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_{1 \leq i \leq n} M_{i,\mathfrak{p}} \longrightarrow \bigoplus\nolimits_{1 \leq i, j \leq n} (M_{i,\mathfrak{p}})_{f_j}\right) \end{aligned}$$

Here we also used Proposition 10.9.11. Since f_1 is a unit in $R_{\mathfrak{p}}$, this reduces us to the case where $f_1 = 1$ by replacing R by $R_{\mathfrak{p}}$, f_i by the image of f_i in $R_{\mathfrak{p}}$, M by $M_{\mathfrak{p}}$, and f_1 by 1.

Assume $f_1 = 1$. Then $\psi_{1j} : (M_1)_{f_j} \to M_j$ is an isomorphism for $j = 2, \ldots, n$. If we use these isomorphisms to identify $M_j = (M_1)_{f_j}$, then we see that $\psi_{ij} : (M_1)_{f_i f_j} \to (M_1)_{f_i f_j}$ is the canonical identification. Thus the complex

$$0 \to M_1 \to \bigoplus\nolimits_{1 \leq i \leq n} (M_1)_{f_i} \longrightarrow \bigoplus\nolimits_{1 \leq i, j \leq n} (M_1)_{f_i f_j}$$

is exact by Lemma 10.24.1. Thus the first map identifies M_1 with M in this case and everything is clear. ☐

10.25. Zerodivisors and total rings of fractions

02LV The local ring at a minimal prime has the following properties.

00EU Lemma 10.25.1. Let $\mathfrak{p}$ be a minimal prime of a ring R. Every element of the maximal ideal of $R_{\mathfrak{p}}$ is nilpotent. If R is reduced then $R_{\mathfrak{p}}$ is a field.

Proof. If some element x of $\mathfrak{p}R_{\mathfrak{p}}$ is not nilpotent, then $D(x) \neq \emptyset$, see Lemma 10.17.2. This contradicts the minimality of $\mathfrak{p}$. If R is reduced, then $\mathfrak{p}R_{\mathfrak{p}} = 0$ and hence it is a field. ☐

00EW Lemma 10.25.2. Let R be a reduced ring. Then

(1) R is a subring of a product of fields,
(2) $R \to \prod_{\mathfrak{p} \text{ minimal}} R_{\mathfrak{p}}$ is an embedding into a product of fields,
(3) $\bigcup_{\mathfrak{p} \text{ minimal}} \mathfrak{p}$ is the set of zerodivisors of R.

Proof. By Lemma 10.25.1 each of the rings $R_{\mathfrak{p}}$ is a field. In particular, the kernel of the ring map $R \to R_{\mathfrak{p}}$ is $\mathfrak{p}$. By Lemma 10.17.2 we have $\bigcap_{\mathfrak{p}} \mathfrak{p} = (0)$. Hence (2) and (1) are true. If $xy = 0$ and $y \neq 0$, then $y \notin \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Hence $x \in \mathfrak{p}$. Thus every zerodivisor of R is contained in $\bigcup_{\mathfrak{p} \text{ minimal}} \mathfrak{p}$. Conversely, suppose that $x \in \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Then x maps to zero in $R_{\mathfrak{p}}$, hence there exists $y \in R$, $y \notin \mathfrak{p}$ such that $xy = 0$. In other words, x is a zerodivisor. This finishes the proof of (3) and the lemma. ☐

The total ring of fractions $Q(R)$ of a ring R was introduced in Example 10.9.8.

02LW Lemma 10.25.3. Let R be a ring. Let $S \subset R$ be a multiplicative subset consisting of nonzerodivisors. Then $Q(R) \cong Q(S^{-1}R)$. In particular $Q(R) \cong Q(Q(R))$.

Proof. If $x \in S^{-1}R$ is a nonzerodivisor, and $x = r/f$ for some $r \in R$, $f \in S$, then r is a nonzerodivisor in R. Whence the lemma. ☐

We can apply glueing results to prove something about total rings of fractions $Q(R)$ which we introduced in Example 10.9.8.

02LX Lemma 10.25.4. Let R be a ring. Assume that R has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ is the set of zerodivisors of R. Then the total ring of fractions $Q(R)$ is equal to $R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$.

Proof. There are natural maps $Q(R) \to R_{\mathfrak{q}_i}$ since any nonzerodivisor is contained in $R \setminus \mathfrak{q}_i$. Hence a natural map $Q(R) \to R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$. For any nonminimal prime $\mathfrak{p} \subset R$ we see that $\mathfrak{p} \not\subset \mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ by Lemma 10.15.2. Hence $\text{Spec}(Q(R)) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ (as subsets of $\text{Spec}(R)$, see Lemma 10.17.5). Therefore $\text{Spec}(Q(R))$ is a finite discrete set and it follows that $Q(R) = A_1 \times \ldots \times A_t$ with $\text{Spec}(A_i) = \{q_i\}$, see Lemma 10.24.3. Moreover A_i is a local ring, which is a localization of R. Hence $A_i \cong R_{\mathfrak{q}_i}$. ☐

10.26. Irreducible components of spectra

00ER We show that irreducible components of the spectrum of a ring correspond to the minimal primes in the ring.

00ES **Lemma 10.26.1.** Let R be a ring.

(1) For a prime $\mathfrak p \subset R$ the closure of $\{\mathfrak p\}$ in the Zariski topology is $V(\mathfrak p)$. In a formula $\overline{\{\mathfrak p\}} = V(\mathfrak p)$.
(2) The irreducible closed subsets of $\text{Spec}(R)$ are exactly the subsets $V(\mathfrak p)$, with $\mathfrak p \subset R$ a prime.
(3) The irreducible components (see Topology, Definition 5.8.1) of $\text{Spec}(R)$ are exactly the subsets $V(\mathfrak p)$, with $\mathfrak p \subset R$ a minimal prime.

Proof. Note that if $\mathfrak p \in V(I)$, then $I \subset \mathfrak p$. Hence, clearly $\overline{\{\mathfrak p\}} = V(\mathfrak p)$. In particular $V(\mathfrak p)$ is the closure of a singleton and hence irreducible. The second assertion implies the third. To show the second, let $V(I) \subset \text{Spec}(R)$ with I a radical ideal. If I is not prime, then choose $a, b \in R$, $a, b \not\in I$ with $ab \in I$. In this case $V(I, a) \cup V(I, b) = V(I)$, but neither $V(I, b) = V(I)$ nor $V(I, a) = V(I)$, by Lemma 10.17.2. Hence $V(I)$ is not irreducible. ☐

In other words, this lemma shows that every irreducible closed subset of $\text{Spec}(R)$ is of the form $V(\mathfrak p)$ for some prime $\mathfrak p$. Since $V(\mathfrak p) = \overline{\{\mathfrak p\}}$ we see that each irreducible closed subset has a unique generic point, see Topology, Definition 5.8.6. In particular, $\text{Spec}(R)$ is a sober topological space. We record this fact in the following lemma.

090M **Lemma 10.26.2.** The spectrum of a ring is a spectral space, see Topology, Definition 5.23.1.

Proof. Formally this follows from Lemma 10.26.1 and Lemma 10.17.11. See also discussion above. ☐

00ET **Lemma 10.26.3.** Let R be a ring. Let $\mathfrak p \subset R$ be a prime.

(1) the set of irreducible closed subsets of $\text{Spec}(R)$ passing through $\mathfrak p$ is in one-to-one correspondence with primes $\mathfrak q \subset R_{\mathfrak p}$.
(2) The set of irreducible components of $\text{Spec}(R)$ passing through $\mathfrak p$ is in one-to-one correspondence with minimal primes $\mathfrak q \subset R_{\mathfrak p}$.

Proof. Follows from Lemma 10.26.1 and the description of $\text{Spec}(R_{\mathfrak p})$ in Lemma 10.17.5 which shows that $\text{Spec}(R_{\mathfrak p})$ corresponds to primes $\mathfrak q$ in R with $\mathfrak q \subset \mathfrak p$. ☐

00EV **Lemma 10.26.4.** Let R be a ring. Let $\mathfrak p$ be a minimal prime of R. Let $W \subset \text{Spec}(R)$ be a quasi-compact open not containing the point $\mathfrak p$. Then there exists an $f \in R$, $f \not\in \mathfrak p$ such that $D(f) \cap W = \emptyset$.

Proof. Since W is quasi-compact we may write it as a finite union of standard affine opens $D(g_i)$, $i = 1, \ldots, n$. Since $\mathfrak p \not\in W$ we have $g_i \in \mathfrak p$ for all i. By Lemma 10.25.1 each g_i is nilpotent in $R_{\mathfrak p}$. Hence we can find an $f \in R$, $f \not\in \mathfrak p$ such that for all i we have $f g_i^{n_i} = 0$ for some $n_i > 0$. Then $D(f)$ works. ☐

04MG **Lemma 10.26.5.** Let R be a ring. Let $X = \text{Spec}(R)$ as a topological space. The following are equivalent

(1) X is profinite,

(2) X is Hausdorff,
(3) X is totally disconnected.
(4) every quasi-compact open of X is closed,
(5) there are no nontrivial inclusions between its prime ideals,
(6) every prime ideal is a maximal ideal,
(7) every prime ideal is minimal,
(8) every standard open $D(f) \subset X$ is closed, and
(9) add more here.

Proof. First proof. It is clear that (5), (6), and (7) are equivalent. It is clear that (4) and (8) are equivalent as every quasi-compact open is a finite union of standard opens. The implication (7) $\Rightarrow$ (4) follows from Lemma 10.26.4. Assume (4) holds. Let $\mathfrak{p}, \mathfrak{p}'$ be distinct primes of R. Choose an $f \in \mathfrak{p}'$, $f \notin \mathfrak{p}$ (if needed switch $\mathfrak{p}$ with $\mathfrak{p}'$). Then $\mathfrak{p}' \notin D(f)$ and $\mathfrak{p} \in D(f)$. By (4) the open $D(f)$ is also closed. Hence $\mathfrak{p}$ and $\mathfrak{p}'$ are in disjoint open neighbourhoods whose union is X. Thus X is Hausdorff and totally disconnected. Thus (4) $\Rightarrow$ (2) and (3). If (3) holds then there cannot be any specializations between points of $\text{Spec}(R)$ and we see that (5) holds. If X is Hausdorff then every point is closed, so (2) implies (6). Thus (2), (3), (4), (5), (6), (7) and (8) are equivalent. Any profinite space is Hausdorff, so (1) implies (2). If X satisfies (2) and (3), then X (being quasi-compact by Lemma 10.17.10) is profinite by Topology, Lemma 5.22.2.

Second proof. Besides the equivalence of (4) and (8) this follows from Lemma 10.26.2 and purely topological facts, see Topology, Lemma 5.23.8. $\square$

10.27. Examples of spectra of rings

00EX In this section we put some examples of spectra.

00EY Example 10.27.1. In this example we describe $X = \text{Spec}(\mathbf{Z}[x]/(x^2-4))$. Let $\mathfrak{p}$ be an arbitrary prime in X. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]/(x^2-4)$ be the natural ring map. Then, $\phi^{-1}(\mathfrak{p})$ is a prime in $\mathbf{Z}$. If $\phi^{-1}(\mathfrak{p}) = (2)$, then since $\mathfrak{p}$ contains 2, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2-4, 2) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ via the map $\mathbf{Z}[x]/(x^2-4) \to \mathbf{Z}[x]/(x^2-4, 2)$. Any prime in $(\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ corresponds to a prime in $(\mathbf{Z}/2\mathbf{Z})[x]$ containing (x^2). Such primes will then contain x. Since $(\mathbf{Z}/2\mathbf{Z}) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x)$ is a field, (x) is a maximal ideal. Since any prime contains (x) and (x) is maximal, the ring contains only one prime (x). Thus, in this case, $\mathfrak{p} = (2, x)$. Now, if $\phi^{-1}(\mathfrak{p}) = (q)$ for $q > 2$, then since $\mathfrak{p}$ contains q, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2-4, q) \cong (\mathbf{Z}/q\mathbf{Z})[x]/(x^2-4)$ via the map $\mathbf{Z}[x]/(x^2-4) \to \mathbf{Z}[x]/(x^2-4, q)$. Any prime in $(\mathbf{Z}/q\mathbf{Z})[x]/(x^2-4)$ corresponds to a prime in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x^2-4) = (x-2)(x+2)$. Hence, these primes must contain either $x-2$ or $x+2$. Since $(\mathbf{Z}/q\mathbf{Z})[x]$ is a PID, all nonzero primes are maximal, and so there are precisely 2 primes in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x-2)(x+2)$, namely $(x-2)$ and $(x+2)$. In conclusion, there exist two primes $(q, x-2)$ and $(q, x+2)$ since $2 \neq -2 \in \mathbf{Z}/(q)$. Finally, we treat the case where $\phi^{-1}(\mathfrak{p}) = (0)$. Notice that $\mathfrak{p}$ corresponds to a prime ideal in $\mathbf{Z}[x]$ that contains $(x^2-4) = (x-2)(x+2)$. Hence, $\mathfrak{p}$ contains either $(x-2)$ or $(x+2)$. Hence, $\mathfrak{p}$ corresponds to a prime in $\mathbf{Z}[x]/(x-2)$ or one in $\mathbf{Z}[x]/(x+2)$ that intersects $\mathbf{Z}$ only at 0, by assumption. Since $\mathbf{Z}[x]/(x-2) \cong \mathbf{Z}$ and $\mathbf{Z}[x]/(x+2) \cong \mathbf{Z}$, this means that $\mathfrak{p}$ must correspond to 0 in one of these rings. Thus, $\mathfrak{p} = (x-2)$ or $\mathfrak{p} = (x+2)$ in the original ring.

00EZ Example 10.27.2. In this example we describe $X = \text{Spec}(\mathbf{Z}[x])$. Fix $\mathfrak{p} \in X$. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]$ and notice that $\phi^{-1}(\mathfrak{p}) \in \text{Spec}(\mathbf{Z})$. If $\phi^{-1}(\mathfrak{p}) = (q)$ for q a prime number $q > 0$, then $\mathfrak{p}$ corresponds to a prime in $(\mathbf{Z}/(q))[x]$, which must be generated by a polynomial that is irreducible in $(\mathbf{Z}/(q))[x]$. If we choose a representative of this polynomial with minimal degree, then it will also be irreducible in $\mathbf{Z}[x]$. Hence, in this case $\mathfrak{p} = (q, f_q)$ where f_q is an irreducible polynomial in $\mathbf{Z}[x]$ that is irreducible when viewed in $(\mathbf{Z}/(q)[x])$. Now, assume that $\phi^{-1}(\mathfrak{p}) = (0)$. In this case, $\mathfrak{p}$ must be generated by nonconstant polynomials which, since $\mathfrak{p}$ is prime, may be assumed to be irreducible in $\mathbf{Z}[x]$. By Gauss' lemma, these polynomials are also irreducible in $\mathbf{Q}[x]$. Since $\mathbf{Q}[x]$ is a Euclidean domain, if there are at least two distinct irreducibles f, g generating $\mathfrak{p}$, then $1 = af + bg$ for $a, b \in \mathbf{Q}[x]$. Multiplying through by a common denominator, we see that $m = \bar{a}f + \bar{b}g$ for $\bar{a}, \bar{b} \in \mathbf{Z}[x]$ and nonzero $m \in \mathbf{Z}$. This is a contradiction. Hence, $\mathfrak{p}$ is generated by one irreducible polynomial in $\mathbf{Z}[x]$.

00F0 Example 10.27.3. In this example we describe $X = \text{Spec}(k[x, y])$ when k is an arbitrary field. Clearly (0) is prime, and any principal ideal generated by an irreducible polynomial will also be a prime since $k[x, y]$ is a unique factorization domain. Now assume $\mathfrak{p}$ is an element of X that is not principal. Since $k[x, y]$ is a Noetherian UFD, the prime ideal $\mathfrak{p}$ can be generated by a finite number of irreducible polynomials $(f_1, \ldots, f_n)$. Now, I claim that if f, g are irreducible polynomials in $k[x, y]$ that are not associates, then $(f, g) \cap k[x] \neq 0$. To do this, it is enough to show that f and g are relatively prime when viewed in $k(x)[y]$. In this case, $k(x)[y]$ is a Euclidean domain, so by applying the Euclidean algorithm and clearing denominators, we obtain $p = af + bg$ for $p, a, b \in k[x]$. Thus, assume this is not the case, that is, that some nonunit $h \in k(x)[y]$ divides both f and g. Then, by Gauss's lemma, for some $a, b \in k(x)$ we have $ah | f$ and $bh | g$ for $ah, bh \in k[x]$. By irreducibility, $ah = f$ and $bh = g$ (since $h \notin k(x)$). So, back in $k(x)[y]$, f, g are associates, as $\frac{a}{b} g = f$. Since $k(x)$ is the fraction field of $k[x]$, we can write $g = \frac{r}{s} f$ for elements $r, s \in k[x]$ sharing no common factors. This implies that $sg = rf$ in $k[x, y]$ and so s must divide f since $k[x, y]$ is a UFD. Hence, $s = 1$ or $s = f$. If $s = f$, then $r = g$, implying $f, g \in k[x]$ and thus must be units in $k(x)$ and relatively prime in $k(x)[y]$, contradicting our hypothesis. If $s = 1$, then $g = rf$, another contradiction. Thus, we must have f, g relatively prime in $k(x)[y]$, a Euclidean domain. Thus, we have reduced to the case $\mathfrak{p}$ contains some irreducible polynomial $p \in k[x] \subset k[x, y]$. By the above, $\mathfrak{p}$ corresponds to a prime in the ring $k[x, y]/(p) = k(\alpha)[y]$, where α is an element algebraic over k with minimum polynomial p. This is a PID, and so any prime ideal corresponds to (0) or an irreducible polynomial in $k(\alpha)[y]$. Thus, $\mathfrak{p}$ is of the form (p) or (p, f) where f is a polynomial in $k[x, y]$ that is irreducible in the quotient $k[x, y]/(p)$.

00F1 Example 10.27.4. Consider the ring

$$R = \{f \in \mathbf{Q}[z] \text{ with } f(0) = f(1)\}.$$

Consider the map

$$\varphi : \mathbf{Q}[A, B] \to R$$

defined by $\varphi(A) = z^2 - z$ and $\varphi(B) = z^3 - z^2$. It is easily checked that $(A^3 - B^2 + AB) \subset \text{Ker}(\varphi)$ and that $A^3 - B^2 + AB$ is irreducible. Assume that φ is surjective; then since R is an integral domain (it is a subring of an integral domain), $\text{Ker}(\varphi)$ must be a prime ideal of $\mathbf{Q}[A, B]$. The prime ideals which contain $(A^3 - B^2 + AB)$

are (A^3-B^2+AB) itself and any maximal ideal (f,g) with $f,g \in \mathbf{Q}[A,B]$ such that f is irreducible mod g. But R is not a field, so the kernel must be (A^3-B^2+AB); hence φ gives an isomorphism $R \to \mathbf{Q}[A,B]/(A^3-B^2+AB)$.

To see that φ is surjective, we must express any $f \in R$ as a $\mathbf{Q}$-coefficient polynomial in $A(z) = z^2 - z$ and $B(z) = z^3 - z^2$. Note the relation $zA(z) = B(z)$. Let $a = f(0) = f(1)$. Then $z(z-1)$ must divide $f(z) - a$, so we can write $f(z) = z(z-1)g(z) + a = A(z)g(z) + a$. If $\deg(g) < 2$, then $h(z) = c_1 z + c_0$ and $f(z) = A(z)(c_1 z + c_0) + a = c_1 B(z) + c_0 A(z) + a$, so we are done. If $\deg(g) \geq 2$, then by the polynomial division algorithm, we can write $g(z) = A(z)h(z) + b_1 z + b_0$ ($\deg(h) \leq \deg(g) - 2$), so $f(z) = A(z)^2 h(z) + b_1 B(z) + b_0 A(z)$. Applying division to $h(z)$ and iterating, we obtain an expression for $f(z)$ as a polynomial in $A(z)$ and $B(z)$; hence φ is surjective.

Now let $a \in \mathbf{Q}$, $a \neq 0, \frac{1}{2}, 1$ and consider

$$R_a = \{f \in \mathbf{Q}[z, \frac{1}{z-a}] \text{ with } f(0) = f(1)\}.$$

This is a finitely generated $\mathbf{Q}$-algebra as well: it is easy to check that the functions $z^2 - z$, $z^3 - z$, and $\frac{a^2-a}{z-a} + z$ generate R_a as an $\mathbf{Q}$-algebra. We have the following inclusions:

$$R \subset R_a \subset \mathbf{Q}[z, \frac{1}{z-a}], \quad R \subset \mathbf{Q}[z] \subset \mathbf{Q}[z, \frac{1}{z-a}].$$

Recall (Lemma 10.17.5) that for a ring T and a multiplicative subset $S \subset T$, the ring map $T \to S^{-1}T$ induces a map on spectra $\text{Spec}(S^{-1}T) \to \text{Spec}(T)$ which is a homeomorphism onto the subset

$$\{\mathfrak{p} \in \text{Spec}(T) \mid S \cap \mathfrak{p} = \emptyset\} \subset \text{Spec}(T).$$

When $S = \{1, f, f^2, \ldots\}$ for some $f \in T$, this is the open set $D(f) \subset T$. We now verify a corresponding property for the ring map $R \to R_a$: we will show that the map $\theta : \text{Spec}(R_a) \to \text{Spec}(R)$ induced by inclusion $R \subset R_a$ is a homeomorphism onto an open subset of $\text{Spec}(R)$ by verifying that θ is an injective local homeomorphism. We do so with respect to an open cover of $\text{Spec}(R_a)$ by two distinguished opens, as we now describe. For any $r \in \mathbf{Q}$, let $\text{ev}_r : R \to \mathbf{Q}$ be the homomorphism given by evaluation at r. Note that for $r = 0$ and $r = 1-a$, this can be extended to a homomorphism $\text{ev}'_r : R_a \to \mathbf{Q}$ (the latter because $\frac{1}{z-a}$ is well-defined at $z = 1-a$, since $a \neq \frac{1}{2}$). However, ev_a does not extend to R_a. Write $\mathfrak{m}_r = \text{Ker}(\text{ev}_r)$. We have

$$\mathfrak{m}_0 = (z^2 - z, z^3 - z),$$

$$\mathfrak{m}_a = ((z-1+a)(z-a), (z^2-1+a)(z-a)), \text{ and}$$

$$\mathfrak{m}_{1-a} = ((z-1+a)(z-a), (z-1+a)(z^2-a)).$$

To verify this, note that the right-hand sides are clearly contained in the left-hand sides. Then check that the right-hand sides are maximal ideals by writing the generators in terms of A and B, and viewing R as $\mathbf{Q}[A,B]/(A^3-B^2+AB)$. Note that $\mathfrak{m}_a$ is not in the image of θ: we have

$$(z^2-z)^2(z-a)\left(\frac{a^2-a}{z-a} + z\right) = (z^2-z)^2(a^2-a) + (z^2-z)^2(z-a)z$$

The left hand side is in $\mathfrak{m}_a R_a$ because $(z^2-z)(z-a)$ is in $\mathfrak{m}_a$ and because $(z^2 - z)(\frac{a^2-a}{z-a} + z)$ is in R_a. Similarly the element $(z^2-z)^2(z-a)z$ is in $\mathfrak{m}_a R_a$ because

$(z^2 - z)$ is in R_a and $(z^2 - z)(z - a)$ is in $\mathfrak{m}_a$. As $a \not\in \{0, 1\}$ we conclude that $(z^2 - z)^2 \in \mathfrak{m}_a R_a$. Hence no ideal I of R_a can satisfy $I \cap R = \mathfrak{m}_a$, as such an I would have to contain $(z^2 - z)^2$, which is in R but not in $\mathfrak{m}_a$. The distinguished open set $D((z - 1 + a)(z - a)) \subset \text{Spec}(R)$ is equal to the complement of the closed set $\{\mathfrak{m}_a, \mathfrak{m}_{1-a}\}$. Then check that $R_{(z-1+a)(z-a)} = (R_a)_{(z-1+a)(z-a)}$; calling this localized ring R', then, it follows that the map $R \to R'$ factors as $R \to R_a \to R'$. By Lemma 10.17.5, then, these maps express $\text{Spec}(R') \subset \text{Spec}(R_a)$ and $\text{Spec}(R') \subset \text{Spec}(R)$ as open subsets; hence $\theta : \text{Spec}(R_a) \to \text{Spec}(R)$, when restricted to $D((z - 1 + a)(z - a))$, is a homeomorphism onto an open subset. Similarly, θ restricted to $D((z^2+z+2a-2)(z-a)) \subset \text{Spec}(R_a)$ is a homeomorphism onto the open subset $D((z^2+z+2a-2)(z-a)) \subset \text{Spec}(R)$. Depending on whether $z^2 + z + 2a - 2$ is irreducible or not over $\mathbf{Q}$, this former distinguished open set has complement equal to one or two closed points along with the closed point $\mathfrak{m}_a$. Furthermore, the ideal in R_a generated by the elements $(z^2+z+2a-a)(z-a)$ and $(z-1+a)(z-a)$ is all of R_a, so these two distinguished open sets cover $\text{Spec}(R_a)$. Hence in order to show that θ is a homeomorphism onto $\text{Spec}(R) - \{\mathfrak{m}_a\}$, it suffices to show that these one or two points can never equal $\mathfrak{m}_{1-a}$. And this is indeed the case, since $1 - a$ is a root of $z^2 + z + 2a - 2$ if and only if $a = 0$ or $a = 1$, both of which do not occur.

Despite this homeomorphism which mimics the behavior of a localization at an element of R, while $\mathbf{Q}[z, \frac{1}{z-a}]$ is the localization of $\mathbf{Q}[z]$ at the maximal ideal $(z-a)$, the ring R_a is not a localization of R: Any localization $S^{-1}R$ results in more units than the original ring R. The units of R are $\mathbf{Q}^\times$, the units of $\mathbf{Q}$. In fact, it is easy to see that the units of R_a are $\mathbf{Q}^*$. Namely, the units of $\mathbf{Q}[z, \frac{1}{z-a}]$ are $c(z-a)^n$ for $c \in \mathbf{Q}^*$ and $n \in \mathbf{Z}$ and it is clear that these are in R_a only if $n = 0$. Hence R_a has no more units than R does, and thus cannot be a localization of R.

We used the fact that $a \neq 0, 1$ to ensure that $\frac{1}{z-a}$ makes sense at $z = 0, 1$. We used the fact that $a \neq 1/2$ in a few places: (1) In order to be able to talk about the kernel of ev_{1-a} on R_a, which ensures that $\mathfrak{m}_{1-a}$ is a point of R_a (i.e., that R_a is missing just one point of R). (2) At the end in order to conclude that $(z-a)^{k+\ell}$ can only be in R for $k = \ell = 0$; indeed, if $a = 1/2$, then this is in R as long as $k + \ell$ is even. Hence there would indeed be more units in R_a than in R, and R_a could possibly be a localization of R.

10.28. A meta-observation about prime ideals

05K7 This section is taken from the CRing project. Let R be a ring and let $S \subset R$ be a multiplicative subset. A consequence of Lemma 10.17.5 is that an ideal $I \subset R$ maximal with respect to the property of not intersecting S is prime. The reason is that $I = R \cap \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of the ring $S^{-1}R$. It turns out that for many properties of ideals, the maximal ones are prime. A general method of seeing this was developed in [LR08]. In this section, we digress to explain this phenomenon.

Let R be a ring. If I is an ideal of R and $a \in R$, we define

$$(I : a) = \{x \in R \mid xa \in I\}.$$

More generally, if $J \subset R$ is an ideal, we define

$$(I : J) = \{x \in R \mid xJ \subset I\}.$$

05K8 Lemma 10.28.1. Let R be a ring. For a principal ideal $J \subset R$, and for any ideal $I \subset J$ we have $I = J(I : J)$.

Proof. Say $J = (a)$. Then $(I : J) = (I : a)$. Since $I \subset J$ we see that any $y \in I$ is of the form $y = xa$ for some $x \in (I : a)$. Hence $I \subset J(I : J)$. Conversely, if $x \in (I : a)$, then $xJ = (xa) \subset I$, which proves the other inclusion. $\square$

Let $\mathcal{F}$ be a collection of ideals of R. We are interested in conditions that will guarantee that the maximal elements in the complement of $\mathcal{F}$ are prime.

05K9 Definition 10.28.2. Let R be a ring. Let $\mathcal{F}$ be a set of ideals of R. We say $\mathcal{F}$ is an Oka family if $R \in \mathcal{F}$ and whenever $I \subset R$ is an ideal and $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$, then $I \in \mathcal{F}$.

Let us give some examples of Oka families. The first example is the basic example discussed in the introduction to this section.

05KA Example 10.28.3. Let R be a ring and let S be a multiplicative subset of R. We claim that $\mathcal{F} = \{I \subset R \mid I \cap S \neq \emptyset\}$ is an Oka family. Namely, suppose that $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$. Then pick $s \in (I, a) \cap S$ and $s' \in (I : a) \cap S$. Then $ss' \in I \cap S$ and hence $I \in \mathcal{F}$. Thus $\mathcal{F}$ is an Oka family.

05KB Example 10.28.4. Let R be a ring, $I \subset R$ an ideal, and $a \in R$. If $(I : a)$ is generated by $a_1, \ldots, a_n$ and (I, a) is generated by $a, b_1, \ldots, b_m$ with $b_1, \ldots, b_m \in I$, then I is generated by $aa_1, \ldots, aa_n, b_1, \ldots, b_m$. To see this, note that if $x \in I$, then $x \in (I, a)$ is a linear combination of $a, b_1, \ldots, b_m$, but the coefficient of a must lie in $(I : a)$. As a result, we deduce that the family of finitely generated ideals is an Oka family.

05KC Example 10.28.5. Let us show that the family of principal ideals of a ring R is an Oka family. Indeed, suppose $I \subset R$ is an ideal, $a \in R$, and (I, a) and $(I : a)$ are principal. Note that $(I : a) = (I : (I, a))$. Setting $J = (I, a)$, we find that J is principal and $(I : J)$ is too. By Lemma 10.28.1 we have $I = J(I : J)$. Thus we find in our situation that since $J = (I, a)$ and $(I : J)$ are principal, I is principal.

05KD Example 10.28.6. Let R be a ring. Let κ be an infinite cardinal. The family of ideals which can be generated by at most κ elements is an Oka family. The argument is analogous to the argument in Example 10.28.4 and is omitted.

0G1N Example 10.28.7. Let A be a ring, $I \subset A$ an ideal, and $a \in A$ an element. There is a short exact sequence $0 \to A/(I : a) \to A/I \to A/(I, a) \to 0$ where the first arrow is given by multiplication by a. Thus if P is a property of A-modules that is stable under extensions and holds for 0, then the family of ideals I such that A/I has P is an Oka family.

05KE Proposition 10.28.8. If $\mathcal{F}$ is an Oka family of ideals, then any maximal element of the complement of $\mathcal{F}$ is prime.

Proof. Suppose $I \notin \mathcal{F}$ is maximal with respect to not being in $\mathcal{F}$ but I is not prime. Note that $I \neq R$ because $R \in \mathcal{F}$. Since I is not prime we can find $a, b \in R - I$ with $ab \in I$. It follows that $(I, a) \neq I$ and $(I : a)$ contains $b \notin I$ so also $(I : a) \neq I$. Thus $(I : a), (I, a)$ both strictly contain I, so they must belong to $\mathcal{F}$. By the Oka condition, we have $I \in \mathcal{F}$, a contradiction. $\square$

At this point we are able to turn most of the examples above into a lemma about prime ideals in a ring.

05KF Lemma 10.28.9. Let R be a ring. Let S be a multiplicative subset of R. An ideal $I \subset R$ which is maximal with respect to the property that $I \cap S = \emptyset$ is prime.

Proof. This is the example discussed in the introduction to this section. For an alternative proof, combine Example 10.28.3 with Proposition 10.28.8. $\square$

05KG Lemma 10.28.10. Let R be a ring.

(1) An ideal $I \subset R$ maximal with respect to not being finitely generated is prime.
(2) If every prime ideal of R is finitely generated, then every ideal of R is finitely generated[2].

Proof. The first assertion is an immediate consequence of Example 10.28.4 and Proposition 10.28.8. For the second, suppose that there exists an ideal $I \subset R$ which is not finitely generated. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that are not finitely generated is not finitely generated; indeed, if $I = \bigcup I_\alpha$ were generated by $a_1, \ldots, a_n$, then all the generators would belong to some I_α and would consequently generate it. By Zorn's lemma, there is an ideal maximal with respect to being not finitely generated. By the first part this ideal is prime. $\square$

05KH Lemma 10.28.11. Let R be a ring.

(1) An ideal $I \subset R$ maximal with respect to not being principal is prime.
(2) If every prime ideal of R is principal, then every ideal of R is principal.

Proof. The first part follows from Example 10.28.5 and Proposition 10.28.8. For the second, suppose that there exists an ideal $I \subset R$ which is not principal. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that not principal is not principal; indeed, if $I = \bigcup I_\alpha$ were generated by a, then a would belong to some I_α and a would generate it. By Zorn's lemma, there is an ideal maximal with respect to not being principal. This ideal is necessarily prime by the first part. $\square$

05KI Lemma 10.28.12. Let R be a ring.

(1) An ideal maximal among the ideals which do not contain a nonzerodivisor is prime.
(2) If R is nonzero and every nonzero prime ideal in R contains a nonzerodivisor, then R is a domain.

Proof. Consider the set S of nonzerodivisors. It is a multiplicative subset of R. Hence any ideal maximal with respect to not intersecting S is prime, see Lemma 10.28.9. Thus, if every nonzero prime ideal contains a nonzerodivisor, then (0) is prime, i.e., R is a domain. $\square$

05KJ Remark 10.28.13. Let R be a ring. Let κ be an infinite cardinal. By applying Example 10.28.6 and Proposition 10.28.8 we see that any ideal maximal with respect to the property of not being generated by κ elements is prime. This result is not so useful because there exists a ring for which every prime ideal of R can be generated by $\aleph_0$ elements, but some ideal cannot. Namely, let k be a field, let T be a set whose cardinality is greater than $\aleph_0$ and let

$$R = k[\{x_n\}_{n \geq 1}, \{z_{t,n}\}_{t \in T, n \geq 0}]/(x_n^2, z_{t,n}^2, x_n z_{t,n} - z_{t,n-1})$$

[2]Later we will say that R is Noetherian.

This is a local ring with unique prime ideal $\mathfrak{m} = (x_n)$. But the ideal $(z_{t,n})$ cannot be generated by countably many elements.

0G2Z Example 10.28.14. Let R be a ring and $X = \text{Spec}(R)$. Since closed subsets of X correspond to radical ideas of R (Lemma 10.17.2) we see that X is a Noetherian topological space if and only if we have ACC for radical ideals. This holds if and only if every radical ideal is the radical of a finitely generated ideal (details omitted). Let

Comment by Lukas Heger of November 12, 2020.

$$\mathcal{F} = \{I \subset R \mid \sqrt{I} = \sqrt{(f_1, \ldots, f_n)} \text{ for some } n \text{ and } f_1, \ldots, f_n \in R\}.$$

The reader can show that $\mathcal{F}$ is an Oka family by using the identity

$$\sqrt{I} = \sqrt{(I, a)(I : a)}$$

which holds for any ideal $I \subset R$ and any element $a \in R$. On the other hand, if we have a totally ordered chain of ideals $\{I_\alpha\}$ none of which are in $\mathcal{F}$, then the union $I = \bigcup I_\alpha$ cannot be in $\mathcal{F}$ either. Otherwise $\sqrt{I} = \sqrt{(f_1, \ldots, f_n)}$, then $f_i^e \in I$ for some e, then $f_i^e \in I_\alpha$ for some α independent of i, then $\sqrt{I_\alpha} = \sqrt{(f_1, \ldots, f_n)}$, contradiction. Thus if the set of ideals not in $\mathcal{F}$ is nonempty, then it has maximal elements and exactly as in Lemma 10.28.10 we conclude that X is a Noetherian topological space if and only if every prime ideal of R is equal to $\sqrt{(f_1, \ldots, f_n)}$ for some $f_1, \ldots, f_n \in R$. If we ever need this result we will carefully state and prove this result here.

10.29. Images of ring maps of finite presentation

00F5 In this section we prove some results on the topology of maps $\text{Spec}(S) \to \text{Spec}(R)$ induced by ring maps $R \to S$, mainly Chevalley's Theorem. In order to do this we will use the notions of constructible sets, quasi-compact sets, retrocompact sets, and so on which are defined in Topology, Section 5.15.

00F6 Lemma 10.29.1. Let $U \subset \text{Spec}(R)$ be open. The following are equivalent:

(1) U is retrocompact in $\text{Spec}(R)$,
(2) U is quasi-compact,
(3) U is a finite union of standard opens, and
(4) there exists a finitely generated ideal $I \subset R$ such that $X \setminus V(I) = U$.

Proof. We have (1) $\Rightarrow$ (2) because $\text{Spec}(R)$ is quasi-compact, see Lemma 10.17.10. We have (2) $\Rightarrow$ (3) because standard opens form a basis for the topology. Proof of (3) $\Rightarrow$ (1). Let $U = \bigcup_{i=1\ldots n} D(f_i)$. To show that U is retrocompact in $\text{Spec}(R)$ it suffices to show that $U \cap V$ is quasi-compact for any quasi-compact open V of $\text{Spec}(R)$. Write $V = \bigcup_{j=1\ldots m} D(g_j)$ which is possible by (2) $\Rightarrow$ (3). Each standard open is homeomorphic to the spectrum of a ring and hence quasi-compact, see Lemmas 10.17.6 and 10.17.10. Thus $U \cap V = (\bigcup_{i=1\ldots n} D(f_i)) \cap (\bigcup_{j=1\ldots m} D(g_j)) = \bigcup_{i,j} D(f_i g_j)$ is a finite union of quasi-compact opens hence quasi-compact. To finish the proof note that (4) is equivalent to (3) by Lemma 10.17.2. $\square$

00F7 Lemma 10.29.2. Let $\varphi : R \to S$ be a ring map. The induced continuous map $f : \text{Spec}(S) \to \text{Spec}(R)$ is quasi-compact. For any constructible set $E \subset \text{Spec}(R)$ the inverse image $f^{-1}(E)$ is constructible in $\text{Spec}(S)$.

Proof. We first show that the inverse image of any quasi-compact open $U \subset \text{Spec}(R)$ is quasi-compact. By Lemma 10.29.1 we may write U as a finite open of standard opens. Thus by Lemma 10.17.4 we see that $f^{-1}(U)$ is a finite union of standard opens. Hence $f^{-1}(U)$ is quasi-compact by Lemma 10.29.1 again. The second assertion now follows from Topology, Lemma 5.15.3. $\square$

0G1P Lemma 10.29.3. Let R be a ring. A subset of $\text{Spec}(R)$ is constructible if and only if it can be written as a finite union of subsets of the form $D(f) \cap V(g_1, \ldots, g_m)$ for $f, g_1, \ldots, g_m \in R$.

Proof. By Lemma 10.29.1 the subset $D(f)$ and the complement of $V(g_1, \ldots, g_m)$ are retro-compact open. Hence $D(f) \cap V(g_1, \ldots, g_m)$ is a constructible subset and so is any finite union of such. Conversely, let $T \subset \text{Spec}(R)$ be constructible. By Topology, Definition 5.15.1, we may assume that $T = U \cap V^c$, where $U, V \subset \text{Spec}(R)$ are retrocompact open. By Lemma 10.29.1 we may write $U = \bigcup_{i=1,\ldots,n} D(f_i)$ and $V = \bigcup_{j=1,\ldots,m} D(g_j)$. Then $T = \bigcup_{i=1,\ldots,n} \big(D(f_i) \cap V(g_1, \ldots, g_m)\big)$. $\square$

00F8 Lemma 10.29.4. Let R be a ring and let $T \subset \text{Spec}(R)$ be constructible. Then there exists a ring map $R \to S$ of finite presentation such that T is the image of $\text{Spec}(S)$ in $\text{Spec}(R)$.

Proof. The spectrum of a finite product of rings is the disjoint union of the spectra, see Lemma 10.21.2. Hence if $T = T_1 \cup T_2$ and the result holds for T_1 and T_2, then the result holds for T. By Lemma 10.29.3 we may assume that $T = D(f) \cap V(g_1, \ldots, g_m)$. In this case T is the image of the map $\text{Spec}((R/(g_1, \ldots, g_m))_f) \to \text{Spec}(R)$, see Lemmas 10.17.6 and 10.17.7. $\square$

00F9 Lemma 10.29.5. Let R be a ring. Let f be an element of R. Let $S = R_f$. Then the image of a constructible subset of $\text{Spec}(S)$ is constructible in $\text{Spec}(R)$.

Proof. We repeatedly use Lemma 10.29.1 without mention. Let U, V be quasi-compact open in $\text{Spec}(S)$. We will show that the image of $U \cap V^c$ is constructible. Under the identification $\text{Spec}(S) = D(f)$ of Lemma 10.17.6 the sets U, V correspond to quasi-compact opens U', V' of $\text{Spec}(R)$. Hence it suffices to show that $U' \cap (V')^c$ is constructible in $\text{Spec}(R)$ which is clear. $\square$

00FA Lemma 10.29.6. Let R be a ring. Let I be a finitely generated ideal of R. Let $S = R/I$. Then the image of a constructible subset of $\text{Spec}(S)$ is constructible in $\text{Spec}(R)$.

Proof. If $I = (f_1, \ldots, f_m)$, then we see that $V(I)$ is the complement of $\bigcup D(f_i)$, see Lemma 10.17.2. Hence it is constructible, by Lemma 10.29.1. Denote the map $R \to S$ by $f \mapsto \overline{f}$. We have to show that if $\overline{U}, \overline{V}$ are retrocompact opens of $\text{Spec}(S)$, then the image of $\overline{U} \cap \overline{V}^c$ in $\text{Spec}(R)$ is constructible. By Lemma 10.29.1 we may write $\overline{U} = \bigcup D(\overline{g_i})$. Setting $U = \bigcup D(g_i)$ we see $\overline{U}$ has image $U \cap V(I)$ which is constructible in $\text{Spec}(R)$. Similarly the image of $\overline{V}$ equals $V \cap V(I)$ for some retrocompact open V of $\text{Spec}(R)$. Hence the image of $\overline{U} \cap \overline{V}^c$ equals $U \cap V(I) \cap V^c$ as desired. $\square$

00FB Lemma 10.29.7. Let R be a ring. The map $\text{Spec}(R[x]) \to \text{Spec}(R)$ is open, and the image of any standard open is a quasi-compact open.

Proof. It suffices to show that the image of a standard open $D(f)$, $f \in R[x]$ is quasi-compact open. The image of $D(f)$ is the image of $\text{Spec}(R[x]_f) \to \text{Spec}(R)$. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $\overline{f}$ be the image of f in $\kappa(\mathfrak{p})[x]$. Recall, see Lemma

10.17.9, that $\mathfrak{p}$ is in the image if and only if $R[x]_f \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]_{\overline{f}}$ is not the zero ring. This is exactly the condition that f does not map to zero in $\kappa(\mathfrak{p})[x]$, in other words, that some coefficient of f is not in $\mathfrak{p}$. Hence we see: if $f = a_d x^d + \ldots + a_0$, then the image of $D(f)$ is $D(a_d) \cup \ldots \cup D(a_0)$. □

We prove a property of characteristic polynomials which will be used below.

00FC Lemma 10.29.8. Let $R \to A$ be a ring homomorphism. Assume $A \cong R^{\oplus n}$ as an R-module. Let $f \in A$. The multiplication map $m_f : A \to A$ is R-linear and hence has a characteristic polynomial $P(T) = T^n + r_{n-1}T^{n-1} + \ldots + r_0 \in R[T]$. For any prime $\mathfrak{p} \in \text{Spec}(R)$, f acts nilpotently on $A \otimes_R \kappa(\mathfrak{p})$ if and only if $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$.

Proof. This follows quite easily once we prove that the characteristic polynomial $\bar{P}(T) \in \kappa(\mathfrak{p})[T]$ of the multiplication map $m_{\bar{f}} : A \otimes_R \kappa(\mathfrak{p}) \to A \otimes_R \kappa(\mathfrak{p})$ which multiplies elements of $A \otimes_R \kappa(\mathfrak{p})$ by $\bar{f}$, the image of f viewed in $\kappa(\mathfrak{p})$, is just the image of $P(T)$ in $\kappa(\mathfrak{p})[T]$. Let (a_{ij}) be the matrix of the map m_f with entries in R, using a basis $e_1, \ldots, e_n$ of A as an R-module. Then, $A \otimes_R \kappa(\mathfrak{p}) \cong (R \otimes_R \kappa(\mathfrak{p}))^{\oplus n} = \kappa(\mathfrak{p})^n$, which is an n-dimensional vector space over $\kappa(\mathfrak{p})$ with basis $e_1 \otimes 1, \ldots, e_n \otimes 1$. The image $\bar{f} = f \otimes 1$, and so the multiplication map $m_{\bar{f}}$ has matrix $(a_{ij} \otimes 1)$. Thus, the characteristic polynomial is precisely the image of $P(T)$.

From linear algebra, we know that a linear transformation acts nilpotently on an n-dimensional vector space if and only if the characteristic polynomial is T^n (since the characteristic polynomial divides some power of the minimal polynomial). Hence, f acts nilpotently on $A \otimes_R \kappa(\mathfrak{p})$ if and only if $\bar{P}(T) = T^n$. This occurs if and only if $r_i \in \mathfrak{p}$ for all $0 \leq i \leq n-1$, that is when $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$. □

00FD Lemma 10.29.9. Let R be a ring. Let $f, g \in R[x]$ be polynomials. Assume the leading coefficient of g is a unit of R. There exists elements $r_i \in R$, $i = 1 \ldots, n$ such that the image of $D(f) \cap V(g)$ in $\text{Spec}(R)$ is $\bigcup_{i=1,\ldots,n} D(r_i)$.

Proof. Write $g = ux^d + a_{d-1}x^{d-1} + \ldots + a_0$, where d is the degree of g, and hence $u \in R^*$. Consider the ring $A = R[x]/(g)$. It is, as an R-module, finite free with basis the images of $1, x, \ldots, x^{d-1}$. Consider multiplication by (the image of) f on A. This is an R-module map. Hence we can let $P(T) \in R[T]$ be the characteristic polynomial of this map. Write $P(T) = T^d + r_{d-1}T^{d-1} + \ldots + r_0$. We claim that $r_0, \ldots, r_{d-1}$ have the desired property. We will use below the property of characteristic polynomials that

$$\mathfrak{p} \in V(r_0, \ldots, r_{d-1}) \Leftrightarrow \text{multiplication by } f \text{ is nilpotent on } A \otimes_R \kappa(\mathfrak{p}).$$

This was proved in Lemma 10.29.8.

Suppose $\mathfrak{q} \in D(f) \cap V(g)$, and let $\mathfrak{p} = \mathfrak{q} \cap R$. Then there is a nonzero map $A \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ which is compatible with multiplication by f. And f acts as a unit on $\kappa(\mathfrak{q})$. Thus we conclude $\mathfrak{p} \notin V(r_0, \ldots, r_{d-1})$.

On the other hand, suppose that $r_i \notin \mathfrak{p}$ for some prime $\mathfrak{p}$ of R and some $0 \leq i \leq d-1$. Then multiplication by f is not nilpotent on the algebra $A \otimes_R \kappa(\mathfrak{p})$. Hence there exists a prime ideal $\overline{\mathfrak{q}} \subset A \otimes_R \kappa(\mathfrak{p})$ not containing the image of f. The inverse image of $\overline{\mathfrak{q}}$ in $R[x]$ is an element of $D(f) \cap V(g)$ mapping to $\mathfrak{p}$. □

00FE Theorem 10.29.10 (Chevalley's Theorem). Suppose that $R \to S$ is of finite presentation. The image of a constructible subset of $\text{Spec}(S)$ in $\text{Spec}(R)$ is constructible.

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We may factor $R \to S$ as $R \to R[x_1] \to R[x_1, x_2] \to \ldots \to R[x_1, \ldots, x_{n-1}] \to S$. Hence we may assume that $S = R[x]/(f_1, \ldots, f_m)$. In this case we factor the map as $R \to R[x] \to S$, and by Lemma 10.29.6 we reduce to the case $S = R[x]$. By Lemma 10.29.1 suffices to show that if $T = (\bigcup_{i=1\ldots n} D(f_i)) \cap V(g_1, \ldots, g_m)$ for $f_i, g_j \in R[x]$ then the image in $\mathrm{Spec}(R)$ is constructible. Since finite unions of constructible sets are constructible, it suffices to deal with the case $n = 1$, i.e., when $T = D(f) \cap V(g_1, \ldots, g_m)$.

Note that if $c \in R$, then we have

$$\mathrm{Spec}(R) = V(c) \amalg D(c) = \mathrm{Spec}(R/(c)) \amalg \mathrm{Spec}(R_c),$$

and correspondingly $\mathrm{Spec}(R[x]) = V(c) \amalg D(c) = \mathrm{Spec}(R/(c)[x]) \amalg \mathrm{Spec}(R_c[x])$. The intersection of $T = D(f) \cap V(g_1, \ldots, g_m)$ with each part still has the same shape, with f, g_i replaced by their images in $R/(c)[x]$, respectively $R_c[x]$. Note that the image of T in $\mathrm{Spec}(R)$ is the union of the image of $T \cap V(c)$ and $T \cap D(c)$. Using Lemmas 10.29.5 and 10.29.6 it suffices to prove the images of both parts are constructible in $\mathrm{Spec}(R/(c))$, respectively $\mathrm{Spec}(R_c)$.

Let us assume we have $T = D(f) \cap V(g_1, \ldots, g_m)$ as above, with $\deg(g_1) \leq \deg(g_2) \leq \ldots \leq \deg(g_m)$. We are going to use induction on m, and on the degrees of the g_i. Let $d = \deg(g_1)$, i.e., $g_1 = cx^{d_1} + l.o.t$ with $c \in R$ not zero. Cutting R up into the pieces $R/(c)$ and R_c we either lower the degree of g_1 (and this is covered by induction) or we reduce to the case where c is invertible. If c is invertible, and $m > 1$, then write $g_2 = c'x^{d_2} + l.o.t.$ In this case consider $g_2' = g_2 - (c'/c)x^{d_2 - d_1} g_1$. Since the ideals $(g_1, g_2, \ldots, g_m)$ and $(g_1, g_2', g_3, \ldots, g_m)$ are equal we see that $T = D(f) \cap V(g_1, g_2', g_3 \ldots, g_m)$. But here the degree of g_2' is strictly less than the degree of g_2 and hence this case is covered by induction.

The bases case for the induction above are the cases (a) $T = D(f) \cap V(g)$ where the leading coefficient of g is invertible, and (b) $T = D(f)$. These two cases are dealt with in Lemmas 10.29.9 and 10.29.7. □

10.30. More on images

00FF In this section we collect a few additional lemmas concerning the image on Spec for ring maps. See also Section 10.41 for example.

00FG Lemma 10.30.1. Let $R \subset S$ be an inclusion of domains. Assume that $R \to S$ is of finite type. There exists a nonzero $f \in R$, and a nonzero $g \in S$ such that $R_f \to S_{fg}$ is of finite presentation.

Proof. By induction on the number of generators of S over R. During the proof we may replace R by R_f and S by S_f for some nonzero $f \in R$.

Suppose that S is generated by a single element over R. Then $S = R[x]/\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset R[x]$. If $\mathfrak{q} = (0)$ there is nothing to prove. If $\mathfrak{q} \neq (0)$, then let $h \in \mathfrak{q}$ be a nonzero element with minimal degree in x. Write $h = fx^d + a_{d-1}x^{d-1} + \ldots + a_0$ with $a_i \in R$ and $f \neq 0$. After inverting f in R and S we may assume that h is monic. We obtain a surjective R-algebra map $R[x]/(h) \to S$. We have $R[x]/(h) = R \oplus Rx \oplus \ldots \oplus Rx^{d-1}$ as an R-module and by minimality of d we see that $R[x]/(h)$ maps injectively into S. Thus $R[x]/(h) \cong S$ is finitely presented over R.

Suppose that S is generated by $n > 1$ elements over R. Say $x_1, \ldots, x_n \in S$ generate S. Denote $S' \subset S$ the subring generated by $x_1, \ldots, x_{n-1}$. By induction hypothesis we see that there exist $f \in R$ and $g \in S'$ nonzero such that $R_f \to S'_{fg}$ is of finite presentation. Next we apply the induction hypothesis to $S'_{fg} \to S_{fg}$ to see that there exist $f' \in S'_{fg}$ and $g' \in S_{fg}$ such that $S'_{fgf'} \to S_{fgf'g'}$ is of finite presentation. We leave it to the reader to conclude. $\square$

00FH Lemma 10.30.2. Let $R \to S$ be a finite type ring map. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \mathrm{Spec}(S)$ be a constructible set. If a point $\xi \in X$ is in $f(E)$, then $\overline{\{\xi\}} \cap f(E)$ contains an open dense subset of $\overline{\{\xi\}}$.

Proof. Let $\xi \in X$ be a point of $f(E)$. Choose a point $\eta \in E$ mapping to ξ. Let $\mathfrak p \subset R$ be the prime corresponding to ξ and let $\mathfrak q \subset S$ be the prime corresponding to η. Consider the diagram

$$\begin{tikzcd} \eta \arrow[r] \arrow[d, mapsto] & E \cap Y' \arrow[r] \arrow[d] & Y' = \mathrm{Spec}(S/\mathfrak q) \arrow[r] \arrow[d] & Y \arrow[d] \\ \xi \arrow[r] & f(E) \cap X' \arrow[r] & X' = \mathrm{Spec}(R/\mathfrak p) \arrow[r] & X \end{tikzcd}$$

By Lemma 10.29.2 the set $E \cap Y'$ is constructible in Y'. It follows that we may replace X by X' and Y by Y'. Hence we may assume that $R \subset S$ is an inclusion of domains, ξ is the generic point of X, and η is the generic point of Y. By Lemma 10.30.1 combined with Chevalley's theorem (Theorem 10.29.10) we see that there exist dense opens $U \subset X$, $V \subset Y$ such that $f(V) \subset U$ and such that $f : V \to U$ maps constructible sets to constructible sets. Note that $E \cap V$ is constructible in V, see Topology, Lemma 5.15.4. Hence $f(E \cap V)$ is constructible in U and contains ξ. By Topology, Lemma 5.15.15 we see that $f(E \cap V)$ contains a dense open $U' \subset U$. $\square$

At the end of this section we present a few more results on images of maps on Spectra that have nothing to do with constructible sets.

00FI Lemma 10.30.3. Let $\varphi : R \to S$ be a ring map. The following are equivalent:

(1) The map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.
(2) For any ideal $I \subset R$ the inverse image of $\sqrt{IS}$ in R is equal to $\sqrt{I}$.
(3) For any radical ideal $I \subset R$ the inverse image of IS in R is equal to I.
(4) For every prime $\mathfrak p$ of R the inverse image of $\mathfrak pS$ in R is $\mathfrak p$.

In this case the same is true after any base change: Given a ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ has the equivalent properties (1), (2), (3) as well.

Proof. If $J \subset S$ is an ideal, then $\sqrt{\varphi^{-1}(J)} = \varphi^{-1}(\sqrt{J})$. This shows that (2) and (3) are equivalent. The implication (3) $\Rightarrow$ (4) is immediate. If $I \subset R$ is a radical ideal, then Lemma 10.17.2 guarantees that $I = \bigcap_{I \subset \mathfrak p} \mathfrak p$. Hence (4) $\Rightarrow$ (2). By Lemma 10.17.9 we have $\mathfrak p = \varphi^{-1}(\mathfrak pS)$ if and only if $\mathfrak p$ is in the image. Hence (1) $\Leftrightarrow$ (4). Thus (1), (2), (3), and (4) are equivalent.

Assume (1) holds. Let $R \to R'$ be a ring map. Let $\mathfrak p' \subset R'$ be a prime ideal lying over the prime $\mathfrak p$ of R. To see that $\mathfrak p'$ is in the image of $\mathrm{Spec}(R' \otimes_R S) \to \mathrm{Spec}(R')$ we have to show that $(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak p')$ is not zero, see Lemma 10.17.9. But we have

$$(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak p') = S \otimes_R \kappa(\mathfrak p) \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak p')$$

which is not zero as $S \otimes_R \kappa(\mathfrak{p})$ is not zero by assumption and $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$ is an extension of fields. $\square$

00FJ Lemma 10.30.4. Let R be a domain. Let $\varphi : R \to S$ be a ring map. The following are equivalent:

(1) The ring map $R \to S$ is injective.
(2) The image $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains a dense set of points.
(3) There exists a prime ideal $\mathfrak{q} \subset S$ whose inverse image in R is (0).

Proof. Let K be the field of fractions of the domain R. Assume that $R \to S$ is injective. Since localization is exact we see that $K \to S \otimes_R K$ is injective. Hence there is a prime mapping to (0) by Lemma 10.17.9.

Note that (0) is dense in $\mathrm{Spec}(R)$, so that the last condition implies the second.

Suppose the second condition holds. Let $f \in R$, $f \neq 0$. As R is a domain we see that $V(f)$ is a proper closed subset of R. By assumption there exists a prime $\mathfrak{q}$ of S such that $\varphi(f) \notin \mathfrak{q}$. Hence $\varphi(f) \neq 0$. Hence $R \to S$ is injective. $\square$

00FK Lemma 10.30.5. Let $R \subset S$ be an injective ring map. Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ hits all the minimal primes.

Proof. Let $\mathfrak{p} \subset R$ be a minimal prime. In this case $R_{\mathfrak{p}}$ has a unique prime ideal. Hence it suffices to show that $S_{\mathfrak{p}}$ is not zero. And this follows from the fact that localization is exact, see Proposition 10.9.12. $\square$

00FL Lemma 10.30.6. Let $R \to S$ be a ring map. The following are equivalent:

(1) The kernel of $R \to S$ consists of nilpotent elements.
(2) The minimal primes of R are in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.
(3) The image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is dense in $\mathrm{Spec}(R)$.

Proof. Let $I = \mathrm{Ker}(R \to S)$. Note that $\sqrt{(0)} = \bigcap_{\mathfrak{q} \subset S} \mathfrak{q}$, see Lemma 10.17.2. Hence $\sqrt{I} = \bigcap_{\mathfrak{q} \subset S} R \cap \mathfrak{q}$. Thus $V(I) = V(\sqrt{I})$ is the closure of the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. This shows that (1) is equivalent to (3). It is clear that (2) implies (3). Finally, assume (1). We may replace R by R/I and S by S/IS without affecting the topology of the spectra and the map. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 10.30.5. $\square$

0CAN Lemma 10.30.7. Let $R \to S$ be a ring map. If a minimal prime $\mathfrak{p} \subset R$ is in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, then it is the image of a minimal prime.

Proof. Say $\mathfrak{p} = \mathfrak{q} \cap R$. Then choose a minimal prime $\mathfrak{r} \subset S$ with $\mathfrak{r} \subset \mathfrak{q}$, see Lemma 10.17.2. By minimality of $\mathfrak{p}$ we see that $\mathfrak{p} = \mathfrak{r} \cap R$. $\square$

10.31. Noetherian rings

00FM A ring R is Noetherian if any ideal of R is finitely generated. This is clearly equivalent to the ascending chain condition for ideals of R. By Lemma 10.28.10 it suffices to check that every prime ideal of R is finitely generated.

00FN Lemma 10.31.1. Any finitely generated ring over a Noetherian ring is Noetherian. Any localization of a Noetherian ring is Noetherian.

Proof. The statement on localizations follows from the fact that any ideal $J \subset S^{-1}R$ is of the form $I \cdot S^{-1}R$. Any quotient R/I of a Noetherian ring R is Noetherian because any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$. Thus it suffices to show that if R is Noetherian so is $R[X]$. Suppose $J_1 \subset J_2 \subset \ldots$ is an ascending chain of ideals in $R[X]$. Consider the ideals $I_{i,d}$ defined as the ideal of elements of R which occur as leading coefficients of degree d polynomials in J_i. Clearly $I_{i,d} \subset I_{i',d'}$ whenever $i \leq i'$ and $d \leq d'$. By the ascending chain condition in R there are at most finitely many distinct ideals among all of the $I_{i,d}$. (Hint: Any infinite set of elements of $\mathbf{N} \times \mathbf{N}$ contains an increasing infinite sequence.) Take i_0 so large that $I_{i,d} = I_{i_0,d}$ for all $i \geq i_0$ and all d. Suppose $f \in J_i$ for some $i \geq i_0$. By induction on the degree $d = \deg(f)$ we show that $f \in J_{i_0}$. Namely, there exists a $g \in J_{i_0}$ whose degree is d and which has the same leading coefficient as f. By induction $f - g \in J_{i_0}$ and we win. $\square$

0306 Lemma 10.31.2. If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \ldots, x_n]]$.

Proof. Since $R[[x_1, \ldots, x_{n+1}]] \cong R[[x_1, \ldots, x_n]][[x_{n+1}]]$ it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be a ideal. We have to show that I is a finitely generated ideal. For each integer d denote $I_d = \{a \in R \mid ax^d + \text{h.o.t.} \in I\}$. Then we see that $I_0 \subset I_1 \subset \ldots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \ldots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d)$, $j = 1, \ldots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{h.o.t}$ then $I_d = (a_{d,j})$. Denote $I' = (\{f_{d,j}\}_{d=0,\ldots,d_0,j=1,\ldots,n_d})$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that

$$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I.$$

Next, we can choose $c_{i,1} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we can choose $c_{i,2} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

And so on. In the end we see that

$$f = \sum c_{d,i} f_{d,i} + \sum\nolimits_i (\sum\nolimits_e c_{i,e} x^e) f_{d_0,i}$$

is contained in I' as desired. $\square$

The following lemma, although easy, is useful because finite type $\mathbf{Z}$-algebras come up quite often in a technique called "absolute Noetherian reduction".

00FO Lemma 10.31.3. Any finite type algebra over a field is Noetherian. Any finite type algebra over $\mathbf{Z}$ is Noetherian.

Proof. This is immediate from Lemma 10.31.1 and the fact that fields are Noetherian rings and that $\mathbf{Z}$ is Noetherian ring (because it is a principal ideal domain). $\square$

00FP Lemma 10.31.4. Let R be a Noetherian ring.

(1) Any finite R-module is of finite presentation.
(2) Any submodule of a finite R-module is finite.
(3) Any finite type R-algebra is of finite presentation over R.

Proof. Let M be a finite R-module. By Lemma 10.5.4 we can find a finite filtration of M whose successive quotients are of the form R/I. Since any ideal is finitely generated, each of the quotients R/I is finitely presented. Hence M is finitely presented by Lemma 10.5.3. This proves (1).

Let $N \subset M$ be a submodule. As M is finite, the quotient M/N is finite. Thus M/N is of finite presentation by part (1). Thus we see that N is finite by Lemma 10.5.3 part (5). This proves part (2).

To see (3) note that any ideal of $R[x_1, \ldots, x_n]$ is finitely generated by Lemma 10.31.1. □

00FQ **Lemma 10.31.5.** *If R is a Noetherian ring then $\mathrm{Spec}(R)$ is a Noetherian topological space, see Topology, Definition 5.9.1.*

Proof. This is because any closed subset of $\mathrm{Spec}(R)$ is uniquely of the form $V(I)$ with I a radical ideal, see Lemma 10.17.2. And this correspondence is inclusion reversing. Thus the result follows from the definitions. □

00FR **Lemma 10.31.6.** *If R is a Noetherian ring then $\mathrm{Spec}(R)$ has finitely many irreducible components. In other words R has finitely many minimal primes.*

Proof. By Lemma 10.31.5 and Topology, Lemma 5.9.2 we see there are finitely many irreducible components. By Lemma 10.26.1 these correspond to minimal primes of R. □

0CY6 **Lemma 10.31.7.** *Let $R \to S$ be a ring map. Let $R \to R'$ be of finite type. If S is Noetherian, then the base change $S' = R' \otimes_R S$ is Noetherian.*

Proof. By Lemma 10.14.2 finite type is stable under base change. Thus $S \to S'$ is of finite type. Since S is Noetherian we can apply Lemma 10.31.1. □

045I **Lemma 10.31.8.** *Let k be a field and let R be a Noetherian k-algebra. If K/k is a finitely generated field extension then $K \otimes_k R$ is Noetherian.*

Proof. Since K/k is a finitely generated field extension, there exists a finitely generated k-algebra $B \subset K$ such that K is the fraction field of B. In other words, $K = S^{-1}B$ with $S = B \setminus \{0\}$. Then $K \otimes_k R = S^{-1}(B \otimes_k R)$. Then $B \otimes_k R$ is Noetherian by Lemma 10.31.7. Finally, $K \otimes_k R = S^{-1}(B \otimes_k R)$ is Noetherian by Lemma 10.31.1. □

Here are some fun lemmas that are sometimes useful.

0BX1 **Lemma 10.31.9.** *Let R be a ring and $\mathfrak{p} \subset R$ be a prime. There exists an $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to R_\mathfrak{p}$ is injective in each of the following cases*

1. *R is a domain,*
2. *R is Noetherian, or*
3. *R is reduced and has finitely many minimal primes.*

Proof. If R is a domain, then $R \subset R_\mathfrak{p}$, hence $f = 1$ works. If R is Noetherian, then the kernel I of $R \to R_\mathfrak{p}$ is a finitely generated ideal and we can find $f \in R$, $f \notin \mathfrak{p}$ such that $IR_f = 0$. For this f the map $R_f \to R_\mathfrak{p}$ is injective and f works. If R is reduced with finitely many minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then we can choose $f \in \bigcap_{\mathfrak{p}_i \not\subset \mathfrak{p}} \mathfrak{p}_i$, $f \notin \mathfrak{p}$. Indeed, if $\mathfrak{p}_i \not\subset \mathfrak{p}$ then there exist $f_i \in \mathfrak{p}_i$, $f_i \notin \mathfrak{p}$ and $f = \prod f_i$ works. For this f we have $R_f \subset R_\mathfrak{p}$ because the minimal primes of R_f correspond to minimal primes of $R_\mathfrak{p}$ and we can apply Lemma 10.25.2 (some details omitted). □

06RN Lemma 10.31.10. Any surjective endomorphism of a Noetherian ring is an isomorphism.

Proof. If $f : R \to R$ were such an endomorphism but not injective, then

$$\mathrm{Ker}(f) \subset \mathrm{Ker}(f \circ f) \subset \mathrm{Ker}(f \circ f \circ f) \subset \ldots$$

would be a strictly increasing chain of ideals. □

10.32. Locally nilpotent ideals

0AMF Here is the definition.

00IL Definition 10.32.1. Let R be a ring. Let $I \subset R$ be an ideal. We say I is locally nilpotent if for every $x \in I$ there exists an $n \in \mathbf{N}$ such that $x^n = 0$. We say I is nilpotent if there exists an $n \in \mathbf{N}$ such that $I^n = 0$.

0EGG Example 10.32.2. Let $R = k[x_n | n \in \mathbf{N}]$ be the polynomial ring in infinitely many variables over a field k. Let I be the ideal generated by the elements x_n^n for $n \in \mathbf{N}$ and $S = R/I$. Then the ideal $J \subset S$ generated by the images of x_n, $n \in \mathbf{N}$ is locally nilpotent, but not nilpotent. Indeed, since S-linear combinations of nilpotents are nilpotent, to prove that J is locally nilpotent it is enough to observe that all its generators are nilpotent (which they obviously are). On the other hand, for each $n \in \mathbf{N}$ it holds that $x_{n+1}^n \notin I$, so that $J^n \neq 0$. It follows that J is not nilpotent.

0544 Lemma 10.32.3. Let $R \to R'$ be a ring map and let $I \subset R$ be a locally nilpotent ideal. Then IR' is a locally nilpotent ideal of R'.

Proof. This follows from the fact that if $x, y \in R'$ are nilpotent, then $x + y$ is nilpotent too. Namely, if $x^n = 0$ and $y^m = 0$, then $(x + y)^{n+m-1} = 0$. □

0AMG Lemma 10.32.4. Let R be a ring and let $I \subset R$ be a locally nilpotent ideal. An element x of R is a unit if and only if the image of x in R/I is a unit.

Proof. If x is a unit in R, then its image is clearly a unit in R/I. It remains to prove the converse. Assume the image of $y \in R$ in R/I is the inverse of the image of x. Then $xy = 1 - z$ for some $z \in I$. This means that $1 \equiv z$ modulo xR. Since z lies in the locally nilpotent ideal I, we have $z^N = 0$ for some sufficiently large N. It follows that $1 = 1^N \equiv z^N = 0$ modulo xR. In other words, x divides 1 and is hence a unit. □

00IM Lemma 10.32.5. Let R be a Noetherian ring. Let I, J be ideals of R. Suppose $J \subset \sqrt{I}$. Then $J^n \subset I$ for some n. In particular, in a Noetherian ring the notions of "locally nilpotent ideal" and "nilpotent ideal" coincide.

Proof. Say $J = (f_1, \ldots, f_s)$. By assumption $f_i^{d_i} \in I$. Take $n = d_1 + d_2 + \ldots + d_s + 1$. □

00J9 Lemma 10.32.6. Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Then $R \to R/I$ induces a bijection on idempotents.

First proof of Lemma 10.32.6. As I is locally nilpotent it is contained in every prime ideal. Hence $\mathrm{Spec}(R/I) = V(I) = \mathrm{Spec}(R)$. Hence the lemma follows from Lemma 10.21.3. □

Second proof of Lemma 10.32.6. Suppose $\overline{e} \in R/I$ is an idempotent. We have to lift $\overline{e}$ to an idempotent of R.

First, choose any lift $f \in R$ of $\overline{e}$, and set $x = f^2 - f$. Then, $x \in I$, so x is nilpotent (since I is locally nilpotent). Let now J be the ideal of R generated by x. Then, J is nilpotent (not just locally nilpotent), since it is generated by the nilpotent x.

Now, assume that we have found a lift $e \in R$ of $\overline{e}$ such that $e^2 - e \in J^k$ for some $k \geq 1$. Let $e' = e - (2e-1)(e^2 - e) = 3e^2 - 2e^3$, which is another lift of $\overline{e}$ (since the idempotency of $\overline{e}$ yields $e^2 - e \in I$). Then

$$(e')^2 - e' = (4e^2 - 4e - 3)(e^2 - e)^2 \in J^{2k}$$

by a simple computation.

We thus have started with a lift e of $\overline{e}$ such that $e^2 - e \in J^k$, and obtained a lift e' of $\overline{e}$ such that $(e')^2 - e' \in J^{2k}$. This way we can successively improve the approximation (starting with $e = f$, which fits the bill for $k = 1$). Eventually, we reach a stage where $J^k = 0$, and at that stage we have a lift e of $\overline{e}$ such that $e^2 - e \in J^k = 0$, that is, this e is idempotent.

We thus have seen that if $\overline{e} \in R/I$ is any idempotent, then there exists a lift of $\overline{e}$ which is an idempotent of R. It remains to prove that this lift is unique. Indeed, let e_1 and e_2 be two such lifts. We need to show that $e_1 = e_2$.

By definition of e_1 and e_2, we have $e_1 \equiv e_2 \mod I$, and both e_1 and e_2 are idempotent. From $e_1 \equiv e_2 \mod I$, we see that $e_1 - e_2 \in I$, so that $e_1 - e_2$ is nilpotent (since I is locally nilpotent). A straightforward computation (using the idempotency of e_1 and e_2) reveals that $(e_1 - e_2)^3 = e_1 - e_2$. Using this and induction, we obtain $(e_1 - e_2)^k = e_1 - e_2$ for any positive odd integer k. Since all high enough k satisfy $(e_1 - e_2)^k = 0$ (since $e_1 - e_2$ is nilpotent), this shows $e_1 - e_2 = 0$, so that $e_1 = e_2$, which completes our proof. $\square$

05BU Lemma 10.32.7. Let A be a possibly noncommutative algebra. Let $e \in A$ be an element such that $x = e^2 - e$ is nilpotent. Then there exists an idempotent of the form $e' = e + x(\sum a_{i,j} e^i x^j) \in A$ with $a_{i,j} \in \mathbf{Z}$.

Proof. Consider the ring $R_n = \mathbf{Z}[e]/((e^2 - e)^n)$. It is clear that if we can prove the result for each R_n then the lemma follows. In R_n consider the ideal $I = (e^2 - e)$ and apply Lemma 10.32.6. $\square$

0CAP Lemma 10.32.8. Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $n \geq 1$ be an integer which is invertible in R/I. Then

(1) the nth power map $1 + I \to 1 + I$, $1 + x \mapsto (1 + x)^n$ is a bijection,
(2) a unit of R is a nth power if and only if its image in R/I is an nth power.

Proof. Let $a \in R$ be a unit whose image in R/I is the same as the image of b^n with $b \in R$. Then b is a unit (Lemma 10.32.4) and $ab^{-n} = 1 + x$ for some $x \in I$. Hence $ab^{-n} = c^n$ by part (1). Thus (2) follows from (1).

Proof of (1). This is true because there is an inverse to the map $1 + x \mapsto (1 + x)^n$. Namely, we can consider the map which sends $1 + x$ to

$$\begin{aligned}(1+x)^{1/n} &= 1 + \binom{1/n}{1}x + \binom{1/n}{2}x^2 + \binom{1/n}{3}x^3 + \ldots \\ &= 1 + \frac{1}{n}x + \frac{1-n}{2n^2}x^2 + \frac{(1-n)(1-2n)}{6n^3}x^3 + \ldots\end{aligned}$$

as in elementary calculus. This makes sense because the series is finite as $x^k = 0$ for all $k \gg 0$ and each coefficient $\binom{1/n}{k} \in \mathbf{Z}[1/n]$ (details omitted; observe that n is invertible in R by Lemma 10.32.4). $\square$

10.33. Curiosity

02JG Lemma 10.24.3 explains what happens if $V(I)$ is open for some ideal $I \subset R$. But what if $\text{Spec}(S^{-1}R)$ is closed in $\text{Spec}(R)$? The next two lemmas give a partial answer. For more information see Section 10.108.

02JH Lemma 10.33.1. Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $\text{Spec}(S^{-1}R) \to \text{Spec}(R)$ is closed. Then $S^{-1}R \cong R/I$ for some ideal $I \subset R$.

Proof. Let $I = \text{Ker}(R \to S^{-1}R)$ so that $V(I)$ contains the image. Say the image is the closed subset $V(I') \subset \text{Spec}(R)$ for some ideal $I' \subset R$. So $V(I') \subset V(I)$. For $f \in I'$ we see that $f/1 \in S^{-1}R$ is contained in every prime ideal. Hence f^n maps to zero in $S^{-1}R$ for some $n \geq 1$ (Lemma 10.17.2). Hence $V(I') = V(I)$. Then this implies every $g \in S$ is invertible mod I. Hence we get ring maps $R/I \to S^{-1}R$ and $S^{-1}R \to R/I$. The first map is injective by choice of I. The second is the map $S^{-1}R \to S^{-1}(R/I) = R/I$ which has kernel $S^{-1}I$ because localization is exact. Since $S^{-1}I = 0$ we see also the second map is injective. Hence $S^{-1}R \cong R/I$. $\square$

02JI Lemma 10.33.2. Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $\text{Spec}(S^{-1}R) \to \text{Spec}(R)$ is closed. If R is Noetherian, or $\text{Spec}(R)$ is a Noetherian topological space, or S is finitely generated as a monoid, then $R \cong S^{-1}R \times R'$ for some ring R'.

Proof. By Lemma 10.33.1 we have $S^{-1}R \cong R/I$ for some ideal $I \subset R$. By Lemma 10.24.3 it suffices to show that $V(I)$ is open. If R is Noetherian then $\text{Spec}(R)$ is a Noetherian topological space, see Lemma 10.31.5. If $\text{Spec}(R)$ is a Noetherian topological space, then the complement $\text{Spec}(R) \setminus V(I)$ is quasi-compact, see Topology, Lemma 5.12.13. Hence there exist finitely many $f_1, \ldots, f_n \in I$ such that $V(I) = V(f_1, \ldots, f_n)$. Since each f_i maps to zero in $S^{-1}R$ there exists a $g \in S$ such that $gf_i = 0$ for $i = 1, \ldots, n$. Hence $D(g) = V(I)$ as desired. In case S is finitely generated as a monoid, say S is generated by $g_1, \ldots, g_m$, then $S^{-1}R \cong R_{g_1 \ldots g_m}$ and we conclude that $V(I) = D(g_1 \ldots g_m)$. $\square$

10.34. Hilbert Nullstellensatz

00FS

00FV Theorem 10.34.1 (Hilbert Nullstellensatz). Let k be a field.

00FW (1) For any maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the field extension $\kappa(\mathfrak{m})/k$ is finite.

00FX (2) Any radical ideal $I \subset k[x_1, \ldots, x_n]$ is the intersection of maximal ideals containing it.

The same is true in any finite type k-algebra.

Proof. It is enough to prove part (1) of the theorem for the case of a polynomial algebra $k[x_1, \ldots, x_n]$, because any finitely generated k-algebra is a quotient of such a polynomial algebra. We prove this by induction on n. The case $n = 0$ is clear. Suppose that $\mathfrak{m}$ is a maximal ideal in $k[x_1, \ldots, x_n]$. Let $\mathfrak{p} \subset k[x_n]$ be the intersection of $\mathfrak{m}$ with $k[x_n]$.

If $\mathfrak{p} \neq (0)$, then $\mathfrak{p}$ is maximal and generated by an irreducible monic polynomial P (because of the Euclidean algorithm in $k[x_n]$). Then $k' = k[x_n]/\mathfrak{p}$ is a finite field extension of k and contained in $\kappa(\mathfrak{m})$. In this case we get a surjection

$$k'[x_1, \ldots, x_{n-1}] \to k'[x_1, \ldots, x_n] = k' \otimes_k k[x_1, \ldots, x_n] \longrightarrow \kappa(\mathfrak{m})$$

and hence we see that $\kappa(\mathfrak{m})$ is a finite extension of k' by induction hypothesis. Thus $\kappa(\mathfrak{m})$ is finite over k as well.

If $\mathfrak{p} = (0)$ we consider the ring extension $k[x_n] \subset k[x_1, \ldots, x_n]/\mathfrak{m}$. This is a finitely generated ring extension, hence of finite presentation by Lemmas 10.31.3 and 10.31.4. Thus the image of $\mathrm{Spec}(k[x_1, \ldots, x_n]/\mathfrak{m})$ in $\mathrm{Spec}(k[x_n])$ is constructible by Theorem 10.29.10. Since the image contains (0) we conclude that it contains a standard open $D(f)$ for some $f \in k[x_n]$ nonzero. Since clearly $D(f)$ is infinite we get a contradiction with the assumption that $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field (and hence has a spectrum consisting of one point).

Proof of (2). Let $I \subset R$ be a radical ideal, with R of finite type over k. Let $f \in R$, $f \notin I$. We have to find a maximal ideal $\mathfrak{m} \subset R$ with $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. The ring $(R/I)_f$ is nonzero, since $1 = 0$ in this ring would mean $f^n \in I$ and since I is radical this would mean $f \in I$ contrary to our assumption on f. Thus we may choose a maximal ideal $\mathfrak{m}'$ in $(R/I)_f$, see Lemma 10.17.2. Let $\mathfrak{m} \subset R$ be the inverse image of $\mathfrak{m}'$ in R. We see that $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. If we show that $\mathfrak{m}$ is a maximal ideal of R, then we are done. We clearly have

$$k \subset R/\mathfrak{m} \subset \kappa(\mathfrak{m}').$$

By part (1) the field extension $\kappa(\mathfrak{m}')/k$ is finite. Hence $R/\mathfrak{m}$ is a field by Fields, Lemma 9.8.10. Thus $\mathfrak{m}$ is maximal and the proof is complete. □

00FY Lemma 10.34.2. Let R be a ring. Let K be a field. If $R \subset K$ and K is of finite type over R, then there exists an $f \in R$ such that R_f is a field, and K/R_f is a finite field extension.

Proof. By Lemma 10.30.2 there exist a nonempty open $U \subset \mathrm{Spec}(R)$ contained in the image $\{(0)\}$ of $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$. Choose $f \in R$, $f \neq 0$ such that $D(f) \subset U$, i.e., $D(f) = \{(0)\}$. Then R_f is a domain whose spectrum has exactly one point and R_f is a field. Then K is a finitely generated algebra over the field R_f and hence a finite field extension of R_f by the Hilbert Nullstellensatz (Theorem 10.34.1). □

10.35. Jacobson rings

00FZ Let R be a ring. The closed points of $\mathrm{Spec}(R)$ are the maximal ideals of R. Often rings which occur naturally in algebraic geometry have lots of maximal ideals. For example finite type algebras over a field or over $\mathbf{Z}$. We will show that these are examples of Jacobson rings.

00G0 Definition 10.35.1. Let R be a ring. We say that R is a Jacobson ring if every radical ideal I is the intersection of the maximal ideals containing it.

00G1 Lemma 10.35.2. Any algebra of finite type over a field is Jacobson.

Proof. This follows from Theorem 10.34.1 and Definition 10.35.1. $\square$

00G2 Lemma 10.35.3. Let R be a ring. If every prime ideal of R is the intersection of the maximal ideals containing it, then R is Jacobson.

Proof. This is immediately clear from the fact that every radical ideal $I \subset R$ is the intersection of the primes containing it. See Lemma 10.17.2. $\square$

00G3 Lemma 10.35.4. A ring R is Jacobson if and only if $\text{Spec}(R)$ is Jacobson, see Topology, Definition 5.18.1.

Proof. Suppose R is Jacobson. Let $Z \subset \text{Spec}(R)$ be a closed subset. We have to show that the set of closed points in Z is dense in Z. Let $U \subset \text{Spec}(R)$ be an open such that $U \cap Z$ is nonempty. We have to show $Z \cap U$ contains a closed point of $\text{Spec}(R)$. We may assume $U = D(f)$ as standard opens form a basis for the topology on $\text{Spec}(R)$. According to Lemma 10.17.2 we may assume that $Z = V(I)$, where I is a radical ideal. We see also that $f \notin I$. By assumption, there exists a maximal ideal $\mathfrak{m} \subset R$ such that $I \subset \mathfrak{m}$ but $f \notin \mathfrak{m}$. Hence $\mathfrak{m} \in D(f) \cap V(I) = U \cap Z$ as desired.

Conversely, suppose that $\text{Spec}(R)$ is Jacobson. Let $I \subset R$ be a radical ideal. Let $J = \cap_{I \subset \mathfrak{m}} \mathfrak{m}$ be the intersection of the maximal ideals containing I. Clearly J is a radical ideal, $V(J) \subset V(I)$, and $V(J)$ is the smallest closed subset of $V(I)$ containing all the closed points of $V(I)$. By assumption we see that $V(J) = V(I)$. But Lemma 10.17.2 shows there is a bijection between Zariski closed sets and radical ideals, hence $I = J$ as desired. $\square$

034J Lemma 10.35.5. Let R be a ring. If R is not Jacobson there exist a prime $\mathfrak{p} \subset R$, an element $f \in R$ such that the following hold

(1) $\mathfrak{p}$ is not a maximal ideal,
(2) $f \notin \mathfrak{p}$,
(3) $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, and
(4) $(R/\mathfrak{p})_f$ is a field.

On the other hand, if R is Jacobson, then for any pair $(\mathfrak{p}, f)$ such that (1) and (2) hold the set $V(\mathfrak{p}) \cap D(f)$ is infinite.

Proof. Assume R is not Jacobson. By Lemma 10.35.4 this means there exists an closed subset $T \subset \text{Spec}(R)$ whose set $T_0 \subset T$ of closed points is not dense in T. Choose an $f \in R$ such that $T_0 \subset V(f)$ but $T \not\subset V(f)$. Note that $T \cap D(f)$ is homeomorphic to $\text{Spec}((R/I)_f)$ if $T = V(I)$, see Lemmas 10.17.7 and 10.17.6. As any ring has a maximal ideal (Lemma 10.17.2) we can choose a closed point t of space $T \cap D(f)$. Then t corresponds to a prime ideal $\mathfrak{p} \subset R$ which is not maximal (as $t \notin T_0$). Thus (1) holds. By construction $f \notin \mathfrak{p}$, hence (2). As t is a closed point of $T \cap D(f)$ we see that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, i.e., (3) holds. Hence we conclude that $(R/\mathfrak{p})_f$ is a domain whose spectrum has one point, hence (4) holds (for example combine Lemmas 10.18.2 and 10.25.1).

Conversely, suppose that R is Jacobson and $(\mathfrak{p}, f)$ satisfy (1) and (2). If $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ then $\mathfrak{p} \neq \mathfrak{q}_i$ implies there exists an element $g \in R$ such that $g \notin \mathfrak{p}$ but $g \in \mathfrak{q}_i$ for all i. Hence $V(\mathfrak{p}) \cap D(fg) = \{\mathfrak{p}\}$ which is impossible since each locally closed subset of $\operatorname{Spec}(R)$ contains at least one closed point as $\operatorname{Spec}(R)$ is a Jacobson topological space. $\square$

00G4 Lemma 10.35.6. The ring $\mathbf{Z}$ is a Jacobson ring. More generally, let R be a ring such that

(1) R is a domain,
(2) R is Noetherian,
(3) any nonzero prime ideal is a maximal ideal, and
(4) R has infinitely many maximal ideals.

Then R is a Jacobson ring.

Proof. Let R satisfy (1), (2), (3) and (4). The statement means that $(0) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$. Since R has infinitely many maximal ideals it suffices to show that any nonzero $x \in R$ is contained in at most finitely many maximal ideals, in other words that $V(x)$ is finite. By Lemma 10.17.7 we see that $V(x)$ is homeomorphic to $\operatorname{Spec}(R/xR)$. By assumption (3) every prime of R/xR is minimal and hence corresponds to an irreducible component of $\operatorname{Spec}(R/xR)$ (Lemma 10.26.1). As R/xR is Noetherian, the topological space $\operatorname{Spec}(R/xR)$ is Noetherian (Lemma 10.31.5) and has finitely many irreducible components (Topology, Lemma 5.9.2). Thus $V(x)$ is finite as desired. $\square$

02CC Example 10.35.7. Let A be an infinite set. For each $\alpha \in A$, let k_α be a field. We claim that $R = \prod_{\alpha \in A} k_\alpha$ is Jacobson. First, note that any element $f \in R$ has the form $f = ue$, with $u \in R$ a unit and $e \in R$ an idempotent (left to the reader). Hence $D(f) = D(e)$, and $R_f = R_e = R/(1-e)$ is a quotient of R. Actually, any ring with this property is Jacobson. Namely, say $\mathfrak{p} \subset R$ is a prime ideal and $f \in R$, $f \notin \mathfrak{p}$. We have to find a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. Because R_f is a quotient of R we see that any maximal ideal of R_f corresponds to a maximal ideal of R not containing f. Hence the result follows by choosing a maximal ideal of R_f containing $\mathfrak{p}R_f$.

00G5 Example 10.35.8. A domain R with finitely many maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$ is not a Jacobson ring, except when it is a field. Namely, in this case (0) is not the intersection of the maximal ideals $(0) \neq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdot \ldots \cdot \mathfrak{m}_n \neq 0$. In particular a discrete valuation ring, or any local ring with at least two prime ideals is not a Jacobson ring.

00GA Lemma 10.35.9. Let $R \to S$ be a ring map. Let $\mathfrak{m} \subset R$ be a maximal ideal. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$ such that $\kappa(\mathfrak{q})/\kappa(\mathfrak{m})$ is an algebraic field extension. Then $\mathfrak{q}$ is a maximal ideal of S.

Proof. Consider the diagram

$$\begin{tikzcd} S \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) \\ R \arrow[u] \arrow[r] & R/\mathfrak{m} \arrow[u] & \end{tikzcd}$$

We see that $\kappa(\mathfrak{m}) \subset S/\mathfrak{q} \subset \kappa(\mathfrak{q})$. Because the field extension $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{q})$ is algebraic, any ring between $\kappa(\mathfrak{m})$ and $\kappa(\mathfrak{q})$ is a field (Fields, Lemma 9.8.10). Thus $S/\mathfrak{q}$ is a field, and a posteriori equal to $\kappa(\mathfrak{q})$. □

00FT Lemma 10.35.10. Suppose that k is a field and suppose that V is a nonzero vector space over k. Assume the dimension of V (which is a cardinal number) is smaller than the cardinality of k. Then for any linear operator $T : V \to V$ there exists some monic polynomial $P(t) \in k[t]$ such that $P(T)$ is not invertible.

Proof. If not then V inherits the structure of a vector space over the field $k(t)$. But the dimension of $k(t)$ over k is at least the cardinality of k for example due to the fact that the elements $\frac{1}{t-\lambda}$ are k-linearly independent. □

Here is another version of Hilbert's Nullstellensatz.

00FU Theorem 10.35.11. Let k be a field. Let S be a k-algebra generated over k by the elements $\{x_i\}_{i\in I}$. Assume the cardinality of I is smaller than the cardinality of k. Then

(1) for all maximal ideals $\mathfrak{m} \subset S$ the field extension $\kappa(\mathfrak{m})/k$ is algebraic, and
(2) S is a Jacobson ring.

Proof. If I is finite then the result follows from the Hilbert Nullstellensatz, Theorem 10.34.1. In the rest of the proof we assume I is infinite. It suffices to prove the result for $\mathfrak{m} \subset k[\{x_i\}_{i\in I}]$ maximal in the polynomial ring on variables x_i, since S is a quotient of this. As I is infinite the set of monomials $x_{i_1}^{e_1} \ldots x_{i_r}^{e_r}$, $i_1, \ldots, i_r \in I$ and $e_1, \ldots, e_r \geq 0$ has cardinality at most equal to the cardinality of I. Because the cardinality of $I \times \ldots \times I$ is the cardinality of I, and also the cardinality of $\bigcup_{n\geq 0} I^n$ has the same cardinality. (If I is finite, then this is not true and in that case this proof only works if k is uncountable.)

To arrive at a contradiction pick $T \in \kappa(\mathfrak{m})$ transcendental over k. Note that the k-linear map $T : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{m})$ given by multiplication by T has the property that $P(T)$ is invertible for all monic polynomials $P(t) \in k[t]$. Also, $\kappa(\mathfrak{m})$ has dimension at most the cardinality of I over k since it is a quotient of the vector space $k[\{x_i\}_{i\in I}]$ over k (whose dimension is $\#I$ as we saw above). This is impossible by Lemma 10.35.10.

To show that S is Jacobson we argue as follows. If not then there exists a prime $\mathfrak{q} \subset S$ and an element $f \in S$, $f \notin \mathfrak{q}$ such that $\mathfrak{q}$ is not maximal and $(S/\mathfrak{q})_f$ is a field, see Lemma 10.35.5. But note that $(S/\mathfrak{q})_f$ is generated by at most $\#I + 1$ elements. Hence the field extension $(S/\mathfrak{q})_f/k$ is algebraic (by the first part of the proof). This implies that $\kappa(\mathfrak{q})$ is an algebraic extension of k hence $\mathfrak{q}$ is maximal by Lemma 10.35.9. This contradiction finishes the proof. □

046V Lemma 10.35.12. Let k be a field. Let S be a k-algebra. For any field extension K/k whose cardinality is larger than the cardinality of S we have

(1) for every maximal ideal $\mathfrak{m}$ of S_K the field $\kappa(\mathfrak{m})$ is algebraic over K, and
(2) S_K is a Jacobson ring.

Proof. Choose $k \subset K$ such that the cardinality of K is greater than the cardinality of S. Since the elements of S generate the K-algebra S_K we see that Theorem 10.35.11 applies. □

02CB Example 10.35.13. The trick in the proof of Theorem 10.35.11 really does not work if k is a countable field and I is countable too. Let k be a countable field. Let x be a variable, and let $k(x)$ be the field of rational functions in x. Consider the polynomial algebra $R = k[x, \{x_f\}_{f \in k[x] - \{0\}}]$. Let $I = (\{fx_f - 1\}_{f \in k[x]-\{0\}})$. Note that I is a proper ideal in R. Choose a maximal ideal $I \subset \mathfrak m$. Then $k \subset R/\mathfrak m$ is isomorphic to $k(x)$, and is not algebraic over k.

00G6 Lemma 10.35.14. Let R be a Jacobson ring. Let $f \in R$. The ring R_f is Jacobson and maximal ideals of R_f correspond to maximal ideals of R not containing f.

Proof. By Topology, Lemma 5.18.5 we see that $D(f) = \text{Spec}(R_f)$ is Jacobson and that closed points of $D(f)$ correspond to closed points in $\text{Spec}(R)$ which happen to lie in $D(f)$. Thus R_f is Jacobson by Lemma 10.35.4. □

00G7 Example 10.35.15. Here is a simple example that shows Lemma 10.35.14 to be false if R is not Jacobson. Consider the ring $R = \mathbf{Z}_{(2)}$, i.e., the localization of $\mathbf{Z}$ at the prime (2). The localization of R at the element 2 is isomorphic to $\mathbf{Q}$, in a formula: $R_2 \cong \mathbf{Q}$. Clearly the map $R \to R_2$ maps the closed point of $\text{Spec}(\mathbf{Q})$ to the generic point of $\text{Spec}(R)$.

00G8 Example 10.35.16. Here is a simple example that shows Lemma 10.35.14 is false if R is Jacobson but we localize at infinitely many elements. Namely, let $R = \mathbf{Z}$ and consider the localization $(R \setminus \{0\})^{-1}R \cong \mathbf{Q}$ of R at the set of all nonzero elements. Clearly the map $\mathbf{Z} \to \mathbf{Q}$ maps the closed point of $\text{Spec}(\mathbf{Q})$ to the generic point of $\text{Spec}(\mathbf{Z})$.

00G9 Lemma 10.35.17. Let R be a Jacobson ring. Let $I \subset R$ be an ideal. The ring R/I is Jacobson and maximal ideals of R/I correspond to maximal ideals of R containing I.

Proof. The proof is the same as the proof of Lemma 10.35.14. □

0CY7 Lemma 10.35.18. Let R be a Jacobson ring. Let K be a field. Let $R \subset K$ and K is of finite type over R. Then R is a field and K/R is a finite field extension.

Proof. First note that R is a domain. By Lemma 10.34.2 we see that R_f is a field and K/R_f is a finite field extension for some nonzero $f \in R$. Hence (0) is a maximal ideal of R_f and by Lemma 10.35.14 we conclude (0) is a maximal ideal of R. □

00GB Proposition 10.35.19. Let R be a Jacobson ring. Let $R \to S$ be a ring map of finite type. Then

(1) The ring S is Jacobson.
(2) The map $\text{Spec}(S) \to \text{Spec}(R)$ transforms closed points to closed points.
(3) For $\mathfrak m' \subset S$ maximal lying over $\mathfrak m \subset R$ the field extension $\kappa(\mathfrak m')/\kappa(\mathfrak m)$ is finite.

Proof. Let $\mathfrak m' \subset S$ be a maximal ideal and $R \cap \mathfrak m' = \mathfrak m$. Then $R/\mathfrak m \to S/\mathfrak m'$ satisfies the conditions of Lemma 10.35.18 by Lemma 10.35.17. Hence $R/\mathfrak m$ is a field and $\mathfrak m$ a maximal ideal and the induced residue field extension is finite. This proves (2) and (3).

If S is not Jacobson, then by Lemma 10.35.5 there exists a non-maximal prime ideal $\mathfrak q$ of S and an $g \in S$, $g \notin \mathfrak q$ such that $(S/\mathfrak q)_g$ is a field. To arrive at a contradiction we show that $\mathfrak q$ is a maximal ideal. Let $\mathfrak p = \mathfrak q \cap R$. Then $R/\mathfrak p \to (S/\mathfrak q)_g$ satisfies the conditions of Lemma 10.35.18 by Lemma 10.35.17. Hence $R/\mathfrak{p}$ is a field and the field extension $\kappa(\mathfrak{p}) \to (S/\mathfrak{q})_g = \kappa(\mathfrak{q})$ is finite, thus algebraic. Then $\mathfrak{q}$ is a maximal ideal of S by Lemma 10.35.9. Contradiction. $\square$

00GC Lemma 10.35.20. Any finite type algebra over $\mathbf{Z}$ is Jacobson.

Proof. Combine Lemma 10.35.6 and Proposition 10.35.19. $\square$

00GD Lemma 10.35.21. Let $R \to S$ be a finite type ring map of Jacobson rings. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \mathrm{Spec}(S)$ be a constructible set. Denote with a subscript $_0$ the set of closed points of a topological space.

(1) We have $f(E)_0 = f(E_0) = X_0 \cap f(E)$.
(2) A point $\xi \in X$ is in $f(E)$ if and only if $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$.

Proof. We have a commutative diagram of continuous maps

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d] \\ f(E) \arrow[r] & X \end{tikzcd}$$

Suppose $x \in f(E)$ is closed in $f(E)$. Then $f^{-1}(\{x\}) \cap E$ is nonempty and closed in E. Applying Topology, Lemma 5.18.5 to both inclusions

$$f^{-1}(\{x\}) \cap E \subset E \subset Y$$

we find there exists a point $y \in f^{-1}(\{x\}) \cap E$ which is closed in Y. In other words, there exists $y \in Y_0$ and $y \in E_0$ mapping to x. Hence $x \in f(E_0)$. This proves that $f(E)_0 \subset f(E_0)$. Proposition 10.35.19 implies that $f(E_0) \subset X_0 \cap f(E)$. The inclusion $X_0 \cap f(E) \subset f(E)_0$ is trivial. This proves the first assertion.

Suppose that $\xi \in f(E)$. According to Lemma 10.30.2 the set $f(E) \cap \overline{\{\xi\}}$ contains a dense open subset of $\overline{\{\xi\}}$. Since X is Jacobson we conclude that $f(E) \cap \overline{\{\xi\}}$ contains a dense set of closed points, see Topology, Lemma 5.18.5. We conclude by part (1) of the lemma.

On the other hand, suppose that $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$. By Lemma 10.29.4 there exists a ring map $S \to S'$ of finite presentation such that E is the image of $Y' := \mathrm{Spec}(S') \to Y$. Then E_0 is the image of Y'_0 by the first part of the lemma applied to the ring map $S \to S'$. Thus we may assume that $E = Y$ by replacing S by S'. Suppose ξ corresponds to $\mathfrak{p} \subset R$. Consider the diagram

$$\begin{tikzcd} S \arrow[r] & S/\mathfrak{p}S \\ R \arrow[r] \arrow[u] & R/\mathfrak{p} \arrow[u] \end{tikzcd}$$

This diagram and the density of $f(Y_0) \cap V(\mathfrak{p})$ in $V(\mathfrak{p})$ shows that the morphism $R/\mathfrak{p} \to S/\mathfrak{p}S$ satisfies condition (2) of Lemma 10.30.4. Hence we conclude there exists a prime $\overline{\mathfrak{q}} \subset S/\mathfrak{p}S$ mapping to (0). In other words the inverse image $\mathfrak{q}$ of $\overline{\mathfrak{q}}$ in S maps to $\mathfrak{p}$ as desired. $\square$

The conclusion of the lemma above is that we can read off the image of f from the set of closed points of the image. This is a little nicer in case the map is of finite presentation because then we know that images of a constructible is constructible. Before we state it we introduce some notation. Denote $\mathrm{Constr}(X)$ the set of constructible sets. Let $R \to S$ be a ring map. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Denote with a subscript $_0$ the set of closed points of a topological space.

00GE Lemma 10.35.22. With notation as above. Assume that R is a Noetherian Jacobson ring. Further assume $R \to S$ is of finite type. There is a commutative diagram

$$\begin{tikzcd} \mathrm{Constr}(Y) \arrow[r, "E\mapsto E_0"] \arrow[d, "E\mapsto f(E)"] & \mathrm{Constr}(Y_0) \arrow[d, "E\mapsto f(E)"] \\ \mathrm{Constr}(X) \arrow[r, "E\mapsto E_0"] & \mathrm{Constr}(X_0) \end{tikzcd}$$

where the horizontal arrows are the bijections from Topology, Lemma 5.18.8.

Proof. Since $R \to S$ is of finite type, it is of finite presentation, see Lemma 10.31.4. Thus the image of a constructible set in X is constructible in Y by Chevalley's theorem (Theorem 10.29.10). Combined with Lemma 10.35.21 the lemma follows. □

To illustrate the use of Jacobson rings, we give the following two examples.

00GF Example 10.35.23. Let k be a field. The space $\mathrm{Spec}(k[x,y]/(xy))$ has two irreducible components: namely the x-axis and the y-axis. As a generalization, let

$$R = k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]/\mathfrak{a},$$

where $\mathfrak{a}$ is the ideal in $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$ generated by the entries of the 2×2 product matrix

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

In this example we will describe $\mathrm{Spec}(R)$.

To prove the statement about $\mathrm{Spec}(k[x,y]/(xy))$ we argue as follows. If $\mathfrak{p} \subset k[x,y]$ is any ideal containing xy, then either x or y would be contained in $\mathfrak{p}$. Hence the minimal such prime ideals are just (x) and (y). In case k is algebraically closed, the max-Spec of these components can then be visualized as the point sets of y- and x-axis.

For the generalization, note that we may identify the closed points of the spectrum of $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}])$ with the space of matrices

$$\left\{(X,Y) \in \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) \mid X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\right\}$$

at least if k is algebraically closed. Now define a group action of $\mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \times \mathrm{GL}(2,k)$ on the space of matrices $\{(X,Y)\}$ by

$$(g_1, g_2, g_3) \times (X, Y) \mapsto ((g_1 X g_2^{-1}, g_2 Y g_3^{-1})).$$

Here, also observe that the algebraic set

$$\mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \subset \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k)$$

is irreducible since it is the max spectrum of the domain

$k[x_{11}, x_{12}, \ldots, z_{21}, z_{22}, (x_{11}x_{22} - x_{12}x_{21})^{-1}, (y_{11}y_{22} - y_{12}y_{21})^{-1}, (z_{11}z_{22} - z_{12}z_{21})^{-1}]$.

Since the image of irreducible an algebraic set is still irreducible, it suffices to classify the orbits of the set $\{(X, Y) \in \mathrm{Mat}(2, k) \times \mathrm{Mat}(2, k) | XY = 0\}$ and take their closures. From standard linear algebra, we are reduced to the following three cases:

(1) $\exists (g_1, g_2)$ such that $g_1 X g_2^{-1} = I_{2\times 2}$. Then Y is necessarily 0, which as an algebraic set is invariant under the group action. It follows that this orbit is contained in the irreducible algebraic set defined by the prime ideal $(y_{11}, y_{12}, y_{21}, y_{22})$. Taking the closure, we see that $(y_{11}, y_{12}, y_{21}, y_{22})$ is actually a component.

(2) $\exists (g_1, g_2)$ such that
$$g_1 X g_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
This case occurs if and only if X is a rank 1 matrix, and furthermore, Y is killed by such an X if and only if
$$x_{11}y_{11} + x_{12}y_{21} = 0; \quad x_{11}y_{12} + x_{12}y_{22} = 0;$$
$$x_{21}y_{11} + x_{22}y_{21} = 0; \quad x_{21}y_{12} + x_{22}y_{22} = 0.$$
Fix a rank 1 X, such non zero Y's satisfying the above equations form an irreducible algebraic set for the following reason($Y = 0$ is contained the previous case): $0 = g_1 X g_2^{-1} g_2 Y$ implies that
$$g_2 Y = \begin{pmatrix} 0 & 0 \\ y'_{21} & y'_{22} \end{pmatrix}.$$
With a further $\mathrm{GL}(2, k)$-action on the right by g_3, $g_2 Y$ can be brought into
$$g_2 Y g_3^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
and thus such Y's form an irreducible algebraic set isomorphic to the image of $\mathrm{GL}(2, k)$ under this action. Finally, notice that the "rank 1" condition for X's forms an open dense subset of the irreducible algebraic set $\det X = x_{11}x_{22} - x_{12}x_{21} = 0$. It now follows that all the five equations define an irreducible component $(x_{11}y_{11} + x_{12}y_{21}, x_{11}y_{12} + x_{12}y_{22}, x_{21}y_{11} + x_{22}y_{21}, x_{21}y_{12} + x_{22}y_{22}, x_{11}x_{22} - x_{12}x_{21})$ in the open subset of the space of pairs of nonzero matrices. It can be shown that the pair of equations $\det X = 0$, $\det Y = 0$ cuts $\mathrm{Spec}(R)$ in an irreducible component with the above locus an open dense subset.

(3) $\exists (g_1, g_2)$ such that $g_1 X g_2^{-1} = 0$, or equivalently, $X = 0$. Then Y can be arbitrary and this component is thus defined by $(x_{11}, x_{12}, x_{21}, x_{22})$.

00GG Example 10.35.24. For another example, consider $R = k[\{t_{ij}\}_{i,j=1}^n]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the entries of the product matrix $T^2 - T$, $T = (t_{ij})$. From linear algebra, we know that under the $GL(n, k)$-action defined by $g, T \mapsto gTg^{-1}$, T is classified by the its rank and each T is conjugate to some $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$, which has r 1's and $n - r$ 0's. Thus each orbit of such a $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ under the group action forms an irreducible component and every idempotent matrix is contained in one such orbit. Next we will show that any two different orbits are necessarily disjoint. For this purpose we only need to cook up polynomial functions that take different values on different orbits. In characteristic 0 cases, such a function can be taken to be $f(t_{ij}) = trace(T) = \sum_{i=1}^n t_{ii}$. In positive characteristic cases, things are slightly more tricky since we might have $trace(T) = 0$ even if $T \neq 0$. For instance, $char = 3$

$$trace \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = 3 = 0$$

Anyway, these components can be separated using other functions. For instance, in the characteristic 3 case, $tr(\wedge^3 T)$ takes value 1 on the components corresponding to $diag(1, 1, 1)$ and 0 on other components.

10.36. Finite and integral ring extensions

00GH Trivial lemmas concerning finite and integral ring maps. We recall the definition.

00GI Definition 10.36.1. Let $\varphi : R \to S$ be a ring map.

1. An element $s \in S$ is integral over R if there exists a monic polynomial $P(x) \in R[x]$ such that $P^\varphi(s) = 0$, where $P^\varphi(x) \in S[x]$ is the image of P under $\varphi : R[x] \to S[x]$.
2. The ring map φ is integral if every $s \in S$ is integral over R.

052I Lemma 10.36.2. Let $\varphi : R \to S$ be a ring map. Let $y \in S$. If there exists a finite R-submodule M of S such that $1 \in M$ and $yM \subset M$, then y is integral over R.

Proof. Consider the map $\varphi : M \to M$, $x \mapsto y \cdot x$. By Lemma 10.16.2 there exists a monic polynomial $P \in R[T]$ with $P(\varphi) = 0$. In the ring S we get $P(y) = P(y) \cdot 1 = P(\varphi)(1) = 0$. □

00GK Lemma 10.36.3. A finite ring extension is integral.

Proof. Let $R \to S$ be finite. Let $y \in S$. Apply Lemma 10.36.2 to $M = S$ to see that y is integral over R. □

00GM Lemma 10.36.4. Let $\varphi : R \to S$ be a ring map. Let $s_1, \ldots, s_n$ be a finite set of elements of S. In this case s_i is integral over R for all $i = 1, \ldots, n$ if and only if there exists an R-subalgebra $S' \subset S$ finite over R containing all of the s_i.

Proof. If each s_i is integral, then the subalgebra generated by $\varphi(R)$ and the s_i is finite over R. Namely, if s_i satisfies a monic equation of degree d_i over R, then this subalgebra is generated as an R-module by the elements $s_1^{e_1} \ldots s_n^{e_n}$ with $0 \leq e_i \leq d_i - 1$. Conversely, suppose given a finite R-subalgebra S' containing all the s_i. Then all of the s_i are integral by Lemma 10.36.3. □

02JJ Lemma 10.36.5. Let $R \to S$ be a ring map. The following are equivalent

1. $R \to S$ is finite,
2. $R \to S$ is integral and of finite type, and
3. there exist $x_1, \ldots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R.

Proof. Clear from Lemma 10.36.4. □

00GN Lemma 10.36.6. Suppose that $R \to S$ and $S \to T$ are integral ring maps. Then $R \to T$ is integral.

Proof. Let $t \in T$. Let $P(x) \in S[x]$ be a monic polynomial such that $P(t) = 0$. Apply Lemma 10.36.4 to the finite set of coefficients of P. Hence t is integral over some subalgebra $S' \subset S$ finite over R. Apply Lemma 10.36.4 again to find a subalgebra $T' \subset T$ finite over S' and containing t. Lemma 10.7.3 applied to $R \to S' \to T'$ shows that T' is finite over R. The integrality of t over R now follows from Lemma 10.36.3. $\square$

00GO Lemma 10.36.7. Let $R \to S$ be a ring homomorphism. The set

$$S' = \{s \in S \mid s \text{ is integral over } R\}$$

is an R-subalgebra of S.

Proof. This is clear from Lemmas 10.36.4 and 10.36.3. $\square$

0CY8 Lemma 10.36.8. Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Let R and S denote the product of the R_i and S_i respectively. Then an element $s = (s_1, \ldots, s_n) \in S$ is integral over R if and only if each s_i is integral over R_i.

Proof. Omitted. $\square$

00GP Definition 10.36.9. Let $R \to S$ be a ring map. The ring $S' \subset S$ of elements integral over R, see Lemma 10.36.7, is called the integral closure of R in S. If $R \subset S$ we say that R is integrally closed in S if $R = S'$.

In particular, we see that $R \to S$ is integral if and only if the integral closure of R in S is all of S.

0CY9 Lemma 10.36.10. Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Denote the integral closure of R_i in S_i by S_i'. Further let R and S denote the product of the R_i and S_i respectively. Then the integral closure of R in S is the product of the S_i'. In particular $R \to S$ is integrally closed if and only if each $R_i \to S_i$ is integrally closed.

Proof. This follows immediately from Lemma 10.36.8. $\square$

0307 Lemma 10.36.11. Integral closure commutes with localization: If $A \to B$ is a ring map, and $S \subset A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subset B$ is the integral closure of A in B.

Proof. Since localization is exact we see that $S^{-1}B' \subset S^{-1}B$. Suppose $x \in B'$ and $f \in S$. Then $x^d + \sum_{i=1,\ldots,d} a_i x^{d-i} = 0$ in B for some $a_i \in A$. Hence also

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} a_i/f^i (x/f)^{d-i} = 0$$

in $S^{-1}B$. In this way we see that $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} (a_i/f_i)(x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f' f_1 \ldots f_d x)^d + \sum\nolimits_{i=1,\ldots,d} f^i (f')^i f_1^i \ldots f_i^{i-1} \ldots f_d^i a_i (f' f_1 \ldots f_d x)^{d-i} = 0$$

for a suitable $f' \in S$. Hence $f' f_1 \ldots f_d x \in B'$ and thus $x/f \in S^{-1}B'$ as desired. $\square$

034K Lemma 10.36.12. Let $\varphi : R \to S$ be a ring map. Let $x \in S$. The following are equivalent:

(1) x is integral over R, and
(2) for every prime ideal $\mathfrak{p} \subset R$ the element $x \in S_\mathfrak{p}$ is integral over $R_\mathfrak{p}$.

Proof. It is clear that (1) implies (2). Assume (2). Consider the R-algebra $S' \subset S$ generated by $\varphi(R)$ and x. Let $\mathfrak{p}$ be a prime ideal of R. Then we know that $x^d + \sum_{i=1,\ldots,d} \varphi(a_i)x^{d-i} = 0$ in $S_\mathfrak{p}$ for some $a_i \in R_\mathfrak{p}$. Hence we see, by looking at which denominators occur, that for some $f \in R$, $f \not\in \mathfrak{p}$ we have $a_i \in R_f$ and $x^d + \sum_{i=1,\ldots,d} \varphi(a_i)x^{d-i} = 0$ in S_f. This implies that S'_f is finite over R_f. Since $\mathfrak{p}$ was arbitrary and $\mathrm{Spec}(R)$ is quasi-compact (Lemma 10.17.10) we can find finitely many elements $f_1, \ldots, f_n \in R$ which generate the unit ideal of R such that S'_{f_i} is finite over R_{f_i}. Hence we conclude from Lemma 10.23.2 that S' is finite over R. Hence x is integral over R by Lemma 10.36.4. □

02JK Lemma 10.36.13. Let $R \to S$ and $R \to R'$ be ring maps. Set $S' = R' \otimes_R S$.

(1) If $R \to S$ is integral so is $R' \to S'$.
(2) If $R \to S$ is finite so is $R' \to S'$.

Proof. We prove (1). Let $s_i \in S$ be generators for S over R. Each of these satisfies a monic polynomial equation P_i over R. Hence the elements $1 \otimes s_i \in S'$ generate S' over R' and satisfy the corresponding polynomial P_i' over R'. Since these elements generate S' over R' we see that S' is integral over R'. Proof of (2) omitted. □

02JL Lemma 10.36.14. Let $R \to S$ be a ring map. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.

(1) If each $R_{f_i} \to S_{f_i}$ is integral, so is $R \to S$.
(2) If each $R_{f_i} \to S_{f_i}$ is finite, so is $R \to S$.

Proof. Proof of (1). Let $s \in S$. Consider the ideal $I \subset R[x]$ of polynomials P such that $P(s) = 0$. Let $J \subset R$ denote the ideal (!) of leading coefficients of elements of I. By assumption and clearing denominators we see that $f_i^{n_i} \in J$ for all i and certain $n_i \geq 0$. Hence J contains 1 and we see s is integral over R. Proof of (2) omitted. □

02JM Lemma 10.36.15. Let $A \to B \to C$ be ring maps.

(1) If $A \to C$ is integral so is $B \to C$.
(2) If $A \to C$ is finite so is $B \to C$.

Proof. Omitted. □

0308 Lemma 10.36.16. Let $A \to B \to C$ be ring maps. Let B' be the integral closure of A in B, let C' be the integral closure of B' in C. Then C' is the integral closure of A in C.

Proof. Omitted. □

00GQ Lemma 10.36.17. Suppose that $R \to S$ is an integral ring extension with $R \subset S$. Then $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. We have to show $\mathfrak{p}S_\mathfrak{p} \neq S_\mathfrak{p}$, see Lemma 10.17.9. The localization $R_\mathfrak{p} \to S_\mathfrak{p}$ is injective (as localization is exact) and integral by Lemma 10.36.11 or 10.36.13. Hence we may replace R, S by $R_\mathfrak{p}$, $S_\mathfrak{p}$ and we may assume R is local with maximal ideal $\mathfrak{m}$ and it suffices to show that $\mathfrak{m}S \neq S$. Suppose $1 = \sum f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$ in order to get a contradiction. Let

$R \subset S' \subset S$ be such that $R \to S'$ is finite and $s_i \in S'$, see Lemma 10.36.4. The equation $1 = \sum f_i s_i$ implies that the finite R-module S' satisfies $S' = \mathfrak{m}S'$. Hence by Nakayama's Lemma 10.20.1 we see $S' = 0$. Contradiction. $\square$

00GR Lemma 10.36.18. Let R be a ring. Let K be a field. If $R \subset K$ and K is integral over R, then R is a field and K is an algebraic extension. If $R \subset K$ and K is finite over R, then R is a field and K is a finite algebraic extension.

Proof. Assume that $R \subset K$ is integral. By Lemma 10.36.17 we see that $\text{Spec}(R)$ has 1 point. Since clearly R is a domain we see that $R = R_{(0)}$ is a field (Lemma 10.25.1). The other assertions are immediate from this. $\square$

00GS Lemma 10.36.19. Let k be a field. Let S be a k-algebra over k.

(1) If S is a domain and finite dimensional over k, then S is a field.
(2) If S is integral over k and a domain, then S is a field.
(3) If S is integral over k then every prime of S is a maximal ideal (see Lemma 10.26.5 for more consequences).

Proof. The statement on primes follows from the statement "integral + domain $\Rightarrow$ field". Let S integral over k and assume S is a domain, Take $s \in S$. By Lemma 10.36.4 we may find a finite dimensional k-subalgebra $k \subset S' \subset S$ containing s. Hence S is a field if we can prove the first statement. Assume S finite dimensional over k and a domain. Pick $s \in S$. Since S is a domain the multiplication map $s : S \to S$ is surjective by dimension reasons. Hence there exists an element $s_1 \in S$ such that $ss_1 = 1$. So S is a field. $\square$

00GT Lemma 10.36.20. Suppose $R \to S$ is integral. Let $\mathfrak{q}, \mathfrak{q}' \in \text{Spec}(S)$ be distinct primes having the same image in $\text{Spec}(R)$. Then neither $\mathfrak{q} \subset \mathfrak{q}'$ nor $\mathfrak{q}' \subset \mathfrak{q}$.

Proof. Let $\mathfrak{p} \subset R$ be the image. By Remark 10.17.8 the primes $\mathfrak{q}, \mathfrak{q}'$ correspond to ideals in $S \otimes_R \kappa(\mathfrak{p})$. Thus the lemma follows from Lemma 10.36.19. $\square$

05DR Lemma 10.36.21. Suppose $R \to S$ is finite. Then the fibres of $\text{Spec}(S) \to \text{Spec}(R)$ are finite.

Proof. By the discussion in Remark 10.17.8 the fibres are the spectra of the rings $S \otimes_R \kappa(\mathfrak{p})$. As $R \to S$ is finite, these fibre rings are finite over $\kappa(\mathfrak{p})$ hence Noetherian by Lemma 10.31.1. By Lemma 10.36.20 every prime of $S \otimes_R \kappa(\mathfrak{p})$ is a minimal prime. Hence by Lemma 10.31.6 there are at most finitely many. $\square$

00GU Lemma 10.36.22. Let $R \to S$ be a ring map such that S is integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}$ be a prime of S mapping to $\mathfrak{p}$. Then there exists a prime $\mathfrak{q}'$ with $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}'$.

Proof. We may replace R by $R/\mathfrak{p}$ and S by $S/\mathfrak{q}$. This reduces us to the situation of having an integral extension of domains $R \subset S$ and a prime $\mathfrak{p}' \subset R$. By Lemma 10.36.17 we win. $\square$

The property expressed in the lemma above is called the "going up property" for the ring map $R \to S$, see Definition 10.41.1.

0564 Lemma 10.36.23. Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. Then M is finitely presented as an R-module if and only if M is finitely presented as an S-module.

Proof. One of the implications follows from Lemma 10.6.4. To see the other assume that M is finitely presented as an S-module. Pick a presentation

$$S^{\oplus m} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

As S is finite as an R-module, the kernel of $S^{\oplus n} \to M$ is a finite R-module. Thus from Lemma 10.5.3 we see that it suffices to prove that S is finitely presented as an R-module.

Pick $y_1, \ldots, y_n \in S$ such that $y_1, \ldots, y_n$ generate S as an R-module. By Lemma 10.36.2 each y_i is integral over R. Choose monic polynomials $P_i(x) \in R[x]$ with $P_i(y_i) = 0$. Consider the ring

$$S' = R[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n))$$

Then we see that S is of finite presentation as an S'-algebra by Lemma 10.6.2. Since $S' \to S$ is surjective, the kernel $J = \text{Ker}(S' \to S)$ is finitely generated as an ideal by Lemma 10.6.3. Hence J is a finite S'-module (immediate from the definitions). Thus $S = \text{Coker}(J \to S')$ is of finite presentation as an S'-module by Lemma 10.5.3. Hence, arguing as in the first paragraph, it suffices to show that S' is of finite presentation as an R-module. Actually, S' is free as an R-module with basis the monomials $x_1^{e_1} \ldots x_n^{e_n}$ for $0 \leq e_i < \deg(P_i)$. Namely, write $R \to S'$ as the composition

$$R \to R[x_1]/(P_1(x_1)) \to R[x_1, x_2]/(P_1(x_1), P_2(x_2)) \to \ldots \to S'$$

This shows that the ith ring in this sequence is free as a module over the $(i-1)$st one with basis $1, x_i, \ldots, x_i^{\deg(P_i)-1}$. The result follows easily from this by induction. Some details omitted. □

052J Lemma 10.36.24. Let R be a ring. Let $x, y \in R$ be nonzerodivisors. Let $R[x/y] \subset R_{xy}$ be the R-subalgebra generated by x/y, and similarly for the subalgebras $R[y/x]$ and $R[x/y, y/x]$. If R is integrally closed in R_x or R_y, then the sequence

$$0 \to R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \to 0$$

is a short exact sequence of R-modules.

Proof. Since $x/y \cdot y/x = 1$ it is clear that the map $R[x/y] \oplus R[y/x] \to R[x/y, y/x]$ is surjective. Let $\alpha \in R[x/y] \cap R[y/x]$. To show exactness in the middle we have to prove that $\alpha \in R$. By assumption we may write

$$\alpha = a_0 + a_1 x/y + \ldots + a_n (x/y)^n = b_0 + b_1 y/x + \ldots + b_m (y/x)^m$$

for some $n, m \geq 0$ and $a_i, b_j \in R$. Pick some $N > \max(n, m)$. Consider the finite R-submodule M of R_{xy} generated by the elements

$$(x/y)^N, (x/y)^{N-1}, \ldots, x/y, 1, y/x, \ldots, (y/x)^{N-1}, (y/x)^N$$

We claim that $\alpha M \subset M$. Namely, it is clear that $(x/y)^i (b_0 + b_1 y/x + \ldots + b_m (y/x)^m) \in M$ for $0 \leq i \leq N$ and that $(y/x)^i (a_0 + a_1 x/y + \ldots + a_n (x/y)^n) \in M$ for $0 \leq i \leq N$. Hence α is integral over R by Lemma 10.36.2. Note that $\alpha \in R_x$, so if R is integrally closed in R_x then $\alpha \in R$ as desired. □

10.37. Normal rings

037B We first introduce the notion of a normal domain, and then we introduce the (very general) notion of a normal ring.

0309 **Definition 10.37.1.** A domain R is called *normal* if it is integrally closed in its field of fractions.

034L **Lemma 10.37.2.** *Let $R \to S$ be a ring map. If S is a normal domain, then the integral closure of R in S is a normal domain.*

Proof. Omitted. $\square$

The following notion is occasionally useful when studying normality.

00GW **Definition 10.37.3.** Let R be a domain.

(1) An element g of the fraction field of R is called *almost integral over R* if there exists an element $r \in R$, $r \neq 0$ such that $rg^n \in R$ for all $n \geq 0$.
(2) The domain R is called *completely normal* if every almost integral element of the fraction field of R is contained in R.

The following lemma shows that a Noetherian domain is normal if and only if it is completely normal.

00GX **Lemma 10.37.4.** *Let R be a domain with fraction field K. If $u, v \in K$ are almost integral over R, then so are $u+v$ and uv. Any element $g \in K$ which is integral over R is almost integral over R. If R is Noetherian then the converse holds as well.*

Proof. If $ru^n \in R$ for all $n \geq 0$ and $v^n r' \in R$ for all $n \geq 0$, then $(uv)^n rr'$ and $(u+v)^n rr'$ are in R for all $n \geq 0$. Hence the first assertion. Suppose $g \in K$ is integral over R. In this case there exists an $d > 0$ such that the ring $R[g]$ is generated by $1, g, \ldots, g^d$ as an R-module. Let $r \in R$ be a common denominator of the elements $1, g, \ldots, g^d \in K$. It is follows that $rR[g] \subset R$, and hence g is almost integral over R.

Suppose R is Noetherian and $g \in K$ is almost integral over R. Let $r \in R$, $r \neq 0$ be as in the definition. Then $R[g] \subset \frac{1}{r}R$ as an R-module. Since R is Noetherian this implies that $R[g]$ is finite over R. Hence g is integral over R, see Lemma 10.36.3. $\square$

00GY **Lemma 10.37.5.** *Any localization of a normal domain is normal.*

Proof. Let R be a normal domain, and let $S \subset R$ be a multiplicative subset. Suppose g is an element of the fraction field of R which is integral over $S^{-1}R$. Let $P = x^d + \sum_{j<d} a_j x^j$ be a polynomial with $a_i \in S^{-1}R$ such that $P(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for all i. Then sg satisfies the monic polynomial $x^d + \sum_{j<d} s^{d-j} a_j x^j$ which has coefficients $s^{d-j}a_j$ in R. Hence $sg \in R$ because R is normal. Hence $g \in S^{-1}R$. $\square$

00GZ **Lemma 10.37.6.** *A principal ideal domain is normal.*

Proof. Let R be a principal ideal domain. Let $g = a/b$ be an element of the fraction field of R integral over R. Because R is a principal ideal domain we may divide out a common factor of a and b and assume $(a, b) = R$. In this case, any equation $(a/b)^n + r_{n-1}(a/b)^{n-1} + \ldots + r_0 = 0$ with $r_i \in R$ would imply $a^n \in (b)$. This contradicts $(a, b) = R$ unless b is a unit in R. $\square$

00H0 Lemma 10.37.7. Let R be a domain with fraction field K. Suppose $f = \sum \alpha_i x^i$ is an element of $K[x]$.

(1) If f is integral over $R[x]$ then all α_i are integral over R, and
(2) If f is almost integral over $R[x]$ then all α_i are almost integral over R.

Proof. We first prove the second statement. Write $f = \alpha_0 + \alpha_1 x + \ldots + \alpha_r x^r$ with $\alpha_r \neq 0$. By assumption there exists $h = b_0 + b_1 x + \ldots + b_s x^s \in R[x]$, $b_s \neq 0$ such that $f^n h \in R[x]$ for all $n \geq 0$. This implies that $b_s \alpha_r^n \in R$ for all $n \geq 0$. Hence α_r is almost integral over R. Since the set of almost integral elements form a subring (Lemma 10.37.4) we deduce that $f - \alpha_r x^r = \alpha_0 + \alpha_1 x + \ldots + \alpha_{r-1} x^{r-1}$ is almost integral over $R[x]$. By induction on r we win.

In order to prove the first statement we will use absolute Noetherian reduction. Namely, write $\alpha_i = a_i/b_i$ and let $P(t) = t^d + \sum_{j<d} f_j t^j$ be a polynomial with coefficients $f_j \in R[x]$ such that $P(f) = 0$. Let $f_j = \sum f_{ji} x^i$. Consider the subring $R_0 \subset R$ generated by the finite list of elements a_i, b_i, f_{ji} of R. It is a domain; let K_0 be its field of fractions. Since R_0 is a finite type $\mathbf{Z}$-algebra it is Noetherian, see Lemma 10.31.3. It is still the case that $f \in K_0[x]$ is integral over $R_0[x]$, because all the identities in R among the elements a_i, b_i, f_{ji} also hold in R_0. By Lemma 10.37.4 the element f is almost integral over $R_0[x]$. By the second statement of the lemma, the elements α_i are almost integral over R_0. And since R_0 is Noetherian, they are integral over R_0, see Lemma 10.37.4. Of course, then they are integral over R. □

030A Lemma 10.37.8. Let R be a normal domain. Then $R[x]$ is a normal domain.

Proof. The result is true if R is a field K because $K[x]$ is a euclidean domain and hence a principal ideal domain and hence normal by Lemma 10.37.6. Let g be an element of the fraction field of $R[x]$ which is integral over $R[x]$. Because g is integral over $K[x]$ where K is the fraction field of R we may write $g = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \ldots + \alpha_0$ with $\alpha_i \in K$. By Lemma 10.37.7 the elements α_i are integral over R and hence are in R. □

0BI0 Lemma 10.37.9. Let R be a Noetherian normal domain. Then $R[[x]]$ is a Noetherian normal domain.

Proof. The power series ring is Noetherian by Lemma 10.31.2. Let $f, g \in R[[x]]$ be nonzero elements such that $w = f/g$ is integral over $R[[x]]$. Let K be the fraction field of R. Since the ring of Laurent series $K((x)) = K[[x]][1/x]$ is a field, we can write $w = a_n x^n + a_{n+1} x^{n+1} + \ldots)$ for some $n \in \mathbf{Z}$, $a_i \in K$, and $a_n \neq 0$. By Lemma 10.37.4 we see there exists a nonzero element $h = b_m x^m + b_{m+1} x^{m+1} + \ldots$ in $R[[x]]$ with $b_m \neq 0$ such that $w^e h \in R[[x]]$ for all $e \geq 1$. We conclude that $n \geq 0$ and that $b_m a_n^e \in R$ for all $e \geq 1$. Since R is Noetherian this implies that $a_n \in R$ by the same lemma. Now, if $a_n, a_{n+1}, \ldots, a_{N-1} \in R$, then we can apply the same argument to $w - a_n x^n - \ldots - a_{N-1} x^{N-1} = a_N x^N + \ldots$. In this way we see that all $a_i \in R$ and the lemma is proved. □

030B Lemma 10.37.10. Let R be a domain. The following are equivalent:

(1) The domain R is a normal domain,
(2) for every prime $\mathfrak{p} \subset R$ the local ring $R_{\mathfrak{p}}$ is a normal domain, and
(3) for every maximal ideal $\mathfrak{m}$ the ring $R_{\mathfrak{m}}$ is a normal domain.

Proof. This follows easily from the fact that for any domain R we have

$$R = \bigcap\nolimits_{\mathfrak m} R_{\mathfrak m}$$

inside the fraction field of R. Namely, if g is an element of the right hand side then the ideal $I = \{x \in R \mid xg \in R\}$ is not contained in any maximal ideal $\mathfrak m$, whence $I = R$. $\square$

Lemma 10.37.10 shows that the following definition is compatible with Definition 10.37.1. (It is the definition from EGA – see [DG67, IV, 5.13.5 and 0, 4.1.4].)

00GV Definition 10.37.11. A ring R is called normal if for every prime $\mathfrak p \subset R$ the localization $R_{\mathfrak p}$ is a normal domain (see Definition 10.37.1).

Note that a normal ring is a reduced ring, as R is a subring of the product of its localizations at all primes (see for example Lemma 10.23.1).

034M Lemma 10.37.12. A normal ring is integrally closed in its total ring of fractions.

Proof. Let R be a normal ring. Let $x \in Q(R)$ be an element of the total ring of fractions of R integral over R. Set $I = \{f \in R, fx \in R\}$. Let $\mathfrak p \subset R$ be a prime. As $R \to R_{\mathfrak p}$ is flat we see that $R_{\mathfrak p} \subset Q(R) \otimes_R R_{\mathfrak p}$. As $R_{\mathfrak p}$ is a normal domain we see that $x \otimes 1$ is an element of $R_{\mathfrak p}$. Hence we can find $a, f \in R$, $f \notin \mathfrak p$ such that $x \otimes 1 = a \otimes 1/f$. This means that $fx - a$ maps to zero in $Q(R) \otimes_R R_{\mathfrak p} = Q(R)_{\mathfrak p}$, which in turn means that there exists an $f' \in R$, $f' \notin \mathfrak p$ such that $f'fx = f'a$ in R. In other words, $ff' \in I$. Thus I is an ideal which isn't contained in any of the prime ideals of R, i.e., $I = R$ and $x \in R$. $\square$

037C Lemma 10.37.13. A localization of a normal ring is a normal ring.

Proof. Omitted. $\square$

00H1 Lemma 10.37.14. Let R be a normal ring. Then $R[x]$ is a normal ring.

Proof. Let $\mathfrak q$ be a prime of $R[x]$. Set $\mathfrak p = R \cap \mathfrak q$. Then we see that $R_{\mathfrak p}[x]$ is a normal domain by Lemma 10.37.8. Hence $(R[x])_{\mathfrak q}$ is a normal domain by Lemma 10.37.5. $\square$

0CYA Lemma 10.37.15. A finite product of normal rings is normal.

Proof. It suffices to show that the product of two normal rings, say R and S, is normal. By Lemma 10.21.3 the prime ideals of $R \times S$ are of the form $\mathfrak p \times S$ and $R \times \mathfrak q$, where $\mathfrak p$ and $\mathfrak q$ are primes of R and S respectively. Localization yields $(R \times S)_{\mathfrak p \times S} = R_{\mathfrak p}$ which is a normal domain by assumption. Similarly for S. $\square$

030C Lemma 10.37.16. Let R be a ring. Assume R is reduced and has finitely many minimal primes. Then the following are equivalent:

1. R is a normal ring,
2. R is integrally closed in its total ring of fractions, and
3. R is a finite product of normal domains.

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) hold in general, see Lemmas 10.37.12 and 10.37.15.

Let $\mathfrak p_1, \ldots, \mathfrak p_n$ be the minimal primes of R. By Lemmas 10.25.2 and 10.25.4 we have $Q(R) = R_{\mathfrak p_1} \times \ldots \times R_{\mathfrak p_n}$, and by Lemma 10.25.1 each factor is a field. Denote $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the ith idempotent of $Q(R)$.

If R is integrally closed in $Q(R)$, then it contains in particular the idempotents e_i, and we see that R is a product of n domains (see Sections 10.22 and 10.24). Each factor is of the form $R/\mathfrak{p}_i$ with field of fractions $R_{\mathfrak{p}_i}$. By Lemma 10.36.10 each map $R/\mathfrak{p}_i \to R_{\mathfrak{p}_i}$ is integrally closed. Hence R is a finite product of normal domains. $\square$

037D Lemma 10.37.17. Let $(R_i, \varphi_{ii'})$ be a directed system (Categories, Definition 10.8.1) of rings. If each R_i is a normal ring so is $R = \mathrm{colim}_i R_i$.

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. Set $\mathfrak{p}_i = R_i \cap \mathfrak{p}$ (usual abuse of notation). Then we see that $R_\mathfrak{p} = \mathrm{colim}_i (R_i)_{\mathfrak{p}_i}$. Since each $(R_i)_{\mathfrak{p}_i}$ is a normal domain we reduce to proving the statement of the lemma for normal domains. If $a, b \in R$ and a/b satisfies a monic polynomial $P(T) \in R[T]$, then we can find a (sufficiently large) $i \in I$ such that a, b come from objects a_i, b_i over R_i, P comes from a monic polynomial $P_i \in R_i[T]$ and $P_i(a_i/b_i) = 0$. Since R_i is normal we see $a_i/b_i \in R_i$ and hence also $a/b \in R$. $\square$

10.38. Going down for integral over normal

037E We first play around a little bit with the notion of elements integral over an ideal, and then we prove the theorem referred to in the section title.

00H2 Definition 10.38.1. Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. We say an element $g \in S$ is integral over I if there exists a monic polynomial $P = x^d + \sum_{j<d} a_j x^j$ with coefficients $a_j \in I^{d-j}$ such that $P^\varphi(g) = 0$ in S.

This is mostly used when $\varphi = \mathrm{id}_R : R \to R$. In this case the set I' of elements integral over I is called the integral closure of I. We will see that I' is an ideal of R (and of course $I \subset I'$).

00H3 Lemma 10.38.2. Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let $A = \sum I^n t^n \subset R[t]$ be the subring of the polynomial ring generated by $R \oplus It \subset R[t]$. An element $s \in S$ is integral over I if and only if the element $st \in S[t]$ is integral over A.

Proof. Suppose st is integral over A. Let $P = x^d + \sum_{j<d} a_j x^j$ be a monic polynomial with coefficients in A such that $P^\varphi(st) = 0$. Let $a'_j \in A$ be the degree $d-j$ part of a_i, in other words $a'_j = a''_j t^{d-j}$ with $a''_j \in I^{d-j}$. For degree reasons we still have $(st)^d + \sum_{j<d} \varphi(a''_j) t^{d-j} (st)^j = 0$. Hence we see that s is integral over I.

Suppose that s is integral over I. Say $P = x^d + \sum_{j<d} a_j x^j$ with $a_j \in I^{d-j}$. Then we immediately find a polynomial $Q = x^d + \sum_{j<d} (a_j t^{d-j}) x^j$ with coefficients in A which proves that st is integral over A. $\square$

00H4 Lemma 10.38.3. Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The set of elements of S which are integral over I form a R-submodule of S. Furthermore, if $s \in S$ is integral over R, and s' is integral over I, then ss' is integral over I.

Proof. Closure under addition is clear from the characterization of Lemma 10.38.2. Any element $s \in S$ which is integral over R corresponds to the degree 0 element s of $S[x]$ which is integral over A (because $R \subset A$). Hence we see that multiplication by s on $S[x]$ preserves the property of being integral over A, by Lemma 10.36.7. $\square$

00H5 Lemma 10.38.4. Suppose $\varphi : R \to S$ is integral. Suppose $I \subset R$ is an ideal. Then every element of IS is integral over I.

Proof. Immediate from Lemma 10.38.3. ☐

00H6 Lemma 10.38.5. Let K be a field. Let $n, m \in \mathbf{N}$ and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in K$. If the polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ divides the polynomial $x^m + b_{m-1}x^{m-1} + \ldots + b_0$ in $K[x]$ then

(1) $a_0, \ldots, a_{n-1}$ are integral over any subring R_0 of K containing the elements $b_0, \ldots, b_{m-1}$, and
(2) each a_i lies in $\sqrt{(b_0, \ldots, b_{m-1})R}$ for any subring $R \subset K$ containing the elements $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}$.

Proof. Let L/K be a field extension such that we can write $x^m + b_{m-1}x^{m-1} + \ldots + b_0 = \prod_{i=1}^m (x - \beta_i)$ with $\beta_i \in L$. See Fields, Section 9.16. Each β_i is integral over R_0. Since each a_i is a homogeneous polynomial in $\beta_1, \ldots, \beta_m$ we deduce the same for the a_i (use Lemma 10.36.7).

Choose $c_0, \ldots, c_{m-n-1} \in K$ such that

$$x^m + b_{m-1}x^{m-1} + \ldots + b_0 =$$
$$(x^n + a_{n-1}x^{n-1} + \ldots + a_0)(x^{m-n} + c_{m-n-1}x^{m-n-1} + \ldots + c_0).$$

By part (1) the elements c_i are integral over R. Consider the integral extension

$$R \subset R' = R[c_0, \ldots, c_{m-n-1}] \subset K$$

By Lemmas 10.36.17 and 10.30.3 we see that $R \cap \sqrt{(b_0, \ldots, b_{m-1})R'} = \sqrt{(b_0, \ldots, b_{m-1})R}$. Thus we may replace R by R' and assume $c_i \in R$. Dividing out the radical $\sqrt{(b_0, \ldots, b_{m-1})}$ we get a reduced ring $\overline{R}$. We have to show that the images $\overline{a}_i \in \overline{R}$ are zero. And in $\overline{R}[x]$ we have the relation

$$x^m = x^m + \overline{b}_{m-1}x^{m-1} + \ldots + \overline{b}_0 =$$
$$(x^n + \overline{a}_{n-1}x^{n-1} + \ldots + \overline{a}_0)(x^{m-n} + \overline{c}_{m-n-1}x^{m-n-1} + \ldots + \overline{c}_0).$$

It is easy to see that this implies $\overline{a}_i = 0$ for all i. Indeed by Lemma 10.25.1 the localization of $\overline{R}$ at a minimal prime $\mathfrak{p}$ is a field and $\overline{R}_\mathfrak{p}[x]$ a UFD. Thus $f = x^n + \sum \overline{a}_i x^i$ is associated to x^n and since f is monic $f = x^n$ in $\overline{R}_\mathfrak{p}[x]$. Then there exists an $s \in \overline{R}$, $s \not\in \mathfrak{p}$ such that $s(f - x^n) = 0$. Therefore all $\overline{a}_i$ lie in $\mathfrak{p}$ and we conclude by Lemma 10.25.2. ☐

00H7 Lemma 10.38.6. Let $R \subset S$ be an inclusion of domains. Assume R is normal. Let $g \in S$ be integral over R. Then the minimal polynomial of g has coefficients in R.

Proof. Let $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ be a polynomial with coefficients in R such that $P(g) = 0$. Let $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be the minimal polynomial for g over the fraction field K of R. Then Q divides P in $K[x]$. By Lemma 10.38.5 we see the a_i are integral over R. Since R is normal this means they are in R. ☐

00H8 Proposition 10.38.7. Let $R \subset S$ be an inclusion of domains. Assume R is normal and S integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}'$ be a prime of S with $\mathfrak{p}' = R \cap \mathfrak{q}'$. Then there exists a prime $\mathfrak{q}$ with $\mathfrak{q} \subset \mathfrak{q}'$ such that $\mathfrak{p} = R \cap \mathfrak{q}$. In other words: the going down property holds for $R \to S$, see Definition 10.41.1.

Proof. Let $\mathfrak{p}$, $\mathfrak{p}'$ and $\mathfrak{q}'$ be as in the statement. We have to show there is a prime $\mathfrak{q}$, with $\mathfrak{q} \subset \mathfrak{q}'$ and $R \cap \mathfrak{q} = \mathfrak{p}$. This is the same as finding a prime of $S_{\mathfrak{q}'}$ mapping to $\mathfrak{p}$. According to Lemma 10.17.9 we have to show that $\mathfrak{p}S_{\mathfrak{q}'} \cap R = \mathfrak{p}$. Pick $z \in \mathfrak{p}S_{\mathfrak{q}'} \cap R$. We may write $z = y/g$ with $y \in \mathfrak{p}S$ and $g \in S$, $g \not\in \mathfrak{q}'$. Written differently we have $zg = y$.

By Lemma 10.38.4 there exists a monic polynomial $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ with $b_i \in \mathfrak{p}$ such that $P(y) = 0$.

By Lemma 10.38.6 the minimal polynomial of g over K has coefficients in R. Write it as $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$. Note that not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak{p}$ since that would imply $g^n = \sum_{j<n} a_j g^j \in \mathfrak{p}S \subset \mathfrak{p}'S \subset \mathfrak{q}'$ which is a contradiction.

Since $y = zg$ we see immediately from the above that $Q' = x^n + za_{n-1}x^{n-1} + \ldots + z^n a_0$ is the minimal polynomial for y. Hence Q' divides P and by Lemma 10.38.5 we see that $z^j a_{n-j} \in \sqrt{(b_0, \ldots, b_{m-1})} \subset \mathfrak{p}$, $j = 1, \ldots, n$. Because not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak{p}$ we conclude $z \in \mathfrak{p}$ as desired. $\square$

10.39. Flat modules and flat ring maps

00H9 One often used result is that if $M = \mathrm{colim}_{i \in \mathcal{I}} M_i$ is a colimit of R-modules and if N is an R-module then

$$M \otimes N = \mathrm{colim}_{i \in \mathcal{I}} M_i \otimes_R N,$$

see Lemma 10.12.9. This property is usually expressed by saying that $\otimes$ commutes with colimits. Another often used result is that if $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence and if M is any R-module, then

$$M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

is still exact, see Lemma 10.12.10. Both of these properties tell us that the functor $N \mapsto M \otimes_R N$ is right exact. See Categories, Section 4.23 and Homology, Section 12.7. An R-module M is flat if $N \mapsto N \otimes_R M$ is also left exact, i.e., if it is exact. Here is the precise definition.

00HB **Definition 10.39.1.** Let R be a ring.

(1) An R-module M is called *flat* if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.
(2) An R-module M is called *faithfully flat* if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.
(3) A ring map $R \to S$ is called *flat* if S is flat as an R-module.
(4) A ring map $R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module.

Here is an example of how you can use the flatness condition.

0BBY **Lemma 10.39.2.** Let R be a ring. Let $I, J \subset R$ be ideals. Let M be a flat R-module. Then $IM \cap JM = (I \cap J)M$.

Proof. Consider the exact sequence $0 \to I \cap J \to R \to R/I \oplus R/J$. Tensoring with the flat module M we obtain an exact sequence

$$0 \to (I \cap J) \otimes_R M \to M \to M/IM \oplus M/JM$$

Since the kernel of $M \to M/IM \oplus M/JM$ is equal to $IM \cap JM$ we conclude. $\square$

05UT **Lemma 10.39.3.** Let R be a ring. Let $\{M_i, \varphi_{ii'}\}$ be a directed system of flat R-modules. Then $\mathrm{colim}_i M_i$ is a flat R-module.

Proof. This follows as $\otimes$ commutes with colimits and because directed colimits are exact, see Lemma 10.8.8. $\square$

00HC Lemma 10.39.4. A composition of (faithfully) flat ring maps is (faithfully) flat. If $R \to R'$ is (faithfully) flat, and M' is a (faithfully) flat R'-module, then M' is a (faithfully) flat R-module.

Proof. The first statement of the lemma is a particular case of the second, so it is clearly enough to prove the latter. Let $R \to R'$ be a flat ring map, and M' a flat R'-module. We need to prove that M' is a flat R-module. Let $N_1 \to N_2 \to N_3$ be an exact complex of R-modules. Then, the complex $R' \otimes_R N_1 \to R' \otimes_R N_2 \to R' \otimes_R N_3$ is exact (since R' is flat as an R-module), and so the complex $M' \otimes_{R'} (R' \otimes_R N_1) \to M' \otimes_{R'} (R' \otimes_R N_2) \to M' \otimes_{R'} (R' \otimes_R N_3)$ is exact (since M' is a flat R'-module). Since $M' \otimes_{R'} (R' \otimes_R N) \cong (M' \otimes_{R'} R') \otimes_R N \cong M' \otimes_R N$ for any R-module N functorially (by Lemmas 10.12.7 and 10.12.3), this complex is isomorphic to the complex $M' \otimes_R N_1 \to M' \otimes_R N_2 \to M' \otimes_R N_3$, which is therefore also exact. This shows that M' is a flat R-module. Tracing this argument backwards, we can show that if $R \to R'$ is faithfully flat, and if M' is faithfully flat as an R'-module, then M' is faithfully flat as an R-module. $\square$

00HD Lemma 10.39.5. Let M be an R-module. The following are equivalent:

00HE (1) M is flat over R.
00HF (2) for every injection of R-modules $N \subset N'$ the map $N \otimes_R M \to N' \otimes_R M$ is injective.
00HG (3) for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.
00HH (4) for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.

Proof. The implications (1) implies (2) implies (3) implies (4) are all trivial. Thus we prove (4) implies (1). Suppose that $N_1 \to N_2 \to N_3$ is exact. Let $K = \mathrm{Ker}(N_2 \to N_3)$ and $Q = \mathrm{Im}(N_2 \to N_3)$. Then we get maps

$$N_1 \otimes_R M \to K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to N_3 \otimes_R M$$

Observe that the first and third arrows are surjective. Thus if we show that the second and fourth arrows are injective, then we are done[3]. Hence it suffices to show that $- \otimes_R M$ transforms injective R-module maps into injective R-module maps.

Assume $K \to N$ is an injective R-module map and let $x \in \mathrm{Ker}(K \otimes_R M \to N \otimes_R M)$. We have to show that x is zero. The R-module K is the union of its finite R-submodules; hence, $K \otimes_R M$ is the colimit of R-modules of the form $K_i \otimes_R M$ where K_i runs over all finite R-submodules of K (because tensor product commutes with colimits). Thus, for some i our x comes from an element $x_i \in K_i \otimes_R M$. Thus we may assume that K is a finite R-module. Assume this. We regard the injection $K \to N$ as an inclusion, so that $K \subset N$.

[3]Here is the argument in more detail: Assume that we know that the second and fourth arrows are injective. Lemma 10.12.10 (applied to the exact sequence $K \to N_2 \to Q \to 0$) yields that the sequence $K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to 0$ is exact. Hence, $\mathrm{Ker}\,(N_2 \otimes_R M \to Q \otimes_R M) = \mathrm{Im}\,(K \otimes_R M \to N_2 \otimes_R M)$. Since $\mathrm{Im}\,(K \otimes_R M \to N_2 \otimes_R M) = \mathrm{Im}\,(N_1 \otimes_R M \to N_2 \otimes_R M)$ (due to the surjectivity of $N_1 \otimes_R M \to K \otimes_R M$) and $\mathrm{Ker}\,(N_2 \otimes_R M \to Q \otimes_R M) = \mathrm{Ker}\,(N_2 \otimes_R M \to N_3 \otimes_R M)$ (due to the injectivity of $Q \otimes_R M \to N_3 \otimes_R M$), this becomes $\mathrm{Ker}\,(N_2 \otimes_R M \to N_3 \otimes_R M) = \mathrm{Im}\,(N_1 \otimes_R M \to N_2 \otimes_R M)$, which shows that the functor $- \otimes_R M$ is exact, whence M is flat.

The R-module N is the union of its finite R-submodules that contain K. Hence, $N \otimes_R M$ is the colimit of R-modules of the form $N_i \otimes_R M$ where N_i runs over all finite R-submodules of N that contain K (again since tensor product commutes with colimits). Notice that this is a colimit over a directed system (since the sum of two finite submodules of N is again finite). Hence, (by Lemma 10.8.4) the element $x \in K \otimes_R M$ maps to zero in at least one of these R-modules $N_i \otimes_R M$ (since x maps to zero in $N \otimes_R M$). Thus we may assume N is a finite R-module.

Assume N is a finite R-module. Write $N = R^{\oplus n}/L$ and $K = L'/L$ for some $L \subset L' \subset R^{\oplus n}$. For any R-submodule $G \subset R^{\oplus n}$, we have a canonical map $G \otimes_R M \to M^{\oplus n}$ obtained by composing $G \otimes_R M \to R^n \otimes_R M = M^{\oplus n}$. It suffices to prove that $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective. Namely, if so, then we see that $K \otimes_R M = L' \otimes_R M/L \otimes_R M \to M^{\oplus n}/L \otimes_R M$ is injective too[4].

Thus it suffices to show that $L \otimes_R M \to M^{\oplus n}$ is injective when $L \subset R^{\oplus n}$ is an R-submodule. We do this by induction on n. The base case $n = 1$ we handle below. For the induction step assume $n > 1$ and set $L' = L \cap R \oplus 0^{\oplus n-1}$. Then $L'' = L/L'$ is a submodule of $R^{\oplus n-1}$. We obtain a diagram

$$\begin{tikzcd} & L' \otimes_R M \arrow[r] \arrow[d] & L \otimes_R M \arrow[r] \arrow[d] & L'' \otimes_R M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & M^{\oplus n} \arrow[r] & M^{\oplus n-1} \arrow[r] & 0 \end{tikzcd}$$

By induction hypothesis and the base case the left and right vertical arrows are injective. The rows are exact. It follows that the middle vertical arrow is injective too.

The base case of the induction above is when $L \subset R$ is an ideal. In other words, we have to show that $I \otimes_R M \to M$ is injective for any ideal I of R. We know this is true when I is finitely generated. However, $I = \bigcup I_\alpha$ is the union of the finitely generated ideals I_α contained in it. In other words, $I = \operatorname{colim} I_\alpha$. Since $\otimes$ commutes with colimits we see that $I \otimes_R M = \operatorname{colim} I_\alpha \otimes_R M$ and since all the morphisms $I_\alpha \otimes_R M \to M$ are injective by assumption, the same is true for $I \otimes_R M \to M$. □

05UU Lemma 10.39.6. Let $\{R_i, \varphi_{ii'}\}$ be a system of rings over the directed set I. Let $R = \operatorname{colim}_i R_i$.

(1) If M is an R-module such that M is flat as an R_i-module for all i, then M is flat as an R-module.
(2) For $i \in I$ let M_i be a flat R_i-module and for $i' \geq i$ let $f_{ii'} : M_i \to M_{i'}$ be a $\varphi_{ii'}$-linear map such that $f_{i'i''} \circ f_{ii'} = f_{ii''}$. Then $M = \operatorname{colim}_{i \in I} M_i$ is a flat R-module.

Proof. Part (1) is a special case of part (2) with $M_i = M$ for all i and $f_{ii'} = \mathrm{id}_M$. Proof of (2). Let $\mathfrak a \subset R$ be a finitely generated ideal. By Lemma 10.39.5 it suffices to show that $\mathfrak a \otimes_R M \to M$ is injective. We can find an $i \in I$ and a finitely generated

[4]This becomes obvious if we identify $L' \otimes_R M$ and $L \otimes_R M$ with submodules of $M^{\oplus n}$ (which is legitimate since the maps $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective and commute with the obvious map $L' \otimes_R M \to L \otimes_R M$).

ideal $\mathfrak{a}' \subset R_i$ such that $\mathfrak{a} = \mathfrak{a}'R$. Then $\mathfrak{a} = \mathrm{colim}_{i' \geq i} \mathfrak{a}'R_{i'}$. Since $\otimes$ commutes with colimits the map $\mathfrak{a} \otimes_R M \to M$ is the colimit of the maps

$$\mathfrak{a}'R_{i'} \otimes_{R_{i'}} M_{i'} \longrightarrow M_{i'}$$

These maps are all injective by assumption. Since colimits over I are exact by Lemma 10.8.8 we win. $\square$

00HI Lemma 10.39.7. Suppose that M is (faithfully) flat over R, and that $R \to R'$ is a ring map. Then $M \otimes_R R'$ is (faithfully) flat over R'.

Proof. For any R'-module N we have a canonical isomorphism $N \otimes_{R'} (R' \otimes_R M) = N \otimes_R M$. Hence the desired exactness properties of the functor $- \otimes_{R'} (R' \otimes_R M)$ follow from the corresponding exactness properties of the functor $- \otimes_R M$. $\square$

00HJ Lemma 10.39.8. Let $R \to R'$ be a faithfully flat ring map. Let M be a module over R, and set $M' = R' \otimes_R M$. Then M is flat over R if and only if M' is flat over R'.

Proof. By Lemma 10.39.7 we see that if M is flat then M' is flat. For the converse, suppose that M' is flat. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. We want to show that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We know that $N_1 \otimes_R R' \to N_2 \otimes_R R' \to N_3 \otimes_R R'$ is exact, because $R \to R'$ is flat. Flatness of M' implies that $N_1 \otimes_R R' \otimes_{R'} M' \to N_2 \otimes_R R' \otimes_{R'} M' \to N_3 \otimes_R R' \otimes_{R'} M'$ is exact. We may write this as $N_1 \otimes_R M \otimes_R R' \to N_2 \otimes_R M \otimes_R R' \to N_3 \otimes_R M \otimes_R R'$. Finally, faithful flatness implies that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. $\square$

0584 Lemma 10.39.9. Let R be a ring. Let $S \to S'$ be a flat map of R-algebras. Let M be a module over S, and set $M' = S' \otimes_S M$.

(1) If M is flat over R, then M' is flat over R.
(2) If $S \to S'$ is faithfully flat, then M is flat over R if and only if M' is flat over R.

Proof. Let $N \to N'$ be an injection of R-modules. By the flatness of $S \to S'$ we have

$$\mathrm{Ker}(N \otimes_R M \to N' \otimes_R M) \otimes_S S' = \mathrm{Ker}(N \otimes_R M' \to N' \otimes_R M')$$

If M is flat over R, then the left hand side is zero and we find that M' is flat over R by the second characterization of flatness in Lemma 10.39.5. If M' is flat over R then we have the vanishing of the right hand side and if in addition $S \to S'$ is faithfully flat, this implies that $\mathrm{Ker}(N \otimes_R M \to N' \otimes_R M)$ is zero which in turn shows that M is flat over R. $\square$

039V Lemma 10.39.10. Let $R \to S$ be a ring map. Let M be an S-module. If M is flat as an R-module and faithfully flat as an S-module, then $R \to S$ is flat.

Proof. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. By assumption $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We may write this as

$$N_1 \otimes_R S \otimes_S M \to N_2 \otimes_R S \otimes_S M \to N_3 \otimes_R S \otimes_S M.$$

By faithful flatness of M over S we conclude that $N_1 \otimes_R S \to N_2 \otimes_R S \to N_3 \otimes_R S$ is exact. Hence $R \to S$ is flat. $\square$

Let R be a ring. Let M be an R-module. Let $\sum f_i x_i = 0$ be a relation in M. We say the relation $\sum f_i x_i$ is trivial if there exist an integer $m \geq 0$, elements $y_j \in M$, $j = 1, \ldots, m$, and elements $a_{ij} \in R$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ such that

$$x_i = \sum\nolimits_j a_{ij} y_j, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i f_i a_{ij}, \forall j.$$

00HK Lemma 10.39.11 (Equational criterion of flatness). A module M over R is flat if and only if every relation in M is trivial.

Proof. Assume M is flat and let $\sum f_i x_i = 0$ be a relation in M. Let $I = (f_1, \ldots, f_n)$, and let $K = \mathrm{Ker}(R^n \to I, (a_1, \ldots, a_n) \mapsto \sum_i a_i f_i)$. So we have the short exact sequence $0 \to K \to R^n \to I \to 0$. Then $\sum f_i \otimes x_i$ is an element of $I \otimes_R M$ which maps to zero in $R \otimes_R M = M$. By flatness $\sum f_i \otimes x_i$ is zero in $I \otimes_R M$. Thus there exists an element of $K \otimes_R M$ mapping to $\sum e_i \otimes x_i \in R^n \otimes_R M$ where e_i is the ith basis element of R^n. Write this element as $\sum k_j \otimes y_j$ and then write the image of k_j in R^n as $\sum a_{ij} e_i$ to get the result.

Assume every relation is trivial, let I be a finitely generated ideal, and let $x = \sum f_i \otimes x_i$ be an element of $I \otimes_R M$ mapping to zero in $R \otimes_R M = M$. This just means exactly that $\sum f_i x_i$ is a relation in M. And the fact that it is trivial implies easily that x is zero, because

$$x = \sum f_i \otimes x_i = \sum f_i \otimes \left(\sum a_{ij} y_j \right) = \sum \left(\sum f_i a_{ij} \right) \otimes y_j = 0$$

$\square$

00HL Lemma 10.39.12. Suppose that R is a ring, $0 \to M'' \to M' \to M \to 0$ a short exact sequence, and N an R-module. If M is flat then $N \otimes_R M'' \to N \otimes_R M'$ is injective, i.e., the sequence

$$0 \to N \otimes_R M'' \to N \otimes_R M' \to N \otimes_R M \to 0$$

is a short exact sequence.

Proof. Let $R^{(I)} \to N$ be a surjection from a free module onto N with kernel K. The result follows from the snake lemma applied to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R N & \to & M' \otimes_R N & \to & M \otimes_R N & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & (M'')^{(I)} & \to & (M')^{(I)} & \to & M^{(I)} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R K & \to & M' \otimes_R K & \to & M \otimes_R K & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the free module $R^{(I)}$ is exact. $\square$

00HM Lemma 10.39.13. Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. If M' and M'' are flat so is M. If M and M'' are flat so is M'.

Proof. We will use the criterion that a module N is flat if for every ideal $I \subset R$ the map $N \otimes_R I \to N$ is injective, see Lemma 10.39.5. Consider an ideal $I \subset R$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & M' \otimes_R I & \to & M \otimes_R I & \to & M'' \otimes_R I & \to & 0 \end{array}$$

with exact rows. This immediately proves the first assertion. The second follows because if M'' is flat then the lower left horizontal arrow is injective by Lemma 10.39.12. $\square$

00HO Lemma 10.39.14. Let R be a ring. Let M be an R-module. The following are equivalent

(1) M is faithfully flat, and
(2) M is flat and for all R-module homomorphisms $\alpha : N \to N'$ we have $\alpha = 0$ if and only if $\alpha \otimes \mathrm{id}_M = 0$.

Proof. If M is faithfully flat, then $0 \to \mathrm{Ker}(\alpha) \to N \to N'$ is exact if and only if the same holds after tensoring with M. This proves (1) implies (2). For the other, assume (2). Let $N_1 \to N_2 \to N_3$ be a complex, and assume the complex $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Take $x \in \mathrm{Ker}(N_2 \to N_3)$, and consider the map $\alpha : R \to N_2/\mathrm{Im}(N_1)$, $r \mapsto rx + \mathrm{Im}(N_1)$. By the exactness of the complex $- \otimes_R M$ we see that $\alpha \otimes \mathrm{id}_M$ is zero. By assumption we get that α is zero. Hence x is in the image of $N_1 \to N_2$. $\square$

00HP Lemma 10.39.15. Let M be a flat R-module. The following are equivalent:

(1) M is faithfully flat,
(2) for every nonzero R-module N, then tensor product $M \otimes_R N$ is nonzero,
(3) for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is nonzero, and
(4) for all maximal ideals $\mathfrak{m}$ of R the tensor product $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.

Proof. Assume M faithfully flat and $N \neq 0$. By Lemma 10.39.14 the nonzero map $1 : N \to N$ induces a nonzero map $M \otimes_R N \to M \otimes_R N$, so $M \otimes_R N \neq 0$. Thus (1) implies (2). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate.

Assume (4). Suppose that $N_1 \to N_2 \to N_3$ is a complex and suppose that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Let H be the cohomology of the complex, so $H = \mathrm{Ker}(N_2 \to N_3)/\mathrm{Im}(N_1 \to N_2)$. To finish the proof we will show $H = 0$. By flatness we see that $H \otimes_R M = 0$. Take $x \in H$ and let $I = \{f \in R \mid fx = 0\}$ be its annihilator. Since $R/I \subset H$ we get $M/IM \subset H \otimes_R M = 0$ by flatness of M. If $I \neq R$ we may choose a maximal ideal $I \subset \mathfrak{m} \subset R$. This immediately gives a contradiction. $\square$

00HQ Lemma 10.39.16. Let $R \to S$ be a flat ring map. The following are equivalent:

(1) $R \to S$ is faithfully flat,
(2) the induced map on Spec is surjective, and
(3) any closed point $x \in \mathrm{Spec}(R)$ is in the image of the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

Proof. This follows quickly from Lemma 10.39.15, because we saw in Remark 10.17.8 that $\mathfrak{p}$ is in the image if and only if the ring $S \otimes_R \kappa(\mathfrak{p})$ is nonzero. $\square$

00HR Lemma 10.39.17. A flat local ring homomorphism of local rings is faithfully flat.

Proof. Immediate from Lemma 10.39.16. □

Flatness meshes well with localization.

00HT Lemma 10.39.18. Let R be a ring. Let $S \subset R$ be a multiplicative subset.

(1) The localization $S^{-1}R$ is a flat R-algebra.
(2) If M is an $S^{-1}R$-module, then M is a flat R-module if and only if M is a flat $S^{-1}R$-module.
(3) Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak p}$ is a flat $R_{\mathfrak p}$-module for all primes $\mathfrak p$ of R.
(4) Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak m}$ is a flat $R_{\mathfrak m}$-module for all maximal ideals $\mathfrak m$ of R.
(5) Suppose $R \to A$ is a ring map, M is an A-module, and $g_1, \ldots, g_m \in A$ are elements generating the unit ideal of A. Then M is flat over R if and only if each localization M_{g_i} is flat over R.
(6) Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak q}$ is a flat $R_{\mathfrak p}$-module (with $\mathfrak p$ the prime of R lying under $\mathfrak q$) for all primes $\mathfrak q$ of A.
(7) Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak m}$ is a flat $R_{\mathfrak p}$-module (with $\mathfrak p = R \cap \mathfrak m$) for all maximal ideals $\mathfrak m$ of A.

Proof. Let us prove the last statement of the lemma. In the proof we will use repeatedly that localization is exact and commutes with tensor product, see Sections 10.9 and 10.12.

Suppose $R \to A$ is a ring map, and M is an A-module. Assume that $M_{\mathfrak m}$ is a flat $R_{\mathfrak p}$-module for all maximal ideals $\mathfrak m$ of A (with $\mathfrak p = R \cap \mathfrak m$). Let $I \subset R$ be an ideal. We have to show the map $I \otimes_R M \to M$ is injective. We can think of this as a map of A-modules. By assumption the localization $(I \otimes_R M)_{\mathfrak m} \to M_{\mathfrak m}$ is injective because $(I \otimes_R M)_{\mathfrak m} = I_{\mathfrak p} \otimes_{R_{\mathfrak p}} M_{\mathfrak m}$. Hence the kernel of $I \otimes_R M \to M$ is zero by Lemma 10.23.1. Hence M is flat over R.

Conversely, assume M is flat over R. Pick a prime $\mathfrak q$ of A lying over the prime $\mathfrak p$ of R. Suppose that $I \subset R_{\mathfrak p}$ is an ideal. We have to show that $I \otimes_{R_{\mathfrak p}} M_{\mathfrak q} \to M_{\mathfrak q}$ is injective. We can write $I = J_{\mathfrak p}$ for some ideal $J \subset R$. Then the map $I \otimes_{R_{\mathfrak p}} M_{\mathfrak q} \to M_{\mathfrak q}$ is just the localization (at $\mathfrak q$) of the map $J \otimes_R M \to M$ which is injective. Since localization is exact we see that $M_{\mathfrak q}$ is a flat $R_{\mathfrak p}$-module.

This proves (7) and (6). The other statements follow in a straightforward way from the last statement (proofs omitted). □

00HS Lemma 10.39.19. Let $R \to S$ be flat. Let $\mathfrak p \subset \mathfrak p'$ be primes of R. Let $\mathfrak q' \subset S$ be a prime of S mapping to $\mathfrak p'$. Then there exists a prime $\mathfrak q \subset \mathfrak q'$ mapping to $\mathfrak p$.

Proof. By Lemma 10.39.18 the local ring map $R_{\mathfrak p'} \to S_{\mathfrak q'}$ is flat. By Lemma 10.39.17 this local ring map is faithfully flat. By Lemma 10.39.16 there is a prime mapping to $\mathfrak p R_{\mathfrak p'}$. The inverse image of this prime in S does the job. □

The property of $R \to S$ described in the lemma is called the "going down property". See Definition 10.41.1.

090N Lemma 10.39.20. Let R be a ring. Let $\{S_i, \varphi_{ii'}\}$ be a directed system of faithfully flat R-algebras. Then $S = \mathrm{colim}_i S_i$ is a faithfully flat R-algebra.

Proof. By Lemma 10.39.3 we see that S is flat. Let $\mathfrak{m} \subset R$ be a maximal ideal. By Lemma 10.39.16 none of the rings $S_i/\mathfrak{m}S_i$ is zero. Hence $S/\mathfrak{m}S = \mathrm{colim}\, S_i/\mathfrak{m}S_i$ is nonzero as well because 1 is not equal to zero. Thus the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains $\mathfrak{m}$ and we see that $R \to S$ is faithfully flat by Lemma 10.39.16. □

10.40. Supports and annihilators

080S Some very basic definitions and lemmas.

00L1 Definition 10.40.1. Let R be a ring and let M be an R-module. The support of M is the set

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$$

0585 Lemma 10.40.2. Let R be a ring. Let M be an R-module. Then

$$M = (0) \Leftrightarrow \mathrm{Supp}(M) = \emptyset.$$

Proof. Actually, Lemma 10.23.1 even shows that $\mathrm{Supp}(M)$ always contains a maximal ideal if M is not zero. □

07T7 Definition 10.40.3. Let R be a ring. Let M be an R-module.

(1) Given an element $m \in M$ the annihilator of m is the ideal

$$\mathrm{Ann}_R(m) = \mathrm{Ann}(m) = \{f \in R \mid fm = 0\}.$$

(2) The annihilator of M is the ideal

$$\mathrm{Ann}_R(M) = \mathrm{Ann}(M) = \{f \in R \mid fm = 0\ \forall m \in M\}.$$

07T8 Lemma 10.40.4. Let $R \to S$ be a flat ring map. Let M be an R-module and $m \in M$. Then $\mathrm{Ann}_R(m)S = \mathrm{Ann}_S(m \otimes 1)$. If M is a finite R-module, then $\mathrm{Ann}_R(M)S = \mathrm{Ann}_S(M \otimes_R S)$.

Proof. Set $I = \mathrm{Ann}_R(m)$. By definition there is an exact sequence $0 \to I \to R \to M$ where the map $R \to M$ sends f to fm. Using flatness we obtain an exact sequence $0 \to I \otimes_R S \to S \to M \otimes_R S$ which proves the first assertion. If $m_1, \ldots, m_n$ is a set of generators of M then $\mathrm{Ann}_R(M) = \bigcap \mathrm{Ann}_R(m_i)$. Similarly $\mathrm{Ann}_S(M \otimes_R S) = \bigcap \mathrm{Ann}_S(m_i \otimes 1)$. Set $I_i = \mathrm{Ann}_R(m_i)$. Then it suffices to show that $\bigcap_{i=1,\ldots,n}(I_iS) = (\bigcap_{i=1,\ldots,n} I_i)S$. This is Lemma 10.39.2. □

00L2 Lemma 10.40.5. Let R be a ring and let M be an R-module. If M is finite, then $\mathrm{Supp}(M)$ is closed. More precisely, if $I = \mathrm{Ann}(M)$ is the annihilator of M, then $V(I) = \mathrm{Supp}(M)$.

Proof. We will show that $V(I) = \mathrm{Supp}(M)$.

Suppose $\mathfrak{p} \in \mathrm{Supp}(M)$. Then $M_{\mathfrak{p}} \neq 0$. Choose an element $m \in M$ whose image in $M_{\mathfrak{p}}$ is nonzero. Then the annihilator of m is contained in $\mathfrak{p}$ by construction of the localization $M_{\mathfrak{p}}$. Hence a fortiori $I = \mathrm{Ann}(M)$ must be contained in $\mathfrak{p}$.

Conversely, suppose that $\mathfrak{p} \not\in \mathrm{Supp}(M)$. Then $M_{\mathfrak{p}} = 0$. Let $x_1, \ldots, x_r \in M$ be generators. By Lemma 10.9.9 there exists an $f \in R$, $f \not\in \mathfrak{p}$ such that $x_i/1 = 0$ in M_f. Hence $f^{n_i}x_i = 0$ for some $n_i \geq 1$. Hence $f^nM = 0$ for $n = \max\{n_i\}$ as desired. □

0BUR Lemma 10.40.6. Let $R \to R'$ be a ring map and let M be a finite R-module. Then $\text{Supp}(M \otimes_R R')$ is the inverse image of $\text{Supp}(M)$.

Proof. Let $\mathfrak{p} \in \text{Supp}(M)$. By Nakayama's lemma (Lemma 10.20.1) we see that

$$M \otimes_R \kappa(\mathfrak{p}) = M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$$

is a nonzero $\kappa(\mathfrak{p})$ vector space. Hence for every prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$ we see that

$$(M \otimes_R R')_{\mathfrak{p}'}/\mathfrak{p}'(M \otimes_R R')_{\mathfrak{p}'} = (M \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = M \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is nonzero. This implies $\mathfrak{p}' \in \text{Supp}(M \otimes_R R')$. For the converse, if $\mathfrak{p}' \subset R'$ is a prime lying over an arbitrary prime $\mathfrak{p} \subset R$, then

$$(M \otimes_R R')_{\mathfrak{p}'} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'}.$$

Hence if $\mathfrak{p}' \in \text{Supp}(M \otimes_R R')$ lies over the prime $\mathfrak{p} \subset R$, then $\mathfrak{p} \in \text{Supp}(M)$. $\square$

07Z5 Lemma 10.40.7. Let R be a ring, let M be an R-module, and let $m \in M$. Then $\mathfrak{p} \in V(\text{Ann}(m))$ if and only if m does not map to zero in $M_\mathfrak{p}$.

Proof. We may replace M by $Rm \subset M$. Then (1) $\text{Ann}(m) = \text{Ann}(M)$ and (2) m does not map to zero in $M_\mathfrak{p}$ if and only if $\mathfrak{p} \in \text{Supp}(M)$. The result now follows from Lemma 10.40.5. $\square$

051B Lemma 10.40.8. Let R be a ring and let M be an R-module. If M is a finitely presented R-module, then $\text{Supp}(M)$ is a closed subset of $\text{Spec}(R)$ whose complement is quasi-compact.

Proof. Choose a presentation

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \to 0$$

Let $A \in \text{Mat}(n \times m, R)$ be the matrix of the first map. By Nakayama's Lemma 10.20.1 we see that

$$M_\mathfrak{p} \neq 0 \Leftrightarrow M \otimes \kappa(\mathfrak{p}) \neq 0 \Leftrightarrow \text{rank}(A \bmod \mathfrak{p}) < n.$$

Hence, if I is the ideal of R generated by the $n \times n$ minors of A, then $\text{Supp}(M) = V(I)$. Since I is finitely generated, say $I = (f_1, \ldots, f_t)$, we see that $\text{Spec}(R) \setminus V(I)$ is a finite union of the standard opens $D(f_i)$, hence quasi-compact. $\square$

00L3 Lemma 10.40.9. Let R be a ring and let M be an R-module.

(1) If M is finite then the support of M/IM is $\text{Supp}(M) \cap V(I)$.
(2) If $N \subset M$, then $\text{Supp}(N) \subset \text{Supp}(M)$.
(3) If Q is a quotient module of M then $\text{Supp}(Q) \subset \text{Supp}(M)$.
(4) If $0 \to N \to M \to Q \to 0$ is a short exact sequence then $\text{Supp}(M) = \text{Supp}(Q) \cup \text{Supp}(N)$.

Proof. The functors $M \mapsto M_\mathfrak{p}$ are exact. This immediately implies all but the first assertion. For the first assertion we need to show that $M_\mathfrak{p} \neq 0$ and $I \subset \mathfrak{p}$ implies $(M/IM)_\mathfrak{p} = M_\mathfrak{p}/IM_\mathfrak{p} \neq 0$. This follows from Nakayama's Lemma 10.20.1. $\square$

10.41. Going up and going down

00HU Suppose $\mathfrak{p}$, $\mathfrak{p}'$ are primes of the ring R. Let $X = \mathrm{Spec}(R)$ with the Zariski topology. Denote $x \in X$ the point corresponding to $\mathfrak{p}$ and $x' \in X$ the point corresponding to $\mathfrak{p}'$. Then we have:

$$x' \rightsquigarrow x \Leftrightarrow \mathfrak{p}' \subset \mathfrak{p}.$$

In words: x is a specialization of x' if and only if $\mathfrak{p}' \subset \mathfrak{p}$. See Topology, Section 5.19 for terminology and notation.

00HV Definition 10.41.1. Let $\varphi : R \to S$ be a ring map.

(1) We say a $\varphi : R \to S$ satisfies going up if given primes $\mathfrak{p} \subset \mathfrak{p}'$ in R and a prime $\mathfrak{q}$ in S lying over $\mathfrak{p}$ there exists a prime $\mathfrak{q}'$ of S such that (a) $\mathfrak{q} \subset \mathfrak{q}'$, and (b) $\mathfrak{q}'$ lies over $\mathfrak{p}'$.
(2) We say a $\varphi : R \to S$ satisfies going down if given primes $\mathfrak{p} \subset \mathfrak{p}'$ in R and a prime $\mathfrak{q}'$ in S lying over $\mathfrak{p}'$ there exists a prime $\mathfrak{q}$ of S such that (a) $\mathfrak{q} \subset \mathfrak{q}'$, and (b) $\mathfrak{q}$ lies over $\mathfrak{p}$.

So far we have see the following cases of this:

(1) An integral ring map satisfies going up, see Lemma 10.36.22.
(2) As a special case finite ring maps satisfy going up.
(3) As a special case quotient maps $R \to R/I$ satisfy going up.
(4) A flat ring map satisfies going down, see Lemma 10.39.19
(5) As a special case any localization satisfies going down.
(6) An extension $R \subset S$ of domains, with R normal and S integral over R satisfies going down, see Proposition 10.38.7.

Here is another case where going down holds.

0407 Lemma 10.41.2. Let $R \to S$ be a ring map. If the induced map $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open, then $R \to S$ satisfies going down.

Proof. Suppose that $\mathfrak{p} \subset \mathfrak{p}' \subset R$ and $\mathfrak{q}' \subset S$ lies over $\mathfrak{p}'$. As φ is open, for every $g \in S$, $g \notin \mathfrak{q}'$ we see that $\mathfrak{p}$ is in the image of $D(g) \subset \mathrm{Spec}(S)$. In other words $S_g \otimes_R \kappa(\mathfrak{p})$ is not zero. Since $S_{\mathfrak{q}'}$ is the directed colimit of these S_g this implies that $S_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p})$ is not zero, see Lemmas 10.9.9 and 10.12.9. Hence $\mathfrak{p}$ is in the image of $\mathrm{Spec}(S_{\mathfrak{q}'}) \to \mathrm{Spec}(R)$ as desired. $\square$

00HW Lemma 10.41.3. Let $R \to S$ be a ring map.

(1) $R \to S$ satisfies going down if and only if generalizations lift along the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Topology, Definition 5.19.4.
(2) $R \to S$ satisfies going up if and only if specializations lift along the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Topology, Definition 5.19.4.

Proof. Omitted. $\square$

00HX Lemma 10.41.4. Suppose $R \to S$ and $S \to T$ are ring maps satisfying going down. Then so does $R \to T$. Similarly for going up.

Proof. According to Lemma 10.41.3 this follows from Topology, Lemma 5.19.5 $\square$

00HY Lemma 10.41.5. Let $R \to S$ be a ring map. Let $T \subset \mathrm{Spec}(R)$ be the image of $\mathrm{Spec}(S)$. If T is stable under specialization, then T is closed.

Proof. We give two proofs.

First proof. Let $\mathfrak{p} \subset R$ be a prime ideal such that the corresponding point of $\text{Spec}(R)$ is in the closure of T. This means that for every $f \in R$, $f \notin \mathfrak{p}$ we have $D(f) \cap T \neq \emptyset$. Note that $D(f) \cap T$ is the image of $\text{Spec}(S_f)$ in $\text{Spec}(R)$. Hence we conclude that $S_f \neq 0$. In other words, $1 \neq 0$ in the ring S_f. Since $S_\mathfrak{p}$ is the directed colimit of the rings S_f we conclude that $1 \neq 0$ in $S_\mathfrak{p}$. In other words, $S_\mathfrak{p} \neq 0$ and considering the image of $\text{Spec}(S_\mathfrak{p}) \to \text{Spec}(S) \to \text{Spec}(R)$ we see there exists a $\mathfrak{p}' \in T$ with $\mathfrak{p}' \subset \mathfrak{p}$. As we assumed T closed under specialization we conclude $\mathfrak{p}$ is a point of T as desired.

Second proof. Let $I = \text{Ker}(R \to S)$. We may replace R by R/I. In this case the ring map $R \to S$ is injective. By Lemma 10.30.5 all the minimal primes of R are contained in the image T. Hence if T is stable under specialization then it contains all primes. □

00HZ Lemma 10.41.6. Let $R \to S$ be a ring map. The following are equivalent:

(1) Going up holds for $R \to S$, and
(2) the map $\text{Spec}(S) \to \text{Spec}(R)$ is closed.

Proof. It is a general fact that specializations lift along a closed map of topological spaces, see Topology, Lemma 5.19.7. Hence the second condition implies the first.

Assume that going up holds for $R \to S$. Let $V(I) \subset \text{Spec}(S)$ be a closed set. We want to show that the image of $V(I)$ in $\text{Spec}(R)$ is closed. The ring map $S \to S/I$ obviously satisfies going up. Hence $R \to S \to S/I$ satisfies going up, by Lemma 10.41.4. Replacing S by S/I it suffices to show the image T of $\text{Spec}(S)$ in $\text{Spec}(R)$ is closed. By Topology, Lemmas 5.19.2 and 5.19.6 this image is stable under specialization. Thus the result follows from Lemma 10.41.5. □

00I0 Lemma 10.41.7. Let R be a ring. Let $E \subset \text{Spec}(R)$ be a constructible subset.

(1) If E is stable under specialization, then E is closed.
(2) If E is stable under generalization, then E is open.

Proof. First proof. The first assertion follows from Lemma 10.41.5 combined with Lemma 10.29.4. The second follows because the complement of a constructible set is constructible (see Topology, Lemma 5.15.2), the first part of the lemma and Topology, Lemma 5.19.2.

Second proof. Since $\text{Spec}(R)$ is a spectral space by Lemma 10.26.2 this is a special case of Topology, Lemma 5.23.6. □

00I1 Proposition 10.41.8. Let $R \to S$ be flat and of finite presentation. Then $\text{Spec}(S) \to \text{Spec}(R)$ is open. More generally this holds for any ring map $R \to S$ of finite presentation which satisfies going down.

Proof. If $R \to S$ is flat, then $R \to S$ satisfies going down by Lemma 10.39.19. Thus to prove the lemma we may assume that $R \to S$ has finite presentation and satisfies going down.

Since the standard opens $D(g) \subset \text{Spec}(S)$, $g \in S$ form a basis for the topology, it suffices to prove that the image of $D(g)$ is open. Recall that $\text{Spec}(S_g) \to \text{Spec}(S)$ is a homeomorphism of $\text{Spec}(S_g)$ onto $D(g)$ (Lemma 10.17.6). Since $S \to S_g$ satisfies going down (see above), we see that $R \to S_g$ satisfies going down by Lemma

10.41.4. Thus after replacing S by S_g we see it suffices to prove the image is open. By Chevalley's theorem (Theorem 10.29.10) the image is a constructible set E. And E is stable under generalization because $R \to S$ satisfies going down, see Topology, Lemmas 5.19.2 and 5.19.6. Hence E is open by Lemma 10.41.7. $\square$

037F Lemma 10.41.9. Let k be a field, and let R, S be k-algebras. Let $S' \subset S$ be a sub k-algebra, and let $f \in S' \otimes_k R$. In the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}((S \otimes_k R)_f) \arrow[rr] \arrow[dr] & & \mathrm{Spec}((S' \otimes_k R)_f) \arrow[dl] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

the images of the diagonal arrows are the same.

Proof. Let $\mathfrak p \subset R$ be in the image of the south-west arrow. This means (Lemma 10.17.9) that

$$(S' \otimes_k R)_f \otimes_R \kappa(\mathfrak p) = (S' \otimes_k \kappa(\mathfrak p))_f$$

is not the zero ring, i.e., $S' \otimes_k \kappa(\mathfrak p)$ is not the zero ring and the image of f in it is not nilpotent. The ring map $S' \otimes_k \kappa(\mathfrak p) \to S \otimes_k \kappa(\mathfrak p)$ is injective. Hence also $S \otimes_k \kappa(\mathfrak p)$ is not the zero ring and the image of f in it is not nilpotent. Hence $(S \otimes_k R)_f \otimes_R \kappa(\mathfrak p)$ is not the zero ring. Thus (Lemma 10.17.9) we see that $\mathfrak p$ is in the image of the south-east arrow as desired. $\square$

037G Lemma 10.41.10. Let k be a field. Let R and S be k-algebras. The map $\mathrm{Spec}(S \otimes_k R) \to \mathrm{Spec}(R)$ is open.

Proof. Let $f \in S \otimes_k R$. It suffices to prove that the image of the standard open $D(f)$ is open. Let $S' \subset S$ be a finite type k-subalgebra such that $f \in S' \otimes_k R$. The map $R \to S' \otimes_k R$ is flat and of finite presentation, hence the image U of $\mathrm{Spec}((S' \otimes_k R)_f) \to \mathrm{Spec}(R)$ is open by Proposition 10.41.8. By Lemma 10.41.9 this is also the image of $D(f)$ and we win. $\square$

Here is a tricky lemma that is sometimes useful.

00EA Lemma 10.41.11. Let $R \to S$ be a ring map. Let $\mathfrak p \subset R$ be a prime. Assume that

(1) there exists a unique prime $\mathfrak q \subset S$ lying over $\mathfrak p$, and
(2) either
  (a) going up holds for $R \to S$, or
  (b) going down holds for $R \to S$ and there is at most one prime of S above every prime of R.

Then $S_{\mathfrak p} = S_{\mathfrak q}$.

Proof. Consider any prime $\mathfrak q' \subset S$ which corresponds to a point of $\mathrm{Spec}(S_{\mathfrak p})$. This means that $\mathfrak p' = R \cap \mathfrak q'$ is contained in $\mathfrak p$. Here is a picture

$$\begin{tikzcd} \mathfrak q' \arrow[r, dash] \arrow[d, dash] & ? \arrow[r, dash] \arrow[d, dash] & S \arrow[d, dash] \\ \mathfrak p' \arrow[r, dash] & \mathfrak p \arrow[r, dash] & R \end{tikzcd}$$

Assume (1) and (2)(a). By going up there exists a prime $\mathfrak{q}'' \subset S$ with $\mathfrak{q}' \subset \mathfrak{q}''$ and $\mathfrak{q}''$ lying over $\mathfrak{p}$. By the uniqueness of $\mathfrak{q}$ we conclude that $\mathfrak{q}'' = \mathfrak{q}$. In other words $\mathfrak{q}'$ defines a point of $\mathrm{Spec}(S_\mathfrak{q})$.

Assume (1) and (2)(b). By going down there exists a prime $\mathfrak{q}'' \subset \mathfrak{q}$ lying over $\mathfrak{p}'$. By the uniqueness of primes lying over $\mathfrak{p}'$ we see that $\mathfrak{q}' = \mathfrak{q}''$. In other words $\mathfrak{q}'$ defines a point of $\mathrm{Spec}(S_\mathfrak{q})$.

In both cases we conclude that the map $\mathrm{Spec}(S_\mathfrak{q}) \to \mathrm{Spec}(S_\mathfrak{p})$ is bijective. Clearly this means all the elements of $S - \mathfrak{q}$ are all invertible in $S_\mathfrak{p}$, in other words $S_\mathfrak{p} = S_\mathfrak{q}$. □

The following lemma is a generalization of going down for flat ring maps.

080T Lemma 10.41.12. Let $R \to S$ be a ring map. Let N be a finite S-module flat over R. Endow $\mathrm{Supp}(N) \subset \mathrm{Spec}(S)$ with the induced topology. Then generalizations lift along $\mathrm{Supp}(N) \to \mathrm{Spec}(R)$.

Proof. The meaning of the statement is as follows. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}' \subset S$ be a prime $\mathfrak{q}' \in \mathrm{Supp}(N)$ Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$, $\mathfrak{q} \in \mathrm{Supp}(N)$ lying over $\mathfrak{p}$. As N is flat over R we see that $N_{\mathfrak{q}'}$ is flat over $R_{\mathfrak{p}'}$, see Lemma 10.39.18. As $N_{\mathfrak{q}'}$ is finite over $S_{\mathfrak{q}'}$ and not zero since $\mathfrak{q}' \in \mathrm{Supp}(N)$ we see that $N_{\mathfrak{q}'} \otimes_{S_{\mathfrak{q}'}} \kappa(\mathfrak{q}')$ is nonzero by Nakayama's Lemma 10.20.1. Thus $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p}')$ is also not zero. We conclude from Lemma 10.39.15 that $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$ is nonzero. Let $J \subset S_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$ be the annihilator of the finite nonzero module $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$. Since J is a proper ideal we can choose a prime $\mathfrak{q} \subset S$ which corresponds to a prime of $S_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})/J$. This prime is in the support of N, lies over $\mathfrak{p}$, and is contained in $\mathfrak{q}'$ as desired. □

10.42. Separable extensions

030I In this section we talk about separability for nonalgebraic field extensions. This is closely related to the concept of geometrically reduced algebras, see Definition 10.43.1.

030O Definition 10.42.1. Let K/k be a field extension.

1. We say K is separably generated over k if there exists a transcendence basis $\{x_i; i \in I\}$ of K/k such that the extension $K/k(x_i; i \in I)$ is a separable algebraic extension.
2. We say K is separable over k if for every subextension $k \subset K' \subset K$ with K' finitely generated over k, the extension K'/k is separably generated.

With this awkward definition it is not clear that a separably generated field extension is itself separable. It will turn out that this is the case, see Lemma 10.44.2.

030P Lemma 10.42.2. Let K/k be a separable field extension. For any subextension $K/K'/k$ the field extension K'/k is separable.

Proof. This is direct from the definition. □

030Q Lemma 10.42.3. Let K/k be a separably generated, and finitely generated field extension. Set $r = \mathrm{trdeg}_k(K)$. Then there exist elements $x_1, \ldots, x_{r+1}$ of K such that

1. $x_1, \ldots, x_r$ is a transcendence basis of K over k,

(2) $K = k(x_1, \ldots, x_{r+1})$, and
(3) x_{r+1} is separable over $k(x_1, \ldots, x_r)$.

Proof. Combine the definition with Fields, Lemma 9.19.1. □

04KM Lemma 10.42.4. Let K/k be a finitely generated field extension. There exists a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is a separably generated field extension.

Proof. This lemma is only interesting when the characteristic of k is $p > 0$. Choose $x_1, \ldots, x_r$ a transcendence basis of K over k. As K is finitely generated over k the extension $k(x_1, \ldots, x_r) \subset K$ is finite. Let $K/K_{sep}/k(x_1, \ldots, x_r)$ be the subextension found in Fields, Lemma 9.14.6. If $K = K_{sep}$ then we are done. We will use induction on $d = [K : K_{sep}]$.

Assume that $d > 1$. Choose a $\beta \in K$ with $\alpha = \beta^p \in K_{sep}$ and $\beta \notin K_{sep}$. Let $P = T^n + a_1 T^{n-1} + \ldots + a_n$ be the minimal polynomial of α over $k(x_1, \ldots, x_r)$. Let k'/k be a finite purely inseparable extension obtained by adjoining pth roots such that each a_i is a pth power in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Such an extension exists; details omitted. Let L be a field fitting into the diagram

$$\begin{tikzcd} K \arrow[r] & L \\ k(x_1, \ldots, x_r) \arrow[u] \arrow[r] & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \arrow[u] \end{tikzcd}$$

We may and do assume L is the compositum of K and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Let $L/L_{sep}/k'(x_1^{1/p}, \ldots, x_r^{1/p})$ be the subextension found in Fields, Lemma 9.14.6. Then L_{sep} is the compositum of K_{sep} and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. The element $\alpha \in L_{sep}$ is a zero of the polynomial P all of whose coefficients are pth powers in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$ and whose roots are pairwise distinct. By Fields, Lemma 9.28.2 we see that $\alpha = (\alpha')^p$ for some $\alpha' \in L_{sep}$. Clearly, this means that β maps to $\alpha' \in L_{sep}$. In other words, we get the tower of fields

$$\begin{tikzcd}
K \arrow[r] & L \\
K_{sep}(\beta) \arrow[u] \arrow[r] & L_{sep} \arrow[u] \\
K_{sep} \arrow[u] \arrow[r] & L_{sep} \arrow[u, equal] \\
k(x_1, \ldots, x_r) \arrow[u] \arrow[r] & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \arrow[u] \\
k \arrow[u] \arrow[r] & k' \arrow[u]
\end{tikzcd}$$

Thus this construction leads to a new situation with $[L : L_{sep}] < [K : K_{sep}]$. By induction we can find $k' \subset k''$ and $L \subset L'$ as in the lemma for the extension L/k'. Then the extensions k''/k and L'/K work for the extension K/k. This proves the lemma. □

10.43. Geometrically reduced algebras

05DS The main result on geometrically reduced algebras is Lemma 10.44.3. We suggest the reader skip to the lemma after reading the definition.

030S Definition 10.43.1. Let k be a field. Let S be a k-algebra. We say S is geometrically reduced over k if for every field extension K/k the K-algebra $K \otimes_k S$ is reduced.

Let k be a field and let S be a reduced k-algebra. To check that S is geometrically reduced it will suffice to check that $\overline{k} \otimes_k S$ is reduced (where $\overline{k}$ denotes the algebraic closure of k). In fact it is enough to check this for finite purely inseparable field extensions k'/k. See Lemma 10.44.3.

030T Lemma 10.43.2. Elementary properties of geometrically reduced algebras. Let k be a field. Let S be a k-algebra.

(1) If S is geometrically reduced over k so is every k-subalgebra.
(2) If all finitely generated k-subalgebras of S are geometrically reduced, then S is geometrically reduced.
(3) A directed colimit of geometrically reduced k-algebras is geometrically reduced.
(4) If S is geometrically reduced over k, then any localization of S is geometrically reduced over k.

Proof. Omitted. The second and third property follow from the fact that tensor product commutes with colimits. □

04KN Lemma 10.43.3. Let k be a field. If R is geometrically reduced over k, and $S \subset R$ is a multiplicative subset, then the localization $S^{-1}R$ is geometrically reduced over k. If R is geometrically reduced over k, then $R[x]$ is geometrically reduced over k.

Proof. Omitted. Hints: A localization of a reduced ring is reduced, and localization commutes with tensor products. □

In the proofs of the following lemmas we will repeatedly use the following observation: Suppose that $R' \subset R$ and $S' \subset S$ are inclusions of k-algebras. Then the map $R' \otimes_k S' \to R \otimes_k S$ is injective.

00I3 Lemma 10.43.4. Let k be a field. Let R, S be k-algebras.

(1) If $R \otimes_k S$ is nonreduced, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ is not reduced.
(2) If $R \otimes_k S$ contains a nonzero zerodivisor, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nonzero zerodivisor.
(3) If $R \otimes_k S$ contains a nontrivial idempotent, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nontrivial idempotent.

Proof. Suppose $z \in R \otimes_k S$ is nilpotent. We may write $z = \sum_{i=1,\ldots,n} x_i \otimes y_i$. Thus we may take R' the k-subalgebra generated by the x_i and S' the k-subalgebra generated by the y_i. The second and third statements are proved in the same way. □

034N Lemma 10.43.5. Let k be a field. Let S be a geometrically reduced k-algebra. Let R be any reduced k-algebra. Then $R \otimes_k S$ is reduced.

Proof. By Lemma 10.43.4 we may assume that R is of finite type over k. Then R, as a reduced Noetherian ring, embeds into a finite product of fields (see Lemmas 10.25.4, 10.31.6, and 10.25.1). Hence we may assume R is a finite product of fields. In this case it follows from Definition 10.43.1 that $R \otimes_k S$ is reduced. □

030U Lemma 10.43.6. Let k be a field. Let S be a reduced k-algebra. Let K/k be either a separable field extension, or a separably generated field extension. Then $K \otimes_k S$ is reduced.

Proof. Assume $k \subset K$ is separable. By Lemma 10.43.4 we may assume that S is of finite type over k and K is finitely generated over k. Then S embeds into a finite product of fields, namely its total ring of fractions (see Lemmas 10.25.1 and 10.25.4). Hence we may actually assume that S is a domain. We choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 10.42.3. Let $P \in k(x_1, \ldots, x_r)[T]$ be the minimal polynomial of x_{r+1}. It is a separable polynomial. It is easy to see that $k[x_1, \ldots, x_r] \otimes_k S = S[x_1, \ldots, x_r]$ is a domain. This implies $k(x_1, \ldots, x_r) \otimes_k S$ is a domain as it is a localization of $S[x_1, \ldots, x_r]$. The ring extension $k(x_1, \ldots, x_r) \otimes_k S \subset K \otimes_k S$ is generated by a single element x_{r+1} with a single equation, namely P. Hence $K \otimes_k S$ embeds into $F[T]/(P)$ where F is the fraction field of $k(x_1, \ldots, x_r) \otimes_k S$. Since P is separable this is a finite product of fields and we win.

At this point we do not yet know that a separably generated field extension is separable, so we have to prove the lemma in this case also. To do this suppose that $\{x_i\}_{i \in I}$ is a separating transcendence basis for K over k. For any finite set of elements $\lambda_j \in K$ there exists a finite subset $T \subset I$ such that $k(\{x_i\}_{i \in T}) \subset k(\{x_i\}_{i \in T} \cup \{\lambda_j\})$ is finite separable. Hence we see that K is a directed colimit of finitely generated and separably generated extensions of k. Thus the argument of the preceding paragraph applies to this case as well. □

07K2 **Lemma 10.43.7.** Let k be a field and let S be a k-algebra. Assume that S is reduced and that $S_\mathfrak{p}$ is geometrically reduced for every minimal prime $\mathfrak{p}$ of S. Then S is geometrically reduced.

Proof. Since S is reduced the map $S \to \prod_{\mathfrak{p} \text{ minimal}} S_\mathfrak{p}$ is injective, see Lemma 10.25.2. If K/k is a field extension, then the maps

$$S \otimes_k K \to (\prod S_\mathfrak{p}) \otimes_k K \to \prod S_\mathfrak{p} \otimes_k K$$

are injective: the first as $k \to K$ is flat and the second by inspection because K is a free k-module. As $S_\mathfrak{p}$ is geometrically reduced the ring on the right is reduced. Thus we see that $S \otimes_k K$ is reduced as a subring of a reduced ring. $\square$

0C2X **Lemma 10.43.8.** Let k'/k be a separable algebraic extension. Then there exists a multiplicative subset $S \subset k' \otimes_k k'$ such that the multiplication map $k' \otimes_k k' \to k'$ is identified with $k' \otimes_k k' \to S^{-1}(k' \otimes_k k')$.

Proof. First assume k'/k is finite separable. Then $k' = k(\alpha)$, see Fields, Lemma 9.19.1. Let $P \in k[x]$ be the minimal polynomial of α over k. Then P is an irreducible, separable, monic polynomial, see Fields, Section 9.12. Then $k'[x]/(P) \to k' \otimes_k k'$, $\sum \alpha_i x^i \mapsto \alpha_i \otimes \alpha^i$ is an isomorphism. We can factor $P = (x - \alpha)Q$ in $k'[x]$ and since P is separable we see that $Q(\alpha) \neq 0$. Then it is clear that the multiplicative set S' generated by Q in $k'[x]/(P)$ works, i.e., that $k' = (S')^{-1}(k'[x]/(P))$. By transport of structure the image S of S' in $k' \otimes_k k'$ works.

In the general case we write $k' = \bigcup k_i$ as the union of its finite subfield extensions over k. For each i there is a multiplicative subset $S_i \subset k_i \otimes_k k_i$ such that $k_i = S_i^{-1}(k_i \otimes_k k_i)$. Then $S = \bigcup S_i \subset k' \otimes_k k'$ works. $\square$

0C2Y **Lemma 10.43.9.** Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically reduced over k if and only if it is geometrically reduced over k'.

Proof. Assume A is geometrically reduced over k'. Let K/k be a field extension. Then $K \otimes_k k'$ is a reduced ring by Lemma 10.43.6. Hence by Lemma 10.43.5 we find that $K \otimes_k A = (K \otimes_k k') \otimes_{k'} A$ is reduced.

Assume A is geometrically reduced over k. Let K/k' be a field extension. Then

$$K \otimes_{k'} A = (K \otimes_k A) \otimes_{(k' \otimes_k k')} k'$$

Since $k' \otimes_k k' \to k'$ is a localization by Lemma 10.43.8, we see that $K \otimes_{k'} A$ is a localization of a reduced algebra, hence reduced. $\square$

10.44. Separable extensions, continued

05DT In this section we continue the discussion started in Section 10.42. Let p be a prime number and let k be a field of characteristic p. In this case we write $k^{1/p}$ for the extension of k gotten by adjoining pth roots of all the elements of k to k. (In other words it is the subfield of an algebraic closure of k generated by the pth roots of elements of k.)

030W **Lemma 10.44.1.** Let k be a field of characteristic $p > 0$. Let K/k be a field extension. The following are equivalent:

(1) K is separable over k,
(2) the ring $K \otimes_k k^{1/p}$ is reduced, and

(3) K is geometrically reduced over k.

Proof. The implication (1) $\Rightarrow$ (3) follows from Lemma 10.43.6. The implication (3) $\Rightarrow$ (2) is immediate.

Assume (2). Let $K/L/k$ be a subextension such that L is a finitely generated field extension of k. We have to show that we can find a separating transcendence basis of L. The assumption implies that $L \otimes_k k^{1/p}$ is reduced. Let $x_1, \ldots, x_r$ be a transcendence basis of L over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$ is minimal. If L is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in L$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be the minimal polynomial of α over $k(x_1, \ldots, x_r)$. After replacing α by $f\alpha$ for some nonzero $f \in k[x_1, \ldots, x_r]$ we may and do assume that P lies in $k[x_1, \ldots, x_r, T]$. Because α is not separable P is a polynomial in T^p, see Fields, Lemma 9.12.1. Let dp be the degree of P as a polynomial in T. Since P is the minimal polynomial of α the monomials

$$x_1^{e_1} \ldots x_r^{e_r} \alpha^e$$

for $e < dp$ are linearly independent over k in L. We claim that the element $\partial P/\partial x_i \in k[x_1, \ldots, x_r, T]$ is not zero for at least one i. Namely, if this was not the case, then P is actually a polynomial in $x_1^p, \ldots, x_r^p, T^p$. In that case we can consider $P^{1/p} \in k^{1/p}[x_1, \ldots, x_r, T]$. This would map to $P^{1/p}(x_1, \ldots, x_r, \alpha)$ which is a nilpotent element of $k^{1/p} \otimes_k L$ and hence zero. On the other hand, $P^{1/p}(x_1, \ldots, x_r, \alpha)$ is a $k^{1/p}$-linear combination the monomials listed above, hence nonzero in $k^{1/p} \otimes_k L$. This is a contradiction which proves our claim.

Thus, after renumbering, we may assume that $\partial P/\partial x_1$ is not zero. As P is an irreducible polynomial in T over $k(x_1, \ldots, x_r)$ it is irreducible as a polynomial in $x_1, \ldots, x_r, T$, hence by Gauss's lemma it is irreducible as a polynomial in x_1 over $k(x_2, \ldots, x_r, T)$. Since the transcendence degree of L is r we see that $x_2, \ldots, x_r, \alpha$ are algebraically independent. Hence $P(X, x_2, \ldots, x_r, \alpha) \in k(x_2, \ldots, x_r, \alpha)[X]$ is irreducible. It follows that x_1 is separably algebraic over $k(x_2, \ldots, x_r, \alpha)$. This means that the degree of inseparability of the finite extension $k(x_2, \ldots, x_r, \alpha) \subset L$ is less than the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$, which is a contradiction. $\square$

030X Lemma 10.44.2. A separably generated field extension is separable.

Proof. Combine Lemma 10.43.6 with Lemma 10.44.1. $\square$

In the following lemma we will use the notion of the perfect closure which is defined in Definition 10.45.5.

030V Lemma 10.44.3. Let k be a field. Let S be a k-algebra. The following are equivalent:

(1) $k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k,
(2) $k^{1/p} \otimes_k S$ is reduced,
(3) $k^{perf} \otimes_k S$ is reduced, where k^{perf} is the perfect closure of k,
(4) $\overline{k} \otimes_k S$ is reduced, where $\overline{k}$ is the algebraic closure of k, and
(5) S is geometrically reduced over k.

Proof. Note that any finite purely inseparable extension k'/k embeds in k^{perf}. Moreover, $k^{1/p}$ embeds into k^{perf} which embeds into $\overline{k}$. Thus it is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and that (3) $\Rightarrow$ (1).

We prove that (1) $\Rightarrow$ (5). Assume $k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k. Let K/k be an extension of fields. We have to show that $K \otimes_k S$ is reduced. By Lemma 10.43.4 we reduce to the case where K/k is a finitely generated field extension. Choose a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

as in Lemma 10.42.4. By assumption $k' \otimes_k S$ is reduced. By Lemma 10.43.6 it follows that $K' \otimes_k S$ is reduced. Hence we conclude that $K \otimes_k S$ is reduced as desired.

Finally we prove that (2) $\Rightarrow$ (5). Assume $k^{1/p} \otimes_k S$ is reduced. Then S is reduced. Moreover, for each localization $S_\mathfrak{p}$ at a minimal prime $\mathfrak{p}$, the ring $k^{1/p} \otimes_k S_\mathfrak{p}$ is a localization of $k^{1/p} \otimes_k S$ hence is reduced. But $S_\mathfrak{p}$ is a field by Lemma 10.25.1, hence $S_\mathfrak{p}$ is geometrically reduced by Lemma 10.44.1. It follows from Lemma 10.43.7 that S is geometrically reduced. $\square$

10.45. Perfect fields

05DU Here is the definition.

030Y Definition 10.45.1. Let k be a field. We say k is perfect if every field extension of k is separable over k.

030Z Lemma 10.45.2. A field k is perfect if and only if it is a field of characteristic 0 or a field of characteristic $p > 0$ such that every element has a pth root.

Proof. The characteristic zero case is clear. Assume the characteristic of k is $p > 0$. If k is perfect, then all the field extensions where we adjoin a pth root of an element of k have to be trivial, hence every element of k has a pth root. Conversely if every element has a pth root, then $k = k^{1/p}$ and every field extension of k is separable by Lemma 10.44.1. $\square$

030R Lemma 10.45.3. Let K/k be a finitely generated field extension. There exists a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is a separable field extension. In this situation we can assume that $K' = k'K$ is the compositum, and also that $K' = (k' \otimes_k K)_{red}$.

Proof. By Lemma 10.42.4 we can find such a diagram with K'/k' separably generated. By Lemma 10.44.2 this implies that K' is separable over k'. The compositum $k'K$ is a subextension of K'/k' and hence $k' \subset k'K$ is separable by Lemma 10.42.2. The ring $(k' \otimes_k K)_{red}$ is a domain as for some $n \gg 0$ the map $x \mapsto x^{p^n}$ maps it into K. Hence it is a field by Lemma 10.36.19. Thus $(k' \otimes_k K)_{red} \to K'$ maps it isomorphically onto $k'K$. $\square$

046W Lemma 10.45.4. For every field k there exists a purely inseparable extension k'/k such that k' is perfect. The field extension k'/k is unique up to unique isomorphism.

Proof. If the characteristic of k is zero, then $k' = k$ is the unique choice. Assume the characteristic of k is $p > 0$. For every $n > 0$ there exists a unique algebraic extension $k \subset k^{1/p^n}$ such that (a) every element $\lambda \in k$ has a p^nth root in k^{1/p^n} and (b) for every element $\mu \in k^{1/p^n}$ we have $\mu^{p^n} \in k$. Namely, consider the ring map $k \to k^{1/p^n} = k$, $x \mapsto x^{p^n}$. This is injective and satisfies (a) and (b). It is clear that $k^{1/p^n} \subset k^{1/p^{n+1}}$ as extensions of k via the map $y \mapsto y^p$. Then we can take $k' = \bigcup k^{1/p^n}$. Some details omitted. $\square$

046X Definition 10.45.5. Let k be a field. The field extension k'/k of Lemma 10.45.4 is called the perfect closure of k. Notation k^{perf}/k.

Note that if k'/k is any algebraic purely inseparable extension, then k' is a subextension of k^{perf}, i.e., $k^{perf}/k'/k$. Namely, $(k')^{perf}$ is isomorphic to k^{perf} by the uniqueness of Lemma 10.45.4.

00I4 Lemma 10.45.6. Let k be a perfect field. Any reduced k algebra is geometrically reduced over k. Let R, S be k-algebras. Assume both R and S are reduced. Then the k-algebra $R \otimes_k S$ is reduced.

Proof. The first statement follows from Lemma 10.44.3. For the second statement use the first statement and Lemma 10.43.5. $\square$

10.46. Universal homeomorphisms

0BR5 Let k'/k be an algebraic purely inseparable field extension. Then for any k-algebra R the ring map $R \to k' \otimes_k R$ induces a homeomorphism of spectra. The reason for this is the slightly more general Lemma 10.46.7 below.

0BR6 Lemma 10.46.1. Let $\varphi : R \to S$ be a surjective map with locally nilpotent kernel. Then φ induces a homeomorphism of spectra and isomorphisms on residue fields. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ is surjective with locally nilpotent kernel.

Proof. By Lemma 10.17.7 the map $\text{Spec}(S) \to \text{Spec}(R)$ is a homeomorphism onto the closed subset $V(\text{Ker}(\varphi))$. Of course $V(\text{Ker}(\varphi)) = \text{Spec}(R)$ because every prime ideal of R contains every nilpotent element of R. This also implies the statement on residue fields. By right exactness of tensor product we see that $\text{Ker}(\varphi)R'$ is the kernel of the surjective map $R' \to R' \otimes_R S$. Hence the final statement by Lemma 10.32.3. $\square$

0BR7 Lemma 10.46.2. Let k'/k be a field extension. The following are equivalent [Alp14, Lemma 3.1.6]

(1) for each $x \in k'$ there exists an $n > 0$ such that $x^n \in k$, and
(2) $k' = k$ or k and k' have characteristic $p > 0$ and either k'/k is a purely inseparable extension or k and k' are algebraic extensions of $\mathbf{F}_p$.

Proof. Observe that each of the possibilities listed in (2) satisfies (1). Thus we assume k'/k satisfies (1) and we prove that we are in one of the cases of (2). Discarding the case $k = k'$ we may assume $k' \neq k$. It is clear that k'/k is algebraic. Hence we may assume that k'/k is a nontrivial finite extension. Let $k'/k'_{sep}/k$ be the separable subextension found in Fields, Lemma 9.14.6. We have to show that

$k = k'_{sep}$ or that k is an algebraic over $\mathbf{F}_p$. Thus we may assume that k'/k is a nontrivial finite separable extension and we have to show k is algebraic over $\mathbf{F}_p$.

Pick $x \in k'$, $x \notin k$. Pick $n, m > 0$ such that $x^n \in k$ and $(x+1)^m \in k$. Let $\overline{k}$ be an algebraic closure of k. We can choose embeddings $\sigma, \tau : k' \to \overline{k}$ with $\sigma(x) \neq \tau(x)$. This follows from the discussion in Fields, Section 9.12 (more precisely, after replacing k' by the k-extension generated by x it follows from Fields, Lemma 9.12.8). Then we see that $\sigma(x) = \zeta\tau(x)$ for some nth root of unity ζ in $\overline{k}$. Similarly, we see that $\sigma(x+1) = \zeta'\tau(x+1)$ for some mth root of unity $\zeta' \in \overline{k}$. Since $\sigma(x+1) \neq \tau(x+1)$ we see $\zeta' \neq 1$. Then

$$\zeta'(\tau(x) + 1) = \zeta'\tau(x+1) = \sigma(x+1) = \sigma(x) + 1 = \zeta\tau(x) + 1$$

implies that

$$\tau(x)(\zeta' - \zeta) = 1 - \zeta'$$

hence $\zeta' \neq \zeta$ and

$$\tau(x) = (1 - \zeta')/(\zeta' - \zeta)$$

Hence every element of k' which is not in k is algebraic over the prime subfield. Since k' is generated over the prime subfield by the elements of k' which are not in k, we conclude that k' (and hence k) is algebraic over the prime subfield.

Finally, if the characteristic of k is 0, the above leads to a contradiction as follows (we encourage the reader to find their own proof). For every rational number y we similarly get a root of unity ζ_y such that $\sigma(x+y) = \zeta_y\tau(x+y)$. Then we find

$$\zeta\tau(x) + y = \zeta_y(\tau(x) + y)$$

and by our formula for $\tau(x)$ above we conclude $\zeta_y \in \mathbf{Q}(\zeta, \zeta')$. Since the number field $\mathbf{Q}(\zeta, \zeta')$ contains only a finite number of roots of unity we find two distinct rational numbers y, y' with $\zeta_y = \zeta_{y'}$. Then we conclude that

$$y - y' = \sigma(x+y) - \sigma(x+y') = \zeta_y(\tau(x+y)) - \zeta_{y'}\tau(x+y') = \zeta_y(y - y')$$

which implies $\zeta_y = 1$ a contradiction. $\square$

0BR8 Lemma 10.46.3. Let $\varphi : R \to S$ be a ring map. If

(1) for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ, and
(2) $\text{Ker}(\varphi)$ is locally nilpotent,

then φ induces a homeomorphism on spectra and induces residue field extensions satisfying the equivalent conditions of Lemma 10.46.2.

Proof. Assume (1) and (2). Let $\mathfrak{q}, \mathfrak{q}'$ be primes of S lying over the same prime ideal $\mathfrak{p}$ of R. Suppose $x \in S$ with $x \in \mathfrak{q}$, $x \notin \mathfrak{q}'$. Then $x^n \in \mathfrak{q}$ and $x^n \notin \mathfrak{q}'$ for all $n > 0$. If $x^n = \varphi(y)$ with $y \in R$ for some $n > 0$ then

$$x^n \in \mathfrak{q} \Rightarrow y \in \mathfrak{p} \Rightarrow x^n \in \mathfrak{q}'$$

which is a contradiction. Hence there does not exist an x as above and we conclude that $\mathfrak{q} = \mathfrak{q}'$, i.e., the map on spectra is injective. By assumption (2) the kernel $I = \text{Ker}(\varphi)$ is contained in every prime, hence $\text{Spec}(R) = \text{Spec}(R/I)$ as topological spaces. As the induced map $R/I \to S$ is integral by assumption (1) Lemma 10.36.17 shows that $\text{Spec}(S) \to \text{Spec}(R/I)$ is surjective. Combining the above we see that $\text{Spec}(S) \to \text{Spec}(R)$ is bijective. If $x \in S$ is arbitrary, and we pick $y \in R$ such that $\varphi(y) = x^n$ for some $n > 0$, then we see that the open $D(x) \subset \text{Spec}(S)$ corresponds to the open $D(y) \subset \mathrm{Spec}(R)$ via the bijection above. Hence we see that the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism.

To see the statement on residue fields, let $\mathfrak{q} \subset S$ be a prime lying over a prime ideal $\mathfrak{p} \subset R$. Let $x \in \kappa(\mathfrak{q})$. If we think of $\kappa(\mathfrak{q})$ as the residue field of the local ring $S_\mathfrak{q}$, then we see that x is the image of some $y/z \in S_\mathfrak{q}$ with $y \in S$, $z \in S$, $z \notin \mathfrak{q}$. Choose $n, m > 0$ such that y^n, z^m are in the image of φ. Then x^{nm} is the residue of $(y/z)^{nm} = (y^n)^m/(z^m)^n$ which is in the image of $R_\mathfrak{p} \to S_\mathfrak{q}$. Hence x^{nm} is in the image of $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$. □

0EUH Lemma 10.46.4. Let $\varphi : R \to S$ be a ring map. Assume

(a) S is generated as an R-algebra by elements x such that $x^2, x^3 \in \varphi(R)$, and
(b) $\mathrm{Ker}(\varphi)$ is locally nilpotent,

Then φ induces isomorphisms on residue fields and a homeomorphism of spectra. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a) and (b).

Proof. Assume (a) and (b). The map on spectra is closed as S is integral over R, see Lemmas 10.41.6 and 10.36.22. The image is dense by Lemma 10.30.6. Thus $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective. If $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ then the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is generated by elements $\alpha \in \kappa(\mathfrak{q})$ whose square and cube are in $\kappa(\mathfrak{p})$. Thus clearly $\alpha \in \kappa(\mathfrak{p})$ and we find that $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$. If $\mathfrak{q}, \mathfrak{q}'$ were two distinct primes lying over $\mathfrak{p}$, then at least one of the generators x of S as in (a) would have distinct images in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q}') = \kappa(\mathfrak{p})$. This would contradict the fact that both x^2 and x^3 do have the same image. This proves that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is injective hence a homeomorphism (by what was already shown).

Since φ induces a homeomorphism on spectra, it is in particular surjective on spectra which is a property preserved under any base change, see Lemma 10.30.3. Therefore for any $R \to R'$ the kernel of the ring map $R' \to R' \otimes_R S$ consists of nilpotent elements, see Lemma 10.30.6, in other words (b) holds for $R' \to R' \otimes_R S$. It is clear that (a) is preserved under base change. □

0545 Lemma 10.46.5. Let p be a prime number. Let $n, m > 0$ be two integers. There exists an integer a such that $(x + y)^{p^a}, p^a(x + y) \in \mathbf{Z}[x^{p^n}, p^n x, y^{p^m}, p^m y]$.

Proof. This is clear for $p^a(x+y)$ as soon as $a \geq n, m$. In fact, pick $a \gg n, m$. Write

$$(x + y)^{p^a} = \sum\nolimits_{i,j \geq 0, i+j=p^a} \binom{p^a}{i, j} x^i y^j$$

For every $i, j \geq 0$ with $i + j = p^a$ write $i = qp^n + r$ with $r \in \{0, \ldots, p^n - 1\}$ and $j = q'p^m + r'$ with $r' \in \{0, \ldots, p^m - 1\}$. The condition $(x+y)^{p^a} \in \mathbf{Z}[x^{p^n}, p^n x, y^{p^m}, p^m y]$ holds if

$$p^{nr+mr'} \text{ divides } \binom{p^a}{i, j}$$

If $r = r' = 0$ then the divisibility holds. If $r \neq 0$, then we write

$$\binom{p^a}{i, j} = \frac{p^a}{i} \binom{p^a - 1}{i - 1, j}$$

Since $r \neq 0$ the rational number p^a/i has p-adic valuation at least $a-(n-1)$ (because i is not divisible by p^n). Thus $\binom{p^a}{i,j}$ is divisible by p^{a-n+1} in this case. Similarly, we see that if $r' \neq 0$, then $\binom{p^a}{i,j}$ is divisible by p^{a-m+1}. Picking $a = np^n + mp^m + n + m$ will work. $\square$

0BR9 Lemma 10.46.6. Let k'/k be a field extension. Let p be a prime number. The following are equivalent

(1) k' is generated as a field extension of k by elements x such that there exists an $n > 0$ with $x^{p^n} \in k$ and $p^n x \in k$, and
(2) $k = k'$ or the characteristic of k and k' is p and k'/k is purely inseparable.

Proof. Let $x \in k'$. If there exists an $n > 0$ with $x^{p^n} \in k$ and $p^n x \in k$ and if the characteristic is not p, then $x \in k$. If the characteristic is p, then we find $x^{p^n} \in k$ and hence x is purely inseparable over k. $\square$

0BRA Lemma 10.46.7. Let $\varphi : R \to S$ be a ring map. Let p be a prime number. Assume

(a) S is generated as an R-algebra by elements x such that there exists an $n > 0$ with $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(R)$, and
(b) $\mathrm{Ker}(\varphi)$ is locally nilpotent,

Then φ induces a homeomorphism of spectra and induces residue field extensions satisfying the equivalent conditions of Lemma 10.46.6. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a) and (b).

Proof. Assume (a) and (b). Note that (b) is equivalent to condition (2) of Lemma 10.46.3. Let $T \subset S$ be the set of elements $x \in S$ such that there exists an integer $n > 0$ such that $x^{p^n}, p^n x \in \varphi(R)$. We claim that $T = S$. This will prove that condition (1) of Lemma 10.46.3 holds and hence φ induces a homeomorphism on spectra. By assumption (a) it suffices to show that $T \subset S$ is an R-sub algebra. If $x \in T$ and $y \in R$, then it is clear that $yx \in T$. Suppose $x, y \in T$ and $n, m > 0$ such that $x^{p^n}, y^{p^m}, p^n x, p^m y \in \varphi(R)$. Then $(xy)^{p^{n+m}}, p^{n+m}xy \in \varphi(R)$ hence $xy \in T$. We have $x + y \in T$ by Lemma 10.46.5 and the claim is proved.

Since φ induces a homeomorphism on spectra, it is in particular surjective on spectra which is a property preserved under any base change, see Lemma 10.30.3. Therefore for any $R \to R'$ the kernel of the ring map $R' \to R' \otimes_R S$ consists of nilpotent elements, see Lemma 10.30.6, in other words (b) holds for $R' \to R' \otimes_R S$. It is clear that (a) is preserved under base change. Finally, the condition on residue fields follows from (a) as generators for S as an R-algebra map to generators for the residue field extensions. $\square$

0BRB Lemma 10.46.8. Let $\varphi : R \to S$ be a ring map. Assume

(1) φ induces an injective map of spectra,
(2) φ induces purely inseparable residue field extensions.

Then for any ring map $R \to R'$ properties (1) and (2) are true for $R' \to R' \otimes_R S$.

Proof. Set $S' = R' \otimes_R S$ so that we have a commutative diagram of continuous maps of spectra of rings

$$\begin{tikzcd} \mathrm{Spec}(S') \arrow[r] \arrow[d] & \mathrm{Spec}(S) \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

Let $\mathfrak{p}' \subset R'$ be a prime ideal lying over $\mathfrak{p} \subset R$. If there is no prime ideal of S lying over $\mathfrak{p}$, then there is no prime ideal of S' lying over $\mathfrak{p}'$. Otherwise, by Remark 10.17.8 there is a unique prime ideal $\mathfrak{r}$ of $F = S \otimes_R \kappa(\mathfrak{p})$ whose residue field is purely inseparable over $\kappa(\mathfrak{p})$. Consider the ring maps

$$\kappa(\mathfrak{p}) \to F \to \kappa(\mathfrak{r})$$

By Lemma 10.25.1 the ideal $\mathfrak{r} \subset F$ is locally nilpotent, hence we may apply Lemma 10.46.1 to the ring map $F \to \kappa(\mathfrak{r})$. We may apply Lemma 10.46.7 to the ring map $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{r})$. Hence the composition and the second arrow in the maps

$$\kappa(\mathfrak{p}') \to \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} F \to \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{r})$$

induces bijections on spectra and purely inseparable residue field extensions. This implies the same thing for the first map. Since

$$\kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} F = \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}) \otimes_R S = \kappa(\mathfrak{p}') \otimes_R S = \kappa(\mathfrak{p}') \otimes_{R'} R' \otimes_R S$$

we conclude by the discussion in Remark 10.17.8. □

0BRC Lemma 10.46.9. Let $\varphi : R \to S$ be a ring map. Assume

(1) φ is integral,
(2) φ induces an injective map of spectra,
(3) φ induces purely inseparable residue field extensions.

Then φ induces a homeomorphism from $\mathrm{Spec}(S)$ onto a closed subset of $\mathrm{Spec}(R)$ and for any ring map $R \to R'$ properties (1), (2), (3) are true for $R' \to R' \otimes_R S$.

Proof. The map on spectra is closed by Lemmas 10.41.6 and 10.36.22. The properties are preserved under base change by Lemmas 10.46.8 and 10.36.13. □

0BRD Lemma 10.46.10. Let $\varphi : R \to S$ be a ring map. Assume

(1) φ is integral,
(2) φ induces an bijective map of spectra,
(3) φ induces purely inseparable residue field extensions.

Then φ induces a homeomorphism on spectra and for any ring map $R \to R'$ properties (1), (2), (3) are true for $R' \to R' \otimes_R S$.

Proof. Follows from Lemmas 10.46.9 and 10.30.3. □

09EF Lemma 10.46.11. Let $\varphi : R \to S$ be a ring map such that

(1) the kernel of φ is locally nilpotent, and
(2) S is generated as an R-algebra by elements x such that there exist $n > 0$ and a polynomial $P(T) \in R[T]$ whose image in $S[T]$ is $(T - x)^n$.

Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism and $R \to S$ induces purely inseparable extensions of residue fields. Moreover, conditions (1) and (2) remain true on arbitrary base change.

Proof. We may replace R by $R/\mathrm{Ker}(\varphi)$, see Lemma 10.46.1. Assumption (2) implies S is generated over R by elements which are integral over R. Hence $R \subset S$ is integral (Lemma 10.36.7). In particular $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective and closed (Lemmas 10.36.17, 10.41.6, and 10.36.22).

Let $x \in S$ be one of the generators in (2), i.e., there exists an $n > 0$ be such that $(T - x)^n \in R[T]$. Let $\mathfrak{p} \subset R$ be a prime. The $\kappa(\mathfrak{p}) \otimes_R S$ ring is nonzero by the above and Lemma 10.17.9. If the characteristic of $\kappa(\mathfrak{p})$ is zero then we see that

$nx \in R$ implies $1 \otimes x$ is in the image of $\kappa(\mathfrak p) \to \kappa(\mathfrak p) \otimes_R S$. Hence $\kappa(\mathfrak p) \to \kappa(\mathfrak p) \otimes_R S$ is an isomorphism. If the characteristic of $\kappa(\mathfrak p)$ is $p > 0$, then write $n = p^k m$ with m prime to p. In $\kappa(\mathfrak p) \otimes_R S[T]$ we have

$$(T - 1 \otimes x)^n = ((T - 1 \otimes x)^{p^k})^m = (T^{p^k} - 1 \otimes x^{p^k})^m$$

and we see that $mx^{p^k} \in R$. This implies that $1 \otimes x^{p^k}$ is in the image of $\kappa(\mathfrak p) \to \kappa(\mathfrak p) \otimes_R S$. Hence Lemma 10.46.7 applies to $\kappa(\mathfrak p) \to \kappa(\mathfrak p) \otimes_R S$. In both cases we conclude that $\kappa(\mathfrak p) \otimes_R S$ has a unique prime ideal with residue field purely inseparable over $\kappa(\mathfrak p)$. By Remark 10.17.8 we conclude that φ is bijective on spectra.

The statement on base change is immediate. □

10.47. Geometrically irreducible algebras

00I2 An algebra S over a field k is geometrically irreducible if the algebra $S \otimes_k k'$ has a unique minimal prime for every field extension k'/k. In this section we develop a bit of theory relevant to this notion.

00I6 Lemma 10.47.1. Let $R \to S$ be a ring map. Assume

(a) $\text{Spec}(R)$ is irreducible,
(b) $R \to S$ is flat,
(c) $R \to S$ is of finite presentation,
(d) the fibre rings $S \otimes_R \kappa(\mathfrak p)$ have irreducible spectra for a dense collection of primes $\mathfrak p$ of R.

Then $\text{Spec}(S)$ is irreducible. This is true more generally with (b) + (c) replaced by "the map $\text{Spec}(S) \to \text{Spec}(R)$ is open".

Proof. The assumptions (b) and (c) imply that the map on spectra is open, see Proposition 10.41.8. Hence the lemma follows from Topology, Lemma 5.8.14. □

00I7 Lemma 10.47.2. Let k be a separably closed field. Let R, S be k-algebras. If R, S have a unique minimal prime, so does $R \otimes_k S$.

Proof. Let $k \subset \overline{k}$ be a perfect closure, see Definition 10.45.5. By assumption $\overline{k}$ is algebraically closed. The ring maps $R \to R \otimes_k \overline{k}$ and $S \to S \otimes_k \overline{k}$ and $R \otimes_k S \to (R \otimes_k S) \otimes_k \overline{k} = (R \otimes_k \overline{k}) \otimes_{\overline{k}} (S \otimes_k \overline{k})$ satisfy the assumptions of Lemma 10.46.7. Hence we may assume k is algebraically closed.

We may replace R and S by their reductions. Hence we may assume that R and S are domains. By Lemma 10.45.6 we see that $R \otimes_k S$ is reduced. Hence its spectrum is reducible if and only if it contains a nonzero zerodivisor. By Lemma 10.43.4 we reduce to the case where R and S are domains of finite type over k algebraically closed.

Note that the ring map $R \to R \otimes_k S$ is of finite presentation and flat. Moreover, for every maximal ideal $\mathfrak m$ of R we have $(R \otimes_k S) \otimes_R R/\mathfrak m \cong S$ because $k \cong R/\mathfrak m$ by the Hilbert Nullstellensatz Theorem 10.34.1. Moreover, the set of maximal ideals is dense in the spectrum of R since $\text{Spec}(R)$ is Jacobson, see Lemma 10.35.2. Hence we see that Lemma 10.47.1 applies to the ring map $R \to R \otimes_k S$ and we conclude that the spectrum of $R \otimes_k S$ is irreducible as desired. □

037K Lemma 10.47.3. Let k be a field. Let R be a k-algebra. The following are equivalent

(1) for every field extension k'/k the spectrum of $R \otimes_k k'$ is irreducible,

(2) for every finite separable field extension k'/k the spectrum of $R \otimes_k k'$ is irreducible,
(3) the spectrum of $R \otimes_k \overline{k}$ is irreducible where $\overline{k}$ is the separable algebraic closure of k, and
(4) the spectrum of $R \otimes_k \overline{k}$ is irreducible where $\overline{k}$ is the algebraic closure of k.

Proof. It is clear that (1) implies (2).

Assume (2) and let $\overline{k}$ is the separable algebraic closure of k. Suppose $\mathfrak{q}_i \subset R \otimes_k \overline{k}$, $i = 1, 2$ are two minimal prime ideals. For every finite subextension $\overline{k}/k'/k$ the extension k'/k is separable and the ring map $R \otimes_k k' \to R \otimes_k \overline{k}$ is flat. Hence $\mathfrak{p}_i = (R \otimes_k k') \cap \mathfrak{q}_i$ are minimal prime ideals (as we have going down for flat ring maps by Lemma 10.39.19). Thus we see that $\mathfrak{p}_1 = \mathfrak{p}_2$ by assumption (2). Since $\overline{k} = \bigcup k'$ we conclude $\mathfrak{q}_1 = \mathfrak{q}_2$. Hence $\mathrm{Spec}(R \otimes_k \overline{k})$ is irreducible.

Assume (3) and let $\overline{k}$ be the algebraic closure of k. Let $\overline{k}/\overline{k}'/k$ be the corresponding separable algebraic closure of k. Then $\overline{k}/\overline{k}'$ is purely inseparable (in positive characteristic) or trivial. Hence $R \otimes_k \overline{k}' \to R \otimes_k \overline{k}$ induces a homeomorphism on spectra, for example by Lemma 10.46.7. Thus we have (4).

Assume (4). Let k'/k be an arbitrary field extension and let $\overline{k}$ be the algebraic closure of k. We may choose a field F such that both k' and $\overline{k}$ are isomorphic to subfields of F. Then

$$R \otimes_k F = (R \otimes_k \overline{k}) \otimes_{\overline{k}} F$$

and hence we see from Lemma 10.47.2 that $R \otimes_k F$ has a unique minimal prime. Finally, the ring map $R \otimes_k k' \to R \otimes_k F$ is flat and injective and hence any minimal prime of $R \otimes_k k'$ is the image of a minimal prime of $R \otimes_k F$ (by Lemma 10.30.5 and going down). We conclude that there is only one such minimal prime and the proof is complete. □

037L Definition 10.47.4. Let k be a field. Let S be a k-algebra. We say S is geometrically irreducible over k if for every field extension k'/k the spectrum of $S \otimes_k k'$ is irreducible[5].

By Lemma 10.47.3 it suffices to check this for finite separable field extensions k'/k or for k' equal to the separable algebraic closure of k.

037M Lemma 10.47.5. Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically irreducible over k if and only if the spectrum of R is irreducible.

Proof. Immediate from the remark following Definition 10.47.4. □

037N Lemma 10.47.6. Let k be a field. Let S be a k-algebra.

(1) If S is geometrically irreducible over k so is every k-subalgebra.
(2) If all finitely generated k-subalgebras of S are geometrically irreducible, then S is geometrically irreducible.
(3) A directed colimit of geometrically irreducible k-algebras is geometrically irreducible.

[5]An irreducible space is nonempty.

Proof. Let $S' \subset S$ be a subalgebra. Then for any extension k'/k the ring map $S' \otimes_k k' \to S \otimes_k k'$ is injective also. Hence (1) follows from Lemma 10.30.5 (and the fact that the image of an irreducible space under a continuous map is irreducible). The second and third property follow from the fact that tensor product commutes with colimits. $\square$

037O Lemma 10.47.7. Let k be a field. Let S be a geometrically irreducible k-algebra. Let R be any k-algebra. The map

$$\mathrm{Spec}(R \otimes_k S) \longrightarrow \mathrm{Spec}(R)$$

induces a bijection on irreducible components.

Proof. Recall that irreducible components correspond to minimal primes (Lemma 10.26.1). As $R \to R \otimes_k S$ is flat we see by going down (Lemma 10.39.19) that any minimal prime of $R \otimes_k S$ lies over a minimal prime of R. Conversely, if $\mathfrak{p} \subset R$ is a (minimal) prime then

$$R \otimes_k S/\mathfrak{p}(R \otimes_k S) = (R/\mathfrak{p}) \otimes_k S \subset \kappa(\mathfrak{p}) \otimes_k S$$

by flatness of $R \to R \otimes_k S$. The ring $\kappa(\mathfrak{p}) \otimes_k S$ has irreducible spectrum by assumption. It follows that $R \otimes_k S/\mathfrak{p}(R \otimes_k S)$ has a single minimal prime (Lemma 10.30.5). In other words, the inverse image of the irreducible set $V(\mathfrak{p})$ is irreducible. Hence the lemma follows. $\square$

Let us make some remarks on the notion of geometrically irreducible field extensions.

037P Lemma 10.47.8. Let K/k be a field extension. If k is algebraically closed in K, then K is geometrically irreducible over k.

Proof. Assume k is algebraically closed in K. By Definition 10.47.4 and Lemma 10.47.3 it suffices to show that the spectrum of $K \otimes_k k'$ is irreducible for every finite separable extension k'/k. Say k' is generated by $\alpha \in k'$ over k, see Fields, Lemma 9.19.1. Let $P = T^d + a_1T^{d-1} + \ldots + a_d \in k[T]$ be the minimal polynomial of α. Then $K \otimes_k k' \cong K[T]/(P)$. The only way the spectrum of $K[T]/(P)$ can be reducible is if P is reducible in $K[T]$. Assume $P = P_1P_2$ is a nontrivial factorization in $K[T]$ to get a contradiction. By Lemma 10.38.5 we see that the coefficients of P_1 and P_2 are algebraic over k. Our assumption implies the coefficients of P_1 and P_2 are in k which contradicts the fact that P is irreducible over k. $\square$

0G30 Lemma 10.47.9. Let K/k be a geometrically irreducible field extension. Let S be a geometrically irreducible K-algebra. Then S is geometrically irreducible over k.

Proof. By Definition 10.47.4 and Lemma 10.47.3 it suffices to show that the spectrum of $S \otimes_k k'$ is irreducible for every finite separable extension k'/k. Since K is geometrically irreducible over k we see that $K' = K \otimes_k k'$ is a finite, separable field extension of K. Hence the spectrum of $S \otimes_k k' = S \otimes_K K'$ is irreducible as S is assumed geometrically irreducible over K. $\square$

0G31 Lemma 10.47.10. Let K/k be a field extension. The following are equivalent

1. (1) K is geometrically irreducible over k, and
2. (2) the induced extension $K(t)/k(t)$ of purely transcendental extensions is geometrically irreducible.

Proof. Assume (1). Denote Ω an algebraic closure of $k(t)$. By Definition 10.47.4 we find that the spectrum of

$$K \otimes_k \Omega = K \otimes_k k(t) \otimes_{k(t)} \Omega$$

is irreducible. Since $K(t)$ is a localization of $K \otimes_k k(T)$ we conclude that the spectrum of $K(t) \otimes_{k(t)} \Omega$ is irreducible. Thus by Lemma 10.47.3 we find that $K(t)/k(t)$ is geometrically irreducible.

Assume (2). Let k'/k be a field extension. We have to show that $K \otimes_k k'$ has a unique minimal prime. We know that the spectrum of

$$K(t) \otimes_{k(t)} k'(t)$$

is irreducible, i.e., has a unique minimal prime. Since there is an injective map $K \otimes_k k' \to K(t) \otimes_{k(t)} k'(t)$ (details omitted) we conclude by Lemmas 10.30.5 and 10.30.7. □

0G32 **Lemma 10.47.11.** *Let $K/L/M$ be a tower of fields with L/M geometrically irreducible. Let $x \in K$ be transcendental over L. Then $L(x)/M(x)$ is geometrically irreducible.*

Proof. This follows from Lemma 10.47.10 because the fields $L(x)$ and $M(x)$ are purely transcendental extensions of L and M. □

0G33 **Lemma 10.47.12.** *Let K/k be a field extension. The following are equivalent*

1. *K/k is geometrically irreducible, and*
2. *every element $\alpha \in K$ separably algebraic over k is in k.*

Proof. Assume (1) and let $\alpha \in K$ be separably algebraic over k. Then $k' = k(\alpha)$ is a finite separable extension of k contained in K. By Lemma 10.47.6 the extension k'/k is geometrically irreducible. In particular, we see that the spectrum of $k' \otimes_k \overline{k}$ is irreducible (and hence if it is a product of fields, then there is exactly one factor). By Fields, Lemma 9.13.4 it follows that $\text{Hom}_k(k', \overline{k})$ has one element which in turn implies that $k' = k$ by Fields, Lemma 9.12.11. Thus (2) holds.

Assume (2). Let $k' \subset K$ be the subfield consisting of elements algebraic over k. By Lemma 10.47.8 the extension K/k' is geometrically irreducible. By assumption k'/k is a purely inseparable extension. By Lemma 10.46.7 the extension k'/k is geometrically irreducible. Hence by Lemma 10.47.9 we see that K/k is geometrically irreducible. □

037Q **Lemma 10.47.13.** *Let K/k be a field extension. Consider the subextension $K/k'/k$ consisting of elements separably algebraic over k. Then K is geometrically irreducible over k'. If K/k is a finitely generated field extension, then $[k' : k] < \infty$.*

Proof. The first statement is immediate from Lemma 10.47.12 and the fact that elements separably algebraic over k' are in k' by the transitivity of separable algebraic extensions, see Fields, Lemma 9.12.12. If K/k is finitely generated, then k' is finite over k by Fields, Lemma 9.26.11. □

04KP **Lemma 10.47.14.** *Let K/k be an extension of fields. Let $\overline{k}/k$ be a separable algebraic closure. Then $\text{Gal}(\overline{k}/k)$ acts transitively on the primes of $\overline{k} \otimes_k K$.*

Proof. Let $K/k'/k$ be the subextension found in Lemma 10.47.13. Note that as $k \subset \overline{k}$ is integral all the prime ideals of $\overline{k} \otimes_k K$ and $\overline{k} \otimes_k k'$ are maximal, see Lemma 10.36.20. By Lemma 10.47.7 the map

$$\mathrm{Spec}(\overline{k} \otimes_k K) \to \mathrm{Spec}(\overline{k} \otimes_k k')$$

is bijective because (1) all primes are minimal primes, (2) $\overline{k} \otimes_k K = (\overline{k} \otimes_k k') \otimes_{k'} K$, and (3) K is geometrically irreducible over k'. Hence it suffices to prove the lemma for the action of $\mathrm{Gal}(\overline{k}/k)$ on the primes of $\overline{k} \otimes_k k'$.

As every prime of $\overline{k} \otimes_k k'$ is maximal, the residue fields are isomorphic to $\overline{k}$. Hence the prime ideals of $\overline{k} \otimes_k k'$ correspond one to one to elements of $\mathrm{Hom}_k(k', \overline{k})$ with $\sigma \in \mathrm{Hom}_k(k', \overline{k})$ corresponding to the kernel $\mathfrak{p}_\sigma$ of $1 \otimes \sigma : \overline{k} \otimes_k k' \to \overline{k}$. In particular $\mathrm{Gal}(\overline{k}/k)$ acts transitively on this set as desired. □

10.48. Geometrically connected algebras

05DV

037R Lemma 10.48.1. Let k be a separably algebraically closed field. Let R, S be k-algebras. If $\mathrm{Spec}(R)$, and $\mathrm{Spec}(S)$ are connected, then so is $\mathrm{Spec}(R \otimes_k S)$.

Proof. Recall that $\mathrm{Spec}(R)$ is connected if and only if R has no nontrivial idempotents, see Lemma 10.21.4. Hence, by Lemma 10.43.4 we may assume R and S are of finite type over k. In this case R and S are Noetherian, and have finitely many minimal primes, see Lemma 10.31.6. Thus we may argue by induction on $n + m$ where n, resp. m is the number of irreducible components of $\mathrm{Spec}(R)$, resp. $\mathrm{Spec}(S)$. Of course the case where either n or m is zero is trivial. If $n = m = 1$, i.e., $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ both have one irreducible component, then the result holds by Lemma 10.47.2. Suppose that $n > 1$. Let $\mathfrak{p} \subset R$ be a minimal prime corresponding to the irreducible closed subset $T \subset \mathrm{Spec}(R)$. Let $T' \subset \mathrm{Spec}(R)$ be the union of the other $n - 1$ irreducible components. Choose an ideal $I \subset R$ such that $T' = V(I) = \mathrm{Spec}(R/I)$ (Lemma 10.17.7). By choosing our minimal prime carefully we may in addition arrange it so that T' is connected, see Topology, Lemma 5.8.17. Then $T \cup T' = \mathrm{Spec}(R)$ and $T \cap T' = V(\mathfrak{p} + I) = \mathrm{Spec}(R/(\mathfrak{p} + I))$ is not empty as $\mathrm{Spec}(R)$ is assumed connected. The inverse image of T in $\mathrm{Spec}(R \otimes_k S)$ is $\mathrm{Spec}(R/\mathfrak{p} \otimes_k S)$, and the inverse of T' in $\mathrm{Spec}(R \otimes_k S)$ is $\mathrm{Spec}(R/I \otimes_k S)$. By induction these are both connected. The inverse image of $T \cap T'$ is $\mathrm{Spec}(R/(\mathfrak{p} + I) \otimes_k S)$ which is nonempty. Hence $\mathrm{Spec}(R \otimes_k S)$ is connected. □

037S Lemma 10.48.2. Let k be a field. Let R be a k-algebra. The following are equivalent

(1) for every field extension k'/k the spectrum of $R \otimes_k k'$ is connected, and
(2) for every finite separable field extension k'/k the spectrum of $R \otimes_k k'$ is connected.

Proof. For any extension of fields k'/k the connectivity of the spectrum of $R \otimes_k k'$ is equivalent to $R \otimes_k k'$ having no nontrivial idempotents, see Lemma 10.21.4. Assume (2). Let $k \subset \overline{k}$ be a separable algebraic closure of k. Using Lemma 10.43.4 we see that (2) is equivalent to $R \otimes_k \overline{k}$ having no nontrivial idempotents. For any field extension k'/k, there exists a field extension $\overline{k}'/\overline{k}$ with $k' \subset \overline{k}'$. By Lemma 10.48.1 we see that $R \otimes_k \overline{k}'$ has no nontrivial idempotents. If $R \otimes_k k'$ has a nontrivial idempotent, then also $R \otimes_k \overline{k}'$, contradiction. □

037T Definition 10.48.3. Let k be a field. Let S be a k-algebra. We say S is geometrically connected over k if for every field extension k'/k the spectrum of $S \otimes_k k'$ is connected.

By Lemma 10.48.2 it suffices to check this for finite separable field extensions k'/k.

037U Lemma 10.48.4. Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically connected over k if and only if the spectrum of R is connected.

Proof. Immediate from the remark following Definition 10.48.3. □

037V Lemma 10.48.5. Let k be a field. Let S be a k-algebra.

(1) If S is geometrically connected over k so is every k-subalgebra.
(2) If all finitely generated k-subalgebras of S are geometrically connected, then S is geometrically connected.
(3) A directed colimit of geometrically connected k-algebras is geometrically connected.

Proof. This follows from the characterization of connectedness in terms of the nonexistence of nontrivial idempotents. The second and third property follow from the fact that tensor product commutes with colimits. □

The following lemma will be superseded by the more general Varieties, Lemma 33.7.4.

037W Lemma 10.48.6. Let k be a field. Let S be a geometrically connected k-algebra. Let R be any k-algebra. The map

$$R \longrightarrow R \otimes_k S$$

induces a bijection on idempotents, and the map

$$\mathrm{Spec}(R \otimes_k S) \longrightarrow \mathrm{Spec}(R)$$

induces a bijection on connected components.

Proof. The second assertion follows from the first combined with Lemma 10.22.2. By Lemmas 10.48.5 and 10.43.4 we may assume that R and S are of finite type over k. Then we see that also $R \otimes_k S$ is of finite type over k. Note that in this case all the rings are Noetherian and hence their spectra have finitely many connected components (since they have finitely many irreducible components, see Lemma 10.31.6). In particular, all connected components in question are open! Hence via Lemma 10.24.3 we see that the first statement of the lemma in this case is equivalent to the second. Let's prove this. As the algebra S is geometrically connected and nonzero we see that all fibres of $X = \mathrm{Spec}(R \otimes_k S) \to \mathrm{Spec}(R) = Y$ are connected and nonempty. Also, as $R \to R \otimes_k S$ is flat of finite presentation the map $X \to Y$ is open (Proposition 10.41.8). Topology, Lemma 5.7.6 shows that $X \to Y$ induces bijection on connected components. □

10.49. Geometrically integral algebras

05DW Here is the definition.

05DX Definition 10.49.1. Let k be a field. Let S be a k-algebra. We say S is geometrically integral over k if for every field extension k'/k the ring of $S \otimes_k k'$ is a domain.

Any question about geometrically integral algebras can be translated in a question about geometrically reduced and irreducible algebras.

05DY **Lemma 10.49.2.** Let k be a field. Let S be a k-algebra. In this case S is geometrically integral over k if and only if S is geometrically irreducible as well as geometrically reduced over k.

Proof. Omitted. □

0FWF **Lemma 10.49.3.** Let k be a field. Let S be a k-algebra. The following are equivalent

(1) S is geometrically integral over k,
(2) for every finite extension k'/k of fields the ring $S \otimes_k k'$ is a domain,
(3) $S \otimes_k \overline{k}$ is a domain where $\overline{k}$ is the algebraic closure of k.

Proof. Follows from Lemmas 10.49.2, 10.44.3, and 10.47.3. □

09P9 **Lemma 10.49.4.** Let k be a field. Let S be a geometrically integral k-algebra. Let R be a k-algebra and an integral domain. Then $R \otimes_k S$ is an integral domain.

Proof. By Lemma 10.43.5 the ring $R \otimes_k S$ is reduced and by Lemma 10.47.7 the ring $R \otimes_k S$ is irreducible (the spectrum has just one irreducible component), so $R \otimes_k S$ is an integral domain. □

10.50. Valuation rings

00I8 Here are some definitions.

00I9 **Definition 10.50.1.** Valuation rings.

(1) Let K be a field. Let A, B be local rings contained in K. We say that B *dominates* A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.
(2) Let A be a ring. We say A is a *valuation ring* if A is a local domain and if A is maximal for the relation of domination among local rings contained in the fraction field of A.
(3) Let A be a valuation ring with fraction field K. If $R \subset K$ is a subring of K, then we say A is *centered* on R if $R \subset A$.

With this definition a field is a valuation ring.

00IA **Lemma 10.50.2.** Let K be a field. Let $A \subset K$ be a local subring. Then there exists a valuation ring with fraction field K dominating A.

Proof. We consider the collection of local subrings of K as a partially ordered set using the relation of domination. Suppose that $\{A_i\}_{i\in I}$ is a totally ordered collection of local subrings of K. Then $B = \bigcup A_i$ is a local subring which dominates all of the A_i. Hence by Zorn's Lemma, it suffices to show that if $A \subset K$ is a local ring whose fraction field is not K, then there exists a local ring $B \subset K$, $B \neq A$ dominating A.

Pick $t \in K$ which is not in the fraction field of A. If t is transcendental over A, then $A[t] \subset K$ and hence $A[t]_{(t,\mathfrak{m})} \subset K$ is a local ring distinct from A dominating A. Suppose t is algebraic over A. Then for some nonzero $a \in A$ the element at is integral over A. In this case the subring $A' \subset K$ generated by A and ta is finite over A. By Lemma 10.36.17 there exists a prime ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$. Then $A'_{\mathfrak{m}'}$ dominates A. If $A = A'_{\mathfrak{m}'}$, then t is in the fraction field of A which we assumed not to be the case. Thus $A \neq A'_{\mathfrak{m}'}$ as desired. □

00IC Lemma 10.50.3. Let A be a valuation ring. Then A is a normal domain.

Proof. Suppose x is in the field of fractions of A and integral over A. Let A' denote the subring of K generated by A and x. Since $A \subset A'$ is an integral extension, we see by Lemma 10.36.17 that there is a prime ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$. Then $A'_{\mathfrak{m}'}$ dominates A. Since A is a valuation ring we conclude that $A = A'_{\mathfrak{m}'}$ and therefore that $x \in A$. $\square$

00IB Lemma 10.50.4. Let A be a valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Let $x \in K$. Then either $x \in A$ or $x^{-1} \in A$ or both.

Proof. Assume that x is not in A. Let A' denote the subring of K generated by A and x. Since A is a valuation ring we see that there is no prime of A' lying over $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal we see that $V(\mathfrak{m}A') = \emptyset$. Then $\mathfrak{m}A' = A'$ by Lemma 10.17.2. Hence we can write $1 = \sum_{i=0}^{d} t_i x^i$ with $t_i \in \mathfrak{m}$. This implies that $(1 - t_0)(x^{-1})^d - \sum t_i (x^{-1})^{d-i} = 0$. In particular we see that x^{-1} is integral over A, and hence $x^{-1} \in A$ by Lemma 10.50.3. $\square$

052K Lemma 10.50.5. Let $A \subset K$ be a subring of a field K such that for all $x \in K$ either $x \in A$ or $x^{-1} \in A$ or both. Then A is a valuation ring with fraction field K.

Proof. If A is not K, then A is not a field and there is a nonzero maximal ideal $\mathfrak{m}$. If $\mathfrak{m}'$ is a second maximal ideal, then choose $x, y \in A$ with $x \in \mathfrak{m}$, $y \notin \mathfrak{m}$, $x \notin \mathfrak{m}'$, and $y \in \mathfrak{m}'$ (see Lemma 10.15.2). Then neither $x/y \in A$ nor $y/x \in A$ contradicting the assumption of the lemma. Thus we see that A is a local ring. Suppose that A' is a local ring contained in K which dominates A. Let $x \in A'$. We have to show that $x \in A$. If not, then $x^{-1} \in A$, and of course $x^{-1} \in \mathfrak{m}_A$. But then $x^{-1} \in \mathfrak{m}_{A'}$ which contradicts $x \in A'$. $\square$

0AS4 Lemma 10.50.6. Let I be a directed set. Let (A_i, φ_{ij}) be a system of valuation rings over I. Then $A = \text{colim}\, A_i$ is a valuation ring.

Proof. It is clear that A is a domain. Let $a, b \in A$. Lemma 10.50.5 tells us we have to show that either $a|b$ or $b|a$ in A. Choose i so large that there exist $a_i, b_i \in A_i$ mapping to a, b. Then Lemma 10.50.4 applied to a_i, b_i in A_i implies the result for a, b in A. $\square$

052L Lemma 10.50.7. Let L/K be an extension of fields. If $B \subset L$ is a valuation ring, then $A = K \cap B$ is a valuation ring.

Proof. We can replace L by the fraction field F of B and K by $K \cap F$. Then the lemma follows from a combination of Lemmas 10.50.4 and 10.50.5. $\square$

0AAV Lemma 10.50.8. Let L/K be an algebraic extension of fields. If $B \subset L$ is a valuation ring with fraction field L and not a field, then $A = K \cap B$ is a valuation ring and not a field.

Proof. By Lemma 10.50.7 the ring A is a valuation ring. If A is a field, then $A = K$. Then $A = K \subset B$ is an integral extension, hence there are no proper inclusions among the primes of B (Lemma 10.36.20). This contradicts the assumption that B is a local domain and not a field. $\square$

088Y Lemma 10.50.9. Let A be a valuation ring. For any prime ideal $\mathfrak{p} \subset A$ the quotient $A/\mathfrak{p}$ is a valuation ring. The same is true for the localization $A_{\mathfrak{p}}$ and in fact any localization of A.

Proof. Use the characterization of valuation rings given in Lemma 10.50.5. □

088Z **Lemma 10.50.10.** Let A' be a valuation ring with residue field K. Let A be a valuation ring with fraction field K. Then $C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A\}$ is a valuation ring.

Proof. Note that $\mathfrak{m}_{A'} \subset C$ and $C/\mathfrak{m}_{A'} = A$. In particular, the fraction field of C is equal to the fraction field of A'. We will use the criterion of Lemma 10.50.5 to prove the lemma. Let x be an element of the fraction field of C. By the lemma we may assume $x \in A'$. If $x \in \mathfrak{m}_{A'}$, then we see $x \in C$. If not, then x is a unit of A' and we also have $x^{-1} \in A'$. Hence either x or x^{-1} maps to an element of A by the lemma again. □

090P **Lemma 10.50.11.** Let A be a normal domain with fraction field K.

(1) For every $x \in K$, $x \notin A$ there exists a valuation ring $A \subset V \subset K$ with fraction field K such that $x \notin V$.
(2) If A is local, we can moreover choose V which dominates A.

In other words, A is the intersection of all valuation rings in K containing A and if A is local, then A is the intersection of all valuation rings in K dominating A.

Proof. Suppose $x \in K$, $x \notin A$. Consider $B = A[x^{-1}]$. Then $x \notin B$. Namely, if $x = a_0 + a_1x^{-1} + \ldots + a_dx^{-d}$ then $x^{d+1} - a_0x^d - \ldots - a_d = 0$ and x is integral over A in contradiction with the fact that A is normal. Thus x^{-1} is not a unit in B. Thus $V(x^{-1}) \subset \text{Spec}(B)$ is not empty (Lemma 10.17.2), and we can choose a prime $\mathfrak{p} \subset B$ with $x^{-1} \in \mathfrak{p}$. Choose a valuation ring $V \subset K$ dominating $B_\mathfrak{p}$ (Lemma 10.50.2). Then $x \notin V$ as $x^{-1} \in \mathfrak{m}_V$.

If A is local, then we claim that $x^{-1}B + \mathfrak{m}_AB \neq B$. Namely, if $1 = (a_0 + a_1x^{-1} + \ldots + a_dx^{-d})x^{-1} + a'_0 + \ldots + a'_dx^{-d}$ with $a_i \in A$ and $a'_i \in \mathfrak{m}_A$, then we'd get

$$(1 - a'_0)x^{d+1} - (a_0 + a'_1)x^d - \ldots - a_d = 0$$

Since $a'_0 \in \mathfrak{m}_A$ we see that $1 - a'_0$ is a unit in A and we conclude that x would be integral over A, a contradiction as before. Then choose the prime $\mathfrak{p} \supset x^{-1}B + \mathfrak{m}_AB$ we find V dominating A. □

An totally ordered abelian group is a pair $(\Gamma, \geq)$ consisting of an abelian group Γ endowed with a total ordering $\geq$ such that $\gamma \geq \gamma' \Rightarrow \gamma + \gamma'' \geq \gamma' + \gamma''$ for all $\gamma, \gamma', \gamma'' \in \Gamma$.

00ID **Lemma 10.50.12.** Let A be a valuation ring with field of fractions K. Set $\Gamma = K^*/A^*$ (with group law written additively). For $\gamma, \gamma' \in \Gamma$ define $\gamma \geq \gamma'$ if and only if $\gamma - \gamma'$ is in the image of $A - \{0\} \to \Gamma$. Then $(\Gamma, \geq)$ is a totally ordered abelian group.

Proof. Omitted, but follows easily from Lemma 10.50.4. Note that in case $A = K$ we obtain the zero group $\Gamma = \{0\}$ endowed with its unique total ordering. □

00IE **Definition 10.50.13.** Let A be a valuation ring.

(1) The totally ordered abelian group $(\Gamma, \geq)$ of Lemma 10.50.12 is called the *value group* of the valuation ring A.
(2) The map $v : A - \{0\} \to \Gamma$ and also $v : K^* \to \Gamma$ is called the *valuation* associated to A.
(3) The valuation ring A is called a *discrete valuation ring* if $\Gamma \cong \mathbf{Z}$.

Note that if $\Gamma \cong \mathbf{Z}$ then there is a unique such isomorphism such that $1 \geq 0$. If the isomorphism is chosen in this way, then the ordering becomes the usual ordering of the integers.

00IF Lemma 10.50.14. Let A be a valuation ring. The valuation $v : A - \{0\} \to \Gamma_{\geq 0}$ has the following properties:

(1) $v(a) = 0 \Leftrightarrow a \in A^*$,
(2) $v(ab) = v(a) + v(b)$,
(3) $v(a + b) \geq \min(v(a), v(b))$.

Proof. Omitted. $\square$

090Q Lemma 10.50.15. Let A be a ring. The following are equivalent

(1) A is a valuation ring,
(2) A is a local domain and every finitely generated ideal of A is principal.

Proof. Assume A is a valuation ring and let $f_1, \ldots, f_n \in A$. Choose i such that $v(f_i)$ is minimal among $v(f_j)$. Then $(f_i) = (f_1, \ldots, f_n)$. Conversely, assume A is a local domain and every finitely generated ideal of A is principal. Pick $f, g \in A$ and write $(f, g) = (h)$. Then $f = ah$ and $g = bh$ and $h = cf + dg$ for some $a, b, c, d \in A$. Thus $ac + bd = 1$ and we see that either a or b is a unit, i.e., either g/f or f/g is an element of A. This shows A is a valuation ring by Lemma 10.50.5. $\square$

00IG Lemma 10.50.16. Let $(\Gamma, \geq)$ be a totally ordered abelian group. Let K be a field. Let $v : K^* \to \Gamma$ be a homomorphism of abelian groups such that $v(a + b) \geq \min(v(a), v(b))$ for $a, b \in K$ with $a, b, a + b$ not zero. Then

$$A = \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

is a valuation ring with value group $\mathrm{Im}(v) \subset \Gamma$, with maximal ideal

$$\mathfrak{m} = \{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

and with group of units

$$A^* = \{x \in K^* \mid v(x) = 0\}.$$

Proof. Omitted. $\square$

Let $(\Gamma, \geq)$ be a totally ordered abelian group. An ideal of Γ is a subset $I \subset \Gamma$ such that all elements of I are ≥ 0 and $\gamma \in I$, $\gamma' \geq \gamma$ implies $\gamma' \in I$. We say that such an ideal is prime if $\gamma + \gamma' \in I, \gamma, \gamma' \geq 0 \Rightarrow \gamma \in I$ or $\gamma' \in I$.

00IH Lemma 10.50.17. Let A be a valuation ring. Ideals in A correspond $1 - 1$ with ideals of Γ. This bijection is inclusion preserving, and maps prime ideals to prime ideals.

Proof. Omitted. $\square$

00II Lemma 10.50.18. A valuation ring is Noetherian if and only if it is a discrete valuation ring or a field.

Proof. Suppose A is a discrete valuation ring with valuation $v : A \setminus \{0\} \to \mathbf{Z}$ normalized so that $\mathrm{Im}(v) = \mathbf{Z}_{\geq 0}$. By Lemma 10.50.17 the ideals of A are the subsets $I_n = \{0\} \cup v^{-1}(\mathbf{Z}_{\geq n})$. It is clear that any element $x \in A$ with $v(x) = n$ generates I_n. Hence A is a PID so certainly Noetherian.

Suppose A is a Noetherian valuation ring with value group Γ. By Lemma 10.50.17 we see the ascending chain condition holds for ideals in Γ. We may assume A is not a field, i.e., there is a $\gamma \in \Gamma$ with $\gamma > 0$. Applying the ascending chain condition to the subsets $\gamma + \Gamma_{\geq 0}$ with $\gamma > 0$ we see there exists a smallest element γ_0 which is bigger than 0. Let $\gamma \in \Gamma$ be an element $\gamma > 0$. Consider the sequence of elements γ, $\gamma - \gamma_0$, $\gamma - 2\gamma_0$, etc. By the ascending chain condition these cannot all be > 0. Let $\gamma - n\gamma_0$ be the last one ≥ 0. By minimality of γ_0 we see that $0 = \gamma - n\gamma_0$. Hence Γ is a cyclic group as desired. $\square$

10.51. More Noetherian rings

00IJ

00IK Lemma 10.51.1. Let R be a Noetherian ring. Any finite R-module is of finite presentation. Any submodule of a finite R-module is finite. The ascending chain condition holds for R-submodules of a finite R-module.

Proof. We first show that any submodule N of a finite R-module M is finite. We do this by induction on the number of generators of M. If this number is 1, then $N = J/I \subset M = R/I$ for some ideals $I \subset J \subset R$. Thus the definition of Noetherian implies the result. If the number of generators of M is greater than 1, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ where M' and M'' have fewer generators. Note that setting $N' = M' \cap N$ and $N'' = \mathrm{Im}(N \to M'')$ gives a similar short exact sequence for N. Hence the result follows from the induction hypothesis since the number of generators of N is at most the number of generators of N' plus the number of generators of N''.

To show that M is finitely presented just apply the previous result to the kernel of a presentation $R^n \to M$.

It is well known and easy to prove that the ascending chain condition for R-submodules of M is equivalent to the condition that every submodule of M is a finite R-module. We omit the proof. $\square$

00IN Lemma 10.51.2 (Artin-Rees). Suppose that R is Noetherian, $I \subset R$ an ideal. Let $N \subset M$ be finite R-modules. There exists a constant $c > 0$ such that $I^n M \cap N = I^{n-c}(I^c M \cap N)$ for all $n \geq c$.

Proof. Consider the ring $S = R \oplus I \oplus I^2 \oplus \ldots = \bigoplus_{n \geq 0} I^n$. Convention: $I^0 = R$. Multiplication maps $I^n \times I^m$ into I^{n+m} by multiplication in R. Note that if $I = (f_1, \ldots, f_t)$ then S is a quotient of the Noetherian ring $R[X_1, \ldots, X_t]$. The map just sends the monomial $X_1^{e_1} \ldots X_t^{e_t}$ to $f_1^{e_1} \ldots f_t^{e_t}$. Thus S is Noetherian. Similarly, consider the module $M \oplus IM \oplus I^2M \oplus \ldots = \bigoplus_{n \geq 0} I^n M$. This is a finitely generated S-module. Namely, if $x_1, \ldots, x_r$ generate M over R, then they also generate $\bigoplus_{n \geq 0} I^n M$ over S. Next, consider the submodule $\bigoplus_{n \geq 0} I^n M \cap N$. This is an S-submodule, as is easily verified. By Lemma 10.51.1 it is finitely generated as an S-module, say by $\xi_j \in \bigoplus_{n \geq 0} I^n M \cap N$, $j = 1, \ldots, s$. We may assume by decomposing each ξ_j into its homogeneous pieces that each $\xi_j \in I^{d_j} M \cap N$ for some d_j. Set $c = \max\{d_j\}$. Then for all $n \geq c$ every element in $I^n M \cap N$ is of the form $\sum h_j \xi_j$ with $h_j \in I^{n-d_j}$. The lemma now follows from this and the trivial observation that $I^{n-d_j}(I^{d_j} M \cap N) \subset I^{n-c}(I^c M \cap N)$. $\square$

00IO Lemma 10.51.3. Suppose that $0 \to K \to M \xrightarrow{f} N$ is an exact sequence of finitely generated modules over a Noetherian ring R. Let $I \subset R$ be an ideal. Then there exists a c such that

$$f^{-1}(I^n N) = K + I^{n-c} f^{-1}(I^c N) \quad \text{and} \quad f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$.

Proof. Apply Lemma 10.51.2 to $\mathrm{Im}(f) \subset N$ and note that $f : I^{n-c}M \to I^{n-c}f(M)$ is surjective. $\square$

00IP Lemma 10.51.4 (Krull's intersection theorem). Let R be a Noetherian local ring. Let $I \subset R$ be a proper ideal. Let M be a finite R-module. Then $\bigcap_{n \geq 0} I^n M = 0$.

Proof. Let $N = \bigcap_{n \geq 0} I^n M$. Then $N = I^n M \cap N$ for all $n \geq 0$. By the Artin-Rees Lemma 10.51.2 we see that $N = I^n M \cap N \subset IN$ for some suitably large n. By Nakayama's Lemma 10.20.1 we see that $N = 0$. $\square$

00IQ Lemma 10.51.5. Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $N = \bigcap_n I^n M$.

(1) For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ there exists a $f \in R$, $f \not\in \mathfrak{p}$ such that $N_f = 0$.
(2) If I is contained in the Jacobson radical of R, then $N = 0$.

Proof. Proof of (1). Let $x_1, \ldots, x_n$ be generators for the module N, see Lemma 10.51.1. For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ we see that the image of N in the localization $M_{\mathfrak{p}}$ is zero, by Lemma 10.51.4. Hence we can find $g_i \in R$, $g_i \not\in \mathfrak{p}$ such that x_i maps to zero in N_{g_i}. Thus $N_{g_1 g_2 \ldots g_n} = 0$.

Part (2) follows from (1) and Lemma 10.23.1. $\square$

00IR Remark 10.51.6. Lemma 10.51.4 in particular implies that $\bigcap_n I^n = (0)$ when $I \subset R$ is a non-unit ideal in a Noetherian local ring R. More generally, let R be a Noetherian ring and $I \subset R$ an ideal. Suppose that $f \in \bigcap_{n \in \mathbf{N}} I^n$. Then Lemma 10.51.5 says that for every prime ideal $I \subset \mathfrak{p}$ there exists a $g \in R$, $g \not\in \mathfrak{p}$ such that f maps to zero in R_g. In algebraic geometry we express this by saying that "f is zero in an open neighbourhood of the closed set $V(I)$ of $\mathrm{Spec}(R)$".

00IS Lemma 10.51.7 (Artin-Tate). Let R be a Noetherian ring. Let S be a finitely generated R-algebra. If $T \subset S$ is an R-subalgebra such that S is finitely generated as a T-module, then T is of finite type over R.

Proof. Choose elements $x_1, \ldots, x_n \in S$ which generate S as an R-algebra. Choose $y_1, \ldots, y_m$ in S which generate S as a T-module. Thus there exist $a_{ij} \in T$ such that $x_i = \sum a_{ij} y_j$. There also exist $b_{ijk} \in T$ such that $y_i y_j = \sum b_{ijk} y_k$. Let $T' \subset T$ be the sub R-algebra generated by a_{ij} and b_{ijk}. This is a finitely generated R-algebra, hence Noetherian. Consider the algebra

$$S' = T'[Y_1, \ldots, Y_m]/(Y_i Y_j - \sum b_{ijk} Y_k).$$

Note that S' is finite over T', namely as a T'-module it is generated by the classes of $1, Y_1, \ldots, Y_m$. Consider the T'-algebra homomorphism $S' \to S$ which maps Y_i to y_i. Because $a_{ij} \in T'$ we see that x_j is in the image of this map. Thus $S' \to S$ is surjective. Therefore S is finite over T' as well. Since T' is Noetherian and we conclude that $T \subset S$ is finite over T' and we win. $\square$

10.52. Length

00IU

02LY Definition 10.52.1. Let R be a ring. For any R-module M we define the length of M over R by the formula

$$\text{length}_R(M) = \sup\{n \mid \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words it is the supremum of the lengths of chains of submodules. There is an obvious notion of when a chain of submodules is a refinement of another. This gives a partial ordering on the collection of all chains of submodules, with the smallest chain having the shape $0 = M_0 \subset M_1 = M$ if M is not zero. We note the obvious fact that if the length of M is finite, then every chain can be refined to a maximal chain. But it is not as obvious that all maximal chains have the same length (as we will see later).

02LZ Lemma 10.52.2. Let R be a ring. Let M be an R-module. If $\text{length}_R(M) < \infty$ then M is a finite R-module.

Proof. Omitted. $\square$

00IV Lemma 10.52.3. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over R then the length of M is the sum of the lengths of M' and M''.

Proof. Given filtrations of M' and M'' of lengths n', n'' it is easy to make a corresponding filtration of M of length $n' + n''$. Thus we see that $\text{length}_R M \geq \text{length}_R M' + \text{length}_R M''$. Conversely, given a filtration $M_0 \subset M_1 \subset \ldots \subset M_n$ of M consider the induced filtrations $M'_i = M_i \cap M'$ and $M''_i = \text{Im}(M_i \to M'')$. Let n' (resp. n'') be the number of steps in the filtration $\{M'_i\}$ (resp. $\{M''_i\}$). If $M'_i = M'_{i+1}$ and $M''_i = M''_{i+1}$ then $M_i = M_{i+1}$. Hence we conclude that $n' + n'' \geq n$. Combined with the earlier result we win. $\square$

00IW Lemma 10.52.4. Let R be a local ring with maximal ideal $\mathfrak{m}$. If M is an R-module and $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$, then $\text{length}_R(M) = \infty$. In other words, if M has finite length then $\mathfrak{m}^n M = 0$ for some n.

Proof. Assume $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$. Choose $x \in M$ and $f_1, \ldots, f_n \in \mathfrak{m}$ such that $f_1 f_2 \ldots f_n x \neq 0$. The first n steps in the filtration

$$0 \subset R f_1 \ldots f_n x \subset R f_1 \ldots f_{n-1} x \subset \ldots \subset R x \subset M$$

are distinct. For example, if $R f_1 x = R f_1 f_2 x$, then $f_1 x = g f_1 f_2 x$ for some g, hence $(1 - g f_2) f_1 x = 0$ hence $f_1 x = 0$ as $1 - g f_2$ is a unit which is a contradiction with the choice of x and $f_1, \ldots, f_n$. Hence the length is infinite. $\square$

00IX Lemma 10.52.5. Let $R \to S$ be a ring map. Let M be an S-module. We always have $\text{length}_R(M) \geq \text{length}_S(M)$. If $R \to S$ is surjective then equality holds.

Proof. A filtration of M by S-submodules gives rise a filtration of M by R-submodules. This proves the inequality. And if $R \to S$ is surjective, then any R-submodule of M is automatically an S-submodule. Hence equality in this case. $\square$

00IY Lemma 10.52.6. Let R be a ring with maximal ideal $\mathfrak{m}$. Suppose that M is an R-module with $\mathfrak{m}M = 0$. Then the length of M as an R-module agrees with the dimension of M as a $R/\mathfrak{m}$ vector space. The length is finite if and only if M is a finite R-module.

Proof. The first part is a special case of Lemma 10.52.5. Thus the length is finite if and only if M has a finite basis as a $R/\mathfrak{m}$-vector space if and only if M has a finite set of generators as an R-module. $\square$

00IZ **Lemma 10.52.7.** Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Then $\text{length}_R(M) \geq \text{length}_{S^{-1}R}(S^{-1}M)$.

Proof. Any submodule $N' \subset S^{-1}M$ is of the form $S^{-1}N$ for some R-submodule $N \subset M$, by Lemma 10.9.15. The lemma follows. $\square$

00J0 **Lemma 10.52.8.** Let R be a ring with finitely generated maximal ideal $\mathfrak{m}$. (For example R Noetherian.) Suppose that M is a finite R-module with $\mathfrak{m}^n M = 0$ for some n. Then $\text{length}_R(M) < \infty$.

Proof. Consider the filtration $0 = \mathfrak{m}^n M \subset \mathfrak{m}^{n-1}M \subset \ldots \subset \mathfrak{m}M \subset M$. All of the subquotients are finitely generated R-modules to which Lemma 10.52.6 applies. We conclude by additivity, see Lemma 10.52.3. $\square$

00J1 **Definition 10.52.9.** Let R be a ring. Let M be an R-module. We say M is *simple* if $M \neq 0$ and every submodule of M is either equal to M or to 0.

00J2 **Lemma 10.52.10.** Let R be a ring. Let M be an R-module. The following are equivalent:

(1) M is simple,
(2) $\text{length}_R(M) = 1$, and
(3) $M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. By Lemma 10.52.6 the module $R/\mathfrak{m}$ has length 1. The equivalence of the first two assertions is tautological. Suppose that M is simple. Choose $x \in M$, $x \neq 0$. As M is simple we have $M = R \cdot x$. Let $I \subset R$ be the annihilator of x, i.e., $I = \{f \in R \mid fx = 0\}$. The map $R/I \to M$, $f \bmod I \mapsto fx$ is an isomorphism, hence R/I is a simple R-module. Since $R/I \neq 0$ we see $I \neq R$. Let $I \subset \mathfrak{m}$ be a maximal ideal containing I. If $I \neq \mathfrak{m}$, then $\mathfrak{m}/I \subset R/I$ is a nontrivial submodule contradicting the simplicity of R/I. Hence we see $I = \mathfrak{m}$ as desired. $\square$

00J3 **Lemma 10.52.11.** Let R be a ring. Let M be a finite length R-module. Choose any maximal chain of submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with $M_i \neq M_{i-1}$, $i = 1, \ldots, n$. Then

(1) $n = \text{length}_R(M)$,
(2) each M_i/M_{i-1} is simple,
(3) each M_i/M_{i-1} is of the form $R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$,
(4) given a maximal ideal $\mathfrak{m} \subset R$ we have
$$\#\{i \mid \mathfrak{m}_i = \mathfrak{m}\} = \text{length}_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. If M_i/M_{i-1} is not simple then we can refine the filtration and the filtration is not maximal. Thus we see that M_i/M_{i-1} is simple. By Lemma 10.52.10 the modules M_i/M_{i-1} have length 1 and are of the form $R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$. By additivity of length, Lemma 10.52.3, we see $n = \text{length}_R(M)$. Since localization is exact, we see that

$$0 = (M_0)_\mathfrak{m} \subset (M_1)_\mathfrak{m} \subset (M_2)_\mathfrak{m} \subset \ldots \subset (M_n)_\mathfrak{m} = M_\mathfrak{m}$$

is a filtration of $M_{\mathfrak m}$ with successive quotients $(M_i/M_{i-1})_{\mathfrak m}$. Thus the last statement follows directly from the fact that given maximal ideals $\mathfrak m$, $\mathfrak m'$ of R we have

$$(R/\mathfrak m')_{\mathfrak m} \cong \begin{cases} 0 & \text{if } \mathfrak m \neq \mathfrak m', \\ R_{\mathfrak m}/\mathfrak m R_{\mathfrak m} & \text{if } \mathfrak m = \mathfrak m' \end{cases}$$

This we leave to the reader. □

02M0 Lemma 10.52.12. Let A be a local ring with maximal ideal $\mathfrak m$. Let B be a semi-local ring with maximal ideals $\mathfrak m_i$, $i = 1, \ldots, n$. Suppose that $A \to B$ is a homomorphism such that each $\mathfrak m_i$ lies over $\mathfrak m$ and such that

$$[\kappa(\mathfrak m_i) : \kappa(\mathfrak m)] < \infty.$$

Let M be a B-module of finite length. Then

$$\text{length}_A(M) = \sum\nolimits_{i=1,\ldots,n} [\kappa(\mathfrak m_i) : \kappa(\mathfrak m)] \text{length}_{B_{\mathfrak m_i}}(M_{\mathfrak m_i}),$$

in particular $\text{length}_A(M) < \infty$.

Proof. Choose a maximal chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M$$

by B-submodules as in Lemma 10.52.11. Then each quotient M_j/M_{j-1} is isomorphic to $\kappa(\mathfrak m_{i(j)})$ for some $i(j) \in \{1, \ldots, n\}$. Moreover $\text{length}_A(\kappa(\mathfrak m_i)) = [\kappa(\mathfrak m_i) : \kappa(\mathfrak m)]$ by Lemma 10.52.6. The lemma follows by additivity of lengths (Lemma 10.52.3). □

02M1 Lemma 10.52.13. Let $A \to B$ be a flat local homomorphism of local rings. Then for any A-module M we have

$$\text{length}_A(M)\text{length}_B(B/\mathfrak m_A B) = \text{length}_B(M \otimes_A B).$$

In particular, if $\text{length}_B(B/\mathfrak m_A B) < \infty$ then M has finite length if and only if $M \otimes_A B$ has finite length.

Proof. The ring map $A \to B$ is faithfully flat by Lemma 10.39.17. Hence if $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a chain of length n in M, then the corresponding chain $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \ldots \subset M_n \otimes_A B = M \otimes_A B$ has length n also. This proves $\text{length}_A(M) = \infty \Rightarrow \text{length}_B(M \otimes_A B) = \infty$. Next, assume $\text{length}_A(M) < \infty$. In this case we see that M has a filtration of length $\ell = \text{length}_A(M)$ whose quotients are $A/\mathfrak m_A$. Arguing as above we see that $M \otimes_A B$ has a filtration of length ℓ whose quotients are isomorphic to $B \otimes_A A/\mathfrak m_A = B/\mathfrak m_A B$. Thus the lemma follows. □

02M2 Lemma 10.52.14. Let $A \to B \to C$ be flat local homomorphisms of local rings. Then

$$\text{length}_B(B/\mathfrak m_A B)\text{length}_C(C/\mathfrak m_B C) = \text{length}_C(C/\mathfrak m_A C)$$

Proof. Follows from Lemma 10.52.13 applied to the ring map $B \to C$ and the B-module $M = B/\mathfrak m_A B$ □

10.53. Artinian rings

00J4 Artinian rings, and especially local Artinian rings, play an important role in algebraic geometry, for example in deformation theory.

00J5 **Definition 10.53.1.** A ring R is *Artinian* if it satisfies the descending chain condition for ideals.

00J6 **Lemma 10.53.2.** *Suppose R is a finite dimensional algebra over a field. Then R is Artinian.*

Proof. The descending chain condition for ideals obviously holds. $\square$

00J7 **Lemma 10.53.3.** *If R is Artinian then R has only finitely many maximal ideals.*

Proof. Suppose that $\mathfrak m_i$, $i = 1, 2, 3, \ldots$ are pairwise distinct maximal ideals. Then $\mathfrak m_1 \supset \mathfrak m_1 \cap \mathfrak m_2 \supset \mathfrak m_1 \cap \mathfrak m_2 \cap \mathfrak m_3 \supset \ldots$ is an infinite descending sequence (because by the Chinese remainder theorem all the maps $R \to \oplus_{i=1}^n R/\mathfrak m_i$ are surjective). $\square$

00J8 **Lemma 10.53.4.** *Let R be Artinian. The Jacobson radical of R is a nilpotent ideal.*

Proof. Let $I \subset R$ be the Jacobson radical. Note that $I \supset I^2 \supset I^3 \supset \ldots$ is a descending sequence. Thus $I^n = I^{n+1}$ for some n. Set $J = \{x \in R \mid xI^n = 0\}$. We have to show $J = R$. If not, choose an ideal $J' \neq J$, $J \subset J'$ minimal (possible by the Artinian property). Then $J' = J + Rx$ for some $x \in R$. By NAK, Lemma 10.20.1, we have $IJ' \subset J$. Hence $xI^{n+1} \subset xI \cdot I^n \subset J \cdot I^n = 0$. Since $I^{n+1} = I^n$ we conclude $x \in J$. Contradiction. $\square$

00JA **Lemma 10.53.5.** *Any ring with finitely many maximal ideals and locally nilpotent Jacobson radical is the product of its localizations at its maximal ideals. Also, all primes are maximal.*

Proof. Let R be a ring with finitely many maximal ideals $\mathfrak m_1, \ldots, \mathfrak m_n$. Let $I = \bigcap_{i=1}^n \mathfrak m_i$ be the Jacobson radical of R. Assume I is locally nilpotent. Let $\mathfrak p$ be a prime ideal of R. Since every prime contains every nilpotent element of R we see $\mathfrak p \supset \mathfrak m_1 \cap \ldots \cap \mathfrak m_n$. Since $\mathfrak m_1 \cap \ldots \cap \mathfrak m_n \supset \mathfrak m_1 \ldots \mathfrak m_n$ we conclude $\mathfrak p \supset \mathfrak m_1 \ldots \mathfrak m_n$. Hence $\mathfrak p \supset \mathfrak m_i$ for some i, and so $\mathfrak p = \mathfrak m_i$. By the Chinese remainder theorem (Lemma 10.15.4) we have $R/I \cong \bigoplus R/\mathfrak m_i$ which is a product of fields. Hence by Lemma 10.32.6 there are idempotents e_i, $i = 1, \ldots, n$ with $e_i \bmod \mathfrak m_j = \delta_{ij}$. Hence $R = \prod Re_i$, and each Re_i is a ring with exactly one maximal ideal. $\square$

00JB **Lemma 10.53.6.** *A ring R is Artinian if and only if it has finite length as a module over itself. Any such ring R is both Artinian and Noetherian, any prime ideal of R is a maximal ideal, and R is equal to the (finite) product of its localizations at its maximal ideals.*

Proof. If R has finite length over itself then it satisfies both the ascending chain condition and the descending chain condition for ideals. Hence it is both Noetherian and Artinian. Any Artinian ring is equal to product of its localizations at maximal ideals by Lemmas 10.53.3, 10.53.4, and 10.53.5.

Suppose that R is Artinian. We will show R has finite length over itself. It suffices to exhibit a chain of submodules whose successive quotients have finite length. By what we said above we may assume that R is local, with maximal ideal $\mathfrak m$. By Lemma 10.53.4 we have $\mathfrak m^n = 0$ for some n. Consider the sequence $0 = \mathfrak m^n \subset$

$\mathfrak{m}^{n-1} \subset \ldots \subset \mathfrak{m} \subset R$. By Lemma 10.52.6 the length of each subquotient $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is the dimension of this as a vector space over $\kappa(\mathfrak{m})$. This has to be finite since otherwise we would have an infinite descending chain of sub vector spaces which would correspond to an infinite descending chain of ideals in R. □

10.54. Homomorphisms essentially of finite type

07DR Some simple remarks on localizations of finite type ring maps.

00QM **Definition 10.54.1.** Let $R \to S$ be a ring map.

1. We say that $R \to S$ is *essentially of finite type* if S is the localization of an R-algebra of finite type.
2. We say that $R \to S$ is *essentially of finite presentation* if S is the localization of an R-algebra of finite presentation.

07DS **Lemma 10.54.2.** *The class of ring maps which are essentially of finite type is preserved under composition. Similarly for essentially of finite presentation.*

Proof. Omitted. □

0AUF **Lemma 10.54.3.** *The class of ring maps which are essentially of finite type is preserved by base change. Similarly for essentially of finite presentation.*

Proof. Omitted. □

07DT **Lemma 10.54.4.** *Let $R \to S$ be a ring map. Assume S is an Artinian local ring with maximal ideal $\mathfrak{m}$. Then*

1. *$R \to S$ is finite if and only if $R \to S/\mathfrak{m}$ is finite,*
2. *$R \to S$ is of finite type if and only if $R \to S/\mathfrak{m}$ is of finite type.*
3. *$R \to S$ is essentially of finite type if and only if the composition $R \to S/\mathfrak{m}$ is essentially of finite type.*

Proof. If $R \to S$ is finite, then $R \to S/\mathfrak{m}$ is finite by Lemma 10.7.3. Conversely, assume $R \to S/\mathfrak{m}$ is finite. As S has finite length over itself (Lemma 10.53.6) we can choose a filtration

$$0 \subset I_1 \subset \ldots \subset I_n = S$$

by ideals such that $I_i/I_{i-1} \cong S/\mathfrak{m}$ as S-modules. Thus S has a filtration by R-submodules I_i such that each successive quotient is a finite R-module. Thus S is a finite R-module by Lemma 10.5.3.

If $R \to S$ is of finite type, then $R \to S/\mathfrak{m}$ is of finite type by Lemma 10.6.2. Conversely, assume that $R \to S/\mathfrak{m}$ is of finite type. Choose $f_1, \ldots, f_n \in S$ which map to generators of $S/\mathfrak{m}$. Then $A = R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ is a ring map such that $A \to S/\mathfrak{m}$ is surjective (in particular finite). Hence $A \to S$ is finite by part (1) and we see that $R \to S$ is of finite type by Lemma 10.6.2.

If $R \to S$ is essentially of finite type, then $R \to S/\mathfrak{m}$ is essentially of finite type by Lemma 10.54.2. Conversely, assume that $R \to S/\mathfrak{m}$ is essentially of finite type. Suppose $S/\mathfrak{m}$ is the localization of $R[x_1, \ldots, x_n]/I$. Choose $f_1, \ldots, f_n \in S$ whose congruence classes modulo $\mathfrak{m}$ correspond to the congruence classes of $x_1, \ldots, x_n$ modulo I. Consider the map $R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ with kernel J. Set $A = R[x_1, \ldots, x_n]/J \subset S$ and $\mathfrak{p} = A \cap \mathfrak{m}$. Note that $A/\mathfrak{p} \subset S/\mathfrak{m}$ is equal to the image of $R[x_1, \ldots, x_n]/I$ in $S/\mathfrak{m}$. Hence $\kappa(\mathfrak{p}) = S/\mathfrak{m}$. Thus $A_\mathfrak{p} \to S$ is finite by part (1). We conclude that S is essentially of finite type by Lemma 10.54.2. □

The following lemma can be proven using properness of projective space instead of the algebraic argument we give here.

0AUG Lemma 10.54.5. Let $\varphi : R \to S$ be essentially of finite type with R and S local (but not necessarily φ local). Then there exists an n and a maximal ideal $\mathfrak m \subset R[x_1, \ldots, x_n]$ lying over $\mathfrak m_R$ such that S is a localization of a quotient of $R[x_1, \ldots, x_n]_{\mathfrak m}$.

Proof. We can write S as a localization of a quotient of $R[x_1, \ldots, x_n]$. Hence it suffices to prove the lemma in case $S = R[x_1, \ldots, x_n]_{\mathfrak q}$ for some prime $\mathfrak q \subset R[x_1, \ldots, x_n]$. If $\mathfrak q + \mathfrak m_R R[x_1, \ldots, x_n] \neq R[x_1, \ldots, x_n]$ then we can find a maximal ideal $\mathfrak m$ as in the statement of the lemma with $\mathfrak q \subset \mathfrak m$ and the result is clear.

Choose a valuation ring $A \subset \kappa(\mathfrak q)$ which dominates the image of $R \to \kappa(\mathfrak q)$ (Lemma 10.50.2). If the image $\lambda_i \in \kappa(\mathfrak q)$ of x_i is contained in A, then $\mathfrak q$ is contained in the inverse image of $\mathfrak m_A$ via $R[x_1, \ldots, x_n] \to A$ which means we are back in the preceding case. Hence there exists an i such that $\lambda_i^{-1} \in A$ and such that $\lambda_j/\lambda_i \in A$ for all $j = 1, \ldots, n$ (because the value group of A is totally ordered, see Lemma 10.50.12). Then we consider the map

$$R[y_0, y_1, \ldots, \hat{y}_i, \ldots, y_n] \to R[x_1, \ldots, x_n]_{\mathfrak q}, \quad y_0 \mapsto 1/x_i, \quad y_j \mapsto x_j/x_i$$

Let $\mathfrak q' \subset R[y_0, \ldots, \hat{y}_i, \ldots, y_n]$ be the inverse image of $\mathfrak q$. Since $y_0 \not\in \mathfrak q'$ it is easy to see that the displayed arrow defines an isomorphism on localizations. On the other hand, the result of the first paragraph applies to $R[y_0, \ldots, \hat{y}_i, \ldots, y_n]$ because y_j maps to an element of A. This finishes the proof. $\square$

10.55. K-groups

00JC Let R be a ring. We will introduce two abelian groups associated to R. The first of the two is denoted $K'_0(R)$ and has the following properties[6]:

(1) For every finite R-module M there is given an element $[M]$ in $K'_0(R)$,
(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite R-modules we have the relation $[M] = [M'] + [M'']$,
(3) the group $K'_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K'_0(R)$ among the generators $[M]$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The actual construction is a bit more annoying since one has to take care that the collection of all finitely generated R-modules is a proper class. However, this problem can be overcome by taking as set of generators of the group $K'_0(R)$ the elements $[R^n/K]$ where n ranges over all integers and K ranges over all submodules $K \subset R^n$. The generators for the subgroup of relations imposed on these elements will be the relations coming from short exact sequences whose terms are of the form R^n/K. The element $[M]$ is defined by choosing n and K such that $M \cong R^n/K$ and putting $[M] = [R^n/K]$. Details left to the reader.

00JD Lemma 10.55.1. If R is an Artinian local ring then the length function defines a natural abelian group homomorphism $\text{length}_R : K'_0(R) \to \mathbf{Z}$.

Proof. The length of any finite R-module is finite, because it is the quotient of R^n which has finite length by Lemma 10.53.6. And the length function is additive, see Lemma 10.52.3. $\square$

[6]The definition makes sense for any ring but is rarely used unless R is Noetherian.

The second of the two is denoted $K_0(R)$ and has the following properties:

(1) For every finite projective R-module M there is given an element $[M]$ in $K_0(R)$,
(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite projective R-modules we have the relation $[M] = [M'] + [M'']$,
(3) the group $K_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K_0(R)$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The construction of this group is done as above.

We note that there is an obvious map $K_0(R) \to K'_0(R)$ which is not an isomorphism in general.

00JE Example 10.55.2. Note that if $R = k$ is a field then we clearly have $K_0(k) = K'_0(k) \cong \mathbf{Z}$ with the isomorphism given by the dimension function (which is also the length function).

0FJ8 Example 10.55.3. Let R be a PID. We claim $K_0(R) = K'_0(R) = \mathbf{Z}$. Namely, any finite projective R-module is finite free. A finite free module has a well defined rank by Lemma 10.15.8. Given a short exact sequence of finite free modules

$$0 \to M' \to M \to M'' \to 0$$

we have $\text{rank}(M) = \text{rank}(M') + \text{rank}(M'')$ because we have $M \cong M' \oplus M'$ in this case (for example we have a splitting by Lemma 10.5.2). We conclude $K_0(R) = \mathbf{Z}$.

The structure theorem for modules of a PID says that any finitely generated R-module is of the form $M = R^{\oplus r} \oplus R/(d_1) \oplus \ldots \oplus R/(d_k)$. Consider the short exact sequence

$$0 \to (d_i) \to R \to R/(d_i) \to 0$$

Since the ideal (d_i) is isomorphic to R as a module (it is free with generator d_i), in $K'_0(R)$ we have $[(d_i)] = [R]$. Then $[R/(d_i)] = [(d_i)] - [R] = 0$. From this it follows that a torsion module has zero class in $K'_0(R)$. Using the rank of the free part gives an identification $K'_0(R) = \mathbf{Z}$ and the canonical homomorphism from $K_0(R) \to K'_0(R)$ is an isomorphism.

00JF Example 10.55.4. Let k be a field. Then $K_0(k[x]) = K'_0(k[x]) = \mathbf{Z}$. This follows from Example 10.55.3 as $R = k[x]$ is a PID.

00JG Example 10.55.5. Let k be a field. Let $R = \{f \in k[x] \mid f(0) = f(1)\}$, compare Example 10.27.4. In this case $K_0(R) \cong k^* \oplus \mathbf{Z}$, but $K'_0(R) = \mathbf{Z}$.

00JH Lemma 10.55.6. Let $R = R_1 \times R_2$. Then $K_0(R) = K_0(R_1) \times K_0(R_2)$ and $K'_0(R) = K'_0(R_1) \times K'_0(R_2)$

Proof. Omitted. □

00JI Lemma 10.55.7. Let R be an Artinian local ring. The map $\text{length}_R : K'_0(R) \to \mathbf{Z}$ of Lemma 10.55.1 is an isomorphism.

Proof. Omitted. □

00JJ Lemma 10.55.8. Let $(R, \mathfrak{m})$ be a local ring. Every finite projective R-module is finite free. The map $\text{rank}_R : K_0(R) \to \mathbf{Z}$ defined by $[M] \to \text{rank}_R(M)$ is well defined and an isomorphism.

Proof. Let P be a finite projective R-module. Choose elements $x_1, \ldots, x_n \in P$ which map to a basis of $P/\mathfrak{m}P$. By Nakayama's Lemma 10.20.1 these elements generate P. The corresponding surjection $u : R^{\oplus n} \to P$ has a splitting as P is projective. Hence $R^{\oplus n} = P \oplus Q$ with $Q = \text{Ker}(u)$. It follows that $Q/\mathfrak{m}Q = 0$, hence Q is zero by Nakayama's lemma. In this way we see that every finite projective R-module is finite free. A finite free module has a well defined rank by Lemma 10.15.8. Given a short exact sequence of finite free R-modules

$$0 \to M' \to M \to M'' \to 0$$

we have $\text{rank}(M) = \text{rank}(M') + \text{rank}(M'')$ because we have $M \cong M' \oplus M'$ in this case (for example we have a splitting by Lemma 10.5.2). We conclude $K_0(R) = \mathbf{Z}$. $\square$

00JK **Lemma 10.55.9.** Let R be a local Artinian ring. There is a commutative diagram

$$\begin{tikzcd} K_0(R) \arrow[r] \arrow[d, "\text{rank}_R"'] & K'_0(R) \arrow[d, "\text{length}_R"] \\ \mathbf{Z} \arrow[r, "\text{length}_R(R)"] & \mathbf{Z} \end{tikzcd}$$

where the vertical maps are isomorphisms by Lemmas 10.55.7 and 10.55.8.

Proof. Let P be a finite projective R-module. We have to show that $\text{length}_R(P) = \text{rank}_R(P)\text{length}_R(R)$. By Lemma 10.55.8 the module P is finite free. So $P \cong R^{\oplus n}$ for some $n \geq 0$. Then $\text{rank}_R(P) = n$ and $\text{length}_R(R^{\oplus n}) = n\text{length}_R(R)$ by additivity of lenghts (Lemma 10.52.3). Thus the result holds. $\square$

10.56. Graded rings

00JL A *graded ring* will be for us a ring S endowed with a direct sum decomposition $S = \bigoplus_{d \geq 0} S_d$ of the underlying abelian group such that $S_d \cdot S_e \subset S_{d+e}$. Note that we do not allow nonzero elements in negative degrees. The *irrelevant ideal* is the ideal $S_+ = \bigoplus_{d > 0} S_d$. A *graded module* will be an S-module M endowed with a direct sum decomposition $M = \bigoplus_{n \in \mathbf{Z}} M_n$ of the underlying abelian group such that $S_d \cdot M_e \subset M_{d+e}$. Note that for modules we do allow nonzero elements in negative degrees. We think of S as a graded S-module by setting $S_{-k} = (0)$ for $k > 0$. An element x (resp. f) of M (resp. S) is called *homogeneous* if $x \in M_d$ (resp. $f \in S_d$) for some d. A *map of graded S-modules* is a map of S-modules $\varphi : M \to M'$ such that $\varphi(M_d) \subset M'_d$. We do not allow maps to shift degrees. Let us denote $\text{GrHom}_0(M, N)$ the S_0-module of homomorphisms of graded modules from M to N.

At this point there are the notions of graded ideal, graded quotient ring, graded submodule, graded quotient module, graded tensor product, etc. We leave it to the reader to find the relevant definitions, and lemmas. For example: A short exact sequence of graded modules is short exact in every degree.

Given a graded ring S, a graded S-module M and $n \in \mathbf{Z}$ we denote $M(n)$ the graded S-module with $M(n)_d = M_{n+d}$. This is called the *twist of M by n*. In particular we get modules $S(n)$, $n \in \mathbf{Z}$ which will play an important role in the study of projective schemes. There are some obvious functorial isomorphisms such as $(M \oplus N)(n) = M(n) \oplus N(n)$, $(M \otimes_S N)(n) = M \otimes_S N(n) = M(n) \otimes_S N$. In addition we can define a graded S-module structure on the S_0-module

$$\mathrm{GrHom}(M, N) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{GrHom}_n(M, N), \quad \mathrm{GrHom}_n(M, N) = \mathrm{GrHom}_0(M, N(n)).$$

We omit the definition of the multiplication.

0EKB Lemma 10.56.1. Let S be a graded ring. Let M be a graded S-module.

(1) If $S_+M = M$ and M is finite, then $M = 0$.
(2) If $N, N' \subset M$ are graded submodules, $M = N + S_+N'$, and N' is finite, then $M = N$.
(3) If $N \to M$ is a map of graded modules, $N/S_+N \to M/S_+M$ is surjective, and M is finite, then $N \to M$ is surjective.
(4) If $x_1, \ldots, x_n \in M$ are homogeneous and generate M/S_+M and M is finite, then $x_1, \ldots, x_n$ generate M.

Proof. Proof of (1). Choose generators $y_1, \ldots, y_r$ of M over S. We may assume that y_i is homogeneous of degree d_i. After renumbering we may assume $d_r = \min(d_i)$. Then the condition that $S_+M = M$ implies $y_r = 0$. Hence $M = 0$ by induction on r. Part (2) follows by applying (1) to M/N. Part (3) follows by applying (2) to the submodules $\mathrm{Im}(N \to M)$ and M. Part (4) follows by applying (3) to the module map $\bigoplus S(-d_i) \to M$, $(s_1, \ldots, s_n) \mapsto \sum s_i x_i$. $\square$

Let S be a graded ring. Let $d \geq 1$ be an integer. We set $S^{(d)} = \bigoplus_{n \geq 0} S_{nd}$. We think of $S^{(d)}$ as a graded ring with degree n summand $(S^{(d)})_n = S_{nd}$. Given a graded S-module M we can similarly consider $M^{(d)} = \bigoplus_{n \in \mathbf{Z}} M_{nd}$ which is a graded $S^{(d)}$-module.

0EGH Lemma 10.56.2. Let S be a graded ring, which is finitely generated over S_0. Then for all sufficiently divisible d the algebra $S^{(d)}$ is generated in degree 1 over S_0.

Proof. Say S is generated by $f_1, \ldots, f_r \in S$ over S_0. After replacing f_i by their homogeneous parts, we may assume f_i is homogeneous of degree $d_i > 0$. Then any element of S_n is a linear combination with coefficients in S_0 of monomials $f_1^{e_1} \ldots f_r^{e_r}$ with $\sum e_i d_i = n$. Let m be a multiple of $\mathrm{lcm}(d_i)$. For any $N \geq r$ if

$$\sum e_i d_i = Nm$$

then for some i we have $e_i \geq m/d_i$ by an elementary argument. Hence every monomial of degree Nm is a product of a monomial of degree m, namely f_i^{m/d_i}, and a monomial of degree $(N-1)m$. It follows that any monomial of degree nrm with $n \geq 2$ is a product of monomials of degree rm. Thus $S^{(rm)}$ is generated in degree 1 over S_0. $\square$

077G Lemma 10.56.3. Let $R \to S$ be a homomorphism of graded rings. Let $S' \subset S$ be the integral closure of R in S. Then

$$S' = \bigoplus\nolimits_{d \geq 0} S' \cap S_d,$$

i.e., S' is a graded R-subalgebra of S.

Proof. We have to show the following: If $s = s_n + s_{n+1} + \ldots + s_m \in S'$, then each homogeneous part $s_j \in S'$. We will prove this by induction on $m - n$ over all homomorphisms $R \to S$ of graded rings. First note that it is immediate that s_0

is integral over R_0 (hence over R) as there is a ring map $S \to S_0$ compatible with the ring map $R \to R_0$. Thus, after replacing s by $s - s_0$, we may assume $n > 0$. Consider the extension of graded rings $R[t, t^{-1}] \to S[t, t^{-1}]$ where t has degree 0. There is a commutative diagram

$$\begin{tikzcd} S[t,t^{-1}] \arrow[r, "s \mapsto t^{\deg(s)} s"'] & S[t,t^{-1}] \\ R[t,t^{-1}] \arrow[u] \arrow[r, "r \mapsto t^{\deg(r)} r"] & R[t,t^{-1}] \arrow[u] \end{tikzcd}$$

where the horizontal maps are ring automorphisms. Hence the integral closure C of $S[t, t^{-1}]$ over $R[t, t^{-1}]$ maps into itself. Thus we see that

$$t^m(s_n + s_{n+1} + \ldots + s_m) - (t^n s_n + t^{n+1} s_{n+1} + \ldots + t^m s_m) \in C$$

which implies by induction hypothesis that each $(t^m - t^i)s_i \in C$ for $i = n, \ldots, m-1$. Note that for any ring A and $m > i \geq n > 0$ we have $A[t, t^{-1}]/(t^m - t^i - 1) \cong A[t]/(t^m - t^i - 1) \supset A$ because $t(t^{m-1} - t^{i-1}) = 1$ in $A[t]/(t^m - t^i - 1)$. Since $t^m - t^i$ maps to 1 we see the image of s_i in the ring $S[t]/(t^m - t^i - 1)$ is integral over $R[t]/(t^m - t^i - 1)$ for $i = n, \ldots, m-1$. Since $R \to R[t]/(t^m - t^i - 1)$ is finite we see that s_i is integral over R by transitivity, see Lemma 10.36.6. Finally, we also conclude that $s_m = s - \sum_{i=n,\ldots,m-1} s_i$ is integral over R. $\square$

10.57. Proj of a graded ring

00JM Let S be a graded ring. A homogeneous ideal is simply an ideal $I \subset S$ which is also a graded submodule of S. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogeneous parts in S then $f_i \in I$ for each i. To check that a homogeneous ideal $\mathfrak{p}$ is prime it suffices to check that if $ab \in \mathfrak{p}$ with a, b homogeneous then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

00JN **Definition 10.57.1.** Let S be a graded ring. We define $\text{Proj}(S)$ to be the set of homogeneous prime ideals $\mathfrak{p}$ of S such that $S_+ \not\subset \mathfrak{p}$. The set $\text{Proj}(S)$ is a subset of $\text{Spec}(S)$ and we endow it with the induced topology. The topological space $\text{Proj}(S)$ is called the homogeneous spectrum of the graded ring S.

Note that by construction there is a continuous map

$$\text{Proj}(S) \longrightarrow \text{Spec}(S_0).$$

Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. Let $f \in S_d$ and assume that $d \geq 1$. We define $S_{(f)}$ to be the subring of S_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$. If M is a graded S-module, then we define the $S_{(f)}$-module $M_{(f)}$ as the sub module of M_f consisting of elements of the form x/f^n with x homogeneous of degree nd.

00JO **Lemma 10.57.2.** Let S be a $\mathbf{Z}$-graded ring containing a homogeneous invertible element of positive degree. Then the set $G \subset \text{Spec}(S)$ of $\mathbf{Z}$-graded primes of S (with induced topology) maps homeomorphically to $\text{Spec}(S_0)$.

Proof. First we show that the map is a bijection by constructing an inverse. Let $f \in S_d$, $d > 0$ be invertible in S. If $\mathfrak p_0$ is a prime of S_0, then $\mathfrak p_0 S$ is a $\mathbf{Z}$-graded ideal of S such that $\mathfrak p_0 S \cap S_0 = \mathfrak p_0$. And if $ab \in \mathfrak p_0 S$ with a, b homogeneous, then $a^d b^d / f^{\deg(a) + \deg(b)} \in \mathfrak p_0$. Thus either $a^d/f^{\deg(a)} \in \mathfrak p_0$ or $b^d/f^{\deg(b)} \in \mathfrak p_0$, in other words either $a^d \in \mathfrak p_0 S$ or $b^d \in \mathfrak p_0 S$. It follows that $\sqrt{\mathfrak p_0 S}$ is a $\mathbf{Z}$-graded prime ideal of S whose intersection with S_0 is $\mathfrak p_0$.

To show that the map is a homeomorphism we show that the image of $G \cap D(g)$ is open. If $g = \sum g_i$ with $g_i \in S_i$, then by the above $G \cap D(g)$ maps onto the set $\bigcup D(g_i^d/f^i)$ which is open. $\square$

For $f \in S$ homogeneous of degree > 0 we define

$$D_+(f) = \{\mathfrak p \in \text{Proj}(S) \mid f \not\in \mathfrak p\}.$$

Finally, for a homogeneous ideal $I \subset S$ we define

$$V_+(I) = \{\mathfrak p \in \text{Proj}(S) \mid I \subset \mathfrak p\}.$$

We will use more generally the notation $V_+(E)$ for any set E of homogeneous elements $E \subset S$.

00JP **Lemma 10.57.3** (Topology on Proj). Let $S = \oplus_{d \geq 0} S_d$ be a graded ring.

(1) The sets $D_+(f)$ are open in $\text{Proj}(S)$.
(2) We have $D_+(ff') = D_+(f) \cap D_+(f')$.
(3) Let $g = g_0 + \ldots + g_m$ be an element of S with $g_i \in S_i$. Then
$$D(g) \cap \text{Proj}(S) = (D(g_0) \cap \text{Proj}(S)) \cup \bigcup\nolimits_{i \geq 1} D_+(g_i).$$
(4) Let $g_0 \in S_0$ be a homogeneous element of degree 0. Then
$$D(g_0) \cap \text{Proj}(S) = \bigcup\nolimits_{f \in S_d,\ d \geq 1} D_+(g_0 f).$$
(5) The open sets $D_+(f)$ form a basis for the topology of $\text{Proj}(S)$.
(6) Let $f \in S$ be homogeneous of positive degree. The ring S_f has a natural $\mathbf{Z}$-grading. The ring maps $S \to S_f \leftarrow S_{(f)}$ induce homeomorphisms
$$D_+(f) \leftarrow \{\mathbf{Z}\text{-graded primes of } S_f\} \to \text{Spec}(S_{(f)}).$$
(7) There exists an S such that $\text{Proj}(S)$ is not quasi-compact.
(8) The sets $V_+(I)$ are closed.
(9) Any closed subset $T \subset \text{Proj}(S)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset S$.
(10) For any graded ideal $I \subset S$ we have $V_+(I) = \emptyset$ if and only if $S_+ \subset \sqrt{I}$.

Proof. Since $D_+(f) = \text{Proj}(S) \cap D(f)$, these sets are open. This proves (1). Also (2) follows as $D(ff') = D(f) \cap D(f')$. Similarly the sets $V_+(I) = \text{Proj}(S) \cap V(I)$ are closed. This proves (8).

Suppose that $T \subset \text{Proj}(S)$ is closed. Then we can write $T = \text{Proj}(S) \cap V(J)$ for some ideal $J \subset S$. By definition of a homogeneous ideal if $g \in J$, $g = g_0 + \ldots + g_m$ with $g_d \in S_d$ then $g_d \in \mathfrak p$ for all $\mathfrak p \in T$. Thus, letting $I \subset S$ be the ideal generated by the homogeneous parts of the elements of J we have $T = V_+(I)$. This proves (9).

The formula for $\text{Proj}(S) \cap D(g)$, with $g \in S$ is direct from the definitions. This proves (3). Consider the formula for $\text{Proj}(S) \cap D(g_0)$. The inclusion of the right hand side in the left hand side is obvious. For the other inclusion, suppose $g_0 \not\in \mathfrak p$ with $\mathfrak p \in \text{Proj}(S)$. If all $g_0 f \in \mathfrak p$ for all homogeneous f of positive degree, then we see that $S_+ \subset \mathfrak p$ which is a contradiction. This gives the other inclusion. This proves (4).

The collection of opens $D(g) \cap \text{Proj}(S)$ forms a basis for the topology since the standard opens $D(g) \subset \text{Spec}(S)$ form a basis for the topology on $\text{Spec}(S)$. By the formulas above we can express $D(g) \cap \text{Proj}(S)$ as a union of opens $D_+(f)$. Hence the collection of opens $D_+(f)$ forms a basis for the topology also. This proves (5).

Proof of (6). First we note that $D_+(f)$ may be identified with a subset (with induced topology) of $D(f) = \text{Spec}(S_f)$ via Lemma 10.17.6. Note that the ring S_f has a $\mathbf{Z}$-grading. The homogeneous elements are of the form r/f^n with $r \in S$ homogeneous and have degree $\deg(r/f^n) = \deg(r) - n\deg(f)$. The subset $D_+(f)$ corresponds exactly to those prime ideals $\mathfrak p \subset S_f$ which are $\mathbf{Z}$-graded ideals (i.e., generated by homogeneous elements). Hence we have to show that the set of $\mathbf{Z}$-graded prime ideals of S_f maps homeomorphically to $\text{Spec}(S_{(f)})$. This follows from Lemma 10.57.2.

Let $S = \mathbf{Z}[X_1, X_2, X_3, \ldots]$ with grading such that each X_i has degree 1. Then it is easy to see that

$$\text{Proj}(S) = \bigcup\nolimits_{i=1}^{\infty} D_+(X_i)$$

does not have a finite refinement. This proves (7).

Let $I \subset S$ be a graded ideal. If $\sqrt{I} \supset S_+$ then $V_+(I) = \emptyset$ since every prime $\mathfrak p \in \text{Proj}(S)$ does not contain S_+ by definition. Conversely, suppose that $S_+ \not\subset \sqrt{I}$. Then we can find an element $f \in S_+$ such that f is not nilpotent modulo I. Clearly this means that one of the homogeneous parts of f is not nilpotent modulo I, in other words we may (and do) assume that f is homogeneous. This implies that $IS_f \neq S_f$, in other words that $(S/I)_f$ is not zero. Hence $(S/I)_{(f)} \neq 0$ since it is a ring which maps into $(S/I)_f$. Pick a prime $\mathfrak q \subset (S/I)_{(f)}$. This corresponds to a graded prime of S/I, not containing the irrelevant ideal $(S/I)_+$. And this in turn corresponds to a graded prime ideal $\mathfrak p$ of S, containing I but not containing S_+ as desired. This proves (10) and finishes the proof. □

00JQ Example 10.57.4. Let R be a ring. If $S = R[X]$ with $\deg(X) = 1$, then the natural map $\text{Proj}(S) \to \text{Spec}(R)$ is a bijection and in fact a homeomorphism. Namely, suppose $\mathfrak p \in \text{Proj}(S)$. Since $S_+ \not\subset \mathfrak p$ we see that $X \not\in \mathfrak p$. Thus if $aX^n \in \mathfrak p$ with $a \in R$ and $n > 0$, then $a \in \mathfrak p$. It follows that $\mathfrak p = \mathfrak p_0 S$ with $\mathfrak p_0 = \mathfrak p \cap R$.

If $\mathfrak p \in \text{Proj}(S)$, then we define $S_{(\mathfrak p)}$ to be the ring whose elements are fractions r/f where $r, f \in S$ are homogeneous elements of the same degree such that $f \not\in \mathfrak p$. As usual we say $r/f = r'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \not\in \mathfrak p$ such that $f''(rf' - r'f) = 0$. Given a graded S-module M we let $M_{(\mathfrak p)}$ be the $S_{(\mathfrak p)}$-module whose elements are fractions x/f with $x \in M$ and $f \in S$ homogeneous of the same degree such that $f \not\in \mathfrak p$. We say $x/f = x'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \not\in \mathfrak p$ such that $f''(xf' - x'f) = 0$.

00JR Lemma 10.57.5. Let S be a graded ring. Let M be a graded S-module. Let $\mathfrak p$ be an element of $\text{Proj}(S)$. Let $f \in S$ be a homogeneous element of positive degree such that $f \not\in \mathfrak p$, i.e., $\mathfrak p \in D_+(f)$. Let $\mathfrak p' \subset S_{(f)}$ be the element of $\text{Spec}(S_{(f)})$

corresponding to $\mathfrak{p}$ as in Lemma 10.57.3. Then $S_{(\mathfrak{p})} = (S_{(f)})_{\mathfrak{p}'}$ and compatibly $M_{(\mathfrak{p})} = (M_{(f)})_{\mathfrak{p}'}$.

Proof. We define a map $\psi : M_{(\mathfrak{p})} \to (M_{(f)})_{\mathfrak{p}'}$. Let $x/g \in M_{(\mathfrak{p})}$. We set

$$\psi(x/g) = (xg^{\deg(f)-1}/f^{\deg(x)})/(g^{\deg(f)}/f^{\deg(g)}).$$

This makes sense since $\deg(x) = \deg(g)$ and since $g^{\deg(f)}/f^{\deg(g)} \notin \mathfrak{p}'$. We omit the verification that ψ is well defined, a module map and an isomorphism. Hint: the inverse sends $(x/f^n)/(g/f^m)$ to $(xf^m)/(gf^n)$. $\square$

Here is a graded variant of Lemma 10.15.2.

00JS Lemma 10.57.6. Suppose S is a graded ring, $\mathfrak{p}_i$, $i = 1, \ldots, r$ homogeneous prime ideals and $I \subset S_+$ a graded ideal. Assume $I \not\subset \mathfrak{p}_i$ for all i. Then there exists a homogeneous element $x \in I$ of positive degree such that $x \notin \mathfrak{p}_i$ for all i.

Proof. We may assume there are no inclusions among the $\mathfrak{p}_i$. The result is true for $r = 1$. Suppose the result holds for $r - 1$. Pick $x \in I$ homogeneous of positive degree such that $x \notin \mathfrak{p}_i$ for all $i = 1, \ldots, r-1$. If $x \notin \mathfrak{p}_r$ we are done. So assume $x \in \mathfrak{p}_r$. If $I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1} \subset \mathfrak{p}_r$ then $I \subset \mathfrak{p}_r$ a contradiction. Pick $y \in I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1}$ homogeneous and $y \notin \mathfrak{p}_r$. Then $x^{\deg(y)} + y^{\deg(x)}$ works. $\square$

00JT Lemma 10.57.7. Let S be a graded ring. Let $\mathfrak{p} \subset S$ be a prime. Let $\mathfrak{q}$ be the homogeneous ideal of S generated by the homogeneous elements of $\mathfrak{p}$. Then $\mathfrak{q}$ is a prime ideal of S.

Proof. Suppose $f, g \in S$ are such that $fg \in \mathfrak{q}$. Let f_d (resp. g_e) be the homogeneous part of f (resp. g) of degree d (resp. e). Assume d, e are maxima such that $f_d \neq 0$ and $g_e \neq 0$. By assumption we can write $fg = \sum a_i f_i$ with $f_i \in \mathfrak{p}$ homogeneous. Say $\deg(f_i) = d_i$. Then $f_d g_e = \sum a'_i f_i$ with a'_i to homogeneous par of degree $d + e - d_i$ of a_i (or 0 if $d + e - d_i < 0$). Hence $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. Hence $f_d \in \mathfrak{q}$ or $g_e \in \mathfrak{q}$. In the first case replace f by $f - f_d$, in the second case replace g by $g - g_e$. Then still $fg \in \mathfrak{q}$ but the discrete invariant $d + e$ has been decreased. Thus we may continue in this fashion until either f or g is zero. This clearly shows that $fg \in \mathfrak{q}$ implies either $f \in \mathfrak{q}$ or $g \in \mathfrak{q}$ as desired. $\square$

00JU Lemma 10.57.8. Let S be a graded ring.

(1) Any minimal prime of S is a homogeneous ideal of S.
(2) Given a homogeneous ideal $I \subset S$ any minimal prime over I is homogeneous.

Proof. The first assertion holds because the prime $\mathfrak{q}$ constructed in Lemma 10.57.7 satisfies $\mathfrak{q} \subset \mathfrak{p}$. The second because we may consider S/I and apply the first part. $\square$

07Z2 Lemma 10.57.9. Let R be a ring. Let S be a graded R-algebra. Let $f \in S_+$ be homogeneous. Assume that S is of finite type over R. Then

(1) the ring $S_{(f)}$ is of finite type over R, and
(2) for any finite graded S-module M the module $M_{(f)}$ is a finite $S_{(f)}$-module.

Proof. Choose $f_1, \ldots, f_n \in S$ which generate S as an R-algebra. We may assume that each f_i is homogeneous (by decomposing each f_i into its homogeneous components). An element of $S_{(f)}$ is a sum of the form

$$\sum\nolimits_{e \deg(f) = \sum e_i \deg(f_i)} \lambda_{e_1 \ldots e_n} f_1^{e_1} \ldots f_n^{e_n} / f^e$$

with $\lambda_{e_1 \ldots e_n} \in R$. Thus $S_{(f)}$ is generated as an R-algebra by the $f_1^{e_1} \ldots f_n^{e_n}/f^e$ with the property that $e \deg(f) = \sum e_i \deg(f_i)$. If $e_i \geq \deg(f)$ then we can write this as

$$f_1^{e_1} \ldots f_n^{e_n}/f^e = f_i^{\deg(f)}/f^{\deg(f_i)} \cdot f_1^{e_1} \ldots f_i^{e_i - \deg(f)} \ldots f_n^{e_n}/f^{e - \deg(f_i)}$$

Thus we only need the elements $f_i^{\deg(f)}/f^{\deg(f_i)}$ as well as the elements $f_1^{e_1} \ldots f_n^{e_n}/f^e$ with $e \deg(f) = \sum e_i \deg(f_i)$ and $e_i < \deg(f)$. This is a finite list and we see that (1) is true.

To see (2) suppose that M is generated by homogeneous elements $x_1, \ldots, x_m$. Then arguing as above we find that $M_{(f)}$ is generated as an $S_{(f)}$-module by the finite list of elements of the form $f_1^{e_1} \ldots f_n^{e_n} x_j / f^e$ with $e \deg(f) = \sum e_i \deg(f_i) + \deg(x_j)$ and $e_i < \deg(f)$. □

052N Lemma 10.57.10. Let R be a ring. Let R' be a finite type R-algebra, and let M be a finite R'-module. There exists a graded R-algebra S, a graded S-module N and an element $f \in S$ homogeneous of degree 1 such that

(1) $R' \cong S_{(f)}$ and $M \cong N_{(f)}$ (as modules),
(2) $S_0 = R$ and S is generated by finitely many elements of degree 1 over R, and
(3) N is a finite S-module.

Proof. We may write $R' = R[x_1, \ldots, x_n]/I$ for some ideal I. For an element $g \in R[x_1, \ldots, x_n]$ denote $\tilde{g} \in R[X_0, \ldots, X_n]$ the element homogeneous of minimal degree such that $g = \tilde{g}(1, x_1, \ldots, x_n)$. Let $\tilde{I} \subset R[X_0, \ldots, X_n]$ generated by all elements $\tilde{g}$, $g \in I$. Set $S = R[X_0, \ldots, X_n]/\tilde{I}$ and denote f the image of X_0 in S. By construction we have an isomorphism

$$S_{(f)} \longrightarrow R', \quad X_i/X_0 \longmapsto x_i.$$

To do the same thing with the module M we choose a presentation

$$M = (R')^{\oplus r} / \sum\nolimits_{j \in J} R' k_j$$

with $k_j = (k_{1j}, \ldots, k_{rj})$. Let $d_{ij} = \deg(\tilde{k}_{ij})$. Set $d_j = \max\{d_{ij}\}$. Set $K_{ij} = X_0^{d_j - d_{ij}} \tilde{k}_{ij}$ which is homogeneous of degree d_j. With this notation we set

$$N = \text{Coker}\left(\bigoplus\nolimits_{j \in J} S(-d_j) \xrightarrow{(K_{ij})} S^{\oplus r}\right)$$

which works. Some details omitted. □

10.58. Noetherian graded rings

00JV A bit of theory on Noetherian graded rings including some material on Hilbert polynomials.

07Z4 Lemma 10.58.1. Let S be a graded ring. A set of homogeneous elements $f_i \in S_+$ generates S as an algebra over S_0 if and only if they generate S_+ as an ideal of S.

Proof. If the f_i generate S as an algebra over S_0 then every element in S_+ is a polynomial without constant term in the f_i and hence S_+ is generated by the f_i as an ideal. Conversely, suppose that $S_+ = \sum Sf_i$. We will prove that any element f of S can be written as a polynomial in the f_i with coefficients in S_0. It suffices to do this for homogeneous elements. Say f has degree d. Then we may perform induction on d. The case $d = 0$ is immediate. If $d > 0$ then $f \in S_+$ hence we can write $f = \sum g_i f_i$ for some $g_i \in S$. As S is graded we can replace g_i by its homogeneous component of degree $d - \deg(f_i)$. By induction we see that each g_i is a polynomial in the f_i and we win. □

00JW **Lemma 10.58.2.** *A graded ring S is Noetherian if and only if S_0 is Noetherian and S_+ is finitely generated as an ideal of S.*

Proof. It is clear that if S is Noetherian then $S_0 = S/S_+$ is Noetherian and S_+ is finitely generated. Conversely, assume S_0 is Noetherian and S_+ finitely generated as an ideal of S. Pick generators $S_+ = (f_1, \ldots, f_n)$. By decomposing the f_i into homogeneous pieces we may assume each f_i is homogeneous. By Lemma 10.58.1 we see that $S_0[X_1, \ldots X_n] \to S$ sending X_i to f_i is surjective. Thus S is Noetherian by Lemma 10.31.1. □

00JX **Definition 10.58.3.** Let A be an abelian group. We say that a function $f : n \mapsto f(n) \in A$ defined for all sufficient large integers n is a *numerical polynomial* if there exists $r \geq 0$, elements $a_0, \ldots, a_r \in A$ such that

$$f(n) = \sum\nolimits_{i=0}^{r} \binom{n}{i} a_i$$

for all $n \gg 0$.

The reason for using the binomial coefficients is the elementary fact that any polynomial $P \in \mathbf{Q}[T]$ all of whose values at integer points are integers, is equal to a sum $P(T) = \sum a_i \binom{T}{i}$ with $a_i \in \mathbf{Z}$. Note that in particular the expressions $\binom{T+1}{i+1}$ are of this form.

00JY **Lemma 10.58.4.** *If $A \to A'$ is a homomorphism of abelian groups and if $f : n \mapsto f(n) \in A$ is a numerical polynomial, then so is the composition.*

Proof. This is immediate from the definitions. □

00JZ **Lemma 10.58.5.** *Suppose that $f : n \mapsto f(n) \in A$ is defined for all n sufficiently large and suppose that $n \mapsto f(n) - f(n-1)$ is a numerical polynomial. Then f is a numerical polynomial.*

Proof. Let $f(n) - f(n-1) = \sum_{i=0}^{r} \binom{n}{i} a_i$ for all $n \gg 0$. Set $g(n) = f(n) - \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$. Then $g(n) - g(n-1) = 0$ for all $n \gg 0$. Hence g is eventually constant, say equal to a_{-1}. We leave it to the reader to show that $a_{-1} + \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$ has the required shape (see remark above the lemma). □

00K0 **Lemma 10.58.6.** *If M is a finitely generated graded S-module, and if S is finitely generated over S_0, then each M_n is a finite S_0-module.*

Proof. Suppose the generators of M are m_i and the generators of S are f_i. By taking homogeneous components we may assume that the m_i and the f_i are homogeneous and we may assume $f_i \in S_+$. In this case it is clear that each M_n is generated over S_0 by the "monomials" $\prod f_i^{e_i} m_j$ whose degree is n. □

00K1 **Proposition 10.58.7.** Suppose that S is a Noetherian graded ring and M a finite graded S-module. Consider the function

$$\mathbf{Z} \longrightarrow K'_0(S_0), \quad n \longmapsto [M_n]$$

see Lemma 10.58.6. If S_+ is generated by elements of degree 1, then this function is a numerical polynomial.

Proof. We prove this by induction on the minimal number of generators of S_1. If this number is 0, then $M_n = 0$ for all $n \gg 0$ and the result holds. To prove the induction step, let $x \in S_1$ be one of a minimal set of generators, such that the induction hypothesis applies to the graded ring $S/(x)$.

First we show the result holds if x is nilpotent on M. This we do by induction on the minimal integer r such that $x^r M = 0$. If $r = 1$, then M is a module over S/xS and the result holds (by the other induction hypothesis). If $r > 1$, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ such that the integers r', r'' are strictly smaller than r. Thus we know the result for M'' and M'. Hence we get the result for M because of the relation $[M_d] = [M'_d] + [M''_d]$ in $K'_0(S_0)$.

If x is not nilpotent on M, let $M' \subset M$ be the largest submodule on which x is nilpotent. Consider the exact sequence $0 \to M' \to M \to M/M' \to 0$ we see again it suffices to prove the result for M/M'. In other words we may assume that multiplication by x is injective.

Let $\overline{M} = M/xM$. Note that the map $x : M \to M$ is not a map of graded S-modules, since it does not map M_d into M_d. Namely, for each d we have the following short exact sequence

$$0 \to M_d \xrightarrow{x} M_{d+1} \to \overline{M}_{d+1} \to 0$$

This proves that $[M_{d+1}] - [M_d] = [\overline{M}_{d+1}]$. Hence we win by Lemma 10.58.5. $\square$

02CD **Remark 10.58.8.** If S is still Noetherian but S is not generated in degree 1, then the function associated to a graded S-module is a periodic polynomial (i.e., it is a numerical polynomial on the congruence classes of integers modulo n for some n).

00K2 **Example 10.58.9.** Suppose that $S = k[X_1, \ldots, X_d]$. By Example 10.55.2 we may identify $K_0(k) = K'_0(k) = \mathbf{Z}$. Hence any finitely generated graded $k[X_1, \ldots, X_d]$-module gives rise to a numerical polynomial $n \mapsto \dim_k(M_n)$.

00K3 **Lemma 10.58.10.** Let k be a field. Suppose that $I \subset k[X_1, \ldots, X_d]$ is a nonzero graded ideal. Let $M = k[X_1, \ldots, X_d]/I$. Then the numerical polynomial $n \mapsto \dim_k(M_n)$ (see Example 10.58.9) has degree $< d - 1$ (or is zero if $d = 1$).

Proof. The numerical polynomial associated to the graded module $k[X_1, \ldots, X_d]$ is $n \mapsto \binom{n-1+d}{d-1}$. For any nonzero homogeneous $f \in I$ of degree e and any degree $n >> e$ we have $I_n \supset f \cdot k[X_1, \ldots, X_d]_{n-e}$ and hence $\dim_k(I_n) \geq \binom{n-e-1+d}{d-1}$. Hence $\dim_k(M_n) \leq \binom{n-1+d}{d-1} - \binom{n-e-1+d}{d-1}$. We win because the last expression has degree $< d - 1$ (or is zero if $d = 1$). $\square$

10.59. Noetherian local rings

00K4

In all of this section $(R, \mathfrak m, \kappa)$ is a Noetherian local ring. We develop some theory on Hilbert functions of modules in this section. Let M be a finite R-module. We define the Hilbert function of M to be the function

$$\varphi_M : n \longmapsto \text{length}_R(\mathfrak m^nM/\mathfrak m^{n+1}M)$$

defined for all integers $n \geq 0$. Another important invariant is the function

$$\chi_M : n \longmapsto \text{length}_R(M/\mathfrak m^{n+1}M)$$

defined for all integers $n \geq 0$. Note that we have by Lemma 10.52.3 that

$$\chi_M(n) = \sum\nolimits_{i=0}^n \varphi_M(i).$$

There is a variant of this construction which uses an ideal of definition.

07DU Definition 10.59.1. Let $(R, \mathfrak m)$ be a local Noetherian ring. An ideal $I \subset R$ such that $\sqrt{I} = \mathfrak m$ is called an ideal of definition of R.

Let $I \subset R$ be an ideal of definition. Because R is Noetherian this means that $\mathfrak m^r \subset I$ for some r, see Lemma 10.32.5. Hence any finite R-module annihilated by a power of I has a finite length, see Lemma 10.52.8. Thus it makes sense to define

$$\varphi_{I,M}(n) = \text{length}_R(I^nM/I^{n+1}M) \quad \text{and} \quad \chi_{I,M}(n) = \text{length}_R(M/I^{n+1}M)$$

for all $n \geq 0$. Again we have that

$$\chi_{I,M}(n) = \sum\nolimits_{i=0}^n \varphi_{I,M}(i).$$

00K5 Lemma 10.59.2. Suppose that $M' \subset M$ are finite R-modules with finite length quotient. Then there exists a constants c_1, c_2 such that for all $n \geq c_2$ we have

$$c_1 + \chi_{I,M'}(n - c_2) \leq \chi_{I,M}(n) \leq c_1 + \chi_{I,M'}(n)$$

Proof. Since M/M' has finite length there is a $c_2 \geq 0$ such that $I^{c_2}M \subset M'$. Let $c_1 = \text{length}_R(M/M')$. For $n \geq c_2$ we have

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\leq c_1 + \text{length}_R(M'/I^{n+1}M') \\ &= c_1 + \chi_{I,M'}(n) \end{aligned}$$

On the other hand, since $I^{c_2}M \subset M'$, we have $I^nM \subset I^{n-c_2}M'$ for $n \geq c_2$. Thus for $n \geq c_2$ we get

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\geq c_1 + \text{length}_R(M'/I^{n+1-c_2}M') \\ &= c_1 + \chi_{I,M'}(n - c_2) \end{aligned}$$

which finishes the proof. □

00K6 Lemma 10.59.3. Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finite R-modules. Then there exists a submodule $N \subset M'$ with finite colength l and $c \geq 0$ such that

$$\chi_{I,M}(n) = \chi_{I,M''}(n) + \chi_{I,N}(n - c) + l$$

and

$$\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n-c)$$

for all $n \geq c$.

Proof. Note that $M/I^nM \to M''/I^nM''$ is surjective with kernel $M'/M' \cap I^nM$. By the Artin-Rees Lemma 10.51.2 there exists a constant c such that $M' \cap I^nM = I^{n-c}(M' \cap I^cM)$. Denote $N = M' \cap I^cM$. Note that $I^cM' \subset N \subset M'$. Hence $\text{length}_R(M'/M' \cap I^nM) = \text{length}_R(M'/N) + \text{length}_R(N/I^{n-c}N)$ for $n \geq c$. From the short exact sequence

$$0 \to M'/M' \cap I^nM \to M/I^nM \to M''/I^nM'' \to 0$$

and additivity of lengths (Lemma 10.52.3) we obtain the equality

$$\chi_{I,M}(n-1) = \chi_{I,M''}(n-1) + \chi_{I,N}(n-c-1) + \text{length}_R(M'/N)$$

for $n \geq c$. We have $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ and similarly for the modules M'' and N. Hence we get $\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n-c)$ for $n \geq c$. □

00K7 Lemma 10.59.4. Suppose that I, I' are two ideals of definition for the Noetherian local ring R. Let M be a finite R-module. There exists a constant a such that $\chi_{I,M}(n) \leq \chi_{I',M}(an)$ for $n \geq 1$.

Proof. There exists an integer $c \geq 1$ such that $(I')^c \subset I$. Hence we get a surjection $M/(I')^{c(n+1)}M \to M/I^{n+1}M$. Whence the result with $a = 2c - 1$. □

00K8 Proposition 10.59.5. Let R be a Noetherian local ring. Let M be a finite R-module. Let $I \subset R$ be an ideal of definition. The Hilbert function $\varphi_{I,M}$ and the function $\chi_{I,M}$ are numerical polynomials.

Proof. Consider the graded ring $S = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots = \bigoplus_{d \geq 0} I^d/I^{d+1}$. Consider the graded S-module $N = M/IM \oplus IM/I^2M \oplus \ldots = \bigoplus_{d \geq 0} I^dM/I^{d+1}M$. This pair (S, N) satisfies the hypotheses of Proposition 10.58.7. Hence the result for $\varphi_{I,M}$ follows from that proposition and Lemma 10.55.1. The result for $\chi_{I,M}$ follows from this and Lemma 10.58.5. □

09CA Definition 10.59.6. Let R be a Noetherian local ring. Let M be a finite R-module. The Hilbert polynomial of M over R is the element $P(t) \in \mathbf{Q}[t]$ such that $P(n) = \varphi_M(n)$ for $n \gg 0$.

By Proposition 10.59.5 we see that the Hilbert polynomial exists.

00K9 Lemma 10.59.7. Let R be a Noetherian local ring. Let M be a finite R-module.

(1) The degree of the numerical polynomial $\varphi_{I,M}$ is independent of the ideal of definition I.
(2) The degree of the numerical polynomial $\chi_{I,M}$ is independent of the ideal of definition I.

Proof. Part (2) follows immediately from Lemma 10.59.4. Part (1) follows from (2) because $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ for $n \geq 1$. □

00KA Definition 10.59.8. Let R be a local Noetherian ring and M a finite R-module. We denote $d(M)$ the element of $\{-\infty, 0, 1, 2, \ldots\}$ defined as follows:

(1) If $M = 0$ we set $d(M) = -\infty$,
(2) if $M \neq 0$ then $d(M)$ is the degree of the numerical polynomial χ_M.

If $\mathfrak{m}^n M \neq 0$ for all n, then we see that $d(M)$ is the degree $+1$ of the Hilbert polynomial of M.

00KB Lemma 10.59.9. Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let M be a finite R-module which does not have finite length. If $M' \subset M$ is a submodule with finite colength, then $\chi_{I,M} - \chi_{I,M'}$ is a polynomial of degree $<$ degree of either polynomial.

Proof. Follows from Lemma 10.59.2 by elementary calculus. □

00KC Lemma 10.59.10. Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules. Then

(1) if M' does not have finite length, then $\chi_{I,M} - \chi_{I,M''} - \chi_{I,M'}$ is a numerical polynomial of degree $<$ the degree of $\chi_{I,M'}$,
(2) $\max\{\deg(\chi_{I,M'}), \deg(\chi_{I,M''})\} = \deg(\chi_{I,M})$, and
(3) $\max\{d(M'), d(M'')\} = d(M)$,

Proof. We first prove (1). Let $N \subset M'$ be as in Lemma 10.59.3. By Lemma 10.59.9 the numerical polynomial $\chi_{I,M'} - \chi_{I,N}$ has degree $<$ the common degree of $\chi_{I,M'}$ and $\chi_{I,N}$. By Lemma 10.59.3 the difference

$$\chi_{I,M}(n) - \chi_{I,M''}(n) - \chi_{I,N}(n-c)$$

is constant for $n \gg 0$. By elementary calculus the difference $\chi_{I,N}(n) - \chi_{I,N}(n-c)$ has degree $<$ the degree of $\chi_{I,N}$ which is bigger than zero (see above). Putting everything together we obtain (1).

Note that the leading coefficients of $\chi_{I,M'}$ and $\chi_{I,M''}$ are nonnegative. Thus the degree of $\chi_{I,M'} + \chi_{I,M''}$ is equal to the maximum of the degrees. Thus if M' does not have finite length, then (2) follows from (1). If M' does have finite length, then $I^nM \to I^nM''$ is an isomorphism for all $n \gg 0$ by Artin-Rees (Lemma 10.51.2). Thus $M/I^nM \to M''/I^nM''$ is a surjection with kernel M' for $n \gg 0$ and we see that $\chi_{I,M}(n) - \chi_{I,M''}(n) = \text{length}(M')$ for all $n \gg 0$. Thus (2) holds in this case also.

Proof of (3). This follows from (2) except if one of M, M', or M'' is zero. We omit the proof in these special cases. □

10.60. Dimension

00KD Please compare with Topology, Section 5.10.

0GIE Definition 10.60.1. Let R be a ring. A chain of prime ideals is a sequence $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ of prime ideals of R such that $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $i = 0, \ldots, n-1$. The length of this chain of prime ideals is n.

Recall that we have an inclusion reversing bijection between prime ideals of a ring R and irreducible closed subsets of $\text{Spec}(R)$, see Lemma 10.26.1.

00KE Definition 10.60.2. The Krull dimension of the ring R is the Krull dimension of the topological space $\text{Spec}(R)$, see Topology, Definition 5.10.1. In other words it is the supremum of the integers $n \geq 0$ such that R has a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n, \quad \mathfrak{p}_i \neq \mathfrak{p}_{i+1}.$$

of length n.

00KF Definition 10.60.3. The height of a prime ideal $\mathfrak{p}$ of a ring R is the dimension of the local ring $R_{\mathfrak{p}}$.

00KG Lemma 10.60.4. The Krull dimension of R is the supremum of the heights of its (maximal) primes.

Proof. This is so because we can always add a maximal ideal at the end of a chain of prime ideals. □

00KH Lemma 10.60.5. A Noetherian ring of dimension 0 is Artinian. Conversely, any Artinian ring is Noetherian of dimension zero.

Proof. Assume R is a Noetherian ring of dimension 0. By Lemma 10.31.5 the space $\text{Spec}(R)$ is Noetherian. By Topology, Lemma 5.9.2 we see that $\text{Spec}(R)$ has finitely many irreducible components, say $\text{Spec}(R) = Z_1 \cup \ldots \cup Z_r$. According to Lemma 10.26.1 each $Z_i = V(\mathfrak{p}_i)$ with $\mathfrak{p}_i$ a minimal ideal. Since the dimension is 0 these $\mathfrak{p}_i$ are also maximal. Thus $\text{Spec}(R)$ is the discrete topological space with elements $\mathfrak{p}_i$. All elements f of the Jacobson radical $\bigcap \mathfrak{p}_i$ are nilpotent since otherwise R_f would not be the zero ring and we would have another prime. By Lemma 10.53.5 R is equal to $\prod R_{\mathfrak{p}_i}$. Since $R_{\mathfrak{p}_i}$ is also Noetherian and dimension 0, the previous arguments show that its radical $\mathfrak{p}_i R_{\mathfrak{p}_i}$ is locally nilpotent. Lemma 10.32.5 gives $\mathfrak{p}_i^n R_{\mathfrak{p}_i} = 0$ for some $n \geq 1$. By Lemma 10.52.8 we conclude that $R_{\mathfrak{p}_i}$ has finite length over R. Hence we conclude that R is Artinian by Lemma 10.53.6.

If R is an Artinian ring then by Lemma 10.53.6 it is Noetherian. All of its primes are maximal by a combination of Lemmas 10.53.3, 10.53.4 and 10.53.5. □

In the following we will use the invariant $d(-)$ defined in Definition 10.59.8. Here is a warm up lemma.

00KI Lemma 10.60.6. Let R be a Noetherian local ring. Then $\dim(R) = 0 \Leftrightarrow d(R) = 0$.

Proof. This is because $d(R) = 0$ if and only if R has finite length as an R-module. See Lemma 10.53.6. □

00KJ Proposition 10.60.7. Let R be a ring. The following are equivalent:

(1) R is Artinian,
(2) R is Noetherian and $\dim(R) = 0$,
(3) R has finite length as a module over itself,
(4) R is a finite product of Artinian local rings,
(5) R is Noetherian and $\text{Spec}(R)$ is a finite discrete topological space,
(6) R is a finite product of Noetherian local rings of dimension 0,
(7) R is a finite product of Noetherian local rings R_i with $d(R_i) = 0$,
(8) R is a finite product of Noetherian local rings R_i whose maximal ideals are nilpotent,
(9) R is Noetherian, has finitely many maximal ideals and its Jacobson radical ideal is nilpotent, and
(10) R is Noetherian and there are no strict inclusions among its primes.

Proof. This is a combination of Lemmas 10.53.5, 10.53.6, 10.60.5, and 10.60.6. □

00KK Lemma 10.60.8. Let R be a local Noetherian ring. The following are equivalent:

00KL (1) $\dim(R) = 1$,
00KM (2) $d(R) = 1$,

00KN (3) there exists an $x \in \mathfrak{m}$, x not nilpotent such that $V(x) = \{\mathfrak{m}\}$,

00KO (4) there exists an $x \in \mathfrak{m}$, x not nilpotent such that $\mathfrak{m} = \sqrt{(x)}$, and

00KP (5) there exists an ideal of definition generated by 1 element, and no ideal of definition is generated by 0 elements.

Proof. First, assume that $\dim(R) = 1$. Let $\mathfrak{p}_i$ be the minimal primes of R. Because the dimension is 1 the only other prime of R is $\mathfrak{m}$. According to Lemma 10.31.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \notin \mathfrak{p}_i$, see Lemma 10.15.2. Thus the only prime containing x is $\mathfrak{m}$ and hence (3).

If (3) then $\mathfrak{m} = \sqrt{(x)}$ by Lemma 10.17.2, and hence (4). The converse is clear as well. The equivalence of (4) and (5) follows from directly the definitions.

Assume (5). Let $I = (x)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of R/I via multiplication by x^n and hence $\text{length}_R(I^n/I^{n+1})$ is bounded. Thus $d(R) = 0$ or $d(R) = 1$, but $d(R) = 0$ is excluded by the assumption that 0 is not an ideal of definition.

Assume (2). To get a contradiction, assume there exist primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$, with both inclusions strict. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 10.59.10. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq 1$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree < 1. In other words, $d(R/(xR + \mathfrak{p})) = 0$, and hence $\dim(R/(xR + \mathfrak{p})) = 0$, by Lemma 10.60.6. But $R/(xR+\mathfrak{p})$ has the distinct primes $\mathfrak{q}/(xR+\mathfrak{p})$ and $\mathfrak{m}/(xR+\mathfrak{p})$ which gives the desired contradiction. $\square$

00KQ Proposition 10.60.9. Let R be a local Noetherian ring. Let $d \geq 0$ be an integer. The following are equivalent:

00KR (1) $\dim(R) = d$,

00KS (2) $d(R) = d$,

00KT (3) there exists an ideal of definition generated by d elements, and no ideal of definition is generated by fewer than d elements.

Proof. This proof is really just the same as the proof of Lemma 10.60.8. We will prove the proposition by induction on d. By Lemmas 10.60.6 and 10.60.8 we may assume that $d > 1$. Denote the minimal number of generators for an ideal of definition of R by $d'(R)$. We will prove the inequalities $\dim(R) \geq d'(R) \geq d(R) \geq \dim(R)$, and hence they are all equal.

First, assume that $\dim(R) = d$. Let $\mathfrak{p}_i$ be the minimal primes of R. According to Lemma 10.31.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \notin \mathfrak{p}_i$, see Lemma 10.15.2. Note that every maximal chain of primes starts with some $\mathfrak{p}_i$, hence the dimension of R/xR is at most $d - 1$. By induction there are $x_2, \ldots, x_d$ which generate an ideal of definition in R/xR. Hence R has an ideal of definition generated by (at most) d elements.

Assume $d'(R) = d$. Let $I = (x_1, \ldots, x_d)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of a direct sum of $\binom{d+n-1}{d-1}$ copies R/I via multiplication by all degree n monomials in $x_1, \ldots, x_d$. Hence $\text{length}_R(I^n/I^{n+1})$ is bounded by a polynomial of degree $d-1$. Thus $d(R) \leq d$.

Assume $d(R) = d$. Consider a chain of primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{q}_2 \subset \ldots \subset \mathfrak{q}_e = \mathfrak{m}$, with all inclusions strict, and $e \geq 2$. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 10.59.10. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq d$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree $< d$. In other words, $d(R/(xR + \mathfrak{p})) \leq d - 1$, and hence $\dim(R/(xR + \mathfrak{p})) \leq d - 1$, by induction. Now $R/(xR+\mathfrak{p})$ has the chain of prime ideals $\mathfrak{q}/(xR+\mathfrak{p}) \subset \mathfrak{q}_2/(xR+\mathfrak{p}) \subset \ldots \subset \mathfrak{q}_e/(xR + \mathfrak{p})$ which gives $e - 1 \leq d - 1$. Since we started with an arbitrary chain of primes this proves that $\dim(R) \leq d(R)$.

Reading back the reader will see we proved the circular inequalities as desired. $\square$

Let $(R, \mathfrak{m})$ be a Noetherian local ring. From the above it is clear that $\mathfrak{m}$ cannot be generated by fewer than $\dim(R)$ variables. By Nakayama's Lemma 10.20.1 the minimal number of generators of $\mathfrak{m}$ equals $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$. Hence we have the following fundamental inequality

$$\dim(R) \leq \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

It turns out that the rings where equality holds have a lot of good properties. They are called regular local rings.

00KU Definition 10.60.10. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d.

(1) A system of parameters of R is a sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ which generates an ideal of definition of R,
(2) if there exist $x_1, \ldots, x_d \in \mathfrak{m}$ such that $\mathfrak{m} = (x_1, \ldots, x_d)$ then we call R a regular local ring and $x_1, \ldots, x_d$ a regular system of parameters.

The following lemmas are clear from the proofs of the lemmas and proposition above, but we spell them out so we have convenient references.

00KV Lemma 10.60.11. Let R be a Noetherian ring. Let $x \in R$.

(1) If $\mathfrak{p}$ is minimal over (x) then the height of $\mathfrak{p}$ is 0 or 1.
(2) If $\mathfrak{p}, \mathfrak{q} \in \text{Spec}(R)$ and $\mathfrak{q}$ is minimal over $(\mathfrak{p}, x)$, then there is no prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$.

Proof. Proof of (1). If $\mathfrak{p}$ is minimal over x, then the only prime ideal of $R_\mathfrak{p}$ containing x is the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. This is true because the primes of $R_\mathfrak{p}$ correspond 1-to-1 with the primes of R contained in $\mathfrak{p}$, see Lemma 10.17.5. Hence Lemma 10.60.8 shows $\dim(R_\mathfrak{p}) = 1$ if x is not nilpotent in $R_\mathfrak{p}$. Of course, if x is nilpotent in $R_\mathfrak{p}$ the argument gives that $\mathfrak{p}R_\mathfrak{p}$ is the only prime ideal and we see that the height is 0.

Proof of (2). By part (1) we see that $\mathfrak{q}/\mathfrak{p}$ is a prime of height 1 or 0 in $R/\mathfrak{p}$. This immediately implies there cannot be a prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$. $\square$

0BBZ Lemma 10.60.12. Let R be a Noetherian ring. Let $f_1, \ldots, f_r \in R$.

(1) If $\mathfrak{p}$ is minimal over $(f_1, \ldots, f_r)$ then the height of $\mathfrak{p}$ is $\leq r$.

(2) If $\mathfrak p, \mathfrak q \in \mathrm{Spec}(R)$ and $\mathfrak q$ is minimal over $(\mathfrak p, f_1, \ldots, f_r)$, then every chain of primes between $\mathfrak p$ and $\mathfrak q$ has length at most r.

Proof. Proof of (1). If $\mathfrak p$ is minimal over $f_1, \ldots, f_r$, then the only prime ideal of $R_{\mathfrak p}$ containing $f_1, \ldots, f_r$ is the maximal ideal $\mathfrak pR_{\mathfrak p}$. This is true because the primes of $R_{\mathfrak p}$ correspond 1-to-1 with the primes of R contained in $\mathfrak p$, see Lemma 10.17.5. Hence Proposition 10.60.9 shows $\dim(R_{\mathfrak p}) \leq r$.

Proof of (2). By part (1) we see that $\mathfrak q/\mathfrak p$ is a prime of height $\leq r$. This immediately implies the statement about chains of primes between $\mathfrak p$ and $\mathfrak q$. □

00KW Lemma 10.60.13. Suppose that R is a Noetherian local ring and $x \in \mathfrak m$ an element of its maximal ideal. Then $\dim R \leq \dim R/xR + 1$. If x is not contained in any of the minimal primes of R then equality holds. (For example if x is a nonzerodivisor.)

Proof. If $x_1, \ldots, x_{\dim R/xR} \in R$ map to elements of R/xR which generate an ideal of definition for R/xR, then $x, x_1, \ldots, x_{\dim R/xR}$ generate an ideal of definition for R. Hence the inequality by Proposition 10.60.9. On the other hand, if x is not contained in any minimal prime of R, then the chains of primes in R/xR all give rise to chains in R which are at least one step away from being maximal. □

02IE Lemma 10.60.14. Let $(R, \mathfrak m)$ be a Noetherian local ring. Suppose $x_1, \ldots, x_d \in \mathfrak m$ generate an ideal of definition and $d = \dim(R)$. Then $\dim(R/(x_1, \ldots, x_i)) = d - i$ for all $i = 1, \ldots, d$.

Proof. Follows either from the proof of Proposition 10.60.9, or by using induction on d and Lemma 10.60.13. □

10.61. Applications of dimension theory

02IF We can use the results on dimension to prove certain rings have infinite spectra and to produce more Jacobson rings.

02IG Lemma 10.61.1. Let R be a Noetherian local domain of dimension ≥ 2. A nonempty open subset $U \subset \mathrm{Spec}(R)$ is infinite.

Proof. To get a contradiction, assume that $U \subset \mathrm{Spec}(R)$ is finite. In this case $(0) \in U$ and $\{(0)\}$ is an open subset of U (because the complement of $\{(0)\}$ is the union of the closures of the other points). Thus we may assume $U = \{(0)\}$. Let $\mathfrak m \subset R$ be the maximal ideal. We can find an $x \in \mathfrak m$, $x \neq 0$ such that $V(x) \cup U = \mathrm{Spec}(R)$. In other words we see that $D(x) = \{(0)\}$. In particular we see that $\dim(R/xR) = \dim(R) - 1 \geq 1$, see Lemma 10.60.13. Let $\overline{y}_2, \ldots, \overline{y}_{\dim(R)} \in R/xR$ generate an ideal of definition of R/xR, see Proposition 10.60.9. Choose lifts $y_2, \ldots, y_{\dim(R)} \in R$, so that $x, y_2, \ldots, y_{\dim(R)}$ generate an ideal of definition in R. This implies that $\dim(R/(y_2)) = \dim(R) - 1$ and $\dim(R/(y_2, x)) = \dim(R) - 2$, see Lemma 10.60.14. Hence there exists a prime $\mathfrak p$ containing y_2 but not x. This contradicts the fact that $D(x) = \{(0)\}$. □

The rings $k[[t]]$ where k is a field, or the ring of p-adic numbers are Noetherian rings of dimension 1 with finitely many primes. This is the maximum dimension for which this can happen.

0ALV Lemma 10.61.2. A Noetherian ring with finitely many primes has dimension ≤ 1.

Proof. Let R be a Noetherian ring with finitely many primes. If R is a local domain, then the lemma follows from Lemma 10.61.1. If R is a domain, then $R_\mathfrak{m}$ has dimension ≤ 1 for all maximal ideals $\mathfrak{m}$ by the local case. Hence $\dim(R) \leq 1$ by Lemma 10.60.4. If R is general, then $\dim(R/\mathfrak{q}) \leq 1$ for every minimal prime $\mathfrak{q}$ of R. Since every prime contains a minimal prime (Lemma 10.17.2), this implies $\dim(R) \leq 1$. $\square$

0ALW **Lemma 10.61.3.** Let S be a nonzero finite type algebra over a field k. Then $\dim(S) = 0$ if and only if S has finitely many primes.

Proof. Recall that $\text{Spec}(S)$ is sober, Noetherian, and Jacobson, see Lemmas 10.26.2, 10.31.5, 10.35.2, and 10.35.4. If it has dimension 0, then every point defines an irreducible component and there are only a finite number of irreducible components (Topology, Lemma 5.9.2). Conversely, if $\text{Spec}(S)$ is finite, then it is discrete by Topology, Lemma 5.18.6 and hence the dimension is 0. $\square$

00KX **Lemma 10.61.4.** Noetherian Jacobson rings.

(1) Any Noetherian domain R of dimension 1 with infinitely many primes is Jacobson.
(2) Any Noetherian ring such that every prime $\mathfrak{p}$ is either maximal or contained in infinitely many prime ideals is Jacobson.

Proof. Part (1) is a reformulation of Lemma 10.35.6.

Let R be a Noetherian ring such that every non-maximal prime $\mathfrak{p}$ is contained in infinitely many prime ideals. Assume $\text{Spec}(R)$ is not Jacobson to get a contradiction. By Lemmas 10.26.1 and 10.31.5 we see that $\text{Spec}(R)$ is a sober, Noetherian topological space. By Topology, Lemma 5.18.3 we see that there exists a non-maximal ideal $\mathfrak{p} \subset R$ such that $\{\mathfrak{p}\}$ is a locally closed subset of $\text{Spec}(R)$. In other words, $\mathfrak{p}$ is not maximal and $\{\mathfrak{p}\}$ is an open subset of $V(\mathfrak{p})$. Consider a prime $\mathfrak{q} \subset R$ with $\mathfrak{p} \subset \mathfrak{q}$. Recall that the topology on the spectrum of $(R/\mathfrak{p})_\mathfrak{q} = R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ is induced from that of $\text{Spec}(R)$, see Lemmas 10.17.5 and 10.17.7. Hence we see that $\{(0)\}$ is a locally closed subset of $\text{Spec}((R/\mathfrak{p})_\mathfrak{q})$. By Lemma 10.61.1 we conclude that $\dim((R/\mathfrak{p})_\mathfrak{q}) = 1$. Since this holds for every $\mathfrak{q} \supset \mathfrak{p}$ we conclude that $\dim(R/\mathfrak{p}) = 1$. At this point we use the assumption that $\mathfrak{p}$ is contained in infinitely many primes to see that $\text{Spec}(R/\mathfrak{p})$ is infinite. Hence by part (1) of the lemma we see that $V(\mathfrak{p}) \cong \text{Spec}(R/\mathfrak{p})$ is the closure of its closed points. This is the desired contradiction since it means that $\{\mathfrak{p}\} \subset V(\mathfrak{p})$ cannot be open. $\square$

10.62. Support and dimension of modules

00KY Some basic results on the support and dimension of modules.

00L0 **Lemma 10.62.1.** Let R be a Noetherian ring, and let M be a finite R-module. There exists a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

First proof. By Lemma 10.5.4 it suffices to do the case $M = R/I$ for some ideal I. Consider the set S of ideals J such that the lemma does not hold for the module R/J, and order it by inclusion. To arrive at a contradiction, assume that S is not empty. Because R is Noetherian, S has a maximal element J. By definition of S, the ideal J cannot be prime. Pick $a, b \in R$ such that $ab \in J$, but neither $a \in J$ nor $b \in J$. Consider the filtration $0 \subset aR/(J \cap aR) \subset R/J$. Note that both the submodule $aR/(J \cap aR)$ and the quotient module $(R/J)/(aR/(J \cap aR))$ are cyclic modules; write them as R/J' and R/J'' so we have a short exact sequence $0 \to R/J' \to R/J \to R/J'' \to 0$. The inclusion $J \subset J'$ is strict as $b \in J'$ and the inclusion $J \subset J''$ is strict as $a \in J''$. Hence by maximality of J, both R/J' and R/J'' have a filtration as above and hence so does R/J. Contradiction. $\square$

Second proof. For an R-module M we say $P(M)$ holds if there exists a filtration as in the statement of the lemma. Observe that P is stable under extensions and holds for 0. By Lemma 10.5.4 it suffices to prove $P(R/I)$ holds for every ideal I. If not then because R is Noetherian, there is a maximal counter example J. By Example 10.28.7 and Proposition 10.28.8 the ideal J is prime which is a contradiction. $\square$

00L4 Lemma 10.62.2. Let R, M, M_i, $\mathfrak{p}_i$ as in Lemma 10.62.1. Then $\text{Supp}(M) = \bigcup V(\mathfrak{p}_i)$ and in particular $\mathfrak{p}_i \in \text{Supp}(M)$.

Proof. This follows from Lemmas 10.40.5 and 10.40.9. $\square$

00L5 Lemma 10.62.3. Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a nonzero finite R-module. Then $\text{Supp}(M) = \{\mathfrak{m}\}$ if and only if M has finite length over R.

Proof. Assume that $\text{Supp}(M) = \{\mathfrak{m}\}$. It suffices to show that all the primes $\mathfrak{p}_i$ in the filtration of Lemma 10.62.1 are the maximal ideal. This is clear by Lemma 10.62.2.

Suppose that M has finite length over R. Then $\mathfrak{m}^n M = 0$ by Lemma 10.52.4. Since some element of $\mathfrak{m}$ maps to a unit in $R_\mathfrak{p}$ for any prime $\mathfrak{p} \neq \mathfrak{m}$ in R we see $M_\mathfrak{p} = 0$. $\square$

00L6 Lemma 10.62.4. Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Then $I^n M = 0$ for some $n \geq 0$ if and only if $\text{Supp}(M) \subset V(I)$.

Proof. Indeed, $I^n M = 0$ is equivalent to $I^n \subset \text{Ann}(M)$. Since R is Noetherian, this is equivalent to $I \subset \sqrt{\text{Ann}(M)}$, see Lemma 10.32.5. This in turn is equivalent to $V(I) \supset V(\text{Ann}(M))$, see Lemma 10.17.2. By Lemma 10.40.5 this is equivalent to $V(I) \supset \text{Supp}(M)$. $\square$

00L7 Lemma 10.62.5. Let R, M, M_i, $\mathfrak{p}_i$ as in Lemma 10.62.1. The minimal elements of the set $\{\mathfrak{p}_i\}$ are the minimal elements of $\text{Supp}(M)$. The number of times a minimal prime $\mathfrak{p}$ occurs is

$$\#\{i \mid \mathfrak{p}_i = \mathfrak{p}\} = \text{length}_{R_\mathfrak{p}} M_\mathfrak{p}.$$

Proof. The first statement follows because $\text{Supp}(M) = \bigcup V(\mathfrak{p}_i)$, see Lemma 10.62.2. Let $\mathfrak{p} \in \text{Supp}(M)$ be minimal. The support of $M_\mathfrak{p}$ is the set consisting of the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Hence by Lemma 10.62.3 the length of $M_\mathfrak{p}$ is finite and > 0. Next we note that $M_\mathfrak{p}$ has a filtration with subquotients $(R/\mathfrak{p}_i)_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p}$. These are zero if $\mathfrak{p}_i \not\subset \mathfrak{p}$ and equal to $\kappa(\mathfrak{p})$ if $\mathfrak{p}_i \subset \mathfrak{p}$ because by minimality of $\mathfrak{p}$ we have $\mathfrak{p}_i = \mathfrak{p}$ in this case. The result follows since $\kappa(\mathfrak{p})$ has length 1. $\square$

00L8 Lemma 10.62.6. Let R be a Noetherian local ring. Let M be a finite R-module. Then $d(M) = \dim(\text{Supp}(M))$ where $d(M)$ is as in Definition 10.59.8.

Proof. Let $M_i, \mathfrak{p}_i$ be as in Lemma 10.62.1. By Lemma 10.59.10 we obtain the equality $d(M) = \max\{d(R/\mathfrak{p}_i)\}$. By Proposition 10.60.9 we have $d(R/\mathfrak{p}_i) = \dim(R/\mathfrak{p}_i)$. Trivially $\dim(R/\mathfrak{p}_i) = \dim V(\mathfrak{p}_i)$. Since all minimal primes of $\text{Supp}(M)$ occur among the $\mathfrak{p}_i$ (Lemma 10.62.5) we win. □

0B51 Lemma 10.62.7. Let R be a Noetherian ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules. Then $\max\{\dim(\text{Supp}(M')), \dim(\text{Supp}(M''))\} = \dim(\text{Supp}(M))$.

Proof. If R is local, this follows immediately from Lemmas 10.62.6 and 10.59.10. A more elementary argument, which works also if R is not local, is to use that $\text{Supp}(M')$, $\text{Supp}(M'')$, and $\text{Supp}(M)$ are closed (Lemma 10.40.5) and that $\text{Supp}(M) = \text{Supp}(M') \cup \text{Supp}(M'')$ (Lemma 10.40.9). □

10.63. Associated primes

00L9 Here is the standard definition. For non-Noetherian rings and non-finite modules it may be more appropriate to use the definition in Section 10.66.

00LA Definition 10.63.1. Let R be a ring. Let M be an R-module. A prime $\mathfrak{p}$ of R is associated to M if there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. The set of all such primes is denoted $\text{Ass}_R(M)$ or $\text{Ass}(M)$.

0586 Lemma 10.63.2. Let R be a ring. Let M be an R-module. Then $\text{Ass}(M) \subset \text{Supp}(M)$.

Proof. If $m \in M$ has annihilator $\mathfrak{p}$, then in particular no element of $R \setminus \mathfrak{p}$ annihilates m. Hence m is a nonzero element of $M_{\mathfrak{p}}$, i.e., $\mathfrak{p} \in \text{Supp}(M)$. □

02M3 Lemma 10.63.3. Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $\text{Ass}(M') \subset \text{Ass}(M)$ and $\text{Ass}(M) \subset \text{Ass}(M') \cup \text{Ass}(M'')$. Also $\text{Ass}(M' \oplus M'') = \text{Ass}(M') \cup \text{Ass}(M'')$.

Proof. If $m' \in M'$, then the annihilator of m' viewed as an element of M' is the same as the annihilator of m' viewed as an element of M. Hence the inclusion $\text{Ass}(M') \subset \text{Ass}(M)$. Let $m \in M$ be an element whose annihilator is a prime ideal $\mathfrak{p}$. If there exists a $g \in R$, $g \notin \mathfrak{p}$ such that $m' = gm \in M'$ then the annihilator of m' is $\mathfrak{p}$. If there does not exist a $g \in R$, $g \notin \mathfrak{p}$ such that $gm \in M'$, then the annilator of the image $m'' \in M''$ of m is $\mathfrak{p}$. This proves the inclusion $\text{Ass}(M) \subset \text{Ass}(M') \cup \text{Ass}(M'')$. We omit the proof of the final statement. □

00LB Lemma 10.63.4. Let R be a ring, and M an R-module. Suppose there exists a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Then $\text{Ass}(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Proof. By induction on the length n of the filtration $\{M_i\}$. Pick $m \in M$ whose annihilator is a prime $\mathfrak{p}$. If $m \in M_{n-1}$ we are done by induction. If not, then m maps to a nonzero element of $M/M_{n-1} \cong R/\mathfrak{p}_n$. Hence we have $\mathfrak{p} \subset \mathfrak{p}_n$. If equality does not hold, then we can find $f \in \mathfrak{p}_n$, $f \notin \mathfrak{p}$. In this case the annihilator of fm is still $\mathfrak{p}$ and $fm \in M_{n-1}$. Thus we win by induction. □

00LC **Lemma 10.63.5.** Let R be a Noetherian ring. Let M be a finite R-module. Then $\text{Ass}(M)$ is finite.

Proof. Immediate from Lemma 10.63.4 and Lemma 10.62.1. $\square$

02CE **Proposition 10.63.6.** Let R be a Noetherian ring. Let M be a finite R-module. The following sets of primes are the same:

(1) The minimal primes in the support of M.
(2) The minimal primes in $\text{Ass}(M)$.
(3) For any filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$ with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ the minimal primes of the set $\{\mathfrak{p}_i\}$.

Proof. Choose a filtration as in (3). In Lemma 10.62.5 we have seen that the sets in (1) and (3) are equal.

Let $\mathfrak{p}$ be a minimal element of the set $\{\mathfrak{p}_i\}$. Let i be minimal such that $\mathfrak{p} = \mathfrak{p}_i$. Pick $m \in M_i$, $m \notin M_{i-1}$. The annihilator of m is contained in $\mathfrak{p}_i = \mathfrak{p}$ and contains $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_i$. By our choice of i and $\mathfrak{p}$ we have $\mathfrak{p}_j \not\subset \mathfrak{p}$ for $j < i$ and hence we have $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1} \not\subset \mathfrak{p}_i$. Pick $f \in \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1}$, $f \notin \mathfrak{p}$. Then fm has annihilator $\mathfrak{p}$. In this way we see that $\mathfrak{p}$ is an associated prime of M. By Lemma 10.63.2 we have $\text{Ass}(M) \subset \text{Supp}(M)$ and hence $\mathfrak{p}$ is minimal in $\text{Ass}(M)$. Thus the set of primes in (1) is contained in the set of primes of (2).

Let $\mathfrak{p}$ be a minimal element of $\text{Ass}(M)$. Since $\text{Ass}(M) \subset \text{Supp}(M)$ there is a minimal element $\mathfrak{q}$ of $\text{Supp}(M)$ with $\mathfrak{q} \subset \mathfrak{p}$. We have just shown that $\mathfrak{q} \in \text{Ass}(M)$. Hence $\mathfrak{q} = \mathfrak{p}$ by minimality of $\mathfrak{p}$. Thus the set of primes in (2) is contained in the set of primes of (1). $\square$

0587 **Lemma 10.63.7.** Let R be a Noetherian ring. Let M be an R-module. Then

$$M = (0) \Leftrightarrow \text{Ass}(M) = \emptyset.$$

Proof. If $M = (0)$, then $\text{Ass}(M) = \emptyset$ by definition. If $M \neq 0$, pick any nonzero finitely generated submodule $M' \subset M$, for example a submodule generated by a single nonzero element. By Lemma 10.40.2 we see that $\text{Supp}(M')$ is nonempty. By Proposition 10.63.6 this implies that $\text{Ass}(M')$ is nonempty. By Lemma 10.63.3 this implies $\text{Ass}(M) \neq \emptyset$. $\square$

05BV **Lemma 10.63.8.** Let R be a Noetherian ring. Let M be an R-module. Any $\mathfrak{p} \in \text{Supp}(M)$ which is minimal among the elements of $\text{Supp}(M)$ is an element of $\text{Ass}(M)$.

Proof. If M is a finite R-module, then this is a consequence of Proposition 10.63.6. In general write $M = \bigcup M_\lambda$ as the union of its finite submodules, and use that $\text{Supp}(M) = \bigcup \text{Supp}(M_\lambda)$ and $\text{Ass}(M) = \bigcup \text{Ass}(M_\lambda)$. $\square$

00LD **Lemma 10.63.9.** Let R be a Noetherian ring. Let M be an R-module. The union $\bigcup_{\mathfrak{q} \in \text{Ass}(M)} \mathfrak{q}$ is the set of elements of R which are zerodivisors on M.

Proof. Any element in any associated prime clearly is a zerodivisor on M. Conversely, suppose $x \in R$ is a zerodivisor on M. Consider the submodule $N = \{m \in M \mid xm = 0\}$. Since N is not zero it has an associated prime $\mathfrak{q}$ by Lemma 10.63.7. Then $x \in \mathfrak{q}$ and $\mathfrak{q}$ is an associated prime of M by Lemma 10.63.3. $\square$

0B52 **Lemma 10.63.10.** Let R is a Noetherian local ring, M a finite R-module, and $f \in \mathfrak m$ an element of the maximal ideal of R. Then

$$\dim(\text{Supp}(M/fM)) \leq \dim(\text{Supp}(M)) \leq \dim(\text{Supp}(M/fM)) + 1$$

If f is not in any of the minimal primes of the support of M (for example if f is a nonzerodivisor on M), then equality holds for the right inequality.

Proof. (The parenthetical statement follows from Lemma 10.63.9.) The first inequality follows from $\text{Supp}(M/fM) \subset \text{Supp}(M)$, see Lemma 10.40.9. For the second inequality, note that $\text{Supp}(M/fM) = \text{Supp}(M) \cap V(f)$, see Lemma 10.40.9. It follows, for example by Lemma 10.62.2 and elementary properties of dimension, that it suffices to show $\dim V(\mathfrak p) \leq \dim(V(\mathfrak p) \cap V(f)) + 1$ for primes $\mathfrak p$ of R. This is a consequence of Lemma 10.60.13. Finally, if f is not contained in any minimal prime of the support of M, then the chains of primes in $\text{Supp}(M/fM)$ all give rise to chains in $\text{Supp}(M)$ which are at least one step away from being maximal. $\square$

05BW **Lemma 10.63.11.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then $\text{Spec}(\varphi)(\text{Ass}_S(M)) \subset \text{Ass}_R(M)$.

Proof. If $\mathfrak q \in \text{Ass}_S(M)$, then there exists an m in M such that the annihilator of m in S is $\mathfrak q$. Then the annihilator of m in R is $\mathfrak q \cap R$. $\square$

05BX **Remark 10.63.12.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $\text{Spec}(\varphi)(\text{Ass}_S(M)) \supset \text{Ass}_R(M)$. For example, consider the ring map $R = k \to S = k[x_1, x_2, x_3, \ldots]/(x_i^2)$ and $M = S$. Then $\text{Ass}_R(M)$ is not empty, but $\text{Ass}_S(S)$ is empty.

05DZ **Lemma 10.63.13.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. If S is Noetherian, then $\text{Spec}(\varphi)(\text{Ass}_S(M)) = \text{Ass}_R(M)$.

Proof. We have already seen in Lemma 10.63.11 that $\text{Spec}(\varphi)(\text{Ass}_S(M)) \subset \text{Ass}_R(M)$. For the converse, choose a prime $\mathfrak p \in \text{Ass}_R(M)$. Let $m \in M$ be an element such that the annihilator of m in R is $\mathfrak p$. Let $I = \{g \in S \mid gm = 0\}$ be the annihilator of m in S. Then $R/\mathfrak p \subset S/I$ is injective. Combining Lemmas 10.30.5 and 10.30.7 we see that there is a prime $\mathfrak q \subset S$ minimal over I mapping to $\mathfrak p$. By Proposition 10.63.6 we see that $\mathfrak q$ is an associated prime of S/I, hence $\mathfrak q$ is an associated prime of M by Lemma 10.63.3 and we win. $\square$

05BY **Lemma 10.63.14.** Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection $\text{Spec}(R/I) \to \text{Spec}(R)$ we have $\text{Ass}_{R/I}(M) = \text{Ass}_R(M)$.

Proof. Omitted. $\square$

0310 **Lemma 10.63.15.** Let R be a ring. Let M be an R-module. Let $\mathfrak p \subset R$ be a prime.

1. If $\mathfrak p \in \text{Ass}(M)$ then $\mathfrak pR_{\mathfrak p} \in \text{Ass}(M_{\mathfrak p})$.
2. If $\mathfrak p$ is finitely generated then the converse holds as well.

Proof. If $\mathfrak p \in \text{Ass}(M)$ there exists an element $m \in M$ whose annihilator is $\mathfrak p$. As localization is exact (Proposition 10.9.12) we see that the annihilator of $m/1$ in $M_{\mathfrak p}$ is $\mathfrak pR_{\mathfrak p}$ hence (1) holds. Assume $\mathfrak pR_{\mathfrak p} \in \text{Ass}(M_{\mathfrak p})$ and $\mathfrak p = (f_1, \ldots, f_n)$. Let m/g be an element of $M_{\mathfrak p}$ whose annihilator is $\mathfrak pR_{\mathfrak p}$. This implies that the annihilator of m is contained in $\mathfrak p$. As $f_i m/g = 0$ in $M_{\mathfrak p}$ we see there exists a $g_i \in R$, $g_i \notin \mathfrak p$ such that $g_i f_i m = 0$ in M. Combined we see the annihilator of $g_1 \ldots g_n m$ is $\mathfrak p$. Hence $\mathfrak p \in \text{Ass}(M)$. $\square$

05BZ **Lemma 10.63.16.** Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ we have

(1) $\mathrm{Ass}_R(S^{-1}M) = \mathrm{Ass}_{S^{-1}R}(S^{-1}M)$,
(2) $\mathrm{Ass}_R(M) \cap \mathrm{Spec}(S^{-1}R) \subset \mathrm{Ass}_R(S^{-1}M)$, and
(3) if R is Noetherian this inclusion is an equality.

Proof. The first equality follows, since if $m \in S^{-1}M$, then the annihilator of m in R is the intersection of the annihilator of m in $S^{-1}R$ with R. The displayed inclusion and equality in the Noetherian case follows from Lemma 10.63.15 since for $\mathfrak{p} \in R$, $S \cap \mathfrak{p} = \emptyset$ we have $M_{\mathfrak{p}} = (S^{-1}M)_{S^{-1}\mathfrak{p}}$. $\square$

05C0 **Lemma 10.63.17.** Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzerodivisor on M. Then

$$\mathrm{Ass}_R(M) = \mathrm{Ass}_R(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we get the inclusion $\mathrm{Ass}(M) \subset \mathrm{Ass}(S^{-1}M)$ from Lemma 10.63.3. Conversely, suppose that $n/s \in S^{-1}M$ is an element whose annihilator is a prime ideal $\mathfrak{p}$. Then the annihilator of $n \in M$ is also $\mathfrak{p}$. $\square$

00LL **Lemma 10.63.18.** Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subset \mathfrak{m}$ be an ideal. Let M be a finite R-module. The following are equivalent:

(1) There exists an $x \in I$ which is not a zerodivisor on M.
(2) We have $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Ass}(M)$.

Proof. If there exists a nonzerodivisor x in I, then x clearly cannot be in any associated prime of M. Conversely, suppose $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Ass}(M)$. In this case we can choose $x \in I$, $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Ass}(M)$ by Lemmas 10.63.5 and 10.15.2. By Lemma 10.63.9 the element x is not a zerodivisor on M. $\square$

0311 **Lemma 10.63.19.** Let R be a ring. Let M be an R-module. If R is Noetherian the map

$$M \longrightarrow \prod\nolimits_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Then if $\mathfrak{p}$ is an associated prime of $Rx \subset M$ we see on the one hand that $\mathfrak{p} \in \mathrm{Ass}(M)$ (Lemma 10.63.3) and on the other hand that $(Rx)_{\mathfrak{p}} \subset M_{\mathfrak{p}}$ is not zero. This contradiction shows that $\mathrm{Ass}(Rx) = \emptyset$. Hence $Rx = 0$ by Lemma 10.63.7. $\square$

This lemma should probably be put somewhere else.

0GEC **Lemma 10.63.20.** Let k be a field. Let S be a finite type k algebra. If $\dim(S) > 0$, then there exists an element $f \in S$ which is a nonzerodivisor and a nonunit.

Proof. By Lemma 10.63.5 the ring S has finitely many associated prime ideals. By Lemma 10.61.3 the ring S has infinitely many maximal ideals. Hence we can choose a maximal ideal $\mathfrak{m} \subset S$ which is not an associated prime of S. By prime avoidance (Lemma 10.15.2), we can choose a nonzero $f \in \mathfrak{m}$ which is not contained in any of the associated primes of S. By Lemma 10.63.9 the element f is a nonzerodivisor and as $f \in \mathfrak{m}$ we see that f is not a unit. $\square$

10.64. Symbolic powers

05G9 Here is the definition.

0313 **Definition 10.64.1.** Let R be a ring. Let $\mathfrak{p}$ be a prime ideal. For $n \geq 0$ the nth *symbolic power* of $\mathfrak{p}$ is the ideal $\mathfrak{p}^{(n)} = \text{Ker}(R \to R_\mathfrak{p}/\mathfrak{p}^nR_\mathfrak{p})$.

Note that $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ but equality does not always hold.

0314 **Lemma 10.64.2.** Let R be a Noetherian ring. Let $\mathfrak{p}$ be a prime ideal. Let $n > 0$. Then $\text{Ass}(R/\mathfrak{p}^{(n)}) = \{\mathfrak{p}\}$.

Proof. If $\mathfrak{q}$ is an associated prime of $R/\mathfrak{p}^{(n)}$ then clearly $\mathfrak{p} \subset \mathfrak{q}$. On the other hand, any element $x \in R$, $x \notin \mathfrak{p}$ is a nonzerodivisor on $R/\mathfrak{p}^{(n)}$. Namely, if $y \in R$ and $xy \in \mathfrak{p}^{(n)} = R \cap \mathfrak{p}^nR_\mathfrak{p}$ then $y \in \mathfrak{p}^nR_\mathfrak{p}$, hence $y \in \mathfrak{p}^{(n)}$. Hence the lemma follows. □

0BC0 **Lemma 10.64.3.** Let $R \to S$ be flat ring map. Let $\mathfrak{p} \subset R$ be a prime such that $\mathfrak{q} = \mathfrak{p}S$ is a prime of S. Then $\mathfrak{p}^{(n)}S = \mathfrak{q}^{(n)}$.

Proof. Since $\mathfrak{p}^{(n)} = \text{Ker}(R \to R_\mathfrak{p}/\mathfrak{p}^nR_\mathfrak{p})$ we see using flatness that $\mathfrak{p}^{(n)}S$ is the kernel of the map $S \to S_\mathfrak{p}/\mathfrak{p}^nS_\mathfrak{p}$. On the other hand $\mathfrak{q}^{(n)}$ is the kernel of the map $S \to S_\mathfrak{q}/\mathfrak{q}^nS_\mathfrak{q} = S_\mathfrak{q}/\mathfrak{p}^nS_\mathfrak{q}$. Hence it suffices to show that

$$S_\mathfrak{p}/\mathfrak{p}^nS_\mathfrak{p} \longrightarrow S_\mathfrak{q}/\mathfrak{p}^nS_\mathfrak{q}$$

is injective. Observe that the right hand module is the localization of the left hand module by elements $f \in S$, $f \notin \mathfrak{q}$. Thus it suffices to show these elements are nonzerodivisors on $S_\mathfrak{p}/\mathfrak{p}^nS_\mathfrak{p}$. By flatness, the module $S_\mathfrak{p}/\mathfrak{p}^nS_\mathfrak{p}$ has a finite filtration whose subquotients are

$$\mathfrak{p}^iS_\mathfrak{p}/\mathfrak{p}^{i+1}S_\mathfrak{p} \cong \mathfrak{p}^iR_\mathfrak{p}/\mathfrak{p}^{i+1}R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{p} \cong V \otimes_{\kappa(\mathfrak{p})} (S/\mathfrak{q})_\mathfrak{p}$$

where V is a $\kappa(\mathfrak{p})$ vector space. Thus f acts invertibly as desired. □

10.65. Relative assassin

05GA **Discussion of relative assassins.** Let $R \to S$ be a ring map. Let N be an S-module. In this situation we can introduce the following sets of primes $\mathfrak{q}$ of S:

(1) A: with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q} \in \text{Ass}_S(N \otimes_R \kappa(\mathfrak{p}))$,
(2) A': with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q}$ is in the image of $\text{Ass}_{S \otimes \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$ under the canonical map $\text{Spec}(S \otimes_R \kappa(\mathfrak{p})) \to \text{Spec}(S)$,
(3) A_{fin}: with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q} \in \text{Ass}_S(N/\mathfrak{p}N)$,
(4) A'_{fin}: for some prime $\mathfrak{p}' \subset R$ we have $\mathfrak{q} \in \text{Ass}_S(N/\mathfrak{p}'N)$,
(5) B: for some R-module M we have $\mathfrak{q} \in \text{Ass}_S(N \otimes_R M)$, and
(6) B_{fin}: for some finite R-module M we have $\mathfrak{q} \in \text{Ass}_S(N \otimes_R M)$.

Let us determine some of the relations between theses sets.

05GB **Lemma 10.65.1.** Let $R \to S$ be a ring map. Let N be an S-module. Let A, A', A_{fin}, B, and B_{fin} be the subsets of $\text{Spec}(S)$ introduced above.

(1) We always have $A = A'$.
(2) We always have $A_{fin} \subset A$, $B_{fin} \subset B$, $A_{fin} \subset A'_{fin} \subset B_{fin}$ and $A \subset B$.
(3) If S is Noetherian, then $A = A_{fin}$ and $B = B_{fin}$.
(4) If N is flat over R, then $A = A_{fin} = A'_{fin}$ and $B = B_{fin}$.
(5) If R is Noetherian and N is flat over R, then all of the sets are equal, i.e., $A = A' = A_{fin} = A'_{fin} = B = B_{fin}$.

Proof. Some of the arguments in the proof will be repeated in the proofs of later lemmas which are more precise than this one (because they deal with a given module M or a given prime $\mathfrak p$ and not with the collection of all of them).

Proof of (1). Let $\mathfrak p$ be a prime of R. Then we have

$$\text{Ass}_S(N \otimes_R \kappa(\mathfrak p)) = \text{Ass}_{S/\mathfrak pS}(N \otimes_R \kappa(\mathfrak p)) = \text{Ass}_{S\otimes_R\kappa(\mathfrak p)}(N \otimes_R \kappa(\mathfrak p))$$

the first equality by Lemma 10.63.14 and the second by Lemma 10.63.16 part (1). This prove that $A = A'$. The inclusion $A_{fin} \subset A'_{fin}$ is clear.

Proof of (2). Each of the inclusions is immediate from the definitions except perhaps $A_{fin} \subset A$ which follows from Lemma 10.63.16 and the fact that we require $\mathfrak p = R\cap\mathfrak q$ in the formulation of A_{fin}.

Proof of (3). The equality $A = A_{fin}$ follows from Lemma 10.63.16 part (3) if S is Noetherian. Let $\mathfrak q = (g_1, \ldots, g_m)$ be a finitely generated prime ideal of S. Say $z \in N \otimes_R M$ is an element whose annihilator is $\mathfrak q$. We may pick a finite submodule $M' \subset M$ such that z is the image of $z' \in N \otimes_R M'$. Then $\text{Ann}_S(z') \subset \mathfrak q = \text{Ann}_S(z)$. Since $N \otimes_R -$ commutes with colimits and since M is the directed colimit of finite R-modules we can find $M' \subset M'' \subset M$ such that the image $z'' \in N \otimes_R M''$ is annihilated by $g_1, \ldots, g_m$. Hence $\text{Ann}_S(z'') = \mathfrak q$. This proves that $B = B_{fin}$ if S is Noetherian.

Proof of (4). If N is flat, then the functor $N \otimes_R -$ is exact. In particular, if $M' \subset M$, then $N \otimes_R M' \subset N \otimes_R M$. Hence if $z \in N \otimes_R M$ is an element whose annihilator $\mathfrak q = \text{Ann}_S(z)$ is a prime, then we can pick any finite R-submodule $M' \subset M$ such that $z \in N \otimes_R M'$ and we see that the annihilator of z as an element of $N \otimes_R M'$ is equal to $\mathfrak q$. Hence $B = B_{fin}$. Let $\mathfrak p'$ be a prime of R and let $\mathfrak q$ be a prime of S which is an associated prime of $N/\mathfrak p'N$. This implies that $\mathfrak p'S \subset \mathfrak q$. As N is flat over R we see that $N/\mathfrak p'N$ is flat over the integral domain $R/\mathfrak p'$. Hence every nonzero element of $R/\mathfrak p'$ is a nonzerodivisor on $N/\mathfrak p'$. Hence none of these elements can map to an element of $\mathfrak q$ and we conclude that $\mathfrak p' = R \cap \mathfrak q$. Hence $A_{fin} = A'_{fin}$. Finally, by Lemma 10.63.17 we see that $\text{Ass}_S(N/\mathfrak p'N) = \text{Ass}_S(N \otimes_R \kappa(\mathfrak p'))$, i.e., $A'_{fin} = A$.

Proof of (5). We only need to prove $A'_{fin} = B_{fin}$ as the other equalities have been proved in (4). To see this let M be a finite R-module. By Lemma 10.62.1 there exists a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak p_i$ for some prime ideal $\mathfrak p_i$ of R. Since N is flat we obtain a filtration by S-submodules

$$0 = N \otimes_R M_0 \subset N \otimes_R M_1 \subset \ldots \subset N \otimes_R M_n = N \otimes_R M$$

such that each subquotient is isomorphic to $N/\mathfrak p_iN$. By Lemma 10.63.3 we conclude that $\text{Ass}_S(N \otimes_R M) \subset \bigcup \text{Ass}_S(N/\mathfrak p_iN)$. Hence we see that $B_{fin} \subset A'_{fin}$. Since the other inclusion is part of (2) we win. □

We define the relative assassin of N over S/R to be the set $A = A'$ above. As a motivation we point out that it depends only on the fibre modules $N \otimes_R \kappa(\mathfrak p)$ over the fibre rings. As in the case of the assassin of a module we warn the reader that this notion makes most sense when the fibre rings $S \otimes_R \kappa(\mathfrak p)$ are Noetherian, for example if $R \to S$ is of finite type.

05GC Definition 10.65.2. Let $R \to S$ be a ring map. Let N be an S-module. The relative assassin of N over S/R is the set

$$\mathrm{Ass}_{S/R}(N) = \{\mathfrak q \subset S \mid \mathfrak q \in \mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak p)) \text{ with } \mathfrak p = R \cap \mathfrak q\}.$$

This is the set named A in Lemma 10.65.1.

The spirit of the next few results is that they are about the relative assassin, even though this may not be apparent.

0312 Lemma 10.65.3. Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. If N is flat as R-module, then

$$\mathrm{Ass}_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak p \in \mathrm{Ass}_R(M)} \mathrm{Ass}_S(N/\mathfrak p N)$$

and if R is Noetherian then we have equality.

Proof. If $\mathfrak p \in \mathrm{Ass}_R(M)$ then there exists an injection $R/\mathfrak p \to M$. As N is flat over R we obtain an injection $R/\mathfrak p \otimes_R N \to M \otimes_R N$. Since $R/\mathfrak p \otimes_R N = N/\mathfrak p N$ we conclude that $\mathrm{Ass}_S(N/\mathfrak p N) \subset \mathrm{Ass}_S(M \otimes_R N)$, see Lemma 10.63.3. Hence the right hand side is contained in the left hand side.

Write $M = \bigcup M_\lambda$ as the union of its finitely generated R-submodules. Then also $N \otimes_R M = \bigcup N \otimes_R M_\lambda$ (as N is R-flat). By definition of associated primes we see that $\mathrm{Ass}_S(N \otimes_R M) = \bigcup \mathrm{Ass}_S(N \otimes_R M_\lambda)$ and $\mathrm{Ass}_R(M) = \bigcup \mathrm{Ass}(M_\lambda)$. Hence we may assume M is finitely generated.

Let $\mathfrak q \in \mathrm{Ass}_S(M \otimes_R N)$, and assume R is Noetherian and M is a finite R-module. To finish the proof we have to show that $\mathfrak q$ is an element of the right hand side. First we observe that $\mathfrak q S_{\mathfrak q} \in \mathrm{Ass}_{S_{\mathfrak q}}((M \otimes_R N)_{\mathfrak q})$, see Lemma 10.63.15. Let $\mathfrak p$ be the corresponding prime of R. Note that

$$(M \otimes_R N)_{\mathfrak q} = M \otimes_R N_{\mathfrak q} = M_{\mathfrak p} \otimes_{R_{\mathfrak p}} N_{\mathfrak q}$$

If $\mathfrak p R_{\mathfrak p} \notin \mathrm{Ass}_{R_{\mathfrak p}}(M_{\mathfrak p})$ then there exists an element $x \in \mathfrak p R_{\mathfrak p}$ which is a nonzerodivisor in $M_{\mathfrak p}$ (see Lemma 10.63.18). Since $N_{\mathfrak q}$ is flat over $R_{\mathfrak p}$ we see that the image of x in $\mathfrak q S_{\mathfrak q}$ is a nonzerodivisor on $(M \otimes_R N)_{\mathfrak q}$. This is a contradiction with the assumption that $\mathfrak q S_{\mathfrak q} \in \mathrm{Ass}_S((M \otimes_R N)_{\mathfrak q})$. Hence we conclude that $\mathfrak p$ is one of the associated primes of M.

Continuing the argument we choose a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak p_i$ for some prime ideal $\mathfrak p_i$ of R, see Lemma 10.62.1. (By Lemma 10.63.4 we have $\mathfrak p_i = \mathfrak p$ for at least one i.) This gives a filtration

$$0 = M_0 \otimes_R N \subset M_1 \otimes_R N \subset \ldots \subset M_n \otimes_R N = M \otimes_R N$$

with subquotients isomorphic to $N/\mathfrak p_i N$. If $\mathfrak p_i \neq \mathfrak p$ then $\mathfrak q$ cannot be associated to the module $N/\mathfrak p_i N$ by the result of the preceding paragraph (as $\mathrm{Ass}_R(R/\mathfrak p_i) = \{\mathfrak p_i\}$). Hence we conclude that $\mathfrak q$ is associated to $N/\mathfrak p N$ as desired. $\square$

05C1 Lemma 10.65.4. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then

$$\mathrm{Ass}_S(N) = \mathrm{Ass}_S(N \otimes_R K) = \mathrm{Ass}_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $\mathrm{Spec}(S \otimes_R K) \subset \mathrm{Spec}(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 10.63.17 combined with Lemma 10.63.16 part (1). □

05C2 **Lemma 10.65.5.** Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. Assume N is flat as R-module. Then

$$\mathrm{Ass}_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak p \in \mathrm{Ass}_R(M)} \mathrm{Ass}_{S \otimes_R \kappa(\mathfrak p)}(N \otimes_R \kappa(\mathfrak p))$$

where we use Remark 10.17.8 to think of the spectra of fibre rings as subsets of $\mathrm{Spec}(S)$. If R is Noetherian then this inclusion is an equality.

Proof. This is equivalent to Lemma 10.65.3 by Lemmas 10.63.14, 10.39.7, and 10.65.4. □

05E0 **Remark 10.65.6.** Let $R \to S$ be a ring map. Let N be an S-module. Let $\mathfrak p$ be a prime of R. Then

$$\mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak p)) = \mathrm{Ass}_{S/\mathfrak pS}(N \otimes_R \kappa(\mathfrak p)) = \mathrm{Ass}_{S \otimes_R \kappa(\mathfrak p)}(N \otimes_R \kappa(\mathfrak p)).$$

The first equality by Lemma 10.63.14 and the second by Lemma 10.63.16 part (1).

10.66. Weakly associated primes

0546 This is a variant on the notion of an associated prime that is useful for non-Noetherian ring and non-finite modules.

0547 **Definition 10.66.1.** Let R be a ring. Let M be an R-module. A prime $\mathfrak p$ of R is *weakly associated* to M if there exists an element $m \in M$ such that $\mathfrak p$ is minimal among the prime ideals containing the annihilator $\mathrm{Ann}(m) = \{f \in R \mid fm = 0\}$. The set of all such primes is denoted $\mathrm{WeakAss}_R(M)$ or $\mathrm{WeakAss}(M)$.

Thus an associated prime is a weakly associated prime. Here is a characterization in terms of the localization at the prime.

0566 **Lemma 10.66.2.** Let R be a ring. Let M be an R-module. Let $\mathfrak p$ be a prime of R. The following are equivalent:

1. $\mathfrak p$ is weakly associated to M,
2. $\mathfrak pR_{\mathfrak p}$ is weakly associated to $M_{\mathfrak p}$, and
3. $M_{\mathfrak p}$ contains an element whose annihilator has radical equal to $\mathfrak pR_{\mathfrak p}$.

Proof. Assume (1). Then there exists an element $m \in M$ such that $\mathfrak p$ is minimal among the primes containing the annihilator $I = \{x \in R \mid xm = 0\}$ of m. As localization is exact, the annihilator of m in $M_{\mathfrak p}$ is $I_{\mathfrak p}$. Hence $\mathfrak pR_{\mathfrak p}$ is a minimal prime of $R_{\mathfrak p}$ containing the annihilator $I_{\mathfrak p}$ of m in $M_{\mathfrak p}$. This implies (2) holds, and also (3) as it implies that $\sqrt{I_{\mathfrak p}} = \mathfrak pR_{\mathfrak p}$.

Applying the implication (1) $\Rightarrow$ (3) to $M_{\mathfrak p}$ over $R_{\mathfrak p}$ we see that (2) $\Rightarrow$ (3).

Finally, assume (3). This means there exists an element $m/f \in M_{\mathfrak p}$ whose annihilator has radical equal to $\mathfrak pR_{\mathfrak p}$. Then the annihilator $I = \{x \in R \mid xm = 0\}$ of m in M is such that $\sqrt{I_{\mathfrak p}} = \mathfrak pR_{\mathfrak p}$. Clearly this means that $\mathfrak p$ contains I and is minimal among the primes containing I, i.e., (1) holds. □

0EMA **Lemma 10.66.3.** For a reduced ring the weakly associated primes of the ring are the minimal primes.

Proof. Let $(R, \mathfrak{m})$ be a reduced local ring. Suppose $x \in R$ is an element whose annihilator has radical $\mathfrak{m}$. If $\mathfrak{m} \neq 0$, then x cannot be a unit, so $x \in \mathfrak{m}$. Then in particular $x^{1+n} = 0$ for some $n \geq 0$. Hence $x = 0$. Which contradicts the assumption that the annihilator of x is contained in $\mathfrak{m}$. Thus we see that $\mathfrak{m} = 0$, i.e., R is a field. By Lemma 10.66.2 this implies the statement of the lemma. $\square$

0548 Lemma 10.66.4. Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $\text{WeakAss}(M') \subset \text{WeakAss}(M)$ and $\text{WeakAss}(M) \subset \text{WeakAss}(M') \cup \text{WeakAss}(M'')$.

Proof. We will use the characterization of weakly associated primes of Lemma 10.66.2. Let $\mathfrak{p}$ be a prime of R. As localization is exact we obtain the short exact sequence $0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0$. Suppose that $m \in M_{\mathfrak{p}}$ is an element whose annihilator has radical $\mathfrak{p}R_{\mathfrak{p}}$. Then either the image $\overline{m}$ of m in $M''_{\mathfrak{p}}$ is zero and $m \in M'_{\mathfrak{p}}$, or the radical of the annihilator of $\overline{m}$ is $\mathfrak{p}R_{\mathfrak{p}}$. This proves that $\text{WeakAss}(M) \subset \text{WeakAss}(M') \cup \text{WeakAss}(M'')$. The inclusion $\text{WeakAss}(M') \subset \text{WeakAss}(M)$ is immediate from the definitions. $\square$

0588 Lemma 10.66.5. Let R be a ring. Let M be an R-module. Then

$$M = (0) \Leftrightarrow \text{WeakAss}(M) = \emptyset$$

Proof. If $M = (0)$ then $\text{WeakAss}(M) = \emptyset$ by definition. Conversely, suppose that $M \neq 0$. Pick a nonzero element $m \in M$. Write $I = \{x \in R \mid xm = 0\}$ the annihilator of m. Then $R/I \subset M$. Hence $\text{WeakAss}(R/I) \subset \text{WeakAss}(M)$ by Lemma 10.66.4. But as $I \neq R$ we have $V(I) = \text{Spec}(R/I)$ contains a minimal prime, see Lemmas 10.17.2 and 10.17.7, and we win. $\square$

0589 Lemma 10.66.6. Let R be a ring. Let M be an R-module. Then

$$\text{Ass}(M) \subset \text{WeakAss}(M) \subset \text{Supp}(M).$$

Proof. The first inclusion is immediate from the definitions. If $\mathfrak{p} \in \text{WeakAss}(M)$, then by Lemma 10.66.2 we have $M_{\mathfrak{p}} \neq 0$, hence $\mathfrak{p} \in \text{Supp}(M)$. $\square$

05C3 Lemma 10.66.7. Let R be a ring. Let M be an R-module. The union $\bigcup_{\mathfrak{q} \in \text{WeakAss}(M)} \mathfrak{q}$ is the set of elements of R which are zerodivisors on M.

Proof. Suppose $f \in \mathfrak{q} \in \text{WeakAss}(M)$. Then there exists an element $m \in M$ such that $\mathfrak{q}$ is minimal over $I = \{x \in R \mid xm = 0\}$. Hence there exists a $g \in R$, $g \notin \mathfrak{q}$ and $n > 0$ such that $f^n gm = 0$. Note that $gm \neq 0$ as $g \notin I$. If we take n minimal as above, then $f(f^{n-1}gm) = 0$ and $f^{n-1}gm \neq 0$, so f is a zerodivisor on M. Conversely, suppose $f \in R$ is a zerodivisor on M. Consider the submodule $N = \{m \in M \mid fm = 0\}$. Since N is not zero it has a weakly associated prime $\mathfrak{q}$ by Lemma 10.66.5. Clearly $f \in \mathfrak{q}$ and by Lemma 10.66.4 $\mathfrak{q}$ is a weakly associated prime of M. $\square$

05C4 Lemma 10.66.8. Let R be a ring. Let M be an R-module. Any $\mathfrak{p} \in \text{Supp}(M)$ which is minimal among the elements of $\text{Supp}(M)$ is an element of $\text{WeakAss}(M)$.

Proof. Note that $\text{Supp}(M_{\mathfrak{p}}) = \{\mathfrak{p}R_{\mathfrak{p}}\}$ in $\text{Spec}(R_{\mathfrak{p}})$. In particular $M_{\mathfrak{p}}$ is nonzero, and hence $\text{WeakAss}(M_{\mathfrak{p}}) \neq \emptyset$ by Lemma 10.66.5. Since $\text{WeakAss}(M_{\mathfrak{p}}) \subset \text{Supp}(M_{\mathfrak{p}})$ by Lemma 10.66.6 we conclude that $\text{WeakAss}(M_{\mathfrak{p}}) = \{\mathfrak{p}R_{\mathfrak{p}}\}$, whence $\mathfrak{p} \in \text{WeakAss}(M)$ by Lemma 10.66.2. $\square$

058A Lemma 10.66.9. Let R be a ring. Let M be an R-module. Let $\mathfrak{p}$ be a prime ideal of R which is finitely generated. Then

$$\mathfrak{p} \in \text{Ass}(M) \Leftrightarrow \mathfrak{p} \in \text{WeakAss}(M).$$

In particular, if R is Noetherian, then $\text{Ass}(M) = \text{WeakAss}(M)$.

Proof. Write $\mathfrak{p} = (g_1, \ldots, g_n)$ for some $g_i \in R$. It is enough the prove the implication "$\Leftarrow$" as the other implication holds in general, see Lemma 10.66.6. Assume $\mathfrak{p} \in \text{WeakAss}(M)$. By Lemma 10.66.2 there exists an element $m \in M_\mathfrak{p}$ such that $I = \{x \in R_\mathfrak{p} \mid xm = 0\}$ has radical $\mathfrak{p}R_\mathfrak{p}$. Hence for each i there exists a smallest $e_i > 0$ such that $g_i^{e_i} m = 0$ in $M_\mathfrak{p}$. If $e_i > 1$ for some i, then we can replace m by $g_i^{e_i - 1} m \neq 0$ and decrease $\sum e_i$. Hence we may assume that the annihilator of $m \in M_\mathfrak{p}$ is $(g_1, \ldots, g_n)R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. By Lemma 10.63.15 we see that $\mathfrak{p} \in \text{Ass}(M)$. $\square$

05C5 Remark 10.66.10. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $\text{Spec}(\varphi)(\text{WeakAss}_S(M)) \subset \text{WeakAss}_R(M)$ contrary to the case of associated primes (see Lemma 10.63.11). An example is to consider the ring map

$$R = k[x_1, x_2, x_3, \ldots] \to S = k[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]/(x_1y_1, x_2y_2, x_3y_3, \ldots)$$

and $M = S$. In this case $\mathfrak{q} = \sum x_i S$ is a minimal prime of S, hence a weakly associated prime of $M = S$ (see Lemma 10.66.8). But on the other hand, for any nonzero element of S the annihilator in R is finitely generated, and hence does not have radical equal to $R \cap \mathfrak{q} = (x_1, x_2, x_3, \ldots)$ (details omitted).

05C6 Lemma 10.66.11. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then we have $\text{Spec}(\varphi)(\text{WeakAss}_S(M)) \supset \text{WeakAss}_R(M)$.

Proof. Let $\mathfrak{p}$ be an element of $\text{WeakAss}_R(M)$. Then there exists an $m \in M_\mathfrak{p}$ whose annihilator $I = \{x \in R_\mathfrak{p} \mid xm = 0\}$ has radical $\mathfrak{p}R_\mathfrak{p}$. Consider the annihilator $J = \{x \in S_\mathfrak{p} \mid xm = 0\}$ of m in $S_\mathfrak{p}$. As $IS_\mathfrak{p} \subset J$ we see that any minimal prime $\mathfrak{q} \subset S_\mathfrak{p}$ over J lies over $\mathfrak{p}$. Moreover such a $\mathfrak{q}$ corresponds to a weakly associated prime of M for example by Lemma 10.66.2. $\square$

05C7 Remark 10.66.12. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : \text{Spec}(S) \to \text{Spec}(R)$ the associated map on spectra. Then we have

$$f(\text{Ass}_S(M)) \subset \text{Ass}_R(M) \subset \text{WeakAss}_R(M) \subset f(\text{WeakAss}_S(M))$$

see Lemmas 10.63.11, 10.66.11, and 10.66.6. In general all of the inclusions may be strict, see Remarks 10.63.12 and 10.66.10. If S is Noetherian, then all the inclusions are equalities as the outer two are equal by Lemma 10.66.9.

05E1 Lemma 10.66.13. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : \text{Spec}(S) \to \text{Spec}(R)$ the associated map on spectra. If φ is a finite ring map, then

$$\text{WeakAss}_R(M) = f(\text{WeakAss}_S(M)).$$

Proof. One of the inclusions has already been proved, see Remark 10.66.12. To prove the other assume $\mathfrak{q} \in \text{WeakAss}_S(M)$ and let $\mathfrak{p}$ be the corresponding prime of R. Let $m \in M$ be an element such that $\mathfrak{q}$ is a minimal prime over $J = \{g \in S \mid gm = 0\}$. Thus the radical of $JS_\mathfrak{q}$ is $\mathfrak{q}S_\mathfrak{q}$. As $R \to S$ is finite there are finitely many primes $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_l$ over $\mathfrak{p}$, see Lemma 10.36.21. Pick $x \in \mathfrak{q}$ with $x \notin \mathfrak{q}_i$ for $i > 1$, see Lemma 10.15.2. By the above there exists an element $y \in S$, $y \notin \mathfrak{q}$

and an integer $t > 0$ such that $yx^t m = 0$. Thus the element $ym \in M$ is annihilated by x^t, hence ym maps to zero in $M_{\mathfrak{q}_i}$, $i = 2, \ldots, l$. To be sure, ym does not map to zero in $S_{\mathfrak{q}}$.

The ring $S_{\mathfrak{p}}$ is semi-local with maximal ideals $\mathfrak{q}_i S_{\mathfrak{p}}$ by going up for finite ring maps, see Lemma 10.36.22. If $f \in \mathfrak{p}R_{\mathfrak{p}}$ then some power of f ends up in $JS_{\mathfrak{q}}$ hence for some $n > 0$ we see that $f^t ym$ maps to zero in $M_{\mathfrak{q}}$. As ym vanishes at the other maximal ideals of $S_{\mathfrak{p}}$ we conclude that $f^t ym$ is zero in $M_{\mathfrak{p}}$, see Lemma 10.23.1. In this way we see that $\mathfrak{p}$ is a minimal prime over the annihilator of ym in R and we win. □

05C8 Lemma 10.66.14. Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ we have $\mathrm{WeakAss}_{R/I}(M) = \mathrm{WeakAss}_R(M)$.

Proof. Special case of Lemma 10.66.13. □

05C9 Lemma 10.66.15. Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ we have $\mathrm{WeakAss}_R(S^{-1}M) = \mathrm{WeakAss}_{S^{-1}R}(S^{-1}M)$ and

$$\mathrm{WeakAss}(M) \cap \mathrm{Spec}(S^{-1}R) = \mathrm{WeakAss}(S^{-1}M).$$

Proof. Suppose that $m \in S^{-1}M$. Let $I = \{x \in R \mid xm = 0\}$ and $I' = \{x' \in S^{-1}R \mid x'm = 0\}$. Then $I' = S^{-1}I$ and $I \cap S = \emptyset$ unless $I = R$ (verifications omitted). Thus primes in $S^{-1}R$ minimal over I' correspond bijectively to primes in R minimal over I and avoiding S. This proves the equality $\mathrm{WeakAss}_R(S^{-1}M) = \mathrm{WeakAss}_{S^{-1}R}(S^{-1}M)$. The second equality follows from Lemma 10.66.2 since for $\mathfrak{p} \in R$, $S \cap \mathfrak{p} = \emptyset$ we have $M_{\mathfrak{p}} = (S^{-1}M)_{S^{-1}\mathfrak{p}}$. □

05CA Lemma 10.66.16. Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzerodivisor on M. Then

$$\mathrm{WeakAss}(M) = \mathrm{WeakAss}(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we obtain $\mathrm{WeakAss}(M) \subset \mathrm{WeakAss}(S^{-1}M)$ from Lemma 10.66.4. Conversely, suppose that $n/s \in S^{-1}M$ is an element with annihilator I and $\mathfrak{p}$ a prime which is minimal over I. Then the annihilator of $n \in M$ is I and $\mathfrak{p}$ is a prime minimal over I. □

05CB Lemma 10.66.17. Let R be a ring. Let M be an R-module. The map

$$M \longrightarrow \prod\nolimits_{\mathfrak{p} \in \mathrm{WeakAss}(M)} M_{\mathfrak{p}}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Set $N = Rx \subset M$. If $\mathfrak{p}$ is a weakly associated prime of N we see on the one hand that $\mathfrak{p} \in \mathrm{WeakAss}(M)$ (Lemma 10.66.4) and on the other hand that $N_{\mathfrak{p}} \subset M_{\mathfrak{p}}$ is not zero. This contradiction shows that $\mathrm{WeakAss}(N) = \emptyset$. Hence $N = 0$, i.e., $x = 0$ by Lemma 10.66.5. □

05CC Lemma 10.66.18. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then

$$\mathrm{WeakAss}_S(N) = \mathrm{WeakAss}_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $\mathrm{Spec}(S \otimes_R K) \subset \mathrm{Spec}(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 10.66.16. □

0CUB Lemma 10.66.19. Let K/k be a field extension. Let R be a k-algebra. Let M be an R-module. Let $\mathfrak q \subset R \otimes_k K$ be a prime lying over $\mathfrak p \subset R$. If $\mathfrak q$ is weakly associated to $M \otimes_k K$, then $\mathfrak p$ is weakly associated to M.

Proof. Let $z \in M \otimes_k K$ be an element such that $\mathfrak q$ is minimal over the annihilator $J \subset R \otimes_k K$ of z. Choose a finitely generated subextension $K/L/k$ such that $z \in M \otimes_k L$. Since $R \otimes_k L \to R \otimes_k K$ is flat we see that $J = I(R \otimes_k K)$ where $I \subset R \otimes_k L$ is the annihilator of z in the smaller ring (Lemma 10.40.4). Thus $\mathfrak q \cap (R \otimes_k L)$ is minimal over I by going down (Lemma 10.39.19). In this way we reduce to the case described in the next paragraph.

Assume K/k is a finitely generated field extension. Let $x_1, \ldots, x_r \in K$ be a transcendence basis of K over k, see Fields, Section 9.26. Set $L = k(x_1, \ldots, x_r)$. Say $[K : L] = n$. Then $R \otimes_k L \to R \otimes_k K$ is a finite ring map. Hence $\mathfrak q \cap (R \otimes_k L)$ is a weakly associated prime of $M \otimes_k K$ viewed as a $R \otimes_k L$-module by Lemma 10.66.13. Since $M \otimes_k K \cong (M \otimes_k L)^{\oplus n}$ as a $R \otimes_k L$-module, we see that $\mathfrak q \cap (R \otimes_k L)$ is a weakly associated prime of $M \otimes_k L$ (for example by using Lemma 10.66.4 and induction). In this way we reduce to the case discussed in the next paragraph.

Assume $K = k(x_1, \ldots, x_r)$ is a purely transcendental field extension. We may replace R by $R_{\mathfrak p}$, M by $M_{\mathfrak p}$ and $\mathfrak q$ by $\mathfrak q(R_{\mathfrak p} \otimes_k K)$. See Lemma 10.66.15. In this way we reduce to the case discussed in the next paragraph.

Assume $K = k(x_1, \ldots, x_r)$ is a purely transcendental field extension and R is local with maximal ideal $\mathfrak p$. We claim that any $f \in R \otimes_k K$, $f \notin \mathfrak p(R \otimes_k K)$ is a nonzerodivisor on $M \otimes_k K$. Namely, let $z \in M \otimes_k K$ be an element. There is a finite R-submodule $M' \subset M$ such that $z \in M' \otimes_k K$ and such that M' is minimal with this property: choose a basis $\{t_\alpha\}$ of K as a k-vector space, write $z = \sum m_\alpha \otimes t_\alpha$ and let M' be the R-submodule generated by the m_α. If $z \in \mathfrak p(M' \otimes_k K) = \mathfrak p M' \otimes_k K$, then $\mathfrak p M' = M'$ and $M' = 0$ by Lemma 10.20.1 a contradiction. Thus z has nonzero image $\overline{z}$ in $M'/\mathfrak p M' \otimes_k K$ But $R/\mathfrak p \otimes_k K$ is a domain as a localization of $\kappa(\mathfrak p)[x_1, \ldots, x_n]$ and $M'/\mathfrak p M' \otimes_k K$ is a free module, hence $f\overline{z} \neq 0$. This proves the claim.

Finally, pick $z \in M \otimes_k K$ such that $\mathfrak q$ is minimal over the annihilator $J \subset R \otimes_k K$ of z. For $f \in \mathfrak p$ there exists an $n \geq 1$ and a $g \in R \otimes_k K$, $g \notin \mathfrak q$ such that $gf^n z \in J$, i.e., $gf^n z = 0$. (This holds because $\mathfrak q$ lies over $\mathfrak p$ and $\mathfrak q$ is minimal over J.) Above we have seen that g is a nonzerodivisor hence $f^n z = 0$. This means that $\mathfrak p$ is a weakly associated prime of $M \otimes_k K$ viewed as an R-module. Since $M \otimes_k K$ is a direct sum of copies of M we conclude that $\mathfrak p$ is a weakly associated prime of M as before. □

10.67. Embedded primes

02M4 Here is the definition.

02M5 Definition 10.67.1. Let R be a ring. Let M be an R-module.

(1) The associated primes of M which are not minimal among the associated primes of M are called the embedded associated primes of M.
(2) The embedded primes of R are the embedded associated primes of R as an R-module.

Here is a way to get rid of these.

02M6 Lemma 10.67.2. Let R be a Noetherian ring. Let M be a finite R-module. Consider the set of R-submodules

$$\{K \subset M \mid \text{Supp}(K) \text{ nowhere dense in } \text{Supp}(M)\}.$$

This set has a maximal element K and the quotient $M' = M/K$ has the following properties

(1) $\text{Supp}(M) = \text{Supp}(M')$,
(2) M' has no embedded associated primes,
(3) for any $f \in R$ which is contained in all embedded associated primes of M we have $M_f \cong M'_f$.

Proof. We will use Lemma 10.63.5 and Proposition 10.63.6 without further mention. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ denote the minimal primes in the support of M. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ denote the embedded associated primes of M. Then $\text{Ass}(M) = \{\mathfrak{q}_j, \mathfrak{p}_i\}$. Let

$$K = \{m \in M \mid \text{Supp}(Rm) \subset \bigcup V(\mathfrak{p}_i)\}$$

It is immediately seen to be a submodule. Since M is finite over a Noetherian ring, we know K is finite too. Hence $\text{Supp}(K)$ is nowhere dense in $\text{Supp}(M)$. Let $K' \subset M$ be another submodule with support nowhere dense in $\text{Supp}(M)$. This means that $K_{\mathfrak{q}_j} = 0$. Hence if $m \in K'$, then m maps to zero in $M_{\mathfrak{q}_j}$ which in turn implies $(Rm)_{\mathfrak{q}_j} = 0$. On the other hand we have $\text{Ass}(Rm) \subset \text{Ass}(M)$. Hence the support of Rm is contained in $\bigcup V(\mathfrak{p}_i)$. Therefore $m \in K$ and thus $K' \subset K$ as m was arbitrary in K'.

Let $M' = M/K$. Since $K_{\mathfrak{q}_j} = 0$ we know $M'_{\mathfrak{q}_j} = M_{\mathfrak{q}_j}$ for all j. Hence M and M' have the same support.

Suppose $\mathfrak{q} = \text{Ann}(\overline{m}) \in \text{Ass}(M')$ where $\overline{m} \in M'$ is the image of $m \in M$. Then $m \notin K$ and hence the support of Rm must contain one of the $\mathfrak{q}_j$. Since $M_{\mathfrak{q}_j} = M'_{\mathfrak{q}_j}$, we know $\overline{m}$ does not map to zero in $M'_{\mathfrak{q}_j}$. Hence $\mathfrak{q} \subset \mathfrak{q}_j$ (actually we have equality), which means that all the associated primes of M' are not embedded.

Let f be an element contained in all $\mathfrak{p}_i$. Then $D(f) \cap \text{supp}(K) = 0$. Hence $M_f = M'_f$ because $K_f = 0$. □

02M7 Lemma 10.67.3. Let R be a Noetherian ring. Let M be a finite R-module. For any $f \in R$ we have $(M')_f = (M_f)'$ where $M \to M'$ and $M_f \to (M_f)'$ are the quotients constructed in Lemma 10.67.2.

Proof. Omitted. □

02M8 Lemma 10.67.4. Let R be a Noetherian ring. Let M be a finite R-module without embedded associated primes. Let $I = \{x \in R \mid xM = 0\}$. Then the ring R/I has no embedded primes.

Proof. We may replace R by R/I. Hence we may assume every nonzero element of R acts nontrivially on M. By Lemma 10.40.5 this implies that $\text{Spec}(R)$ equals the support of M. Suppose that $\mathfrak{p}$ is an embedded prime of R. Let $x \in R$ be an element whose annihilator is $\mathfrak{p}$. Consider the nonzero module $N = xM \subset M$. It is annihilated by $\mathfrak{p}$. Hence any associated prime $\mathfrak{q}$ of N contains $\mathfrak{p}$ and is also an associated prime of M. Then $\mathfrak{q}$ would be an embedded associated prime of M which contradicts the assumption of the lemma. □

10.68. Regular sequences

0AUH In this section we develop some basic properties of regular sequences.

00LF **Definition 10.68.1.** Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_r$ of R is called an *M-regular sequence* if the following conditions hold:

1. f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$ for each $i = 1, \ldots, r$, and
2. the module $M/(f_1, \ldots, f_r)M$ is not zero.

If I is an ideal of R and $f_1, \ldots, f_r \in I$ then we call $f_1, \ldots, f_r$ an *M-regular sequence in I*. If $M = R$, we call $f_1, \ldots, f_r$ simply a *regular sequence* (in I).

Please pay attention to the fact that the definition depends on the order of the elements $f_1, \ldots, f_r$ (see examples below). Some papers/books drop the requirement that the module $M/(f_1, \ldots, f_r)M$ is nonzero. This has the advantage that being a regular sequence is preserved under localization. However, we will use this definition mainly to define the depth of a module in case R is local; in that case the f_i are required to be in the maximal ideal – a condition which is not preserved under going from R to a localization $R_{\mathfrak{p}}$.

00LG **Example 10.68.2.** Let k be a field. In the ring $k[x, y, z]$ the sequence $x, y(1-x), z(1-x)$ is regular but the sequence $y(1-x), z(1-x), x$ is not.

00LH **Example 10.68.3.** Let k be a field. Consider the ring $k[x, y, w_0, w_1, w_2, \ldots]/I$ where I is generated by yw_i, $i = 0, 1, 2, \ldots$ and $w_i - xw_{i+1}$, $i = 0, 1, 2, \ldots$. The sequence x, y is regular, but y is a zerodivisor. Moreover you can localize at the maximal ideal (x, y, w_i) and still get an example.

00LJ **Lemma 10.68.4.** Let R be a local Noetherian ring. Let M be a finite R-module. Let $x_1, \ldots, x_c$ be an M-regular sequence. Then any permutation of the x_i is a regular sequence as well.

Proof. First we do the case $c = 2$. Consider $K \subset M$ the kernel of $x_2 : M \to M$. For any $z \in K$ we know that $z = x_1 z'$ for some $z' \in M$ because x_2 is a nonzerodivisor on $M/x_1 M$. Because x_1 is a nonzerodivisor on M we see that $x_2 z' = 0$ as well. Hence $x_1 : K \to K$ is surjective. Thus $K = 0$ by Nakayama's Lemma 10.20.1. Next, consider multiplication by x_1 on $M/x_2 M$. If $z \in M$ maps to an element $\overline{z} \in M/x_2 M$ in the kernel of this map, then $x_1 z = x_2 y$ for some $y \in M$. But then since x_1, x_2 is a regular sequence we see that $y = x_1 y'$ for some $y' \in M$. Hence $x_1(z - x_2 y') = 0$ and hence $z = x_2 y'$ and hence $\overline{z} = 0$ as desired.

For the general case, observe that any permutation is a composition of transpositions of adjacent indices. Hence it suffices to prove that

$$x_1, \ldots, x_{i-2}, x_i, x_{i-1}, x_{i+1}, \ldots, x_c$$

is an M-regular sequence. This follows from the case we just did applied to the module $M/(x_1, \ldots, x_{i-2})$ and the length 2 regular sequence x_{i-1}, x_i. $\square$

00LM Lemma 10.68.5. Let R, S be local rings. Let $R \to S$ be a flat local ring homomorphism. Let $x_1, \ldots, x_r$ be a sequence in R. Let M be an R-module. The following are equivalent

(1) $x_1, \ldots, x_r$ is an M-regular sequence in R, and
(2) the images of $x_1, \ldots, x_r$ in S form a $M \otimes_R S$-regular sequence.

Proof. This is so because $R \to S$ is faithfully flat by Lemma 10.39.17. $\square$

061L Lemma 10.68.6. Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak{p}$ be a prime. Let $x_1, \ldots, x_r$ be a sequence in R whose image in $R_{\mathfrak{p}}$ forms an $M_{\mathfrak{p}}$-regular sequence. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the image of $x_1, \ldots, x_r$ in R_g forms an M_g-regular sequence.

Proof. Set

$$K_i = \operatorname{Ker}\left(x_i : M/(x_1, \ldots, x_{i-1})M \to M/(x_1, \ldots, x_{i-1})M\right).$$

This is a finite R-module whose localization at $\mathfrak{p}$ is zero by assumption. Hence there exists a $g \in R$, $g \notin \mathfrak{p}$ such that $(K_i)_g = 0$ for all $i = 1, \ldots, r$. This g works. $\square$

065K Lemma 10.68.7. Let A be a ring. Let I be an ideal generated by a regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A.

Proof. This follows immediately from the definitions. $\square$

0F1T Lemma 10.68.8. Let R be a ring. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. Let $f_1, \ldots, f_r \in R$. If $f_1, \ldots, f_r$ is M_1-regular and M_3-regular, then $f_1, \ldots, f_r$ is M_2-regular.

Proof. By Lemma 10.4.1, if $f_1 : M_1 \to M_1$ and $f_1 : M_3 \to M_3$ are injective, then so is $f_1 : M_2 \to M_2$ and we obtain a short exact sequence

$$0 \to M_1/f_1M_1 \to M_2/f_1M_2 \to M_3/f_1M_3 \to 0$$

The lemma follows from this and induction on r. Some details omitted. $\square$

07DV Lemma 10.68.9. Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_r \in R$ and $e_1, \ldots, e_r > 0$ integers. Then $f_1, \ldots, f_r$ is an M-regular sequence if and only if $f_1^{e_1}, \ldots, f_r^{e_r}$ is an M-regular sequence.

Proof. We will prove this by induction on r. If $r = 1$ this follows from the following two easy facts: (a) a power of a nonzerodivisor on M is a nonzerodivisor on M and (b) a divisor of a nonzerodivisor on M is a nonzerodivisor on M. If $r > 1$, then by induction applied to M/f_1M we have that $f_1, f_2, \ldots, f_r$ is an M-regular sequence if and only if $f_1, f_2^{e_2}, \ldots, f_r^{e_r}$ is an M-regular sequence. Thus it suffices to show, given $e > 0$, that $f_1^e, f_2, \ldots, f_r$ is an M-regular sequence if and only if $f_1, \ldots, f_r$ is an M-regular sequence. We will prove this by induction on e. The case $e = 1$ is trivial. Since f_1 is a nonzerodivisor under both assumptions (by the case $r = 1$) we have a short exact sequence

$$0 \to M/f_1M \xrightarrow{f_1^{e-1}} M/f_1^eM \to M/f_1^{e-1}M \to 0$$

Suppose that $f_1, f_2, \ldots, f_r$ is an M-regular sequence. Then by induction the elements $f_2, \ldots, f_r$ are M/f_1M and $M/f_1^{e-1}M$-regular sequences. By Lemma 10.68.8 $f_2, \ldots, f_r$ is M/f_1^eM-regular. Hence $f_1^e, f_2, \ldots, f_r$ is M-regular. Conversely, suppose that $f_1^e, f_2, \ldots, f_r$ is an M-regular sequence. Then $f_2 : M/f_1^eM \to M/f_1^eM$ is injective, hence $f_2 : M/f_1M \to M/f_1M$ is injective, hence by induction(!) $f_2 : M/f_1^{e-1}M \to M/f_1^{e-1}M$ is injective, hence

$$0 \to M/(f_1, f_2)M \xrightarrow{f_1^{e-1}} M/(f_1^e, f_2)M \to M/(f_1^{e-1}, f_2)M \to 0$$

is a short exact sequence by Lemma 10.4.1. This proves the converse for $r = 2$. If $r > 2$, then we have $f_3 : M/(f_1^e, f_2)M \to M/(f_1^e, f_2)M$ is injective, hence $f_3 : M/(f_1, f_2)M \to M/(f_1, f_2)M$ is injective, and so on. Some details omitted. □

07DW Lemma 10.68.10. Let R be a ring. Let $f_1, \ldots, f_r \in R$ which do not generate the unit ideal. The following are equivalent:

(1) any permutation of $f_1, \ldots, f_r$ is a regular sequence,
(2) any subsequence of $f_1, \ldots, f_r$ (in the given order) is a regular sequence, and
(3) $f_1x_1, \ldots, f_rx_r$ is a regular sequence in the polynomial ring $R[x_1, \ldots, x_r]$.

Proof. It is clear that (1) implies (2). We prove (2) implies (1) by induction on r. The case $r = 1$ is trivial. The case $r = 2$ says that if $a, b \in R$ are a regular sequence and b is a nonzerodivisor, then b, a is a regular sequence. This is clear because the kernel of $a : R/(b) \to R/(b)$ is isomorphic to the kernel of $b : R/(a) \to R/(a)$ if both a and b are nonzerodivisors. The case $r > 2$. Assume (2) holds and say we want to prove $f_{\sigma(1)}, \ldots, f_{\sigma(r)}$ is a regular sequence for some permutation σ. We already know that $f_{\sigma(1)}, \ldots, f_{\sigma(r-1)}$ is a regular sequence by induction. Hence it suffices to show that f_s where $s = \sigma(r)$ is a nonzerodivisor modulo $f_1, \ldots, \hat{f}_s, \ldots, f_r$. If $s = r$ we are done. If $s < r$, then note that f_s and f_r are both nonzerodivisors in the ring $R/(f_1, \ldots, \hat{f}_s, \ldots, f_{r-1})$ (by induction hypothesis again). Since we know f_s, f_r is a regular sequence in that ring we conclude by the case of sequence of length 2 that f_r, f_s is too.

Note that $R[x_1, \ldots, x_r]/(f_1x_1, \ldots, f_ix_i)$ as an R-module is a direct sum of the modules

$$R/I_E \cdot x_1^{e_1} \ldots x_r^{e_r}$$

indexed by multi-indices $E = (e_1, \ldots, e_r)$ where I_E is the ideal generated by f_j for $1 \leq j \leq i$ with $e_j > 0$. Hence $f_{i+1}x_i$ is a nonzerodivisor on this if and only if f_{i+1} is a nonzerodivisor on R/I_E for all E. Taking E with all positive entries, we see that f_{i+1} is a nonzerodivisor on $R/(f_1, \ldots, f_i)$. Thus (3) implies (2). Conversely, if (2) holds, then any subsequence of $f_1, \ldots, f_i, f_{i+1}$ is a regular sequence in particular f_{i+1} is a nonzerodivisor on all R/I_E. In this way we see that (2) implies (3). □

10.69. Quasi-regular sequences

061M We introduce the notion of quasi-regular sequence which is slightly weaker than that of a regular sequence and easier to use. Let R be a ring and let $f_1, \ldots, f_c \in R$. Set $J = (f_1, \ldots, f_c)$. Let M be an R-module. Then there is a canonical map

061N (10.69.0.1) $$M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^nM/J^{n+1}M$$

of graded $R/J[X_1, \ldots, X_c]$-modules defined by the rule

$$\overline{m} \otimes X_1^{e_1} \ldots X_c^{e_c} \longmapsto f_1^{e_1} \ldots f_c^{e_c} m \bmod J^{e_1+\ldots+e_c+1}M.$$

Note that (10.69.0.1) is always surjective.

061P Definition 10.69.1. Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_c$ of R is called M-quasi-regular if (10.69.0.1) is an isomorphism. If $M = R$, we call $f_1, \ldots, f_c$ simply a quasi-regular sequence.

So if $f_1, \ldots, f_c$ is a quasi-regular sequence, then

$$R/J[X_1, \ldots, X_c] = \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

where $J = (f_1, \ldots, f_c)$. It is clear that being a quasi-regular sequence is independent of the order of $f_1, \ldots, f_c$.

00LN Lemma 10.69.2. Let R be a ring.

(1) A regular sequence $f_1, \ldots, f_c$ of R is a quasi-regular sequence.
(2) Suppose that M is an R-module and that $f_1, \ldots, f_c$ is an M-regular sequence. Then $f_1, \ldots, f_c$ is an M-quasi-regular sequence.

Proof. Set $J = (f_1, \ldots, f_c)$. We prove the first assertion by induction on c. We have to show that given any relation $\sum_{|I|=n} a_I f^I \in J^{n+1}$ with $a_I \in R$ we actually have $a_I \in J$ for all multi-indices I. Since any element of J^{n+1} is of the form $\sum_{|I|=n} b_I f^I$ with $b_I \in J$ we may assume, after replacing a_I by $a_I - b_I$, the relation reads $\sum_{|I|=n} a_I f^I = 0$. We can rewrite this as

$$\sum\nolimits_{e=0}^{n} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Here and below the "primed" multi-indices I' are required to be of the form $I' = (i_1, \ldots, i_{c-1}, 0)$. We will show by descending induction on $l \in \{0, \ldots, n\}$ that if we have a relation

$$\sum\nolimits_{e=0}^{l} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

then $a_{I',e} \in J$ for all I', e. Namely, set $J' = (f_1, \ldots, f_{c-1})$. Observe that $\sum_{|I'|=n-l} a_{I',l} f^{I'}$ is mapped into $(J')^{n-l+1}$ by f_c^l. By induction hypothesis (for the induction on c) we see that $f_c^l a_{I',l} \in J'$. Because f_c is not a zerodivisor on R/J' (as $f_1, \ldots, f_c$ is a regular sequence) we conclude that $a_{I',l} \in J'$. This allows us to rewrite the term $(\sum_{|I'|=n-l} a_{I',l} f^{I'}) f_c^l$ in the form $(\sum_{|I'|=n-l+1} f_c b_{I',l-1} f^{I'}) f_c^{l-1}$. This gives a new relation of the form

$$\left(\sum\nolimits_{|I'|=n-l+1} (a_{I',l-1} + f_c b_{I',l-1}) f^{I'} \right) f_c^{l-1} + \sum\nolimits_{e=0}^{l-2} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Now by the induction hypothesis (on l this time) we see that all $a_{I',l-1} + f_c b_{I',l-1} \in J$ and all $a_{I',e} \in J$ for $e \leq l-2$. This, combined with $a_{I',l} \in J' \subset J$ seen above, finishes the proof of the induction step.

The second assertion means that given any formal expression $F = \sum_{|I|=n} m_I X^I$, $m_I \in M$ with $\sum m_I f^I \in J^{n+1}M$, then all the coefficients m_I are in J. This is proved in exactly the same way as we prove the corresponding result for the first assertion above. $\square$

065L Lemma 10.69.3. Let $R \to R'$ be a flat ring map. Let M be an R-module. Suppose that $f_1, \ldots, f_r \in R$ form an M-quasi-regular sequence. Then the images of $f_1, \ldots, f_r$ in R' form a $M \otimes_R R'$-quasi-regular sequence.

Proof. Set $J = (f_1, \ldots, f_r)$, $J' = JR'$ and $M' = M \otimes_R R'$. We have to show the canonical map $\mu : R'/J'[X_1, \ldots X_r] \otimes_{R'/J'} M'/J'M' \to \bigoplus (J')^n M'/(J')^{n+1}M'$ is an isomorphism. Because $R \to R'$ is flat the sequences $0 \to J^nM \to M$ and $0 \to J^{n+1}M \to J^nM \to J^nM/J^{n+1}M \to 0$ remain exact on tensoring with R'. This first implies that $J^nM \otimes_R R' = (J')^nM'$ and then that $(J')^nM'/(J')^{n+1}M' = J^nM/J^{n+1}M \otimes_R R'$. Thus μ is the tensor product of (10.69.0.1), which is an isomorphism by assumption, with $\mathrm{id}_{R'}$ and we conclude. ☐

061Q Lemma 10.69.4. Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak p$ be a prime. Let $x_1, \ldots, x_c$ be a sequence in R whose image in $R_{\mathfrak p}$ forms an $M_{\mathfrak p}$-quasi-regular sequence. Then there exists a $g \in R$, $g \not\in \mathfrak p$ such that the image of $x_1, \ldots, x_c$ in R_g forms an M_g-quasi-regular sequence.

Proof. Consider the kernel K of the map (10.69.0.1). As $M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c]$ is a finite $R/J[X_1, \ldots, X_c]$-module and as $R/J[X_1, \ldots, X_c]$ is Noetherian, we see that K is also a finite $R/J[X_1, \ldots, X_c]$-module. Pick homogeneous generators $k_1, \ldots, k_t \in K$. By assumption for each $i = 1, \ldots, t$ there exists a $g_i \in R$, $g_i \not\in \mathfrak p$ such that $g_ik_i = 0$. Hence $g = g_1 \ldots g_t$ works. ☐

061R Lemma 10.69.5. Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_c \in R$ be an M-quasi-regular sequence. For any i the sequence $\overline{f}_{i+1}, \ldots, \overline{f}_c$ of $\overline{R} = R/(f_1, \ldots, f_i)$ is an $\overline{M} = M/(f_1, \ldots, f_i)M$-quasi-regular sequence.

Proof. It suffices to prove this for $i = 1$. Set $\overline{J} = (\overline{f}_2, \ldots, \overline{f}_c) \subset \overline{R}$. Then

$$\begin{aligned} \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M} &= (J^nM + f_1M)/(J^{n+1}M + f_1M) \\ &= J^nM/(J^{n+1}M + J^nM \cap f_1M). \end{aligned}$$

Thus, in order to prove the lemma it suffices to show that $J^{n+1}M + J^nM \cap f_1M = J^{n+1}M + f_1J^{n-1}M$ because that will show that $\bigoplus_{n \geq 0} \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M}$ is the quotient of $\bigoplus_{n \geq 0} J^nM/J^{n+1} \cong M/JM[X_1, \ldots, X_c]$ by X_1. Actually, we have $J^nM \cap f_1M = f_1J^{n-1}M$. Namely, if $m \not\in J^{n-1}M$, then $f_1m \not\in J^nM$ because $\bigoplus J^nM/J^{n+1}M$ is the polynomial algebra $M/J[X_1, \ldots, X_c]$ by assumption. ☐

061S Lemma 10.69.6. Let $(R, \mathfrak m)$ be a local Noetherian ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_c \in \mathfrak m$ be an M-quasi-regular sequence. Then $f_1, \ldots, f_c$ is an M-regular sequence.

Proof. Set $J = (f_1, \ldots, f_c)$. Let us show that f_1 is a nonzerodivisor on M. Suppose $x \in M$ is not zero. By Krull's intersection theorem there exists an integer r such that $x \in J^rM$ but $x \not\in J^{r+1}M$, see Lemma 10.51.4. Then $f_1x \in J^{r+1}M$ is an element whose class in $J^{r+1}M/J^{r+2}M$ is nonzero by the assumed structure of $\bigoplus J^nM/J^{n+1}M$. Whence $f_1x \neq 0$.

Now we can finish the proof by induction on c using Lemma 10.69.5. ☐

061T Remark 10.69.7 (Other types of regular sequences). In the paper [Kab71] the author discusses two more regularity conditions for sequences $x_1, \ldots, x_r$ of elements of a ring R. Namely, we say the sequence is Koszul-regular if $H_i(K_\bullet(R, x_\bullet)) = 0$

for $i \geq 1$ where $K_\bullet(R, x_\bullet)$ is the Koszul complex. The sequence is called H_1-regular if $H_1(K_\bullet(R, x_\bullet)) = 0$. One has the implications regular $\Rightarrow$ Koszul-regular $\Rightarrow$ H_1-regular $\Rightarrow$ quasi-regular. By examples the author shows that these implications cannot be reversed in general even if R is a (non-Noetherian) local ring and the sequence generates the maximal ideal of R. We introduce these notions in more detail in More on Algebra, Section 15.30.

065M Remark 10.69.8. Let k be a field. Consider the ring

$$A = k[x, y, w, z_0, z_1, z_2, \ldots]/(y^2z_0 - wx, z_0 - yz_1, z_1 - yz_2, \ldots)$$

In this ring x is a nonzerodivisor and the image of y in A/xA gives a quasi-regular sequence. But it is not true that x, y is a quasi-regular sequence in A because $(x, y)/(x, y)^2$ isn't free of rank two over $A/(x, y)$ due to the fact that $wx = 0$ in $(x, y)/(x, y)^2$ but w isn't zero in $A/(x, y)$. Hence the analogue of Lemma 10.68.7 does not hold for quasi-regular sequences.

065N Lemma 10.69.9. Let R be a ring. Let $J = (f_1, \ldots, f_r)$ be an ideal of R. Let M be an R-module. Set $\overline{R} = R/\bigcap_{n\geq 0} J^n$, $\overline{M} = M/\bigcap_{n\geq 0} J^nM$, and denote $\overline{f}_i$ the image of f_i in $\overline{R}$. Then $f_1, \ldots, f_r$ is M-quasi-regular if and only if $\overline{f}_1, \ldots, \overline{f}_r$ is $\overline{M}$-quasi-regular.

Proof. This is true because $J^nM/J^{n+1}M \cong \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M}$. $\square$

10.70. Blow up algebras

052P In this section we make some elementary observations about blowing up.

052Q Definition 10.70.1. Let R be a ring. Let $I \subset R$ be an ideal.

(1) The blowup algebra, or the Rees algebra, associated to the pair (R, I) is the graded R-algebra
$$\text{Bl}_I(R) = \bigoplus\nolimits_{n\geq 0} I^n = R \oplus I \oplus I^2 \oplus \ldots$$
where the summand I^n is placed in degree n.

(2) Let $a \in I$ be an element. Denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra. Then the affine blowup algebra $R[\frac{I}{a}]$ is the algebra $(\text{Bl}_I(R))_{(a^{(1)})}$ constructed in Section 10.57.

In other words, an element of $R[\frac{I}{a}]$ is represented by an expression of the form x/a^n with $x \in I^n$. Two representatives x/a^n and y/a^m define the same element if and only if $a^k(a^mx - a^ny) = 0$ for some $k \geq 0$.

07Z3 Lemma 10.70.2. Let R be a ring, $I \subset R$ an ideal, and $a \in I$. Let $R' = R[\frac{I}{a}]$ be the affine blowup algebra. Then

(1) the image of a in R' is a nonzerodivisor,
(2) $IR' = aR'$, and
(3) $(R')_a = R_a$.

Proof. Immediate from the description of $R[\frac{I}{a}]$ above. $\square$

0BIP Lemma 10.70.3. Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal and $a \in I$. Set $J = IS$ and let $b \in J$ be the image of a. Then $S[\frac{J}{b}]$ is the quotient of $S \otimes_R R[\frac{I}{a}]$ by the ideal of elements annihilated by some power of b.

Proof. Let S' be the quotient of $S \otimes_R R[\frac{I}{a}]$ by its b-power torsion elements. The ring map

$$S \otimes_R R[\tfrac{I}{a}] \longrightarrow S[\tfrac{J}{b}]$$

is surjective and annihilates a-power torsion as b is a nonzerodivisor in $S[\frac{J}{b}]$. Hence we obtain a surjective map $S' \to S[\frac{J}{b}]$. To see that the kernel is trivial, we construct an inverse map. Namely, let $z = y/b^n$ be an element of $S[\frac{J}{b}]$, i.e., $y \in J^n$. Write $y = \sum x_i s_i$ with $x_i \in I^n$ and $s_i \in S$. We map z to the class of $\sum s_i \otimes x_i/a^n$ in S'. This is well defined because an element of the kernel of the map $S \otimes_R I^n \to J^n$ is annihilated by a^n, hence maps to zero in S'. $\square$

0G8Q Example 10.70.4. Let R be a ring. Let $P = R[t_1, \ldots, t_n]$ be the polynomial algebra. Let $I = (t_1, \ldots, t_n) \subset P$. With notation as in Definition 10.70.1 there is an isomorphism

$$P[T_1, \ldots, T_n]/(t_iT_j - t_jT_i) \longrightarrow \text{Bl}_I(P)$$

sending T_i to $t_i^{(1)}$. We leave it to the reader to show that this map is well defined. Since I is generated by $t_1, \ldots, t_n$ we see that our map is surjective. To see that our map is injective one has to show: for each $e \geq 1$ the P-module I^e is generated by the monomials $t^E = t_1^{e_1} \ldots x_n^{e_n}$ for multiindices $E = (e_1, \ldots, e_n)$ of degree $|E| = e$ subject only to the relations $t_it^E = t_jt^{E'}$ when $|E| = |E'| = e$ and $e_a + \delta_{ai} = e'_a + \delta_{aj},\ a = 1, \ldots, n$ (Kronecker delta). We omit the details.

0G8R Example 10.70.5. Let R be a ring. Let $P = R[t_1, \ldots, t_n]$ be the polynomial algebra. Let $I = (t_1, \ldots, t_n) \subset P$. Let $a = t_1$. With notation as in Definition 10.70.1 there is an isomorphism

$$P[x_2, \ldots, x_n]/(t_1x_2 - t_2, \ldots, t_1x_n - t_n) \longrightarrow P[\tfrac{I}{a}] = P[\tfrac{I}{t_1}]$$

sending x_i to t_i/t_1. We leave it to the reader to show that this map is well defined. Since I is generated by $t_1, \ldots, t_n$ we see that our map is surjective. To see that our map is injective, the reader can argue that the source and target of our map are t_1-torsion free and that the map is an isomorphism after inverting t_1, see Lemma 10.70.2. Alternatively, the reader can use the description of the Rees algebra in Example 10.70.4. We omit the details.

0G8S Lemma 10.70.6. Let R be a ring. Let $I = (a_1, \ldots, a_n)$ be an ideal of R. Let $a = a_1$. Then there is a surjection

$$R[x_2, \ldots, x_n]/(ax_2 - a_2, \ldots, ax_n - a_n) \longrightarrow R[\tfrac{I}{a}]$$

whose kernel is the a-power torsion in the source.

Proof. Consider the ring map $P = \mathbf{Z}[t_1, \ldots, t_n] \to R$ sending t_i to a_i. Set $J = (t_1, \ldots, t_n)$. By Example 10.70.5 we have $P[\frac{J}{t_1}] = P[x_2, \ldots, x_n]/(t_1x_2 - t_2, \ldots, t_1x_n - t_n)$. Apply Lemma 10.70.3 to the map $P \to A$ to conclude. $\square$

080U Lemma 10.70.7. Let R be a ring, $I \subset R$ an ideal, and $a \in I$. Set $R' = R[\frac{I}{a}]$. If $f \in R$ is such that $V(f) = V(I)$, then f maps to a nonzerodivisor in R' and $R'_f = R'_a = R_a$.

Proof. We will use the results of Lemma 10.70.2 without further mention. The assumption $V(f) = V(I)$ implies $V(fR') = V(IR') = V(aR')$. Hence $a^n = fb$ and $f^m = ac$ for some $b, c \in R'$. The lemma follows. $\square$

0BBI Lemma 10.70.8. Let R be a ring, $I \subset R$ an ideal, $a \in I$, and $f \in R$. Set $R' = R[\frac{I}{a}]$ and $R'' = R[\frac{fI}{fa}]$. Then there is a surjective R-algebra map $R' \to R''$ whose kernel is the set of f-power torsion elements of R'.

Proof. The map is given by sending x/a^n for $x \in I^n$ to $f^n x/(fa)^n$. It is straightforward to check this map is well defined and surjective. Since af is a nonzero divisor in R'' (Lemma 10.70.2) we see that the set of f-power torsion elements are mapped to zero. Conversely, if $x \in R'$ and $f^n x \neq 0$ for all $n > 0$, then $(af)^n x \neq 0$ for all n as a is a nonzero divisor in R'. It follows that the image of x in R'' is not zero by the description of R'' following Definition 10.70.1. □

052S Lemma 10.70.9. If R is reduced then every (affine) blowup algebra of R is reduced.

Proof. Let $I \subset R$ be an ideal and $a \in I$. Suppose x/a^n with $x \in I^n$ is a nilpotent element of $R[\frac{I}{a}]$. Then $(x/a^n)^m = 0$. Hence $a^N x^m = 0$ in R for some $N \geq 0$. After increasing N if necessary we may assume $N = me$ for some $e \geq 0$. Then $(a^e x)^m = 0$ and since R is reduced we find $a^e x = 0$. This means that $x/a^n = 0$ in $R[\frac{I}{a}]$. □

052R Lemma 10.70.10. Let R be a domain, $I \subset R$ an ideal, and $a \in I$ a nonzero element. Then the affine blowup algebra $R[\frac{I}{a}]$ is a domain.

Proof. Suppose x/a^n, y/a^m with $x \in I^n$, $y \in I^m$ are elements of $R[\frac{I}{a}]$ whose product is zero. Then $a^N xy = 0$ in R. Since R is a domain we conclude that either $x = 0$ or $y = 0$. □

052T Lemma 10.70.11. Let R be a ring. Let $I \subset R$ be an ideal. Let $a \in I$. If a is not contained in any minimal prime of R, then $\text{Spec}(R[\frac{I}{a}]) \to \text{Spec}(R)$ has dense image.

Proof. If $a^k x = 0$ for $x \in R$, then x is contained in all the minimal primes of R and hence nilpotent, see Lemma 10.17.2. Thus the kernel of $R \to R[\frac{I}{a}]$ consists of nilpotent elements. Hence the result follows from Lemma 10.30.6. □

052M Lemma 10.70.12. Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let $R \subset A \subset K$ be a valuation ring which dominates R. Then

$$A = \text{colim}\, R[\tfrac{I}{a}]$$

is a directed colimit of affine blowups $R \to R[\frac{I}{a}]$ with the following properties

(1) $a \in I \subset \mathfrak{m}$,
(2) I is finitely generated, and
(3) the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero.

Proof. Any blowup algebra $R[\frac{I}{a}]$ is a domain contained in K see Lemma 10.70.10. The lemma simply says that A is the directed union of the ones where $a \in I$ have properties (1), (2), (3). If $R[\frac{I}{a}] \subset A$ and $R[\frac{J}{b}] \subset A$, then we have

$$R[\tfrac{I}{a}] \cup R[\tfrac{J}{b}] \subset R[\tfrac{IJ}{ab}] \subset A$$

The first inclusion because $x/a^n = b^n x/(ab)^n$ and the second one because if $z \in (IJ)^n$, then $z = \sum x_i y_i$ with $x_i \in I^n$ and $y_i \in J^n$ and hence $z/(ab)^n = \sum (x_i/a^n)(y_i/b^n)$ is contained in A.

Consider a finite subset $E \subset A$. Say $E = \{e_1, \ldots, e_n\}$. Choose a nonzero $a \in R$ such that we can write $e_i = f_i/a$ for all $i = 1, \ldots, n$. Set $I = (f_1, \ldots, f_n, a)$. We claim that $R[\frac{I}{a}] \subset A$. This is clear as an element of $R[\frac{I}{a}]$ can be represented as a polynomial in the elements e_i. The lemma follows immediately from this observation. $\square$

10.71. Ext groups

00LO In this section we do a tiny bit of homological algebra, in order to establish some fundamental properties of depth over Noetherian local rings.

00LP Lemma 10.71.1. Let R be a ring. Let M be an R-module.

(1) There exists an exact complex
$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0.$$
with F_i free R-modules.

(2) If R is Noetherian and M finite over R, then we can choose the complex such that F_i is finite free. In other words, we can find an exact complex
$$\ldots \to R^{\oplus n_2} \to R^{\oplus n_1} \to R^{\oplus n_0} \to M \to 0.$$

Proof. Let us explain only the Noetherian case. As a first step choose a surjection $R^{n_0} \to M$. Then having constructed an exact complex of length e we simply choose a surjection $R^{n_{e+1}} \to \mathrm{Ker}(R^{n_e} \to R^{n_{e-1}})$ which is possible because R is Noetherian. $\square$

00LQ Definition 10.71.2. Let R be a ring. Let M be an R-module.

(1) A (left) resolution $F_\bullet \to M$ of M is an exact complex
$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$
of R-modules.

(2) A resolution of M by free R-modules is a resolution $F_\bullet \to M$ where each F_i is a free R-module.

(3) A resolution of M by finite free R-modules is a resolution $F_\bullet \to M$ where each F_i is a finite free R-module.

We often use the notation $F_\bullet$ to denote a complex of R-modules
$$\ldots \to F_i \to F_{i-1} \to \ldots$$
In this case we often use d_i or $d_{F,i}$ to denote the map $F_i \to F_{i-1}$. In this section we are always going to assume that F_0 is the last nonzero term in the complex. The ith homology group of the complex $F_\bullet$ is the group $H_i = \mathrm{Ker}(d_{F,i})/\operatorname{Im}(d_{F,i+1})$. A map of complexes $\alpha : F_\bullet \to G_\bullet$ is given by maps $\alpha_i : F_i \to G_i$ such that $\alpha_{i-1} \circ d_{F,i} = d_{G,i-1} \circ \alpha_i$. Such a map induces a map on homology $H_i(\alpha) : H_i(F_\bullet) \to H_i(G_\bullet)$. If $\alpha, \beta : F_\bullet \to G_\bullet$ are maps of complexes, then a homotopy between α and β is given by a collection of maps $h_i : F_i \to G_{i+1}$ such that $\alpha_i - \beta_i = d_{G,i+1} \circ h_i + h_{i-1} \circ d_{F,i}$. Two maps $\alpha, \beta : F_\bullet \to G_\bullet$ are said to be homotopic if a homotopy between α and β exists.

We will use a very similar notation regarding complexes of the form $F^\bullet$ which look like
$$\ldots \to F^i \xrightarrow{d^i} F^{i+1} \to \ldots$$

There are maps of complexes, homotopies, etc. In this case we set $H^i(F^\bullet) = \text{Ker}(d^i)/\text{Im}(d^{i-1})$ and we call it the ith cohomology group.

00LR Lemma 10.71.3. Any two homotopic maps of complexes induce the same maps on (co)homology groups.

Proof. Omitted. $\square$

00LS Lemma 10.71.4. Let R be a ring. Let $M \to N$ be a map of R-modules. Let $N_\bullet \to N$ be an arbitrary resolution. Let

$$\ldots \to F_2 \to F_1 \to F_0 \to M$$

be a complex of R-modules where each F_i is a free R-module. Then

(1) there exists a map of complexes $F_\bullet \to N_\bullet$ such that

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & M \arrow[d] \\ N_0 \arrow[r] & N \end{tikzcd}$$

is commutative, and

(2) any two maps $\alpha, \beta : F_\bullet \to N_\bullet$ as in (1) are homotopic.

Proof. Proof of (1). Because F_0 is free we can find a map $F_0 \to N_0$ lifting the map $F_0 \to M \to N$. We obtain an induced map $F_1 \to F_0 \to N_0$ which ends up in the image of $N_1 \to N_0$. Since F_1 is free we may lift this to a map $F_1 \to N_1$. This in turn induces a map $F_2 \to F_1 \to N_1$ which maps to zero into N_0. Since $N_\bullet$ is exact we see that the image of this map is contained in the image of $N_2 \to N_1$. Hence we may lift to get a map $F_2 \to N_2$. Repeat.

Proof of (2). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : F_0 \to N_0$ is contained in the image of $N_1 \to N_0$. Hence we may lift γ_0 to a map $h_0 : F_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{F,1}$. By our choice of h_0 we see that the image of γ_1' is contained in the kernel of $N_1 \to N_0$. Since $N_\bullet$ is exact we may lift γ_1' to a map $h_1 : F_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. $\square$

At this point we are ready to define the groups $\text{Ext}^i_R(M, N)$. Namely, choose a resolution $F_\bullet$ of M by free R-modules, see Lemma 10.71.1. Consider the (cohomological) complex

$$\text{Hom}_R(F_\bullet, N) : \text{Hom}_R(F_0, N) \to \text{Hom}_R(F_1, N) \to \text{Hom}_R(F_2, N) \to \ldots$$

We define $\text{Ext}^i_R(M, N)$ for $i \geq 0$ to be the ith cohomology group of this complex[7]. For $i < 0$ we set $\text{Ext}^i_R(M, N) = 0$. Before we continue we point out that

$$\text{Ext}^0_R(M, N) = \text{Ker}(\text{Hom}_R(F_0, N) \to \text{Hom}_R(F_1, N)) = \text{Hom}_R(M, N)$$

because we can apply part (1) of Lemma 10.10.1 to the exact sequence $F_1 \to F_0 \to M \to 0$. The following lemma explains in what sense this is well defined.

[7]At this point it would perhaps be more appropriate to say “an” in stead of “the” Ext-group.

00LT **Lemma 10.71.5.** Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1, and $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = \text{Coker}(d_{F,1}) \to M_2 = \text{Coker}(d_{G,1})$, see Lemma 10.71.4. Then the induced maps

$$H^i(\alpha) : H^i(\text{Hom}_R(F_\bullet, N)) \longrightarrow H^i(\text{Hom}_R(G_\bullet, N))$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H^i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. Another map $\beta : F_\bullet \to G_\bullet$ inducing φ is homotopic to α by Lemma 10.71.4. Hence the maps $\text{Hom}_R(F_\bullet, N) \to \text{Hom}_R(G_\bullet, N)$ are homotopic. Hence the independence result follows from Lemma 10.71.3.

Suppose that φ is an isomorphism. Let $\psi : M_2 \to M_1$ be an inverse. Choose $\beta : G_\bullet \to F_\bullet$ be a map inducing $\psi : M_2 = \text{Coker}(d_{G,1}) \to M_1 = \text{Coker}(d_{F,1})$, see Lemma 10.71.4. OK, and now consider the map $H^i(\alpha) \circ H^i(\beta) = H^i(\alpha \circ \beta)$. By the above the map $H^i(\alpha \circ \beta)$ is the same as the map $H^i(\text{id}_{G_\bullet}) = \text{id}$. Similarly for the composition $H^i(\beta) \circ H^i(\alpha)$. Hence $H^i(\alpha)$ and $H^i(\beta)$ are inverses of each other. $\square$

00LU **Lemma 10.71.6.** Let R be a ring. Let M be an R-module. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence. Then we get a long exact sequence

$$\begin{aligned} 0 &\to \text{Hom}_R(M, N') \to \text{Hom}_R(M, N) \to \text{Hom}_R(M, N'') \\ &\to \text{Ext}^1_R(M, N') \to \text{Ext}^1_R(M, N) \to \text{Ext}^1_R(M, N'') \to \ldots \end{aligned}$$

Proof. Pick a free resolution $F_\bullet \to M$. Since each of the F_i are free we see that we get a short exact sequence of complexes

$$0 \to \text{Hom}_R(F_\bullet, N') \to \text{Hom}_R(F_\bullet, N) \to \text{Hom}_R(F_\bullet, N'') \to 0$$

Thus we get the long exact sequence from the snake lemma applied to this. $\square$

065P **Lemma 10.71.7.** Let R be a ring. Let N be an R-module. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Then we get a long exact sequence

$$\begin{aligned} 0 &\to \text{Hom}_R(M'', N) \to \text{Hom}_R(M, N) \to \text{Hom}_R(M', N) \\ &\to \text{Ext}^1_R(M'', N) \to \text{Ext}^1_R(M, N) \to \text{Ext}^1_R(M', N) \to \ldots \end{aligned}$$

Proof. Pick sets of generators $\{m'_{i'}\}_{i' \in I'}$ and $\{m''_{i''}\}_{i'' \in I''}$ of M' and M''. For each $i'' \in I''$ choose a lift $\tilde{m}''_{i''} \in M$ of the element $m''_{i''} \in M''$. Set $F' = \bigoplus_{i' \in I'} R$, $F'' = \bigoplus_{i'' \in I''} R$ and $F = F' \oplus F''$. Mapping the generators of these free modules to the corresponding chosen generators gives surjective R-module maps $F' \to M'$, $F'' \to M''$, and $F \to M$. We obtain a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F' & \to & F & \to & F'' & \to & 0 \end{array}$$

By the snake lemma we see that the sequence of kernels $0 \to K' \to K \to K'' \to 0$ is short exact sequence of R-modules. Hence we can continue this process indefinitely. In other words we obtain a short exact sequence of resolutions fitting into the diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F'_\bullet & \to & F_\bullet & \to & F''_\bullet & \to & 0 \end{array}$$

Because each of the sequences $0 \to F'_n \to F_n \to F''_n \to 0$ is split exact (by construction) we obtain a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(F''_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(F'_\bullet, N) \to 0$$

by applying the $\mathrm{Hom}_R(-, N)$ functor. Thus we get the long exact sequence from the snake lemma applied to this. $\square$

00LV Lemma 10.71.8. Let R be a ring. Let M, N be R-modules. Any $x \in R$ such that either $xN = 0$, or $xM = 0$ annihilates each of the modules $\mathrm{Ext}^i_R(M, N)$.

Proof. Pick a free resolution $F_\bullet$ of M. Since $\mathrm{Ext}^i_R(M, N)$ is defined as the cohomology of the complex $\mathrm{Hom}_R(F_\bullet, N)$ the lemma is clear when $xN = 0$. If $xM = 0$, then we see that multiplication by x on $F_\bullet$ lifts the zero map on M. Hence by Lemma 10.71.5 we see that it induces the same map on Ext groups as the zero map. $\square$

08YR Lemma 10.71.9. Let R be a Noetherian ring. Let M, N be finite R-modules. Then $\mathrm{Ext}^i_R(M, N)$ is a finite R-module for all i.

Proof. This holds because $\mathrm{Ext}^i_R(M, N)$ is computed as the cohomology groups of a complex $\mathrm{Hom}_R(F_\bullet, N)$ with each F_n a finite free R-module, see Lemma 10.71.1. $\square$

10.72. Depth

00LE Here is our definition.

00LI Definition 10.72.1. Let R be a ring, and $I \subset R$ an ideal. Let M be a finite R-module. The I-depth of M, denoted $\mathrm{depth}_I(M)$, is defined as follows:

(1) if $IM \neq M$, then $\mathrm{depth}_I(M)$ is the supremum in $\{0, 1, 2, \ldots, \infty\}$ of the lengths of M-regular sequences in I,
(2) if $IM = M$ we set $\mathrm{depth}_I(M) = \infty$.

If $(R, \mathfrak m)$ is local we call $\mathrm{depth}_{\mathfrak m}(M)$ simply the depth of M.

Explanation. By Definition 10.68.1 the empty sequence is not a regular sequence on the zero module, but for practical purposes it turns out to be convenient to set the depth of the 0 module equal to $+\infty$. Note that if $I = R$, then $\mathrm{depth}_I(M) = \infty$ for all finite R-modules M. If I is contained in the Jacobson radical of R (e.g., if R is local and $I \subset \mathfrak m_R$), then $M \neq 0 \Rightarrow IM \neq M$ by Nakayama's lemma. A module M has I-depth 0 if and only if M is nonzero and I does not contain a nonzerodivisor on M.

Example 10.68.2 shows depth does not behave well even if the ring is Noetherian, and Example 10.68.3 shows that it does not behave well if the ring is local but non-Noetherian. We will see depth behaves well if the ring is local Noetherian.

0AUI Lemma 10.72.2. Let R be a ring, $I \subset R$ an ideal, and M a finite R-module. Then $\mathrm{depth}_I(M)$ is equal to the supremum of the lengths of sequences $f_1, \ldots, f_r \in I$ such that f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$.

Proof. Suppose that $IM = M$. Then Lemma 10.20.1 shows there exists an $f \in I$ such that $f : M \to M$ is id_M. Hence $f, 0, 0, 0, \ldots$ is an infinite sequence of successive nonzerodivisors and we see agreement holds in this case. If $IM \neq M$, then we see that a sequence as in the lemma is an M-regular sequence and we conclude that agreement holds as well. $\square$

00LK **Lemma 10.72.3.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a nonzero finite R-module. Then $\dim(\text{Supp}(M)) \geq \text{depth}(M)$.

Proof. The proof is by induction on $\dim(\text{Supp}(M))$. If $\dim(\text{Supp}(M)) = 0$, then $\text{Supp}(M) = \{\mathfrak{m}\}$, whence $\text{Ass}(M) = \{\mathfrak{m}\}$ (by Lemmas 10.63.2 and 10.63.7), and hence the depth of M is zero for example by Lemma 10.63.18. For the induction step we assume $\dim(\text{Supp}(M)) > 0$. Let $f_1, \ldots, f_d$ be a sequence of elements of $\mathfrak{m}$ such that f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$. According to Lemma 10.72.2 it suffices to prove $\dim(\text{Supp}(M)) \geq d$. We may assume $d > 0$ otherwise the lemma holds. By Lemma 10.63.10 we have $\dim(\text{Supp}(M/f_1M)) = \dim(\text{Supp}(M)) - 1$. By induction we conclude $\dim(\text{Supp}(M/f_1M)) \geq d - 1$ as desired. $\square$

0AUJ **Lemma 10.72.4.** Let R be a Noetherian ring, $I \subset R$ an ideal, and M a finite nonzero R-module such that $IM \neq M$. Then $\text{depth}_I(M) < \infty$.

Proof. Since M/IM is nonzero we can choose $\mathfrak{p} \in \text{Supp}(M/IM)$ by Lemma 10.40.2. Then $(M/IM)_\mathfrak{p} \neq 0$ which implies $I \subset \mathfrak{p}$ and moreover implies $M_\mathfrak{p} \neq IM_\mathfrak{p}$ as localization is exact. Let $f_1, \ldots, f_r \in I$ be an M-regular sequence. Then $M_\mathfrak{p}/(f_1, \ldots, f_r)M_\mathfrak{p}$ is nonzero as $(f_1, \ldots, f_r) \subset I$. As localization is flat we see that the images of $f_1, \ldots, f_r$ form a $M_\mathfrak{p}$-regular sequence in $I_\mathfrak{p}$. Since this works for every M-regular sequence in I we conclude that $\text{depth}_I(M) \leq \text{depth}_{I_\mathfrak{p}}(M_\mathfrak{p})$. The latter is $\leq \text{depth}(M_\mathfrak{p})$ which is $< \infty$ by Lemma 10.72.3. $\square$

00LW **Lemma 10.72.5.** Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a nonzero finite R-module. Then $\text{depth}(M)$ is equal to the smallest integer i such that $\text{Ext}^i_R(R/\mathfrak{m}, M)$ is nonzero.

Proof. Let $\delta(M)$ denote the depth of M and let $i(M)$ denote the smallest integer i such that $\text{Ext}^i_R(R/\mathfrak{m}, M)$ is nonzero. We will see in a moment that $i(M) < \infty$. By Lemma 10.63.18 we have $\delta(M) = 0$ if and only if $i(M) = 0$, because $\mathfrak{m} \in \text{Ass}(M)$ exactly means that $i(M) = 0$. Hence if $\delta(M)$ or $i(M)$ is > 0, then we may choose $x \in \mathfrak{m}$ such that (a) x is a nonzerodivisor on M, and (b) $\text{depth}(M/xM) = \delta(M) - 1$. Consider the long exact sequence of Ext-groups associated to the short exact sequence $0 \to M \to M \to M/xM \to 0$ by Lemma 10.71.6:

$$\begin{aligned} 0 &\to \text{Hom}_R(\kappa, M) \to \text{Hom}_R(\kappa, M) \to \text{Hom}_R(\kappa, M/xM) \\ &\to \text{Ext}^1_R(\kappa, M) \to \text{Ext}^1_R(\kappa, M) \to \text{Ext}^1_R(\kappa, M/xM) \to \ldots \end{aligned}$$

Since $x \in \mathfrak{m}$ all the maps $\text{Ext}^i_R(\kappa, M) \to \text{Ext}^i_R(\kappa, M)$ are zero, see Lemma 10.71.8. Thus it is clear that $i(M/xM) = i(M) - 1$. Induction on $\delta(M)$ finishes the proof. $\square$

00LX **Lemma 10.72.6.** Let R be a local Noetherian ring. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of nonzero finite R-modules.

(1) $\text{depth}(N) \geq \min\{\text{depth}(N'), \text{depth}(N'')\}$
(2) $\text{depth}(N'') \geq \min\{\text{depth}(N), \text{depth}(N') - 1\}$
(3) $\text{depth}(N') \geq \min\{\text{depth}(N), \text{depth}(N'') + 1\}$

Proof. Use the characterization of depth using the Ext groups $\text{Ext}^i(\kappa, N)$, see Lemma 10.72.5, and use the long exact cohomology sequence

$$\begin{aligned} 0 &\to \text{Hom}_R(\kappa, N') \to \text{Hom}_R(\kappa, N) \to \text{Hom}_R(\kappa, N'') \\ &\to \text{Ext}^1_R(\kappa, N') \to \text{Ext}^1_R(\kappa, N) \to \text{Ext}^1_R(\kappa, N'') \to \ldots \end{aligned}$$

from Lemma 10.71.6. $\square$

090R Lemma 10.72.7. Let R be a local Noetherian ring and M a nonzero finite R-module.

(1) If $x \in \mathfrak{m}$ is a nonzerodivisor on M, then $\text{depth}(M/xM) = \text{depth}(M) - 1$.
(2) Any M-regular sequence $x_1, \ldots, x_r$ can be extended to an M-regular sequence of length $\text{depth}(M)$.

Proof. Part (2) is a formal consequence of part (1). Let $x \in R$ be as in (1). By the short exact sequence $0 \to M \to M \to M/xM \to 0$ and Lemma 10.72.6 we see that the depth drops by at most 1. On the other hand, if $x_1, \ldots, x_r \in \mathfrak{m}$ is a regular sequence for M/xM, then $x, x_1, \ldots, x_r$ is a regular sequence for M. Hence we see that the depth drops by at least 1. $\square$

0CN5 Lemma 10.72.8. Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finite R-module. Let $x \in \mathfrak{m}$, $\mathfrak{p} \in \text{Ass}(M)$, and $\mathfrak{q}$ minimal over $\mathfrak{p} + (x)$. Then $\mathfrak{q} \in \text{Ass}(M/x^nM)$ for some $n \geq 1$.

Proof. Pick a submodule $N \subset M$ with $N \cong R/\mathfrak{p}$. By the Artin-Rees lemma (Lemma 10.51.2) we can pick $n > 0$ such that $N \cap x^nM \subset xN$. Let $\overline{N} \subset M/x^nM$ be the image of $N \to M \to M/x^nM$. By Lemma 10.63.3 it suffices to show $\mathfrak{q} \in \text{Ass}(\overline{N})$. By our choice of n there is a surjection $\overline{N} \to N/xN = R/\mathfrak{p} + (x)$ and hence $\mathfrak{q}$ is in the support of $\overline{N}$. Since $\overline{N}$ is annihilated by x^n and $\mathfrak{p}$ we see that $\mathfrak{q}$ is minimal among the primes in the support of $\overline{N}$. Thus $\mathfrak{q}$ is an associated prime of $\overline{N}$ by Lemma 10.63.8. $\square$

0BK4 Lemma 10.72.9. Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finite R-module. For $\mathfrak{p} \in \text{Ass}(M)$ we have $\dim(R/\mathfrak{p}) \geq \text{depth}(M)$.

Proof. If $\mathfrak{m} \in \text{Ass}(M)$ then there is a nonzero element $x \in M$ which is annihilated by all elements of $\mathfrak{m}$. Thus $\text{depth}(M) = 0$. In particular the lemma holds in this case.

If $\text{depth}(M) = 1$, then by the first paragraph we find that $\mathfrak{m} \notin \text{Ass}(M)$. Hence $\dim(R/\mathfrak{p}) \geq 1$ for all $\mathfrak{p} \in \text{Ass}(M)$ and the lemma is true in this case as well.

We will prove the lemma in general by induction on $\text{depth}(M)$ which we may and do assume to be > 1. Pick $x \in \mathfrak{m}$ which is a nonzerodivisor on M. Note $x \notin \mathfrak{p}$ (Lemma 10.63.9). By Lemma 10.60.13 we have $\dim(R/\mathfrak{p} + (x)) = \dim(R/\mathfrak{p}) - 1$. Thus there exists a prime $\mathfrak{q}$ minimal over $\mathfrak{p} + (x)$ with $\dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) - 1$ (small argument omitted; hint: the dimension of a Noetherian local ring A is the maximum of the dimensions of $A/\mathfrak{r}$ taken over the minimal primes $\mathfrak{r}$ of A). Pick n as in Lemma 10.72.8 so that $\mathfrak{q}$ is an associated prime of M/x^nM. We may apply induction hypothesis to M/x^nM and $\mathfrak{q}$ because $\text{depth}(M/x^nM) = \text{depth}(M) - 1$ by Lemma 10.72.7. We find $\dim(R/\mathfrak{q}) \geq \text{depth}(M/x^nM)$ and we win. $\square$

0FCC Lemma 10.72.10. Let R be a local Noetherian ring and M a finite R-module. For a prime ideal $\mathfrak{p} \subset R$ we have $\text{depth}(M_\mathfrak{p}) + \dim(R/\mathfrak{p}) \geq \text{depth}(M)$.

Proof. If $M_\mathfrak{p} = 0$, then $\text{depth}(M_\mathfrak{p}) = \infty$ and the lemma holds. If $\text{depth}(M) \leq \dim(R/\mathfrak{p})$, then the lemma is true. If $\text{depth}(M) > \dim(R/\mathfrak{p})$, then $\mathfrak{p}$ is not contained in any associated prime $\mathfrak{q}$ of M by Lemma 10.72.9. Hence we can find an $x \in \mathfrak{p}$ not contained in any associated prime of M by Lemma 10.15.2 and Lemma 10.63.5. Then x is a nonzerodivisor on M, see Lemma 10.63.9. Hence $\text{depth}(M/xM) = \text{depth}(M) - 1$ and $\text{depth}(M_\mathfrak{p}/xM_\mathfrak{p}) = \text{depth}(M_\mathfrak{p}) - 1$ provided $M_\mathfrak{p}$ is nonzero, see Lemma 10.72.7. Thus we conclude by induction on $\text{depth}(M)$. $\square$

0AUK Lemma 10.72.11. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $R \to S$ be a finite ring map. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of S. Let N be a finite S-module. Then

$$\min_{i=1,\ldots,n} \text{depth}(N_{\mathfrak{m}_i}) = \text{depth}_{\mathfrak{m}}(N)$$

Proof. By Lemmas 10.36.20, 10.36.22, and Lemma 10.36.21 the maximal ideals of S are exactly the primes of S lying over $\mathfrak{m}$ and there are finitely many of them. Hence the statement of the lemma makes sense. We will prove the lemma by induction on $k = \min_{i=1,\ldots,n} \text{depth}(N_{\mathfrak{m}_i})$. If $k = 0$, then $\text{depth}(N_{\mathfrak{m}_i}) = 0$ for some i. By Lemma 10.72.5 this means $\mathfrak{m}_i S_{\mathfrak{m}_i}$ is an associated prime of $N_{\mathfrak{m}_i}$ and hence $\mathfrak{m}_i$ is an associated prime of N (Lemma 10.63.16). By Lemma 10.63.13 we see that $\mathfrak{m}$ is an associated prime of N as an R-module. Whence $\text{depth}_{\mathfrak{m}}(N) = 0$. This proves the base case. If $k > 0$, then we see that $\mathfrak{m}_i \notin \text{Ass}_S(N)$. Hence $\mathfrak{m} \notin \text{Ass}_R(N)$, again by Lemma 10.63.13. Thus we can find $f \in \mathfrak{m}$ which is not a zerodivisor on N, see Lemma 10.63.18. By Lemma 10.72.7 all the depths drop exactly by 1 when passing from N to N/fN and the induction hypothesis does the rest. □

10.73. Functorialities for Ext

087M In this section we briefly discuss the functoriality of Ext with respect to change of ring, etc. Here is a list of items to work out.

(1) Given $R \to R'$, an R-module M and an R'-module N' the R-module $\text{Ext}^i_R(M, N')$ has a natural R'-module structure. Moreover, there is a canonical R'-linear map $\text{Ext}^i_{R'}(M \otimes_R R', N') \to \text{Ext}^i_R(M, N')$.
(2) Given $R \to R'$ and R-modules M, N there is a natural R-module map $\text{Ext}^i_R(M, N) \to \text{Ext}^i_R(M, N \otimes_R R')$.

087N Lemma 10.73.1. Given a flat ring map $R \to R'$, an R-module M, and an R'-module N' the natural map

$$\text{Ext}^i_{R'}(M \otimes_R R', N') \to \text{Ext}^i_R(M, N')$$

is an isomorphism for $i \geq 0$.

Proof. Choose a free resolution $F_\bullet$ of M. Since $R \to R'$ is flat we see that $F_\bullet \otimes_R R'$ is a free resolution of $M \otimes_R R'$ over R'. The statement is that the map

$$\text{Hom}_{R'}(F_\bullet \otimes_R R', N') \to \text{Hom}_R(F_\bullet, N')$$

induces an isomorphism on homology groups, which is true because it is an isomorphism of complexes by Lemma 10.14.3. □

10.74. An application of Ext groups

02HN Here it is.

02HO Lemma 10.74.1. Let R be a Noetherian ring. Let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let $N \to M$ be a homomorphism of finite R-modules. Suppose that there exists arbitrarily large n such that $N/I^nN \to M/I^nM$ is a split injection. Then $N \to M$ is a split injection.

Proof. Assume $\varphi : N \to M$ satisfies the assumptions of the lemma. Note that this implies that $\text{Ker}(\varphi) \subset I^nN$ for arbitrarily large n. Hence by Lemma 10.51.5 we see that φ is injection. Let $Q = M/N$ so that we have a short exact sequence

$$0 \to N \to M \to Q \to 0.$$

Let

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to Q \to 0$$

be a finite free resolution of Q. We can choose a map $\alpha : F_0 \to M$ lifting the map $F_0 \to Q$. This induces a map $\beta : F_1 \to N$ such that $\beta \circ d_2 = 0$. The extension above is split if and only if there exists a map $\gamma : F_0 \to N$ such that $\beta = \gamma \circ d_1$. In other words, the class of β in $\mathrm{Ext}^1_R(Q, N)$ is the obstruction to splitting the short exact sequence above.

Suppose n is a large integer such that $N/I^nN \to M/I^nM$ is a split injection. This implies

$$0 \to N/I^nN \to M/I^nM \to Q/I^nQ \to 0.$$

is still short exact. Also, the sequence

$$F_1/I^nF_1 \xrightarrow{d_1} F_0/I^nF_0 \to Q/I^nQ \to 0$$

is still exact. Arguing as above we see that the map $\overline{\beta} : F_1/I^nF_1 \to N/I^nN$ induced by β is equal to $\overline{\gamma_n} \circ d_1$ for some map $\overline{\gamma_n} : F_0/I^nF_0 \to N/I^nN$. Since F_0 is free we can lift $\overline{\gamma_n}$ to a map $\gamma_n : F_0 \to N$ and then we see that $\beta - \gamma_n \circ d_1$ is a map from F_1 into I^nN. In other words we conclude that

$$\beta \in \mathrm{Im}\Big(\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)\Big) + I^n \mathrm{Hom}_R(F_1, N).$$

for this n.

Since we have this property for arbitrarily large n by assumption we conclude that the image of β in the cokernel of $\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$ is zero by Lemma 10.51.5. Hence β is in the image of the map $\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$ as desired. □

10.75. Tor groups and flatness

00LY In this section we use some of the homological algebra developed in the previous section to explain what Tor groups are. Namely, suppose that R is a ring and that M, N are two R-modules. Choose a resolution $F_\bullet$ of M by free R-modules. See Lemma 10.71.1. Consider the homological complex

$$F_\bullet \otimes_R N : \ldots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N$$

We define $\mathrm{Tor}_i^R(M, N)$ to be the ith homology group of this complex. The following lemma explains in what sense this is well defined.

00LZ **Lemma 10.75.1.** *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1 and that $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = \mathrm{Coker}(d_{F,1}) \to M_2 = \mathrm{Coker}(d_{G,1})$, see Lemma 10.71.4. Then the induced maps*

$$H_i(\alpha) : H_i(F_\bullet \otimes_R N) \longrightarrow H_i(G_\bullet \otimes_R N)$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H_i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. The proof of this lemma is identical to the proof of Lemma 10.71.5. □

Not only does this lemma imply that the Tor modules are well defined, but it also provides for the functoriality of the constructions $(M, N) \mapsto \text{Tor}_i^R(M, N)$ in the first variable. Of course the functoriality in the second variable is evident. We leave it to the reader to see that each of the Tor_i^R is in fact a functor

$$\text{Mod}_R \times \text{Mod}_R \to \text{Mod}_R.$$

Here Mod_R denotes the category of R-modules, and for the definition of the product category see Categories, Definition 4.2.20. Namely, given morphisms of R-modules $M_1 \to M_2$ and $N_1 \to N_2$ we get a commutative diagram

$$\begin{tikzcd} \text{Tor}_i^R(M_1, N_1) \arrow[r] \arrow[d] & \text{Tor}_i^R(M_1, N_2) \arrow[d] \\ \text{Tor}_i^R(M_2, N_1) \arrow[r] & \text{Tor}_i^R(M_2, N_2) \end{tikzcd}$$

00M0 Lemma 10.75.2. Let R be a ring and let M be an R-module. Suppose that $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of R-modules. There exists a long exact sequence

$$\text{Tor}_1^R(M, N') \to \text{Tor}_1^R(M, N) \to \text{Tor}_1^R(M, N'') \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

Proof. The proof of this is the same as the proof of Lemma 10.71.6. $\square$

Consider a homological double complex of R-modules

$$\begin{tikzcd} \ldots \arrow[r, "d"] & A_{2,0} \arrow[r, "d"] & A_{1,0} \arrow[r, "d"] & A_{0,0} \\ \ldots \arrow[r, "d"] & A_{2,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,1} \arrow[u, "\delta"] \\ \ldots \arrow[r, "d"] & A_{2,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,2} \arrow[u, "\delta"] \\ & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] \end{tikzcd}$$

This means that $d_{i,j} : A_{i,j} \to A_{i-1,j}$ and $\delta_{i,j} : A_{i,j} \to A_{i,j-1}$ have the following properties

(1) Any composition of two $d_{i,j}$ is zero. In other words the rows of the double complex are complexes.
(2) Any composition of two $\delta_{i,j}$ is zero. In other words the columns of the double complex are complexes.
(3) For any pair (i, j) we have $\delta_{i-1,j} \circ d_{i,j} = d_{i,j-1} \circ \delta_{i,j}$. In other words, all the squares commute.

The correct thing to do is to associate a spectral sequence to any such double complex. However, for the moment we can get away with doing something slightly easier.

Namely, for the purposes of this section only, given a double complex $(A_{\bullet,\bullet}, d, \delta)$ set $R(A)_j = \text{Coker}(A_{1,j} \to A_{0,j})$ and $U(A)_i = \text{Coker}(A_{i,1} \to A_{i,0})$. (The letters R and U are meant to suggest Right and Up.) We endow $R(A)_\bullet$ with the structure of a complex using the maps δ. Similarly we endow $U(A)_\bullet$ with the structure of a complex using the maps d. In other words we obtain the following huge commutative diagram

$$
\begin{tikzcd}
\ldots \arrow[r, "d"] & U(A)_2 \arrow[r, "d"] & U(A)_1 \arrow[r, "d"] & U(A)_0 & \\
\ldots \arrow[r, "d"] & A_{2,0} \arrow[r, "d"] \arrow[u] & A_{1,0} \arrow[r, "d"] \arrow[u] & A_{0,0} \arrow[r] \arrow[u] & R(A)_0 \\
\ldots \arrow[r, "d"] & A_{2,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,1} \arrow[r] \arrow[u, "\delta"] & R(A)_1 \arrow[u, "\delta"] \\
\ldots \arrow[r, "d"] & A_{2,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,2} \arrow[r] \arrow[u, "\delta"] & R(A)_2 \arrow[u, "\delta"] \\
& \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"]
\end{tikzcd}
$$

(This is no longer a double complex of course.) It is clear what a morphism $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ of double complexes is, and it is clear that this induces morphisms of complexes $R(\Phi) : R(A)_\bullet \to R(B)_\bullet$ and $U(\Phi) : U(A)_\bullet \to U(B)_\bullet$.

00M1 **Lemma 10.75.3.** Let $(A_{\bullet,\bullet}, d, \delta)$ be a double complex such that

1. Each row $A_{\bullet,j}$ is a resolution of $R(A)_j$.
2. Each column $A_{i,\bullet}$ is a resolution of $U(A)_i$.

Then there are canonical isomorphisms

$$H_i(R(A)_\bullet) \cong H_i(U(A)_\bullet).$$

The isomorphisms are functorial with respect to morphisms of double complexes with the properties above.

Proof. We will show that $H_i(R(A)_\bullet))$ and $H_i(U(A)_\bullet)$ are canonically isomorphic to a third group. Namely

$$\mathbf{H}_i(A) := \frac{\{(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mid d(a_{i,0}) = \delta(a_{i-1,1}), \ldots, d(a_{1,i-1}) = \delta(a_{0,i})\}}{\{d(a_{i+1,0}) + \delta(a_{i,1}), d(a_{i,1}) + \delta(a_{i-1,2}), \ldots, d(a_{1,i}) + \delta(a_{0,i+1})\}}$$

Here we use the notational convention that $a_{i,j}$ denotes an element of $A_{i,j}$. In other words, an element of $\mathbf{H}_i$ is represented by a zig-zag, represented as follows for $i = 2$

$$
\begin{tikzcd}
a_{2,0} \arrow[r, mapsto] & d(a_{2,0}) = \delta(a_{1,1}) & & \\
& a_{1,1} \arrow[u, mapsto] \arrow[rr, mapsto] & & d(a_{1,1}) = \delta(a_{0,2}) \\
& & & a_{0,2} \arrow[u, mapsto]
\end{tikzcd}
$$

Naturally, we divide out by "trivial" zig-zags, namely the submodule generated by elements of the form $(0, \ldots, 0, -\delta(a_{t+1,t-i}), d(a_{t+1,t-i}), 0, \ldots, 0)$. Note that there are canonical homomorphisms

$$\mathbf{H}_i(A) \to H_i(R(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{0,i}$$

and

$$\mathbf{H}_i(A) \to H_i(U(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{i,0}$$

First we show that these maps are surjective. Suppose that $\overline{r} \in H_i(R(A)_\bullet)$. Let $r \in R(A)_i$ be a cocycle representing the class of $\overline{r}$. Let $a_{0,i} \in A_{0,i}$ be an element which maps to r. Because $\delta(r) = 0$, we see that $\delta(a_{0,i})$ is in the image of d. Hence there exists an element $a_{1,i-1} \in A_{1,i-1}$ such that $d(a_{1,i-1}) = \delta(a_{0,i})$. This in turn implies that $\delta(a_{1,i-1})$ is in the kernel of d (because $d(\delta(a_{1,i-1})) = \delta(d(a_{1,i-1})) = \delta(\delta(a_{0,i})) = 0$. By exactness of the rows we find an element $a_{2,i-2}$ such that $d(a_{2,i-2}) = \delta(a_{1,i-1})$. And so on until a full zig-zag is found. Of course surjectivity of $\mathbf{H}_i \to H_i(U(A))$ is shown similarly.

To prove injectivity we argue in exactly the same way. Namely, suppose we are given a zig-zag $(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i})$ which maps to zero in $H_i(R(A)_\bullet)$. This means that $a_{0,i}$ maps to an element of $\text{Coker}(A_{i,1} \to A_{i,0})$ which is in the image of $\delta : \text{Coker}(A_{i+1,1} \to A_{i+1,0}) \to \text{Coker}(A_{i,1} \to A_{i,0})$. In other words, $a_{0,i}$ is in the image of $\delta \oplus d : A_{0,i+1} \oplus A_{1,i} \to A_{0,i}$. From the definition of trivial zig-zags we see that we may modify our zig-zag by a trivial one and assume that $a_{0,i} = 0$. This immediately implies that $d(a_{1,i-1}) = 0$. As the rows are exact this implies that $a_{1,i-1}$ is in the image of $d : A_{2,i-1} \to A_{1,i-1}$. Thus we may modify our zig-zag once again by a trivial zig-zag and assume that our zig-zag looks like $(a_{i,0}, a_{i-1,1}, \ldots, a_{2,i-2}, 0, 0)$. Continuing like this we obtain the desired injectivity.

If $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ is a morphism of double complexes both of which satisfy the conditions of the lemma, then we clearly obtain a commutative diagram

$$\begin{tikzcd}
H_i(U(A)_\bullet) \arrow[d] & \mathbf{H}_i(A) \arrow[l] \arrow[r] \arrow[d] & H_i(R(A)_\bullet) \arrow[d] \\
H_i(U(B)_\bullet) & \mathbf{H}_i(B) \arrow[l] \arrow[r] & H_i(R(B)_\bullet)
\end{tikzcd}$$

This proves the functoriality. $\square$

00M2 Remark 10.75.4. The isomorphism constructed above is the "correct" one only up to signs. A good part of homological algebra is concerned with choosing signs for various maps and showing commutativity of diagrams with intervention of suitable signs. For the moment we will simply use the isomorphism as given in the proof above, and worry about signs later.

00M3 Lemma 10.75.5. Let R be a ring. For any $i \geq 0$ the functors $\text{Mod}_R \times \text{Mod}_R \to \text{Mod}_R$, $(M, N) \mapsto \text{Tor}_i^R(M, N)$ and $(M, N) \mapsto \text{Tor}_i^R(N, M)$ are canonically isomorphic.

Proof. Let $F_\bullet$ be a free resolution of the module M and let $G_\bullet$ be a free resolution of the module N. Consider the double complex $(A_{i,j}, d, \delta)$ defined as follows:

(1) set $A_{i,j} = F_i \otimes_R G_j$,
(2) set $d_{i,j} : F_i \otimes_R G_j \to F_{i-1} \otimes G_j$ equal to $d_{F,i} \otimes \text{id}$, and
(3) set $\delta_{i,j} : F_i \otimes_R G_j \to F_i \otimes G_{j-1}$ equal to $\text{id} \otimes d_{G,j}$.

This double complex is usually simply denoted $F_\bullet \otimes_R G_\bullet$.

Since each G_j is free, and hence flat we see that each row of the double complex is exact except in homological degree 0. Since each F_i is free and hence flat we see that each column of the double complex is exact except in homological degree 0. Hence the double complex satisfies the conditions of Lemma 10.75.3.

To see what the lemma says we compute $R(A)_\bullet$ and $U(A)_\bullet$. Namely,

$$\begin{aligned} R(A)_i &= \mathrm{Coker}(A_{1,i} \to A_{0,i}) \\ &= \mathrm{Coker}(F_1 \otimes_R G_i \to F_0 \otimes_R G_i) \\ &= \mathrm{Coker}(F_1 \to F_0) \otimes_R G_i \\ &= M \otimes_R G_i \end{aligned}$$

In fact these isomorphisms are compatible with the differentials δ and we see that $R(A)_\bullet = M \otimes_R G_\bullet$ as homological complexes. In exactly the same way we see that $U(A)_\bullet = F_\bullet \otimes_R N$. We get

$$\begin{aligned} \mathrm{Tor}_i^R(M, N) &= H_i(F_\bullet \otimes_R N) \\ &= H_i(U(A)_\bullet) \\ &= H_i(R(A)_\bullet) \\ &= H_i(M \otimes_R G_\bullet) \\ &= H_i(G_\bullet \otimes_R M) \\ &= \mathrm{Tor}_i^R(N, M) \end{aligned}$$

Here the third equality is Lemma 10.75.3, and the fifth equality uses the isomorphism $V \otimes W = W \otimes V$ of the tensor product.

Functoriality. Suppose that we have R-modules M_ν, N_ν, $\nu = 1, 2$. Let $\varphi : M_1 \to M_2$ and $\psi : N_1 \to N_2$ be morphisms of R-modules. Suppose that we have free resolutions $F_{\nu,\bullet}$ for M_ν and free resolutions $G_{\nu,\bullet}$ for N_ν. By Lemma 10.71.4 we may choose maps of complexes $\alpha : F_{1,\bullet} \to F_{2,\bullet}$ and $\beta : G_{1,\bullet} \to G_{2,\bullet}$ compatible with φ and ψ. We claim that the pair (α, β) induces a morphism of double complexes

$$\alpha \otimes \beta : F_{1,\bullet} \otimes_R G_{1,\bullet} \longrightarrow F_{2,\bullet} \otimes_R G_{2,\bullet}$$

This is really a very straightforward check using the rule that $F_{1,i} \otimes_R G_{1,j} \to F_{2,i} \otimes_R G_{2,j}$ is given by $\alpha_i \otimes \beta_j$ where α_i, resp. β_j is the degree i, resp. j component of α, resp. β. The reader also readily verifies that the induced maps $R(F_{1,\bullet} \otimes_R G_{1,\bullet})_\bullet \to R(F_{2,\bullet} \otimes_R G_{2,\bullet})_\bullet$ agrees with the map $M_1 \otimes_R G_{1,\bullet} \to M_2 \otimes_R G_{2,\bullet}$ induced by $\varphi \otimes \beta$. Similarly for the map induced on the $U(-)_\bullet$ complexes. Thus the statement on functoriality follows from the statement on functoriality in Lemma 10.75.3. □

00M4 Remark 10.75.6. An interesting case occurs when $M = N$ in the above. In this case we get a canonical map $\mathrm{Tor}_i^R(M, M) \to \mathrm{Tor}_i^R(M, M)$. Note that this map is not the identity, because even when $i = 0$ this map is not the identity! For example, if V is a vector space of dimension n over a field, then the switch map $V \otimes_k V \to V \otimes_k V$ has $(n^2 + n)/2$ eigenvalues $+1$ and $(n^2 - n)/2$ eigenvalues -1. In characteristic 2 it is not even diagonalizable. Note that even changing the sign of the map will not get rid of this.

0AZ4 Lemma 10.75.7. Let R be a Noetherian ring. Let M, N be finite R-modules. Then $\mathrm{Tor}_p^R(M, N)$ is a finite R-module for all p.

Proof. This holds because $\text{Tor}_p^R(M, N)$ is computed as the cohomology groups of a complex $F_\bullet \otimes_R N$ with each F_n a finite free R-module, see Lemma 10.71.1. $\square$

00M5 **Lemma 10.75.8.** Let R be a ring. Let M be an R-module. The following are equivalent:

(1) The module M is flat over R.
(2) For all $i > 0$ the functor $\text{Tor}_i^R(M, -)$ is zero.
(3) The functor $\text{Tor}_1^R(M, -)$ is zero.
(4) For all ideals $I \subset R$ we have $\text{Tor}_1^R(M, R/I) = 0$.
(5) For all finitely generated ideals $I \subset R$ we have $\text{Tor}_1^R(M, R/I) = 0$.

Proof. Suppose M is flat. Let N be an R-module. Let $F_\bullet$ be a free resolution of N. Then $F_\bullet \otimes_R M$ is a resolution of $N \otimes_R M$, by flatness of M. Hence all higher Tor groups vanish.

It now suffices to show that the last condition implies that M is flat. Let $I \subset R$ be an ideal. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Apply Lemma 10.75.2. We get an exact sequence

$$\text{Tor}_1^R(M, R/I) \to M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0$$

Since obviously $M \otimes_R R = M$ we conclude that the last hypothesis implies that $M \otimes_R I \to M$ is injective for every finitely generated ideal I. Thus M is flat by Lemma 10.39.5. $\square$

00M6 **Remark 10.75.9.** The proof of Lemma 10.75.8 actually shows that

$$\text{Tor}_1^R(M, R/I) = \text{Ker}(I \otimes_R M \to M).$$

10.76. Functorialities for Tor

00M7 In this section we briefly discuss the functoriality of Tor with respect to change of ring, etc. Here is a list of items to work out.

(1) Given a ring map $R \to R'$, an R-module M and an R'-module N' the R-modules $\text{Tor}_i^R(M, N')$ have a natural R'-module structure.
(2) Given a ring map $R \to R'$ and R-modules M, N there is a natural R-module map $\text{Tor}_i^R(M, N) \to \text{Tor}_i^{R'}(M \otimes_R R', N \otimes_R R')$.
(3) Given a ring map $R \to R'$ an R-module M and an R'-module N' there exists a natural R'-module map $\text{Tor}_i^R(M, N') \to \text{Tor}_i^{R'}(M \otimes_R R', N')$.

00M8 **Lemma 10.76.1.** Given a flat ring map $R \to R'$ and R-modules M, N the natural R-module map $\text{Tor}_i^R(M, N) \otimes_R R' \to \text{Tor}_i^{R'}(M \otimes_R R', N \otimes_R R')$ is an isomorphism for all i.

Proof. Omitted. This is true because a free resolution $F_\bullet$ of M over R stays exact when tensoring with R' over R and hence $(F_\bullet \otimes_R N) \otimes_R R'$ computes the Tor groups over R'. $\square$

The following lemma does not seem to fit anywhere else.

0BNF **Lemma 10.76.2.** Let R be a ring. Let $M = \text{colim}\, M_i$ be a filtered colimit of R-modules. Let N be an R-module. Then $\text{Tor}_n^R(M, N) = \text{colim}\, \text{Tor}_n^R(M_i, N)$ for all n.

Proof. Choose a free resolution $F_\bullet$ of N. Then $F_\bullet \otimes_R M = \operatorname{colim} F_\bullet \otimes_R M_i$ as complexes by Lemma 10.12.9. Thus the result by Lemma 10.8.8. $\square$

10.77. Projective modules

05CD Some lemmas on projective modules.

05CE Definition 10.77.1. Let R be a ring. An R-module P is projective if and only if the functor $\mathrm{Hom}_R(P, -) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor.

The functor $\mathrm{Hom}_R(M, -)$ is left exact for any R-module M, see Lemma 10.10.1. Hence the condition for P to be projective really signifies that given a surjection of R-modules $N \to N'$ the map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, N')$ is surjective.

05CF Lemma 10.77.2. Let R be a ring. Let P be an R-module. The following are equivalent

(1) P is projective,
(2) P is a direct summand of a free R-module, and
(3) $\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M.

Proof. Assume P is projective. Choose a surjection $\pi : F \to P$ where F is a free R-module. As P is projective there exists a $i \in \mathrm{Hom}_R(P, F)$ such that $\pi \circ i = \mathrm{id}_P$. In other words $F \cong \mathrm{Ker}(\pi) \oplus i(P)$ and we see that P is a direct summand of F.

Conversely, assume that $P \oplus Q = F$ is a free R-module. Note that the free module $F = \bigoplus_{i \in I} R$ is projective as $\mathrm{Hom}_R(F, M) = \prod_{i \in I} M$ and the functor $M \mapsto \prod_{i \in I} M$ is exact. Then $\mathrm{Hom}_R(F, -) = \mathrm{Hom}_R(P, -) \times \mathrm{Hom}_R(Q, -)$ as functors, hence both P and Q are projective.

Assume $P \oplus Q = F$ is a free R-module. Then we have a free resolution $F_\bullet$ of the form

$$\ldots F \xrightarrow{a} F \xrightarrow{b} F \to P \to 0$$

where the maps a, b alternate and are equal to the projector onto P and Q. Hence the complex $\mathrm{Hom}_R(F_\bullet, M)$ is split exact in degrees ≥ 1, whence we see the vanishing in (3).

Assume $\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M. Pick a free resolution $F_\bullet \to P$. Set $M = \mathrm{Im}(F_1 \to F_0) = \mathrm{Ker}(F_0 \to P)$. Consider the element $\xi \in \mathrm{Ext}^1_R(P, M)$ given by the class of the quotient map $\pi : F_1 \to M$. Since ξ is zero there exists a map $s : F_0 \to M$ such that $\pi = s \circ (F_1 \to F_0)$. Clearly, this means that

$$F_0 = \mathrm{Ker}(s) \oplus \mathrm{Ker}(F_0 \to P) = P \oplus \mathrm{Ker}(F_0 \to P)$$

and we win. $\square$

0G8T Lemma 10.77.3. Let R be a Noetherian ring. Let P be a finite R-module. If $\mathrm{Ext}^1_R(P, M) = 0$ for every finite R-module M, then P is projective.

This lemma can be strengthened: There is a version for finitely presented R-modules if R is not assumed Noetherian. There is a version with M running through all finite length modules in the Noetherian case.

Proof. Choose a surjection $R^{\oplus n} \to P$ with kernel M. Since $\mathrm{Ext}^1_R(P, M) = 0$ this surjection is split and we conclude by Lemma 10.77.2. $\square$

065Q Lemma 10.77.4. A direct sum of projective modules is projective.

Proof. This is true by the characterization of projectives as direct summands of free modules in Lemma 10.77.2. ☐

07LV Lemma 10.77.5. Let R be a ring. Let $I \subset R$ be a nilpotent ideal. Let $\overline{P}$ be a projective R/I-module. Then there exists a projective R-module P such that $P/IP \cong \overline{P}$.

Proof. By Lemma 10.77.2 we can choose a set A and a direct sum decomposition $\bigoplus_{\alpha \in A} R/I = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Write $F = \bigoplus_{\alpha \in A} R$ for the free R-module on A. Choose a lift $p : F \to F$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$ of $\bigoplus_{\alpha \in A} R/I$. Note that $p^2 - p \in \text{End}_R(F)$ is a nilpotent endomorphism of F (as I is nilpotent and the matrix entries of $p^2 - p$ are in I; more precisely, if $I^n = 0$, then $(p^2 - p)^n = 0$). Hence by Lemma 10.32.7 we can modify our choice of p and assume that p is a projector. Set $P = \text{Im}(p)$. ☐

0D47 Lemma 10.77.6. Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $\overline{P}$ be a finite projective R/I-module. Then there exists a finite projective R-module P such that $P/IP \cong \overline{P}$.

Proof. Recall that $\overline{P}$ is a direct summand of a free R/I-module $\bigoplus_{\alpha \in A} R/I$ by Lemma 10.77.2. As $\overline{P}$ is finite, it follows that $\overline{P}$ is contained in $\bigoplus_{\alpha \in A'} R/I$ for some $A' \subset A$ finite. Hence we may assume we have a direct sum decomposition $(R/I)^{\oplus n} = \overline{P} \oplus \overline{K}$ for some n and some R/I-module $\overline{K}$. Choose a lift $p \in \text{Mat}(n \times n, R)$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$ of $(R/I)^{\oplus n}$. Note that $p^2 - p \in \text{Mat}(n \times n, R)$ is nilpotent: as I is locally nilpotent and the matrix entries c_{ij} of $p^2 - p$ are in I we have $c_{ij}^t = 0$ for some $t > 0$ and then $(p^2 - p)^{tn^2} = 0$ (by looking at the matrix coefficients). Hence by Lemma 10.32.7 we can modify our choice of p and assume that p is a projector. Set $P = \text{Im}(p)$. ☐

05CG Lemma 10.77.7. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume

(1) I is nilpotent,
(2) M/IM is a projective R/I-module,
(3) M is a flat R-module.

Then M is a projective R-module.

Proof. By Lemma 10.77.5 we can find a projective R-module P and an isomorphism $P/IP \to M/IM$. We are going to show that M is isomorphic to P which will finish the proof. Because P is projective we can lift the map $P \to P/IP \to M/IM$ to an R-module map $P \to M$ which is an isomorphism modulo I. Since $I^n = 0$ for some n, we can use the filtrations

$$0 = I^n M \subset I^{n-1} M \subset \ldots \subset IM \subset M$$
$$0 = I^n P \subset I^{n-1} P \subset \ldots \subset IP \subset P$$

to see that it suffices to show that the induced maps $I^a P/I^{a+1} P \to I^a M/I^{a+1} M$ are bijective. Since both P and M are flat R-modules we can identify this with the map

$$I^a/I^{a+1} \otimes_{R/I} P/IP \longrightarrow I^a/I^{a+1} \otimes_{R/I} M/IM$$

induced by $P \to M$. Since we chose $P \to M$ such that the induced map $P/IP \to M/IM$ is an isomorphism, we win. ☐

10.78. Finite projective modules

00NV

00NW Definition 10.78.1. Let R be a ring and M an R-module.

(1) We say that M is locally free if we can cover $\mathrm{Spec}(R)$ by standard opens $D(f_i)$, $i \in I$ such that M_{f_i} is a free R_{f_i}-module for all $i \in I$.
(2) We say that M is finite locally free if we can choose the covering such that each M_{f_i} is finite free.
(3) We say that M is finite locally free of rank r if we can choose the covering such that each M_{f_i} is isomorphic to $R_{f_i}^{\oplus r}$.

Note that a finite locally free R-module is automatically finitely presented by Lemma 10.23.2. Moreover, if M is a finite locally free module of rank r over a ring R and if R is nonzero, then r is uniquely determined by Lemma 10.15.8 (because at least one of the localizations R_{f_i} is a nonzero ring).

00NX Lemma 10.78.2. Let R be a ring and let M be an R-module. The following are equivalent

(1) M is finitely presented and R-flat,
(2) M is finite projective,
(3) M is a direct summand of a finite free R-module,
(4) M is finitely presented and for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the localization $M_\mathfrak{p}$ is free,
(5) M is finitely presented and for all maximal ideals $\mathfrak{m} \subset R$ the localization $M_\mathfrak{m}$ is free,
(6) M is finite and locally free,
(7) M is finite locally free, and
(8) M is finite, for every prime $\mathfrak{p}$ the module $M_\mathfrak{p}$ is free, and the function

$$\rho_M : \mathrm{Spec}(R) \to \mathbf{Z}, \quad \mathfrak{p} \longmapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$$

is locally constant in the Zariski topology.

Proof. First suppose M is finite projective, i.e., (2) holds. Take a surjection $R^n \to M$ and let K be the kernel. Since M is projective, $0 \to K \to R^n \to M \to 0$ splits. Hence (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) follows from the fact that a direct summand of a projective is projective, see Lemma 10.77.2.

Assume (3), so we can write $K \oplus M \cong R^{\oplus n}$. So K is a direct summand of R^n and thus finitely generated. This shows $M = R^{\oplus n}/K$ is finitely presented. In other words, (3) $\Rightarrow$ (1).

Assume M is finitely presented and flat, i.e., (1) holds. We will prove that (7) holds. Pick any prime $\mathfrak{p}$ and $x_1, \ldots, x_r \in M$ which map to a basis of $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's lemma (in the form of Lemma 10.20.2) these elements generate M_g for some $g \in R$, $g \notin \mathfrak{p}$. The corresponding surjection $\varphi : R_g^{\oplus r} \to M_g$ has the following two properties: (a) $\mathrm{Ker}(\varphi)$ is a finite R_g-module (see Lemma 10.5.3) and (b) $\mathrm{Ker}(\varphi) \otimes \kappa(\mathfrak{p}) = 0$ by flatness of M_g over R_g (see Lemma 10.39.12). Hence by Nakayama's lemma again there exists a $g' \in R_g$ such that $\mathrm{Ker}(\varphi)_{g'} = 0$. In other words, $M_{gg'}$ is free.

A finite locally free module is a finite module, see Lemma 10.23.2, hence (7) $\Rightarrow$ (6). It is clear that (6) $\Rightarrow$ (7) and that (7) $\Rightarrow$ (8).

A finite locally free module is a finitely presented module, see Lemma 10.23.2, hence (7) $\Rightarrow$ (4). Of course (4) implies (5). Since we may check flatness locally (see Lemma 10.39.18) we conclude that (5) implies (1). At this point we have

$$\begin{tikzcd}
(2) \arrow[r, Leftrightarrow] & (3) \arrow[r, Rightarrow] & (1) \arrow[r, Rightarrow] & (7) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & (6) \\
 & & (5) \arrow[u, Rightarrow] & (4) \arrow[l, Rightarrow] & (8)
\end{tikzcd}$$

Suppose that M satisfies (1), (4), (5), (6), and (7). We will prove that (3) holds. It suffices to show that M is projective. We have to show that $\mathrm{Hom}_R(M, -)$ is exact. Let $0 \to N'' \to N \to N' \to 0$ be a short exact sequence of R-module. We have to show that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact. As M is finite locally free there exist a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ such that M_{f_i} is finite free. By Lemma 10.10.2 we see that

$$0 \to \mathrm{Hom}_R(M, N'')_{f_i} \to \mathrm{Hom}_R(M, N)_{f_i} \to \mathrm{Hom}_R(M, N')_{f_i} \to 0$$

is equal to $0 \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N''_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N'_{f_i}) \to 0$ which is exact as M_{f_i} is free and as the localization $0 \to N''_{f_i} \to N_{f_i} \to N'_{f_i} \to 0$ is exact (as localization is exact). Whence we see that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact by Lemma 10.23.2.

Finally, assume that (8) holds. Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 10.20.1 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. To see the claim it suffices to show that the induced map on localizations $\Psi_\mathfrak{p} : R_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in D(fg)$, see Lemma 10.23.1. By our choice of f the map $\Psi_\mathfrak{p}$ is surjective. By assumption (8) we have $M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus \rho_M(\mathfrak{p})}$ and by our choice of g we have $\rho_M(\mathfrak{p}) = r$. Hence $\Psi_\mathfrak{p}$ determines a surjection $R_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus r}$ whence is an isomorphism by Lemma 10.16.4. (Of course this last fact follows from a simple matrix argument also.) $\square$

0FWG **Lemma 10.78.3.** Let R be a reduced ring and let M be an R-module. Then the equivalent conditions of Lemma 10.78.2 are also equivalent to

(9) M is finite and the function $\rho_M : \mathrm{Spec}(R) \to \mathbf{Z}$, $\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$ is locally constant in the Zariski topology.

Proof. Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 10.20.1 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. Since Ψ is surjective, it suffices to show that Ψ is injective. Since R_{fg} is reduced, it suffices to show that Ψ is injective after localization at all minimal primes $\mathfrak{p}$ of R_{fg}, see Lemma 10.25.2. However, we know that $R_\mathfrak{p} = \kappa(\mathfrak{p})$ by Lemma 10.25.1 and $\rho_M(\mathfrak{p}) = r$ hence $\Psi_\mathfrak{p} : R_\mathfrak{p}^{\oplus r} \to M \otimes_R \kappa(\mathfrak{p})$ is an isomorphism as a surjective map of finite dimensional vector spaces of the same dimension. □

00NY Remark 10.78.4. It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_\mathfrak{m} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0\ \forall x, |x| < \epsilon\}$.

00NZ Lemma 10.78.5. (Warning: see Remark 10.78.4.) Suppose R is a local ring, and M is a finite flat R-module. Then M is finite free.

Proof. Follows from the equational criterion of flatness, see Lemma 10.39.11. Namely, suppose that $x_1, \ldots, x_r \in M$ map to a basis of $M/\mathfrak{m}M$. By Nakayama's Lemma 10.20.1 these elements generate M. We want to show there is no relation among the x_i. Instead, we will show by induction on n that if $x_1, \ldots, x_n \in M$ are linearly independent in the vector space $M/\mathfrak{m}M$ then they are independent over R.

The base case of the induction is where we have $x \in M$, $x \notin \mathfrak{m}M$ and a relation $fx = 0$. By the equational criterion there exist $y_j \in M$ and $a_j \in R$ such that $x = \sum a_j y_j$ and $fa_j = 0$ for all j. Since $x \notin \mathfrak{m}M$ we see that at least one a_j is a unit and hence $f = 0$.

Suppose that $\sum f_i x_i$ is a relation among $x_1, \ldots, x_n$. By our choice of x_i we have $f_i \in \mathfrak{m}$. According to the equational criterion of flatness there exist $a_{ij} \in R$ and $y_j \in M$ such that $x_i = \sum a_{ij} y_j$ and $\sum f_i a_{ij} = 0$. Since $x_n \notin \mathfrak{m}M$ we see that $a_{nj} \notin \mathfrak{m}$ for at least one j. Since $\sum f_i a_{ij} = 0$ we get $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. The relation $\sum f_i x_i = 0$ now can be rewritten as $\sum_{i=1}^{n-1} f_i(x_i + (-a_{ij}/a_{nj})x_n) = 0$. Note that the elements $x_i + (-a_{ij}/a_{nj})x_n$ map to $n-1$ linearly independent elements of $M/\mathfrak{m}M$. By induction assumption we get that all the f_i, $i \leq n-1$ have to be zero, and also $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. This proves the induction step. □

00O1 Lemma 10.78.6. Let $R \to S$ be a flat local homomorphism of local rings. Let M be a finite R-module. Then M is finite projective over R if and only if $M \otimes_R S$ is finite projective over S.

Proof. By Lemma 10.78.2 being finite projective over a local ring is the same thing as being finite free. Suppose that $M \otimes_R S$ is a finite free S-module. Pick $x_1, \ldots, x_r \in M$ whose images in $M/\mathfrak{m}_R M$ form a basis over $\kappa(\mathfrak{m})$. Then we see that $x_1 \otimes 1, \ldots, x_r \otimes 1$ are a basis for $M \otimes_R S$. This implies that the map $R^{\oplus r} \to M, (a_i) \mapsto \sum a_i x_i$ becomes an isomorphism after tensoring with S. By faithful flatness of $R \to S$, see Lemma 10.39.17 we see that it is an isomorphism. □

02M9 Lemma 10.78.7. Let R be a semi-local ring. Let M be a finite locally free module. If M has constant rank, then M is free. In particular, if R has connected spectrum, then M is free.

Proof. Omitted. Hints: First show that $M/\mathfrak{m}_i M$ has the same dimension d for all maximal ideal $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R using the rank is constant. Next, show that there exist elements $x_1, \ldots, x_d \in M$ which form a basis for each $M/\mathfrak{m}_iM$ by the Chinese remainder theorem. Finally show that $x_1, \ldots, x_d$ is a basis for M. □

Here is a technical lemma that is used in the chapter on groupoids.

03C1 Lemma 10.78.8. Let R be a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field. Let $R \to S$ be a ring map. Let M be an S-module and let $N \subset M$ be an R-submodule. Assume

(1) S is semi-local and $\mathfrak{m}S$ is contained in the Jacobson radical of S,
(2) M is a finite free S-module, and
(3) N generates M as an S-module.

Then N contains an S-basis of M.

Proof. Assume M is free of rank n. Let $I \subset S$ be the Jacobson radical. By Nakayama's Lemma 10.20.1 a sequence of elements $m_1, \ldots, m_n$ is a basis for M if and only if $\overline{m}_i \in M/IM$ generate M/IM. Hence we may replace M by M/IM, N by $N/(N \cap IM)$, R by $R/\mathfrak{m}$, and S by S/IS. In this case we see that S is a finite product of fields $S = k_1 \times \ldots \times k_r$ and $M = k_1^{\oplus n} \times \ldots \times k_r^{\oplus n}$. The fact that $N \subset M$ generates M as an S-module means that there exist $x_j \in N$ such that a linear combination $\sum a_j x_j$ with $a_j \in S$ has a nonzero component in each factor $k_i^{\oplus n}$. Because $R = k$ is an infinite field, this means that also some linear combination $y = \sum c_j x_j$ with $c_j \in k$ has a nonzero component in each factor. Hence $y \in N$ generates a free direct summand $Sy \subset M$. By induction on n the result holds for M/Sy and the submodule $\overline{N} = N/(N \cap Sy)$. In other words there exist $\overline{y}_2, \ldots, \overline{y}_n$ in $\overline{N}$ which (freely) generate M/Sy. Then $y, y_2, \ldots, y_n$ (freely) generate M and we win. □

0DVB Lemma 10.78.9. Let R be ring. Let L, M, N be R-modules. The canonical map

$$\mathrm{Hom}_R(M, N) \otimes_R L \to \mathrm{Hom}_R(M, N \otimes_R L)$$

is an isomorphism if M is finite projective.

Proof. By Lemma 10.78.2 we see that M is finitely presented as well as finite locally free. By Lemmas 10.10.2 and 10.12.16 formation of the left and right hand side of the arrow commutes with localization. We may check that our map is an isomorphism after localization, see Lemma 10.23.2. Thus we may assume M is finite free. In this case the lemma is immediate. □

10.79. Open loci defined by module maps

05GD The set of primes where a given module map is surjective, or an isomorphism is sometimes open. In the case of finite projective modules we can look at the rank of the map.

05GE Lemma 10.79.1. Let R be a ring. Let $\varphi : M \to N$ be a map of R-modules with N a finite R-module. Then we have the equality

$$\begin{aligned} U &= \{\mathfrak{p} \subset R \mid \varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p} \text{ is surjective}\} \\ &= \{\mathfrak{p} \subset R \mid \varphi \otimes \kappa(\mathfrak{p}) : M \otimes \kappa(\mathfrak{p}) \to N \otimes \kappa(\mathfrak{p}) \text{ is surjective}\} \end{aligned}$$

and U is an open subset of $\mathrm{Spec}(R)$. Moreover, for any $f \in R$ such that $D(f) \subset U$ the map $M_f \to N_f$ is surjective.

Proof. The equality in the displayed formula follows from Nakayama's lemma. Nakayama's lemma also implies that U is open. See Lemma 10.20.1 especially part (3). If $D(f) \subset U$, then $M_f \to N_f$ is surjective on all localizations at primes of R_f, and hence it is surjective by Lemma 10.23.1. $\square$

05GF Lemma 10.79.2. Let R be a ring. Let $\varphi : M \to N$ be a map of R-modules with M finite and N finitely presented. Then

$$U = \{\mathfrak p \subset R \mid \varphi_{\mathfrak p} : M_{\mathfrak p} \to N_{\mathfrak p} \text{ is an isomorphism}\}$$

is an open subset of $\mathrm{Spec}(R)$.

Proof. Let $\mathfrak p \in U$. Pick a presentation $N = R^{\oplus n}/\sum_{j=1,\ldots,m} Rk_j$. Denote e_i the image in N of the ith basis vector of $R^{\oplus n}$. For each $i \in \{1, \ldots, n\}$ choose an element $m_i \in M_{\mathfrak p}$ such that $\varphi(m_i) = f_i e_i$ for some $f_i \in R$, $f_i \not\in \mathfrak p$. This is possible as $\varphi_{\mathfrak p}$ is an isomorphism. Set $f = f_1 \ldots f_n$ and let $\psi : R_f^{\oplus n} \to M_f$ be the map which maps the ith basis vector to m_i/f_i. Note that $\varphi_f \circ \psi$ is the localization at f of the given map $R^{\oplus n} \to N$. As $\varphi_{\mathfrak p}$ is an isomorphism we see that $\psi(k_j)$ is an element of M which maps to zero in $M_{\mathfrak p}$. Hence we see that there exist $g_j \in R$, $g_j \not\in \mathfrak p$ such that $g_j\psi(k_j) = 0$. Setting $g = g_1 \ldots g_m$, we see that ψ_g factors through N_{fg} to give a map $\chi : N_{fg} \to M_{fg}$. By construction χ is a right inverse to φ_{fg}. It follows that $\chi_{\mathfrak p}$ is an isomorphism. By Lemma 10.79.1 there is an $h \in R$, $h \not\in \mathfrak p$ such that $\chi_h : N_{fgh} \to M_{fgh}$ is surjective. Hence φ_{fgh} and χ_h are mutually inverse maps, which implies that $D(fgh) \subset U$ as desired. $\square$

0GWM Lemma 10.79.3. Let R be a ring. Let $\mathfrak p \subset R$ be a prime. Let M be a finitely presented R-module. If $M_{\mathfrak p}$ is free, then there is an $f \in R$, $f \not\in \mathfrak p$ such that M_f is a free R_f-module.

Proof. Choose a basis $x_1, \ldots, x_n \in M_{\mathfrak p}$. We can choose an $f \in R$, $f \not\in \mathfrak p$ such that x_i is the image of some $y_i \in M_f$. After replacing y_i by $f^m y_i$ for $m \gg 0$ we may assume $y_i \in M$. Namely, this replaces $x_1, \ldots, x_n$ by $f^m x_1, \ldots, f^m x_n$ which is still a basis as f maps to a unit in $R_{\mathfrak p}$. Hence we obtain a homomorphism $\varphi = (y_1, \ldots, y_n) : R^{\oplus n} \to M$ of R-modules whose localization at $\mathfrak p$ is an isomorphism. By Lemma 10.79.2 we can find an $f \in R$, $f \not\in \mathfrak p$ such that $\varphi_{\mathfrak q}$ is an isomorphism for all primes $\mathfrak q \subset R$ with $f \not\in \mathfrak q$. Then it follows from Lemma 10.23.1 that φ_f is an isomorphism and the proof is complete. $\square$

00O0 Lemma 10.79.4. Let R be a ring. Let $\varphi : P_1 \to P_2$ be a map of finite projective modules. Then

(1) The set U of primes $\mathfrak p \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak p)$ is injective is open and for any $f \in R$ such that $D(f) \subset U$ we have
   (a) $P_{1,f} \to P_{2,f}$ is injective, and
   (b) the module $\mathrm{Coker}(\varphi)_f$ is finite projective over R_f.
(2) The set W of primes $\mathfrak p \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak p)$ is surjective is open and for any $f \in R$ such that $D(f) \subset W$ we have
   (a) $P_{1,f} \to P_{2,f}$ is surjective, and
   (b) the module $\mathrm{Ker}(\varphi)_f$ is finite projective over R_f.
(3) The set V of primes $\mathfrak p \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak p)$ is an isomorphism is open and for any $f \in R$ such that $D(f) \subset V$ the map $\varphi : P_{1,f} \to P_{2,f}$ is an isomorphism of modules over R_f.

Proof. To prove the set U is open we may work locally on $\mathrm{Spec}(R)$. Thus we may replace R by a suitable localization and assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$, see Lemma 10.78.2. In this case injectivity of $\varphi \otimes \kappa(\mathfrak p)$ is equivalent to $n_1 \leq n_2$ and some $n_1 \times n_1$ minor f of the matrix of φ being invertible in $\kappa(\mathfrak p)$. Thus $D(f) \subset U$. This argument also shows that $P_{1,\mathfrak p} \to P_{2,\mathfrak p}$ is injective for $\mathfrak p \in U$.

Now suppose $D(f) \subset U$. By the remark in the previous paragraph and Lemma 10.23.1 we see that $P_{1,f} \to P_{2,f}$ is injective, i.e., (1)(a) holds. By Lemma 10.78.2 to prove (1)(b) it suffices to prove that $\mathrm{Coker}(\varphi)$ is finite projective locally on $D(f)$. Thus, as we saw above, we may assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$ and that some minor of the matrix of φ is invertible in R. If the minor in question corresponds to the first n_1 basis vectors of R^{n_2}, then using the last $n_2 - n_1$ basis vectors we get a map $R^{n_2 - n_1} \to R^{n_2} \to \mathrm{Coker}(\varphi)$ which is easily seen to be an isomorphism.

Openness of W and (2)(a) for $D(f) \subset W$ follow from Lemma 10.79.1. Since $P_{2,f}$ is projective over R_f we see that $\varphi_f : P_{1,f} \to P_{2,f}$ has a section and it follows that $\mathrm{Ker}(\varphi)_f$ is a direct summand of $P_{2,f}$. Therefore $\mathrm{Ker}(\varphi)_f$ is finite projective. Thus (2)(b) holds as well.

It is clear that $V = U \cap W$ is open and the other statement in (3) follows from (1)(a) and (2)(a). $\square$

10.80. Faithfully flat descent for projectivity of modules

058B

In the next few sections we prove, following Raynaud and Gruson [GR71], that the projectivity of modules descends along faithfully flat ring maps. The idea of the proof is to use dévissage à la Kaplansky [Kap58] to reduce to the case of countably generated modules. Given a well-behaved filtration of a module M, dévissage allows us to express M as a direct sum of successive quotients of the filtering submodules (see Section 10.84). Using this technique, we prove that a projective module is a direct sum of countably generated modules (Theorem 10.84.5). To prove descent of projectivity for countably generated modules, we introduce a "Mittag-Leffler" condition on modules, prove that a countably generated module is projective if and only if it is flat and Mittag-Leffler (Theorem 10.93.3), and then show that the property of being a Mittag-Leffler module descends (Lemma 10.95.1). Finally, given an arbitrary module M whose base change by a faithfully flat ring map is projective, we filter M by submodules whose successive quotients are countably generated projective modules, and then by dévissage conclude M is a direct sum of projectives, hence projective itself (Theorem 10.95.6).

We note that there is an error in the proof of faithfully flat descent of projectivity in [GR71]. There, descent of projectivity along faithfully flat ring maps is deduced from descent of projectivity along a more general type of ring map ([GR71, Example 3.1.4(1) of Part II]). However, the proof of descent along this more general type of map is incorrect. In [Gru73], Gruson explains what went wrong, although he does not provide a fix for the case of interest. Patching this hole in the proof of faithfully flat descent of projectivity comes down to proving that the property of being a Mittag-Leffler module descends along faithfully flat ring maps. We do this in Lemma 10.95.1.

10.81. Characterizing flatness

058C In this section we discuss criteria for flatness. The main result in this section is Lazard's theorem (Theorem 10.81.4 below), which says that a flat module is the colimit of a directed system of free finite modules. We remind the reader of the "equational criterion for flatness", see Lemma 10.39.11. It turns out that this can be massaged into a seemingly much stronger property.

058D Lemma 10.81.1. Let M be an R-module. The following are equivalent:

(1) M is flat.
(2) If $f : R^n \to M$ is a module map and $x \in \mathrm{Ker}(f)$, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $x \in \mathrm{Ker}(h)$.
(3) Suppose $f : R^n \to M$ is a module map, $N \subset \mathrm{Ker}(f)$ any submodule, and $h : R^n \to R^m$ a map such that $N \subset \mathrm{Ker}(h)$ and f factors through h. Then given any $x \in \mathrm{Ker}(f)$ we can find a map $h' : R^n \to R^{m'}$ such that $N + Rx \subset \mathrm{Ker}(h')$ and f factors through h'.
(4) If $f : R^n \to M$ is a module map and $N \subset \mathrm{Ker}(f)$ is a finitely generated submodule, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $N \subset \mathrm{Ker}(h)$.

Proof. That (1) is equivalent to (2) is just a reformulation of the equational criterion for flatness[8]. To show (2) implies (3), let $g : R^m \to M$ be the map such that f factors as $f = g \circ h$. By (2) find $h'' : R^m \to R^{m'}$ such that h'' kills $h(x)$ and $g : R^m \to M$ factors through h''. Then taking $h' = h'' \circ h$ works. (3) implies (4) by induction on the number of generators of $N \subset \mathrm{Ker}(f)$ in (4). Clearly (4) implies (2). $\square$

058E Lemma 10.81.2. Let M be an R-module. Then M is flat if and only if the following condition holds: if P is a finitely presented R-module and $f : P \to M$ a module map, then there is a free finite R-module F and module maps $h : P \to F$ and $g : F \to M$ such that $f = g \circ h$.

Proof. This is just a reformulation of condition (4) from Lemma 10.81.1. $\square$

058F Lemma 10.81.3. Let M be an R-module. Then M is flat if and only if the following condition holds: for every finitely presented R-module P, if $N \to M$ is a surjective R-module map, then the induced map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, M)$ is surjective.

Proof. First suppose M is flat. We must show that if P is finitely presented, then given a map $f : P \to M$, it factors through the map $N \to M$. By Lemma 10.81.2 the map f factors through a map $F \to M$ where F is free and finite. Since F is free, this map factors through $N \to M$. Thus f factors through $N \to M$.

Conversely, suppose the condition of the lemma holds. Let $f : P \to M$ be a map from a finitely presented module P. Choose a free module N with a surjection $N \to M$ onto M. Then f factors through $N \to M$, and since P is finitely generated,

[8] In fact, a module map $f : R^n \to M$ corresponds to a choice of elements $x_1, x_2, \ldots, x_n$ of M (namely, the images of the standard basis elements $e_1, e_2, \ldots, e_n$); furthermore, an element $x \in \mathrm{Ker}(f)$ corresponds to a relation between these $x_1, x_2, \ldots, x_n$ (namely, the relation $\sum_i f_i x_i = 0$, where the f_i are the coordinates of x). The module map h (represented as an $m \times n$-matrix) corresponds to the matrix (a_{ij}) from Lemma 10.39.11, and the y_j of Lemma 10.39.11 are the images of the standard basis vectors of R^m under g.

f factors through a free finite submodule of N. Thus M satisfies the condition of Lemma 10.81.2, hence is flat. □

058G Theorem 10.81.4 (Lazard's theorem). Let M be an R-module. Then M is flat if and only if it is the colimit of a directed system of free finite R-modules.

Proof. A colimit of a directed system of flat modules is flat, as taking directed colimits is exact and commutes with tensor product. Hence if M is the colimit of a directed system of free finite modules then M is flat.

For the converse, first recall that any module M can be written as the colimit of a directed system of finitely presented modules, in the following way. Choose a surjection $f : R^I \to M$ for some set I, and let K be the kernel. Let E be the set of ordered pairs (J, N) where J is a finite subset of I and N is a finitely generated submodule of $R^J \cap K$. Then E is made into a directed partially ordered set by defining $(J, N) \leq (J', N')$ if and only if $J \subset J'$ and $N \subset N'$. Define $M_e = R^J/N$ for $e = (J, N)$, and define $f_{ee'} : M_e \to M_{e'}$ to be the natural map for $e \leq e'$. Then $(M_e, f_{ee'})$ is a directed system and the natural maps $f_e : M_e \to M$ induce an isomorphism $\mathrm{colim}_{e\in E} M_e \xrightarrow{\cong} M$.

Now suppose M is flat. Let $I = M \times \mathbf{Z}$, write (x_i) for the canonical basis of R^I, and take in the above discussion $f : R^I \to M$ to be the map sending x_i to the projection of i onto M. To prove the theorem it suffices to show that the $e \in E$ such that M_e is free form a cofinal subset of E. So let $e = (J, N) \in E$ be arbitrary. By Lemma 10.81.2 there is a free finite module F and maps $h : R^J/N \to F$ and $g : F \to M$ such that the natural map $f_e : R^J/N \to M$ factors as $R^J/N \xrightarrow{h} F \xrightarrow{g} M$. We are going to realize F as $M_{e'}$ for some $e' \geq e$.

Let $\{b_1, \ldots, b_n\}$ be a finite basis of F. Choose n distinct elements $i_1, \ldots, i_n \in I$ such that $i_\ell \notin J$ for all ℓ, and such that the image of x_{i_ℓ} under $f : R^I \to M$ equals the image of b_ℓ under $g : F \to M$. This is possible since every element of M can be written as $f(x_i)$ for infinitely many distinct $i \in I$ (by our choice of I). Now let $J' = J \cup \{i_1, \ldots, i_n\}$, and define $R^{J'} \to F$ by $x_i \mapsto h(x_i)$ for $i \in J$ and $x_{i_\ell} \mapsto b_\ell$ for $\ell = 1, \ldots, n$. Let $N' = \mathrm{Ker}(R^{J'} \to F)$. Observe:

(1) The square

$$\begin{tikzcd} R^{J'} \arrow[r] \arrow[d, hook] & F \arrow[d, "g"] \\ R^I \arrow[r, "f"'] & M \end{tikzcd}$$

is commutative, hence $N' \subset K = \mathrm{Ker}(f)$;

(2) $R^{J'} \to F$ is a surjection onto a free finite module, hence it splits and so N' is finitely generated;

(3) $J \subset J'$ and $N \subset N'$.

By (1) and (2) $e' = (J', N')$ is in E, by (3) $e' \geq e$, and by construction $M_{e'} = R^{J'}/N' \cong F$ is free. □

10.82. Universally injective module maps

058H

Next we discuss universally injective module maps, which are in a sense complementary to flat modules (see Lemma 10.82.5). We follow Lazard's thesis [Laz69]; also see [Lam99].

058I Definition 10.82.1. Let $f : M \to N$ be a map of R-modules. Then f is called universally injective if for every R-module Q, the map $f \otimes_R \text{id}_Q : M \otimes_R Q \to N \otimes_R Q$ is injective. A sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is called universally exact if it is exact and $M_1 \to M_2$ is universally injective.

058J Example 10.82.2. Examples of universally exact sequences.

(1) A split short exact sequence is universally exact since tensoring commutes with taking direct sums.
(2) The colimit of a directed system of universally exact sequences is universally exact. This follows from the fact that taking directed colimits is exact and that tensoring commutes with taking colimits. In particular the colimit of a directed system of split exact sequences is universally exact. We will see below that, conversely, any universally exact sequence arises in this way.

Next we give a list of criteria for a short exact sequence to be universally exact. They are analogues of criteria for flatness given above. Parts (3)-(6) below correspond, respectively, to the criteria for flatness given in Lemmas 10.39.11, 10.81.1, 10.81.3, and Theorem 10.81.4.

058K Theorem 10.82.3. Let

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

be an exact sequence of R-modules. The following are equivalent:

(1) The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is universally exact.
(2) For every finitely presented R-module Q, the sequence
$$0 \to M_1 \otimes_R Q \to M_2 \otimes_R Q \to M_3 \otimes_R Q \to 0$$
is exact.
(3) Given elements $x_i \in M_1$ $(i = 1, \ldots, n)$, $y_j \in M_2$ $(j = 1, \ldots, m)$, and $a_{ij} \in R$ $(i = 1, \ldots, n, j = 1, \ldots, m)$ such that for all i
$$f_1(x_i) = \sum\nolimits_j a_{ij} y_j,$$
there exists $z_j \in M_1$ $(j = 1, \ldots, m)$ such that for all i,
$$x_i = \sum\nolimits_j a_{ij} z_j.$$
(4) Given a commutative diagram of R-module maps
$$\begin{tikzcd} R^n \arrow[r] \arrow[d] & R^m \arrow[d] \\ M_1 \arrow[r, "f_1"] & M_2 \end{tikzcd}$$
where m and n are integers, there exists a map $R^m \to M_1$ making the top triangle commute.
(5) For every finitely presented R-module P, the R-module map $\text{Hom}_R(P, M_2) \to \text{Hom}_R(P, M_3)$ is surjective.

(6) The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is the colimit of a directed system of split exact sequences of the form
$$0 \to M_1 \to M_{2,i} \to M_{3,i} \to 0$$
where the $M_{3,i}$ are finitely presented.

Proof. Obviously (1) implies (2).

Next we show (2) implies (3). Let $f_1(x_i) = \sum_j a_{ij} y_j$ be relations as in (3). Let (d_j) be a basis for R^m, (e_i) a basis for R^n, and $R^m \to R^n$ the map given by $d_j \mapsto \sum_i a_{ij} e_i$. Let Q be the cokernel of $R^m \to R^n$. Then tensoring $R^m \to R^n \to Q \to 0$ by the map $f_1 : M_1 \to M_2$, we get a commutative diagram

$$\begin{tikzcd}
M_1^{\oplus m} \arrow[r] \arrow[d] & M_1^{\oplus n} \arrow[r] \arrow[d] & M_1 \otimes_R Q \arrow[r] \arrow[d] & 0 \\
M_2^{\oplus m} \arrow[r] & M_2^{\oplus n} \arrow[r] & M_2 \otimes_R Q \arrow[r] & 0
\end{tikzcd}$$

where $M_1^{\oplus m} \to M_1^{\oplus n}$ is given by
$$(z_1, \ldots, z_m) \mapsto (\sum\nolimits_j a_{1j} z_j, \ldots, \sum\nolimits_j a_{nj} z_j),$$
and $M_2^{\oplus m} \to M_2^{\oplus n}$ is given similarly. We want to show $x = (x_1, \ldots, x_n) \in M_1^{\oplus n}$ is in the image of $M_1^{\oplus m} \to M_1^{\oplus n}$. By (2) the map $M_1 \otimes Q \to M_2 \otimes Q$ is injective, hence by exactness of the top row it is enough to show x maps to 0 in $M_2 \otimes Q$, and so by exactness of the bottom row it is enough to show the image of x in $M_2^{\oplus n}$ is in the image of $M_2^{\oplus m} \to M_2^{\oplus n}$. This is true by assumption.

Condition (4) is just a translation of (3) into diagram form.

Next we show (4) implies (5). Let $\varphi : P \to M_3$ be a map from a finitely presented R-module P. We must show that φ lifts to a map $P \to M_2$. Choose a presentation of P,
$$R^n \xrightarrow{g_1} R^m \xrightarrow{g_2} P \to 0.$$
Using freeness of R^n and R^m, we can construct $h_2 : R^m \to M_2$ and then $h_1 : R^n \to M_1$ such that the following diagram commutes

$$\begin{tikzcd}
 & R^n \arrow[r, "g_1"] \arrow[d, "h_1"] & R^m \arrow[r, "g_2"] \arrow[d, "h_2"] & P \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f_1"] & M_2 \arrow[r, "f_2"] & M_3 \arrow[r] & 0.
\end{tikzcd}$$

By (4) there is a map $k_1 : R^m \to M_1$ such that $k_1 \circ g_1 = h_1$. Now define $h'_2 : R^m \to M_2$ by $h'_2 = h_2 - f_1 \circ k_1$. Then
$$h'_2 \circ g_1 = h_2 \circ g_1 - f_1 \circ k_1 \circ g_1 = h_2 \circ g_1 - f_1 \circ h_1 = 0.$$
Hence by passing to the quotient h'_2 defines a map $\varphi' : P \to M_2$ such that $\varphi' \circ g_2 = h'_2$. In a diagram, we have

$$\begin{tikzcd}
R^m \arrow[r, "g_2"] \arrow[d, "h'_2"'] & P \arrow[dl, "\varphi'"'] \arrow[d, "\varphi"] \\
M_2 \arrow[r, "f_2"] & M_3.
\end{tikzcd}$$

where the top triangle commutes. We claim that φ' is the desired lift, i.e. that $f_2 \circ \varphi' = \varphi$. From the definitions we have

$$f_2 \circ \varphi' \circ g_2 = f_2 \circ h_2' = f_2 \circ h_2 - f_2 \circ f_1 \circ k_1 = f_2 \circ h_2 = \varphi \circ g_2.$$

Since g_2 is surjective, this finishes the proof.

Now we show (5) implies (6). Write M_3 as the colimit of a directed system of finitely presented modules $M_{3,i}$, see Lemma 10.11.3. Let $M_{2,i}$ be the fiber product of $M_{3,i}$ and M_2 over M_3—by definition this is the submodule of $M_2 \times M_{3,i}$ consisting of elements whose two projections onto M_3 are equal. Let $M_{1,i}$ be the kernel of the projection $M_{2,i} \to M_{3,i}$. Then we have a directed system of exact sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0,$$

and for each i a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_{1,i} \arrow[r] \arrow[d] & M_{2,i} \arrow[r] \arrow[d] & M_{3,i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0
\end{tikzcd}$$

compatible with the directed system. From the definition of the fiber product $M_{2,i}$, it follows that the map $M_{1,i} \to M_1$ is an isomorphism. By (5) there is a map $M_{3,i} \to M_2$ lifting $M_{3,i} \to M_3$, and by the universal property of the fiber product this gives rise to a section of $M_{2,i} \to M_{3,i}$. Hence the sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0$$

split. Passing to the colimit, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{colim} M_{1,i} \arrow[r] \arrow[d, "\cong"] & \operatorname{colim} M_{2,i} \arrow[r] \arrow[d] & \operatorname{colim} M_{3,i} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0
\end{tikzcd}$$

with exact rows and outer vertical maps isomorphisms. Hence $\operatorname{colim} M_{2,i} \to M_2$ is also an isomorphism and (6) holds.

Condition (6) implies (1) by Example 10.82.2 (2). □

The previous theorem shows that a universally exact sequence is always a colimit of split short exact sequences. If the cokernel of a universally injective map is finitely presented, then in fact the map itself splits:

058L **Lemma 10.82.4.** Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Suppose M_3 is of finite presentation. Then

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact if and only if it is split.

Proof. A split short exact sequence is always universally exact, see Example 10.82.2. Conversely, if the sequence is universally exact, then by Theorem 10.82.3 (5) applied to $P = M_3$, the map $M_2 \to M_3$ admits a section. □

The following lemma shows how universally injective maps are complementary to flat modules.

058M **Lemma 10.82.5.** Let M be an R-module. Then M is flat if and only if any exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M \to 0$$

is universally exact.

Proof. This follows from Lemma 10.81.3 and Theorem 10.82.3 (5). $\square$

058N **Example 10.82.6.** Non-split and non-flat universally exact sequences.

(1) In spite of Lemma 10.82.4, it is possible to have a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact but non-split. For instance, take $R = \mathbf{Z}$, let $M_1 = \bigoplus_{n=1}^{\infty} \mathbf{Z}$, let $M_2 = \prod_{n=1}^{\infty} \mathbf{Z}$, and let M_3 be the cokernel of the inclusion $M_1 \to M_2$. Then M_1, M_2, M_3 are all flat since they are torsion-free (More on Algebra, Lemma 15.22.11), so by Lemma 10.82.5,

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact. However there can be no section $s : M_3 \to M_2$. In fact, if x is the image of $(2, 2^2, 2^3, \ldots) \in M_2$ in M_3, then any module map $s : M_3 \to M_2$ must kill x. This is because $x \in 2^n M_3$ for any $n \geq 1$, hence $s(x)$ is divisible by 2^n for all $n \geq 1$ and so must be 0.

(2) In spite of Lemma 10.82.5, it is possible to have a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact but with M_1, M_2, M_3 all non-flat. In fact if M is any non-flat module, just take the split exact sequence

$$0 \to M \to M \oplus M \to M \to 0.$$

For instance over $R = \mathbf{Z}$, take M to be any torsion module.

(3) Taking the direct sum of an exact sequence as in (1) with one as in (2), we get a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact, non-split, and such that M_1, M_2, M_3 are all non-flat.

058P **Lemma 10.82.7.** Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules, and suppose M_2 is flat. Then M_1 and M_3 are flat.

Proof. Let $0 \to N \to N' \to N'' \to 0$ be a short exact sequence of R-modules. Consider the commutative diagram

$$\begin{tikzcd}
M_1 \otimes_R N \arrow[r] \arrow[d] & M_2 \otimes_R N \arrow[r] \arrow[d] & M_3 \otimes_R N \arrow[d] \\
M_1 \otimes_R N' \arrow[r] \arrow[d] & M_2 \otimes_R N' \arrow[r] \arrow[d] & M_3 \otimes_R N' \arrow[d] \\
M_1 \otimes_R N'' \arrow[r] & M_2 \otimes_R N'' \arrow[r] & M_3 \otimes_R N''
\end{tikzcd}$$

(we have dropped the 0's on the boundary). By assumption the rows give short exact sequences and the arrow $M_2 \otimes N \to M_2 \otimes N'$ is injective. Clearly this implies that $M_1 \otimes N \to M_1 \otimes N'$ is injective and we see that M_1 is flat. In particular the left and middle columns give rise to short exact sequences. It follows from a diagram chase that the arrow $M_3 \otimes N \to M_3 \otimes N'$ is injective. Hence M_3 is flat. □

05CH Lemma 10.82.8. Let R be a ring. Let $M \to M'$ be a universally injective R-module map. Then for any R-module N the map $M \otimes_R N \to M' \otimes_R N$ is universally injective.

Proof. Omitted. □

05CI Lemma 10.82.9. Let R be a ring. A composition of universally injective R-module maps is universally injective.

Proof. Omitted. □

05CJ Lemma 10.82.10. Let R be a ring. Let $M \to M'$ and $M' \to M''$ be R-module maps. If their composition $M \to M''$ is universally injective, then $M \to M'$ is universally injective.

Proof. Omitted. □

05CK Lemma 10.82.11. Let $R \to S$ be a faithfully flat ring map. Then $R \to S$ is universally injective as a map of R-modules. In particular $R \cap IS = I$ for any ideal $I \subset R$.

Proof. Let N be an R-module. We have to show that $N \to N \otimes_R S$ is injective. As S is faithfully flat as an R-module, it suffices to prove this after tensoring with S. Hence it suffices to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$, $n \otimes s \mapsto n \otimes 1 \otimes s$ is injective. This is true because there is a retraction, namely, $n \otimes s \otimes s' \mapsto n \otimes ss'$. □

05CL Lemma 10.82.12. Let $R \to S$ be a ring map. Let $M \to M'$ be a map of S-modules. The following are equivalent

(1) $M \to M'$ is universally injective as a map of R-modules,
(2) for each prime $\mathfrak{q}$ of S the map $M_{\mathfrak{q}} \to M'_{\mathfrak{q}}$ is universally injective as a map of R-modules,
(3) for each maximal ideal $\mathfrak{m}$ of S the map $M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is universally injective as a map of R-modules,
(4) for each prime $\mathfrak{q}$ of S the map $M_{\mathfrak{q}} \to M'_{\mathfrak{q}}$ is universally injective as a map of $R_{\mathfrak{p}}$-modules, where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R, and
(5) for each maximal ideal $\mathfrak{m}$ of S the map $M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is universally injective as a map of $R_{\mathfrak{p}}$-modules, where $\mathfrak{p}$ is the inverse image of $\mathfrak{m}$ in R.

Proof. Let N be an R-module. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Then we have

$$(M \otimes_R N)_{\mathfrak{q}} = M_{\mathfrak{q}} \otimes_R N = M_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}.$$

Moreover, the same thing holds for M' and localization is exact. Also, if N is an $R_{\mathfrak{p}}$-module, then $N_{\mathfrak{p}} = N$. Using this the equivalences can be proved in a straightforward manner.

For example, suppose that (5) holds. Let $K = \mathrm{Ker}(M \otimes_R N \to M' \otimes_R N)$. By the remarks above we see that $K_{\mathfrak{m}} = 0$ for each maximal ideal $\mathfrak{m}$ of S. Hence $K = 0$ by Lemma 10.23.1. Thus (1) holds. Conversely, suppose that (1) holds. Take any $\mathfrak{q} \subset S$ lying over $\mathfrak{p} \subset R$. Take any module N over $R_\mathfrak{p}$. Then by assumption $\mathrm{Ker}(M \otimes_R N \to M' \otimes_R N) = 0$. Hence by the formulae above and the fact that $N = N_\mathfrak{p}$ we see that $\mathrm{Ker}(M_\mathfrak{q} \otimes_{R_\mathfrak{p}} N \to M'_\mathfrak{q} \otimes_{R_\mathfrak{p}} N) = 0$. In other words (4) holds. Of course (4) $\Rightarrow$ (5) is immediate. Hence (1), (4) and (5) are all equivalent. We omit the proof of the other equivalences. $\square$

05CM Lemma 10.82.13. Let $\varphi : A \to B$ be a ring map. Let $S \subset A$ and $S' \subset B$ be multiplicative subsets such that $\varphi(S) \subset S'$. Let $M \to M'$ be a map of B-modules.

(1) If $M \to M'$ is universally injective as a map of A-modules, then $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules and as a map of $S^{-1}A$-modules.
(2) If M and M' are $(S')^{-1}B$-modules, then $M \to M'$ is universally injective as a map of A-modules if and only if it is universally injective as a map of $S^{-1}A$-modules.

Proof. You can prove this using Lemma 10.82.12 but you can also prove it directly as follows. Assume $M \to M'$ is A-universally injective. Let Q be an A-module. Then $Q \otimes_A M \to Q \otimes_A M'$ is injective. Since localization is exact we see that $(S')^{-1}(Q \otimes_A M) \to (S')^{-1}(Q \otimes_A M')$ is injective. As $(S')^{-1}(Q \otimes_A M) = Q \otimes_A (S')^{-1}M$ and similarly for M' we see that $Q \otimes_A (S')^{-1}M \to Q \otimes_A (S')^{-1}M'$ is injective, hence $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules. This proves the first part of (1). To see (2) we can use the following two facts: (a) if Q is an $S^{-1}A$-module, then $Q \otimes_A S^{-1}A = Q$, i.e., tensoring with Q over A is the same thing as tensoring with Q over $S^{-1}A$, (b) if M is any A-module on which the elements of S are invertible, then $M \otimes_A Q = M \otimes_{S^{-1}A} S^{-1}Q$. Part (2) follows from this immediately. $\square$

0AS5 Lemma 10.82.14. Let R be a ring and let $M \to M'$ be a map of R-modules. If M' is flat, then $M \to M'$ is universally injective if and only if $M/IM \to M'/IM'$ is injective for every finitely generated ideal I of R.

Proof. It suffices to show that $M \otimes_R Q \to M' \otimes_R Q$ is injective for every finite R-module Q, see Theorem 10.82.3. Then Q has a finite filtration $0 = Q_0 \subset Q_1 \subset \ldots \subset Q_n = Q$ by submodules whose subquotients are isomorphic to cyclic modules R/I_i, see Lemma 10.5.4. Since M' is flat, we obtain a filtration

$$\begin{tikzcd} M \otimes Q_1 \arrow[r] \arrow[d] & M \otimes Q_2 \arrow[r] \arrow[d] & \ldots \arrow[r] & M \otimes Q \arrow[d] \\ M' \otimes Q_1 \arrow[r, hook] & M' \otimes Q_2 \arrow[r, hook] & \ldots \arrow[r, hook] & M' \otimes Q \end{tikzcd}$$

of $M' \otimes_R Q$ by submodules $M' \otimes_R Q_i$ whose successive quotients are $M' \otimes_R R/I_i = M'/I_iM'$. A simple induction argument shows that it suffices to check $M/I_iM \to M'/I_iM'$ is injective. Note that the collection of finitely generated ideals $I'_i \subset I_i$ is a directed set. Thus $M/I_iM = \mathrm{colim}\, M/I'_iM$ is a filtered colimit, similarly for M', the maps $M/I'_iM \to M'/I'_iM'$ are injective by assumption, and since filtered colimits are exact (Lemma 10.8.8) we conclude. $\square$

10.83. Descent for finite projective modules

058Q In this section we give an elementary proof of the fact that the property of being a finite projective module descends along faithfully flat ring maps. The proof does not apply when we drop the finiteness condition. However, the method is indicative of the one we shall use to prove descent for the property of being a countably generated projective module—see the comments at the end of this section.

058R Lemma 10.83.1. Let M be an R-module. Then M is finite projective if and only if M is finitely presented and flat.

Proof. This is part of Lemma 10.78.2. However, at this point we can give a more elegant proof of the implication (1) $\Rightarrow$ (2) of that lemma as follows. If M is finitely presented and flat, then take a surjection $R^n \to M$. By Lemma 10.81.3 applied to $P = M$, the map $R^n \to M$ admits a section. So M is a direct summand of a free module and hence projective. $\square$

Here are some properties of modules that descend.

03C4 Lemma 10.83.2. Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. Then

(1) if the S-module $M \otimes_R S$ is of finite type, then M is of finite type,
(2) if the S-module $M \otimes_R S$ is of finite presentation, then M is of finite presentation,
(3) if the S-module $M \otimes_R S$ is flat, then M is flat, and
(4) add more here as needed.

Proof. Assume $M \otimes_R S$ is of finite type. Let $y_1, \ldots, y_m$ be generators of $M \otimes_R S$ over S. Write $y_j = \sum x_i \otimes f_i$ for some $x_1, \ldots, x_n \in M$. Then we see that the map $\varphi : R^{\oplus n} \to M$ has the property that $\varphi \otimes \mathrm{id}_S : S^{\oplus n} \to M \otimes_R S$ is surjective. Since $R \to S$ is faithfully flat we see that φ is surjective, and M is finitely generated.

Assume $M \otimes_R S$ is of finite presentation. By (1) we see that M is of finite type. Choose a surjection $R^{\oplus n} \to M$ and denote K the kernel. As $R \to S$ is flat we see that $K \otimes_R S$ is the kernel of the base change $S^{\oplus n} \to M \otimes_R S$. As $M \otimes_R S$ is of finite presentation we conclude that $K \otimes_R S$ is of finite type. Hence by (1) we see that K is of finite type and hence M is of finite presentation.

Part (3) is Lemma 10.39.8. $\square$

058S Proposition 10.83.3. Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is finite projective, then M is finite projective.

Proof. Follows from Lemmas 10.83.1 and 10.83.2. $\square$

The next few sections are about removing the finiteness assumption by using dévissage to reduce to the countably generated case. In the countably generated case, the strategy is to find a characterization of countably generated projective modules analogous to Lemma 10.83.1, and then to prove directly that this characterization descends. We do this by introducing the notion of a Mittag-Leffler module and proving that if a module M is countably generated, then it is projective if and only if it is flat and Mittag-Leffler (Theorem 10.93.3). When M is finitely generated, this statement reduces to Lemma 10.83.1 (since, according to Example 10.91.1 (1), a finitely generated module is Mittag-Leffler if and only if it is finitely presented).

10.84. Transfinite dévissage of modules

058T In this section we introduce a dévissage technique for decomposing a module into a direct sum. The main result is that a projective module is a direct sum of countably generated modules (Theorem 10.84.5 below). We follow [Kap58].

058U Definition 10.84.1. Let M be an R-module. A direct sum dévissage of M is a family of submodules $(M_\alpha)_{\alpha \in S}$, indexed by an ordinal S and increasing (with respect to inclusion), such that:

(0) $M_0 = 0$;
(1) $M = \bigcup_\alpha M_\alpha$;
(2) if $\alpha \in S$ is a limit ordinal, then $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$;
(3) if $\alpha + 1 \in S$, then M_α is a direct summand of $M_{\alpha+1}$.

If moreover

(4) $M_{\alpha+1}/M_\alpha$ is countably generated for $\alpha + 1 \in S$,

then $(M_\alpha)_{\alpha \in S}$ is called a Kaplansky dévissage of M.

The terminology is justified by the following lemma.

058V Lemma 10.84.2. Let M be an R-module. If $(M_\alpha)_{\alpha \in S}$ is a direct sum dévissage of M, then $M \cong \bigoplus_{\alpha+1 \in S} M_{\alpha+1}/M_\alpha$.

Proof. By property (3) of a direct sum dévissage, there is an inclusion $M_{\alpha+1}/M_\alpha \to M$ for each $\alpha \in S$. Consider the map

$$f : \bigoplus\nolimits_{\alpha+1 \in S} M_{\alpha+1}/M_\alpha \to M$$

given by the sum of these inclusions. Further consider the restrictions

$$f_\beta : \bigoplus\nolimits_{\alpha+1 \leq \beta} M_{\alpha+1}/M_\alpha \longrightarrow M$$

for $\beta \in S$. Transfinite induction on S shows that the image of f_β is M_β. For $\beta = 0$ this is true by (0). If $\beta + 1$ is a successor ordinal and it is true for β, then it is true for $\beta + 1$ by (3). And if β is a limit ordinal and it is true for $\alpha < \beta$, then it is true for β by (2). Hence f is surjective by (1).

Transfinite induction on S also shows that the restrictions f_β are injective. For $\beta = 0$ it is true. If $\beta+1$ is a successor ordinal and f_β is injective, then let x be in the kernel and write $x = (x_{\alpha+1})_{\alpha+1 \leq \beta+1}$ in terms of its components $x_{\alpha+1} \in M_{\alpha+1}/M_\alpha$. By property (3) and the fact that the image of f_β is M_β both $(x_{\alpha+1})_{\alpha+1 \leq \beta}$ and $x_{\beta+1}$ map to 0. Hence $x_{\beta+1} = 0$ and, by the assumption that the restriction f_β is injective also $x_{\alpha+1} = 0$ for every $\alpha + 1 \leq \beta$. So $x = 0$ and $f_{\beta+1}$ is injective. If β is a limit ordinal consider an element x of the kernel. Then x is already contained in the domain of f_α for some $\alpha < \beta$. Thus $x = 0$ which finishes the induction. We conclude that f is injective since f_β is for each $\beta \in S$. □

058W Lemma 10.84.3. Let M be an R-module. Then M is a direct sum of countably generated R-modules if and only if it admits a Kaplansky dévissage.

Proof. The lemma takes care of the "if" direction. Conversely, suppose $M = \bigoplus_{i \in I} N_i$ where each N_i is a countably generated R-module. Well-order I so that we can think of it as an ordinal. Then setting $M_i = \bigoplus_{j < i} N_j$ gives a Kaplansky dévissage $(M_i)_{i \in I}$ of M. □

058X **Theorem 10.84.4.** Suppose M is a direct sum of countably generated R-modules. If P is a direct summand of M, then P is also a direct sum of countably generated R-modules.

Proof. Write $M = P \oplus Q$. We are going to construct a Kaplansky dévissage $(M_\alpha)_{\alpha \in S}$ of M which, in addition to the defining properties (0)-(4), satisfies:

(5) Each M_α is a direct summand of M;
(6) $M_\alpha = P_\alpha \oplus Q_\alpha$, where $P_\alpha = P \cap M_\alpha$ and $Q = Q \cap M_\alpha$.

(Note: if properties (0)-(2) hold, then in fact property (3) is equivalent to property (5).)

To see how this implies the theorem, it is enough to show that $(P_\alpha)_{\alpha \in S}$ forms a Kaplansky dévissage of P. Properties (0), (1), and (2) are clear. By (5) and (6) for (M_α), each P_α is a direct summand of M. Since $P_\alpha \subset P_{\alpha+1}$, this implies P_α is a direct summand of $P_{\alpha+1}$; hence (3) holds for (P_α). For (4), note that

$$M_{\alpha+1}/M_\alpha \cong P_{\alpha+1}/P_\alpha \oplus Q_{\alpha+1}/Q_\alpha,$$

so $P_{\alpha+1}/P_\alpha$ is countably generated because this is true of $M_{\alpha+1}/M_\alpha$.

It remains to construct the M_α. Write $M = \bigoplus_{i \in I} N_i$ where each N_i is a countably generated R-module. Choose a well-ordering of I. By transfinite recursion we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that M_α is a direct sum of some subset of the N_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$. Since each M_β for $\beta < \alpha$ is a direct sum of a subset of the N_i, the same will be true of M_α. If $\alpha + 1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. If not, choose the smallest $j \in I$ such that N_j is not contained in M_α. We will construct an infinite matrix $(x_{mn}), m, n = 1, 2, 3, \ldots$ such that:

(1) N_j is contained in the submodule of M generated by the entries x_{mn};
(2) if we write any entry $x_{k\ell}$ in terms of its P- and Q-components, $x_{k\ell} = y_{k\ell} + z_{k\ell}$, then the matrix (x_{mn}) contains a set of generators for each N_i for which $y_{k\ell}$ or $z_{k\ell}$ has nonzero component.

Then we define $M_{\alpha+1}$ to be the submodule of M generated by M_α and all x_{mn}; by property (2) of the matrix (x_{mn}), $M_{\alpha+1}$ will be a direct sum of some subset of the N_i. To construct the matrix (x_{mn}), let $x_{11}, x_{12}, x_{13}, \ldots$ be a countable set of generators for N_j. Then if $x_{11} = y_{11} + z_{11}$ is the decomposition into P- and Q-components, let $x_{21}, x_{22}, x_{23}, \ldots$ be a countable set of generators for the sum of the N_i for which y_{11} or z_{11} have nonzero component. Repeat this process on x_{12} to get elements $x_{31}, x_{32}, \ldots$, the third row of our matrix. Repeat on x_{21} to get the fourth row, on x_{13} to get the fifth, and so on, going down along successive anti-diagonals as indicated below:

$$\left(\begin{array}{lllll} x_{11} & x_{12} & x_{13} & x_{14} & \ldots \\ x_{21} & x_{22} & x_{23} & \ldots & \\ x_{31} & x_{32} & \ldots & & \\ x_{41} & \ldots & & & \\ \ldots & & & & \end{array}\right).$$

Transfinite induction on I (using the fact that we constructed $M_{\alpha+1}$ to contain N_j for the smallest j such that N_j is not contained in M_α) shows that for each $i \in I$, N_i is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $i \in I$ there is $\alpha \in S$ such that N_i is contained in M_α. This means $(M_\alpha)_{\alpha \in S}$ satisfies property (1) of a Kaplansky dévissage of M. The family $(M_\alpha)_{\alpha \in S}$ moreover satisfies the other defining properties, and also (5) and (6) above: properties (0), (2), (4), and (6) are clear by construction; property (5) is true because each M_α is by construction a direct sum of some N_i; and (3) is implied by (5) and the fact that $M_\alpha \subset M_{\alpha+1}$. □

As a corollary we get the result for projective modules stated at the beginning of the section.

058Y Theorem 10.84.5. If P is a projective R-module, then P is a direct sum of countably generated projective R-modules.

Proof. A module is projective if and only if it is a direct summand of a free module, so this follows from Theorem 10.84.4. □

10.85. Projective modules over a local ring

058Z In this section we prove a very cute result: a projective module M over a local ring is free (Theorem 10.85.4 below). Note that with the additional assumption that M is finite, this result is Lemma 10.78.5. In general we have:

0590 Lemma 10.85.1. Let R be a ring. Then every projective R-module is free if and only if every countably generated projective R-module is free.

Proof. Follows immediately from Theorem 10.84.5. □

Here is a criterion for a countably generated module to be free.

0591 Lemma 10.85.2. Let M be a countably generated R-module with the following property: if $M = N \oplus N'$ with N' a finite free R-module, then any element of N is contained in a free direct summand of N. Then M is free.

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. We inductively construct finite free direct summands $F_1, F_2, \ldots$ of M such that for all n we have that $F_1 \oplus \ldots \oplus F_n$ is a direct summand of M which contains $x_1, \ldots, x_n$. Namely, given $F_1, \ldots, F_n$ with the desired properties, write

$$M = F_1 \oplus \ldots \oplus F_n \oplus N$$

and let $x \in N$ be the image of x_{n+1}. Then we can find a free direct summand $F_{n+1} \subset N$ containing x by the assumption in the statement of the lemma. Of course we can replace F_{n+1} by a finite free direct summand of F_{n+1} and the induction step is complete. Then $M = \bigoplus_{i=1}^{\infty} F_i$ is free. □

0592 Lemma 10.85.3. Let P be a projective module over a local ring R. Then any element of P is contained in a free direct summand of P.

Proof. Since P is projective it is a direct summand of some free R-module F, say $F = P \oplus Q$. Let $x \in P$ be the element that we wish to show is contained in a free direct summand of P. Let B be a basis of F such that the number of basis elements needed in the expression of x is minimal, say $x = \sum_{i=1}^{n} a_i e_i$ for some $e_i \in B$ and $a_i \in R$. Then no a_j can be expressed as a linear combination of the other a_i; for if

$a_j = \sum_{i \neq j} a_i b_i$ for some $b_i \in R$, then replacing e_i by $e_i + b_i e_j$ for $i \neq j$ and leaving unchanged the other elements of B, we get a new basis for F in terms of which x has a shorter expression.

Let $e_i = y_i + z_i, y_i \in P, z_i \in Q$ be the decomposition of e_i into its P- and Q-components. Write $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$, where t_i is a linear combination of elements in B other than $e_1, \ldots, e_n$. To finish the proof it suffices to show that the matrix (b_{ij}) is invertible. For then the map $F \to F$ sending $e_i \mapsto y_i$ for $i = 1, \ldots, n$ and fixing $B \setminus \{e_1, \ldots, e_n\}$ is an isomorphism, so that $y_1, \ldots, y_n$ together with $B \setminus \{e_1, \ldots, e_n\}$ form a basis for F. Then the submodule N spanned by $y_1, \ldots, y_n$ is a free submodule of P; N is a direct summand of P since $N \subset P$ and both N and P are direct summands of F; and $x \in N$ since $x \in P$ implies $x = \sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$.

Now we prove that (b_{ij}) is invertible. Plugging $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$ into $\sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$ and equating the coefficients of e_j gives $a_j = \sum_{i=1}^n a_i b_{ij}$. But as noted above, our choice of B guarantees that no a_j can be written as a linear combination of the other a_i. Thus b_{ij} is a non-unit for $i \neq j$, and $1 - b_{ii}$ is a non-unit—so in particular b_{ii} is a unit—for all i. But a matrix over a local ring having units along the diagonal and non-units elsewhere is invertible, as its determinant is a unit. □

0593 Theorem 10.85.4. If P is a projective module over a local ring R, then P is free.

Proof. Follows from Lemmas 10.85.1, 10.85.2, and 10.85.3. □

10.86. Mittag-Leffler systems

0594 The purpose of this section is to define Mittag-Leffler systems and why this is a useful notion.

In the following, I will be a directed set, see Categories, Definition 4.21.1. Let $(A_i, \varphi_{ji} : A_j \to A_i)$ be an inverse system of sets or of modules indexed by I, see Categories, Definition 4.21.4. This is a directed inverse system as we assumed I directed (Categories, Definition 4.21.4). For each $i \in I$, the images $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ form a decreasing directed family of subsets (or submodules) of A_i. Let $A'_i = \bigcap_{j \geq i} \varphi_{ji}(A_j)$. Then $\varphi_{ji}(A'_j) \subset A'_i$ for $j \geq i$, hence by restricting we get a directed inverse system $(A'_i, \varphi_{ji}|_{A'_j})$. From the construction of the limit of an inverse system in the category of sets or modules, we have $\lim A_i = \lim A'_i$. The Mittag-Leffler condition on (A_i, φ_{ji}) is that A'_i equals $\varphi_{ji}(A_j)$ for some $j \geq i$ (and hence equals $\varphi_{ki}(A_k)$ for all $k \geq j$):

0595 Definition 10.86.1. Let (A_i, φ_{ji}) be a directed inverse system of sets over I. Then we say (A_i, φ_{ji}) is Mittag-Leffler if for each $i \in I$, the family $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ stabilizes. Explicitly, this means that for each $i \in I$, there exists $j \geq i$ such that for $k \geq j$ we have $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$. If (A_i, φ_{ji}) is a directed inverse system of modules over a ring R, we say that it is Mittag-Leffler if the underlying inverse system of sets is Mittag-Leffler.

0596 Example 10.86.2. If (A_i, φ_{ji}) is a directed inverse system of sets or of modules and the maps φ_{ji} are surjective, then clearly the system is Mittag-Leffler. Conversely, suppose (A_i, φ_{ji}) is Mittag-Leffler. Let $A'_i \subset A_i$ be the stable image of $\varphi_{ji}(A_j)$ for $j \geq i$. Then $\varphi_{ji}|_{A'_j} : A'_j \to A'_i$ is surjective for $j \geq i$ and $\lim A_i = \lim A'_i$. Hence the limit of the Mittag-Leffler system (A_i, φ_{ji}) can also be written as the limit of a directed inverse system over I with surjective maps.

0597 Lemma 10.86.3. Let (A_i, φ_{ji}) be a directed inverse system over I. Suppose I is countable. If (A_i, φ_{ji}) is Mittag-Leffler and the A_i are nonempty, then $\lim A_i$ is nonempty.

Proof. Let $i_1, i_2, i_3, \ldots$ be an enumeration of the elements of I. Define inductively a sequence of elements $j_n \in I$ for $n = 1, 2, 3, \ldots$ by the conditions: $j_1 = i_1$, and $j_n \geq i_n$ and $j_n \geq j_m$ for $m < n$. Then the sequence j_n is increasing and forms a cofinal subset of I. Hence we may assume $I = \{1, 2, 3, \ldots\}$. So by Example 10.86.2 we are reduced to showing that the limit of an inverse system of nonempty sets with surjective maps indexed by the positive integers is nonempty. This is obvious. $\square$

The Mittag-Leffler condition will be important for us because of the following exactness property.

0598 Lemma 10.86.4. Let

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

be an exact sequence of directed inverse systems of abelian groups over I. Suppose I is countable. If (A_i) is Mittag-Leffler, then

$$0 \to \lim A_i \to \lim B_i \to \lim C_i \to 0$$

is exact.

Proof. Taking limits of directed inverse systems is left exact, hence we only need to prove surjectivity of $\lim B_i \to \lim C_i$. So let $(c_i) \in \lim C_i$. For each $i \in I$, let $E_i = g_i^{-1}(c_i)$, which is nonempty since $g_i : B_i \to C_i$ is surjective. The system of maps $\varphi_{ji} : B_j \to B_i$ for (B_i) restrict to maps $E_j \to E_i$ which make (E_i) into an inverse system of nonempty sets. It is enough to show that (E_i) is Mittag-Leffler. For then Lemma 10.86.3 would show $\lim E_i$ is nonempty, and taking any element of $\lim E_i$ would give an element of $\lim B_i$ mapping to (c_i).

By the injection $f_i : A_i \to B_i$ we will regard A_i as a subset of B_i. Since (A_i) is Mittag-Leffler, if $i \in I$ then there exists $j \geq i$ such that $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$ for $k \geq j$. We claim that also $\varphi_{ki}(E_k) = \varphi_{ji}(E_j)$ for $k \geq j$. Always $\varphi_{ki}(E_k) \subset \varphi_{ji}(E_j)$ for $k \geq j$. For the reverse inclusion let $e_j \in E_j$, and we need to find $x_k \in E_k$ such that $\varphi_{ki}(x_k) = \varphi_{ji}(e_j)$. Let $e'_k \in E_k$ be any element, and set $e'_j = \varphi_{kj}(e'_k)$. Then $g_j(e_j - e'_j) = c_j - c_j = 0$, hence $e_j - e'_j = a_j \in A_j$. Since $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$, there exists $a_k \in A_k$ such that $\varphi_{ki}(a_k) = \varphi_{ji}(a_j)$. Hence

$$\varphi_{ki}(e'_k + a_k) = \varphi_{ji}(e'_j) + \varphi_{ji}(a_j) = \varphi_{ji}(e_j),$$

so we can take $x_k = e'_k + a_k$. $\square$

10.87. Inverse systems

03C9 In many papers (and in this section) the term inverse system is used to indicate an inverse system over the partially ordered set $(\mathbf{N}, \geq)$. We briefly discuss such systems in this section. This material will be discussed more broadly in Homology, Section 12.31. Suppose we are given a ring R and a sequence of R-modules

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

with maps as indicated. By composing successive maps we obtain maps $\varphi_{ii'} : M_i \to M_{i'}$ whenever $i \geq i'$ such that moreover $\varphi_{ii''} = \varphi_{i'i''} \circ \varphi_{ii'}$ whenever $i \geq i' \geq i''$. Conversely, given the system of maps $\varphi_{ii'}$ we can set $\varphi_i = \varphi_{i(i-1)}$ and recover the maps displayed above. In this case

$$\lim M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

compare with Categories, Section 4.15. As explained in Homology, Section 12.31 this is actually a limit in the category of R-modules, as defined in Categories, Section 4.14.

03CA Lemma 10.87.1. Let R be a ring. Let $0 \to K_i \to L_i \to M_i \to 0$ be short exact sequences of R-modules, $i \geq 1$ which fit into maps of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K_i \arrow[r] & L_i \arrow[r] & M_i \arrow[r] & 0 \\ 0 \arrow[r] & K_{i+1} \arrow[r] \arrow[u] & L_{i+1} \arrow[r] \arrow[u] & M_{i+1} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

If for every i there exists a $c = c(i) \geq i$ such that $\mathrm{Im}(K_c \to K_i) = \mathrm{Im}(K_j \to K_i)$ for all $j \geq c$, then the sequence

$$0 \to \lim K_i \to \lim L_i \to \lim M_i \to 0$$

is exact.

Proof. This is a special case of the more general Lemma 10.86.4. □

10.88. Mittag-Leffler modules

0599 A Mittag-Leffler module is (very roughly) a module which can be written as a directed limit whose dual is a Mittag-Leffler system. To be able to give a precise definition we need to do a bit of work.

059A Definition 10.88.1. Let (M_i, f_{ij}) be a directed system of R-modules. We say that (M_i, f_{ij}) is a Mittag-Leffler directed system of modules if each M_i is an R-module of finite presentation and if for every R-module N, the inverse system

$$(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$$

is Mittag-Leffler.

We are going to characterize those R-modules that are colimits of Mittag-Leffler directed systems of modules.

059B Definition 10.88.2. Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then we say g dominates f if for any R-module Q, we have $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$.

It is enough to check this condition for finitely presented modules.

059C Lemma 10.88.3. Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then g dominates f if and only if for any finitely presented R-module Q, we have $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$.

Proof. Suppose $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$ for all finitely presented modules Q. If Q is an arbitrary module, write $Q = \mathrm{colim}_{i \in I} Q_i$ as a colimit of a directed system of finitely presented modules Q_i. Then $\mathrm{Ker}(f \otimes_R \mathrm{id}_{Q_i}) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_{Q_i})$ for all i. Since taking directed colimits is exact and commutes with tensor product, it follows that $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$. □

0AUM **Lemma 10.88.4.** Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Consider the pushout of f and g,

$$\begin{tikzcd} M \arrow[r, "f"'] \arrow[d, "g"'] & N \arrow[d, "g'"] \\ M' \arrow[r, "f'"] & N' \end{tikzcd}$$

Then g dominates f if and only if f' is universally injective.

Proof. Recall that N' is $M' \oplus N$ modulo the submodule consisting of elements $(g(x), -f(x))$ for $x \in M$. From the construction of N' we have a short exact sequence

$$0 \to \text{Ker}(f) \cap \text{Ker}(g) \to \text{Ker}(f) \to \text{Ker}(f') \to 0.$$

Since tensoring commutes with taking pushouts, we have such a short exact sequence

$$0 \to \text{Ker}(f \otimes \text{id}_Q) \cap \text{Ker}(g \otimes \text{id}_Q) \to \text{Ker}(f \otimes \text{id}_Q) \to \text{Ker}(f' \otimes \text{id}_Q) \to 0$$

for every R-module Q. So f' is universally injective if and only if $\text{Ker}(f \otimes \text{id}_Q) \subset \text{Ker}(g \otimes \text{id}_Q)$ for every Q, if and only if g dominates f. $\square$

The above definition of domination is sometimes related to the usual notion of domination of maps as the following lemma shows.

059D **Lemma 10.88.5.** Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Suppose $\text{Coker}(f)$ is of finite presentation. Then g dominates f if and only if g factors through f, i.e. there exists a module map $h : N \to M'$ such that $g = h \circ f$.

Proof. Consider the pushout of f and g as in the statement of Lemma 10.88.4. From the construction of the pushout it follows that $\text{Coker}(f') = \text{Coker}(f)$, so $\text{Coker}(f')$ is of finite presentation. Then by Lemma 10.82.4, f' is universally injective if and only if

$$0 \to M' \xrightarrow{f'} N' \to \text{Coker}(f') \to 0$$

splits. This is the case if and only if there is a map $h' : N' \to M'$ such that $h' \circ f' = \text{id}_{M'}$. From the universal property of the pushout, the existence of such an h' is equivalent to g factoring through f. $\square$

059E **Proposition 10.88.6.** Let M be an R-module. Let (M_i, f_{ij}) be a directed system of finitely presented R-modules, indexed by I, such that $M = \text{colim}\, M_i$. Let $f_i : M_i \to M$ be the canonical map. The following are equivalent:

(1) For every finitely presented R-module P and module map $f : P \to M$, there exists a finitely presented R-module Q and a module map $g : P \to Q$ such that g and f dominate each other, i.e., $\text{Ker}(f \otimes_R \text{id}_N) = \text{Ker}(g \otimes_R \text{id}_N)$ for every R-module N.
(2) For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ dominates $f_i : M_i \to M$.
(3) For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$.
(4) For every R-module N, the inverse system $(\text{Hom}_R(M_i, N), \text{Hom}_R(f_{ij}, N))$ is Mittag-Leffler.
(5) For $N = \prod_{s \in I} M_s$, the inverse system $(\text{Hom}_R(M_i, N), \text{Hom}_R(f_{ij}, N))$ is Mittag-Leffler.

Proof. First we prove the equivalence of (1) and (2). Suppose (1) holds and let $i \in I$. Corresponding to the map $f_i : M_i \to M$, we can choose $g : M_i \to Q$ as in (1). Since M_i and Q are of finite presentation, so is $\text{Coker}(g)$. Then by Lemma 10.88.5, $f_i : M_i \to M$ factors through $g : M_i \to Q$, say $f_i = h \circ g$ for some $h : Q \to M$. Then since Q is finitely presented, h factors through $M_j \to M$ for some $j \geq i$, say $h = f_j \circ h'$ for some $h' : Q \to M_j$. In total we have a commutative diagram

$$\begin{tikzcd} & M & \\ M_i \arrow[ur, "f_i"] \arrow[rr, "f_{ij}"] \arrow[dr, "g"'] & & M_j \arrow[ul, "f_j"'] \\ & Q \arrow[ur, "h'"'] & \end{tikzcd}$$

Thus f_{ij} dominates g. But g dominates f_i, so f_{ij} dominates f_i.

Conversely, suppose (2) holds. Let P be of finite presentation and $f : P \to M$ a module map. Then f factors through $f_i : M_i \to M$ for some $i \in I$, say $f = f_i \circ g'$ for some $g' : P \to M_i$. Choose by (2) a $j \geq i$ such that f_{ij} dominates f_i. We have a commutative diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g'"'] & M \\ M_i \arrow[ur, "f_i"] \arrow[r, "f_{ij}"'] & M_j \arrow[u, "f_j"'] \end{tikzcd}$$

From the diagram and the fact that f_{ij} dominates f_i, we find that f and $f_{ij} \circ g'$ dominate each other. Hence taking $g = f_{ij} \circ g' : P \to M_j$ works.

Next we prove (2) is equivalent to (3). Let $i \in I$. It is always true that f_i dominates f_{ik} for $k \geq i$, since f_i factors through f_{ik}. If (2) holds, choose $j \geq i$ such that f_{ij} dominates f_i. Then since domination is a transitive relation, f_{ij} dominates f_{ik} for $k \geq i$. All M_i are of finite presentation, so $\text{Coker}(f_{ik})$ is of finite presentation for $k \geq i$. By Lemma 10.88.5, f_{ij} factors through f_{ik} for all $k \geq i$. Thus (2) implies (3). On the other hand, if (3) holds then for any R-module N, $f_{ij} \otimes_R \text{id}_N$ factors through $f_{ik} \otimes_R \text{id}_N$ for $k \geq i$. So $\text{Ker}(f_{ik} \otimes_R \text{id}_N) \subset \text{Ker}(f_{ij} \otimes_R \text{id}_N)$ for $k \geq i$. But $\text{Ker}(f_i \otimes_R \text{id}_N : M_i \otimes_R N \to M \otimes_R N)$ is the union of $\text{Ker}(f_{ik} \otimes_R \text{id}_N)$ for $k \geq i$. Thus $\text{Ker}(f_i \otimes_R \text{id}_N) \subset \text{Ker}(f_{ij} \otimes_R \text{id}_N)$ for any R-module N, which by definition means f_{ij} dominates f_i.

It is trivial that (3) implies (4) implies (5). We show (5) implies (3). Let $N = \prod_{s \in I} M_s$. If (5) holds, then given $i \in I$ choose $j \geq i$ such that

$$\text{Im}(\text{Hom}(M_j, N) \to \text{Hom}(M_i, N)) = \text{Im}(\text{Hom}(M_k, N) \to \text{Hom}(M_i, N))$$

for all $k \geq j$. Passing the product over $s \in I$ outside of the Hom's and looking at the maps on each component of the product, this says

$$\text{Im}(\text{Hom}(M_j, M_s) \to \text{Hom}(M_i, M_s)) = \text{Im}(\text{Hom}(M_k, M_s) \to \text{Hom}(M_i, M_s))$$

for all $k \geq j$ and $s \in I$. Taking $s = j$ we have

$$\text{Im}(\text{Hom}(M_j, M_j) \to \text{Hom}(M_i, M_j)) = \text{Im}(\text{Hom}(M_k, M_j) \to \text{Hom}(M_i, M_j))$$

for all $k \geq j$. Since f_{ij} is the image of $\text{id} \in \text{Hom}(M_j, M_j)$ under $\text{Hom}(M_j, M_j) \to \text{Hom}(M_i, M_j)$, this shows that for any $k \geq j$ there is $h \in \text{Hom}(M_k, M_j)$ such that $f_{ij} = h \circ f_{ik}$. If $j \geq k$ then we can take $h = f_{kj}$. Hence (3) holds. $\square$

059F Definition 10.88.7. Let M be an R-module. We say that M is Mittag-Leffler if the equivalent conditions of Proposition 10.88.6 hold.

In particular a finitely presented module is Mittag-Leffler.

059G Remark 10.88.8. Let M be a flat R-module. By Lazard's theorem (Theorem 10.81.4) we can write $M = \text{colim}\, M_i$ as the colimit of a directed system (M_i, f_{ij}) where the M_i are free finite R-modules. For M to be Mittag-Leffler, it is enough for the inverse system of duals $(\text{Hom}_R(M_i, R), \text{Hom}_R(f_{ij}, R))$ to be Mittag-Leffler. This follows from criterion (4) of Proposition 10.88.6 and the fact that for a free finite R-module F, there is a functorial isomorphism $\text{Hom}_R(F, R) \otimes_R N \cong \text{Hom}_R(F, N)$ for any R-module N.

05CN Lemma 10.88.9. If R is a ring and M, N are Mittag-Leffler modules over R, then $M \otimes_R N$ is a Mittag-Leffler module.

Proof. Write $M = \text{colim}_{i \in I} M_i$ and $N = \text{colim}_{j \in J} N_j$ as directed colimits of finitely presented R-modules. Denote $f_{ii'} : M_i \to M_{i'}$ and $g_{jj'} : N_j \to N_{j'}$ the transition maps. Then $M_i \otimes_R N_j$ is a finitely presented R-module (see Lemma 10.12.14), and $M \otimes_R N = \text{colim}_{(i,j) \in I \times J} M_i \otimes_R M_j$. Pick $(i, j) \in I \times J$. By the definition of a Mittag-Leffler module we have Proposition 10.88.6 (3) for both systems. In other words there exist $i' \geq i$ and $j' \geq j$ such that for every choice of $i'' \geq i$ and $j'' \geq j$ there exist maps $a : M_{i''} \to M_{i'}$ and $b : M_{j''} \to M_{j'}$ such that $f_{ii'} = a \circ f_{ii''}$ and $g_{jj'} = b \circ g_{jj''}$. Then it is clear that $a \otimes b : M_{i''} \otimes_R N_{j''} \to M_{i'} \otimes_R N_{j'}$ serves the same purpose for the system $(M_i \otimes_R N_j, f_{ii'} \otimes g_{jj'})$. Thus by the characterization Proposition 10.88.6 (3) we conclude that $M \otimes_R N$ is Mittag-Leffler. $\square$

05CP Lemma 10.88.10. Let R be a ring and M an R-module. Then M is Mittag-Leffler if and only if for every finite free R-module F and module map $f : F \to M$, there exists a finitely presented R-module Q and a module map $g : F \to Q$ such that g and f dominate each other, i.e., $\text{Ker}(f \otimes_R \text{id}_N) = \text{Ker}(g \otimes_R \text{id}_N)$ for every R-module N.

Proof. Since the condition is clear weaker than condition (1) of Proposition 10.88.6 we see that a Mittag-Leffler module satisfies the condition. Conversely, suppose that M satisfies the condition and that $f : P \to M$ is an R-module map from a finitely presented R-module P into M. Choose a surjection $F \to P$ where F is a finite free R-module. By assumption we can find a map $F \to Q$ where Q is a finitely presented R-module such that $F \to Q$ and $F \to M$ dominate each other. In particular, the kernel of $F \to Q$ contains the kernel of $F \to P$, hence we obtain an R-module map $g : P \to Q$ such that $F \to Q$ is equal to the composition $F \to P \to Q$. Let N be any R-module and consider the commutative diagram

$$\begin{tikzcd}
F \otimes_R N \arrow[r] \arrow[d] & Q \otimes_R N \\
P \otimes_R N \arrow[r] \arrow[ur] & M \otimes_R N
\end{tikzcd}$$

By assumption the kernels of $F \otimes_R N \to Q \otimes_R N$ and $F \otimes_R N \to M \otimes_R N$ are equal. Hence, as $F \otimes_R N \to P \otimes_R N$ is surjective, also the kernels of $P \otimes_R N \to Q \otimes_R N$ and $P \otimes_R N \to M \otimes_R N$ are equal. □

05CQ Lemma 10.88.11. Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. If M is a Mittag-Leffler module over S then M is a Mittag-Leffler module over R.

Proof. Assume M is a Mittag-Leffler module over S. Write $M = \operatorname{colim} M_i$ as a directed colimit of finitely presented S-modules M_i. As M is Mittag-Leffler over S there exists for each i an index $j \geq i$ such that for all $k \geq j$ there is a factorization $f_{ij} = h \circ f_{ik}$ (where h depends on i, the choice of j and k). Note that by Lemma 10.36.23 the modules M_i are also finitely presented as R-modules. Moreover, all the maps f_{ij}, f_{ik}, h are maps of R-modules. Thus we see that the system (M_i, f_{ij}) satisfies the same condition when viewed as a system of R-modules. Thus M is Mittag-Leffler as an R-module. □

05CR Lemma 10.88.12. Let R be a ring. Let $S = R/I$ for some finitely generated ideal I. Let M be an S-module. Then M is a Mittag-Leffler module over R if and only if M is a Mittag-Leffler module over S.

Proof. One implication follows from Lemma 10.88.11. To prove the other, assume M is Mittag-Leffler as an R-module. Write $M = \operatorname{colim} M_i$ as a directed colimit of finitely presented S-modules. As I is finitely generated, the ring S is finite and finitely presented as an R-algebra, hence the modules M_i are finitely presented as R-modules, see Lemma 10.36.23. Next, let N be any S-module. Note that for each i we have $\operatorname{Hom}_R(M_i, N) = \operatorname{Hom}_S(M_i, N)$ as $R \to S$ is surjective. Hence the condition that the inverse system $(\operatorname{Hom}_R(M_i, N))_i$ satisfies Mittag-Leffler, implies that the system $(\operatorname{Hom}_S(M_i, N))_i$ satisfies Mittag-Leffler. Thus M is Mittag-Leffler over S by definition. □

05CS Remark 10.88.13. Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module which is Mittag-Leffler as an R-module. Then it is in general not the case that M is Mittag-Leffler as an S-module. For example suppose that S is the ring of dual numbers over R, i.e., $S = R \oplus R\epsilon$ with $\epsilon^2 = 0$. Then an S-module consists of an R-module M endowed with a square zero R-linear endomorphism $\epsilon : M \to M$. Now suppose that M_0 is an R-module which is not Mittag-Leffler. Choose a presentation $F_1 \xrightarrow{u} F_0 \to M_0 \to 0$ with F_1 and F_0 free R-modules. Set $M = F_1 \oplus F_0$ with

$$\epsilon = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} : M \longrightarrow M.$$

Then $M/\epsilon M \cong F_1 \oplus M_0$ is not Mittag-Leffler over $R = S/\epsilon S$, hence not Mittag-Leffler over S (see Lemma 10.88.12). On the other hand, $M/\epsilon M = M \otimes_S S/\epsilon S$ which would be Mittag-Leffler over S if M was, see Lemma 10.88.9.

10.89. Interchanging direct products with tensor

059H Let M be an R-module and let $(Q_\alpha)_{\alpha \in A}$ be a family of R-modules. Then there is a canonical map $M \otimes_R \left(\prod_{\alpha \in A} Q_\alpha\right) \to \prod_{\alpha \in A}(M \otimes_R Q_\alpha)$ given on pure tensors by $x \otimes (q_\alpha) \mapsto (x \otimes q_\alpha)$. This map is not necessarily injective or surjective, as the following example shows.

059I Example 10.89.1. Take $R = \mathbf{Z}$, $M = \mathbf{Q}$, and consider the family $Q_n = \mathbf{Z}/n$ for $n \geq 1$. Then $\prod_n (M \otimes Q_n) = 0$. However there is an injection $\mathbf{Q} \to M \otimes (\prod_n Q_n)$ obtained by tensoring the injection $\mathbf{Z} \to \prod_n Q_n$ by M, so $M \otimes (\prod_n Q_n)$ is nonzero. Thus $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n)$ is not injective.

On the other hand, take again $R = \mathbf{Z}$, $M = \mathbf{Q}$, and let $Q_n = \mathbf{Z}$ for $n \geq 1$. The image of $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n) = \prod_n M$ consists precisely of sequences of the form $(a_n/m)_{n \geq 1}$ with $a_n \in \mathbf{Z}$ and m some nonzero integer. Hence the map is not surjective.

We determine below the precise conditions needed on M for the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ to be surjective, bijective, or injective for all choices of $(Q_\alpha)_{\alpha \in A}$. This is relevant because the modules for which it is injective turn out to be exactly Mittag-Leffler modules (Proposition 10.89.5). In what follows, if M is an R-module and A a set, we write M^A for the product $\prod_{\alpha \in A} M$.

059J Proposition 10.89.2. Let M be an R-module. The following are equivalent:

(1) M is finitely generated.
(2) For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective.
(3) For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is surjective.
(4) For every set A, the canonical map $M \otimes_R R^A \to M^A$ is surjective.

Proof. First we prove (1) implies (2). Choose a surjection $R^n \to M$ and consider the commutative diagram

$$\begin{tikzcd} R^n \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[d] & \prod_\alpha (R^n \otimes_R Q_\alpha) \arrow[d] \\ M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha). \end{tikzcd}$$

The top arrow is an isomorphism and the vertical arrows are surjections. We conclude that the bottom arrow is a surjection.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). In fact for (1) to hold it suffices that the element $d = (x)_{x \in M}$ of M^M is in the image of the map $f : M \otimes_R R^M \to M^M$. In this case $d = \sum_{i=1}^n f(x_i \otimes a_i)$ for some $x_i \in M$ and $a_i \in R^M$. If for $x \in M$ we write $p_x : M^M \to M$ for the projection onto the x-th factor, then

$$x = p_x(d) = \sum\nolimits_{i=1}^n p_x(f(x_i \otimes a_i)) = \sum\nolimits_{i=1}^n p_x(a_i) x_i.$$

Thus $x_1, \ldots, x_n$ generate M. $\square$

059K Proposition 10.89.3. Let M be an R-module. The following are equivalent:

(1) M is finitely presented.
(2) For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is bijective.
(3) For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is bijective.
(4) For every set A, the canonical map $M \otimes_R R^A \to M^A$ is bijective.

Proof. First we prove (1) implies (2). Choose a presentation $R^m \to R^n \to M$ and consider the commutative diagram

$$\begin{tikzcd} R^m \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d, "\cong"] & R^n \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d, "\cong"] & M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & 0 \\ \prod_\alpha (R^m \otimes_R Q_\alpha) \arrow[r] & \prod_\alpha (R^n \otimes_R Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \arrow[r] & 0. \end{tikzcd}$$

The first two vertical arrows are isomorphisms and the rows are exact. This implies that the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective and, by a diagram chase, also injective. Hence (2) holds.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). From Proposition 10.89.2, if (4) holds we already know that M is finitely generated. So we can choose a surjection $F \to M$ where F is free and finite. Let K be the kernel. We must show K is finitely generated. For any set A, we have a commutative diagram

$$\begin{tikzcd} & K \otimes_R R^A \arrow[r] \arrow[d, "f_3"'] & F \otimes_R R^A \arrow[r] \arrow[d, "f_2"', "\cong"] & M \otimes_R R^A \arrow[r] \arrow[d, "f_1"', "\cong"] & 0 \\ 0 \arrow[r] & K^A \arrow[r] & F^A \arrow[r] & M^A \arrow[r] & 0. \end{tikzcd}$$

The map f_1 is an isomorphism by assumption, the map f_2 is a isomorphism since F is free and finite, and the rows are exact. A diagram chase shows that f_3 is surjective, hence by Proposition 10.89.2 we get that K is finitely generated. □

We need the following lemma for the next proposition.

059L **Lemma 10.89.4.** Let M be an R-module, P a finitely presented R-module, and $f : P \to M$ a map. Let Q be an R-module and suppose $x \in \text{Ker}(P \otimes Q \to M \otimes Q)$. Then there exists a finitely presented R-module P' and a map $f' : P \to P'$ such that f factors through f' and $x \in \text{Ker}(P \otimes Q \to P' \otimes Q)$.

Proof. Write M as a colimit $M = \text{colim}_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Since P is finitely presented, the map $f : P \to M$ factors through $M_j \to M$ for some $j \in I$. Upon tensoring by Q we have a commutative diagram

$$\begin{tikzcd} & M_j \otimes Q \arrow[dr] & \\ P \otimes Q \arrow[ur] \arrow[rr] & & M \otimes Q. \end{tikzcd}$$

The image y of x in $M_j \otimes Q$ is in the kernel of $M_j \otimes Q \to M \otimes Q$. Since $M \otimes Q = \text{colim}_{i \in I}(M_i \otimes Q)$, this means y maps to 0 in $M_{j'} \otimes Q$ for some $j' \geq j$. Thus we may take $P' = M_{j'}$ and f' to be the composite $P \to M_j \to M_{j'}$. □

059M **Proposition 10.89.5.** Let M be an R-module. The following are equivalent:

(1) M is Mittag-Leffler.
(2) For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is injective.

Proof. First we prove (1) implies (2). Suppose M is Mittag-Leffler and let x be in the kernel of $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. Write M as a colimit $M = \mathrm{colim}_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Then $M \otimes_R (\prod_\alpha Q_\alpha)$ is the colimit of $M_i \otimes_R (\prod_\alpha Q_\alpha)$. So x is the image of an element $x_i \in M_i \otimes_R (\prod_\alpha Q_\alpha)$. We must show that x_i maps to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$ for some $j \geq i$. Since M is Mittag-Leffler, we may choose $j \geq i$ such that $M_i \to M_j$ and $M_i \to M$ dominate each other. Then consider the commutative diagram

$$\begin{tikzcd}
M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \\
M_i \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] \arrow[d] & \prod_\alpha (M_i \otimes_R Q_\alpha) \arrow[u] \arrow[d] \\
M_j \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] & \prod_\alpha (M_j \otimes_R Q_\alpha)
\end{tikzcd}$$

whose bottom two horizontal maps are isomorphisms, according to Proposition 10.89.3. Since x_i maps to 0 in $\prod_\alpha (M \otimes_R Q_\alpha)$, its image in $\prod_\alpha (M_i \otimes_R Q_\alpha)$ is in the kernel of the map $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. But this kernel equals the kernel of $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M_j \otimes_R Q_\alpha)$ according to the choice of j. Thus x_i maps to 0 in $\prod_\alpha (M_j \otimes_R Q_\alpha)$ and hence to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$.

Now suppose (2) holds. We prove M satisfies formulation (1) of being Mittag-Leffler from Proposition 10.88.6. Let $f : P \to M$ be a map from a finitely presented module P to M. Choose a set B of representatives of the isomorphism classes of finitely presented R-modules. Let A be the set of pairs (Q, x) where $Q \in B$ and $x \in \mathrm{Ker}(P \otimes Q \to M \otimes Q)$. For $\alpha = (Q, x) \in A$, we write Q_α for Q and x_α for x. Consider the commutative diagram

$$\begin{tikzcd}
M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \\
P \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] & \prod_\alpha (P \otimes_R Q_\alpha) \arrow[u]
\end{tikzcd}$$

The top arrow is an injection by assumption, and the bottom arrow is an isomorphism by Proposition 10.89.3. Let $x \in P \otimes_R (\prod_\alpha Q_\alpha)$ be the element corresponding to $(x_\alpha) \in \prod_\alpha (P \otimes_R Q_\alpha)$ under this isomorphism. Then $x \in \mathrm{Ker}(P \otimes_R (\prod_\alpha Q_\alpha) \to M \otimes_R (\prod_\alpha Q_\alpha))$ since the top arrow in the diagram is injective. By Lemma 10.89.4, we get a finitely presented module P' and a map $f' : P \to P'$ such that $f : P \to M$ factors through f' and $x \in \mathrm{Ker}(P \otimes_R (\prod_\alpha Q_\alpha) \to P' \otimes_R (\prod_\alpha Q_\alpha))$. We have a commutative diagram

$$\begin{tikzcd}
P' \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] & \prod_\alpha (P' \otimes_R Q_\alpha) \\
P \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] & \prod_\alpha (P \otimes_R Q_\alpha) \arrow[u]
\end{tikzcd}$$

where both the top and bottom arrows are isomorphisms by Proposition 10.89.3. Thus since x is in the kernel of the left vertical map, (x_α) is in the kernel of the right vertical map. This means $x_\alpha \in \mathrm{Ker}(P \otimes_R Q_\alpha \to P' \otimes_R Q_\alpha)$ for every $\alpha \in A$. By the definition of A this means $\mathrm{Ker}(P \otimes_R Q \to P' \otimes_R Q) \supset \mathrm{Ker}(P \otimes_R Q \to M \otimes_R Q)$ for all finitely presented Q and, since $f : P \to M$ factors through $f' : P \to P'$, actually equality holds. By Lemma 10.88.3, f and f' dominate each other. $\square$

0AS6 Lemma 10.89.6. Let M be a flat Mittag-Leffler module over R. Let F be an R-module and let $x \in F \otimes_R M$. Then there exists a smallest submodule $F' \subset F$ such that $x \in F' \otimes_R M$. Also, F' is a finite R-module.

Proof. Since M is flat we have $F' \otimes_R M \subset F \otimes_R M$ if $F' \subset F$ is a submodule, hence the statement makes sense. Let $I = \{F' \subset F \mid x \in F' \otimes_R M\}$ and for $i \in I$ denote $F_i \subset F$ the corresponding submodule. Then x maps to zero under the map

$$F \otimes_R M \longrightarrow \prod (F/F_i \otimes_R M)$$

whence by Proposition 10.89.5 x maps to zero under the map

$$F \otimes_R M \longrightarrow \left(\prod F/F_i\right) \otimes_R M$$

Since M is flat the kernel of this arrow is $(\bigcap F_i) \otimes_R M$ which proves that $F' = \bigcap F_i$. To see that F' is a finite module, suppose that $x = \sum_{j=1,\ldots,m} f_j \otimes m_j$ with $f_j \in F'$ and $m_j \in M$. Then $x \in F'' \otimes_R M$ where $F'' \subset F'$ is the submodule generated by $f_1, \ldots, f_m$. Of course then $F'' = F'$ and we conclude the final statement holds. $\square$

059N Lemma 10.89.7. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules. Then:

(1) If M_2 is Mittag-Leffler, then M_1 is Mittag-Leffler.
(2) If M_1 and M_3 are Mittag-Leffler, then M_2 is Mittag-Leffler.

Proof. For any family $(Q_\alpha)_{\alpha \in A}$ of R-modules we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & M_2 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & M_3 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_\alpha (M_1 \otimes Q_\alpha) \arrow[r] & \prod_\alpha (M_2 \otimes Q_\alpha) \arrow[r] & \prod_\alpha (M_3 \otimes Q_\alpha) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Thus (1) and (2) follow from Proposition 10.89.5. $\square$

0EGI Lemma 10.89.8. Let $M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. If M_1 is finitely generated and M_2 is Mittag-Leffler, then M_3 is Mittag-Leffler.

Proof. For any family $(Q_\alpha)_{\alpha \in A}$ of R-modules, since tensor product is right exact, we have a commutative diagram

$$\begin{tikzcd}
M_1 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & M_2 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & M_3 \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & 0 \\
\prod_\alpha (M_1 \otimes Q_\alpha) \arrow[r] & \prod_\alpha (M_2 \otimes Q_\alpha) \arrow[r] & \prod_\alpha (M_3 \otimes Q_\alpha) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By Proposition 10.89.2 the left vertical arrow is surjective. By Proposition 10.89.5 the middle vertical arrow is injective. A diagram chase shows the right vertical arrow is injective. Hence M_3 is Mittag-Leffler by Proposition 10.89.5. $\square$

0AS7 **Lemma 10.89.9.** If $M = \mathrm{colim}\, M_i$ is the colimit of a directed system of Mittag-Leffler R-modules M_i with universally injective transition maps, then M is Mittag-Leffler.

Proof. Let $(Q_\alpha)_{\alpha \in A}$ be a family of R-modules. We have to show that $M \otimes_R (\prod Q_\alpha) \to \prod M \otimes_R Q_\alpha$ is injective and we know that $M_i \otimes_R (\prod Q_\alpha) \to \prod M_i \otimes_R Q_\alpha$ is injective for each i, see Proposition 10.89.5. Since $\otimes$ commutes with filtered colimits, it suffices to show that $\prod M_i \otimes_R Q_\alpha \to \prod M \otimes_R Q_\alpha$ is injective. This is clear as each of the maps $M_i \otimes_R Q_\alpha \to M \otimes_R Q_\alpha$ is injective by our assumption that the transition maps are universally injective. $\square$

059P **Lemma 10.89.10.** If $M = \bigoplus_{i \in I} M_i$ is a direct sum of R-modules, then M is Mittag-Leffler if and only if each M_i is Mittag-Leffler.

Proof. The "only if" direction follows from Lemma 10.89.7 (1) and the fact that a split short exact sequence is universally exact. The converse follows from Lemma 10.89.9 but we can also argue it directly as follows. First note that if I is finite then this follows from Lemma 10.89.7 (2). For general I, if all M_i are Mittag-Leffler then we prove the same of M by verifying condition (1) of Proposition 10.88.6. Let $f : P \to M$ be a map from a finitely presented module P. Then f factors as $P \xrightarrow{f'} \bigoplus_{i' \in I'} M_{i'} \hookrightarrow \bigoplus_{i \in I} M_i$ for some finite subset I' of I. By the finite case $\bigoplus_{i' \in I'} M_{i'}$ is Mittag-Leffler and hence there exists a finitely presented module Q and a map $g : P \to Q$ such that g and f' dominate each other. Then also g and f dominate each other. $\square$

05CT **Lemma 10.89.11.** Let $R \to S$ be a ring map. Let M be an S-module. If S is Mittag-Leffler as an R-module, and M is flat and Mittag-Leffler as an S-module, then M is Mittag-Leffler as an R-module.

Proof. We deduce this from the characterization of Proposition 10.89.5. Namely, suppose that Q_α is a family of R-modules. Consider the composition

$$\begin{tikzcd} M \otimes_R \prod_\alpha Q_\alpha = M \otimes_S S \otimes_R \prod_\alpha Q_\alpha \arrow[d] \\ M \otimes_S \prod_\alpha (S \otimes_R Q_\alpha) \arrow[d] \\ \prod_\alpha (M \otimes_S S \otimes_R Q_\alpha) = \prod_\alpha (M \otimes_R Q_\alpha) \end{tikzcd}$$

The first arrow is injective as M is flat over S and S is Mittag-Leffler over R and the second arrow is injective as M is Mittag-Leffler over S. Hence M is Mittag-Leffler over R. $\square$

10.90. Coherent rings

05CU We use the discussion on interchanging $\prod$ and $\otimes$ to determine for which rings products of flat modules are flat. It turns out that these are the so-called coherent rings. You may be more familiar with the notion of a coherent $\mathcal{O}_X$-module on a ringed space, see Modules, Section 17.12.

05CV **Definition 10.90.1.** Let R be a ring. Let M be an R-module.

(1) We say M is a coherent module if it is finitely generated and every finitely generated submodule of M is finitely presented over R.
(2) We say R is a coherent ring if it is coherent as a module over itself.

Thus a ring is coherent if and only if every finitely generated ideal is finitely presented as a module.

0EWV Example 10.90.2. A valuation ring is a coherent ring. Namely, every nonzero finitely generated ideal is principal (Lemma 10.50.15), hence free as a valuation ring is a domain, hence finitely presented.

The category of coherent modules is abelian.

05CW Lemma 10.90.3. Let R be a ring.

(1) A finite submodule of a coherent module is coherent.
(2) Let $\varphi : N \to M$ be a homomorphism from a finite module to a coherent module. Then $\mathrm{Ker}(\varphi)$ is finite, $\mathrm{Im}(\varphi)$ is coherent, and $\mathrm{Coker}(\varphi)$ is coherent.
(3) Let $\varphi : N \to M$ be a homomorphism of coherent modules. Then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are coherent modules.
(4) Given a short exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ if two out of three are coherent so is the third.

Proof. The first statement is immediate from the definition.

Let $\varphi : N \to M$ satisfy the assumptions of (2). First, $\mathrm{Im}(\varphi)$ is finite, hence coherent by (1). In particular $\mathrm{Im}(\varphi)$ is finitely presented, so applying Lemma 10.5.3 to the exact sequence $0 \to \mathrm{Ker}(\varphi) \to N \to \mathrm{Im}(\varphi) \to 0$ we see that $\mathrm{Ker}(\varphi)$ is finite. To prove that $\mathrm{Coker}(\varphi)$ is coherent, let $E \subset \mathrm{Coker}(\varphi)$ be a finite subomdule, and let E' be its inverse image in M. From the exact sequence $0 \to \mathrm{Ker}(\varphi) \to E' \to E \to 0$ and since $\mathrm{Ker}(\varphi)$ is finite we conclude by Lemma 10.5.3 that $E' \subset M$ is finite, hence finitely presented because M is coherent. The same exact sequence then shows that E is finitely presented, whence our claim.

Part (3) follows immediately from (1) and (2).

Let $0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_3 \to 0$ be a short exact sequence of R-modules as in (4). It remains to prove that if M_1 and M_3 are coherent so is M_2. By Lemma 10.5.3 we see that M_2 is finite. Let $N_2 \subset M_2$ be a finite submodule. Put $N_3 = p(N_2) \subset M_3$ and $N_1 = i^{-1}(N_2) \subset M_1$. We have an exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$. Clearly N_3 is finite (as a quotient of N_2), hence finitely presented (as a finite submodule of M_3). It follows by Lemma 10.5.3 (5) that N_1 is finite, hence finitely presented (as a finite submodule of M_1). We conclude by Lemma 10.5.3 (2) that M_2 is finitely presented. □

05CX Lemma 10.90.4. Let R be a ring. If R is coherent, then a module is coherent if and only if it is finitely presented.

Proof. It is clear that a coherent module is finitely presented (over any ring). Conversely, if R is coherent, then $R^{\oplus n}$ is coherent and so is the cokernel of any map $R^{\oplus m} \to R^{\oplus n}$, see Lemma 10.90.3. □

05CY Lemma 10.90.5. A Noetherian ring is a coherent ring.

Proof. By Lemma 10.31.4 any finite R-module is finitely presented. In particular any ideal of R is finitely presented. □

05CZ Proposition 10.90.6. Let R be a ring. The following are equivalent This is [Cha60, Theorem 2.1].

(1) R is coherent,
(2) any product of flat R-modules is flat, and
(3) for every set A the module R^A is flat.

Proof. Assume R coherent, and let Q_α, $\alpha \in A$ be a set of flat R-modules. We have to show that $I \otimes_R \prod_\alpha Q_\alpha \to \prod Q_\alpha$ is injective for every finitely generated ideal I of R, see Lemma 10.39.5. Since R is coherent I is an R-module of finite presentation. Hence $I \otimes_R \prod_\alpha Q_\alpha = \prod I \otimes_R Q_\alpha$ by Proposition 10.89.3. The desired injectivity follows as $I \otimes_R Q_\alpha \to Q_\alpha$ is injective by flatness of Q_α.

The implication (2) ⇒ (3) is trivial.

Assume that the R-module R^A is flat for every set A. Let I be a finitely generated ideal in R. Then $I \otimes_R R^A \to R^A$ is injective by assumption. By Proposition 10.89.2 and the finiteness of I the image is equal to I^A. Hence $I \otimes_R R^A = I^A$ for every set A and we conclude that I is finitely presented by Proposition 10.89.3. □

10.91. Examples and non-examples of Mittag-Leffler modules

059Q We end this section with some examples and non-examples of Mittag-Leffler modules.

059R Example 10.91.1. Mittag-Leffler modules.

(1) Any finitely presented module is Mittag-Leffler. This follows, for instance, from Proposition 10.88.6 (1). In general, it is true that a finitely generated module is Mittag-Leffler if and only it is finitely presented. This follows from Propositions 10.89.2, 10.89.3, and 10.89.5.
(2) A free module is Mittag-Leffler since it satisfies condition (1) of Proposition 10.88.6.
(3) By the previous example together with Lemma 10.89.10, projective modules are Mittag-Leffler.

We also want to add to our list of examples power series rings over a Noetherian ring R. This will be a consequence the following lemma.

059S Lemma 10.91.2. Let M be a flat R-module. The following are equivalent

(1) M is Mittag-Leffler, and
(2) if F is a finite free R-module and $x \in F \otimes_R M$, then there exists a smallest submodule F' of F such that $x \in F' \otimes_R M$.

Proof. The implication (1) ⇒ (2) is a special case of Lemma 10.89.6. Assume (2). By Theorem 10.81.4 we can write M as the colimit $M = \mathrm{colim}_{i \in I} M_i$ of a directed system (M_i, f_{ij}) of finite free R-modules. By Remark 10.88.8, it suffices to show that the inverse system $(\mathrm{Hom}_R(M_i, R), \mathrm{Hom}_R(f_{ij}, R))$ is Mittag-Leffler. In other words, fix $i \in I$ and for $j \geq i$ let Q_j be the image of $\mathrm{Hom}_R(M_j, R) \to \mathrm{Hom}_R(M_i, R)$; we must show that the Q_j stabilize.

Since M_i is free and finite, we can make the identification $\mathrm{Hom}_R(M_i, M_j) = \mathrm{Hom}_R(M_i, R) \otimes_R M_j$ for all j. Using the fact that the M_j are free, it follows that for $j \geq i$, Q_j is the smallest submodule of $\mathrm{Hom}_R(M_i, R)$ such that $f_{ij} \in Q_j \otimes_R M_j$.

Under the identification $\mathrm{Hom}_R(M_i, M) = \mathrm{Hom}_R(M_i, R) \otimes_R M$, the canonical map $f_i : M_i \to M$ is in $\mathrm{Hom}_R(M_i, R) \otimes_R M$. By the assumption on M, there exists a smallest submodule Q of $\mathrm{Hom}_R(M_i, R)$ such that $f_i \in Q \otimes_R M$. We are going to show that the Q_j stabilize to Q.

For $j \geq i$ we have a commutative diagram

$$\begin{tikzcd} Q_j \otimes_R M_j \arrow[r] \arrow[d] & \mathrm{Hom}_R(M_i, R) \otimes_R M_j \arrow[d] \\ Q_j \otimes_R M \arrow[r] & \mathrm{Hom}_R(M_i, R) \otimes_R M. \end{tikzcd}$$

Since $f_{ij} \in Q_j \otimes_R M_j$ maps to $f_i \in \mathrm{Hom}_R(M_i, R) \otimes_R M$, it follows that $f_i \in Q_j \otimes_R M$. Hence, by the choice of Q, we have $Q \subset Q_j$ for all $j \geq i$.

Since the Q_j are decreasing and $Q \subset Q_j$ for all $j \geq i$, to show that the Q_j stabilize to Q it suffices to find a $j \geq i$ such that $Q_j \subset Q$. As an element of

$$\mathrm{Hom}_R(M_i, R) \otimes_R M = \mathrm{colim}_{j \in J}(\mathrm{Hom}_R(M_i, R) \otimes_R M_j),$$

f_i is the colimit of f_{ij} for $j \geq i$, and f_i also lies in the submodule

$$\mathrm{colim}_{j \in J}(Q \otimes_R M_j) \subset \mathrm{colim}_{j \in J}(\mathrm{Hom}_R(M_i, R) \otimes_R M_j).$$

It follows that for some $j \geq i$, f_{ij} lies in $Q \otimes_R M_j$. Since Q_j is the smallest submodule of $\mathrm{Hom}_R(M_i, R)$ with $f_{ij} \in Q_j \otimes_R M_j$, we conclude $Q_j \subset Q$. □

05D0 Lemma 10.91.3. Let R be a Noetherian ring and A a set. Then $M = R^A$ is a flat and Mittag-Leffler R-module.

Proof. Combining Lemma 10.90.5 and Proposition 10.90.6 we see that M is flat over R. We show that M satisfies the condition of Lemma 10.91.2. Let F be a free finite R-module. If F' is any submodule of F then it is finitely presented since R is Noetherian. So by Proposition 10.89.3 we have a commutative diagram

$$\begin{tikzcd} F' \otimes_R M \arrow[r] \arrow[d, "\cong"] & F \otimes_R M \arrow[d, "\cong"] \\ (F')^A \arrow[r] & F^A \end{tikzcd}$$

by which we can identify the map $F' \otimes_R M \to F \otimes_R M$ with $(F')^A \to F^A$. Hence if $x \in F \otimes_R M$ corresponds to $(x_\alpha) \in F^A$, then the submodule of F' of F generated by the x_α is the smallest submodule of F such that $x \in F' \otimes_R M$. □

059T Lemma 10.91.4. Let R be a Noetherian ring and n a positive integer. Then the R-module $M = R[[t_1, \ldots, t_n]]$ is flat and Mittag-Leffler.

Proof. As an R-module, we have $M = R^A$ for a (countable) set A. Hence this lemma is a special case of Lemma 10.91.3. □

059U Example 10.91.5. Non Mittag-Leffler modules.

(1) By Example 10.89.1 and Proposition 10.89.5, $\mathbf{Q}$ is not a Mittag-Leffler $\mathbf{Z}$-module.

(2) We prove below (Theorem 10.93.3) that for a flat and countably generated module, projectivity is equivalent to being Mittag-Leffler. Thus any flat, countably generated, non-projective module M is an example of a non-Mittag-Leffler module. For such an example, see Remark 10.78.4.
(3) Let k be a field. Let $R = k[[x]]$. The R-module $M = \prod_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler. Namely, consider the element $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$ defined by $\xi_{2^m} = x^{2^{m-1}}$ and $\xi_n = 0$ else, so

$$\xi = (0, x, 0, x^2, 0, 0, 0, x^4, 0, 0, 0, 0, 0, 0, 0, x^8, \ldots)$$

Then the annihilator of ξ in $M/x^{2^m}M$ is generated $x^{2^{m-1}}$ for $m \gg 0$. But if M was Mittag-Leffler, then there would exist a finite R-module Q and an element $\xi' \in Q$ such that the annihilator of ξ' in Q/x^lQ agrees with the annihilator of ξ in M/x^lM for all $l \geq 1$, see Proposition 10.88.6 (1). Now you can prove there exists an integer $a \geq 0$ such that the annihilator of ξ' in Q/x^lQ is generated by either x^a or x^{l-a} for all $l \gg 0$ (depending on whether $\xi' \in Q$ is torsion or not). The combination of the above would give for all $l = 2^m >> 0$ the equality $a = l/2$ or $l - a = l/2$ which is nonsensical.
(4) The same argument shows that (x)-adic completion of $\bigoplus_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler over $R = k[[x]]$ (hint: ξ is actually an element of this completion).
(5) Let $R = k[a, b]/(a^2, ab, b^2)$. Let S be the finitely presented R-algebra with presentation $S = R[t]/(at - b)$. Then as an R-module S is countably generated and indecomposable (details omitted). On the other hand, R is Artinian local, hence complete local, hence a henselian local ring, see Lemma 10.153.9. If S was Mittag-Leffler as an R-module, then it would be a direct sum of finite R-modules by Lemma 10.153.13. Thus we conclude that S is not Mittag-Leffler as an R-module.

10.92. Countably generated Mittag-Leffler modules

05D1 It turns out that countably generated Mittag-Leffler modules have a particularly simple structure.

059W Lemma 10.92.1. Let M be an R-module. Write $M = \mathrm{colim}_{i \in I} M_i$ where (M_i, f_{ij}) is a directed system of finitely presented R-modules. If M is Mittag-Leffler and countably generated, then there is a directed countable subset $I' \subset I$ such that $M \cong \mathrm{colim}_{i \in I'} M_i$.

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. For each x_n choose $i \in I$ such that x_n is in the image of the canonical map $f_i : M_i \to M$; let $I'_0 \subset I$ be the set of all these i. Now since M is Mittag-Leffler, for each $i \in I'_0$ we can choose $j \in I$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$ (condition (3) of Proposition 10.88.6); let I'_1 be the union of I'_0 with all of these j. Since I'_1 is a countable, we can enlarge it to a countable directed set $I'_2 \subset I$. Now we can apply the same procedure to I'_2 as we did to I'_0 to get a new countable set $I'_3 \subset I$. Then we enlarge I'_3 to a countable directed set I'_4. Continuing in this way—adding in a j as in Proposition 10.88.6 (3) for each $i \in I'_\ell$ if ℓ is odd and enlarging I'_ℓ to a directed set if ℓ is even—we get a sequence of subsets $I'_\ell \subset I$ for $\ell \geq 0$. The union $I' = \bigcup I'_\ell$ satisfies:

(1) I' is countable and directed;
(2) each x_n is in the image of $f_i : M_i \to M$ for some $i \in I'$;
(3) if $i \in I'$, then there is $j \in I'$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \in I$ with $k \geq i$. In particular $\text{Ker}(f_{ik}) \subset \text{Ker}(f_{ij})$ for $k \geq i$.

We claim that the canonical map $\text{colim}_{i \in I'} M_i \to \text{colim}_{i \in I} M_i = M$ is an isomorphism. By (2) it is surjective. For injectivity, suppose $x \in \text{colim}_{i \in I'} M_i$ maps to 0 in $\text{colim}_{i \in I} M_i$. Representing x by an element $\tilde{x} \in M_i$ for some $i \in I'$, this means that $f_{ik}(\tilde{x}) = 0$ for some $k \in I, k \geq i$. But then by (3) there is $j \in I', j \geq i$, such that $f_{ij}(\tilde{x}) = 0$. Hence $x = 0$ in $\text{colim}_{i \in I'} M_i$. □

Lemma 10.92.1 implies that a countably generated Mittag-Leffler module M over R is the colimit of a system

$$M_1 \to M_2 \to M_3 \to M_4 \to \ldots$$

with each M_n a finitely presented R-module. To see this argue as in the proof of Lemma 10.86.3 to see that a countable directed set has a cofinal subset isomorphic to $(\mathbf{N}, \geq)$. Suppose $R = k[x_1, x_2, x_3, \ldots]$ and $M = R/(x_i)$. Then M is finitely generated but not finitely presented, hence not Mittag-Leffler (see Example 10.91.1 part (1)). But of course you can write $M = \text{colim}_n M_n$ by taking $M_n = R/(x_1, \ldots, x_n)$, hence the condition that you can write M as such a limit does not imply that M is Mittag-Leffler.

05D2 **Lemma 10.92.2.** *Let R be a ring. Let M be an R-module. Assume M is Mittag-Leffler and countably generated. For any R-module map $f : P \to M$ with P finitely generated there exists an endomorphism $\alpha : M \to M$ such that*

(1) $\alpha : M \to M$ factors through a finitely presented R-module, and
(2) $\alpha \circ f = f$.

Proof. Write $M = \text{colim}_{i \in I} M_i$ as a directed colimit of finitely presented R-modules with I countable, see Lemma 10.92.1. The transition maps are denoted f_{ij} and we use $f_i : M_i \to M$ to denote the canonical maps into M. Set $N = \prod_{s \in I} M_s$. Denote

$$M_i^* = \text{Hom}_R(M_i, N) = \prod\nolimits_{s \in I} \text{Hom}_R(M_i, M_s)$$

so that (M_i^*) is an inverse system of R-modules over I. Note that $\text{Hom}_R(M, N) = \lim M_i^*$. As M is Mittag-Leffler, we find for every $i \in I$ an index $k(i) \geq i$ such that

$$E_i := \bigcap\nolimits_{i' \geq i} \text{Im}(M_{i'}^* \to M_i^*) = \text{Im}(M_{k(i)}^* \to M_i^*)$$

Choose and fix $j \in I$ such that $\text{Im}(P \to M) \subset \text{Im}(M_j \to M)$. This is possible as P is finitely generated. Set $k = k(j)$. Let $x = (0, \ldots, 0, \text{id}_{M_k}, 0, \ldots, 0) \in M_k^*$ and note that this maps to $y = (0, \ldots, 0, f_{jk}, 0, \ldots, 0) \in M_j^*$. By our choice of k we see that $y \in E_j$. By Example 10.86.2 the transition maps $E_i \to E_j$ are surjective for each $i \geq j$ and $\lim E_i = \lim M_i^* = \text{Hom}_R(M, N)$. Hence Lemma 10.86.3 guarantees there exists an element $z \in \text{Hom}_R(M, N)$ which maps to y in $E_j \subset M_j^*$. Let z_k be the kth component of z. Then $z_k : M \to M_k$ is a homomorphism such that

$$\begin{tikzcd} M \arrow[r, "z_k"] & M_k \\ M_j \arrow[u, "f_j"] \arrow[ur, "f_{jk}"'] & \end{tikzcd}$$

commutes. Let $\alpha : M \to M$ be the composition $f_k \circ z_k : M \to M_k \to M$. Then α factors through a finitely presented module by construction and $\alpha \circ f_j = f_j$. Since the image of f is contained in the image of f_j this also implies that $\alpha \circ f = f$. □

We will see later (see Lemma 10.153.13) that Lemma 10.92.2 means that a countably generated Mittag-Leffler module over a henselian local ring is a direct sum of finitely presented modules.

10.93. Characterizing projective modules

059V The goal of this section is to prove that a module is projective if and only if it is flat, Mittag-Leffler, and a direct sum of countably generated modules (Theorem 10.93.3 below).

059X **Lemma 10.93.1.** Let M be an R-module. If M is flat, Mittag-Leffler, and countably generated, then M is projective.

Proof. By Lazard's theorem (Theorem 10.81.4), we can write $M = \text{colim}_{i \in I} M_i$ for a directed system of finite free R-modules (M_i, f_{ij}) indexed by a set I. By Lemma 10.92.1, we may assume I is countable. Now let

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence of R-modules. We must show that applying $\text{Hom}_R(M, -)$ preserves exactness. Since M_i is finite free,

$$0 \to \text{Hom}_R(M_i, N_1) \to \text{Hom}_R(M_i, N_2) \to \text{Hom}_R(M_i, N_3) \to 0$$

is exact for each i. Since M is Mittag-Leffler, $(\text{Hom}_R(M_i, N_1))$ is a Mittag-Leffler inverse system. So by Lemma 10.86.4,

$$0 \to \lim_{i \in I} \text{Hom}_R(M_i, N_1) \to \lim_{i \in I} \text{Hom}_R(M_i, N_2) \to \lim_{i \in I} \text{Hom}_R(M_i, N_3) \to 0$$

is exact. But for any R-module N there is a functorial isomorphism $\text{Hom}_R(M, N) \cong \lim_{i \in I} \text{Hom}_R(M_i, N)$, so

$$0 \to \text{Hom}_R(M, N_1) \to \text{Hom}_R(M, N_2) \to \text{Hom}_R(M, N_3) \to 0$$

is exact. □

059Y **Remark 10.93.2.** Lemma 10.93.1 does not hold without the countable generation assumption. For example, the $\mathbf{Z}$-module $M = \mathbf{Z}[[x]]$ is flat and Mittag-Leffler but not projective. It is Mittag-Leffler by Lemma 10.91.4. Subgroups of free abelian groups are free, hence a projective $\mathbf{Z}$-module is in fact free and so are its submodules. Thus to show M is not projective it suffices to produce a non-free submodule. Fix a prime p and consider the submodule N consisting of power series $f(x) = \sum a_i x^i$ such that for every integer $m \geq 1$, p^m divides a_i for all but finitely many i. Then $\sum a_i p^i x^i$ is in N for all $a_i \in \mathbf{Z}$, so N is uncountable. Thus if N were free it would have uncountable rank and the dimension of N/pN over $\mathbf{Z}/p$ would be uncountable. This is not true as the elements $x^i \in N/pN$ for $i \geq 0$ span N/pN.

059Z **Theorem 10.93.3.** Let M be an R-module. Then M is projective if and only it satisfies:

(1) M is flat,
(2) M is Mittag-Leffler,
(3) M is a direct sum of countably generated R-modules.

Proof. First suppose M is projective. Then M is a direct summand of a free module, so M is flat and Mittag-Leffler since these properties pass to direct summands. By Kaplansky's theorem (Theorem 10.84.5), M satisfies (3).

Conversely, suppose M satisfies (1)-(3). Since being flat and Mittag-Leffler passes to direct summands, M is a direct sum of flat, Mittag-Leffler, countably generated R-modules. Lemma 10.93.1 implies M is a direct sum of projective modules. Hence M is projective. $\square$

05A0 Lemma 10.93.4. Let $f : M \to N$ be universally injective map of R-modules. Suppose M is a direct sum of countably generated R-modules, and suppose N is flat and Mittag-Leffler. Then M is projective.

Proof. By Lemmas 10.82.7 and 10.89.7, M is flat and Mittag-Leffler, so the conclusion follows from Theorem 10.93.3. $\square$

05A1 Lemma 10.93.5. Let R be a Noetherian ring and let M be a R-module. Suppose M is a direct sum of countably generated R-modules, and suppose there is a universally injective map $M \to R[[t_1, \ldots, t_n]]$ for some n. Then M is projective.

Proof. Follows from Lemmas 10.93.4 and 10.91.4. $\square$

10.94. Ascending properties of modules

05A2 All of the properties of a module in Theorem 10.93.3 ascend along arbitrary ring maps:

05A3 Lemma 10.94.1. Let $R \to S$ be a ring map. Let M be an R-module. Then:

(1) If M is flat, then the S-module $M \otimes_R S$ is flat.
(2) If M is Mittag-Leffler, then the S-module $M \otimes_R S$ is Mittag-Leffler.
(3) If M is a direct sum of countably generated R-modules, then the S-module $M \otimes_R S$ is a direct sum of countably generated S-modules.
(4) If M is projective, then the S-module $M \otimes_R S$ is projective.

Proof. All are obvious except (2). For this, use formulation (3) of being Mittag-Leffler from Proposition 10.88.6 and the fact that tensoring commutes with taking colimits. $\square$

10.95. Descending properties of modules

05A4 We address the faithfully flat descent of the properties from Theorem 10.93.3 that characterize projectivity. In the presence of flatness, the property of being a Mittag-Leffler module descends:

05A5 Lemma 10.95.1. Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is Mittag-Leffler, then M is Mittag-Leffler. Email from Juan Pablo Acosta Lopez dated 12/20/14.

Proof. Write $M = \mathrm{colim}_{i \in I} M_i$ as a directed colimit of finitely presented R-modules M_i. Using Proposition 10.88.6, we see that we have to prove that for each $i \in I$ there exists $i \leq j$, $j \in I$ such that $M_i \to M_j$ dominates $M_i \to M$.

Take N the pushout

$$\begin{tikzcd} M_i \arrow[r] \arrow[d] & M_j \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

Then the lemma is equivalent to the existence of j such that $M_j \to N$ is universally injective, see Lemma 10.88.4. Observe that the tensorization by S

$$\begin{tikzcd} M_i \otimes_R S \arrow[r] \arrow[d] & M_j \otimes_R S \arrow[d] \\ M \otimes_R S \arrow[r] & N \otimes_R S \end{tikzcd}$$

Is a pushout diagram. So because $M \otimes_R S = \mathrm{colim}_{i\in I} M_i \otimes_R S$ expresses $M \otimes_R S$ as a colimit of S-modules of finite presentation, and $M \otimes_R S$ is Mittag-Leffler, there exists $j \geq i$ such that $M_j \otimes_R S \to N \otimes_R S$ is universally injective. So using that $R \to S$ is faithfully flat we conclude that $M_j \to N$ is universally injective too. $\square$

0GVD Lemma 10.95.2. Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is countably generated, then M is countably generated.

Proof. Say $M \otimes_R S$ is generated by the elements y_i, $i = 1, 2, 3, \ldots$. Write $y_i = \sum_{j=1,\ldots,n_i} x_{ij} \otimes s_{ij}$ for some $n_i \geq 0$, $x_{ij} \in M$ and $s_{ij} \in S$. Denote $M' \subset M$ the submodule generated by the countable collection of elements x_{ij}. Then $M' \otimes_R S \to M \otimes_R S$ is surjective as the image contains the generators y_i. Since S is faithfully flat over R we conclude that $M' = M$ as desired. $\square$

At this point the faithfully flat descent of countably generated projective modules follows easily.

05A6 Lemma 10.95.3. Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is countably generated and projective, then M is countably generated and projective.

Proof. Follows from Lemmas 10.83.2, 10.95.1, and 10.95.2 and Theorem 10.93.3. $\square$

All that remains is to use dévissage to reduce descent of projectivity in the general case to the countably generated case. First, two simple lemmas.

05A7 Lemma 10.95.4. Let $R \to S$ be a ring map, let M be an R-module, and let Q be a countably generated S-submodule of $M \otimes_R S$. Then there exists a countably generated R-submodule P of M such that $\mathrm{Im}(P \otimes_R S \to M \otimes_R S)$ contains Q.

Proof. Let $y_1, y_2, \ldots$ be generators for Q and write $y_j = \sum_k x_{jk} \otimes s_{jk}$ for some $x_{jk} \in M$ and $s_{jk} \in S$. Then take P be the submodule of M generated by the x_{jk}. $\square$

05A8 Lemma 10.95.5. Let $R \to S$ be a ring map, and let M be an R-module. Suppose $M \otimes_R S = \bigoplus_{i\in I} Q_i$ is a direct sum of countably generated S-modules Q_i. If N is a countably generated submodule of M, then there is a countably generated submodule N' of M such that $N' \supset N$ and $\mathrm{Im}(N' \otimes_R S \to M \otimes_R S) = \bigoplus_{i\in I'} Q_i$ for some subset $I' \subset I$.

Proof. Let $N'_0 = N$. We construct by induction an increasing sequence of countably generated submodules $N'_\ell \subset M$ for $\ell = 0, 1, 2, \ldots$ such that: if I'_ℓ is the set of $i \in I$ such that the projection of $\mathrm{Im}(N'_\ell \otimes_R S \to M \otimes_R S)$ onto Q_i is nonzero, then $\mathrm{Im}(N'_{\ell+1} \otimes_R S \to M \otimes_R S)$ contains Q_i for all $i \in I'_\ell$. To construct $N'_{\ell+1}$ from N'_ℓ, let Q be the sum of (the countably many) Q_i for $i \in I'_\ell$, choose P as in

Lemma 10.95.4, and then let $N'_{\ell+1} = N'_\ell + P$. Having constructed the N'_ℓ, just take $N' = \bigcup_\ell N'_\ell$ and $I' = \bigcup_\ell I'_\ell$. ☐

05A9 Theorem 10.95.6. Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is projective, then M is projective.

Proof. We are going to construct a Kaplansky dévissage of M to show that it is a direct sum of projective modules and hence projective. By Theorem 10.84.5 we can write $M \otimes_R S = \bigoplus_{i \in I} Q_i$ as a direct sum of countably generated S-modules Q_i. Choose a well-ordering on M. Using transfinite recursion we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that $M_\alpha \otimes_R S$ is a direct sum of some subset of the Q_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\beta = \bigcup_{\beta < \alpha} M_\beta$. Since each $M_\beta \otimes_R S$ for $\beta < \alpha$ is a direct sum of a subset of the Q_i, the same will be true of $M_\alpha \otimes_R S$. If $\alpha + 1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. Otherwise choose the smallest $x \in M$ (with respect to the fixed well-ordering) such that $x \notin M_\alpha$. Since S is flat over R, $(M/M_\alpha) \otimes_R S = M \otimes_R S / M_\alpha \otimes_R S$, so since $M_\alpha \otimes_R S$ is a direct sum of some Q_i, the same is true of $(M/M_\alpha) \otimes_R S$. By Lemma 10.95.5, we can find a countably generated R-submodule P of M/M_α containing the image of x in M/M_α and such that $P \otimes_R S$ (which equals $\mathrm{Im}(P \otimes_R S \to M \otimes_R S)$ since S is flat over R) is a direct sum of some Q_i. Since $M \otimes_R S = \bigoplus_{i \in I} Q_i$ is projective and projectivity passes to direct summands, $P \otimes_R S$ is also projective. Thus by Lemma 10.95.3, P is projective. Finally we define $M_{\alpha+1}$ to be the preimage of P in M, so that $M_{\alpha+1}/M_\alpha = P$ is countably generated and projective. In particular M_α is a direct summand of $M_{\alpha+1}$ since projectivity of $M_{\alpha+1}/M_\alpha$ implies the sequence $0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$ splits.

Transfinite induction on M (using the fact that we constructed $M_{\alpha+1}$ to contain the smallest $x \in M$ not contained in M_α) shows that each $x \in M$ is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $x \in M$ there is $\alpha \in S$ such that $x \in M_\alpha$. This means $(M_\alpha)_{\alpha \in S}$ satisfies property (1) of a Kaplansky dévissage of M. The other properties are clear by construction. We conclude $M = \bigoplus_{\alpha+1 \in S} M_{\alpha+1}/M_\alpha$. Since each $M_{\alpha+1}/M_\alpha$ is projective by construction, M is projective. ☐

10.96. Completion

00M9 Suppose that R is a ring and I is an ideal. We define the completion of R with respect to I to be the limit

$$R^\wedge = \lim_n R/I^n.$$

An element of $R^\wedge$ is given by a sequence of elements $f_n \in R/I^n$ such that $f_n \equiv f_{n+1} \bmod I^n$ for all n. We will view $R^\wedge$ as an R-algebra. Similarly, if M is an R-module then we define the completion of M with respect to I to be the limit

$$M^\wedge = \lim_n M/I^n M.$$

An element of $M^\wedge$ is given by a sequence of elements $m_n \in M/I^n M$ such that $m_n \equiv m_{n+1} \bmod I^n M$ for all n. We will view $M^\wedge$ as an $R^\wedge$-module. From this description it is clear that there are always canonical maps

$$M \longrightarrow M^\wedge \quad \text{and} \quad M \otimes_R R^\wedge \longrightarrow M^\wedge.$$

Moreover, given a map $\varphi : M \to N$ of modules we get an induced map $\varphi^\wedge : M^\wedge \to N^\wedge$ on completions making the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ M^\wedge \arrow[r] & N^\wedge \end{tikzcd}$$

commute. In general completion is not an exact functor, see Examples, Section 110.9. Here are some initial positive results.

0315 Lemma 10.96.1. Let R be a ring. Let $I \subset R$ be an ideal. Let $\varphi : M \to N$ be a map of R-modules.

(1) If $M/IM \to N/IN$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.
(2) If $M \to N$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.
(3) If $0 \to K \to M \to N \to 0$ is a short exact sequence of R-modules and N is flat, then $0 \to K^\wedge \to M^\wedge \to N^\wedge \to 0$ is a short exact sequence.
(4) The map $M \otimes_R R^\wedge \to M^\wedge$ is surjective for any finite R-module M.

Proof. Assume $M/IM \to N/IN$ is surjective. Then the map $M/I^nM \to N/I^nN$ is surjective for each $n \geq 1$ by Nakayama's lemma. More precisely, apply Lemma 10.20.1 part (11) to the map $M/I^nM \to N/I^nN$ over the ring R/I^n and the nilpotent ideal I/I^n to see this. Set $K_n = \{x \in M \mid \varphi(x) \in I^nN\}$. Thus we get short exact sequences

$$0 \to K_n/I^nM \to M/I^nM \to N/I^nN \to 0$$

We claim that the canonical map $K_{n+1}/I^{n+1}M \to K_n/I^nM$ is surjective. Namely, if $x \in K_n$ write $\varphi(x) = \sum z_j n_j$ with $z_j \in I^n$, $n_j \in N$. By assumption we can write $n_j = \varphi(m_j) + \sum z_{jk} n_{jk}$ with $m_j \in M$, $z_{jk} \in I$ and $n_{jk} \in N$. Hence

$$\varphi(x - \sum z_j m_j) = \sum z_j z_{jk} n_{jk}.$$

This means that $x' = x - \sum z_j m_j \in K_{n+1}$ maps to $x \bmod I^nM$ which proves the claim. Now we may apply Lemma 10.87.1 to the inverse system of short exact sequences above to see (1). Part (2) is a special case of (1). If the assumptions of (3) hold, then for each n the sequence

$$0 \to K/I^nK \to M/I^nM \to N/I^nN \to 0$$

is short exact by Lemma 10.39.12. Hence we can directly apply Lemma 10.87.1 to conclude (3) is true. To see (4) choose generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ is surjective. Hence by (2) we see $(R^\wedge)^{\oplus n} \to M^\wedge$, $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ is surjective. Assertion (4) follows from this. $\square$

0317 Definition 10.96.2. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. We say M is I-adically complete if the map

$$M \longrightarrow M^\wedge = \lim_n M/I^nM$$

is an isomorphism[9]. We say R is I-adically complete if R is I-adically complete as an R-module.

[9] This includes the condition that $\bigcap I^nM = 0$.

It is not true that the completion of an R-module M with respect to I is I-adically complete. For an example see Examples, Section 110.7. If the ideal is finitely generated, then the completion is complete.

05GG Lemma 10.96.3. Let R be a ring. Let I be a finitely generated ideal of R. Let M be an R-module. Then [Mat78, Theorem 15]. The slick proof given here is from an email of Bjorn Poonen dated Nov 5, 2016.

1. the completion $M^\wedge$ is I-adically complete, and
2. $I^n M^\wedge = \text{Ker}(M^\wedge \to M/I^n M) = (I^n M)^\wedge$ for all $n \geq 1$.

In particular $R^\wedge$ is I-adically complete, $I^n R^\wedge = (I^n)^\wedge$, and $R^\wedge/I^n R^\wedge = R/I^n$.

Proof. Since I is finitely generated, I^n is finitely generated, say by $f_1, \ldots, f_r$. Applying Lemma 10.96.1 part (2) to the surjection $(f_1, \ldots, f_r) : M^{\oplus r} \to I^n M$ yields a surjection

$$(M^\wedge)^{\oplus r} \xrightarrow{(f_1,\ldots,f_r)} (I^n M)^\wedge = \lim_{m \geq n} I^n M/I^m M = \text{Ker}(M^\wedge \to M/I^n M).$$

On the other hand, the image of $(f_1, \ldots, f_r) : (M^\wedge)^{\oplus r} \to M^\wedge$ is $I^n M^\wedge$. Thus $M^\wedge/I^n M^\wedge \simeq M/I^n M$. Taking inverse limits yields $(M^\wedge)^\wedge \simeq M^\wedge$; that is, $M^\wedge$ is I-adically complete. □

0BNG Lemma 10.96.4. Let R be a ring. Let $I \subset R$ be an ideal. Let $0 \to M \to N \to Q \to 0$ be an exact sequence of R-modules such that Q is annihilated by a power of I. Then completion produces an exact sequence $0 \to M^\wedge \to N^\wedge \to Q \to 0$.

Proof. Say $I^c Q = 0$. Then $Q/I^n Q = Q$ for $n \geq c$. On the other hand, it is clear that $I^n M \subset M \cap I^n N \subset I^{n-c} M$ for $n \geq c$. Thus $M^\wedge = \lim M/(M \cap I^n N)$. Apply Lemma 10.87.1 to the system of exact sequences

$$0 \to M/(M \cap I^n N) \to N/I^n N \to Q \to 0$$

for $n \geq c$ to conclude. □

0318 Lemma 10.96.5. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Denote $K_n = \text{Ker}(M^\wedge \to M/I^n M)$. Then $M^\wedge$ is I-adically complete if and only if K_n is equal to $I^n M^\wedge$ for all $n \geq 1$. Taken from an unpublished note of Lenstra and de Smit.

Proof. The module $I^n M^\wedge$ is contained in K_n. Thus for each $n \geq 1$ there is a canonical exact sequence

$$0 \to K_n/I^n M^\wedge \to M^\wedge/I^n M^\wedge \to M/I^n M \to 0.$$

As $I^n M^\wedge$ maps onto $I^n M/I^{n+1} M$ we see that $K_{n+1} + I^n M^\wedge = K_n$. Thus the inverse system $\{K_n/I^n M^\wedge\}_{n \geq 1}$ has surjective transition maps. By Lemma 10.87.1 we see that there is a short exact sequence

$$0 \to \lim_n K_n/I^n M^\wedge \to (M^\wedge)^\wedge \to M^\wedge \to 0$$

Hence $M^\wedge$ is complete if and only if $K_n/I^n M^\wedge = 0$ for all $n \geq 1$. □

05GI Lemma 10.96.6. Let R be a ring, let $I \subset R$ be an ideal, and let $R^\wedge = \lim R/I^n$.

1. any element of $R^\wedge$ which maps to a unit of R/I is a unit,
2. any element of $1 + I$ maps to an invertible element of $R^\wedge$,
3. any element of $1 + IR^\wedge$ is invertible in $R^\wedge$, and
4. the ideals $IR^\wedge$ and $\text{Ker}(R^\wedge \to R/I)$ are contained in the Jacobson radical of $R^\wedge$.

Proof. Let $x \in R^\wedge$ map to a unit x_1 in R/I. Then x maps to a unit x_n in R/I^n for every n by Lemma 10.32.4. Hence $y = (x_n^{-1}) \in \lim R/I^n = R^\wedge$ is an inverse to x. Parts (2) and (3) follow immediately from (1). Part (4) follows from (1) and Lemma 10.19.1. $\square$

090S **Lemma 10.96.7.** Let A be a ring. Let $I = (f_1, \ldots, f_r)$ be a finitely generated ideal. If $M \to \lim M/f_i^nM$ is surjective for each i, then $M \to \lim M/I^nM$ is surjective.

Proof. Note that $\lim M/I^nM = \lim M/(f_1^n, \ldots, f_r^n)M$ as $I^n \supset (f_1^n, \ldots, f_r^n) \supset I^{rn}$. An element ξ of $\lim M/(f_1^n, \ldots, f_r^n)M$ can be symbolically written as

$$\xi = \sum\nolimits_{n \geq 0} \sum\nolimits_i f_i^n x_{n,i}$$

with $x_{n,i} \in M$. If $M \to \lim M/f_i^nM$ is surjective, then there is an $x_i \in M$ mapping to $\sum x_{n,i}f_i^n$ in $\lim M/f_i^nM$. Then $x = \sum x_i$ maps to ξ in $\lim M/I^nM$. $\square$

090T **Lemma 10.96.8.** Let A be a ring. Let $I \subset J \subset A$ be ideals. If M is J-adically complete and I is finitely generated, then M is I-adically complete.

Proof. Assume M is J-adically complete and I is finitely generated. We have $\bigcap I^nM = 0$ because $\bigcap J^nM = 0$. By Lemma 10.96.7 it suffices to prove the surjectivity of $M \to \lim M/I^nM$ in case I is generated by a single element. Say $I = (f)$. Let $x_n \in M$ with $x_{n+1} - x_n \in f^nM$. We have to show there exists an $x \in M$ such that $x_n - x \in f^nM$ for all n. As $x_{n+1} - x_n \in J^nM$ and as M is J-adically complete, there exists an element $x \in M$ such that $x_n - x \in J^nM$. Replacing x_n by $x_n - x$ we may assume that $x_n \in J^nM$. To finish the proof we will show that this implies $x_n \in I^nM$. Namely, write $x_n - x_{n+1} = f^nz_n$. Then

$$x_n = f^n(z_n + fz_{n+1} + f^2z_{n+2} + \ldots)$$

The sum $z_n + fz_{n+1} + f^2z_{n+2} + \ldots$ converges in M as $f^c \in J^c$. The sum $f^n(z_n + fz_{n+1} + f^2z_{n+2} + \ldots)$ converges in M to x_n because the partial sums equal $x_n - x_{n+c}$ and $x_{n+c} \in J^{n+c}M$. $\square$

0319 **Lemma 10.96.9.** Let R be a ring. Let I, J be ideals of R. Assume there exist integers $c, d > 0$ such that $I^c \subset J$ and $J^d \subset I$. Then completion with respect to I agrees with completion with respect to J for any R-module. In particular an R-module M is I-adically complete if and only if it is J-adically complete.

Proof. Consider the system of maps $M/I^nM \to M/J^{\lfloor n/d \rfloor}M$ and the system of maps $M/J^mM \to M/I^{\lfloor m/c \rfloor}M$ to get mutually inverse maps between the completions. $\square$

031A **Lemma 10.96.10.** Let R be a ring. Let I be an ideal of R. Let M be an I-adically complete R-module, and let $K \subset M$ be an R-submodule. The following are equivalent

(1) $K = \bigcap(K + I^nM)$ and
(2) M/K is I-adically complete.

Proof. Set $N = M/K$. By Lemma 10.96.1 the map $M = M^\wedge \to N^\wedge$ is surjective. Hence $N \to N^\wedge$ is surjective. It is easy to see that the kernel of $N \to N^\wedge$ is the module $\bigcap(K + I^nM)/K$. $\square$

031B Lemma 10.96.11. Let R be a ring. Let I be an ideal of R. Let M be an R-module. If (a) R is I-adically complete, (b) M is a finite R-module, and (c) $\bigcap I^n M = (0)$, then M is I-adically complete.

Proof. By Lemma 10.96.1 the map $M = M \otimes_R R = M \otimes_R R^\wedge \to M^\wedge$ is surjective. The kernel of this map is $\bigcap I^n M$ hence zero by assumption. Hence $M \cong M^\wedge$ and M is complete. □

031D Lemma 10.96.12. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume

(1) R is I-adically complete,
(2) $\bigcap_{n \geq 1} I^n M = (0)$, and
(3) M/IM is a finite R/I-module.

Then M is a finite R-module.

Proof. Let $x_1, \ldots, x_n \in M$ be elements whose images in M/IM generate M/IM as a R/I-module. Denote $M' \subset M$ the R-submodule generated by $x_1, \ldots, x_n$. By Lemma 10.96.1 the map $(M')^\wedge \to M^\wedge$ is surjective. Since $\bigcap I^n M = 0$ we see in particular that $\bigcap I^n M' = (0)$. Hence by Lemma 10.96.11 we see that M' is complete, and we conclude that $M' \to M^\wedge$ is surjective. Finally, the kernel of $M \to M^\wedge$ is zero since it is equal to $\bigcap I^n M = (0)$. Hence we conclude that $M \cong M' \cong M^\wedge$ is finitely generated. □

10.97. Completion for Noetherian rings

0BNH In this section we discuss completion with respect to ideals in Noetherian rings.

00MA Lemma 10.97.1. Let I be an ideal of a Noetherian ring R. Denote $^\wedge$ completion with respect to I.

(1) If $K \to N$ is an injective map of finite R-modules, then the map on completions $K^\wedge \to N^\wedge$ is injective.
(2) If $0 \to K \to N \to M \to 0$ is a short exact sequence of finite R-modules, then $0 \to K^\wedge \to N^\wedge \to M^\wedge \to 0$ is a short exact sequence.
(3) If M is a finite R-module, then $M^\wedge = M \otimes_R R^\wedge$.

Proof. Setting $M = N/K$ we find that part (1) follows from part (2). Let $0 \to K \to N \to M \to 0$ be as in (2). For each n we get the short exact sequence

$$0 \to K/(I^n N \cap K) \to N/I^n N \to M/I^n M \to 0.$$

By Lemma 10.87.1 we obtain the exact sequence

$$0 \to \lim K/(I^n N \cap K) \to N^\wedge \to M^\wedge \to 0.$$

By the Artin-Rees Lemma 10.51.2 we may choose c such that $I^n K \subset I^n N \cap K \subset I^{n-c} K$ for $n \geq c$. Hence $K^\wedge = \lim K/I^n K = \lim K/(I^n N \cap K)$ and we conclude that (2) is true.

Let M be as in (3) and let $0 \to K \to R^{\oplus t} \to M \to 0$ be a presentation of M. We get a commutative diagram

$$\begin{tikzcd}
 & K \otimes_R R^\wedge \arrow[r] \arrow[d] & R^{\oplus t} \otimes_R R^\wedge \arrow[r] \arrow[d] & M \otimes_R R^\wedge \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^\wedge \arrow[r] & (R^{\oplus t})^\wedge \arrow[r] & M^\wedge \arrow[r] & 0
\end{tikzcd}$$

The top row is exact, see Section 10.39. The bottom row is exact by part (2). By Lemma 10.96.1 the vertical arrows are surjective. The middle vertical arrow is an isomorphism. We conclude (3) holds by the Snake Lemma 10.4.1. ☐

00MB Lemma 10.97.2. Let I be a ideal of a Noetherian ring R. Denote $^\wedge$ completion with respect to I.

(1) The ring map $R \to R^\wedge$ is flat.
(2) The functor $M \mapsto M^\wedge$ is exact on the category of finitely generated R-modules.

Proof. Consider $J \otimes_R R^\wedge \to R \otimes_R R^\wedge = R^\wedge$ where J is an arbitrary ideal of R. According to Lemma 10.97.1 this is identified with $J^\wedge \to R^\wedge$ and $J^\wedge \to R^\wedge$ is injective. Part (1) follows from Lemma 10.39.5. Part (2) is a reformulation of Lemma 10.97.1 part (2). ☐

00MC Lemma 10.97.3. Let $(R, \mathfrak m)$ be a Noetherian local ring. Let $I \subset \mathfrak m$ be an ideal. Denote $R^\wedge$ the completion of R with respect to I. The ring map $R \to R^\wedge$ is faithfully flat. In particular the completion with respect to $\mathfrak m$, namely $\lim_n R/\mathfrak m^n$ is faithfully flat.

Proof. By Lemma 10.97.2 it is flat. The composition $R \to R^\wedge \to R/\mathfrak m$ where the last map is the projection map $R^\wedge \to R/I$ combined with $R/I \to R/\mathfrak m$ shows that $\mathfrak m$ is in the image of $\mathrm{Spec}(R^\wedge) \to \mathrm{Spec}(R)$. Hence the map is faithfully flat by Lemma 10.39.15. ☐

031C Lemma 10.97.4. Let R be a Noetherian ring. Let I be an ideal of R. Let M be an R-module. Then the completion $M^\wedge$ of M with respect to I is I-adically complete, $I^nM^\wedge = (I^nM)^\wedge$, and $M^\wedge/I^nM^\wedge = M/I^nM$.

Proof. This is a special case of Lemma 10.96.3 because I is a finitely generated ideal. ☐

05GH Lemma 10.97.5. Let I be an ideal of a ring R. Assume

(1) R/I is a Noetherian ring,
(2) I is finitely generated.

Then the completion $R^\wedge$ of R with respect to I is a Noetherian ring complete with respect to $IR^\wedge$.

Proof. By Lemma 10.96.3 we see that $R^\wedge$ is I-adically complete. Hence it is also $IR^\wedge$-adically complete. Since $R^\wedge/IR^\wedge = R/I$ is Noetherian we see that after replacing R by $R^\wedge$ we may in addition to assumptions (1) and (2) assume that also R is I-adically complete.

Let $f_1, \ldots, f_t$ be generators of I. Then there is a surjection of rings $R/I[T_1, \ldots, T_t] \to \bigoplus I^n/I^{n+1}$ mapping T_i to the element $\overline{f}_i \in I/I^2$. Hence $\bigoplus I^n/I^{n+1}$ is a Noetherian ring. Let $J \subset R$ be an ideal. Consider the ideal

$$\bigoplus J \cap I^n/J \cap I^{n+1} \subset \bigoplus I^n/I^{n+1}.$$

Let $\overline{g}_1, \ldots, \overline{g}_m$ be generators of this ideal. We may choose $\overline{g}_j$ to be a homogeneous element of degree d_j and we may pick $g_j \in J \cap I^{d_j}$ mapping to $\overline{g}_j \in J \cap I^{d_j}/J \cap I^{d_j+1}$. We claim that $g_1, \ldots, g_m$ generate J.

Let $x \in J \cap I^n$. There exist $a_j \in I^{\max(0, n-d_j)}$ such that $x - \sum a_j g_j \in J \cap I^{n+1}$. The reason is that $J \cap I^n / J \cap I^{n+1}$ is equal to $\sum \overline{g}_j I^{n-d_j}/I^{n-d_j+1}$ by our choice of $g_1, \ldots, g_m$. Hence starting with $x \in J$ we can find a sequence of vectors $(a_{1,n}, \ldots, a_{m,n})_{n \geq 0}$ with $a_{j,n} \in I^{\max(0, n-d_j)}$ such that

$$x = \sum\nolimits_{n=0,\ldots,N} \sum\nolimits_{j=1,\ldots,m} a_{j,n} g_j \bmod I^{N+1}$$

Setting $A_j = \sum_{n \geq 0} a_{j,n}$ we see that $x = \sum A_j g_j$ as R is complete. Hence J is finitely generated and we win. ☐

0316 Lemma 10.97.6. Let R be a Noetherian ring. Let I be an ideal of R. The completion $R^\wedge$ of R with respect to I is Noetherian.

Proof. This is a consequence of Lemma 10.97.5. It can also be seen directly as follows. Choose generators $f_1, \ldots, f_n$ of I. Consider the map

$$R[[x_1, \ldots, x_n]] \longrightarrow R^\wedge, \quad x_i \longmapsto f_i.$$

This is a well defined and surjective ring map (details omitted). Since $R[[x_1, \ldots, x_n]]$ is Noetherian (see Lemma 10.31.2) we win. ☐

Suppose $R \to S$ is a local homomorphism of local rings $(R, \mathfrak m)$ and $(S, \mathfrak n)$. Let $S^\wedge$ be the completion of S with respect to $\mathfrak n$. In general $S^\wedge$ is not the $\mathfrak m$-adic completion of S. If $\mathfrak n^t \subset \mathfrak m S$ for some $t \geq 1$ then we do have $S^\wedge = \lim S/\mathfrak m^n S$ by Lemma 10.96.9. In some cases this even implies that $S^\wedge$ is finite over $R^\wedge$.

0394 Lemma 10.97.7. Let $R \to S$ be a local homomorphism of local rings $(R, \mathfrak m)$ and $(S, \mathfrak n)$. Let $R^\wedge$, resp. $S^\wedge$ be the completion of R, resp. S with respect to $\mathfrak m$, resp. $\mathfrak n$. If $\mathfrak m$ and $\mathfrak n$ are finitely generated and $\dim_{\kappa(\mathfrak m)} S/\mathfrak m S < \infty$, then

(1) $S^\wedge$ is equal to the $\mathfrak m$-adic completion of S, and
(2) $S^\wedge$ is a finite $R^\wedge$-module.

Proof. We have $\mathfrak m S \subset \mathfrak n$ because $R \to S$ is a local ring map. The assumption $\dim_{\kappa(\mathfrak m)} S/\mathfrak m S < \infty$ implies that $S/\mathfrak m S$ is an Artinian ring, see Lemma 10.53.2. Hence has dimension 0, see Lemma 10.60.5, hence $\mathfrak n = \sqrt{\mathfrak m S}$. This and the fact that $\mathfrak n$ is finitely generated implies that $\mathfrak n^t \subset \mathfrak m S$ for some $t \geq 1$. By Lemma 10.96.9 we see that $S^\wedge$ can be identified with the $\mathfrak m$-adic completion of S. As $\mathfrak m$ is finitely generated we see from Lemma 10.96.3 that $S^\wedge$ and $R^\wedge$ are $\mathfrak m$-adically complete. At this point we may apply Lemma 10.96.12 to $S^\wedge$ as an $R^\wedge$-module to conclude. ☐

07N9 Lemma 10.97.8. Let R be a Noetherian ring. Let $R \to S$ be a finite ring map. Let $\mathfrak p \subset R$ be a prime and let $\mathfrak q_1, \ldots, \mathfrak q_m$ be the primes of S lying over $\mathfrak p$ (Lemma 10.36.21). Then

$$R_{\mathfrak p}^\wedge \otimes_R S = (S_{\mathfrak p})^\wedge = S_{\mathfrak q_1}^\wedge \times \ldots \times S_{\mathfrak q_m}^\wedge$$

where the $(S_{\mathfrak p})^\wedge$ is the completion with respect to $\mathfrak p$ and the local rings $R_{\mathfrak p}$ and $S_{\mathfrak q_i}$ are completed with respect to their maximal ideals.

Proof. The first equality follows from Lemma 10.97.1. We may replace R by the localization $R_{\mathfrak p}$ and S by $S_{\mathfrak p} = S \otimes_R R_{\mathfrak p}$. Hence we may assume that R is a local Noetherian ring and that $\mathfrak p = \mathfrak m$ is its maximal ideal. The $\mathfrak q_i S_{\mathfrak q_i}$-adic completion $S_{\mathfrak q_i}^\wedge$ is equal to the $\mathfrak m$-adic completion by Lemma 10.97.7. For every $n \geq 1$ prime ideals of $S/\mathfrak m^n S$ are in 1-to-1 correspondence with the maximal ideals $\mathfrak q_1, \ldots, \mathfrak q_m$ of S (by going up for S over R, see Lemma 10.36.22). Hence $S/\mathfrak m^n S = \prod S_{\mathfrak q_i}/\mathfrak m^n S_{\mathfrak q_i}$

by Lemma 10.53.6 (using for example Proposition 10.60.7 to see that $S/\mathfrak{m}^nS$ is Artinian). Hence the $\mathfrak{m}$-adic completion $S^\wedge$ of S is equal to $\prod S_{\mathfrak{q}_i}^\wedge$. Finally, we have $R^\wedge \otimes_R S = S^\wedge$ by Lemma 10.97.1. $\square$

05D3 Lemma 10.97.9. Let R be a ring. Let $I \subset R$ be an ideal. Let $0 \to K \to P \to M \to 0$ be a short exact sequence of R-modules. If M is flat over R and M/IM is a projective R/I-module, then the sequence of I-adic completions

$$0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$$

is a split exact sequence.

Proof. As M is flat, each of the sequences

$$0 \to K/I^nK \to P/I^nP \to M/I^nM \to 0$$

is short exact, see Lemma 10.39.12 and the sequence $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is a short exact sequence, see Lemma 10.96.1. It suffices to show that we can find splittings $s_n : M/I^nM \to P/I^nP$ such that $s_{n+1} \bmod I^n = s_n$. We will construct these s_n by induction on n. Pick any splitting s_1, which exists as M/IM is a projective R/I-module. Assume given s_n for some $n > 0$. Set $P_{n+1} = \{x \in P \mid x \bmod I^nP \in \mathrm{Im}(s_n)\}$. The map $\pi : P_{n+1}/I^{n+1}P_{n+1} \to M/I^{n+1}M$ is surjective (details omitted). As $M/I^{n+1}M$ is projective as a R/I^{n+1}-module by Lemma 10.77.7 we may choose a section $t : M/I^{n+1}M \to P_{n+1}/I^{n+1}P_{n+1}$ of π. Setting s_{n+1} equal to the composition of t with the canonical map $P_{n+1}/I^{n+1}P_{n+1} \to P/I^{n+1}P$ works. $\square$

0DYC Lemma 10.97.10. Let A be a Noetherian ring. Let $I, J \subset A$ be ideals. If A is I-adically complete and A/I is J-adically complete, then A is J-adically complete.

Proof. Let B be the $(I + J)$-adic completion of A. By Lemma 10.97.2 B/IB is the J-adic completion of A/I hence isomorphic to A/I by assumption. Moreover B is I-adically complete by Lemma 10.96.8. Hence B is a finite A-module by Lemma 10.96.12. By Nakayama's lemma (Lemma 10.20.1 using I is in the Jacobson radical of A by Lemma 10.96.6) we find that $A \to B$ is surjective. The map $A \to B$ is flat by Lemma 10.97.2. The image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains $V(I)$ and as I is contained in the Jacobson radical of A we find $A \to B$ is faithfully flat (Lemma 10.39.16). Thus $A \to B$ is injective. Thus A is complete with respect to $I + J$, hence a fortiori complete with respect to J. $\square$

10.98. Taking limits of modules

09B7 In this section we discuss what happens when we take a limit of modules.

0G1Q Lemma 10.98.1. Let $I \subset A$ be a finitely generated ideal of a ring. Let (M_n) be an inverse system of A-modules with $I^nM_n = 0$. Then $M = \lim M_n$ is I-adically complete.

Proof. We have $M \to M/I^nM \to M_n$. Taking the limit we get $M \to M^\wedge \to M$. Hence M is a direct summand of $M^\wedge$. Since $M^\wedge$ is I-adically complete by Lemma 10.96.3, so is M. $\square$

09B8 Lemma 10.98.2. Let $I \subset A$ be a finitely generated ideal of a ring. Let (M_n) be an inverse system of A-modules with $M_n = M_{n+1}/I^nM_{n+1}$. Then $M/I^nM = M_n$ and M is I-adically complete.

Proof. By Lemma 10.98.1 we see that M is I-adically complete. Since the transition maps are surjective, the maps $M \to M_n$ are surjective. Consider the inverse system of short exact sequences

$$0 \to N_n \to M \to M_n \to 0$$

defining N_n. Since $M_n = M_{n+1}/I^nM_{n+1}$ the map $N_{n+1} + I^nM \to N_n$ is surjective. Hence $N_{n+1}/(N_{n+1} \cap I^{n+1}M) \to N_n/(N_n \cap I^nM)$ is surjective. Taking the inverse limit of the short exact sequences

$$0 \to N_n/(N_n \cap I^nM) \to M/I^nM \to M_n \to 0$$

we obtain an exact sequence

$$0 \to \lim N_n/(N_n \cap I^nM) \to M^\wedge \to M$$

Since M is I-adically complete we conclude that $\lim N_n/(N_n \cap I^nM) = 0$ and hence by the surjectivity of the transition maps we get $N_n/(N_n \cap I^nM) = 0$ for all n. Thus $M_n = M/I^nM$ as desired. □

0EKC Lemma 10.98.3. Let A be a Noetherian graded ring. Let $I \subset A_+$ be a homogeneous ideal. Let (N_n) be an inverse system of finite graded A-modules with $N_n = N_{n+1}/I^nN_{n+1}$. Then there is a finite graded A-module N such that $N_n = N/I^nN$ as graded modules for all n.

Proof. Pick r and homogeneous elements $x_{1,1}, \ldots, x_{1,r} \in N_1$ of degrees $d_1, \ldots, d_r$ generating N_1. Since the transition maps are surjective, we can pick a compatible system of homogeneous elements $x_{n,i} \in N_n$ lifting $x_{1,i}$. By the graded Nakayama lemma (Lemma 10.56.1) we see that N_n is generated by the elements $x_{n,1}, \ldots, x_{n,r}$ sitting in degrees $d_1, \ldots, d_r$. Thus for $m \leq n$ we see that $N_n \to N_n/I^mN_n$ is an isomorphism in degrees $< \min(d_i) + m$ (as I^mN_n is zero in those degrees). Thus the inverse system of degree d parts

$$\ldots = N_{2+d-\min(d_i),d} = N_{1+d-\min(d_i),d} = N_{d-\min(d_i),d} \to N_{-1+d-\min(d_i),d} \to \ldots$$

stabilizes as indicated. Let N be the graded A-module whose dth graded part is this stabilization. In particular, we have the elements $x_i = \lim x_{n,i}$ in N. We claim the x_i generate N: any $x \in N_d$ is a linear combination of $x_1, \ldots, x_r$ because we can check this in $N_{d-\min(d_i),d}$ where it holds as $x_{d-\min(d_i),i}$ generate $N_{d-\min(d_i)}$. Finally, the reader checks that the surjective map $N/I^nN \to N_n$ is an isomorphism by checking to see what happens in each degree as before. Details omitted. □

0EKD Lemma 10.98.4. Let A be a graded ring. Let $I \subset A_+$ be a homogeneous ideal. Denote $A' = \lim A/I^n$. Let (G_n) be an inverse system of graded A-modules with G_n annihilated by I^n. Let M be a graded A-module and let $\varphi_n : M \to G_n$ be a compatible system of graded A-module maps. If the induced map

$$\varphi : M \otimes_A A' \longrightarrow \lim G_n$$

is an isomorphism, then $M_d \to \lim G_{n,d}$ is an isomorphism for all $d \in \mathbf{Z}$.

Proof. By convention graded rings are in degrees ≥ 0 and graded modules may have nonzero parts of any degree, see Section 10.56. The map φ exists because $\lim G_n$ is a module over A' as G_n is annihilated by I^n. Another useful thing to keep in mind is that we have

$$\bigoplus\nolimits_{d \in \mathbf{Z}} \lim G_{n,d} \subset \lim G_n \subset \prod\nolimits_{d \in \mathbf{Z}} \lim G_{n,d}$$

where a subscript ${}_d$ indicates the dth graded part.

Injective. Let $x \in M_d$. If $x \mapsto 0$ in $\lim G_{n,d}$ then $x \otimes 1 = 0$ in $M \otimes_A A'$. Then we can find a finitely generated submodule $M' \subset M$ with $x \in M'$ such that $x \otimes 1$ is zero in $M' \otimes_A A'$. Say M' is generated by homogeneous elements sitting in degrees $d_1, \ldots, d_r$. Let $n = d - \min(d_i) + 1$. Since A' has a map to A/I^n and since $A \to A/I^n$ is an isomorphism in degrees $\leq n-1$ we see that $M' \to M' \otimes_A A'$ is injective in degrees $\leq n-1$. Thus $x = 0$ as desired.

Surjective. Let $y \in \lim G_{n,d}$. Choose a finite sum $\sum x_i \otimes f'_i$ in $M \otimes_A A'$ mapping to y. We may assume x_i is homogeneous, say of degree d_i. Observe that although A' is not a graded ring, it is a limit of the graded rings A/I^nA and moreover, in any given degree the transition maps eventually become isomorphisms (see above). This gives

$$A = \bigoplus\nolimits_{d \geq 0} A_d \subset A' \subset \prod\nolimits_{d \geq 0} A_d$$

Thus we can write

$$f'_i = \sum\nolimits_{j = 0, \ldots, d - d_i - 1} f_{i,j} + f_i + g'_i$$

with $f_{i,j} \in A_j$, $f_i \in A_{d - d_i}$, and $g'_i \in A'$ mapping to zero in $\prod_{j \leq d - d_i} A_j$. Now if we compute $\varphi_n(\sum_{i,j} f_{i,j} x_i) \in G_n$, then we get a sum of homogeneous elements of degree $< d$. Hence $\varphi(\sum x_i \otimes f_{i,j})$ maps to zero in $\lim G_{n,d}$. Similarly, a computation shows the element $\varphi(\sum x_i \otimes g'_i)$ maps to zero in $\prod_{d' \leq d} \lim G_{n,d'}$. Since we know that $\varphi(\sum x_i \otimes f'_i)$ is y, we conclude that $\sum f_i x_i \in M_d$ maps to y as desired. □

10.99. Criteria for flatness

00MD In this section we prove some important technical lemmas in the Noetherian case. We will (partially) generalize these to the non-Noetherian case in Section 10.128.

00ME Lemma 10.99.1. Suppose that $R \to S$ is a local homomorphism of Noetherian local rings. Denote $\mathfrak m$ the maximal ideal of R. Let M be a flat R-module and N a finite S-module. Let $u : N \to M$ be a map of R-modules. If $\overline{u} : N/\mathfrak m N \to M/\mathfrak m M$ is injective then u is injective. In this case $M/u(N)$ is flat over R.

Proof. First we claim that $u_n : N/\mathfrak m^n N \to M/\mathfrak m^n M$ is injective for all $n \geq 1$. We proceed by induction, the base case is that $\overline{u} = u_1$ is injective. By our assumption that M is flat over R we have a short exact sequence $0 \to M \otimes_R \mathfrak m^n/\mathfrak m^{n+1} \to M/\mathfrak m^{n+1}M \to M/\mathfrak m^n M \to 0$. Also, $M \otimes_R \mathfrak m^n/\mathfrak m^{n+1} = M/\mathfrak m M \otimes_{R/\mathfrak m} \mathfrak m^n/\mathfrak m^{n+1}$. We have a similar exact sequence $N \otimes_R \mathfrak m^n/\mathfrak m^{n+1} \to N/\mathfrak m^{n+1}N \to N/\mathfrak m^n N \to 0$ for N except we do not have the zero on the left. We also have $N \otimes_R \mathfrak m^n/\mathfrak m^{n+1} = N/\mathfrak m N \otimes_{R/\mathfrak m} \mathfrak m^n/\mathfrak m^{n+1}$. Thus the map u_{n+1} is injective as both u_n and the map $\overline{u} \otimes \text{id}_{\mathfrak m^n/\mathfrak m^{n+1}}$ are.

By Krull's intersection theorem (Lemma 10.51.4) applied to N over the ring S and the ideal $\mathfrak m S$ we have $\bigcap \mathfrak m^n N = 0$. Thus the injectivity of u_n for all n implies u is injective.

To show that $M/u(N)$ is flat over R, it suffices to show that $\text{Tor}_1^R(M/u(N), R/I) = 0$ for every ideal $I \subset R$, see Lemma 10.75.8. From the short exact sequence

$$0 \to N \xrightarrow{u} M \to M/u(N) \to 0$$

and the flatness of M we obtain an exact sequence of Tors

$$0 \to \mathrm{Tor}_1^R(M/u(N), R/I) \to N/IN \to M/IM$$

See Lemma 10.75.2. Thus it suffices to show that N/IN injects into M/IM. Note that $R/I \to S/IS$ is a local homomorphism of Noetherian local rings, $N/IN \to M/IM$ is a map of R/I-modules, N/IN is finite over S/IS, and M/IM is flat over R/I and $u \bmod I : N/IN \to M/IM$ is injective modulo $\mathfrak{m}$. Thus we may apply the first part of the proof to $u \bmod I$ and we conclude. □

00MF Lemma 10.99.2. Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f \in S$ is a nonzerodivisor in $S/\mathfrak{m}S$. Then S/fS is flat over R, and f is a nonzerodivisor in S.

Proof. Follows directly from Lemma 10.99.1. □

00MG Lemma 10.99.3. Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$. Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.

Proof. Induction and Lemma 10.99.2. □

00MH Lemma 10.99.4. Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let M be a finite S-module. Suppose that (a) $M/\mathfrak{m}M$ is a free $S/\mathfrak{m}S$-module, and (b) M is flat over R. Then M is free and S is flat over R.

Proof. Let $\overline{x}_1, \ldots, \overline{x}_n$ be a basis for the free module $M/\mathfrak{m}M$. Choose $x_1, \ldots, x_n \in M$ with x_i mapping to $\overline{x}_i$. Let $u : S^{\oplus n} \to M$ be the map which maps the ith standard basis vector to x_i. By Lemma 10.99.1 we see that u is injective. On the other hand, by Nakayama's Lemma 10.20.1 the map is surjective. The lemma follows. □

00MI Lemma 10.99.5. Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ be a finite complex of finite S-modules. Assume that each F_i is R-flat, and that the complex $0 \to F_e/\mathfrak{m}F_e \to F_{e-1}/\mathfrak{m}F_{e-1} \to \ldots \to F_0/\mathfrak{m}F_0$ is exact. Then $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ is exact, and moreover the module $\mathrm{Coker}(F_1 \to F_0)$ is R-flat.

Proof. By induction on e. If $e = 1$, then this is exactly Lemma 10.99.1. If $e > 1$, we see by Lemma 10.99.1 that $F_e \to F_{e-1}$ is injective and that $C = \mathrm{Coker}(F_e \to F_{e-1})$ is a finite S-module flat over R. Hence we can apply the induction hypothesis to the complex $0 \to C \to F_{e-2} \to \ldots \to F_0$. We deduce that $C \to F_{e-2}$ is injective and the exactness of the complex follows, as well as the flatness of the cokernel of $F_1 \to F_0$. □

In the rest of this section we prove two versions of what is called the "local criterion of flatness". Note also the interesting Lemma 10.128.1 below.

00MJ Lemma 10.99.6. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Let M be an R-module. If $\mathrm{Tor}_1^R(\kappa, M) = 0$, then for every finite length R-module N we have $\mathrm{Tor}_1^R(N, M) = 0$.

Proof. By descending induction on the length of N. If the length of N is 1, then $N \cong \kappa$ and we are done. If the length of N is more than 1, then we can fit N into a short exact sequence $0 \to N' \to N \to N'' \to 0$ where N', N'' are finite length R-modules of smaller length. The vanishing of $\text{Tor}_1^R(N, M)$ follows from the vanishing of $\text{Tor}_1^R(N', M)$ and $\text{Tor}_1^R(N'', M)$ (induction hypothesis) and the long exact sequence of Tor groups, see Lemma 10.75.2. $\square$

00MK Lemma 10.99.7 (Local criterion for flatness). Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R, and let $\kappa = R/\mathfrak{m}$. Let M be a finite S-module. If $\text{Tor}_1^R(\kappa, M) = 0$, then M is flat over R.

Proof. Let $I \subset R$ be an ideal. By Lemma 10.39.5 it suffices to show that $I \otimes_R M \to M$ is injective. By Remark 10.75.9 we see that this kernel is equal to $\text{Tor}_1^R(M, R/I)$. By Lemma 10.99.6 we see that $J \otimes_R M \to M$ is injective for all ideals of finite colength.

Choose $n >> 0$ and consider the following short exact sequence

$$0 \to I \cap \mathfrak{m}^n \to I \oplus \mathfrak{m}^n \to I + \mathfrak{m}^n \to 0$$

This is a sub sequence of the short exact sequence $0 \to R \to R^{\oplus 2} \to R \to 0$. Thus we get the diagram

$$\begin{tikzcd}
(I \cap \mathfrak{m}^n) \otimes_R M \arrow[r] \arrow[d] & I \otimes_R M \oplus \mathfrak{m}^n \otimes_R M \arrow[r] \arrow[d] & (I + \mathfrak{m}^n) \otimes_R M \arrow[d] \\
M \arrow[r] & M \oplus M \arrow[r] & M
\end{tikzcd}$$

Note that $I + \mathfrak{m}^n$ and $\mathfrak{m}^n$ are ideals of finite colength. Thus a diagram chase shows that $\text{Ker}((I \cap \mathfrak{m}^n) \otimes_R M \to M) \to \text{Ker}(I \otimes_R M \to M)$ is surjective. We conclude in particular that $K = \text{Ker}(I \otimes_R M \to M)$ is contained in the image of $(I \cap \mathfrak{m}^n) \otimes_R M$ in $I \otimes_R M$. By Artin-Rees, Lemma 10.51.2 we see that K is contained in $\mathfrak{m}^{n-c}(I \otimes_R M)$ for some $c > 0$ and all $n >> 0$. Since $I \otimes_R M$ is a finite S-module (!) and since S is Noetherian, we see that this implies $K = 0$. Namely, the above implies K maps to zero in the $\mathfrak{m}S$-adic completion of $I \otimes_R M$. But the map from S to its $\mathfrak{m}S$-adic completion is faithfully flat by Lemma 10.97.3. Hence $K = 0$, as desired. $\square$

In the following we often encounter the conditions "M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$". The following lemma gives some consequences of these conditions (it is a generalization of Lemma 10.99.6).

051C Lemma 10.99.8. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. If M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$ then

(1) M/I^nM is flat over R/I^n for all $n \geq 1$, and
(2) for any module N which is annihilated by I^m for some $m \geq 0$ we have $\text{Tor}_1^R(N, M) = 0$.

In particular, if I is nilpotent, then M is flat over R.

Proof. Assume M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$. Let N be an R/I-module. Choose a short exact sequence

$$0 \to K \to \bigoplus\nolimits_{i \in I} R/I \to N \to 0$$

By the long exact sequence of Tor and the vanishing of $\text{Tor}_1^R(R/I, M)$ we get

$$0 \to \text{Tor}_1^R(N, M) \to K \otimes_R M \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_R M \to N \otimes_R M \to 0$$

But since K, $\bigoplus_{i \in I} R/I$, and N are all annihilated by I we see that

$$K \otimes_R M = K \otimes_{R/I} M/IM,$$
$$(\bigoplus\nolimits_{i \in I} R/I) \otimes_R M = (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM,$$
$$N \otimes_R M = N \otimes_{R/I} M/IM.$$

As M/IM is flat over R/I we conclude that

$$0 \to K \otimes_{R/I} M/IM \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM \to N \otimes_{R/} M/IM \to 0$$

is exact. Combining this with the above we conclude that $\text{Tor}_1^R(N, M) = 0$ for any R-module N annihilated by I.

In particular, if we apply this to the module I/I^2, then we conclude that the sequence

$$0 \to I^2 \otimes_R M \to I \otimes_R M \to I/I^2 \otimes_R M \to 0$$

is short exact. This implies that $I^2 \otimes_R M \to M$ is injective and it implies that $I/I^2 \otimes_{R/I} M/IM = IM/I^2M$.

Let us prove that M/I^2M is flat over R/I^2. Let $I^2 \subset J$ be an ideal. We have to show that $J/I^2 \otimes_{R/I^2} M/I^2M \to M/I^2M$ is injective, see Lemma 10.39.5. As M/IM is flat over R/I we know that the map $(I+J)/I \otimes_{R/I} M/IM \to M/IM$ is injective. The sequence

$$(I \cap J)/I^2 \otimes_{R/I^2} M/I^2M \to J/I^2 \otimes_{R/I^2} M/I^2M \to (I+J)/I \otimes_{R/I} M/IM \to 0$$

is exact, as you get it by tensoring the exact sequence $0 \to (I \cap J) \to J \to (I+J)/I \to 0$ by M/I^2M. Hence suffices to prove the injectivity of the map $(I \cap J)/I^2 \otimes_{R/I} M/IM \to IM/I^2M$. However, the map $(I \cap J)/I^2 \to I/I^2$ is injective and as M/IM is flat over R/I the map $(I \cap J)/I^2 \otimes_{R/I} M/IM \to I/I^2 \otimes_{R/I} M/IM$ is injective. Since we have previously seen that $I/I^2 \otimes_{R/I} M/IM = IM/I^2M$ we obtain the desired injectivity.

Hence we have proven that the assumptions imply: (a) $\text{Tor}_1^R(N, M) = 0$ for all N annihilated by I, (b) $I^2 \otimes_R M \to M$ is injective, and (c) M/I^2M is flat over R/I^2. Thus we can continue by induction to get the same results for I^n for all $n \geq 1$. □

0AS8 Lemma 10.99.9. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module.

(1) If M/IM is flat over R/I and $M \otimes_R I/I^2 \to IM/I^2M$ is injective, then M/I^2M is flat over R/I^2.
(2) If M/IM is flat over R/I and $M \otimes_R I^n/I^{n+1} \to I^nM/I^{n+1}M$ is injective for $n = 1, \ldots, k$, then $M/I^{k+1}M$ is flat over R/I^{k+1}.

Proof. The first statement is a consequence of Lemma 10.99.8 applied with R replaced by R/I^2 and M replaced by M/I^2M using that

$$\text{Tor}_1^{R/I^2}(M/I^2M, R/I) = \text{Ker}(M \otimes_R I/I^2 \to IM/I^2M),$$

see Remark 10.75.9. The second statement follows in the same manner using induction on n to show that $M/I^{n+1}M$ is flat over R/I^{n+1} for $n = 1, \ldots, k$. Here we use that

$$\mathrm{Tor}_1^{R/I^{n+1}}(M/I^{n+1}M, R/I) = \mathrm{Ker}(M \otimes_R I^n/I^{n+1} \to I^nM/I^{n+1}M)$$

for every n. $\square$

00ML Lemma 10.99.10 (Variant of the local criterion). Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $I \neq R$ be an ideal in R. Let M be a finite S-module. If $\mathrm{Tor}_1^R(M, R/I) = 0$ and M/IM is flat over R/I, then M is flat over R.

Proof. First proof: By Lemma 10.99.8 we see that $\mathrm{Tor}_1^R(\kappa, M)$ is zero where κ is the residue field of R. Hence we see that M is flat over R by Lemma 10.99.7.

Second proof: Let $\mathfrak{m}$ be the maximal ideal of R. We will show that $\mathfrak{m} \otimes_R M \to M$ is injective, and then apply Lemma 10.99.7. Suppose that $\sum f_i \otimes x_i \in \mathfrak{m} \otimes_R M$ and that $\sum f_i x_i = 0$ in M. By the equational criterion for flatness Lemma 10.39.11 applied to M/IM over R/I we see there exist $\overline{a}_{ij} \in R/I$ and $\overline{y}_j \in M/IM$ such that $x_i \bmod IM = \sum_j \overline{a}_{ij}\overline{y}_j$ and $0 = \sum_i (f_i \bmod I)\overline{a}_{ij}$. Let $a_{ij} \in R$ be a lift of $\overline{a}_{ij}$ and similarly let $y_j \in M$ be a lift of $\overline{y}_j$. Then we see that

$$\begin{aligned} \sum f_i \otimes x_i &= \sum f_i \otimes x_i + \sum f_i a_{ij} \otimes y_j - \sum f_i \otimes a_{ij} y_j \\ &= \sum f_i \otimes (x_i - \sum a_{ij} y_j) + \sum (\sum f_i a_{ij}) \otimes y_j \end{aligned}$$

Since $x_i - \sum a_{ij} y_j \in IM$ and $\sum f_i a_{ij} \in I$ we see that there exists an element in $I \otimes_R M$ which maps to our given element $\sum f_i \otimes x_i$ in $\mathfrak{m} \otimes_R M$. But $I \otimes_R M \to M$ is injective by assumption (see Remark 10.75.9) and we win. $\square$

In particular, in the situation of Lemma 10.99.10, suppose that $I = (x)$ is generated by a single element x which is a nonzerodivisor in R. Then $\mathrm{Tor}_1^R(M, R/(x)) = (0)$ if and only if x is a nonzerodivisor on M.

0523 Lemma 10.99.11. Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Assume

(1) R is a Noetherian ring,
(2) S is a Noetherian ring,
(3) M is a finite S-module, and
(4) for each $n \geq 1$ the module M/I^nM is flat over R/I^n.

Then for every $\mathfrak{q} \in V(IS)$ the localization $M_\mathfrak{q}$ is flat over R. In particular, if S is local and IS is contained in its maximal ideal, then M is flat over R.

Proof. We are going to use Lemma 10.99.10. By assumption M/IM is flat over R/I. Hence it suffices to check that $\mathrm{Tor}_1^R(M, R/I)$ is zero on localization at $\mathfrak{q}$. By Remark 10.75.9 this Tor group is equal to $K = \mathrm{Ker}(I \otimes_R M \to M)$. We know for each $n \geq 1$ that the kernel $\mathrm{Ker}(I/I^n \otimes_{R/I^n} M/I^nM \to M/I^nM)$ is zero. Since there is a module map $I/I^n \otimes_{R/I^n} M/I^nM \to (I \otimes_R M)/I^{n-1}(I \otimes_R M)$ we conclude that $K \subset I^{n-1}(I \otimes_R M)$ for each n. By the Artin-Rees lemma, and more precisely Lemma 10.51.5 we conclude that $K_\mathfrak{q} = 0$, as desired. $\square$

00MM Lemma 10.99.12. Let $R \to R' \to R''$ be ring maps. Let M be an R-module. Suppose that $M \otimes_R R'$ is flat over R'. Then the natural map $\text{Tor}_1^R(M, R') \otimes_{R'} R'' \to \text{Tor}_1^R(M, R'')$ is onto.

Proof. Let $F_\bullet$ be a free resolution of M over R. The complex $F_2 \otimes_R R' \to F_1 \otimes_R R' \to F_0 \otimes_R R'$ computes $\text{Tor}_1^R(M, R')$. The complex $F_2 \otimes_R R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R''$ computes $\text{Tor}_1^R(M, R'')$. Note that $F_i \otimes_R R' \otimes_{R'} R'' = F_i \otimes_R R''$. Let $K' = \text{Ker}(F_1 \otimes_R R' \to F_0 \otimes_R R')$ and similarly $K'' = \text{Ker}(F_1 \otimes_R R'' \to F_0 \otimes_R R'')$. Thus we have an exact sequence

$$0 \to K' \to F_1 \otimes_R R' \to F_0 \otimes_R R' \to M \otimes_R R' \to 0.$$

By the assumption that $M \otimes_R R'$ is flat over R', the sequence

$$K' \otimes_{R'} R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R'' \to M \otimes_R R'' \to 0$$

is still exact. This means that $K' \otimes_{R'} R'' \to K''$ is surjective. Since $\text{Tor}_1^R(M, R')$ is a quotient of K' and $\text{Tor}_1^R(M, R'')$ is a quotient of K'' we win. □

00MN Lemma 10.99.13. Let $R \to R'$ be a ring map. Let $I \subset R$ be an ideal and $I' = IR'$. Let M be an R-module and set $M' = M \otimes_R R'$. The natural map $\text{Tor}_1^R(R'/I', M) \to \text{Tor}_1^{R'}(R'/I', M')$ is surjective.

Proof. Let $F_2 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M over R. Set $F_i' = F_i \otimes_R R'$. The sequence $F_2' \to F_1' \to F_0' \to M' \to 0$ may no longer be exact at F_1'. A free resolution of M' over R' therefore looks like

$$F_2' \oplus F_2'' \to F_1' \to F_0' \to M' \to 0$$

for a suitable free module F_2'' over R'. Next, note that $F_i \otimes_R R'/I' = F_i'/IF_i' = F_i'/I'F_i'$. So the complex $F_2'/I'F_2' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\text{Tor}_1^R(M, R'/I')$. On the other hand $F_i' \otimes_{R'} R'/I' = F_i'/I'F_i'$ and similarly for F_2''. Thus the complex $F_2'/I'F_2' \oplus F_2''/I'F_2'' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\text{Tor}_1^{R'}(M', R'/I')$. Since the vertical map on complexes

$$\begin{tikzcd}
F_2'/I'F_2' \arrow[r] \arrow[d] & F_1'/I'F_1' \arrow[r] \arrow[d] & F_0'/I'F_0' \arrow[d] \\
F_2'/I'F_2' \oplus F_2''/I'F_2'' \arrow[r] & F_1'/I'F_1' \arrow[r] & F_0'/I'F_0'
\end{tikzcd}$$

clearly induces a surjection on cohomology we win. □

00MO Lemma 10.99.14. Let

$$\begin{tikzcd}
S \arrow[r] & S' \\
R \arrow[r] \arrow[u] & R' \arrow[u]
\end{tikzcd}$$

be a commutative diagram of local homomorphisms of local Noetherian rings. Let $I \subset R$ be a proper ideal. Let M be a finite S-module. Denote $I' = IR'$ and $M' = M \otimes_S S'$. Assume that

(1) S' is a localization of the tensor product $S \otimes_R R'$,
(2) M/IM is flat over R/I,
(3) $\text{Tor}_1^R(M, R/I) \to \text{Tor}_1^{R'}(M', R'/I')$ is zero.

Then M' is flat over R'.

Proof. Since S' is a localization of $S \otimes_R R'$ we see that M' is a localization of $M \otimes_R R'$. Note that by Lemma 10.39.7 the module $M/IM \otimes_{R/I} R'/I' = M \otimes_R R'/I'(M \otimes_R R')$ is flat over R'/I'. Hence also $M'/I'M'$ is flat over R'/I' as the localization of a flat module is flat. By Lemma 10.99.10 it suffices to show that $\text{Tor}_1^{R'}(M', R'/I')$ is zero. Since M' is a localization of $M \otimes_R R'$, the last assumption implies that it suffices to show that $\text{Tor}_1^R(M, R/I) \otimes_R R' \to \text{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective.

By Lemma 10.99.13 we see that $\text{Tor}_1^R(M, R'/I') \to \text{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective. So now it suffices to show that $\text{Tor}_1^R(M, R/I) \otimes_R R' \to \text{Tor}_1^R(M, R'/I')$ is surjective. This follows from Lemma 10.99.12 by looking at the ring maps $R \to R/I \to R'/I'$ and the module M. $\square$

Please compare the lemma below to Lemma 10.101.8 (the case of a nilpotent ideal) and Lemma 10.128.8 (the case of finitely presented algebras).

00MP **Lemma 10.99.15** (Critère de platitude par fibres; Noetherian case). *Let R, S, S' be Noetherian local rings and let $R \to S \to S'$ be local ring homomorphisms. Let $\mathfrak{m} \subset R$ be the maximal ideal. Let M be an S'-module. Assume*

1. *The module M is finite over S'.*
2. *The module M is not zero.*
3. *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
4. *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. Set $I = \mathfrak{m}S \subset S$. Then we see that M/IM is a flat S/I-module because of (3). Since $\mathfrak{m} \otimes_R S' \to I \otimes_S S'$ is surjective we see that also $\mathfrak{m} \otimes_R M \to I \otimes_S M$ is surjective. Consider

$$\mathfrak{m} \otimes_R M \to I \otimes_S M \to M.$$

As M is flat over R the composition is injective and so both arrows are injective. In particular $\text{Tor}_1^S(S/I, M) = 0$ see Remark 10.75.9. By Lemma 10.99.10 we conclude that M is flat over S. Note that since $M/\mathfrak{m}_{S'}M$ is not zero by Nakayama's Lemma 10.20.1 we see that actually M is faithfully flat over S by Lemma 10.39.15 (since it forces $M/\mathfrak{m}_S M \neq 0$).

Consider the exact sequence $0 \to \mathfrak{m} \to R \to \kappa \to 0$. This gives an exact sequence $0 \to \text{Tor}_1^R(\kappa, S) \to \mathfrak{m} \otimes_R S \to I \to 0$. Since M is flat over S this gives an exact sequence $0 \to \text{Tor}_1^R(\kappa, S) \otimes_S M \to \mathfrak{m} \otimes_R M \to I \otimes_S M \to 0$. By the above this implies that $\text{Tor}_1^R(\kappa, S) \otimes_S M = 0$. Since M is faithfully flat over S this implies that $\text{Tor}_1^R(\kappa, S) = 0$ and we conclude that S is flat over R by Lemma 10.99.7. $\square$

10.100. Base change and flatness

051D Some lemmas which deal with what happens with flatness when doing a base change.

00MQ **Lemma 10.100.1.** *Let*

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd}$$

be a commutative diagram of local homomorphisms of local rings. Assume that S' is a localization of the tensor product $S \otimes_R R'$. Let M be an S-module and set $M' = S' \otimes_S M$.

(1) If M is flat over R then M' is flat over R'.
(2) If M' is flat over R' and $R \to R'$ is flat then M is flat over R.

In particular we have

(3) If S is flat over R then S' is flat over R'.
(4) If $R' \to S'$ and $R \to R'$ are flat then S is flat over R.

Proof. Proof of (1). If M is flat over R, then $M \otimes_R R'$ is flat over R' by Lemma 10.39.7. If $W \subset S \otimes_R R'$ is the multiplicative subset such that $W^{-1}(S \otimes_R R') = S'$ then $M' = W^{-1}(M \otimes_R R')$. Hence M' is flat over R' as the localization of a flat module, see Lemma 10.39.18 part (5). This proves (1) and in particular, we see that (3) holds.

Proof of (2). Suppose that M' is flat over R' and $R \to R'$ is flat. By (3) applied to the diagram reflected in the northwest diagonal we see that $S \to S'$ is flat. Thus $S \to S'$ is faithfully flat by Lemma 10.39.17. We are going to use the criterion of Lemma 10.39.5 (3) to show that M is flat. Let $I \subset R$ be an ideal. If $I \otimes_R M \to M$ has a kernel, so does $(I \otimes_R M) \otimes_S S' \to M \otimes_S S' = M'$. Note that $I \otimes_R R' = IR'$ as $R \to R'$ is flat, and that

$$(I \otimes_R M) \otimes_S S' = (I \otimes_R R') \otimes_{R'} (M \otimes_S S') = IR' \otimes_{R'} M'.$$

From flatness of M' over R' we conclude that this maps injectively into M'. This concludes the proof of (2), and hence (4) is true as well. ☐

Here is yet another application of the local criterion of flatness.

0GEB Lemma 10.100.2. Consider a commutative diagram of local rings and local homomorphisms

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd}$$

Let M be a finite S-module. Assume that

(1) the horizontal arrows are flat ring maps
(2) M is flat over R,
(3) $\mathfrak{m}_R R' = \mathfrak{m}_{R'}$,
(4) R' and S' are Noetherian.

Then $M' = M \otimes_S S'$ is flat over R'.

Proof. Since $\mathfrak{m}_R \subset R$ and $R \to R'$ is flat, we get $\mathfrak{m}_R \otimes_R R' = \mathfrak{m}_R R' = \mathfrak{m}_{R'}$ by assumption (3). Observe that M' is a finite S'-module which is flat over R by Lemma 10.39.9. Thus $\mathfrak{m}_R \otimes_R M' \to M'$ is injective. Then we get

$$\mathfrak{m}_R \otimes_R M' = \mathfrak{m}_R \otimes_R R' \otimes_{R'} M' = \mathfrak{m}_{R'} \otimes_{R'} M'$$

Thus $\mathfrak{m}_{R'} \otimes_{R'} M' \to M'$ is injective. This shows that $\text{Tor}_1^{R'}(\kappa_{R'}, M') = 0$ (Remark 10.75.9). Thus M' is flat over R' by Lemma 10.99.7. ☐

10.101. Flatness criteria over Artinian rings

051E We discuss some flatness criteria for modules over Artinian rings. Note that an Artinian local ring has a nilpotent maximal ideal so that the following two lemmas apply to Artinian local rings.

051F Lemma 10.101.1. Let $(R, \mathfrak{m})$ be a local ring with nilpotent maximal ideal $\mathfrak{m}$. Let M be a flat R-module. If A is a set and $x_\alpha \in M$, $\alpha \in A$ is a collection of elements of M, then the following are equivalent:

(1) $\{\overline{x}_\alpha\}_{\alpha \in A}$ forms a basis for the vector space $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, and
(2) $\{x_\alpha\}_{\alpha \in A}$ forms a basis for M over R.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). By Nakayama's Lemma 10.20.1 the elements x_α generate M. Then one gets a short exact sequence

$$0 \to K \to \bigoplus\nolimits_{\alpha \in A} R \to M \to 0$$

Tensoring with $R/\mathfrak{m}$ and using Lemma 10.39.12 we obtain $K/\mathfrak{m}K = 0$. By Nakayama's Lemma 10.20.1 we conclude $K = 0$. $\square$

051G Lemma 10.101.2. Let R be a local ring with nilpotent maximal ideal. Let M be an R-module. The following are equivalent

(1) M is flat over R,
(2) M is a free R-module, and
(3) M is a projective R-module.

Proof. Since any projective module is flat (as a direct summand of a free module) and every free module is projective, it suffices to prove that a flat module is free. Let M be a flat module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be elements such that $\overline{x_\alpha} \in M/\mathfrak{m}M$ forms a basis over the residue field of R. By Lemma 10.101.1 the x_α are a basis for M over R and we win. $\square$

051H Lemma 10.101.3. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be a collection of elements of M. Assume

(1) I is nilpotent,
(2) $\{\overline{x}_\alpha\}_{\alpha \in A}$ forms a basis for M/IM over R/I, and
(3) $\text{Tor}_1^R(R/I, M) = 0$.

Then M is free on $\{x_\alpha\}_{\alpha \in A}$ over R.

Proof. Let R, I, M, $\{x_\alpha\}_{\alpha \in A}$ be as in the lemma and satisfy assumptions (1), (2), and (3). By Nakayama's Lemma 10.20.1 the elements x_α generate M over R. The assumption $\text{Tor}_1^R(R/I, M) = 0$ implies that we have a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Let $\sum f_\alpha x_\alpha = 0$ be a relation in M. By choice of x_α we see that $f_\alpha \in I$. Hence we conclude that $\sum f_\alpha \otimes x_\alpha = 0$ in $I \otimes_R M$. The map $I \otimes_R M \to I/I^2 \otimes_{R/I} M/IM$ and the fact that $\{x_\alpha\}_{\alpha \in A}$ forms a basis for M/IM implies that $f_\alpha \in I^2$! Hence we conclude that there are no relations among the images of the x_α in M/I^2M. In other words, we see that M/I^2M is free with basis the images of the x_α. Using the map $I \otimes_R M \to I/I^3 \otimes_{R/I^2} M/I^2M$ we then conclude that $f_\alpha \in I^3$! And so on. Since $I^n = 0$ for some n by assumption (1) we win. $\square$

051I Lemma 10.101.4. Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume

(1) M/IM is flat over R/I, and
(2) $R' \otimes_R M$ is flat over R'.

Set $I_2 = \varphi^{-1}(\varphi(I^2)R')$. Then M/I_2M is flat over R/I_2.

Proof. We may replace R, M, and R' by R/I_2, M/I_2M, and $R'/\varphi(I)^2R'$. Then $I^2 = 0$ and φ is injective. By Lemma 10.99.8 and the fact that $I^2 = 0$ it suffices to prove that $\text{Tor}_1^R(R/I, M) = K = \text{Ker}(I \otimes_R M \to M)$ is zero. Set $M' = M \otimes_R R'$ and $I' = IR'$. By assumption the map $I' \otimes_{R'} M' \to M'$ is injective. Hence K maps to zero in

$$I' \otimes_{R'} M' = I' \otimes_R M = I' \otimes_{R/I} M/IM.$$

Then $I \to I'$ is an injective map of R/I-modules. Since M/IM is flat over R/I the map

$$I \otimes_{R/I} M/IM \longrightarrow I' \otimes_{R/I} M/IM$$

is injective. This implies that K is zero in $I \otimes_R M = I \otimes_{R/I} M/IM$ as desired. □

051J Lemma 10.101.5. Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume

(1) I is nilpotent,
(2) $R \to R'$ is injective,
(3) M/IM is flat over R/I, and
(4) $R' \otimes_R M$ is flat over R'.

Then M is flat over R.

Proof. Define inductively $I_1 = I$ and $I_{n+1} = \varphi^{-1}(\varphi(I_n)^2R')$ for $n \geq 1$. Note that by Lemma 10.101.4 we find that M/I_nM is flat over R/I_n for each $n \geq 1$. It is clear that $\varphi(I_n) \subset \varphi(I)^{2^n}R'$. Since I is nilpotent we see that $\varphi(I_n) = 0$ for some n. As φ is injective we conclude that $I_n = 0$ for some n and we win. □

Here is the local Artinian version of the local criterion for flatness.

051K Lemma 10.101.6. Let R be an Artinian local ring. Let M be an R-module. Let $I \subset R$ be a proper ideal. The following are equivalent

(1) M is flat over R, and
(2) M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$.

Proof. The implication (1) ⇒ (2) follows immediately from the definitions. Assume M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$. By Lemma 10.101.2 this implies that M/IM is free over R/I. Pick a set A and elements $x_\alpha \in M$ such that the images in M/IM form a basis. By Lemma 10.101.3 we conclude that M is free and in particular flat. □

It turns out that flatness descends along injective homomorphism whose source is an Artinian ring.

051L Lemma 10.101.7. Let $R \to S$ be a ring map. Let M be an R-module. Assume

(1) R is Artinian
(2) $R \to S$ is injective, and
(3) $M \otimes_R S$ is a flat S-module.

Then M is a flat R-module.

Proof. First proof: Let $I \subset R$ be the Jacobson radical of R. Then I is nilpotent and M/IM is flat over R/I as R/I is a product of fields, see Section 10.53. Hence M is flat by an application of Lemma 10.101.5.

Second proof: By Lemma 10.53.6 we may write $R = \prod R_i$ as a finite product of local Artinian rings. This induces similar product decompositions for both R and S. Hence we reduce to the case where R is local Artinian (details omitted).

Assume that $R \to S$, M are as in the lemma satisfying (1), (2), and (3) and in addition that R is local with maximal ideal $\mathfrak{m}$. Let A be a set and $x_\alpha \in A$ be elements such that $\overline{x}_\alpha$ forms a basis for $M/\mathfrak{m}M$ over $R/\mathfrak{m}$. By Nakayama's Lemma 10.20.1 we see that the elements x_α generate M as an R-module. Set $N = S \otimes_R M$ and $I = \mathfrak{m}S$. Then $\{1 \otimes x_\alpha\}_{\alpha \in A}$ is a family of elements of N which form a basis for N/IN. Moreover, since N is flat over S we have $\text{Tor}_1^S(S/I, N) = 0$. Thus we conclude from Lemma 10.101.3 that N is free on $\{1 \otimes x_\alpha\}_{\alpha \in A}$. The injectivity of $R \to S$ then guarantees that there cannot be a nontrivial relation among the x_α with coefficients in R. □

Please compare the lemma below to Lemma 10.99.15 (the case of Noetherian local rings), Lemma 10.128.8 (the case of finitely presented algebras), and Lemma 10.128.10 (the case of locally nilpotent ideals).

06A5 Lemma 10.101.8 (Critère de platitude par fibres: Nilpotent case). Let

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

be a commutative diagram in the category of rings. Let $I \subset R$ be a nilpotent ideal and M an S'-module. Assume

(1) The module M/IM is a flat S/IS-module.
(2) The module M is a flat R-module.

Then M is a flat S-module and $S_\mathfrak{q}$ is flat over R for every $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero.

Proof. As M is flat over R tensoring with the short exact sequence $0 \to I \to R \to R/I \to 0$ gives a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Note that $I \otimes_R M \to IS \otimes_S M$ is surjective. Combined with the above this means both maps in

$$I \otimes_R M \to IS \otimes_S M \to M$$

are injective. Hence $\text{Tor}_1^S(IS, M) = 0$ (see Remark 10.75.9) and we conclude that M is a flat S-module by Lemma 10.99.8. To finish we need to show that $S_\mathfrak{q}$ is flat over R for any prime $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero. This follows from Lemma 10.39.15 and 10.39.10. □

10.102. What makes a complex exact?

00MR Some of this material can be found in the paper [BE73] by Buchsbaum and Eisenbud.

00MS Situation 10.102.1. Here R is a ring, and we have a complex

$$0 \to R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} R^{n_0}$$

In other words we require $\varphi_i \circ \varphi_{i+1} = 0$ for $i = 1, \ldots, e-1$.

00MT Lemma 10.102.2. Suppose R is a ring. Let

$$\ldots \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots$$

be a complex of finite free R-modules. Suppose that for some i some matrix coefficient of the map φ_i is invertible. Then the displayed complex is isomorphic to the direct sum of a complex

$$\ldots \to R^{n_{i+2}} \xrightarrow{\varphi_{i+2}} R^{n_{i+1}} \to R^{n_i - 1} \to R^{n_{i-1}-1} \to R^{n_{i-2}} \xrightarrow{\varphi_{i-2}} R^{n_{i-3}} \to \ldots$$

and the complex $\ldots \to 0 \to R \to R \to 0 \to \ldots$ where the map $R \to R$ is the identity map.

Proof. The assumption means, after a change of basis of R^{n_i} and $R^{n_{i-1}}$ that the first basis vector of R^{n_i} is mapped via φ_i to the first basis vector of $R^{n_{i-1}}$. Let e_j denote the jth basis vector of R^{n_i} and f_k the kth basis vector of $R^{n_{i-1}}$. Write $\varphi_i(e_j) = \sum a_{jk} f_k$. So $a_{1k} = 0$ unless $k = 1$ and $a_{11} = 1$. Change basis on R^{n_i} again by setting $e'_j = e_j - a_{j1}e_1$ for $j > 1$. After this change of coordinates we have $a_{j1} = 0$ for $j > 1$. Note the image of $R^{n_{i+1}} \to R^{n_i}$ is contained in the subspace spanned by e_j, $j > 1$. Note also that $R^{n_{i-1}} \to R^{n_{i-2}}$ has to annihilate f_1 since it is in the image. These conditions and the shape of the matrix (a_{jk}) for φ_i imply the lemma. $\square$

In Situation 10.102.1 we say a complex of the form

$$0 \to \ldots \to 0 \to R \xrightarrow{1} R \to 0 \to \ldots \to 0$$

or of the form

$$0 \to \ldots \to 0 \to R$$

is trivial. More precisely, we say $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is trivial if either there exists an $e \geq i \geq 1$ with $n_i = n_{i-1} = 1$, $\varphi_i = \text{id}_R$, and $n_j = 0$ for $j \notin \{i, i-1\}$ or $n_0 = 1$ and $n_i = 0$ for $i > 0$. The lemma above clearly says that any finite complex of finite free modules over a local ring is up to direct sums with trivial complexes the same as a complex all of whose maps have all matrix coefficients in the maximal ideal.

00MY Lemma 10.102.3. In Situation 10.102.1. Suppose R is a local Noetherian ring with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m} \in \text{Ass}(R)$, in other words R has depth 0. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Then the complex is isomorphic to a direct sum of trivial complexes.

Proof. Pick $x \in R$, $x \neq 0$, with $\mathfrak{m}x = 0$. Let i be the biggest index such that $n_i > 0$. If $i = 0$, then the statement is true. If $i > 0$ denote f_1 the first basis vector of R^{n_i}. Since xf_1 is not mapped to zero by exactness of the complex we deduce that some matrix coefficient of the map $R^{n_i} \to R^{n_{i-1}}$ is not in $\mathfrak{m}$. Lemma 10.102.2 then allows us to decrease $n_e + \ldots + n_1$. Induction finishes the proof. $\square$

00MU Lemma 10.102.4. In Situation 10.102.1. Let R be a Artinian local ring. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Then the complex is isomorphic to a direct sum of trivial complexes.

Proof. This is a special case of Lemma 10.102.3 because an Artinian local ring has depth 0. □

Below we define the rank of a map of finite free modules. This is just one possible definition of rank. It is just the definition that works in this section; there are others that may be more convenient in other settings.

00MV Definition 10.102.5. Let R be a ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules.

(1) The rank of φ is the maximal r such that $\wedge^r\varphi : \wedge^r R^m \to \wedge^r R^n$ is nonzero.
(2) We let $I(\varphi) \subset R$ be the ideal generated by the $r \times r$ minors of the matrix of φ, where r is the rank as defined above.

The rank of $\varphi : R^m \to R^n$ is 0 if and only if $\varphi = 0$ and in this case $I(\varphi) = R$.

00MW Lemma 10.102.6. In Situation 10.102.1, suppose the complex is isomorphic to a direct sum of trivial complexes. Then we have

(1) the maps φ_i have rank $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i-1}n_{e-1} + (-1)^{e-i}n_e$,
(2) for all i, $1 \leq i \leq e-1$ we have $\text{rank}(\varphi_{i+1}) + \text{rank}(\varphi_i) = n_i$,
(3) each $I(\varphi_i) = R$.

Proof. We may assume the complex is the direct sum of trivial complexes. Then for each i we can split the standard basis elements of R^{n_i} into those that map to a basis element of $R^{n_{i-1}}$ and those that are mapped to zero (and these are mapped onto by basis elements of $R^{n_{i+1}}$ if $i > 0$). Using descending induction starting with $i = e$ it is easy to prove that there are r_{i+1}-basis elements of R^{n_i} which are mapped to zero and r_i which are mapped to basis elements of $R^{n_{i-1}}$. From this the result follows. □

00MZ Lemma 10.102.7. In Situation 10.102.1. Suppose R is a local ring with maximal ideal $\mathfrak{m}$. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Let $x \in \mathfrak{m}$ be a nonzerodivisor. The complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is exact at $(R/xR)^{n_e}, \ldots, (R/xR)^{n_2}$.

Proof. Denote $F_\bullet$ the complex with terms $F_i = R^{n_i}$ and differential given by φ_i. Then we have a short exact sequence of complexes

$$0 \to F_\bullet \xrightarrow{x} F_\bullet \to F_\bullet/xF_\bullet \to 0$$

Applying the snake lemma we get a long exact sequence

$$H_i(F_\bullet) \xrightarrow{x} H_i(F_\bullet) \to H_i(F_\bullet/xF_\bullet) \to H_{i-1}(F_\bullet) \xrightarrow{x} H_{i-1}(F_\bullet)$$

The lemma follows. □

00N0 Lemma 10.102.8 (Acyclicity lemma). Let R be a local Noetherian ring. Let $0 \to M_e \to M_{e-1} \to \ldots \to M_0$ be a complex of finite R-modules. Assume $\text{depth}(M_i) \geq i$. Let i be the largest index such that the complex is not exact at M_i. If $i > 0$ then $\text{Ker}(M_i \to M_{i-1})/\text{Im}(M_{i+1} \to M_i)$ has depth ≥ 1. [PS73, Lemma 1.8]

Proof. Let $H = \mathrm{Ker}(M_i \to M_{i-1})/\mathrm{Im}(M_{i+1} \to M_i)$ be the cohomology group in question. We may break the complex into short exact sequences $0 \to M_e \to M_{e-1} \to K_{e-2} \to 0$, $0 \to K_j \to M_j \to K_{j-1} \to 0$, for $i+2 \leq j \leq e-2$, $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$, $0 \to K_i \to M_i \to M_{i-1}$, and $0 \to B_i \to K_i \to H \to 0$. We proceed up through these complexes to prove the statements about depths, repeatedly using Lemma 10.72.6. First of all, since $\mathrm{depth}(M_e) \geq e$, and $\mathrm{depth}(M_{e-1}) \geq e-1$ we deduce that $\mathrm{depth}(K_{e-2}) \geq e-1$. At this point the sequences $0 \to K_j \to M_j \to K_{j-1} \to 0$ for $i+2 \leq j \leq e-2$ imply similarly that $\mathrm{depth}(K_{j-1}) \geq j$ for $i+2 \leq j \leq e-2$. The sequence $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$ then shows that $\mathrm{depth}(B_i) \geq i+1$. The sequence $0 \to K_i \to M_i \to M_{i-1}$ shows that $\mathrm{depth}(K_i) \geq 1$ since M_i has depth $\geq i \geq 1$ by assumption. The sequence $0 \to B_i \to K_i \to H \to 0$ then implies the result. □

00N1 Proposition 10.102.9. In Situation 10.102.1, suppose R is a local Noetherian ring. The following are equivalent [BE73, Corollary 1]

(1) $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$, and
(2) for all i, $1 \leq i \leq e$ the following two conditions are satisfied:
   (a) $\mathrm{rank}(\varphi_i) = r_i$ where $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i-1} n_{e-1} + (-1)^{e-i} n_e$,
   (b) $I(\varphi_i) = R$, or $I(\varphi_i)$ contains a regular sequence of length i.

Proof. If for some i some matrix coefficient of φ_i is not in $\mathfrak m$, then we apply Lemma 10.102.2. It is easy to see that the proposition for a complex and for the same complex with a trivial complex added to it are equivalent. Thus we may assume that all matrix entries of each φ_i are elements of the maximal ideal. We may also assume that $e \geq 1$.

Assume the complex is exact at $R^{n_e}, \ldots, R^{n_1}$. Let $\mathfrak q \in \mathrm{Ass}(R)$. Note that the ring $R_{\mathfrak q}$ has depth 0 and that the complex remains exact after localization at $\mathfrak q$. We apply Lemmas 10.102.3 and 10.102.6 to the localized complex over $R_{\mathfrak q}$. We conclude that $\varphi_{i,\mathfrak q}$ has rank r_i for all i. Since $R \to \bigoplus_{\mathfrak q \in \mathrm{Ass}(R)} R_{\mathfrak q}$ is injective (Lemma 10.63.19), we conclude that φ_i has rank r_i over R by the definition of rank as given in Definition 10.102.5. Therefore we see that $I(\varphi_i)_{\mathfrak q} = I(\varphi_{i,\mathfrak q})$ as the ranks do not change. Since all of the ideals $I(\varphi_i)_{\mathfrak q}$, $e \geq i \geq 1$ are equal to $R_{\mathfrak q}$ (by the lemmas referenced above) we conclude none of the ideals $I(\varphi_i)$ is contained in $\mathfrak q$. This implies that $I(\varphi_e)I(\varphi_{e-1})\ldots I(\varphi_1)$ is not contained in any of the associated primes of R. By Lemma 10.15.2 we may choose $x \in I(\varphi_e)I(\varphi_{e-1})\ldots I(\varphi_1)$, $x \notin \mathfrak q$ for all $\mathfrak q \in \mathrm{Ass}(R)$. Observe that x is a nonzerodivisor (Lemma 10.63.9). According to Lemma 10.102.7 the complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is exact at $(R/xR)^{n_e}, \ldots, (R/xR)^{n_2}$. By induction on e all the ideals $I(\varphi_i)/xR$ have a regular sequence of length $i-1$. This proves that $I(\varphi_i)$ contains a regular sequence of length i.

Assume (2)(a) and (2)(b) hold. We claim that for any prime $\mathfrak p \subset R$ conditions (2)(a) and (2)(b) hold for the complex $0 \to R_{\mathfrak p}^{n_e} \to R_{\mathfrak p}^{n_{e-1}} \to \ldots \to R_{\mathfrak p}^{n_0}$ with maps $\varphi_{i,\mathfrak p}$ over $R_{\mathfrak p}$. Namely, since $I(\varphi_i)$ contains a nonzero divisor, the image of $I(\varphi_i)$ in $R_{\mathfrak p}$ is nonzero. This implies that the rank of $\varphi_{i,\mathfrak p}$ is the same as the rank of φ_i: the rank as defined above of a matrix φ over a ring R can only drop when passing to an R-algebra R' and this happens if and only if $I(\varphi)$ maps to zero in R'. Thus (2)(a) holds. Having said this we know that $I(\varphi_{i,\mathfrak p}) = I(\varphi_i)_{\mathfrak p}$ and we see that (2)(b) is preserved under localization as well. By induction on the dimension of R we may assume the complex is exact when localized at any nonmaximal prime $\mathfrak{p}$ of R. Thus $\mathrm{Ker}(\varphi_i)/\mathrm{Im}(\varphi_{i+1})$ has support contained in $\{\mathfrak{m}\}$ and hence if nonzero has depth 0. As $I(\varphi_i) \subset \mathfrak{m}$ for all i because of what was said in the first paragraph of the proof, we see that (2)(b) implies $\mathrm{depth}(R) \geq e$. By Lemma 10.102.8 we see that the complex is exact at $R^{n_e}, \ldots, R^{n_1}$ concluding the proof. □

0GLM Remark 10.102.10. If in Proposition 10.102.9 the equivalent conditions (1) and (2) are satisfied, then there exists a j such that $I(\varphi_i) = R$ if and only if $i \geq j$. As in the proof of the proposition, it suffices to see this when all the matrices have coefficients in the maximal ideal $\mathfrak{m}$ of R. In this case we see that $I(\varphi_j) = R$ if and only if $\varphi_j = 0$. But if $\varphi_j = 0$, then we get arbitrarily long exact complexes $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_j} \to 0 \to 0 \to \ldots \to 0$ and hence by the proposition we see that $I(\varphi_i)$ for $i > j$ has to be R (since otherwise it is a proper ideal of a Noetherian local ring containing arbitrary long regular sequences which is impossible).

10.103. Cohen-Macaulay modules

00N2 Here we show that Cohen-Macaulay modules have good properties. We postpone using Ext groups to establish the connection with duality and so on.

00N3 Definition 10.103.1. Let R be a Noetherian local ring. Let M be a finite R-module. We say M is Cohen-Macaulay if $\dim(\mathrm{Supp}(M)) = \mathrm{depth}(M)$.

A first goal will be to establish Proposition 10.103.4. We do this by a (perhaps nonstandard) sequence of elementary lemmas involving almost none of the earlier results on depth. Let us introduce some notation.

Let R be a local Noetherian ring. Let M be a Cohen-Macaulay module, and let $f_1, \ldots, f_d$ be an M-regular sequence with $d = \dim(\mathrm{Supp}(M))$. We say that $g \in \mathfrak{m}$ is good with respect to $(M, f_1, \ldots, f_d)$ if for all $i = 0, 1, \ldots, d-1$ we have $\dim(\mathrm{Supp}(M) \cap V(g, f_1, \ldots, f_i)) = d - i - 1$. This is equivalent to the condition that $\dim(\mathrm{Supp}(M/(f_1, \ldots, f_i)M) \cap V(g)) = d - i - 1$ for $i = 0, 1, \ldots, d-1$.

00N4 Lemma 10.103.2. Notation and assumptions as above. If g is good with respect to $(M, f_1, \ldots, f_d)$, then (a) g is a nonzerodivisor on M, and (b) M/gM is Cohen-Macaulay with maximal regular sequence $f_1, \ldots, f_{d-1}$.

Proof. We prove the lemma by induction on d. If $d = 0$, then M is finite and there is no case to which the lemma applies. If $d = 1$, then we have to show that $g : M \to M$ is injective. The kernel K has support $\{\mathfrak{m}\}$ because by assumption $\dim \mathrm{Supp}(M) \cap V(g) = 0$. Hence K has finite length. Hence $f_1 : K \to K$ injective implies the length of the image is the length of K, and hence $f_1 K = K$, which by Nakayama's Lemma 10.20.1 implies $K = 0$. Also, $\dim \mathrm{Supp}(M/gM) = 0$ and so M/gM is Cohen-Macaulay of depth 0.

Assume $d > 1$. Observe that g is good for $(M/f_1M, f_2, \ldots, f_d)$, as is easily seen from the definition. By induction, we have that (a) g is a nonzerodivisor on M/f_1M and (b) $M/(g, f_1)M$ is Cohen-Macaulay with maximal regular sequence $f_2, \ldots, f_{d-1}$. By Lemma 10.68.4 we see that g, f_1 is an M-regular sequence. Hence g is a nonzerodivisor on M and $f_1, \ldots, f_{d-1}$ is an M/gM-regular sequence. □

00N5 Lemma 10.103.3. Let R be a Noetherian local ring. Let M be a Cohen-Macaulay module over R. Suppose $g \in \mathfrak{m}$ is such that $\dim(\mathrm{Supp}(M) \cap V(g)) = \dim(\mathrm{Supp}(M)) -$

1. Then (a) g is a nonzerodivisor on M, and (b) M/gM is Cohen-Macaulay of depth one less.

Proof. Choose a M-regular sequence $f_1, \ldots, f_d$ with $d = \dim(\text{Supp}(M))$. If g is good with respect to $(M, f_1, \ldots, f_d)$ we win by Lemma 10.103.2. In particular the lemma holds if $d = 1$. (The case $d = 0$ does not occur.) Assume $d > 1$. Choose an element $h \in R$ such that (i) h is good with respect to $(M, f_1, \ldots, f_d)$, and (ii) $\dim(\text{Supp}(M) \cap V(h, g)) = d - 2$. To see h exists, let $\{\mathfrak{q}_j\}$ be the (finite) set of minimal primes of the closed sets $\text{Supp}(M)$, $\text{Supp}(M) \cap V(f_1, \ldots, f_i)$, $i = 1, \ldots, d-1$, and $\text{Supp}(M) \cap V(g)$. None of these $\mathfrak{q}_j$ is equal to $\mathfrak{m}$ and hence we may find $h \in \mathfrak{m}$, $h \notin \mathfrak{q}_j$ by Lemma 10.15.2. It is clear that h satisfies (i) and (ii). From Lemma 10.103.2 we conclude that M/hM is Cohen-Macaulay. By (ii) we see that the pair $(M/hM, g)$ satisfies the induction hypothesis. Hence $M/(h,g)M$ is Cohen-Macaulay and $g : M/hM \to M/hM$ is injective. By Lemma 10.68.4 we see that $g : M \to M$ and $h : M/gM \to M/gM$ are injective. Combined with the fact that $M/(g,h)M$ is Cohen-Macaulay this finishes the proof. □

00N6 Proposition 10.103.4. Let R be a Noetherian local ring, with maximal ideal $\mathfrak{m}$. Let M be a Cohen-Macaulay module over R whose support has dimension d. Suppose that $g_1, \ldots, g_c$ are elements of $\mathfrak{m}$ such that $\dim(\text{Supp}(M/(g_1, \ldots, g_c)M)) = d - c$. Then $g_1, \ldots, g_c$ is an M-regular sequence, and can be extended to a maximal M-regular sequence.

Proof. Let $Z = \text{Supp}(M) \subset \text{Spec}(R)$. By Lemma 10.60.13 in the chain $Z \supset Z \cap V(g_1) \supset \ldots \supset Z \cap V(g_1, \ldots, g_c)$ each step decreases the dimension at most by 1. Hence by assumption each step decreases the dimension by exactly 1 each time. Thus we may successively apply Lemma 10.103.3 to the modules $M/(g_1, \ldots, g_i)$ and the element g_{i+1}.

To extend $g_1, \ldots, g_c$ by one element if $c < d$ we simply choose an element $g_{c+1} \in \mathfrak{m}$ which is not in any of the finitely many minimal primes of $Z \cap V(g_1, \ldots, g_c)$, using Lemma 10.15.2. □

Having proved Proposition 10.103.4 we continue the development of standard theory.

0C6G Lemma 10.103.5. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. Let $x \in \mathfrak{m}$ be a nonzerodivisor on M. Then M is Cohen-Macaulay if and only if M/xM is Cohen-Macaulay.

Proof. By Lemma 10.72.7 we have $\text{depth}(M/xM) = \text{depth}(M) - 1$. By Lemma 10.63.10 we have $\dim(\text{Supp}(M/xM)) = \dim(\text{Supp}(M)) - 1$. □

0AAD Lemma 10.103.6. Let $R \to S$ be a surjective homomorphism of Noetherian local rings. Let N be a finite S-module. Then N is Cohen-Macaulay as an S-module if and only if N is Cohen-Macaulay as an R-module.

Proof. Omitted. □

0BUS Lemma 10.103.7. Let R be a Noetherian local ring. Let M be a finite Cohen-Macaulay R-module. If $\mathfrak{p} \in \text{Ass}(M)$, then $\dim(R/\mathfrak{p}) = \dim(\text{Supp}(M))$ and $\mathfrak{p}$ is a minimal prime in the support of M. In particular, M has no embedded associated primes. [DG67, Chapter 0, Proposition 16.5.4]

Proof. By Lemma 10.72.9 we have $\text{depth}(M) \leq \dim(R/\mathfrak{p})$. Of course $\dim(R/\mathfrak{p}) \leq \dim(\text{Supp}(M))$ as $\mathfrak{p} \in \text{Supp}(M)$ (Lemma 10.63.2). Thus we have equality in both inequalities as M is Cohen-Macaulay. Then $\mathfrak{p}$ must be minimal in $\text{Supp}(M)$ otherwise we would have $\dim(R/\mathfrak{p}) < \dim(\text{Supp}(M))$. Finally, minimal primes in the support of M are equal to the minimal elements of $\text{Ass}(M)$ (Proposition 10.63.6) hence M has no embedded associated primes (Definition 10.67.1). □

00NF Definition 10.103.8. Let R be a Noetherian local ring. A finite module M over R is called a maximal Cohen-Macaulay module if $\text{depth}(M) = \dim(R)$.

In other words, a maximal Cohen-Macaulay module over a Noetherian local ring is a finite module with the largest possible depth over that ring. Equivalently, a maximal Cohen-Macaulay module over a Noetherian local ring R is a Cohen-Macaulay module of dimension equal to the dimension of the ring. In particular, if M is a Cohen-Macaulay R-module with $\text{Spec}(R) = \text{Supp}(M)$, then M is maximal Cohen-Macaulay. Thus the following two lemmas are on maximal Cohen-Macaulay modules.

0AAE Lemma 10.103.9. Let R be a Noetherian local ring. Assume there exists a Cohen-Macaulay module M with $\text{Spec}(R) = \text{Supp}(M)$. Then any maximal chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ has length $n = \dim(R)$.

Proof. We will prove this by induction on $\dim(R)$. If $\dim(R) = 0$, then the statement is clear. Assume $\dim(R) > 0$. Then $n > 0$. Choose an element $x \in \mathfrak{p}_1$, with x not in any of the minimal primes of R, and in particular $x \notin \mathfrak{p}_0$. (See Lemma 10.15.2.) Then $\dim(R/xR) = \dim(R) - 1$ by Lemma 10.60.13. The module M/xM is Cohen-Macaulay over R/xR by Proposition 10.103.4 and Lemma 10.103.6. The support of M/xM is $\text{Spec}(R/xR)$ by Lemma 10.40.9. After replacing x by x^n for some n, we may assume that $\mathfrak{p}_1$ is an associated prime of M/xM, see Lemma 10.72.8. By Lemma 10.103.7 we conclude that $\mathfrak{p}_1/(x)$ is a minimal prime of R/xR. It follows that the chain $\mathfrak{p}_1/(x) \subset \ldots \subset \mathfrak{p}_n/(x)$ is a maximal chain of primes in R/xR. By induction we find that this chain has length $\dim(R/xR) = \dim(R) - 1$ as desired. □

0AAF Lemma 10.103.10. Suppose R is a Noetherian local ring. Assume there exists a Cohen-Macaulay module M with $\text{Spec}(R) = \text{Supp}(M)$. Then for a prime $\mathfrak{p} \subset R$ we have

$$\dim(R) = \dim(R_{\mathfrak{p}}) + \dim(R/\mathfrak{p}).$$

Proof. Follows immediately from Lemma 10.103.9. □

0AAG Lemma 10.103.11. Suppose R is a Noetherian local ring. Let M be a Cohen-Macaulay module over R. For any prime $\mathfrak{p} \subset R$ the module $M_{\mathfrak{p}}$ is Cohen-Macaulay over $R_{\mathfrak{p}}$.

Proof. We may and do assume $\mathfrak{p} \neq \mathfrak{m}$ and M not zero. Choose a maximal chain of primes $\mathfrak{p} = \mathfrak{p}_c \subset \mathfrak{p}_{c-1} \subset \ldots \subset \mathfrak{p}_1 \subset \mathfrak{m}$. If we prove the result for $M_{\mathfrak{p}_1}$ over $R_{\mathfrak{p}_1}$, then the lemma will follow by induction on c. Thus we may assume that there is no prime strictly between $\mathfrak{p}$ and $\mathfrak{m}$. Note that $\dim(\text{Supp}(M_{\mathfrak{p}})) \leq \dim(\text{Supp}(M)) - 1$ because any chain of primes in the support of $M_{\mathfrak{p}}$ can be extended by one more prime (namely $\mathfrak{m}$) in the support of M. On the other hand, we have $\text{depth}(M_{\mathfrak{p}}) \geq \text{depth}(M) - \dim(R/\mathfrak{p}) = \text{depth}(M) - 1$ by Lemma 10.72.10 and our choice of

$\mathfrak{p}$. Thus $\text{depth}(M_\mathfrak{p}) \geq \dim(\text{Supp}(M_\mathfrak{p}))$ as desired (the other inequality is Lemma 10.72.3). □

0AAH Definition 10.103.12. Let R be a Noetherian ring. Let M be a finite R-module. We say M is Cohen-Macaulay if $M_\mathfrak{p}$ is a Cohen-Macaulay module over $R_\mathfrak{p}$ for all primes $\mathfrak{p}$ of R.

By Lemma 10.103.11 it suffices to check this in the maximal ideals of R.

0AAI Lemma 10.103.13. Let R be a Noetherian ring. Let M be a Cohen-Macaulay module over R. Then $M \otimes_R R[x_1, \ldots, x_n]$ is a Cohen-Macaulay module over $R[x_1, \ldots, x_n]$.

Proof. By induction on the number of variables it suffices to prove this for $M[x] = M \otimes_R R[x]$ over $R[x]$. Let $\mathfrak{m} \subset R[x]$ be a maximal ideal, and let $\mathfrak{p} = R \cap \mathfrak{m}$. Let $f_1, \ldots, f_d$ be a $M_\mathfrak{p}$-regular sequence in the maximal ideal of $R_\mathfrak{p}$ of length $d = \dim(\text{Supp}(M_\mathfrak{p}))$. Note that since $R[x]$ is flat over R the localization $R[x]_\mathfrak{m}$ is flat over $R_\mathfrak{p}$. Hence, by Lemma 10.68.5, the sequence $f_1, \ldots, f_d$ is a $M[x]_\mathfrak{m}$-regular sequence of length d in $R[x]_\mathfrak{m}$. The quotient

$$Q = M[x]_\mathfrak{m}/(f_1, \ldots, f_d)M[x]_\mathfrak{m} = M_\mathfrak{p}/(f_1, \ldots, f_d)M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R[x]_\mathfrak{m}$$

has support equal to the primes lying over $\mathfrak{p}$ because $R_\mathfrak{p} \to R[x]_\mathfrak{m}$ is flat and the support of $M_\mathfrak{p}/(f_1, \ldots, f_d)M_\mathfrak{p}$ is equal to $\{\mathfrak{p}\}$ (details omitted; hint: follows from Lemmas 10.40.4 and 10.40.5). Hence the dimension is 1. To finish the proof it suffices to find an $f \in \mathfrak{m}$ which is a nonzerodivisor on Q. Since $\mathfrak{m}$ is a maximal ideal, the field extension $\kappa(\mathfrak{m})/\kappa(\mathfrak{p})$ is finite (Theorem 10.34.1). Hence we can find $f \in \mathfrak{m}$ which viewed as a polynomial in x has leading coefficient not in $\mathfrak{p}$. Such an f acts as a nonzerodivisor on

$$M_\mathfrak{p}/(f_1, \ldots, f_d)M_\mathfrak{p} \otimes_R R[x] = \bigoplus\nolimits_{n \geq 0} M_\mathfrak{p}/(f_1, \ldots, f_d)M_\mathfrak{p} \cdot x^n$$

and hence acts as a nonzerodivisor on Q. □

10.104. Cohen-Macaulay rings

00N7 Most of the results of this section are special cases of the results in Section 10.103.

00N8 Definition 10.104.1. A Noetherian local ring R is called Cohen-Macaulay if it is Cohen-Macaulay as a module over itself.

Note that this is equivalent to requiring the existence of a R-regular sequence $x_1, \ldots, x_d$ of the maximal ideal such that $R/(x_1, \ldots, x_d)$ has dimension 0. We will usually just say "regular sequence" and not "R-regular sequence".

02JN Lemma 10.104.2. Let R be a Noetherian local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$. Let $x_1, \ldots, x_c \in \mathfrak{m}$ be elements. Then

$$x_1, \ldots, x_c \text{ is a regular sequence } \Leftrightarrow \dim(R/(x_1, \ldots, x_c)) = \dim(R) - c$$

If so $x_1, \ldots, x_c$ can be extended to a regular sequence of length $\dim(R)$ and each quotient $R/(x_1, \ldots, x_i)$ is a Cohen-Macaulay ring of dimension $\dim(R) - i$.

Proof. Special case of Proposition 10.103.4. □

00N9 Lemma 10.104.3. Let R be Noetherian local. Suppose R is Cohen-Macaulay of dimension d. Any maximal chain of ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ has length $n = d$.

Proof. Special case of Lemma 10.103.9. □

00NA Lemma 10.104.4. Suppose R is a Noetherian local Cohen-Macaulay ring of dimension d. For any prime $\mathfrak p \subset R$ we have

$$\dim(R) = \dim(R_{\mathfrak p}) + \dim(R/\mathfrak p).$$

Proof. Follows immediately from Lemma 10.104.3. (Also, this is a special case of Lemma 10.103.10.) □

00NB Lemma 10.104.5. Suppose R is a Cohen-Macaulay local ring. For any prime $\mathfrak p \subset R$ the ring $R_{\mathfrak p}$ is Cohen-Macaulay as well.

Proof. Special case of Lemma 10.103.11. □

00NC Definition 10.104.6. A Noetherian ring R is called Cohen-Macaulay if all its local rings are Cohen-Macaulay.

00ND Lemma 10.104.7. Suppose R is a Noetherian Cohen-Macaulay ring. Any polynomial algebra over R is Cohen-Macaulay.

Proof. Special case of Lemma 10.103.13. □

00NE Lemma 10.104.8. Let R be a Noetherian local Cohen-Macaulay ring of dimension d. Let $0 \to K \to R^{\oplus n} \to M \to 0$ be an exact sequence of R-modules. Then either $M = 0$, or $\text{depth}(K) > \text{depth}(M)$, or $\text{depth}(K) = \text{depth}(M) = d$.

Proof. This is a special case of Lemma 10.72.6. □

00NG Lemma 10.104.9. Let R be a local Noetherian Cohen-Macaulay ring of dimension d. Let M be a finite R-module of depth e. There exists an exact complex

$$0 \to K \to F_{d-e-1} \to \ldots \to F_0 \to M \to 0$$

with each F_i finite free and K maximal Cohen-Macaulay.

Proof. Immediate from the definition and Lemma 10.104.8. □

06LC Lemma 10.104.10. Let $\varphi : A \to B$ be a map of local rings. Assume that B is Noetherian and Cohen-Macaulay and that $\mathfrak m_B = \sqrt{\varphi(\mathfrak m_A)B}$. Then there exists a sequence of elements $f_1, \ldots, f_{\dim(B)}$ in A such that $\varphi(f_1), \ldots, \varphi(f_{\dim(B)})$ is a regular sequence in B.

Proof. By induction on $\dim(B)$ it suffices to prove: If $\dim(B) \geq 1$, then we can find an element f of A which maps to a nonzerodivisor in B. By Lemma 10.104.2 it suffices to find $f \in A$ whose image in B is not contained in any of the finitely many minimal primes $\mathfrak q_1, \ldots, \mathfrak q_r$ of B. By the assumption that $\mathfrak m_B = \sqrt{\varphi(\mathfrak m_A)B}$ we see that $\mathfrak m_A \not\subset \varphi^{-1}(\mathfrak q_i)$. Hence we can find f by Lemma 10.15.2. □

10.105. Catenary rings

00NH Compare with Topology, Section 5.11.

00NI Definition 10.105.1. A ring R is said to be catenary if for any pair of prime ideals $\mathfrak p \subset \mathfrak q$, there exists an integer bounding the lengths of all finite chains of prime ideals $\mathfrak p = \mathfrak p_0 \subset \mathfrak p_1 \subset \ldots \subset \mathfrak p_e = \mathfrak q$ and all maximal such chains have the same length.

02IH Lemma 10.105.2. A ring R is catenary if and only if the topological space $\mathrm{Spec}(R)$ is catenary (see Topology, Definition 5.11.4).

Proof. Immediate from the definition and the characterization of irreducible closed subsets in Lemma 10.26.1. □

In general it is not the case that a finitely generated R-algebra is catenary if R is. Thus we make the following definition.

00NL Definition 10.105.3. A Noetherian ring R is said to be universally catenary if every R algebra of finite type is catenary.

We restrict to Noetherian rings as it is not clear this definition is the right one for non-Noetherian rings. By Lemma 10.105.7 to check a Noetherian ring R is universally catenary, it suffices to check each polynomial algebra $R[x_1, \ldots, x_n]$ is catenary.

00NJ Lemma 10.105.4. Any localization of a catenary ring is catenary. Any localization of a Noetherian universally catenary ring is universally catenary.

Proof. Let A be a ring and let $S \subset A$ be a multiplicative subset. The description of $\mathrm{Spec}(S^{-1}A)$ in Lemma 10.17.5 shows that if A is catenary, then so is $S^{-1}A$. If $S^{-1}A \to C$ is of finite type, then $C = S^{-1}B$ for some finite type ring map $A \to B$. Hence if A is Noetherian and universally catenary, then B is catenary and we see that C is catenary too. This proves the lemma. □

0ECE Lemma 10.105.5. Let A be a Noetherian universally catenary ring. Any A-algebra essentially of finite type over A is universally catenary.

Proof. If B is a finite type A-algebra, then B is Noetherian by Lemma 10.31.1. Any finite type B-algebra is a finite type A-algebra and hence catenary by our assumption that A is universally catenary. Thus B is universally catenary. Any localization of B is universally catenary by Lemma 10.105.4 and this finishes the proof. □

0AUN Lemma 10.105.6. Let R be a ring. The following are equivalent

(1) R is catenary,
(2) $R_{\mathfrak p}$ is catenary for all prime ideals $\mathfrak p$,
(3) $R_{\mathfrak m}$ is catenary for all maximal ideals $\mathfrak m$.

Assume R is Noetherian. The following are equivalent

(1) R is universally catenary,
(2) $R_{\mathfrak p}$ is universally catenary for all prime ideals $\mathfrak p$,
(3) $R_{\mathfrak m}$ is universally catenary for all maximal ideals $\mathfrak m$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 10.105.4 in both cases. The implication (2) $\Rightarrow$ (3) is immediate in both cases. Assume $R_{\mathfrak m}$ is catenary for all maximal ideals $\mathfrak m$ of R. If $\mathfrak p \subset \mathfrak q$ are primes in R, then choose a maximal ideal $\mathfrak q \subset \mathfrak m$. Chains of primes ideals between $\mathfrak p$ and $\mathfrak q$ are in 1-to-1 correspondence with chains of prime ideals between $\mathfrak p R_{\mathfrak m}$ and $\mathfrak q R_{\mathfrak m}$ hence we see R is catenary. Assume R is Noetherian and $R_{\mathfrak m}$ is universally catenary for all maximal ideals $\mathfrak m$ of R. Let $R \to S$ be a finite type ring map. Let $\mathfrak q$ be a prime ideal of S lying over the prime $\mathfrak p \subset R$. Choose a maximal ideal $\mathfrak p \subset \mathfrak m$ in R. Then $R_{\mathfrak p}$ is a localization of $R_{\mathfrak m}$ hence universally catenary by Lemma 10.105.4. Then $S_{\mathfrak p}$ is catenary as a finite type ring over $R_{\mathfrak p}$. Hence $S_{\mathfrak q}$ is catenary as a localization. Thus S is catenary by the first case treated above. □

00NK Lemma 10.105.7. Any quotient of a catenary ring is catenary. Any quotient of a Noetherian universally catenary ring is universally catenary.

Proof. Let A be a ring and let $I \subset A$ be an ideal. The description of $\text{Spec}(A/I)$ in Lemma 10.17.7 shows that if A is catenary, then so is A/I. The second statement is a special case of Lemma 10.105.5. □

0AUP Lemma 10.105.8. Let R be a Noetherian ring.

(1) R is catenary if and only if $R/\mathfrak p$ is catenary for every minimal prime $\mathfrak p$.
(2) R is universally catenary if and only if $R/\mathfrak p$ is universally catenary for every minimal prime $\mathfrak p$.

Proof. If $\mathfrak a \subset \mathfrak b$ is an inclusion of primes of R, then we can find a minimal prime $\mathfrak p \subset \mathfrak a$ and the first assertion is clear. We omit the proof of the second. □

00NM Lemma 10.105.9. A Noetherian Cohen-Macaulay ring is universally catenary. More generally, if R is a Noetherian ring and M is a Cohen-Macaulay R-module with $\text{Supp}(M) = \text{Spec}(R)$, then R is universally catenary.

Proof. Since a polynomial algebra over R is Cohen-Macaulay, by Lemma 10.104.7, it suffices to show that a Cohen-Macaulay ring is catenary. Let R be Cohen-Macaulay and $\mathfrak p \subset \mathfrak q$ primes of R. By definition $R_{\mathfrak q}$ and $R_{\mathfrak p}$ are Cohen-Macaulay. Take a maximal chain of primes $\mathfrak p = \mathfrak p_0 \subset \mathfrak p_1 \subset \ldots \subset \mathfrak p_n = \mathfrak q$. Next choose a maximal chain of primes $\mathfrak q_0 \subset \mathfrak q_1 \subset \ldots \subset \mathfrak q_m = \mathfrak p$. By Lemma 10.104.3 we have $n + m = \dim(R_{\mathfrak q})$. And we have $m = \dim(R_{\mathfrak p})$ by the same lemma. Hence $n = \dim(R_{\mathfrak q}) - \dim(R_{\mathfrak p})$ is independent of choices.

To prove the more general statement, argue exactly as above but using Lemmas 10.103.13 and 10.103.9. □

0ECF Lemma 10.105.10. Let $(A, \mathfrak m)$ be a Noetherian local ring. The following are equivalent

(1) A is catenary, and
(2) $\mathfrak p \mapsto \dim(A/\mathfrak p)$ is a dimension function on $\text{Spec}(A)$.

Proof. If A is catenary, then $\text{Spec}(A)$ has a dimension function δ by Topology, Lemma 5.20.4 (and Lemma 10.105.2). We may assume $\delta(\mathfrak m) = 0$. Then we see that

$$\delta(\mathfrak p) = \text{codim}(V(\mathfrak m), V(\mathfrak p)) = \dim(A/\mathfrak p)$$

by Topology, Lemma 5.20.2. In this way we see that (1) implies (2). The reverse implication follows from Topology, Lemma 5.20.2 as well. □

10.106. Regular local rings

00NN It is not that easy to show that all prime localizations of a regular local ring are regular. In fact, quite a bit of the material developed so far is geared towards a proof of this fact. See Proposition 10.110.5, and trace back the references.

00NO Lemma 10.106.1. Let $(R, \mathfrak m, \kappa)$ be a regular local ring of dimension d. The graded ring $\bigoplus \mathfrak m^n/\mathfrak m^{n+1}$ is isomorphic to the graded polynomial algebra $\kappa[X_1, \ldots, X_d]$.

Proof. Let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$, see Definition 10.60.10. There is a surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$, which maps X_i to the class of x_i in $\mathfrak{m}/\mathfrak{m}^2$. Since $d(R) = d$ by Proposition 10.60.9 we know that the numerical polynomial $n \mapsto \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$ has degree $d - 1$. By Lemma 10.58.10 we conclude that the surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism. $\square$

00NP **Lemma 10.106.2.** Any regular local ring is a domain.

Proof. We will use that $\bigcap \mathfrak{m}^n = 0$ by Lemma 10.51.4. Let $f, g \in R$ such that $fg = 0$. Suppose that $f \in \mathfrak{m}^a$ and $g \in \mathfrak{m}^b$, with a, b maximal. Since $fg = 0 \in \mathfrak{m}^{a+b+1}$ we see from the result of Lemma 10.106.1 that either $f \in \mathfrak{m}^{a+1}$ or $g \in \mathfrak{m}^{b+1}$. Contradiction. $\square$

00NQ **Lemma 10.106.3.** Let R be a regular local ring and let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$. Then $x_1, \ldots, x_d$ is a regular sequence, and each $R/(x_1, \ldots, x_c)$ is a regular local ring of dimension $d - c$. In particular R is Cohen-Macaulay.

Proof. Note that R/x_1R is a Noetherian local ring of dimension $\geq d - 1$ by Lemma 10.60.13 with $x_2, \ldots, x_d$ generating the maximal ideal. Hence it is a regular local ring by definition. Since R is a domain by Lemma 10.106.2 x_1 is a nonzerodivisor. $\square$

00NR **Lemma 10.106.4.** Let R be a regular local ring. Let $I \subset R$ be an ideal such that R/I is a regular local ring as well. Then there exists a minimal set of generators $x_1, \ldots, x_d$ for the maximal ideal $\mathfrak{m}$ of R such that $I = (x_1, \ldots, x_c)$ for some $0 \leq c \leq d$.

Proof. Say $\dim(R) = d$ and $\dim(R/I) = d - c$. Denote $\overline{\mathfrak{m}} = \mathfrak{m}/I$ the maximal ideal of R/I. Let $\kappa = R/\mathfrak{m}$. We have

$$\dim_\kappa((I + \mathfrak{m}^2)/\mathfrak{m}^2) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) - \dim(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = d - (d - c) = c$$

by the definition of a regular local ring. Hence we can choose $x_1, \ldots, x_c \in I$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent and supplement with $x_{c+1}, \ldots, x_d$ to get a minimal system of generators of $\mathfrak{m}$. The induced map $R/(x_1, \ldots, x_c) \to R/I$ is a surjection between regular local rings of the same dimension (Lemma 10.106.3). It follows that the kernel is zero, i.e., $I = (x_1, \ldots, x_c)$. Namely, if not then we would have $\dim(R/I) < \dim(R/(x_1, \ldots, x_c))$ by Lemmas 10.106.2 and 10.60.13. $\square$

00NS **Lemma 10.106.5.** Let R be a Noetherian local ring. Let $x \in \mathfrak{m}$. Let M be a finite R-module such that x is a nonzerodivisor on M and M/xM is free over R/xR. Then M is free over R.

Proof. Let $m_1, \ldots, m_r$ be elements of M which map to a R/xR-basis of M/xM. By Nakayama's Lemma 10.20.1 $m_1, \ldots, m_r$ generate M. If $\sum a_i m_i = 0$ is a relation, then $a_i \in xR$ for all i. Hence $a_i = b_i x$ for some $b_i \in R$. Hence the kernel K of $R^r \to M$ satisfies $xK = K$ and hence is zero by Nakayama's lemma. $\square$

00NT **Lemma 10.106.6.** Let R be a regular local ring. Any maximal Cohen-Macaulay module over R is free.

Proof. Let M be a maximal Cohen-Macaulay module over R. Let $x \in \mathfrak{m}$ be part of a regular sequence generating $\mathfrak{m}$. Then x is a nonzerodivisor on M by Proposition 10.103.4, and M/xM is a maximal Cohen-Macaulay module over R/xR. By induction on $\dim(R)$ we see that M/xM is free. We win by Lemma 10.106.5. $\square$

00NU **Lemma 10.106.7.** Suppose R is a Noetherian local ring. Let $x \in \mathfrak{m}$ be a nonzerodivisor such that R/xR is a regular local ring. Then R is a regular local ring. More generally, if $x_1, \ldots, x_r$ is a regular sequence in R such that $R/(x_1, \ldots, x_r)$ is a regular local ring, then R is a regular local ring.

Proof. This is true because x together with the lifts of a system of minimal generators of the maximal ideal of R/xR will give $\dim(R)$ generators of $\mathfrak{m}$. Use Lemma 10.60.13. The last statement follows from the first and induction. $\square$

07DX **Lemma 10.106.8.** Let $(R_i, \varphi_{ii'})$ be a directed system of local rings whose transition maps are local ring maps. If each R_i is a regular local ring and $R = \text{colim}\, R_i$ is Noetherian, then R is a regular local ring.

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal; it is the colimit of the maximal ideal $\mathfrak{m}_i \subset R_i$. We prove the lemma by induction on $d = \dim \mathfrak{m}/\mathfrak{m}^2$. If $d = 0$, then $R = R/\mathfrak{m}$ is a field and R is a regular local ring. If $d > 0$ pick an $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$. For some i we can find an $x_i \in \mathfrak{m}_i$ mapping to x. Note that $R/xR = \text{colim}_{i' \geq i} R_{i'}/x_iR_{i'}$ is a Noetherian local ring. By Lemma 10.106.3 we see that $R_{i'}/x_iR_{i'}$ is a regular local ring. Hence by induction we see that R/xR is a regular local ring. Since each R_i is a domain (Lemma 10.106.1) we see that R is a domain. Hence x is a nonzerodivisor and we conclude that R is a regular local ring by Lemma 10.106.7. $\square$

10.107. Epimorphisms of rings

04VM In any category there is a notion of an epimorphism. Some of this material is taken from [Laz69] and [Maz68].

04VN **Lemma 10.107.1.** Let $R \to S$ be a ring map. The following are equivalent

(1) $R \to S$ is an epimorphism,
(2) the two ring maps $S \to S \otimes_R S$ are equal,
(3) either of the ring maps $S \to S \otimes_R S$ is an isomorphism, and
(4) the ring map $S \otimes_R S \to S$ is an isomorphism.

Proof. Omitted. $\square$

04VP **Lemma 10.107.2.** The composition of two epimorphisms of rings is an epimorphism.

Proof. Omitted. Hint: This is true in any category. $\square$

04VQ **Lemma 10.107.3.** If $R \to S$ is an epimorphism of rings and $R \to R'$ is any ring map, then $R' \to R' \otimes_R S$ is an epimorphism.

Proof. Omitted. Hint: True in any category with pushouts. $\square$

04VR **Lemma 10.107.4.** If $A \to B \to C$ are ring maps and $A \to C$ is an epimorphism, so is $B \to C$.

Proof. Omitted. Hint: This is true in any category. $\square$

This means in particular, that if $R \to S$ is an epimorphism with image $\overline{R} \subset S$, then $\overline{R} \to S$ is an epimorphism. Hence while proving results for epimorphisms we may often assume the map is injective. The following lemma means in particular that every localization is an epimorphism.

04VS Lemma 10.107.5. Let $R \to S$ be a ring map. The following are equivalent:

(1) $R \to S$ is an epimorphism, and
(2) $R_{\mathfrak p} \to S_{\mathfrak p}$ is an epimorphism for each prime $\mathfrak p$ of R.

Proof. Since $S_{\mathfrak p} = R_{\mathfrak p} \otimes_R S$ (see Lemma 10.12.15) we see that (1) implies (2) by Lemma 10.107.3. Conversely, assume that (2) holds. Let $a, b : S \to A$ be two ring maps from S to a ring A equalizing the map $R \to S$. By assumption we see that for every prime $\mathfrak p$ of R the induced maps $a_{\mathfrak p}, b_{\mathfrak p} : S_{\mathfrak p} \to A_{\mathfrak p}$ are the same. Hence $a = b$ as $A \subset \prod_{\mathfrak p} A_{\mathfrak p}$, see Lemma 10.23.1. $\square$

04VT Lemma 10.107.6. Let $R \to S$ be a ring map. The following are equivalent

(1) $R \to S$ is an epimorphism and finite, and
(2) $R \to S$ is surjective.

Proof. (This lemma seems to have been reproved many times in the literature, and has many different proofs.) It is clear that a surjective ring map is an epimorphism. Suppose that $R \to S$ is a finite ring map such that $S \otimes_R S \to S$ is an isomorphism. Our goal is to show that $R \to S$ is surjective. Assume S/R is not zero. The exact sequence $R \to S \to S/R \to 0$ leads to an exact sequence

$$R \otimes_R S \to S \otimes_R S \to S/R \otimes_R S \to 0.$$

Our assumption implies that the first arrow is an isomorphism, hence we conclude that $S/R \otimes_R S = 0$. Hence also $S/R \otimes_R S/R = 0$. By Lemma 10.5.4 there exists a surjection of R-modules $S/R \to R/I$ for some proper ideal $I \subset R$. Hence there exists a surjection $S/R \otimes_R S/R \to R/I \otimes_R R/I = R/I \neq 0$, contradiction. $\square$

04VU Lemma 10.107.7. A faithfully flat epimorphism is an isomorphism.

Proof. This is clear from Lemma 10.107.1 part (3) as the map $S \to S \otimes_R S$ is the map $R \to S$ tensored with S. $\square$

04VV Lemma 10.107.8. If $k \to S$ is an epimorphism and k is a field, then $S = k$ or $S = 0$.

Proof. This is clear from the result of Lemma 10.107.7 (as any nonzero algebra over k is faithfully flat), or by arguing directly that $R \to R \otimes_k R$ cannot be surjective unless $\dim_k(R) \leq 1$. $\square$

04VW Lemma 10.107.9. Let $R \to S$ be an epimorphism of rings. Then

(1) $\text{Spec}(S) \to \text{Spec}(R)$ is injective, and
(2) for $\mathfrak q \subset S$ lying over $\mathfrak p \subset R$ we have $\kappa(\mathfrak p) = \kappa(\mathfrak q)$.

Proof. Let $\mathfrak p$ be a prime of R. The fibre of the map is the spectrum of the fibre ring $S \otimes_R \kappa(\mathfrak p)$. By Lemma 10.107.3 the map $\kappa(\mathfrak p) \to S \otimes_R \kappa(\mathfrak p)$ is an epimorphism, and hence by Lemma 10.107.8 we have either $S \otimes_R \kappa(\mathfrak p) = 0$ or $S \otimes_R \kappa(\mathfrak p) = \kappa(\mathfrak p)$ which proves (1) and (2). $\square$

04VX Lemma 10.107.10. Let R be a ring. Let M, N be R-modules. Let $\{x_i\}_{i\in I}$ be a set of generators of M. Let $\{y_j\}_{j\in J}$ be a set of generators of N. Let $\{m_j\}_{j\in J}$ be a family of elements of M with $m_j = 0$ for all but finitely many j. Then

$$\sum\nolimits_{j\in J} m_j \otimes y_j = 0 \text{ in } M \otimes_R N$$

is equivalent to the following: There exist $a_{i,j} \in R$ with $a_{i,j} = 0$ for all but finitely many pairs (i, j) such that

$$m_j = \sum\nolimits_{i\in I} a_{i,j} x_i \quad \text{for all } j \in J,$$
$$0 = \sum\nolimits_{j\in J} a_{i,j} y_j \quad \text{for all } i \in I.$$

Proof. The sufficiency is immediate. Suppose that $\sum_{j\in J} m_j \otimes y_j = 0$. Consider the short exact sequence

$$0 \to K \to \bigoplus\nolimits_{j\in J} R \to N \to 0$$

where the jth basis vector of $\bigoplus_{j\in J} R$ maps to y_j. Tensor this with M to get the exact sequence

$$K \otimes_R M \to \bigoplus\nolimits_{j\in J} M \to N \otimes_R M \to 0.$$

The assumption implies that there exist elements $k_i \in K$ such that $\sum k_i \otimes x_i$ maps to the element $(m_j)_{j\in J}$ of the middle. Writing $k_i = (a_{i,j})_{j\in J}$ and we obtain what we want. $\square$

04VY Lemma 10.107.11. Let $\varphi : R \to S$ be a ring map. Let $g \in S$. The following are equivalent:

(1) $g \otimes 1 = 1 \otimes g$ in $S \otimes_R S$, and
(2) there exist $n \geq 0$ and elements $y_i, z_j \in S$ and $x_{i,j} \in R$ for $1 \leq i, j \leq n$ such that
  (a) $g = \sum_{i,j\leq n} x_{i,j} y_i z_j$,
  (b) for each j we have $\sum x_{i,j} y_i \in \varphi(R)$, and
  (c) for each i we have $\sum x_{i,j} z_j \in \varphi(R)$.

Proof. It is clear that (2) implies (1). Conversely, suppose that $g \otimes 1 = 1 \otimes g$. Choose generators $\{s_i\}_{i\in I}$ of S as an R-module with $0, 1 \in I$ and $s_0 = 1$ and $s_1 = g$. Apply Lemma 10.107.10 to the relation $g \otimes s_0 + (-1) \otimes s_1 = 0$. We see that there exist $a_{i,j} \in R$ such that $g = \sum_i a_{i,0} s_i$, $-1 = \sum_i a_{i,1} s_i$, and for $j \neq 0, 1$ we have $0 = \sum_i a_{i,j} s_i$, and moreover for all i we have $\sum_j a_{i,j} s_j = 0$. Then we have

$$\sum\nolimits_{i,j\neq 0} a_{i,j} s_i s_j = -g + a_{0,0}$$

and for each $j \neq 0$ we have $\sum_{i\neq 0} a_{i,j} s_i \in R$. This proves that $-g + a_{0,0}$ can be written as in (2). It follows that g can be written as in (2). Details omitted. Hint: Show that the set of elements of S which have an expression as in (2) form an R-subalgebra of S. $\square$

04VZ Remark 10.107.12. Let $R \to S$ be a ring map. Sometimes the set of elements $g \in S$ such that $g \otimes 1 = 1 \otimes g$ is called the epicenter of S. It is an R-algebra. By the construction of Lemma 10.107.11 we get for each g in the epicenter a matrix factorization

$$(g) = Y X Z$$

with $X \in \text{Mat}(n \times n, R)$, $Y \in \text{Mat}(1 \times n, S)$, and $Z \in \text{Mat}(n \times 1, S)$. Namely, let $x_{i,j}, y_i, z_j$ be as in part (2) of the lemma. Set $X = (x_{i,j})$, let y be the row vector whose entries are the y_i and let z be the column vector whose entries are the z_j. With this notation conditions (b) and (c) of Lemma 10.107.11 mean exactly that $YX \in \text{Mat}(1 \times n, R)$, $XZ \in \text{Mat}(n \times 1, R)$. It turns out to be very convenient to consider the triple of matrices (X, YX, XZ). Given $n \in \mathbf{N}$ and a triple (P, U, V) we say that (P, U, V) is a n-triple associated to g if there exists a matrix factorization as above such that $P = X$, $U = YX$ and $V = XZ$.

04W0 Lemma 10.107.13. Let $R \to S$ be an epimorphism of rings. Then the cardinality of S is at most the cardinality of R. In a formula: $|S| \leq |R|$.

Proof. The condition that $R \to S$ is an epimorphism means that each $g \in S$ satisfies $g \otimes 1 = 1 \otimes g$, see Lemma 10.107.1. We are going to use the notation introduced in Remark 10.107.12. Suppose that $g, g' \in S$ and suppose that (P, U, V) is an n-triple which is associated to both g and g'. Then we claim that $g = g'$. Namely, write $(P, U, V) = (X, YX, XZ)$ for a matrix factorization $(g) = YXZ$ of g and write $(P, U, V) = (X', Y'X', X'Z')$ for a matrix factorization $(g') = Y'X'Z'$ of g'. Then we see that

$$(g) = YXZ = UZ = Y'X'Z = Y'PZ = Y'XZ = Y'V = Y'X'Z' = (g')$$

and hence $g = g'$. This implies that the cardinality of S is bounded by the number of possible triples, which has cardinality at most $\sup_{n \in \mathbf{N}} |R|^n$. If R is infinite then this is at most $|R|$, see [Kun83, Ch. I, 10.13].

If R is a finite ring then the argument above only proves that S is at worst countable. In fact in this case R is Artinian and the map $R \to S$ is surjective. We omit the proof of this case. $\square$

08YS Lemma 10.107.14. Let $R \to S$ be an epimorphism of rings. Let N_1, N_2 be S-modules. Then $\text{Hom}_S(N_1, N_2) = \text{Hom}_R(N_1, N_2)$. In other words, the restriction functor $\text{Mod}_S \to \text{Mod}_R$ is fully faithful.

Proof. Let $\varphi : N_1 \to N_2$ be an R-linear map. For any $x \in N_1$ consider the map $S \otimes_R S \to N_2$ defined by the rule $g \otimes g' \mapsto g\varphi(g'x)$. Since both maps $S \to S \otimes_R S$ are isomorphisms (Lemma 10.107.1), we conclude that $g\varphi(g'x) = gg'\varphi(x) = \varphi(gg'x)$. Thus φ is S-linear. $\square$

10.108. Pure ideals

04PQ The material in this section is discussed in many papers, see for example [Laz67], [Bko70], and [DM83].

04PR Definition 10.108.1. Let R be a ring. We say that $I \subset R$ is pure if the quotient ring R/I is flat over R.

04PS Lemma 10.108.2. Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent:

(1) I is pure,
(2) for every ideal $J \subset R$ we have $J \cap I = IJ$,
(3) for every finitely generated ideal $J \subset R$ we have $J \cap I = JI$,
(4) for every $x \in R$ we have $(x) \cap I = xI$,
(5) for every $x \in I$ we have $x = yx$ for some $y \in I$,

(6) for every $x_1, \ldots, x_n \in I$ there exists a $y \in I$ such that $x_i = yx_i$ for all $i = 1, \ldots, n$,
(7) for every prime $\mathfrak{p}$ of R we have $IR_\mathfrak{p} = 0$ or $IR_\mathfrak{p} = R_\mathfrak{p}$,
(8) $\text{Supp}(I) = \text{Spec}(R) \setminus V(I)$,
(9) I is the kernel of the map $R \to (1 + I)^{-1}R$,
(10) $R/I \cong S^{-1}R$ as R-algebras for some multiplicative subset S of R, and
(11) $R/I \cong (1 + I)^{-1}R$ as R-algebras.

Proof. For any ideal J of R we have the short exact sequence $0 \to J \to R \to R/J \to 0$. Tensoring with R/I we get an exact sequence $J \otimes_R R/I \to R/I \to R/I + J \to 0$ and $J \otimes_R R/I = J/JI$. Thus the equivalence of (1), (2), and (3) follows from Lemma 10.39.5. Moreover, these imply (4).

The implication (4) $\Rightarrow$ (5) is trivial. Assume (5) and let $x_1, \ldots, x_n \in I$. Choose $y_i \in I$ such that $x_i = y_i x_i$. Let $y \in I$ be the element such that $1 - y = \prod_{i=1,\ldots,n}(1 - y_i)$. Then $x_i = yx_i$ for all $i = 1, \ldots, n$. Hence (6) holds, and it follows that (5) $\Leftrightarrow$ (6).

Assume (5). Let $x \in I$. Then $x = yx$ for some $y \in I$. Hence $x(1 - y) = 0$, which shows that x maps to zero in $(1 + I)^{-1}R$. Of course the kernel of the map $R \to (1+I)^{-1}R$ is always contained in I. Hence we see that (5) implies (9). Assume (9). Then for any $x \in I$ we see that $x(1 - y) = 0$ for some $y \in I$. In other words, $x = yx$. We conclude that (5) is equivalent to (9).

Assume (5). Let $\mathfrak{p}$ be a prime of R. If $\mathfrak{p} \not\in V(I)$, then $IR_\mathfrak{p} = R_\mathfrak{p}$. If $\mathfrak{p} \in V(I)$, in other words, if $I \subset \mathfrak{p}$, then $x \in I$ implies $x(1 - y) = 0$ for some $y \in I$, implies x maps to zero in $R_\mathfrak{p}$, i.e., $IR_\mathfrak{p} = 0$. Thus we see that (7) holds.

Assume (7). Then $(R/I)_\mathfrak{p}$ is either 0 or $R_\mathfrak{p}$ for any prime $\mathfrak{p}$ of R. Hence by Lemma 10.39.18 we see that (1) holds. At this point we see that all of (1) – (7) and (9) are equivalent.

As $IR_\mathfrak{p} = I_\mathfrak{p}$ we see that (7) implies (8). Finally, if (8) holds, then this means exactly that $I_\mathfrak{p}$ is the zero module if and only if $\mathfrak{p} \in V(I)$, which is clearly saying that (7) holds. Now (1) – (9) are equivalent.

Assume (1) – (9) hold. Then $R/I \subset (1 + I)^{-1}R$ by (9) and the map $R/I \to (1 + I)^{-1}R$ is also surjective by the description of localizations at primes afforded by (7). Hence (11) holds.

The implication (11) $\Rightarrow$ (10) is trivial. And (10) implies that (1) holds because a localization of R is flat over R, see Lemma 10.39.18. $\square$

04PT Lemma 10.108.3. Let R be a ring. If $I, J \subset R$ are pure ideals, then $V(I) = V(J)$ implies $I = J$.

Proof. For example, by property (7) of Lemma 10.108.2 we see that $I = \text{Ker}(R \to \prod_{\mathfrak{p} \in V(I)} R_\mathfrak{p})$ can be recovered from the closed subset associated to it. $\square$

04PU Lemma 10.108.4. Let R be a ring. The rule $I \mapsto V(I)$ determines a bijection

$$\{I \subset R \text{ pure}\} \leftrightarrow \{Z \subset \text{Spec}(R) \text{ closed and closed under generalizations}\}$$

Proof. Let I be a pure ideal. Then since $R \to R/I$ is flat, by going down generalizations lift along the map $\text{Spec}(R/I) \to \text{Spec}(R)$. Hence $V(I)$ is closed under generalizations. This shows that the map is well defined. By Lemma 10.108.3 the map is injective. Suppose that $Z \subset \text{Spec}(R)$ is closed and closed under generalizations.

Let $J \subset R$ be the radical ideal such that $Z = V(J)$. Let $I = \{x \in R : x \in xJ\}$. Note that I is an ideal: if $x, y \in I$ then there exist $f, g \in J$ such that $x = xf$ and $y = yg$. Then

$$x + y = (x + y)(f + g - fg)$$

Verification left to the reader. We claim that I is pure and that $V(I) = V(J)$. If the claim is true then the map of the lemma is surjective and the lemma holds.

Note that $I \subset J$, so that $V(J) \subset V(I)$. Let $I \subset \mathfrak{p}$ be a prime. Consider the multiplicative subset $S = (R \setminus \mathfrak{p})(1 + J)$. By definition of I and $I \subset \mathfrak{p}$ we see that $0 \notin S$. Hence we can find a prime $\mathfrak{q}$ of R which is disjoint from S, see Lemmas 10.9.4 and 10.17.5. Hence $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \cap (1 + J) = \emptyset$. This implies that $\mathfrak{q} + J$ is a proper ideal of R. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{q} + J$. Then we get $\mathfrak{m} \in V(J)$ and hence $\mathfrak{q} \in V(J) = Z$ as Z was assumed to be closed under generalization. This in turn implies $\mathfrak{p} \in V(J)$ as $\mathfrak{q} \subset \mathfrak{p}$. Thus we see that $V(I) = V(J)$.

Finally, since $V(I) = V(J)$ (and J radical) we see that $J = \sqrt{I}$. Pick $x \in I$, so that $x = xy$ for some $y \in J$ by definition. Then $x = xy = xy^2 = \ldots = xy^n$. Since $y^n \in I$ for some $n > 0$ we conclude that property (5) of Lemma 10.108.2 holds and we see that I is indeed pure. $\square$

05KK Lemma 10.108.5. Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent

(1) I is pure and finitely generated,
(2) I is generated by an idempotent,
(3) I is pure and $V(I)$ is open, and
(4) R/I is a projective R-module.

Proof. If (1) holds, then $I = I \cap I = I^2$ by Lemma 10.108.2. Hence I is generated by an idempotent by Lemma 10.21.5. Thus (1) $\Rightarrow$ (2). If (2) holds, then $I = (e)$ and $R = (1-e) \oplus (e)$ as an R-module hence R/I is flat and I is pure and $V(I) = D(1-e)$ is open. Thus (2) $\Rightarrow$ (1) + (3). Finally, assume (3). Then $V(I)$ is open and closed, hence $V(I) = D(1 - e)$ for some idempotent e of R, see Lemma 10.21.3. The ideal $J = (e)$ is a pure ideal such that $V(J) = V(I)$ hence $I = J$ by Lemma 10.108.3. In this way we see that (3) $\Rightarrow$ (2). By Lemma 10.78.2 we see that (4) is equivalent to the assertion that I is pure and R/I finitely presented. Moreover, R/I is finitely presented if and only if I is finitely generated, see Lemma 10.5.3. Hence (4) is equivalent to (1). $\square$

We can use the above to characterize those rings for which every finite flat module is finitely presented.

052U Lemma 10.108.6. Let R be a ring. The following are equivalent:

(1) every $Z \subset \mathrm{Spec}(R)$ which is closed and closed under generalizations is also open, and
(2) any finite flat R-module is finite locally free.

Proof. If any finite flat R-module is finite locally free then the support of R/I where I is a pure ideal is open. Hence the implication (2) $\Rightarrow$ (1) follows from Lemma 10.108.3.

For the converse assume that R satisfies (1). Let M be a finite flat R-module. The support $Z = \mathrm{Supp}(M)$ of M is closed, see Lemma 10.40.5. On the other hand, if

$\mathfrak{p} \subset \mathfrak{p}'$, then by Lemma 10.78.5 the module $M_{\mathfrak{p}'}$ is free, and $M_{\mathfrak{p}} = M_{\mathfrak{p}'} \otimes_{R_{\mathfrak{p}'}} R_{\mathfrak{p}}$ Hence $\mathfrak{p}' \in \text{Supp}(M) \Rightarrow \mathfrak{p} \in \text{Supp}(M)$, in other words, the support is closed under generalization. As R satisfies (1) we see that the support of M is open and closed. Suppose that M is generated by r elements $m_1, \ldots, m_r$. The modules $\wedge^i(M)$, $i = 1, \ldots, r$ are finite flat R-modules also, because $\wedge^i(M)_{\mathfrak{p}} = \wedge^i(M_{\mathfrak{p}})$ is free over $R_{\mathfrak{p}}$. Note that $\text{Supp}(\wedge^{i+1}(M)) \subset \text{Supp}(\wedge^i(M))$. Thus we see that there exists a decomposition

$$\text{Spec}(R) = U_0 \amalg U_1 \amalg \ldots \amalg U_r$$

by open and closed subsets such that the support of $\wedge^i(M)$ is $U_r \cup \ldots \cup U_i$ for all $i = 0, \ldots, r$. Let $\mathfrak{p}$ be a prime of R, and say $\mathfrak{p} \in U_i$. Note that $\wedge^i(M) \otimes_R \kappa(\mathfrak{p}) = \wedge^i(M \otimes_R \kappa(\mathfrak{p}))$. Hence, after possibly renumbering $m_1, \ldots, m_r$ we may assume that $m_1, \ldots, m_i$ generate $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's Lemma 10.20.1 we get a surjection

$$R_f^{\oplus i} \longrightarrow M_f, \quad (a_1, \ldots, a_i) \longmapsto \sum a_i m_i$$

for some $f \in R$, $f \notin \mathfrak{p}$. We may also assume that $D(f) \subset U_i$. This means that $\wedge^i(M_f) = \wedge^i(M)_f$ is a flat R_f module whose support is all of $\text{Spec}(R_f)$. By the above it is generated by a single element, namely $m_1 \wedge \ldots \wedge m_i$. Hence $\wedge^i(M)_f \cong R_f/J$ for some pure ideal $J \subset R_f$ with $V(J) = \text{Spec}(R_f)$. Clearly this means that $J = (0)$, see Lemma 10.108.3. Thus $m_1 \wedge \ldots \wedge m_i$ is a basis for $\wedge^i(M_f)$ and it follows that the displayed map is injective as well as surjective. This proves that M is finite locally free as desired. $\square$

10.109. Rings of finite global dimension

00O2 The following lemma is often used to compare different projective resolutions of a given module.

00O3 **Lemma 10.109.1** (Schanuel's lemma). *Let R be a ring. Let M be an R-module. Suppose that*

$$0 \to K \xrightarrow{c_1} P_1 \xrightarrow{p_1} M \to 0 \quad \text{and} \quad 0 \to L \xrightarrow{c_2} P_2 \xrightarrow{p_2} M \to 0$$

are two short exact sequences, with P_i projective. Then $K \oplus P_2 \cong L \oplus P_1$. More precisely, there exist a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \oplus P_2 \arrow[r, "{(c_1, \text{id})}"'] \arrow[d] & P_1 \oplus P_2 \arrow[r, "{(p_1, 0)}"'] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \oplus L \arrow[r, "{(\text{id}, c_2)}"] & P_1 \oplus P_2 \arrow[r, "{(0, p_2)}"] & M \arrow[r] & 0
\end{tikzcd}$$

whose vertical arrows are isomorphisms.

Proof. Consider the module N defined by the short exact sequence $0 \to N \to P_1 \oplus P_2 \to M \to 0$, where the last map is the sum of the two maps $P_i \to M$. It is easy to see that the projection $N \to P_1$ is surjective with kernel L, and that $N \to P_2$ is surjective with kernel K. Since P_i are projective we have $N \cong K \oplus P_2 \cong L \oplus P_1$. This proves the first statement.

To prove the second statement (and to reprove the first), choose $a : P_1 \to P_2$ and $b : P_2 \to P_1$ such that $p_1 = p_2 \circ a$ and $p_2 = p_1 \circ b$. This is possible because P_1 and

P_2 are projective. Then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \oplus P_2 \arrow[r, "(c_1, \mathrm{id})"'] & P_1 \oplus P_2 \arrow[r, "(p_1, 0)"'] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u] \arrow[d] & P_1 \oplus P_2 \arrow[r, "(p_1, p_2)"'] \arrow[u, "T"] \arrow[d, "S"] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \oplus L \arrow[r, "(\mathrm{id}, c_2)"] & P_1 \oplus P_2 \arrow[r, "(0, p_2)"] & M \arrow[r] & 0
\end{tikzcd}$$

with T and S given by the matrices

$$S = \begin{pmatrix} \mathrm{id} & 0 \\ a & \mathrm{id} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \mathrm{id} & b \\ 0 & \mathrm{id} \end{pmatrix}$$

Then S, T and the maps $N \to P_1 \oplus L$ and $N \to K \oplus P_2$ are isomorphisms as desired. ☐

00O4 Definition 10.109.2. Let R be a ring. Let M be an R-module. We say M has finite projective dimension if it has a finite length resolution by projective R-modules. The minimal length of such a resolution is called the projective dimension of M.

It is clear that the projective dimension of M is 0 if and only if M is a projective module. The following lemma explains to what extent the projective dimension is independent of the choice of a projective resolution.

00O5 Lemma 10.109.3. Let R be a ring. Suppose that M is an R-module of projective dimension d. Suppose that $F_e \to F_{e-1} \to \ldots \to F_0 \to M \to 0$ is exact with F_i projective and $e \geq d-1$. Then the kernel of $F_e \to F_{e-1}$ is projective (or the kernel of $F_0 \to M$ is projective in case $e = 0$).

Proof. We prove this by induction on d. If $d = 0$, then M is projective. In this case there is a splitting $F_0 = \mathrm{Ker}(F_0 \to M) \oplus M$, and hence $\mathrm{Ker}(F_0 \to M)$ is projective. This finishes the proof if $e = 0$, and if $e > 0$, then replacing M by $\mathrm{Ker}(F_0 \to M)$ we decrease e.

Next assume $d > 0$. Let $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ be a minimal length finite resolution with P_i projective. According to Schanuel's Lemma 10.109.1 we have $P_0 \oplus \mathrm{Ker}(F_0 \to M) \cong F_0 \oplus \mathrm{Ker}(P_0 \to M)$. This proves the case $d = 1$, $e = 0$, because then the right hand side is $F_0 \oplus P_1$ which is projective. Hence now we may assume $e > 0$. The module $F_0 \oplus \mathrm{Ker}(P_0 \to M)$ has the finite projective resolution

$$0 \to P_d \to P_{d-1} \to \ldots \to P_2 \to P_1 \oplus F_0 \to \mathrm{Ker}(P_0 \to M) \oplus F_0 \to 0$$

of length $d-1$. By induction applied to the exact sequence

$$F_e \to F_{e-1} \to \ldots \to F_2 \to P_0 \oplus F_1 \to P_0 \oplus \mathrm{Ker}(F_0 \to M) \to 0$$

of length $e-1$ we conclude $\mathrm{Ker}(F_e \to F_{e-1})$ is projective (if $e \geq 2$) or that $\mathrm{Ker}(F_1 \oplus P_0 \to F_0 \oplus P_0)$ is projective. This implies the lemma. ☐

0CXC Lemma 10.109.4. Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent

(1) M has projective dimension $\leq d$,

(2) there exists a resolution $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ with P_i projective,
(3) for some resolution $\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$ with P_i projective we have $\text{Ker}(P_{d-1} \to P_{d-2})$ is projective if $d \geq 2$, or $\text{Ker}(P_0 \to M)$ is projective if $d = 1$, or M is projective if $d = 0$,
(4) for any resolution $\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$ with P_i projective we have $\text{Ker}(P_{d-1} \to P_{d-2})$ is projective if $d \geq 2$, or $\text{Ker}(P_0 \to M)$ is projective if $d = 1$, or M is projective if $d = 0$.

Proof. The equivalence of (1) and (2) is the definition of projective dimension, see Definition 10.109.2. We have (2) $\Rightarrow$ (4) by Lemma 10.109.3. The implications (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are immediate. $\square$

0CXD Lemma 10.109.5. Let R be a local ring. Let M be an R-module. Let $d \geq 0$. The equivalent conditions (1) – (4) of Lemma 10.109.4 are also equivalent to

(5) there exists a resolution $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ with P_i free.

Proof. Follows from Lemma 10.109.4 and Theorem 10.85.4. $\square$

0CXE Lemma 10.109.6. Let R be a Noetherian ring. Let M be a finite R-module. Let $d \geq 0$. The equivalent conditions (1) – (4) of Lemma 10.109.4 are also equivalent to

(6) there exists a resolution $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ with P_i finite projective.

Proof. Choose a resolution $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$ with F_i finite free (Lemma 10.71.1). By Lemma 10.109.4 we see that $P_d = \text{Ker}(F_{d-1} \to F_{d-2})$ is projective at least if $d \geq 2$. Then P_d is a finite R-module as R is Noetherian and $P_d \subset F_{d-1}$ which is finite free. Whence $0 \to P_d \to F_{d-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ is the desired resolution. $\square$

0CXF Lemma 10.109.7. Let R be a local Noetherian ring. Let M be a finite R-module. Let $d \geq 0$. The equivalent conditions (1) – (4) of Lemma 10.109.4, condition (5) of Lemma 10.109.5, and condition (6) of Lemma 10.109.6 are also equivalent to

(7) there exists a resolution $0 \to F_d \to F_{d-1} \to \ldots \to F_0 \to M \to 0$ with F_i finite free.

Proof. This follows from Lemmas 10.109.4, 10.109.5, and 10.109.6 and because a finite projective module over a local ring is finite free, see Lemma 10.78.2. $\square$

065R Lemma 10.109.8. Let R be a ring. Let M be an R-module. Let $n \geq 0$. The following are equivalent

(1) M has projective dimension $\leq n$,
(2) $\text{Ext}^i_R(M, N) = 0$ for all R-modules N and all $i \geq n + 1$, and
(3) $\text{Ext}^{n+1}_R(M, N) = 0$ for all R-modules N.

Proof. Assume (1). Choose a free resolution $F_\bullet \to M$ of M. Denote $d_e : F_e \to F_{e-1}$. By Lemma 10.109.3 we see that $P_e = \text{Ker}(d_e)$ is projective for $e \geq n - 1$. This implies that $F_e \cong P_e \oplus P_{e-1}$ for $e \geq n$ where d_e maps the summand P_{e-1} isomorphically to P_{e-1} in F_{e-1}. Hence, for any R-module N the complex

$\mathrm{Hom}_R(F_\bullet, N)$ is split exact in degrees $\geq n+1$. Whence (2) holds. The implication (2) $\Rightarrow$ (3) is trivial.

Assume (3) holds. If $n = 0$ then M is projective by Lemma 10.77.2 and we see that (1) holds. If $n > 0$ choose a free R-module F and a surjection $F \to M$ with kernel K. By Lemma 10.71.7 and the vanishing of $\mathrm{Ext}^i_R(F, N)$ for all $i > 0$ by part (1) we see that $\mathrm{Ext}^n_R(K, N) = 0$ for all R-modules N. Hence by induction we see that K has projective dimension $\leq n-1$. Then M has projective dimension $\leq n$ as any finite projective resolution of K gives a projective resolution of length one more for M by adding F to the front. ☐

065S Lemma 10.109.9. Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules.

(1) If M has projective dimension $\leq n$ and M'' has projective dimension $\leq n+1$, then M' has projective dimension $\leq n$.
(2) If M' and M'' have projective dimension $\leq n$ then M has projective dimension $\leq n$.
(3) If M' has projective dimension $\leq n$ and M has projective dimension $\leq n+1$ then M'' has projective dimension $\leq n+1$.

Proof. Combine the characterization of projective dimension in Lemma 10.109.8 with the long exact sequence of ext groups in Lemma 10.71.7. ☐

00O6 Definition 10.109.10. Let R be a ring. The ring R is said to have finite global dimension if there exists an integer n such that every R-module has a resolution by projective R-modules of length at most n. The minimal such n is then called the global dimension of R.

The argument in the proof of the following lemma can be found in the paper [Aus55] by Auslander.

0D1U Lemma 10.109.11. Let R be a ring. Suppose we have a module $M = \bigcup_{e \in E} M_e$ where the M_e are submodules well-ordered by inclusion. Assume the quotients $M_e/\bigcup_{e'<e} M_{e'}$ have projective dimension $\leq n$. Then M has projective dimension $\leq n$.

Proof. We will prove this by induction on n.

Base case: $n = 0$. Then $P_e = M_e/\bigcup_{e'<e} M_{e'}$ is projective. Thus we may choose a section $P_e \to M_e$ of the projection $M_e \to P_e$. We claim that the induced map $\psi : \bigoplus_{e \in E} P_e \to M$ is an isomorphism. Namely, if $x = \sum x_e \in \bigoplus P_e$ is nonzero, then we let e_{max} be maximal such that $x_{e_{max}}$ is nonzero and we conclude that $y = \psi(x) = \psi(\sum x_e)$ is nonzero because $y \in M_{e_{max}}$ has nonzero image $x_{e_{max}}$ in $P_{e_{max}}$. On the other hand, let $y \in M$. Then $y \in M_e$ for some e. We show that $y \in \mathrm{Im}(\psi)$ by transfinite induction on e. Let $x_e \in P_e$ be the image of y. Then $y - \psi(x_e) \in \bigcup_{e'<e} M_{e'}$. By induction hypothesis we conclude that $y - \psi(x_e) \in \mathrm{Im}(\psi)$ hence $y \in \mathrm{Im}(\psi)$. Thus the claim is true and ψ is an isomorphism. We conclude that M is projective as a direct sum of projectives, see Lemma 10.77.4.

If $n > 0$, then for $e \in E$ we denote F_e the free R-module on the set of elements of M_e. Then we have a system of short exact sequences

$$0 \to K_e \to F_e \to M_e \to 0$$

over the well-ordered set E. Note that the transition maps $F_{e'} \to F_e$ and $K_{e'} \to K_e$ are injective too. Set $F = \bigcup F_e$ and $K = \bigcup K_e$. Then

$$0 \to K_e/\bigcup\nolimits_{e'<e} K_{e'} \to F_e/\bigcup\nolimits_{e'<e} F_{e'} \to M_e/\bigcup\nolimits_{e'<e} M_{e'} \to 0$$

is a short exact sequence of R-modules too and $F_e/\bigcup_{e'<e} F_{e'}$ is the free R-module on the set of elements in M_e which are not contained in $\bigcup_{e'<e} M_{e'}$. Hence by Lemma 10.109.9 we see that the projective dimension of $K_e/\bigcup_{e'<e} K_{e'}$ is at most $n-1$. By induction we conclude that K has projective dimension at most $n-1$. Whence M has projective dimension at most n and we win. $\square$

065T Lemma 10.109.12. Let R be a ring. The following are equivalent

(1) R has finite global dimension $\leq n$,
(2) every finite R-module has projective dimension $\leq n$, and
(3) every cyclic R-module R/I has projective dimension $\leq n$.

Proof. It is clear that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Assume (3). Choose a set $E \subset M$ of generators of M. Choose a well ordering on E. For $e \in E$ denote M_e the submodule of M generated by the elements $e' \in E$ with $e' \leq e$. Then $M = \bigcup_{e \in E} M_e$. Note that for each $e \in E$ the quotient

$$M_e/\bigcup\nolimits_{e'<e} M_{e'}$$

is either zero or generated by one element, hence has projective dimension $\leq n$ by (3). By Lemma 10.109.11 this means that M has projective dimension $\leq n$. $\square$

00O8 Lemma 10.109.13. Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset.

(1) If M has projective dimension $\leq n$, then $S^{-1}M$ has projective dimension $\leq n$ over $S^{-1}R$.
(2) If R has finite global dimension $\leq n$, then $S^{-1}R$ has finite global dimension $\leq n$.

Proof. Let $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ be a projective resolution. As localization is exact, see Proposition 10.9.12, and as each $S^{-1}P_i$ is a projective $S^{-1}R$-module, see Lemma 10.94.1, we see that $0 \to S^{-1}P_n \to \ldots \to S^{-1}P_0 \to S^{-1}M \to 0$ is a projective resolution of $S^{-1}M$. This proves (1). Let M' be an $S^{-1}R$-module. Note that $M' = S^{-1}M'$. Hence we see that (2) follows from (1). $\square$

10.110. Regular rings and global dimension

065U We can use the material on rings of finite global dimension to give another characterization of regular local rings.

00O7 Proposition 10.110.1. Let R be a regular local ring of dimension d. Every finite R-module M of depth e has a finite free resolution

$$0 \to F_{d-e} \to \ldots \to F_0 \to M \to 0.$$

In particular a regular local ring has global dimension $\leq d$.

Proof. The first part holds in view of Lemma 10.106.6 and Lemma 10.104.9. The last part follows from this and Lemma 10.109.12. $\square$

00O9 Lemma 10.110.2. Let R be a Noetherian ring. Then R has finite global dimension if and only if there exists an integer n such that for all maximal ideals $\mathfrak{m}$ of R the ring $R_\mathfrak{m}$ has global dimension $\leq n$.

Proof. We saw, Lemma 10.109.13 that if R has finite global dimension n, then all the localizations $R_\mathfrak{m}$ have finite global dimension at most n. Conversely, suppose that all the $R_\mathfrak{m}$ have global dimension $\leq n$. Let M be a finite R-module. Let $0 \to K_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$ be a resolution with F_i finite free. Then K_n is a finite R-module. According to Lemma 10.109.3 and the assumption all the modules $K_n \otimes_R R_\mathfrak{m}$ are projective. Hence by Lemma 10.78.2 the module K_n is finite projective. □

00OA Lemma 10.110.3. Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case the projective dimension of κ is $\geq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.

Proof. Let $x_1, \ldots, x_n$ be elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. Consider the Koszul complex on $x_1, \ldots, x_n$. This is the complex

$$0 \to \wedge^n R^n \to \wedge^{n-1} R^n \to \wedge^{n-2} R^n \to \ldots \to \wedge^i R^n \to \ldots \to R^n \to R$$

with maps given by

$$e_{j_1} \wedge \ldots \wedge e_{j_i} \longmapsto \sum_{a=1}^{i} (-1)^{a+1} x_{j_a} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_a} \wedge \ldots \wedge e_{j_i}$$

It is easy to see that this is a complex $K_\bullet(R, x_\bullet)$. Note that the cokernel of the last map of $K_\bullet(R, x_\bullet)$ is κ by Lemma 10.20.1 part (8).

If κ has finite projective dimension d, then we can find a resolution $F_\bullet \to \kappa$ by finite free R-modules of length d (Lemma 10.109.7). By Lemma 10.102.2 we may assume all the maps in the complex $F_\bullet$ have the property that $\mathrm{Im}(F_i \to F_{i-1}) \subset \mathfrak{m}F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. By Lemma 10.71.4 we can find a map of complexes $\alpha : K_\bullet(R, x_\bullet) \to F_\bullet$ inducing the identity on κ. We will prove by induction that the maps $\alpha_i : \wedge^i R^n = K_i(R, x_\bullet) \to F_i$ have the property that $\alpha_i \otimes \kappa : \wedge^i \kappa^n \to F_i \otimes \kappa$ are injective. This shows that $F_n \neq 0$ and hence $d \geq n$ as desired.

The result is clear for $i = 0$ because the composition $R \xrightarrow{\alpha_0} F_0 \to \kappa$ is nonzero. Note that F_0 must have rank 1 since otherwise the map $F_1 \to F_0$ whose cokernel is a single copy of κ cannot have image contained in $\mathfrak{m}F_0$.

Next we check the case $i = 1$ as we feel that it is instructive; the reader can skip this as the induction step will deduce the $i = 1$ case from the case $i = 0$. We saw above that $F_0 = R$ and $F_1 \to F_0 = R$ has image $\mathfrak{m}$. We have a commutative diagram

$$\begin{matrix} R^n & = & K_1(R, x_\bullet) & \to & K_0(R, x_\bullet) & = & R \\ & & \downarrow & & \downarrow & & \downarrow \\ & & F_1 & \to & F_0 & = & R \end{matrix}$$

where the rightmost vertical arrow is given by multiplication by a unit. Hence we see that the image of the composition $R^n \to F_1 \to F_0 = R$ is also equal to $\mathfrak{m}$. Thus the map $R^n \otimes \kappa \to F_1 \otimes \kappa$ has to be injective since $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = n$.

Let $i \geq 1$ and assume injectivity of $\alpha_j \otimes \kappa$ has been proved for all $j \leq i-1$. Consider the commutative diagram

$$\begin{array}{ccccccc} \wedge^i R^n & = & K_i(R, x_\bullet) & \to & K_{i-1}(R, x_\bullet) & = & \wedge^{i-1} R^n \\ & & \downarrow & & \downarrow & & \\ & & F_i & \to & F_{i-1} & & \end{array}$$

We know that $\wedge^{i-1}\kappa^n \to F_{i-1} \otimes \kappa$ is injective. This proves that $\wedge^{i-1}\kappa^n \otimes_\kappa \mathfrak{m}/\mathfrak{m}^2 \to F_{i-1} \otimes \mathfrak{m}/\mathfrak{m}^2$ is injective. Also, by our choice of the complex, F_i maps into $\mathfrak{m}F_{i-1}$, and similarly for the Koszul complex. Hence we get a commutative diagram

$$\begin{array}{ccc} \wedge^i \kappa^n & \to & \wedge^{i-1}\kappa^n \otimes \mathfrak{m}/\mathfrak{m}^2 \\ \downarrow & & \downarrow \\ F_i \otimes \kappa & \to & F_{i-1} \otimes \mathfrak{m}/\mathfrak{m}^2 \end{array}$$

At this point it suffices to verify the map $\wedge^i\kappa^n \to \wedge^{i-1}\kappa^n \otimes \mathfrak{m}/\mathfrak{m}^2$ is injective, which can be done by hand. □

00OB Lemma 10.110.4. Let R be a Noetherian local ring. Suppose that the residue field κ has finite projective dimension n over R. In this case $\dim(R) \geq n$.

Proof. Let $F_\bullet$ be a finite resolution of κ by finite free R-modules (Lemma 10.109.7). By Lemma 10.102.2 we may assume all the maps in the complex $F_\bullet$ have to property that $\text{Im}(F_i \to F_{i-1}) \subset \mathfrak{m}F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. Say $F_n \neq 0$ and $F_i = 0$ for $i > n$, so that the projective dimension of κ is n. By Proposition 10.102.9 we see that $\text{depth}_{I(\varphi_n)}(R) \geq n$ since $I(\varphi_n)$ cannot equal R by our choice of the complex. Thus by Lemma 10.72.3 also $\dim(R) \geq n$. □

00OC Proposition 10.110.5. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. The following are equivalent

(1) κ has finite projective dimension as an R-module,
(2) R has finite global dimension,
(3) R is a regular local ring.

Moreover, in this case the global dimension of R equals $\dim(R) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$.

Proof. We have (3) ⇒ (2) by Proposition 10.110.1. The implication (2) ⇒ (1) is trivial. Assume (1). By Lemmas 10.110.3 and 10.110.4 we see that $\dim(R) \geq \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Thus R is regular, see Definition 10.60.10 and the discussion preceding it. Assume the equivalent conditions (1) – (3) hold. By Proposition 10.110.1 the global dimension of R is at most $\dim(R)$ and by Lemma 10.110.3 it is at least $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Thus the stated equality holds. □

0AFS Lemma 10.110.6. A Noetherian local ring R is a regular local ring if and only if it has finite global dimension. In this case $R_\mathfrak{p}$ is a regular local ring for all primes $\mathfrak{p}$.

Proof. By Propositions 10.110.5 and 10.110.1 we see that a Noetherian local ring is a regular local ring if and only if it has finite global dimension. Furthermore, any localization $R_\mathfrak{p}$ has finite global dimension, see Lemma 10.109.13, and hence is a regular local ring. □

By Lemma 10.110.6 it makes sense to make the following definition, because it does not conflict with the earlier definition of a regular local ring.

00OD Definition 10.110.7. A Noetherian ring R is said to be regular if all the localizations $R_{\mathfrak p}$ at primes are regular local rings.

It is enough to require the local rings at maximal ideals to be regular. Note that this is not the same as asking R to have finite global dimension, even assuming R is Noetherian. This is because there is an example of a regular Noetherian ring which does not have finite global dimension, namely because it does not have finite dimension.

00OE Lemma 10.110.8. Let R be a Noetherian ring. The following are equivalent:

(1) R has finite global dimension n,
(2) R is a regular ring of dimension n,
(3) there exists an integer n such that all the localizations $R_{\mathfrak m}$ at maximal ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak m$, and
(4) there exists an integer n such that all the localizations $R_{\mathfrak p}$ at prime ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak p$.

Proof. This follows from the discussion above. More precisely, it follows by combining Definition 10.110.7 with Lemma 10.110.2 and Proposition 10.110.5. $\square$

00OF Lemma 10.110.9. Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume that $R \to S$ is flat and that S is regular. Then R is regular.

Proof. Let $\mathfrak m \subset R$ be the maximal ideal and let $\kappa = R/\mathfrak m$ be the residue field. Let $d = \dim S$. Choose any resolution $F_\bullet \to \kappa$ with each F_i a finite free R-module. Set $K_d = \text{Ker}(F_{d-1} \to F_{d-2})$. By flatness of $R \to S$ the complex $0 \to K_d \otimes_R S \to F_{d-1} \otimes_R S \to \ldots \to F_0 \otimes_R S \to \kappa \otimes_R S \to 0$ is still exact. Because the global dimension of S is d, see Proposition 10.110.1, we see that $K_d \otimes_R S$ is a finite free S-module (see also Lemma 10.109.3). By Lemma 10.78.6 we see that K_d is a finite free R-module. Hence κ has finite projective dimension and R is regular by Proposition 10.110.5. $\square$

10.111. Auslander-Buchsbaum

090U The following result can be found in [AB57].

090V Proposition 10.111.1. Let R be a Noetherian local ring. Let M be a nonzero finite R-module which has finite projective dimension $\text{pd}_R(M)$. Then we have

$$\text{depth}(R) = \text{pd}_R(M) + \text{depth}(M)$$

Proof. We prove this by induction on $\text{depth}(M)$. The most interesting case is the case $\text{depth}(M) = 0$. In this case, let

$$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0} \to M \to 0$$

be a minimal finite free resolution, so $e = \text{pd}_R(M)$. By Lemma 10.102.2 we may assume all matrix coefficients of the maps in the complex are contained in the maximal ideal of R. Then on the one hand, by Proposition 10.102.9 we see that $\text{depth}(R) \geq e$. On the other hand, breaking the long exact sequence into short exact sequences

$$\begin{aligned} 0 \to R^{n_e} \to R^{n_{e-1}} \to K_{e-2} \to 0, \\ 0 \to K_{e-2} \to R^{n_{e-2}} \to K_{e-3} \to 0, \\ \ldots, \\ 0 \to K_0 \to R^{n_0} \to M \to 0 \end{aligned}$$

we see, using Lemma 10.72.6, that

$$\begin{aligned} \text{depth}(K_{e-2}) \geq \text{depth}(R) - 1, \\ \text{depth}(K_{e-3}) \geq \text{depth}(R) - 2, \\ \ldots, \\ \text{depth}(K_0) \geq \text{depth}(R) - (e-1), \\ \text{depth}(M) \geq \text{depth}(R) - e \end{aligned}$$

and since $\text{depth}(M) = 0$ we conclude $\text{depth}(R) \leq e$. This finishes the proof of the case $\text{depth}(M) = 0$.

Induction step. If $\text{depth}(M) > 0$, then we pick $x \in \mathfrak{m}$ which is a nonzerodivisor on both M and R. This is possible, because either $\text{pd}_R(M) > 0$ and $\text{depth}(R) > 0$ by the aforementioned Proposition 10.102.9 or $\text{pd}_R(M) = 0$ in which case M is finite free hence also $\text{depth}(R) = \text{depth}(M) > 0$. Thus $\text{depth}(R \oplus M) > 0$ by Lemma 10.72.6 (for example) and we can find an $x \in \mathfrak{m}$ which is a nonzerodivisor on both R and M. Let

$$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0} \to M \to 0$$

be a minimal resolution as above. An application of the snake lemma shows that

$$0 \to (R/xR)^{n_e} \to (R/xR)^{n_{e-1}} \to \ldots \to (R/xR)^{n_0} \to M/xM \to 0$$

is a minimal resolution too. Thus $\text{pd}_R(M) = \text{pd}_{R/xR}(M/xM)$. By Lemma 10.72.7 we have $\text{depth}(R/xR) = \text{depth}(R) - 1$ and $\text{depth}(M/xM) = \text{depth}(M) - 1$. Till now depths have all been depths as R modules, but we observe that $\text{depth}_R(M/xM) = \text{depth}_{R/xR}(M/xM)$ and similarly for R/xR. By induction hypothesis we see that the Auslander-Buchsbaum formula holds for M/xM over R/xR. Since the depths of both R/xR and M/xM have decreased by one and the projective dimension has not changed we conclude. □

10.112. Homomorphisms and dimension

00OG This section contains a collection of easy results relating dimensions of rings when there are maps between them.

00OH **Lemma 10.112.1.** Suppose $R \to S$ is a ring map satisfying either going up, see Definition 10.41.1, or going down see Definition 10.41.1. Assume in addition that $\text{Spec}(S) \to \text{Spec}(R)$ is surjective. Then $\dim(R) \leq \dim(S)$.

Proof. Assume going up. Take any chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e$ of prime ideals in R. By surjectivity we may choose a prime $\mathfrak{q}_0$ mapping to $\mathfrak{p}_0$. By going up we may extend this to a chain of length e of primes $\mathfrak{q}_i$ lying over $\mathfrak{p}_i$. Thus $\dim(S) \geq \dim(R)$. The case of going down is exactly the same. See also Topology, Lemma 5.19.9 for a purely topological version. □

00OI Lemma 10.112.2. Suppose that $R \to S$ is a ring map with the going up property, see Definition 10.41.1. If $\mathfrak{q} \subset S$ is a maximal ideal. Then the inverse image of $\mathfrak{q}$ in R is a maximal ideal too.

Proof. Trivial. $\square$

00OJ Lemma 10.112.3. Suppose that $R \to S$ is a ring map such that S is integral over R. Then $\dim(R) \geq \dim(S)$, and every closed point of $\mathrm{Spec}(S)$ maps to a closed point of $\mathrm{Spec}(R)$.

Proof. Immediate from Lemmas 10.36.20 and 10.112.2 and the definitions. $\square$

00OK Lemma 10.112.4. Suppose $R \subset S$ and S integral over R. Then $\dim(R) = \dim(S)$.

Proof. This is a combination of Lemmas 10.36.22, 10.36.17, 10.112.1, and 10.112.3. $\square$

00OL Definition 10.112.5. Suppose that $R \to S$ is a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. The local ring of the fibre at $\mathfrak{q}$ is the local ring

$$S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} = (S/\mathfrak{p}S)_{\mathfrak{q}} = (S \otimes_R \kappa(\mathfrak{p}))_{\mathfrak{q}}$$

00OM Lemma 10.112.6. Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Then

$$\dim(S_{\mathfrak{q}}) \leq \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}).$$

Proof. We use the characterization of dimension of Proposition 10.60.9. Let $x_1, \ldots, x_d$ be elements of $\mathfrak{p}$ generating an ideal of definition of $R_{\mathfrak{p}}$ with $d = \dim(R_{\mathfrak{p}})$. Let $y_1, \ldots, y_e$ be elements of $\mathfrak{q}$ generating an ideal of definition of $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ with $e = \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$. It is clear that $S_{\mathfrak{q}}/(x_1, \ldots, x_d, y_1, \ldots, y_e)$ has a nilpotent maximal ideal. Hence $x_1, \ldots, x_d, y_1, \ldots, y_e$ generate an ideal of definition of $S_{\mathfrak{q}}$. $\square$

00ON Lemma 10.112.7. Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Assume the going down property holds for $R \to S$ (for example if $R \to S$ is flat, see Lemma 10.39.19). Then

$$\dim(S_{\mathfrak{q}}) = \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}).$$

Proof. By Lemma 10.112.6 we have an inequality $\dim(S_{\mathfrak{q}}) \leq \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$. To get equality, choose a chain of primes $\mathfrak{p}S \subset \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_d = \mathfrak{q}$ with $d = \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$. On the other hand, choose a chain of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{p}$ with $e = \dim(R_{\mathfrak{p}})$. By the going down theorem we may choose $\mathfrak{q}_{-1} \subset \mathfrak{q}_0$ lying over $\mathfrak{p}_{e-1}$. And then we may choose $\mathfrak{q}_{-2} \subset \mathfrak{q}_{e-1}$ lying over $\mathfrak{p}_{e-2}$. Inductively we keep going until we get a chain $\mathfrak{q}_{-e} \subset \ldots \subset \mathfrak{q}_d$ of length $e + d$. $\square$

031E Lemma 10.112.8. Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume

(1) R is regular,
(2) $S/\mathfrak{m}_R S$ is regular, and
(3) $R \to S$ is flat.

Then S is regular.

Proof. By Lemma 10.112.7 we have $\dim(S) = \dim(R) + \dim(S/\mathfrak m_R S)$. Pick generators $x_1, \ldots, x_d \in \mathfrak m_R$ with $d = \dim(R)$, and pick $y_1, \ldots, y_e \in \mathfrak m_S$ which generate the maximal ideal of $S/\mathfrak m_R S$ with $e = \dim(S/\mathfrak m_R S)$. Then we see that $x_1, \ldots, x_d, y_1, \ldots, y_e$ are elements which generate the maximal ideal of S and $e+d = \dim(S)$. □

The lemma below will later be used to show that rings of finite type over a field are Cohen-Macaulay if and only if they are quasi-finite flat over a polynomial ring. It is a partial converse to Lemma 10.128.1.

00R5 **Lemma 10.112.9.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R Cohen-Macaulay. If S is finite flat over R, or if S is flat over R and $\dim(S) \leq \dim(R)$, then S is Cohen-Macaulay and $\dim(R) = \dim(S)$.*

Proof. Let $x_1, \ldots, x_d \in \mathfrak m_R$ be a regular sequence of length $d = \dim(R)$. By Lemma 10.68.5 this maps to a regular sequence in S. Hence S is Cohen-Macaulay if $\dim(S) \leq d$. This is true if S is finite flat over R by Lemma 10.112.4. And in the second case we assumed it. □

10.113. The dimension formula

02II Recall the definitions of catenary (Definition 10.105.1) and universally catenary (Definition 10.105.3).

02IJ **Lemma 10.113.1.** *Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over the prime $\mathfrak p$ of R. Assume that*

1. *R is Noetherian,*
2. *$R \to S$ is of finite type,*
3. *R, S are domains, and*
4. *$R \subset S$.*

Then we have

$$\text{height}(\mathfrak q) \leq \text{height}(\mathfrak p) + \text{trdeg}_R(S) - \text{trdeg}_{\kappa(\mathfrak p)}\kappa(\mathfrak q)$$

with equality if R is universally catenary.

Proof. Suppose that $R \subset S' \subset S$ is a finitely generated R-subalgebra of S. In this case set $\mathfrak q' = S' \cap \mathfrak q$. The lemma for the ring maps $R \to S'$ and $S' \to S$ implies the lemma for $R \to S$ by additivity of transcendence degree in towers of fields (Fields, Lemma 9.26.5). Hence we can use induction on the number of generators of S over R and reduce to the case where S is generated by one element over R.

Case I: $S = R[x]$ is a polynomial algebra over R. In this case we have $\text{trdeg}_R(S) = 1$. Also $R \to S$ is flat and hence

$$\dim(S_{\mathfrak q}) = \dim(R_{\mathfrak p}) + \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q})$$

see Lemma 10.112.7. Let $\mathfrak r = \mathfrak p S$. Then $\text{trdeg}_{\kappa(\mathfrak p)}\kappa(\mathfrak q) = 1$ is equivalent to $\mathfrak q = \mathfrak r$, and implies that $\dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}) = 0$. In the same vein $\text{trdeg}_{\kappa(\mathfrak p)}\kappa(\mathfrak q) = 0$ is equivalent to having a strict inclusion $\mathfrak r \subset \mathfrak q$, which implies that $\dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}) = 1$. Thus we are done with case I with equality in every instance.

Case II: $S = R[x]/\mathfrak n$ with $\mathfrak n \neq 0$. In this case we have $\text{trdeg}_R(S) = 0$. Denote $\mathfrak q' \subset R[x]$ the prime corresponding to $\mathfrak q$. Thus we have

$$S_{\mathfrak q} = (R[x])_{\mathfrak q'}/\mathfrak n(R[x])_{\mathfrak q'}$$

By the previous case we have $\dim((R[x])_{\mathfrak q'}) = \dim(R_{\mathfrak p}) + 1 - \text{trdeg}_{\kappa(\mathfrak p)}\kappa(\mathfrak q)$. Since $\mathfrak n \neq 0$ we see that the dimension of $S_{\mathfrak q}$ decreases by at least one, see Lemma 10.60.13, which proves the inequality of the lemma. To see the equality in case R is universally catenary note that $\mathfrak n \subset R[x]$ is a height one prime as it corresponds to a nonzero prime in $F[x]$ where F is the fraction field of R. Hence any maximal chain of primes in $S_{\mathfrak q} = R[x]_{\mathfrak q'}/\mathfrak n R[x]_{\mathfrak q'}$ corresponds to a maximal chain of primes with length 1 greater between $\mathfrak q'$ and (0) in $R[x]$. If R is universally catenary these all have the same length equal to the height of $\mathfrak q'$. This proves that $\dim(S_{\mathfrak q}) = \dim(R[x]_{\mathfrak q'}) - 1$ and this implies equality holds as desired. □

The following lemma says that generically finite maps tend to be quasi-finite in codimension 1.

02MA Lemma 10.113.2. Let $A \to B$ be a ring map. Assume

(1) $A \subset B$ is an extension of domains,
(2) the induced extension of fraction fields is finite,
(3) A is Noetherian, and
(4) $A \to B$ is of finite type.

Let $\mathfrak p \subset A$ be a prime of height 1. Then there are at most finitely many primes of B lying over $\mathfrak p$ and they all have height 1.

Proof. By the dimension formula (Lemma 10.113.1) for any prime $\mathfrak q$ lying over $\mathfrak p$ we have

$$\dim(B_{\mathfrak q}) \leq \dim(A_{\mathfrak p}) - \text{trdeg}_{\kappa(\mathfrak p)}\kappa(\mathfrak q).$$

As the domain $B_{\mathfrak q}$ has at least 2 prime ideals we see that $\dim(B_{\mathfrak q}) \geq 1$. We conclude that $\dim(B_{\mathfrak q}) = 1$ and that the extension $\kappa(\mathfrak p) \subset \kappa(\mathfrak q)$ is algebraic. Hence $\mathfrak q$ defines a closed point of its fibre $\text{Spec}(B \otimes_A \kappa(\mathfrak p))$, see Lemma 10.35.9. Since $B \otimes_A \kappa(\mathfrak p)$ is a Noetherian ring the fibre $\text{Spec}(B \otimes_A \kappa(\mathfrak p))$ is a Noetherian topological space, see Lemma 10.31.5. A Noetherian topological space consisting of closed points is finite, see for example Topology, Lemma 5.9.2. □

10.114. Dimension of finite type algebras over fields

00OO In this section we compute the dimension of a polynomial ring over a field. We also prove that the dimension of a finite type domain over a field is the dimension of its local rings at maximal ideals. We will establish the connection with the transcendence degree over the ground field in Section 10.116.

00OP Lemma 10.114.1. Let $\mathfrak m$ be a maximal ideal in $k[x_1, \ldots, x_n]$. The ideal $\mathfrak m$ is generated by n elements. The dimension of $k[x_1, \ldots, x_n]_{\mathfrak m}$ is n. Hence $k[x_1, \ldots, x_n]_{\mathfrak m}$ is a regular local ring of dimension n.

Proof. By the Hilbert Nullstellensatz (Theorem 10.34.1) we know the residue field $\kappa = \kappa(\mathfrak m)$ is a finite extension of k. Denote $\alpha_i \in \kappa$ the image of x_i. Denote $\kappa_i = k(\alpha_1, \ldots, \alpha_i) \subset \kappa$, $i = 1, \ldots, n$ and $\kappa_0 = k$. Note that $\kappa_i = k[\alpha_1, \ldots, \alpha_i]$ by field theory. Define inductively elements $f_i \in \mathfrak m \cap k[x_1, \ldots, x_i]$ as follows: Let $P_i(T) \in \kappa_{i-1}[T]$ be the monic minimal polynomial of α_i over κ_{i-1}. Let $Q_i(T) \in k[x_1, \ldots, x_{i-1}][T]$ be a monic lift of $P_i(T)$ (of the same degree). Set $f_i = Q_i(x_i)$. Note that if $d_i = \deg_T(P_i) = \deg_T(Q_i) = \deg_{x_i}(f_i)$ then $d_1 d_2 \ldots d_i = [\kappa_i : k]$ by Fields, Lemmas 9.7.7 and 9.9.2.

We claim that for all $i = 0, 1, \ldots, n$ there is an isomorphism

$$\psi_i : k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \cong \kappa_i.$$

By construction the composition $k[x_1, \ldots, x_i] \to k[x_1, \ldots, x_n] \to \kappa$ is surjective onto κ_i and $f_1, \ldots, f_i$ are in the kernel. This gives a surjective homomorphism. We prove ψ_i is injective by induction. It is clear for $i = 0$. Given the statement for i we prove it for $i+1$. The ring extension $k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \to k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is generated by 1 element over a field and one irreducible equation. By elementary field theory $k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is a field, and hence ψ_i is injective.

This implies that $\mathfrak m = (f_1, \ldots, f_n)$. Moreover, we also conclude that

$$k[x_1, \ldots, x_n]/(f_1, \ldots, f_i) \cong \kappa_i[x_{i+1}, \ldots, x_n].$$

Hence $(f_1, \ldots, f_i)$ is a prime ideal. Thus

$$(0) \subset (f_1) \subset (f_1, f_2) \subset \ldots \subset (f_1, \ldots, f_n) = \mathfrak m$$

is a chain of primes of length n. The lemma follows. □

00OQ Proposition 10.114.2. A polynomial algebra in n variables over a field is a regular ring. It has global dimension n. All localizations at maximal ideals are regular local rings of dimension n.

Proof. By Lemma 10.114.1 all localizations $k[x_1, \ldots, x_n]_{\mathfrak m}$ at maximal ideals are regular local rings of dimension n. Hence we conclude by Lemma 10.110.8. □

00OR Lemma 10.114.3. Let k be a field. Let $\mathfrak p \subset \mathfrak q \subset k[x_1, \ldots, x_n]$ be a pair of primes. Any maximal chain of primes between $\mathfrak p$ and $\mathfrak q$ has length $\text{height}(\mathfrak q) - \text{height}(\mathfrak p)$.

Proof. By Proposition 10.114.2 any local ring of $k[x_1, \ldots, x_n]$ is regular. Hence all local rings are Cohen-Macaulay, see Lemma 10.106.3. The local rings at maximal ideals have dimension n hence every maximal chain of primes in $k[x_1, \ldots, x_n]$ has length n, see Lemma 10.104.3. Hence every maximal chain of primes between (0) and $\mathfrak p$ has length $\text{height}(\mathfrak p)$, see Lemma 10.104.4 for example. Putting these together leads to the assertion of the lemma. □

00OS Lemma 10.114.4. Let k be a field. Let S be a finite type k-algebra which is an integral domain. Then $\dim(S) = \dim(S_{\mathfrak m})$ for any maximal ideal $\mathfrak m$ of S. In words: every maximal chain of primes has length equal to the dimension of S.

Proof. Write $S = k[x_1, \ldots, x_n]/\mathfrak p$. By Proposition 10.114.2 and Lemma 10.114.3 all the maximal chains of primes in S (which necessarily end with a maximal ideal) have length $n - \text{height}(\mathfrak p)$. Thus this number is the dimension of S and of $S_{\mathfrak m}$ for any maximal ideal $\mathfrak m$ of S. □

Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 5.10.1.

00OT Lemma 10.114.5. Let k be a field. Let S be a finite type k-algebra. Let $X = \text{Spec}(S)$. Let $\mathfrak p \subset S$ be a prime ideal and let $x \in X$ be the corresponding point. The following numbers are equal

(1) $\dim_x(X)$,
(2) $\max \dim(Z)$ where the maximum is over those irreducible components Z of X passing through x, and

(3) $\min \dim(S_\mathfrak m)$ where the minimum is over maximal ideals $\mathfrak m$ with $\mathfrak p \subset \mathfrak m$.

Proof. Let $X = \bigcup_{i \in I} Z_i$ be the decomposition of X into its irreducible components. There are finitely many of them (see Lemmas 10.31.3 and 10.31.5). Let $I' = \{i \mid x \in Z_i\}$, and let $T = \bigcup_{i \not\in I'} Z_i$. Then $U = X \setminus T$ is an open subset of X containing the point x. The number (2) is $\max_{i \in I'} \dim(Z_i)$. For any open $W \subset U$ with $x \in W$ the irreducible components of W are the irreducible sets $W_i = Z_i \cap W$ for $i \in I'$ and x is contained in each of these. Note that each W_i, $i \in I'$ contains a closed point because X is Jacobson, see Section 10.35. Since $W_i \subset Z_i$ we have $\dim(W_i) \leq \dim(Z_i)$. The existence of a closed point implies, via Lemma 10.114.4, that there is a chain of irreducible closed subsets of length equal to $\dim(Z_i)$ in the open W_i. Thus $\dim(W_i) = \dim(Z_i)$ for any $i \in I'$. Hence $\dim(W)$ is equal to the number (2). This proves that (1) = (2).

Let $\mathfrak m \supset \mathfrak p$ be any maximal ideal containing $\mathfrak p$. Let $x_0 \in X$ be the corresponding point. First of all, x_0 is contained in all the irreducible components Z_i, $i \in I'$. Let $\mathfrak q_i$ denote the minimal primes of S corresponding to the irreducible components Z_i. For each i such that $x_0 \in Z_i$ (which is equivalent to $\mathfrak m \supset \mathfrak q_i$) we have a surjection

$$S_\mathfrak m \longrightarrow S_\mathfrak m/\mathfrak q_i S_\mathfrak m = (S/\mathfrak q_i)_\mathfrak m$$

Moreover, the primes $\mathfrak q_i S_\mathfrak m$ so obtained exhaust the minimal primes of the Noetherian local ring $S_\mathfrak m$, see Lemma 10.26.3. We conclude, using Lemma 10.114.4, that the dimension of $S_\mathfrak m$ is the maximum of the dimensions of the Z_i passing through x_0. To finish the proof of the lemma it suffices to show that we can choose x_0 such that $x_0 \in Z_i \Rightarrow i \in I'$. Because S is Jacobson (as we saw above) it is enough to show that $V(\mathfrak p) \setminus T$ (with T as above) is nonempty. And this is clear since it contains the point x (i.e. $\mathfrak p$). □

00OU Lemma 10.114.6. Let k be a field. Let S be a finite type k-algebra. Let $X = \text{Spec}(S)$. Let $\mathfrak m \subset S$ be a maximal ideal and let $x \in X$ be the associated closed point. Then $\dim_x(X) = \dim(S_\mathfrak m)$.

Proof. This is a special case of Lemma 10.114.5. □

00OV Lemma 10.114.7. Let k be a field. Let S be a finite type k algebra. Assume that S is Cohen-Macaulay. Then $\text{Spec}(S) = \coprod T_d$ is a finite disjoint union of open and closed subsets T_d with T_d equidimensional (see Topology, Definition 5.10.5) of dimension d. Equivalently, S is a product of rings S_d, $d = 0, \ldots, \dim(S)$ such that every maximal ideal $\mathfrak m$ of S_d has height d.

Proof. The equivalence of the two statements follows from Lemma 10.24.3. Let $\mathfrak m \subset S$ be a maximal ideal. Every maximal chain of primes in $S_\mathfrak m$ has the same length equal to $\dim(S_\mathfrak m)$, see Lemma 10.104.3. Hence, the dimension of the irreducible components passing through the point corresponding to $\mathfrak m$ all have dimension equal to $\dim(S_\mathfrak m)$, see Lemma 10.114.4. Since $\text{Spec}(S)$ is a Jacobson topological space the intersection of any two irreducible components of it contains a closed point if nonempty, see Lemmas 10.35.2 and 10.35.4. Thus we have shown that any two irreducible components that meet have the same dimension. The lemma follows easily from this, and the fact that $\text{Spec}(S)$ has a finite number of irreducible components (see Lemmas 10.31.3 and 10.31.5). □

10.115. Noether normalization

00OW In this section we prove variants of the Noether normalization lemma. The key ingredient we will use is contained in the following two lemmas.

051M **Lemma 10.115.1.** Let $n \in \mathbf{N}$. Let N be a finite nonempty set of multi-indices $\nu = (\nu_1, \ldots, \nu_n)$. Given $e = (e_1, \ldots, e_n)$ we set $e \cdot \nu = \sum e_i \nu_i$. Then for $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n$ we have: If $\nu, \nu' \in N$ then

$$(e \cdot \nu = e \cdot \nu') \Leftrightarrow (\nu = \nu')$$

Proof. Say $N = \{\nu_j\}$ with $\nu_j = (\nu_{j1}, \ldots, \nu_{jn})$. Let $A_i = \max_j \nu_{ji} - \min_j \nu_{ji}$. If for each i we have $e_{i-1} > A_i e_i + A_{i+1} e_{i+1} + \ldots + A_n e_n$ then the lemma holds. For suppose that $e \cdot (\nu - \nu') = 0$. Then for $n \geq 2$,

$$e_1(\nu_1 - \nu'_1) = \sum\nolimits_{i=2}^n e_i(\nu'_i - \nu_i).$$

We may assume that $(\nu_1 - \nu'_1) \geq 0$. If $(\nu_1 - \nu'_1) > 0$, then

$$e_1(\nu_1 - \nu'_1) \geq e_1 > A_2 e_2 + \ldots + A_n e_n \geq \sum\nolimits_{i=2}^n e_i |\nu'_i - \nu_i| \geq \sum\nolimits_{i=2}^n e_i(\nu'_i - \nu_i).$$

This contradiction implies that $\nu'_1 = \nu_1$. By induction, $\nu'_i = \nu_i$ for $2 \leq i \leq n$. $\square$

051N **Lemma 10.115.2.** Let R be a ring. Let $g \in R[x_1, \ldots, x_n]$ be an element which is nonconstant, i.e., $g \notin R$. For $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n = 1$ the polynomial

$$g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n) = a x_n^d + \text{lower order terms in } x_n$$

where $d > 0$ and $a \in R$ is one of the nonzero coefficients of g.

Proof. Write $g = \sum_{\nu \in N} a_\nu x^\nu$ with $a_\nu \in R$ not zero. Here N is a finite set of multi-indices as in Lemma 10.115.1 and $x^\nu = x_1^{\nu_1} \ldots x_n^{\nu_n}$. Note that the leading term in

$$(x_1 + x_n^{e_1})^{\nu_1} \ldots (x_{n-1} + x_n^{e_{n-1}})^{\nu_{n-1}} x_n^{\nu_n} \quad \text{is} \quad x_n^{e_1\nu_1 + \ldots + e_{n-1}\nu_{n-1} + \nu_n}.$$

Hence the lemma follows from Lemma 10.115.1 which guarantees that there is exactly one nonzero term $a_\nu x^\nu$ of g which gives rise to the leading term of $g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n)$, i.e., $a = a_\nu$ for the unique $\nu \in N$ such that $e \cdot \nu$ is maximal. $\square$

00OX **Lemma 10.115.3.** Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some proper ideal I. If $I \neq 0$, then there exist $y_1, \ldots, y_{n-1} \in k[x_1, \ldots, x_n]$ such that S is finite over $k[y_1, \ldots, y_{n-1}]$. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.

Proof. Pick $f \in I$, $f \neq 0$. It suffices to show the lemma for $k[x_1, \ldots, x_n]/(f)$ since S is a quotient of that ring. We will take $y_i = x_i - x_n^{e_i}$, $i = 1, \ldots, n-1$ for suitable integers e_i. When does this work? It suffices to show that $\overline{x_n} \in k[x_1, \ldots, x_n]/(f)$ is integral over the ring $k[y_1, \ldots, y_{n-1}]$. The equation for $\overline{x_n}$ over this ring is

$$f(y_1 + x_n^{e_1}, \ldots, y_{n-1} + x_n^{e_{n-1}}, x_n) = 0.$$

Hence we are done if we can show there exists integers e_i such that the leading coefficient with respect to x_n of the equation above is a nonzero element of k. This can be achieved for example by choosing $e_1 \gg e_2 \gg \ldots \gg e_{n-1}$, see Lemma 10.115.2. $\square$

00OY Lemma 10.115.4. Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some ideal I. If $I \neq (1)$, there exist $r \geq 0$, and $y_1, \ldots, y_r \in k[x_1, \ldots, x_n]$ such that (a) the map $k[y_1, \ldots, y_r] \to S$ is injective, and (b) the map $k[y_1, \ldots, y_r] \to S$ is finite. In this case the integer r is the dimension of S. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.

Proof. By induction on n, with $n = 0$ being trivial. If $I = 0$, then take $r = n$ and $y_i = x_i$. If $I \neq 0$, then choose $y_1, \ldots, y_{n-1}$ as in Lemma 10.115.3. Let $S' \subset S$ be the subring generated by the images of the y_i. By induction we can choose r and $z_1, \ldots, z_r \in k[y_1, \ldots, y_{n-1}]$ such that (a), (b) hold for $k[z_1, \ldots, z_r] \to S'$. Since $S' \to S$ is injective and finite we see (a), (b) hold for $k[z_1, \ldots, z_r] \to S$. The last assertion follows from Lemma 10.112.4. □

00OZ Lemma 10.115.5. Let k be a field. Let S be a finite type k algebra and denote $X = \text{Spec}(S)$. Let $\mathfrak{q}$ be a prime of S, and let $x \in X$ be the corresponding point. There exists a $g \in S$, $g \notin \mathfrak{q}$ such that $\dim(S_g) = \dim_x(X) =: d$ and such that there exists a finite injective map $k[y_1, \ldots, y_d] \to S_g$.

Proof. Note that by definition $\dim_x(X)$ is the minimum of the dimensions of S_g for $g \in S$, $g \notin \mathfrak{q}$, i.e., the minimum is attained. Thus the lemma follows from Lemma 10.115.4. □

051P Lemma 10.115.6. Let k be a field. Let $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ be a prime ideal. Set $r = \text{trdeg}_k\, \kappa(\mathfrak{q})$. Then there exists a finite ring map $\varphi : k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ such that $\varphi^{-1}(\mathfrak{q}) = (y_{r+1}, \ldots, y_n)$.

Proof. By induction on n. The case $n = 0$ is clear. Assume $n > 0$. If $r = n$, then $\mathfrak{q} = (0)$ and the result is clear. Choose a nonzero $f \in \mathfrak{q}$. Of course f is nonconstant. After applying an automorphism of the form

$$k[x_1, \ldots, x_n] \longrightarrow k[x_1, \ldots, x_n], \quad x_n \mapsto x_n, \quad x_i \mapsto x_i + x_n^{e_i} \ (i < n)$$

we may assume that f is monic in x_n over $k[x_1, \ldots, x_n]$, see Lemma 10.115.2. Hence the ring map

$$k[y_1, \ldots, y_n] \longrightarrow k[x_1, \ldots, x_n], \quad y_n \mapsto f, \quad y_i \mapsto x_i \ (i < n)$$

is finite. Moreover $y_n \in \mathfrak{q} \cap k[y_1, \ldots, y_n]$ by construction. Thus $\mathfrak{q} \cap k[y_1, \ldots, y_n] = \mathfrak{p}k[y_1, \ldots, y_n] + (y_n)$ where $\mathfrak{p} \subset k[y_1, \ldots, y_{n-1}]$ is a prime ideal. Note that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite, and hence $r = \text{trdeg}_k\ \kappa(\mathfrak{p})$. Apply the induction hypothesis to the pair $(k[y_1, \ldots, y_{n-1}], \mathfrak{p})$ and we obtain a finite ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ such that $\mathfrak{p} \cap k[z_1, \ldots, z_{n-1}] = (z_{r+1}, \ldots, z_{n-1})$. We extend the ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ to a ring map $k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n]$ by mapping z_n to y_n. The composition of the ring maps

$$k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$$

solves the problem. □

07NA Lemma 10.115.7. Let $R \to S$ be an injective finite type ring map. Assume R is a domain. Then there exists an integer d and a factorization

$$R \to R[y_1, \ldots, y_d] \to S' \to S$$

by injective maps such that S' is finite over $R[y_1, \ldots, y_d]$ and such that $S'_f \cong S_f$ for some nonzero $f \in R$.

Proof. Pick $x_1, \ldots, x_n \in S$ which generate S over R. Let K be the fraction field of R and $S_K = S \otimes_R K$. By Lemma 10.115.4 we can find $y_1, \ldots, y_d \in S$ such that $K[y_1, \ldots, y_d] \to S_K$ is a finite injective map. Note that $y_i \in S$ because we may pick the y_j in the $\mathbf{Z}$-algebra generated by $x_1, \ldots, x_n$. As a finite ring map is integral (see Lemma 10.36.3) we can find monic $P_i \in K[y_1, \ldots, y_d][T]$ such that $P_i(x_i) = 0$ in S_K. Let $f \in R$ be a nonzero element such that $fP_i \in R[y_1, \ldots, y_d][T]$ for all i. Then $fP_i(x_i)$ maps to zero in S_K. Hence after replacing f by another nonzero element of R we may also assume $fP_i(x_i)$ is zero in S. Set $x_i' = fx_i$ and let $S' \subset S$ be the R-subalgebra generated by $y_1, \ldots, y_d$ and $x_1', \ldots, x_n'$. Note that x_i' is integral over $R[y_1, \ldots, y_d]$ as we have $Q_i(x_i') = 0$ where $Q_i = f^{\deg_T(P_i)}P_i(T/f)$ which is a monic polynomial in T with coefficients in $R[y_1, \ldots, y_d]$ by our choice of f. Hence $R[y_1, \ldots, y_d] \subset S'$ is finite by Lemma 10.36.5. Since $S' \subset S$ we have $S'_f \subset S_f$ (localization is exact). On the other hand, the elements $x_i = x_i'/f$ in S'_f generate S_f over R_f and hence $S'_f \to S_f$ is surjective. Whence $S'_f \cong S_f$ and we win. $\square$

10.116. Dimension of finite type algebras over fields, reprise

07NB This section is a continuation of Section 10.114. In this section we establish the connection between dimension and transcendence degree over the ground field for finite type domains over a field.

00P0 Lemma 10.116.1. Let k be a field. Let S be a finite type k algebra which is an integral domain. Let K be the field of fractions of S. Let $r = \text{trdeg}(K/k)$ be the transcendence degree of K over k. Then $\dim(S) = r$. Moreover, the local ring of S at every maximal ideal has dimension r.

Proof. We may write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Lemma 10.114.3 all local rings of S at maximal ideals have the same dimension. Apply Lemma 10.115.4. We get a finite injective ring map

$$k[y_1, \ldots, y_d] \to S$$

with $d = \dim(S)$. Clearly, $k(y_1, \ldots, y_d) \subset K$ is a finite extension and we win. $\square$

06RP Lemma 10.116.2. Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset \mathfrak{q}' \subset S$ be distinct prime ideals. Then $\text{trdeg}_k\ \kappa(\mathfrak{q}') < \text{trdeg}_k\ \kappa(\mathfrak{q})$.

Proof. By Lemma 10.116.1 we have $\dim V(\mathfrak{q}) = \text{trdeg}_k\ \kappa(\mathfrak{q})$ and similarly for $\mathfrak{q}'$. Hence the result follows as the strict inclusion $V(\mathfrak{q}') \subset V(\mathfrak{q})$ implies a strict inequality of dimensions. $\square$

The following lemma generalizes Lemma 10.114.6.

00P1 Lemma 10.116.3. Let k be a field. Let S be a finite type k algebra. Let $X = \text{Spec}(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $x \in X$ be the corresponding point. Then we have

$$\dim_x(X) = \dim(S_\mathfrak{p}) + \text{trdeg}_k\ \kappa(\mathfrak{p}).$$

Proof. By Lemma 10.116.1 we know that $r = \text{trdeg}_k\ \kappa(\mathfrak{p})$ is equal to the dimension of $V(\mathfrak{p})$. Pick any maximal chain of primes $\mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ starting with $\mathfrak{p}$ in S. This has length r by Lemma 10.114.4. Let $\mathfrak{q}_j$, $j \in J$ be the minimal primes of S which are contained in $\mathfrak{p}$. These correspond $1-1$ to minimal primes in $S_\mathfrak{p}$ via the rule $\mathfrak{q}_j \mapsto \mathfrak{q}_jS_\mathfrak{p}$. By Lemma 10.114.5 we know that $\dim_x(X)$ is equal to the maximum of the dimensions of the rings $S/\mathfrak{q}_j$. For each j pick a maximal chain of primes $\mathfrak{q}_j \subset \mathfrak{p}'_1 \subset \ldots \subset \mathfrak{p}'_{s(j)} = \mathfrak{p}$. Then $\dim(S_\mathfrak{p}) = \max_{j \in J} s(j)$. Now, each chain

$$\mathfrak{q}_i \subset \mathfrak{p}'_1 \subset \ldots \subset \mathfrak{p}'_{s(j)} = \mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$$

is a maximal chain in $S/\mathfrak{q}_j$, and by what was said before we have $\dim_x(X) = \max_{j \in J} r + s(j)$. The lemma follows. □

The following lemma says that the codimension of one finite type Spec in another is the difference of heights.

00P2 Lemma 10.116.4. Let k be a field. Let $S' \to S$ be a surjection of finite type k algebras. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $\mathfrak{p}'$ be the corresponding prime ideal of S'. Let $X = \mathrm{Spec}(S)$, resp. $X' = \mathrm{Spec}(S')$, and let $x \in X$, resp. $x' \in X'$ be the point corresponding to $\mathfrak{p}$, resp. $\mathfrak{p}'$. Then

$$\dim_{x'} X' - \dim_x X = \mathrm{height}(\mathfrak{p}') - \mathrm{height}(\mathfrak{p}).$$

Proof. Immediate from Lemma 10.116.3. □

00P3 Lemma 10.116.5. Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension. Then $\dim(S) = \dim(K \otimes_k S)$.

Proof. By Lemma 10.115.4 there exists a finite injective map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S)$. Since K is flat over k we also get a finite injective map $K[y_1, \ldots, y_d] \to K \otimes_k S$. The result follows from Lemma 10.112.4. □

00P4 Lemma 10.116.6. Let k be a field. Let S be a finite type k-algebra. Set $X = \mathrm{Spec}(S)$. Let K/k be a field extension. Set $S_K = K \otimes_k S$, and $X_K = \mathrm{Spec}(S_K)$. Let $\mathfrak{q} \subset S$ be a prime corresponding to $x \in X$ and let $\mathfrak{q}_K \subset S_K$ be a prime corresponding to $x_K \in X_K$ lying over $\mathfrak{q}$. Then $\dim_x X = \dim_{x_K} X_K$.

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $K \otimes_k S = K[x_1, \ldots, x_n]/(K \otimes_k I)$. Let $\mathfrak{q}'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ be the corresponding primes. Consider the following commutative diagram of Noetherian local rings

$$\begin{tikzcd}
K[x_1, \ldots, x_n]_{\mathfrak{q}'_K} \arrow[r] & (K \otimes_k S)_{\mathfrak{q}_K} \\
k[x_1, \ldots, x_n]_{\mathfrak{q}'} \arrow[r] \arrow[u] & S_\mathfrak{q} \arrow[u]
\end{tikzcd}$$

Both vertical arrows are flat because they are localizations of the flat ring maps $S \to S_K$ and $k[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$. Moreover, the vertical arrows have the same fibre rings. Hence, we see from Lemma 10.112.7 that $\mathrm{height}(\mathfrak{q}') - \mathrm{height}(\mathfrak{q}) = \mathrm{height}(\mathfrak{q}'_K) - \mathrm{height}(\mathfrak{q}_K)$. Denote $x' \in X' = \mathrm{Spec}(k[x_1, \ldots, x_n])$ and $x'_K \in X'_K = \mathrm{Spec}(K[x_1, \ldots, x_n])$ the points corresponding to $\mathfrak{q}'$ and $\mathfrak{q}'_K$. By Lemma 10.116.4 and what we showed above we have

$$\begin{aligned}
n - \dim_x X &= \dim_{x'} X' - \dim_x X \\
&= \mathrm{height}(\mathfrak{q}') - \mathrm{height}(\mathfrak{q}) \\
&= \mathrm{height}(\mathfrak{q}'_K) - \mathrm{height}(\mathfrak{q}_K) \\
&= \dim_{x'_K} X'_K - \dim_{x_K} X_K \\
&= n - \dim_{x_K} X_K
\end{aligned}$$

and the lemma follows. □

0CWE Lemma 10.116.7. Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension. Set $S_K = K \otimes_k S$. Let $\mathfrak q \subset S$ be a prime and let $\mathfrak q_K \subset S_K$ be a prime lying over $\mathfrak q$. Then

$$\dim(S_K \otimes_S \kappa(\mathfrak q))_{\mathfrak q_K} = \dim(S_K)_{\mathfrak q_K} - \dim S_{\mathfrak q} = \text{trdeg}_k \kappa(\mathfrak q) - \text{trdeg}_K \kappa(\mathfrak q_K)$$

Moreover, given $\mathfrak q$ we can always choose $\mathfrak q_K$ such that the number above is zero.

Proof. Observe that $S_{\mathfrak q} \to (S_K)_{\mathfrak q_K}$ is a flat local homomorphism of local Noetherian rings with special fibre $(S_K \otimes_S \kappa(\mathfrak q))_{\mathfrak q_K}$. Hence the first equality by Lemma 10.112.7. The second equality follows from the fact that we have $\dim_x X = \dim_{x_K} X_K$ with notation as in Lemma 10.116.6 and we have $\dim_x X = \dim S_{\mathfrak q} + \text{trdeg}_k \kappa(\mathfrak q)$ by Lemma 10.116.3 and similarly for $\dim_{x_K} X_K$. If we choose $\mathfrak q_K$ minimal over $\mathfrak q S_K$, then the dimension of the fibre ring will be zero. □

10.117. Dimension of graded algebras over a field

00P5 Here is a basic result.

00P6 Lemma 10.117.1. Let k be a field. Let S be a graded k-algebra generated over k by finitely many elements of degree 1. Assume $S_0 = k$. Let $P(T) \in \mathbf{Q}[T]$ be the polynomial such that $\dim(S_d) = P(d)$ for all $d \gg 0$. See Proposition 10.58.7. Then

(1) The irrelevant ideal S_+ is a maximal ideal $\mathfrak m$.
(2) Any minimal prime of S is a homogeneous ideal and is contained in $S_+ = \mathfrak m$.
(3) We have $\dim(S) = \deg(P) + 1 = \dim_x \text{Spec}(S)$ (with the convention that $\deg(0) = -1$) where x is the point corresponding to the maximal ideal $S_+ = \mathfrak m$.
(4) The Hilbert function of the local ring $R = S_{\mathfrak m}$ is equal to the Hilbert function of S.

Proof. The first statement is obvious. The second follows from Lemma 10.57.8. By (2) every irreducible component passes through x. Thus we have $\dim(S) = \dim_x \text{Spec}(S) = \dim(S_{\mathfrak m})$ by Lemma 10.114.5. Since $\mathfrak m^d/\mathfrak m^{d+1} \cong \mathfrak m^d S_{\mathfrak m}/\mathfrak m^{d+1} S_{\mathfrak m}$ we see that the Hilbert function of the local ring $S_{\mathfrak m}$ is equal to the Hilbert function of S, which is (4). We conclude the last equality of (3) by Proposition 10.60.9. □

10.118. Generic flatness

051Q Basically this says that a finite type algebra over a domain becomes flat after inverting a single element of the domain. There are several versions of this result (in increasing order of strength).

051R Lemma 10.118.1. Let $R \to S$ be a ring map. Let M be an S-module. Assume

(1) R is Noetherian,
(2) R is a domain,
(3) $R \to S$ is of finite type, and
(4) M is a finite type S-module.

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Let K be the fraction field of R. Set $S_K = K \otimes_R S$. This is an algebra of finite type over K. We will argue by induction on $d = \dim(S_K)$ (which is finite for example by Noether normalization, see Section 10.115). Fix $d \geq 0$. Assume we know that the lemma holds in all cases where $\dim(S_K) < d$.

Suppose given $R \to S$ and M as in the lemma with $\dim(S_K) = d$. By Lemma 10.62.1 there exists a filtration $0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ so that M_i/M_{i-1} is isomorphic to $S/\mathfrak q$ for some prime $\mathfrak q$ of S. Note that $\dim((S/\mathfrak q)_K) \leq \dim(S_K)$. Also, note that an extension of free modules is free (see basic notion 50). Thus we may assume $M = S$ and that S is a domain of finite type over R.

If $R \to S$ has a nontrivial kernel, then take a nonzero $f \in R$ in this kernel. In this case $S_f = 0$ and the lemma holds. (This is really the case $d = -1$ and the start of the induction.) Hence we may assume that $R \to S$ is a finite type extension of Noetherian domains.

Apply Lemma 10.115.7 and replace R by R_f (with f as in the lemma) to get a factorization

$$R \subset R[y_1, \ldots, y_d] \subset S$$

where the second extension is finite. Choose $z_1, \ldots, z_r \in S$ which form a basis for the fraction field of S over the fraction field of $R[y_1, \ldots, y_d]$. This gives a short exact sequence

$$0 \to R[y_1, \ldots, y_d]^{\oplus r} \xrightarrow{(z_1, \ldots, z_r)} S \to N \to 0$$

By construction N is a finite $R[y_1, \ldots, y_d]$-module whose support does not contain the generic point (0) of $\text{Spec}(R[y_1, \ldots, y_d])$. By Lemma 10.40.5 there exists a nonzero $g \in R[y_1, \ldots, y_d]$ such that g annihilates N, so we may view N as a finite module over $S' = R[y_1, \ldots, y_d]/(g)$. Since $\dim(S'_K) < d$ by induction there exists a nonzero $f \in R$ such that N_f is a free R_f-module. Since $(R[y_1, \ldots, y_d])_f \cong R_f[y_1, \ldots, y_d]$ is free also we conclude by the already mentioned fact that an extension of free modules is free. $\square$

051S Lemma 10.118.2. Let $R \to S$ be a ring map. Let M be an S-module. Assume

(1) R is a domain,
(2) $R \to S$ is of finite presentation, and
(3) M is an S-module of finite presentation.

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. For $g \in R[x_1, \ldots, x_n]$ denote $\overline{g}$ its image in S. We may write $M = S^{\oplus t}/\sum S n_i$ for some $n_i \in S^{\oplus t}$. Write $n_i = (\overline{g}_{i1}, \ldots, \overline{g}_{it})$ for some $g_{ij} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated by all the coefficients of all the elements $g_i, g_{ij} \in R[x_1, \ldots, x_n]$. Define $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Define $M_0 = S_0^{\oplus t}/\sum S_0 n_i$. Then R_0 is a domain of finite type over $\mathbf{Z}$ and hence Noetherian (see Lemma 10.31.1). Moreover via the injection $R_0 \to R$ we have $S \cong R \otimes_{R_0} S_0$ and $M \cong R \otimes_{R_0} M_0$. Applying Lemma 10.118.1 we obtain a nonzero $f \in R_0$ such that $(M_0)_f$ is a free $(R_0)_f$-module. Hence $M_f = R_f \otimes_{(R_0)_f} (M_0)_f$ is a free R_f-module. $\square$

051T Lemma 10.118.3. Let $R \to S$ be a ring map. Let M be an S-module. Assume

(1) R is a domain,
(2) $R \to S$ is of finite type, and
(3) M is a finite type S-module.

Then there exists a nonzero $f \in R$ such that

(a) M_f and S_f are free as R_f-modules, and
(b) S_f is a finitely presented R_f-algebra and M_f is a finitely presented S_f-module.

Proof. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Assume $S = R[x_1, \ldots, x_n]$. Choose elements $m_1, \ldots, m_t$ which generate M. This gives a short exact sequence

$$0 \to N \to S^{\oplus t} \xrightarrow{(m_1, \ldots, m_t)} M \to 0.$$

Denote K the fraction field of R. Denote $S_K = K \otimes_R S = K[x_1, \ldots, x_n]$, and similarly $N_K = K \otimes_R N$, $M_K = K \otimes_R M$. As $R \to K$ is flat the sequence remains exact after tensoring with K. As $S_K = K[x_1, \ldots, x_n]$ is a Noetherian ring (see Lemma 10.31.1) we can find finitely many elements $n'_1, \ldots, n'_s \in N_K$ which generate it. Choose $n_1, \ldots, n_r \in N$ such that $n'_i = \sum a_{ij} n_j$ for some $a_{ij} \in K$. Set

$$M' = S^{\oplus t} / \sum\nolimits_{i = 1, \ldots, r} S n_i$$

By construction M' is a finitely presented S-module, and there is a surjection $M' \to M$ which induces an isomorphism $M'_K \cong M_K$. We may apply Lemma 10.118.2 to $R \to S$ and M' and we find an $f \in R$ such that M'_f is a free R_f-module. Thus $M'_f \to M_f$ is a surjection of modules over the domain R_f where the source is a free module and which becomes an isomorphism upon tensoring with K. Thus it is injective as $M'_f \subset M'_K$ as it is free over the domain R_f. Hence $M'_f \to M_f$ is an isomorphism and the result is proved.

For the general case, choose a surjection $R[x_1, \ldots, x_n] \to S$. Think of both S and M as finite modules over $R[x_1, \ldots, x_n]$. By the special case proved above there exists a nonzero $f \in R$ such that both S_f and M_f are free as R_f-modules and finitely presented as $R_f[x_1, \ldots, x_n]$-modules. Clearly this implies that S_f is a finitely presented R_f-algebra and that M_f is a finitely presented S_f-module. □

Let $R \to S$ be a ring map. Let M be an S-module. Consider the following condition on an element $f \in R$:

051U (10.118.3.1)
$$\begin{cases} S_f & \text{is of finite presentation over } R_f \\ M_f & \text{is of finite presentation as } S_f\text{-module} \\ S_f, M_f & \text{are free as } R_f\text{-modules} \end{cases}$$

We define

051V (10.118.3.2)
$$U(R \to S, M) = \bigcup\nolimits_{f \in R \text{ with } (10.118.3.1)} D(f)$$

which is an open subset of $\text{Spec}(R)$.

051W **Lemma 10.118.4.** Let $R \to S$ be a ring map. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of S-modules. Then

$$U(R \to S, M_1) \cap U(R \to S, M_3) \subset U(R \to S, M_2).$$

Proof. Let $u \in U(R \to S, M_1) \cap U(R \to S, M_3)$. Choose $f_1, f_3 \in R$ such that $u \in D(f_1)$, $u \in D(f_3)$ and such that (10.118.3.1) holds for f_1 and M_1 and for f_3 and M_3. Then set $f = f_1 f_3$. Then $u \in D(f)$ and (10.118.3.1) holds for f and both M_1 and M_3. An extension of free modules is free, and an extension of finitely presented modules is finitely presented (Lemma 10.5.3). Hence we see that (10.118.3.1) holds for f and M_2. Thus $u \in U(R \to S, M_2)$ and we win. □

051X Lemma 10.118.5. Let $R \to S$ be a ring map. Let M be an S-module. Let $f \in R$. Using the identification $\text{Spec}(R_f) = D(f)$ we have $U(R_f \to S_f, M_f) = D(f) \cap U(R \to S, M)$.

Proof. Suppose that $u \in U(R_f \to S_f, M_f)$. Then there exists an element $g \in R_f$ such that $u \in D(g)$ and such that (10.118.3.1) holds for the pair $((R_f)_g \to (S_f)_g, (M_f)_g)$. Write $g = a/f^n$ for some $a \in R$. Set $h = af$. Then $R_h = (R_f)_g$, $S_h = (S_f)_g$, and $M_h = (M_f)_g$. Moreover $u \in D(h)$. Hence $u \in U(R \to S, M)$. Conversely, suppose that $u \in D(f) \cap U(R \to S, M)$. Then there exists an element $g \in R$ such that $u \in D(g)$ and such that (10.118.3.1) holds for the pair $(R_g \to S_g, M_g)$. Then it is clear that (10.118.3.1) also holds for the pair $(R_{fg} \to S_{fg}, M_{fg}) = ((R_f)_g \to (S_f)_g, (M_f)_g)$. Hence $u \in U(R_f \to S_f, M_f)$ and we win. $\square$

051Y Lemma 10.118.6. Let $R \to S$ be a ring map. Let M be an S-module. Let $U \subset \text{Spec}(R)$ be a dense open. Assume there is a covering $U = \bigcup_{i \in I} D(f_i)$ of opens such that $U(R_{f_i} \to S_{f_i}, M_{f_i})$ is dense in $D(f_i)$ for each $i \in I$. Then $U(R \to S, M)$ is dense in $\text{Spec}(R)$.

Proof. In view of Lemma 10.118.5 this is a purely topological statement. Namely, by that lemma we see that $U(R \to S, M) \cap D(f_i)$ is dense in $D(f_i)$ for each $i \in I$. By Topology, Lemma 5.21.4 we see that $U(R \to S, M) \cap U$ is dense in U. Since U is dense in $\text{Spec}(R)$ we conclude that $U(R \to S, M)$ is dense in $\text{Spec}(R)$. $\square$

051Z Lemma 10.118.7. Let $R \to S$ be a ring map. Let M be an S-module. Assume

(1) $R \to S$ is of finite type,
(2) M is a finite S-module, and
(3) R is reduced.

Then there exists a subset $U \subset \text{Spec}(R)$ such that

(1) U is open and dense in $\text{Spec}(R)$,
(2) for every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that we have
   (a) M_f and S_f are free over R_f,
   (b) S_f is a finitely presented R_f-algebra, and
   (c) M_f is a finitely presented S_f-module.

Proof. Note that the lemma is equivalent to the statement that the open $U(R \to S, M)$, see Equation (10.118.3.2), is dense in $\text{Spec}(R)$. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Proof of the case $S = R[x_1, \ldots, x_n]$ and M any finite module over S. Note that in this case $S_f = R_f[x_1, \ldots, x_n]$ is free and of finite presentation over R_f, so we do not have to worry about the conditions regarding S, only those that concern M. We will use induction on n.

There exists a finite filtration

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t = M$$

such that $M_i/M_{i-1} \cong S/J_i$ for some ideal $J_i \subset S$, see Lemma 10.5.4. Since a finite intersection of dense opens is dense open, we see from Lemma 10.118.4 that it suffices to prove the lemma for each of the modules R/J_i. Hence we may assume that $M = S/J$ for some ideal J of $S = R[x_1, \ldots, x_n]$.

Let $I \subset R$ be the ideal generated by the coefficients of elements of J. Let $U_1 = \text{Spec}(R) \setminus V(I)$ and let

$$U_2 = \text{Spec}(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $\text{Spec}(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 10.118.6 we see that $U(R \to S, M)$ is dense in $\text{Spec}(R)$. Hence we may assume either (a) $I = R$, or (b) $V(I) = \text{Spec}(R)$.

In case (b) we actually have $I = 0$ as R is reduced! Hence $J = 0$ and $M = S$ and the lemma holds in this case.

In case (a) we have to do a little bit more work. Note that every element of I is actually the coefficient of a monomial of an element of J, because the set of coefficients of elements of J forms an ideal (details omitted). Hence we find an element

$$g = \sum\nolimits_{K \in E} a_K x^K \in J$$

where E is a finite set of multi-indices $K = (k_1, \ldots, k_n)$ with at least one coefficient a_{K_0} a unit in R. Actually we can find one which has a coefficient equal to 1 as $1 \in I$ in case (a). Let $m = \#\{K \in E \mid a_K \text{ is not a unit}\}$. Note that $0 \leq m \leq \#E - 1$. We will argue by induction on m.

The case $m = 0$. In this case all the coefficients a_K, $K \in E$ of g are units and $E \neq \emptyset$. If $E = \{K_0\}$ is a singleton and $K_0 = (0, \ldots, 0)$, then g is a unit and $J = S$ so the result holds for sure. (This happens in particular when $n = 0$ and it provides the base case of the induction on n.) If not $E = \{(0, \ldots, 0)\}$, then at least one K is not equal to $(0, \ldots, 0)$, i.e., $g \notin R$. At this point we employ the usual trick of Noether normalization. Namely, we consider

$$G(y_1, \ldots, y_n) = g(y_1 + y_n^{e_1}, y_2 + y_n^{e_2}, \ldots, y_{n-1} + y_n^{e_{n-1}}, y_n)$$

with $0 \ll e_{n-1} \ll e_{n-2} \ll \ldots \ll e_1$. By Lemma 10.115.2 it follows that $G(y_1, \ldots, y_n)$ as a polynomial in y_n looks like

$$a_K y_n^{k_n + \sum_{i=1, \ldots, n-1} e_i k_i} + \text{lower order terms in } y_n$$

As a_K is a unit we conclude that $M = R[x_1, \ldots, x_n]/J$ is finite over $R[y_1, \ldots, y_{n-1}]$. Hence $U(R \to R[x_1, \ldots, x_n], M) = U(R \to R[y_1, \ldots, y_{n-1}], M)$ and we win by induction on n.

The case $m > 0$. Pick a multi-index $K \in E$ such that a_K is not a unit. As before set $U_1 = \text{Spec}(R_{a_K}) = \text{Spec}(R) \setminus V(a_K)$ and set

$$U_2 = \text{Spec}(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $\text{Spec}(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 10.118.6 we see that $U(R \to S, M)$ is dense in $\text{Spec}(R)$. Hence we may assume either (a) $a_K R = R$, or (b) $V(a_K) = \text{Spec}(R)$. In case (a) the number m drops, as a_K has turned into a unit. In case (b), since R is reduced, we conclude that $a_K = 0$. Hence the set E decreases so the number m drops as well. In both cases we win by induction on m.

At this point we have proven the lemma in case $S = R[x_1, \ldots, x_n]$. Assume that $(R \to S, M)$ is an arbitrary pair satisfying the conditions of the lemma. Choose a surjection $R[x_1, \ldots, x_n] \to S$. Observe that, with the notation introduced in (10.118.3.2), we have

$$U(R \to S, M) = U(R \to R[x_1, \ldots, x_n], S) \cap U(R \to R[x_1, \ldots, x_n], M)$$

Hence as we've just finished proving the right two opens are dense also the open on the left is dense. □

10.119. Around Krull-Akizuki

00P7 One application of Krull-Akizuki is to show that there are plenty of discrete valuation rings. More generally in this section we show how to construct discrete valuation rings dominating Noetherian local rings.

First we show how to dominate a Noetherian local domain by a 1-dimensional Noetherian local domain by blowing up the maximal ideal.

00P8 **Lemma 10.119.1.** Let R be a local Noetherian domain with fraction field K. Assume R is not a field. Then there exist $R \subset R' \subset K$ with

1. R' local Noetherian of dimension 1,
2. $R \to R'$ a local ring map, i.e., R' dominates R, and
3. $R \to R'$ essentially of finite type.

Proof. Choose any valuation ring $A \subset K$ dominating R (which exist by Lemma 10.50.2). Denote v the corresponding valuation. Let $x_1, \ldots, x_r$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of R. We may and do assume that $v(x_r) = \min\{v(x_1), \ldots, v(x_r)\}$. Consider the ring

$$S = R[x_1/x_r, x_2/x_r, \ldots, x_{r-1}/x_r] \subset K.$$

Note that $\mathfrak{m}S = x_rS$ is a principal ideal. Note that $S \subset A$ and that $v(x_r) > 0$, hence we see that $x_rS \neq S$. Choose a minimal prime $\mathfrak{q}$ over x_rS. Then $\text{height}(\mathfrak{q}) = 1$ by Lemma 10.60.11 and $\mathfrak{q}$ lies over $\mathfrak{m}$. Hence we see that $R' = S_{\mathfrak{q}}$ is a solution. □

0BHZ **Lemma 10.119.2** (Kollár). Let $(R, \mathfrak{m})$ be a local Noetherian ring. Then exactly one of the following holds:

1. $(R, \mathfrak{m})$ is Artinian,
2. $(R, \mathfrak{m})$ is regular of dimension 1,
3. $\text{depth}(R) \geq 2$, or
4. there exists a finite ring map $R \to R'$ which is not an isomorphism whose kernel and cokernel are annihilated by a power of $\mathfrak{m}$ such that $\mathfrak{m}$ is not an associated prime of R' and $R' \neq 0$.

This is taken from a forthcoming paper by János Kollár entitled "Variants of normality for Noetherian schemes".

Proof. Observe that $(R, \mathfrak{m})$ is not Artinian if and only if $V(\mathfrak{m}) \subset \text{Spec}(R)$ is nowhere dense. See Proposition 10.60.7. We assume this from now on.

Let $J \subset R$ be the largest ideal killed by a power of $\mathfrak{m}$. If $J \neq 0$ then $R \to R/J$ shows that $(R, \mathfrak{m})$ is as in (4).

Otherwise $J = 0$. In particular $\mathfrak{m}$ is not an associated prime of R and we see that there is a nonzerodivisor $x \in \mathfrak{m}$ by Lemma 10.63.18. If $\mathfrak{m}$ is not an associated prime of R/xR then $\text{depth}(R) \geq 2$ by the same lemma. Thus we are left with the case when there is a $y \in R$, $y \notin xR$ such that $y\mathfrak{m} \subset xR$.

If $y\mathfrak{m} \subset x\mathfrak{m}$ then we can consider the map $\varphi : \mathfrak{m} \to \mathfrak{m}$, $f \mapsto yf/x$ (well defined as x is a nonzerodivisor). By the determinantal trick of Lemma 10.16.2 there exists a monic polynomial P with coefficients in R such that $P(\varphi) = 0$. We conclude that $P(y/x) = 0$ in R_x. Let $R' \subset R_x$ be the ring generated by R and y/x. Then $R \subset R'$ and R'/R is a finite R-module annihilated by a power of $\mathfrak{m}$. Thus R is as in (4).

Otherwise there is a $t \in \mathfrak{m}$ such that $yt = ux$ for some unit u of R. After replacing t by $u^{-1}t$ we get $yt = x$. In particular y is a nonzerodivisor. For any $t' \in \mathfrak{m}$ we have $yt' = xs$ for some $s \in R$. Thus $y(t' - st) = xs - xs = 0$. Since y is not a zero-divisor this implies that $t' = ts$ and so $\mathfrak{m} = (t)$. Thus $(R, \mathfrak{m})$ is regular of dimension 1. □

00P9 Lemma 10.119.3. Let R be a local ring with maximal ideal $\mathfrak{m}$. Assume R is Noetherian, has dimension 1, and that $\dim(\mathfrak{m}/\mathfrak{m}^2) > 1$. Then there exists a ring map $R \to R'$ such that

(1) $R \to R'$ is finite,
(2) $R \to R'$ is not an isomorphism,
(3) the kernel and cokernel of $R \to R'$ are annihilated by a power of $\mathfrak{m}$, and
(4) $\mathfrak{m}$ is not an associated prime of R'.

Proof. This follows from Lemma 10.119.2 and the fact that R is not Artinian, not regular, and does not have depth ≥ 2 (the last part because the depth does not exceed the dimension by Lemma 10.72.3). □

00PA Example 10.119.4. Consider the Noetherian local ring

$$R = k[[x, y]]/(y^2)$$

It has dimension 1 and it is Cohen-Macaulay. An example of an extension as in Lemma 10.119.3 is the extension

$$k[[x, y]]/(y^2) \subset k[[x, z]]/(z^2), \quad y \mapsto xz$$

in other words it is gotten by adjoining y/x to R. The effect of repeating the construction $n > 1$ times is to adjoin the element y/x^n.

00PB Example 10.119.5. Let k be a field of characteristic $p > 0$ such that k has infinite degree over its subfield k^p of pth powers. For example $k = \mathbf{F}_p(t_1, t_2, t_3, \ldots)$. Consider the ring

$$A = \left\{ \sum a_i x^i \in k[[x]] \text{ such that } [k^p(a_0, a_1, a_2, \ldots) : k^p] < \infty \right\}$$

Then A is a discrete valuation ring and its completion is $A^\wedge = k[[x]]$. Note that the induced extension of fraction fields of $A \subset k[[x]]$ is infinite purely inseparable. Choose any $f \in k[[x]]$, $f \notin A$. Let $R = A[f] \subset k[[x]]$. Then R is a Noetherian local domain of dimension 1 whose completion $R^\wedge$ is nonreduced (think!).

00PC Remark 10.119.6. Suppose that R is a 1-dimensional semi-local Noetherian domain. If there is a maximal ideal $\mathfrak{m} \subset R$ such that $R_\mathfrak{m}$ is not regular, then we may apply Lemma 10.119.3 to $(R, \mathfrak{m})$ to get a finite ring extension $R \subset R_1$. (For example one can do this so that $\text{Spec}(R_1) \to \text{Spec}(R)$ is the blowup of $\text{Spec}(R)$ in the ideal $\mathfrak{m}$.) Of course R_1 is a 1-dimensional semi-local Noetherian domain with the same fraction field as R. If R_1 is not a regular semi-local ring, then we may repeat the construction to get $R_1 \subset R_2$. Thus we get a sequence

$$R \subset R_1 \subset R_2 \subset R_3 \subset \ldots$$

of finite ring extensions which may stop if R_n is regular for some n. Resolution of singularities would be the claim that eventually R_n is indeed regular. In reality this is not the case. Namely, there exists a characteristic 0 Noetherian local domain A of dimension 1 whose completion is nonreduced, see [FR70, Proposition 3.1] or our Examples, Section 110.16. For an example in characteristic $p > 0$ see Example 10.119.5. Since the construction of blowing up commutes with completion it is easy to see the sequence never stabilizes. See [Ben73] for a discussion (mostly in positive characteristic). On the other hand, if the completion of R in all of its maximal ideals is reduced, then the procedure stops (insert future reference here).

00PD Lemma 10.119.7. Let A be a ring. The following are equivalent.

(1) The ring A is a discrete valuation ring.
(2) The ring A is a valuation ring and Noetherian but not a field.
(3) The ring A is a regular local ring of dimension 1.
(4) The ring A is a Noetherian local domain with maximal ideal $\mathfrak{m}$ generated by a single nonzero element.
(5) The ring A is a Noetherian local normal domain of dimension 1.

In this case if π is a generator of the maximal ideal of A, then every element of A can be uniquely written as $u\pi^n$, where $u \in A$ is a unit.

Proof. The equivalence of (1) and (2) is Lemma 10.50.18. Moreover, in the proof of Lemma 10.50.18 we saw that if A is a discrete valuation ring, then A is a PID, hence (3). Note that a regular local ring is a domain (see Lemma 10.106.2). Using this the equivalence of (3) and (4) follows from dimension theory, see Section 10.60.

Assume (3) and let π be a generator of the maximal ideal $\mathfrak{m}$. For all $n \geq 0$ we have $\dim_{A/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = 1$ because it is generated by π^n (and it cannot be zero). In particular $\mathfrak{m}^n = (\pi^n)$ and the graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the polynomial ring $A/\mathfrak{m}[T]$. For $x \in A \setminus \{0\}$ define $v(x) = \max\{n \mid x \in \mathfrak{m}^n\}$. In other words $x = u\pi^{v(x)}$ with $u \in A^*$. By the remarks above we have $v(xy) = v(x) + v(y)$ for all $x, y \in A \setminus \{0\}$. We extend this to the field of fractions K of A by setting $v(a/b) = v(a) - v(b)$ (well defined by multiplicativity shown above). Then it is clear that A is the set of elements of K which have valuation ≥ 0. Hence we see that A is a valuation ring by Lemma 10.50.16.

A valuation ring is a normal domain by Lemma 10.50.3. Hence we see that the equivalent conditions (1) – (3) imply (5). Assume (5). Suppose that $\mathfrak{m}$ cannot be generated by 1 element to get a contradiction. Then Lemma 10.119.3 implies there is a finite ring map $A \to A'$ which is an isomorphism after inverting any nonzero element of $\mathfrak{m}$ but not an isomorphism. In particular we may identify A' with a subset of the fraction field of A. Since $A \to A'$ is finite it is integral (see Lemma 10.36.3). Since A is normal we get $A = A'$ a contradiction. $\square$

09DZ Definition 10.119.8. Let A be a discrete valuation ring. A uniformizer is an element $\pi \in A$ which generates the maximal ideal of A.

By Lemma 10.119.7 any two uniformizers of a discrete valuation ring are associates.

00PE Lemma 10.119.9. Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is local Noetherian of dimension 1. For any nonzero $x \in R$ we have $\text{length}_R(R/xR) < \infty$ and

$$\text{length}_R(M/xM) \leq r \cdot \text{length}_R(R/xR).$$

Proof. If x is a unit then the result is true. Hence we may assume $x \in \mathfrak{m}$ the maximal ideal of R. Since x is not zero and R is a domain we have $\dim(R/xR) = 0$, and hence R/xR has finite length. Consider $M \subset K^{\oplus r}$ as in the lemma. We may assume that the elements of M generate $K^{\oplus r}$ as a K-vector space after replacing $K^{\oplus r}$ by a smaller subspace if necessary.

Suppose first that M is a finite R-module. In that case we can clear denominators and assume $M \subset R^{\oplus r}$. Since M generates $K^{\oplus r}$ as a vectors space we see that $R^{\oplus r}/M$ has finite length. In particular there exists an integer $c \geq 0$ such that $x^c R^{\oplus r} \subset M$. Note that $M \supset xM \supset x^2M \supset \ldots$ is a sequence of modules with successive quotients each isomorphic to M/xM. Hence we see that

$$n\text{length}_R(M/xM) = \text{length}_R(M/x^nM).$$

The same argument for $M = R^{\oplus r}$ shows that

$$n\text{length}_R(R^{\oplus r}/xR^{\oplus r}) = \text{length}_R(R^{\oplus r}/x^nR^{\oplus r}).$$

By our choice of c above we see that x^nM is sandwiched between $x^nR^{\oplus r}$ and $x^{n+c}R^{\oplus r}$. This easily gives that

$$r(n+c)\text{length}_R(R/xR) \geq n\text{length}_R(M/xM) \geq r(n-c)\text{length}_R(R/xR)$$

Hence in the finite case we actually get the result of the lemma with equality.

Suppose now that M is not finite. Suppose that the length of M/xM is $\geq k$ for some natural number k. Then we can find

$$0 \subset N_0 \subset N_1 \subset N_2 \subset \ldots N_k \subset M/xM$$

with $N_i \neq N_{i+1}$ for $i = 0, \ldots k-1$. Choose an element $m_i \in M$ whose congruence class mod xM falls into N_i but not into N_{i-1} for $i = 1, \ldots, k$. Consider the finite R-module $M' = Rm_1 + \ldots + Rm_k \subset M$. Let $N_i' \subset M'/xM'$ be the inverse image of N_i. It is clear that $N_i' \neq N_{i+1}'$ by our choice of m_i. Hence we see that $\text{length}_R(M'/xM') \geq k$. By the finite case we conclude $k \leq r\text{length}_R(R/xR)$ as desired. $\square$

Here is a first application.

031F **Lemma 10.119.10.** Let $R \to S$ be a homomorphism of domains inducing an injection of fraction fields $K \subset L$. If R is Noetherian local of dimension 1 and $[L : K] < \infty$ then

(1) each prime ideal $\mathfrak{n}_i$ of S lying over the maximal ideal $\mathfrak{m}$ of R is maximal,
(2) there are finitely many of these, and
(3) $[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m})] < \infty$ for each i.

Proof. Pick $x \in \mathfrak{m}$ nonzero. Apply Lemma 10.119.9 to the submodule $S \subset L \cong K^{\oplus n}$ where $n = [L : K]$. Thus the ring S/xS has finite length over R. It follows that $S/\mathfrak{m}S$ has finite length over $\kappa(\mathfrak{m})$. In other words, $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S$ is finite (Lemma 10.52.6). Thus $S/\mathfrak{m}S$ is Artinian (Lemma 10.53.2). The structural results on Artinian rings implies parts (1) and (2), see for example Lemma 10.53.6. Part (3) is implied by the finiteness established above. $\square$

00PF **Lemma 10.119.11.** Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is Noetherian of dimension 1. For any nonzero $x \in R$ we have $\text{length}_R(M/xM) < \infty$.

Proof. Since R has dimension 1 we see that x is contained in finitely many primes $\mathfrak m_i$, $i = 1, \ldots, n$, each maximal. Since R is Noetherian we see that R/xR is Artinian and $R/xR = \prod_{i=1,\ldots,n} (R/xR)_{\mathfrak m_i}$ by Proposition 10.60.7 and Lemma 10.53.6. Hence M/xM similarly decomposes as the product $M/xM = \prod (M/xM)_{\mathfrak m_i}$ of its localizations at the $\mathfrak m_i$. By Lemma 10.119.9 applied to $M_{\mathfrak m_i}$ over $R_{\mathfrak m_i}$ we see each $M_{\mathfrak m_i}/xM_{\mathfrak m_i} = (M/xM)_{\mathfrak m_i}$ has finite length over $R_{\mathfrak m_i}$. Thus M/xM has finite length over R as the above implies M/xM has a finite filtration by R-submodules whose successive quotients are isomorphic to the residue fields $\kappa(\mathfrak m_i)$. $\square$

00PG **Lemma 10.119.12** (Krull-Akizuki). Let R be a domain with fraction field K. Let L/K be a finite extension of fields. Assume R is Noetherian and $\dim(R) = 1$. In this case any ring A with $R \subset A \subset L$ is Noetherian.

Proof. To begin we may assume that L is the fraction field of A by replacing L by the fraction field of A if necessary. Let $I \subset A$ be a nonzero ideal. Clearly I generates L as a K-vector space. Hence we see that $I \cap R \neq (0)$. Pick any nonzero $x \in I \cap R$. Then we get $I/xA \subset A/xA$. By Lemma 10.119.11 the R-module A/xA has finite length as an R-module. Hence I/xA has finite length as an R-module. Hence I is finitely generated as an ideal in A. $\square$

00PH **Lemma 10.119.13.** Let R be a Noetherian local domain with fraction field K. Assume that R is not a field. Let L/K be a finitely generated field extension. Then there exists discrete valuation ring A with fraction field L which dominates R.

Proof. If L is not finite over K choose a transcendence basis $x_1, \ldots, x_r$ of L over K and replace R by $R[x_1, \ldots, x_r]$ localized at the maximal ideal generated by $\mathfrak m_R$ and $x_1, \ldots, x_r$. Thus we may assume $K \subset L$ finite.

By Lemma 10.119.1 we may assume $\dim(R) = 1$.

Let $A \subset L$ be the integral closure of R in L. By Lemma 10.119.12 this is Noetherian. By Lemma 10.36.17 there is a prime ideal $\mathfrak q \subset A$ lying over the maximal ideal of R. By Lemma 10.119.7 the ring $A_{\mathfrak q}$ is a discrete valuation ring dominating R as desired. $\square$

10.120. Factorization

034O Here are some notions and relations between them that are typically taught in a first year course on algebra at the undergraduate level.

034P **Definition 10.120.1.** Let R be a domain.

1. Elements $x, y \in R$ are called *associates* if there exists a unit $u \in R^*$ such that $x = uy$.
2. An element $x \in R$ is called *irreducible* if it is nonzero, not a unit and whenever $x = yz$, $y, z \in R$, then y is either a unit or an associate of x.
3. An element $x \in R$ is called *prime* if the ideal generated by x is a prime ideal.

034Q **Lemma 10.120.2.** Let R be a domain. Let $x, y \in R$. Then x, y are associates if and only if $(x) = (y)$.

Proof. If $x = uy$ for some unit $u \in R$, then $(x) \subset (y)$ and $y = u^{-1}x$ so also $(y) \subset (x)$. Conversely, suppose that $(x) = (y)$. Then $x = fy$ and $y = gx$ for some $f, g \in A$. Then $x = fgx$ and since R is a domain $fg = 1$. Thus x and y are associates. ☐

034R Lemma 10.120.3. Let R be a domain. Consider the following conditions:

(1) The ring R satisfies the ascending chain condition for principal ideals.
(2) Every nonzero, nonunit element $a \in R$ has a factorization $a = b_1 \ldots b_k$ with each b_i an irreducible element of R.

Then (1) implies (2).

Proof. Let x be a nonzero element, not a unit, which does not have a factorization into irreducibles. Set $x_1 = x$. We can write $x = yz$ where neither y nor z is irreducible or a unit. Then either y does not have a factorization into irreducibles, in which case we set $x_2 = y$, or z does not have a factorization into irreducibles, in which case we set $x_2 = z$. Continuing in this fashion we find a sequence

$$x_1 | x_2 | x_3 | \ldots$$

of elements of R with x_n / x_{n+1} not a unit. This gives a strictly increasing sequence of principal ideals $(x_1) \subset (x_2) \subset (x_3) \subset \ldots$ thereby finishing the proof. ☐

034S Definition 10.120.4. A unique factorization domain, abbreviated UFD, is a domain R such that if $x \in R$ is a nonzero, nonunit, then x has a factorization into irreducibles, and if

$$x = a_1 \ldots a_m = b_1 \ldots b_n$$

are factorizations into irreducibles then $n = m$ and there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that a_i and $b_{\sigma(i)}$ are associates.

034T Lemma 10.120.5. Let R be a domain. Assume every nonzero, nonunit factors into irreducibles. Then R is a UFD if and only if every irreducible element is prime.

Proof. Assume R is a UFD and let $x \in R$ be an irreducible element. Say $ab \in (x)$, i.e., $ab = cx$. Choose factorizations $a = a_1 \ldots a_n$, $b = b_1 \ldots b_m$, and $c = c_1 \ldots c_r$. By uniqueness of the factorization

$$a_1 \ldots a_n b_1 \ldots b_m = c_1 \ldots c_r x$$

we find that x is an associate of one of the elements $a_1, \ldots, b_m$. In other words, either $a \in (x)$ or $b \in (x)$ and we conclude that x is prime.

Assume every irreducible element is prime. We have to prove that factorization into irreducibles is unique up to permutation and taking associates. Say $a_1 \ldots a_m = b_1 \ldots b_n$ with a_i and b_j irreducible. Since a_1 is prime, we see that $b_j \in (a_1)$ for some j. After renumbering we may assume $b_1 \in (a_1)$. Then $b_1 = a_1 u$ and since b_1 is irreducible we see that u is a unit. Hence a_1 and b_1 are associates and $a_2 \ldots a_n = ub_2 \ldots b_m$. By induction on $n + m$ we see that $n = m$ and a_i associate to $b_{\sigma(i)}$ for $i = 2, \ldots, n$ as desired. ☐

0AFT Lemma 10.120.6. Let R be a Noetherian domain. Then R is a UFD if and only if every height 1 prime ideal is principal.

Proof. Assume R is a UFD and let $\mathfrak{p}$ be a height 1 prime ideal. Take $x \in \mathfrak{p}$ nonzero and let $x = a_1 \ldots a_n$ be a factorization into irreducibles. Since $\mathfrak{p}$ is prime we see that $a_i \in \mathfrak{p}$ for some i. By Lemma 10.120.5 the ideal (a_i) is prime. Since $\mathfrak{p}$ has height 1 we conclude that $(a_i) = \mathfrak{p}$.

Assume every height 1 prime is principal. Since R is Noetherian every nonzero nonunit element x has a factorization into irreducibles, see Lemma 10.120.3. It suffices to prove that an irreducible element x is prime, see Lemma 10.120.5. Let $(x) \subset \mathfrak{p}$ be a prime minimal over (x). Then $\mathfrak{p}$ has height 1 by Lemma 10.60.11. By assumption $\mathfrak{p} = (y)$. Hence $x = yz$ and z is a unit as x is irreducible. Thus $(x) = (y)$ and we see that x is prime. □

0AFU Lemma 10.120.7 (Nagata's criterion for factoriality). Let A be a domain. Let $S \subset A$ be a multiplicative subset generated by prime elements. Let $x \in A$ be irreducible. Then [Nag57b, Lemma 2]

1. the image of x in $S^{-1}A$ is irreducible or a unit, and
2. x is prime if and only if the image of x in $S^{-1}A$ is a unit or a prime element in $S^{-1}A$.

Moreover, then A is a UFD if and only if every element of A has a factorization into irreducibles and $S^{-1}A$ is a UFD.

Proof. Say $x = \alpha\beta$ for $\alpha, \beta \in S^{-1}A$. Then $\alpha = a/s$ and $\beta = b/s'$ for $a, b \in A$, $s, s' \in S$. Thus we get $ss'x = ab$. By assumption we can write $ss' = p_1 \ldots p_r$ for some prime elements p_i. For each i the element p_i divides either a or b. Dividing we find a factorization $x = a'b'$ and $a = s''a'$, $b = s'''b'$ for some $s'', s''' \in S$. As x is irreducible, either a' or b' is a unit. Tracing back we find that either α or β is a unit. This proves (1).

Suppose x is prime. Then $A/(x)$ is a domain. Hence $S^{-1}A/xS^{-1}A = S^{-1}(A/(x))$ is a domain or zero. Thus x maps to a prime element or a unit.

Suppose that the image of x in $S^{-1}A$ is a unit. Then $yx = s$ for some $s \in S$ and $y \in A$. By assumption $s = p_1 \ldots p_r$ with p_i a prime element. For each i either p_i divides y or p_i divides x. In the second case p_i and x are associates (as x is irreducible) and we are done. But if the first case happens for all $i = 1, \ldots, r$, then x is a unit which is a contradiction.

Suppose that the image of x in $S^{-1}A$ is a prime element. Assume $a, b \in A$ and $ab \in (x)$. Then $sa = xy$ or $sb = xy$ for some $s \in S$ and $y \in A$. Say the first case happens. By assumption $s = p_1 \ldots p_r$ with p_i a prime element. For each i either p_i divides y or p_i divides x. In the second case p_i and x are associates (as x is irreducible) and we are done. If the first case happens for all $i = 1, \ldots, r$, then $a \in (x)$ as desired. This completes the proof of (2).

The final statement of the lemma follows from (1) and (2) and Lemma 10.120.5. □

0BUD Lemma 10.120.8. A UFD satisfies the ascending chain condition for principal ideals.

Proof. Consider an ascending chain $(a_1) \subset (a_2) \subset (a_3) \subset \ldots$ of principal ideals in R. Write $a_1 = p_1^{e_1} \ldots p_r^{e_r}$ with p_i prime. Then we see that a_n is an associate of $p_1^{c_1} \ldots p_r^{c_r}$ for some $0 \leq c_i \leq e_i$. Since there are only finitely many possibilities we conclude. □

0BUE Lemma 10.120.9. Let R be a domain. Assume R has the ascending chain condition for principal ideals. Then the same property holds for a polynomial ring over R.

Proof. Consider an ascending chain $(f_1) \subset (f_2) \subset (f_3) \subset \ldots$ of principal ideals in $R[x]$. Since f_{n+1} divides f_n we see that the degrees decrease in the sequence. Thus f_n has fixed degree $d \geq 0$ for all $n \gg 0$. Let a_n be the leading coefficient of f_n. The condition $f_n \in (f_{n+1})$ implies that a_{n+1} divides a_n for all n. By our assumption on R we see that a_{n+1} and a_n are associates for all n large enough (Lemma 10.120.2). Thus for large n we see that $f_n = uf_{n+1}$ where $u \in R$ (for reasons of degree) is a unit (as a_n and a_{n+1} are associates). □

0BC1 Lemma 10.120.10. A polynomial ring over a UFD is a UFD. In particular, if k is a field, then $k[x_1, \ldots, x_n]$ is a UFD.

Proof. Let R be a UFD. Then R satisfies the ascending chain condition for principal ideals (Lemma 10.120.8), hence $R[x]$ satisfies the ascending chain condition for principal ideals (Lemma 10.120.9), and hence every element of $R[x]$ has a factorization into irreducibles (Lemma 10.120.3). Let $S \subset R$ be the multiplicative subset generated by prime elements. Since every nonunit of R is a product of prime elements we see that $K = S^{-1}R$ is the fraction field of R. Observe that every prime element of R maps to a prime element of $R[x]$ and that $S^{-1}(R[x]) = S^{-1}R[x] = K[x]$ is a UFD (and even a PID). Thus we may apply Lemma 10.120.7 to conclude. □

0AFV Lemma 10.120.11. A unique factorization domain is normal.

Proof. Let R be a UFD. Let x be an element of the fraction field of R which is integral over R. Say $x^d - a_1x^{d-1} - \ldots - a_d = 0$ with $a_i \in R$. We can write $x = up_1^{e_1} \ldots p_r^{e_r}$ with u a unit, $e_i \in \mathbf{Z}$, and $p_1, \ldots, p_r$ irreducible elements which are not associates. To prove the lemma we have to show $e_i \geq 0$. If not, say $e_1 < 0$, then for $N \gg 0$ we get

$$u^d p_2^{de_2+N} \ldots p_r^{de_r+N} = p_1^{-de_1} p_2^N \ldots p_r^N (\sum\nolimits_{i=1,\ldots,d} a_i x^{d-i}) \in (p_1)$$

which contradicts uniqueness of factorization in R. □

034U Definition 10.120.12. A principal ideal domain, abbreviated PID, is a domain R such that every ideal is a principal ideal.

034V Lemma 10.120.13. A principal ideal domain is a unique factorization domain.

Proof. As a PID is Noetherian this follows from Lemma 10.120.6. □

034W Definition 10.120.14. A Dedekind domain is a domain R such that every nonzero ideal $I \subset R$ can be written as a product

$$I = \mathfrak{p}_1 \ldots \mathfrak{p}_r$$

of nonzero prime ideals uniquely up to permutation of the $\mathfrak{p}_i$.

0AUQ Lemma 10.120.15. A PID is a Dedekind domain.

Proof. Let R be a PID. Since every nonzero ideal of R is principal, and R is a UFD (Lemma 10.120.13), this follows from the fact that every irreducible element in R is prime (Lemma 10.120.5) so that factorizations of elements turn into factorizations into primes. □

09ME Lemma 10.120.16. Let A be a ring. Let I and J be nonzero ideals of A such that $IJ = (f)$ for some nonzerodivisor $f \in A$. Then I and J are finitely generated ideals and finitely locally free of rank 1 as A-modules.

Proof. It suffices to show that I and J are finite locally free A-modules of rank 1, see Lemma 10.78.2. To do this, write $f = \sum_{i=1,\ldots,n} x_i y_i$ with $x_i \in I$ and $y_i \in J$. We can also write $x_i y_i = a_i f$ for some $a_i \in A$. Since f is a nonzerodivisor we see that $\sum a_i = 1$. Thus it suffices to show that each I_{a_i} and J_{a_i} is free of rank 1 over A_{a_i}. After replacing A by A_{a_i} we conclude that $f = xy$ for some $x \in I$ and $y \in J$. Note that both x and y are nonzerodivisors. We claim that $I = (x)$ and $J = (y)$ which finishes the proof. Namely, if $x' \in I$, then $x'y = af = axy$ for some $a \in A$. Hence $x' = ax$ and we win. □

034X Lemma 10.120.17. Let R be a ring. The following are equivalent

(1) R is a Dedekind domain,
(2) R is a Noetherian domain, and for every maximal ideal $\mathfrak{m}$ the local ring $R_{\mathfrak{m}}$ is a discrete valuation ring, and
(3) R is a Noetherian, normal domain, and $\dim(R) \leq 1$.

Proof. Assume (1). The argument is nontrivial because we did not assume that R was Noetherian in our definition of a Dedekind domain. Let $\mathfrak{p} \subset R$ be a prime ideal. Observe that $\mathfrak{p} \neq \mathfrak{p}^2$ by uniqueness of the factorizations in the definition. Pick $x \in \mathfrak{p}$ with $x \notin \mathfrak{p}^2$. Let $y \in \mathfrak{p}$ be a second element (for example $y = 0$). Write $(x, y) = \mathfrak{p}_1 \ldots \mathfrak{p}_r$. Since $(x, y) \subset \mathfrak{p}$ at least one of the primes $\mathfrak{p}_i$ is contained in $\mathfrak{p}$. But as $x \notin \mathfrak{p}^2$ there is at most one. Thus exactly one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ is contained in $\mathfrak{p}$, say $\mathfrak{p}_1 \subset \mathfrak{p}$. We conclude that $(x, y)R_{\mathfrak{p}} = \mathfrak{p}_1 R_{\mathfrak{p}}$ is prime for every choice of y. We claim that $(x)R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$. Namely, pick $y \in \mathfrak{p}$. By the above applied with y^2 we see that $(x, y^2)R_{\mathfrak{p}}$ is prime. Hence $y \in (x, y^2)R_{\mathfrak{p}}$, i.e., $y = ax + by^2$ in $R_{\mathfrak{p}}$. Thus $(1 - by)y = ax \in (x)R_{\mathfrak{p}}$, i.e., $y \in (x)R_{\mathfrak{p}}$ as desired.

Writing $(x) = \mathfrak{p}_1 \ldots \mathfrak{p}_r$ anew with $\mathfrak{p}_1 \subset \mathfrak{p}$ we conclude that $\mathfrak{p}_1 R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$, i.e., $\mathfrak{p}_1 = \mathfrak{p}$. Moreover, $\mathfrak{p}_1 = \mathfrak{p}$ is a finitely generated ideal of R by Lemma 10.120.16. We conclude that R is Noetherian by Lemma 10.28.10. Moreover, it follows that $R_{\mathfrak{m}}$ is a discrete valuation ring for every prime ideal $\mathfrak{p}$, see Lemma 10.119.7.

The equivalence of (2) and (3) follows from Lemmas 10.37.10 and 10.119.7. Assume (2) and (3) are satisfied. Let $I \subset R$ be an ideal. We will construct a factorization of I. If I is prime, then there is nothing to prove. If not, pick $I \subset \mathfrak{p}$ with $\mathfrak{p} \subset R$ maximal. Let $J = \{x \in R \mid x\mathfrak{p} \subset I\}$. We claim $J\mathfrak{p} = I$. It suffices to check this after localization at the maximal ideals $\mathfrak{m}$ of R (the formation of J commutes with localization and we use Lemma 10.23.1). Then either $\mathfrak{p}R_{\mathfrak{m}} = R_{\mathfrak{m}}$ and the result is clear, or $\mathfrak{p}R_{\mathfrak{m}} = \mathfrak{m}R_{\mathfrak{m}}$. In the last case $\mathfrak{p}R_{\mathfrak{m}} = (\pi)$ and the case where $\mathfrak{p}$ is principal is immediate. By Noetherian induction the ideal J has a factorization and we obtain the desired factorization of I. We omit the proof of uniqueness of the factorization. □

The following is a variant of the Krull-Akizuki lemma.

09IG Lemma 10.120.18. Let A be a Noetherian domain of dimension 1 with fraction field K. Let L/K be a finite extension. Let B be the integral closure of A in L. Then B is a Dedekind domain and $\text{Spec}(B) \to \text{Spec}(A)$ is surjective, has finite fibres, and induces finite residue field extensions.

Proof. By Krull-Akizuki (Lemma 10.119.12) the ring B is Noetherian. By Lemma 10.112.4 $\dim(B) = 1$. Thus B is a Dedekind domain by Lemma 10.120.17. Surjectivity of the map on spectra follows from Lemma 10.36.17. The last two statements follow from Lemma 10.119.10. □

10.121. Orders of vanishing

02MB

02MC **Lemma 10.121.1.** Let R be a semi-local Noetherian ring of dimension 1. If $a, b \in R$ are nonzerodivisors then

$$\text{length}_R(R/(ab)) = \text{length}_R(R/(a)) + \text{length}_R(R/(b))$$

and these lengths are finite.

Proof. We saw the finiteness in Lemma 10.119.11. Additivity holds since there is a short exact sequence $0 \to R/(a) \to R/(ab) \to R/(b) \to 0$ where the first map is given by multiplication by b. (Use length is additive, see Lemma 10.52.3.) $\square$

02MD **Definition 10.121.2.** Suppose that K is a field, and $R \subset K$ is a local[10] Noetherian subring of dimension 1 with fraction field K. In this case we define the *order of vanishing along R*

$$\text{ord}_R : K^* \longrightarrow \mathbf{Z}$$

by the rule

$$\text{ord}_R(x) = \text{length}_R(R/(x))$$

if $x \in R$ and we set $\text{ord}_R(x/y) = \text{ord}_R(x) - \text{ord}_R(y)$ for $x, y \in R$ both nonzero.

We can use the order of vanishing to compare lattices in a vector space. Here is the definition.

02ME **Definition 10.121.3.** Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. A *lattice in V* is a finite R-submodule $M \subset V$ such that $V = K \otimes_R M$.

The condition $V = K \otimes_R M$ signifies that M contains a basis for the vector space V. We remark that in many places in the literature the notion of a lattice may be defined only in case the ring R is a discrete valuation ring. If R is a discrete valuation ring then any lattice is a free R-module, and this may not be the case in general.

02MF **Lemma 10.121.4.** Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space.

1. If M is a lattice in V and $M \subset M' \subset V$ is an R-submodule of V containing M then the following are equivalent
   (a) M' is a lattice,
   (b) $\text{length}_R(M'/M)$ is finite, and
   (c) M' is finitely generated.
2. If M is a lattice in V and $M' \subset M$ is an R-submodule of M then M' is a lattice if and only if $\text{length}_R(M/M')$ is finite.
3. If M, M' are lattices in V, then so are $M \cap M'$ and $M + M'$.
4. If $M \subset M' \subset M'' \subset V$ are lattices in V then
$$\text{length}_R(M''/M) = \text{length}_R(M'/M) + \text{length}_R(M''/M').$$

[10] We could also define this when R is only semi-local but this is probably never really what you want!

(5) If M, M', N, N' are lattices in V and $N \subset M \cap M'$, $M + M' \subset N'$, then we have

$$\begin{aligned} & \text{length}_R(M/M \cap M') - \text{length}_R(M'/M \cap M') \\ = \; & \text{length}_R(M/N) - \text{length}_R(M'/N) \\ = \; & \text{length}_R(M + M'/M') - \text{length}_R(M + M'/M) \\ = \; & \text{length}_R(N'/M') - \text{length}_R(N'/M) \end{aligned}$$

Proof. Proof of (1). Assume (1)(a). Say $y_1, \ldots, y_m$ generate M'. Then each $y_i = x_i/f_i$ for some $x_i \in M$ and nonzero $f_i \in R$. Hence we see that $f_1 \ldots f_m M' \subset M$. Since R is Noetherian local of dimension 1 we see that $\mathfrak{m}^n \subset (f_1 \ldots f_m)$ for some n (for example combine Lemmas 10.60.13 and Proposition 10.60.7 or combine Lemmas 10.119.9 and 10.52.4). In other words $\mathfrak{m}^n M' \subset M$ for some n Hence $\text{length}(M'/M) < \infty$ by Lemma 10.52.8, in other words (1)(b) holds. Assume (1)(b). Then M'/M is a finite R-module (see Lemma 10.52.2). Hence M' is a finite R-module as an extension of finite R-modules. Hence (1)(c). The implication (1)(c) $\Rightarrow$ (1)(a) follows from the remark following Definition 10.121.3.

Proof of (2). Suppose M is a lattice in V and $M' \subset M$ is an R-submodule. We have seen in (1) that if M' is a lattice, then $\text{length}_R(M/M') < \infty$. Conversely, assume that $\text{length}_R(M/M') < \infty$. Then M' is finitely generated as R is Noetherian and for some n we have $\mathfrak{m}^n M \subset M'$ (Lemma 10.52.4). Hence it follows that M' contains a basis for V, and M' is a lattice.

Proof of (3). Assume M, M' are lattices in V. Since R is Noetherian the submodule $M \cap M'$ of M is finite. As M is a lattice we can find $x_1, \ldots, x_n \in M$ which form a K-basis for V. Because M' is a lattice we can write $x_i = y_i/f_i$ with $y_i \in M'$ and $f_i \in R$. Hence $f_i x_i \in M \cap M'$. Hence $M \cap M'$ is a lattice also. The fact that $M + M'$ is a lattice follows from part (1).

Part (4) follows from additivity of lengths (Lemma 10.52.3) and the exact sequence

$$0 \to M'/M \to M''/M \to M''/M' \to 0$$

Part (5) follows from repeatedly applying part (4). $\square$

02MG Definition 10.121.5. Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let M, M' be two lattices in V. The distance between M and M' is the integer

$$d(M, M') = \text{length}_R(M/M \cap M') - \text{length}_R(M'/M \cap M')$$

of Lemma 10.121.4 part (5).

In particular, if $M' \subset M$, then $d(M, M') = \text{length}_R(M/M')$.

02MH Lemma 10.121.6. Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. This distance function has the property that

$$d(M, M'') = d(M, M') + d(M', M'')$$

whenever given three lattices M, M', M'' of V. In particular we have $d(M, M') = -d(M', M)$.

Proof. Omitted. $\square$

02MI Lemma 10.121.7. Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let $\varphi : V \to V$ be a K-linear isomorphism. For any lattice $M \subset V$ we have

$$d(M, \varphi(M)) = \mathrm{ord}_R(\det(\varphi))$$

Proof. We can see that the integer $d(M, \varphi(M))$ does not depend on the lattice M as follows. Suppose that M' is a second such lattice. Then we see that

$$\begin{aligned} d(M, \varphi(M)) &= d(M, M') + d(M', \varphi(M)) \\ &= d(M, M') + d(\varphi(M'), \varphi(M)) + d(M', \varphi(M')) \end{aligned}$$

Since φ is an isomorphism we see that $d(\varphi(M'), \varphi(M)) = d(M', M) = -d(M, M')$, and hence $d(M, \varphi(M)) = d(M', \varphi(M'))$. Moreover, both sides of the equation (of the lemma) are additive in φ, i.e.,

$$\mathrm{ord}_R(\det(\varphi \circ \psi)) = \mathrm{ord}_R(\det(\varphi)) + \mathrm{ord}_R(\det(\psi))$$

and also

$$\begin{aligned} d(M, \varphi(\psi((M))) &= d(M, \psi(M)) + d(\psi(M), \varphi(\psi(M))) \\ &= d(M, \psi(M)) + d(M, \varphi(M)) \end{aligned}$$

by the independence shown above. Hence it suffices to prove the lemma for generators of $\mathrm{GL}(V)$. Choose an isomorphism $K^{\oplus n} \cong V$. Then $\mathrm{GL}(V) = \mathrm{GL}_n(K)$ is generated by elementary matrices E. The result is clear for E equal to the identity matrix. If $E = E_{ij}(\lambda)$ with $i \neq j$, $\lambda \in K$, $\lambda \neq 0$, for example

$$E_{12}(\lambda) = \begin{pmatrix} 1 & \lambda & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then with respect to a different basis we get $E_{12}(1)$. The result is clear for $E = E_{12}(1)$ by taking as lattice $R^{\oplus n} \subset K^{\oplus n}$. Finally, if $E = E_i(a)$, with $a \in K^*$ for example

$$E_1(a) = \begin{pmatrix} a & 0 & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then $E_1(a)(R^{\oplus b}) = aR \oplus R^{\oplus n-1}$ and it is clear that $d(R^{\oplus n}, aR \oplus R^{\oplus n-1}) = \mathrm{ord}_R(a)$ as desired. □

02MJ Lemma 10.121.8. Let $A \to B$ be a ring map. Assume

(1) A is a Noetherian local domain of dimension 1,
(2) $A \subset B$ is a finite extension of domains.

Let L/K be the corresponding finite extension of fraction fields. Let $y \in L^*$ and $x = \mathrm{Nm}_{L/K}(y)$. In this situation B is semi-local. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$\mathrm{ord}_A(x) = \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] \mathrm{ord}_{B_{\mathfrak{m}_i}}(y)$$

where ord is defined as in Definition 10.121.2.

Proof. The ring B is semi-local by Lemma 10.113.2. Write $y = b/b'$ for some $b, b' \in B$. By the additivity of ord and multiplicativity of Nm it suffices to prove the lemma for $y = b$ or $y = b'$. In other words we may assume $y \in B$. In this case the right hand side of the formula is

$$\sum [\kappa(\mathfrak m_i) : \kappa(\mathfrak m_A)] \text{length}_{B_{\mathfrak m_i}}((B/yB)_{\mathfrak m_i})$$

By Lemma 10.52.12 this is equal to $\text{length}_A(B/yB)$. By Lemma 10.121.7 we have

$$\text{length}_A(B/yB) = d(B, yB) = \text{ord}_A(\det_K(L \xrightarrow{y} L)).$$

Since $x = \text{Nm}_{L/K}(y) = \det_K(L \xrightarrow{y} L)$ by definition the lemma is proved. □

10.122. Quasi-finite maps

02MK Consider a ring map $R \to S$ of finite type. A map $\text{Spec}(S) \to \text{Spec}(R)$ is quasi-finite at a point if that point is isolated in its fibre. This means that the fibre is zero dimensional at that point. In this section we study the basic properties of this important but technical notion. More advanced material can be found in the next section.

00PJ **Lemma 10.122.1.** Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak q$ be a prime of S. The following are equivalent:

(1) $\mathfrak q$ is an isolated point of $\text{Spec}(S)$,
(2) $S_{\mathfrak q}$ is finite over k,
(3) there exists a $g \in S$, $g \notin \mathfrak q$ such that $D(g) = \{\mathfrak q\}$,
(4) $\dim_{\mathfrak q} \text{Spec}(S) = 0$,
(5) $\mathfrak q$ is a closed point of $\text{Spec}(S)$ and $\dim(S_{\mathfrak q}) = 0$, and
(6) the field extension $\kappa(\mathfrak q)/k$ is finite and $\dim(S_{\mathfrak q}) = 0$.

In this case $S = S_{\mathfrak q} \times S'$ for some finite type k-algebra S'. Also, the element g as in (3) has the property $S_{\mathfrak q} = S_g$.

Proof. Suppose $\mathfrak q$ is an isolated point of $\text{Spec}(S)$, i.e., $\{\mathfrak q\}$ is open in $\text{Spec}(S)$. Because $\text{Spec}(S)$ is a Jacobson space (see Lemmas 10.35.2 and 10.35.4) we see that $\mathfrak q$ is a closed point. Hence $\{\mathfrak q\}$ is open and closed in $\text{Spec}(S)$. By Lemmas 10.21.3 and 10.24.3 we may write $S = S_1 \times S_2$ with $\mathfrak q$ corresponding to the only point $\text{Spec}(S_1)$. Hence $S_1 = S_{\mathfrak q}$ is a zero dimensional ring of finite type over k. Hence it is finite over k for example by Lemma 10.115.4. We have proved (1) implies (2).

Suppose $S_{\mathfrak q}$ is finite over k. Then $S_{\mathfrak q}$ is Artinian local, see Lemma 10.53.2. So $\text{Spec}(S_{\mathfrak q}) = \{\mathfrak q S_{\mathfrak q}\}$ by Lemma 10.53.6. Consider the exact sequence $0 \to K \to S \to S_{\mathfrak q} \to Q \to 0$. It is clear that $K_{\mathfrak q} = Q_{\mathfrak q} = 0$. Also, K is a finite S-module as S is Noetherian and Q is a finite S-module since $S_{\mathfrak q}$ is finite over k. Hence there exists $g \in S$, $g \notin \mathfrak q$ such that $K_g = Q_g = 0$. Thus $S_{\mathfrak q} = S_g$ and $D(g) = \{\mathfrak q\}$. We have proved that (2) implies (3).

Suppose $D(g) = \{\mathfrak q\}$. Since $D(g)$ is open by construction of the topology on $\text{Spec}(S)$ we see that $\mathfrak q$ is an isolated point of $\text{Spec}(S)$. We have proved that (3) implies (1). In other words (1), (2) and (3) are equivalent.

Assume $\dim_{\mathfrak q} \text{Spec}(S) = 0$. This means that there is some open neighbourhood of $\mathfrak q$ in $\text{Spec}(S)$ which has dimension zero. Then there is an open neighbourhood of the form $D(g)$ which has dimension zero. Since S_g is Noetherian we conclude that S_g is Artinian and $D(g) = \text{Spec}(S_g)$ is a finite discrete set, see Proposition 10.60.7. Thus $\mathfrak q$ is an isolated point of $D(g)$ and, by the equivalence of (1) and (2) above applied to $\mathfrak{q}S_g \subset S_g$, we see that $S_\mathfrak{q} = (S_g)_{\mathfrak{q}S_g}$ is finite over k. Hence (4) implies (2). It is clear that (1) implies (4). Thus (1) – (4) are all equivalent.

Lemma 10.114.6 gives the implication (5) $\Rightarrow$ (4). The implication (4) $\Rightarrow$ (6) follows from Lemma 10.116.3. The implication (6) $\Rightarrow$ (5) follows from Lemma 10.35.9. At this point we know (1) – (6) are equivalent.

The two statements at the end of the lemma we saw during the course of the proof of the equivalence of (1), (2) and (3) above. ☐

00PK Lemma 10.122.2. Let $R \to S$ be a ring map of finite type. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $F = \text{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ be the fibre of $\text{Spec}(S) \to \text{Spec}(R)$, see Remark 10.17.8. Denote $\overline{\mathfrak{q}} \in F$ the point corresponding to $\mathfrak{q}$. The following are equivalent

(1) $\overline{\mathfrak{q}}$ is an isolated point of F,
(2) $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is finite over $\kappa(\mathfrak{p})$,
(3) there exists a $g \in S$, $g \notin \mathfrak{q}$ such that the only prime of $D(g)$ mapping to $\mathfrak{p}$ is $\mathfrak{q}$,
(4) $\dim_{\overline{\mathfrak{q}}}(F) = 0$,
(5) $\overline{\mathfrak{q}}$ is a closed point of F and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$, and
(6) the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$.

Proof. Note that $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = (S \otimes_R \kappa(\mathfrak{p}))_{\overline{\mathfrak{q}}}$. Moreover $S \otimes_R \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$. The conditions correspond exactly to the conditions of Lemma 10.122.1 for the $\kappa(\mathfrak{p})$-algebra $S \otimes_R \kappa(\mathfrak{p})$ and the prime $\overline{\mathfrak{q}}$, hence they are equivalent. ☐

00PL Definition 10.122.3. Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime.

(1) If the equivalent conditions of Lemma 10.122.2 are satisfied then we say $R \to S$ is quasi-finite at $\mathfrak{q}$.
(2) We say a ring map $A \to B$ is quasi-finite if it is of finite type and quasi-finite at all primes of B.

00PM Lemma 10.122.4. Let $R \to S$ be a finite type ring map. Then $R \to S$ is quasi-finite if and only if for all primes $\mathfrak{p} \subset R$ the fibre $S \otimes_R \kappa(\mathfrak{p})$ is finite over $\kappa(\mathfrak{p})$.

Proof. If the fibres are finite then the map is clearly quasi-finite. For the converse, note that $S \otimes_R \kappa(\mathfrak{p})$ is a $\kappa(\mathfrak{p})$-algebra of finite type and of dimension 0. Hence it is finite over $\kappa(\mathfrak{p})$ for example by Lemma 10.115.4. ☐

077H Lemma 10.122.5. Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $f \in R$, $f \notin \mathfrak{p}$ and $g \in S$, $g \notin \mathfrak{q}$. Then $R \to S$ is quasi-finite at $\mathfrak{q}$ if and only if $R_f \to S_{fg}$ is quasi-finite at $\mathfrak{q}S_{fg}$.

Proof. The fibre of $\text{Spec}(S_{fg}) \to \text{Spec}(R_f)$ is homeomorphic to an open subset of the fibre of $\text{Spec}(S) \to \text{Spec}(R)$. Hence the lemma follows from part (1) of the equivalent conditions of Lemma 10.122.2. ☐

00PN Lemma 10.122.6. Let

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd}[arrows=dash] \mathfrak{q} \arrow[r] & \mathfrak{q}' \\ \mathfrak{p} \arrow[u] \arrow[r] & \mathfrak{p}' \arrow[u] \end{tikzcd}$$

be a commutative diagram of rings with primes as indicated. Assume $R \to S$ of finite type, and $S \otimes_R R' \to S'$ surjective. If $R \to S$ is quasi-finite at $\mathfrak{q}$, then $R' \to S'$ is quasi-finite at $\mathfrak{q}'$.

Proof. Write $S \otimes_R \kappa(\mathfrak{p}) = S_1 \times S_2$ with S_1 finite over $\kappa(\mathfrak{p})$ and such that $\mathfrak{q}$ corresponds to a point of S_1 as in Lemma 10.122.1. This product decomposition induces a corresponding product decomposition for any $S \otimes_R \kappa(\mathfrak{p})$-algebra. In particular, we obtain $S' \otimes_{R'} \kappa(\mathfrak{p}') = S'_1 \times S'_2$. Because $S \otimes_R R' \to S'$ is surjective the canonical map $(S \otimes_R \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}') \to S' \otimes_{R'} \kappa(\mathfrak{p}')$ is surjective and hence $S_i \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}') \to S'_i$ is surjective. It follows that S'_1 is finite over $\kappa(\mathfrak{p}')$. The map $S' \otimes_{R'} \kappa(\mathfrak{p}') \to \kappa(\mathfrak{q}')$ factors through S'_1 (i.e. it annihilates the factor S'_2) because the map $S \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ factors through S_1 (i.e. it annihilates the factor S_2). Thus $\mathfrak{q}'$ corresponds to a point of $\text{Spec}(S'_1)$ in the disjoint union decomposition of the fibre: $\text{Spec}(S' \otimes_{R'} \kappa(\mathfrak{p}')) = \text{Spec}(S'_1) \amalg \text{Spec}(S'_2)$, see Lemma 10.21.2. Since S'_1 is finite over a field, it is Artinian ring, and hence $\text{Spec}(S'_1)$ is a finite discrete set. (See Proposition 10.60.7.) We conclude $\mathfrak{q}'$ is isolated in its fibre as desired. $\square$

00PO Lemma 10.122.7. A composition of quasi-finite ring maps is quasi-finite.

Proof. Suppose $A \to B$ and $B \to C$ are quasi-finite ring maps. By Lemma 10.6.2 we see that $A \to C$ is of finite type. Let $\mathfrak{r} \subset C$ be a prime of C lying over $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$. Since $A \to B$ and $B \to C$ are quasi-finite at $\mathfrak{q}$ and $\mathfrak{r}$ respectively, then there exist $b \in B$ and $c \in C$ such that $\mathfrak{q}$ is the only prime of $D(b)$ which maps to $\mathfrak{p}$ and similarly $\mathfrak{r}$ is the only prime of $D(c)$ which maps to $\mathfrak{q}$. If $c' \in C$ is the image of $b \in B$, then $\mathfrak{r}$ is the only prime of $D(cc')$ which maps to $\mathfrak{p}$. Therefore $A \to C$ is quasi-finite at $\mathfrak{r}$. $\square$

00PP Lemma 10.122.8. Let $R \to S$ be a ring map of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$.

(1) The set $\{\mathfrak{q}' \mid R' \to S' \text{ quasi-finite at } \mathfrak{q}'\}$ is the inverse image of the corresponding set of $\text{Spec}(S)$ under the canonical map $\text{Spec}(S') \to \text{Spec}(S)$.
(2) If $\text{Spec}(R') \to \text{Spec}(R)$ is surjective, then $R \to S$ is quasi-finite if and only if $R' \to S'$ is quasi-finite.
(3) Any base change of a quasi-finite ring map is quasi-finite.

Proof. Let $\mathfrak{p}' \subset R'$ be a prime lying over $\mathfrak{p} \subset R$. Then the fibre ring $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is the base change of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ by the field extension $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$. Hence the first assertion follows from the invariance of dimension under field extension (Lemma 10.116.6) and Lemma 10.122.1. The stability of quasi-finite maps under base change follows from this and the stability of finite type property under base change. The second assertion follows since the assumption implies that given a prime $\mathfrak{q} \subset S$ we can find a prime $\mathfrak{q}' \subset S'$ lying over it. $\square$

0C6H Lemma 10.122.9. Let $A \to B$ and $B \to C$ be ring homomorphisms such that $A \to C$ is of finite type. Let $\mathfrak{r}$ be a prime of C lying over $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$. If $A \to C$ is quasi-finite at $\mathfrak{r}$, then $B \to C$ is quasi-finite at $\mathfrak{r}$.

Proof. Observe that $B \to C$ is of finite type (Lemma 10.6.2) so that the statement makes sense. Let us use characterization (3) of Lemma 10.122.2. If $A \to C$ is quasi-finite at $\mathfrak{r}$, then there exists some $c \in C$ such that

$$\{\mathfrak{r}' \subset C \text{ lying over } \mathfrak{p}\} \cap D(c) = \{\mathfrak{r}\}.$$

Since the primes $\mathfrak{r}' \subset C$ lying over $\mathfrak{q}$ form a subset of the primes $\mathfrak{r}' \subset C$ lying over $\mathfrak{p}$ we conclude $B \to C$ is quasi-finite at $\mathfrak{r}$. $\square$

The following lemma is not quite about quasi-finite ring maps, but it does not seem to fit anywhere else so well.

02ML Lemma 10.122.10. Let $R \to S$ be a ring map of finite type. Let $\mathfrak{p} \subset R$ be a minimal prime. Assume that there are at most finitely many primes of S lying over $\mathfrak{p}$. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the ring map $R_g \to S_g$ is finite.

Proof. Let $x_1, \ldots, x_n$ be generators of S over R. Since $\mathfrak{p}$ is a minimal prime we have that $\mathfrak{p}R_\mathfrak{p}$ is a locally nilpotent ideal, see Lemma 10.25.1. Hence $\mathfrak{p}S_\mathfrak{p}$ is a locally nilpotent ideal, see Lemma 10.32.3. By assumption the finite type $\kappa(\mathfrak{p})$-algebra $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ has finitely many primes. Hence (for example by Lemmas 10.61.3 and 10.115.4) $\kappa(\mathfrak{p}) \to S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ is a finite ring map. Thus we may find monic polynomials $P_i \in R_\mathfrak{p}[X]$ such that $P_i(x_i)$ maps to zero in $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$. By what we said above there exist $e_i \geq 1$ such that $P(x_i)^{e_i} = 0$ in $S_\mathfrak{p}$. Let $g_1 \in R$, $g_1 \notin \mathfrak{p}$ be an element such that P_i has coefficients in $R[1/g_1]$ for all i. Next, let $g_2 \in R$, $g_2 \notin \mathfrak{p}$ be an element such that $P(x_i)^{e_i} = 0$ in $S_{g_1 g_2}$. Setting $g = g_1 g_2$ we win. $\square$

10.123. Zariski's Main Theorem

00PI In this section our aim is to prove the algebraic version of Zariski's Main theorem. This theorem will be the basis of many further developments in the theory of schemes and morphisms of schemes later in the Stacks project.

Let $R \to S$ be a ring map of finite type. Our goal in this section is to show that the set of points of $\text{Spec}(S)$ where the map is quasi-finite is open (Theorem 10.123.12). In fact, it will turn out that there exists a finite ring map $R \to S'$ such that in some sense the quasi-finite locus of S/R is open in $\text{Spec}(S')$ (but we will not prove this in the algebra chapter since we do not develop the language of schemes here – for the case where $R \to S$ is quasi-finite see Lemma 10.123.14). These statements are somewhat tricky to prove and we do it by a long list of lemmas concerning integral and finite extensions of rings. This material may be found in [Ray70], and [Pes66]. We also found notes by Thierry Coquand helpful.

00PQ Lemma 10.123.1. Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Then $\varphi(a_n)t$ is integral over R.

Proof. Namely, multiply the equation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$ with $\varphi(a_n)^{n-1}$ and write it as $\varphi(a_0 a_n^{n-1}) + \varphi(a_1 a_n^{n-2})(\varphi(a_n)t) + \ldots + (\varphi(a_n)t)^n = 0$. $\square$

The following lemma is in some sense the key lemma in this section.

00PT Lemma 10.123.2. Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume that (a) t is integral over $R[x]$, and (b) there exists a monic $p \in R[x]$ such that $t\varphi(p) \in \text{Im}(\varphi)$. Then there exists a $q \in R[x]$ such that $t - \varphi(q)$ is integral over R.

Proof. Write $t\varphi(p) = \varphi(r)$ for some $r \in R[x]$. Using euclidean division, write $r = qp + r'$ with $q, r' \in R[x]$ and $\deg(r') < \deg(p)$. We may replace t by $t - \varphi(q)$ which is still integral over $R[x]$, so that we obtain $t\varphi(p) = \varphi(r')$. In the ring S_t we may write this as $\varphi(p) - (1/t)\varphi(r') = 0$. This implies that $\varphi(x)$ gives an element of the localization S_t which is integral over $\varphi(R)[1/t] \subset S_t$. On the other hand, t is integral over the subring $\varphi(R)[\varphi(x)] \subset S$. Combined we conclude that t is integral over the subring $\varphi(R)[1/t] \subset S_t$, see Lemma 10.36.6. In other words there exists an equation of the form

$$t^d + \sum\nolimits_{i<d} \left(\sum\nolimits_{j=0,\ldots,n_i} \varphi(r_{i,j})/t^j\right) t^i = 0$$

in S_t with $r_{i,j} \in R$. This means that $t^{d+N} + \sum_{i<d} \sum_{j=0,\ldots,n_i} \varphi(r_{i,j}) t^{i+N-j} = 0$ in S for some N large enough. In other words t is integral over R. $\square$

00PV Lemma 10.123.3. Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume t is integral over $R[x]$. Let $p \in R[x]$, $p = a_0 + a_1 x + \ldots + a_k x^k$ such that $t\varphi(p) \in \mathrm{Im}(\varphi)$. Then there exists a $q \in R[x]$ and $n \geq 0$ such that $\varphi(a_k)^n t - \varphi(q)$ is integral over R.

Proof. Let R' and S' be the localization of R and S at the element a_k. Let $\varphi' : R'[x] \to S'$ be the localization of φ. Let $t' \in S'$ be the image of t. Set $p' = p/a_k \in R'[x]$. Then $t'\varphi'(p') \in \mathrm{Im}(\varphi')$ since $t\varphi(p) \in \mathrm{Im}(\varphi)$. As p' is monic, by Lemma 10.123.2 there exists a $q' \in R'[x]$ such that $t' - \varphi'(q')$ is integral over R'. We may choose an $n \geq 0$ and an element $q \in R[x]$ such that $a_k^n q'$ is the image of q. Then $\varphi(a_k)^n t - \varphi(q)$ is an element of S whose image in S' is integral over R'. By Lemma 10.36.11 there exists an $m \geq 0$ such that $\varphi(a_k)^m(\varphi(a_k)^n t - \varphi(q))$ is integral over R. Thus $\varphi(a_k)^{m+n} t - \varphi(a_k^m q)$ is integral over R as desired. $\square$

00PW Situation 10.123.4. Let R be a ring. Let $\varphi : R[x] \to S$ be finite. Let

$$J = \{g \in S \mid gS \subset \mathrm{Im}(\varphi)\}$$

be the "conductor ideal" of φ. Assume $\varphi(R) \subset S$ integrally closed in S.

00PX Lemma 10.123.5. In Situation 10.123.4. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1 x + \ldots + a_k x^k) \in J$. Then there exists an $m \geq 0$ such that $u\varphi(a_k)^m \in J$.

Proof. Assume that S is generated by $t_1, \ldots, t_n$ as an $R[x]$-module. In this case $J = \{g \in S \mid gt_i \in \mathrm{Im}(\varphi) \text{ for all } i\}$. Note that each element ut_i is integral over $R[x]$, see Lemma 10.36.3. We have $\varphi(a_0 + a_1 x + \ldots + a_k x^k) ut_i \in \mathrm{Im}(\varphi)$. By Lemma 10.123.3, for each i there exists an integer n_i and an element $q_i \in R[x]$ such that $\varphi(a_k^{n_i}) ut_i - \varphi(q_i)$ is integral over R. By assumption this element is in $\varphi(R)$ and hence $\varphi(a_k^{n_i}) ut_i \in \mathrm{Im}(\varphi)$. It follows that $m = \max\{n_1, \ldots, n_n\}$ works. $\square$

00PY Lemma 10.123.6. In Situation 10.123.4. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1 x + \ldots + a_k x^k) \in \sqrt{J}$. Then $u\varphi(a_i) \in \sqrt{J}$ for all i.

Proof. Under the assumptions of the lemma we have $u^n \varphi(a_0 + a_1 x + \ldots + a_k x^k)^n \in J$ for some $n \geq 1$. By Lemma 10.123.5 we deduce $u^n \varphi(a_k^{nm}) \in J$ for some $m \geq 1$. Thus $u\varphi(a_k) \in \sqrt{J}$, and so $u\varphi(a_0 + a_1 x + \ldots + a_k x^k) - u\varphi(a_k x^k) = u\varphi(a_0 + a_1 x + \ldots + a_{k-1} x^{k-1}) \in \sqrt{J}$. We win by induction on k. $\square$

This lemma suggests the following definition.

00PZ Definition 10.123.7. Given an inclusion of rings $R \subset S$ and an element $x \in S$ we say that x is strongly transcendental over R if whenever $u(a_0 + a_1 x + \ldots + a_k x^k) = 0$ with $u \in S$ and $a_i \in R$, then we have $ua_i = 0$ for all i.

Note that if S is a domain then this is the same as saying that x as an element of the fraction field of S is transcendental over the fraction field of R.

00Q0 Lemma 10.123.8. Suppose $R \subset S$ is an inclusion of reduced rings and suppose that $x \in S$ is strongly transcendental over R. Let $\mathfrak q \subset S$ be a minimal prime and let $\mathfrak p = R \cap \mathfrak q$. Then the image of x in $S/\mathfrak q$ is strongly transcendental over the subring $R/\mathfrak p$.

Proof. Suppose $u(a_0 + a_1 x + \ldots + a_k x^k) \in \mathfrak q$. By Lemma 10.25.1 the local ring $S_{\mathfrak q}$ is a field, and hence $u(a_0 + a_1 x + \ldots + a_k x^k)$ is zero in $S_{\mathfrak q}$. Thus $uu'(a_0 + a_1 x + \ldots + a_k x^k) = 0$ for some $u' \in S$, $u' \notin \mathfrak q$. Since x is strongly transcendental over R we get $uu'a_i = 0$ for all i. This in turn implies that $ua_i \in \mathfrak q$. $\square$

00Q1 Lemma 10.123.9. Suppose $R \subset S$ is an inclusion of domains and let $x \in S$. Assume x is (strongly) transcendental over R and that S is finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.

Proof. As a first case, assume that R is normal, see Definition 10.37.11. By Lemma 10.37.14 we see that $R[x]$ is normal. Take a prime $\mathfrak q \subset S$, and set $\mathfrak p = R \cap \mathfrak q$. Assume that the extension $\kappa(\mathfrak p) \subset \kappa(\mathfrak q)$ is finite. This would be the case if $R \to S$ is quasi-finite at $\mathfrak q$. Let $\mathfrak r = R[x] \cap \mathfrak q$. Then since $\kappa(\mathfrak p) \subset \kappa(\mathfrak r) \subset \kappa(\mathfrak q)$ we see that the extension $\kappa(\mathfrak p) \subset \kappa(\mathfrak r)$ is finite too. Thus the inclusion $\mathfrak r \supset \mathfrak p R[x]$ is strict. By going down for $R[x] \subset S$, see Proposition 10.38.7, we find a prime $\mathfrak q' \subset \mathfrak q$, lying over the prime $\mathfrak p R[x]$. Hence the fibre $\text{Spec}(S \otimes_R \kappa(\mathfrak p))$ contains a point not equal to $\mathfrak q$, namely $\mathfrak q'$, whose closure contains $\mathfrak q$ and hence $\mathfrak q$ is not isolated in its fibre.

If R is not normal, let $R \subset R' \subset K$ be the integral closure R' of R in its field of fractions K. Let $S \subset S' \subset L$ be the subring S' of the field of fractions L of S generated by R' and S. Note that by construction the map $S \otimes_R R' \to S'$ is surjective. This implies that $R'[x] \subset S'$ is finite. Also, the map $S \subset S'$ induces a surjection on Spec, see Lemma 10.36.17. We conclude by Lemma 10.122.6 and the normal case we just discussed. $\square$

00Q2 Lemma 10.123.10. Suppose $R \subset S$ is an inclusion of reduced rings. Assume $x \in S$ be strongly transcendental over R, and S finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.

Proof. Let $\mathfrak q \subset S$ be any prime. Choose a minimal prime $\mathfrak q' \subset \mathfrak q$. According to Lemmas 10.123.8 and 10.123.9 the extension $R/(R \cap \mathfrak q') \subset S/\mathfrak q'$ is not quasi-finite at the prime corresponding to $\mathfrak q$. By Lemma 10.122.6 the extension $R \to S$ is not quasi-finite at $\mathfrak q$. $\square$

00Q8 Lemma 10.123.11. Let R be a ring. Let $S = R[x]/I$. Let $\mathfrak q \subset S$ be a prime. Assume $R \to S$ is quasi-finite at $\mathfrak q$. Let $S' \subset S$ be the integral closure of R in S. Then there exists an element $g \in S'$, $g \notin \mathfrak q$ such that $S'_g \cong S_g$.

Proof. Let $\mathfrak p$ be the image of $\mathfrak q$ in $\text{Spec}(R)$. There exists an $f \in I$, $f = a_n x^n + \ldots + a_0$ such that $a_i \notin \mathfrak p$ for some i. Namely, otherwise the fibre ring $S \otimes_R \kappa(\mathfrak p)$ would be $\kappa(\mathfrak p)[x]$ and the map would not be quasi-finite at any prime lying over $\mathfrak p$. We conclude there exists a relation $b_m x^m + \ldots + b_0 = 0$ with $b_j \in S'$, $j = 0, \ldots, m$ and $b_j \notin \mathfrak q \cap S'$ for some j. We prove the lemma by induction on m. The base case is $m = 0$ is vacuous (because the statements $b_0 = 0$ and $b_0 \notin \mathfrak q$ are contradictory).

The case $b_m \notin \mathfrak q$. In this case x is integral over S'_{b_m}, in fact $b_m x \in S'$ by Lemma 10.123.1. Hence the injective map $S'_{b_m} \to S_{b_m}$ is also surjective, i.e., an isomorphism as desired.

The case $b_m \in \mathfrak{q}$. In this case we have $b_m x \in S'$ by Lemma 10.123.1. Set $b'_{m-1} = b_m x + b_{m-1}$. Then

$$b'_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \ldots + b_0 = 0$$

Since b'_{m-1} is congruent to b_{m-1} modulo $S' \cap \mathfrak{q}$ we see that it is still the case that one of $b'_{m-1}, b_{m-2}, \ldots, b_0$ is not in $S' \cap \mathfrak{q}$. Thus we win by induction on m. □

00Q9 Theorem 10.123.12 (Zariski's Main Theorem). Let R be a ring. Let $R \to S$ be a finite type R-algebra. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak{q} \subset S$ be a prime of S. If $R \to S$ is quasi-finite at $\mathfrak{q}$ then there exists a $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$.

Proof. There exist finitely many elements $x_1, \ldots, x_n \in S$ such that S is finite over the R-sub algebra generated by $x_1, \ldots, x_n$. (For example generators of S over R.) We prove the proposition by induction on the minimal such number n.

The case $n = 0$ is trivial, because in this case $S' = S$, see Lemma 10.36.3.

The case $n = 1$. We may replace R by its integral closure in S (Lemma 10.122.9 guarantees that $R \to S$ is still quasi-finite at $\mathfrak{q}$). Thus we may assume $R \subset S$ is integrally closed in S, in other words $R = S'$. Consider the map $\varphi : R[x] \to S$, $x \mapsto x_1$. (We will see that φ is not injective below.) By assumption φ is finite. Hence we are in Situation 10.123.4. Let $J \subset S$ be the "conductor ideal" defined in Situation 10.123.4. Consider the diagram

$$\begin{tikzcd} R[x] \arrow[r] & S \arrow[r] & S/\sqrt{J} & R/(R \cap \sqrt{J})[x] \arrow[l] \\ & R \arrow[ul] \arrow[u] \arrow[r] & R/(R \cap \sqrt{J}) \arrow[u] \arrow[ur] & \end{tikzcd}$$

According to Lemma 10.123.6 the image of x in the quotient $S/\sqrt{J}$ is strongly transcendental over $R/(R \cap \sqrt{J})$. Hence by Lemma 10.123.10 the ring map $R/(R \cap \sqrt{J}) \to S/\sqrt{J}$ is not quasi-finite at any prime of $S/\sqrt{J}$. By Lemma 10.122.6 we deduce that $\mathfrak{q}$ does not lie in $V(J) \subset \text{Spec}(S)$. Thus there exists an element $s \in J$, $s \notin \mathfrak{q}$. By definition of J we may write $s = \varphi(f)$ for some polynomial $f \in R[x]$. Let $I = \text{Ker}(\varphi : R[x] \to S)$. Since $\varphi(f) \in J$ we get $(R[x]/I)_f \cong S_{\varphi(f)}$. Also $s \notin \mathfrak{q}$ means that $f \notin \varphi^{-1}(\mathfrak{q})$. Thus $\varphi^{-1}(\mathfrak{q})$ is a prime of $R[x]/I$ at which $R \to R[x]/I$ is quasi-finite, see Lemma 10.122.5. Note that R is integrally closed in $R[x]/I$ since R is integrally closed in S. By Lemma 10.123.11 there exists an element $h \in R$, $h \notin R \cap \mathfrak{q}$ such that $R_h \cong (R[x]/I)_h$. Thus $(R[x]/I)_{fh} = S_{\varphi(fh)}$ is isomorphic to a principal localization $R_{h'}$ of R for some $h' \in R$, $h' \notin \mathfrak{q}$.

The case $n > 1$. Consider the subring $R' \subset S$ which is the integral closure of $R[x_1, \ldots, x_{n-1}]$ in S. By Lemma 10.122.9 the extension S/R' is quasi-finite at $\mathfrak{q}$. Also, note that S is finite over $R'[x_n]$. By the case $n = 1$ above, there exists a $g' \in R'$, $g' \notin \mathfrak{q}$ such that $(R')_{g'} \cong S_{g'}$. At this point we cannot apply induction to $R \to R'$ since R' may not be finite type over R. Since S is finitely generated over R we deduce in particular that $(R')_{g'}$ is finitely generated over R. Say the elements g', and $y_1/(g')^{n_1}, \ldots, y_N/(g')^{n_N}$ with $y_i \in R'$ generate $(R')_{g'}$ over R. Let R'' be the R-sub algebra of R' generated by $x_1, \ldots, x_{n-1}, y_1, \ldots, y_N, g'$. This has the property $(R'')_{g'} \cong S_{g'}$. Surjectivity because of how we chose y_i, injectivity because $R'' \subset R'$, and localization is exact. Note that R'' is finite over $R[x_1, \ldots, x_{n-1}]$

because of our choice of R', see Lemma 10.36.4. Let $\mathfrak{q}'' = R'' \cap \mathfrak{q}$. Since $(R'')_{\mathfrak{q}''} = S_{\mathfrak{q}}$ we see that $R \to R''$ is quasi-finite at $\mathfrak{q}''$, see Lemma 10.122.2. We apply our induction hypothesis to $R \to R''$, $\mathfrak{q}''$ and $x_1, \ldots, x_{n-1} \in R''$ and we find a subring $R''' \subset R''$ which is integral over R and an element $g'' \in R'''$, $g'' \notin \mathfrak{q}''$ such that $(R''')_{g''} \cong (R'')_{g''}$. Write the image of g' in $(R'')_{g''}$ as $g'''/(g'')^n$ for some $g''' \in R'''$. Set $g = g''g''' \in R'''$. Then it is clear that $g \notin \mathfrak{q}$ and $(R''')_g \cong S_g$. Since by construction we have $R''' \subset S'$ we also have $S'_g \cong S_g$ as desired. □

00QA Lemma 10.123.13. Let $R \to S$ be a finite type ring map. The set of points $\mathfrak{q}$ of $\mathrm{Spec}(S)$ at which S/R is quasi-finite is open in $\mathrm{Spec}(S)$.

Proof. Let $\mathfrak{q} \subset S$ be a point at which the ring map is quasi-finite. By Theorem 10.123.12 there exists an integral ring extension $R \to S'$, $S' \subset S$ and an element $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$. Since S and hence S_g are of finite type over R we may find finitely many elements $y_1, \ldots, y_N$ of S' such that $S''_g \cong S_g$ where $S'' \subset S'$ is the sub R-algebra generated by $g, y_1, \ldots, y_N$. Since S'' is finite over R (see Lemma 10.36.4) we see that S'' is quasi-finite over R (see Lemma 10.122.4). It is easy to see that this implies that S''_g is quasi-finite over R, for example because the property of being quasi-finite at a prime depends only on the local ring at the prime. Thus we see that S_g is quasi-finite over R. By the same token this implies that $R \to S$ is quasi-finite at every prime of S which lies in $D(g)$. □

00QB Lemma 10.123.14. Let $R \to S$ be a finite type ring map. Suppose that S is quasi-finite over R. Let $S' \subset S$ be the integral closure of R in S. Then

(1) $\mathrm{Spec}(S) \to \mathrm{Spec}(S')$ is a homeomorphism onto an open subset,
(2) if $g \in S'$ and $D(g)$ is contained in the image of the map, then $S'_g \cong S_g$, and
(3) there exists a finite R-algebra $S'' \subset S'$ such that (1) and (2) hold for the ring map $S'' \to S$.

Proof. Because S/R is quasi-finite we may apply Theorem 10.123.12 to each point $\mathfrak{q}$ of $\mathrm{Spec}(S)$. Since $\mathrm{Spec}(S)$ is quasi-compact, see Lemma 10.17.10, we may choose a finite number of $g_i \in S'$, $i = 1, \ldots, n$ such that $S'_{g_i} = S_{g_i}$, and such that $g_1, \ldots, g_n$ generate the unit ideal in S (in other words the standard opens of $\mathrm{Spec}(S)$ associated to $g_1, \ldots, g_n$ cover all of $\mathrm{Spec}(S)$).

Suppose that $D(g) \subset \mathrm{Spec}(S')$ is contained in the image. Then $D(g) \subset \bigcup D(g_i)$. In other words, $g_1, \ldots, g_n$ generate the unit ideal of S'_g. Note that $S'_{gg_i} \cong S_{gg_i}$ by our choice of g_i. Hence $S'_g \cong S_g$ by Lemma 10.23.2.

We construct a finite algebra $S'' \subset S'$ as in (3). To do this note that each $S'_{g_i} \cong S_{g_i}$ is a finite type R-algebra. For each i pick some elements $y_{ij} \in S'$ such that each S'_{g_i} is generated as R-algebra by $1/g_i$ and the elements y_{ij}. Then set S'' equal to the sub R-algebra of S' generated by all g_i and all the y_{ij}. Details omitted. □

10.124. Applications of Zariski's Main Theorem

03GB Here is an immediate application characterizing the finite maps of 1-dimensional semi-local rings among the quasi-finite ones as those where equality always holds in the formula of Lemma 10.121.8.

02MM Lemma 10.124.1. Let $A \subset B$ be an extension of domains. Assume

(1) A is a local Noetherian ring of dimension 1,
(2) $A \to B$ is of finite type, and
(3) the induced extension L/K of fraction fields is finite.

Then B is semi-local. Let $x \in \mathfrak{m}_A$, $x \neq 0$. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$[L : K]\text{ord}_A(x) \geq \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)]\text{ord}_{B_{\mathfrak{m}_i}}(x)$$

where ord is defined as in Definition 10.121.2. We have equality if and only if $A \to B$ is finite.

Proof. The ring B is semi-local by Lemma 10.113.2. Let B' be the integral closure of A in B. By Lemma 10.123.14 we can find a finite A-subalgebra $C \subset B'$ such that on setting $\mathfrak{n}_i = C \cap \mathfrak{m}_i$ we have $C_{\mathfrak{n}_i} \cong B_{\mathfrak{m}_i}$ and the primes $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are pairwise distinct. The ring C is semi-local by Lemma 10.113.2. Let $\mathfrak{p}_j$, $j = 1, \ldots, m$ be the other maximal ideals of C (the "missing points"). By Lemma 10.121.8 we have

$$\text{ord}_A(x^{[L:K]}) = \sum\nolimits_i [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_A)]\text{ord}_{C_{\mathfrak{n}_i}}(x) + \sum\nolimits_j [\kappa(\mathfrak{p}_j) : \kappa(\mathfrak{m}_A)]\text{ord}_{C_{\mathfrak{p}_j}}(x)$$

hence the inequality follows. In case of equality we conclude that $m = 0$ (no "missing points"). Hence $C \subset B$ is an inclusion of semi-local rings inducing a bijection on maximal ideals and an isomorphism on all localizations at maximal ideals. So if $b \in B$, then $I = \{x \in C \mid xb \in C\}$ is an ideal of C which is not contained in any of the maximal ideals of C, and hence $I = C$, hence $b \in C$. Thus $B = C$ and B is finite over A. ☐

Here is a more standard application of Zariski's main theorem to the structure of local homomorphisms of local rings.

052V Lemma 10.124.2. Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume

(1) $R \to S$ is essentially of finite type,
(2) $\kappa(\mathfrak{m}_R) \subset \kappa(\mathfrak{m}_S)$ is finite, and
(3) $\dim(S/\mathfrak{m}_R S) = 0$.

Then S is the localization of a finite R-algebra.

Proof. Let S' be a finite type R-algebra such that $S = S'_{\mathfrak{q}'}$ for some prime $\mathfrak{q}'$ of S'. By Definition 10.122.3 we see that $R \to S'$ is quasi-finite at $\mathfrak{q}'$. After replacing S' by $S'_{g'}$ for some $g' \in S'$, $g' \notin \mathfrak{q}'$ we may assume that $R \to S'$ is quasi-finite, see Lemma 10.123.13. Then by Lemma 10.123.14 there exists a finite R-algebra S'' and elements $g' \in S'$, $g' \notin \mathfrak{q}'$ and $g'' \in S''$ such that $S'_{g'} \cong S''_{g''}$ as R-algebras. This proves the lemma. ☐

07NC Lemma 10.124.3. Let $R \to S$ be a ring map, $\mathfrak{q}$ a prime of S lying over $\mathfrak{p}$ in R. If

(1) R is Noetherian,
(2) $R \to S$ is of finite type, and
(3) $R \to S$ is quasi-finite at $\mathfrak{q}$,

then $R_{\mathfrak{p}}^\wedge \otimes_R S = S_{\mathfrak{q}}^\wedge \times B$ for some $R_{\mathfrak{p}}^\wedge$-algebra B.

Proof. There exists a finite R-algebra $S' \subset S$ and an element $g \in S'$, $g \notin \mathfrak{q}' = S' \cap \mathfrak{q}$ such that $S'_g = S_g$ and in particular $S'_{\mathfrak{q}'} = S_{\mathfrak{q}}$, see Lemma 10.123.14. We have

$$R_{\mathfrak{p}}^\wedge \otimes_R S' = (S'_{\mathfrak{q}'})^\wedge \times B'$$

by Lemma 10.97.8. Observe that under this product decomposition g maps to a pair (u, b') with $u \in (S'_{\mathfrak q'})^\wedge$ a unit because $g \not\in \mathfrak q'$. The product decomposition for $R_{\mathfrak p}^\wedge \otimes_R S'$ induces a product decomposition

$$R_{\mathfrak p}^\wedge \otimes_R S = A \times B$$

Since $S'_g = S_g$ we also have $(R_{\mathfrak p}^\wedge \otimes_R S')_g = (R_{\mathfrak p}^\wedge \otimes_R S)_g$ and since $g \mapsto (u, b')$ where u is a unit we see that $(S'_{\mathfrak q'})^\wedge = A$. Since the isomorphism $S'_{\mathfrak q'} = S_{\mathfrak q}$ determines an isomorphism on completions this also tells us that $A = S_{\mathfrak q}^\wedge$. This finishes the proof, except that we should perform the sanity check that the induced map $\phi : R_{\mathfrak p}^\wedge \otimes_R S \to A = S_{\mathfrak q}^\wedge$ is the natural one. For elements of the form $x \otimes 1$ with $x \in R_{\mathfrak p}^\wedge$ this is clear as the natural map $R_{\mathfrak p}^\wedge \to S_{\mathfrak q}^\wedge$ factors through $(S'_{\mathfrak q'})^\wedge$. For elements of the form $1 \otimes y$ with $y \in S$ we can argue that for some $n \geq 1$ the element $g^n y$ is the image of some $y' \in S'$. Thus $\phi(1 \otimes g^n y)$ is the image of y' by the composition $S' \to (S'_{\mathfrak q'})^\wedge \to S_{\mathfrak q}^\wedge$ which is equal to the image of $g^n y$ by the map $S \to S_{\mathfrak q}^\wedge$. Since g maps to a unit this also implies that $\phi(1 \otimes y)$ has the correct value, i.e., the image of y by $S \to S_{\mathfrak q}^\wedge$. □

10.125. Dimension of fibres

00QC We study the behaviour of dimensions of fibres, using Zariski's main theorem. Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 5.10.1.

00QD Definition 10.125.1. Suppose that $R \to S$ is of finite type, and let $\mathfrak q \subset S$ be a prime lying over a prime $\mathfrak p$ of R. We define the relative dimension of S/R at $\mathfrak q$, denoted $\dim_{\mathfrak q}(S/R)$, to be the dimension of $\text{Spec}(S \otimes_R \kappa(\mathfrak p))$ at the point corresponding to $\mathfrak q$. We let $\dim(S/R)$ be the supremum of $\dim_{\mathfrak q}(S/R)$ over all $\mathfrak q$. This is called the relative dimension of S/R.

In particular, $R \to S$ is quasi-finite at $\mathfrak q$ if and only if $\dim_{\mathfrak q}(S/R) = 0$. The following lemma is more or less a reformulation of Zariski's Main Theorem.

00QE Lemma 10.125.2. Let $R \to S$ be a finite type ring map. Let $\mathfrak q \subset S$ be a prime. Suppose that $\dim_{\mathfrak q}(S/R) = n$. There exists a $g \in S$, $g \not\in \mathfrak q$ such that S_g is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$.

Proof. The ring $\overline{S} = S \otimes_R \kappa(\mathfrak p)$ is of finite type over $\kappa(\mathfrak p)$. Let $\overline{\mathfrak q}$ be the prime of $\overline{S}$ corresponding to $\mathfrak q$. By definition of the dimension of a topological space at a point there exists an open $U \subset \text{Spec}(\overline{S})$ with $\overline{q} \in U$ and $\dim(U) = n$. Since the topology on $\text{Spec}(\overline{S})$ is induced from the topology on $\text{Spec}(S)$ (see Remark 10.17.8), we can find a $g \in S$, $g \not\in \mathfrak q$ with image $\overline{g} \in \overline{S}$ such that $D(\overline{g}) \subset U$. Thus after replacing S by S_g we see that $\dim(\overline{S}) = n$.

Next, choose generators $x_1, \ldots, x_N$ for S as an R-algebra. By Lemma 10.115.4 there exist elements $y_1, \ldots, y_n$ in the $\mathbf{Z}$-subalgebra of S generated by $x_1, \ldots, x_N$ such that the map $R[t_1, \ldots, t_n] \to S$, $t_i \mapsto y_i$ has the property that $\kappa(\mathfrak p)[t_1 \ldots, t_n] \to \overline{S}$ is finite. In particular, S is quasi-finite over $R[t_1, \ldots, t_n]$ at $\mathfrak q$. Hence, by Lemma 10.123.13 we may replace S by S_g for some $g \in S$, $g \not\in \mathfrak q$ such that $R[t_1, \ldots, t_n] \to S$ is quasi-finite. □

0520 Lemma 10.125.3. Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$ of R. Assume

(1) $R \to S$ is of finite type,
(2) $\dim_{\mathfrak q}(S/R) = n$, and
(3) $\text{trdeg}_{\kappa(\mathfrak p)}\kappa(\mathfrak q) = r$.

Then there exist $f \in R$, $f \not\in \mathfrak p$, $g \in S$, $g \not\in \mathfrak q$ and a quasi-finite ring map

$$\varphi : R_f[x_1, \ldots, x_n] \longrightarrow S_g$$

such that $\varphi^{-1}(\mathfrak qS_g) = (\mathfrak p, x_{r+1}, \ldots, x_n)R_f[x_{r+1}, \ldots, x_n]$

Proof. After replacing S by a principal localization we may assume there exists a quasi-finite ring map $\varphi : R[t_1, \ldots, t_n] \to S$, see Lemma 10.125.2. Set $\mathfrak q' = \varphi^{-1}(\mathfrak q)$. Let $\overline{\mathfrak q}' \subset \kappa(\mathfrak p)[t_1, \ldots, t_n]$ be the prime corresponding to $\mathfrak q'$. By Lemma 10.115.6 there exists a finite ring map $\kappa(\mathfrak p)[x_1, \ldots, x_n] \to \kappa(\mathfrak p)[t_1, \ldots, t_n]$ such that the inverse image of $\overline{\mathfrak q}'$ is $(x_{r+1}, \ldots, x_n)$. Let $\overline{h}_i \in \kappa(\mathfrak p)[t_1, \ldots, t_n]$ be the image of x_i. We can find an element $f \in R$, $f \not\in \mathfrak p$ and $h_i \in R_f[t_1, \ldots, t_n]$ which map to $\overline{h}_i$ in $\kappa(\mathfrak p)[t_1, \ldots, t_n]$. Then the ring map

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n]$$

becomes finite after tensoring with $\kappa(\mathfrak p)$. In particular, $R_f[t_1, \ldots, t_n]$ is quasi-finite over $R_f[x_1, \ldots, x_n]$ at the prime $\mathfrak q'R_f[t_1, \ldots, t_n]$. Hence, by Lemma 10.123.13 there exists a $g \in R_f[t_1, \ldots, t_n]$, $g \not\in \mathfrak q'R_f[t_1, \ldots, t_n]$ such that $R_f[x_1, \ldots, x_n] \to R_f[t_1, \ldots, t_n, 1/g]$ is quasi-finite. Thus we see that the composition

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n, 1/g] \longrightarrow S_{\varphi(g)}$$

is quasi-finite and we win. ☐

00QF Lemma 10.125.4. Let $R \to S$ be a finite type ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. If $R \to S$ is quasi-finite at $\mathfrak q$, then $\dim(S_{\mathfrak q}) \leq \dim(R_{\mathfrak p})$.

Proof. If $R_{\mathfrak p}$ is Noetherian (and hence $S_{\mathfrak q}$ Noetherian since it is essentially of finite type over $R_{\mathfrak p}$) then this follows immediately from Lemma 10.112.6 and the definitions. In the general case, let S' be the integral closure of $R_{\mathfrak p}$ in $S_{\mathfrak p}$. By Zariski's Main Theorem 10.123.12 we have $S_{\mathfrak q} = S'_{\mathfrak q'}$ for some $\mathfrak q' \subset S'$ lying over $\mathfrak q$. By Lemma 10.112.3 we have $\dim(S') \leq \dim(R_{\mathfrak p})$ and hence a fortiori $\dim(S_{\mathfrak q}) = \dim(S'_{\mathfrak q'}) \leq \dim(R_{\mathfrak p})$. ☐

00QG Lemma 10.125.5. Let k be a field. Let S be a finite type k-algebra. Suppose there is a quasi-finite k-algebra map $k[t_1, \ldots, t_n] \subset S$. Then $\dim(S) \leq n$.

Proof. By Lemma 10.114.1 the dimension of any local ring of $k[t_1, \ldots, t_n]$ is at most n. Thus the result follows from Lemma 10.125.4. ☐

00QH Lemma 10.125.6. Let $R \to S$ be a finite type ring map. Let $\mathfrak q \subset S$ be a prime. Suppose that $\dim_{\mathfrak q}(S/R) = n$. There exists an open neighbourhood V of $\mathfrak q$ in $\text{Spec}(S)$ such that $\dim_{\mathfrak q'}(S/R) \leq n$ for all $\mathfrak q' \in V$.

Proof. By Lemma 10.125.2 we see that we may assume that S is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$. Considering the fibres, we reduce to Lemma 10.125.5. ☐

In other words, the lemma says that the set of points where the fibre has dimension $\leq n$ is open in $\text{Spec}(S)$. The next lemma says that formation of this open commutes with base change. If the ring map is of finite presentation then this set is quasi-compact open (see below).

00QI Lemma 10.125.7. Let $R \to S$ be a finite type ring map. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$ and denote $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ the associated map on spectra. Let $n \geq 0$. The inverse image $f^{-1}(\{\mathfrak q \in \mathrm{Spec}(S) \mid \dim_{\mathfrak q}(S/R) \leq n\})$ is equal to $\{\mathfrak q' \in \mathrm{Spec}(S') \mid \dim_{\mathfrak q'}(S'/R') \leq n\}$.

Proof. The condition is formulated in terms of dimensions of fibre rings which are of finite type over a field. Combined with Lemma 10.116.6 this yields the lemma. □

00QJ Lemma 10.125.8. Let $R \to S$ be a ring homomorphism of finite presentation. Let $n \geq 0$. The set

$$V_n = \{\mathfrak q \in \mathrm{Spec}(S) \mid \dim_{\mathfrak q}(S/R) \leq n\}$$

is a quasi-compact open subset of $\mathrm{Spec}(S)$.

Proof. It is open by Lemma 10.125.6. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a presentation of S. Let R_0 be the $\mathbf{Z}$-subalgebra of R generated by the coefficients of the polynomials f_i. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Then $S = R \otimes_{R_0} S_0$. By Lemma 10.125.7 V_n is the inverse image of an open $V_{0,n}$ under the quasi-compact continuous map $\mathrm{Spec}(S) \to \mathrm{Spec}(S_0)$. Since S_0 is Noetherian we see that $V_{0,n}$ is quasi-compact. □

00QK Lemma 10.125.9. Let R be a valuation ring with residue field k and field of fractions K. Let S be a domain containing R such that S is of finite type over R. If $S \otimes_R k$ is not the zero ring then

$$\dim(S \otimes_R k) = \dim(S \otimes_R K)$$

In fact, $\mathrm{Spec}(S \otimes_R k)$ is equidimensional.

Proof. It suffices to show that $\dim_{\mathfrak q}(S/k)$ is equal to $\dim(S \otimes_R K)$ for every prime $\mathfrak q$ of S containing $\mathfrak m_R S$. Pick such a prime. By Lemma 10.125.6 the inequality $\dim_{\mathfrak q}(S/k) \geq \dim(S \otimes_R K)$ holds. Set $n = \dim_{\mathfrak q}(S/k)$. By Lemma 10.125.2 after replacing S by S_g for some $g \in S$, $g \notin \mathfrak q$ there exists a quasi-finite ring map $R[t_1, \ldots, t_n] \to S$. If $\dim(S \otimes_R K) < n$, then $K[t_1, \ldots, t_n] \to S \otimes_R K$ has a nonzero kernel. Say $f = \sum a_I t_1^{i_1} \ldots t_n^{i_n}$. After dividing f by a nonzero coefficient of f with minimal valuation, we may assume $f \in R[t_1, \ldots, t_n]$ and some a_I does not map to zero in k. Hence the ring map $k[t_1, \ldots, t_n] \to S \otimes_R k$ has a nonzero kernel which implies that $\dim(S \otimes_R k) < n$. Contradiction. □

10.126. Algebras and modules of finite presentation

05N4 In this section we discuss some standard results where the key feature is that the assumption involves a finite type or finite presentation assumption.

00QP Lemma 10.126.1. Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite type if and only if $R' \to S'$ is of finite type.

Proof. It is clear that if $R \to S$ is of finite type then $R' \to S'$ is of finite type. Assume that $R' \to S'$ is of finite type. Say $y_1, \ldots, y_m$ generate S' over R'. Write $y_j = \sum_i a_{ij} \otimes x_{ji}$ for some $a_{ij} \in R'$ and $x_{ji} \in S$. Let $A \subset S$ be the R-subalgebra generated by the x_{ij}. By flatness we have $A' := R' \otimes_R A \subset S'$, and by construction $y_j \in A'$. Hence $A' = S'$. By faithful flatness $A = S$. □

00QQ Lemma 10.126.2. Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite presentation if and only if $R' \to S'$ is of finite presentation.

Proof. It is clear that if $R \to S$ is of finite presentation then $R' \to S'$ is of finite presentation. Assume that $R' \to S'$ is of finite presentation. By Lemma 10.126.1 we see that $R \to S$ is of finite type. Write $S = R[x_1, \ldots, x_n]/I$. By flatness $S' = R'[x_1, \ldots, x_n]/R' \otimes I$. Say $g_1, \ldots, g_m$ generate $R' \otimes I$ over $R'[x_1, \ldots, x_n]$. Write $g_j = \sum_i a_{ij} \otimes f_{ji}$ for some $a_{ij} \in R'$ and $f_{ji} \in I$. Let $J \subset I$ be the ideal generated by the f_{ij}. By flatness we have $R' \otimes_R J \subset R' \otimes_R I$, and both are ideals over $R'[x_1, \ldots, x_n]$. By construction $g_j \in R' \otimes_R J$. Hence $R' \otimes_R J = R' \otimes_R I$. By faithful flatness $J = I$. $\square$

05N5 Lemma 10.126.3. Let R be a ring. Let $I \subset R$ be an ideal. Let $S \subset R$ be a multiplicative subset. Set $R' = S^{-1}(R/I) = S^{-1}R/S^{-1}I$.

(1) For any finite R'-module M' there exists a finite R-module M such that $S^{-1}(M/IM) \cong M'$.
(2) For any finitely presented R'-module M' there exists a finitely presented R-module M such that $S^{-1}(M/IM) \cong M'$.

Proof. Proof of (1). Choose a short exact sequence $0 \to K' \to (R')^{\oplus n} \to M' \to 0$. Let $K \subset R^{\oplus n}$ be the inverse image of K' under the map $R^{\oplus n} \to (R')^{\oplus n}$. Then $M = R^{\oplus n}/K$ works.

Proof of (2). Choose a presentation $(R')^{\oplus m} \to (R')^{\oplus n} \to M' \to 0$. Suppose that the first map is given by the matrix $A' = (a'_{ij})$ and the second map is determined by generators $x'_i \in M'$, $i = 1, \ldots, n$. As $R' = S^{-1}(R/I)$ we can choose $s \in S$ and a matrix $A = (a_{ij})$ with coefficients in R such that $a'_{ij} = a_{ij}/s \bmod S^{-1}I$. Let M be the finitely presented R-module with presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$ where the first map is given by the matrix A and the second map is determined by generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $M \to M'$, $x_i \mapsto x'_i$ induces an isomorphism $S^{-1}(M/IM) \cong M'$. $\square$

05N6 Lemma 10.126.4. Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module.

(1) If $S^{-1}M$ is a finite $S^{-1}R$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.
(2) If $S^{-1}M$ is a finitely presented $S^{-1}R$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.

Proof. Proof of (1). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let M' be the R-submodule of M generated by $x_1, \ldots, x_n$.

Proof of (2). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let $K = \text{Ker}(R^{\oplus n} \to M)$ where the map is given by the rule $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. By Lemma 10.5.3 we see that $S^{-1}K$ is a finite $S^{-1}R$-module. By (1) we can find a finite submodule $K' \subset K$ with $S^{-1}K' = S^{-1}K$. Take $M' = \text{Coker}(K' \to R^{\oplus n})$. $\square$

05GJ Lemma 10.126.5. Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let M be an R-module.

(1) If $M_{\mathfrak p}$ is a finite $R_{\mathfrak p}$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $M'_{\mathfrak p} \to M_{\mathfrak p}$.
(2) If $M_{\mathfrak p}$ is a finitely presented $R_{\mathfrak p}$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $M'_{\mathfrak p} \to M_{\mathfrak p}$.

Proof. This is a special case of Lemma 10.126.4 □

00QR Lemma 10.126.6. Let $\varphi : R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. Assume

(1) S is of finite presentation over R,
(2) φ induces an isomorphism $R_{\mathfrak p} \cong S_{\mathfrak q}$.

Then there exist $f \in R$, $f \notin \mathfrak p$ and an R_f-algebra C such that $S_f \cong R_f \times C$ as R_f-algebras.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Let $a_i \in R_{\mathfrak p}$ be an element mapping to the image of x_i in $S_{\mathfrak q}$. Write $a_i = b_i/f$ for some $f \in R$, $f \notin \mathfrak p$. After replacing R by R_f and x_i by $x_i - a_i$ we may assume that $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ such that x_i maps to zero in $S_{\mathfrak q}$. Then if c_j denotes the constant term of g_j we conclude that c_j maps to zero in $R_{\mathfrak p}$. After another replacement of R we may assume that the constant coefficients c_j of the g_j are zero. Thus we obtain an R-algebra map $S \to R$, $x_i \mapsto 0$ whose kernel is the ideal $(x_1, \ldots, x_n)$.

Note that $\mathfrak q = \mathfrak pS + (x_1, \ldots, x_n)$. Write $g_j = \sum a_{ji}x_i + h.o.t.$. Since $S_{\mathfrak q} = R_{\mathfrak p}$ we have $\mathfrak p \otimes \kappa(\mathfrak p) = \mathfrak q \otimes \kappa(\mathfrak q)$. It follows that $m \times n$ matrix $A = (a_{ji})$ defines a surjective map $\kappa(\mathfrak p)^{\oplus m} \to \kappa(\mathfrak p)^{\oplus n}$. Thus after inverting some element of R not in $\mathfrak p$ we may assume there are $b_{ij} \in R$ such that $\sum b_{ij}g_j = x_i + h.o.t.$. We conclude that $(x_1, \ldots, x_n) = (x_1, \ldots, x_n)^2$ in S. It follows from Lemma 10.21.5 that $(x_1, \ldots, x_n)$ is generated by an idempotent e. Setting $C = eS$ finishes the proof. □

00QS Lemma 10.126.7. Let R be a ring. Let S, S' be of finite presentation over R. Let $\mathfrak q \subset S$ and $\mathfrak q' \subset S'$ be primes. If $S_{\mathfrak q} \cong S'_{\mathfrak q'}$ as R-algebras, then there exist $g \in S$, $g \notin \mathfrak q$ and $g' \in S'$, $g' \notin \mathfrak q'$ such that $S_g \cong S'_{g'}$ as R-algebras.

Proof. Let $\psi : S_{\mathfrak q} \to S'_{\mathfrak q'}$ be the isomorphism of the hypothesis of the lemma. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and $S' = R[y_1, \ldots, y_m]/J$. For each $i = 1, \ldots, n$ choose a fraction h_i/g_i with $h_i, g_i \in R[y_1, \ldots, y_m]$ and $g_i \bmod J$ not in $\mathfrak q'$ which represents the image of x_i under ψ. After replacing S' by $S'_{g_1 \ldots g_n}$ and $R[y_1, \ldots, y_m, y_{m+1}]$ (mapping y_{m+1} to $1/(g_1 \ldots g_n)$) we may assume that $\psi(x_i)$ is the image of some $h_i \in R[y_1, \ldots, y_m]$. Consider the elements $f_j(h_1, \ldots, h_n) \in R[y_1, \ldots, y_m]$. Since ψ kills each f_j we see that there exists a $g \in R[y_1, \ldots, y_m]$, $g \bmod J \notin \mathfrak q'$ such that $gf_j(h_1, \ldots, h_n) \in J$ for each $j = 1, \ldots, r$. After replacing S' by S'_g and $R[y_1, \ldots, y_m, y_{m+1}]$ as before we may assume that $f_j(h_1, \ldots, h_n) \in J$. Thus we obtain a ring map $S \to S'$, $x_i \mapsto h_i$ which induces ψ on local rings. By Lemma 10.6.2 the map $S \to S'$ is of finite presentation. By Lemma 10.126.6 we may assume that $S' = S \times C$. Thus localizing S' at the idempotent corresponding to the factor C we obtain the result. □

0G8U Lemma 10.126.8. Let R be a ring. Let $I \subset R$ be a nilpotent ideal. Let S be an R-algebra such that $R/I \to S/IS$ is of finite type. Then $R \to S$ is of finite type.

Proof. Choose $s_1, \ldots, s_n \in S$ whose images in S/IS generate S/IS as an algebra over R/I. By Lemma 10.20.1 part (11) we see that the R-algebra map $R[x_1, \ldots, x_n \to S$, $x_i \mapsto s_i$ is surjective and we conclude. □

07RD Lemma 10.126.9. Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $S \to S'$ be an R-algebra map such that $S \to S'/IS'$ is surjective and such that S' is of finite type over R. Then $S \to S'$ is surjective.

Proof. Write $S' = R[x_1, \ldots, x_m]/K$ for some ideal K. By assumption there exist $g_j = x_j + \sum \delta_{j,J} x^J \in R[x_1, \ldots, x_n]$ with $\delta_{j,J} \in I$ and with $g_j \bmod K \in \mathrm{Im}(S \to S')$. Hence it suffices to show that $g_1, \ldots, g_m$ generate $R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be a finitely generated $\mathbf{Z}$-subalgebra of R containing at least the $\delta_{j,J}$. Then $R_0 \cap I$ is a nilpotent ideal (by Lemma 10.32.5). It follows that $R_0[x_1, \ldots, x_n]$ is generated by $g_1, \ldots, g_m$ (because $x_j \mapsto g_j$ defines an automorphism of $R_0[x_1, \ldots, x_m]$; details omitted). Since R is the union of the subrings R_0 we win. □

087P Lemma 10.126.10. Let R be a ring. Let $I \subset R$ be an ideal. Let $S \to S'$ be an R-algebra map. Let $IS \subset \mathfrak{q} \subset S$ be a prime ideal. Assume that

(1) $S \to S'$ is surjective,
(2) $S_\mathfrak{q}/IS_\mathfrak{q} \to S'_\mathfrak{q}/IS'_\mathfrak{q}$ is an isomorphism,
(3) S is of finite type over R,
(4) S' of finite presentation over R, and
(5) $S'_\mathfrak{q}$ is flat over R.

Then $S_g \to S'_g$ is an isomorphism for some $g \in S$, $g \notin \mathfrak{q}$.

Proof. Let $J = \mathrm{Ker}(S \to S')$. By Lemma 10.6.2 J is a finitely generated ideal. Since $S'_\mathfrak{q}$ is flat over R we see that $J_\mathfrak{q}/IJ_\mathfrak{q} \subset S_\mathfrak{q}/IS_\mathfrak{q}$ (apply Lemma 10.39.12 to $0 \to J \to S \to S' \to 0$). By assumption (2) we see that $J_\mathfrak{q}/IJ_\mathfrak{q}$ is zero. By Nakayama's lemma (Lemma 10.20.1) we see that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $J_g = 0$. Hence $S_g \cong S'_g$ as desired. □

07RE Lemma 10.126.11. Let R be a ring. Let $I \subset R$ be an ideal. Let $S \to S'$ be an R-algebra map. Assume that

(1) I is locally nilpotent,
(2) $S/IS \to S'/IS'$ is an isomorphism,
(3) S is of finite type over R,
(4) S' of finite presentation over R, and
(5) S' is flat over R.

Then $S \to S'$ is an isomorphism.

Proof. By Lemma 10.126.9 the map $S \to S'$ is surjective. As I is locally nilpotent, so are the ideals IS and IS' (Lemma 10.32.3). Hence every prime ideal $\mathfrak{q}$ of S contains IS and (trivially) $S_\mathfrak{q}/IS_\mathfrak{q} \cong S'_\mathfrak{q}/IS'_\mathfrak{q}$. Thus Lemma 10.126.10 applies and we see that $S_\mathfrak{q} \to S'_\mathfrak{q}$ is an isomorphism for every prime $\mathfrak{q} \subset S$. It follows that $S \to S'$ is injective for example by Lemma 10.23.1. □

10.127. Colimits and maps of finite presentation

00QL In this section we prove some preliminary lemmas which will eventually help us prove result using absolute Noetherian reduction. In Categories, Section 4.19 we discuss filtered colimits in general. Here is an example of this very general notion.

0BUF Lemma 10.127.1. Let $R \to A$ be a ring map. Consider the category $\mathcal{I}$ of all diagrams of R-algebra maps $A' \to A$ with A' finitely presented over R. Then $\mathcal{I}$ is filtered, and the colimit of the A' over $\mathcal{I}$ is isomorphic to A.

Proof. The category[11] $\mathcal{I}$ is nonempty as $R \to R$ is an object of it. Consider a pair of objects $A' \to A$, $A'' \to A$ of $\mathcal{I}$. Then $A' \otimes_R A'' \to A$ is in $\mathcal{I}$ (use Lemmas 10.6.2 and 10.14.2). The ring maps $A' \to A' \otimes_R A''$ and $A'' \to A' \otimes_R A''$ define arrows in $\mathcal{I}$ thereby proving the second defining property of a filtered category, see Categories, Definition 4.19.1. Finally, suppose that we have two morphisms $\sigma, \tau : A' \to A''$ in $\mathcal{I}$. If $x_1, \ldots, x_r \in A'$ are generators of A' as an R-algebra, then we can consider $A''' = A''/(\sigma(x_i) - \tau(x_i))$. This is a finitely presented R-algebra and the given R-algebra map $A'' \to A$ factors through the surjection $\nu : A'' \to A'''$. Thus ν is a morphism in $\mathcal{I}$ equalizing σ and τ as desired.

The fact that our index category is cofiltered means that we may compute the value of $B = \mathrm{colim}_{A' \to A} A'$ in the category of sets (some details omitted; compare with the discussion in Categories, Section 4.19). To see that $B \to A$ is surjective, for every $a \in A$ we can use $R[x] \to A$, $x \mapsto a$ to see that a is in the image of $B \to A$. Conversely, if $b \in B$ is mapped to zero in A, then we can find $A' \to A$ in $\mathcal{I}$ and $a' \in A'$ which maps to b. Then $A'/(a') \to A$ is in $\mathcal{I}$ as well and the map $A' \to B$ factors as $A' \to A'/(a') \to B$ which shows that $b = 0$ as desired. $\square$

Often it is easier to think about colimits over preordered sets. Let $(\Lambda, \geq)$ a preordered set. A system of rings over Λ is given by a ring R_λ for every $\lambda \in \Lambda$, and a morphism $R_\lambda \to R_\mu$ whenever $\lambda \leq \mu$. These morphisms have to satisfy the rule that $R_\lambda \to R_\mu \to R_\nu$ is equal to the map $R_\lambda \to R_\nu$ for all $\lambda \leq \mu \leq \nu$. See Categories, Section 4.21. We will often assume that $(I, \leq)$ is directed, which means that Λ is nonempty and given $\lambda, \mu \in \Lambda$ there exists a $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$. Recall that the colimit $\mathrm{colim}_\lambda R_\lambda$ is sometimes called a "direct limit" in this case (but we will not use this terminology).

Note that Categories, Lemma 4.21.5 tells us that colimits over filtered index categories are the same thing as colimits over directed sets.

00QN Lemma 10.127.2. Let $R \to A$ be a ring map. There exists a directed system A_λ of R-algebras of finite presentation such that $A = \mathrm{colim}_\lambda A_\lambda$. If A is of finite type over R we may arrange it so that all the transition maps in the system of A_λ are surjective.

Proof. The first proof is that this follows from Lemma 10.127.1 and Categories, Lemma 4.21.5.

Second proof. Compare with the proof of Lemma 10.11.3. Consider any finite subset $S \subset A$, and any finite collection of polynomial relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in A$ and each $e \in E$ consists of a polynomial $f_e \in R[X_s; s \in S]$ such that $f_e(x_s) = 0$. Let $A_{S,E} = R[X_s; s \in S]/(f_e; e \in E)$ which is a finitely presented R-algebra. There are canonical maps $A_{S,E} \to A$. If $S \subset S'$ and if the elements of E correspond, via the map $R[X_s; s \in S] \to R[X_s; s \in S']$, to a subset of E', then there is an obvious map $A_{S,E} \to A_{S',E'}$ commuting with the maps to A. Thus, setting Λ equal the set of pairs (S, E) with ordering by inclusion

[11] To avoid set theoretical difficulties we consider only $A' \to A$ such that A' is a quotient of $R[x_1, x_2, x_3, \ldots]$.

as above, we get a directed partially ordered set. It is clear that the colimit of this directed system is A.

For the last statement, suppose $A = R[x_1, \ldots, x_n]/I$. In this case, consider the subset $\Lambda' \subset \Lambda$ consisting of those systems (S, E) above with $S = \{x_1, \ldots, x_n\}$. It is easy to see that still $A = \mathrm{colim}_{\lambda' \in \Lambda'} A_{\lambda'}$. Moreover, the transition maps are clearly surjective. □

It turns out that we can characterize ring maps of finite presentation as follows. This in some sense says that the algebras of finite presentation are the "compact" objects in the category of R-algebras.

00QO Lemma 10.127.3. Let $\varphi : R \to S$ be a ring map. The following are equivalent

(1) φ is of finite presentation,
(2) for every directed system A_λ of R-algebras the map
$$\mathrm{colim}_\lambda \mathrm{Hom}_R(S, A_\lambda) \longrightarrow \mathrm{Hom}_R(S, \mathrm{colim}_\lambda A_\lambda)$$
is bijective, and
(3) for every directed system A_λ of R-algebras the map
$$\mathrm{colim}_\lambda \mathrm{Hom}_R(S, A_\lambda) \longrightarrow \mathrm{Hom}_R(S, \mathrm{colim}_\lambda A_\lambda)$$
is surjective.

Proof. Assume (1) and write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $A = \mathrm{colim}\, A_\lambda$. Observe that an R-algebra homomorphism $S \to A$ or $S \to A_\lambda$ is determined by the images of $x_1, \ldots, x_n$. Hence it is clear that $\mathrm{colim}_\lambda \mathrm{Hom}_R(S, A_\lambda) \to \mathrm{Hom}_R(S, A)$ is injective. To see that it is surjective, let $\chi : S \to A$ be an R-algebra homomorphism. Then each x_i maps to some element in the image of some A_{λ_i}. We may pick $\mu \geq \lambda_i$, $i = 1, \ldots, n$ and assume $\chi(x_i)$ is the image of $y_i \in A_\mu$ for $i = 1, \ldots, n$. Consider $z_j = f_j(y_1, \ldots, y_n) \in A_\mu$. Since χ is a homomorphism the image of z_j in $A = \mathrm{colim}_\lambda A_\lambda$ is zero. Hence there exists a $\mu_j \geq \mu$ such that z_j maps to zero in A_{μ_j}. Pick $\nu \geq \mu_j$, $j = 1, \ldots, m$. Then the images of $z_1, \ldots, z_m$ are zero in A_ν. This exactly means that the y_i map to elements $y'_i \in A_\nu$ which satisfy the relations $f_j(y'_1, \ldots, y'_n) = 0$. Thus we obtain a ring map $S \to A_\nu$. This shows that (1) implies (2).

It is clear that (2) implies (3). Assume (3). By Lemma 10.127.2 we may write $S = \mathrm{colim}_\lambda S_\lambda$ with S_λ of finite presentation over R. Then the identity map factors as
$$S \to S_\lambda \to S$$
for some λ. This implies that S is finitely presented over S_λ by Lemma 10.6.2 part (4) applied to $S \to S_\lambda \to S$. Applying part (2) of the same lemma to $R \to S_\lambda \to S$ we conclude that S is of finite presentation over R. □

Using the basic material above we can give a criterion of when an algebra A is a filtered colimit of given type of algebra as follows.

07C3 Lemma 10.127.4. Let $R \to \Lambda$ be a ring map. Let $\mathcal{E}$ be a set of R-algebras such that each $A \in \mathcal{E}$ is of finite presentation over R. Then the following two statements are equivalent

(1) Λ is a filtered colimit of elements of $\mathcal{E}$, and

(2) for any R algebra map $A \to \Lambda$ with A of finite presentation over R we can find a factorization $A \to B \to \Lambda$ with $B \in \mathcal{E}$.

Proof. Suppose that $\mathcal{I} \to \mathcal{E}$, $i \mapsto A_i$ is a filtered diagram such that $\Lambda = \mathrm{colim}_i A_i$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation over R. Then we get a factorization $A \to A_i \to \Lambda$ by applying Lemma 10.127.3. Thus (1) implies (2).

Consider the category $\mathcal{I}$ of Lemma 10.127.1. By Categories, Lemma 4.19.3 the full subcategory $\mathcal{J}$ consisting of those $A \to \Lambda$ with $A \in \mathcal{E}$ is cofinal in $\mathcal{I}$ and is a filtered category. Then Λ is also the colimit over $\mathcal{J}$ by Categories, Lemma 4.17.2. □

But more is true. Namely, given $R = \mathrm{colim}_\lambda R_\lambda$ we see that the category of finitely presented R-modules is equivalent to the limit of the category of finitely presented R_λ-modules. Similarly for the categories of finitely presented R-algebras.

05LI Lemma 10.127.5. Let A be a ring and let M, N be A-modules. Suppose that $R = \mathrm{colim}_{i \in I} R_i$ is a directed colimit of A-algebras.

(1) If M is a finite A-module, and $u, u' : M \to N$ are A-module maps such that $u \otimes 1 = u' \otimes 1 : M \otimes_A R \to N \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.
(2) If N is a finite A-module and $u : M \to N$ is an A-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.
(3) If N is a finitely presented A-module, and $v : N \otimes_A R \to M \otimes_A R$ is an R-module map, then there exists an i and an R_i-module map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ such that $v = v_i \otimes 1$.
(4) If M is a finite A-module, N is a finitely presented A-module, and $u : M \to N$ is an A-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism.

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in M$ be generators. Since $N \otimes_A R = \mathrm{colim}_i N \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $M \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in N$ be generators. Since $N \otimes_A R = \mathrm{colim}_i N \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $N \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.

To prove (3) let $y_1, \ldots, y_m \in N$ be generators. Let $K = \mathrm{Ker}(A^{\oplus m} \to N)$ where the map is given by the rule $(a_1, \ldots, a_m) \mapsto \sum a_j x_j$. Let $k_1, \ldots, k_t$ be generators for K. Say $k_s = (k_{s1}, \ldots, k_{sm})$. Since $M \otimes_A R = \mathrm{colim}_i M \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $M \otimes_A R$ equal $v(y_j \otimes 1)$. We want to use the z_j to define the map $v_i : N \otimes_A R_i \to M \otimes_A R_i$. Since $K \otimes_A R_i \to R_i^{\oplus m} \to N \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = \sum_j k_{sj} z_j$ is zero in $M \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $M \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that M is finite, and that N is finitely presented. Let $v : N \otimes_A R \to M \otimes_A R$ be an inverse to $u \otimes 1$. Apply part (3) to get a map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ for some i. Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \mathrm{id}_{M \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \mathrm{id}_{N \otimes_R R_i}$. □

05N7 **Lemma 10.127.6.** Suppose that $R = \mathrm{colim}_{\lambda \in \Lambda} R_\lambda$ is a directed colimit of rings. Then the category of finitely presented R-modules is the colimit of the categories of finitely presented R_λ-modules. More precisely

(1) Given a finitely presented R-module M there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-module M_λ such that $M \cong M_\lambda \otimes_{R_\lambda} R$.
(2) Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and an R-module map $\varphi : M_\lambda \otimes_{R_\lambda} R \to N_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-module map $\varphi_\mu : M_\lambda \otimes_{R_\lambda} R_\mu \to N_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.
(3) Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and R-module maps $\varphi_\lambda, \psi_\lambda : M_\lambda \to N_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.

Proof. To prove (1) choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Suppose that the first map is given by the matrix $A = (a_{ij})$. We can choose a $\lambda \in \Lambda$ and a matrix $A_\lambda = (a_{\lambda,ij})$ with coefficients in R_λ which maps to A in R. Then we simply let M_λ be the R_λ-module with presentation $R_\lambda^{\oplus m} \to R_\lambda^{\oplus n} \to M_\lambda \to 0$ where the first arrow is given by A_λ.

Parts (2) and (3) follow from Lemma 10.127.5. $\square$

05N8 **Lemma 10.127.7.** Let A be a ring and let B, C be A-algebras. Suppose that $R = \mathrm{colim}_{i \in I} R_i$ is a directed colimit of A-algebras.

(1) If B is a finite type A-algebra, and $u, u' : B \to C$ are A-algebra maps such that $u \otimes 1 = u' \otimes 1 : B \otimes_A R \to C \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.
(2) If C is a finite type A-algebra and $u : B \to C$ is an A-algebra map such that $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.
(3) If C is of finite presentation over A and $v : C \otimes_A R \to B \otimes_A R$ is an R-algebra map, then there exists an i and an R_i-algebra map $v_i : C \otimes_A R_i \to B \otimes_A R_i$ such that $v = v_i \otimes 1$.
(4) If B is a finite type A-algebra, C is a finitely presented A-algebra, and $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is an isomorphism.

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in B$ be generators. Since $B \otimes_A R = \mathrm{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $B \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in C$ be generators. Since $B \otimes_A R = \mathrm{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $C \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.

To prove (3) let $c_1, \ldots, c_m \in C$ be generators. Let $K = \mathrm{Ker}(A[x_1, \ldots, x_m] \to N)$ where the map is given by the rule $x_j \mapsto \sum c_j$. Let $f_1, \ldots, f_t$ be generators for K as an ideal in $A[x_1, \ldots, x_m]$. We think of $f_j = f_j(x_1, \ldots, x_m)$ as a polynomial. Since $B \otimes_A R = \mathrm{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $B \otimes_A R$ equal $v(c_j \otimes 1)$. We want to use the z_j to define a map $v_i : C \otimes_A R_i \to B \otimes_A R_i$. Since $K \otimes_A R_i \to R_i[x_1, \ldots, x_m] \to C \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = f_j(z_1, \ldots, z_m)$ is zero in $B \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $B \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that B is a finite type A-algebra, and that C is a finitely presented A-algebra. Let $v : B \otimes_A R \to C \otimes_A R$ be an inverse to $u \otimes 1$. Let $v_i : C \otimes_A R_i \to B \otimes_A R_i$ be as in part (3). Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \text{id}_{B \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \text{id}_{C \otimes_R R_i}$. □

05N9 Lemma 10.127.8. Suppose that $R = \text{colim}_{\lambda \in \Lambda} R_\lambda$ is a directed colimit of rings. Then the category of finitely presented R-algebras is the colimit of the categories of finitely presented R_λ-algebras. More precisely

(1) Given a finitely presented R-algebra A there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-algebra A_λ such that $A \cong A_\lambda \otimes_{R_\lambda} R$.
(2) Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and an R-algebra map $\varphi : A_\lambda \otimes_{R_\lambda} R \to B_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-algebra map $\varphi_\mu : A_\lambda \otimes_{R_\lambda} R_\mu \to B_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.
(3) Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and R_λ-algebra maps $\varphi_\lambda, \psi_\lambda : A_\lambda \to B_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.

Proof. To prove (1) choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We can choose a $\lambda \in \Lambda$ and elements $f_{\lambda,j} \in R_\lambda[x_1, \ldots, x_n]$ mapping to $f_j \in R[x_1, \ldots, x_n]$. Then we simply let $A_\lambda = R_\lambda[x_1, \ldots, x_n]/(f_{\lambda,1}, \ldots, f_{\lambda,m})$.

Parts (2) and (3) follow from Lemma 10.127.7. □

00QT Lemma 10.127.9. Suppose $R \to S$ is a local homomorphism of local rings. There exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.
(2) Each R_λ is essentially of finite type over $\mathbf{Z}$.
(3) Each S_λ is essentially of finite type over R_λ.

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let

$$\Lambda = \{(A, B) \mid A \subset R, B \subset S, \#A < \infty, \#B < \infty, \varphi(A) \subset B\}.$$

As partial ordering we take the inclusion relation. For each $\lambda = (A, B) \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by b, $b \in B$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{tikzcd} B \arrow[r] & S'_\lambda \arrow[r] & S_\lambda \arrow[r] & S \\ A \arrow[r] \arrow[u] & R'_\lambda \arrow[r] \arrow[u] & R_\lambda \arrow[r] \arrow[u] & R \arrow[u] \end{tikzcd}.$$

The transition maps are clear. We leave the proofs of the other assertions to the reader. □

00QU Lemma 10.127.10. Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.
(2) Each R_λ is essentially of finite type over $\mathbf{Z}$.
(3) Each S_λ is essentially of finite type over R_λ.
(4) For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let $x_1, \ldots, x_n \in S$ be elements such that S is a localization of the sub R-algebra of S generated by $x_1, \ldots, x_n$. In other words, S is a quotient of a localization of the polynomial ring $R[x_1, \ldots, x_n]$.

Let $\Lambda = \{A \subset R \mid \#A < \infty\}$ be the set of finite subsets of R. As partial ordering we take the inclusion relation. For each $\lambda = A \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by $\varphi(a)$, $a \in A$ and the elements $x_1, \ldots, x_n$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{tikzcd}
\varphi(A) \amalg \{x_i\} \arrow[r] & S'_\lambda \arrow[r] & S_\lambda \arrow[r] & S \\
A \arrow[u] \arrow[r] & R'_\lambda \arrow[u] \arrow[r] & R_\lambda \arrow[u] \arrow[r] & R \arrow[u]
\end{tikzcd}$$

It is clear that if $A \subset B$ corresponds to $\lambda \leq \mu$ in Λ, then there are canonical maps $R_\lambda \to R_\mu$, and $S_\lambda \to S_\mu$ and we obtain a system over the directed set Λ.

The assertion that $R = \operatorname{colim} R_\lambda$ is clear because all the maps $R_\lambda \to R$ are injective and any element of R eventually is in the image. The same argument works for $S = \operatorname{colim} S_\lambda$. Assertions (2), (3) are true by construction. The final assertion holds because clearly the maps $S'_\lambda \otimes_{R'_\lambda} R'_\mu \to S'_\mu$ are surjective. $\square$

00QV Lemma 10.127.11. Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphism $R_\lambda \to S_\lambda$ of local rings such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.
(2) Each R_λ is essentially of finite type over $\mathbf{Z}$.
(3) Each S_λ is essentially of finite type over R_λ.
(4) For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.

Proof. By assumption we may choose an isomorphism $\Phi : (R[x_1, \ldots, x_n]/I)_\mathfrak{q} \to S$ where $I \subset R[x_1, \ldots, x_n]$ is a finitely generated ideal, and $\mathfrak{q} \subset R[x_1, \ldots, x_n]/I$ is a prime. (Note that $R \cap \mathfrak{q}$ is equal to the maximal ideal $\mathfrak{m}$ of R.) We also choose generators $f_1, \ldots, f_m \in I$ for the ideal I. Write R in any way as a colimit $R = \operatorname{colim} R_\lambda$ over a directed set $(\Lambda, \leq)$, with each R_λ local and essentially of finite type over $\mathbf{Z}$. There exists some $\lambda_0 \in \Lambda$ such that f_j is the image of some $f_{j,\lambda_0} \in R_{\lambda_0}[x_1, \ldots, x_n]$. For all $\lambda \geq \lambda_0$ denote $f_{j,\lambda} \in R_\lambda[x_1, \ldots, x_n]$ the image of f_{j,λ_0}. Thus we obtain a system of ring maps

$$R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}) \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \to S$$

Set $\mathfrak{q}_\lambda$ the inverse image of $\mathfrak{q}$. Set $S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}))_{\mathfrak{q}_\lambda}$. We leave it to the reader to see that this works. $\square$

00QW Remark 10.127.12. Suppose that $R \to S$ is a local homomorphism of local rings, which is essentially of finite presentation. Take any system $(\Lambda, \leq)$, $R_\lambda \to S_\lambda$ with the properties listed in Lemma 10.127.10. What may happen is that this is the "wrong" system, namely, it may happen that property (4) of Lemma 10.127.11 is not satisfied. Here is an example. Let k be a field. Consider the ring

$$R = k[[z, y_1, y_2, \ldots]]/(y_i^2 - zy_{i+1}).$$

Set $S = R/zR$. As system take $\Lambda = \mathbf{N}$ and $R_n = k[[z, y_1, \ldots, y_n]]/(\{y_i^2 - zy_{i+1}\}_{i \leq n-1})$ and $S_n = R_n/(z, y_n^2)$. All the maps $S_n \otimes_{R_n} R_{n+1} \to S_{n+1}$ are not localizations (i.e., isomorphisms in this case) since $1 \otimes y_{n+1}^2$ maps to zero. If we take instead $S'_n = R_n/zR_n$ then the maps $S'_n \otimes_{R_n} R_{n+1} \to S'_{n+1}$ are isomorphisms. The moral of this remark is that we do have to be a little careful in choosing the systems.

00QX Lemma 10.127.13. Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings together with S_λ-modules M_λ, such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.
(2) Each R_λ is essentially of finite type over $\mathbf{Z}$.
(3) Each S_λ is essentially of finite type over R_λ.
(4) Each M_λ is finite over S_λ.
(5) For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.
(6) For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to M_\mu$ is an isomorphism.

Proof. As in the proof of Lemma 10.127.11 we may first write $R = \operatorname{colim} R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = \operatorname{colim}_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}))_{\mathfrak{q}_\lambda}$$

Choose a presentation

$$S^{\oplus s} \to S^{\oplus t} \to M \to 0$$

of M over S. Let $A \in \text{Mat}(t \times s, S)$ be the matrix of the presentation. For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ we can find a matrix $A_{\lambda_2} \in \text{Mat}(t \times s, S_{\lambda_2})$ which maps to A. For all $\lambda \geq \lambda_2$ we let $M_\lambda = \text{Coker}(S_\lambda^{\oplus s} \xrightarrow{A_\lambda} S_\lambda^{\oplus t})$. We leave it to the reader to see that this works. $\square$

00QY Lemma 10.127.14. Suppose $R \to S$ is a ring map. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.
(2) Each R_λ is of finite type over $\mathbf{Z}$.
(3) Each S_λ is of finite type over R_λ.

Proof. This is the non-local version of Lemma 10.127.9. Proof is similar and left to the reader. $\square$

0BTG Lemma 10.127.15. Suppose $R \to S$ is a ring map. Assume that S is integral over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.
(2) Each R_λ is of finite type over $\mathbf{Z}$.
(3) Each S_λ is finite over R_λ.

Proof. Consider the set Λ of pairs (E, F) where $E \subset R$ is a finite subset, $F \subset S$ is a finite subset, and every element $f \in F$ is the root of a monic $P(X) \in R[X]$ whose coefficients are in E. Say $(E, F) \leq (E', F')$ if $E \subset E'$ and $F \subset F'$. Given $\lambda = (E, F) \in \Lambda$ set $R_\lambda \subset R$ equal to the $\mathbf{Z}$-subalgebra of R generated by E and $S_\lambda \subset S$ equal to the $\mathbf{Z}$-subalgebra generated by F and the image of E in S. It is clear that $R = \operatorname{colim} R_\lambda$. We have $S = \operatorname{colim} S_\lambda$ as every element of S is integral over S. The ring maps $R_\lambda \to S_\lambda$ are finite by Lemma 10.36.5 and the fact that S_λ is generated over R_λ by the elements of F which are integral over R_λ by our condition on the pairs (E, F). The lemma follows. □

00QZ Lemma 10.127.16. Suppose $R \to S$ is a ring map. Assume that S is of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.
(2) Each R_λ is of finite type over $\mathbf{Z}$.
(3) Each S_λ is of finite type over R_λ.
(4) For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.

Proof. This is the non-local version of Lemma 10.127.10. Proof is similar and left to the reader. □

00R0 Lemma 10.127.17. Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.
(2) Each R_λ is of finite type over $\mathbf{Z}$.
(3) Each S_λ is of finite type over R_λ.
(4) For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.

Proof. This is the non-local version of Lemma 10.127.11. Proof is similar and left to the reader. □

00R1 Lemma 10.127.18. Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ together with S_λ-modules M_λ, such that

(1) The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.
(2) Each R_λ is of finite type over $\mathbf{Z}$.
(3) Each S_λ is of finite type over R_λ.
(4) Each M_λ is finite over S_λ.
(5) For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.
(6) For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to M_\mu$ is an isomorphism.

In particular, for every $\lambda \in \Lambda$ we have

$$M = M_\lambda \otimes_{S_\lambda} S = M_\lambda \otimes_{R_\lambda} R.$$

Proof. This is the non-local version of Lemma 10.127.13. Proof is similar and left to the reader. ☐

10.128. More flatness criteria

00R3 The following lemma is often used in algebraic geometry to show that a finite morphism from a normal surface to a smooth surface is flat. It is a partial converse to Lemma 10.112.9 because an injective finite local ring map certainly satisfies condition (3).

00R4 Lemma 10.128.1. Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume

(1) R is regular,
(2) S Cohen-Macaulay,
(3) $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)$.

Then $R \to S$ is flat.

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial, because then R is a field. Assume $\dim(R) > 0$. By (3) this implies that $\dim(S) > 0$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of S. Note that $\mathfrak{q}_i \not\supset \mathfrak{m}_R S$ since

$$\dim(S/\mathfrak{q}_i) = \dim(S) > \dim(S/\mathfrak{m}_R S)$$

the first equality by Lemma 10.104.3 and the inequality by (3). Thus $\mathfrak{p}_i = R \cap \mathfrak{q}_i$ is not equal to $\mathfrak{m}_R$. Pick $x \in \mathfrak{m}_R$, $x \notin \mathfrak{m}_R^2$, and $x \notin \mathfrak{p}_i$, see Lemma 10.15.2. Hence we see that x is not contained in any of the minimal primes of S. Hence x is a nonzerodivisor on S by (2), see Lemma 10.104.2 and S/xS is Cohen-Macaulay with $\dim(S/xS) = \dim(S) - 1$. By (1) and Lemma 10.106.3 the ring R/xR is regular with $\dim(R/xR) = \dim(R) - 1$. By induction we see that $R/xR \to S/xS$ is flat. Hence we conclude by Lemma 10.99.10 and the remark following it. ☐

07DY Lemma 10.128.2. Let $R \to S$ be a homomorphism of Noetherian local rings. Assume that R is a regular local ring and that a regular system of parameters maps to a regular sequence in S. Then $R \to S$ is flat.

Proof. Suppose that $x_1, \ldots, x_d$ are a system of parameters of R which map to a regular sequence in S. Note that $S/(x_1, \ldots, x_d)S$ is flat over $R/(x_1, \ldots, x_d)$ as the latter is a field. Then x_d is a nonzerodivisor in $S/(x_1, \ldots, x_{d-1})S$ hence $S/(x_1, \ldots, x_{d-1})S$ is flat over $R/(x_1, \ldots, x_{d-1})$ by the local criterion of flatness (see Lemma 10.99.10 and remarks following). Then x_{d-1} is a nonzerodivisor in $S/(x_1, \ldots, x_{d-2})S$ hence $S/(x_1, \ldots, x_{d-2})S$ is flat over $R/(x_1, \ldots, x_{d-2})$ by the local criterion of flatness (see Lemma 10.99.10 and remarks following). Continue till one reaches the conclusion that S is flat over R. ☐

The following lemma is the key to proving that results for finitely presented modules over finitely presented rings over a base ring follow from the corresponding results for finite modules in the Noetherian case.

00R6 Lemma 10.128.3. Let $R \to S$, M, Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 10.127.13. Assume that M is flat over R. Then for some $\lambda \in \Lambda$ the module M_λ is flat over R_λ.

Proof. Pick some $\lambda \in \Lambda$ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) = \mathrm{Ker}(\mathfrak{m}_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 10.75.9. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because M is flat over R we have that $0 = \mathrm{Ker}(\mathfrak{m}_\lambda R \otimes_R M \to M)$. Since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in $\mathfrak{m}_\lambda R_{\lambda'} \otimes_{R_{\lambda'}} M_{\lambda'}$. Hence we see that

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) \longrightarrow \mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/\mathfrak{m}_\lambda R_{\lambda'})$$

is zero. Note that $M_\lambda \otimes_{R_\lambda} R_\lambda/\mathfrak{m}_\lambda$ is flat over $R_\lambda/\mathfrak{m}_\lambda$ because this last ring is a field. Hence we may apply Lemma 10.99.14 to get that $M_{\lambda'}$ is flat over $R_{\lambda'}$. $\square$

Using the lemma above we can start to reprove the results of Section 10.99 in the non-Noetherian case.

046Y **Lemma 10.128.4.** Suppose that $R \to S$ is a local homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Let $u : M \to N$ be a map of S-modules. Assume

(1) S is essentially of finite presentation over R,
(2) M, N are finitely presented over S,
(3) N is flat over R, and
(4) $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective.

Then u is injective, and $N/u(M)$ is flat over R.

Proof. By Lemma 10.127.13 and its proof we can find a system $R_\lambda \to S_\lambda$ of local ring maps together with maps of S_λ-modules $u_\lambda : M_\lambda \to N_\lambda$ satisfying the conclusions (1) – (6) for both N and M of that lemma and such that the colimit of the maps u_λ is u. By Lemma 10.128.3 we may assume that N_λ is flat over R_λ for all sufficiently large λ. Denote $\mathfrak{m}_\lambda \subset R_\lambda$ the maximal ideal and $\kappa_\lambda = R_\lambda/\mathfrak{m}_\lambda$, resp. $\kappa = R/\mathfrak{m}$ the residue fields.

Consider the map

$$\Psi_\lambda : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa \longrightarrow M/\mathfrak{m}M.$$

Since $S_\lambda/\mathfrak{m}_\lambda S_\lambda$ is essentially of finite type over the field κ_λ we see that the tensor product $S_\lambda/\mathfrak{m}_\lambda S_\lambda \otimes_{\kappa_\lambda} \kappa$ is essentially of finite type over κ. Hence it is a Noetherian ring and we conclude the kernel of Ψ_λ is finitely generated. Since $M/\mathfrak{m}M$ is the colimit of the system $M_\lambda/\mathfrak{m}_\lambda M_\lambda$ and κ is the colimit of the fields κ_λ there exists a $\lambda' > \lambda$ such that the kernel of Ψ_λ is generated by the kernel of

$$\Psi_{\lambda,\lambda'} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'} \longrightarrow M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

By construction there exists a multiplicative subset $W \subset S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ such that $S_{\lambda'} = W^{-1}(S_\lambda \otimes_{R_\lambda} R_{\lambda'})$ and

$$W^{-1}(M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'}) = M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

Now suppose that x is an element of the kernel of

$$\Psi_{\lambda'} : M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa \longrightarrow M/\mathfrak{m}M.$$

Then for some $w \in W$ we have $wx \in M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes \kappa$. Hence $wx \in \mathrm{Ker}(\Psi_\lambda)$. Hence wx is a linear combination of elements in the kernel of $\Psi_{\lambda,\lambda'}$. Hence $wx = 0$ in $M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa$, hence $x = 0$ because w is invertible in $S_{\lambda'}$. We conclude that the kernel of $\Psi_{\lambda'}$ is zero for all sufficiently large λ'!

By the result of the preceding paragraph we may assume that the kernel of Ψ_λ is zero for all λ sufficiently large, which implies that the map $M_\lambda/\mathfrak{m}_\lambda M_\lambda \to M/\mathfrak{m}M$ is injective. Combined with $\overline{u}$ being injective this formally implies that also $\overline{u_\lambda} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \to N_\lambda/\mathfrak{m}_\lambda N_\lambda$ is injective. By Lemma 10.99.1 we conclude that (for all sufficiently large λ) the map u_λ is injective and that $N_\lambda/u_\lambda(M_\lambda)$ is flat over R_λ. The lemma follows. $\square$

046Z Lemma 10.128.5. Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose

(1) S is essentially of finite presentation over R,
(2) S is flat over R, and
(3) $f \in S$ is a nonzerodivisor in $S/\mathfrak{m}S$.

Then S/fS is flat over R, and f is a nonzerodivisor in S.

Proof. Follows directly from Lemma 10.128.4. $\square$

0470 Lemma 10.128.6. Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose

(1) $R \to S$ is essentially of finite presentation,
(2) $R \to S$ is flat, and
(3) $f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$.

Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.

Proof. Induction and Lemma 10.128.5. $\square$

Here is the version of the local criterion of flatness for the case of local ring maps which are locally of finite presentation.

0471 Lemma 10.128.7. Let $R \to S$ be a local homomorphism of local rings. Let $I \neq R$ be an ideal in R. Let M be an S-module. Assume

(1) S is essentially of finite presentation over R,
(2) M is of finite presentation over S,
(3) $\mathrm{Tor}_1^R(M, R/I) = 0$, and
(4) M/IM is flat over R/I.

Then M is flat over R.

Proof. Let Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 10.127.13. Denote $I_\lambda \subset R_\lambda$ the inverse image of I. In this case the system $R/I \to S/IS$, M/IM, $R_\lambda \to S_\lambda/I_\lambda S_\lambda$, and $M_\lambda/I_\lambda M_\lambda$ satisfies the conclusions of Lemma 10.127.13 as well. Hence by Lemma 10.128.3 we may assume (after shrinking the index set Λ) that $M_\lambda/I_\lambda M_\lambda$ is flat for all λ. Pick some λ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) = \mathrm{Ker}(I_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 10.75.9. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because $\mathrm{Tor}_1^R(M, R/I) = 0$ and since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in $\text{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$. The composition of the maps

$$\begin{tikzcd} R_{\lambda'} \otimes_{R_\lambda} \text{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) \arrow[d, "\text{surjective by Lemma 10.99.12}"] \\ \text{Tor}_1^{R_\lambda}(M_\lambda, R_{\lambda'}/I_\lambda R_{\lambda'}) \arrow[d, "\text{surjective up to localization by Lemma 10.99.13}"] \\ \text{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_\lambda R_{\lambda'}) \arrow[d, "\text{surjective by Lemma 10.99.12}"] \\ \text{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'}). \end{tikzcd}$$

is surjective up to a localization by the reasons indicated. The localization is necessary since $M_{\lambda'}$ is not equal to $M_\lambda \otimes_{R_\lambda} R_{\lambda'}$. Namely, it is equal to $M_\lambda \otimes_{S_\lambda} S_{\lambda'}$ and $S_{\lambda'}$ is the localization of $S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ whence the statement up to a localization (or tensoring with $S_{\lambda'}$). Note that Lemma 10.99.12 applies to the first and third arrows because $M_\lambda/I_\lambda M_\lambda$ is flat over R_λ/I_λ and because $M_{\lambda'}/I_\lambda M_{\lambda'}$ is flat over $R_{\lambda'}/I_\lambda R_{\lambda'}$ as it is a base change of the flat module $M_\lambda/I_\lambda M_\lambda$. The composition maps the generators ξ_i to zero as we explained above. We finally conclude that $\text{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$ is zero. This implies that $M_{\lambda'}$ is flat over $R_{\lambda'}$ by Lemma 10.99.10. □

Please compare the lemma below to Lemma 10.99.15 (the case of Noetherian local rings) and Lemma 10.101.8 (the case of a nilpotent ideal in the base).

00R7 **Lemma 10.128.8** (Critère de platitude par fibres). Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak{m} \subset R$ be the maximal ideal. Assume

(1) The ring maps $R \to S$ and $R \to S'$ are essentially of finite presentation.
(2) The module M is of finite presentation over S'.
(3) The module M is not zero.
(4) The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.
(5) The module M is a flat R-module.

Then S is flat over R and M is a flat S-module.

Proof. As in the proof of Lemma 10.127.11 we may first write $R = \text{colim}\, R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Denote $\mathfrak{p}_\lambda$ the maximal ideal of R_λ. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = \text{colim}_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{u,\lambda}))_{\mathfrak{q}_\lambda}$$

For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ there exist $g_{j,\lambda_2} \in R_{\lambda_2}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with images $\overline{g}_{j,\lambda_2} \in S_{\lambda_2}[y_1, \ldots, y_m]$ such that

$$S' = \text{colim}_{\lambda \geq \lambda_2} S'_\lambda, \text{ with } S'_\lambda = (S_\lambda[y_1, \ldots, y_m]/(\overline{g}_{1,\lambda}, \ldots, \overline{g}_{v,\lambda}))_{\overline{\mathfrak{q}}'_\lambda}$$

Note that this also implies that

$$S'_\lambda = (R_\lambda[x_1, \ldots, x_n, y_1, \ldots, y_m]/(g_{1,\lambda}, \ldots, g_{v,\lambda}))_{\mathfrak{q}'_\lambda}$$

Choose a presentation

$$(S')^{\oplus s} \to (S')^{\oplus t} \to M \to 0$$

of M over S'. Let $A \in \text{Mat}(t \times s, S')$ be the matrix of the presentation. For some $\lambda_3 \in \Lambda$, $\lambda_3 \geq \lambda_2$ we can find a matrix $A_{\lambda_3} \in \text{Mat}(t \times s, S_{\lambda_3})$ which maps to A. For all $\lambda \geq \lambda_3$ we let $M_\lambda = \text{Coker}((S'_\lambda)^{\oplus s} \xrightarrow{A_\lambda} (S'_\lambda)^{\oplus t})$.

With these choices, we have for each $\lambda_3 \leq \lambda \leq \mu$ that $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is a localization, $S'_\lambda \otimes_{S_\lambda} S_\mu \to S'_\mu$ is a localization, and the map $M_\lambda \otimes_{S'_\lambda} S'_\mu \to M_\mu$ is an isomorphism. This also implies that $S'_\lambda \otimes_{R_\lambda} R_\mu \to S'_\mu$ is a localization. Thus, since M is flat over R we see by Lemma 10.128.3 that for all λ big enough the module M_λ is flat over R_λ. Moreover, note that $\mathfrak m = \text{colim}\, \mathfrak p_\lambda$, $S/\mathfrak m S = \text{colim}\, S_\lambda/\mathfrak p_\lambda S_\lambda$, $S'/\mathfrak m S' = \text{colim}\, S'_\lambda/\mathfrak p_\lambda S'_\lambda$, and $M/\mathfrak m M = \text{colim}\, M_\lambda/\mathfrak p_\lambda M_\lambda$. Also, for each $\lambda_3 \leq \lambda \leq \mu$ we see (from the properties listed above) that

$$S'_\lambda/\mathfrak p_\lambda S'_\lambda \otimes_{S_\lambda/\mathfrak p_\lambda S_\lambda} S_\mu/\mathfrak p_\mu S_\mu \longrightarrow S'_\mu/\mathfrak p_\mu S'_\mu$$

is a localization, and the map

$$M_\lambda/\mathfrak p_\lambda M_\lambda \otimes_{S'_\lambda/\mathfrak p_\lambda S'_\lambda} S'_\mu/\mathfrak p_\mu S'_\mu \longrightarrow M_\mu/\mathfrak p_\mu M_\mu$$

is an isomorphism. Hence the system $(S_\lambda/\mathfrak p_\lambda S_\lambda \to S'_\lambda/\mathfrak p_\lambda S'_\lambda, M_\lambda/\mathfrak p_\lambda M_\lambda)$ is a system as in Lemma 10.127.13 as well. We may apply Lemma 10.128.3 again because $M/\mathfrak m M$ is assumed flat over $S/\mathfrak m S$ and we see that $M_\lambda/\mathfrak p_\lambda M_\lambda$ is flat over $S_\lambda/\mathfrak p_\lambda S_\lambda$ for all λ big enough. Thus for λ big enough the data $R_\lambda \to S_\lambda \to S'_\lambda, M_\lambda$ satisfies the hypotheses of Lemma 10.99.15. Pick such a λ. Then $S = S_\lambda \otimes_{R_\lambda} R$ is flat over R, and $M = M_\lambda \otimes_{S_\lambda} S$ is flat over S (since the base change of a flat module is flat). $\square$

The following is an easy consequence of the "critère de platitude par fibres" Lemma 10.128.8. For more results of this kind see More on Flatness, Section 38.1.

05UV Lemma 10.128.9. Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak m \subset R$ be the maximal ideal. Assume

(1) $R \to S'$ is essentially of finite presentation,
(2) $R \to S$ is essentially of finite type,
(3) M is of finite presentation over S',
(4) M is not zero,
(5) $M/\mathfrak m M$ is a flat $S/\mathfrak m S$-module, and
(6) M is a flat R-module.

Then S is essentially of finite presentation and flat over R and M is a flat S-module.

Proof. As S is essentially of finite presentation over R we can write $S = C_{\overline{\mathfrak q}}$ for some finite type R-algebra C. Write $C = R[x_1, \ldots, x_n]/I$. Denote $\mathfrak q \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\overline{\mathfrak q}$. Then we see that $S = B/J$ where $B = R[x_1, \ldots, x_n]_{\mathfrak q}$ is essentially of finite presentation over R and $J = IB$. We can find $f_1, \ldots, f_k \in J$ such that the images $\overline{f}_i \in B/\mathfrak m B$ generate the image $\overline{J}$ of J in the Noetherian ring $B/\mathfrak m B$. Hence there exist finitely generated ideals $J' \subset J$ such that $B/J' \to B/J$ induces an isomorphism

$$(B/J') \otimes_R R/\mathfrak m \longrightarrow B/J \otimes_R R/\mathfrak m = S/\mathfrak m S.$$

For any J' as above we see that Lemma 10.128.8 applies to the ring maps

$$R \longrightarrow B/J' \longrightarrow S'$$

and the module M. Hence we conclude that B/J' is flat over R for any choice J' as above. Now, if $J' \subset J' \subset J$ are two finitely generated ideals as above, then we conclude that $B/J' \to B/J''$ is a surjective map between flat R-algebras which are essentially of finite presentation which is an isomorphism modulo $\mathfrak{m}$. Hence Lemma 10.128.4 implies that $B/J' = B/J''$, i.e., $J' = J''$. Clearly this means that J is finitely generated, i.e., S is essentially of finite presentation over R. Thus we may apply Lemma 10.128.8 to $R \to S \to S'$ and we win. $\square$

0CEL Lemma 10.128.10 (Critère de platitude par fibres: locally nilpotent case). Let

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

be a commutative diagram in the category of rings. Let $I \subset R$ be a locally nilpotent ideal and M an S'-module. Assume

(1) $R \to S$ is of finite type,
(2) $R \to S'$ is of finite presentation,
(3) M is a finitely presented S'-module,
(4) M/IM is flat as a S/IS-module, and
(5) M is flat as an R-module.

Then M is a flat S-module and $S_\mathfrak{q}$ is flat and essentially of finite presentation over R for every $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero.

Proof. If $M \otimes_S \kappa(\mathfrak{q})$ is nonzero, then $S' \otimes_S \kappa(\mathfrak{q})$ is nonzero and hence there exists a prime $\mathfrak{q}' \subset S'$ lying over $\mathfrak{q}$ (Lemma 10.17.9). Let $\mathfrak{p} \subset R$ be the image of $\mathfrak{q}$ in $\text{Spec}(R)$. Then $I \subset \mathfrak{p}$ as I is locally nilpotent hence $M/\mathfrak{p}M$ is flat over $S/\mathfrak{p}S$. Hence we may apply Lemma 10.128.9 to $R_\mathfrak{p} \to S_\mathfrak{q} \to S'_{\mathfrak{q}'}$ and $M_{\mathfrak{q}'}$. We conclude that $M_{\mathfrak{q}'}$ is flat over S and $S_\mathfrak{q}$ is flat and essentially of finite presentation over R. Since $\mathfrak{q}'$ was an arbitrary prime of S' we also see that M is flat over S (Lemma 10.39.18). $\square$

10.129. Openness of the flat locus

00R8 We use Lemma 10.128.3 to reduce to the Noetherian case. The Noetherian case is handled using the characterization of exact complexes given in Section 10.102.

00R9 Lemma 10.129.1. Let k be a field. Let S be a finite type k-algebra. Let $f_1, \ldots, f_i$ be elements of S. Assume that S is Cohen-Macaulay and equidimensional of dimension d, and that $\dim V(f_1, \ldots, f_i) \leq d - i$. Then equality holds and $f_1, \ldots, f_i$ forms a regular sequence in $S_\mathfrak{q}$ for every prime $\mathfrak{q}$ of $V(f_1, \ldots, f_i)$.

Proof. If S is Cohen-Macaulay and equidimensional of dimension d, then we have $\dim(S_\mathfrak{m}) = d$ for all maximal ideals $\mathfrak{m}$ of S, see Lemma 10.114.7. By Proposition 10.103.4 we see that for all maximal ideals $\mathfrak{m} \in V(f_1, \ldots, f_i)$ the sequence is a regular sequence in $S_\mathfrak{m}$ and the local ring $S_\mathfrak{m}/(f_1, \ldots, f_i)$ is Cohen-Macaulay of dimension $d - i$. This actually means that $S/(f_1, \ldots, f_i)$ is Cohen-Macaulay and equidimensional of dimension $d - i$. $\square$

00RA Lemma 10.129.2. Let $R \to S$ be a finite type ring map. Let d be an integer such that all fibres $S \otimes_R \kappa(\mathfrak p)$ are Cohen-Macaulay and equidimensional of dimension d. Let $f_1, \ldots, f_i$ be elements of S. The set

$$\{\mathfrak q \in V(f_1, \ldots, f_i) \mid f_1, \ldots, f_i \text{ are a regular sequence in } S_{\mathfrak q}/\mathfrak p S_{\mathfrak q} \text{ where } \mathfrak p = R \cap \mathfrak q\}$$

is open in $V(f_1, \ldots, f_i)$.

Proof. Write $\overline{S} = S/(f_1, \ldots, f_i)$. Suppose $\mathfrak q$ is an element of the set defined in the lemma, and $\mathfrak p$ is the corresponding prime of R. We will use relative dimension as defined in Definition 10.125.1. First, note that $d = \dim_{\mathfrak q}(S/R) = \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}) + \text{trdeg}_{\kappa(\mathfrak p)}\ \kappa(\mathfrak q)$ by Lemma 10.116.3. Since $f_1, \ldots, f_i$ form a regular sequence in the Noetherian local ring $S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$ Lemma 10.60.13 tells us that $\dim(\overline{S}_{\mathfrak q}/\mathfrak p\overline{S}_{\mathfrak q}) = \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}) - i$. We conclude that $\dim_{\mathfrak q}(\overline{S}/R) = \dim(\overline{S}_{\mathfrak q}/\mathfrak p\overline{S}_{\mathfrak q}) + \text{trdeg}_{\kappa(\mathfrak p)}\ \kappa(\mathfrak q) = d - i$ by Lemma 10.116.3. By Lemma 10.125.6 we have $\dim_{\mathfrak q'}(\overline{S}/R) \leq d - i$ for all $\mathfrak q' \in V(f_1, \ldots, f_i) = \text{Spec}(\overline{S})$ in a neighbourhood of $\mathfrak q$. Thus after replacing S by S_g for some $g \in S$, $g \notin \mathfrak q$ we may assume that the inequality holds for all $\mathfrak q'$. The result follows from Lemma 10.129.1. □

00RB Lemma 10.129.3. Let $R \to S$ be a ring map. Consider a finite homological complex of finite free S-modules:

$$F_\bullet : 0 \to S^{n_e} \xrightarrow{\varphi_e} S^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} S^{n_i} \xrightarrow{\varphi_i} S^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} S^{n_0}$$

For every prime $\mathfrak q$ of S consider the complex $\overline{F}_{\bullet,\mathfrak q} = F_{\bullet,\mathfrak q} \otimes_R \kappa(\mathfrak p)$ where $\mathfrak p$ is inverse image of $\mathfrak q$ in R. Assume R is Noetherian and there exists an integer d such that $R \to S$ is finite type, flat with fibres $S \otimes_R \kappa(\mathfrak p)$ Cohen-Macaulay of dimension d. The set

$$\{\mathfrak q \in \text{Spec}(S) \mid \overline{F}_{\bullet,\mathfrak q} \text{ is exact}\}$$

is open in $\text{Spec}(S)$.

Proof. Let $\mathfrak q$ be an element of the set defined in the lemma. We are going to use Proposition 10.102.9 to show there exists a $g \in S$, $g \notin \mathfrak q$ such that $D(g)$ is contained in the set defined in the lemma. In other words, we are going to show that after replacing S by S_g, the set of the lemma is all of $\text{Spec}(S)$. Thus during the proof we will, finitely often, replace S by such a localization. Recall that Proposition 10.102.9 characterizes exactness of complexes in terms of ranks of the maps φ_i and the ideals $I(\varphi_i)$, in case the ring is local. We first address the rank condition. Set $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i}n_e$. Note that $r_i + r_{i+1} = n_i$ and note that r_i is the expected rank of φ_i (in the exact case).

By Lemma 10.99.5 we see that if $\overline{F}_{\bullet,\mathfrak q}$ is exact, then the localization $F_{\bullet,\mathfrak q}$ is exact. In particular the complex $F_\bullet$ becomes exact after localizing by an element $g \in S$, $g \notin \mathfrak q$. In this case Proposition 10.102.9 applied to all localizations of S at prime ideals implies that all $(r_i + 1) \times (r_i + 1)$-minors of φ_i are zero. Thus we see that the rank of φ_i is at most r_i.

Let $I_i \subset S$ denote the ideal generated by the $r_i \times r_i$-minors of the matrix of φ_i. By Proposition 10.102.9 the complex $\overline{F}_{\bullet,\mathfrak q}$ is exact if and only if for every $1 \leq i \leq e$ we have either $(I_i)_{\mathfrak q} = S_{\mathfrak q}$ or $(I_i)_{\mathfrak q}$ contains a $S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$-regular sequence of length i. Namely, by our choice of r_i above and by the bound on the ranks of the φ_i this is the only way the conditions of Proposition 10.102.9 can be satisfied.

If $(I_i)_{\mathfrak q} = S_{\mathfrak q}$, then after localizing S at some element $g \not\in \mathfrak q$ we may assume that $I_i = S$. Clearly, this is an open condition.

If $(I_i)_{\mathfrak q} \neq S_{\mathfrak q}$, then we have a sequence $f_1, \ldots, f_i \in (I_i)_{\mathfrak q}$ which form a regular sequence in $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$. Note that for any prime $\mathfrak q' \subset S$ such that $(f_1, \ldots, f_i) \not\subset \mathfrak q'$ we have $(I_i)_{\mathfrak q'} = S_{\mathfrak q'}$. Thus the result follows from Lemma 10.129.2. $\square$

00RC Theorem 10.129.4. Let R be a ring. Let $R \to S$ be a ring map of finite presentation. Let M be a finitely presented S-module. The set

$$\{\mathfrak q \in \mathrm{Spec}(S) \mid M_{\mathfrak q} \text{ is flat over } R\}$$

is open in $\mathrm{Spec}(S)$.

Proof. Let $\mathfrak q \in \mathrm{Spec}(S)$ be a prime. Let $\mathfrak p \subset R$ be the inverse image of $\mathfrak q$ in R. Note that $M_{\mathfrak q}$ is flat over R if and only if it is flat over $R_{\mathfrak p}$. Let us assume that $M_{\mathfrak q}$ is flat over R. We claim that there exists a $g \in S$, $g \not\in \mathfrak q$ such that M_g is flat over R.

We first reduce to the case where R and S are of finite type over $\mathbf{Z}$. Choose a directed set Λ and a system $(R_\lambda \to S_\lambda, M_\lambda)$ as in Lemma 10.127.18. Set $\mathfrak p_\lambda$ equal to the inverse image of $\mathfrak p$ in R_λ. Set $\mathfrak q_\lambda$ equal to the inverse image of $\mathfrak q$ in S_λ. Then the system

$$((R_\lambda)_{\mathfrak p_\lambda}, (S_\lambda)_{\mathfrak q_\lambda}, (M_\lambda)_{\mathfrak q_\lambda})$$

is a system as in Lemma 10.127.13. Hence by Lemma 10.128.3 we see that for some λ the module M_λ is flat over R_λ at the prime $\mathfrak q_\lambda$. Suppose we can prove our claim for the system $(R_\lambda \to S_\lambda, M_\lambda, \mathfrak q_\lambda)$. In other words, suppose that we can find a $g \in S_\lambda$, $g \not\in \mathfrak q_\lambda$ such that $(M_\lambda)_g$ is flat over R_λ. By Lemma 10.127.18 we have $M = M_\lambda \otimes_{R_\lambda} R$ and hence also $M_g = (M_\lambda)_g \otimes_{R_\lambda} R$. Thus by Lemma 10.39.7 we deduce the claim for the system $(R \to S, M, \mathfrak q)$.

At this point we may assume that R and S are of finite type over $\mathbf{Z}$. We may write S as a quotient of a polynomial ring $R[x_1, \ldots, x_n]$. Of course, we may replace S by $R[x_1, \ldots, x_n]$ and assume that S is a polynomial ring over R. In particular we see that $R \to S$ is flat and all fibres rings $S \otimes_R \kappa(\mathfrak p)$ have global dimension n.

Choose a resolution $F_\bullet$ of M over S with each F_i finite free, see Lemma 10.71.1. Let $K_n = \mathrm{Ker}(F_{n-1} \to F_{n-2})$. Note that $(K_n)_{\mathfrak q}$ is flat over R, since each F_i is flat over R and by assumption on M, see Lemma 10.39.13. In addition, the sequence

$$0 \to K_n/\mathfrak pK_n \to F_{n-1}/\mathfrak pF_{n-1} \to \ldots \to F_0/\mathfrak pF_0 \to M/\mathfrak pM \to 0$$

is exact upon localizing at $\mathfrak q$, because of vanishing of $\mathrm{Tor}_i^{R_{\mathfrak p}}(\kappa(\mathfrak p), M_{\mathfrak q})$. Since the global dimension of $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$ is n we conclude that $K_n/\mathfrak pK_n$ localized at $\mathfrak q$ is a finite free module over $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$. By Lemma 10.99.4 $(K_n)_{\mathfrak q}$ is free over $S_{\mathfrak q}$. In particular, there exists a $g \in S$, $g \not\in \mathfrak q$ such that $(K_n)_g$ is finite free over S_g.

By Lemma 10.129.3 there exists a further localization S_g such that the complex

$$0 \to K_n \to F_{n-1} \to \ldots \to F_0$$

is exact on all fibres of $R \to S$. By Lemma 10.99.5 this implies that the cokernel of $F_1 \to F_0$ is flat. This proves the theorem in the Noetherian case. $\square$

10.130. Openness of Cohen-Macaulay loci

00RD In this section we characterize the Cohen-Macaulay property of finite type algebras in terms of flatness. We then use this to prove the set of points where such an algebra is Cohen-Macaulay is open.

00RE Lemma 10.130.1. Let S be a finite type algebra over a field k. Let $\varphi : k[y_1, \ldots, y_d] \to S$ be a quasi-finite ring map. As subsets of $\mathrm{Spec}(S)$ we have

$$\{\mathfrak q \mid S_{\mathfrak q} \text{ flat over } k[y_1, \ldots, y_d]\} = \{\mathfrak q \mid S_{\mathfrak q} \text{ CM and } \dim_{\mathfrak q}(S/k) = d\}$$

For notation see Definition 10.125.1.

Proof. Let $\mathfrak q \subset S$ be a prime. Denote $\mathfrak p = k[y_1, \ldots, y_d] \cap \mathfrak q$. Note that always $\dim(S_{\mathfrak q}) \leq \dim(k[y_1, \ldots, y_d]_{\mathfrak p})$ by Lemma 10.125.4 for example. Moreover, the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite and hence $\mathrm{trdeg}_k(\kappa(\mathfrak p)) = \mathrm{trdeg}_k(\kappa(\mathfrak q))$.

Let $\mathfrak q$ be an element of the left hand side. Then Lemma 10.112.9 applies and we conclude that $S_{\mathfrak q}$ is Cohen-Macaulay and $\dim(S_{\mathfrak q}) = \dim(k[y_1, \ldots, y_d]_{\mathfrak p})$. Combined with the equality of transcendence degrees above and Lemma 10.116.3 this implies that $\dim_{\mathfrak q}(S/k) = d$. Hence $\mathfrak q$ is an element of the right hand side.

Let $\mathfrak q$ be an element of the right hand side. By the equality of transcendence degrees above, the assumption that $\dim_{\mathfrak q}(S/k) = d$ and Lemma 10.116.3 we conclude that $\dim(S_{\mathfrak q}) = \dim(k[y_1, \ldots, y_d]_{\mathfrak p})$. Hence Lemma 10.128.1 applies and we see that $\mathfrak q$ is an element of the left hand side. □

00RF Lemma 10.130.2. Let S be a finite type algebra over a field k. The set of primes $\mathfrak q$ such that $S_{\mathfrak q}$ is Cohen-Macaulay is open in S.

This lemma is a special case of Lemma 10.130.4 below, so you can skip straight to the proof of that lemma if you like.

Proof. Let $\mathfrak q \subset S$ be a prime such that $S_{\mathfrak q}$ is Cohen-Macaulay. We have to show there exists a $g \in S$, $g \notin \mathfrak q$ such that the ring S_g is Cohen-Macaulay. For any $g \in S$, $g \notin \mathfrak q$ we may replace S by S_g and $\mathfrak q$ by $\mathfrak q S_g$. Combining this with Lemmas 10.115.5 and 10.116.3 we may assume that there exists a finite injective ring map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S_{\mathfrak q}) + \mathrm{trdeg}_k(\kappa(\mathfrak q))$. Set $\mathfrak p = k[y_1, \ldots, y_d] \cap \mathfrak q$. By construction we see that $\mathfrak q$ is an element of the right hand side of the displayed equality of Lemma 10.130.1. Hence it is also an element of the left hand side.

By Theorem 10.129.4 we see that for some $g \in S$, $g \notin \mathfrak q$ the ring S_g is flat over $k[y_1, \ldots, y_d]$. Hence by the equality of Lemma 10.130.1 again we conclude that all local rings of S_g are Cohen-Macaulay as desired. □

00RG Lemma 10.130.3. Let k be a field. Let S be a finite type k algebra. The set of Cohen-Macaulay primes forms a dense open $U \subset \mathrm{Spec}(S)$.

Proof. The set is open by Lemma 10.130.2. It contains all minimal primes $\mathfrak q \subset S$ since the local ring at a minimal prime $S_{\mathfrak q}$ has dimension zero and hence is Cohen-Macaulay. □

00RH Lemma 10.130.4. Let R be a ring. Let $R \to S$ be of finite presentation and flat. For any $d \geq 0$ the set

$$\left\{\begin{matrix} \mathfrak q \in \mathrm{Spec}(S) \text{ such that setting } \mathfrak p = R \cap \mathfrak q \text{ the fibre ring} \\ S_{\mathfrak q}/\mathfrak p S_{\mathfrak q} \text{ is Cohen-Macaulay and } \dim_{\mathfrak q}(S/R) = d \end{matrix}\right\}$$

is open in $\mathrm{Spec}(S)$.

Proof. Let $\mathfrak{q}$ be an element of the set indicated, with $\mathfrak{p}$ the corresponding prime of R. We have to find a $g \in S$, $g \notin \mathfrak{q}$ such that all fibre rings of $R \to S_g$ are Cohen-Macaulay. During the course of the proof we may (finitely many times) replace S by S_g for a $g \in S$, $g \notin \mathfrak{q}$. Thus by Lemma 10.125.2 we may assume there is a quasi-finite ring map $R[t_1, \ldots, t_d] \to S$ with $d = \dim_{\mathfrak{q}}(S/R)$. Let $\mathfrak{q}' = R[t_1, \ldots, t_d] \cap \mathfrak{q}$. By Lemma 10.130.1 we see that the ring map

$$R[t_1, \ldots, t_d]_{\mathfrak{q}'}/\mathfrak{p}R[t_1, \ldots, t_d]_{\mathfrak{q}'} \longrightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$$

is flat. Hence by the critère de platitude par fibres Lemma 10.128.8 we see that $R[t_1, \ldots, t_d]_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is flat. Hence by Theorem 10.129.4 we see that for some $g \in S$, $g \notin \mathfrak{q}$ the ring map $R[t_1, \ldots, t_d] \to S_g$ is flat. Replacing S by S_g we see that for every prime $\mathfrak{r} \subset S$, setting $\mathfrak{r}' = R[t_1, \ldots, t_d] \cap \mathfrak{r}$ and $\mathfrak{p}' = R \cap \mathfrak{r}$ the local ring map $R[t_1, \ldots, t_d]_{\mathfrak{r}'} \to S_{\mathfrak{r}}$ is flat. Hence also the base change

$$R[t_1, \ldots, t_d]_{\mathfrak{r}'}/\mathfrak{p}'R[t_1, \ldots, t_d]_{\mathfrak{r}'} \longrightarrow S_{\mathfrak{r}}/\mathfrak{p}'S_{\mathfrak{r}}$$

is flat. Hence by Lemma 10.130.1 applied with $k = \kappa(\mathfrak{p}')$ we see $\mathfrak{r}$ is in the set of the lemma as desired. $\square$

00RI Lemma 10.130.5. Let R be a ring. Let $R \to S$ be flat of finite presentation. The set of primes $\mathfrak{q}$ such that the fibre ring $S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$, with $\mathfrak{p} = R \cap \mathfrak{q}$ is Cohen-Macaulay is open and dense in every fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

Proof. The set, call it W, is open by Lemma 10.130.4. It is dense in the fibres because the intersection of W with a fibre is the corresponding set of the fibre to which Lemma 10.130.3 applies. $\square$

00RJ Lemma 10.130.6. Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak{q} \subset S$ be a prime of S. Let $\mathfrak{q}_K \subset S_K$ be a prime of S_K lying over $\mathfrak{q}$. Then $S_{\mathfrak{q}}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak{q}_K}$ is Cohen-Macaulay.

Proof. During the course of the proof we may (finitely many times) replace S by S_g for any $g \in S$, $g \notin \mathfrak{q}$. Hence using Lemma 10.115.5 we may assume that $\dim(S) = \dim_{\mathfrak{q}}(S/k) =: d$ and find a finite injective map $k[x_1, \ldots, x_d] \to S$. Note that this also induces a finite injective map $K[x_1, \ldots, x_d] \to S_K$ by base change. By Lemma 10.116.6 we have $\dim_{\mathfrak{q}_K}(S_K/K) = d$. Set $\mathfrak{p} = k[x_1, \ldots, x_d] \cap \mathfrak{q}$ and $\mathfrak{p}_K = K[x_1, \ldots, x_d] \cap \mathfrak{q}_K$. Consider the following commutative diagram of Noetherian local rings

$$\begin{tikzcd} S_{\mathfrak{q}} \arrow[r] & (S_K)_{\mathfrak{q}_K} \\ k[x_1, \ldots, x_d]_{\mathfrak{p}} \arrow[u] \arrow[r] & K[x_1, \ldots, x_d]_{\mathfrak{p}_K} \arrow[u] \end{tikzcd}$$

By Lemma 10.130.1 we have to show that the left vertical arrow is flat if and only if the right vertical arrow is flat. Because the bottom arrow is flat this equivalence holds by Lemma 10.100.1. $\square$

00RK Lemma 10.130.7. Let R be a ring. Let $R \to S$ be of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$. Denote $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ the map associated to the ring map $S \to S'$. Set W equal to the set of primes $\mathfrak{q}$ such that the fibre ring $S_{\mathfrak q} \otimes_R \kappa(\mathfrak p)$, $\mathfrak p = R \cap \mathfrak q$ is Cohen-Macaulay, and let W' denote the analogue for S'/R'. Then $W' = f^{-1}(W)$.

Proof. Trivial from Lemma 10.130.6 and the definitions. $\square$

00RL Lemma 10.130.8. Let R be a ring. Let $R \to S$ be a ring map which is (a) flat, (b) of finite presentation, (c) has Cohen-Macaulay fibres. Then we can write $S = S_0 \times \ldots \times S_n$ as a product of R-algebras S_d such that each S_d satisfies (a), (b), (c) and has all fibres equidimensional of dimension d.

Proof. For each integer d denote $W_d \subset \mathrm{Spec}(S)$ the set defined in Lemma 10.130.4. Clearly we have $\mathrm{Spec}(S) = \coprod W_d$, and each W_d is open by the lemma we just quoted. Hence the result follows from Lemma 10.24.3. $\square$

10.131. Differentials

00RM In this section we define the module of differentials of a ring map.

00RN Definition 10.131.1. Let $\varphi : R \to S$ be a ring map and let M be an S-module. A derivation, or more precisely an R-derivation into M is a map $D : S \to M$ which is additive, annihilates elements of $\varphi(R)$, and satisfies the Leibniz rule: $D(ab) = aD(b) + bD(a)$.

Note that $D(ra) = rD(a)$ if $r \in R$ and $a \in S$. An equivalent definition is that an R-derivation is an R-linear map $D : S \to M$ which satisfies the Leibniz rule. The set of all R-derivations forms an S-module: Given two R-derivations D, D' the sum $D + D' : S \to M$, $a \mapsto D(a) + D'(a)$ is an R-derivation, and given an R-derivation D and an element $c \in S$ the scalar multiple $cD : S \to M$, $a \mapsto cD(a)$ is an R-derivation. We denote this S-module

$$\mathrm{Der}_R(S, M).$$

Also, if $\alpha : M \to N$ is an S-module map, then the composition $\alpha \circ D$ is an R-derivation into N. In this way the assignment $M \mapsto \mathrm{Der}_R(S, M)$ is a covariant functor.

Consider the following map of free S-modules

$$\bigoplus\nolimits_{(a,b)\in S^2} S[(a,b)] \oplus \bigoplus\nolimits_{(f,g)\in S^2} S[(f,g)] \oplus \bigoplus\nolimits_{r\in R} S[r] \longrightarrow \bigoplus\nolimits_{a\in S} S[a]$$

defined by the rules

$$[(a,b)] \longmapsto [a+b] - [a] - [b], \quad [(f,g)] \longmapsto [fg] - f[g] - g[f], \quad [r] \longmapsto [\varphi(r)]$$

with obvious notation. Let $\Omega_{S/R}$ be the cokernel of this map. There is a map $\mathrm{d} : S \to \Omega_{S/R}$ which maps a to the class $\mathrm{d}a$ of $[a]$ in the cokernel. This is an R-derivation by the relations imposed on $\Omega_{S/R}$, in other words

$$\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b, \quad \mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f, \quad \mathrm{d}\varphi(r) = 0$$

where $a, b, f, g \in S$ and $r \in R$.

07BK Definition 10.131.2. The pair $(\Omega_{S/R}, \mathrm{d})$ is called the module of Kähler differentials or the module of differentials of S over R.

00RO **Lemma 10.131.3.** The module of differentials of S over R has the following universal property. The map

$$\mathrm{Hom}_S(\Omega_{S/R}, M) \longrightarrow \mathrm{Der}_R(S, M), \quad \alpha \longmapsto \alpha \circ \mathrm{d}$$

is an isomorphism of functors.

Proof. By definition an R-derivation is a rule which associates to each $a \in S$ an element $D(a) \in M$. Thus D gives rise to a map $[D] : \bigoplus S[a] \to M$. However, the conditions of being an R-derivation exactly mean that $[D]$ annihilates the image of the map in the displayed presentation of $\Omega_{S/R}$ above. $\square$

00RP **Lemma 10.131.4.** Suppose that $R \to S$ is surjective. Then $\Omega_{S/R} = 0$.

Proof. You can see this either because all R-derivations clearly have to be zero, or because the map in the presentation of $\Omega_{S/R}$ is surjective. $\square$

Suppose that

00RQ (10.131.4.1)
$$\begin{tikzcd} S \arrow[r, "\varphi"'] & S' \\ R \arrow[u, "\alpha"] \arrow[r, "\psi"] & R' \arrow[u, "\beta"'] \end{tikzcd}$$

is a commutative diagram of rings. In this case there is a natural map of modules of differentials fitting into the commutative diagram

$$\begin{tikzcd} \Omega_{S/R} \arrow[r] & \Omega_{S'/R'} \\ S \arrow[u, "\mathrm{d}"] \arrow[r, "\varphi"] & S' \arrow[u, "\mathrm{d}"'] \end{tikzcd}$$

To construct the map just use the obvious map between the presentations for $\Omega_{S/R}$ and $\Omega_{S'/R'}$. Namely,

0H2F (10.131.4.2)
$$\begin{tikzcd} \bigoplus S'[(a',b')] \oplus \bigoplus S'[(f',g')] \oplus \bigoplus S'[r'] \arrow[r] & \bigoplus S'[a'] \\ \bigoplus S[(a,b)] \oplus \bigoplus S[(f,g)] \oplus \bigoplus S[r] \arrow[u, "\begin{smallmatrix} {[(a,b)] \mapsto [(\varphi(a),\varphi(b))]} \\ {[(f,g)] \mapsto [(\varphi(f),\varphi(g))]} \\ {[r] \mapsto [\psi(r)]} \end{smallmatrix}"] \arrow[r] & \bigoplus S[a] \arrow[u, "{[a] \mapsto [\varphi(a)]}"'] \end{tikzcd}$$

The result is simply that $f\mathrm{d}g \in \Omega_{S/R}$ is mapped to $\varphi(f)\mathrm{d}\varphi(g)$.

031G **Lemma 10.131.5.** Let I be a directed set. Let $(R_i \to S_i, \varphi_{ii'})$ be a system of ring maps over I, see Categories, Section 4.21. Then we have

$$\Omega_{S/R} = \mathrm{colim}_i\, \Omega_{S_i/R_i}.$$

where $R \to S = \mathrm{colim}(R_i \to S_i)$.

Proof. This is clear from the defining presentation of $\Omega_{S/R}$ and the functoriality of this described above. $\square$

00RR Lemma 10.131.6. In diagram (10.131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$. Then $\Omega_{S/R} \to \Omega_{S'/R'}$ is surjective with kernel generated as an S-module by the elements $\mathrm{d}a$, where $a \in S$ is such that $\varphi(a) \in \beta(R')$. (This includes in particular the elements $\mathrm{d}(i)$, $i \in I$.)

First proof. Consider the map of presentations (10.131.4.2). Clearly the right vertical map of free modules is surjective. Thus the map is surjective. Suppose that some element η of $\Omega_{S/R}$ maps to zero in $\Omega_{S'/R'}$. Write η as the image of $\sum s_i[a_i]$ for some $s_i, a_i \in S$. Then we see that $\sum \varphi(s_i)[\varphi(a_i)]$ is the image of an element

$$\theta = \sum s'_j[a'_j, b'_j] + \sum s'_k[f'_k, g'_k] + \sum s'_l[r'_l]$$

in the upper left corner of the diagram. Since φ is surjective, the terms $s'_j[a'_j, b'_j]$ and $s'_k[f'_k, g'_k]$ are in the image of elements in the lower right corner. Thus, modifying η and θ by substracting the images of these elements, we may assume $\theta = \sum s'_l[r'_l]$. In other words, we see $\sum \varphi(s_i)[\varphi(a_i)]$ is of the form $\sum s'_l[\beta(r'_l)]$. Pick $a' \in S'$. Next, we may assume that we have some $a' \in S'$ such that $a' = \varphi(a_i)$ for all i and $a' = \beta(r'_l)$ for all l. This is clear from the direct sum decomposition of the upper right corner of the diagram. Choose $a \in S$ with $\varphi(a) = a'$. Then we can write $a_i = a + x_i$ for some $x_i \in I$. Thus we may assume that all a_i are equal to a by using the relations that are allowed. But then we may assume our element is of the form $s[a]$. We still know that $\varphi(s)[a'] = \sum \varphi(s'_l)[\beta(r'_l)]$. Hence either $\varphi(s) = 0$ and we're done, or $a' = \varphi(a)$ is in the image of β and we're done as well. □

Second proof. We will use the universal property of modules of differentials given in Lemma 10.131.3 without further mention.

In (10.131.4.1) let $R'' = S \times_{S'} R'$. Then we have following diagram:

$$\begin{tikzcd} S \arrow[r] & S \arrow[r] & S' \\ R \arrow[r] \arrow[u] & R'' \arrow[r] \arrow[u] & R' \arrow[u] \end{tikzcd}$$

Let M be an S-module. It follows immediately from the definitions that an R-derivation $D : S \to M$ is an R''-derivation if and only if it annihilates the elements in the image of $R'' \to S$. The universal property translates this into the statement that the natural map $\Omega_{S/R} \to \Omega_{S/R''}$ is surjective with kernel generated as an S-module by the image of R''.

From the previous paragraph we see that it suffices to show that $\Omega_{S/R} \to \Omega_{S'/R'}$ is an isomorphism when $S \to S'$ is surjective and $R = S \times_{S'} R'$. Let M' be an S'-module. Observe that any R'-derivation $D' : S' \to M'$ gives an R-derivation by precomposing with $S \to S'$. Conversely, suppose M is an S-module and $D : S \to M$ is an R-derivation. If $i \in I$, then there exist an $a \in R$ with $\alpha(a) = i$ (as $R = S \times_{S'} R'$). It follows that $D(i) = 0$ and hence $0 = D(is) = iD(s)$ for all $s \in S$. Thus the image of D is contained in the submodule $M' \subset M$ of elements annihilated by I and moreover the induced map $S \to M'$ factors through an R'-derivation $S' \to M'$. It is an exercise to use the universal property to see that this means $\Omega_{S/R} \to \Omega_{S'/R'}$ is an isomorphism; details omitted. □

00RS Lemma 10.131.7. Let $A \to B \to C$ be ring maps. Then there is a canonical exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules.

Proof. We get a diagram (10.131.4.1) by putting $R = A$, $S = C$, $R' = B$, and $S' = C$. By Lemma 10.131.6 the map $\Omega_{C/A} \to \Omega_{C/B}$ is surjective, and the kernel is generated by the elements $\mathrm{d}(c)$, where $c \in C$ is in the image of $B \to C$. The lemma follows. $\square$

00RT Lemma 10.131.8. Let $\varphi : A \to B$ be a ring map.

(1) If $S \subset A$ is a multiplicative subset mapping to invertible elements of B, then $\Omega_{B/A} = \Omega_{B/S^{-1}A}$.
(2) If $S \subset B$ is a multiplicative subset then $S^{-1}\Omega_{B/A} = \Omega_{S^{-1}B/A}$.

Proof. To show the equality of (1) it is enough to show that any A-derivation $D : B \to M$ annihilates the elements $\varphi(s)^{-1}$. This is clear from the Leibniz rule applied to $1 = \varphi(s)\varphi(s)^{-1}$. To show (2) note that there is an obvious map $S^{-1}\Omega_{B/A} \to \Omega_{S^{-1}B/A}$. To show it is an isomorphism it is enough to show that there is a A-derivation d' of $S^{-1}B$ into $S^{-1}\Omega_{B/A}$. To define it we simply set $\mathrm{d}'(b/s) = (1/s)\mathrm{d}b - (1/s^2)b\mathrm{d}s$. Details omitted. $\square$

00RU Lemma 10.131.9. In diagram (10.131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Then there is a canonical exact sequence of S'-modules

$$I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

The leftmost map is characterized by the rule that $f \in I$ maps to $\mathrm{d}f \otimes 1$.

Proof. The middle term is $\Omega_{S/R} \otimes_S S/I$. For $f \in I$ denote $\overline{f}$ the image of f in I/I^2. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1f_2 \otimes 1 = 0$ if $f_1, f_2 \in I$. And this is clear from the Leibniz rule $\mathrm{d}f_1f_2 \otimes 1 = (f_1\mathrm{d}f_2 + f_2\mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_1 \otimes f_2 = 0$. A similar computation show this map is $S' = S/I$-linear.

The map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$ is the canonical S'-linear map associated to the S-linear map $\Omega_{S/R} \to \Omega_{S'/R}$. It is surjective because $\Omega_{S/R} \to \Omega_{S'/R}$ is surjective by Lemma 10.131.6.

The composite of the two maps is zero because $\mathrm{d}f$ maps to zero in $\Omega_{S'/R}$ for $f \in I$. Note that exactness just says that the kernel of $\Omega_{S/R} \to \Omega_{S'/R}$ is generated as an S-submodule by the submodule $I\Omega_{S/R}$ together with the elements $\mathrm{d}f$, with $f \in I$. We know by Lemma 10.131.6 that this kernel is generated by the elements $\mathrm{d}(a)$ where $\varphi(a) = \beta(r)$ for some $r \in R$. But then $a = \alpha(r) + a - \alpha(r)$, so $\mathrm{d}(a) = \mathrm{d}(a - \alpha(r))$. And $a - \alpha(r) \in I$ since $\varphi(a - \alpha(r)) = \varphi(a) - \varphi(\alpha(r)) = \beta(r) - \beta(r) = 0$. We conclude the elements $\mathrm{d}f$ with $f \in I$ already generate the kernel as an S-module, as desired. $\square$

02HP Lemma 10.131.10. In diagram (10.131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Moreover, assume that there exists an R-algebra map $S' \to S$ which is a right inverse to $S \to S'$. Then the exact sequence of S'-modules of Lemma 10.131.9 turns into a short exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

which is even a split short exact sequence.

Proof. Let $\beta : S' \to S$ be the right inverse to the surjection $\alpha : S \to S'$, so $S = I \oplus \beta(S')$. Clearly we can use $\beta : \Omega_{S'/R} \to \Omega_{S/R}$, to get a right inverse to the map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$. On the other hand, consider the map

$$D : S \longrightarrow I/I^2, \quad x \longmapsto x - \beta(\alpha(x))$$

It is easy to show that D is an R-derivation (omitted). Moreover $xD(s) = 0$ if $x \in I, s \in S$. Hence, by the universal property D induces a map $\tau : \Omega_{S/R} \otimes_S S' \to I/I^2$. We omit the verification that it is a left inverse to $\text{d} : I/I^2 \to \Omega_{S/R} \otimes_S S'$. Hence we win. $\square$

02HQ Lemma 10.131.11. Let $R \to S$ be a ring map. Let $I \subset S$ be an ideal. Let $n \geq 1$ be an integer. Set $S' = S/I^{n+1}$. The map $\Omega_{S/R} \to \Omega_{S'/R}$ induces an isomorphism

$$\Omega_{S/R} \otimes_S S/I^n \longrightarrow \Omega_{S'/R} \otimes_{S'} S/I^n.$$

Proof. This follows from Lemma 10.131.9 and the fact that $\text{d}(I^{n+1}) \subset I^n\Omega_{S/R}$ by the Leibniz rule for d. $\square$

00RV Lemma 10.131.12. Suppose that we have ring maps $R \to R'$ and $R \to S$. Set $S' = S \otimes_R R'$, so that we obtain a diagram (10.131.4.1). Then the canonical map defined above induces an isomorphism $\Omega_{S/R} \otimes_R R' = \Omega_{S'/R'}$.

Proof. Let $\text{d}' : S' = S \otimes_R R' \to \Omega_{S/R} \otimes_R R'$ denote the map $\text{d}'(\sum a_i \otimes x_i) = \sum \text{d}(a_i) \otimes x_i$. It exists because the map $S \times R' \to \Omega_{S/R} \otimes_R R'$, $(a, x) \mapsto \text{d}a \otimes_R x$ is R-bilinear. This is an R'-derivation, as can be verified by a simple computation. We will show that $(\Omega_{S/R} \otimes_R R', \text{d}')$ satisfies the universal property. Let $D : S' \to M'$ be an R' derivation into an S'-module. The composition $S \to S' \to M'$ is an R-derivation, hence we get an S-linear map $\varphi_D : \Omega_{S/R} \to M'$. We may tensor this with R' and get the map $\varphi'_D : \Omega_{S/R} \otimes_R R' \to M'$, $\varphi'_D(\eta \otimes x) = x\varphi_D(\eta)$. It is clear that $D = \varphi'_D \circ \text{d}'$. $\square$

The multiplication map $S \otimes_R S \to S$ is the R-algebra map which maps $a \otimes b$ to ab in S. It is also an S-algebra map, if we think of $S \otimes_R S$ as an S-algebra via either of the maps $S \to S \otimes_R S$.

00RW Lemma 10.131.13. Let $R \to S$ be a ring map. Let $J = \text{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. There is a canonical isomorphism of S-modules $\Omega_{S/R} \to J/J^2$, $a\text{d}b \mapsto a \otimes b - ab \otimes 1$.

First proof. Apply Lemma 10.131.10 to the commutative diagram

$$\begin{tikzcd} S \otimes_R S \arrow[r] & S \\ S \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

where the left vertical arrow is $a \mapsto a \otimes 1$. We get the exact sequence $0 \to J/J^2 \to \Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S \to \Omega_{S/S} \to 0$. By Lemma 10.131.4 the term $\Omega_{S/S}$ is 0, and we obtain an isomorphism between the other two terms. We have $\Omega_{S \otimes_R S/S} = \Omega_{S/R} \otimes_S (S \otimes_R S)$ by Lemma 10.131.12 as $S \to S \otimes_R S$ is the base change of $R \to S$ and hence

$$\Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S = \Omega_{S/R} \otimes_S (S \otimes_R S) \otimes_{S \otimes_R S} S = \Omega_{S/R}$$

We omit the verification that the map is given by the rule of the lemma. $\square$

Second proof. First we show that the rule $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$ is well defined. In order to do this we have to show that $\mathrm{d}r$ and $a\mathrm{d}b + b\mathrm{d}a - d(ab)$ map to zero. The first because $r \otimes 1 - 1 \otimes r = 0$ by definition of the tensor product. The second because

$$(a \otimes b - ab \otimes 1) + (b \otimes a - ba \otimes 1) - (1 \otimes ab - ab \otimes 1) = (a \otimes 1 - 1 \otimes a)(1 \otimes b - b \otimes 1)$$

is in J^2.

We construct a map in the other direction. We may think of $S \to S \otimes_R S$, $a \mapsto a \otimes 1$ as the base change of $R \to S$. Hence we have $\Omega_{S \otimes_R S/S} = \Omega_{S/R} \otimes_S (S \otimes_R S)$, by Lemma 10.131.12. At this point the sequence of Lemma 10.131.9 gives a map

$$J/J^2 \to \Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S = (\Omega_{S/R} \otimes_S (S \otimes_R S)) \otimes_{S \otimes_R S} S = \Omega_{S/R}.$$

We leave it to the reader to see it is the inverse of the map above. $\square$

00RX Lemma 10.131.14. If $S = R[x_1, \ldots, x_n]$, then $\Omega_{S/R}$ is a finite free S-module with basis $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$.

Proof. We first show that $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ generate $\Omega_{S/R}$ as an S-module. To prove this we show that $\mathrm{d}g$ can be expressed as a sum $\sum g_i \mathrm{d}x_i$ for any $g \in R[x_1, \ldots, x_n]$. We do this by induction on the (total) degree of g. It is clear if the degree of g is 0, because then $\mathrm{d}g = 0$. If the degree of g is > 0, then we may write g as $c + \sum g_i x_i$ with $c \in R$ and $\deg(g_i) < \deg(g)$. By the Leibniz rule we have $\mathrm{d}g = \sum g_i \mathrm{d}x_i + \sum x_i \mathrm{d}g_i$, and hence we win by induction.

Consider the R-derivation $\partial/\partial x_i : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]$. (We leave it to the reader to define this; the defining property being that $\partial/\partial x_i(x_j) = \delta_{ij}$.) By the universal property this corresponds to an S-module map $l_i : \Omega_{S/R} \to R[x_1, \ldots, x_n]$ which maps $\mathrm{d}x_i$ to 1 and $\mathrm{d}x_j$ to 0 for $j \neq i$. Thus it is clear that there are no S-linear relations among the elements $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. $\square$

00RY Lemma 10.131.15. Suppose $R \to S$ is of finite presentation. Then $\Omega_{S/R}$ is a finitely presented S-module.

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Write $I = (f_1, \ldots, f_m)$. According to Lemma 10.131.9 there is an exact sequence of S-modules

$$I/I^2 \to \Omega_{R[x_1, \ldots, x_n]/R} \otimes_{R[x_1, \ldots, x_n]} S \to \Omega_{S/R} \to 0$$

The result follows from the fact that I/I^2 is a finite S-module (generated by the images of the f_i), and that the middle term is finite free by Lemma 10.131.14. $\square$

00RZ Lemma 10.131.16. Suppose $R \to S$ is of finite type. Then $\Omega_{S/R}$ is finitely generated S-module.

Proof. This is very similar to, but easier than the proof of Lemma 10.131.15. $\square$

10.132. The de Rham complex

0FKF Let $A \to B$ be a ring map. Denote $\mathrm{d} : B \to \Omega_{B/A}$ the module of differentials with its universal A-derivation constructed in Section 10.131. Let $\Omega^i_{B/A} = \wedge^i_B(\Omega_{B/A})$ for $i \geq 0$ be the ith exterior power as in Section 10.13. The de Rham complex of B over A is the complex

$$\Omega^0_{B/A} \to \Omega^1_{B/A} \to \Omega^2_{B/A} \to \ldots$$

with A-linear differentials constructed and described below.

The map $\mathrm{d} : \Omega^0_{B/A} \to \Omega^1_{B/A}$ is the universal derivation $\mathrm{d} : B \to \Omega_{B/A}$. Observe that this is indeed A-linear.

For $p \geq 1$ we claim there is a unique A-linear map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ such that

0FKG (10.132.0.1)
$$\mathrm{d}\left(b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p\right) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$$

Recall that $\Omega_{B/A}$ is generated as a B-module by the elements $\mathrm{d}b$. Thus $\Omega^p_{B/A}$ is generated as an A-module by the element $b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and it follows that the map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ if it exists is unique.

Construction of $\mathrm{d} : \Omega^1_{B/A} \to \Omega^2_{B/A}$. By Definition 10.131.2 the elements $\mathrm{d}b$ freely generate $\Omega_{B/A}$ as a B-module subject to the relations $\mathrm{d}a = 0$ for $a \in A$ and $\mathrm{d}(b' + b'') = \mathrm{d}b' + \mathrm{d}b''$ and $\mathrm{d}(b'b'') = b'\mathrm{d}b'' + b''\mathrm{d}b'$ for $b', b'' \in B$. Hence to show that the rule

$$\sum b'_i \mathrm{d}b_i \longmapsto \sum \mathrm{d}b'_i \wedge \mathrm{d}b_i$$

is well defined we have to show that the elements

$$b\mathrm{d}a, \quad \text{and} \quad b\mathrm{d}(b' + b'') - b\mathrm{d}b' - b\mathrm{d}b'' \quad \text{and} \quad b\mathrm{d}(b'b'') - bb'\mathrm{d}b'' - bb''\mathrm{d}b'$$

for $a \in A$ and $b, b', b'' \in B$ are mapped to zero. This is clear by direct computation using the Leibniz rule for d.

Observe that the composition $\Omega^0_{B/A} \to \Omega^1_{B/A} \to \Omega^2_{B/A}$ is zero as $\mathrm{d}(\mathrm{d}(b)) = \mathrm{d}(1\mathrm{d}b) = \mathrm{d}(1) \wedge \mathrm{d}(b) = 0 \wedge \mathrm{d}b = 0$. Here $\mathrm{d}(1) = 0$ as $1 \in B$ is in the image of $A \to B$. We will use this below.

Construction of $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ for $p \geq 2$. We will show the A-linear map

$$\gamma : \Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \longrightarrow \Omega^{p+1}_{B/A}$$

defined by the formula

$$\omega_1 \otimes \ldots \otimes \omega_p \longmapsto \sum (-1)^{i+1} \omega_1 \wedge \ldots \wedge \mathrm{d}(\omega_i) \wedge \ldots \wedge \omega_p$$

factors over the natural surjection $\Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \to \Omega^p_{B/A}$ to give the desired map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$. According to Lemma 10.13.4 the kernel of $\Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \to \Omega^p_{B/A}$ is generated as an A-module by the elements $\omega_1 \otimes \ldots \otimes \omega_p$ with $\omega_i = \omega_j$ for some $i \neq j$ and $\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p - \omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p$ for some $f \in B$. A direct computation shows the first type of element is mapped to 0 by γ, in other words, γ is alternating. To finish we have to show that

$$\gamma(\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p) = \gamma(\omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p)$$

for $f \in B$. By A-linearity and the alternating property, it is enough to show this for $p = 2$, $i = 1$, $j = 2$, $\omega_1 = b\mathrm{d}b'$ and $\omega_2 = c\mathrm{d}c'$ for $b, b', c, c' \in B$. Thus we need to show that

$$\begin{aligned} &\mathrm{d}(fb) \wedge \mathrm{d}b' \wedge c\mathrm{d}c' - fb\mathrm{d}b' \wedge \mathrm{d}c \wedge \mathrm{d}c' \\ &= \mathrm{d}b \wedge \mathrm{d}b' \wedge fc\mathrm{d}c' - b\mathrm{d}b' \wedge \mathrm{d}(fc) \wedge \mathrm{d}c' \end{aligned}$$

in other words that

$$(c\mathrm{d}(fb) + fb\mathrm{d}c - fc\mathrm{d}b - b\mathrm{d}(fc)) \wedge \mathrm{d}b' \wedge \mathrm{d}c' = 0.$$

This follows from the Leibniz rule. Observe that the value of γ on the element $b_0\mathrm{d}b_1 \otimes \mathrm{d}b_2 \otimes \ldots \otimes \mathrm{d}b_p$ is $\mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and hence (10.132.0.1) will be satisfied for the map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ so obtained.

Finally, since $\Omega^p_{B/A}$ is additively generated by the elements $b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and since $\mathrm{d}(b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p) = \mathrm{d}b_0 \wedge \ldots \wedge \mathrm{d}b_p$ we see in exactly the same manner that the composition $\Omega^p_{B/A} \to \Omega^{p+1}_{B/A} \to \Omega^{p+2}_{B/A}$ is zero for $p \geq 1$. Thus the de Rham complex is indeed a complex.

Given just a ring R we set $\Omega_R = \Omega_{R/\mathbf{Z}}$. This is sometimes called the absolute module of differentials of R; this makes sense: if Ω_R is the module of differentials where we only assume the Leibniz rule and not the vanishing of d1, then the Leibniz rule gives $\mathrm{d}1 = \mathrm{d}(1 \cdot 1) = 1\mathrm{d}1 + 1\mathrm{d}1 = 2\mathrm{d}1$ and hence $\mathrm{d}1 = 0$ in Ω_R. In this case the absolute de Rham complex of R is the corresponding complex

$$\Omega^0_R \to \Omega^1_R \to \Omega^2_R \to \ldots$$

where we set $\Omega^i_R = \Omega^i_{R/\mathbf{Z}}$ and so on.

Suppose we have a commutative diagram of rings

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

There is a natural map of de Rham complexes

$$\Omega^\bullet_{B/A} \longrightarrow \Omega^\bullet_{B'/A'}$$

Namely, in degree 0 this is the map $B \to B'$, in degree 1 this is the map $\Omega_{B/A} \to \Omega_{B'/A'}$ constructed in Section 10.131, and for $p \geq 2$ it is the induced map $\Omega^p_{B/A} = \wedge^p_B(\Omega_{B/A}) \to \wedge^p_{B'}(\Omega_{B'/A'}) = \Omega^p_{B'/A'}$. The compatibility with differentials follows from the characterization of the differentials by the formula (10.132.0.1).

07HY Lemma 10.132.1. Let $A \to B$ be a ring map. Let $\pi : \Omega_{B/A} \to \Omega$ be a surjective B-module map. Denote $\mathrm{d} : B \to \Omega$ the composition of π with the universal derivation $\mathrm{d}_{B/A} : B \to \Omega_{B/A}$. Set $\Omega^i = \wedge^i_B(\Omega)$. Assume that the kernel of π is generated, as a B-module, by elements $\omega \in \Omega_{B/A}$ such that $\mathrm{d}_{B/A}(\omega) \in \Omega^2_{B/A}$ maps to zero in Ω^2. Then there is a de Rham complex

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to \ldots$$

whose differential is defined by the rule

$$\mathrm{d} : \Omega^p \to \Omega^{p+1}, \quad \mathrm{d}\left(f_0\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_p\right) = \mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_p$$

Proof. We will show that there exists a commutative diagram

$$\begin{tikzcd} \Omega^0_{B/A} \arrow[r, "\mathrm{d}_{B/A}"'] \arrow[d] & \Omega^1_{B/A} \arrow[r, "\mathrm{d}_{B/A}"'] \arrow[d, "\pi"'] & \Omega^2_{B/A} \arrow[r, "\mathrm{d}_{B/A}"'] \arrow[d, "\wedge^2\pi"'] & \ldots \\ \Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}"] & \Omega^2 \arrow[r, "\mathrm{d}"] & \ldots \end{tikzcd}$$

the description of the map d will follow from the construction of the differentials $\mathrm{d}_{B/A} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ of the de Rham complex of B over A given above. Since the left most vertical arrow is an isomorphism we have the first square. Because π is surjective, to get the second square it suffices to show that $\mathrm{d}_{B/A}$ maps the kernel of π into the kernel of $\wedge^2\pi$. We are given that any element of the kernel of π is of the form $\sum b_i\omega_i$ with $\pi(\omega_i) = 0$ and $\wedge^2\pi(\mathrm{d}_{B/A}(\omega_i)) = 0$. By the Leibniz rule for $\mathrm{d}_{B/A}$ we have $\mathrm{d}_{B/A}(\sum b_i\omega_i) = \sum b_i\mathrm{d}_{B/A}(\omega_i) + \sum \mathrm{d}_{B/A}(b_i) \wedge \omega_i$. Hence this maps to zero under $\wedge^2\pi$.

For $i > 1$ we note that $\wedge^i\pi$ is surjective with kernel the image of $\mathrm{Ker}(\pi) \wedge \Omega^{i-1}_{B/A} \to \Omega^i_{B/A}$. For $\omega_1 \in \mathrm{Ker}(\pi)$ and $\omega_2 \in \Omega^{i-1}_{B/A}$ we have

$$\mathrm{d}_{B/A}(\omega_1 \wedge \omega_2) = \mathrm{d}_{B/A}(\omega_1) \wedge \omega_2 - \omega_1 \wedge \mathrm{d}_{B/A}(\omega_2)$$

which is in the kernel of $\wedge^{i+1}\pi$ by what we just proved above. Hence we get the $(i+1)$st square in the diagram above. This concludes the proof. □

10.133. Finite order differential operators

09CH In this section we introduce differential operators of finite order.

09CI Definition 10.133.1. Let $R \to S$ be a ring map. Let M, N be S-modules. Let $k \geq 0$ be an integer. We inductively define a differential operator $D : M \to N$ of order k to be an R-linear map such that for all $g \in S$ the map $m \mapsto D(gm) - gD(m)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an S-linear map.

If $D : M \to N$ is a differential operator of order k, then for all $g \in S$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\mathrm{Diff}^k(M, N) = \mathrm{Diff}^k_{S/R}(M, N)$$

is an S-module. We have

$$\mathrm{Diff}^0(M, N) \subset \mathrm{Diff}^1(M, N) \subset \mathrm{Diff}^2(M, N) \subset \ldots$$

09CJ Lemma 10.133.2. Let $R \to S$ be a ring map. Let L, M, N be S-modules. If $D : L \to M$ and $D' : M \to N$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.

Proof. Let $g \in S$. Then the map which sends $x \in L$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. □

09CK Lemma 10.133.3. Let $R \to S$ be a ring map. Let M be an S-module. Let $k \geq 0$. There exists an S-module $P^k_{S/R}(M)$ and a canonical isomorphism

$$\mathrm{Diff}^k_{S/R}(M, N) = \mathrm{Hom}_S(P^k_{S/R}(M), N)$$

functorial in the S-module N.

Proof. The existence of $P^k_{S/R}(M)$ follows from general category theoretic arguments (insert future reference here), but we will also give a construction. Set $F = \bigoplus_{m \in M} S[m]$ where $[m]$ is a symbol indicating the basis element in the summand corresponding to m. Given any differential operator $D : M \to N$ we obtain an S-linear map $L_D : F \to N$ sending $[m]$ to $D(m)$. If D has order 0, then L_D annihilates the elements

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where $g_0 \in S$ and $m, m' \in M$. If D has order 1, then L_D annihilates the elements

$$[m + m'] - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where $f \in R$, $g_0, g_1 \in S$, and $m \in M$. If D has order k, then L_D annihilates the elements $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the elements

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : F \to N$ is an S-linear map annihilating all the elements listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $P^k_{S/R}(M)$ is the quotient of F by the submodule generated by these elements. □

09CL Definition 10.133.4. Let $R \to S$ be a ring map. Let M be an S-module. The module $P^k_{S/R}(M)$ constructed in Lemma 10.133.3 is called the module of principal parts of order k of M.

Note that the inclusions

$$\mathrm{Diff}^0(M, N) \subset \mathrm{Diff}^1(M, N) \subset \mathrm{Diff}^2(M, N) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 4.3.5) to surjections

$$\ldots \to P^2_{S/R}(M) \to P^1_{S/R}(M) \to P^0_{S/R}(M) = M$$

09CM Example 10.133.5. Let $R \to S$ be a ring map and let N be an S-module. Observe that $\mathrm{Diff}^1(S, N) = \mathrm{Der}_R(S, N) \oplus N$. Namely, if $D : S \to N$ is a differential operator of order 1 then $\sigma_D : S \to N$ defined by $\sigma_D(g) := D(g) - gD(1)$ is an R-derivation and $D = \sigma_D + \lambda_{D(1)}$ where $\lambda_x : S \to N$ is the linear map sending g to gx. It follows that $P^1_{S/R} = \Omega_{S/R} \oplus S$ by the universal property of $\Omega_{S/R}$.

09CN Lemma 10.133.6. Let $R \to S$ be a ring map. Let M be an S-module. There is a canonical short exact sequence

$$0 \to \Omega_{S/R} \otimes_S M \to P^1_{S/R}(M) \to M \to 0$$

functorial in M called the sequence of principal parts.

Proof. The map $P^1_{S/R}(M) \to M$ is given above. Let N be an S-module and let $D : M \to N$ be a differential operator of order 1. For $m \in M$ the map

$$g \longmapsto D(gm) - gD(m)$$

is an R-derivation $S \to N$ by the axioms for differential operators of order 1. Thus it corresponds to a linear map $D_m : \Omega_{S/R} \to N$ determined by the rule $adb \mapsto aD(bm) - abD(m)$ (see Lemma 10.131.3). The map

$$\Omega_{S/R} \times M \longrightarrow N, \quad (\eta, m) \longmapsto D_m(\eta)$$

is S-bilinear (details omitted) and hence determines an S-linear map

$$\sigma_D : \Omega_{S/R} \otimes_S M \to N$$

In this way we obtain a map $\text{Diff}^1(M, N) \to \text{Hom}_S(\Omega_{S/R} \otimes_S M, N)$, $D \mapsto \sigma_D$ functorial in N. By the Yoneda lemma this corresponds a map $\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M)$. It is immediate from the construction that this map is functorial in M. The sequence

$$\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M) \to M \to 0$$

is exact because for every module N the sequence

$$0 \to \text{Hom}_S(M, N) \to \text{Diff}^1(M, N) \to \text{Hom}_S(\Omega_{S/R} \otimes_S M, N)$$

is exact by inspection.

To see that $\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M)$ is injective we argue as follows. Choose an exact sequence

$$0 \to M' \to F \to M \to 0$$

with F a free S-module. This induces an exact sequence

$$0 \to \text{Diff}^1(M, N) \to \text{Diff}^1(F, N) \to \text{Diff}^1(M', N)$$

for all N. This proves that in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{S/R} \otimes_S M' \arrow[r] \arrow[d] & P^1_{S/R}(M') \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{S/R} \otimes_S F \arrow[r] \arrow[d] & P^1_{S/R}(F) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{S/R} \otimes_S M \arrow[r] \arrow[d] & P^1_{S/R}(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the middle column is exact. The left column is exact by right exactness of $\Omega_{S/R} \otimes_S -$. By the snake lemma (see Section 10.4) it suffices to prove exactness on the left for the free module F. Using that $P^1_{S/R}(-)$ commutes with direct sums we reduce to the case $M = S$. This case is a consequence of the discussion in Example 10.133.5. $\square$

09CP Remark 10.133.7. Suppose given a commutative diagram of rings

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

a B-module M, a B'-module M', and a B-linear map $M \to M'$. Then we get a compatible system of module maps

$$\begin{tikzcd} \ldots \arrow[r] & P^2_{B'/A'}(M') \arrow[r] & P^1_{B'/A'}(M') \arrow[r] & P^0_{B'/A'}(M') \\ \ldots \arrow[r] & P^2_{B/A}(M) \arrow[r] \arrow[u] & P^1_{B/A}(M) \arrow[r] \arrow[u] & P^0_{B/A}(M) \arrow[u] \end{tikzcd}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $P^k_{B/A}(M)$ in terms of generators and relations in the proof of Lemma 10.133.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 10.133.6.

0G34 Lemma 10.133.8. Let $A \to B$ be a ring map. The differentials $\mathrm{d} : \Omega^i_{B/A} \to \Omega^{i+1}_{B/A}$ are differential operators of order 1.

Proof. Given $b \in B$ we have to show that $\mathrm{d} \circ b - b \circ \mathrm{d}$ is a linear operator. Thus we have to show that

$$\mathrm{d} \circ b \circ b' - b \circ \mathrm{d} \circ b' - b' \circ \mathrm{d} \circ b + b' \circ b \circ \mathrm{d} = 0$$

To see this it suffices to check this on additive generators for $\Omega^i_{B/A}$. Thus it suffices to show that

$$\mathrm{d}(bb'b_0\mathrm{d}b_1\wedge\ldots\wedge\mathrm{d}b_i) - b\mathrm{d}(b'b_0\mathrm{d}b_1\wedge\ldots\wedge\mathrm{d}b_i) - b'\mathrm{d}(bb_0\mathrm{d}b_1\wedge\ldots\wedge\mathrm{d}b_i) + bb'\mathrm{d}(b_0\mathrm{d}b_1\wedge\ldots\wedge\mathrm{d}b_i)$$

is zero. This is a pleasant calculation using the Leibniz rule which is left to the reader. □

0G35 Lemma 10.133.9. Let $A \to B$ be a ring map. Let $g_i \in B$, $i \in I$ be a set of generators for B as an A-algebra. Let M, N be B-modules. Let $D : M \to N$ be an A-linear map. In order to show that D is a differential operator of order k it suffices to show that $D \circ g_i - g_i \circ D$ is a differential operator of order $k-1$ for $i \in I$.

Proof. Namely, we claim that the set of elements $g \in B$ such that $D \circ g - g \circ D$ is a differential operator of order $k-1$ is an A-subalgebra of B. This follows from the relations

$$D \circ (g + g') - (g + g') \circ D = (D \circ g - g \circ D) + (D \circ g' - g' \circ D)$$

and

$$D \circ gg' - gg' \circ D = (D \circ g - g \circ D) \circ g' + g \circ (D \circ g' - g' \circ D)$$

Strictly speaking, to conclude for products we also use Lemma 10.133.2. □

0G36 Lemma 10.133.10. Let $A \to B$ be a ring map. Let M, N be B-modules. Let $S \subset B$ be a multiplicative subset. Any differential operator $D : M \to N$ of order k extends uniquely to a differential operator $E : S^{-1}M \to S^{-1}N$ of order k.

Proof. By induction on k. If $k = 0$, then D is B-linear and hence we get the extension by the functoriality of localization. Given $b \in B$ the operator $L_b : m \mapsto D(bm) - bD(m)$ has order $k - 1$. Hence it has a unique extension to a differential operator $E_b : S^{-1}M \to S^{-1}N$ of order $k-1$ by induction. Moreover, a computation shows that $L_{b'b} = L_{b'} \circ b + b' \circ L_b$ hence by uniqueness we obtain $E_{b'b} = E_{b'} \circ b + b' \circ E_b$. Similarly, we obtain $E_{b'} \circ b - b \circ E_{b'} = E_b \circ b' - b' \circ E_b$. Now for $m \in M$ and $g \in S$ we set

$$E(m/g) = (1/g)(D(m) - E_g(m/g))$$

To show that this is well defined it suffices to show that for $g' \in S$ if we use the representative $g'm/g'g$ we get the same result. We compute

$$\begin{aligned}(1/g'g)(D(g'm) - E_{g'g}(g'm/gg')) &= (1/gg')(g'D(m) + E_{g'}(m) - E_{g'g}(g'm/gg')) \\ &= (1/g'g)(g'D(m) - g'E_g(m/g))\end{aligned}$$

which is the same as before. It is clear that E is R-linear as D and E_g are R-linear. Taking $g = 1$ and using that $E_1 = 0$ we see that E extends D. By Lemma 10.133.9 it now suffices to show that $E \circ b - b \circ E$ for $b \in B$ and $E \circ 1/g' - 1/g' \circ E$ for $g' \in S$ are differential operators of order $k - 1$ in order to show that E is a differential operator of order k. For the first, choose an element m/g in $S^{-1}M$ and observe that

$$\begin{aligned}E(bm/g) - bE(m/g) &= (1/g)(D(bm) - bD(m) - E_g(bm/g) + bE_g(m/g)) \\ &= (1/g)(L_b(m) - E_b(m) + gE_b(m/g)) \\ &= E_b(m/g)\end{aligned}$$

which is a differential operator of order $k - 1$. Finally, we have

$$\begin{aligned}E(m/g'g) - (1/g')E(m/g) &= (1/g'g)(D(m) - E_{g'g}(m/g'g)) - (1/g'g)(D(m) - E_g(m/g)) \\ &= -(1/g')E_{g'}(m/g'g)\end{aligned}$$

which also is a differential operator of order $k-1$ as the composition of linear maps (multiplication by $1/g'$ and signs) and $E_{g'}$. We omit the proof of uniqueness. $\square$

0G37 **Lemma 10.133.11.** Let $R \to A$ and $R \to B$ be ring maps. Let M and M' be A-modules. Let $D : M \to M'$ be a differential operator of order k with respect to $R \to A$. Let N be any B-module. Then the map

$$D \otimes \mathrm{id}_N : M \otimes_R N \to M' \otimes_R N$$

is a differential operator of order k with respect to $B \to A \otimes_R B$.

Proof. It is clear that $D' = D \otimes \mathrm{id}_N$ is B-linear. By Lemma 10.133.9 it suffices to show that

$$D' \circ a \otimes 1 - a \otimes 1 \circ D' = (D \circ a - a \circ D) \otimes \mathrm{id}_N$$

is a differential operator of order $k - 1$ which follows by induction on k. $\square$

10.134. The naive cotangent complex

00S0 Let $R \to S$ be a ring map. Denote $R[S]$ the polynomial ring whose variables are the elements $s \in S$. Let's denote $[s] \in R[S]$ the variable corresponding to $s \in S$. Thus $R[S]$ is a free R-module on the basis elements $[s_1] \ldots [s_n]$ where $s_1, \ldots, s_n$ ranges over all unordered sequences of elements of S. There is a canonical surjection

07BL (10.134.0.1) $$R[S] \longrightarrow S, \quad [s] \longmapsto s$$

whose kernel we denote $I \subset R[S]$. It is a simple observation that I is generated by the elements $[s + s'] - [s] - [s']$, $[s][s'] - [ss']$ and $[r] - r$. According to Lemma 10.131.9 there is a canonical map

07BM (10.134.0.2) $$I/I^2 \longrightarrow \Omega_{R[S]/R} \otimes_{R[S]} S$$

whose cokernel is canonically isomorphic to $\Omega_{S/R}$. Observe that the S-module $\Omega_{R[S]/R} \otimes_{R[S]} S$ is free on the generators $\text{d}[s]$.

07BN Definition 10.134.1. Let $R \to S$ be a ring map. The *naive cotangent complex* $NL_{S/R}$ is the chain complex (10.134.0.2)

$$NL_{S/R} = \left(I/I^2 \longrightarrow \Omega_{R[S]/R} \otimes_{R[S]} S\right)$$

with I/I^2 placed in (homological) degree 1 and $\Omega_{R[S]/R} \otimes_{R[S]} S$ placed in degree 0. We will denote $H_1(L_{S/R}) = H_1(NL_{S/R})$[12] the homology in degree 1.

Before we continue let us say a few words about the actual cotangent complex (Cotangent, Section 92.3). Given a ring map $R \to S$ there exists a canonical simplicial R-algebra $P_\bullet$ whose terms are polynomial algebras and which comes equipped with a canonical homotopy equivalence

$$P_\bullet \longrightarrow S$$

The cotangent complex $L_{S/R}$ of S over R is defined as the chain complex associated to the cosimplicial module

$$\Omega_{P_\bullet/R} \otimes_{P_\bullet} S$$

The naive cotangent complex as defined above is canonically isomorphic to the truncation $\tau_{\leq 1} L_{S/R}$ (see Homology, Section 12.15 and Cotangent, Section 92.11). In particular, it is indeed the case that $H_1(NL_{S/R}) = H_1(L_{S/R})$ so our definition is compatible with the one using the cotangent complex. Moreover, $H_0(L_{S/R}) = H_0(NL_{S/R}) = \Omega_{S/R}$ as we've seen above.

Let $R \to S$ be a ring map. A *presentation of S over R* is a surjection $\alpha : P \to S$ of R-algebras where P is a polynomial algebra (on a set of variables). Often, when S is of finite type over R we will indicate this by saying: "Let $R[x_1, \ldots, x_n] \to S$ be a presentation of S/R", or "Let $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ be a presentation of S/R" if we want to indicate that I is the kernel of the presentation. Note that the map $R[S] \to S$ used to define the naive cotangent complex is an example of a presentation.

Note that for every presentation α we obtain a two term chain complex of S-modules

$$NL(\alpha) : I/I^2 \longrightarrow \Omega_{P/R} \otimes_P S.$$

Here the term I/I^2 is placed in degree 1 and the term $\Omega_{P/R} \otimes S$ is placed in degree 0. The class of $f \in I$ in I/I^2 is mapped to $\text{d}f \otimes 1$ in $\Omega_{P/R} \otimes S$. The cokernel of this complex is canonically $\Omega_{S/R}$, see Lemma 10.131.9. We call the complex $NL(\alpha)$ the *naive cotangent complex associated to the presentation* $\alpha : P \to S$ of S/R. Note that if $P = R[S]$ with its canonical surjection onto S, then we recover $NL_{S/R}$. If $P = R[x_1, \ldots, x_n]$ then will sometimes use the notation $I/I^2 \to \bigoplus_{i=1,\ldots,n} S\text{d}x_i$ to denote this complex.

[12] This module is sometimes denoted $\Gamma_{S/R}$ in the literature.

Suppose we are given a commutative diagram

06RQ (10.134.1.1)
$$\begin{tikzcd} S \arrow[r, "\phi"'] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd}$$

of rings. Let $\alpha : P \to S$ be a presentation of S over R and let $\alpha' : P' \to S'$ be a presentation of S' over R'. A morphism of presentations from $\alpha : P \to S$ to $\alpha' : P' \to S'$ is defined to be an R-algebra map

$$\varphi : P \to P'$$

such that $\phi \circ \alpha = \alpha' \circ \varphi$. Note that in this case $\varphi(I) \subset I'$, where $I = \mathrm{Ker}(\alpha)$ and $I' = \mathrm{Ker}(\alpha')$. Thus φ induces a map of S-modules $I/I^2 \to I'/(I')^2$ and by functoriality of differentials also an S-module map $\Omega_{P/R} \otimes S \to \Omega_{P'/R'} \otimes S'$. These maps are compatible with the differentials of $NL(\alpha)$ and $NL(\alpha')$ and we obtain a map of naive cotangent complexes

$$NL(\alpha) \longrightarrow NL(\alpha').$$

It is often convenient to consider the induced map $NL(\alpha) \otimes_S S' \to NL(\alpha')$.

In the special case that $P = R[S]$ and $P' = R'[S']$ the map $\phi : S \to S'$ induces a canonical ring map $\varphi : P \to P'$ by the rule $[s] \mapsto [\phi(s)]$. Hence the construction above determines canonical(!) maps of chain complexes

$$NL_{S/R} \longrightarrow NL_{S'/R'}, \quad \text{and} \quad NL_{S/R} \otimes_S S' \longrightarrow NL_{S'/R'}$$

associated to the diagram (10.134.1.1). Note that this construction is compatible with composition: given a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\phi"'] & S' \arrow[r, "\phi'"'] & S'' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & R'' \arrow[u] \end{tikzcd}$$

we see that the composition of

$$NL_{S/R} \longrightarrow NL_{S'/R'} \longrightarrow NL_{S''/R''}$$

is the map $NL_{S/R} \to NL_{S''/R''}$ given by the outer square.

It turns out that $NL(\alpha)$ is homotopy equivalent to $NL_{S/R}$ and that the maps constructed above are well defined up to homotopy (homotopies of maps of complexes are discussed in Homology, Section 12.13 but we also spell out the exact meaning of the statements in the lemma below in its proof).

00S1 **Lemma 10.134.2.** Suppose given a diagram (10.134.1.1). Let $\alpha : P \to S$ and $\alpha' : P' \to S'$ be presentations.

1. There exists a morphism of presentations from α to α'.
2. Any two morphisms of presentations induce homotopic morphisms of complexes $NL(\alpha) \to NL(\alpha')$.
3. The construction is compatible with compositions of morphisms of presentations (see proof for exact statement).

(4) If $R \to R'$ and $S \to S'$ are isomorphisms, then for any map φ of presentations from α to α' the induced map $NL(\alpha) \to NL(\alpha')$ is a homotopy equivalence and a quasi-isomorphism.

In particular, comparing α to the canonical presentation (10.134.0.1) we conclude there is a quasi-isomorphism $NL(\alpha) \to NL_{S/R}$ well defined up to homotopy and compatible with all functorialities (up to homotopy).

Proof. Since P is a polynomial algebra over R we can write $P = R[x_a, a \in A]$ for some set A. As α' is surjective, we can choose for every $a \in A$ an element $f_a \in P'$ such that $\alpha'(f_a) = \phi(\alpha(x_a))$. Let $\varphi : P = R[x_a, a \in A] \to P'$ be the unique R-algebra map such that $\varphi(x_a) = f_a$. This gives the morphism in (1).

Let φ and φ' morphisms of presentations from α to α'. Let $I = \mathrm{Ker}(\alpha)$ and $I' = \mathrm{Ker}(\alpha')$. We have to construct the diagonal map h in the diagram

$$\begin{tikzcd} I/I^2 \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "\varphi_1"'] \arrow[d, shift left, "\varphi'_1"] & \Omega_{P/R} \otimes_P S \arrow[dl, "h"'] \arrow[d, shift right, "\varphi_0"'] \arrow[d, shift left, "\varphi'_0"] \\ I'/(I')^2 \arrow[r, "\mathrm{d}"] & \Omega_{P'/R'} \otimes_{P'} S' \end{tikzcd}$$

where the vertical maps are induced by φ, φ' such that

$$\varphi_1 - \varphi'_1 = h \circ \mathrm{d} \quad \text{and} \quad \varphi_0 - \varphi'_0 = \mathrm{d} \circ h$$

Consider the map $\varphi - \varphi' : P \to P'$. Since both φ and φ' are compatible with α and α' we obtain $\varphi - \varphi' : P \to I'$. This implies that $\varphi, \varphi' : P \to P'$ induce the same P-module structure on $I'/(I')^2$, since $\varphi(p)i' - \varphi'(p)i' = (\varphi - \varphi')(p)i' \in (I')^2$. Also $\varphi - \varphi'$ is R-linear and

$$(\varphi - \varphi')(fg) = \varphi(f)(\varphi - \varphi')(g) + (\varphi - \varphi')(f)\varphi'(g)$$

Hence the induced map $D : P \to I'/(I')^2$ is a R-derivation. Thus we obtain a canonical map $h : \Omega_{P/R} \otimes_P S \to I'/(I')^2$ such that $D = h \circ \mathrm{d}$. A calculation (omitted) shows that h is the desired homotopy.

Suppose that we have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\phi"'] & S' \arrow[r, "\phi'"'] & S'' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & R'' \arrow[u] \end{tikzcd}$$

and that

(1) $\alpha : P \to S$,
(2) $\alpha' : P' \to S'$, and
(3) $\alpha'' : P'' \to S''$

are presentations. Suppose that

(1) $\varphi : P \to P'$ is a morphism of presentations from α to α' and
(2) $\varphi' : P' \to P''$ is a morphism of presentations from α' to α''.

Then it is immediate that $\varphi' \circ \varphi : P \to P''$ is a morphism of presentations from α to α'' and that the induced map $NL(\alpha) \to NL(\alpha'')$ of naive cotangent complexes is the composition of the maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha') \to NL(\alpha'')$ induced by φ and φ'.

In the simple case of complexes with 2 terms a quasi-isomorphism is just a map that induces an isomorphism on both the cokernel and the kernel of the maps between the terms. Note that homotopic maps of 2 term complexes (as explained above) define the same maps on kernel and cokernel. Hence if φ is a map from a presentation α of S over R to itself, then the induced map $NL(\alpha) \to NL(\alpha)$ is a quasi-isomorphism being homotopic to the identity by part (2). To prove (4) in full generality, consider a morphism φ' from α' to α which exists by (1). The compositions $NL(\alpha) \to NL(\alpha') \to NL(\alpha)$ and $NL(\alpha') \to NL(\alpha) \to NL(\alpha')$ are homotopic to the identity maps by (3), hence these maps are homotopy equivalences by definition. It follows formally that both maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha') \to NL(\alpha)$ are quasi-isomorphisms. Some details omitted. $\square$

08Q1 Lemma 10.134.3. Let $A \to B$ be a polynomial algebra. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(0 \to \Omega_{B/A})$ with $\Omega_{B/A}$ in degree 0.

Proof. Follows from Lemma 10.134.2 and the fact that $\mathrm{id}_B : B \to B$ is a presentation of B over A with zero kernel. $\square$

The following lemma is part of the motivation for introducing the naive cotangent complex. The cotangent complex extends this to a genuine long exact cohomology sequence. If $B \to C$ is a local complete intersection, then one can extend the sequence with a zero on the left, see More on Algebra, Lemma 15.33.6.

00S2 Lemma 10.134.4 (Jacobi-Zariski sequence). Let $A \to B \to C$ be ring maps. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_t, t \in T] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{A[x_s]/A} \otimes C \arrow[r] & \Omega_{A[x_s, y_t]/A} \otimes C \arrow[r] & \Omega_{B[y_t]/B} \otimes C \arrow[r] & 0 \\
 & I/I^2 \otimes C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J/J^2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows. We get the following exact sequence of homology groups

$$H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules extending the sequence of Lemma 10.131.7. If $\mathrm{Tor}_1^B(\Omega_{B/A}, C) = 0$, then $H_1(NL_{B/A} \otimes_B C) = H_1(L_{B/A}) \otimes_B C$.

Proof. The precise definition of the maps is omitted. The exactness of the top row follows as the $\mathrm{d}x_s$, $\mathrm{d}y_t$ form a basis for the middle module. The map γ factors

$$A[x_s, y_t] \to B[y_t] \to C$$

with surjective first arrow and second arrow equal to β. Thus we see that $K \to J$ is surjective. Moreover, the kernel of the first displayed arrow is $IA[x_s, y_t]$. Hence $I/I^2 \otimes C$ surjects onto the kernel of $K/K^2 \to J/J^2$. Finally, we can use Lemma 10.134.2 to identify the terms as homology groups of the naive cotangent complexes. The final assertion follows as the degree 0 term of the complex $NL_{B/A}$ is a free B-module. $\square$

07VC Remark 10.134.5. Let $A \to B$ and $\phi : B \to C$ be ring maps. Then the composition $NL_{B/A} \to NL_{C/A} \to NL_{C/B}$ is homotopy equivalent to zero. Namely, this composition is the functoriality of the naive cotangent complex for the square

$$\begin{tikzcd} B \arrow[r, "\phi"'] & C \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

Write $J = \mathrm{Ker}(B[C] \to C)$. An explicit homotopy is given by the map $\Omega_{A[B]/A} \otimes_A B \to J/J^2$ which maps the basis element $\mathrm{d}[b]$ to the class of $[\phi(b)] - b$ in J/J^2.

07BP Lemma 10.134.6. Let $A \to B$ be a surjective ring map with kernel I. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(I/I^2 \to 0)$ with I/I^2 in degree 1. In particular $H_1(L_{B/A}) = I/I^2$.

Proof. Follows from Lemma 10.134.2 and the fact that $A \to B$ is a presentation of B over A. $\square$

065V Lemma 10.134.7. Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is). Denote $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(B \to C)$. Then the sequence

$$I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is exact.

Proof. Follows from Lemma 10.134.4 and the description of the naive cotangent complexes $NL_{C/B}$ and $NL_{C/A}$ in Lemma 10.134.6. $\square$

00S4 Lemma 10.134.8 (Flat base change). Let $R \to S$ be a ring map. Let $\alpha : P \to S$ be a presentation. Let $R \to R'$ be a flat ring map. Let $\alpha' : P \otimes_R R' \to S' = S \otimes_R R'$ be the induced presentation. Then $NL(\alpha) \otimes_R R' = NL(\alpha) \otimes_S S' = NL(\alpha')$. In particular, the canonical map

$$NL_{S/R} \otimes_S S' \longrightarrow NL_{S \otimes_R R'/R'}$$

is a homotopy equivalence if $R \to R'$ is flat.

Proof. This is true because $\mathrm{Ker}(\alpha') = R' \otimes_R \mathrm{Ker}(\alpha)$ since $R \to R'$ is flat. $\square$

07BQ Lemma 10.134.9. Let $R_i \to S_i$ be a system of ring maps over the directed set I. Set $R = \mathrm{colim}\, R_i$ and $S = \mathrm{colim}\, S_i$. Then $NL_{S/R} = \mathrm{colim}\, NL_{S_i/R_i}$.

Proof. Recall that $NL_{S/R}$ is the complex $I/I^2 \to \bigoplus_{s \in S} S\mathrm{d}[s]$ where $I \subset R[S]$ is the kernel of the canonical presentation $R[S] \to S$. Now it is clear that $R[S] = \mathrm{colim}\, R_i[S_i]$ and similarly that $I = \mathrm{colim}\, I_i$ where $I_i = \mathrm{Ker}(R_i[S_i] \to S_i)$. Hence the lemma is clear. $\square$

07BR Lemma 10.134.10. If $S \subset A$ is a multiplicative subset of A, then $NL_{S^{-1}A/A}$ is homotopy equivalent to the zero complex.

Proof. Since $A \to S^{-1}A$ is flat we see that $NL_{S^{-1}A/A} \otimes_A S^{-1}A \to NL_{S^{-1}A/S^{-1}A}$ is a homotopy equivalence by flat base change (Lemma 10.134.8). Since the source of the arrow is isomorphic to $NL_{S^{-1}A/A}$ and the target of the arrow is zero (by Lemma 10.134.6) we win. $\square$

07BS Lemma 10.134.11. Let $S \subset A$ is a multiplicative subset of A. Let $S^{-1}A \to B$ be a ring map. Then $NL_{B/A} \to NL_{B/S^{-1}A}$ is a homotopy equivalence.

Proof. Choose a presentation $\alpha : P \to B$ of B over A. Then $\beta : S^{-1}P \to B$ is a presentation of B over $S^{-1}A$. A direct computation shows that we have $NL(\alpha) = NL(\beta)$ which proves the lemma as the naive cotangent complex is well defined up to homotopy by Lemma 10.134.2. $\square$

08JZ Lemma 10.134.12. Let $A \to B$ be a ring map. Let $g \in B$. Suppose $\alpha : P \to B$ is a presentation with kernel I. Then a presentation of B_g over A is the map

$$\beta : P[x] \longrightarrow B_g$$

extending α and sending x to $1/g$. The kernel J of β is generated by I and the element $fx - 1$ where $f \in P$ is an element mapped to $g \in B$ by α. In this situation we have

(1) $J/J^2 = (I/I^2)_g \oplus B_g(fx - 1)$,
(2) $\Omega_{P[x]/A} \otimes_{P[x]} B_g = \Omega_{P/A} \otimes_P B_g \oplus B_g \text{d}x$,
(3) $NL(\beta) \cong NL(\alpha) \otimes_B B_g \oplus (B_g \xrightarrow{g} B_g)$

Hence the canonical map $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ is a homotopy equivalence.

Proof. Since $P[x]/(I, fx - 1) = B[x]/(gx - 1) = B_g$ we get the statement about I and $fx - 1$ generating J. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{P/A} \otimes B_g \arrow[r] & \Omega_{P[x]/A} \otimes B_g \arrow[r] & \Omega_{B[x]/B} \otimes B_g \arrow[r] & 0 \\
 & (I/I^2)_g \arrow[r] \arrow[u] & J/J^2 \arrow[r] \arrow[u] & (gx-1)/(gx-1)^2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows of Lemma 10.134.4. The B_g-module $\Omega_{B[x]/B} \otimes B_g$ is free of rank 1 on $\text{d}x$. The element $\text{d}x$ in the B_g-module $\Omega_{P[x]/A} \otimes B_g$ provides a splitting for the top row. The element $gx - 1 \in (gx - 1)/(gx - 1)^2$ is mapped to $g\text{d}x$ in $\Omega_{B[x]/B} \otimes B_g$ and hence $(gx - 1)/(gx - 1)^2$ is free of rank 1 over B_g. (This can also be seen by arguing that $gx - 1$ is a nonzerodivisor in $B[x]$ because it is a polynomial with invertible constant term and any nonzerodivisor gives a quasi-regular sequence of length 1 by Lemma 10.69.2.)

Let us prove $(I/I^2)_g \to J/J^2$ injective. Consider the P-algebra map

$$\pi : P[x] \to (P/I^2)_f = P_f/I_f^2$$

sending x to $1/f$. Since J is generated by I and $fx - 1$ we see that $\pi(J) \subset (I/I^2)_f = (I/I^2)_g$. Since this is an ideal of square zero we see that $\pi(J^2) = 0$. If $a \in I$ maps to an element of J^2 in J, then $\pi(a) = 0$, which implies that a maps to zero in I_f/I_f^2. This proves the desired injectivity.

Thus we have a short exact sequence of two term complexes

$$0 \to NL(\alpha) \otimes_B B_g \to NL(\beta) \to (B_g \xrightarrow{g} B_g) \to 0$$

Such a short exact sequence can always be split in the category of complexes. In our particular case we can take as splittings

$$J/J^2 = (I/I^2)_g \oplus B_g(fx - 1) \quad \text{and} \quad \Omega_{P[x]/A} \otimes B_g = \Omega_{P/A} \otimes B_g \oplus B_g(g^{-2}\text{d}f + \text{d}x)$$

This works because $\text{d}(fx - 1) = x\text{d}f + f\text{d}x = g(g^{-2}\text{d}f + \text{d}x)$ in $\Omega_{P[x]/A} \otimes B_g$. $\square$

00S7 **Lemma 10.134.13.** Let $A \to B$ be a ring map. Let $S \subset B$ be a multiplicative subset. The canonical map $NL_{B/A} \otimes_B S^{-1}B \to NL_{S^{-1}B/A}$ is a quasi-isomorphism.

Proof. We have $S^{-1}B = \mathrm{colim}_{g \in S} B_g$ where we think of S as a directed set (ordering by divisibility), see Lemma 10.9.9. By Lemma 10.134.12 each of the maps $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ are quasi-isomorphisms. The lemma follows from Lemma 10.134.9. □

00S3 **Lemma 10.134.14.** Let R be a ring. Let $A_1 \to A_0$, and $B_1 \to B_0$ be two term complexes. Suppose that there exist morphisms of complexes $\varphi : A_\bullet \to B_\bullet$ and $\psi : B_\bullet \to A_\bullet$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity maps. Then $A_1 \oplus B_0 \cong B_1 \oplus A_0$ as R-modules.

Proof. Choose a map $h : A_0 \to A_1$ such that

$$\mathrm{id}_{A_1} - \psi_1 \circ \varphi_1 = h \circ d_A \text{ and } \mathrm{id}_{A_0} - \psi_0 \circ \varphi_0 = d_A \circ h.$$

Similarly, choose a map $h' : B_0 \to B_1$ such that

$$\mathrm{id}_{B_1} - \varphi_1 \circ \psi_1 = h' \circ d_B \text{ and } \mathrm{id}_{B_0} - \varphi_0 \circ \psi_0 = d_B \circ h'.$$

A trivial computation shows that

$$\begin{pmatrix} \mathrm{id}_{A_1} & -h' \circ \psi_1 + h \circ \psi_0 \\ 0 & \mathrm{id}_{B_0} \end{pmatrix} = \begin{pmatrix} \psi_1 & h \\ -d_B & \varphi_0 \end{pmatrix} \begin{pmatrix} \varphi_1 & -h' \\ d_A & \psi_0 \end{pmatrix}$$

This shows that both matrices on the right hand side are invertible and proves the lemma. □

00S5 **Lemma 10.134.15.** Let $R \to S$ be a ring map of finite type. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S$ we have

$$I/I^2 \oplus S^{\oplus m} \cong J/J^2 \oplus S^{\oplus n}$$

as S-modules where $I = \mathrm{Ker}(\alpha)$ and $J = \mathrm{Ker}(\beta)$.

Proof. See Lemmas 10.134.2 and 10.134.14. □

00S6 **Lemma 10.134.16.** Let $R \to S$ be a ring map of finite type. Let $g \in S$. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S_g$ we have

$$(I/I^2)_g \oplus S_g^{\oplus m} \cong J/J^2 \oplus S_g^{\oplus n}$$

as S_g-modules where $I = \mathrm{Ker}(\alpha)$ and $J = \mathrm{Ker}(\beta)$.

Proof. Let $\beta' : R[x_1, \ldots, x_n, x] \to S_g$ be the presentation of Lemma 10.134.12 constructed starting with α. Then we know that $NL(\alpha) \otimes_S S_g$ is homotopy equivalent to $NL(\beta')$. We know that $NL(\beta)$ and $NL(\beta')$ are homotopy equivalent by Lemma 10.134.2. We conclude that $NL(\alpha) \otimes_S S_g$ is homotopy equivalent to $NL(\beta)$. Finally, we apply Lemma 10.134.15. □

10.135. Local complete intersections

00S8 The property of being a local complete intersection is an intrinsic property of a Noetherian local ring. This will be discussed in Divided Power Algebra, Section 23.8. However, for the moment we just define this property for finite type algebras over a field.

00S9 **Definition 10.135.1.** Let k be a field. Let S be a finite type k-algebra.

(1) We say that S is a global complete intersection over k if there exists a presentation $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\dim(S) = n - c$.
(2) We say that S is a local complete intersection over k if there exists a covering $\text{Spec}(S) = \bigcup D(g_i)$ such that each of the rings S_{g_i} is a global complete intersection over k.

We will also use the convention that the zero ring is a global complete intersection over k.

Suppose S is a global complete intersection $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as in Definition 10.135.1. For a maximal ideal $\mathfrak m \subset k[x_1, \ldots, x_n]$ we have $\dim(k[x_1, \ldots, x_n]_{\mathfrak m}) = n$ (Lemma 10.114.1). If $(f_1, \ldots, f_c) \subset \mathfrak m$, then we conclude that $\dim(S_{\mathfrak m}) \geq n - c$ by Lemma 10.60.13. Since $\dim(S) = n - c$ by Definition 10.135.1 we conclude that $\dim(S_{\mathfrak m}) = n - c$ for all maximal ideals of S and that $\text{Spec}(S)$ is equidimensional (Topology, Definition 5.10.5) of dimension $n-c$, see Lemma 10.114.5. We will often use this without further mention.

00SA Lemma 10.135.2. Let k be a field. Let S be a finite type k-algebra. Let $g \in S$.
(1) If S is a global complete intersection so is S_g.
(2) If S is a local complete intersection so is S_g.

Proof. The second statement follows immediately from the first. Proof of the first statement. If S_g is the zero ring, then it is true. Assume S_g is nonzero. Write $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $n - c = \dim(S)$ as in Definition 10.135.1. By the remarks following the definition $\dim(S_g) = n-c$. Let $g' \in k[x_1, \ldots, x_n]$ be an element whose residue class corresponds to g. Then $S_g = k[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}g' - 1)$ as desired. □

00SB Lemma 10.135.3. Let k be a field. Let S be a finite type k-algebra. If S is a local complete intersection, then S is a Cohen-Macaulay ring.

Proof. Choose a maximal prime $\mathfrak m$ of S. We have to show that $S_{\mathfrak m}$ is Cohen-Macaulay. By assumption we may assume $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim(S) = n - c$. Let $\mathfrak m' \subset k[x_1, \ldots, x_n]$ be the maximal ideal corresponding to $\mathfrak m$. According to Proposition 10.114.2 the local ring $k[x_1, \ldots, x_n]_{\mathfrak m'}$ is regular local of dimension n. In particular it is Cohen-Macaulay by Lemma 10.106.3. By Lemma 10.60.13 applied c times the local ring $S_{\mathfrak m} = k[x_1, \ldots, x_n]_{\mathfrak m'}/(f_1, \ldots, f_c)$ has dimension $\geq n - c$. By assumption $\dim(S_{\mathfrak m}) \leq n - c$. Thus we get equality. This implies that $f_1, \ldots, f_c$ is a regular sequence in $k[x_1, \ldots, x_n]_{\mathfrak m'}$ and that $S_{\mathfrak m}$ is Cohen-Macaulay, see Proposition 10.103.4. □

The following is the technical key to the rest of the material in this section. An important feature of this lemma is that we may choose any presentation for the ring S, but that condition (1) does not depend on this choice.

00SC Lemma 10.135.4. Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak q$ be a prime of S. Choose any presentation $S = k[x_1, \ldots, x_n]/I$. Let $\mathfrak q'$ be the prime of $k[x_1, \ldots, x_n]$ corresponding to $\mathfrak q$. Set $c = \text{height}(\mathfrak q') - \text{height}(\mathfrak q)$, in other words $\dim_{\mathfrak q}(S) = n - c$ (see Lemma 10.116.4). The following are equivalent
(1) There exists a $g \in S$, $g \notin \mathfrak q$ such that S_g is a global complete intersection over k.
(2) The ideal $I_{\mathfrak q'} \subset k[x_1, \ldots, x_n]_{\mathfrak q'}$ can be generated by c elements.
(3) The conormal module $(I/I^2)_{\mathfrak q}$ can be generated by c elements over $S_{\mathfrak q}$.

(4) The conormal module $(I/I^2)_{\mathfrak q}$ is a free $S_{\mathfrak q}$-module of rank c.
(5) The ideal $I_{\mathfrak q'}$ can be generated by a regular sequence in the regular local ring $k[x_1, \ldots, x_n]_{\mathfrak q'}$.

In this case any c elements of $I_{\mathfrak q'}$ which generate $I_{\mathfrak q'}/\mathfrak q' I_{\mathfrak q'}$ form a regular sequence in the local ring $k[x_1, \ldots, x_n]_{\mathfrak q'}$.

Proof. Set $R = k[x_1, \ldots, x_n]_{\mathfrak q'}$. This is a Cohen-Macaulay local ring of dimension height$(\mathfrak q')$, see for example Lemma 10.135.3. Moreover, $\overline{R} = R/IR = R/I_{\mathfrak q'} = S_{\mathfrak q}$ is a quotient of dimension height$(\mathfrak q)$. Let $f_1, \ldots, f_c \in I_{\mathfrak q'}$ be elements which generate $(I/I^2)_{\mathfrak q}$. By Lemma 10.20.1 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak q'}$. Since the dimensions work out, we conclude by Proposition 10.103.4 that $f_1, \ldots, f_c$ is a regular sequence in R. By Lemma 10.69.2 we see that $(I/I^2)_{\mathfrak q}$ is free. These arguments show that (2), (3), (4) are equivalent and that they imply the last statement of the lemma, and therefore they imply (5).

If (5) holds, say $I_{\mathfrak q'}$ is generated by a regular sequence of length e, then $\text{height}(\mathfrak q) = \dim(S_{\mathfrak q}) = \dim(k[x_1, \ldots, x_n]_{\mathfrak q'}) - e = \text{height}(\mathfrak q') - e$ by dimension theory, see Section 10.60. We conclude that $e = c$. Thus (5) implies (2).

We continue with the notation introduced in the first paragraph. For each f_i we may find $d_i \in k[x_1, \ldots, x_n]$, $d_i \notin \mathfrak q'$ such that $f_i' = d_i f_i \in k[x_1, \ldots, x_n]$. Then it is still true that $I_{\mathfrak q'} = (f_1', \ldots, f_c')R$. Hence there exists a $g' \in k[x_1, \ldots, x_n]$, $g' \notin \mathfrak q'$ such that $I_{g'} = (f_1', \ldots, f_c')$. Moreover, pick $g'' \in k[x_1, \ldots, x_n]$, $g'' \notin \mathfrak q'$ such that $\dim(S_{g''}) = \dim_{\mathfrak q} \text{Spec}(S)$. By Lemma 10.116.4 this dimension is equal to $n - c$. Finally, set g equal to the image of $g'g''$ in S. Then we see that

$$S_g \cong k[x_1, \ldots, x_n, x_{n+1}]/(f_1', \ldots, f_c', x_{n+1}g'g'' - 1)$$

and by our choice of g'' this ring has dimension $n - c$. Therefore it is a global complete intersection. Thus each of (2), (3), and (4) implies (1).

Assume (1). Let $S_g \cong k[y_1, \ldots, y_m]/(f_1, \ldots, f_t)$ be a presentation of S_g as a global complete intersection. Write $J = (f_1, \ldots, f_t)$. Let $\mathfrak q'' \subset k[y_1, \ldots, y_m]$ be the prime corresponding to $\mathfrak q S_g$. Note that $t = m - \dim(S_g) = \text{height}(\mathfrak q'') - \text{height}(\mathfrak q)$, see Lemma 10.116.4 for the last equality. As seen in the proof of Lemma 10.135.3 (and also above) the elements $f_1, \ldots, f_t$ form a regular sequence in the local ring $k[y_1, \ldots, y_m]_{\mathfrak q''}$. By Lemma 10.69.2 we see that $(J/J^2)_{\mathfrak q}$ is free of rank t. By Lemma 10.134.16 we have

$$J/J^2 \oplus S_g^n \cong (I/I^2)_g \oplus S_g^m$$

Thus $(I/I^2)_{\mathfrak q}$ is free of rank $t + n - m = m - \dim(S_g) + n - m = n - \dim(S_g) = \text{height}(\mathfrak q') - \text{height}(\mathfrak q) = c$. Thus we obtain (4). □

The result of Lemma 10.135.4 suggests the following definition.

00SD Definition 10.135.5. Let k be a field. Let S be a local k-algebra essentially of finite type over k. We say S is a complete intersection (over k) if there exists a local k-algebra R and elements $f_1, \ldots, f_c \in \mathfrak m_R$ such that

(1) R is essentially of finite type over k,
(2) R is a regular local ring,
(3) $f_1, \ldots, f_c$ form a regular sequence in R, and
(4) $S \cong R/(f_1, \ldots, f_c)$ as k-algebras.

By the Cohen structure theorem (see Theorem 10.160.8) any complete Noetherian local ring may be written as the quotient of some regular complete local ring. Hence we may use the definition above to define the notion of a complete intersection ring for any complete Noetherian local ring. We will discuss this in Divided Power Algebra, Section 23.8. In the meantime the following lemma shows that such a definition makes sense.

00SE Lemma 10.135.6. Let $A \to B \to C$ be surjective local ring homomorphisms. Assume A and B are regular local rings. The following are equivalent

(1) $\text{Ker}(A \to C)$ is generated by a regular sequence,
(2) $\text{Ker}(A \to C)$ is generated by $\dim(A) - \dim(C)$ elements,
(3) $\text{Ker}(B \to C)$ is generated by a regular sequence, and
(4) $\text{Ker}(B \to C)$ is generated by $\dim(B) - \dim(C)$ elements.

Proof. A regular local ring is Cohen-Macaulay, see Lemma 10.106.3. Hence the equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4), see Proposition 10.103.4. By Lemma 10.106.4 the ideal $\text{Ker}(A \to B)$ can be generated by $\dim(A) - \dim(B)$ elements. Hence we see that (4) implies (2).

It remains to show that (1) implies (4). We do this by induction on $\dim(A) - \dim(B)$. The case $\dim(A) - \dim(B) = 0$ is trivial. Assume $\dim(A) > \dim(B)$. Write $I = \text{Ker}(A \to C)$ and $J = \text{Ker}(A \to B)$. Note that $J \subset I$. Our assumption is that the minimal number of generators of I is $\dim(A) - \dim(C)$. Let $\mathfrak{m} \subset A$ be the maximal ideal. Consider the maps

$$J/\mathfrak{m}J \to I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2$$

By Lemma 10.106.4 and its proof the composition is injective. Take any element $x \in J$ which is not zero in $J/\mathfrak{m}J$. By the above and Nakayama's lemma x is an element of a minimal set of generators of I. Hence we may replace A by A/xA and I by I/xA which decreases both $\dim(A)$ and the minimal number of generators of I by 1. Thus we win. □

00SF Lemma 10.135.7. Let k be a field. Let S be a local k-algebra essentially of finite type over k. The following are equivalent:

(1) S is a complete intersection over k,
(2) for any surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $\text{Ker}(R \to S)$ can be generated by a regular sequence,
(3) for some surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $\text{Ker}(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements,
(4) there exists a global complete intersection A over k and a prime $\mathfrak{a}$ of A such that $S \cong A_{\mathfrak{a}}$, and
(5) there exists a local complete intersection A over k and a prime $\mathfrak{a}$ of A such that $S \cong A_{\mathfrak{a}}$.

Proof. It is clear that (2) implies (1) and (1) implies (3). It is also clear that (4) implies (5). Let us show that (3) implies (4). Thus we assume there exists a surjection $R \to S$ with R a regular local ring essentially of finite presentation over k such that the ideal $\text{Ker}(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements. We may write $R = (k[x_1, \ldots, x_n]/J)_{\mathfrak{q}}$ for some $J \subset k[x_1, \ldots, x_n]$ and some prime $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ with $J \subset \mathfrak{q}$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of the map $k[x_1, \ldots, x_n] \to S$ so that $S \cong (k[x_1, \ldots, x_n]/I)_{\mathfrak{q}}$. By assumption $(I/J)_{\mathfrak{q}}$ is generated by $\dim(R) - \dim(S)$ elements. We conclude that $I_{\mathfrak{q}}$ can be generated by $\dim(k[x_1, \ldots, x_n]_{\mathfrak{q}}) - \dim(S)$ elements by Lemma 10.135.6. From Lemma 10.135.4 we see that for some $g \in k[x_1, \ldots, x_n]$, $g \notin \mathfrak{q}$ the algebra $(k[x_1, \ldots, x_n]/I)_g$ is a global complete intersection and S is isomorphic to a local ring of it.

To finish the proof of the lemma we have to show that (5) implies (2). Assume (5) and let $\pi : R \to S$ be a surjection with R a regular local k-algebra essentially of finite type over k. By assumption we have $S = A_{\mathfrak{a}}$ for some local complete intersection A over k. Choose a presentation $R = (k[y_1, \ldots, y_m]/J)_{\mathfrak{q}}$ with $J \subset \mathfrak{q} \subset k[y_1, \ldots, y_m]$. We may and do assume that J is the kernel of the map $k[y_1, \ldots, y_m] \to R$. Let $I \subset k[y_1, \ldots, y_m]$ be the kernel of the map $k[y_1, \ldots, y_m] \to S = A_{\mathfrak{a}}$. Then $J \subset I$ and $(I/J)_{\mathfrak{q}}$ is the kernel of the surjection $\pi : R \to S$. So $S = (k[y_1, \ldots, y_m]/I)_{\mathfrak{q}}$.

By Lemma 10.126.7 we see that there exist $g \in A$, $g \notin \mathfrak{a}$ and $g' \in k[y_1, \ldots, y_m]$, $g' \notin \mathfrak{q}$ such that $A_g \cong (k[y_1, \ldots, y_m]/I)_{g'}$. After replacing A by A_g and $k[y_1, \ldots, y_m]$ by $k[y_1, \ldots, y_{m+1}]$ we may assume that $A \cong k[y_1, \ldots, y_m]/I$. Consider the surjective maps of local rings

$$k[y_1, \ldots, y_m]_{\mathfrak{q}} \to R \to S.$$

We have to show that the kernel of $R \to S$ is generated by a regular sequence. By Lemma 10.135.4 we know that $k[y_1, \ldots, y_m]_{\mathfrak{q}} \to A_{\mathfrak{a}} = S$ has this property (as A is a local complete intersection over k). We win by Lemma 10.135.6. □

00SG Lemma 10.135.8. Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q}$ be a prime of S. The following are equivalent:

(1) The local ring $S_{\mathfrak{q}}$ is a complete intersection ring (Definition 10.135.5).
(2) There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a local complete intersection over k.
(3) There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a global complete intersection over k.
(4) For any presentation $S = k[x_1, \ldots, x_n]/I$ with $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$ any of the equivalent conditions (1) – (5) of Lemma 10.135.4 hold.

Proof. This is a combination of Lemmas 10.135.4 and 10.135.7 and the definitions. □

00SH Lemma 10.135.9. Let k be a field. Let S be a finite type k-algebra. The following are equivalent:

(1) The ring S is a local complete intersection over k.
(2) All local rings of S are complete intersection rings over k.
(3) All localizations of S at maximal ideals are complete intersection rings over k.

Proof. This follows from Lemma 10.135.8, the fact that $\mathrm{Spec}(S)$ is quasi-compact and the definitions. □

The following lemma says that being a complete intersection is preserved under change of base field (in a strong sense).

00SI Lemma 10.135.10. Let K/k be a field extension. Let S be a finite type algebra over k. Let $\mathfrak{q}_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak{q}$ be the corresponding prime of S. Then $S_\mathfrak{q}$ is a complete intersection over k (Definition 10.135.5) if and only if $(S_K)_{\mathfrak{q}_K}$ is a complete intersection over K.

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $S_K = K[x_1, \ldots, x_n]/I_K$ where $I_K = K \otimes_k I$. Let $\mathfrak{q}'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ be the corresponding prime. We will show that the equivalent conditions of Lemma 10.135.4 hold for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak{q})$ if and only if they hold for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak{q}_K)$. The lemma will follow from this (see Lemma 10.135.8).

By Lemma 10.116.6 we have $\dim_\mathfrak{q} S = \dim_{\mathfrak{q}_K} S_K$. Hence the integer c occurring in Lemma 10.135.4 is the same for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak{q})$ as for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak{q}_K)$. On the other hand we have

$$\begin{aligned} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}') \otimes_{\kappa(\mathfrak{q}')} \kappa(\mathfrak{q}'_K) &= I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= I \otimes_{k[x_1,\ldots,x_n]} K[x_1, \ldots, x_n] \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= (K \otimes_k I) \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K). \end{aligned}$$

Therefore, $\dim_{\kappa(\mathfrak{q}')} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}') = \dim_{\kappa(\mathfrak{q}'_K)} I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K)$. Thus it follows from Nakayama's Lemma 10.20.1 that the minimal number of generators of $I_{\mathfrak{q}'}$ is the same as the minimal number of generators of $(I_K)_{\mathfrak{q}'_K}$. Thus the lemma follows from characterization (2) of Lemma 10.135.4. □

00SJ Lemma 10.135.11. Let $k \to K$ be a field extension. Let S be a finite type k-algebra. Then S is a local complete intersection over k if and only if $S \otimes_k K$ is a local complete intersection over K.

Proof. This follows from a combination of Lemmas 10.135.9 and 10.135.10. But we also give a different proof here (based on the same principles).

Set $S' = S \otimes_k K$. Let $\alpha : k[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : K[x_1, \ldots, x_n] \to S'$ be the induced presentation with kernel I'.

Suppose that S is a local complete intersection. Pick a prime $\mathfrak{q} \subset S'$. Denote $\mathfrak{q}'$ the corresponding prime of $K[x_1, \ldots, x_n]$, $\mathfrak{p}$ the corresponding prime of S, and $\mathfrak{p}'$ the corresponding prime of $k[x_1, \ldots, x_n]$. Consider the following diagram of Noetherian local rings

$$\begin{tikzcd} S'_\mathfrak{q} & K[x_1, \ldots, x_n]_{\mathfrak{q}'} \arrow[l] \\ S_\mathfrak{p} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak{p}'} \arrow[l] \arrow[u] \end{tikzcd}$$

By Lemma 10.135.4 we know that $S_\mathfrak{p}$ is cut out by some regular sequence $f_1, \ldots, f_c$ in $k[x_1, \ldots, x_n]_{\mathfrak{p}'}$. Since the right vertical arrow is flat we see that the images of $f_1, \ldots, f_c$ form a regular sequence in $K[x_1, \ldots, x_n]_{\mathfrak{q}'}$. Because tensoring with K over k is an exact functor we have $S'_\mathfrak{q} = K[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_c)$. Hence by Lemma 10.135.4 again we see that S' is a local complete intersection in a neighbourhood of $\mathfrak{q}$. Since $\mathfrak{q}$ was arbitrary we see that S' is a local complete intersection over K.

Suppose that S' is a local complete intersection. Pick a maximal ideal $\mathfrak{m}$ of S. Let $\mathfrak{m}'$ denote the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Denote $\kappa = \kappa(\mathfrak{m})$ the residue field. By Remark 10.17.8 the primes of S' lying over $\mathfrak{m}$ correspond to primes in $K \otimes_k \kappa$. By the Hilbert-Nullstellensatz Theorem 10.34.1 we have $[\kappa : k] < \infty$. Hence $K \otimes_k \kappa$ is finite nonzero over K. Hence $K \otimes_k \kappa$ has a finite number > 0 of primes which are all maximal, each of which has a residue field finite over K (see Section 10.53). Hence there are finitely many > 0 prime ideals $\mathfrak{n} \subset S'$ lying over $\mathfrak{m}$, each of which is maximal and has a residue field which is finite over K. Pick one, say $\mathfrak{n} \subset S'$, and let $\mathfrak{n}' \subset K[x_1, \ldots, x_n]$ denote the corresponding prime ideal of $K[x_1, \ldots, x_n]$. Note that since $V(\mathfrak{m}S')$ is finite, we see that $\mathfrak{n}$ is an isolated closed point of it, and we deduce that $\mathfrak{m}S'_{\mathfrak{n}}$ is an ideal of definition of $S'_{\mathfrak{n}}$. This implies that $\dim(S_{\mathfrak{m}}) = \dim(S'_{\mathfrak{n}})$ for example by Lemma 10.112.7. (This can also be seen using Lemma 10.116.6.) Consider the corresponding diagram of Noetherian local rings

$$\begin{tikzcd} S'_{\mathfrak{n}} & K[x_1, \ldots, x_n]_{\mathfrak{n}'} \arrow[l] \\ S_{\mathfrak{m}} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak{m}'} \arrow[l] \arrow[u] \end{tikzcd}$$

According to Lemma 10.134.8 we have $NL(\alpha) \otimes_S S' = NL(\alpha')$, in particular $I'/(I')^2 = I/I^2 \otimes_S S'$. Thus $(I/I^2)_{\mathfrak{m}} \otimes_{S_{\mathfrak{m}}} \kappa$ and $(I'/(I')^2)_{\mathfrak{n}} \otimes_{S'_{\mathfrak{n}}} \kappa(\mathfrak{n})$ have the same dimension. Since $(I'/(I')^2)_{\mathfrak{n}}$ is free of rank $n - \dim S'_{\mathfrak{n}}$ we deduce that $(I/I^2)_{\mathfrak{m}}$ can be generated by $n - \dim S'_{\mathfrak{n}} = n - \dim S_{\mathfrak{m}}$ elements. By Lemma 10.135.4 we see that S is a local complete intersection in a neighbourhood of $\mathfrak{m}$. Since $\mathfrak{m}$ was any maximal ideal we conclude that S is a local complete intersection. $\square$

We end with a lemma which we will later use to prove that given ring maps $T \to A \to B$ where B is syntomic over T, and B is syntomic over A, then A is syntomic over T.

02JP **Lemma 10.135.12.** Let

$$\begin{tikzcd} B & S \arrow[l] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

be a commutative square of local rings. Assume

(1) R and $\overline{S} = S/\mathfrak{m}_R S$ are regular local rings,
(2) $A = R/I$ and $B = S/J$ for some ideals I, J,
(3) $J \subset S$ and $\overline{J} = J/\mathfrak{m}_R \cap J \subset \overline{S}$ are generated by regular sequences, and
(4) $A \to B$ and $R \to S$ are flat.

Then I is generated by a regular sequence.

Proof. Set $\overline{B} = B/\mathfrak{m}_R B = B/\mathfrak{m}_A B$ so that $\overline{B} = \overline{S}/\overline{J}$. Let $f_1, \ldots, f_{\overline{c}} \in J$ be elements such that $\overline{f}_1, \ldots, \overline{f}_{\overline{c}} \in \overline{J}$ form a regular sequence generating $\overline{J}$. Note that $\overline{c} = \dim(\overline{S}) - \dim(\overline{B})$, see Lemma 10.135.6. By Lemma 10.99.3 the ring $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R. Hence $S/(f_1, \ldots, f_{\overline{c}}) + IS$ is flat over A. The map $S/(f_1, \ldots, f_{\overline{c}}) + IS \to B$ is therefore a surjection of finite S/IS-modules flat over A which is an isomorphism modulo $\mathfrak{m}_A$, and hence an isomorphism by Lemma 10.99.1. In other words, $J = (f_1, \ldots, f_{\overline{c}}) + IS$.

By Lemma 10.135.6 again the ideal J is generated by a regular sequence of $c = \dim(S) - \dim(B)$ elements. Hence $J/\mathfrak{m}_S J$ is a vector space of dimension c. By the description of J above there exist $g_1, \ldots, g_{c-\overline{c}} \in I$ such that J is generated by $f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}}$ (use Nakayama's Lemma 10.20.1). Consider the ring $A' = R/(g_1, \ldots, g_{c-\overline{c}})$ and the surjection $A' \to A$. We see from the above that $B = S/(f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}})$ is flat over A' (as $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R). Hence $A' \to B$ is injective (as it is faithfully flat, see Lemma 10.39.17). Since this map factors through A we get $A' = A$. Note that $\dim(B) = \dim(A) + \dim(\overline{B})$, and $\dim(S) = \dim(R) + \dim(\overline{S})$, see Lemma 10.112.7. Hence $c - \overline{c} = \dim(R) - \dim(A)$ by elementary algebra. Thus $I = (g_1, \ldots, g_{c-\overline{c}})$ is generated by a regular sequence according to Lemma 10.135.6. $\square$

10.136. Syntomic morphisms

00SK Syntomic ring maps are flat finitely presented ring maps all of whose fibers are local complete intersections. We discuss general local complete intersection ring maps in More on Algebra, Section 15.33.

00SL Definition 10.136.1. A ring map $R \to S$ is called syntomic, or we say S is a flat local complete intersection over R if it is flat, of finite presentation, and if all of its fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are local complete intersections, see Definition 10.135.1.

Clearly, an algebra over a field is syntomic over the field if and only if it is a local complete intersection. Here is a pleasing feature of this definition.

00SM Lemma 10.136.2. Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is syntomic if and only if $R' \to S'$ is syntomic.

Proof. By Lemma 10.126.2 and Lemma 10.39.8 this holds for the property of being flat and for the property of being of finite presentation. The map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective, see Lemma 10.39.16. Thus it suffices to show given primes $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$ that $S \otimes_R \kappa(\mathfrak{p})$ is a local complete intersection if and only if $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is a local complete intersection. Note that $S' \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$. Thus Lemma 10.135.11 applies. $\square$

00SN Lemma 10.136.3. Any base change of a syntomic map is syntomic.

Proof. This is true for being flat, for being of finite presentation, and for having local complete intersections as fibres by Lemmas 10.39.7, 10.6.2 and 10.135.11. $\square$

00SO Lemma 10.136.4. Let $R \to S$ be a ring map. Suppose we have $g_1, \ldots g_m \in S$ which generate the unit ideal such that each $R \to S_{g_i}$ is syntomic. Then $R \to S$ is syntomic.

Proof. This is true for being flat and for being of finite presentation by Lemmas 10.39.18 and 10.23.3. The property of having fibre rings which are local complete intersections is local on S by its very definition, see Definition 10.135.1. $\square$

00SP Definition 10.136.5. Let $R \to S$ be a ring map. We say that $R \to S$ is a relative global complete intersection if there exists a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and every nonempty fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ has dimension $n - c$. We will say "let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection" to indicate this situation.

The following lemma is occasionally useful to find global presentations.

07CF Lemma 10.136.6. Let S be a finitely presented R-algebra which has a presentation $S = R[x_1, \ldots, x_n]/I$ such that I/I^2 is free over S. Then S has a presentation $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$ such that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the classes of $f_1, \ldots, f_c$.

Proof. Note that I is a finitely generated ideal by Lemma 10.6.3. Let $f_1, \ldots, f_c \in I$ be elements which map to a basis of I/I^2. By Nakayama's lemma (Lemma 10.20.1) there exists a $g \in 1 + I$ such that

$$g \cdot I \subset (f_1, \ldots, f_c)$$

and $I_g \cong (f_1, \ldots, f_c)_g$. Hence we see that

$$S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)[1/g] \cong R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, gx_{n+1} - 1)$$

as desired. It follows that $f_1, \ldots, f_c, gx_{n+1} - 1$ form a basis for $(f_1, \ldots, f_c, gx_{n+1} - 1)/(f_1, \ldots, f_c, gx_{n+1} - 1)^2$ for example by applying Lemma 10.134.12. □

00SQ Example 10.136.7. Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_nc_m \end{array}$$

In other words, this is the unique ring map of polynomial rings as indicated such that the polynomial factorization

$$x^{n+m} + a_1x^{n+m-1} + \ldots + a_{n+m} = (x^n + b_1x^{n-1} + \ldots + b_n)(x^m + c_1x^{m-1} + \ldots + c_m)$$

holds. Note that S is generated by $n+m$ elements over R (namely, b_i, c_j) and that there are $n + m$ equations (namely $a_k = a_k(b_i, c_j)$). In order to show that S is a relative global complete intersection over R it suffices to prove that all fibres have dimension 0.

To prove this, let $R \to k$ be a ring map into a field k. Say a_i maps to $\alpha_i \in k$. Consider the fibre ring $S_k = k \otimes_R S$. Let $k \to K$ be a field extension. A k-algebra map of $S_k \to K$ is the same thing as finding $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m \in K$ such that

$$x^{n+m} + \alpha_1x^{n+m-1} + \ldots + \alpha_{n+m} = (x^n + \beta_1x^{n-1} + \ldots + \beta_n)(x^m + \gamma_1x^{m-1} + \ldots + \gamma_m).$$

Hence we see there are at most finitely many choices of such $n+m$-tuples in K. This proves that all fibres have finitely many closed points (use Hilbert's Nullstellensatz to see they all correspond to solutions in $\overline{k}$ for example) and hence that $R \to S$ is a relative global complete intersection.

Another way to argue this is to show $\mathbf{Z}[a_1, \ldots, a_{n+m}] \to \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$ is actually also a finite ring map. Namely, by Lemma 10.38.5 each of b_i, c_j is integral over R, and hence $R \to S$ is finite by Lemma 10.36.4.

00SR Example 10.136.8. Consider the ring map

$$\begin{array}{ccc} R = \mathbf{Z}[a_1, \ldots, a_n] & \longrightarrow & S = \mathbf{Z}[\alpha_1, \ldots, \alpha_n] \\ a_1 & \longmapsto & \alpha_1 + \ldots + \alpha_n \\ \ldots & \ldots & \ldots \\ a_n & \longmapsto & \alpha_1 \ldots \alpha_n \end{array}$$

In other words this is the unique ring map of polynomial rings as indicated such that

$$x^n + a_1 x^{n-1} + \ldots + a_n = \prod\nolimits_{i=1}^n (x + \alpha_i)$$

holds in $\mathbf{Z}[\alpha_i, x]$. Another way to say this is that a_i maps to the ith elementary symmetric function in $\alpha_1, \ldots, \alpha_n$. Note that S is generated by n elements over R subject to n equations. Hence to show that S is a relative global complete intersection over R we have to show that the fibre rings $S \otimes_R \kappa(\mathfrak p)$ have dimension 0. This follows as in Example 10.136.7 because the ring map $\mathbf{Z}[a_1, \ldots, a_n] \to \mathbf{Z}[\alpha_1, \ldots, \alpha_n]$ is actually finite since each $\alpha_i \in S$ satisfies the monic equation $x^n - a_1 x^{n-1} + \ldots + (-1)^n a_n$ over R.

00SS Lemma 10.136.9. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 10.136.5)

(1) For any $R \to R'$ the base change $R' \otimes_R S = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection.
(2) For any $g \in S$ which is the image of $h \in R[x_1, \ldots, x_n]$ the ring $S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$ is a relative global complete intersection.
(3) If $R \to S$ factors as $R \to R_f \to S$ for some $f \in R$. Then the ring $S = R_f[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R_f.

Proof. By Lemma 10.116.5 the fibres of a base change have the same dimension as the fibres of the original map. Moreover $R' \otimes_R R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Thus (1) follows. The proof of (2) is that the localization at one element can be described as $S_g \cong S[x_{n+1}]/(gx_{n+1} - 1)$. Assertion (3) follows from (1) since under the assumptions of (3) we have $R_f \otimes_R S \cong S$. □

00ST Lemma 10.136.10. Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. We will find $h \in R[x_1, \ldots, x_n]$ which maps to $g \in S$ such that

$$S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$$

is a relative global complete intersection with a presentation as in Definition 10.136.5 in each of the following cases:

(1) Let $I \subset R$ be an ideal. If the fibres of $\mathrm{Spec}(S/IS) \to \mathrm{Spec}(R/I)$ have dimension $n - c$, then we can find (h, g) as above such that g maps to $1 \in S/IS$.
(2) Let $\mathfrak p \subset R$ be a prime. If $\dim(S \otimes_R \kappa(\mathfrak p)) = n - c$, then we can find (h, g) as above such that g maps to a unit of $S \otimes_R \kappa(\mathfrak p)$.
(3) Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. If $\dim_{\mathfrak q}(S/R) = n - c$, then we can find (h, g) as above such that $g \notin \mathfrak q$.

Proof. Ad (1). By Lemma 10.125.6 there exists an open subset $W \subset \text{Spec}(S)$ containing $V(IS)$ such that all fibres of $W \to \text{Spec}(R)$ have dimension $\leq n-c$. Say $W = \text{Spec}(S) \setminus V(J)$. Then $V(J) \cap V(IS) = \emptyset$ hence we can find a $g \in J$ which maps to $1 \in S/IS$. Let $h \in R[x_1, \ldots, x_n]$ be any preimage of g.

Ad (2). By Lemma 10.125.6 there exists an open subset $W \subset \text{Spec}(S)$ containing $\text{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ such that all fibres of $W \to \text{Spec}(R)$ have dimension $\leq n-c$. Say $W = \text{Spec}(S) \setminus V(J)$. Then $V(J \cdot S \otimes_R \kappa(\mathfrak{p})) = \emptyset$. Hence we can find a $g \in J$ which maps to a unit in $S \otimes_R \kappa(\mathfrak{p})$ (details omitted). Let $h \in R[x_1, \ldots, x_n]$ be any preimage of g.

Ad (3). By Lemma 10.125.6 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that all nonempty fibres of $R \to S_g$ have dimension $\leq n-c$. Let $h \in R[x_1, \ldots, x_n]$ be any element that maps to g. $\square$

The following lemma says we can do absolute Noetherian approximation for relative global complete intersections.

00SU **Lemma 10.136.11.** Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 10.136.5). There exist a finite type $\mathbf{Z}$-subalgebra $R_0 \subset R$ such that $f_i \in R_0[x_1, \ldots, x_n]$ and such that

$$S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

is a relative global complete intersection.

Proof. Let $R_0 \subset R$ be the $\mathbf{Z}$-algebra of R generated by all the coefficients of the polynomials $f_1, \ldots, f_c$. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Clearly, $S = R \otimes_{R_0} S_0$. Pick a prime $\mathfrak{q} \subset S$ and denote $\mathfrak{p} \subset R$, $\mathfrak{q}_0 \subset S_0$, and $\mathfrak{p}_0 \subset R_0$ the primes it lies over. Because $\dim(S \otimes_R \kappa(\mathfrak{p})) = n-c$ we also have $\dim(S_0 \otimes_{R_0} \kappa(\mathfrak{p}_0)) = n-c$, see Lemma 10.116.5. By Lemma 10.125.6 there exists a $g \in S_0$, $g \notin \mathfrak{q}_0$ such that all nonempty fibres of $R_0 \to (S_0)_g$ have dimension $\leq n-c$. As $\mathfrak{q}$ was arbitrary and $\text{Spec}(S)$ quasi-compact, we can find finitely many $g_1, \ldots, g_m \in S_0$ such that (a) for $j = 1, \ldots, m$ the nonempty fibres of $R_0 \to (S_0)_{g_j}$ have dimension $\leq n-c$ and (b) the image of $\text{Spec}(S) \to \text{Spec}(S_0)$ is contained in $D(g_1) \cup \ldots \cup D(g_m)$. In other words, the images of $g_1, \ldots, g_m$ in $S = R \otimes_{R_0} S_0$ generate the unit ideal. After increasing R_0 we may assume that $g_1, \ldots, g_m$ generate the unit ideal in S_0. By (a) the nonempty fibres of $R_0 \to S_0$ all have dimension $\leq n-c$ and we conclude. $\square$

00SV **Lemma 10.136.12.** Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 10.136.5). For every prime $\mathfrak{q}$ of S, let $\mathfrak{q}'$ denote the corresponding prime of $R[x_1, \ldots, x_n]$. Then

(1) $f_1, \ldots, f_c$ is a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$,
(2) each of the rings $R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i)$ is flat over R, and
(3) the S-module $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the elements $f_i \bmod (f_1, \ldots, f_c)^2$.

Proof. By Lemma 10.69.2 part (3) follows from part (1).

Assume R is Noetherian. Let $\mathfrak{p} = R \cap \mathfrak{q}'$. By Lemma 10.135.4 for example we see that $f_1, \ldots, f_c$ form a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p})$. Moreover, the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$ is flat over $R_{\mathfrak{p}}$. Since R, and hence $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$ is Noetherian we see from Lemma 10.99.3 that (1) and (2) hold.

Let R be general. Write $R = \mathrm{colim}_{\lambda \in \Lambda} R_\lambda$ as the filtered colimit of finite type $\mathbf{Z}$-subalgebras (compare with Section 10.127). We may assume that $f_1, \ldots, f_c \in R_\lambda[x_1, \ldots, x_n]$ for all λ. Let $R_0 \subset R$ be as in Lemma 10.136.11. Then we may assume $R_0 \subset R_\lambda$ for all λ. It follows that $S_\lambda = R_\lambda[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection (as base change of S_0 via $R_0 \to R_\lambda$, see Lemma 10.136.9). Denote $\mathfrak p_\lambda$, $\mathfrak q_\lambda$, $\mathfrak q'_\lambda$ the prime of R_λ, S_λ, $R_\lambda[x_1, \ldots, x_n]$ induced by $\mathfrak p$, $\mathfrak q$, $\mathfrak q'$. With this notation, we have (1) and (2) for each λ. Since

$$R[x_1, \ldots, x_n]_{\mathfrak q'}/(f_1, \ldots, f_i) = \mathrm{colim}\, R_\lambda[x_1, \ldots, x_n]_{\mathfrak q'_\lambda}/(f_1, \ldots, f_i)$$

we deduce flatness in (2) over R from Lemma 10.39.6. Since we have

$$\begin{aligned} & R[x_1, \ldots, x_n]_{\mathfrak q'}/(f_1, \ldots, f_i) \xrightarrow{f_{i+1}} R[x_1, \ldots, x_n]_{\mathfrak q'}/(f_1, \ldots, f_i) \\ = \mathrm{colim} & \left(R_\lambda[x_1, \ldots, x_n]_{\mathfrak q'_\lambda}/(f_1, \ldots, f_i) \xrightarrow{f_{i+1}} R_\lambda[x_1, \ldots, x_n]_{\mathfrak q'_\lambda}/(f_1, \ldots, f_i) \right) \end{aligned}$$

and since filtered colimits are exact (Lemma 10.8.8) we conclude that we have (1). ☐

00SW Lemma 10.136.13. A relative global complete intersection is syntomic, i.e., flat.

Proof. Let $R \to S$ be a relative global complete intersection. The fibres are global complete intersections, and S is of finite presentation over R. Thus the only thing to prove is that $R \to S$ is flat. This is true by (2) of Lemma 10.136.12. ☐

03HS Lemma 10.136.14. Suppose that A is a ring, and $P(x) = x^n + b_1 x^{n-1} + \ldots + b_n \in A[x]$ is a monic polynomial over A. Then there exists a syntomic, finite locally free, faithfully flat ring extension $A \subset A'$ such that $P(x) = \prod_{i=1,\ldots,n}(x - \beta_i)$ for certain $\beta_i \in A'$.

Proof. Take $A' = A \otimes_R S$, where R and S are as in Example 10.136.8, where $R \to A$ maps a_i to b_i, and let $\beta_i = -1 \otimes \alpha_i$. Observe that $R \to S$ is syntomic (Lemma 10.136.13), $R \to S$ is finite by construction, and R is Noetherian (so any finite R-module is finitely presented). Hence S is finite locally free as an R-module by Lemma 10.78.2. We omit the verification that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, which shows that S is faithfully flat over R (Lemma 10.39.16). These properties are inherited by the base change $A \to A'$; some details omitted. ☐

00SY Lemma 10.136.15. Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$ of R. The following are equivalent:

(1) There exists an element $g \in S$, $g \notin \mathfrak q$ such that $R \to S_g$ is syntomic.
(2) There exists an element $g \in S$, $g \notin \mathfrak q$ such that S_g is a relative global complete intersection over R.
(3) There exists an element $g \in S$, $g \notin \mathfrak q$, such that $R \to S_g$ is of finite presentation, the local ring map $R_{\mathfrak p} \to S_{\mathfrak q}$ is flat, and the local ring $S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$ is a complete intersection ring over $\kappa(\mathfrak p)$ (see Definition 10.135.5).

Proof. The implication (1) ⇒ (3) is Lemma 10.135.8. The implication (2) ⇒ (1) is Lemma 10.136.13. It remains to show that (3) implies (2).

Assume (3). After replacing S by S_g for some $g \in S$, $g \notin \mathfrak q$ we may assume S is finitely presented over R. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. Let $\mathfrak q' \subset R[x_1, \ldots, x_n]$ be the prime corresponding to $\mathfrak q$. Write $\kappa(\mathfrak p) = k$. Note that $S \otimes_R k = k[x_1, \ldots, x_n]/\overline{I}$ where $\overline{I} \subset k[x_1, \ldots, x_n]$ is the ideal generated by the image of I. Let $\overline{\mathfrak q}' \subset k[x_1, \ldots, x_n]$ be the prime ideal generated by the image of $\mathfrak q'$. By Lemma 10.135.8 the equivalent conditions of Lemma 10.135.4 hold for $\overline{I}$ and $\overline{\mathfrak q}'$. Say the dimension of $\overline{I}_{\overline{\mathfrak q}'}/\overline{\mathfrak q}'\overline{I}_{\overline{\mathfrak q}'}$ over $\kappa(\overline{\mathfrak q}')$ is c. Pick $f_1, \ldots, f_c \in I$ mapping to a basis of this vector space. The images $\overline{f}_j \in \overline{I}$ generate $\overline{I}_{\overline{\mathfrak q}'}$ (by Lemma 10.135.4). Set $S' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Let J be the kernel of the surjection $S' \to S$. Since S is of finite presentation J is a finitely generated ideal (Lemma 10.6.2). Consider the short exact sequence

$$0 \to J \to S' \to S \to 0$$

As $S_{\mathfrak q}$ is flat over R we see that $J_{\mathfrak q'} \otimes_R k \to S'_{\mathfrak q'} \otimes_R k$ is injective (Lemma 10.39.12). However, by construction $S'_{\mathfrak q'} \otimes_R k$ maps isomorphically to $S_{\mathfrak q} \otimes_R k$. Hence we conclude that $J_{\mathfrak q'} \otimes_R k = J_{\mathfrak q'}/\mathfrak p J_{\mathfrak q'} = 0$. By Nakayama's lemma (Lemma 10.20.1) we conclude that there exists a $g \in R[x_1, \ldots, x_n]$, $g \notin \mathfrak q'$ such that $J_g = 0$. In other words $S'_g \cong S_g$. After further localizing we see that S' (and hence S) becomes a relative global complete intersection by Lemma 10.136.10 as desired. □

07BT Lemma 10.136.16. Let R be a ring. Let $S = R[x_1, \ldots, x_n]/I$ for some finitely generated ideal I. If $g \in S$ is such that S_g is syntomic over R, then $(I/I^2)_g$ is a finite projective S_g-module.

Proof. By Lemma 10.136.15 there exist finitely many elements $g_1, \ldots, g_m \in S$ which generate the unit ideal in S_g such that each S_{gg_j} is a relative global complete intersection over R. Since it suffices to prove that $(I/I^2)_{gg_j}$ is finite projective, see Lemma 10.78.2, we may assume that S_g is a relative global complete intersection. In this case the result follows from Lemmas 10.134.16 and 10.136.12. □

00SZ Lemma 10.136.17. Let $R \to S$, $S \to S'$ be ring maps.

(1) If $R \to S$ and $S \to S'$ are syntomic, then $R \to S'$ is syntomic.
(2) If $R \to S$ and $S \to S'$ are relative global complete intersections, then $R \to S'$ is a relative global complete intersection.

Proof. Proof of (2). Say $R \to S$ and $S \to S'$ are relative global complete intersections and we have presentations $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(h_1, \ldots, h_d)$ as in Definition 10.136.5. Then

$$S' \cong R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, h'_1, \ldots, h'_d)$$

for some lifts $h'_j \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ of the h_j. Hence it suffices to bound the dimensions of the fibre rings. Thus we may assume $R = k$ is a field. In this case we see that we have a ring, namely S, which is of finite type over k and equidimensional of dimension $n - c$, and a finite type ring map $S \to S'$ all of whose nonempty fibre rings are equidimensional of dimension $m - d$. Then, by Lemma 10.112.6 for example applied to localizations at maximal ideals of S', we see that $\dim(S') \leq n - c + m - d$ as desired.

We will reduce part (1) to part (2). Assume $R \to S$ and $S \to S'$ are syntomic. Let $\mathfrak q' \subset S$ be a prime ideal lying over $\mathfrak q \subset S$. By Lemma 10.136.15 there exists a $g' \in S'$, $g' \notin \mathfrak q'$ such that $S \to S'_{g'}$ is a relative global complete intersection. Similarly, we find $g \in S$, $g \notin \mathfrak q$ such that $R \to S_g$ is a relative global complete intersection. By Lemma 10.136.9 the ring map $S_g \to S_{gg'}$ is a relative global complete intersection. By part (2) we see that $R \to S_{gg'}$ is a relative global complete intersection and $gg' \notin \mathfrak q'$. Since $\mathfrak q'$ was arbitrary combining Lemmas 10.136.15 and 10.136.4 we see that $R \to S'$ is syntomic (this also uses that the spectrum of S' is quasi-compact, see Lemma 10.17.10). $\square$

The following lemma will be improved later, see Smoothing Ring Maps, Proposition 16.3.2.

00T0 Lemma 10.136.18. Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a syntomic map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some relative global complete intersection S_i over R.

Proof. By Lemma 10.136.15 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is a relative global complete intersection over R/I. Hence we may assume that $\overline{S}$ is a relative global complete intersection. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 10.136.5. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Note that $S/IS \cong \overline{S}$. By Lemma 10.136.10 we can find $g \in S$ mapping to 1 in $\overline{S}$ such that S_g is a relative global complete intersection over R. Since $\overline{S} \cong S_g/IS_g$ this finishes the proof. $\square$

10.137. Smooth ring maps

00T1 Let us motivate the definition of a smooth ring map by an example. Suppose R is a ring and $S = R[x, y]/(f)$ for some nonzero $f \in R[x, y]$. In this case there is an exact sequence

$$S \to S\mathrm{d}x \oplus S\mathrm{d}y \to \Omega_{S/R} \to 0$$

where the first arrow maps 1 to $\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$ see Section 10.134. We conclude that $\Omega_{S/R}$ is locally free of rank 1 if the partial derivatives of f generate the unit ideal in S. In this case S is smooth of relative dimension 1 over R. But it can happen that $\Omega_{S/R}$ is locally free of rank 2 namely if both partial derivatives of f are zero. For example if for a prime p we have $p = 0$ in R and $f = x^p + y^p$ then this happens. Here $R \to S$ is a relative global complete intersection of relative dimension 1 which is not smooth. Hence, in order to check that a ring map is smooth it is not sufficient to check whether the module of differentials is free. The correct condition is the following.

00T2 Definition 10.137.1. A ring map $R \to S$ is smooth if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module placed in degree 0.

In particular, if $R \to S$ is smooth then the module $\Omega_{S/R}$ is a finite projective S-module. Moreover, by Lemma 10.137.2 the naive cotangent complex of any presentation has the same structure. Thus, for a surjection $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I the map

$$I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$$

is a split injection. In other words $\bigoplus_{i=1}^n S\mathrm{d}x_i \cong I/I^2 \oplus \Omega_{S/R}$ as S-modules. This implies that I/I^2 is a finite projective S-module too!

05GK Lemma 10.137.2. Let $R \to S$ be a ring map of finite presentation. If for some presentation α of S over R the naive cotangent complex $NL(\alpha)$ is quasi-isomorphic to a finite projective S-module placed in degree 0, then this holds for any presentation.

Proof. Immediate from Lemma 10.134.2. □

00T3 Lemma 10.137.3. Let $R \to S$ be a smooth ring map. Any localization S_g is smooth over R. If $f \in R$ maps to an invertible element of S, then $R_f \to S$ is smooth.

Proof. By Lemma 10.134.13 the naive cotangent complex for S_g over R is the base change of the naive cotangent complex of S over R. The assumption is that the naive cotangent complex of S/R is $\Omega_{S/R}$ and that this is a finite projective S-module. Hence so is its base change. Thus S_g is smooth over R.

The second assertion follows in the same way from Lemma 10.134.11. □

00T4 Lemma 10.137.4. Let $R \to S$ be a smooth ring map. Let $R \to R'$ be any ring map. Then the base change $R' \to S' = R' \otimes_R S$ is smooth.

Proof. Let $\alpha : R[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : R'[x_1, \ldots, x_n] \to R' \otimes_R S$ be the induced presentation. Let $I' = \text{Ker}(\alpha')$. Since $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ is exact, the sequence $R' \otimes_R I \to R'[x_1, \ldots, x_n] \to R' \otimes_R S \to 0$ is exact. Thus $R' \otimes_R I \to I'$ is surjective. By Definition 10.137.1 there is a short exact sequence

$$0 \to I/I^2 \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \to \Omega_{S/R} \to 0$$

and the S-module $\Omega_{S/R}$ is finite projective. In particular I/I^2 is a direct summand of $\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$. Consider the commutative diagram

$$\begin{tikzcd} R' \otimes_R (I/I^2) \arrow[r] \arrow[d] & R' \otimes_R (\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S) \arrow[d] \\ I'/(I')^2 \arrow[r] & \Omega_{R'[x_1,\ldots,x_n]/R'} \otimes_{R'[x_1,\ldots,x_n]} (R' \otimes_R S) \end{tikzcd}$$

Since the right vertical map is an isomorphism we see that the left vertical map is injective and surjective by what was said above. Thus we conclude that $NL(\alpha')$ is quasi-isomorphic to $\Omega_{S'/R'} \cong S' \otimes_S \Omega_{S/R}$. And this is finite projective since it is the base change of a finite projective module. □

00T5 Lemma 10.137.5. Let k be a field. Let S be a smooth k-algebra. Then S is a local complete intersection.

Proof. By Lemmas 10.137.4 and 10.135.11 it suffices to prove this when k is algebraically closed. Choose a presentation $\alpha : k[x_1, \ldots, x_n] \to S$ with kernel I. Let $\mathfrak{m}$ be a maximal ideal of S, and let $\mathfrak{m}' \supset I$ be the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. We will show that condition (5) of Lemma 10.135.4 holds (with $\mathfrak{m}$ instead of $\mathfrak{q}$). We may write $\mathfrak{m}' = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in k$, because k is algebraically closed, see Theorem 10.34.1. By our assumption that $k \to S$ is smooth the S-module map $\text{d} : I/I^2 \to \bigoplus_{i=1}^n S\text{d}x_i$ is a split injection. Hence the corresponding map $I/\mathfrak{m}'I \to \bigoplus \kappa(\mathfrak{m}')\text{d}x_i$ is injective. Say $\dim_{\kappa(\mathfrak{m}')}(I/\mathfrak{m}'I) = c$ and pick $f_1, \ldots, f_c \in I$ which map to a $\kappa(\mathfrak{m}')$-basis of $I/\mathfrak{m}'I$. By Nakayama's Lemma 10.20.1 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak{m}'}$ over $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$. Consider the commutative diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d] & I/I^2 \arrow[r] \arrow[d] & I/\mathfrak m'I \arrow[d] \\ \Omega_{k[x_1, \ldots, x_n]/k} \arrow[r] & \bigoplus S\text{d}x_i \arrow[r, "\text{d}x_i \mapsto x_i - a_i"] & \mathfrak m'/(\mathfrak m')^2 \end{tikzcd}$$

(proof commutativity omitted). The middle vertical map is the one defining the naive cotangent complex of α. Note that the right lower horizontal arrow induces an isomorphism $\bigoplus \kappa(\mathfrak m')\text{d}x_i \to \mathfrak m'/(\mathfrak m')^2$. Hence our generators $f_1, \ldots, f_c$ of $I_{\mathfrak m'}$ map to a collection of elements in $k[x_1, \ldots, x_n]_{\mathfrak m'}$ whose classes in $\mathfrak m'/(\mathfrak m')^2$ are linearly independent over $\kappa(\mathfrak m')$. Therefore they form a regular sequence in the ring $k[x_1, \ldots, x_n]_{\mathfrak m'}$ by Lemma 10.106.3. This verifies condition (5) of Lemma 10.135.4 hence S_g is a global complete intersection over k for some $g \in S$, $g \notin \mathfrak m$. As this works for any maximal ideal of S we conclude that S is a local complete intersection over k. □

00T6 Definition 10.137.6. Let R be a ring. Given integers $n \geq c \geq 0$ and $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ we say

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

is a standard smooth algebra over R if the polynomial

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

maps to an invertible element in S.

00T7 Lemma 10.137.7. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R[x_1, \ldots, x_n]/I$ be a standard smooth algebra. Then

(1) the ring map $R \to S$ is smooth,
(2) the S-module $\Omega_{S/R}$ is free on $\text{d}x_{c+1}, \ldots, \text{d}x_n$,
(3) the S-module I/I^2 is free on the classes of $f_1, \ldots, f_c$,
(4) for any $g \in S$ the ring map $R \to S_g$ is standard smooth,
(5) for any ring map $R \to R'$ the base change $R' \to R' \otimes_R S$ is standard smooth,
(6) if $f \in R$ maps to an invertible element in S, then $R_f \to S$ is standard smooth, and
(7) the ring S is a relative global complete intersection over R.

Proof. Consider the naive cotangent complex of the given presentation

$$(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \longrightarrow \bigoplus\nolimits_{i=1}^n S\text{d}x_i$$

Let us compose this map with the projection onto the first c direct summands of the direct sum. According to the definition of a standard smooth algebra the classes $f_i \bmod (f_1, \ldots, f_c)^2$ map to a basis of $\bigoplus_{i=1}^c S\text{d}x_i$. We conclude that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free of rank c with a basis given by the elements f_i mod $(f_1, \ldots, f_c)^2$, and that the homology in degree 0, i.e., $\Omega_{S/R}$, of the naive cotangent complex is a free S-module with basis the images of $\text{d}x_{c+j}$, $j = 1, \ldots, n - c$. In particular, this proves $R \to S$ is smooth.

The proofs of (4) and (6) are omitted. But see the example below and the proof of Lemma 10.136.9.

Let $\varphi : R \to R'$ be any ring map. Denote $S' = R'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_c^\varphi)$ where f^φ is the polynomial obtained from $f \in R[x_1, \ldots, x_n]$ by applying φ to all the coefficients. Then $S' \cong R' \otimes_R S$. Moreover, the determinant of Definition 10.137.6 for S'/R' is equal to g^φ. Its image in S' is therefore the image of g via $R[x_1, \ldots, x_n] \to S \to S'$ and hence invertible. This proves (5).

To prove (7) it suffices to show that $S \otimes_R \kappa(\mathfrak p)$ has dimension $n - c$ for every prime $\mathfrak p \subset R$. By (5) it suffices to prove that any standard smooth algebra $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over a field k has dimension $n - c$. We already know that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a local complete intersection by Lemma 10.137.5. Hence, since I/I^2 is free of rank c we see that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ has dimension $n - c$, by Lemma 10.135.4 for example. □

00T8 Example 10.137.8. Let R be a ring. Let $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$. Let

$$h = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}.$$

Set $S = R[x_1, \ldots, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}h - 1)$. This is an example of a standard smooth algebra, except that the presentation is wrong and the variables should be in the following order: $x_1, \ldots, x_c, x_{n+1}, x_{c+1}, \ldots, x_n$.

00T9 Lemma 10.137.9. A composition of standard smooth ring maps is standard smooth.

Proof. Suppose that $R \to S$ and $S \to S'$ are standard smooth. We choose presentations $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$. Choose elements $g_j' \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to the g_j. In this way we see $S' = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, g_1', \ldots, g_d')$. To show that S' is standard smooth it suffices to verify that the determinant

$$\det \begin{pmatrix} \partial f_1/\partial x_1 & \ldots & \partial f_c/\partial x_1 & \partial g_1/\partial x_1 & \ldots & \partial g_d/\partial x_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \ldots & \partial f_c/\partial x_c & \partial g_1/\partial x_c & \ldots & \partial g_d/\partial x_c \\ 0 & \ldots & 0 & \partial g_1/\partial y_1 & \ldots & \partial g_d/\partial y_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \partial g_1/\partial y_d & \ldots & \partial g_d/\partial y_d \end{pmatrix}$$

is invertible in S'. This is clear since it is the product of the two determinants which were assumed to be invertible by hypothesis. □

00TA Lemma 10.137.10. Let $R \to S$ be a smooth ring map. There exists an open covering of $\mathrm{Spec}(S)$ by standard opens $D(g)$ such that each S_g is standard smooth over R. In particular $R \to S$ is syntomic.

Proof. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel $I = (f_1, \ldots, f_m)$. For every subset $E \subset \{1, \ldots, m\}$ consider the open subset U_E where the classes $f_e, e \in E$ freely generate the finite projective S-module I/I^2, see Lemma 10.79.4. We may cover $\mathrm{Spec}(S)$ by standard opens $D(g)$ each completely contained in one of the opens U_E. For such a g we look at the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \longrightarrow S_g$$

mapping x_{n+1} to $1/g$. Setting $J = \text{Ker}(\beta)$ we use Lemma 10.134.12 to see that $J/J^2 \cong (I/I^2)_g \oplus S_g$ is free. We may and do replace S by S_g. Then using Lemma 10.136.6 we may assume we have a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel $I = (f_1, \ldots, f_c)$ such that I/I^2 is free on the classes of $f_1, \ldots, f_c$.

Using the presentation α obtained at the end of the previous paragraph, we more or less repeat this argument with the basis elements $\text{d}x_1, \ldots, \text{d}x_n$ of $\Omega_{R[x_1,\ldots,x_n]/R}$. Namely, for any subset $E \subset \{1, \ldots, n\}$ of cardinality c we may consider the open subset U_E of $\text{Spec}(S)$ where the differential of $NL(\alpha)$ composed with the projection

$$S^{\oplus c} \cong I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \longrightarrow \bigoplus\nolimits_{i \in E} S\text{d}x_i$$

is an isomorphism. Again we may find a covering of $\text{Spec}(S)$ by (finitely many) standard opens $D(g)$ such that each $D(g)$ is completely contained in one of the opens U_E. By renumbering, we may assume $E = \{1, \ldots, c\}$. For a g with $D(g) \subset U_E$ we look at the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \to S_g$$

mapping x_{n+1} to $1/g$. Setting $J = \text{Ker}(\beta)$ we conclude from Lemma 10.134.12 that $J = (f_1, \ldots, f_c, fx_{n+1} - 1)$ where $\alpha(f) = g$ and that the composition

$$J/J^2 \longrightarrow \Omega_{R[x_1,\ldots,x_{n+1}]/R} \otimes_{R[x_1,\ldots,x_{n+1}]} S_g \longrightarrow \bigoplus\nolimits_{i=1}^{c} S_g\text{d}x_i \oplus S_g\text{d}x_{n+1}$$

is an isomorphism. Reordering the coordinates as $x_1, \ldots, x_c, x_{n+1}, x_{c+1}, \ldots, x_n$ we conclude that S_g is standard smooth over R as desired.

This finishes the proof as standard smooth algebras are syntomic (Lemmas 10.137.7 and 10.136.13) and being syntomic over R is local on S (Lemma 10.136.4). □

00TB Definition 10.137.11. Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S. We say $R \to S$ is smooth at $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth.

For ring maps of finite presentation we can characterize this as follows.

07BU Lemma 10.137.12. Let $R \to S$ be of finite presentation. Let $\mathfrak{q}$ be a prime of S. The following are equivalent

(1) $R \to S$ is smooth at $\mathfrak{q}$,
(2) $H_1(L_{S/R})_\mathfrak{q} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a finite free $S_\mathfrak{q}$-module,
(3) $H_1(L_{S/R})_\mathfrak{q} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a projective $S_\mathfrak{q}$-module, and
(4) $H_1(L_{S/R})_\mathfrak{q} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a flat $S_\mathfrak{q}$-module.

Proof. We will use without further mention that formation of the naive cotangent complex commutes with localization, see Section 10.134, especially Lemma 10.134.13. Note that $\Omega_{S/R}$ is a finitely presented S-module, see Lemma 10.131.15. Hence (2), (3), and (4) are equivalent by Lemma 10.78.2. It is clear that (1) implies the equivalent conditions (2), (3), and (4). Assume (2) holds. Writing $S_\mathfrak{q}$ as the colimit of principal localizations we see from Lemma 10.127.6 that we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $(\Omega_{S/R})_g$ is finite free. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I. We may work with $NL(\alpha)$ instead of $NL_{S/R}$, see Lemma 10.134.2. The surjection

$$\Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S \to \Omega_{S/R} \to 0$$

has a right inverse after inverting g because $(\Omega_{S/R})_g$ is projective. Hence the image of $\text{d} : (I/I^2)_g \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S_g$ is a direct summand and this map has a right inverse too. We conclude that $H_1(L_{S/R})_g$ is a quotient of $(I/I^2)_g$. In particular $H_1(L_{S/R})_g$ is a finite S_g-module. Thus the vanishing of $H_1(L_{S/R})_{\mathfrak q}$ implies the vanishing of $H_1(L_{S/R})_{gg'}$ for some $g' \in S$, $g' \notin \mathfrak q$. Then $R \to S_{gg'}$ is smooth by definition. $\square$

00TC Lemma 10.137.13. Let $R \to S$ be a ring map. Then $R \to S$ is smooth if and only if $R \to S$ is smooth at every prime $\mathfrak q$ of S.

Proof. The direct implication is trivial. Suppose that $R \to S$ is smooth at every prime $\mathfrak q$ of S. Since $\text{Spec}(S)$ is quasi-compact, see Lemma 10.17.10, there exists a finite covering $\text{Spec}(S) = \bigcup D(g_i)$ such that each S_{g_i} is smooth. By Lemma 10.23.3 this implies that S is of finite presentation over R. According to Lemma 10.134.13 we see that $NL_{S/R} \otimes_S S_{g_i}$ is quasi-isomorphic to a finite projective S_{g_i}-module. By Lemma 10.78.2 this implies that $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module. $\square$

00TD Lemma 10.137.14. A composition of smooth ring maps is smooth.

Proof. You can prove this in many different ways. One way is to use the snake lemma (Lemma 10.4.1), the Jacobi-Zariski sequence (Lemma 10.134.4), combined with the characterization of projective modules as being direct summands of free modules (Lemma 10.77.2). Another proof can be obtained by combining Lemmas 10.137.10, 10.137.9 and 10.137.13. $\square$

0GIF Lemma 10.137.15. Let R be a ring. Let $S = S' \times S''$ be a product of R-algebras. Then S is smooth over R if and only if both S' and S'' are smooth over R.

Proof. Omitted. Hints: By Lemma 10.137.13 we can check smoothness one prime at a time. Since $\text{Spec}(S)$ is the disjoint union of $\text{Spec}(S')$ and $\text{Spec}(S'')$ by Lemma 10.21.2 we find that smoothness of $R \to S$ at $\mathfrak q$ corresponds to either smoothness of $R \to S'$ at the corresponding prime or smoothness of $R \to S''$ at the corresponding prime. $\square$

00TE Lemma 10.137.16. Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Let $\mathfrak q \subset S$ be a prime. Then $R \to S$ is smooth at $\mathfrak q$ if and only if there exists a subset $I \subset \{1, \ldots, n\}$ of cardinality c such that the polynomial

$$g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i\in I}.$$

does not map to an element of $\mathfrak q$.

Proof. By Lemma 10.136.12 we see that the naive cotangent complex associated to the given presentation of S is the complex

$$\bigoplus\nolimits_{j=1}^{c} S \cdot f_j \longrightarrow \bigoplus\nolimits_{i=1}^{n} S \cdot \text{d}x_i, \quad f_j \longmapsto \sum \frac{\partial f_j}{\partial x_i}\text{d}x_i.$$

The maximal minors of the matrix giving the map are exactly the polynomials g_I.

Assume g_I maps to $g \in S$, with $g \notin \mathfrak q$. Then the algebra S_g is smooth over R. Namely, its naive cotangent complex is quasi-isomorphic to the complex above localized at g, see Lemma 10.134.13. And by construction it is quasi-isomorphic to a free rank $n - c$ module in degree 0.

Conversely, suppose that all g_I end up in $\mathfrak q$. In this case the complex above tensored with $\kappa(\mathfrak q)$ does not have maximal rank, and hence there is no localization by an element $g \in S$, $g \notin \mathfrak q$ where this map becomes a split injection. By Lemma 10.134.13 again there is no such localization which is smooth over R. $\square$

00TF Lemma 10.137.17. Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$ of R. Assume

(1) there exists a $g \in S$, $g \notin \mathfrak q$ such that $R \to S_g$ is of finite presentation,
(2) the local ring homomorphism $R_{\mathfrak p} \to S_{\mathfrak q}$ is flat,
(3) the fibre $S \otimes_R \kappa(\mathfrak p)$ is smooth over $\kappa(\mathfrak p)$ at the prime corresponding to $\mathfrak q$.

Then $R \to S$ is smooth at $\mathfrak q$.

Proof. By Lemmas 10.136.15 and 10.137.5 we see that there exists a $g \in S$ such that S_g is a relative global complete intersection. Replacing S by S_g we may assume $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection. For any subset $I \subset \{1, \ldots, n\}$ of cardinality c consider the polynomial $g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c, i \in I}$ of Lemma 10.137.16. Note that the image $\overline{g}_I$ of g_I in the polynomial ring $\kappa(\mathfrak p)[x_1, \ldots, x_n]$ is the determinant of the partial derivatives of the images $\overline{f}_j$ of the f_j in the ring $\kappa(\mathfrak p)[x_1, \ldots, x_n]$. Thus the lemma follows by applying Lemma 10.137.16 both to $R \to S$ and to $\kappa(\mathfrak p) \to S \otimes_R \kappa(\mathfrak p)$. $\square$

Note that the sets U, V in the following lemma are open by definition.

00TG Lemma 10.137.18. Let $R \to S$ be a ring map of finite presentation. Let $R \to R'$ be a flat ring map. Denote $S' = R' \otimes_R S$ the base change. Let $U \subset \text{Spec}(S)$ be the set of primes at which $R \to S$ is smooth. Let $V \subset \text{Spec}(S')$ the set of primes at which $R' \to S'$ is smooth. Then V is the inverse image of U under the map $f : \text{Spec}(S') \to \text{Spec}(S)$.

Proof. By Lemma 10.134.8 we see that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This already implies that $f^{-1}(U) \subset V$.

Let $\mathfrak q' \subset S'$ be a prime lying over $\mathfrak q \subset S$. Assume $\mathfrak q' \in V$. We have to show that $\mathfrak q \in U$. Since $S \to S'$ is flat, we see that $S_{\mathfrak q} \to S'_{\mathfrak q'}$ is faithfully flat (Lemma 10.39.17). Thus the vanishing of $H_1(L_{S'/R'})_{\mathfrak q'}$ implies the vanishing of $H_1(L_{S/R})_{\mathfrak q}$. By Lemma 10.78.6 applied to the $S_{\mathfrak q}$-module $(\Omega_{S/R})_{\mathfrak q}$ and the map $S_{\mathfrak q} \to S'_{\mathfrak q'}$ we see that $(\Omega_{S/R})_{\mathfrak q}$ is projective. Hence $R \to S$ is smooth at $\mathfrak q$ by Lemma 10.137.12. $\square$

02UQ Lemma 10.137.19. Let K/k be a field extension. Let S be a finite type algebra over k. Let $\mathfrak q_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak q$ be the corresponding prime of S. Then S is smooth over k at $\mathfrak q$ if and only if S_K is smooth at $\mathfrak q_K$ over K.

Proof. This is a special case of Lemma 10.137.18. $\square$

04B1 Lemma 10.137.20. Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a smooth ring map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some (standard) smooth ring S_i over R.

Proof. By Lemma 10.137.10 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is standard smooth over R/I. Hence we may assume that $\overline{S}$ is standard smooth over R/I. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 10.137.6. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_j}{\partial x_i})_{i,j=1,\ldots,c}$ as in Example 10.137.8. This proves the lemma. $\square$

10.138. Formally smooth maps

00TH In this section we define formally smooth ring maps. It will turn out that a ring map of finite presentation is formally smooth if and only if it is smooth, see Proposition 10.138.13.

00TI **Definition 10.138.1.** Let $R \to S$ be a ring map. We say S is *formally smooth over R* if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

00TJ **Lemma 10.138.2.** Let $R \to S$ be a formally smooth ring map. Let $R \to R'$ be any ring map. Then the base change $S' = R' \otimes_R S$ is formally smooth over R'.

Proof. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rrd, dashed] & R' \otimes_R S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[r] \arrow[u] & R' \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

as in Definition 10.138.1 be given. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

031H **Lemma 10.138.3.** A composition of formally smooth ring maps is formally smooth.

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) □

00TK **Lemma 10.138.4.** A polynomial ring over R is formally smooth over R.

Proof. Suppose we have a diagram as in Definition 10.138.1 with $S = R[x_j; j \in J]$. Then there exists a dotted arrow simply by choosing lifts $a_j \in A$ of the elements in A/I to which the elements x_j map to under the top horizontal arrow. □

00TL **Lemma 10.138.5.** Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if there exists an R-algebra map $\sigma : S \to P/J^2$ which is a right inverse to the surjection $P/J^2 \to S$.

Proof. Assume $R \to S$ is formally smooth. Consider the commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & P/J \\ R \arrow[r] \arrow[u] & P/J^2 \arrow[u] \end{tikzcd}$$

By assumption the dotted arrow exists. This proves that σ exists.

Conversely, suppose we have a σ as in the lemma. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 10.138.1 be given. Because P is formally smooth by Lemma 10.138.4, there exists an R-algebra homomorphism $\psi : P \to A$ which lifts the map $P \to S \to A/I$. Clearly $\psi(J) \subset I$ and since $I^2 = 0$ we conclude that $\psi(J^2) = 0$. Hence ψ factors as $\overline{\psi} : P/J^2 \to A$. The desired dotted arrow is the composition $\overline{\psi} \circ \sigma : S \to A$. □

00TM Remark 10.138.6. Lemma 10.138.5 holds more generally whenever P is formally smooth over R.

031I Lemma 10.138.7. Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if the sequence

$$0 \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

of Lemma 10.131.9 is a split exact sequence.

Proof. Assume S is formally smooth over R. By Lemma 10.138.5 this means there exists an R-algebra map $S \to P/J^2$ which is a right inverse to the canonical map $P/J^2 \to S$. By Lemma 10.131.11 we have $\Omega_{P/R} \otimes_P S = \Omega_{(P/J^2)/R} \otimes_{P/J^2} S$. By Lemma 10.131.10 the sequence is split.

Assume the exact sequence of the lemma is split exact. Choose a splitting $\sigma : \Omega_{S/R} \to \Omega_{P/R} \otimes_P S$. For each $\lambda \in S$ choose $x_\lambda \in P$ which maps to λ. Next, for each $\lambda \in S$ choose $f_\lambda \in J$ such that

$$\mathrm{d}f_\lambda = \mathrm{d}x_\lambda - \sigma(\mathrm{d}\lambda)$$

in the middle term of the exact sequence. We claim that $s : \lambda \mapsto x_\lambda - f_\lambda \mod J^2$ is an R-algebra homomorphism $s : S \to P/J^2$. To prove this we will repeatedly use that if $h \in J$ and $\mathrm{d}h = 0$ in $\Omega_{P/R} \otimes_R S$, then $h \in J^2$. Let $\lambda, \mu \in S$. Then $\sigma(\mathrm{d}\lambda + \mathrm{d}\mu - \mathrm{d}(\lambda + \mu)) = 0$. This implies

$$\mathrm{d}(x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu}) = 0$$

which means that $x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu} \in J^2$, which in turn means that $s(\lambda) + s(\mu) = s(\lambda + \mu)$. Similarly, we have $\sigma(\lambda\mathrm{d}\mu + \mu\mathrm{d}\lambda - \mathrm{d}\lambda\mu) = 0$ which implies that

$$\mu(\mathrm{d}x_\lambda - \mathrm{d}f_\lambda) + \lambda(\mathrm{d}x_\mu - \mathrm{d}f_\mu) - \mathrm{d}x_{\lambda\mu} + \mathrm{d}f_{\lambda\mu} = 0$$

in the middle term of the exact sequence. Moreover we have

$$\mathrm{d}(x_\lambda x_\mu) = x_\lambda \mathrm{d}x_\mu + x_\mu \mathrm{d}x_\lambda = \lambda \mathrm{d}x_\mu + \mu \mathrm{d}x_\lambda$$

in the middle term again. Combined these equations mean that $x_\lambda x_\mu - x_{\lambda\mu} - \mu f_\lambda - \lambda f_\mu + f_{\lambda\mu} \in J^2$, hence $(x_\lambda - f_\lambda)(x_\mu - f_\mu) - (x_{\lambda\mu} - f_{\lambda\mu}) \in J^2$ as $f_\lambda f_\mu \in J^2$, which means that $s(\lambda)s(\mu) = s(\lambda\mu)$. If $\lambda \in R$, then $\mathrm{d}\lambda = 0$ and we see that $\mathrm{d}f_\lambda = \mathrm{d}x_\lambda$, hence $\lambda - x_\lambda + f_\lambda \in J^2$ and hence $s(\lambda) = \lambda$ as desired. At this point we can apply Lemma 10.138.5 to conclude that S/R is formally smooth. □

031J Proposition 10.138.8. Let $R \to S$ be a ring map. Consider a formally smooth R-algebra P and a surjection $P \to S$ with kernel J. The following are equivalent

(1) S is formally smooth over R,
(2) for some $P \to S$ as above there exists a section to $P/J^2 \to S$,
(3) for all $P \to S$ as above there exists a section to $P/J^2 \to S$,
(4) for some $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact,
(5) for all $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact, and
(6) the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a projective S-module placed in degree 0.

Proof. It is clear that (1) implies (3) implies (2), see first part of the proof of Lemma 10.138.5. It is also true that (3) implies (5) implies (4) and that (2) implies (4), see first part of the proof of Lemma 10.138.7. Finally, Lemma 10.138.7 applied to the canonical surjection $R[S] \to S$ (10.134.0.1) shows that (1) implies (6).

Assume (4) and let's prove (6). Consider the sequence of Lemma 10.134.4 associated to the ring maps $R \to P \to S$. By the implication (1) $\Rightarrow$ (6) proved above we see that $NL_{P/R} \otimes_R S$ is quasi-isomorphic to $\Omega_{P/R} \otimes_P S$ placed in degree 0. Hence $H_1(NL_{P/R} \otimes_P S) = 0$. Since $P \to S$ is surjective we see that $NL_{S/P}$ is homotopy equivalent to J/J^2 placed in degree 1 (Lemma 10.134.6). Thus we obtain the exact sequence $0 \to H_1(L_{S/R}) \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$. By assumption we see that $H_1(L_{S/R}) = 0$ and that $\Omega_{S/R}$ is a projective S-module. Thus (6) follows.

Finally, let's prove that (6) implies (1). The assumption means that the complex $J/J^2 \to \Omega_{P/R} \otimes S$ where $P = R[S]$ and $P \to S$ is the canonical surjection (10.134.0.1). Hence Lemma 10.138.7 shows that S is formally smooth over R. ☐

031K Lemma 10.138.9. Let $A \to B \to C$ be ring maps. Assume $B \to C$ is formally smooth. Then the sequence

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 10.131.7 is a split short exact sequence.

Proof. Follows from Proposition 10.138.8 and Lemma 10.134.4. ☐

06A6 Lemma 10.138.10. Let $A \to B \to C$ be ring maps with $A \to C$ formally smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 10.131.9 is split exact.

Proof. Follows from Proposition 10.138.8, Lemma 10.134.4, and Lemma 10.131.9. ☐

06A7 Lemma 10.138.11. Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ formally smooth. Denote $I = \text{Ker}(A \to C)$ and $J = \text{Ker}(B \to C)$. Then the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 10.134.7 is split exact.

Proof. Since $A \to B$ is formally smooth there exists a ring map $\sigma : B \to A/I^2$ whose composition with $A \to B$ equals the quotient map $A \to A/I^2$. Then σ induces a map $J/J^2 \to I/I^2$ which is inverse to the map $I/I^2 \to J/J^2$. $\square$

031L **Lemma 10.138.12.** Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Assume

(1) $I^2 = 0$,
(2) $R \to S$ is flat, and
(3) $R/I \to S/IS$ is formally smooth.

Then $R \to S$ is formally smooth.

Proof. Assume (1), (2) and (3). Let $P = R[\{x_t\}_{t\in T}] \to S$ be a surjection of R-algebras with kernel J. Thus $0 \to J \to P \to S \to 0$ is a short exact sequence of flat R-modules. This implies that $I \otimes_R S = IS$, $I \otimes_R P = IP$ and $I \otimes_R J = IJ$ as well as $J \cap IP = IJ$. We will use throughout the proof that

$$\Omega_{(S/IS)/(R/I)} = \Omega_{S/R} \otimes_S (S/IS) = \Omega_{S/R} \otimes_R R/I = \Omega_{S/R}/I\Omega_{S/R}$$

and similarly for P (see Lemma 10.131.12). By Lemma 10.138.7 the sequence

031M $$0 \to J/(IJ + J^2) \to \Omega_{P/R} \otimes_P S/IS \to \Omega_{S/R} \otimes_S S/IS \to 0 \quad (10.138.12.1)$$

is split exact. Of course the middle term is $\bigoplus_{t\in T} S/IS\mathrm{d}x_t$. Choose a splitting $\sigma : \Omega_{P/R} \otimes_P S/IS \to J/(IJ + J^2)$. For each $t \in T$ choose an element $f_t \in J$ which maps to $\sigma(\mathrm{d}x_t)$ in $J/(IJ + J^2)$. This determines a unique S-module map

$$\tilde\sigma : \Omega_{P/R} \otimes_R S = \bigoplus S\mathrm{d}x_t \longrightarrow J/J^2$$

with the property that $\tilde\sigma(\mathrm{d}x_t) = f_t$. As σ is a section to d the difference

$$\Delta = \mathrm{id}_{J/J^2} - \tilde\sigma \circ \mathrm{d}$$

is a self map $J/J^2 \to J/J^2$ whose image is contained in $(IJ+J^2)/J^2$. In particular $\Delta((IJ + J^2)/J^2) = 0$ because $I^2 = 0$. This means that Δ factors as

$$J/J^2 \to J/(IJ + J^2) \xrightarrow{\overline{\Delta}} (IJ + J^2)/J^2 \to J/J^2$$

where $\overline{\Delta}$ is a S/IS-module map. Using again that the sequence (10.138.12.1) is split, we can find a S/IS-module map $\overline{\delta} : \Omega_{P/R} \otimes_P S/IS \to (IJ + J^2)/J^2$ such that $\overline{\delta} \circ d$ is equal to $\overline{\Delta}$. In the same manner as above the map $\overline{\delta}$ determines an S-module map $\delta : \Omega_{P/R} \otimes_P S \to J/J^2$. After replacing $\tilde\sigma$ by $\tilde\sigma + \delta$ a simple computation shows that $\Delta = 0$. In other words $\tilde\sigma$ is a section of $J/J^2 \to \Omega_{P/R} \otimes_P S$. By Lemma 10.138.7 we conclude that $R \to S$ is formally smooth. $\square$

00TN **Proposition 10.138.13.** Let $R \to S$ be a ring map. The following are equivalent

(1) $R \to S$ is of finite presentation and formally smooth,
(2) $R \to S$ is smooth.

Proof. Follows from Proposition 10.138.8 and Definition 10.137.1. (Note that $\Omega_{S/R}$ is a finitely presented S-module if $R \to S$ is of finite presentation, see Lemma 10.131.15.) $\square$

00TP **Lemma 10.138.14.** Let $R \to S$ be a smooth ring map. Then there exists a subring $R_0 \subset R$ of finite type over $\mathbf{Z}$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$.

Proof. We are going to use that smooth is equivalent to finite presentation and formally smooth, see Proposition 10.138.13. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and denote $I = (f_1, \ldots, f_m)$. Choose a right inverse $\sigma : S \to R[x_1, \ldots, x_n]/I^2$ to the projection to S as in Lemma 10.138.5. Choose $h_i \in R[x_1, \ldots, x_n]$ such that $\sigma(x_i \bmod I) = h_i \bmod I^2$. The fact that σ is an R-algebra homomorphism $R[x_1, \ldots, x_n]/I \to R[x_1, \ldots, x_n]/I^2$ is equivalent to the condition that

$$f_j(h_1, \ldots, h_n) = \sum\nolimits_{j_1 j_2} a_{j_1 j_2} f_{j_1} f_{j_2}$$

for certain $a_{kl} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated over $\mathbf{Z}$ by all the coefficients of the polynomials f_j, h_i, a_{kl}. Set $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, with $I_0 = (f_1, \ldots, f_m)$. Let $\sigma_0 : S_0 \to R_0[x_1, \ldots, x_n]/I_0^2$ defined by the rule $x_i \mapsto h_i \bmod I_0^2$; this works since the a_{lk} are defined over R_0 and satisfy the same relations. Thus by Lemma 10.138.5 the ring S_0 is formally smooth over R_0. $\square$

0CAQ **Lemma 10.138.15.** Let $A = \operatorname{colim} A_i$ be a filtered colimit of rings. Let $A \to B$ be a smooth ring map. There exists an i and a smooth ring map $A_i \to B_i$ such that $B = B_i \otimes_{A_i} A$.

Proof. Follows from Lemma 10.138.14 since $R_0 \to A$ will factor through A_i for some i by Lemma 10.127.3. $\square$

06CM **Lemma 10.138.16.** Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = S \otimes_R R'$. Then $R \to S$ is formally smooth if and only if $R' \to S'$ is formally smooth.

Proof. If $R \to S$ is formally smooth, then $R' \to S'$ is formally smooth by Lemma 10.138.2. To prove the converse, assume $R' \to S'$ is formally smooth. Note that $N \otimes_R R' = N \otimes_S S'$ for any S-module N. In particular $S \to S'$ is faithfully flat also. Choose a polynomial ring $P = R[\{x_i\}_{i \in I}]$ and a surjection of R-algebras $P \to S$ with kernel J. Note that $P' = P \otimes_R R'$ is a polynomial algebra over R'. Since $R \to R'$ is flat the kernel J' of the surjection $P' \to S'$ is $J \otimes_R R'$. Hence the split exact sequence (see Lemma 10.138.7)

$$0 \to J'/(J')^2 \to \Omega_{P'/R'} \otimes_{P'} S' \to \Omega_{S'/R'} \to 0$$

is the base change via $S \to S'$ of the corresponding sequence

$$J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

see Lemma 10.131.9. As $S \to S'$ is faithfully flat we conclude two things: (1) this sequence (without $'$) is exact too, and (2) $\Omega_{S/R}$ is a projective S-module. Namely, $\Omega_{S'/R'}$ is projective as a direct sum of the free module $\Omega_{P'/R'} \otimes_{P'} S'$ and $\Omega_{S/R} \otimes_S S' = \Omega_{S'/R'}$ by what we said above. Thus (2) follows by descent of projectivity through faithfully flat ring maps, see Theorem 10.95.6. Hence the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$ is exact also and we win by applying Lemma 10.138.7 once more. $\square$

It turns out that smooth ring maps satisfy the following strong lifting property.

07K4 **Lemma 10.138.17.** Let $R \to S$ be a smooth ring map. Given a commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is a locally nilpotent ideal, a dotted arrow exists which makes the diagram commute.

Proof. By Lemma 10.138.14 we can extend the diagram to a commutative diagram

$$\begin{tikzcd} S_0 \arrow[r] & S \arrow[r] \arrow[rd, dashed] & A/I \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with $R_0 \to S_0$ smooth, R_0 of finite type over $\mathbf{Z}$, and $S = S_0 \otimes_{R_0} R$. Let $x_1, \ldots, x_n \in S_0$ be generators of S_0 over R_0. Let $a_1, \ldots, a_n$ be elements of A which map to the same elements in A/I as the elements $x_1, \ldots, x_n$. Denote $A_0 \subset A$ the subring generated by the image of R_0 and the elements $a_1, \ldots, a_n$. Set $I_0 = A_0 \cap I$. Then $A_0/I_0 \subset A/I$ and $S_0 \to A/I$ maps into A_0/I_0. Thus it suffices to find the dotted arrow in the diagram

$$\begin{tikzcd} S_0 \arrow[r] \arrow[rd, dashed] & A_0/I_0 \\ R_0 \arrow[u] \arrow[r] & A_0 \arrow[u] \end{tikzcd}$$

The ring A_0 is of finite type over $\mathbf{Z}$ by construction. Hence A_0 is Noetherian, whence I_0 is nilpotent, see Lemma 10.32.5. Say $I_0^n = 0$. By Proposition 10.138.13 we can successively lift the R_0-algebra map $S_0 \to A_0/I_0$ to $S_0 \to A_0/I_0^2$, $S_0 \to A_0/I_0^3$, ..., and finally $S_0 \to A_0/I_0^n = A_0$. □

10.139. Smoothness and differentials

05D4 Some results on differentials and smooth ring maps.

04B2 **Lemma 10.139.1.** Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 10.131.7 is exact.

Proof. This follows from the more general Lemma 10.138.9 because a smooth ring map is formally smooth, see Proposition 10.138.13. But it also follows directly from Lemma 10.134.4 since $H_1(L_{C/B}) = 0$ is part of the definition of smoothness of $B \to C$. □

06A8 **Lemma 10.139.2.** Let $A \to B \to C$ be ring maps with $A \to C$ smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 10.131.9 is split exact.

Proof. This follows from the more general Lemma 10.138.10 because a smooth ring map is formally smooth, see Proposition 10.138.13. $\square$

06A9 **Lemma 10.139.3.** Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ smooth. Denote $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(B \to C)$. Then the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 10.134.7 is exact.

Proof. This follows from the more general Lemma 10.138.11 because a smooth ring map is formally smooth, see Proposition 10.138.13. $\square$

05D5 **Lemma 10.139.4.** Let $\varphi : R \to S$ be a smooth ring map. Let $\sigma : S \to R$ be a left inverse to φ. Set $I = \mathrm{Ker}(\sigma)$. Then

(1) I/I^2 is a finite locally free R-module, and
(2) if I/I^2 is free, then $S^\wedge \cong R[[t_1, \ldots, t_d]]$ as R-algebras, where $S^\wedge$ is the I-adic completion of S.

Proof. By Lemma 10.131.10 applied to $R \to S \to R$ we see that $I/I^2 = \Omega_{S/R} \otimes_{S,\sigma} R$. Since by definition of a smooth morphism the module $\Omega_{S/R}$ is finite locally free over S we deduce that (1) holds. If I/I^2 is free, then choose $f_1, \ldots, f_d \in I$ whose images in I/I^2 form an R-basis. Consider the R-algebra map defined by

$$\Psi : R[[x_1, \ldots, x_d]] \longrightarrow S^\wedge, \quad x_i \longmapsto f_i.$$

Denote $P = R[[x_1, \ldots, x_d]]$ and $J = (x_1, \ldots, x_d) \subset P$. We write $\Psi_n : P/J^n \to S/I^n$ for the induced map of quotient rings. Note that $S/I^2 = \varphi(R) \oplus I/I^2$. Thus Ψ_2 is an isomorphism. Denote $\sigma_2 : S/I^2 \to P/J^2$ the inverse of Ψ_2. We will prove by induction on n that for all $n > 2$ there exists an inverse $\sigma_n : S/I^n \to P/J^n$ of Ψ_n. Namely, as S is formally smooth over R (by Proposition 10.138.13) we see that in the solid diagram

$$\begin{tikzcd} S \arrow[r, dotted] \arrow[rd, "\sigma_{n-1}"'] & P/J^n \arrow[d] \\ & P/J^{n-1} \end{tikzcd}$$

of R-algebras we can fill in the dotted arrow by some R-algebra map $\tau : S \to P/J^n$ making the diagram commute. This induces an R-algebra map $\overline{\tau} : S/I^n \to P/J^n$ which is equal to σ_{n-1} modulo J^n. By construction the map Ψ_n is surjective and now $\overline{\tau} \circ \Psi_n$ is an R-algebra endomorphism of P/J^n which maps x_i to $x_i + \delta_{i,n}$ with $\delta_{i,n} \in J^{n-1}/J^n$. It follows that Ψ_n is an isomorphism and hence it has an inverse σ_n. This proves the lemma. $\square$

10.140. Smooth algebras over fields

00TQ Warning: The following two lemmas do not hold over nonperfect fields in general.

00TR **Lemma 10.140.1.** Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. Then

$$\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

Proof. Consider the exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) \to \Omega_{\kappa(\mathfrak{m})/k} \to 0$$

of Lemma 10.131.9. We would like to show that the first map is an isomorphism. Since k is algebraically closed the composition $k \to \kappa(\mathfrak{m})$ is an isomorphism by Theorem 10.34.1. So the surjection $S \to \kappa(\mathfrak{m})$ splits as a map of k-algebras, and Lemma 10.131.10 shows that the sequence above is exact on the left. Since $\Omega_{\kappa(\mathfrak{m})/k} = 0$, we win. □

00TS **Lemma 10.140.2.** Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. The following are equivalent:

(1) The ring $S_{\mathfrak{m}}$ is a regular local ring.
(2) We have $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) \leq \dim(S_{\mathfrak{m}})$.
(3) We have $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim(S_{\mathfrak{m}})$.
(4) There exists a $g \in S$, $g \notin \mathfrak{m}$ such that S_g is smooth over k. In other words S/k is smooth at $\mathfrak{m}$.

Proof. Note that (1), (2) and (3) are equivalent by Lemma 10.140.1 and Definition 10.110.7.

Assume that S is smooth at $\mathfrak{m}$. By Lemma 10.137.10 we see that S_g is standard smooth over k for a suitable $g \in S$, $g \notin \mathfrak{m}$. Hence by Lemma 10.137.7 we see that $\Omega_{S_g/k}$ is free of rank $\dim(S_g)$. Hence by Lemma 10.140.1 we see that $\dim(S_{\mathfrak{m}}) = \dim(\mathfrak{m}/\mathfrak{m}^2)$ in other words $S_{\mathfrak{m}}$ is regular.

Conversely, suppose that $S_{\mathfrak{m}}$ is regular. Let $d = \dim(S_{\mathfrak{m}}) = \dim \mathfrak{m}/\mathfrak{m}^2$. Choose a presentation $S = k[x_1, \ldots, x_n]/I$ such that x_i maps to an element of $\mathfrak{m}$ for all i. In other words, $\mathfrak{m}'' = (x_1, \ldots, x_n)$ is the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Note that we have a short exact sequence

$$I/\mathfrak{m}''I \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}/(\mathfrak{m})^2 \to 0$$

Pick $c = n - d$ elements $f_1, \ldots, f_c \in I$ such that their images in $\mathfrak{m}''/(\mathfrak{m}'')^2$ span the kernel of the map to $\mathfrak{m}/\mathfrak{m}^2$. This is clearly possible. Denote $J = (f_1, \ldots, f_c)$. So $J \subset I$. Denote $S' = k[x_1, \ldots, x_n]/J$ so there is a surjection $S' \to S$. Denote $\mathfrak{m}' = \mathfrak{m}''S'$ the corresponding maximal ideal of S'. Hence we have

$$\begin{tikzcd} k[x_1, \ldots, x_n] \arrow[r] & S' \arrow[r] & S \\ \mathfrak{m}'' \arrow[r] \arrow[u] & \mathfrak{m}' \arrow[r] \arrow[u] & \mathfrak{m} \arrow[u] \end{tikzcd}$$

By our choice of J the exact sequence

$$J/\mathfrak{m}''J \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}'/(\mathfrak{m}')^2 \to 0$$

shows that $\dim(\mathfrak{m}'/(\mathfrak{m}')^2) = d$. Since $S'_{\mathfrak{m}'}$ surjects onto $S_{\mathfrak{m}}$ we see that $\dim(S_{\mathfrak{m}'}) \geq d$. Hence by the discussion preceding Definition 10.60.10 we conclude that $S'_{\mathfrak{m}'}$ is regular of dimension d as well. Because S' was cut out by $c = n - d$ equations we conclude that there exists a $g' \in S'$, $g' \notin \mathfrak{m}'$ such that $S'_{g'}$ is a global complete intersection over k, see Lemma 10.135.4. Also the map $S'_{\mathfrak{m}'} \to S_{\mathfrak{m}}$ is a surjection of Noetherian local domains of the same dimension and hence an isomorphism. Hence $S' \to S$ is surjective with finitely generated kernel and becomes an isomorphism after localizing at $\mathfrak{m}'$. Thus we can find $g' \in S'$, $g \notin \mathfrak{m}'$ such that $S'_{g'} \to S_{g'}$

is an isomorphism. All in all we conclude that after replacing S by a principal localization we may assume that S is a global complete intersection.

At this point we may write $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim S = n - c$. Recall that the naive cotangent complex of this algebra is given by

$$\bigoplus S \cdot f_j \to \bigoplus S \cdot \mathrm{d}x_i$$

see Lemma 10.136.12. By Lemma 10.137.16 in order to show that S is smooth at $\mathfrak m$ we have to show that one of the $c \times c$ minors g_I of the matrix "A" giving the map above does not vanish at $\mathfrak m$. By Lemma 10.140.1 the matrix A mod $\mathfrak m$ has rank c. Thus we win. □

00TT Lemma 10.140.3. Let k be any field. Let S be a finite type k-algebra. Let $X = \mathrm{Spec}(S)$. Let $\mathfrak q \subset S$ be a prime corresponding to $x \in X$. The following are equivalent:

(1) The k-algebra S is smooth at $\mathfrak q$ over k.
(2) We have $\dim_{\kappa(\mathfrak q)} \Omega_{S/k} \otimes_S \kappa(\mathfrak q) \leq \dim_x X$.
(3) We have $\dim_{\kappa(\mathfrak q)} \Omega_{S/k} \otimes_S \kappa(\mathfrak q) = \dim_x X$.

Moreover, in this case the local ring $S_{\mathfrak q}$ is regular.

Proof. If S is smooth at $\mathfrak q$ over k, then there exists a $g \in S$, $g \notin \mathfrak q$ such that S_g is standard smooth over k, see Lemma 10.137.10. A standard smooth algebra over k has a module of differentials which is free of rank equal to the dimension, see Lemma 10.137.7 (use that a relative global complete intersection over a field has dimension equal to the number of variables minus the number of equations). Thus we see that (1) implies (3). To finish the proof of the lemma it suffices to show that (2) implies (1) and that it implies that $S_{\mathfrak q}$ is regular.

Assume (2). By Nakayama's Lemma 10.20.1 we see that $\Omega_{S/k, \mathfrak q}$ can be generated by $\leq \dim_x X$ elements. We may replace S by S_g for some $g \in S$, $g \notin \mathfrak q$ such that $\Omega_{S/k}$ is generated by at most $\dim_x X$ elements. Let K/k be an algebraically closed field extension such that there exists a k-algebra map $\psi : \kappa(\mathfrak q) \to K$. Consider $S_K = K \otimes_k S$. Let $\mathfrak m \subset S_K$ be the maximal ideal corresponding to the surjection

$$S_K = K \otimes_k S \longrightarrow K \otimes_k \kappa(\mathfrak q) \xrightarrow{\mathrm{id}_K \otimes \psi} K.$$

Note that $\mathfrak m \cap S = \mathfrak q$, in other words $\mathfrak m$ lies over $\mathfrak q$. By Lemma 10.116.6 the dimension of $X_K = \mathrm{Spec}(S_K)$ at the point corresponding to $\mathfrak m$ is $\dim_x X$. By Lemma 10.114.6 this is equal to $\dim((S_K)_{\mathfrak m})$. By Lemma 10.131.12 the module of differentials of S_K over K is the base change of $\Omega_{S/k}$, hence also generated by at most $\dim_x X = \dim((S_K)_{\mathfrak m})$ elements. By Lemma 10.140.2 we see that S_K is smooth at $\mathfrak m$ over K. By Lemma 10.137.18 this implies that S is smooth at $\mathfrak q$ over k. This proves (1). Moreover, we know by Lemma 10.140.2 that the local ring $(S_K)_{\mathfrak m}$ is regular. Since $S_{\mathfrak q} \to (S_K)_{\mathfrak m}$ is flat we conclude from Lemma 10.110.9 that $S_{\mathfrak q}$ is regular. □

The following lemma can be significantly generalized (in several different ways).

00TU Lemma 10.140.4. Let k be a field. Let R be a Noetherian local ring containing k. Assume that the residue field $\kappa = R/\mathfrak m$ is a finitely generated separable extension of k. Then the map

$$\mathrm{d} : \mathfrak m/\mathfrak m^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak m)$$

is injective.

Proof. We may replace R by $R/\mathfrak{m}^2$. Hence we may assume that $\mathfrak{m}^2 = 0$. By assumption we may write $\kappa = k(\overline{x}_1, \ldots, \overline{x}_r, \overline{y})$ where $\overline{x}_1, \ldots, \overline{x}_r$ is a transcendence basis of κ over k and $\overline{y}$ is separable algebraic over $k(\overline{x}_1, \ldots, \overline{x}_r)$. Say its minimal equation is $P(\overline{y}) = 0$ with $P(T) = T^d + \sum_{i<d} a_i T^i$, with $a_i \in k(\overline{x}_1, \ldots, \overline{x}_r)$ and $P'(\overline{y}) \neq 0$. Choose any lifts $x_i \in R$ of the elements $\overline{x}_i \in \kappa$. This gives a commutative diagram

$$\begin{tikzcd} R \arrow[r] & \kappa \\ & k(\overline{x}_1, \ldots, \overline{x}_r) \arrow[ul, "\varphi"] \arrow[u] \end{tikzcd}$$

of k-algebras. We want to extend the left upwards arrow φ to a k-algebra map from κ to R. To do this choose any $y \in R$ lifting $\overline{y}$. To see that it defines a k-algebra map defined on $\kappa \cong k(\overline{x}_1, \ldots, \overline{x}_r)[T]/(P)$ all we have to show is that we may choose y such that $P^\varphi(y) = 0$. If not then we compute for $\delta \in \mathfrak{m}$ that

$$P(y + \delta) = P(y) + P'(y)\delta$$

because $\mathfrak{m}^2 = 0$. Since $P'(y)\delta = P'(\overline{y})\delta$ we see that we can adjust our choice as desired. This shows that $R \cong \kappa \oplus \mathfrak{m}$ as k-algebras! From a direct computation of $\Omega_{\kappa \oplus \mathfrak{m}/k}$ the lemma follows. $\square$

00TV Lemma 10.140.5. Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. Assume $\kappa(\mathfrak{q})$ is separable over k. The following are equivalent:

(1) The algebra S is smooth at $\mathfrak{q}$ over k.
(2) The ring $S_\mathfrak{q}$ is regular.

Proof. Denote $R = S_\mathfrak{q}$ and denote its maximal by $\mathfrak{m}$ and its residue field κ. By Lemma 10.140.4 and 10.131.9 we see that there is a short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0$$

Note that $\Omega_{R/k} = \Omega_{S/k, \mathfrak{q}}$, see Lemma 10.131.8. Moreover, since κ is separable over k we have $\dim_\kappa \Omega_{\kappa/k} = \text{trdeg}_k(\kappa)$. Hence we get

$$\dim_\kappa \Omega_{R/k} \otimes_R \kappa = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2 + \text{trdeg}_k(\kappa) \geq \dim R + \text{trdeg}_k(\kappa) = \dim_\mathfrak{q} S$$

(see Lemma 10.116.3 for the last equality) with equality if and only if R is regular. Thus we win by applying Lemma 10.140.3. $\square$

00TW Lemma 10.140.6. Let $R \to S$ be a $\mathbf{Q}$-algebra map. Let $f \in S$ be such that $\Omega_{S/R} = S\text{d}f \oplus C$ for some S-submodule C. Then

(1) f is not nilpotent, and
(2) if S is a Noetherian local ring, then f is a nonzerodivisor in S.

Proof. For $a \in S$ write $\text{d}(a) = \theta(a)\text{d}f + c(a)$ for some $\theta(a) \in S$ and $c(a) \in C$. Consider the R-derivation $S \to S$, $a \mapsto \theta(a)$. Note that $\theta(f) = 1$.

If $f^n = 0$ with $n > 1$ minimal, then $0 = \theta(f^n) = nf^{n-1}$ contradicting the minimality of n. We conclude that f is not nilpotent.

Suppose $fa = 0$. If f is a unit then $a = 0$ and we win. Assume f is not a unit. Then $0 = \theta(fa) = f\theta(a) + a$ by the Leibniz rule and hence $a \in (f)$. By induction suppose we have shown $fa = 0 \Rightarrow a \in (f^n)$. Then writing $a = f^n b$ we get $0 =$

$\theta(f^{n+1}b) = (n+1)f^nb + f^{n+1}\theta(b)$. Hence $a = f^nb = -f^{n+1}\theta(b)/(n+1) \in (f^{n+1})$. Since in the Noetherian local ring S we have $\bigcap(f^n) = 0$, see Lemma 10.51.4 we win. □

The following is probably quite useless in applications.

00TX Lemma 10.140.7. Let k be a field of characteristic 0. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. The following are equivalent:

(1) The algebra S is smooth at $\mathfrak{q}$ over k.
(2) The $S_\mathfrak{q}$-module $\Omega_{S/k,\mathfrak{q}}$ is (finite) free.
(3) The ring $S_\mathfrak{q}$ is regular.

Proof. In characteristic zero any field extension is separable and hence the equivalence of (1) and (3) follows from Lemma 10.140.5. Also (1) implies (2) by definition of smooth algebras. Assume that $\Omega_{S/k,\mathfrak{q}}$ is free over $S_\mathfrak{q}$. We are going to use the notation and observations made in the proof of Lemma 10.140.5. So $R = S_\mathfrak{q}$ with maximal ideal $\mathfrak{m}$ and residue field κ. Our goal is to prove R is regular.

If $\mathfrak{m}/\mathfrak{m}^2 = 0$, then $\mathfrak{m} = 0$ and $R \cong \kappa$. Hence R is regular and we win.

If $\mathfrak{m}/\mathfrak{m}^2 \neq 0$, then choose any $f \in \mathfrak{m}$ whose image in $\mathfrak{m}/\mathfrak{m}^2$ is not zero. By Lemma 10.140.4 we see that $\mathrm{d}f$ has nonzero image in $\Omega_{R/k}/\mathfrak{m}\Omega_{R/k}$. By assumption $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$ is finite free and hence by Nakayama's Lemma 10.20.1 we see that $\mathrm{d}f$ generates a direct summand. We apply Lemma 10.140.6 to deduce that f is a nonzerodivisor in R. Furthermore, by Lemma 10.131.9 we get an exact sequence

$$(f)/(f^2) \to \Omega_{R/k} \otimes_R R/fR \to \Omega_{(R/fR)/k} \to 0$$

This implies that $\Omega_{(R/fR)/k}$ is finite free as well. Hence by induction we see that R/fR is a regular local ring. Since $f \in \mathfrak{m}$ was a nonzerodivisor we conclude that R is regular, see Lemma 10.106.7. □

00TY Example 10.140.8. Lemma 10.140.7 does not hold in characteristic $p > 0$. The standard examples are the ring maps

$$\mathbf{F}_p \longrightarrow \mathbf{F}_p[x]/(x^p)$$

whose module of differentials is free but is clearly not smooth, and the ring map $(p > 2)$

$$\mathbf{F}_p(t) \to \mathbf{F}_p(t)[x, y]/(x^p + y^2 + \alpha)$$

which is not smooth at the prime $\mathfrak{q} = (y, x^p + \alpha)$ but is regular.

Using the material above we can characterize smoothness at the generic point in terms of field extensions.

07ND Lemma 10.140.9. Let $R \to S$ be an injective finite type ring map with R and S domains. Then $R \to S$ is smooth at $\mathfrak{q} = (0)$ if and only if the induced extension L/K of fraction fields is separable.

Proof. Assume $R \to S$ is smooth at (0). We may replace S by S_g for some nonzero $g \in S$ and assume that $R \to S$ is smooth. Then $K \to S \otimes_R K$ is smooth (Lemma 10.137.4). Moreover, for any field extension K'/K the ring map $K' \to S \otimes_R K'$ is smooth as well. Hence $S \otimes_R K'$ is a regular ring by Lemma 10.140.3, in particular reduced. It follows that $S \otimes_R K$ is a geometrically reduced over K. Hence L is geometrically reduced over K, see Lemma 10.43.3. Hence L/K is separable by Lemma 10.44.1.

Conversely, assume that L/K is separable. We may assume $R \to S$ is of finite presentation, see Lemma 10.30.1. It suffices to prove that $K \to S \otimes_R K$ is smooth at (0), see Lemma 10.137.18. This follows from Lemma 10.140.5, the fact that a field is a regular ring, and the assumption that L/K is separable. □

10.141. Smooth ring maps in the Noetherian case

02HR

02HS Definition 10.141.1. Let $\varphi : B' \to B$ be a ring map. We say φ is a small extension if B' and B are local Artinian rings, φ is surjective and $I = \mathrm{Ker}(\varphi)$ has length 1 as a B'-module.

Clearly this means that $I^2 = 0$ and that $I = (x)$ for some $x \in B'$ such that $\mathfrak{m}'x = 0$ where $\mathfrak{m}' \subset B'$ is the maximal ideal.

02HT Lemma 10.141.2. Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime ideal of S lying over $\mathfrak{p} \subset R$. Assume R is Noetherian and $R \to S$ of finite type. The following are equivalent:

(1) $R \to S$ is smooth at $\mathfrak{q}$,
(2) for every surjection of local R-algebras $(B', \mathfrak{m}') \to (B, \mathfrak{m})$ with $\mathrm{Ker}(B' \to B)$ having square zero and every solid commutative diagram
$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & B \\ R \arrow[u] \arrow[r] & B' \arrow[u] \end{tikzcd}$$
such that $\mathfrak{q} = S \cap \mathfrak{m}$ there exists a dotted arrow making the diagram commute,
(3) same as in (2) but with $B' \to B$ ranging over small extensions, and
(4) same as in (2) but with $B' \to B$ ranging over small extensions such that in addition $S \to B$ induces an isomorphism $\kappa(\mathfrak{q}) \cong \kappa(\mathfrak{m})$.

Proof. Assume (1). This means there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. By Proposition 10.138.13 we know that $R \to S_g$ is formally smooth. Note that given any diagram as in (2) the map $S \to B$ factors automatically through $S_\mathfrak{q}$ and a fortiori through S_g. The formal smoothness of S_g over R gives us a morphism $S_g \to B'$ fitting into a similar diagram with S_g at the upper left corner. Composing with $S \to S_g$ gives the desired arrow. In other words, we have shown that (1) implies (2).

Clearly (2) implies (3) and (3) implies (4).

Assume (4). We are going to show that (1) holds, thereby finishing the proof of the lemma. Choose a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. This is possible as S is of finite type over R and therefore of finite presentation (see Lemma 10.31.4). Set $I = (f_1, \ldots, f_m)$. Consider the naive cotangent complex
$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{j=1}^{m} S\mathrm{d}x_j$$
of this presentation (see Section 10.134). It suffices to show that when we localize this complex at $\mathfrak{q}$ then the map becomes a split injection, see Lemma 10.137.12.

Denote $S' = R[x_1, \ldots, x_n]/I^2$. By Lemma 10.131.11 we have

$$S \otimes_{S'} \Omega_{S'/R} = S \otimes_{R[x_1,\ldots,x_n]} \Omega_{R[x_1,\ldots,x_n]/R} = \bigoplus\nolimits_{j=1}^{m} S\mathrm{d}x_j.$$

Thus the map

$$\mathrm{d} : I/I^2 \longrightarrow S \otimes_{S'} \Omega_{S'/R}$$

is the same as the map in the naive cotangent complex above. In particular the truth of the assertion we are trying to prove depends only on the three rings $R \to S' \to S$. Let $\mathfrak{q}' \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak{q}$. Since localization commutes with taking modules of differentials (Lemma 10.131.8) we see that it suffices to show that the map

02HU (10.141.2.1)
$$\mathrm{d} : I_{\mathfrak{q}'}/I_{\mathfrak{q}'}^2 \longrightarrow S_{\mathfrak{q}} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}$$

coming from $R \to S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is a split injection.

Let $N \in \mathbf{N}$ be an integer. Consider the ring

$$B'_N = S'_{\mathfrak{q}'}/(\mathfrak{q}')^N S'_{\mathfrak{q}'} = (S'/(\mathfrak{q}')^N S')_{\mathfrak{q}'}$$

and its quotient $B_N = B'_N/IB'_N$. Note that $B_N \cong S_{\mathfrak{q}}/\mathfrak{q}^N S_{\mathfrak{q}}$. Observe that B'_N is an Artinian local ring since it is the quotient of a local Noetherian ring by a power of its maximal ideal. Consider a filtration of the kernel I_N of $B'_N \to B_N$ by B'_N-submodules

$$0 \subset J_{N,1} \subset J_{N,2} \subset \ldots \subset J_{N,n(N)} = I_N$$

such that each successive quotient $J_{N,i}/J_{N,i-1}$ has length 1. (As B'_N is Artinian such a filtration exists.) This gives a sequence of small extensions

$$B'_N \to B'_N/J_{N,1} \to B'_N/J_{N,2} \to \ldots \to B'_N/J_{N,n(N)} = B'_N/I_N = B_N = S_{\mathfrak{q}}/\mathfrak{q}^N S_{\mathfrak{q}}$$

Applying condition (4) successively to these small extensions starting with the map $S \to B_N$ we see there exists a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd] & B_N \\ R \arrow[u] \arrow[r] & B'_N \arrow[u] \end{tikzcd}$$

Clearly the ring map $S \to B'_N$ factors as $S \to S_{\mathfrak{q}} \to B'_N$ where $S_{\mathfrak{q}} \to B'_N$ is a local homomorphism of local rings. Moreover, since the maximal ideal of B'_N to the Nth power is zero we conclude that $S_{\mathfrak{q}} \to B'_N$ factors through $S_{\mathfrak{q}}/(\mathfrak{q})^N S_{\mathfrak{q}} = B_N$. In other words we have shown that for all $N \in \mathbf{N}$ the surjection of R-algebras $B'_N \to B_N$ has a splitting.

Consider the presentation

$$I_N \to B_N \otimes_{B'_N} \Omega_{B'_N/R} \to \Omega_{B_N/R} \to 0$$

coming from the surjection $B'_N \to B_N$ with kernel I_N (see Lemma 10.131.9). By the above the R-algebra map $B'_N \to B_N$ has a right inverse. Hence by Lemma 10.131.10 we see that the sequence above is split exact! Thus for every N the map

$$I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R}$$

is a split injection. The rest of the proof is gotten by unwinding what this means exactly. Note that

$$I_N = I_{\mathfrak{q}'}/(I_{\mathfrak{q}'}^2 + (\mathfrak{q}')^N \cap I_{\mathfrak{q}'})$$

By Artin-Rees (Lemma 10.51.2) we find a $c \geq 0$ such that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} I_N = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} I_{\mathfrak{q}'}/I_{\mathfrak{q}'}^2$$

for all $N \geq c$ (these tensor product are just a fancy way of dividing by $\mathfrak{q}^{N-c}$). We may of course assume $c \geq 1$. By Lemma 10.131.11 we see that

$$S'_{\mathfrak{q}'}/(\mathfrak{q}')^{N-c}S'_{\mathfrak{q}'} \otimes_{S'_{\mathfrak{q}'}} \Omega_{B'_N/R} = S'_{\mathfrak{q}'}/(\mathfrak{q}')^{N-c}S'_{\mathfrak{q}'} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}$$

we can further tensor this by $B_N = S_\mathfrak{q}/\mathfrak{q}^N$ to see that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S'_{\mathfrak{q}'}} \Omega_{B'_N/R} = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}.$$

Since a split injection remains a split injection after tensoring with anything we see that

$$S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}} (10.141.2.1) = S_\mathfrak{q}/\mathfrak{q}^{N-c}S_\mathfrak{q} \otimes_{S_\mathfrak{q}/\mathfrak{q}^N S_\mathfrak{q}} (I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R})$$

is a split injection for all $N \geq c$. By Lemma 10.74.1 we see that (10.141.2.1) is a split injection. This finishes the proof. □

10.142. Overview of results on smooth ring maps

00TZ Here is a list of results on smooth ring maps that we proved in the preceding sections. For more precise statements and definitions please consult the references given.

(1) A ring map $R \to S$ is smooth if it is of finite presentation and the naive cotangent complex of S/R is quasi-isomorphic to a finite projective S-module in degree 0, see Definition 10.137.1.
(2) If S is smooth over R, then $\Omega_{S/R}$ is a finite projective S-module, see discussion following Definition 10.137.1.
(3) The property of being smooth is local on S, see Lemma 10.137.13.
(4) The property of being smooth is stable under base change, see Lemma 10.137.4.
(5) The property of being smooth is stable under composition, see Lemma 10.137.14.
(6) A smooth ring map is syntomic, in particular flat, see Lemma 10.137.10.
(7) A finitely presented, flat ring map with smooth fibre rings is smooth, see Lemma 10.137.17.
(8) A finitely presented ring map $R \to S$ is smooth if and only if it is formally smooth, see Proposition 10.138.13.
(9) If $R \to S$ is a finite type ring map with R Noetherian then to check that $R \to S$ is smooth it suffices to check the lifting property of formal smoothness along small extensions of Artinian local rings, see Lemma 10.141.2.
(10) A smooth ring map $R \to S$ is the base change of a smooth ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, see Lemma 10.138.14.
(11) Formation of the set of points where a ring map is smooth commutes with flat base change, see Lemma 10.137.18.
(12) If S is of finite type over an algebraically closed field k, and $\mathfrak{m} \subset S$ a maximal ideal, then the following are equivalent
    (a) S is smooth over k in a neighbourhood of $\mathfrak{m}$,
    (b) $S_\mathfrak{m}$ is a regular local ring,

(c) $\dim(S_{\mathfrak m}) = \dim_{\kappa(m)} \Omega_{S/k} \otimes_S \kappa(\mathfrak m)$.

see Lemma 10.140.2.

(13) If S is of finite type over a field k, and $\mathfrak q \subset S$ a prime ideal, then the following are equivalent
   (a) S is smooth over k in a neighbourhood of $\mathfrak q$,
   (b) $\dim_{\mathfrak q}(S/k) = \dim_{\kappa(\mathfrak q)} \Omega_{S/k} \otimes_S \kappa(\mathfrak q)$.

   see Lemma 10.140.3.
(14) If S is smooth over a field, then all its local rings are regular, see Lemma 10.140.3.
(15) If S is of finite type over a field k, $\mathfrak q \subset S$ a prime ideal, the field extension $\kappa(\mathfrak q)/k$ is separable and $S_{\mathfrak q}$ is regular, then S is smooth over k at $\mathfrak q$, see Lemma 10.140.5.
(16) If S is of finite type over a field k, if k has characteristic 0, if $\mathfrak q \subset S$ a prime ideal, and if $\Omega_{S/k,\mathfrak q}$ is free, then S is smooth over k at $\mathfrak q$, see Lemma 10.140.7.

Some of these results were proved using the notion of a standard smooth ring map, see Definition 10.137.6. This is the analogue of what a relative global complete intersection map is for the case of syntomic morphisms. It is also the easiest way to make examples.

10.143. Étale ring maps

00U0 An étale ring map is a smooth ring map whose relative dimension is equal to zero. This is the same as the following slightly more direct definition.

00U1 Definition 10.143.1. Let $R \to S$ be a ring map. We say $R \to S$ is étale if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to zero. Given a prime $\mathfrak q$ of S we say that $R \to S$ is étale at $\mathfrak q$ if there exists a $g \in S$, $g \notin \mathfrak q$ such that $R \to S_g$ is étale.

In particular we see that $\Omega_{S/R} = 0$ if S is étale over R. If $R \to S$ is smooth, then $R \to S$ is étale if and only if $\Omega_{S/R} = 0$. From our results on smooth ring maps we automatically get a whole host of results for étale maps. We summarize these in Lemma 10.143.3 below. But before we do so we prove that any étale ring map is standard smooth.

00U9 Lemma 10.143.2. Any étale ring map is standard smooth. More precisely, if $R \to S$ is étale, then there exists a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that the image of $\det(\partial f_j/\partial x_i)$ is invertible in S.

Proof. Let $R \to S$ be étale. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. As $R \to S$ is étale we know that

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} S\mathrm{d}x_i$$

is an isomorphism, in particular I/I^2 is a free S-module. Thus by Lemma 10.136.6 we may assume (after possibly changing the presentation), that $I = (f_1, \ldots, f_c)$ such that the classes $f_i \bmod I^2$ form a basis of I/I^2. It follows immediately from the fact that the displayed map above is an isomorphism that $c = n$ and that $\det(\partial f_j/\partial x_i)$ is invertible in S. $\square$

00U2 Lemma 10.143.3. Results on étale ring maps.

(1) The ring map $R \to R_f$ is étale for any ring R and any $f \in R$.

(2) Compositions of étale ring maps are étale.
(3) A base change of an étale ring map is étale.
(4) The property of being étale is local: Given a ring map $R \to S$ and elements $g_1, \ldots, g_m \in S$ which generate the unit ideal such that $R \to S_{g_j}$ is étale for $j = 1, \ldots, m$ then $R \to S$ is étale.
(5) Given $R \to S$ of finite presentation, and a flat ring map $R \to R'$, set $S' = R' \otimes_R S$. The set of primes where $R' \to S'$ is étale is the inverse image via $\text{Spec}(S') \to \text{Spec}(S)$ of the set of primes where $R \to S$ is étale.
(6) An étale ring map is syntomic, in particular flat.
(7) If S is finite type over a field k, then S is étale over k if and only if $\Omega_{S/k} = 0$.
(8) Any étale ring map $R \to S$ is the base change of an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$.
(9) Let $A = \text{colim}\, A_i$ be a filtered colimit of rings. Let $A \to B$ be an étale ring map. Then there exists an étale ring map $A_i \to B_i$ for some i such that $B \cong A \otimes_{A_i} B_i$.
(10) Let A be a ring. Let S be a multiplicative subset of A. Let $S^{-1}A \to B'$ be étale. Then there exists an étale ring map $A \to B$ such that $B' \cong S^{-1}B$.
(11) Let A be a ring. Let $B = B' \times B''$ be a product of A-algebras. Then B is étale over A if and only if both B' and B'' are étale over A.

Proof. In each case we use the corresponding result for smooth ring maps with a small argument added to show that $\Omega_{S/R}$ is zero.

Proof of (1). The ring map $R \to R_f$ is smooth and $\Omega_{R_f/R} = 0$.

Proof of (2). The composition $A \to C$ of smooth maps $A \to B$ and $B \to C$ is smooth, see Lemma 10.137.14. By Lemma 10.131.7 we see that $\Omega_{C/A}$ is zero as both $\Omega_{C/B}$ and $\Omega_{B/A}$ are zero.

Proof of (3). Let $R \to S$ be étale and $R \to R'$ be arbitrary. Then $R' \to S' = R' \otimes_R S$ is smooth, see Lemma 10.137.4. Since $\Omega_{S'/R'} = S' \otimes_S \Omega_{S/R}$ by Lemma 10.131.12 we conclude that $\Omega_{S'/R'} = 0$. Hence $R' \to S'$ is étale.

Proof of (4). Assume the hypotheses of (4). By Lemma 10.137.13 we see that $R \to S$ is smooth. We are also given that $\Omega_{S_{g_i}/R} = (\Omega_{S/R})_{g_i} = 0$ for all i. Then $\Omega_{S/R} = 0$, see Lemma 10.23.2.

Proof of (5). The result for smooth maps is Lemma 10.137.18. In the proof of that lemma we used that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This reduces us to showing that if M is a finitely presented S-module the set of primes $\mathfrak{q}'$ of S' such that $(M \otimes_S S')_{\mathfrak{q}'} = 0$ is the inverse image of the set of primes $\mathfrak{q}$ of S such that $M_{\mathfrak{q}} = 0$. This follows from Lemma 10.40.6.

Proof of (6). Follows directly from the corresponding result for smooth ring maps (Lemma 10.137.10).

Proof of (7). Follows from Lemma 10.140.3 and the definitions.

Proof of (8). Lemma 10.138.14 gives the result for smooth ring maps. The resulting smooth ring map $R_0 \to S_0$ satisfies the hypotheses of Lemma 10.130.8, and hence we may replace S_0 by the factor of relative dimension 0 over R_0.

Proof of (9). Follows from (8) since $R_0 \to A$ will factor through A_i for some i by Lemma 10.127.3.

Proof of (10). Follows from (9), (1), and (2) since $S^{-1}A$ is a filtered colimit of principal localizations of A.

Proof of (11). Use Lemma 10.137.15 to see the result for smoothness and then use that $\Omega_{B/A}$ is zero if and only if both $\Omega_{B'/A}$ and $\Omega_{B''/A}$ are zero. □

Next we work out in more detail what it means to be étale over a field.

00U3 Lemma 10.143.4. Let k be a field. A ring map $k \to S$ is étale if and only if S is isomorphic as a k-algebra to a finite product of finite separable extensions of k.

Proof. We are going to use without further mention: if $S = S_1 \times \ldots \times S_n$ is a finite product of k-algebras, then S is étale over k if and only if each S_i is étale over k. See Lemma 10.143.3 part (11).

If k'/k is a finite separable field extension then we can write $k' = k(\alpha) \cong k[x]/(f)$. Here f is the minimal polynomial of the element α. Since k' is separable over k we have $\gcd(f, f') = 1$. This implies that $\mathrm{d} : k' \cdot f \to k' \cdot \mathrm{d}x$ is an isomorphism. Hence $k \to k'$ is étale. Thus if S is a finite product of finite separable extension of k, then S is étale over k.

Conversely, suppose that $k \to S$ is étale. Then S is smooth over k and $\Omega_{S/k} = 0$. By Lemma 10.140.3 we see that $\dim_{\mathfrak m} \mathrm{Spec}(S) = 0$ for every maximal ideal $\mathfrak m$ of S. Thus $\dim(S) = 0$. By Proposition 10.60.7 we find that S is a finite product of Artinian local rings. By the already used Lemma 10.140.3 these local rings are fields. Hence we may assume $S = k'$ is a field. By the Hilbert Nullstellensatz (Theorem 10.34.1) we see that the extension k'/k is finite. The smoothness of $k \to k'$ implies by Lemma 10.140.9 that k'/k is a separable extension and the proof is complete. □

00U4 Lemma 10.143.5. Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p$ in R. If S/R is étale at $\mathfrak q$ then

(1) we have $\mathfrak pS_{\mathfrak q} = \mathfrak qS_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and
(2) the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite separable.

Proof. First we may replace S by S_g for some $g \in S$, $g \not\in \mathfrak q$ and assume that $R \to S$ is étale. Then the lemma follows from Lemma 10.143.4 by unwinding the fact that $S \otimes_R \kappa(\mathfrak p)$ is étale over $\kappa(\mathfrak p)$. □

00U5 Lemma 10.143.6. An étale ring map is quasi-finite.

Proof. Let $R \to S$ be an étale ring map. By definition $R \to S$ is of finite type. For any prime $\mathfrak p \subset R$ the fibre ring $S \otimes_R \kappa(\mathfrak p)$ is étale over $\kappa(\mathfrak p)$ and hence a finite products of fields finite separable over $\kappa(\mathfrak p)$, in particular finite over $\kappa(\mathfrak p)$. Thus $R \to S$ is quasi-finite by Lemma 10.122.4. □

00U6 Lemma 10.143.7. Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over a prime $\mathfrak p$ of R. If

(1) $R \to S$ is of finite presentation,
(2) $R_{\mathfrak p} \to S_{\mathfrak q}$ is flat
(3) $\mathfrak pS_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and
(4) the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite separable,

then $R \to S$ is étale at $\mathfrak q$.

Proof. Apply Lemma 10.122.2 to find a $g \in S$, $g \notin \mathfrak{q}$ such that $\mathfrak{q}$ is the only prime of S_g lying over $\mathfrak{p}$. We may and do replace S by S_g. Then $S \otimes_R \kappa(\mathfrak{p})$ has a unique prime, hence is a local ring, hence is equal to $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \cong \kappa(\mathfrak{q})$. By Lemma 10.137.17 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. Replace S by S_g again we may assume that $R \to S$ is smooth. By Lemma 10.137.10 we may even assume that $R \to S$ is standard smooth, say $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Since $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$ has dimension 0 we conclude that $n = c$, i.e., $R \to S$ is étale. □

Here is a completely new phenomenon.

00U7 **Lemma 10.143.8.** *Let $R \to S$ and $R \to S'$ be étale. Then any R-algebra map $S' \to S$ is étale.*

Proof. First of all we note that $S' \to S$ is of finite presentation by Lemma 10.6.2. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the primes $\mathfrak{q}' \subset S'$ and $\mathfrak{p} \subset R$. By Lemma 10.143.5 the ring map $S'_{\mathfrak{q}'}/\mathfrak{p}S'_{\mathfrak{q}'} \to S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is a map finite separable extensions of $\kappa(\mathfrak{p})$. In particular it is flat. Hence by Lemma 10.128.8 we see that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is flat. Thus $S' \to S$ is flat. Moreover, the above also shows that $\mathfrak{q}'S_\mathfrak{q}$ is the maximal ideal of $S_\mathfrak{q}$ and that the residue field extension of $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite separable. Hence from Lemma 10.143.7 we conclude that $S' \to S$ is étale at $\mathfrak{q}$. Since being étale is local (see Lemma 10.143.3) we win. □

00U8 **Lemma 10.143.9.** *Let $\varphi : R \to S$ be a ring map. If $R \to S$ is surjective, flat and finitely presented then there exist an idempotent $e \in R$ such that $S = R_e$.*

First proof. Let I be the kernel of φ. We have that I is finitely generated by Lemma 10.6.3 since φ is of finite presentation. Moreover, since S is flat over R, tensoring the exact sequence $0 \to I \to R \to S \to 0$ over R with S gives $I/I^2 = 0$. Now we conclude by Lemma 10.21.5. □

Second proof. Since $\text{Spec}(S) \to \text{Spec}(R)$ is a homeomorphism onto a closed subset (see Lemma 10.17.7) and is open (see Proposition 10.41.8) we see that the image is $D(e)$ for some idempotent $e \in R$ (see Lemma 10.21.3). Thus $R_e \to S$ induces a bijection on spectra. Now this map induces an isomorphism on all local rings for example by Lemmas 10.78.5 and 10.20.1. Then it follows that $R_e \to S$ is also injective, for example see Lemma 10.23.1. □

04D1 **Lemma 10.143.10.** *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be an étale ring map. Then there exists an étale ring map $R \to S$ such that $\overline{S} \cong S/IS$ as R/I-algebras.*

Proof. By Lemma 10.143.2 we can write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$ as in Definition 10.137.6 with $\overline{\Delta} = \det(\frac{\partial \overline{f}_i}{\partial x_j})_{i,j=1,\ldots,n}$ invertible in $\overline{S}$. Just take some lifts f_i and set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_n, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_i}{\partial x_j})_{i,j=1,\ldots,n}$ as in Example 10.137.8. This proves the lemma. □

05YT **Lemma 10.143.11.** *Consider a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & J & \to & B' & \to & B & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & I & \to & A' & \to & A & \to & 0 \end{array}$$

with exact rows where $B' \to B$ and $A' \to A$ are surjective ring maps whose kernels are ideals of square zero. If $A \to B$ is étale, and $J = I \otimes_A B$, then $A' \to B'$ is étale.

Proof. By Lemma 10.143.10 there exists an étale ring map $A' \to C$ such that $C/IC = B$. Then $A' \to C$ is formally smooth (by Proposition 10.138.13) hence we get an A'-algebra map $\varphi : C \to B'$. Since $A' \to C$ is flat we have $I \otimes_A B = I \otimes_A C/IC = IC$. Hence the assumption that $J = I \otimes_A B$ implies that φ induces an isomorphism $IC \to J$ and an isomorphism $C/IC \to B'/IB'$, whence φ is an isomorphism. $\square$

00UA Example 10.143.12. Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_n c_m \end{array}$$

of Example 10.136.7. Write symbolically

$$S = R[b_1, \ldots, c_m]/(\{a_k(b_i, c_j) - a_k\}_{k=1,\ldots,n+m})$$

where for example $a_1(b_i, c_j) = b_1 + c_1$. The matrix of partial derivatives is

$$\begin{pmatrix} 1 & c_1 & \ldots & c_m & 0 & \ldots & \ldots & 0 \\ 0 & 1 & c_1 & \ldots & c_m & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & c_1 & c_2 & \ldots & c_m \\ 1 & b_1 & \ldots & b_{n-1} & b_n & 0 & \ldots & 0 \\ 0 & 1 & b_1 & \ldots & b_{n-1} & b_n & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & 1 & b_1 & \ldots & b_n \end{pmatrix}$$

The determinant Δ of this matrix is better known as the resultant of the polynomials $g = x^n + b_1x^{n-1} + \ldots + b_n$ and $h = x^m + c_1x^{m-1} + \ldots + c_m$, and the matrix above is known as the Sylvester matrix associated to g, h. In a formula $\Delta = \text{Res}_x(g, h)$. The Sylvester matrix is the transpose of the matrix of the linear map

$$\begin{array}{rcl} S[x]_{<m} \oplus S[x]_{<n} & \longrightarrow & S[x]_{<n+m} \\ a \oplus b & \longmapsto & ag + bh \end{array}$$

Let $\mathfrak{q} \subset S$ be any prime. By the above the following are equivalent:

(1) $R \to S$ is étale at $\mathfrak{q}$,
(2) $\Delta = \text{Res}_x(g, h) \notin \mathfrak{q}$,
(3) the images $\overline{g}, \overline{h} \in \kappa(\mathfrak{q})[x]$ of the polynomials g, h are relatively prime in $\kappa(\mathfrak{q})[x]$.

The equivalence of (2) and (3) holds because the image of the Sylvester matrix in $\text{Mat}(n + m, \kappa(\mathfrak{q}))$ has a kernel if and only if the polynomials $\overline{g}, \overline{h}$ have a factor in common. We conclude that the ring map

$$R \longrightarrow S[\frac{1}{\Delta}] = S[\frac{1}{\text{Res}_x(g, h)}]$$

is étale.

00UH Lemma 10.143.13. Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $\mathfrak p$ be a prime of R. Let $f \bmod \mathfrak p = \overline{g}\overline{h}$ be a factorization of the image of f in $\kappa(\mathfrak p)[x]$. If $\gcd(\overline{g}, \overline{h}) = 1$, then there exist

(1) an étale ring map $R \to R'$,
(2) a prime $\mathfrak p' \subset R'$ lying over $\mathfrak p$, and
(3) a factorization $f = gh$ in $R'[x]$

such that

(1) $\kappa(\mathfrak p) = \kappa(\mathfrak p')$,
(2) $\overline{g} = g \bmod \mathfrak p'$, $\overline{h} = h \bmod \mathfrak p'$, and
(3) the polynomials g, h generate the unit ideal in $R'[x]$.

Proof. Suppose $\overline{g} = \overline{b}_0 x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = \overline{c}_0 x^m + \overline{c}_1 x^{m-1} + \ldots + \overline{c}_m$ with $\overline{b}_0, \overline{c}_0 \in \kappa(\mathfrak p)$ nonzero. After localizing R at some element of R not contained in $\mathfrak p$ we may assume $\overline{b}_0$ is the image of an invertible element $b_0 \in R$. Replacing $\overline{g}$ by $\overline{g}/b_0$ and $\overline{h}$ by $b_0\overline{h}$ we reduce to the case where $\overline{g}$, $\overline{h}$ are monic (verification omitted). Say $\overline{g} = x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = x^m + \overline{c}_1 x^{m-1} + \ldots + \overline{c}_m$. Write $f = x^{n+m} + a_1 x^{n+m-1} + \ldots + a_{n+m}$. Consider the fibre product

$$R' = R \otimes_{\mathbf{Z}[a_1, \ldots, a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

where the map $\mathbf{Z}[a_k] \to \mathbf{Z}[b_i, c_j]$ is as in Examples 10.136.7 and 10.143.12. By construction there is an R-algebra map

$$R' = R \otimes_{\mathbf{Z}[a_1, \ldots, a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \longrightarrow \kappa(\mathfrak p)$$

which maps b_i to $\overline{b}_i$ and c_j to $\overline{c}_j$. Denote $\mathfrak p' \subset R'$ the kernel of this map. Since by assumption the polynomials $\overline{g}, \overline{h}$ are relatively prime we see that the element $\Delta = \text{Res}_x(g, h) \in \mathbf{Z}[b_i, c_j]$ (see Example 10.143.12) does not map to zero in $\kappa(\mathfrak p)$ under the displayed map. We conclude that $R \to R'$ is étale at $\mathfrak p'$. In fact a solution to the problem posed in the lemma is the ring map $R \to R'[1/\Delta]$ and the prime $\mathfrak p' R'[1/\Delta]$. Because $\text{Res}_x(f, g)$ is invertible in this ring the Sylvester matrix is invertible over $R'[1/\Delta]$ and hence $1 = ag + bh$ for some $a, b \in R'[1/\Delta][x]$ see Example 10.143.12. $\square$

10.144. Local structure of étale ring maps

0G1A Lemma 10.143.2 tells us that it does not really make sense to define a standard étale morphism to be a standard smooth morphism of relative dimension 0. As a model for an étale morphism we take the example given by a finite separable extension k'/k of fields. Namely, we can always find an element $\alpha \in k'$ such that $k' = k(\alpha)$ and such that the minimal polynomial $f(x) \in k[x]$ of α has derivative f' which is relatively prime to f.

00UB Definition 10.144.1. Let R be a ring. Let $g, f \in R[x]$. Assume that f is monic and the derivative f' is invertible in the localization $R[x]_g/(f)$. In this case the ring map $R \to R[x]_g/(f)$ is said to be standard étale.

In Proposition 10.144.4 we show that every étale ring map is locally standard étale.

00UC Lemma 10.144.2. Let $R \to R[x]_g/(f)$ be standard étale.

(1) The ring map $R \to R[x]_g/(f)$ is étale.
(2) For any ring map $R \to R'$ the base change $R' \to R'[x]_g/(f)$ of the standard étale ring map $R \to R[x]_g/(f)$ is standard étale.

(3) Any principal localization of $R[x]_g/(f)$ is standard étale over R.
(4) A composition of standard étale maps is not standard étale in general.

Proof. Omitted. Here is an example for (4). The ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2}$ is standard étale. The ring map $\mathbf{F}_{2^2} \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is standard étale. But the ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is not standard étale. $\square$

Standard étale morphisms are a convenient way to produce étale maps. Here is an example.

00UD Lemma 10.144.3. Let R be a ring. Let $\mathfrak{p}$ be a prime of R. Let $L/\kappa(\mathfrak{p})$ be a finite separable field extension. There exists an étale ring map $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}')/\kappa(\mathfrak{p})$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$.

Proof. By the theorem of the primitive element we may write $L = \kappa(\mathfrak{p})[\alpha]$. Let $\overline{f} \in \kappa(\mathfrak{p})[x]$ denote the minimal polynomial for α (in particular this is monic). After replacing α by $c\alpha$ for some $c \in R$, $c \notin \mathfrak{p}$ we may assume all the coefficients of $\overline{f}$ are in the image of $R \to \kappa(\mathfrak{p})$ (verification omitted). Thus we can find a monic polynomial $f \in R[x]$ which maps to $\overline{f}$ in $\kappa(\mathfrak{p})[x]$. Since $\kappa(\mathfrak{p}) \subset L$ is separable, we see that $\gcd(\overline{f}, \overline{f}') = 1$. Hence there is an element $\gamma \in L$ such that $\overline{f}'(\alpha)\gamma = 1$. Thus we get a R-algebra map

$$\begin{array}{rcl} R[x, 1/f']/(f) & \longrightarrow & L \\ x & \longmapsto & \alpha \\ 1/f' & \longmapsto & \gamma \end{array}$$

The left hand side is a standard étale algebra R' over R and the kernel of the ring map gives the desired prime. $\square$

00UE Proposition 10.144.4. Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is étale at $\mathfrak{q}$, then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is standard étale.

Proof. The following proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 10.143.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is étale. Thus we may assume that S is étale over R.

Step 2. By Lemma 10.143.3 there exists an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that $R = R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 10.143.6. By Lemma 10.123.14 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \notin \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that of course S' is not étale over R in general.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is étale at $\mathfrak{q}$ (but no longer necessarily étale at all primes).

Step 4. Let $\mathfrak{p} \subset R$ be the prime corresponding to $\mathfrak{q}$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. This is a finite algebra over $\kappa(\mathfrak{p})$. Hence it is Artinian (see Lemma

10.53.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak p) = \prod\nolimits_{i=1}^n A_i$$

see Proposition 10.60.7. One of the factors, say A_1, is the local ring $S_\mathfrak q/\mathfrak p S_\mathfrak q$ which is isomorphic to $\kappa(\mathfrak q)$, see Lemma 10.143.5. The other factors correspond to the other primes, say $\mathfrak q_2, \ldots, \mathfrak q_n$ of S lying over $\mathfrak p$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak q)$ which generates the finite separable field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak p)^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak q)$ over $\kappa(\mathfrak p)$. Consider the element

$$\overline{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^n A_i = S \otimes_R \kappa(\mathfrak p).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\overline{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

$$\begin{tikzcd} R[x] \arrow[r] & S' \arrow[r] & S \\ R \arrow[u] \arrow[ru] \arrow[rru] & & \end{tikzcd}$$

By construction the primes $\mathfrak q_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak p, x)$ of $R[x]$, whereas the prime $\mathfrak q$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak q' \subset S'$ the prime of S' corresponding to $\mathfrak q$. By the above $\mathfrak q$ is the only prime of S lying over $\mathfrak q'$. Thus we see that $S_\mathfrak q = S_{\mathfrak q'}$, see Lemma 10.41.11 (we have going up for $S' \to S$ by Lemma 10.36.22 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak q'} \to S_\mathfrak q$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_\mathfrak p \to S'_{\mathfrak q'} \to S_\mathfrak q$$

The second map is finite and injective. We have $S_\mathfrak q/\mathfrak p S_\mathfrak q = \kappa(\mathfrak q)$, see Lemma 10.143.5. Hence a fortiori $S_\mathfrak q/\mathfrak q' S_\mathfrak q = \kappa(\mathfrak q)$. Since

$$\kappa(\mathfrak p) \subset \kappa(\mathfrak q') \subset \kappa(\mathfrak q)$$

and since α is in the image of $\kappa(\mathfrak q')$ in $\kappa(\mathfrak q)$ we conclude that $\kappa(\mathfrak q') = \kappa(\mathfrak q)$. Hence by Nakayama's Lemma 10.20.1 applied to the $S'_{\mathfrak q'}$-module map $S'_{\mathfrak q'} \to S_\mathfrak q$, the map $S'_{\mathfrak q'} \to S_\mathfrak q$ is surjective. In other words, $S'_{\mathfrak q'} \cong S_\mathfrak q$.

Step 7. By Lemma 10.126.7 there exist $g \in S$, $g \notin \mathfrak q$ and $g' \in S'$, $g' \notin \mathfrak q'$ such that $S'_{g'} \cong S_g$. As R is Noetherian the ring S' is finite over R because it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is étale over R at $\mathfrak q$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak p) = \kappa(\mathfrak p)[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak p)[x]$ is the ideal generated by I in $\kappa(\mathfrak p)[x]$. As $\kappa(\mathfrak p)[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak p)[x]$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak p)$ we may assume that $\overline{h}$ is the image of some $h \in I \subset R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak p) = A_1 \times \ldots \times A_n$ with

$A_1 = \kappa(\mathfrak q)$ a finite separable extension of $\kappa(\mathfrak p)$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak p)[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak p)[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak p)[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak q)$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak p)[x]$. We may factor

$$\overline{m} = \overline{k}\overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak p)[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l \deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak p)[x]$ factors as

$$\overline{f} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n} = \overline{h}_1(\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1 - 1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n}) = \overline{h}_1 \overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak q$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak q) = \kappa(\mathfrak p)[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak p)[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w}\frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1 \frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 10.144.2) and $\varphi(g) \notin \mathfrak q$. Pick an element $g' \in R[x]/(f)$ such that also $\varphi(g') \notin \mathfrak q$ and $S_{\varphi(g')}$ is étale over R (which exists since S is étale over R at $\mathfrak q$). Then the ring map $R[x]_{gg'}/(f) \to S_{\varphi(gg')}$ is a surjective map of étale algebras over R. Hence it is étale by Lemma 10.143.8. Hence it is a localization by Lemma 10.143.9. Thus a localization of S at an element not in $\mathfrak q$ is isomorphic to a localization of a standard étale algebra over R which is what we wanted to show. ☐

The following two lemmas say that the étale topology is coarser than the topology generated by Zariski coverings and finite flat morphisms. They should be skipped on a first reading.

00UF Lemma 10.144.5. Let $R \to S$ be a standard étale morphism. There exists a ring map $R \to S'$ with the following properties

(1) $R \to S'$ is finite, finitely presented, and flat (in other words S' is finite projective as an R-module),
(2) $\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ is surjective,
(3) for every prime $\mathfrak q \subset S$, lying over $\mathfrak p \subset R$ and every prime $\mathfrak q' \subset S'$ lying over $\mathfrak p$ there exists a $g' \in S'$, $g' \notin \mathfrak q'$ such that the ring map $R \to S'_{g'}$ factors through a map $\varphi : S \to S'_{g'}$ with $\varphi^{-1}(\mathfrak q' S'_{g'}) = \mathfrak q$.

Proof. Let $S = R[x]_g/(f)$ be a presentation of S as in Definition 10.144.1. Write $f = x^n + a_1x^{n-1} + \ldots + a_n$ with $a_i \in R$. By Lemma 10.136.14 there exists a finite locally free and faithfully flat ring map $R \to S'$ such that $f = \prod(x - \alpha_i)$ for certain $\alpha_i \in S'$. Hence $R \to S'$ satisfies conditions (1), (2). Let $\mathfrak q \subset R[x]/(f)$ be a prime ideal with $g \notin \mathfrak q$ (i.e., it corresponds to a prime of S). Let $\mathfrak p = R \cap \mathfrak q$ and let $\mathfrak q' \subset S'$ be a prime lying over $\mathfrak p$. Note that there are n maps of R-algebras

$$\begin{array}{rcl} \varphi_i : R[x]/(f) & \longrightarrow & S' \\ x & \longmapsto & \alpha_i \end{array}$$

To finish the proof we have to show that for some i we have (a) the image of $\varphi_i(g)$ in $\kappa(\mathfrak q')$ is not zero, and (b) $\varphi_i^{-1}(\mathfrak q') = \mathfrak q$. Because then we can just take $g' = \varphi_i(g)$, and $\varphi = \varphi_i$ for that i.

Let $\overline{f}$ denote the image of f in $\kappa(\mathfrak p)[x]$. Note that as a point of $\text{Spec}(\kappa(\mathfrak p)[x]/(\overline{f}))$ the prime $\mathfrak q$ corresponds to an irreducible factor f_1 of $\overline{f}$. Moreover, $g \notin \mathfrak q$ means that f_1 does not divide the image $\overline{g}$ of g in $\kappa(\mathfrak p)[x]$. Denote $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ the images of $\alpha_1, \ldots, \alpha_n$ in $\kappa(\mathfrak q')$. Note that the polynomial $\overline{f}$ splits completely in $\kappa(\mathfrak q')[x]$, namely

$$\overline{f} = \prod\nolimits_i (x - \overline{\alpha}_i)$$

Moreover $\varphi_i(g)$ reduces to $\overline{g}(\overline{\alpha}_i)$. It follows we may pick i such that $f_1(\overline{\alpha}_i) = 0$ and $\overline{g}(\overline{\alpha}_i) \neq 0$. For this i properties (a) and (b) hold. Some details omitted. □

00UG **Lemma 10.144.6.** Let $R \to S$ be a ring map. Assume that

1. $R \to S$ is étale, and
2. $\text{Spec}(S) \to \text{Spec}(R)$ is surjective.

Then there exists a ring map $R \to S'$ such that

1. $R \to S'$ is finite, finitely presented, and flat (in other words it is finite projective as an R-module),
2. $\text{Spec}(S') \to \text{Spec}(R)$ is surjective,
3. for every prime $\mathfrak q' \subset S'$ there exists a $g' \in S'$, $g' \notin \mathfrak q'$ such that the ring map $R \to S'_{g'}$ factors as $R \to S \to S'_{g'}$.

Proof. By Proposition 10.144.4 and the quasi-compactness of $\text{Spec}(S)$ (see Lemma 10.17.10) we can find $g_1, \ldots, g_n \in S$ generating the unit ideal of S such that each $R \to S_{g_i}$ is standard étale. If we prove the lemma for the ring map $R \to \prod_{i=1,\ldots,n} S_{g_i}$ then the lemma follows for the ring map $R \to S$. Hence we may assume that $S = \prod_{i=1,\ldots,n} S_i$ is a finite product of standard étale morphisms.

For each i choose a ring map $R \to S'_i$ as in Lemma 10.144.5 adapted to the standard étale morphism $R \to S_i$. Set $S' = S'_1 \otimes_R \ldots \otimes_R S'_n$; we will use the R-algebra maps $S'_i \to S'$ without further mention below. We claim this works. Properties (1) and (2) are immediate. For property (3) suppose that $\mathfrak q' \subset S'$ is a prime. Denote $\mathfrak p$ its image in $\text{Spec}(R)$. Choose $i \in \{1, \ldots, n\}$ such that $\mathfrak p$ is in the image of $\text{Spec}(S_i) \to \text{Spec}(R)$; this is possible by assumption. Set $\mathfrak q'_i \subset S'_i$ the image of $\mathfrak q'$ in the spectrum of S'_i. By construction of S'_i there exists a $g'_i \in S'_i$ such that $R \to (S'_i)_{g'_i}$ factors as $R \to S_i \to (S'_i)_{g'_i}$. Hence also $R \to S'_{g'_i}$ factors as

$$R \to S_i \to (S'_i)_{g'_i} \to S'_{g'_i}$$

as desired. □

10.145. Étale local structure of quasi-finite ring maps

0G1B The following lemmas say roughly that after an étale extension a quasi-finite ring map becomes finite. To help interpret the results recall that the locus where a finite type ring map is quasi-finite is open (see Lemma 10.123.13) and that formation of this locus commutes with arbitrary base change (see Lemma 10.122.8).

00UI **Lemma 10.145.1.** *Let $R \to S' \to S$ be ring maps. Let $\mathfrak p \subset R$ be a prime. Let $g \in S'$ be an element. Assume*

1. *$R \to S'$ is integral,*
2. *$R \to S$ is finite type,*
3. *$S'_g \cong S_g$, and*
4. *g invertible in $S' \otimes_R \kappa(\mathfrak p)$.*

Then there exists a $f \in R$, $f \notin \mathfrak p$ such that $R_f \to S_f$ is finite.

Proof. By assumption the image T of $V(g) \subset \mathrm{Spec}(S')$ under the morphism $\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ does not contain $\mathfrak p$. By Section 10.41 especially, Lemma 10.41.6 we see T is closed. Pick $f \in R$, $f \notin \mathfrak p$ such that $T \cap D(f) = \emptyset$. Then we see that g becomes invertible in S'_f. Hence $S'_f \cong S_f$. Thus S_f is both of finite type and integral over R_f, hence finite. □

00UJ **Lemma 10.145.2.** *Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p \subset R$. Assume $R \to S$ finite type and quasi-finite at $\mathfrak q$. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak p' \subset R'$ lying over $\mathfrak p$,*
3. *a product decomposition*

$$R' \otimes_R S = A \times B$$

with the following properties

1. *$\kappa(\mathfrak p) = \kappa(\mathfrak p')$,*
2. *$R' \to A$ is finite,*
3. *A has exactly one prime $\mathfrak r$ lying over $\mathfrak p'$, and*
4. *$\mathfrak r$ lies over $\mathfrak q$.*

Proof. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak q' = S' \cap \mathfrak q$. By Zariski's Main Theorem 10.123.12 there exists a $g \in S'$, $g \notin \mathfrak q'$ such that $S'_g \cong S_g$. Consider the fibre rings $F = S \otimes_R \kappa(\mathfrak p)$ and $F' = S' \otimes_R \kappa(\mathfrak p)$. Denote $\overline{\mathfrak q}'$ the prime of F' corresponding to $\mathfrak q'$. Since F' is integral over $\kappa(\mathfrak p)$ we see that $\overline{\mathfrak q}'$ is a closed point of $\mathrm{Spec}(F')$, see Lemma 10.36.19. Note that $\mathfrak q$ defines an isolated closed point $\overline{\mathfrak q}$ of $\mathrm{Spec}(F)$ (see Definition 10.122.3). Since $S'_g \cong S_g$ we have $F'_g \cong F_g$, so $\overline{\mathfrak q}$ and $\overline{\mathfrak q}'$ have isomorphic open neighbourhoods in $\mathrm{Spec}(F)$ and $\mathrm{Spec}(F')$. We conclude the set $\{\overline{\mathfrak q}'\} \subset \mathrm{Spec}(F')$ is open. Combined with $\mathfrak q'$ being closed (shown above) we conclude that $\overline{\mathfrak q}'$ defines an isolated closed point of $\mathrm{Spec}(F')$ as well.

An additional small remark is that under the map $\mathrm{Spec}(F) \to \mathrm{Spec}(F')$ the point $\overline{\mathfrak q}$ is the only point mapping to $\overline{\mathfrak q}'$. This follows from the discussion above.

By Lemma 10.24.3 we may write $F' = F'_1 \times F'_2$ with $\mathrm{Spec}(F'_1) = \{\overline{\mathfrak q}'\}$. Since $F' = S' \otimes_R \kappa(\mathfrak p)$, there exists an $s' \in S'$ which maps to the element $(r, 0) \in F'_1 \times F'_2 = F'$ for some $r \in R$, $r \notin \mathfrak p$. In fact, what we will use about s' is that it is an element of S', not contained in $\mathfrak q'$, and contained in any other prime lying over $\mathfrak p$.

Let $f(x) \in R[x]$ be a monic polynomial such that $f(s') = 0$. Denote $\overline{f} \in \kappa(\mathfrak p)[x]$ the image. We can factor it as $\overline{f} = x^e\overline{h}$ where $\overline{h}(0) \neq 0$. After replacing f by xf if necessary, we may assume $e \geq 1$. By Lemma 10.143.13 we can find an étale ring extension $R \to R'$, a prime $\mathfrak p'$ lying over $\mathfrak p$, and a factorization $f = hi$ in $R'[x]$ such that $\kappa(\mathfrak p) = \kappa(\mathfrak p')$, $\overline{h} = h \bmod \mathfrak p'$, $x^e = i \bmod \mathfrak p'$, and we can write $ah + bi = 1$ in $R'[x]$ (for suitable a, b).

Consider the elements $h(s'), i(s') \in R' \otimes_R S'$. By construction we have $h(s')i(s') = f(s') = 0$. On the other hand they generate the unit ideal since $a(s')h(s') + b(s')i(s') = 1$. Thus we see that $R' \otimes_R S'$ is the product of the localizations at these elements:

$$R' \otimes_R S' = (R' \otimes_R S')_{i(s')} \times (R' \otimes_R S')_{h(s')} = S'_1 \times S'_2$$

Moreover this product decomposition is compatible with the product decomposition we found for the fibre ring F'; this comes from our choices of s', i, h which guarantee that $\overline{\mathfrak q}'$ is the only prime of F' which does not contain the image of $i(s')$ in F'. Here we use that the fibre ring of $R' \otimes_R S'$ over R' at $\mathfrak p'$ is the same as F' due to the fact that $\kappa(\mathfrak p) = \kappa(\mathfrak p')$. It follows that S'_1 has exactly one prime, say $\mathfrak r'$, lying over $\mathfrak p'$ and that this prime lies over $\mathfrak q'$. Hence the element $g \in S'$ maps to an element of S'_1 not contained in $\mathfrak r'$.

The base change $R' \otimes_R S$ inherits a similar product decomposition

$$R' \otimes_R S = (R' \otimes_R S)_{i(s')} \times (R' \otimes_R S)_{h(s')} = S_1 \times S_2$$

It follows from the above that S_1 has exactly one prime, say $\mathfrak r$, lying over $\mathfrak p'$ (consider the fibre ring as above), and that this prime lies over $\mathfrak q$.

Now we may apply Lemma 10.145.1 to the ring maps $R' \to S'_1 \to S_1$, the prime $\mathfrak p'$ and the element g to see that after replacing R' by a principal localization we can assume that S_1 is finite over R' as desired. $\square$

00UK Lemma 10.145.3. Let $R \to S$ be a ring map. Let $\mathfrak p \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists

(1) an étale ring map $R \to R'$,
(2) a prime $\mathfrak p' \subset R'$ lying over $\mathfrak p$,
(3) a product decomposition

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) we have $\kappa(\mathfrak p) = \kappa(\mathfrak p')$,
(2) each A_i is finite over R',
(3) each A_i has exactly one prime $\mathfrak r_i$ lying over $\mathfrak p'$, and
(4) $R' \to B$ not quasi-finite at any prime lying over $\mathfrak p'$.

Proof. Denote $F = S \otimes_R \kappa(\mathfrak p)$ the fibre ring of S/R at the prime $\mathfrak p$. As F is of finite type over $\kappa(\mathfrak p)$ it is Noetherian and hence $\mathrm{Spec}(F)$ has finitely many isolated closed points. If there are no isolated closed points, i.e., no primes $\mathfrak q$ of S over $\mathfrak p$ such that S/R is quasi-finite at $\mathfrak q$, then the lemma holds. If there exists at least one such prime $\mathfrak{q}$, then we may apply Lemma 10.145.2. This gives a diagram

$$\begin{tikzcd} S \arrow[r] & R' \otimes_R S \arrow[r, equal] & A_1 \times B' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[ru] & \end{tikzcd}$$

as in said lemma. Since the residue fields at $\mathfrak{p}$ and $\mathfrak{p}'$ are the same, the fibre rings of S/R and $(A_1 \times B')/R'$ are the same. Hence, by induction on the number of isolated closed points of the fibre we may assume that the lemma holds for $R' \to B'$ and $\mathfrak{p}'$. Thus we get an étale ring map $R' \to R''$, a prime $\mathfrak{p}'' \subset R''$ and a decomposition

$$R'' \otimes_{R'} B' = A_2 \times \ldots \times A_n \times B$$

We omit the verification that the ring map $R \to R''$, the prime $\mathfrak{p}''$ and the resulting decomposition

$$R'' \otimes_R S = (R'' \otimes_{R'} A_1) \times A_2 \times \ldots \times A_n \times B$$

is a solution to the problem posed in the lemma. □

00UL Lemma 10.145.4. Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists

(1) an étale ring map $R \to R'$,
(2) a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,
(3) a product decomposition

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) each A_i is finite over R',
(2) each A_i has exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}'$,
(3) the finite field extensions $\kappa(\mathfrak{r}_i)/\kappa(\mathfrak{p}')$ are purely inseparable, and
(4) $R' \to B$ not quasi-finite at any prime lying over $\mathfrak{p}'$.

Proof. The strategy of the proof is to make two étale ring extensions: first we control the residue fields, then we apply Lemma 10.145.3.

Denote $F = S \otimes_R \kappa(\mathfrak{p})$ the fibre ring of S/R at the prime $\mathfrak{p}$. As in the proof of Lemma 10.145.3 there are finitely may primes, say $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of S lying over R at which the ring map $R \to S$ is quasi-finite. Let $\kappa(\mathfrak{p}) \subset L_i \subset \kappa(\mathfrak{q}_i)$ be the subfield such that $\kappa(\mathfrak{p}) \subset L_i$ is separable, and the field extension $\kappa(\mathfrak{q}_i)/L_i$ is purely inseparable. Let $L/\kappa(\mathfrak{p})$ be a finite Galois extension into which L_i embeds for $i = 1, \ldots, n$. By Lemma 10.144.3 we can find an étale ring extension $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}')/\kappa(\mathfrak{p})$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$. Thus the fibre ring of $R' \otimes_R S$ at $\mathfrak{p}'$ is isomorphic to $F \otimes_{\kappa(\mathfrak{p})} L$. The primes lying over $\mathfrak{q}_i$ correspond to primes of $\kappa(\mathfrak{q}_i) \otimes_{\kappa(\mathfrak{p})} L$ which is a product of fields purely inseparable over L by our choice of L and elementary field theory. These are also the only primes over $\mathfrak{p}'$ at which $R' \to R' \otimes_R S$ is quasi-finite, by Lemma 10.122.8. Hence after replacing R by R', $\mathfrak{p}$ by $\mathfrak{p}'$, and S by $R' \otimes_R S$ we may assume that for all primes $\mathfrak{q}$ lying over $\mathfrak{p}$ for which S/R is quasi-finite the field extensions $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ are purely inseparable.

Next apply Lemma 10.145.3. The result is what we want since the field extensions do not change under this étale ring extension. □

10.146. Local homomorphisms

053J Some lemmas which don't have a natural section to go into. The first lemma says, loosely speaking, that an étale map of local rings is an isomorphism modulo all powers of a nonunit principal ideal.

0GSD Lemma 10.146.1. Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume S is the localization of an étale ring extension of R and that $\kappa(\mathfrak{m}_R) \to \kappa(\mathfrak{m}_S)$ is an isomorphism. Then there exists an $t \in \mathfrak{m}_R$ such that $R/t^nR \to S/t^nS$ is an isomorphsm for all $n \geq 1$. [Lin82, Lemma on page 321], [Ces22, Lemma 4.1.5]

Proof. Write $S = T_\mathfrak{q}$ for some étale R-algebra T and prime ideal $\mathfrak{q} \subset T$ lying over $\mathfrak{m}_R$. By Proposition 10.144.4 we may assume $R \to T$ is standard étale. Write $T = R[x]_g/(f)$ as in Definition 10.144.1. By our assumption on residue fields, we may choose $a \in R$ such that x and a have the same image in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{m}_S) = \kappa(\mathfrak{m}_R)$. Then after replacing x by $x - a$ we may assume that $\mathfrak{q}$ is generated by x and $\mathfrak{m}_R$ in T. In particular $t = f(0) \in \mathfrak{m}_R$. We will show that $t = f(0)$ works.

Write $f = x^d + \sum_{i=1,\ldots,d-1} a_i x^i + t$. Since $R \to T$ is standard étale we find that a_1 is a unit in R: the derivative of f is invertible in T in particular is not contained in $\mathfrak{q}$. Let $h = a_1 + a_2 x + \ldots + a_{d-1}x^{d-2} + x^{d-1} \in R[x]$ so that $f = t + xh$ in $R[x]$. We see that $h \notin \mathfrak{q}$ and hence we may replace T by $R[x]_{hg}/(f)$. After this replacement we see that

$$T/tT = (R/tR)[x]_{hg}/(f) = (R/tR)[x]_{hg}/(xh) = (R/tR)[x]_{hg}/(x)$$

is a quotient of R/tR. By Lemma 10.126.9 we conclude that $R/t^nR \to T/t^nT$ is surjective for all $n \geq 1$. On the other hand, we know that the flat local ring map $R/t^nR \to S/t^nS$ factors through $R/t^nR \to T/t^nT$ for all n, hence these maps are also injective (a flat local homomorphism of local rings is faithfully flat and hence injective, see Lemmas 10.39.17 and 10.82.11). As S is the localization of T we see that S/t^nS is the localization of $T/t^nT = R/t^nR$ at a prime lying over the maximal ideal, but this ring is already local and the proof is complete. □

053K Lemma 10.146.2. Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume S is the localization of an étale ring extension of R. Then there exists a finite, finitely presented, faithfully flat ring map $R \to S'$ such that for every maximal ideal $\mathfrak{m}'$ of S' there is a factorization

$$R \to S \to S'_{\mathfrak{m}'}.$$

of the ring map $R \to S'_{\mathfrak{m}'}$.

Proof. Write $S = T_\mathfrak{q}$ for some étale R-algebra T. By Proposition 10.144.4 we may assume T is standard étale. Apply Lemma 10.144.5 to the ring map $R \to T$ to get $R \to S'$. Then in particular for every maximal ideal $\mathfrak{m}'$ of S' we get a factorization $\varphi : T \to S'_{g'}$ for some $g' \notin \mathfrak{m}'$ such that $\mathfrak{q} = \varphi^{-1}(\mathfrak{m}' S'_{g'})$. Thus φ induces the desired local ring map $S \to S'_{\mathfrak{m}'}$. □

10.147. Integral closure and smooth base change

03GC

03GD Lemma 10.147.1. Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $R \to B$ be a ring map. If $h \in B[x]/(f)$ is integral over R, then the element $f'h$ can be written as $f'h = \sum_i b_i x^i$ with $b_i \in B$ integral over R.

Proof. Say $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ with $r_i \in R$. There exists a finite free ring extension $B \subset B'$ such that $f = (x - \alpha_1) \ldots (x - \alpha_d)$ for some $\alpha_i \in B'$, see Lemma 10.136.14. Note that each α_i is integral over R. We may represent $h = h_0 + h_1 x + \ldots + h_{d-1} x^{d-1}$ with $h_i \in B$. Then it is a universal fact that

$$f'h = \sum\nolimits_{i=1,\ldots,d} h(\alpha_i)(x - \alpha_1) \ldots \widehat{(x - \alpha_i)} \ldots (x - \alpha_d)$$

as elements of $B'[x]/(f)$. You prove this by evaluating both sides at the points α_i over the ring $B_{univ} = \mathbf{Z}[\alpha_i, h_j]$ (some details omitted). By our assumption that h satisfies $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ we see that

$$h(\alpha_i)^e + r_1 h(\alpha_i)^{e-1} + \ldots + r_e = 0$$

in B'. Hence $h(\alpha_i)$ is integral over R. Using the formula above we see that $f'h \equiv \sum_{j=0,\ldots,d-1} b'_j x^j$ in $B'[x]/(f)$ with $b'_j \in B'$ integral over R. However, since $f'h \in B[x]/(f)$ and since $1, x, \ldots, x^{d-1}$ is a B'-basis for $B'[x]/(f)$ we see that $b'_j \in B$ as desired. $\square$

03GE **Lemma 10.147.2.** Let $R \to S$ be an étale ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.

Proof. The map $S \otimes_R A \to A'$ is injective because $A \subset B$ and $R \to S$ is flat. We are going to use repeatedly that taking integral closure commutes with localization, see Lemma 10.36.11. Hence we may localize on S, by Lemma 10.23.2 (the criterion for checking whether an S-module map is an isomorphism). Thus we may assume that $S = R[x]_g/(f) = (R[x]/(f))_g$ is standard étale over R, see Proposition 10.144.4. Applying localization one more time we see that A' is $(A'')_g$ where A'' is the integral closure of $R[x]/(f)$ in $B[x]/(f)$. Suppose that $a \in A''$. It suffices to show that a is in $S \otimes_R A$. By Lemma 10.147.1 we see that $f'a = \sum a_i x^i$ with $a_i \in A$. Since f' is invertible in S (by definition of a standard étale ring map) we conclude that $a \in S \otimes_R A$ as desired. $\square$

03GF **Example 10.147.3.** Let p be a prime number. The ring extension

$$R = \mathbf{Z}[1/p] \subset R' = \mathbf{Z}[1/p][x]/(x^{p-1} + \ldots + x + 1)$$

has the following property: For $d < p$ there exist elements $\alpha_0, \ldots, \alpha_{d-1} \in R'$ such that

$$\prod\nolimits_{0 \leq i < j < d} (\alpha_i - \alpha_j)$$

is a unit in R'. Namely, take α_i equal to the class of x^i in R' for $i = 0, \ldots, p-1$. Then we have

$$T^p - 1 = \prod\nolimits_{i=0,\ldots,p-1} (T - \alpha_i)$$

in $R'[T]$. Namely, the ring $\mathbf{Q}[x]/(x^{p-1} + \ldots + x + 1)$ is a field because the cyclotomic polynomial $x^{p-1} + \ldots + x + 1$ is irreducible over $\mathbf{Q}$ and the α_i are pairwise distinct roots of $T^p - 1$, whence the equality. Taking derivatives on both sides and substituting $T = \alpha_i$ we obtain

$$p\alpha_i^{p-1} = (\alpha_i - \alpha_1) \ldots \widehat{(\alpha_i - \alpha_i)} \ldots (\alpha_i - \alpha_1)$$

and we see this is invertible in R'.

03GG Lemma 10.147.4. Let $R \to S$ be a smooth ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.

Proof. Arguing as in the proof of Lemma 10.147.2 we may localize on S. Hence we may assume that $R \to S$ is a standard smooth ring map, see Lemma 10.137.10. By definition of a standard smooth ring map we see that S is étale over a polynomial ring $R[x_1, \ldots, x_n]$. Since we have seen the result in the case of an étale ring extension (Lemma 10.147.2) this reduces us to the case where $S = R[x]$. Thus we have to show

$$f = \sum b_i x^i \text{ integral over } R[x] \Leftrightarrow \text{each } b_i \text{ integral over } R.$$

The implication from right to left holds because the set of elements in $B[x]$ integral over $R[x]$ is a ring (Lemma 10.36.7) and contains x.

Suppose that $f \in B[x]$ is integral over $R[x]$, and assume that $f = \sum_{i<d} b_i x^i$ has degree $< d$. Since integral closure and localization commute, it suffices to show there exist distinct primes p, q such that each b_i is integral both over $R[1/p]$ and over $R[1/q]$. Hence, we can find a finite free ring extension $R \subset R'$ such that R' contains $\alpha_1, \ldots, \alpha_d$ with the property that $\prod_{i<j}(\alpha_i - \alpha_j)$ is a unit in R', see Example 10.147.3. In this case we have the universal equality

$$f = \sum_i f(\alpha_i) \frac{(x - \alpha_1) \ldots \widehat{(x - \alpha_i)} \ldots (x - \alpha_d)}{(\alpha_i - \alpha_1) \ldots \widehat{(\alpha_i - \alpha_i)} \ldots (\alpha_i - \alpha_d)}.$$

OK, and the elements $f(\alpha_i)$ are integral over R' since $(R' \otimes_R B)[x] \to R' \otimes_R B$, $h \mapsto h(\alpha_i)$ is a ring map. Hence we see that the coefficients of f in $(R' \otimes_R B)[x]$ are integral over R'. Since R' is finite over R (hence integral over R) we see that they are integral over R also, as desired. □

0CBF Lemma 10.147.5. Let $R \to S$ and $R \to B$ be ring maps. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. If S is a filtered colimit of smooth R-algebras, then the canonical map $S \otimes_R A \to A'$ is an isomorphism.

Proof. This follows from the straightforward fact that taking tensor products and taking integral closures commutes with filtered colimits and Lemma 10.147.4. □

10.148. Formally unramified maps

00UM It turns out to be logically more efficient to define the notion of a formally unramified map before introducing the notion of a formally étale one.

00UN Definition 10.148.1. Let $R \to S$ be a ring map. We say S is formally unramified over R if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there exists at most one dotted arrow making the diagram commute.

00UO Lemma 10.148.2. Let $R \to S$ be a ring map. The following are equivalent:

(1) $R \to S$ is formally unramified,
(2) the module of differentials $\Omega_{S/R}$ is zero.

Proof. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. Let $A_{univ} = S \otimes_R S/J^2$. Recall that $I_{univ} = J/J^2$ is isomorphic to $\Omega_{S/R}$, see Lemma 10.131.13. Moreover, the two R-algebra maps $\sigma_1, \sigma_2 : S \to A_{univ}$, $\sigma_1(s) = s \otimes 1 \bmod J^2$, and $\sigma_2(s) = 1 \otimes s \bmod J^2$ differ by the universal derivation $\mathrm{d} : S \to \Omega_{S/R} = I_{univ}$.

Assume $R \to S$ formally unramified. Then we see that $\sigma_1 = \sigma_2$. Hence $\mathrm{d}(s) = 0$ for all $s \in S$. Hence $\Omega_{S/R} = 0$.

Assume that $\Omega_{S/R} = 0$. Let $A, I, R \to A, S \to A/I$ be a solid diagram as in Definition 10.148.1. Let $\tau_1, \tau_2 : S \to A$ be two dotted arrows making the diagram commute. Consider the R-algebra map $A_{univ} \to A$ defined by the rule $s_1 \otimes s_2 \mapsto \tau_1(s_1)\tau_2(s_2)$. We omit the verification that this is well defined. Since $A_{univ} \cong S$ as $I_{univ} = \Omega_{S/R} = 0$ we conclude that $\tau_1 = \tau_2$. □

04E8 Lemma 10.148.3. Let $R \to S$ be a ring map. The following are equivalent:

(1) $R \to S$ is formally unramified,
(2) $R \to S_{\mathfrak q}$ is formally unramified for all primes $\mathfrak q$ of S, and
(3) $R_{\mathfrak p} \to S_{\mathfrak q}$ is formally unramified for all primes $\mathfrak q$ of S with $\mathfrak p = R \cap \mathfrak q$.

Proof. We have seen in Lemma 10.148.2 that (1) is equivalent to $\Omega_{S/R} = 0$. Similarly, by Lemma 10.131.8 we see that (2) and (3) are equivalent to $(\Omega_{S/R})_{\mathfrak q} = 0$ for all $\mathfrak q$. Hence the equivalence follows from Lemma 10.23.1. □

04E9 Lemma 10.148.4. Let $A \to B$ be a formally unramified ring map.

(1) For $S \subset A$ a multiplicative subset, $S^{-1}A \to S^{-1}B$ is formally unramified.
(2) For $S \subset B$ a multiplicative subset, $A \to S^{-1}B$ is formally unramified.

Proof. Follows from Lemma 10.148.3. (You can also deduce it from Lemma 10.148.2 combined with Lemma 10.131.8.) □

07QE Lemma 10.148.5. Let R be a ring. Let I be a directed set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally unramified, then $S = \mathrm{colim}_{i \in I} S_i$ is formally unramified over R

Proof. Consider a diagram as in Definition 10.148.1. By assumption there exists at most one R-algebra map $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Since every element of S is in the image of one of the maps $S_i \to S$ we see that there is at most one map $S \to A$ fitting into the diagram. □

10.149. Conormal modules and universal thickenings

04EA It turns out that one can define the first infinitesimal neighbourhood not just for a closed immersion of schemes, but already for any formally unramified morphism. This is based on the following algebraic fact.

04EB Lemma 10.149.1. Let $R \to S$ be a formally unramified ring map. There exists a surjection of R-algebras $S' \to S$ whose kernel is an ideal of square zero with the following universal property: Given any commutative diagram

$$\begin{tikzcd} S \arrow[r, "a"'] & A/I \\ R \arrow[u] \arrow[r, "b"] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there is a unique R-algebra map $a' : S' \to A$ such that $S' \to A \to A/I$ is equal to $S' \to S \to A/I$.

Proof. Choose a set of generators $z_i \in S$, $i \in I$ for S as an R-algebra. Let $P = R[\{x_i\}_{i \in I}]$ denote the polynomial ring on generators x_i, $i \in I$. Consider the R-algebra map $P \to S$ which maps x_i to z_i. Let $J = \mathrm{Ker}(P \to S)$. Consider the map

$$\mathrm{d} : J/J^2 \longrightarrow \Omega_{P/R} \otimes_P S$$

see Lemma 10.131.9. This is surjective since $\Omega_{S/R} = 0$ by assumption, see Lemma 10.148.2. Note that $\Omega_{P/R}$ is free on $\mathrm{d}x_i$, and hence the module $\Omega_{P/R} \otimes_P S$ is free over S. Thus we may choose a splitting of the surjection above and write

$$J/J^2 = K \oplus \Omega_{P/R} \otimes_P S$$

Let $J^2 \subset J' \subset J$ be the ideal of P such that J'/J^2 is the second summand in the decomposition above. Set $S' = P/J'$. We obtain a short exact sequence

$$0 \to J/J' \to S' \to S \to 0$$

and we see that $J/J' \cong K$ is a square zero ideal in S'. Hence

$$\begin{tikzcd} S \arrow[r, "1"'] & S \\ R \arrow[u] \arrow[r] & S' \arrow[u] \end{tikzcd}$$

is a diagram as above. In fact we claim that this is an initial object in the category of diagrams. Namely, let $(I \subset A, a, b)$ be an arbitrary diagram. We may choose an R-algebra map $\beta : P \to A$ such that

$$\begin{tikzcd} S \arrow[r, "1"'] & S \arrow[r, "a"'] & A/I \\ R \arrow[u] \arrow[r] \arrow[rr, bend right, "b"'] & P \arrow[u] \arrow[r, "\beta"] & A \arrow[u] \end{tikzcd}$$

is commutative. Now it may not be the case that $\beta(J') = 0$, in other words it may not be true that β factors through $S' = P/J'$. But what is clear is that $\beta(J') \subset I$ and since $\beta(J) \subset I$ and $I^2 = 0$ we have $\beta(J^2) = 0$. Thus the "obstruction" to finding a morphism from $(J/J' \subset S', 1, R \to S')$ to $(I \subset A, a, b)$ is the corresponding S-linear map $\overline{\beta} : J'/J^2 \to I$. The choice in picking β lies in the choice of $\beta(x_i)$. A different choice of β, say β', is gotten by taking $\beta'(x_i) = \beta(x_i) + \delta_i$ with $\delta_i \in I$. In this case, for $g \in J'$, we obtain

$$\beta'(g) = \beta(g) + \sum\nolimits_i \delta_i \frac{\partial g}{\partial x_i}.$$

Since the map $\mathrm{d}|_{J'/J^2} : J'/J^2 \to \Omega_{P/R} \otimes_P S$ given by $g \mapsto \frac{\partial g}{\partial x_i} \mathrm{d}x_i$ is an isomorphism by construction, we see that there is a unique choice of $\delta_i \in I$ such that $\beta'(g) = 0$

for all $g \in J'$. (Namely, δ_i is $-\overline{\beta}(g)$ where $g \in J'/J^2$ is the unique element with $\frac{\partial g}{\partial x_j} = 1$ if $i = j$ and 0 else.) The uniqueness of the solution implies the uniqueness required in the lemma. $\square$

In the situation of Lemma 10.149.1 the R-algebra map $S' \to S$ is unique up to unique isomorphism.

04EC Definition 10.149.2. Let $R \to S$ be a formally unramified ring map.

(1) The universal first order thickening of S over R is the surjection of R-algebras $S' \to S$ of Lemma 10.149.1.
(2) The conormal module of $R \to S$ is the kernel I of the universal first order thickening $S' \to S$, seen as an S-module.

We often denote the conormal module $C_{S/R}$ in this situation.

04ED Lemma 10.149.3. Let $I \subset R$ be an ideal of a ring. The universal first order thickening of R/I over R is the surjection $R/I^2 \to R/I$. The conormal module of R/I over R is $C_{(R/I)/R} = I/I^2$.

Proof. Omitted. $\square$

04EE Lemma 10.149.4. Let $A \to B$ be a formally unramified ring map. Let $\varphi : B' \to B$ be the universal first order thickening of B over A.

(1) Let $S \subset A$ be a multiplicative subset. Then $S^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/S^{-1}A}$.
(2) Let $S \subset B$ be a multiplicative subset. Then $S' = \varphi^{-1}(S)$ is a multiplicative subset in B' and $(S')^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over A. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/A}$.

Note that the lemma makes sense by Lemma 10.148.4.

Proof. With notation and assumptions as in (1). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. Note that $S^{-1}B' \to S^{-1}B$ is a surjection of $S^{-1}A$-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to S^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{tikzcd} B \arrow[r] & S^{-1}B \arrow[r] & D/I \\ A \arrow[u] \arrow[r] & S^{-1}A \arrow[u] \arrow[r] & D \arrow[u] \end{tikzcd}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S in D is mapped to invertible elements of D, and hence we obtain a compatible map $S^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $S^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above.

With notation and assumptions as in (2). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over A. Note that $(S')^{-1}B' \to S^{-1}B$ is a surjection of A-algebras whose kernel has square zero. Hence by definition we obtain a map

$(S^{-1}B)' \to (S')^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{tikzcd} B \arrow[r] & S^{-1}B \arrow[r] & D/I \\ A \arrow[u] \arrow[r] & A \arrow[u] \arrow[r] & D \arrow[u] \end{tikzcd}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S' in D is mapped to invertible elements of D, and hence we obtain a compatible map $(S')^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $(S')^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above. □

04EF Lemma 10.149.5. Let $R \to A \to B$ be ring maps. Assume $A \to B$ formally unramified. Let $B' \to B$ be the universal first order thickening of B over A. Then B' is formally unramified over A, and the canonical map $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ is an isomorphism.

Proof. We are going to use the construction of B' from the proof of Lemma 10.149.1 although in principle it should be possible to deduce these results formally from the definition. Namely, we choose a presentation $B = P/J$, where $P = A[x_i]$ is a polynomial ring over A. Next, we choose elements $f_i \in J$ such that $\mathrm{d}f_i = \mathrm{d}x_i \otimes 1$ in $\Omega_{P/A} \otimes_P B$. Having made these choices we have $B' = P/J'$ with $J' = (f_i) + J^2$, see proof of Lemma 10.149.1.

Consider the canonical exact sequence

$$J'/(J')^2 \to \Omega_{P/A} \otimes_P B' \to \Omega_{B'/A} \to 0$$

see Lemma 10.131.9. By construction the classes of the $f_i \in J'$ map to elements of the module $\Omega_{P/A} \otimes_P B'$ which generate it modulo J'/J^2 by construction. Since J'/J^2 is a nilpotent ideal, we see that these elements generate the module altogether (by Nakayama's Lemma 10.20.1). This proves that $\Omega_{B'/A} = 0$ and hence that B' is formally unramified over A, see Lemma 10.148.2.

Since P is a polynomial ring over A we have $\Omega_{P/R} = \Omega_{A/R} \otimes_A P \oplus \bigoplus P\mathrm{d}x_i$. We are going to use this decomposition. Consider the following exact sequence

$$J'/(J')^2 \to \Omega_{P/R} \otimes_P B' \to \Omega_{B'/R} \to 0$$

see Lemma 10.131.9. We may tensor this with B and obtain the exact sequence

$$J'/(J')^2 \otimes_{B'} B \to \Omega_{P/R} \otimes_P B \to \Omega_{B'/R} \otimes_{B'} B \to 0$$

If we remember that $J' = (f_i) + J^2$ then we see that the first arrow annihilates the submodule $J^2/(J')^2$. In terms of the direct sum decomposition $\Omega_{P/R} \otimes_P B = \Omega_{A/R} \otimes_A B \oplus \bigoplus B\mathrm{d}x_i$ given we see that the submodule $(f_i)/(J')^2 \otimes_{B'} B$ maps isomorphically onto the summand $\bigoplus B\mathrm{d}x_i$. Hence what is left of this exact sequence is an isomorphism $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ as desired. □

10.150. Formally étale maps

00UP

00UQ Definition 10.150.1. Let $R \to S$ be a ring map. We say S is formally étale over R if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there exists a unique dotted arrow making the diagram commute.

Clearly a ring map is formally étale if and only if it is both formally smooth and formally unramified.

00UR Lemma 10.150.2. Let $R \to S$ be a ring map of finite presentation. The following are equivalent:

(1) $R \to S$ is formally étale,
(2) $R \to S$ is étale.

Proof. Assume that $R \to S$ is formally étale. Then $R \to S$ is smooth by Proposition 10.138.13. By Lemma 10.148.2 we have $\Omega_{S/R} = 0$. Hence $R \to S$ is étale by definition.

Assume that $R \to S$ is étale. Then $R \to S$ is formally smooth by Proposition 10.138.13. By Lemma 10.148.2 it is formally unramified. Hence $R \to S$ is formally étale. $\square$

031N Lemma 10.150.3. Let R be a ring. Let I be a directed set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally étale, then $S = \text{colim}_{i \in I} S_i$ is formally étale over R

Proof. Consider a diagram as in Definition 10.150.1. By assumption we get unique R-algebra maps $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Hence these are compatible with the transition maps $\varphi_{ii'}$ and define a lift $S \to A$. This proves existence. The uniqueness is clear by restricting to each S_i. $\square$

04EG Lemma 10.150.4. Let R be a ring. Let $S \subset R$ be any multiplicative subset. Then the ring map $R \to S^{-1}R$ is formally étale.

Proof. Let $I \subset A$ be an ideal of square zero. What we are saying here is that given a ring map $\varphi : R \to A$ such that $\varphi(f) \mod I$ is invertible for all $f \in S$ we have also that $\varphi(f)$ is invertible in A for all $f \in S$. This is true because A^* is the inverse image of $(A/I)^*$ under the canonical map $A \to A/I$. $\square$

0H1D Lemma 10.150.5. Let $R \to S$ be a ring map. Let $J \subset S$ be an ideal such that $R \to S/J$ is surjective; let $I \subset R$ be the kernel. If $R \to S$ is formally étale, then $\bigoplus I^n/I^{n+1} \to \bigoplus J^n/J^{n+1}$ is an isomorphism of graded rings.

Proof. Using the lifting property inductively we find dotted arrows

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & S/J = R/I \\ R \arrow[u] \arrow[r] & R/I^2 \arrow[u] \end{tikzcd} \quad \begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & R/I^2 \\ R \arrow[u] \arrow[r] & R/I^3 \arrow[u] \end{tikzcd} \quad \begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & R/I^3 \\ R \arrow[u] \arrow[r] & R/I^4 \arrow[u] \end{tikzcd}$$

The corresponding maps $S/J^n \to R/I^n$ are isomorphisms since the compositions $S/J^n \to R/I^n \to S/J^n$ are (inductively) the identity by the uniqueness in the lifting property of formally étale ring maps. □

10.151. Unramified ring maps

00US The definition of a G-unramified ring map is the one from EGA. The definition of an unramified ring map is the one from [Ray70].

00UT **Definition 10.151.1.** Let $R \to S$ be a ring map.

1. We say $R \to S$ is *unramified* if $R \to S$ is of finite type and $\Omega_{S/R} = 0$.
2. We say $R \to S$ is *G-unramified* if $R \to S$ is of finite presentation and $\Omega_{S/R} = 0$.
3. Given a prime $\mathfrak q$ of S we say that S is *unramified at $\mathfrak q$* if there exists a $g \in S$, $g \notin \mathfrak q$ such that $R \to S_g$ is unramified.
4. Given a prime $\mathfrak q$ of S we say that S is *G-unramified at $\mathfrak q$* if there exists a $g \in S$, $g \notin \mathfrak q$ such that $R \to S_g$ is G-unramified.

Of course a G-unramified map is unramified.

00UU **Lemma 10.151.2.** Let $R \to S$ be a ring map. The following are equivalent

1. $R \to S$ is formally unramified and of finite type, and
2. $R \to S$ is unramified.

Moreover, also the following are equivalent

1. $R \to S$ is formally unramified and of finite presentation, and
2. $R \to S$ is G-unramified.

Proof. Follows from Lemma 10.148.2 and the definitions. □

00UV **Lemma 10.151.3.** Properties of unramified and G-unramified ring maps.

1. The base change of an unramified ring map is unramified. The base change of a G-unramified ring map is G-unramified.
2. The composition of unramified ring maps is unramified. The composition of G-unramified ring maps is G-unramified.
3. Any principal localization $R \to R_f$ is G-unramified and unramified.
4. If $I \subset R$ is an ideal, then $R \to R/I$ is unramified. If $I \subset R$ is a finitely generated ideal, then $R \to R/I$ is G-unramified.
5. An étale ring map is G-unramified and unramified.
6. If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak q \subset S$ is a prime and $(\Omega_{S/R})_{\mathfrak q} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak q$.
7. If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak q \subset S$ is a prime and $\Omega_{S/R} \otimes_S \kappa(\mathfrak q) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak q$.
8. If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak q \subset S$ is a prime lying over $\mathfrak p \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak p)/\kappa(\mathfrak p)})_{\mathfrak q} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak q$.

(9) If $R \to S$ is of finite type (resp. presentation), $\mathfrak q \subset S$ is a prime lying over $\mathfrak p \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak p)/\kappa(\mathfrak p)}) \otimes_{S \otimes_R \kappa(\mathfrak p)} \kappa(\mathfrak q) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak q$.
(10) If $R \to S$ is a ring map, $g_1, \ldots, g_m \in S$ generate the unit ideal and $R \to S_{g_j}$ is unramified (resp. G-unramified) for $j = 1, \ldots, m$, then $R \to S$ is unramified (resp. G-unramified).
(11) If $R \to S$ is a ring map which is unramified (resp. G-unramified) at every prime of S, then $R \to S$ is unramified (resp. G-unramified).
(12) If $R \to S$ is G-unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and a G-unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that $S = R \otimes_{R_0} S_0$.
(13) If $R \to S$ is unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and an unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$.

Proof. We prove each point, in order.

Ad (1). Follows from Lemmas 10.131.12 and 10.14.2.

Ad (2). Follows from Lemmas 10.131.7 and 10.14.2.

Ad (3). Follows by direct computation of $\Omega_{R_f/R}$ which we omit.

Ad (4). We have $\Omega_{(R/I)/R} = 0$, see Lemma 10.131.4, and the ring map $R \to R/I$ is of finite type. If I is a finitely generated ideal then $R \to R/I$ is of finite presentation.

Ad (5). See discussion following Definition 10.143.1.

Ad (6). In this case $\Omega_{S/R}$ is a finite S-module (see Lemma 10.131.16) and hence there exists a $g \in S$, $g \notin \mathfrak q$ such that $(\Omega_{S/R})_g = 0$. By Lemma 10.131.8 this means that $\Omega_{S_g/R} = 0$ and hence $R \to S_g$ is unramified as desired.

Ad (7). Use Nakayama's lemma (Lemma 10.20.1) to see that the condition is equivalent to the condition of (6).

Ad (8) and (9). These are equivalent in the same manner that (6) and (7) are equivalent. Moreover $\Omega_{S \otimes_R \kappa(\mathfrak p)/\kappa(\mathfrak p)} = \Omega_{S/R} \otimes_S (S \otimes_R \kappa(\mathfrak p))$ by Lemma 10.131.12. Hence we see that (9) is equivalent to (7) since the $\kappa(\mathfrak q)$ vector spaces in both are canonically isomorphic.

Ad (10). Follows from Lemmas 10.23.2 and 10.131.8.

Ad (11). Follows from (6) and (7) and the fact that the spectrum of S is quasi-compact.

Ad (12). Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij} \mathrm{d}g_j + \sum a_{ijk} g_j \mathrm{d}x_k$$

in $\Omega_{R[x_1, \ldots, x_n]/R}$ for some $h_{ij}, a_{ijk} \in R[x_1, \ldots, x_n]$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_i, h_{ij}, a_{ijk}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. This works.

Ad (13). Write $S = R[x_1, \ldots, x_n]/I$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij} \mathrm{d}g_{ij} + \sum g'_{ik} \mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij} \in R[x_1, \ldots, x_n]$ and $g_{ij}, g'_{ik} \in I$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_{ij}, h_{ij}, g'_{ik}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_{ij}, g'_{ik})$. This works. $\square$

02FL Lemma 10.151.4. Let $R \to S$ be a ring map. If $R \to S$ is unramified, then there exists an idempotent $e \in S \otimes_R S$ such that $S \otimes_R S \to S$ is isomorphic to $S \otimes_R S \to (S \otimes_R S)_e$.

Proof. Let $J = \text{Ker}(S \otimes_R S \to S)$. By assumption $J/J^2 = 0$, see Lemma 10.131.13. Since S is of finite type over R we see that J is finitely generated, namely by $x_i \otimes 1 - 1 \otimes x_i$, where x_i generate S over R. We win by Lemma 10.21.5. $\square$

00UW Lemma 10.151.5. Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p$ in R. If S/R is unramified at $\mathfrak q$ then

(1) we have $\mathfrak p S_{\mathfrak q} = \mathfrak q S_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and
(2) the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite separable.

Proof. We may first replace S by S_g for some $g \in S$, $g \notin \mathfrak q$ and assume that $R \to S$ is unramified. The base change $S \otimes_R \kappa(\mathfrak p)$ is unramified over $\kappa(\mathfrak p)$ by Lemma 10.151.3. By Lemma 10.140.3 it is smooth hence étale over $\kappa(\mathfrak p)$. Hence we see that $S \otimes_R \kappa(\mathfrak p) = (R \setminus \mathfrak p)^{-1} S/\mathfrak p S$ is a product of finite separable field extensions of $\kappa(\mathfrak p)$ by Lemma 10.143.4. This implies the lemma. $\square$

02UR Lemma 10.151.6. Let $R \to S$ be a finite type ring map. Let $\mathfrak q$ be a prime of S. If $R \to S$ is unramified at $\mathfrak q$ then $R \to S$ is quasi-finite at $\mathfrak q$. In particular, an unramified ring map is quasi-finite.

Proof. An unramified ring map is of finite type. Thus it is clear that the second statement follows from the first. To see the first statement apply the characterization of Lemma 10.122.2 part (2) using Lemma 10.151.5. $\square$

02FM Lemma 10.151.7. Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over a prime $\mathfrak p$ of R. If

(1) $R \to S$ is of finite type,
(2) $\mathfrak p S_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and
(3) the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite separable,

then $R \to S$ is unramified at $\mathfrak q$.

Proof. By Lemma 10.151.3 (8) it suffices to show that $\Omega_{S \otimes_R \kappa(\mathfrak p)/\kappa(\mathfrak p)}$ is zero when localized at $\mathfrak q$. Hence we may replace S by $S \otimes_R \kappa(\mathfrak p)$ and R by $\kappa(\mathfrak p)$. In other words, we may assume that $R = k$ is a field and S is a finite type k-algebra. In this case the hypotheses imply that $S_{\mathfrak q} \cong \kappa(\mathfrak q)$. Thus $(\Omega_{S/k})_{\mathfrak q} = \Omega_{S_{\mathfrak q}/k} = \Omega_{\kappa(\mathfrak q)/k}$ is zero as desired (the first equality is Lemma 10.131.8). $\square$

08WD Lemma 10.151.8. Let $R \to S$ be a ring map. The following are equivalent

(1) $R \to S$ is étale,
(2) $R \to S$ is flat and G-unramified, and
(3) $R \to S$ is flat, unramified, and of finite presentation.

Proof. Parts (2) and (3) are equivalent by definition. The implication (1) $\Rightarrow$ (3) follows from the fact that étale ring maps are of finite presentation, Lemma 10.143.3 (flatness of étale maps), and Lemma 10.151.3 (étale maps are unramified). Conversely, the characterization of étale ring maps in Lemma 10.143.7 and the structure of unramified ring maps in Lemma 10.151.5 shows that (3) implies (1). (This uses that $R \to S$ is étale if $R \to S$ is étale at every prime $\mathfrak{q} \subset S$, see Lemma 10.143.3.) $\square$

0G1C Lemma 10.151.9. Let k be a field. Let

$$\varphi : k[x_1, \ldots, x_n] \to A, \quad x_i \longmapsto a_i$$

be a finite type ring map. Then φ is étale if and only if we have the following two conditions: (a) the local rings of A at maximal ideals have dimension n, and (b) the elements $\mathrm{d}(a_1), \ldots, \mathrm{d}(a_n)$ generate $\Omega_{A/k}$ as an A-module.

Proof. Assume (a) and (b). Condition (b) implies that $\Omega_{A/k[x_1,\ldots,x_n]} = 0$ and hence φ is unramified. Thus it suffices to prove that φ is flat, see Lemma 10.151.8. Let $\mathfrak{m} \subset A$ be a maximal ideal. Set $X = \mathrm{Spec}(A)$ and denote $x \in X$ the closed point corresponding to $\mathfrak{m}$. Then $\dim(A_\mathfrak{m})$ is $\dim_x X$, see Lemma 10.114.6. Thus by Lemma 10.140.3 we see that if (a) and (b) hold, then $A_\mathfrak{m}$ is a regular local ring for every maximal ideal $\mathfrak{m}$. Then $k[x_1, \ldots, x_n]_{\varphi^{-1}(\mathfrak{m})} \to A_\mathfrak{m}$ is flat by Lemma 10.128.1 (and the fact that a regular local ring is CM, see Lemma 10.106.3). Thus φ is flat by Lemma 10.39.18.

Assume φ is étale. Then $\Omega_{A/k[x_1,\ldots,x_n]} = 0$ and hence (b) holds. On the other hand, étale ring maps are flat (Lemma 10.143.3) and quasi-finite (Lemma 10.143.6). Hence for every maximal ideal $\mathfrak{m}$ of A we my apply Lemma 10.112.7 to $k[x_1, \ldots, x_n]_{\varphi^{-1}(\mathfrak{m})} \to A_\mathfrak{m}$ to see that $\dim(A_\mathfrak{m}) = n$ and hence (a) holds. $\square$

10.152. Local structure of unramified ring maps

0G1D An unramified morphism is locally (in a suitable sense) the composition of a closed immersion and an étale morphism. The algebraic underpinnings of this fact are discussed in this section.

0395 Proposition 10.152.1. Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is unramified at $\mathfrak{q}$, then there exist

(1) a $g \in S$, $g \notin \mathfrak{q}$,
(2) a standard étale ring map $R \to S'$, and
(3) a surjective R-algebra map $S' \to S_g$.

Proof. This proof is the "same" as the proof of Proposition 10.144.4. The proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 10.151.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified. Thus we may assume that S is unramified over R.

Step 2. By Lemma 10.151.3 there exists an unramified ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 10.151.6. By Lemma 10.123.14 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \notin \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that S' may not be unramified over R.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is unramified at $\mathfrak{q}$ (but no longer necessarily unramified at all primes).

Step 4. Let $\mathfrak p \subset R$ be the prime corresponding to $\mathfrak q$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak p)$. This is a finite algebra over $\kappa(\mathfrak p)$. Hence it is Artinian (see Lemma 10.53.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak p) = \prod\nolimits_{i=1}^n A_i$$

see Proposition 10.60.7. One of the factors, say A_1, is the local ring $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$ which is isomorphic to $\kappa(\mathfrak q)$, see Lemma 10.151.5. The other factors correspond to the other primes, say $\mathfrak q_2, \ldots, \mathfrak q_n$ of S lying over $\mathfrak p$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak q)$ which generates the finite separable field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak p)^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak q)$ over $\kappa(\mathfrak p)$. Consider the element

$$\bar t = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^n A_i = S \otimes_R \kappa(\mathfrak p).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\bar t$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

$$\begin{tikzcd} R[x] \arrow[r] & S' \arrow[r] & S \\ R \arrow[u] \arrow[ru] \arrow[rru] & & \end{tikzcd}$$

By construction the primes $\mathfrak q_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak p, x)$ of $R[x]$, whereas the prime $\mathfrak q$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak q' \subset S'$ the prime of S' corresponding to $\mathfrak q$. By the above $\mathfrak q$ is the only prime of S lying over $\mathfrak q'$. Thus we see that $S_{\mathfrak q} = S_{\mathfrak q'}$, see Lemma 10.41.11 (we have going up for $S' \to S$ by Lemma 10.36.22 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak q'} \to S_{\mathfrak q}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_{\mathfrak p} \to S'_{\mathfrak q'} \to S_{\mathfrak q}$$

The second map is finite and injective. We have $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q} = \kappa(\mathfrak q)$, see Lemma 10.151.5. Hence a fortiori $S_{\mathfrak q}/\mathfrak q'S_{\mathfrak q} = \kappa(\mathfrak q)$. Since

$$\kappa(\mathfrak p) \subset \kappa(\mathfrak q') \subset \kappa(\mathfrak q)$$

and since α is in the image of $\kappa(\mathfrak q')$ in $\kappa(\mathfrak q)$ we conclude that $\kappa(\mathfrak q') = \kappa(\mathfrak q)$. Hence by Nakayama's Lemma 10.20.1 applied to the $S'_{\mathfrak q'}$-module map $S'_{\mathfrak q'} \to S_{\mathfrak q}$, the map $S'_{\mathfrak q'} \to S_{\mathfrak q}$ is surjective. In other words, $S'_{\mathfrak q'} \cong S_{\mathfrak q}$.

Step 7. By Lemma 10.126.7 there exist $g \in S$, $g \notin \mathfrak q$ and $g' \in S'$, $g' \notin \mathfrak q'$ such that $S'_{g'} \cong S_g$. As R is Noetherian the ring S' is finite over R because it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is unramified over R at $\mathfrak q$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak p) = \kappa(\mathfrak p)[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak p)[x]$ is the ideal generated by I in $\kappa(\mathfrak p)[x]$. As $\kappa(\mathfrak p)[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak p)$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak p)$ we may assume that $\overline{h}$ is the image of some $h \in R[x]$. (The problem is that we do not know if we may choose h

monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak p) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak q)$ a finite separable extension of $\kappa(\mathfrak p)$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak p)[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak p)[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak p)[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak q)$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak p)[x]$. We may factor

$$\overline{m} = \overline{k} \overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak p)[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l \deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak p)[x]$ factors as

$$\overline{f} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n} = \overline{h}_1 (\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1 - 1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n}) = \overline{h}_1 \overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak q$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak q) = \kappa(\mathfrak p)[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak p)[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w} \frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1 \frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 10.144.2) and $\varphi(g) \notin \mathfrak q$. Thus the map $(R[x]/(f))_g \to S_{\varphi(g)}$ is the desired surjection. □

00UX Lemma 10.152.2. Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over $\mathfrak p \subset R$. Assume that $R \to S$ is of finite type and unramified at $\mathfrak q$. Then there exist

1. an étale ring map $R \to R'$,
2. a prime $\mathfrak p' \subset R'$ lying over $\mathfrak p$.
3. a product decomposition
$$R' \otimes_R S = A \times B$$

with the following properties

1. $R' \to A$ is surjective, and
2. $\mathfrak p' A$ is a prime of A lying over $\mathfrak p'$ and over $\mathfrak q$.

Proof. We may replace $(R \to S, \mathfrak p, \mathfrak q)$ with any base change $(R' \to R' \otimes_R S, \mathfrak p', \mathfrak q')$ by an étale ring map $R \to R'$ with a prime $\mathfrak p'$ lying over $\mathfrak p$, and a choice of $\mathfrak q'$ lying over both $\mathfrak q$ and $\mathfrak p'$. Note also that given $R \to R'$ and $\mathfrak p'$ a suitable $\mathfrak q'$ can always be found.

The assumption that $R \to S$ is of finite type means that we may apply Lemma 10.145.4. Thus we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Then clearly $\mathfrak{q} = \mathfrak{r}_i$ for some i, since an unramified morphism is quasi-finite (see Lemma 10.151.6). Say $\mathfrak{q} = \mathfrak{r}_1$. By Lemma 10.151.5 we see that $\kappa(\mathfrak{r}_1)/\kappa(\mathfrak{p})$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_1)_{\mathfrak{r}_1}$ is the maximal ideal. Also, by Lemma 10.41.11 (which applies to $R \to A_1$ because a finite ring map satisfies going up by Lemma 10.36.22) we have $(A_1)_{\mathfrak{r}_1} = (A_1)_{\mathfrak{p}}$. It follows from Nakayama's Lemma 10.20.1 that the map of local rings $R_{\mathfrak{p}} \to (A_1)_{\mathfrak{p}} = (A_1)_{\mathfrak{r}_1}$ is surjective. Since A_1 is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to (A_1)_f$ is surjective. After replacing R by R_f we win. □

00UY Lemma 10.152.3. Let $R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. If $R \to S$ is unramified then there exist

(1) an étale ring map $R \to R'$,
(2) a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$.
(3) a product decomposition

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) $R' \to A_i$ is surjective,
(2) $\mathfrak{p}' A_i$ is a prime of A_i lying over $\mathfrak{p}'$, and
(3) there is no prime of B lying over $\mathfrak{p}'$.

Proof. We may apply Lemma 10.145.4. Thus, after an étale base change, we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable, and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Since $R \to S$ is quasi-finite (see Lemma 10.151.6) we see there is no prime of B lying over $\mathfrak{p}$. By Lemma 10.151.5 we see that $\kappa(\mathfrak{r}_i)/\kappa(\mathfrak{p})$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_i)_{\mathfrak{r}_i}$ is the maximal ideal. Also, by Lemma 10.41.11 (which applies to $R \to A_i$ because a finite ring map satisfies going up by Lemma 10.36.22) we have $(A_i)_{\mathfrak{r}_i} = (A_i)_{\mathfrak{p}}$. It follows from Nakayama's Lemma 10.20.1 that the map of local rings $R_{\mathfrak{p}} \to (A_i)_{\mathfrak{p}} = (A_i)_{\mathfrak{r}_i}$ is surjective. Since A_i is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to (A_i)_f$ is surjective. After replacing R by R_f we win. □

10.153. Henselian local rings

04GE In this section we discuss a bit the notion of a henselian local ring. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For $a \in R$ we denote $\overline{a}$ the image of a in κ. For a polynomial $f \in R[T]$ we often denote $\overline{f}$ the image of f in $\kappa[T]$. Given a polynomial $f \in R[T]$ we denote f' the derivative of f with respect to T. Note that $\overline{f}' = \overline{f'}$.

04GF Definition 10.153.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

(1) We say R is henselian if for every monic $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$.
(2) We say R is strictly henselian if R is henselian and its residue field is separably algebraically closed.

Note that the condition $\overline{f'}(a_0) \neq 0$ is equivalent to the condition that a_0 is a simple root of the polynomial $\overline{f}$. In fact, it implies that the lift $a \in R$, if it exists, is unique.

06RR Lemma 10.153.2. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $f \in R[T]$. Let $a, b \in R$ such that $f(a) = f(b) = 0$, $a = b \bmod \mathfrak{m}$, and $f'(a) \notin \mathfrak{m}$. Then $a = b$.

Proof. Write $f(x+y) - f(x) = f'(x)y + g(x,y)y^2$ in $R[x,y]$ (this is possible as one sees by expanding $f(x+y)$; details omitted). Then we see that $0 = f(b) - f(a) = f(a+(b-a)) - f(a) = f'(a)(b-a) + c(b-a)^2$ for some $c \in R$. By assumption $f'(a)$ is a unit in R. Hence $(b-a)(1 + f'(a)^{-1}c(b-a)) = 0$. By assumption $b - a \in \mathfrak{m}$, hence $1 + f'(a)^{-1}c(b-a)$ is a unit in R. Hence $b - a = 0$ in R. $\square$

Here is the characterization of henselian local rings.

04GG Lemma 10.153.3. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent

(1) R is henselian,
(2) for every $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$,
(3) for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,
(4) for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,
(5) for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,
(6) for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,
(7) for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a section $\tau : S \to R$ of $R \to S$,
(8) for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a section $\tau : S \to R$ of $R \to S$ with $\mathfrak{q} = \tau^{-1}(\mathfrak{m})$,
(9) any finite R-algebra is a product of local rings,
(10) any finite R-algebra is a finite product of local rings,
(11) any finite type R-algebra S can be written as $A \times B$ with $R \to A$ finite and $R \to B$ not quasi-finite at any prime lying over $\mathfrak{m}$,
(12) any finite type R-algebra S can be written as $A \times B$ with $R \to A$ finite such that each irreducible component of $\mathrm{Spec}(B \otimes_R \kappa)$ has dimension ≥ 1, and
(13) any quasi-finite R-algebra S can be written as $S = A \times B$ with $R \to A$ finite such that $B \otimes_R \kappa = 0$.

Proof. Here is a list of the easier implications:

(1) 2$\Rightarrow$1 because in (2) we consider all polynomials and in (1) only monic ones,
(2) 5$\Rightarrow$3 because in (5) we consider all polynomials and in (3) only monic ones,
(3) 6$\Rightarrow$4 because in (6) we consider all polynomials and in (4) only monic ones,
(4) 4$\Rightarrow$3 is obvious,

(5) 6⇒5 is obvious,
(6) 8⇒7 is obvious,
(7) 10⇒9 is obvious,
(8) 11⇔12 by definition of being quasi-finite at a prime,
(9) 11⇒13 by definition of being quasi-finite,

Proof of 1⇒8. Assume (1). Let $R \to S$ be étale, and let $\mathfrak q \subset S$ be a prime ideal such that $\kappa(\mathfrak q) \cong \kappa$. By Proposition 10.144.4 we can find a $g \in S$, $g \not\in \mathfrak q$ such that $R \to S_g$ is standard étale. After replacing S by S_g we may assume that $S = R[t]_g/(f)$ is standard étale. Since the prime $\mathfrak q$ has residue field κ it corresponds to a root a_0 of $\overline{f}$ which is not a root of $\overline{g}$. By definition of a standard étale algebra this also means that $\overline{f'}(a_0) \neq 0$. Since also f is monic by definition of a standard étale algebra again we may use that R is henselian to conclude that there exists an $a \in R$ with $a_0 = \overline{a}$ such that $f(a) = 0$. This implies that $g(a)$ is a unit of R and we obtain the desired map $\tau : S = R[t]_g/(f) \to R$ by the rule $t \mapsto a$. By construction $\tau^{-1}(\mathfrak q) = \mathfrak m$. This proves (8) holds.

Proof of 7⇒8. (This is really unimportant and should be skipped.) Assume (7) holds and assume $R \to S$ is étale. Let $\mathfrak q_1, \ldots, \mathfrak q_r$ be the other primes of S lying over $\mathfrak m$. Then we can find a $g \in S$, $g \not\in \mathfrak q$ and $g \in \mathfrak q_i$ for $i = 1, \ldots, r$. Namely, we can argue that $\bigcap_{i=1}^r \mathfrak q_i \not\subset \mathfrak q$ since otherwise $\mathfrak q_i \subset \mathfrak q$ for some i, but this cannot happen as the fiber of an étale morphism is discrete (use Lemma 10.143.4 for example). Apply (7) to the étale ring map $R \to S_g$ and the prime $\mathfrak q S_g$. This gives a section $\tau_g : S_g \to R$ such that the composition $\tau : S \to S_g \to R$ has the property $\tau^{-1}(\mathfrak m) = \mathfrak q$. Minor details omitted.

Proof of 8⇒11. Assume (8) and let $R \to S$ be a finite type ring map. Apply Lemma 10.145.3. We find an étale ring map $R \to R'$ and a prime $\mathfrak m' \subset R'$ lying over $\mathfrak m$ with $\kappa = \kappa(\mathfrak m')$ such that $R' \otimes_R S = A' \times B'$ with A' finite over R' and B' not quasi-finite over R' at any prime lying over $\mathfrak m'$. Apply (8) to get a section $\tau : R' \to R$ with $\mathfrak m = \tau^{-1}(\mathfrak m')$. Then use that

$$S = (S \otimes_R R') \otimes_{R', \tau} R = (A' \times B') \otimes_{R', \tau} R = (A' \otimes_{R', \tau} R) \times (B' \otimes_{R', \tau} R)$$

which gives a decomposition as in (11).

Proof of 8⇒10. Assume (8) and let $R \to S$ be a finite ring map. Apply Lemma 10.145.3. We find an étale ring map $R \to R'$ and a prime $\mathfrak m' \subset R'$ lying over $\mathfrak m$ with $\kappa = \kappa(\mathfrak m')$ such that $R' \otimes_R S = A'_1 \times \ldots \times A'_n \times B'$ with A'_i finite over R' having exactly one prime over $\mathfrak m'$ and B' not quasi-finite over R' at any prime lying over $\mathfrak m'$. Apply (8) to get a section $\tau : R' \to R$ with $\mathfrak m' = \tau^{-1}(\mathfrak m)$. Then we obtain

$$\begin{aligned} S &= (S \otimes_R R') \otimes_{R', \tau} R \\ &= (A'_1 \times \ldots \times A'_n \times B') \otimes_{R', \tau} R \\ &= (A'_1 \otimes_{R', \tau} R) \times \ldots \times (A'_1 \otimes_{R', \tau} R) \times (B' \otimes_{R', \tau} R) \\ &= A_1 \times \ldots \times A_n \times B \end{aligned}$$

The factor B is finite over R but $R \to B$ is not quasi-finite at any prime lying over $\mathfrak m$. Hence $B = 0$. The factors A_i are finite R-algebras having exactly one prime lying over $\mathfrak m$, hence they are local rings. This proves that S is a finite product of local rings.

Proof of 9⇒10. This holds because if S is finite over the local ring R, then it has at most finitely many maximal ideals. Namely, by going up for $R \to S$ the maximal ideals of S all lie over $\mathfrak{m}$, and $S/\mathfrak{m}S$ is Artinian hence has finitely many primes.

Proof of 10⇒1. Assume (10). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S = R[T]/(f)$ is a finite R-algebra. Applying (10) we get $S = A_1 \times \ldots \times A_r$ is a finite product of local R-algebras. In particular we see that $S/\mathfrak{m}S = \prod A_i/\mathfrak{m}A_i$ is the decomposition of $\kappa[T]/(\overline{f})$ as a product of local rings. This means that one of the factors, say $A_1/\mathfrak{m}A_1$ is the quotient $\kappa[T]/(\overline{f}) \to \kappa[T]/(T - a_0)$. Since A_1 is a summand of the finite free R-module S it is a finite free R-module itself. As $A_1/\mathfrak{m}A_1$ is a κ-vector space of dimension 1 we see that $A_1 \cong R$ as an R-module. Clearly this means that $R \to A_1$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A_1 \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 13⇒1. Assume (13). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S_1 = R[T]/(f)$ is a finite R-algebra. Let $g \in R[T]$ be any element such that $\overline{g} = \overline{f}/(T - a_0)$. Then $S = (S_1)_g$ is a quasi-finite R-algebra such that $S \otimes_R \kappa \cong \kappa[T]_{\overline{g}}/(\overline{f}) \cong \kappa[T]/(T - a_0) \cong \kappa$. Applying (13) to S we get $S = A \times B$ with A finite over R and $B \otimes_R \kappa = 0$. In particular we see that $\kappa \cong S/\mathfrak{m}S = A/\mathfrak{m}A$. Since A is a summand of the flat R-algebra S we see that it is finite flat, hence free over R. As $A/\mathfrak{m}A$ is a κ-vector space of dimension 1 we see that $A \cong R$ as an R-module. Clearly this means that $R \to A$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 8⇒2. Assume (8). Let $f \in R[T]$ be any polynomial and let $a_0 \in \kappa$ be a simple root. Then the algebra $S = R[T]_{f'}/(f)$ is étale over R. Let $\mathfrak{q} \subset S$ be the prime generated by $\mathfrak{m}$ and $T - b$ where $b \in R$ is any element such that $\overline{b} = a_0$. Apply (8) to S and $\mathfrak{q}$ to get $\tau : S \to R$. Then the image $\tau(T) = a \in R$ works in (2).

At this point we see that (1), (2), (7), (8), (9), (10), (11), (12), (13) are all equivalent. The weakest assertion of (3), (4), (5) and (6) is (3) and the strongest is (6). Hence we still have to prove that (3) implies (1) and (1) implies (6).

Proof of 3⇒1. Assume (3). Let $f \in R[T]$ be monic and let $a_0 \in \kappa$ be a simple root of $\overline{f}$. This gives a factorization $\overline{f} = (T - a_0)h_0$ with $h_0(a_0) \neq 0$, so $\gcd(T - a_0, h_0) = 1$. Apply (3) to get a factorization $f = gh$ with $\overline{g} = T - a_0$ and $\overline{h} = h_0$. Set $S = R[T]/(f)$ which is a finite free R-algebra. We will write g, h also for the images of g and h in S. Then $gS + hS = S$ by Nakayama's Lemma 10.20.1 as the equality holds modulo $\mathfrak{m}$. Since $gh = f = 0$ in S this also implies that $gS \cap hS = 0$. Hence by the Chinese Remainder theorem we obtain $S = S/(g) \times S/(h)$. This implies that $A = S/(g)$ is a summand of a finite free R-module, hence finite free. Moreover, the rank of A is 1 as $A/\mathfrak{m}A = \kappa[T]/(T - a_0)$. Thus the map $R \to A$ is an isomorphism. Setting $a \in R$ equal to the image of T under the maps $R[T] \to S \to A \to R$ gives an element of R with $f(a) = 0$ and $\overline{a} = a_0$.

Proof of 1⇒6. Assume (1) or equivalently all of (1), (2), (7), (8), (9), (10), (11), (12), (13). Let $f \in R[T]$ be a polynomial. Suppose that $\overline{f} = g_0 h_0$ is a factorization with $\gcd(g_0, h_0) = 1$. We may and do assume that g_0 is monic. Consider $S = R[T]/(f)$. Because we have the factorization we see that the coefficients of f generate the unit ideal in R. This implies that S has finite fibres over R, hence is quasi-finite over R. It also implies that S is flat over R by Lemma 10.128.5. Combining (13) and (10) we may write $S = A_1 \times \ldots \times A_n \times B$ where each A_i is local and finite over R, and $B \otimes_R \kappa = 0$. After reordering the factors $A_1, \ldots, A_n$ we may assume that

$$\kappa[T]/(g_0) = A_1/\mathfrak{m}A_1 \times \ldots \times A_r/\mathfrak{m}A_r,\ \kappa[T]/(h_0) = A_{r+1}/\mathfrak{m}A_{r+1} \times \ldots \times A_n/\mathfrak{m}A_n$$

as quotients of $\kappa[T]$. The finite flat R-algebra $A = A_1 \times \ldots \times A_r$ is free as an R-module, see Lemma 10.78.5. Its rank is $\deg_T(g_0)$. Let $g \in R[T]$ be the characteristic polynomial of the R-linear operator $T : A \to A$. Then g is a monic polynomial of degree $\deg_T(g) = \deg_T(g_0)$ and moreover $\overline{g} = g_0$. By Cayley-Hamilton (Lemma 10.16.1) we see that $g(T_A) = 0$ where T_A indicates the image of T in A. Hence we obtain a well defined surjective map $R[T]/(g) \to A$ which is an isomorphism by Nakayama's Lemma 10.20.1. The map $R[T] \to A$ factors through $R[T]/(f)$ by construction hence we may write $f = gh$ for some h. This finishes the proof. □

04GH Lemma 10.153.4. Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring.

(1) If $R \to S$ is a finite ring map then S is a finite product of henselian local rings each finite over R.
(2) If $R \to S$ is a finite ring map and S is local, then S is a henselian local ring and $R \to S$ is a (finite) local ring map.
(3) If $R \to S$ is a finite type ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ at which $R \to S$ is quasi-finite, then $S_\mathfrak{q}$ is henselian and finite over R.
(4) If $R \to S$ is quasi-finite then $S_\mathfrak{q}$ is henselian and finite over R for every prime $\mathfrak{q}$ lying over $\mathfrak{m}$.

Proof. Part (2) implies part (1) since S as in part (1) is a finite product of its localizations at the primes lying over $\mathfrak{m}$ by Lemma 10.153.3 part (10). Part (2) also follows from Lemma 10.153.3 part (10) since any finite S-algebra is also a finite R-algebra (of course any finite ring map between local rings is local).

Let $R \to S$ and $\mathfrak{q}$ be as in (3). Write $S = A \times B$ with A finite over R and B not quasi-finite over R at any prime lying over $\mathfrak{m}$, see Lemma 10.153.3 part (11). Hence $S_\mathfrak{q}$ is a localization of A at a maximal ideal and we deduce (3) from (1). Part (4) follows from part (3). □

04GJ Lemma 10.153.5. Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.

Proof. This is a combination of parts (11) and (10) of Lemma 10.153.3. □

06DD Lemma 10.153.6. Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $\kappa \subset \kappa(\mathfrak{m}_{A_i})$ finite purely inseparable and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 10.153.5. The field extension $\kappa(\mathfrak{m}_{A_i})/\kappa$ is finite and κ is separably algebraically closed, hence it is finite purely inseparable. □

04GK Lemma 10.153.7. Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. The category of finite étale ring extensions $R \to S$ is equivalent to the category of finite étale algebras $\kappa \to \overline{S}$ via the functor $S \mapsto S/\mathfrak{m}S$.

Proof. Denote $\mathcal{C} \to \mathcal{D}$ the functor of categories of the statement. Suppose that $R \to S$ is finite étale. Then we may write

$$S = A_1 \times \ldots \times A_n$$

with A_i local and finite étale over S, use either Lemma 10.153.5 or Lemma 10.153.3 part (10). In particular $A_i/\mathfrak{m}A_i$ is a finite separable field extension of κ, see Lemma 10.143.5. Thus we see that every object of $\mathcal{C}$ and $\mathcal{D}$ decomposes canonically into irreducible pieces which correspond via the given functor. Next, suppose that S_1, S_2 are finite étale over R such that $\kappa_1 = S_1/\mathfrak{m}S_1$ and $\kappa_2 = S_2/\mathfrak{m}S_2$ are fields (finite separable over κ). Then $S_1 \otimes_R S_2$ is finite étale over R and we may write

$$S_1 \otimes_R S_2 = A_1 \times \ldots \times A_n$$

as before. Then we see that $\mathrm{Hom}_R(S_1, S_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $S_2 \to A_i$ is an isomorphism. To see this use that given any R-algebra map $\varphi : S_1 \to S_2$ the map $\varphi \times 1 : S_1 \otimes_R S_2 \to S_2$ is surjective, and hence is equal to projection onto one of the factors A_i. But in exactly the same way we see that $\mathrm{Hom}_\kappa(\kappa_1, \kappa_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $\kappa_2 \to A_i/\mathfrak{m}A_i$ is an isomorphism. By the discussion above these sets of indices match, and we conclude that our functor is fully faithful. Finally, let κ'/κ be a finite separable field extension. By Lemma 10.144.3 there exists an étale ring map $R \to S$ and a prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ such that $\kappa \subset \kappa(\mathfrak{q})$ is isomorphic to the given extension. By part (1) we may write $S = A_1 \times \ldots \times A_n \times B$. Since $R \to S$ is quasi-finite we see that there exists no prime of B over $\mathfrak{m}$. Hence $S_\mathfrak{q}$ is equal to A_i for some i. Hence $R \to A_i$ is finite étale and produces the given residue field extension. Thus the functor is essentially surjective and we win. □

04GL **Lemma 10.153.8.** Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Let $R \to S$ be an unramified ring map. Then

$$S = A_1 \times \ldots \times A_n \times B$$

with each $R \to A_i$ surjective and no prime of B lying over $\mathfrak{m}$.

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 10.153.5. Now we see that $R \to A_i$ is finite unramified and A_i local. Hence the maximal ideal of A_i is $\mathfrak{m}A_i$ and its residue field $A_i/\mathfrak{m}A_i$ is a finite separable extension of κ, see Lemma 10.151.5. However, the condition that R is strictly henselian means that κ is separably algebraically closed, so $\kappa = A_i/\mathfrak{m}A_i$. By Nakayama's Lemma 10.20.1 we conclude that $R \to A_i$ is surjective as desired. □

04GM **Lemma 10.153.9.** Let $(R, \mathfrak{m}, \kappa)$ be a complete local ring, see Definition 10.160.1. Then R is henselian.

Proof. Let $f \in R[T]$ be monic. Denote $f_n \in R/\mathfrak{m}^{n+1}[T]$ the image. Denote f'_n the derivative of f_n with respect to T. Let $a_0 \in \kappa$ be a simple root of f_0. We lift this to a solution of f over R inductively as follows: Suppose given $a_n \in R/\mathfrak{m}^{n+1}$ such that $a_n \bmod \mathfrak{m} = a_0$ and $f_n(a_n) = 0$. Pick any element $b \in R/\mathfrak{m}^{n+2}$ such that $a_n = b \bmod \mathfrak{m}^{n+1}$. Then $f_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Set

$$a_{n+1} = b - f_{n+1}(b)/f'_{n+1}(b)$$

(Newton's method). This makes sense as $f'_{n+1}(b) \in R/\mathfrak{m}^{n+1}$ is invertible by the condition on a_0. Then we compute $f_{n+1}(a_{n+1}) = f_{n+1}(b) - f_{n+1}(b) = 0$ in $R/\mathfrak{m}^{n+2}$. Since the system of elements $a_n \in R/\mathfrak{m}^{n+1}$ so constructed is compatible we get an element $a \in \lim R/\mathfrak{m}^n = R$ (here we use that R is complete). Moreover, $f(a) = 0$ since it maps to zero in each $R/\mathfrak{m}^n$. Finally $\overline{a} = a_0$ and we win. $\square$

06RS Lemma 10.153.10. Let $(R, \mathfrak{m})$ be a local ring of dimension 0. Then R is henselian.

Proof. Let $R \to S$ be a finite ring map. By Lemma 10.153.3 it suffices to show that S is a product of local rings. By Lemma 10.36.21 S has finitely many primes $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ which all lie over $\mathfrak{m}$. There are no inclusions among these primes, see Lemma 10.36.20, hence they are all maximal. Every element of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$ is nilpotent by Lemma 10.17.2. It follows S is the product of the localizations of S at the primes $\mathfrak{m}_i$ by Lemma 10.53.5. $\square$

The following lemma will be the key to the uniqueness and functorial properties of henselization and strict henselization.

08HQ Lemma 10.153.11. Let $R \to S$ be a ring map with S henselian local. Given

(1) an étale ring map $R \to A$,
(2) a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,
(3) a $\kappa(\mathfrak{p})$-algebra map $\tau : \kappa(\mathfrak{q}) \to S/\mathfrak{m}_S$,

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{m}_S)$ and $f \bmod \mathfrak{q} = \tau$.

Proof. Consider $A \otimes_R S$. This is an étale algebra over S, see Lemma 10.143.3. Moreover, the kernel

$$\mathfrak{q}' = \mathrm{Ker}(A \otimes_R S \to \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{m}_S) \to \kappa(\mathfrak{m}_S))$$

of the map using the map given in (3) is a prime ideal lying over $\mathfrak{m}_S$ with residue field equal to the residue field of S. Hence by Lemma 10.153.3 there exists a unique splitting $\tau : A \otimes_R S \to S$ with $\tau^{-1}(\mathfrak{m}_S) = \mathfrak{q}'$. Set f equal to the composition $A \to A \otimes_R S \to S$. $\square$

04GX Lemma 10.153.12. Let $\varphi : R \to S$ be a local homomorphism of strictly henselian local rings. Let $P_1, \ldots, P_n \in R[x_1, \ldots, x_n]$ be polynomials such that $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R. Then the map

$$R^n \longrightarrow S^n, \quad (h_1, \ldots, h_n) \longmapsto (\varphi(h_1), \ldots, \varphi(h_n))$$

induces a bijection between

$$\{(r_1, \ldots, r_n) \in R^n \mid P_i(r_1, \ldots, r_n) = 0,\ i = 1, \ldots, n\}$$

and

$$\{(s_1, \ldots, s_n) \in S^n \mid P_i^\varphi(s_2, \ldots, s_n) = 0,\ i = 1, \ldots, n\}$$

where $P_i^\varphi \in S[x_1, \ldots, x_n]$ are the images of the P_i under φ.

Proof. The first solution set is canonically isomorphic to the set

$$\mathrm{Hom}_R(R[x_1, \ldots, x_n]/(P_1, \ldots, P_n), R).$$

As R is henselian the map $R \to R/\mathfrak{m}_R$ induces a bijection between this set and the set of solutions in the residue field $R/\mathfrak{m}_R$, see Lemma 10.153.3. The same is true for S. Now since $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R and $R/\mathfrak{m}_R$ is separably algebraically closed we see that $R/\mathfrak{m}_R[x_1, \ldots, x_n]/(\overline{P}_1, \ldots, \overline{P}_n)$ is a finite product of copies of $R/\mathfrak{m}_R$ where $\overline{P}_i$ is the image of P_i in $R/\mathfrak{m}_R[x_1, \ldots, x_n]$. Hence the tensor product

$$R/\mathfrak{m}_R[x_1, \ldots, x_n]/(\overline{P}_1, \ldots, \overline{P}_n) \otimes_{R/\mathfrak{m}_R} S/\mathfrak{m}_S = S/\mathfrak{m}_S[x_1, \ldots, x_n]/(\overline{P}_1^{\varphi}, \ldots, \overline{P}_n^{\varphi})$$

is also a finite product of copies of $S/\mathfrak{m}_S$ with the same index set. This proves the lemma. □

05D6 Lemma 10.153.13. Let R be a henselian local ring. Any countably generated Mittag-Leffler module over R is a direct sum of finitely presented R-modules.

Proof. Let M be a countably generated and Mittag-Leffler R-module. We claim that for any element $x \in M$ there exists a direct sum decomposition $M = N \oplus K$ with $x \in N$, the module N finitely presented, and K Mittag-Leffler.

Suppose the claim is true. Choose generators $x_1, x_2, x_3, \ldots$ of M. By the claim we can inductively find direct sum decompositions

$$M = N_1 \oplus N_2 \oplus \ldots \oplus N_n \oplus K_n$$

with N_i finitely presented, $x_1, \ldots, x_n \in N_1 \oplus \ldots \oplus N_n$, and K_n Mittag-Leffler. Repeating ad infinitum we see that $M = \bigoplus N_i$.

We still have to prove the claim. Let $x \in M$. By Lemma 10.92.2 there exists an endomorphism $\alpha : M \to M$ such that α factors through a finitely presented module, and $\alpha(x) = x$. Say α factors as

$$M \xrightarrow{\pi} P \xrightarrow{i} M$$

Set $a = \pi \circ \alpha \circ i : P \to P$, so $i \circ a \circ \pi = \alpha^3$. By Lemma 10.16.2 there exists a monic polynomial $P \in R[T]$ such that $P(a) = 0$. Note that this implies formally that $\alpha^2 P(\alpha) = 0$. Hence we may think of M as a module over $R[T]/(T^2P)$. Assume that $x \neq 0$. Then $\alpha(x) = x$ implies that $0 = \alpha^2 P(\alpha)x = P(1)x$ hence $P(1) = 0$ in R/I where $I = \{r \in R \mid rx = 0\}$ is the annihilator of x. As $x \neq 0$ we see $I \subset \mathfrak{m}_R$, hence 1 is a root of $\overline{P} = P \bmod \mathfrak{m}_R \in R/\mathfrak{m}_R[T]$. As R is henselian we can find a factorization

$$T^2 P = (T^2 Q_1) Q_2$$

for some $Q_1, Q_2 \in R[T]$ with $Q_2 = (T-1)^e \bmod \mathfrak{m}_R R[T]$ and $Q_1(1) \neq 0 \bmod \mathfrak{m}_R$, see Lemma 10.153.3. Let $N = \mathrm{Im}(\alpha^2 Q_1(\alpha) : M \to M)$ and $K = \mathrm{Im}(Q_2(\alpha) : M \to M)$. As T^2Q_1 and Q_2 generate the unit ideal of $R[T]$ we get a direct sum decomposition $M = N \oplus K$. Moreover, Q_2 acts as zero on N and T^2Q_1 acts as zero on K. Note that N is a quotient of P hence is finitely generated. Also $x \in N$ because $\alpha^2 Q_1(\alpha)x = Q_1(1)x$ and $Q_1(1)$ is a unit in R. By Lemma 10.89.10 the modules N and K are Mittag-Leffler. Finally, the finitely generated module N is finitely presented as a finitely generated Mittag-Leffler module is finitely presented, see Example 10.91.1 part (1). □

10.154. Filtered colimits of étale ring maps

0BSG This section is a precursor to the section on ind-étale ring maps (Pro-étale Cohomology, Section 61.7). The material will also be useful to prove uniqueness properties of the henselization and strict henselization of a local ring.

0BSH Lemma 10.154.1. Let $R \to A$ and $R \to R'$ be ring maps. If A is a filtered colimit of étale ring maps, then so is $R' \to R' \otimes_R A$.

Proof. This is true because colimits commute with tensor products and étale ring maps are preserved under base change (Lemma 10.143.3). ☐

0BSI Lemma 10.154.2. Let $A \to B \to C$ be ring maps. If $A \to B$ is a filtered colimit of étale ring maps and $B \to C$ is a filtered colimit of étale ring maps, then $A \to C$ is a filtered colimit of étale ring maps.

Proof. We will use the criterion of Lemma 10.127.4. Let $A \to P \to C$ be a factorization of $A \to C$ with P of finite presentation over A. Write $B = \mathrm{colim}_{i \in I} B_i$ where I is a directed set and where B_i is an étale A-algebra. Write $C = \mathrm{colim}_{j \in J} C_j$ where J is a directed set and where C_j is an étale B-algebra. We can factor $P \to C$ as $P \to C_j \to C$ for some j by Lemma 10.127.3. By Lemma 10.143.3 we can find an $i \in I$ and an étale ring map $B_i \to C'_j$ such that $C_j = B \otimes_{B_i} C'_j$. Then $C_j = \mathrm{colim}_{i' \geq i} B_{i'} \otimes_{B_i} C'_j$ and again we see that $P \to C_j$ factors as $P \to B_{i'} \otimes_{B_i} C'_j \to C$. As $A \to C' = B_{i'} \otimes_{B_i} C'_j$ is étale as compositions and tensor products of étale ring maps are étale. Hence we have factored $P \to C$ as $P \to C' \to C$ with C' étale over A and the criterion of Lemma 10.127.4 applies. ☐

0BSJ Lemma 10.154.3. Let R be a ring. Let $A = \mathrm{colim}\, A_i$ be a filtered colimit of R-algebras such that each A_i is a filtered colimit of étale R-algebras. Then A is a filtered colimit of étale R-algebras.

Proof. Write $A_i = \mathrm{colim}_{j \in J_i} A_j$ where J_i is a directed set and A_j is an étale R-algebra. For each $i \leq i'$ and $j \in J_i$ there exists an $j' \in J_{i'}$ and an R-algebra map $\varphi_{jj'} : A_j \to A_{j'}$ making the diagram

$$\begin{tikzcd} A_i \arrow[r] & A_{i'} \\ A_j \arrow[u] \arrow[r, "\varphi_{jj'}"] & A_{j'} \arrow[u] \end{tikzcd}$$

commute. This is true because $R \to A_j$ is of finite presentation so that Lemma 10.127.3 applies. Let $\mathcal{J}$ be the category with objects $\coprod_{i \in I} J_i$ and morphisms triples $(j, j', \varphi_{jj'})$ as above (and obvious composition law). Then $\mathcal{J}$ is a filtered category and $A = \mathrm{colim}_{\mathcal{J}} A_j$. Details omitted. ☐

0GIM Lemma 10.154.4. Let I be a directed set. Let $i \mapsto (R_i \to A_i)$ be a system of arrows of rings over I. Set $R = \mathrm{colim}\, R_i$ and $A = \mathrm{colim}\, A_i$. If each A_i is a filtered colimit of étale R_i-algebras, then A is a filtered colimit of étale R-algebras.

Proof. This is true because $A = A \otimes_R R = \mathrm{colim}\, A_i \otimes_{R_i} R$ and hence we can apply Lemma 10.154.3 because $R \to A_i \otimes_{R_i} R$ is a filtered colimit of étale ring maps by Lemma 10.154.1. ☐

08HS Lemma 10.154.5. Let R be a ring. Let $A \to B$ be an R-algebra homomorphism. If A and B are filtered colimits of étale R-algebras, then B is a filtered colimit of étale A-algebras.

Proof. Write $A = \mathrm{colim}\, A_i$ and $B = \mathrm{colim}\, B_j$ as filtered colimits with A_i and B_j étale over R. For each i we can find a j such that $A_i \to B$ factors through B_j, see Lemma 10.127.3. The factorization $A_i \to B_j$ is étale by Lemma 10.143.8. Since $A \to A \otimes_{A_i} B_j$ is étale (Lemma 10.143.3) it suffices to prove that $B = \mathrm{colim}\, A \otimes_{A_i} B_j$ where the colimit is over pairs (i, j) and factorizations $A_i \to B_j \to B$ of $A_i \to B$

(this is a directed system; details omitted). This is clear because colimits commute with tensor products and hence $\operatorname{colim} A \otimes_{A_i} B_j = A \otimes_A B = B$. $\square$

08HR Lemma 10.154.6. Let $R \to S$ be a ring map with S henselian local. Given

(1) an R-algebra A which is a filtered colimit of étale R-algebras,
(2) a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,
(3) a $\kappa(\mathfrak{p})$-algebra map $\tau : \kappa(\mathfrak{q}) \to S/\mathfrak{m}_S$,

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{m}_S)$ and $f \bmod \mathfrak{q} = \tau$.

Proof. Write $A = \operatorname{colim} A_i$ as a filtered colimit of étale R-algebras. Set $\mathfrak{q}_i = A_i \cap \mathfrak{q}$. We obtain $f_i : A_i \to S$ by applying Lemma 10.153.11. Set $f = \operatorname{colim} f_i$. $\square$

08HT Lemma 10.154.7. Let R be a ring. Given a commutative diagram of ring maps

$$\begin{tikzcd} S \arrow[r] & K \\ R \arrow[u] \arrow[r] & S' \arrow[u] \end{tikzcd}$$

where S, S' are henselian local, S, S' are filtered colimits of étale R-algebras, K is a field and the arrows $S \to K$ and $S' \to K$ identify K with the residue field of both S and S'. Then there exists an unique R-algebra isomorphism $S \to S'$ compatible with the maps to K.

Proof. Follows immediately from Lemma 10.154.6. $\square$

The following lemma is not strictly speaking about colimits of étale ring maps.

04GI Lemma 10.154.8. A filtered colimit of (strictly) henselian local rings along local homomorphisms is (strictly) henselian.

Proof. Categories, Lemma 4.21.5 says that this is really just a question about a colimit of (strictly) henselian local rings over a directed set. Let $(R_i, \varphi_{ii'})$ be such a system with each $\varphi_{ii'}$ local. Then $R = \operatorname{colim}_i R_i$ is local, and its residue field κ is $\operatorname{colim} \kappa_i$ (argument omitted). It is easy to see that $\operatorname{colim} \kappa_i$ is separably algebraically closed if each κ_i is so; thus it suffices to prove R is henselian if each R_i is henselian. Suppose that $f \in R[T]$ is monic and that $a_0 \in \kappa$ is a simple root of $\overline{f}$. Then for some large enough i there exists an $f_i \in R_i[T]$ mapping to f and an $a_{0,i} \in \kappa_i$ mapping to a_0. Since $\overline{f_i}(a_{0,i}) \in \kappa_i$, resp. $\overline{f_i'}(a_{0,i}) \in \kappa_i$ maps to $0 = \overline{f}(a_0) \in \kappa$, resp. $0 \neq \overline{f'}(a_0) \in \kappa$ we conclude that $a_{0,i}$ is a simple root of $\overline{f_i}$. As R_i is henselian we can find $a_i \in R_i$ such that $f_i(a_i) = 0$ and $a_{0,i} = \overline{a_i}$. Then the image $a \in R$ of a_i is the desired solution. Thus R is henselian. $\square$

10.155. Henselization and strict henselization

0BSK In this section we construct the henselization. We encourage the reader to keep in mind the uniqueness already proved in Lemma 10.154.7 and the functorial behaviour pointed out in Lemma 10.154.6 while reading this material.

04GN Lemma 10.155.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. There exists a local ring map $R \to R^h$ with the following properties

(1) R^h is henselian,
(2) R^h is a filtered colimit of étale R-algebras,

(3) $\mathfrak{m}R^h$ is the maximal ideal of R^h, and
(4) $\kappa = R^h/\mathfrak{m}R^h$.

Proof. Consider the category of pairs $(S, \mathfrak{q})$ where $R \to S$ is an étale ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$. A morphism of pairs $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. We set

$$R^h = \text{colim}_{(S,\mathfrak{q})} S.$$

Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the pair $(R, \mathfrak{m})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. For any pair $(S, \mathfrak{q})$ the prime ideal $\mathfrak{q}$ is maximal with residue field κ since the composition $\kappa \to S/\mathfrak{q} \to \kappa(\mathfrak{q})$ is an isomorphism. Suppose that $(S, \mathfrak{q})$ and $(S', \mathfrak{q}')$ are two objects. Set $S'' = S \otimes_R S'$ and $\mathfrak{q}'' = \mathfrak{q}S'' + \mathfrak{q}'S''$. Then $S''/\mathfrak{q}'' = S/\mathfrak{q} \otimes_R S'/\mathfrak{q}' = \kappa$ by what we said above. Moreover, $R \to S''$ is étale by Lemma 10.143.3. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 10.143.8. Consider

$$S'' = (S' \otimes_{\varphi, S, \psi} S') \otimes_{S' \otimes_R S'} S'$$

with prime ideal

$$\mathfrak{q}'' = (\mathfrak{q}' \otimes S' + S' \otimes \mathfrak{q}') \otimes S' + (S' \otimes_{\varphi, S, \psi} S') \otimes \mathfrak{q}'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. Moreover, the canonical map $S' \to S''$ (using the right most factor for example) equalizes φ and ψ. This proves part (3) of Categories, Definition 4.19.1. Hence we conclude that R^h consists of triples $(S, \mathfrak{q}, f)$ with $f \in S$, and two such triples $(S, \mathfrak{q}, f)$, $(S', \mathfrak{q}', f')$ define the same element of R^h if and only if there exists a pair $(S'', \mathfrak{q}'')$ and morphisms of pairs $\varphi : (S, \mathfrak{q}) \to (S'', \mathfrak{q}'')$ and $\varphi' : (S', \mathfrak{q}') \to (S'', \mathfrak{q}'')$ such that $\varphi(f) = \varphi'(f')$.

Suppose that $x \in R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over $\mathfrak{m}$. Then $\mathfrak{q} \not\subset \mathfrak{q}_i$ as we have seen above that $\mathfrak{q}$ is maximal. Thus, since $\mathfrak{q}$ is a prime ideal, we can find a $g \in S$, $g \not\in \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$. Consider the morphism of pairs $(S, \mathfrak{q}) \to (S_g, \mathfrak{q}S_g)$. In this way we see that we may always assume that x is given by a triple $(S, \mathfrak{q}, f)$ where $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{m}$, i.e., $\sqrt{\mathfrak{m}S} = \mathfrak{q}$. But since $R \to S$ is étale, we have $\mathfrak{m}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}}$, see Lemma 10.143.5. Hence we actually get that $\mathfrak{m}S = \mathfrak{q}$.

Suppose that $x \not\in \mathfrak{m}R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$ with $\mathfrak{m}S = \mathfrak{q}$. Then $f \not\in \mathfrak{m}S$, i.e., $f \not\in \mathfrak{q}$. Hence $(S, \mathfrak{q}) \to (S_f, \mathfrak{q}S_f)$ is a morphism of pairs such that the image of f becomes invertible. Hence x is invertible with inverse represented by the triple $(S_f, \mathfrak{q}S_f, 1/f)$. We conclude that R^h is a local ring with maximal ideal $\mathfrak{m}R^h$. The residue field is κ since we can define $R^h/\mathfrak{m}R^h \to \kappa$ by mapping a triple $(S, \mathfrak{q}, f)$ to the residue class of f modulo $\mathfrak{q}$.

We still have to show that R^h is henselian. Namely, suppose that $P \in R^h[T]$ is a monic polynomial and $a_0 \in \kappa$ is a simple root of the reduction $\overline{P} \in \kappa[T]$. Then we can find a pair $(S, \mathfrak{q})$ such that P is the image of a monic polynomial $Q \in S[T]$. Since $S \to R^h$ induces an isomorphism of residue fields we see that $S' = S[T]/(Q)$ has a prime ideal $\mathfrak{q}' = (\mathfrak{q}, T - a_0)$ at which $S \to S'$ is standard étale. Moreover, $\kappa = \kappa(\mathfrak{q}')$. Pick $g \in S'$, $g \not\in \mathfrak{q}'$ such that $S'' = S'_g$ is étale over S. Then $(S, \mathfrak{q}) \to (S'', \mathfrak{q}'S'')$ is a morphism of pairs. Now that triple $(S'', \mathfrak{q}'S'', \text{class of } T)$ determines an element $a \in R^h$ with the properties $P(a) = 0$, and $\overline{a} = a_0$ as desired. $\square$

04GP Lemma 10.155.2. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\kappa \subset \kappa^{sep}$ be a separable algebraic closure. There exists a commutative diagram

$$\begin{tikzcd} \kappa \arrow[r] & \kappa \arrow[r] & \kappa^{sep} \\ R \arrow[r] \arrow[u] & R^h \arrow[r] \arrow[u] & R^{sh} \arrow[u] \end{tikzcd}$$

with the following properties

(1) the map $R^h \to R^{sh}$ is local
(2) R^{sh} is strictly henselian,
(3) R^{sh} is a filtered colimit of étale R-algebras,
(4) $\mathfrak{m}R^{sh}$ is the maximal ideal of R^{sh}, and
(5) $\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$.

Proof. This is proved by exactly the same proof as used for Lemma 10.155.1. The only difference is that, instead of pairs, one uses triples $(S, \mathfrak{q}, \alpha)$ where $R \to S$ étale, $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$, and $\alpha : \kappa(\mathfrak{q}) \to \kappa^{sep}$ is an embedding of extensions of κ. $\square$

04GQ Definition 10.155.3. Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

(1) The local ring map $R \to R^h$ constructed in Lemma 10.155.1 is called the henselization of R.
(2) Given a separable algebraic closure $\kappa \subset \kappa^{sep}$ the local ring map $R \to R^{sh}$ constructed in Lemma 10.155.2 is called the strict henselization of R with respect to $\kappa \subset \kappa^{sep}$.
(3) A local ring map $R \to R^{sh}$ is called a strict henselization of R if it is isomorphic to one of the local ring maps constructed in Lemma 10.155.2

The maps $R \to R^h \to R^{sh}$ are flat local ring homomorphisms. By Lemma 10.154.7 the R-algebras R^h and R^{sh} are well defined up to unique isomorphism by the conditions that they are henselian local, filtered colimits of étale R-algebras with residue field κ and κ^{sep}. In the rest of this section we mostly just discuss functoriality of the (strict) henselizations. We will discuss more intricate results concerning the relationship between R and its henselization in More on Algebra, Section 15.45.

0BSL Remark 10.155.4. We can also construct R^{sh} from R^h. Namely, for any finite separable subextension $\kappa^{sep}/\kappa'/\kappa$ there exists a unique (up to unique isomorphism) finite étale local ring extension $R^h \subset R^h(\kappa')$ whose residue field extension reproduces the given extension, see Lemma 10.153.7. Hence we can set

$$R^{sh} = \bigcup\nolimits_{\kappa \subset \kappa' \subset \kappa^{sep}} R^h(\kappa')$$

The arrows in this system, compatible with the arrows on the level of residue fields, exist by Lemma 10.153.7. This will produce a henselian local ring by Lemma 10.154.8 since each of the rings $R^h(\kappa')$ is henselian by Lemma 10.153.4. By construction the residue field extension induced by $R^h \to R^{sh}$ is the field extension κ^{sep}/κ. Hence R^{sh} so constructed is strictly henselian. By Lemma 10.154.2 the R-algebra R^{sh} is a colimit of étale R-algebras. Hence the uniqueness of Lemma 10.154.7 shows that R^{sh} is the strict henselization.

04GR Lemma 10.155.5. Let $R \to S$ be a local map of local rings. Let $S \to S^h$ be the henselization. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying over $\mathfrak{m}_R$ such that $R/\mathfrak{m}_R \cong \kappa(\mathfrak{q})$. Then there exists a unique morphism of rings $f : A \to S^h$ fitting into the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"'] & S^h \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$.

Proof. This is a special case of Lemma 10.153.11. □

04GS Lemma 10.155.6. Let $R \to S$ be a local map of local rings. Let $R \to R^h$ and $S \to S^h$ be the henselizations. There exists a unique local ring map $R^h \to S^h$ fitting into the commutative diagram

$$\begin{tikzcd} R^h \arrow[r, "f"'] & S^h \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

Proof. Follows immediately from Lemma 10.154.6. □

Here is a slightly different construction of the henselization.

04GV Lemma 10.155.7. Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Consider the category of pairs $(S, \mathfrak{q})$ where $R \to S$ is étale and $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. This category is filtered and

$$(R_{\mathfrak{p}})^h = \mathrm{colim}_{(S,\mathfrak{q})} S = \mathrm{colim}_{(S,\mathfrak{q})} S_{\mathfrak{q}}$$

canonically.

Proof. A morphism of pairs $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the pair $(R, \mathfrak{p})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. Suppose that $(S, \mathfrak{q})$ and $(S', \mathfrak{q}')$ are two pairs. Note that $\mathfrak{q}$, resp. $\mathfrak{q}'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$ with residue fields $\kappa(\mathfrak{p})$, hence they correspond to maximal ideals of $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$. Set $S'' = S \otimes_R S'$. By the above there exists a unique prime $\mathfrak{q}'' \subset S''$ lying over $\mathfrak{q}$ and over $\mathfrak{q}'$ whose residue field is $\kappa(\mathfrak{p})$. The ring map $R \to S''$ is étale by Lemma 10.143.3. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 10.143.8. Consider

$$S'' = (S' \otimes_{\varphi, S, \psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak{p}$ is

$$F'' = (F' \otimes_{\varphi, F, \psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak{p})} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of pairs the map $F' \to \kappa(\mathfrak{p})$ corresponding to $\mathfrak{p}'$ extends to a map $F'' \to \kappa(\mathfrak{p})$ and in turn corresponds to a prime ideal $\mathfrak{q}'' \subset S''$ whose residue field is $\kappa(\mathfrak{p})$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of pairs $(S', \mathfrak{q}') \to (S'', \mathfrak{q}'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 4.19.1. Hence we conclude that the category is filtered.

Recall that in the proof of Lemma 10.155.1 we constructed $(R_\mathfrak{p})^h$ as the corresponding colimit but starting with $R_\mathfrak{p}$ and its maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Now, given any pair $(S, \mathfrak{q})$ for $(R, \mathfrak{p})$ we obtain a pair $(S_\mathfrak{p}, \mathfrak{q}S_\mathfrak{p})$ for $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$. Moreover, in this situation

$$S_\mathfrak{p} = \operatorname{colim}_{f \in R, f \not\in \mathfrak{p}} S_f.$$

Hence in order to show the equalities of the lemma, it suffices to show that any pair $(S_{loc}, \mathfrak{q}_{loc})$ for $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ is of the form $(S_\mathfrak{p}, \mathfrak{q}S_\mathfrak{p})$ for some pair $(S, \mathfrak{q})$ over $(R, \mathfrak{p})$ (some details omitted). This follows from Lemma 10.143.3. □

08HU Lemma 10.155.8. Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $R \to R^h$ and $S \to S^h$ be the henselizations of $R_\mathfrak{p}$ and $S_\mathfrak{q}$. The local ring map $R^h \to S^h$ of Lemma 10.155.6 identifies S^h with the henselization of $R^h \otimes_R S$ at the unique prime lying over $\mathfrak{m}^h$ and $\mathfrak{q}$.

Proof. By Lemma 10.155.7 we see that R^h, resp. S^h are filtered colimits of étale R, resp. S-algebras. Hence we see that $R^h \otimes_R S$ is a filtered colimit of étale S-algebras A_i (Lemma 10.143.3). By Lemma 10.154.5 we see that S^h is a filtered colimit of étale $R^h \otimes_R S$-algebras. Since moreover S^h is a henselian local ring with residue field equal to $\kappa(\mathfrak{q})$, the statement follows from the uniqueness result of Lemma 10.154.7. □

04GT Lemma 10.155.9. Let $\varphi : R \to S$ be a local map of local rings. Let $S/\mathfrak{m}_S \subset \kappa^{sep}$ be a separable algebraic closure. Let $S \to S^{sh}$ be the strict henselization of S with respect to $S/\mathfrak{m}_S \subset \kappa^{sep}$. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying over $\mathfrak{m}_R$. Given any commutative diagram

$$\begin{tikzcd} \kappa(\mathfrak{q}) \arrow[r, "\phi"'] & \kappa^{sep} \\ R/\mathfrak{m}_R \arrow[u] \arrow[r, "\varphi"] & S/\mathfrak{m}_S \arrow[u] \end{tikzcd}$$

there exists a unique morphism of rings $f : A \to S^{sh}$ fitting into the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"'] & S^{sh} \\ R \arrow[u] \arrow[r, "\varphi"] & S \arrow[u] \end{tikzcd}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$ and the induced map $\kappa(\mathfrak{q}) \to \kappa^{sep}$ is the given one.

Proof. This is a special case of Lemma 10.153.11. □

04GU Lemma 10.155.10. Let $R \to S$ be a local map of local rings. Choose separable algebraic closures $R/\mathfrak{m}_R \subset \kappa_1^{sep}$ and $S/\mathfrak{m}_S \subset \kappa_2^{sep}$. Let $R \to R^{sh}$ and $S \to S^{sh}$ be the corresponding strict henselizations. Given any commutative diagram

$$\begin{tikzcd} \kappa_1^{sep} \arrow[r, "\phi"'] & \kappa_2^{sep} \\ R/\mathfrak m_R \arrow[u] \arrow[r, "\varphi"] & S/\mathfrak m_S \arrow[u] \end{tikzcd}$$

There exists a unique local ring map $R^{sh} \to S^{sh}$ fitting into the commutative diagram

$$\begin{tikzcd} R^{sh} \arrow[r, "f"'] & S^{sh} \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

and inducing ϕ on the residue fields of R^{sh} and S^{sh}.

Proof. Follows immediately from Lemma 10.154.6. $\square$

04GW Lemma 10.155.11. Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Let $\kappa(\mathfrak p) \subset \kappa^{sep}$ be a separable algebraic closure. Consider the category of triples $(S, \mathfrak q, \phi)$ where $R \to S$ is étale, $\mathfrak q$ is a prime lying over $\mathfrak p$, and $\phi : \kappa(\mathfrak q) \to \kappa^{sep}$ is a $\kappa(\mathfrak p)$-algebra map. This category is filtered and

$$(R_{\mathfrak p})^{sh} = \operatorname{colim}_{(S,\mathfrak q,\phi)} S = \operatorname{colim}_{(S,\mathfrak q,\phi)} S_{\mathfrak q}$$

canonically.

Proof. A morphism of triples $(S, \mathfrak q, \phi) \to (S', \mathfrak q', \phi')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak q') = \mathfrak q$ and such that $\phi' \circ \varphi = \phi$. Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the triple $(R, \mathfrak p, \kappa(\mathfrak p) \subset \kappa^{sep})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. Suppose that $(S, \mathfrak q, \phi)$ and $(S', \mathfrak q', \phi')$ are two triples. Note that $\mathfrak q$, resp. $\mathfrak q'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak p)$, resp. $S' \otimes \kappa(\mathfrak p)$ with residue fields finite separable over $\kappa(\mathfrak p)$ and ϕ, resp. ϕ' correspond to maps into κ^{sep}. Hence this data corresponds to $\kappa(\mathfrak p)$-algebra maps

$$\phi : S \otimes_R \kappa(\mathfrak p) \longrightarrow \kappa^{sep}, \quad \phi' : S' \otimes_R \kappa(\mathfrak p) \longrightarrow \kappa^{sep}.$$

Set $S'' = S \otimes_R S'$. Combining the maps the above we get a unique $\kappa(\mathfrak p)$-algebra map

$$\phi'' = \phi \otimes \phi' : S'' \otimes_R \kappa(\mathfrak p) \longrightarrow \kappa^{sep}$$

whose kernel corresponds to a prime $\mathfrak q'' \subset S''$ lying over $\mathfrak q$ and over $\mathfrak q'$, and whose residue field maps via ϕ'' to the compositum of $\phi(\kappa(\mathfrak q))$ and $\phi'(\kappa(\mathfrak q'))$ in κ^{sep}. The ring map $R \to S''$ is étale by Lemma 10.143.3. Hence $(S'', \mathfrak q'', \phi'')$ is a triple dominating both $(S, \mathfrak q, \phi)$ and $(S', \mathfrak q', \phi')$. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak q, \phi) \to (S', \mathfrak q', \phi')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 10.143.8. Consider

$$S'' = (S' \otimes_{\varphi, S, \psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak p$ is

$$F'' = (F' \otimes_{\varphi, F, \psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak p)} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of triples the map $\phi' : F' \to \kappa^{sep}$ extends to a map $\phi'' : F'' \to \kappa^{sep}$ which in turn corresponds to a prime ideal $\mathfrak q'' \subset S''$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of triples $(S', \mathfrak q', \phi') \to (S'', \mathfrak q'', \phi'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 4.19.1. Hence we conclude that the category is filtered.

We still have to show that the colimit R_{colim} of the system is equal to the strict henselization of $R_\mathfrak p$ with respect to κ^{sep}. To see this note that the system of triples $(S, \mathfrak q, \phi)$ contains as a subsystem the pairs $(S, \mathfrak q)$ of Lemma 10.155.7. Hence R_{colim} contains $R_\mathfrak p^h$ by the result of that lemma. Moreover, it is clear that $R_\mathfrak p^h \subset R_{colim}$ is a directed colimit of étale ring extensions. It follows that R_{colim} is henselian by Lemmas 10.153.4 and 10.154.8. Finally, by Lemma 10.144.3 we see that the residue field of R_{colim} is equal to κ^{sep}. Hence we conclude that R_{colim} is strictly henselian and hence equals the strict henselization of $R_\mathfrak p$ as desired. Some details omitted. □

08HV Lemma 10.155.12. Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. Choose separable algebraic closures $\kappa(\mathfrak p) \subset \kappa_1^{sep}$ and $\kappa(\mathfrak q) \subset \kappa_2^{sep}$. Let R^{sh} and S^{sh} be the corresponding strict henselizations of $R_\mathfrak p$ and $S_\mathfrak q$. Given any commutative diagram

$$\begin{tikzcd} \kappa_1^{sep} \arrow[r, "\phi"'] & \kappa_2^{sep} \\ \kappa(\mathfrak p) \arrow[u] \arrow[r, "\varphi"] & \kappa(\mathfrak q) \arrow[u] \end{tikzcd}$$

The local ring map $R^{sh} \to S^{sh}$ of Lemma 10.155.10 identifies S^{sh} with the strict henselization of $R^{sh} \otimes_R S$ at a prime lying over $\mathfrak q$ and the maximal ideal $\mathfrak m^{sh} \subset R^{sh}$.

Proof. The proof is identical to the proof of Lemma 10.155.8 except that it uses Lemma 10.155.11 instead of Lemma 10.155.7. □

0C2Z Lemma 10.155.13. Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$ such that $\kappa(\mathfrak p) \to \kappa(\mathfrak q)$ is an isomorphism. Choose a separable algebraic closure κ^{sep} of $\kappa(\mathfrak p) = \kappa(\mathfrak q)$. Then

$$(S_\mathfrak q)^{sh} = (S_\mathfrak q)^h \otimes_{(R_\mathfrak p)^h} (R_\mathfrak p)^{sh}$$

Proof. This follows from the alternative construction of the strict henselization of a local ring in Remark 10.155.4 and the fact that the residue fields are equal. Some details omitted. □

10.156. Henselization and quasi-finite ring maps

0GIN In this section we prove some results concerning the functorial maps between (strict) henselizations for quasi-finite ring maps.

05WP Lemma 10.156.1. Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over $\mathfrak p$ in R. Assume $R \to S$ is quasi-finite at $\mathfrak q$. The commutative diagram

$$\begin{tikzcd} R_{\mathfrak p}^h \arrow[r] & S_{\mathfrak q}^h \\ R_{\mathfrak p} \arrow[r] \arrow[u] & S_{\mathfrak q} \arrow[u] \end{tikzcd}$$

of Lemma 10.155.6 identifies $S_{\mathfrak q}^h$ with the localization of $R_{\mathfrak p}^h \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$ at the prime generated by $\mathfrak q$. Moreover, the ring map $R_{\mathfrak p}^h \to S_{\mathfrak q}^h$ is finite.

Proof. Note that $R_{\mathfrak p}^h \otimes_R S$ is quasi-finite over $R_{\mathfrak p}^h$ at the prime ideal corresponding to $\mathfrak q$, see Lemma 10.122.6. Hence the localization S' of $R_{\mathfrak p}^h \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$ is henselian and finite over $R_{\mathfrak p}^h$, see Lemma 10.153.4. As a localization S' is a filtered colimit of étale $R_{\mathfrak p}^h \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$-algebras. By Lemma 10.155.8 we see that $S_{\mathfrak q}^h$ is the henselization of $R_{\mathfrak p}^h \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$. Thus $S' = S_{\mathfrak q}^h$ by the uniqueness result of Lemma 10.154.7. ☐

05WQ Lemma 10.156.2. Let R be a local ring with henselization R^h. Let $I \subset \mathfrak m_R$. Then R^h/IR^h is the henselization of R/I.

Proof. This is a special case of Lemma 10.156.1. ☐

05WR Lemma 10.156.3. Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over $\mathfrak p$ in R. Assume $R \to S$ is quasi-finite at $\mathfrak q$. Let $\kappa_2^{sep}/\kappa(\mathfrak q)$ be a separable algebraic closure and denote $\kappa_1^{sep} \subset \kappa_2^{sep}$ the subfield of elements separable algebraic over $\kappa(\mathfrak q)$ (Fields, Lemma 9.14.6). The commutative diagram

$$\begin{tikzcd} R_{\mathfrak p}^{sh} \arrow[r] & S_{\mathfrak q}^{sh} \\ R_{\mathfrak p} \arrow[r] \arrow[u] & S_{\mathfrak q} \arrow[u] \end{tikzcd}$$

of Lemma 10.155.10 identifies $S_{\mathfrak q}^{sh}$ with the localization of $R_{\mathfrak p}^{sh} \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$ at the prime ideal which is the kernel of the map

$$R_{\mathfrak p}^{sh} \otimes_{R_{\mathfrak p}} S_{\mathfrak q} \longrightarrow \kappa_1^{sep} \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak q) \longrightarrow \kappa_2^{sep}$$

Moreover, the ring map $R_{\mathfrak p}^{sh} \to S_{\mathfrak q}^{sh}$ is a finite local homomorphism of local rings whose residue field extension is the extension $\kappa_2^{sep}/\kappa_1^{sep}$ which is both finite and purely inseparable.

Proof. Since $R \to S$ is quasi-finite at $\mathfrak q$ we see that the extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite, see Definition 10.122.3 and Lemma 10.122.2. Hence κ_1^{sep} is a separable algebraic closure of $\kappa(\mathfrak p)$ (small detail omitted). In particular Lemma 10.155.10 does really apply. Next, the compositum of $\kappa(\mathfrak p)$ and κ_1^{sep} in κ_2^{sep} is separably algebraically closed and hence equal to κ_2^{sep}. We conclude that $\kappa_2^{sep}/\kappa_1^{sep}$ is finite. By construction the extension $\kappa_2^{sep}/\kappa_1^{sep}$ is purely inseparable. The ring map $R_{\mathfrak p}^{sh} \to S_{\mathfrak q}^{sh}$ is indeed local and induces the residue field extension $\kappa_2^{sep}/\kappa_1^{sep}$ which is indeed finite purely inseparable.

Note that $R_{\mathfrak p}^{sh} \otimes_R S$ is quasi-finite over $R_{\mathfrak p}^{sh}$ at the prime ideal $\mathfrak q'$ given in the statement of the lemma, see Lemma 10.122.6. Hence the localization S' of $R_{\mathfrak p}^{sh} \otimes_{R_{\mathfrak p}}$

$S_\mathfrak{q}$ at $\mathfrak{q}'$ is henselian and finite over $R_\mathfrak{p}^{sh}$, see Lemma 10.153.4. Note that the residue field of S' is κ_2^{sep} as the map $\kappa_1^{sep} \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \to \kappa_2^{sep}$ is surjective by the discussion in the previous paragraph. Furthermore, as a localization S' is a filtered colimit of étale $R_\mathfrak{p}^{sh} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$-algebras. By Lemma 10.155.12 we see that $S_\mathfrak{q}^{sh}$ is the strict henselization of $R_\mathfrak{p}^{sh} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ at $\mathfrak{q}'$. Thus $S' = S_\mathfrak{q}^{sh}$ by the uniqueness result of Lemma 10.154.7. □

05WS Lemma 10.156.4. Let R be a local ring with strict henselization R^{sh}. Let $I \subset \mathfrak{m}_R$. Then R^{sh}/IR^{sh} is a strict henselization of R/I.

Proof. This is a special case of Lemma 10.156.3. □

092Y Lemma 10.156.5. Let $A \to B$ and $A \to C$ be local homomorphisms of local rings. If $A \to C$ is integral and either $\kappa(\mathfrak{m}_C)/\kappa(\mathfrak{m}_A)$ or $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ is purely inseparable, then $D = B \otimes_A C$ is a local ring and $B \to D$ and $C \to D$ are local.

Proof. Any maximal ideal of D lies over the maximal ideal of B by going up for the integral ring map $B \to D$ (Lemma 10.36.22). Now $D/\mathfrak{m}_B D = \kappa(\mathfrak{m}_B) \otimes_A C = \kappa(\mathfrak{m}_B) \otimes_{\kappa(\mathfrak{m}_A)} C/\mathfrak{m}_A C$. The spectrum of $C/\mathfrak{m}_A C$ consists of a single point, namely $\mathfrak{m}_C$. Thus the spectrum of $D/\mathfrak{m}_B D$ is the same as the spectrum of $\kappa(\mathfrak{m}_B) \otimes_{\kappa(\mathfrak{m}_A)} \kappa(\mathfrak{m}_C)$ which is a single point by our assumption that either $\kappa(\mathfrak{m}_C)/\kappa(\mathfrak{m}_A)$ or $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ is purely inseparable. This proves that D is local and that the ring maps $B \to D$ and $C \to D$ are local. □

0GIP Lemma 10.156.6. Let $A \to B$ and $A \to C$ be ring maps. Let κ be a separably algebraically closed field and let $B \otimes_A C \to \kappa$ be a ring homomorphism. Denote

$$\begin{tikzcd} B^{sh} \arrow[r] & (B \otimes_A C)^{sh} \\ A^{sh} \arrow[r] \arrow[u] & C^{sh} \arrow[u] \end{tikzcd}$$

the corresponding maps of strict henselizations (see proof). If

(1) $A \to B$ is quasi-finite at the prime $\mathfrak{p}_B = \mathrm{Ker}(B \to \kappa)$, or
(2) B is a filtered colimit of quasi-finite A-algebras, or
(3) $B_{\mathfrak{p}_B}$ is a filtered colimit of quasi-finite algebras over $A_{\mathfrak{p}_A}$, or
(4) B is integral over A,

then $B^{sh} \otimes_{A^{sh}} C^{sh} \to (B \otimes_A C)^{sh}$ is an isomorphism.

Proof. Write $D = B \otimes_A C$. Denote $\mathfrak{p}_A = \mathrm{Ker}(A \to \kappa)$ and similarly for $\mathfrak{p}_B$, $\mathfrak{p}_C$, and $\mathfrak{p}_D$. Denote $\kappa_A \subset \kappa$ the separable algebraic closure of $\kappa(\mathfrak{p}_A)$ in κ and similarly for κ_B, κ_C, and κ_D. Denote A^{sh} the strict henselization of $A_{\mathfrak{p}_A}$ constructed using the separable algebraic closure $\kappa_A/\kappa(\mathfrak{p}_A)$. Similarly for B^{sh}, C^{sh}, and D^{sh}. We obtain the commutative diagram of the lemma from the functoriality of Lemma 10.155.10.

Consider the map

$$c : B^{sh} \otimes_{A^{sh}} C^{sh} \to D^{sh} = (B \otimes_A C)^{sh}$$

we obtain from the commutative diagram. If $A \to B$ is quasi-finite at $\mathfrak{p}_B = \mathrm{Ker}(B \to \kappa)$, then the ring map $C \to D$ is quasi-finite at $\mathfrak{p}_D$ by Lemma 10.122.6. Hence by Lemma 10.156.3 (and Lemma 10.36.13) the ring map c is a homomorphism of finite C^{sh}-algebras and

$$B^{sh} = (B \otimes_A A^{sh})_\mathfrak{q} \quad \text{and} \quad D^{sh} = (D \otimes_C C^{sh})_\mathfrak{r} = (B \otimes_A C^{sh})_\mathfrak{r}$$

for some primes $\mathfrak q$ and $\mathfrak r$. Since

$$B^{sh} \otimes_{A^{sh}} C^{sh} = (B \otimes_A A^{sh})_{\mathfrak q} \otimes_{A^{sh}} C^{sh} = \text{a localization of } B \otimes_A C^{sh}$$

we conclude that source and target of c are both localizations of $B \otimes_A C^{sh}$ (compatibly with the map). Hence it suffices to show that $B^{sh} \otimes_{A^{sh}} C^{sh}$ is local (small detail omitted). This follows from Lemma 10.156.5 and the fact that $A^{sh} \to B^{sh}$ is finite with purely inseparable residue field extension by the already used Lemma 10.156.3. This proves case (1) of the lemma.

In case (2) write $B = \operatorname{colim} B_i$ as a filtered colimit of quasi-finite A-algebras. We correspondingly get $D = \operatorname{colim} D_i$ with $D_i = B_i \otimes_A C$. Observe that $B^{sh} = \operatorname{colim} B_i^{sh}$. Namely, the ring $\operatorname{colim} B_i^{sh}$ is a strictly henselian local ring by Lemma 10.154.8. Also $\operatorname{colim} B_i^{sh}$ is a filtered colimit of étale B-algebras by Lemma 10.154.4. Finally, the residue field of $\operatorname{colim} B_i^{sh}$ is a separable algebraic closure of $\kappa(\mathfrak p_B)$ (details omitted). Hence we conclude that $B^{sh} = \operatorname{colim} B_i^{sh}$, see discussion following Definition 10.155.3. Similarly, we have $D^{sh} = \operatorname{colim} D_i^{sh}$. Then we conclude by case (1) because

$$D^{sh} = \operatorname{colim} D_i^{sh} = \operatorname{colim} B_i^{sh} \otimes_{A^{sh}} C^{sh} = B^{sh} \otimes_{A^{sh}} C^{sh}$$

since filtered colimit commute with tensor products.

Case (3). We may replace A, B, C by their localizations at $\mathfrak p_A$, $\mathfrak p_B$, and $\mathfrak p_C$. Thus (3) follows from (2).

Since an integral ring map is a filtered colimit of finite ring maps, we see that (4) follows from (2) as well. $\square$

10.157. Serre's criterion for normality

031O We introduce the following properties of Noetherian rings.

031P Definition 10.157.1. Let R be a Noetherian ring. Let $k \geq 0$ be an integer.

(1) We say R has property (R_k) if for every prime $\mathfrak p$ of height $\leq k$ the local ring $R_{\mathfrak p}$ is regular. We also say that R is regular in codimension $\leq k$.
(2) We say R has property (S_k) if for every prime $\mathfrak p$ the local ring $R_{\mathfrak p}$ has depth at least $\min\{k, \dim(R_{\mathfrak p})\}$.
(3) Let M be a finite R-module. We say M has property (S_k) if for every prime $\mathfrak p$ the module $M_{\mathfrak p}$ has depth at least $\min\{k, \dim(\operatorname{Supp}(M_{\mathfrak p}))\}$.

Any Noetherian ring has property (S_0) and so does any finite module over it. Our convention that the depth of the zero module is ∞ (see Section 10.72) and the dimension of the empty set is $-\infty$ (see Topology, Section 5.10) guarantees that the zero module has property (S_k) for all k.

031Q Lemma 10.157.2. Let R be a Noetherian ring. Let M be a finite R-module. The following are equivalent:

(1) M has no embedded associated prime, and
(2) M has property (S_1).

Proof. Let $\mathfrak p$ be an embedded associated prime of M. Then there exists another associated prime $\mathfrak q$ of M such that $\mathfrak p \supset \mathfrak q$. In particular this implies that $\dim(\operatorname{Supp}(M_{\mathfrak p})) \geq 1$ (since $\mathfrak q$ is in the support as well). On the other hand $\mathfrak pR_{\mathfrak p}$ is associated to $M_{\mathfrak p}$ (Lemma 10.63.15) and hence $\operatorname{depth}(M_{\mathfrak p}) = 0$ (see Lemma

10.63.18). In other words (S_1) does not hold. Conversely, if (S_1) does not hold then there exists a prime $\mathfrak p$ such that $\dim(\text{Supp}(M_{\mathfrak p})) \geq 1$ and $\text{depth}(M_{\mathfrak p}) = 0$. Since $\text{depth}(M_{\mathfrak p}) = 0$, we see that $\mathfrak p \in \text{Ass}(M)$ by the two Lemmas 10.63.15 and 10.63.18. Since $\dim(\text{Supp}(M_{\mathfrak p})) \geq 1$, there is a prime $\mathfrak q \in \text{Supp}(M)$ with $\mathfrak q \subset \mathfrak p$, $\mathfrak q \neq \mathfrak p$. We can take such a $\mathfrak q$ that is minimal in $\text{Supp}(M)$. Then by Proposition 10.63.6 we have $\mathfrak q \in \text{Ass}(M)$ and hence $\mathfrak p$ is an embedded associated prime. □

031R Lemma 10.157.3. Let R be a Noetherian ring. The following are equivalent:

(1) R is reduced, and
(2) R has properties (R_0) and (S_1).

Proof. Suppose that R is reduced. Then $R_{\mathfrak p}$ is a field for every minimal prime $\mathfrak p$ of R, according to Lemma 10.25.1. Hence we have (R_0). Let $\mathfrak p$ be a prime of height ≥ 1. Then $A = R_{\mathfrak p}$ is a reduced local ring of dimension ≥ 1. Hence its maximal ideal $\mathfrak m$ is not an associated prime since this would mean there exists an $x \in \mathfrak m$ with annihilator $\mathfrak m$ so $x^2 = 0$. Hence the depth of $A = R_{\mathfrak p}$ is at least one, by Lemma 10.63.9. This shows that (S_1) holds.

Conversely, assume that R satisfies (R_0) and (S_1). If $\mathfrak p$ is a minimal prime of R, then $R_{\mathfrak p}$ is a field by (R_0), and hence is reduced. If $\mathfrak p$ is not minimal, then we see that $R_{\mathfrak p}$ has depth ≥ 1 by (S_1) and we conclude there exists an element $t \in \mathfrak pR_{\mathfrak p}$ such that $R_{\mathfrak p} \to R_{\mathfrak p}[1/t]$ is injective. Now $R_{\mathfrak p}[1/t]$ is contained in the product of its localizations at prime ideals, see Lemma 10.23.1. This implies that $R_{\mathfrak p}$ is a subring of a product of localizations of R at $\mathfrak q \supset \mathfrak p$ with $t \notin \mathfrak q$. Since theses primes have smaller height by induction on the height we conclude that R is reduced. □

031S Lemma 10.157.4 (Serre's criterion for normality). Let R be a Noetherian ring. The following are equivalent: [DG67, IV, Theorem 5.8.6]

(1) R is a normal ring, and
(2) R has properties (R_1) and (S_2).

Proof. Proof of (1) ⇒ (2). Assume R is normal, i.e., all localizations $R_{\mathfrak p}$ at primes are normal domains. In particular we see that R has (R_0) and (S_1) by Lemma 10.157.3. Hence it suffices to show that a local Noetherian normal domain R of dimension d has depth $\geq \min(2, d)$ and is regular if $d = 1$. The assertion if $d = 1$ follows from Lemma 10.119.7.

Let R be a local Noetherian normal domain with maximal ideal $\mathfrak m$ and dimension $d \geq 2$. Apply Lemma 10.119.2 to R. It is clear that R does not fall into cases (1) or (2) of the lemma. Let $R \to R'$ as in (4) of the lemma. Since R is a domain we have $R \subset R'$. Since $\mathfrak m$ is not an associated prime of R' there exists an $x \in \mathfrak m$ which is a nonzerodivisor on R'. Then $R_x = R'_x$ so R and R' are domains with the same fraction field. But finiteness of $R \subset R'$ implies every element of R' is integral over R (Lemma 10.36.3) and we conclude that $R = R'$ as R is normal. This means (4) does not happen. Thus we get the remaining possibility (3), i.e., $\text{depth}(R) \geq 2$ as desired.

Proof of (2) ⇒ (1). Assume R satisfies (R_1) and (S_2). By Lemma 10.157.3 we conclude that R is reduced. Hence it suffices to show that if R is a reduced local Noetherian ring of dimension d satisfying (S_2) and (R_1) then R is a normal domain. If $d = 0$, the result is clear. If $d = 1$, then the result follows from Lemma 10.119.7.

Let R be a reduced local Noetherian ring with maximal ideal $\mathfrak{m}$ and dimension $d \geq 2$ which satisfies (R_1) and (S_2). By Lemma 10.37.16 it suffices to show that R is integrally closed in its total ring of fractions $Q(R)$. Pick $x \in Q(R)$ which is integral over R. Then $R' = R[x]$ is a finite ring extension of R (Lemma 10.36.5). Because $\dim(R_\mathfrak{p}) < d$ for every nonmaximal prime $\mathfrak{p} \subset R$ we have $R_\mathfrak{p} = R'_\mathfrak{p}$ by induction. Hence the support of R'/R is $\{\mathfrak{m}\}$. It follows that R'/R is annihilated by a power of $\mathfrak{m}$ (Lemma 10.62.4). By Lemma 10.119.2 this contradicts the assumption that the depth of R is $\geq 2 = \min(2, d)$ and the proof is complete. $\square$

0567 Lemma 10.157.5. A regular ring is normal.

Proof. Let R be a regular ring. By Lemma 10.157.4 it suffices to prove that R is (R_1) and (S_2). As a regular local ring is Cohen-Macaulay, see Lemma 10.106.3, it is clear that R is (S_2). Property (R_1) is immediate. $\square$

031T Lemma 10.157.6. Let R be a Noetherian normal domain with fraction field K. Then

(1) for any nonzero $a \in R$ the quotient R/aR has no embedded primes, and all its associated primes have height 1
(2)
$$R = \bigcap\nolimits_{\text{height}(\mathfrak{p})=1} R_\mathfrak{p}$$
(3) For any nonzero $x \in K$ the quotient $R/(R \cap xR)$ has no embedded primes, and all its associates primes have height 1.

Proof. By Lemma 10.157.4 we see that R has (S_2). Hence for any nonzero element $a \in R$ we see that R/aR has (S_1) (use Lemma 10.72.6 for example) Hence R/aR has no embedded primes (Lemma 10.157.2). We conclude the associated primes of R/aR are exactly the minimal primes $\mathfrak{p}$ over (a), which have height 1 as a is not zero (Lemma 10.60.11). This proves (1).

Thus, given $b \in R$ we have $b \in aR$ if and only if $b \in aR_\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ over (a) (see Lemma 10.63.19). These primes all have height 1 as seen above so $b/a \in R$ if and only if $b/a \in R_\mathfrak{p}$ for all height 1 primes. Hence (2) holds.

For (3) write $x = a/b$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes over (ab). These all have height 1 by the above. Then we see that $R \cap xR = \bigcap_{i=1,\ldots,r}(R \cap xR_{\mathfrak{p}_i})$ by part (2) of the lemma. Hence $R/(R \cap xR)$ is a submodule of $\bigoplus R/(R \cap xR_{\mathfrak{p}_i})$. As $R_{\mathfrak{p}_i}$ is a discrete valuation ring (by property (R_1) for the Noetherian normal domain R, see Lemma 10.157.4) we have $xR_{\mathfrak{p}_i} = \mathfrak{p}_i^{e_i} R_{\mathfrak{p}_i}$ for some $e_i \in \mathbf{Z}$. Hence the direct sum is equal to $\bigoplus_{e_i > 0} R/\mathfrak{p}_i^{(e_i)}$, see Definition 10.64.1. By Lemma 10.64.2 the only associated prime of the module $R/\mathfrak{p}^{(n)}$ is $\mathfrak{p}$. Hence the set of associate primes of $R/(R \cap xR)$ is a subset of $\{\mathfrak{p}_i\}$ and there are no inclusion relations among them. This proves (3). $\square$

10.158. Formal smoothness of fields

031U In this section we show that field extensions are formally smooth if and only if they are separable. However, we first prove finitely generated field extensions are separable algebraic if and only if they are formally unramified.

090W Lemma 10.158.1. Let K/k be a finitely generated field extension. The following are equivalent

(1) K is a finite separable field extension of k,
(2) $\Omega_{K/k} = 0$,
(3) K is formally unramified over k,
(4) K is unramified over k,
(5) K is formally étale over k,
(6) K is étale over k.

Proof. The equivalence of (2) and (3) is Lemma 10.148.2. By Lemma 10.143.4 we see that (1) is equivalent to (6). Property (6) implies (5) and (4) which both in turn imply (3) (Lemmas 10.150.2, 10.151.3, and 10.151.2). Thus it suffices to show that (2) implies (1). Choose a finitely generated k-subalgebra $A \subset K$ such that K is the fraction field of the domain A. Set $S = A \setminus \{0\}$. Since $0 = \Omega_{K/k} = S^{-1}\Omega_{A/k}$ (Lemma 10.131.8) and since $\Omega_{A/k}$ is finitely generated (Lemma 10.131.16), we can replace A by a localization A_f to reduce to the case that $\Omega_{A/k} = 0$ (details omitted). Then A is unramified over k, hence K/k is finite separable for example by Lemma 10.151.5 applied with $\mathfrak q = (0)$. $\square$

031W Lemma 10.158.2. Let k be a perfect field of characteristic $p > 0$. Let K/k be an extension. Let $a \in K$. Then $\mathrm{d}a = 0$ in $\Omega_{K/k}$ if and only if a is a pth power.

Proof. By Lemma 10.131.5 we see that there exists a subfield $k \subset L \subset K$ such that L/k is a finitely generated field extension and such that $\mathrm{d}a$ is zero in $\Omega_{L/k}$. Hence we may assume that K is a finitely generated field extension of k.

Choose a transcendence basis $x_1, \ldots, x_r \in K$ such that K is finite separable over $k(x_1, \ldots, x_r)$. This is possible by the definitions, see Definitions 10.45.1 and 10.42.1. We remark that the result holds for the purely transcendental subfield $k(x_1, \ldots, x_r) \subset K$. Namely,

$$\Omega_{k(x_1,\ldots,x_r)/k} = \bigoplus\nolimits_{i=1}^{r} k(x_1, \ldots, x_r)\mathrm{d}x_i$$

and any rational function all of whose partial derivatives are zero is a pth power. Moreover, we also have

$$\Omega_{K/k} = \bigoplus\nolimits_{i=1}^{r} K\mathrm{d}x_i$$

since $k(x_1, \ldots, x_r) \subset K$ is finite separable (computation omitted). Suppose $a \in K$ is an element such that $\mathrm{d}a = 0$ in the module of differentials. By our choice of x_i we see that the minimal polynomial $P(T) \in k(x_1, \ldots, x_r)[T]$ of a is separable. Write

$$P(T) = T^d + \sum\nolimits_{i=1}^{d} a_i T^{d-i}$$

and hence

$$0 = \mathrm{d}P(a) = \sum\nolimits_{i=1}^{d} a^{d-i}\mathrm{d}a_i$$

in $\Omega_{K/k}$. By the description of $\Omega_{K/k}$ above and the fact that P was the minimal polynomial of a, we see that this implies $\mathrm{d}a_i = 0$. Hence $a_i = b_i^p$ for each i. Therefore by Fields, Lemma 9.28.2 we see that a is a pth power. $\square$

07DZ Lemma 10.158.3. Let k be a field of characteristic $p > 0$. Let $a_1, \ldots, a_n \in k$ be elements such that $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Then the field extension $k(a_1^{1/p}, \ldots, a_n^{1/p})$ has degree p^n over k.

Proof. By induction on n. If $n = 1$ the result is Lemma 10.158.2. For the induction step, suppose that $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$ has degree p^{n-1} over k. We have to show that a_n does not map to a pth power in $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$. If it does then we can write

$$\begin{aligned} a_n &= \left(\sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\leq i_j\leq p-1} \lambda_I a_1^{i_1/p} \ldots a_{n-1}^{i_{n-1}/p}\right)^p \\ &= \sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\leq i_j\leq p-1} \lambda_I^p a_1^{i_1} \ldots a_{n-1}^{i_{n-1}} \end{aligned}$$

Applying d we see that $\text{d}a_n$ is linearly dependent on $\text{d}a_i$, $i < n$. This is a contradiction. □

031X **Lemma 10.158.4.** Let k be a field of characteristic $p > 0$. The following are equivalent:

(1) the field extension K/k is separable (see Definition 10.42.1), and
(2) the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective.

Proof. Write K as a directed colimit $K = \text{colim}_i K_i$ of finitely generated field extensions K_i/k. By definition K is separable if and only if each K_i is separable over k, and by Lemma 10.131.5 we see that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective if and only if each $K_i \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K_i/\mathbf{F}_p}$ is injective. Hence we may assume that K/k is a finitely generated field extension.

Assume K/k is a finitely generated field extension which is separable. Choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 10.42.3. In this case there exists an irreducible polynomial $G(X_1, \ldots, X_{r+1}) \in k[X_1, \ldots, X_{r+1}]$ such that $G(x_1, \ldots, x_{r+1}) = 0$ and such that $\partial G/\partial X_{r+1}$ is not identically zero. Moreover K is the field of fractions of the domain. $S = K[X_1, \ldots, X_{r+1}]/(G)$. Write

$$G = \sum a_I X^I, \quad X^I = X_1^{i_1} \ldots X_{r+1}^{i_{r+1}}.$$

Using the presentation of S above we see that

$$\Omega_{S/\mathbf{F}_p} = \frac{S \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} S\text{d}X_i}{\langle \sum X^I \text{d}a_I + \sum \partial G/\partial X_i \text{d}X_i \rangle}$$

Since $\Omega_{K/\mathbf{F}_p}$ is the localization of the S-module $\Omega_{S/\mathbf{F}_p}$ (see Lemma 10.131.8) we conclude that

$$\Omega_{K/\mathbf{F}_p} = \frac{K \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} K\text{d}X_i}{\langle \sum X^I \text{d}a_I + \sum \partial G/\partial X_i \text{d}X_i \rangle}$$

Now, since the polynomial $\partial G/\partial X_{r+1}$ is not identically zero we conclude that the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{S/\mathbf{F}_p}$ is injective as desired.

Assume K/k is a finitely generated field extension and that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective. (This part of the proof is the same as the argument proving Lemma 10.44.1.) Let $x_1, \ldots, x_r$ be a transcendence basis of K over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset K$ is minimal. If K is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in K$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be its minimal polynomial. Because α is not separable actually P is a polynomial in T^p. Clear denominators to get an irreducible polynomial

$$G(X_1, \ldots, X_r, T) = \sum a_{I,i} X^I T^i \in k[X_1, \ldots, X_r, T]$$

such that $G(x_1, \ldots, x_r, \alpha) = 0$ in L. Note that this means $k[X_1, \ldots, X_r, T]/(G) \subset L$. We may assume that for some pair (I_0, i_0) the coefficient $a_{I_0,i_0} = 1$. We claim that $\mathrm{d}G/\mathrm{d}X_i$ is not identically zero for at least one i. Namely, if this is not the case, then G is actually a polynomial in $X_1^p, \ldots, X_r^p, T^p$. Then this means that

$$\sum\nolimits_{(I,i)\neq(I_0,i_0)} x^I\alpha^i \mathrm{d}a_{I,i}$$

is zero in $\Omega_{K/\mathbf{F}_p}$. Note that there is no k-linear relation among the elements

$$\{x^I\alpha^i \mid a_{I,i} \neq 0 \text{ and } (I,i) \neq (I_0,i_0)\}$$

of K. Hence the assumption that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective this implies that $\mathrm{d}a_{I,i} = 0$ in $\Omega_{k/\mathbf{F}_p}$ for all (I,i). By Lemma 10.158.2 we see that each $a_{I,i}$ is a pth power, which implies that G is a pth power contradicting the irreducibility of G. Thus, after renumbering, we may assume that $\mathrm{d}G/\mathrm{d}X_1$ is not zero. Then we see that x_1 is separably algebraic over $k(x_2, \ldots, x_r, \alpha)$, and that $x_2, \ldots, x_r, \alpha$ is a transcendence basis of L over k. This means that the degree of inseparability of the finite extension $k(x_2, \ldots, x_r, \alpha) \subset L$ is less than the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$, which is a contradiction. □

031Y Lemma 10.158.5. Let K/k be an extension of fields. If K is formally smooth over k, then K is a separable extension of k.

Proof. Assume K is formally smooth over k. By Lemma 10.138.9 we see that $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective. Hence K is separable over k by Lemma 10.158.4. □

031Z Lemma 10.158.6. Let K/k be an extension of fields. Then K is formally smooth over k if and only if $H_1(L_{K/k}) = 0$.

Proof. This follows from Proposition 10.138.8 and the fact that a vector spaces is free (hence projective). □

0320 Lemma 10.158.7. Let K/k be an extension of fields.

(1) If K is purely transcendental over k, then K is formally smooth over k.
(2) If K is separable algebraic over k, then K is formally smooth over k.
(3) If K is separable over k, then K is formally smooth over k.

Proof. For (1) write $K = k(x_j; j \in J)$. Suppose that A is a k-algebra, and $I \subset A$ is an ideal of square zero. Let $\varphi : K \to A/I$ be a k-algebra map. Let $a_j \in A$ be an element such that $a_j \mod I = \varphi(x_j)$. Then it is easy to see that there is a unique k-algebra map $K \to A$ which maps x_j to a_j and which reduces to φ mod I. Hence $k \subset K$ is formally smooth.

In case (2) we see that $k \subset K$ is a colimit of étale ring extensions. An étale ring map is formally étale (Lemma 10.150.2). Hence this case follows from Lemma 10.150.3 and the trivial observation that a formally étale ring map is formally smooth.

In case (3), write $K = \operatorname{colim} K_i$ as the filtered colimit of its finitely generated sub k-extensions. By Definition 10.42.1 each K_i is separable algebraic over a purely transcendental extension of k. Hence K_i/k is formally smooth by cases (1) and (2) and Lemma 10.138.3. Thus $H_1(L_{K_i/k}) = 0$ by Lemma 10.158.6. Hence $H_1(L_{K/k}) = 0$ by Lemma 10.134.9. Hence K/k is formally smooth by Lemma 10.158.6 again. □

0321 Lemma 10.158.8. Let k be a field.

(1) If the characteristic of k is zero, then any extension field of k is formally smooth over k.
(2) If the characteristic of k is $p > 0$, then K/k is formally smooth if and only if it is a separable field extension.

Proof. Combine Lemmas 10.158.5 and 10.158.7. □

Here we put together all the different characterizations of separable field extensions.

0322 **Proposition 10.158.9.** Let K/k be a field extension. If the characteristic of k is zero then

(1) K is separable over k,
(2) K is geometrically reduced over k,
(3) K is formally smooth over k,
(4) $H_1(L_{K/k}) = 0$, and
(5) the map $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective.

If the characteristic of k is $p > 0$, then the following are equivalent:

(1) K is separable over k,
(2) the ring $K \otimes_k k^{1/p}$ is reduced,
(3) K is geometrically reduced over k,
(4) the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective,
(5) $H_1(L_{K/k}) = 0$, and
(6) K is formally smooth over k.

Proof. This is a combination of Lemmas 10.44.1, 10.158.8 10.158.5, and 10.158.4. □

Here is yet another characterization of finitely generated separable field extensions.

037X **Lemma 10.158.10.** Let K/k be a finitely generated field extension. Then K is separable over k if and only if K is the localization of a smooth k-algebra.

Proof. Choose a finite type k-algebra R which is a domain whose fraction field is K. Lemma 10.140.9 says that $k \to R$ is smooth at (0) if and only if K/k is separable. This proves the lemma. □

07BV **Lemma 10.158.11.** Let K/k be a field extension. Then K is a filtered colimit of global complete intersection algebras over k. If K/k is separable, then K is a filtered colimit of smooth algebras over k.

Proof. Suppose that $E \subset K$ is a finite subset. It suffices to show that there exists a k subalgebra $A \subset K$ which contains E and which is a global complete intersection (resp. smooth) over k. The separable/smooth case follows from Lemma 10.158.10. In general let $L \subset K$ be the subfield generated by E. Pick a transcendence basis $x_1, \ldots, x_d \in L$ over k. The extension $L/k(x_1, \ldots, x_d)$ is finite. Say $L = k(x_1, \ldots, x_d)[y_1, \ldots, y_r]$. Pick inductively polynomials $P_i \in k(x_1, \ldots, x_d)[Y_1, \ldots, Y_r]$ such that $P_i = P_i(Y_1, \ldots, Y_i)$ is monic in Y_i over $k(x_1, \ldots, x_d)[Y_1, \ldots, Y_{i-1}]$ and maps to the minimum polynomial of y_i in $k(x_1, \ldots, x_d)[y_1, \ldots, y_{i-1}][Y_i]$. Then it is clear that $P_1, \ldots, P_r$ is a regular sequence in $k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]$ and that $L = k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$. If $h \in k[x_1, \ldots, x_d]$ is a polynomial such that $P_i \in k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$, then we see that $P_1, \ldots, P_r$ is a regular sequence in $k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$ and $A = k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$

is a global complete intersection. After adjusting our choice of h we may assume $E \subset A$ and we win. $\square$

10.159. Constructing flat ring maps

03C2 The following lemma is occasionally useful.

03C3 **Lemma 10.159.1.** Let $(R, \mathfrak{m}, k)$ be a local ring. Let K/k be a field extension. There exists a local ring $(R', \mathfrak{m}', k')$, a flat local ring map $R \to R'$ such that $\mathfrak{m}' = \mathfrak{m}R'$ and such that k' is isomorphic to K as an extension of k.

Proof. Suppose that $k' = k(\alpha)$ is a monogenic extension of k. Then k' is the residue field of a flat local extension $R \subset R'$ as in the lemma. Namely, if α is transcendental over k, then we let R' be the localization of $R[x]$ at the prime $\mathfrak{m}R[x]$. If α is algebraic with minimal polynomial $T^d + \sum \overline{\lambda}_i T^{d-i}$, then we let $R' = R[T]/(T^d + \sum \lambda_i T^{d-i})$.

Consider the collection of triples $(k', R \to R', \phi)$, where $k \subset k' \subset K$ is a subfield, $R \to R'$ is a local ring map as in the lemma, and $\phi : R' \to k'$ induces an isomorphism $R'/\mathfrak{m}R' \cong k'$ of k-extensions. These form a "big" category $\mathcal{C}$ with morphisms $(k_1, R_1, \phi_1) \to (k_2, R_2, \phi_2)$ given by ring maps $\psi : R_1 \to R_2$ such that

$$\begin{tikzcd} R_1 \arrow[r, "\phi_1"'] \arrow[d, "\psi"'] & k_1 \arrow[r] & K \arrow[d, equal] \\ R_2 \arrow[r, "\phi_2"] & k_2 \arrow[r] & K \end{tikzcd}$$

commutes. This implies that $k_1 \subset k_2$.

Suppose that I is a directed set, and $((R_i, k_i, \phi_i), \psi_{ii'})$ is a system over I, see Categories, Section 4.21. In this case we can consider

$$R' = \operatorname{colim}_{i \in I} R_i$$

This is a local ring with maximal ideal $\mathfrak{m}R'$, and residue field $k' = \bigcup_{i \in I} k_i$. Moreover, the ring map $R \to R'$ is flat as it is a colimit of flat maps (and tensor products commute with directed colimits). Hence we see that (R', k', ϕ') is an "upper bound" for the system.

An almost trivial application of Zorn's Lemma would finish the proof if $\mathcal{C}$ was a set, but it isn't. (Actually, you can make this work by finding a reasonable bound on the cardinals of the local rings occurring.) To get around this problem we choose a well ordering on K. For $x \in K$ we let $K(x)$ be the subfield of K generated by all elements of K which are $\leq x$. By transfinite recursion on $x \in K$ we will produce ring maps $R \subset R(x)$ as in the lemma with residue field extension $K(x)/k$. Moreover, by construction we will have that $R(x)$ will contain $R(y)$ for all $y \leq x$. Namely, if x has a predecessor x', then $K(x) = K(x')[x]$ and hence we can let $R(x') \subset R(x)$ be the local ring extension constructed in the first paragraph of the proof. If x does not have a predecessor, then we first set $R'(x) = \operatorname{colim}_{x' < x} R(x')$ as in the third paragraph of the proof. The residue field of $R'(x)$ is $K'(x) = \bigcup_{x' < x} K(x')$. Since $K(x) = K'(x)[x]$ we see that we can use the construction of the first paragraph of the proof to produce $R'(x) \subset R(x)$. This finishes the proof of the lemma. $\square$

09E0 **Lemma 10.159.2.** Let $(R, \mathfrak{m}, k)$ be a local ring. If $k \subset K$ is a separable algebraic extension, then there exists a directed set I and a system of finite étale extensions

$R \subset R_i$, $i \in I$ of local rings such that $R' = \operatorname{colim} R_i$ has residue field K (as extension of k).

Proof. Let $R \subset R'$ be the extension constructed in the proof of Lemma 10.159.1. By construction $R' = \operatorname{colim}_{\alpha \in A} R_\alpha$ where A is a well-ordered set and the transition maps $R_\alpha \to R_{\alpha+1}$ are finite étale and $R_\alpha = \operatorname{colim}_{\beta < \alpha} R_\beta$ if α is not a successor. We will prove the result by transfinite induction.

Suppose the result holds for R_α, i.e., $R_\alpha = \operatorname{colim} R_i$ with R_i finite étale over R. Since $R_\alpha \to R_{\alpha+1}$ is finite étale there exists an i and a finite étale extension $R_i \to R_{i,1}$ such that $R_{\alpha+1} = R_\alpha \otimes_{R_i} R_{i,1}$. Thus $R_{\alpha+1} = \operatorname{colim}_{i' \geq i} R_{i'} \otimes_{R_i} R_{i,1}$ and the result holds for $\alpha + 1$. Suppose α is not a successor and the result holds for R_β for all $\beta < \alpha$. Since every finite subset $E \subset R_\alpha$ is contained in R_β for some $\beta < \alpha$ and we see that E is contained in a finite étale subextension by assumption. Thus the result holds for R_α. $\square$

07NE Lemma 10.159.3. Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime and let $L/\kappa(\mathfrak{p})$ be a finite extension of fields. Then there exists a finite free ring map $R \to S$ such that $\mathfrak{q} = \mathfrak{p}S$ is prime and $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is isomorphic to the given extension $L/\kappa(\mathfrak{p})$.

Proof. By induction of the degree of $\kappa(\mathfrak{p}) \subset L$. If the degree is 1, then we take $R = S$. In general, if there exists a sub extension $\kappa(\mathfrak{p}) \subset L' \subset L$ then we win by induction on the degree (by first constructing $R \subset S'$ corresponding to $L'/\kappa(\mathfrak{p})$ and then construction $S' \subset S$ corresponding to L/L'). Thus we may assume that $L \supset \kappa(\mathfrak{p})$ is generated by a single element $\alpha \in L$. Let $X^d + \sum_{i<d} a_i X^i$ be the minimal polynomial of α over $\kappa(\mathfrak{p})$, so $a_i \in \kappa(\mathfrak{p})$. We may write a_i as the image of f_i/g for some $f_i, g \in R$ and $g \notin \mathfrak{p}$. After replacing α by $g\alpha$ (and correspondingly replacing a_i by $g^{d-i}a_i$) we may assume that a_i is the image of some $f_i \in R$. Then we simply take $S = R[x]/(x^d + \sum f_i x^i)$. $\square$

0GIL Lemma 10.159.4. Let A be a ring. Let $\kappa = \max(|A|, \aleph_0)$. Then every flat A-algebra B is the filtered colimit of its flat A-subalgebras $B' \subset B$ of cardinality $|B'| \leq \kappa$. (Observe that B' is faithfully flat over A if B is faithfully flat over A.)

Proof. If B has cardinality $\leq \kappa$ then this is true. Let $E \subset B$ be an A-subalgebra with $|E| \leq \kappa$. We will show that E is contained in a flat A-subalgebra B' with $|B'| \leq \kappa$. The lemma follows because (a) every finite subset of B is contained in an A-subalgebra of cardinality at most κ and (b) every pair of A-subalgebras of B of cardinality at most κ is contained in an A-subalgebra of cardinality at most κ. Details omitted.

We will inductively construct a sequence of A-subalgebras

$$E = E_0 \subset E_1 \subset E_2 \subset \ldots$$

each having cardinality $\leq \kappa$ and we will show that $B' = \bigcup E_k$ is flat over A to finish the proof.

The construction is as follows. Set $E_0 = E$. Given E_k for $k \geq 0$ we consider the set S_k of relations between elements of E_k with coefficients in A. Thus an element $s \in S_k$ is given by an integer $n \geq 1$ and $a_1, \ldots, a_n \in A$, and $e_1, \ldots, e_n \in E_k$ such that $\sum a_i e_i = 0$ in E_k. The flatness of $A \to B$ implies by Lemma 10.39.11 that for every $s = (n, a_1, \ldots, a_n, e_1, \ldots, e_n) \in S_k$ we may choose

$$(m_s, b_{s,1}, \ldots, b_{s,m_s}, a_{s,11}, \ldots, a_{s,nm_s})$$

where $m_s \geq 0$ is an integer, $b_{s,j} \in B$, $a_{s,ij} \in A$, and

$$e_i = \sum\nolimits_j a_{s,ij} b_{s,j}, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i a_i a_{s,ij}, \forall j.$$

Given these choicse, we let $E_{k+1} \subset B$ be the A-subalgebra generated by

(1) E_k and
(2) the elements $b_{s,1}, \ldots, b_{s,m_s}$ for every $s \in S_k$.

Some set theory (omitted) shows that E_{k+1} has at most cardinality κ (this uses that we inductively know $|E_k| \leq \kappa$ and consequently the cardinality of S_k is also at most κ).

To show that $B' = \bigcup E_k$ is flat over A we consider a relation $\sum_{i=1,\ldots,n} a_i b'_i = 0$ in B' with coefficients in A. Choose k large enough so that $b'_i \in E_k$ for $i = 1, \ldots, n$. Then $(n, a_1, \ldots, a_n, b'_1, \ldots, b'_n) \in S_k$ and hence we see that the relation is trivial in E_{k+1} and a fortiori in B'. Thus $A \to B'$ is flat by Lemma 10.39.11. □

10.160. The Cohen structure theorem

0323 Here is a fundamental notion in commutative algebra.

0324 Definition 10.160.1. Let $(R, \mathfrak{m})$ be a local ring. We say R is a complete local ring if the canonical map

$$R \longrightarrow \lim\nolimits_n R/\mathfrak{m}^n$$

to the completion of R with respect to $\mathfrak{m}$ is an isomorphism[13].

Note that an Artinian local ring R is a complete local ring because $\mathfrak{m}_R^n = 0$ for some $n > 0$. In this section we mostly focus on Noetherian complete local rings.

0325 Lemma 10.160.2. Let R be a Noetherian complete local ring. Any quotient of R is also a Noetherian complete local ring. Given a finite ring map $R \to S$, then S is a product of Noetherian complete local rings.

Proof. The ring S is Noetherian by Lemma 10.31.1. As an R-module S is complete by Lemma 10.97.1. Hence S is the product of the completions at its maximal ideals by Lemma 10.97.8. □

032B Lemma 10.160.3. Let $(R, \mathfrak{m})$ be a complete local ring. If $\mathfrak{m}$ is a finitely generated ideal then R is Noetherian.

Proof. See Lemma 10.97.5. □

0326 Definition 10.160.4. Let $(R, \mathfrak{m})$ be a complete local ring. A subring $\Lambda \subset R$ is called a coefficient ring if the following conditions hold:

(1) Λ is a complete local ring with maximal ideal $\Lambda \cap \mathfrak{m}$,
(2) the residue field of Λ maps isomorphically to the residue field of R, and
(3) $\Lambda \cap \mathfrak{m} = p\Lambda$, where p is the characteristic of the residue field of R.

Let us make some remarks on this definition. We split the discussion into the following cases:

[13]This includes the condition that $\bigcap \mathfrak{m}^n = (0)$; in some texts this may be indicated by saying that R is complete and separated. Warning: It can happen that the completion $\lim_n R/\mathfrak{m}^n$ of a local ring is non-complete, see Examples, Lemma 110.7.1. This does not happen when $\mathfrak{m}$ is finitely generated, see Lemma 10.96.3 in which case the completion is Noetherian, see Lemma 10.97.5.

(1) The local ring R contains a field. This happens if either $\mathbf{Q} \subset R$, or $pR = 0$ where p is the characteristic of $R/\mathfrak{m}$. In this case a coefficient ring Λ is a field contained in R which maps isomorphically to $R/\mathfrak{m}$.
(2) The characteristic of $R/\mathfrak{m}$ is $p > 0$ but no power of p is zero in R. In this case Λ is a complete discrete valuation ring with uniformizer p and residue field $R/\mathfrak{m}$.
(3) The characteristic of $R/\mathfrak{m}$ is $p > 0$, and for some $n > 1$ we have $p^{n-1} \neq 0$, $p^n = 0$ in R. In this case Λ is an Artinian local ring whose maximal ideal is generated by p and which has residue field $R/\mathfrak{m}$.

The complete discrete valuation rings with uniformizer p above play a special role and we baptize them as follows.

0327 Definition 10.160.5. A Cohen ring is a complete discrete valuation ring with uniformizer p a prime number.

0328 Lemma 10.160.6. Let p be a prime number. Let k be a field of characteristic p. There exists a Cohen ring Λ with $\Lambda/p\Lambda \cong k$.

Proof. First note that the p-adic integers $\mathbf{Z}_p$ form a Cohen ring for $\mathbf{F}_p$. Let k be an arbitrary field of characteristic p. Let $\mathbf{Z}_p \to R$ be a flat local ring map such that $\mathfrak{m}_R = pR$ and $R/pR = k$, see Lemma 10.159.1. By Lemma 10.97.5 the completion $\Lambda = R^\wedge$ is Noetherian. It is a complete Noetherian local ring with maximal ideal (p) as $\Lambda/p\Lambda = R/pR$ is a field (use Lemma 10.96.3). Since $\mathbf{Z}_p \to R \to \Lambda$ is flat (by Lemma 10.97.2) we see that p is a nonzerodivisor in Λ. Hence Λ has dimension ≥ 1 (Lemma 10.60.13) and we conclude that Λ is regular of dimension 1, i.e., a discrete valuation ring by Lemma 10.119.7. We conclude Λ is a Cohen ring for k. □

0329 Lemma 10.160.7. Let $p > 0$ be a prime. Let Λ be a Cohen ring with residue field of characteristic p. For every $n \geq 1$ the ring map

$$\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$$

is formally smooth.

Proof. If $n = 1$, this follows from Proposition 10.158.9. For general n we argue by induction on n. Namely, if $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth, then we can apply Lemma 10.138.12 to the ring map $\mathbf{Z}/p^{n+1}\mathbf{Z} \to \Lambda/p^{n+1}\Lambda$ and the ideal $I = (p^n) \subset \mathbf{Z}/p^{n+1}\mathbf{Z}$. □

032A Theorem 10.160.8 (Cohen structure theorem). Let $(R, \mathfrak{m})$ be a complete local ring.

(1) R has a coefficient ring (see Definition 10.160.4),
(2) if $\mathfrak{m}$ is a finitely generated ideal, then R is isomorphic to a quotient

$$\Lambda[[x_1, \ldots, x_n]]/I$$

where Λ is either a field or a Cohen ring.

Proof. Let us prove a coefficient ring exists. First we prove this in case the characteristic of the residue field κ is zero. Namely, in this case we will prove by induction on $n > 0$ that there exists a section

$$\varphi_n : \kappa \longrightarrow R/\mathfrak{m}^n$$

to the canonical map $R/\mathfrak{m}^n \to \kappa = R/\mathfrak{m}$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The field extension $\kappa/\mathbf{Q}$ is formally smooth by Proposition 10.158.9. Hence we can find the dotted arrow in the following diagram

$$\begin{tikzcd} R/\mathfrak{m}^{n-1} & R/\mathfrak{m}^n \arrow[l] \\ \kappa \arrow[u, "\varphi_{n-1}"] \arrow[ur, dotted] & \mathbf{Q} \arrow[l] \arrow[u] \end{tikzcd}$$

This proves the induction step. Putting these maps together

$$\lim_n\ \varphi_n : \kappa \longrightarrow R = \lim_n\ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

Next, we prove the existence of a coefficient ring in the case where the characteristic of the residue field κ is $p > 0$. Namely, choose a Cohen ring Λ with $\kappa = \Lambda/p\Lambda$, see Lemma 10.160.6. In this case we will prove by induction on $n > 0$ that there exists a map

$$\varphi_n : \Lambda/p^n\Lambda \longrightarrow R/\mathfrak{m}^n$$

whose composition with the reduction map $R/\mathfrak{m}^n \to \kappa$ produces the given isomorphism $\Lambda/p\Lambda = \kappa$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The ring map $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth by Lemma 10.160.7. Hence we can find the dotted arrow in the following diagram

$$\begin{tikzcd} R/\mathfrak{m}^{n-1} & R/\mathfrak{m}^n \arrow[l] \\ \Lambda/p^n\Lambda \arrow[u, "\varphi_{n-1}"] \arrow[ur, dotted] & \mathbf{Z}/p^n\mathbf{Z} \arrow[l] \arrow[u] \end{tikzcd}$$

This proves the induction step. Putting these maps together

$$\lim_n\ \varphi_n : \Lambda = \lim_n\ \Lambda/p^n\Lambda \longrightarrow R = \lim_n\ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

The final statement of the theorem follows readily. Namely, if $y_1, \ldots, y_n$ are generators of the ideal $\mathfrak{m}$, then we can use the map $\Lambda \to R$ just constructed to get a map

$$\Lambda[[x_1, \ldots, x_n]] \longrightarrow R, \quad x_i \longmapsto y_i.$$

Since both sides are $(x_1, \ldots, x_n)$-adically complete this map is surjective by Lemma 10.96.1 as it is surjective modulo $(x_1, \ldots, x_n)$ by construction. □

032C Remark 10.160.9. If k is a field then the power series ring $k[[X_1, \ldots, X_d]]$ is a Noetherian complete local regular ring of dimension d. If Λ is a Cohen ring then $\Lambda[[X_1, \ldots, X_d]]$ is a complete local Noetherian regular ring of dimension $d{+}1$. Hence the Cohen structure theorem implies that any Noetherian complete local ring is a quotient of a regular local ring. In particular we see that a Noetherian complete local ring is universally catenary, see Lemma 10.105.9 and Lemma 10.106.3.

0C0S Lemma 10.160.10. Let $(R, \mathfrak{m})$ be a Noetherian complete local ring. Assume R is regular.

(1) If R contains either $\mathbf{F}_p$ or $\mathbf{Q}$, then R is isomorphic to a power series ring over its residue field.

(2) If k is a field and $k \to R$ is a ring map inducing an isomorphism $k \to R/\mathfrak{m}$, then R is isomorphic as a k-algebra to a power series ring over k.

Proof. In case (1), by the Cohen structure theorem (Theorem 10.160.8) there exists a coefficient ring which must be a field mapping isomorphically to the residue field. Thus it suffices to prove (2). In case (2) we pick $f_1, \ldots, f_d \in \mathfrak{m}$ which map to a basis of $\mathfrak{m}/\mathfrak{m}^2$ and we consider the continuous k-algebra map $k[[x_1, \ldots, x_d]] \to R$ sending x_i to f_i. As both source and target are $(x_1, \ldots, x_d)$-adically complete, this map is surjective by Lemma 10.96.1. On the other hand, it has to be injective because otherwise the dimension of R would be $< d$ by Lemma 10.60.13. □

032D Lemma 10.160.11. Let $(R, \mathfrak{m})$ be a Noetherian complete local domain. Then there exists a $R_0 \subset R$ with the following properties

(1) R_0 is a regular complete local ring,
(2) $R_0 \subset R$ is finite and induces an isomorphism on residue fields,
(3) R_0 is either isomorphic to $k[[X_1, \ldots, X_d]]$ where k is a field or $\Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.

Proof. Let Λ be a coefficient ring of R. Since R is a domain we see that either Λ is a field or Λ is a Cohen ring.

Case I: $\Lambda = k$ is a field. Let $d = \dim(R)$. Choose $x_1, \ldots, x_d \in \mathfrak{m}$ which generate an ideal of definition $I \subset R$. (See Section 10.60.) By Lemma 10.96.9 we see that R is I-adically complete as well. Consider the map $R_0 = k[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 10.60.) Hence we conclude that $R_0 \to R$ is finite by Lemma 10.96.12. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 10.112.3), and the lemma is proved.

Case II: Λ is a Cohen ring. Let $d + 1 = \dim(R)$. Let $p > 0$ be the characteristic of the residue field k. As R is a domain we see that p is a nonzerodivisor in R. Hence $\dim(R/pR) = d$, see Lemma 10.60.13. Choose $x_1, \ldots, x_d \in R$ which generate an ideal of definition in R/pR. Then $I = (p, x_1, \ldots, x_d)$ is an ideal of definition of R. By Lemma 10.96.9 we see that R is I-adically complete as well. Consider the map $R_0 = \Lambda[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (p, X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 10.60.) Hence we conclude that $R_0 \to R$ is finite by Lemma 10.96.12. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 10.112.3), and the lemma is proved. □

10.161. Japanese rings

0BI1 In this section we begin to discuss finiteness of integral closure.

032F Definition 10.161.1. Let R be a domain with field of fractions K. [DG67, Chapter 0, Definition 23.1.1]

(1) We say R is N-1 if the integral closure of R in K is a finite R-module.
(2) We say R is N-2 or Japanese if for any finite extension L/K of fields the integral closure of R in L is finite over R.

The main interest in these notions is for Noetherian rings, but here is a non-Noetherian example.

0350 Example 10.161.2. Let k be a field. The domain $R = k[x_1, x_2, x_3, \ldots]$ is N-2, but not Noetherian. The reason is the following. Suppose that $R \subset L$ and the field L is a finite extension of the fraction field of R. Then there exists an integer n such that L comes from a finite extension $L_0/k(x_1, \ldots, x_n)$ by adjoining the (transcendental) elements x_{n+1}, x_{n+2}, etc. Let S_0 be the integral closure of $k[x_1, \ldots, x_n]$ in L_0. By Proposition 10.162.16 below it is true that S_0 is finite over $k[x_1, \ldots, x_n]$. Moreover, the integral closure of R in L is $S = S_0[x_{n+1}, x_{n+2}, \ldots]$ (use Lemma 10.37.8) and hence finite over R. The same argument works for $R = \mathbf{Z}[x_1, x_2, x_3, \ldots]$.

032G Lemma 10.161.3. Let R be a domain. If R is N-1 then so is any localization of R. Same for N-2.

Proof. These statements hold because taking integral closure commutes with localization, see Lemma 10.36.11. $\square$

032H Lemma 10.161.4. Let R be a domain. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. If each domain R_{f_i} is N-1 then so is R. Same for N-2.

Proof. Assume R_{f_i} is N-2 (or N-1). Let L be a finite extension of the fraction field of R (equal to the fraction field in the N-1 case). Let S be the integral closure of R in L. By Lemma 10.36.11 we see that S_{f_i} is the integral closure of R_{f_i} in L. Hence S_{f_i} is finite over R_{f_i} by assumption. Thus S is finite over R by Lemma 10.23.2. $\square$

032I Lemma 10.161.5. Let R be a domain. Let $R \subset S$ be a quasi-finite extension of domains (for example finite). Assume R is N-2 and Noetherian. Then S is N-2.

Proof. Let L/K be the induced extension of fraction fields. Note that this is a finite field extension (for example by Lemma 10.122.2 (2) applied to the fibre $S \otimes_R K$, and the definition of a quasi-finite ring map). Let S' be the integral closure of R in S. Then S' is contained in the integral closure of R in L which is finite over R by assumption. As R is Noetherian this implies S' is finite over R. By Lemma 10.123.14 there exist elements $g_1, \ldots, g_n \in S'$ such that $S'_{g_i} \cong S_{g_i}$ and such that $g_1, \ldots, g_n$ generate the unit ideal in S. Hence it suffices to show that S' is N-2 by Lemmas 10.161.3 and 10.161.4. Thus we have reduced to the case where S is finite over R.

Assume $R \subset S$ with hypotheses as in the lemma and moreover that S is finite over R. Let M be a finite field extension of the fraction field of S. Then M is also a finite field extension of K and we conclude that the integral closure T of R in M is finite over R. By Lemma 10.36.16 we see that T is also the integral closure of S in M and we win by Lemma 10.36.15. $\square$

032J Lemma 10.161.6. Let R be a Noetherian domain. If $R[z, z^{-1}]$ is N-1, then so is R.

Proof. Let R' be the integral closure of R in its field of fractions K. Let S' be the integral closure of $R[z, z^{-1}]$ in its field of fractions. Clearly $R' \subset S'$. Since $K[z, z^{-1}]$ is a normal domain we see that $S' \subset K[z, z^{-1}]$. Suppose that $f_1, \ldots, f_n \in S'$ generate S' as $R[z, z^{-1}]$-module. Say $f_i = \sum a_{ij} z^j$ (finite sum), with $a_{ij} \in K$. For any $x \in R'$ we can write

$$x = \sum h_i f_i$$

with $h_i \in R[z, z^{-1}]$. Thus we see that R' is contained in the finite R-submodule $\sum R a_{ij} \subset K$. Since R is Noetherian we conclude that R' is a finite R-module. $\square$

032K Lemma 10.161.7. Let R be a Noetherian domain, and let $R \subset S$ be a finite extension of domains. If S is N-1, then so is R. If S is N-2, then so is R.

Proof. Omitted. (Hint: Integral closures of R in extension fields are contained in integral closures of S in extension fields.) □

032L Lemma 10.161.8. Let R be a Noetherian normal domain with fraction field K. Let L/K be a finite separable field extension. Then the integral closure of R in L is finite over R.

Proof. Consider the trace pairing (Fields, Definition 9.20.6)

$$L \times L \longrightarrow K, \quad (x, y) \longmapsto \langle x, y\rangle := \text{Trace}_{L/K}(xy).$$

Since L/K is separable this is nondegenerate (Fields, Lemma 9.20.7). Moreover, if $x \in L$ is integral over R, then $\text{Trace}_{L/K}(x)$ is in R. This is true because the minimal polynomial of x over K has coefficients in R (Lemma 10.38.6) and because $\text{Trace}_{L/K}(x)$ is an integer multiple of one of these coefficients (Fields, Lemma 9.20.3). Pick $x_1, \ldots, x_n \in L$ which are integral over R and which form a K-basis of L. Then the integral closure $S \subset L$ is contained in the R-module

$$M = \{y \in L \mid \langle x_i, y\rangle \in R,\ i = 1, \ldots, n\}$$

By linear algebra we see that $M \cong R^{\oplus n}$ as an R-module. Hence $S \subset R^{\oplus n}$ is a finitely generated R-module as R is Noetherian. □

03B7 Example 10.161.9. Lemma 10.161.8 does not work if the ring is not Noetherian. For example consider the action of $G = \{+1, -1\}$ on $A = \mathbf{C}[x_1, x_2, x_3, \ldots]$ where -1 acts by mapping x_i to $-x_i$. The invariant ring $R = A^G$ is the $\mathbf{C}$-algebra generated by all $x_i x_j$. Hence $R \subset A$ is not finite. But R is a normal domain with fraction field $K = L^G$ the G-invariants in the fraction field L of A. And clearly A is the integral closure of R in L.

The following lemma can sometimes be used as a substitute for Lemma 10.161.8 in case of purely inseparable extensions.

0AE0 Lemma 10.161.10. Let R be a Noetherian normal domain with fraction field K of characteristic $p > 0$. Let $a \in K$ be an element such that there exists a derivation $D : R \to R$ with $D(a) \neq 0$. Then the integral closure of R in $L = K[x]/(x^p - a)$ is finite over R.

Proof. After replacing x by fx and a by $f^p a$ for some $f \in R$ we may assume $a \in R$. Hence also $D(a) \in R$. We will show by induction on $i \leq p - 1$ that if

$$y = a_0 + a_1 x + \ldots + a_i x^i, \quad a_j \in K$$

is integral over R, then $D(a)^i a_j \in R$. Thus the integral closure is contained in the finite R-module with basis $D(a)^{-p+1} x^j$, $j = 0, \ldots, p - 1$. Since R is Noetherian this proves the lemma.

If $i = 0$, then $y = a_0$ is integral over R if and only if $a_0 \in R$ and the statement is true. Suppose the statement holds for some $i < p - 1$ and suppose that

$$y = a_0 + a_1 x + \ldots + a_{i+1} x^{i+1}, \quad a_j \in K$$

is integral over R. Then

$$y^p = a_0^p + a_1^p a + \ldots + a_{i+1}^p a^{i+1}$$

is an element of R (as it is in K and integral over R). Applying D we obtain

$$(a_1^p + 2a_2^p a + \ldots + (i+1)a_{i+1}^p a^i)D(a)$$

is in R. Hence it follows that

$$D(a)a_1 + 2D(a)a_2 x + \ldots + (i+1)D(a)a_{i+1}x^i$$

is integral over R. By induction we find $D(a)^{i+1}a_j \in R$ for $j = 1, \ldots, i+1$. (Here we use that $1, \ldots, i+1$ are invertible.) Hence $D(a)^{i+1}a_0$ is also in R because it is the difference of y and $\sum_{j>0} D(a)^{i+1}a_j x^j$ which are integral over R (since x is integral over R as $a \in R$). $\square$

032M Lemma 10.161.11. A Noetherian domain whose fraction field has characteristic zero is N-1 if and only if it is N-2 (i.e., Japanese).

Proof. This is clear from Lemma 10.161.8 since every field extension in characteristic zero is separable. $\square$

032N Lemma 10.161.12. Let R be a Noetherian domain with fraction field K of characteristic $p > 0$. Then R is N-2 if and only if for every finite purely inseparable extension L/K the integral closure of R in L is finite over R.

Proof. Assume the integral closure of R in every finite purely inseparable field extension of K is finite. Let L/K be any finite extension. We have to show the integral closure of R in L is finite over R. Choose a finite normal field extension M/K containing L. As R is Noetherian it suffices to show that the integral closure of R in M is finite over R. By Fields, Lemma 9.27.3 there exists a subextension $M/M_{insep}/K$ such that M_{insep}/K is purely inseparable, and M/M_{insep} is separable. By assumption the integral closure R' of R in M_{insep} is finite over R. By Lemma 10.161.8 the integral closure R'' of R' in M is finite over R'. Then R'' is finite over R by Lemma 10.7.3. Since R'' is also the integral closure of R in M (see Lemma 10.36.16) we win. $\square$

032O Lemma 10.161.13. Let R be a Noetherian domain. If R is N-1 then $R[x]$ is N-1. If R is N-2 then $R[x]$ is N-2.

Proof. Assume R is N-1. Let R' be the integral closure of R which is finite over R. Hence also $R'[x]$ is finite over $R[x]$. The ring $R'[x]$ is normal (see Lemma 10.37.8), hence N-1. This proves the first assertion.

For the second assertion, by Lemma 10.161.7 it suffices to show that $R'[x]$ is N-2. In other words we may and do assume that R is a normal N-2 domain. In characteristic zero we are done by Lemma 10.161.11. In characteristic $p > 0$ we have to show that the integral closure of $R[x]$ is finite in any finite purely inseparable extension of $L/K(x)$ where K is the fraction field of R. There exists a finite purely inseparable field extension L'/K and $q = p^e$ such that $L \subset L'(x^{1/q})$; some details omitted. As $R[x]$ is Noetherian it suffices to show that the integral closure of $R[x]$ in $L'(x^{1/q})$ is finite over $R[x]$. And this integral closure is equal to $R'[x^{1/q}]$ with $R \subset R' \subset L'$ the integral closure of R in L'. Since R is N-2 we see that R' is finite over R and hence $R'[x^{1/q}]$ is finite over $R[x]$. $\square$

0332 Lemma 10.161.14. Let R be a Noetherian domain. If there exists an $f \in R$ such that R_f is normal then

$$U = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_{\mathfrak{p}} \text{ is normal}\}$$

is open in $\mathrm{Spec}(R)$.

Proof. It is clear that the standard open $D(f)$ is contained in U. By Serre's criterion Lemma 10.157.4 we see that $\mathfrak p \not\in U$ implies that for some $\mathfrak q \subset \mathfrak p$ we have either

(1) Case I: $\mathrm{depth}(R_{\mathfrak q}) < 2$ and $\dim(R_{\mathfrak q}) \geq 2$, and
(2) Case II: $R_{\mathfrak q}$ is not regular and $\dim(R_{\mathfrak q}) = 1$.

This in particular also means that $R_{\mathfrak q}$ is not normal, and hence $f \in \mathfrak q$. In case I we see that $\mathrm{depth}(R_{\mathfrak q}) = \mathrm{depth}(R_{\mathfrak q}/fR_{\mathfrak q}) + 1$. Hence such a prime $\mathfrak q$ is the same thing as an embedded associated prime of R/fR. In case II $\mathfrak q$ is an associated prime of R/fR of height 1. Thus there is a finite set E of such primes $\mathfrak q$ (see Lemma 10.63.5) and

$$\mathrm{Spec}(R) \setminus U = \bigcup\nolimits_{\mathfrak q \in E} V(\mathfrak q)$$

as desired. □

0333 Lemma 10.161.15. Let R be a Noetherian domain. Then R is N-1 if and only if the following two conditions hold

(1) there exists a nonzero $f \in R$ such that R_f is normal, and
(2) for every maximal ideal $\mathfrak m \subset R$ the local ring $R_{\mathfrak m}$ is N-1.

Proof. First assume R is N-1. Let R' be the integral closure of R in its field of fractions K. By assumption we can find $x_1, \ldots, x_n$ in R' which generate R' as an R-module. Since $R' \subset K$ we can find $f_i \in R$ nonzero such that $f_i x_i \in R$. Then $R_f \cong R'_f$ where $f = f_1 \ldots f_n$. Hence R_f is normal and we have (1). Part (2) follows from Lemma 10.161.3.

Assume (1) and (2). Let K be the fraction field of R. Suppose that $R \subset R' \subset K$ is a finite extension of R contained in K. Note that $R_f = R'_f$ since R_f is already normal. Hence by Lemma 10.161.14 the set of primes $\mathfrak p' \in \mathrm{Spec}(R')$ with $R'_{\mathfrak p'}$ non-normal is closed in $\mathrm{Spec}(R')$. Since $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is closed the image of this set is closed in $\mathrm{Spec}(R)$. For such a ring R' denote $Z_{R'} \subset \mathrm{Spec}(R)$ this image.

Pick a maximal ideal $\mathfrak m \subset R$. Let $R_{\mathfrak m} \subset R'_{\mathfrak m}$ be the integral closure of the local ring in K. By assumption this is a finite ring extension. By Lemma 10.36.11 we can find finitely many elements $x_1, \ldots, x_n \in K$ integral over R such that $R'_{\mathfrak m}$ is generated by $x_1, \ldots, x_n$ over $R_{\mathfrak m}$. Let $R' = R[x_1, \ldots, x_n] \subset K$. With this choice it is clear that $\mathfrak m \not\in Z_{R'}$.

As $\mathrm{Spec}(R)$ is quasi-compact, the above shows that we can find a finite collection $R \subset R'_i \subset K$ such that $\bigcap Z_{R'_i} = \emptyset$. Let R' be the subring of K generated by all of these. It is finite over R. Also $Z_{R'} = \emptyset$. Namely, every prime $\mathfrak p'$ lies over a prime $\mathfrak p'_i$ such that $(R'_i)_{\mathfrak p'_i}$ is normal. This implies that $R'_{\mathfrak p'} = (R'_i)_{\mathfrak p'_i}$ is normal too. Hence R' is normal, in other words R' is the integral closure of R in K. □

032P Lemma 10.161.16 (Tate). Let R be a ring. Let $x \in R$. Assume [DG67, Theorem 23.1.3]

(1) R is a normal Noetherian domain,
(2) R/xR is a domain and N-2,
(3) $R \cong \lim_n R/x^nR$ is complete with respect to x.

Then R is N-2.

Proof. We may assume $x \neq 0$ since otherwise the lemma is trivial. Let K be the fraction field of R. If the characteristic of K is zero the lemma follows from (1), see Lemma 10.161.11. Hence we may assume that the characteristic of K is $p > 0$, and we may apply Lemma 10.161.12. Thus given L/K a finite purely inseparable field extension we have to show that the integral closure S of R in L is finite over R.

Let q be a power of p such that $L^q \subset K$. By enlarging L if necessary we may assume there exists an element $y \in L$ such that $y^q = x$. Since $R \to S$ induces a homeomorphism of spectra (see Lemma 10.46.7) there is a unique prime ideal $\mathfrak q \subset S$ lying over the prime ideal $\mathfrak p = xR$. It is clear that

$$\mathfrak q = \{f \in S \mid f^q \in \mathfrak p\} = yS$$

since $y^q = x$. Observe that $R_{\mathfrak p}$ is a discrete valuation ring by Lemma 10.119.7. Then $S_{\mathfrak q}$ is Noetherian by Krull-Akizuki (Lemma 10.119.12). Whereupon we conclude $S_{\mathfrak q}$ is a discrete valuation ring by Lemma 10.119.7 once again. By Lemma 10.119.10 we see that $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is a finite field extension. Hence the integral closure $S' \subset \kappa(\mathfrak q)$ of R/xR is finite over R/xR by assumption (2). Since $S/yS \subset S'$ this implies that S/yS is finite over R. Note that S/y^nS has a finite filtration whose subquotients are the modules $y^iS/y^{i+1}S \cong S/yS$. Hence we see that each S/y^nS is finite over R. In particular S/xS is finite over R. Also, it is clear that $\bigcap x^nS = (0)$ since an element in the intersection has qth power contained in $\bigcap x^nR = (0)$ (Lemma 10.51.4). Thus we may apply Lemma 10.96.12 to conclude that S is finite over R, and we win. □

032Q Lemma 10.161.17. Let R be a ring. If R is Noetherian, a domain, and N-2, then so is $R[[x]]$.

Proof. Observe that $R[[x]]$ is Noetherian by Lemma 10.31.2. Let $R' \supset R$ be the integral closure of R in its fraction field. Because R is N-2 this is finite over R. Hence $R'[[x]]$ is finite over $R[[x]]$. By Lemma 10.37.9 we see that $R'[[x]]$ is a normal domain. Apply Lemma 10.161.16 to the element $x \in R'[[x]]$ to see that $R'[[x]]$ is N-2. Then Lemma 10.161.7 shows that $R[[x]]$ is N-2. □

10.162. Nagata rings

032E Here is the definition.

032R Definition 10.162.1. Let R be a ring.

(1) We say R is universally Japanese if for any finite type ring map $R \to S$ with S a domain we have that S is N-2 (i.e., Japanese).
(2) We say that R is a Nagata ring if R is Noetherian and for every prime ideal $\mathfrak p$ the ring $R/\mathfrak p$ is N-2.

It is clear that a Noetherian universally Japanese ring is a Nagata ring. It is our goal to show that a Nagata ring is universally Japanese. This is not obvious at all, and requires some work. But first, here is a useful lemma.

03GH Lemma 10.162.2. Let R be a Nagata ring. Let $R \to S$ be essentially of finite type with S reduced. Then the integral closure of R in S is finite over R.

Proof. As S is essentially of finite type over R it is Noetherian and has finitely many minimal primes $\mathfrak q_1, \ldots, \mathfrak q_m$, see Lemma 10.31.6. Since S is reduced we have

$S \subset \prod S_{\mathfrak{q}_i}$ and each $S_{\mathfrak{q}_i} = K_i$ is a field, see Lemmas 10.25.4 and 10.25.1. It suffices to show that the integral closure A'_i of R in each K_i is finite over R. This is true because R is Noetherian and $A \subset \prod A'_i$. Let $\mathfrak{p}_i \subset R$ be the prime of R corresponding to $\mathfrak{q}_i$. As S is essentially of finite type over R we see that $K_i = S_{\mathfrak{q}_i} = \kappa(\mathfrak{q}_i)$ is a finitely generated field extension of $\kappa(\mathfrak{p}_i)$. Hence the algebraic closure L_i of $\kappa(\mathfrak{p}_i)$ in K_i is finite over $\kappa(\mathfrak{p}_i)$, see Fields, Lemma 9.26.11. It is clear that A'_i is the integral closure of $R/\mathfrak{p}_i$ in L_i, and hence we win by definition of a Nagata ring. □

0351 Lemma 10.162.3. Let R be a ring. To check that R is universally Japanese it suffices to show: If $R \to S$ is of finite type, and S a domain then S is N-1.

Proof. Namely, assume the condition of the lemma. Let $R \to S$ be a finite type ring map with S a domain. Let L be a finite extension of the fraction field of S. Then there exists a finite ring extension $S \subset S' \subset L$ such that L is the fraction field of S'. By assumption S' is N-1, and hence the integral closure S'' of S' in L is finite over S'. Thus S'' is finite over S (Lemma 10.7.3) and S'' is the integral closure of S in L (Lemma 10.36.16). We conclude that R is universally Japanese. □

032S Lemma 10.162.4. If R is universally Japanese then any algebra essentially of finite type over R is universally Japanese.

Proof. The case of an algebra of finite type over R is immediate from the definition. The general case follows on applying Lemma 10.161.3. □

032T Lemma 10.162.5. Let R be a Nagata ring. If $R \to S$ is a quasi-finite ring map (for example finite) then S is a Nagata ring also.

Proof. First note that S is Noetherian as R is Noetherian and a quasi-finite ring map is of finite type. Let $\mathfrak{q} \subset S$ be a prime ideal, and set $\mathfrak{p} = R \cap \mathfrak{q}$. Then $R/\mathfrak{p} \subset S/\mathfrak{q}$ is quasi-finite and hence we conclude that $S/\mathfrak{q}$ is N-2 by Lemma 10.161.5 as desired. □

032U Lemma 10.162.6. A localization of a Nagata ring is a Nagata ring.

Proof. Clear from Lemma 10.161.3. □

032V Lemma 10.162.7. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.

(1) If each R_{f_i} is universally Japanese then so is R.
(2) If each R_{f_i} is Nagata then so is R.

Proof. Let $\varphi : R \to S$ be a finite type ring map so that S is a domain. Then $\varphi(f_1), \ldots, \varphi(f_n)$ generate the unit ideal in S. Hence if each $S_{f_i} = S_{\varphi(f_i)}$ is N-1 then so is S, see Lemma 10.161.4. This proves (1).

If each R_{f_i} is Nagata, then each R_{f_i} is Noetherian and hence R is Noetherian, see Lemma 10.23.2. And if $\mathfrak{p} \subset R$ is a prime, then we see each $R_{f_i}/\mathfrak{p}R_{f_i} = (R/\mathfrak{p})_{f_i}$ is N-2 and hence we conclude $R/\mathfrak{p}$ is N-2 by Lemma 10.161.4. This proves (2). □

032W Lemma 10.162.8. A Noetherian complete local ring is a Nagata ring.

Proof. Let R be a complete local Noetherian ring. Let $\mathfrak{p} \subset R$ be a prime. Then $R/\mathfrak{p}$ is also a complete local Noetherian ring, see Lemma 10.160.2. Hence it suffices to show that a Noetherian complete local domain R is N-2. By Lemmas 10.161.5 and 10.160.11 we reduce to the case $R = k[[X_1, \ldots, X_d]]$ where k is a field or $R = \Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.

In the case $k[[X_1, \ldots, X_d]]$ we reduce to the statement that a field is N-2 by Lemma 10.161.17. This is clear. In the case $\Lambda[[X_1, \ldots, X_d]]$ we reduce to the statement that a Cohen ring Λ is N-2. Applying Lemma 10.161.16 once more with $x = p \in \Lambda$ we reduce yet again to the case of a field. Thus we win. $\square$

032X Definition 10.162.9. Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say R is analytically unramified if its completion $R^\wedge = \lim_n R/\mathfrak{m}^n$ is reduced. A prime ideal $\mathfrak{p} \subset R$ is said to be analytically unramified if $R/\mathfrak{p}$ is analytically unramified.

At this point we know the following are true for any Noetherian local ring R: The map $R \to R^\wedge$ is a faithfully flat local ring homomorphism (Lemma 10.97.3). The completion $R^\wedge$ is Noetherian (Lemma 10.97.5) and complete (Lemma 10.97.4). Hence the completion $R^\wedge$ is a Nagata ring (Lemma 10.162.8). Moreover, we have seen in Section 10.160 that $R^\wedge$ is a quotient of a regular local ring (Theorem 10.160.8), and hence universally catenary (Remark 10.160.9).

032Y Lemma 10.162.10. Let $(R, \mathfrak{m})$ be a Noetherian local ring.

(1) If R is analytically unramified, then R is reduced.
(2) If R is analytically unramified, then each minimal prime of R is analytically unramified.
(3) If R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified, then R is analytically unramified.
(4) If R is analytically unramified, then the integral closure of R in its total ring of fractions $Q(R)$ is finite over R.
(5) If R is a domain and analytically unramified, then R is N-1.

Proof. In this proof we will use the remarks immediately following Definition 10.162.9. As $R \to R^\wedge$ is a faithfully flat local ring homomorphism it is injective and (1) follows.

Let $\mathfrak{q}$ be a minimal prime of R, and assume R is analytically unramified. Then $\mathfrak{q}$ is an associated prime of R (see Proposition 10.63.6). Hence there exists an $f \in R$ such that $\{x \in R \mid fx = 0\} = \mathfrak{q}$. Note that $(R/\mathfrak{q})^\wedge = R^\wedge/\mathfrak{q}^\wedge$, and that $\{x \in R^\wedge \mid fx = 0\} = \mathfrak{q}^\wedge$, because completion is exact (Lemma 10.97.2). If $x \in R^\wedge$ is such that $x^2 \in \mathfrak{q}^\wedge$, then $fx^2 = 0$ hence $(fx)^2 = 0$ hence $fx = 0$ hence $x \in \mathfrak{q}^\wedge$. Thus $\mathfrak{q}$ is analytically unramified and (2) holds.

Assume R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified. Then $R \to R/\mathfrak{q}_1 \times \ldots \times R/\mathfrak{q}_t$ is injective. Since completion is exact (see Lemma 10.97.2) we see that $R^\wedge \subset (R/\mathfrak{q}_1)^\wedge \times \ldots \times (R/\mathfrak{q}_t)^\wedge$. Hence (3) is clear.

Assume R is analytically unramified. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of $R^\wedge$. Then we see that

$$Q(R^\wedge) = R^\wedge_{\mathfrak{p}_1} \times \ldots \times R^\wedge_{\mathfrak{p}_s}$$

with each $R^\wedge_{\mathfrak{p}_i}$ a field as $R^\wedge$ is reduced (see Lemma 10.25.4). Hence the integral closure S of $R^\wedge$ in $Q(R^\wedge)$ is equal to $S = S_1 \times \ldots \times S_s$ with S_i the integral closure of $R^\wedge/\mathfrak{p}_i$ in its fraction field. In particular S is finite over $R^\wedge$. Denote R' the integral closure of R in $Q(R)$. As $R \to R^\wedge$ is flat we see that $R' \otimes_R R^\wedge \subset Q(R) \otimes_R R^\wedge \subset Q(R^\wedge)$. Moreover $R' \otimes_R R^\wedge$ is integral over $R^\wedge$ (Lemma 10.36.13). Hence $R' \otimes_R R^\wedge \subset S$ is a $R^\wedge$-submodule. As $R^\wedge$ is Noetherian it is a finite $R^\wedge$-module. Thus we may find $f_1, \ldots, f_n \in R'$ such that $R' \otimes_R R^\wedge$ is generated by the elements $f_i \otimes 1$ as a $R^\wedge$-module. By faithful flatness we see that R' is generated by $f_1, \ldots, f_n$ as an R-module. This proves (4).

Part (5) is a special case of part (4). □

032Z Lemma 10.162.11. Let R be a Noetherian local ring. Let $\mathfrak p \subset R$ be a prime. Assume

1. $R_\mathfrak p$ is a discrete valuation ring, and
2. $\mathfrak p$ is analytically unramified.

Then for any associated prime $\mathfrak q$ of $R^\wedge/\mathfrak pR^\wedge$ the local ring $(R^\wedge)_\mathfrak q$ is a discrete valuation ring.

Proof. Assumption (2) says that $R^\wedge/\mathfrak pR^\wedge$ is a reduced ring. Hence an associated prime $\mathfrak q \subset R^\wedge$ of $R^\wedge/\mathfrak pR^\wedge$ is the same thing as a minimal prime over $\mathfrak pR^\wedge$. In particular we see that the maximal ideal of $(R^\wedge)_\mathfrak q$ is $\mathfrak p(R^\wedge)_\mathfrak q$. Choose $x \in R$ such that $xR_\mathfrak p = \mathfrak pR_\mathfrak p$. By the above we see that $x \in (R^\wedge)_\mathfrak q$ generates the maximal ideal. As $R \to R^\wedge$ is faithfully flat we see that x is a nonzerodivisor in $(R^\wedge)_\mathfrak q$. Hence we win. □

0330 Lemma 10.162.12. Let $(R, \mathfrak m)$ be a Noetherian local domain. Let $x \in \mathfrak m$. Assume

1. $x \neq 0$,
2. R/xR has no embedded primes, and
3. for each associated prime $\mathfrak p \subset R$ of R/xR we have
   (a) the local ring $R_\mathfrak p$ is regular, and
   (b) $\mathfrak p$ is analytically unramified.

Then R is analytically unramified.

Proof. Let $\mathfrak p_1, \ldots, \mathfrak p_t$ be the associated primes of the R-module R/xR. Since R/xR has no embedded primes we see that each $\mathfrak p_i$ has height 1, and is a minimal prime over (x). For each i, let $\mathfrak q_{i1}, \ldots, \mathfrak q_{is_i}$ be the associated primes of the $R^\wedge$-module $R^\wedge/\mathfrak p_iR^\wedge$. By Lemma 10.162.11 we see that $(R^\wedge)_{\mathfrak q_{ij}}$ is regular. By Lemma 10.65.3 we see that

$$\mathrm{Ass}_{R^\wedge}(R^\wedge/xR^\wedge) = \bigcup\nolimits_{\mathfrak p \in \mathrm{Ass}_R(R/xR)} \mathrm{Ass}_{R^\wedge}(R^\wedge/\mathfrak pR^\wedge) = \{\mathfrak q_{ij}\}.$$

Let $y \in R^\wedge$ with $y^2 = 0$. As $(R^\wedge)_{\mathfrak q_{ij}}$ is regular, and hence a domain (Lemma 10.106.2) we see that y maps to zero in $(R^\wedge)_{\mathfrak q_{ij}}$. Hence y maps to zero in $R^\wedge/xR^\wedge$ by Lemma 10.63.19. Hence $y = xy'$. Since x is a nonzerodivisor (as $R \to R^\wedge$ is flat) we see that $(y')^2 = 0$. Hence we conclude that $y \in \bigcap x^nR^\wedge = (0)$ (Lemma 10.51.4). □

0331 Lemma 10.162.13. Let $(R, \mathfrak m)$ be a local ring. If R is Noetherian, a domain, and Nagata, then R is analytically unramified.

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial. Hence we assume $\dim(R) = d$ and that the lemma holds for all Noetherian Nagata domains of dimension $< d$.

Let $R \subset S$ be the integral closure of R in the field of fractions of R. By assumption S is a finite R-module. By Lemma 10.162.5 we see that S is Nagata. By Lemma 10.112.4 we see $\dim(R) = \dim(S)$. Let $\mathfrak m_1, \ldots, \mathfrak m_t$ be the maximal ideals of S. Each of these lies over the maximal ideal $\mathfrak m$ of R. Moreover

$$(\mathfrak m_1 \cap \ldots \cap \mathfrak m_t)^n \subset \mathfrak mS$$

for sufficiently large n as $S/\mathfrak{m}S$ is Artinian. By Lemma 10.97.2 $R^\wedge \to S^\wedge$ is an injective map, and by the Chinese Remainder Lemma 10.15.4 combined with Lemma 10.96.9 we have $S^\wedge = \prod S_i^\wedge$ where $S_i^\wedge$ is the completion of S with respect to the maximal ideal $\mathfrak{m}_i$. Hence it suffices to show that $S_{\mathfrak{m}_i}$ is analytically unramified. In other words, we have reduced to the case where R is a Noetherian normal Nagata domain.

Assume R is a Noetherian, normal, local Nagata domain. Pick a nonzero $x \in \mathfrak{m}$ in the maximal ideal. We are going to apply Lemma 10.162.12. We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is clear. We have that R/xR has no embedded primes by Lemma 10.157.6. Thus property (2) holds. The same lemma also tells us each associated prime $\mathfrak{p}$ of R/xR has height 1. Hence $R_\mathfrak{p}$ is a 1-dimensional normal domain hence regular (Lemma 10.119.7). Thus (3)(a) holds. Finally (3)(b) holds by induction hypothesis, since $R/\mathfrak{p}$ is Nagata (by Lemma 10.162.5 or directly from the definition). Thus we conclude R is analytically unramified. □

0BI2 Lemma 10.162.14. Let $(R, \mathfrak{m})$ be a Noetherian local ring. The following are equivalent

(1) R is Nagata,
(2) for $R \to S$ finite with S a domain and $\mathfrak{m}' \subset S$ maximal the local ring $S_{\mathfrak{m}'}$ is analytically unramified,
(3) for $(R, \mathfrak{m}) \to (S, \mathfrak{m}')$ finite local homomorphism with S a domain, then S is analytically unramified.

Proof. Assume R is Nagata and let $R \to S$ and $\mathfrak{m}' \subset S$ be as in (2). Then S is Nagata by Lemma 10.162.5. Hence the local ring $S_{\mathfrak{m}'}$ is Nagata (Lemma 10.162.6). Thus it is analytically unramified by Lemma 10.162.13. It is clear that (2) implies (3).

Assume (3) holds. Let $\mathfrak{p} \subset R$ be a prime ideal and let $L/\kappa(\mathfrak{p})$ be a finite extension of fields. To prove (1) we have to show that the integral closure of $R/\mathfrak{p}$ is finite over $R/\mathfrak{p}$. Choose $x_1, \ldots, x_n \in L$ which generate L over $\kappa(\mathfrak{p})$. For each i let $P_i(T) = T^{d_i} + a_{i,1}T^{d_i - 1} + \ldots + a_{i,d_i}$ be the minimal polynomial for x_i over $\kappa(\mathfrak{p})$. After replacing x_i by $f_i x_i$ for a suitable $f_i \in R$, $f_i \not\in \mathfrak{p}$ we may assume $a_{i,j} \in R/\mathfrak{p}$. In fact, after further multiplying by elements of $\mathfrak{m}$, we may assume $a_{i,j} \in \mathfrak{m}/\mathfrak{p} \subset R/\mathfrak{p}$ for all i, j. Having done this let $S = R/\mathfrak{p}[x_1, \ldots, x_n] \subset L$. Then S is finite over R, a domain, and $S/\mathfrak{m}S$ is a quotient of $R/\mathfrak{m}[T_1, \ldots, T_n]/(T_1^{d_1}, \ldots, T_n^{d_n})$. Hence S is local. By (3) S is analytically unramified and by Lemma 10.162.10 we find that its integral closure S' in L is finite over S. Since S' is also the integral closure of $R/\mathfrak{p}$ in L we win. □

The following proposition says in particular that an algebra of finite type over a Nagata ring is a Nagata ring.

0334 Proposition 10.162.15 (Nagata). Let R be a ring. The following are equivalent:

(1) R is a Nagata ring,
(2) any finite type R-algebra is Nagata, and
(3) R is universally Japanese and Noetherian.

Proof. It is clear that a Noetherian universally Japanese ring is universally Nagata (i.e., condition (2) holds). Let R be a Nagata ring. We will show that any finitely generated R-algebra S is Nagata. This will prove the proposition.

Step 1. There exists a sequence of ring maps $R = R_0 \to R_1 \to R_2 \to \ldots \to R_n = S$ such that each $R_i \to R_{i+1}$ is generated by a single element. Hence by induction it suffices to prove S is Nagata if $S \cong R[x]/I$.

Step 2. Let $\mathfrak q \subset S$ be a prime of S, and let $\mathfrak p \subset R$ be the corresponding prime of R. We have to show that $S/\mathfrak q$ is N-2. Hence we have reduced to the proving the following: (*) Given a Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 3. Let R be a Nagata domain and $R \subset S$ a monogenic extension of domains. Let $R \subset R'$ be the integral closure of R in its fraction field. Let S' be the subring of the fraction field of S generated by R' and S. As R' is finite over R (by the Nagata property) also S' is finite over S. Since S is Noetherian it suffices to prove that S' is N-2 (Lemma 10.161.7). Hence we have reduced to proving the following: (**) Given a normal Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 4: Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains. Suppose the induced extension of fraction fields of R and S is purely transcendental. In this case $S = R[x]$. By Lemma 10.161.13 we see that S is N-2. Hence we have reduced to proving the following: (**) Given a normal Nagata domain R and a monogenic extension $R \subset S$ of domains inducing a finite extension of fraction fields then S is N-2.

Step 5. Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains inducing a finite extension of fraction fields L/K. Choose an element $x \in S$ which generates S as an R-algebra. Let M/L be a finite extension of fields. Let R' be the integral closure of R in M. Then the integral closure S' of S in M is equal to the integral closure of $R'[x]$ in M. Also the fraction field of R' is M and $R \subset R'$ is finite (by the Nagata property of R). This implies that R' is a Nagata ring (Lemma 10.162.5). To show that S' is finite over S is the same as showing that S' is finite over $R'[x]$. Replace R by R' and S by $R'[x]$ to reduce to the following statement: (***) Given a normal Nagata domain R with fraction field K, and $x \in K$, the ring $S \subset K$ generated by R and x is N-1.

Step 6. Let R be a normal Nagata domain with fraction field K. Let $x = b/a \in K$. We have to show that the ring $S \subset K$ generated by R and x is N-1. Note that $S_a \cong R_a$ is normal. Hence by Lemma 10.161.15 it suffices to show that $S_{\mathfrak m}$ is N-1 for every maximal ideal $\mathfrak m$ of S.

With assumptions as in the preceding paragraph, pick such a maximal ideal and set $\mathfrak n = R \cap \mathfrak m$. The residue field extension $\kappa(\mathfrak m)/\kappa(\mathfrak n)$ is finite (Theorem 10.34.1) and generated by the image of x. Hence there exists a monic polynomial $f(X) = X^d + \sum_{i=1,\ldots,d} a_i X^{d-i}$ with $f(x) \in \mathfrak m$. Let K''/K be a finite extension of fields such that $f(X)$ splits completely in $K''[X]$. Let R' be the integral closure of R in K''. Let $S' \subset K''$ be the subring generated by R' and x. As R is Nagata we see R' is finite over R and Nagata (Lemma 10.162.5). Moreover, S' is finite over S. If for every maximal ideal $\mathfrak m'$ of S' the local ring $S'_{\mathfrak m'}$ is N-1, then $S'_{\mathfrak m}$ is N-1 by Lemma 10.161.15, which in turn implies that $S_{\mathfrak m}$ is N-1 by Lemma 10.161.7.

After replacing R by R' and S by S', and $\mathfrak m$ by any of the maximal ideals $\mathfrak m'$ lying over $\mathfrak m$ we reach the situation where the polynomial f above split completely: $f(X) = \prod_{i=1,\ldots,d}(X - a_i)$ with $a_i \in R$. Since $f(x) \in \mathfrak m$ we see that $x - a_i \in \mathfrak m$ for some i. Finally, after replacing x by $x - a_i$ we may assume that $x \in \mathfrak m$.

To recapitulate: R is a normal Nagata domain with fraction field K, $x \in K$ and S is the subring of K generated by x and R, finally $\mathfrak m \subset S$ is a maximal ideal with $x \in \mathfrak m$. We have to show $S_{\mathfrak m}$ is N-1.

We will show that Lemma 10.162.12 applies to the local ring $S_{\mathfrak m}$ and the element x. This will imply that $S_{\mathfrak m}$ is analytically unramified, whereupon we see that it is N-1 by Lemma 10.162.10.

We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is trivial. Let $I = \text{Ker}(R[X] \to S)$ where $X \mapsto x$. We claim that I is generated by all linear forms $aX - b$ such that $ax = b$ in K. Clearly all these linear forms are in I. If $g = a_dX^d + \ldots a_1X + a_0 \in I$, then we see that a_dx is integral over R (Lemma 10.123.1) and hence $b := a_dx \in R$ as R is normal. Then $g - (a_dX - b)X^{d-1} \in I$ and we win by induction on the degree. As a consequence we see that

$$S/xS = R[X]/(X, I) = R/J$$

where

$$J = \{b \in R \mid ax = b \text{ for some } a \in R\} = xR \cap R$$

By Lemma 10.157.6 we see that $S/xS = R/J$ has no embedded primes as an R-module, hence as an R/J-module, hence as an S/xS-module, hence as an S-module. This proves property (2). Take such an associated prime $\mathfrak q \subset S$ with the property $\mathfrak q \subset \mathfrak m$ (so that it is an associated prime of $S_{\mathfrak m}/xS_{\mathfrak m}$ – it does not matter for the arguments). Then $\mathfrak q$ is minimal over xS and hence has height 1. By the sequence of equalities above we see that $\mathfrak p = R \cap \mathfrak q$ is an associated prime of R/J, and so has height 1 (see Lemma 10.157.6). Thus $R_{\mathfrak p}$ is a discrete valuation ring and therefore $R_{\mathfrak p} \subset S_{\mathfrak q}$ is an equality. This shows that $S_{\mathfrak q}$ is regular. This proves property (3)(a). Finally, $(S/\mathfrak q)_{\mathfrak m}$ is a localization of $S/\mathfrak q$, which is a quotient of $S/xS = R/J$. Hence $(S/\mathfrak q)_{\mathfrak m}$ is a localization of a quotient of the Nagata ring R, hence Nagata (Lemmas 10.162.5 and 10.162.6) and hence analytically unramified (Lemma 10.162.13). This shows (3)(b) holds and we are done. □

0335 Proposition 10.162.16. The following types of rings are Nagata and in particular universally Japanese:

(1) fields,
(2) Noetherian complete local rings,
(3) $\mathbf{Z}$,
(4) Dedekind domains with fraction field of characteristic zero,
(5) finite type ring extensions of any of the above.

Proof. The Noetherian complete local ring case is Lemma 10.162.8. In the other cases you just check if $R/\mathfrak p$ is N-2 for every prime ideal $\mathfrak p$ of the ring. This is clear whenever $R/\mathfrak p$ is a field, i.e., $\mathfrak p$ is maximal. Hence for the Dedekind ring case we only need to check it when $\mathfrak p = (0)$. But since we assume the fraction field has characteristic zero Lemma 10.161.11 kicks in. □

09E1 Example 10.162.17. A discrete valuation ring is Nagata if and only if it is N-2 (because the quotient by the maximal ideal is a field and hence N-2). The discrete valuation ring A of Example 10.119.5 is not Nagata, i.e., it is not N-2. Namely, the finite extension $A \subset R = A[f]$ is not N-1. To see this say $f = \sum a_i x^i$. For every $n \geq 1$ set $g_n = \sum_{i<n} a_i x^i \in A$. Then $h_n = (f - g_n)/x^n$ is an element of the fraction field of R and $h_n^p \in k^p[[x]] \subset A$. Hence the integral closure R' of R contains $h_1, h_2, h_3, \ldots$. Now, if R' were finite over R and hence A, then $f = x^n h_n + g_n$ would be contained in the submodule $A + x^n R'$ for all n. By Artin-Rees this would imply $f \in A$ (Lemma 10.51.4), a contradiction.

09E2 Lemma 10.162.18. Let $(A, \mathfrak m)$ be a Noetherian local domain which is Nagata and has fraction field of characteristic p. If $a \in A$ has a pth root in $A^\wedge$, then a has a pth root in A.

Proof. Consider the ring extension $A \subset B = A[x]/(x^p - a)$. If a does not have a pth root in A, then B is a domain whose completion isn't reduced. This contradicts our earlier results, as B is a Nagata ring (Proposition 10.162.15) and hence analytically unramified by Lemma 10.162.13. ☐

10.163. Ascending properties

0336 In this section we start proving some algebraic facts concerning the "ascent" of properties of rings. To do this for depth of rings one uses the following result on ascending depth of modules, see [DG67, IV, Proposition 6.3.1].

0338 Lemma 10.163.1. We have [DG67, IV, Proposition 6.3.1]

$$\text{depth}(M \otimes_R N) = \text{depth}(M) + \text{depth}(N/\mathfrak m_R N)$$

where $R \to S$ is a local homomorphism of local Noetherian rings, M is a finite R-module, and N is a finite S-module flat over R.

Proof. In the statement and in the proof below, we take the depth of M as an R-module, the depth of $M \otimes_R N$ as an S-module, and the depth of $N/\mathfrak m_R N$ as an $S/\mathfrak m_R S$-module. Denote n the right hand side. First assume that n is zero. Then both $\text{depth}(M) = 0$ and $\text{depth}(N/\mathfrak m_R N) = 0$. This means there is a $z \in M$ whose annihilator is $\mathfrak m_R$ and a $\overline{y} \in N/\mathfrak m_R N$ whose annihilator is $\mathfrak m_S/\mathfrak m_R S$. Let $y \in N$ be a lift of $\overline{y}$. Since N is flat over R the map $z : R/\mathfrak m_R \to M$ produces an injective map $N/\mathfrak m_R N \to M \otimes_R N$. Hence the annihilator of $z \otimes y$ is $\mathfrak m_S$. Thus $\text{depth}(M \otimes_R N) = 0$ as well.

Assume $n > 0$. If $\text{depth}(N/\mathfrak m_R N) > 0$, then we may choose $f \in \mathfrak m_S$ mapping to $\overline{f} \in S/\mathfrak m_R S$ which is a nonzerodivisor on $N/\mathfrak m_R N$. Then $\text{depth}(N/\mathfrak m_R N) = \text{depth}(N/(f, \mathfrak m_R)N) + 1$ by Lemma 10.72.7. According to Lemma 10.99.1 the element $f \in S$ is a nonzerodivisor on N and N/fN is flat over R. Hence by induction on n we have

$$\text{depth}(M \otimes_R N/fN) = \text{depth}(M) + \text{depth}(N/(f, \mathfrak m_R)N).$$

Because N/fN is flat over R the sequence

$$0 \to M \otimes_R N \to M \otimes_R N \to M \otimes_R N/fN \to 0$$

is exact where the first map is multiplication by f (Lemma 10.39.12). Hence by Lemma 10.72.7 we find that $\text{depth}(M \otimes_R N) = \text{depth}(M \otimes_R N/fN) + 1$ and we conclude that equality holds in the formula of the lemma.

If $n > 0$, but $\text{depth}(N/\mathfrak{m}_R N) = 0$, then we can choose $f \in \mathfrak{m}_R$ which is a nonzerodivisor on M. As N is flat over R it is also the case that f is a nonzerodivisor on $M \otimes_R N$. By induction on n again we have

$$\text{depth}(M/fM \otimes_R N) = \text{depth}(M/fM) + \text{depth}(N/\mathfrak{m}_R N).$$

In this case $\text{depth}(M \otimes_R N) = \text{depth}(M/fM \otimes_R N) + 1$ and $\text{depth}(M) = \text{depth}(M/fM) + 1$ by Lemma 10.72.7 and we conclude that equality holds in the formula of the lemma. □

0337 Lemma 10.163.2. Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Then

$$\text{depth}(S) = \text{depth}(R) + \text{depth}(S/\mathfrak{m}_R S).$$

Proof. This is a special case of Lemma 10.163.1. □

045J Lemma 10.163.3. Let $R \to S$ be a flat local homomorphism of local Noetherian rings. Then the following are equivalent

(1) S is Cohen-Macaulay, and
(2) R and $S/\mathfrak{m}_R S$ are Cohen-Macaulay.

Proof. Follows from the definitions and Lemmas 10.163.2 and 10.112.7. □

0339 Lemma 10.163.4. Let $\varphi : R \to S$ be a ring map. Assume

(1) R is Noetherian,
(2) S is Noetherian,
(3) φ is flat,
(4) the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are (S_k), and
(5) R has property (S_k).

Then S has property (S_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. By Lemma 10.163.2 we have

$$\text{depth}(S_\mathfrak{q}) = \text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{depth}(R_\mathfrak{p}).$$

On the other hand, we have

$$\dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \geq \dim(S_\mathfrak{q})$$

by Lemma 10.112.6. (Actually equality holds, by Lemma 10.112.7 but strictly speaking we do not need this.) Finally, as the fibre rings of the map are assumed (S_k) we see that $\text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \geq \min(k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}))$. Thus the lemma follows by the following string of inequalities

$$\begin{array}{rcl} \text{depth}(S_\mathfrak{q}) & = & \text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{depth}(R_\mathfrak{p}) \\ & \geq & \min(k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})) + \min(k, \dim(R_\mathfrak{p})) \\ & = & \min(2k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + k, k + \dim(R_\mathfrak{p}), \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \dim(R_\mathfrak{p})) \\ & \geq & \min(k, \dim(S_\mathfrak{q})) \end{array}$$

as desired. □

033A Lemma 10.163.5. Let $\varphi : R \to S$ be a ring map. Assume

(1) R is Noetherian,
(2) S is Noetherian
(3) φ is flat,

(4) the fibre rings $S \otimes_R \kappa(\mathfrak p)$ have property (R_k), and
(5) R has property (R_k).

Then S has property (R_k).

Proof. Let $\mathfrak q$ be a prime of S lying over a prime $\mathfrak p$ of R. Assume that $\dim(S_{\mathfrak q}) \leq k$. Since $\dim(S_{\mathfrak q}) = \dim(R_{\mathfrak p}) + \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q})$ by Lemma 10.112.7 we see that $\dim(R_{\mathfrak p}) \leq k$ and $\dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}) \leq k$. Hence $R_{\mathfrak p}$ and $S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$ are regular by assumption. It follows that $S_{\mathfrak q}$ is regular by Lemma 10.112.8. □

0C21 Lemma 10.163.6. Let $\varphi : R \to S$ be a ring map. Assume

(1) R is Noetherian,
(2) S is Noetherian
(3) φ is flat,
(4) the fibre rings $S \otimes_R \kappa(\mathfrak p)$ are reduced,
(5) R is reduced.

Then S is reduced.

Proof. For Noetherian rings reduced is the same as having properties (S_1) and (R_0), see Lemma 10.157.3. Thus we know R and the fibre rings have these properties. Hence we may apply Lemmas 10.163.4 and 10.163.5 and we see that S is (S_1) and (R_0), in other words reduced by Lemma 10.157.3 again. □

033B Lemma 10.163.7. Let $\varphi : R \to S$ be a ring map. Assume

(1) φ is smooth,
(2) R is reduced.

Then S is reduced.

Proof. Observe that $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 10.142). In particular, the fibres are reduced. Thus if R is Noetherian, then S is Noetherian and we get the result from Lemma 10.163.6.

In the general case we may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 10.142. Now, if $x \in S$ is an element with $x^2 = 0$, then we can enlarge R_0 and assume that x comes from an element $x_0 \in S_0$. After enlarging R_0 once more we may assume that $x_0^2 = 0$ in S_0. However, since $R_0 \subset R$ is reduced we see that S_0 is reduced and hence $x_0 = 0$ as desired. □

0C22 Lemma 10.163.8. Let $\varphi : R \to S$ be a ring map. Assume

(1) R is Noetherian,
(2) S is Noetherian,
(3) φ is flat,
(4) the fibre rings $S \otimes_R \kappa(\mathfrak p)$ are normal, and
(5) R is normal.

Then S is normal.

Proof. For a Noetherian ring being normal is the same as having properties (S_2) and (R_1), see Lemma 10.157.4. Thus we know R and the fibre rings have these properties. Hence we may apply Lemmas 10.163.4 and 10.163.5 and we see that S is (S_2) and (R_1), in other words normal by Lemma 10.157.4 again. □

033C Lemma 10.163.9. Let $\varphi : R \to S$ be a ring map. Assume

(1) φ is smooth,
(2) R is normal.

Then S is normal.

Proof. Observe that $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 10.142). In particular, the fibres are normal. Thus if R is Noetherian, then S is Noetherian and we get the result from Lemma 10.163.8.

The general case. First note that R is reduced and hence S is reduced by Lemma 10.163.7. Let $\mathfrak{q}$ be a prime of S and let $\mathfrak{p}$ be the corresponding prime of R. Note that $R_{\mathfrak{p}}$ is a normal domain. We have to show that $S_{\mathfrak{q}}$ is a normal domain. To do this we may replace R by $R_{\mathfrak{p}}$ and S by $S_{\mathfrak{p}}$. Hence we may assume that R is a normal domain.

Assume $R \to S$ smooth, and R a normal domain. We may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 10.142. As R_0 is a Nagata domain (see Proposition 10.162.16) we see that its integral closure R_0' is finite over R_0. Moreover, as R is a normal domain it is clear that $R_0' \subset R$. Hence we may replace R_0 by R_0' and S_0 by $R_0' \otimes_{R_0} S_0$ and assume that R_0 is a normal Noetherian domain. By the first paragraph of the proof we conclude that S_0 is a normal ring (it need not be a domain of course). In this way we see that $R = \bigcup R_\lambda$ is the union of normal Noetherian domains and correspondingly $S = \operatorname{colim} R_\lambda \otimes_{R_0} S_0$ is the colimit of normal rings. This implies that S is a normal ring. Some details omitted. $\square$

07NF Lemma 10.163.10. Let $\varphi : R \to S$ be a ring map. Assume

(1) φ is smooth,
(2) R is a regular ring.

Then S is regular.

Proof. This follows from Lemma 10.163.5 applied for all (R_k) using Lemma 10.140.3 to see that the hypotheses are satisfied. $\square$

10.164. Descending properties

033D In this section we start proving some algebraic facts concerning the "descent" of properties of rings. It turns out that it is often "easier" to descend properties than it is to ascend them. In other words, the assumption on the ring map $R \to S$ are often weaker than the assumptions in the corresponding lemma of the preceding section. However, we warn the reader that the results on descent are often useless unless the corresponding ascent can also be shown! Here is a typical result which illustrates this phenomenon.

033E Lemma 10.164.1. Let $R \to S$ be a ring map. Assume that

(1) $R \to S$ is faithfully flat, and
(2) S is Noetherian.

Then R is Noetherian.

Proof. Let $I_0 \subset I_1 \subset I_2 \subset \ldots$ be a growing sequence of ideals of R. By assumption we have $I_nS = I_{n+1}S = I_{n+2}S = \ldots$ for some n. Since $R \to S$ is flat we have $I_kS = I_k \otimes_R S$. Hence, as $R \to S$ is faithfully flat we see that $I_nS = I_{n+1}S = I_{n+2}S = \ldots$ implies that $I_n = I_{n+1} = I_{n+2} = \ldots$ as desired. $\square$

033F Lemma 10.164.2. Let $R \to S$ be a ring map. Assume that

(1) $R \to S$ is faithfully flat, and
(2) S is reduced.

Then R is reduced.

Proof. This is clear as $R \to S$ is injective. □

033G Lemma 10.164.3. Let $R \to S$ be a ring map. Assume that

(1) $R \to S$ is faithfully flat, and
(2) S is a normal ring.

Then R is a normal ring.

Proof. Since S is reduced it follows that R is reduced. Let $\mathfrak{p}$ be a prime of R. We have to show that $R_\mathfrak{p}$ is a normal domain. Since $S_\mathfrak{p}$ is faithfully over $R_\mathfrak{p}$ too we may assume that R is local with maximal ideal $\mathfrak{m}$. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Then we see that $R \to S_\mathfrak{q}$ is faithfully flat (Lemma 10.39.17). Hence we may assume S is local as well. In particular S is a normal domain. Since $R \to S$ is faithfully flat and S is a normal domain we see that R is a domain. Next, suppose that a/b is integral over R with $a, b \in R$. Then $a/b \in S$ as S is normal. Hence $a \in bS$. This means that $a : R \to R/bR$ becomes the zero map after base change to S. By faithful flatness we see that $a \in bR$, so $a/b \in R$. Hence R is normal. □

07NG Lemma 10.164.4. Let $R \to S$ be a ring map. Assume that

(1) $R \to S$ is faithfully flat, and
(2) S is a regular ring.

Then R is a regular ring.

Proof. We see that R is Noetherian by Lemma 10.164.1. Let $\mathfrak{p} \subset R$ be a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$. Then Lemma 10.110.9 applies to $R_\mathfrak{p} \to S_\mathfrak{q}$ and we conclude that $R_\mathfrak{p}$ is regular. Since $\mathfrak{p}$ was arbitrary we see R is regular. □

0352 Lemma 10.164.5. Let $R \to S$ be a ring map. Assume that

(1) $R \to S$ is faithfully flat, and
(2) S is Noetherian and has property (S_k).

Then R is Noetherian and has property (S_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 10.164.1. Let $\mathfrak{p} \subset R$ be a prime ideal. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 10.112.7) and $\text{depth}(A) = \text{depth}(B)$ (Lemma 10.163.2). Hence since B has (S_k) we see that A has (S_k). □

0353 Lemma 10.164.6. Let $R \to S$ be a ring map. Assume that

(1) $R \to S$ is faithfully flat, and
(2) S is Noetherian and has property (R_k).

Then R is Noetherian and has property (R_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 10.164.1. Let $\mathfrak{p} \subset R$ be a prime ideal and assume $\dim(R_\mathfrak{p}) \leq k$. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 10.112.7). As S has (R_k) we conclude that B is a regular local ring. By Lemma 10.110.9 we conclude that A is regular. □

0354 **Lemma 10.164.7.** Let $R \to S$ be a ring map. Assume that

(1) $R \to S$ is smooth and surjective on spectra, and
(2) S is a Nagata ring.

Then R is a Nagata ring.

Proof. Recall that a Nagata ring is the same thing as a Noetherian universally Japanese ring (Proposition 10.162.15). We have already seen that R is Noetherian in Lemma 10.164.1. Let $R \to A$ be a finite type ring map into a domain. According to Lemma 10.162.3 it suffices to check that A is N-1. It is clear that $B = A \otimes_R S$ is a finite type S-algebra and hence Nagata (Proposition 10.162.15). Since $A \to B$ is smooth (Lemma 10.137.4) we see that B is reduced (Lemma 10.163.7). Since B is Noetherian it has only a finite number of minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ (see Lemma 10.31.6). As $A \to B$ is flat each of these lies over $(0) \subset A$ (by going down, see Lemma 10.39.19) The total ring of fractions $Q(B)$ is the product of the $L_i = \kappa(\mathfrak{q}_i)$ (Lemmas 10.25.4 and 10.25.1). Moreover, the integral closure B' of B in $Q(B)$ is the product of the integral closures B'_i of the $B/\mathfrak{q}_i$ in the factors L_i (compare with Lemma 10.37.16). Since B is universally Japanese the ring extensions $B/\mathfrak{q}_i \subset B'_i$ are finite and we conclude that $B' = \prod B'_i$ is finite over B. Since $A \to B$ is flat we see that any nonzerodivisor on A maps to a nonzerodivisor on B. The corresponding map

$$Q(A) \otimes_A B = (A \setminus \{0\})^{-1} A \otimes_A B = (A \setminus \{0\})^{-1} B \to Q(B)$$

is injective (we used Lemma 10.12.15). Via this map A' maps into B'. This induces a map

$$A' \otimes_A B \longrightarrow B'$$

which is injective (by the above and the flatness of $A \to B$). Since B' is a finite B-module and B is Noetherian we see that $A' \otimes_A B$ is a finite B-module. Hence there exist finitely many elements $x_i \in A'$ such that the elements $x_i \otimes 1$ generate $A' \otimes_A B$ as a B-module. Finally, by faithful flatness of $A \to B$ we conclude that the x_i also generated A' as an A-module, and we win. □

0355 **Remark 10.164.8.** The property of being "universally catenary" does not descend; not even along étale ring maps. In Examples, Section 110.18 there is a construction of a finite ring map $A \to B$ with A local Noetherian and not universally catenary, B semi-local with two maximal ideals $\mathfrak{m}$, $\mathfrak{n}$ with $B_\mathfrak{m}$ and $B_\mathfrak{n}$ regular of dimension 2 and 1 respectively, and the same residue fields as that of A. Moreover, $\mathfrak{m}_A$ generates the maximal ideal in both $B_\mathfrak{m}$ and $B_\mathfrak{n}$ (so $A \to B$ is unramified as well as finite). By Lemma 10.152.3 there exists a local étale ring map $A \to A'$ such that $B \otimes_A A' = B_1 \times B_2$ decomposes with $A' \to B_i$ surjective. This shows that A' has two minimal primes $\mathfrak{q}_i$ with $A'/\mathfrak{q}_i \cong B_i$. Since B_i is regular local (since it is étale over either $B_\mathfrak{m}$ or $B_\mathfrak{n}$) we conclude that A' is universally catenary.

10.165. Geometrically normal algebras

037Y In this section we put some applications of ascent and descent of properties of rings.

037Z **Lemma 10.165.1.** Let k be a field. Let A be a k-algebra. The following properties of A are equivalent:

(1) $k' \otimes_k A$ is a normal ring for every field extension k'/k,
(2) $k' \otimes_k A$ is a normal ring for every finitely generated field extension k'/k,
(3) $k' \otimes_k A$ is a normal ring for every finite purely inseparable extension k'/k,
(4) $k^{perf} \otimes_k A$ is a normal ring.

Here normal ring is defined in Definition 10.37.11.

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4).

If k'/k is a finite purely inseparable extension, then there is an embedding $k' \to k^{perf}$ of k-extensions. The ring map $k' \otimes_k A \to k^{perf} \otimes_k A$ is faithfully flat, hence $k' \otimes_k A$ is normal if $k^{perf} \otimes_k A$ is normal by Lemma 10.164.3. In this way we see that (4) $\Rightarrow$ (3).

Assume (2) and let k'/k be any field extension. Then we can write $k' = \text{colim}_i k_i$ as a directed colimit of finitely generated field extensions. Hence we see that $k' \otimes_k A = \text{colim}_i k_i \otimes_k A$ is a directed colimit of normal rings. Thus we see that $k' \otimes_k A$ is a normal ring by Lemma 10.37.17. Hence (1) holds.

Assume (3) and let K/k be a finitely generated field extension. By Lemma 10.45.3 we can find a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Lemma 10.158.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a normal ring because we assumed (3). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a normal ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 10.137.4) and ascent of normality along smooth maps (Lemma 10.163.9). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a normal ring as it is a localization of a normal ring (Lemma 10.37.13). Step 4. Finally $K \otimes_k A$ is a normal ring by descent of normality along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 10.164.3). This proves the lemma. $\square$

0380 **Definition 10.165.2.** Let k be a field. A k-algebra R is called *geometrically normal* over k if the equivalent conditions of Lemma 10.165.1 hold.

06DE **Lemma 10.165.3.** Let k be a field. A localization of a geometrically normal k-algebra is geometrically normal.

Proof. This is clear as being a normal ring is checked at the localizations at prime ideals. $\square$

0C30 **Lemma 10.165.4.** Let k be a field. Let K/k be a separable field extension. Then K is geometrically normal over k.

Proof. This is true because $k^{perf} \otimes_k K$ is a field. Namely, it is reduced by Lemma 10.43.6. By Lemma 10.45.4 (or by Definition 10.45.5) the field extension k^{perf}/k is purely inseparable. Hence by Lemma 10.46.10 the ring $k^{perf} \otimes_k K$ has a unique prime ideal. A reduced ring with a unique prime ideal is a field. $\square$

06DF Lemma 10.165.5. Let k be a field. Let A, B be k-algebras. Assume A is geometrically normal over k and B is a normal ring. Then $A \otimes_k B$ is a normal ring.

Proof. Let $\mathfrak{r}$ be a prime ideal of $A \otimes_k B$. Denote $\mathfrak{p}$, resp. $\mathfrak{q}$ the corresponding prime of A, resp. B. Then $(A \otimes_k B)_{\mathfrak{r}}$ is a localization of $A_{\mathfrak{p}} \otimes_k B_{\mathfrak{q}}$. Hence it suffices to prove the result for the ring $A_{\mathfrak{p}} \otimes_k B_{\mathfrak{q}}$, see Lemma 10.37.13 and Lemma 10.165.3. Thus we may assume A and B are domains.

Assume that A and B are domains with fractions fields K and L. Note that B is the filtered colimit of its finite type normal k-sub algebras (as k is a Nagata ring, see Proposition 10.162.16, and hence the integral closure of a finite type k-sub algebra is still a finite type k-sub algebra by Proposition 10.162.15). By Lemma 10.37.17 we reduce to the case that B is of finite type over k.

Assume that A and B are domains with fractions fields K and L and B of finite type over k. In this case the ring $K \otimes_k B$ is of finite type over K, hence Noetherian (Lemma 10.31.1). In particular $K \otimes_k B$ has finitely many minimal primes (Lemma 10.31.6). Since $A \to A \otimes_k B$ is flat, this implies that $A \otimes_k B$ has finitely many minimal primes (by going down for flat ring maps – Lemma 10.39.19 – these primes all lie over $(0) \subset A$). Thus it suffices to prove that $A \otimes_k B$ is integrally closed in its total ring of fractions (Lemma 10.37.16).

We claim that $K \otimes_k B$ and $A \otimes_k L$ are both normal rings. If this is true then any element x of $Q(A \otimes_k B)$ which is integral over $A \otimes_k B$ is (by Lemma 10.37.12) contained in $K \otimes_k B \cap A \otimes_k L = A \otimes_k B$ and we're done. Since $A \otimes_K L$ is a normal ring by assumption, it suffices to prove that $K \otimes_k B$ is normal.

As A is geometrically normal over k we see K is geometrically normal over k (Lemma 10.165.3) hence K is geometrically reduced over k. Hence $K = \bigcup K_i$ is the union of finitely generated field extensions of k which are geometrically reduced (Lemma 10.43.2). Each K_i is the localization of a smooth k-algebra (Lemma 10.158.10). So $K_i \otimes_k B$ is the localization of a smooth B-algebra hence normal (Lemma 10.163.9). Thus $K \otimes_k B$ is a normal ring (Lemma 10.37.17) and we win. $\square$

0C31 Lemma 10.165.6. Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically normal over k if and only if it is geometrically normal over k'.

Proof. Let L/k be a finite purely inseparable field extension. Then $L' = k' \otimes_k L$ is a field (see material in Fields, Section 9.28) and $A \otimes_k L = A \otimes_{k'} L'$. Hence if A is geometrically normal over k', then A is geometrically normal over k.

Assume A is geometrically normal over k. Let K/k' be a field extension. Then

$$K \otimes_{k'} A = (K \otimes_k A) \otimes_{(k' \otimes_k k')} k'$$

Since $k' \otimes_k k' \to k'$ is a localization by Lemma 10.43.8, we see that $K \otimes_{k'} A$ is a localization of a normal ring, hence normal. $\square$

10.166. Geometrically regular algebras

045K Let k be a field. Let A be a Noetherian k-algebra. Let K/k be a finitely generated field extension. Then the ring $K \otimes_k A$ is Noetherian as well, see Lemma 10.31.8. Thus the following lemma makes sense.

0381 Lemma 10.166.1. Let k be a field. Let A be a k-algebra. Assume A is Noetherian. The following properties of A are equivalent:

(1) $k' \otimes_k A$ is regular for every finitely generated field extension k'/k, and
(2) $k' \otimes_k A$ is regular for every finite purely inseparable extension k'/k.

Here regular ring is as in Definition 10.110.7.

Proof. The lemma makes sense by the remarks preceding the lemma. It is clear that (1) $\Rightarrow$ (2).

Assume (2) and let K/k be a finitely generated field extension. By Lemma 10.45.3 we can find a diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Lemma 10.158.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a regular ring because we assumed (2). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a regular ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 10.137.4) and ascent of regularity along smooth maps (Lemma 10.163.10). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a regular ring as it is a localization of a regular ring (immediate from the definition). Step 4. Finally $K \otimes_k A$ is a regular ring by descent of regularity along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 10.164.4). This proves the lemma. $\square$

0382 Definition 10.166.2. Let k be a field. Let R be a Noetherian k-algebra. The k-algebra R is called geometrically regular over k if the equivalent conditions of Lemma 10.166.1 hold.

It is clear from the definition that $K \otimes_k R$ is a geometrically regular algebra over K for any finitely generated field extension K of k. We will see later (More on Algebra, Proposition 15.35.1) that it suffices to check $R \otimes_k k'$ is regular whenever $k \subset k' \subset k^{1/p}$ (finite).

07NH Lemma 10.166.3. Let k be a field. Let $A \to B$ be a faithfully flat k-algebra map. If B is geometrically regular over k, so is A.

Proof. Assume B is geometrically regular over k. Let k'/k be a finite, purely inseparable extension. Then $A \otimes_k k' \to B \otimes_k k'$ is faithfully flat as a base change of $A \to B$ (by Lemmas 10.30.3 and 10.39.7) and $B \otimes_k k'$ is regular by our assumption on B over k. Then $A \otimes_k k'$ is regular by Lemma 10.164.4. $\square$

07QF Lemma 10.166.4. Let k be a field. Let $A \to B$ be a smooth ring map of k-algebras. If A is geometrically regular over k, then B is geometrically regular over k.

Proof. Let k'/k be a finitely generated field extension. Then $A \otimes_k k' \to B \otimes_k k'$ is a smooth ring map (Lemma 10.137.4) and $A \otimes_k k'$ is regular. Hence $B \otimes_k k'$ is regular by Lemma 10.163.10. $\square$

07QG Lemma 10.166.5. Let k be a field. Let A be an algebra over k. Let $k = \operatorname{colim} k_i$ be a directed colimit of subfields. If A is geometrically regular over each k_i, then A is geometrically regular over k.

Proof. Let k'/k be a finite purely inseparable field extension. We can get k' by adjoining finitely many variables to k and imposing finitely many polynomial relations. Hence we see that there exists an i and a finite purely inseparable field extension k_i'/k_i such that $k_i = k \otimes_{k_i} k_i'$. Thus $A \otimes_k k' = A \otimes_{k_i} k_i'$ and the lemma is clear. $\square$

07QH Lemma 10.166.6. Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically regular over k if and only if it is geometrically regular over k'.

Proof. Let L/k be a finite purely inseparable field extension. Then $L' = k' \otimes_k L$ is a field (see material in Fields, Section 9.28) and $A \otimes_k L = A \otimes_{k'} L'$. Hence if A is geometrically regular over k', then A is geometrically regular over k.

Assume A is geometrically regular over k. Since k' is the filtered colimit of finite extensions of k we may assume by Lemma 10.166.5 that k'/k is finite separable. Consider the ring maps

$$k' \to A \otimes_k k' \to A.$$

Note that $A \otimes_k k'$ is geometrically regular over k' as a base change of A to k'. Note that $A \otimes_k k' \to A$ is the base change of $k' \otimes_k k' \to k'$ by the map $k' \to A$. Since k'/k is an étale extension of rings, we see that $k' \otimes_k k' \to k'$ is étale (Lemma 10.143.3). Hence A is geometrically regular over k' by Lemma 10.166.4. $\square$

10.167. Geometrically Cohen-Macaulay algebras

045L This section is a bit of a misnomer, since Cohen-Macaulay algebras are automatically geometrically Cohen-Macaulay. Namely, see Lemma 10.130.6 and Lemma 10.167.2 below.

045M Lemma 10.167.1. Let k be a field and let K/k and L/k be two field extensions such that one of them is a field extension of finite type. Then $K \otimes_k L$ is a Noetherian Cohen-Macaulay ring.

Proof. The ring $K \otimes_k L$ is Noetherian by Lemma 10.31.8. Say K is a finite extension of the purely transcendental extension $k(t_1, \ldots, t_r)$. Then $k(t_1, \ldots, t_r) \otimes_k L \to K \otimes_k L$ is a finite free ring map. By Lemma 10.112.9 it suffices to show that $k(t_1, \ldots, t_r) \otimes_k L$ is Cohen-Macaulay. This is clear because it is a localization of the polynomial ring $L[t_1, \ldots, t_r]$. (See for example Lemma 10.104.7 for the fact that a polynomial ring is Cohen-Macaulay.) $\square$

045N Lemma 10.167.2. Let k be a field. Let S be a Noetherian k-algebra. Let K/k be a finitely generated field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak{q} \subset S$ be a prime of S. Let $\mathfrak{q}_K \subset S_K$ be a prime of S_K lying over $\mathfrak{q}$. Then $S_{\mathfrak{q}}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak{q}_K}$ is Cohen-Macaulay.

Proof. By Lemma 10.31.8 the ring S_K is Noetherian. Hence $S_{\mathfrak{q}} \to (S_K)_{\mathfrak{q}_K}$ is a flat local homomorphism of Noetherian local rings. Note that the fibre

$$(S_K)_{\mathfrak{q}_K}/\mathfrak{q}(S_K)_{\mathfrak{q}_K} \cong (\kappa(\mathfrak{q}) \otimes_k K)_{\mathfrak{q}'}$$

is the localization of the Cohen-Macaulay (Lemma 10.167.1) ring $\kappa(\mathfrak{q}) \otimes_k K$ at a suitable prime ideal $\mathfrak{q}'$. Hence the lemma follows from Lemma 10.163.3. □

10.168. Colimits and maps of finite presentation, II

07RF This section is a continuation of Section 10.127.

We start with an application of the openness of flatness. It says that we can approximate flat modules by flat modules which is useful.

02JO Lemma 10.168.1. Let $R \to S$ be a ring map. Let M be an S-module. Assume that

(1) $R \to S$ is of finite presentation,
(2) M is a finitely presented S-module, and
(3) M is flat over R.

In this case we have the following:

(1) There exists a finite type $\mathbf{Z}$-algebra R_0 and a finite type ring map $R_0 \to S_0$ and a finite S_0-module M_0 such that M_0 is flat over R_0, together with a ring maps $R_0 \to R$ and $S_0 \to S$ and an S_0-module map $M_0 \to M$ such that $S \cong R \otimes_{R_0} S_0$ and $M = S \otimes_{S_0} M_0$.
(2) If $R = \mathrm{colim}_{\lambda \in \Lambda} R_\lambda$ is written as a directed colimit, then there exists a λ and a ring map $R_\lambda \to S_\lambda$ of finite presentation, and an S_λ-module M_λ of finite presentation such that M_λ is flat over R_λ and such that $S = R \otimes_{R_\lambda} S_\lambda$ and $M = S \otimes_{S_\lambda} M_\lambda$.
(3) If
$$(R \to S, M) = \mathrm{colim}_{\lambda \in \Lambda}(R_\lambda \to S_\lambda, M_\lambda)$$
is written as a directed colimit such that
  (a) $R_\mu \otimes_{R_\lambda} S_\lambda \to S_\mu$ and $S_\mu \otimes_{S_\lambda} M_\lambda \to M_\mu$ are isomorphisms for $\mu \geq \lambda$,
  (b) $R_\lambda \to S_\lambda$ is of finite presentation,
  (c) M_λ is a finitely presented S_λ-module,
then for all sufficiently large λ the module M_λ is flat over R_λ.

Proof. We first write $(R \to S, M)$ as the directed colimit of a system $(R_\lambda \to S_\lambda, M_\lambda)$ as in as in Lemma 10.127.18. Let $\mathfrak{q} \subset S$ be a prime. Let $\mathfrak{p} \subset R$, $\mathfrak{q}_\lambda \subset S_\lambda$, and $\mathfrak{p}_\lambda \subset R_\lambda$ the corresponding primes. As seen in the proof of Theorem 10.129.4
$$((R_\lambda)_{\mathfrak{p}_\lambda}, (S_\lambda)_{\mathfrak{q}_\lambda}, (M_\lambda)_{\mathfrak{q}_\lambda})$$
is a system as in Lemma 10.127.13, and hence by Lemma 10.128.3 we see that for some $\lambda_\mathfrak{q} \in \Lambda$ for all $\lambda \geq \lambda_\mathfrak{q}$ the module M_λ is flat over R_λ at the prime $\mathfrak{q}_\lambda$.

By Theorem 10.129.4 we get an open subset $U_\lambda \subset \mathrm{Spec}(S_\lambda)$ such that M_λ flat over R_λ at all the primes of U_λ. Denote $V_\lambda \subset \mathrm{Spec}(S)$ the inverse image of U_λ under the map $\mathrm{Spec}(S) \to \mathrm{Spec}(S_\lambda)$. The argument above shows that for every $\mathfrak{q} \in \mathrm{Spec}(S)$ there exists a $\lambda_\mathfrak{q}$ such that $\mathfrak{q} \in V_\lambda$ for all $\lambda \geq \lambda_\mathfrak{q}$. Since $\mathrm{Spec}(S)$ is quasi-compact we see this implies there exists a single $\lambda_0 \in \Lambda$ such that $V_{\lambda_0} = \mathrm{Spec}(S)$.

The complement $\mathrm{Spec}(S_{\lambda_0}) \setminus U_{\lambda_0}$ is $V(I)$ for some ideal $I \subset S_{\lambda_0}$. As $V_{\lambda_0} = \mathrm{Spec}(S)$ we see that $IS = S$. Choose $f_1, \ldots, f_r \in I$ and $s_1, \ldots, s_n \in S$ such that $\sum f_i s_i = 1$. Since $\mathrm{colim}\, S_\lambda = S$, after increasing λ_0 we may assume there exist $s_{i,\lambda_0} \in S_{\lambda_0}$ such that $\sum f_i s_{i,\lambda_0} = 1$. Hence for this λ_0 we have $U_{\lambda_0} = \mathrm{Spec}(S_{\lambda_0})$. This proves (1).

Proof of (2). Let $(R_0 \to S_0, M_0)$ be as in (1) and suppose that $R = \mathrm{colim}\, R_\lambda$. Since R_0 is a finite type $\mathbf{Z}$ algebra, there exists a λ and a map $R_0 \to R_\lambda$ such that

$R_0 \to R_\lambda \to R$ is the given map $R_0 \to R$ (see Lemma 10.127.3). Then, part (2) follows by taking $S_\lambda = R_\lambda \otimes_{R_0} S_0$ and $M_\lambda = S_\lambda \otimes_{S_0} M_0$.

Finally, we come to the proof of (3). Let $(R_\lambda \to S_\lambda, M_\lambda)$ be as in (3). Choose $(R_0 \to S_0, M_0)$ and $R_0 \to R$ as in (1). As in the proof of (2), there exists a λ_0 and a ring map $R_0 \to R_{\lambda_0}$ such that $R_0 \to R_{\lambda_0} \to R$ is the given map $R_0 \to R$. Since S_0 is of finite presentation over R_0 and since $S = \operatorname{colim} S_\lambda$ we see that for some $\lambda_1 \geq \lambda_0$ we get an R_0-algebra map $S_0 \to S_{\lambda_1}$ such that the composition $S_0 \to S_{\lambda_1} \to S$ is the given map $S_0 \to S$ (see Lemma 10.127.3). For all $\lambda \geq \lambda_1$ this gives maps

$$\Psi_\lambda : R_\lambda \otimes_{R_0} S_0 \longrightarrow R_\lambda \otimes_{R_{\lambda_1}} S_{\lambda_1} \cong S_\lambda$$

the last isomorphism by assumption. By construction $\operatorname{colim}_\lambda \Psi_\lambda$ is an isomorphism. Hence Ψ_λ is an isomorphism for all λ large enough by Lemma 10.127.8. In the same vein, there exists a $\lambda_2 \geq \lambda_1$ and an S_0-module map $M_0 \to M_{\lambda_2}$ such that $M_0 \to M_{\lambda_2} \to M$ is the given map $M_0 \to M$ (see Lemma 10.127.5). For $\lambda \geq \lambda_2$ there is an induced map

$$S_\lambda \otimes_{S_0} M_0 \longrightarrow S_\lambda \otimes_{S_{\lambda_2}} M_{\lambda_2} \cong M_\lambda$$

and for λ large enough this map is an isomorphism by Lemma 10.127.6. This implies (3) because M_0 is flat over R_0. □

034Y Lemma 10.168.2. Let $R \to A \to B$ be ring maps. Assume $A \to B$ faithfully flat of finite presentation. Then there exists a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & A_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\ R \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$.

Proof. We first prove the lemma with R replaced $\mathbf{Z}$. By Lemma 10.168.1 there exists a diagram

$$\begin{tikzcd} A_0 \arrow[r] & A \\ B_0 \arrow[r] \arrow[u] & B \arrow[u] \end{tikzcd}$$

where A_0 is of finite type over $\mathbf{Z}$, B_0 is flat of finite presentation over A_0 such that $B = A \otimes_{A_0} B_0$. As $A_0 \to B_0$ is flat of finite presentation we see that the image of $\operatorname{Spec}(B_0) \to \operatorname{Spec}(A_0)$ is open, see Proposition 10.41.8. Hence the complement of the image is $V(I_0)$ for some ideal $I_0 \subset A_0$. As $A \to B$ is faithfully flat the map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective, see Lemma 10.39.16. Now we use that the base change of the image is the image of the base change. Hence $I_0 A = A$. Pick a relation $\sum f_i r_i = 1$, with $r_i \in A$, $f_i \in I_0$. Then after enlarging A_0 to contain the elements r_i (and correspondingly enlarging B_0) we see that $A_0 \to B_0$ is surjective on spectra also, i.e., faithfully flat.

Thus the lemma holds in case $R = \mathbf{Z}$. In the general case, take the solution $A_0' \to B_0'$ just obtained and set $A_0 = A_0' \otimes_{\mathbf{Z}} R$, $B_0 = B_0' \otimes_{\mathbf{Z}} R$. □

07RG **Lemma 10.168.3.** Let $A = \text{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume

(1) $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is finite,
(2) C_0 is of finite type over B_0.

Then there exists an $i \geq 0$ such that the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is finite.

Proof. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Pick monic polynomials $P_j \in A \otimes_{A_0} B_0[T]$ such that $P_j(1 \otimes x_j) = 0$ in $A \otimes_{A_0} C_0$. For some $i \geq 0$ we can find $P_{j,i} \in A_i \otimes_{A_0} B_0[T]$ mapping to P_j. Since $\otimes$ commutes with colimits we see that $P_{j,i}(1 \otimes x_j)$ is zero in $A_i \otimes_{A_0} C_0$ after possibly increasing i. Then this i works. $\square$

07RH **Lemma 10.168.4.** Let $A = \text{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume

(1) $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is surjective,
(2) C_0 is of finite type over B_0.

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is surjective.

Proof. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Pick $b_j \in A \otimes_{A_0} B_0$ mapping to $1 \otimes x_j$ in $A \otimes_{A_0} C_0$. For some $i \geq 0$ we can find $b_{j,i} \in A_i \otimes_{A_0} B_0$ mapping to b_j. Then this i works. $\square$

0C4F **Lemma 10.168.5.** Let $A = \text{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume

(1) $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is unramified,
(2) C_0 is of finite type over B_0.

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is unramified.

Proof. Set $B_i = A_i \otimes_{A_0} B_0$, $C_i = A_i \otimes_{A_0} C_0$, $B = A \otimes_{A_0} B_0$, and $C = A \otimes_{A_0} C_0$. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Then $\text{d}x_1, \ldots, \text{d}x_m$ generate Ω_{C_0/B_0} over C_0 and their images generate Ω_{C_i/B_i} over C_i (Lemmas 10.131.14 and 10.131.9). Observe that $0 = \Omega_{C/B} = \text{colim}\, \Omega_{C_i/B_i}$ (Lemma 10.131.5). Thus there is an i such that $\text{d}x_1, \ldots, \text{d}x_m$ map to zero and hence $\Omega_{C_i/B_i} = 0$ as desired. $\square$

0C32 **Lemma 10.168.6.** Let $A = \text{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume

(1) $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is an isomorphism,
(2) $B_0 \to C_0$ is of finite presentation.

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is an isomorphism.

Proof. By Lemma 10.168.4 there exists an i such that $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is surjective. Since the map is of finite presentation the kernel is a finitely generated ideal. Let $g_1, \ldots, g_r \in A_i \otimes_{A_0} B_0$ generate the kernel. Then we may pick $i' \geq i$ such that g_j map to zero in $A_{i'} \otimes_{A_0} B_0$. Then $A_{i'} \otimes_{A_0} B_0 \to A_{i'} \otimes_{A_0} C_0$ is an isomorphism. $\square$

07RI **Lemma 10.168.7.** Let $A = \text{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume

(1) $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is étale,
(2) $B_0 \to C_0$ is of finite presentation.

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is étale.

Proof. Write $C_0 = B_0[x_1, \ldots, x_n]/(f_{1,0}, \ldots, f_{m,0})$. Write $B_i = A_i \otimes_{A_0} B_0$ and $C_i = A_i \otimes_{A_0} C_0$. Note that $C_i = B_i[x_1, \ldots, x_n]/(f_{1,i}, \ldots, f_{m,i})$ where $f_{j,i}$ is the image of $f_{j,0}$ in the polynomial ring over B_i. Write $B = A \otimes_{A_0} B_0$ and $C = A \otimes_{A_0} C_0$. Note that $C = B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where f_j is the image of $f_{j,0}$ in the polynomial ring over B. The assumption is that the map

$$\mathrm{d} : (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2 \longrightarrow \bigoplus C \mathrm{d}x_k$$

is an isomorphism. Thus for sufficiently large i we can find elements

$$\xi_{k,i} \in (f_{1,i}, \ldots, f_{m,i})/(f_{1,i}, \ldots, f_{m,i})^2$$

with $\mathrm{d}\xi_{k,i} = \mathrm{d}x_k$ in $\bigoplus C_i \mathrm{d}x_k$. Moreover, on increasing i if necessary, we see that $\sum (\partial f_{j,i}/\partial x_k)\xi_{k,i} = f_{j,i} \bmod (f_{1,i}, \ldots, f_{m,i})^2$ since this is true in the limit. Then this i works. □

0C0B Lemma 10.168.8. Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume

(1) $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is smooth,
(2) $B_0 \to C_0$ is of finite presentation.

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is smooth.

Proof. Write $C_0 = B_0[x_1, \ldots, x_n]/(f_{1,0}, \ldots, f_{m,0})$. Write $B_i = A_i \otimes_{A_0} B_0$ and $C_i = A_i \otimes_{A_0} C_0$. Note that $C_i = B_i[x_1, \ldots, x_n]/(f_{1,i}, \ldots, f_{m,i})$ where $f_{j,i}$ is the image of $f_{j,0}$ in the polynomial ring over B_i. Write $B = A \otimes_{A_0} B_0$ and $C = A \otimes_{A_0} C_0$. Note that $C = B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where f_j is the image of $f_{j,0}$ in the polynomial ring over B. The assumption is that the map

$$\mathrm{d} : (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2 \longrightarrow \bigoplus C \mathrm{d}x_k$$

is a split injection. Let $\xi_k \in (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2$ be elements such that $\sum (\partial f_j/\partial x_k)\xi_k = f_j \bmod (f_1, \ldots, f_m)^2$. Then for sufficiently large i we can find elements

$$\xi_{k,i} \in (f_{1,i}, \ldots, f_{m,i})/(f_{1,i}, \ldots, f_{m,i})^2$$

with $\sum (\partial f_{j,i}/\partial x_k)\xi_{k,i} = f_{j,i} \bmod (f_{1,i}, \ldots, f_{m,i})^2$ since this is true in the limit. Then this i works. □

0C33 Lemma 10.168.9. Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume

(1) $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is syntomic (resp. a relative global complete intersection),
(2) C_0 is of finite presentation over B_0.

Then there exists an $i \geq 0$ such that the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is syntomic (resp. a relative global complete intersection).

Proof. Assume $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is a relative global complete intersection. By Lemma 10.136.11 there exists a finite type $\mathbf{Z}$-algebra R, a ring map $R \to A \otimes_{A_0} B_0$, a relative global complete intersection $R \to S$, and an isomorphism

$$(A \otimes_{A_0} B_0) \otimes_R S \longrightarrow A \otimes_{A_0} C_0$$

Because R is of finite type (and hence finite presentation) over $\mathbf{Z}$, there exists an i and a map $R \to A_i \otimes_{A_0} B_0$ lifting the map $R \to A \otimes_{A_0} B_0$, see Lemma 10.127.3. Using the same lemma, there exists an $i' \geq i$ such that $(A_i \otimes_{A_0} B_0) \otimes_R S \to A \otimes_{A_0} C_0$

comes from a map $(A_i \otimes_{A_0} B_0) \otimes_R S \to A_{i'} \otimes_{A_0} C_0$. Thus we may assume, after replacing i by i', that the displayed map comes from an $A_i \otimes_{A_0} B_0$-algebra map

$$(A_i \otimes_{A_0} B_0) \otimes_R S \longrightarrow A_i \otimes_{A_0} C_0$$

By Lemma 10.168.6 after increasing i this map is an isomorphism. This finishes the proof in this case because the base change of a relative global complete intersection is a relative global complete intersection by Lemma 10.136.9.

Assume $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is syntomic. Then there exist elements $g_1, \ldots, g_m$ in $A \otimes_{A_0} C_0$ generating the unit ideal such that $A \otimes_{A_0} B_0 \to (A \otimes_{A_0} C_0)_{g_j}$ is a relative global complete intersection, see Lemma 10.136.15. We can find an i and elements $g_{i,j} \in A_i \otimes_{A_0} C_0$ mapping to g_j. After increasing i we may assume $g_{i,1}, \ldots, g_{i,m}$ generate the unit ideal of $A_i \otimes_{A_0} C_0$. The result of the previous paragraph implies that, after increasing i, we may assume the maps $A_i \otimes_{A_0} B_0 \to (A_i \otimes_{A_0} C_0)_{g_{i,j}}$ are relative global complete intersections. Then $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is syntomic by Lemma 10.136.4 (and the already used Lemma 10.136.15). $\square$

The following lemma is an application of the results above which doesn't seem to fit well anywhere else.

034Z Lemma 10.168.10. Let $R \to S$ be a faithfully flat ring map of finite presentation. Then there exists a commutative diagram

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where $R \to S'$ is quasi-finite, faithfully flat and of finite presentation.

Proof. As a first step we reduce this lemma to the case where R is of finite type over $\mathbf{Z}$. By Lemma 10.168.2 there exists a diagram

$$\begin{tikzcd} S_0 \arrow[r] & S \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

where R_0 is of finite type over $\mathbf{Z}$, and S_0 is faithfully flat of finite presentation over R_0 such that $S = R \otimes_{R_0} S_0$. If we prove the lemma for the ring map $R_0 \to S_0$, then the lemma follows for $R \to S$ by base change, as the base change of a quasi-finite ring map is quasi-finite, see Lemma 10.122.8. (Of course we also use that base changes of flat maps are flat and base changes of maps of finite presentation are of finite presentation.)

Assume $R \to S$ is a faithfully flat ring map of finite presentation and that R is Noetherian (which we may assume by the preceding paragraph). Let $W \subset \text{Spec}(S)$ be the open set of Lemma 10.130.4. As $R \to S$ is faithfully flat the map $\text{Spec}(S) \to \text{Spec}(R)$ is surjective, see Lemma 10.39.16. By Lemma 10.130.5 the map $W \to \text{Spec}(R)$ is also surjective. Hence by replacing S with a product $S_{g_1} \times \ldots \times S_{g_m}$ we may assume $W = \text{Spec}(S)$; here we use that $\text{Spec}(R)$ is quasi-compact (Lemma 10.17.10), and that the map $\text{Spec}(S) \to \text{Spec}(R)$ is open (Proposition 10.41.8). Suppose that $\mathfrak{p} \subset R$ is a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a maximal ideal of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. The Noetherian local ring $\overline{S}_{\mathfrak q} = S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$ is Cohen-Macaulay, say of dimension d. We may choose $f_1, \ldots, f_d$ in the maximal ideal of $S_{\mathfrak q}$ which map to a regular sequence in $\overline{S}_{\mathfrak q}$. Choose a common denominator $g \in S$, $g \not\in \mathfrak q$ of $f_1, \ldots, f_d$, and consider the R-algebra

$$S' = S_g/(f_1, \ldots, f_d).$$

By construction there is a prime ideal $\mathfrak q' \subset S'$ lying over $\mathfrak p$ and corresponding to $\mathfrak q$ (via $S_g \to S'_g$). Also by construction the ring map $R \to S'$ is quasi-finite at $\mathfrak q$ as the local ring

$$S'_{\mathfrak q'}/\mathfrak p S'_{\mathfrak q'} = S_{\mathfrak q}/(f_1, \ldots, f_d) + \mathfrak p S_{\mathfrak q} = \overline{S}_{\mathfrak q}/(\overline{f}_1, \ldots, \overline{f}_d)$$

has dimension zero, see Lemma 10.122.2. Also by construction $R \to S'$ is of finite presentation. Finally, by Lemma 10.99.3 the local ring map $R_{\mathfrak p} \to S'_{\mathfrak q'}$ is flat (this is where we use that R is Noetherian). Hence, by openness of flatness (Theorem 10.129.4), and openness of quasi-finiteness (Lemma 10.123.13) we may after replacing g by gg' for a suitable $g' \in S$, $g' \not\in \mathfrak q$ assume that $R \to S'$ is flat and quasi-finite. The image $\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ is open and contains $\mathfrak p$. In other words we have shown a ring S' as in the statement of the lemma exists (except possibly the faithfulness part) whose image contains any given prime. Using one more time the quasi-compactness of $\mathrm{Spec}(R)$ we see that a finite product of such rings does the job. □

10.169. Other chapters

CHAPTER 11

Brauer groups

073W 11.1. Introduction

073X A reference is the lectures by Serre in the Seminaire Cartan, see [Ser55a]. Serre in turn refers to [Deu68] and [ANT44]. We changed some of the proofs, in particular we used a fun argument of Rieffel to prove Wedderburn's theorem. Very likely this change is not an improvement and we strongly encourage the reader to read the original exposition by Serre.

11.2. Noncommutative algebras

073Y Let k be a field. In this chapter an algebra A over k is a possibly noncommutative ring A together with a ring map $k \to A$ such that k maps into the center of A and such that 1 maps to an identity element of A. An A-module is a right A-module such that the identity of A acts as the identity.

073Z Definition 11.2.1. Let A be a k-algebra. We say A is finite if $\dim_k(A) < \infty$. In this case we write $[A : k] = \dim_k(A)$.

0740 Definition 11.2.2. A skew field is a possibly noncommutative ring with an identity element 1, with $1 \neq 0$, in which every nonzero element has a multiplicative inverse.

A skew field is a k-algebra for some k (e.g., for the prime field contained in it). We will use below that any module over a skew field is free because a maximal linearly independent set of vectors forms a basis and exists by Zorn's lemma.

0741 Definition 11.2.3. Let A be a k-algebra. We say an A-module M is simple if it is nonzero and the only A-submodules are 0 and M. We say A is simple if the only two-sided ideals of A are 0 and A.

0742 Definition 11.2.4. A k-algebra A is central if the center of A is the image of $k \to A$.

0743 Definition 11.2.5. Given a k-algebra A we denote A^{op} the k-algebra we get by reversing the order of multiplication in A. This is called the opposite algebra.

11.3. Wedderburn's theorem

0744 The following cute argument can be found in a paper of Rieffel, see [Rie65]. The proof could not be simpler (quote from Carl Faith's review).

0745 Lemma 11.3.1. Let A be a possibly noncommutative ring with 1 which contains no nontrivial two-sided ideal. Let M be a nonzero right ideal in A, and view M as a right A-module. Then A coincides with the bicommutant of M.

Proof. Let $A' = \mathrm{End}_A(M)$, so M is a left A'-module. Set $A'' = \mathrm{End}_{A'}(M)$ (the bicommutant of M). We view M as a right A''-module[1]. Let $R : A \to A''$ be the natural homomorphism such that $mR(a) = ma$. Then R is injective, since $R(1) = \mathrm{id}_M$ and A contains no nontrivial two-sided ideal. We claim that $R(M)$ is a right ideal in A''. Namely, $R(m)a'' = R(ma'')$ for $a'' \in A''$ and m in M, because left multiplication of M by any element n of M represents an element of A', and so $(nm)a'' = n(ma'')$ for all n in M. Finally, the product ideal AM is a two-sided ideal, and so $A = AM$. Thus $R(A) = R(A)R(M)$, so that $R(A)$ is a right ideal in A''. But $R(A)$ contains the identity element of A'', and so $R(A) = A''$. □

0746 **Lemma 11.3.2.** Let A be a k-algebra. If A is finite, then

1. A has a simple module,
2. any nonzero module contains a simple submodule,
3. a simple module over A has finite dimension over k, and
4. if M is a simple A-module, then $\mathrm{End}_A(M)$ is a skew field.

Proof. Of course (1) follows from (2) since A is a nonzero A-module. For (2), any submodule of minimal (finite) dimension as a k-vector space will be simple. There exists a finite dimensional one because a cyclic submodule is one. If M is simple, then $mA \subset M$ is a sub-module, hence we see (3). Any nonzero element of $\mathrm{End}_A(M)$ is an isomorphism, hence (4) holds. □

0747 **Theorem 11.3.3.** Let A be a simple finite k-algebra. Then A is a matrix algebra over a finite k-algebra K which is a skew field.

Proof. We may choose a simple submodule $M \subset A$ and then the k-algebra $K = \mathrm{End}_A(M)$ is a skew field, see Lemma 11.3.2. By Lemma 11.3.1 we see that $A = \mathrm{End}_K(M)$. Since K is a skew field and M is finitely generated (since $\dim_k(M) < \infty$) we see that M is finite free as a left K-module. It follows immediately that $A \cong \mathrm{Mat}(n \times n, K^{op})$. □

11.4. Lemmas on algebras

0748 Let A be a k-algebra. Let $B \subset A$ be a subalgebra. The *centralizer of B in A* is the subalgebra

$$C = \{y \in A \mid xy = yx \text{ for all } x \in B\}.$$

It is a k-algebra.

0749 **Lemma 11.4.1.** Let A, A' be k-algebras. Let $B \subset A$, $B' \subset A'$ be subalgebras with centralizers C, C'. Then the centralizer of $B \otimes_k B'$ in $A \otimes_k A'$ is $C \otimes_k C'$.

Proof. Denote $C'' \subset A \otimes_k A'$ the centralizer of $B \otimes_k B'$. It is clear that $C \otimes_k C' \subset C''$. Conversely, every element of C'' commutes with $B \otimes 1$ hence is contained in $C \otimes_k A'$. Similarly $C'' \subset A \otimes_k C'$. Thus $C'' \subset C \otimes_k A' \cap A \otimes_k C' = C \otimes_k C'$. □

074A **Lemma 11.4.2.** Let A be a finite simple k-algebra. Then the center k' of A is a finite field extension of k.

[1] This means that given $a'' \in A''$ and $m \in M$ we have a product $ma'' \in M$. In particular, the multiplication in A'' is the opposite of what you'd get if you wrote elements of A'' as endomorphisms acting on the left.

Proof. Write $A = \text{Mat}(n \times n, K)$ for some skew field K finite over k, see Theorem 11.3.3. By Lemma 11.4.1 the center of A is $k \otimes_k k'$ where $k' \subset K$ is the center of K. Since the center of a skew field is a field, we win. $\square$

074B **Lemma 11.4.3.** Let V be a k vector space. Let K be a central k-algebra which is a skew field. Let $W \subset V \otimes_k K$ be a two-sided K-sub vector space. Then W is generated as a left K-vector space by $W \cap (V \otimes 1)$.

Proof. Let $V' \subset V$ be the k-sub vector space generated by $v \in V$ such that $v \otimes 1 \in W$. Then $V' \otimes_k K \subset W$ and we have

$$W/(V' \otimes_k K) \subset (V/V') \otimes_k K.$$

If $\overline{v} \in V/V'$ is a nonzero vector such that $\overline{v} \otimes 1$ is contained in $W/(V' \otimes_k K)$, then we see that $v \otimes 1 \in W$ where $v \in V$ lifts $\overline{v}$. This contradicts our construction of V'. Hence we may replace V by V/V' and W by $W/(V' \otimes_k K)$ and it suffices to prove that $W \cap (V \otimes 1)$ is nonzero if W is nonzero.

To see this let $w \in W$ be a nonzero element which can be written as $w = \sum_{i=1,\ldots,n} v_i \otimes k_i$ with n minimal. We may right multiply with k_1^{-1} and assume that $k_1 = 1$. If $n = 1$, then we win because $v_1 \otimes 1 \in W$. If $n > 1$, then we see that for any $c \in K$

$$cw - wc = \sum\nolimits_{i=2,\ldots,n} v_i \otimes (ck_i - k_i c) \in W$$

and hence $ck_i - k_i c = 0$ by minimality of n. This implies that k_i is in the center of K which is k by assumption. Hence $w = (v_1 + \sum k_i v_i) \otimes 1$ contradicting the minimality of n. $\square$

074C **Lemma 11.4.4.** Let A be a k-algebra. Let K be a central k-algebra which is a skew field. Then any two-sided ideal $I \subset A \otimes_k K$ is of the form $J \otimes_k K$ for some two-sided ideal $J \subset A$. In particular, if A is simple, then so is $A \otimes_k K$.

Proof. Set $J = \{a \in A \mid a \otimes 1 \in I\}$. This is a two-sided ideal of A. And $I = J \otimes_k K$ by Lemma 11.4.3. $\square$

074D **Lemma 11.4.5.** Let R be a possibly noncommutative ring. Let $n \geq 1$ be an integer. Let $R_n = \text{Mat}(n \times n, R)$.

1. The functors $M \mapsto M^{\oplus n}$ and $N \mapsto Ne_{11}$ define quasi-inverse equivalences of categories $\text{Mod}_R \leftrightarrow \text{Mod}_{R_n}$.
2. A two-sided ideal of R_n is of the form IR_n for some two-sided ideal I of R.
3. The center of R_n is equal to the center of R.

Proof. Part (1) proves itself. If $J \subset R_n$ is a two-sided ideal, then $J = \bigoplus e_{ii} J e_{jj}$ and all of the summands $e_{ii} J e_{jj}$ are equal to each other and are a two-sided ideal I of R. This proves (2). Part (3) is clear. $\square$

074E **Lemma 11.4.6.** Let A be a finite simple k-algebra.

1. There exists exactly one simple A-module M up to isomorphism.
2. Any finite A-module is a direct sum of copies of a simple module.
3. Two finite A-modules are isomorphic if and only if they have the same dimension over k.
4. If $A = \text{Mat}(n \times n, K)$ with K a finite skew field extension of k, then $M = K^{\oplus n}$ is a simple A-module and $\text{End}_A(M) = K^{op}$.

(5) If M is a simple A-module, then $L = \text{End}_A(M)$ is a skew field finite over k acting on the left on M, we have $A = \text{End}_L(M)$, and the centers of A and L agree. Also $[A : k][L : k] = \dim_k(M)^2$.
(6) For a finite A-module N the algebra $B = \text{End}_A(N)$ is a matrix algebra over the skew field L of (5). Moreover $\text{End}_B(N) = A$.

Proof. By Theorem 11.3.3 we can write $A = \text{Mat}(n\times n, K)$ for some finite skew field extension K of k. By Lemma 11.4.5 the category of modules over A is equivalent to the category of modules over K. Thus (1), (2), and (3) hold because every module over K is free. Part (4) holds because the equivalence transforms the K-module K to $M = K^{\oplus n}$. Using $M = K^{\oplus n}$ in (5) we see that $L = K^{op}$. The statement about the center of $L = K^{op}$ follows from Lemma 11.4.5. The statement about $\text{End}_L(M)$ follows from the explicit form of M. The formula of dimensions is clear. Part (6) follows as N is isomorphic to a direct sum of copies of a simple module. □

074F Lemma 11.4.7. Let A, A' be two simple k-algebras one of which is finite and central over k. Then $A \otimes_k A'$ is simple.

Proof. Suppose that A' is finite and central over k. Write $A' = \text{Mat}(n \times n, K')$, see Theorem 11.3.3. Then the center of K' is k and we conclude that $A \otimes_k K'$ is simple by Lemma 11.4.4. Hence $A \otimes_k A' = \text{Mat}(n \times n, A \otimes_k K')$ is simple by Lemma 11.4.5. □

074G Lemma 11.4.8. The tensor product of finite central simple algebras over k is finite, central, and simple.

Proof. Combine Lemmas 11.4.1 and 11.4.7. □

074H Lemma 11.4.9. Let A be a finite central simple algebra over k. Let k'/k be a field extension. Then $A' = A \otimes_k k'$ is a finite central simple algebra over k'.

Proof. Combine Lemmas 11.4.1 and 11.4.7. □

074I Lemma 11.4.10. Let A be a finite central simple algebra over k. Then $A \otimes_k A^{op} \cong \text{Mat}(n \times n, k)$ where $n = [A : k]$.

Proof. By Lemma 11.4.8 the algebra $A \otimes_k A^{op}$ is simple. Hence the map

$$A \otimes_k A^{op} \longrightarrow \text{End}_k(A), \quad a \otimes a' \longmapsto (x \mapsto axa')$$

is injective. Since both sides of the arrow have the same dimension we win. □

11.5. The Brauer group of a field

074J Let k be a field. Consider two finite central simple algebras A and B over k. We say A and B are similar if there exist $n, m > 0$ such that $\text{Mat}(n\times n, A) \cong \text{Mat}(m\times m, B)$ as k-algebras.

074K Lemma 11.5.1. Similarity.

(1) Similarity defines an equivalence relation on the set of isomorphism classes of finite central simple algebras over k.
(2) Every similarity class contains a unique (up to isomorphism) finite central skew field extension of k.
(3) If $A = \text{Mat}(n \times n, K)$ and $B = \text{Mat}(m \times m, K')$ for some finite central skew fields K, K' over k then A and B are similar if and only if $K \cong K'$ as k-algebras.

Proof. Note that by Wedderburn's theorem (Theorem 11.3.3) we can always write a finite central simple algebra as a matrix algebra over a finite central skew field. Hence it suffices to prove the third assertion. To see this it suffices to show that if $A = \text{Mat}(n \times n, K) \cong \text{Mat}(m \times m, K') = B$ then $K \cong K'$. To see this note that for a simple module M of A we have $\text{End}_A(M) = K^{op}$, see Lemma 11.4.6. Hence $A \cong B$ implies $K^{op} \cong (K')^{op}$ and we win. $\square$

Given two finite central simple k-algebras A, B the tensor product $A \otimes_k B$ is another, see Lemma 11.4.8. Moreover if A is similar to A', then $A \otimes_k B$ is similar to $A' \otimes_k B$ because tensor products and taking matrix algebras commute. Hence tensor product defines an operation on equivalence classes of finite central simple algebras which is clearly associative and commutative. Finally, Lemma 11.4.10 shows that $A \otimes_k A^{op}$ is isomorphic to a matrix algebra, i.e., that $A \otimes_k A^{op}$ is in the similarity class of k. Thus we obtain an abelian group.

074L **Definition 11.5.2.** Let k be a field. The *Brauer group* of k is the abelian group of similarity classes of finite central simple k-algebras defined above. Notation $\text{Br}(k)$.

For any map of fields $k \to k'$ we obtain a group homomorphism

$$\text{Br}(k) \longrightarrow \text{Br}(k'), \quad A \longmapsto A \otimes_k k'$$

see Lemma 11.4.9. In other words, $\text{Br}(-)$ is a functor from the category of fields to the category of abelian groups. Observe that the Brauer group of a field is zero if and only if every finite central skew field extension $k \subset K$ is trivial.

074M **Lemma 11.5.3.** *The Brauer group of an algebraically closed field is zero.*

Proof. Let $k \subset K$ be a finite central skew field extension. For any element $x \in K$ the subring $k[x] \subset K$ is a commutative finite integral k-sub algebra, hence a field, see Algebra, Lemma 10.36.19. Since k is algebraically closed we conclude that $k[x] = k$. Since x was arbitrary we conclude $k = K$. $\square$

074N **Lemma 11.5.4.** *Let A be a finite central simple algebra over a field k. Then $[A : k]$ is a square.*

Proof. This is true because $A \otimes_k \overline{k}$ is a matrix algebra over $\overline{k}$ by Lemma 11.5.3. $\square$

11.6. Skolem-Noether

074P

074Q **Theorem 11.6.1.** *Let A be a finite central simple k-algebra. Let B be a simple k-algebra. Let $f, g : B \to A$ be two k-algebra homomorphisms. Then there exists an invertible element $x \in A$ such that $f(b) = xg(b)x^{-1}$ for all $b \in B$.*

Proof. Choose a simple A-module M. Set $L = \text{End}_A(M)$. Then L is a skew field with center k which acts on the left on M, see Lemmas 11.3.2 and 11.4.6. Then M has two $B \otimes_k L^{op}$-module structures defined by $m \cdot_1 (b \otimes l) = lmf(b)$ and $m \cdot_2 (b \otimes l) = lmg(b)$. The k-algebra $B \otimes_k L^{op}$ is simple by Lemma 11.4.7. Since B is simple, the existence of a k-algebra homomorphism $B \to A$ implies that B is finite. Thus $B \otimes_k L^{op}$ is finite simple and we conclude the two $B \otimes_k L^{op}$-module structures on M are isomorphic by Lemma 11.4.6. Hence we find $\varphi : M \to M$ intertwining these operations. In particular φ is in the commutant of L which implies that φ is multiplication by some $x \in A$, see Lemma 11.4.6. Working out the definitions we see that x is a solution to our problem. $\square$

074R Lemma 11.6.2. Let A be a finite central simple k-algebra. Any automorphism of A is inner. In particular, any automorphism of $\text{Mat}(n \times n, k)$ is inner.

Proof. Note that A is a finite central simple algebra over the center of A which is a finite field extension of k, see Lemma 11.4.2. Hence the Skolem-Noether theorem (Theorem 11.6.1) applies. □

11.7. The centralizer theorem

074S

074T Theorem 11.7.1. Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. Then

(1) the centralizer C of B in A is simple,
(2) $[A : k] = [B : k][C : k]$, and
(3) the centralizer of C in A is B.

Proof. Throughout this proof we use the results of Lemma 11.4.6 freely. Choose a simple A-module M. Set $L = \text{End}_A(M)$. Then L is a skew field with center k which acts on the left on M and $A = \text{End}_L(M)$. Then M is a right $B \otimes_k L^{op}$-module and $C = \text{End}_{B \otimes_k L^{op}}(M)$. Since the algebra $B \otimes_k L^{op}$ is simple by Lemma 11.4.7 we see that C is simple (by Lemma 11.4.6 again).

Write $B \otimes_k L^{op} = \text{Mat}(m \times m, K)$ for some skew field K finite over k. Then $C = \text{Mat}(n \times n, K^{op})$ if M is isomorphic to a direct sum of n copies of the simple $B \otimes_k L^{op}$-module $K^{\oplus m}$ (the lemma again). Thus we have $\dim_k(M) = nm[K : k]$, $[B : k][L : k] = m^2[K : k]$, $[C : k] = n^2[K : k]$, and $[A : k][L : k] = \dim_k(M)^2$ (by the lemma again). We conclude that (2) holds.

Part (3) follows because of (2) applied to $C \subset A$ shows that $[B : k] = [C' : k]$ where C' is the centralizer of C in A (and the obvious fact that $B \subset C'$). □

074U Lemma 11.7.2. Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. If B is a central k-algebra, then $A = B \otimes_k C$ where C is the (central simple) centralizer of B in A.

Proof. We have $\dim_k(A) = \dim_k(B \otimes_k C)$ by Theorem 11.7.1. By Lemma 11.4.7 the tensor product is simple. Hence the natural map $B \otimes_k C \to A$ is injective hence an isomorphism. □

074V Lemma 11.7.3. Let A be a finite central simple algebra over k. If $K \subset A$ is a subfield, then the following are equivalent

(1) $[A : k] = [K : k]^2$,
(2) K is its own centralizer, and
(3) K is a maximal commutative subring.

Proof. Theorem 11.7.1 shows that (1) and (2) are equivalent. It is clear that (3) and (2) are equivalent. □

074W Lemma 11.7.4. Let A be a finite central skew field over k. Then every maximal subfield $K \subset A$ satisfies $[A : k] = [K : k]^2$.

Proof. Special case of Lemma 11.7.3. □

11.8. Splitting fields

074X

074Y Definition 11.8.1. Let A be a finite central simple k-algebra. We say a field extension k'/k splits A, or k' is a splitting field for A if $A \otimes_k k'$ is a matrix algebra over k'.

Another way to say this is that the class of A maps to zero under the map $\text{Br}(k) \to \text{Br}(k')$.

074Z Theorem 11.8.2. Let A be a finite central simple k-algebra. Let k'/k be a finite field extension. The following are equivalent

(1) k' splits A, and
(2) there exists a finite central simple algebra B similar to A such that $k' \subset B$ and $[B : k] = [k' : k]^2$.

Proof. Assume (2). It suffices to show that $B \otimes_k k'$ is a matrix algebra. We know that $B \otimes_k B^{op} \cong \text{End}_k(B)$. Since k' is the centralizer of k' in B^{op} by Lemma 11.7.3 we see that $B \otimes_k k'$ is the centralizer of $k \otimes k'$ in $B \otimes_k B^{op} = \text{End}_k(B)$. Of course this centralizer is just $\text{End}_{k'}(B)$ where we view B as a k' vector space via the embedding $k' \to B$. Thus the result.

Assume (1). This means that we have an isomorphism $A \otimes_k k' \cong \text{End}_{k'}(V)$ for some k'-vector space V. Let B be the commutant of A in $\text{End}_k(V)$. Note that k' sits in B. By Lemma 11.7.2 the classes of A and B add up to zero in $\text{Br}(k)$. From the dimension formula in Theorem 11.7.1 we see that

$$[B : k][A : k] = \dim_k(V)^2 = [k' : k]^2 \dim_{k'}(V)^2 = [k' : k]^2[A : k].$$

Hence $[B : k] = [k' : k]^2$. Thus we have proved the result for the opposite to the Brauer class of A. However, k' splits the Brauer class of A if and only if it splits the Brauer class of the opposite algebra, so we win anyway. ☐

0750 Lemma 11.8.3. A maximal subfield of a finite central skew field K over k is a splitting field for K.

Proof. Combine Lemma 11.7.4 with Theorem 11.8.2. ☐

0751 Lemma 11.8.4. Consider a finite central skew field K over k. Let $d^2 = [K : k]$. For any finite splitting field k' for K the degree $[k' : k]$ is divisible by d.

Proof. By Theorem 11.8.2 there exists a finite central simple algebra B in the Brauer class of K such that $[B : k] = [k' : k]^2$. By Lemma 11.5.1 we see that $B = \text{Mat}(n \times n, K)$ for some n. Then $[k' : k]^2 = n^2 d^2$ whence the result. ☐

0752 Proposition 11.8.5. Consider a finite central skew field K over k. There exists a maximal subfield $k \subset k' \subset K$ which is separable over k. In particular, every Brauer class has a finite separable spitting field.

Proof. Since every Brauer class is represented by a finite central skew field over k, we see that the second statement follows from the first by Lemma 11.8.3.

To prove the first statement, suppose that we are given a separable subfield $k' \subset K$. Then the centralizer K' of k' in K has center k', and the problem reduces to finding a maximal subfield of K' separable over k'. Thus it suffices to prove, if $k \neq K$, that we can find an element $x \in K$, $x \notin k$ which is separable over k. This statement is clear in characteristic zero. Hence we may assume that k has characteristic $p > 0$. If the ground field k is finite then, the result is clear as well (because extensions of finite fields are always separable). Thus we may assume that k is an infinite field of positive characteristic.

To get a contradiction assume no element of K is separable over k. By the discussion in Fields, Section 9.28 this means the minimal polynomial of any $x \in K$ is of the form $T^q - a$ where q is a power of p and $a \in k$. Since it is clear that every element of K has a minimal polynomial of degree $\leq \dim_k(K)$ we conclude that there exists a fixed p-power q such that $x^q \in k$ for all $x \in K$.

Consider the map

$$(-)^q : K \longrightarrow K$$

and write it out in terms of a k-basis $\{a_1, \ldots, a_n\}$ of K with $a_1 = 1$. So

$$(\sum x_i a_i)^q = \sum f_i(x_1, \ldots, x_n) a_i.$$

Since multiplication on K is k-bilinear we see that each f_i is a polynomial in $x_1, \ldots, x_n$ (details omitted). The choice of q above and the fact that k is infinite shows that f_i is identically zero for $i \geq 2$. Hence we see that it remains zero on extending k to its algebraic closure $\overline{k}$. But the algebra $K \otimes_k \overline{k}$ is a matrix algebra (for example by Lemmas 11.4.9 and 11.5.3), which implies there are some elements whose qth power is not central (e.g., e_{11}). This is the desired contradiction. □

The results above allow us to characterize finite central simple algebras as follows.

0753 Lemma 11.8.6. Let k be a field. For a k-algebra A the following are equivalent

(1) A is finite central simple k-algebra,
(2) A is a finite dimensional k-vector space, k is the center of A, and A has no nontrivial two-sided ideal,
(3) there exists $d \geq 1$ such that $A \otimes_k \bar{k} \cong \text{Mat}(d \times d, \bar{k})$,
(4) there exists $d \geq 1$ such that $A \otimes_k k^{sep} \cong \text{Mat}(d \times d, k^{sep})$,
(5) there exist $d \geq 1$ and a finite Galois extension k'/k such that $A \otimes_k k' \cong \text{Mat}(d \times d, k')$,
(6) there exist $n \geq 1$ and a finite central skew field K over k such that $A \cong \text{Mat}(n \times n, K)$.

The integer d is called the degree of A.

Proof. The equivalence of (1) and (2) is a consequence of the definitions, see Section 11.2. Assume (1). By Proposition 11.8.5 there exists a separable splitting field $k \subset k'$ for A. Of course, then a Galois closure of k'/k is a splitting field also. Thus we see that (1) implies (5). It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3). Assume (3). Then $A \otimes_k \overline{k}$ is a finite central simple $\overline{k}$-algebra for example by Lemma 11.4.5. This trivially implies that A is a finite central simple k-algebra. Finally, the equivalence of (1) and (6) is Wedderburn's theorem, see Theorem 11.3.3. □

11.9. Other chapters

Preliminaries

(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

Topics in Scheme Theory

(42) Chow Homology
(43) Intersection Theory
(44) Picard Schemes of Curves
(45) Weil Cohomology Theories
(46) Adequate Modules
(47) Dualizing Complexes
(48) Duality for Schemes
(49) Discriminants and Differents
(50) de Rham Cohomology
(51) Local Cohomology
(52) Algebraic and Formal Geometry
(53) Algebraic Curves
(54) Resolution of Surfaces
(55) Semistable Reduction
(56) Functors and Morphisms
(57) Derived Categories of Varieties
(58) Fundamental Groups of Schemes
(59) Étale Cohomology
(60) Crystalline Cohomology
(61) Pro-étale Cohomology
(62) Relative Cycles
(63) More Étale Cohomology
(64) The Trace Formula

Algebraic Spaces

(65) Algebraic Spaces
(66) Properties of Algebraic Spaces
(67) Morphisms of Algebraic Spaces
(68) Decent Algebraic Spaces
(69) Cohomology of Algebraic Spaces
(70) Limits of Algebraic Spaces
(71) Divisors on Algebraic Spaces
(72) Algebraic Spaces over Fields
(73) Topologies on Algebraic Spaces
(74) Descent and Algebraic Spaces
(75) Derived Categories of Spaces
(76) More on Morphisms of Spaces
(77) Flatness on Algebraic Spaces
(78) Groupoids in Algebraic Spaces
(79) More on Groupoids in Spaces
(80) Bootstrap
(81) Pushouts of Algebraic Spaces

Topics in Geometry

(82) Chow Groups of Spaces
(83) Quotients of Groupoids
(84) More on Cohomology of Spaces
(85) Simplicial Spaces
(86) Duality for Spaces
(87) Formal Algebraic Spaces
(88) Algebraization of Formal Spaces
(89) Resolution of Surfaces Revisited

Deformation Theory

(90) Formal Deformation Theory
(91) Deformation Theory
(92) The Cotangent Complex

CHAPTER 12

Homological Algebra

00ZU 12.1. Introduction

00ZV Basic homological algebra will be explained in this document. We add as needed in the other parts, since there is clearly an infinite amount of this stuff around. A reference is [ML63].

12.2. Basic notions

00ZW The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) Nothing yet.

12.3. Preadditive and additive categories

09SE Here is the definition of a preadditive category.

00ZY Definition 12.3.1. A category $\mathcal{A}$ is called preadditive if each morphism set $\text{Mor}_\mathcal{A}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\text{Mor}(x, y) \times \text{Mor}(y, z) \longrightarrow \text{Mor}(x, z)$$

are bilinear. A functor $F : \mathcal{A} \to \mathcal{B}$ of preadditive categories is called additive if and only if $F : \text{Mor}(x, y) \to \text{Mor}(F(x), F(y))$ is a homomorphism of abelian groups for all $x, y \in \text{Ob}(\mathcal{A})$.

In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

00ZZ Lemma 12.3.2. Let $\mathcal{A}$ be a preadditive category. Let x be an object of $\mathcal{A}$. The following are equivalent

(1) x is an initial object,
(2) x is a final object, and
(3) $\text{id}_x = 0$ in $\text{Mor}_\mathcal{A}(x, x)$.

Furthermore, if such an object 0 exists, then a morphism $\alpha : x \to y$ factors through 0 if and only if $\alpha = 0$.

Proof. First assume that x is either (1) initial or (2) final. In both cases, it follows that $\text{Mor}(x, x)$ is a trivial abelian group containing id_x, thus $\text{id}_x = 0$ in $\text{Mor}(x, x)$, which shows that each of (1) and (2) implies (3).

Now assume that $\text{id}_x = 0$ in $\text{Mor}(x, x)$. Let y be an arbitrary object of $\mathcal{A}$ and let $f \in \text{Mor}(x, y)$. Denote $C : \text{Mor}(x, x) \times \text{Mor}(x, y) \to \text{Mor}(x, y)$ the composition map. Then $f = C(0, f)$ and since C is bilinear we have $C(0, f) = 0$. Thus $f = 0$. Hence x is initial in $\mathcal{A}$. A similar argument for $f \in \text{Mor}(y, x)$ can be used to show that x is also final. Thus (3) implies both (1) and (2). $\square$

0100 Definition 12.3.3. In a preadditive category $\mathcal{A}$ we call zero object, and we denote it 0 any final and initial object as in Lemma 12.3.2 above.

0101 Lemma 12.3.4. Let $\mathcal{A}$ be a preadditive category. Let $x, y \in \mathrm{Ob}(\mathcal{A})$. If the product $x \times y$ exists, then so does the coproduct $x \amalg y$. If the coproduct $x \amalg y$ exists, then so does the product $x \times y$. In this case also $x \amalg y \cong x \times y$.

Proof. Suppose that $z = x \times y$ with projections $p : z \to x$ and $q : z \to y$. Denote $i : x \to z$ the morphism corresponding to $(1, 0)$. Denote $j : y \to z$ the morphism corresponding to $(0, 1)$. Thus we have the commutative diagram

$$\begin{tikzcd} x \arrow[rr, "1"] \arrow[dr, "i"] & & x \\ & z \arrow[ur, "p"] \arrow[dr, "q"] & \\ y \arrow[ur, "j"] \arrow[rr, "1"] & & y \end{tikzcd}$$

where the diagonal compositions are zero. It follows that $i \circ p + j \circ q : z \to z$ is the identity since it is a morphism which upon composing with p gives p and upon composing with q gives q. Suppose given morphisms $a : x \to w$ and $b : y \to w$. Then we can form the map $a \circ p + b \circ q : z \to w$. In this way we get a bijection $\mathrm{Mor}(z, w) = \mathrm{Mor}(x, w) \times \mathrm{Mor}(y, w)$ which show that $z = x \amalg y$.

We leave it to the reader to construct the morphisms p, q given a coproduct $x \amalg y$ instead of a product. $\square$

0102 Definition 12.3.5. Given a pair of objects x, y in a preadditive category $\mathcal{A}$, the direct sum $x \oplus y$ of x and y is the direct product $x \times y$ endowed with the morphisms i, j, p, q as in Lemma 12.3.4 above.

0103 Remark 12.3.6. Note that the proof of Lemma 12.3.4 shows that given p and q the morphisms i, j are uniquely determined by the rules $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$. Moreover, we automatically have $i \circ p + j \circ q = \mathrm{id}_{x \oplus y}$. Similarly, given i, j the morphisms p and q are uniquely determined. Finally, given objects x, y, z and morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, then z is the direct sum of x and y with the four morphisms equal to i, j, p, q.

0105 Lemma 12.3.7. Let $\mathcal{A}$, $\mathcal{B}$ be preadditive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.

Proof. Suppose F is additive. A direct sum z of x and y is characterized by having morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, according to Remark 12.3.6. Clearly $F(x), F(y), F(z)$ and the morphisms $F(i), F(j), F(p), F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. Hence, F transforms direct sums to direct sums.

To see that F transforms zero to zero, use the characterization (3) of the zero object in Lemma 12.3.2. $\square$

0104 Definition 12.3.8. A category $\mathcal{A}$ is called additive if it is preadditive and finite products exist, in other words it has a zero object and direct sums.

Namely the empty product is a finite product and if it exists, then it is a final object.

0106 Definition 12.3.9. Let $\mathcal{A}$ be a preadditive category. Let $f : x \to y$ be a morphism.

(1) A kernel of f is a morphism $i : z \to x$ such that (a) $f \circ i = 0$ and (b) for any $i' : z' \to x$ such that $f \circ i' = 0$ there exists a unique morphism $g : z' \to z$ such that $i' = i \circ g$.
(2) If the kernel of f exists, then we denote this $\text{Ker}(f) \to x$.
(3) A cokernel of f is a morphism $p : y \to z$ such that (a) $p \circ f = 0$ and (b) for any $p' : y \to z'$ such that $p' \circ f = 0$ there exists a unique morphism $g : z \to z'$ such that $p' = g \circ p$.
(4) If a cokernel of f exists we denote this $y \to \text{Coker}(f)$.
(5) If a kernel of f exists, then a coimage of f is a cokernel for the morphism $\text{Ker}(f) \to x$.
(6) If a kernel and coimage exist then we denote this $x \to \text{Coim}(f)$.
(7) If a cokernel of f exists, then the image of f is a kernel of the morphism $y \to \text{Coker}(f)$.
(8) If a cokernel and image of f exist then we denote this $\text{Im}(f) \to y$.

In the above definition, we have spoken of "the kernel" and "the cokernel", tacitly using their uniqueness up to unique isomorphism. This follows from the Yoneda lemma (Categories, Section 4.3) because the kernel of $f : x \to y$ represents the functor sending an object z to the set $\text{Ker}(\text{Mor}_\mathcal{A}(z, x) \to \text{Mor}_\mathcal{A}(z, y))$. The case of cokernels is dual.

We first relate the direct sum to kernels as follows.

09QG Lemma 12.3.10. Let $\mathcal{C}$ be a preadditive category. Let $x \oplus y$ with morphisms i, j, p, q as in Lemma 12.3.4 be a direct sum in $\mathcal{C}$. Then $i : x \to x \oplus y$ is a kernel of $q : x \oplus y \to y$. Dually, p is a cokernel for j.

Proof. Let $f : z' \to x \oplus y$ be a morphism such that $q \circ f = 0$. We have to show that there exists a unique morphism $g : z' \to x$ such that $f = i \circ g$. Since $i \circ p + j \circ q$ is the identity on $x \oplus y$ we see that

$$f = (i \circ p + j \circ q) \circ f = i \circ p \circ f$$

and hence $g = p \circ f$ works. Uniqueness holds because $p \circ i$ is the identity on x. The proof of the second statement is dual. □

0E43 Lemma 12.3.11. Let $\mathcal{C}$ be a preadditive category. Let $f : x \to y$ be a morphism in $\mathcal{C}$.

(1) If a kernel of f exists, then this kernel is a monomorphism.
(2) If a cokernel of f exists, then this cokernel is an epimorphism.
(3) If a kernel and coimage of f exist, then the coimage is an epimorphism.
(4) If a cokernel and image of f exist, then the image is a monomorphism.

Proof. Part (1) follows easily from the uniqueness required in the definition of a kernel. The proof of (2) is dual. Part (3) follows from (2), since the coimage is a cokernel. Similarly, (4) follows from (1). □

0107 Lemma 12.3.12. Let $f : x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then f can be factored uniquely as $x \to \text{Coim}(f) \to \text{Im}(f) \to y$.

Proof. There is a canonical morphism $\mathrm{Coim}(f) \to y$ because $\mathrm{Ker}(f) \to x \to y$ is zero. The composition $\mathrm{Coim}(f) \to y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $x \to y \to \mathrm{Coker}(f)$ which is zero (the uniqueness follows from Lemma 12.3.11 (3)). Hence $\mathrm{Coim}(f) \to y$ factors uniquely through $\mathrm{Im}(f) \to y$, which gives us the desired map. □

0108 Example 12.3.13. Let k be a field. Consider the category of filtered vector spaces over k. (See Definition 12.19.1.) Consider the filtered vector spaces (V, F) and (W, F) with $V = W = k$ and

$$F^iV = \begin{cases} V & \text{if} \quad i < 0 \\ 0 & \text{if} \quad i \geq 0 \end{cases} \text{ and } F^iW = \begin{cases} W & \text{if} \quad i \leq 0 \\ 0 & \text{if} \quad i > 0 \end{cases}$$

The map $f : V \to W$ corresponding to id_k on the underlying vector spaces has trivial kernel and cokernel but is not an isomorphism. Note also that $\mathrm{Coim}(f) = V$ and $\mathrm{Im}(f) = W$. This means that the category of filtered vector spaces over k is not abelian.

12.4. Karoubian categories

09SF Skip this section on a first reading.

09SG Definition 12.4.1. Let $\mathcal{C}$ be a preadditive category. We say $\mathcal{C}$ is Karoubian if every idempotent endomorphism of an object of $\mathcal{C}$ has a kernel.

The dual notion would be that every idempotent endomorphism of an object has a cokernel. However, in view of the (dual of the) following lemma that would be an equivalent notion.

09SH Lemma 12.4.2. Let $\mathcal{C}$ be a preadditive category. The following are equivalent

(1) $\mathcal{C}$ is Karoubian,
(2) every idempotent endomorphism of an object of $\mathcal{C}$ has a cokernel, and
(3) given an idempotent endomorphism $p : z \to z$ of $\mathcal{C}$ there exists a direct sum decomposition $z = x \oplus y$ such that p corresponds to the projection onto y.

Proof. Assume (1) and let $p : z \to z$ be as in (3). Let $x = \mathrm{Ker}(p)$ and $y = \mathrm{Ker}(1-p)$. There are maps $x \to z$ and $y \to z$. Since $(1 - p)p = 0$ we see that $p : z \to z$ factors through y, hence we obtain a morphism $z \to y$. Similarly we obtain a morphism $z \to x$. We omit the verification that these four morphisms induce an isomorphism $x = y \oplus z$ as in Remark 12.3.6. Thus (1) ⇒ (3). The implication (2) ⇒ (3) is dual. Finally, condition (3) implies (1) and (2) by Lemma 12.3.10. □

05QV Lemma 12.4.3. Let $\mathcal{D}$ be a preadditive category.

(1) If $\mathcal{D}$ has countable products and kernels of maps which have a right inverse, then $\mathcal{D}$ is Karoubian.
(2) If $\mathcal{D}$ has countable coproducts and cokernels of maps which have a left inverse, then $\mathcal{D}$ is Karoubian.

Proof. Let X be an object of $\mathcal{D}$ and let $e : X \to X$ be an idempotent. The functor

$$W \longmapsto \mathrm{Ker}(\mathrm{Mor}_{\mathcal{D}}(W, X) \xrightarrow{e} \mathrm{Mor}_{\mathcal{D}}(W, X))$$

if representable if and only if e has a kernel. Note that for any abelian group A and idempotent endomorphism $e : A \to A$ we have

$$\mathrm{Ker}(e : A \to A) = \mathrm{Ker}(\Phi : \prod\nolimits_{n \in \mathbf{N}} A \to \prod\nolimits_{n \in \mathbf{N}} A)$$

where

$$\Phi(a_1, a_2, a_3, \ldots) = (ea_1 + (1 - e)a_2, ea_2 + (1 - e)a_3, \ldots)$$

Moreover, Φ has the right inverse

$$\Psi(a_1, a_2, a_3, \ldots) = (a_1, (1 - e)a_1 + ea_2, (1 - e)a_2 + ea_3, \ldots).$$

Hence (1) holds. The proof of (2) is dual (using the dual definition of a Karoubian category, namely condition (2) of Lemma 12.4.2). □

12.5. Abelian categories

00ZX An abelian category is a category satisfying just enough axioms so the snake lemma holds. An axiom (that is sometimes forgotten) is that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ of Lemma 12.3.12 is always an isomorphism. Example 12.3.13 shows that it is necessary.

0109 Definition 12.5.1. A category $\mathcal{A}$ is abelian if it is additive, if all kernels and cokernels exist, and if the natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for all morphisms f of $\mathcal{A}$.

010A Lemma 12.5.2. Let $\mathcal{A}$ be a preadditive category. The additions on sets of morphisms make $\mathcal{A}^{opp}$ into a preadditive category. Furthermore, $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, and $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian.

Proof. The first statement is straightforward. To see that $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, recall that additivity can be characterized by the existence of a zero object and direct sums, which are both preserved when passing to the opposite category. Finally, to see that $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian, observes that kernels, cokernels, images and coimages in $\mathcal{A}^{opp}$ correspond to cokernels, kernels, coimages and images in $\mathcal{A}$, respectively. □

010B Definition 12.5.3. Let $f : x \to y$ be a morphism in an abelian category.

(1) We say f is injective if $\mathrm{Ker}(f) = 0$.
(2) We say f is surjective if $\mathrm{Coker}(f) = 0$.

If $x \to y$ is injective, then we say that x is a subobject of y and we use the notation $x \subset y$. If $x \to y$ is surjective, then we say that y is a quotient of x.

010C Lemma 12.5.4. Let $f : x \to y$ be a morphism in an abelian category $\mathcal{A}$. Then

(1) f is injective if and only if f is a monomorphism, and
(2) f is surjective if and only if f is an epimorphism.

Proof. Proof of (1). Recall that $\mathrm{Ker}(f)$ is an object representing the functor sending z to $\mathrm{Ker}(\mathrm{Mor}_\mathcal{A}(z, x) \to \mathrm{Mor}_\mathcal{A}(z, y))$, see Definition 12.3.9. Thus $\mathrm{Ker}(f)$ is 0 if and only if $\mathrm{Mor}_\mathcal{A}(z, x) \to \mathrm{Mor}_\mathcal{A}(z, y)$ is injective for all z if and only if f is a monomorphism. The proof of (2) is similar. □

In an abelian category, if $x \subset y$ is a subobject, then we denote

$$y/x = \mathrm{Coker}(x \to y).$$

010D Lemma 12.5.5. Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.

Proof. To show that finite limits exist it suffices to show that finite products and equalizers exist, see Categories, Lemma 4.18.4. Finite products exist by definition and the equalizer of $a, b : x \to y$ is the kernel of $a - b$. The argument for finite colimits is similar but dual to this. $\square$

05PJ Example 12.5.6. Let $\mathcal{A}$ be an abelian category. Pushouts and fibre products in $\mathcal{A}$ have the following simple descriptions:

(1) If $a : x \to y$, $b : z \to y$ are morphisms in $\mathcal{A}$, then we have the fibre product: $x \times_y z = \mathrm{Ker}((a, -b) : x \oplus z \to y)$.
(2) If $a : y \to x$, $b : y \to z$ are morphisms in $\mathcal{A}$, then we have the pushout: $x \amalg_y z = \mathrm{Coker}((a, -b) : y \to x \oplus z)$.

010E Definition 12.5.7. Let $\mathcal{A}$ be an additive category. Consider a sequence of morphisms

$$\ldots \to x \to y \to z \to \ldots \quad \text{or} \quad x_1 \to x_2 \to \ldots \to x_n$$

in $\mathcal{A}$. We say such a sequence is a complex if the composition of any two consecutive (drawn) arrows is zero. If $\mathcal{A}$ is abelian then we say a complex of the first type above is exact at y if $\mathrm{Im}(x \to y) = \mathrm{Ker}(y \to z)$ and we say a complex of the second kind is exact at x_i where $1 < i < n$ if $\mathrm{Im}(x_{i-1} \to x_i) = \mathrm{Ker}(x_i \to x_{i+1})$. We a sequence as above is exact or is an exact sequence or is an exact complex if it is a complex and exact at every object (in the first case) or exact at x_i for all $1 < i < n$ (in the second case). There are variants of these notions for sequences of the form

$$\ldots \to x_{-3} \to x_{-2} \to x_{-1} \quad \text{and} \quad x_1 \to x_2 \to x_3 \to \ldots$$

A short exact sequence is an exact complex of the form

$$0 \to A \to B \to C \to 0.$$

In the following lemma we assume the reader knows what it means for a sequence of abelian groups to be exact.

05AA Lemma 12.5.8. Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a complex of $\mathcal{A}$.

(1) $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if
$$0 \to \mathrm{Hom}_{\mathcal{A}}(M_3, N) \to \mathrm{Hom}_{\mathcal{A}}(M_2, N) \to \mathrm{Hom}_{\mathcal{A}}(M_1, N)$$
is an exact sequence of abelian groups for all objects N of $\mathcal{A}$, and
(2) $0 \to M_1 \to M_2 \to M_3$ is exact if and only if
$$0 \to \mathrm{Hom}_{\mathcal{A}}(N, M_1) \to \mathrm{Hom}_{\mathcal{A}}(N, M_2) \to \mathrm{Hom}_{\mathcal{A}}(N, M_3)$$
is an exact sequence of abelian groups for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: See Algebra, Lemma 10.10.1. $\square$

010F Definition 12.5.9. Let $\mathcal{A}$ be an abelian category. Let $i : A \to B$ and $q : B \to C$ be morphisms of $\mathcal{A}$ such that $0 \to A \to B \to C \to 0$ is a short exact sequence. We say the short exact sequence is split if there exist morphisms $j : C \to B$ and $p : B \to A$ such that (B, i, j, p, q) is the direct sum of A and C.

010G Lemma 12.5.10. Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to B \to C \to 0$ be a short exact sequence.

(1) Given a morphism $s : C \to B$ left inverse to $B \to C$, there exists a unique $\pi : B \to A$ such that (s, π) splits the short exact sequence as in Definition 12.5.9.
(2) Given a morphism $\pi : B \to A$ right inverse to $A \to B$, there exists a unique $s : C \to B$ such that (s, π) splits the short exact sequence as in Definition 12.5.9.

Proof. Omitted. □

08N2 Lemma 12.5.11. Let $\mathcal{A}$ be an abelian category. Let

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x \arrow[r, "k"] & z \end{tikzcd}$$

be a commutative diagram.

(1) The diagram is cartesian if and only if
$$0 \to w \xrightarrow{(g,f)} x \oplus y \xrightarrow{(k,-h)} z$$
is exact.
(2) The diagram is cocartesian if and only if
$$w \xrightarrow{(g,-f)} x \oplus y \xrightarrow{(k,h)} z \to 0$$
is exact.

Proof. Let $u = (g, f) : w \to x \oplus y$ and $v = (k, -h) : x \oplus y \to z$. Let $p : x \oplus y \to x$ and $q : x \oplus y \to y$ be the canonical projections. Let $i : \mathrm{Ker}(v) \to x \oplus y$ be the canonical injection. By Example 12.5.6, the diagram is cartesian if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $f \circ r = q \circ i$ and $g \circ r = p \circ i$. The sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z$ is exact if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $u \circ r = i$. But given $r : \mathrm{Ker}(v) \to w$, we have $f \circ r = q \circ i$ and $g \circ r = p \circ i$ if and only if $q \circ u \circ r = f \circ r = q \circ i$ and $p \circ u \circ r = g \circ r = p \circ i$, hence if and only if $u \circ r = i$. This proves (1), and then (2) follows by duality. □

08N3 Lemma 12.5.12. Let $\mathcal{A}$ be an abelian category. Let

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x \arrow[r, "k"] & z \end{tikzcd}$$

be a commutative diagram.

(1) If the diagram is cartesian, then the morphism $\mathrm{Ker}(f) \to \mathrm{Ker}(k)$ induced by g is an isomorphism.
(2) If the diagram is cocartesian, then the morphism $\mathrm{Coker}(f) \to \mathrm{Coker}(k)$ induced by h is an isomorphism.

Proof. Suppose the diagram is cartesian. Let $e : \mathrm{Ker}(f) \to \mathrm{Ker}(k)$ be induced by g. Let $i : \mathrm{Ker}(f) \to w$ and $j : \mathrm{Ker}(k) \to x$ be the canonical injections. There exists $t : \mathrm{Ker}(k) \to w$ with $f \circ t = 0$ and $g \circ t = j$. Hence, there exists $u : \mathrm{Ker}(k) \to \mathrm{Ker}(f)$ with $i \circ u = t$. It follows $g \circ i \circ u \circ e = g \circ t \circ e = j \circ e = g \circ i$ and $f \circ i \circ u \circ e = 0 = f \circ i$, hence

$i \circ u \circ e = i$. Since i is a monomorphism this implies $u \circ e = \mathrm{id}_{\mathrm{Ker}(f)}$. Furthermore, we have $j \circ e \circ u = g \circ i \circ u = g \circ t = j$. Since j is a monomorphism this implies $e \circ u = \mathrm{id}_{\mathrm{Ker}(k)}$. This proves (1). Now, (2) follows by duality. □

08N4 Lemma 12.5.13. Let $\mathcal{A}$ be an abelian category. Let

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ x \arrow[r, "k"] & z \end{tikzcd}$$

be a commutative diagram.

(1) If the diagram is cartesian and k is an epimorphism, then the diagram is cocartesian and f is an epimorphism.
(2) If the diagram is cocartesian and g is a monomorphism, then the diagram is cartesian and h is a monomorphism.

Proof. Suppose the diagram is cartesian and k is an epimorphism. Let $u = (g, f) : w \to x \oplus y$ and let $v = (k, -h) : x \oplus y \to z$. As k is an epimorphism, v is an epimorphism, too. Therefore and by Lemma 12.5.11, the sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z \to 0$ is exact. Thus, the diagram is cocartesian by Lemma 12.5.11. Finally, f is an epimorphism by Lemma 12.5.12 and Lemma 12.5.4. This proves (1), and (2) follows by duality. □

05PK Lemma 12.5.14. Let $\mathcal{A}$ be an abelian category.

(1) If $x \to y$ is surjective, then for every $z \to y$ the projection $x \times_y z \to z$ is surjective.
(2) If $x \to y$ is injective, then for every $x \to z$ the morphism $z \to z \amalg_x y$ is injective.

Proof. Immediately from Lemma 12.5.4 and Lemma 12.5.13. □

08N5 Lemma 12.5.15. Let $\mathcal{A}$ be an abelian category. Let $f : x \to y$ and $g : y \to z$ be morphisms with $g \circ f = 0$. Then, the following statements are equivalent:

(1) The sequence $x \xrightarrow{f} y \xrightarrow{g} z$ is exact.
(2) For every $h : w \to y$ with $g \circ h = 0$ there exist an object v, an epimorphism $k : v \to w$ and a morphism $l : v \to x$ with $h \circ k = f \circ l$.

Proof. Let $i : \mathrm{Ker}(g) \to y$ be the canonical injection. Let $p : x \to \mathrm{Coim}(f)$ be the canonical projection. Let $j : \mathrm{Im}(f) \to \mathrm{Ker}(g)$ be the canonical injection.

Suppose (1) holds. Let $h : w \to y$ with $g \circ h = 0$. There exists $c : w \to \mathrm{Ker}(g)$ with $i \circ c = h$. Let $v = x \times_{\mathrm{Ker}(g)} w$ with canonical projections $k : v \to w$ and $l : v \to x$, so that $c \circ k = j \circ p \circ l$. Then, $h \circ k = i \circ c \circ k = i \circ j \circ p \circ l = f \circ l$. As $j \circ p$ is an epimorphism by hypothesis, k is an epimorphism by Lemma 12.5.13. This implies (2).

Suppose (2) holds. Then, $g \circ i = 0$. So, there are an object w, an epimorphism $k : w \to \mathrm{Ker}(g)$ and a morphism $l : w \to x$ with $f \circ l = i \circ k$. It follows $i \circ j \circ p \circ l = f \circ l = i \circ k$. Since i is a monomorphism we see that $j \circ p \circ l = k$ is an epimorphism. So, j is an epimorphisms and thus an isomorphism. This implies (1). □

08N6 **Lemma 12.5.16.** Let $\mathcal{A}$ be an abelian category. Let

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\alpha"] & y \arrow[r, "g"] \arrow[d, "\beta"] & z \arrow[d, "\gamma"] \\ u \arrow[r, "k"] & v \arrow[r, "l"] & w \end{tikzcd}$$

be a commutative diagram.

(1) If the first row is exact and k is a monomorphism, then the induced sequence $\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ is exact.
(2) If the second row is exact and g is an epimorphism, then the induced sequence $\mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma)$ is exact.

Proof. Suppose the first row is exact and k is a monomorphism. Let $a : \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta)$ and $b : \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ be the induced morphisms. Let $h : \mathrm{Ker}(\alpha) \to x$, $i : \mathrm{Ker}(\beta) \to y$ and $j : \mathrm{Ker}(\gamma) \to z$ be the canonical injections. As j is a monomorphism we have $b \circ a = 0$. Let $c : s \to \mathrm{Ker}(\beta)$ with $b \circ c = 0$. Then, $g \circ i \circ c = j \circ b \circ c = 0$. By Lemma 12.5.15 there are an object t, an epimorphism $d : t \to s$ and a morphism $e : t \to x$ with $i \circ c \circ d = f \circ e$. Then, $k \circ \alpha \circ e = \beta \circ f \circ e = \beta \circ i \circ c \circ d = 0$. As k is a monomorphism we get $\alpha \circ e = 0$. So, there exists $m : t \to \mathrm{Ker}(\alpha)$ with $h \circ m = e$. It follows $i \circ a \circ m = f \circ h \circ m = f \circ e = i \circ c \circ d$. As i is a monomorphism we get $a \circ m = c \circ d$. Thus, Lemma 12.5.15 implies (1), and then (2) follows by duality. $\square$

010H **Lemma 12.5.17.** Let $\mathcal{A}$ be an abelian category. Let

$$\begin{tikzcd} & x \arrow[r, "f"] \arrow[d, "\alpha"] & y \arrow[r, "g"] \arrow[d, "\beta"] & z \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & u \arrow[r, "k"] & v \arrow[r, "l"] & w & \end{tikzcd}$$

be a commutative diagram with exact rows.

(1) There exists a unique morphism $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ such that the diagram

$$\begin{tikzcd} y \arrow[d, "\beta"] & y \times_z \mathrm{Ker}(\gamma) \arrow[l, "\pi'"'] \arrow[r, "\pi"] & \mathrm{Ker}(\gamma) \arrow[d, "\delta"] \\ v \arrow[r, "\iota'"] & \mathrm{Coker}(\alpha) \amalg_u v & \mathrm{Coker}(\alpha) \arrow[l, "\iota"'] \end{tikzcd}$$

commutes, where π and π' are the canonical projections and ι and ι' are the canonical coprojections.
(2) The induced sequence

$$\mathrm{Ker}(\alpha) \xrightarrow{f'} \mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha) \xrightarrow{k'} \mathrm{Coker}(\beta) \xrightarrow{l'} \mathrm{Coker}(\gamma)$$

is exact. If f is injective then so is f', and if l is surjective then so is l'.

Proof. As π is an epimorphism and ι is a monomorphism by Lemma 12.5.13, uniqueness of δ is clear. Let $p = y \times_z \mathrm{Ker}(\gamma)$ and $q = \mathrm{Coker}(\alpha) \amalg_u v$. Let

$h : \mathrm{Ker}(\beta) \to y$, $i : \mathrm{Ker}(\gamma) \to z$ and $j : \mathrm{Ker}(\pi) \to p$ be the canonical injections. Let $\pi'' : u \to \mathrm{Coker}(\alpha)$ be the canonical projection. Keeping in mind Lemma 12.5.13 we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\pi) \arrow[r, "j"] & p \arrow[r, "\pi"] \arrow[d, "\pi'"'] & \mathrm{Ker}(\gamma) \arrow[r] \arrow[d, "i"'] & 0 \\
 & x \arrow[r, "f"] \arrow[d, "\alpha"'] & y \arrow[r, "g"] \arrow[d, "\beta"'] & z \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & u \arrow[r, "k"] \arrow[d, "\pi''"'] & v \arrow[r, "l"] \arrow[d, "\iota'"'] & w & \\
0 \arrow[r] & \mathrm{Coker}(\alpha) \arrow[r, "\iota"] & q & &
\end{tikzcd}$$

As $l \circ \beta \circ \pi' = \gamma \circ i \circ \pi = 0$ and as the third row of the diagram above is exact, there is an $a : p \to u$ with $k \circ a = \beta \circ \pi'$. As the upper right quadrangle of the diagram above is cartesian, Lemma 12.5.12 yields an epimorphism $b : x \to \mathrm{Ker}(\pi)$ with $\pi' \circ j \circ b = f$. It follows $k \circ a \circ j \circ b = \beta \circ \pi' \circ j \circ b = \beta \circ f = k \circ \alpha$. As k is a monomorphism this implies $a \circ j \circ b = \alpha$. It follows $\pi'' \circ a \circ j \circ b = \pi'' \circ \alpha = 0$. As b is an epimorphism this implies $\pi'' \circ a \circ j = 0$. Therefore, as the top row of the diagram above is exact, there exists $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ with $\delta \circ \pi = \pi'' \circ a$. It follows $\iota \circ \delta \circ \pi = \iota \circ \pi'' \circ a = \iota' \circ k \circ a = \iota' \circ \beta \circ \pi'$ as desired.

As the upper right quadrangle in the diagram above is cartesian there is a $c : \mathrm{Ker}(\beta) \to p$ with $\pi' \circ c = h$ and $\pi \circ c = g'$. It follows $\iota \circ \delta \circ g' = \iota \circ \delta \circ \pi \circ c = \iota' \circ \beta \circ \pi' \circ c = \iota' \circ \beta \circ h = 0$. As ι is a monomorphism this implies $\delta \circ g' = 0$.

Next, let $d : r \to \mathrm{Ker}(\gamma)$ with $\delta \circ d = 0$. Applying Lemma 12.5.15 to the exact sequence $p \xrightarrow{\pi} \mathrm{Ker}(\gamma) \to 0$ and d yields an object s, an epimorphism $m : s \to r$ and a morphism $n : s \to p$ with $\pi \circ n = d \circ m$. As $\pi'' \circ a \circ n = \delta \circ d \circ m = 0$, applying Lemma 12.5.15 to the exact sequence $x \xrightarrow{\alpha} u \xrightarrow{p} \mathrm{Coker}(\alpha)$ and $a \circ n$ yields an object t, an epimorphism $\varepsilon : t \to s$ and a morphism $\zeta : t \to x$ with $a \circ n \circ \varepsilon = \alpha \circ \zeta$. It holds $\beta \circ \pi' \circ n \circ \varepsilon = k \circ \alpha \circ \zeta = \beta \circ f \circ \zeta$. Let $\eta = \pi' \circ n \circ \varepsilon - f \circ \zeta : t \to y$. Then, $\beta \circ \eta = 0$. It follows that there is a $\vartheta : t \to \mathrm{Ker}(\beta)$ with $\eta = h \circ \vartheta$. It holds $i \circ g' \circ \vartheta = g \circ h \circ \vartheta = g \circ \pi' \circ n \circ \varepsilon - g \circ f \circ \zeta = i \circ \pi \circ n \circ \varepsilon = i \circ d \circ m \circ \varepsilon$. As i is a monomorphism we get $g' \circ \vartheta = d \circ m \circ \varepsilon$. Thus, as $m \circ \varepsilon$ is an epimorphism, Lemma 12.5.15 implies that $\mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha)$ is exact. Then, the claim follows by Lemma 12.5.16 and duality. $\square$

08N7 **Lemma 12.5.18.** Let $\mathcal{A}$ be an abelian category. Let

$$\begin{tikzcd}
 & & x \arrow[rr] \arrow[dl] \arrow[dd, "\alpha"] & & y \arrow[rr] \arrow[dl] \arrow[dd, "\beta"] & & z \arrow[rr] \arrow[dl] \arrow[dd, "\gamma"] & & 0 \\
 & x' \arrow[rr] \arrow[dd, "\alpha'"] & & y' \arrow[rr] \arrow[dd, "\beta'"] & & z' \arrow[rr] \arrow[dd, "\gamma'"] & & 0 & \\
 & 0 \arrow[r] & u \arrow[rr] \arrow[dl] & & v \arrow[rr] \arrow[dl] & & w \arrow[dl] & & \\
0 \arrow[r] & u' \arrow[rr] & & v' \arrow[rr] & & w' & & &
\end{tikzcd}$$

be a commutative diagram with exact rows. Then, the induced diagram

$$\begin{tikzcd}
\mathrm{Ker}(\alpha) \arrow[r] \arrow[d] & \mathrm{Ker}(\beta) \arrow[r] \arrow[d] & \mathrm{Ker}(\gamma) \arrow[r, "\delta"] \arrow[d] & \mathrm{Coker}(\alpha) \arrow[r] \arrow[d] & \mathrm{Coker}(\beta) \arrow[r] \arrow[d] & \mathrm{Coker}(\gamma) \arrow[d] \\
\mathrm{Ker}(\alpha') \arrow[r] & \mathrm{Ker}(\beta') \arrow[r] & \mathrm{Ker}(\gamma') \arrow[r, "\delta'"] & \mathrm{Coker}(\alpha') \arrow[r] & \mathrm{Coker}(\beta') \arrow[r] & \mathrm{Coker}(\gamma')
\end{tikzcd}$$

commutes.

Proof. Omitted. □

05QA **Lemma 12.5.19.** Let $\mathcal{A}$ be an abelian category. Let

$$\begin{tikzcd}
w \arrow[r] \arrow[d, "\alpha"] & x \arrow[r] \arrow[d, "\beta"] & y \arrow[r] \arrow[d, "\gamma"] & z \arrow[d, "\delta"] \\
w' \arrow[r] & x' \arrow[r] & y' \arrow[r] & z'
\end{tikzcd}$$

be a commutative diagram with exact rows.

(1) If α, γ are surjective and δ is injective, then β is surjective.
(2) If β, δ are injective and α is surjective, then γ is injective.

Proof. Assume α, γ are surjective and δ is injective. We may replace w' by $\mathrm{Im}(w' \to x')$, i.e., we may assume that $w' \to x'$ is injective. We may replace z by $\mathrm{Im}(y \to z)$, i.e., we may assume that $y \to z$ is surjective. Then we may apply Lemma 12.5.17 to

$$\begin{tikzcd}
 & \mathrm{Ker}(y \to z) \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(y' \to z') \arrow[r] & y' \arrow[r] & z' &
\end{tikzcd}$$

to conclude that $\mathrm{Ker}(y \to z) \to \mathrm{Ker}(y' \to z')$ is surjective. Finally, we apply Lemma 12.5.17 to

$$\begin{tikzcd}
 & w \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & \mathrm{Ker}(y \to z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & w' \arrow[r] & x' \arrow[r] & \mathrm{Ker}(y' \to z') &
\end{tikzcd}$$

to conclude that $x \to x'$ is surjective. This proves (1). The proof of (2) is dual to this. □

05QB **Lemma 12.5.20.** Let $\mathcal{A}$ be an abelian category. Let [ES52, Lemma 4.5 page 16]

$$\begin{tikzcd} v \arrow[r] \arrow[d, "\alpha"] & w \arrow[r] \arrow[d, "\beta"] & x \arrow[r] \arrow[d, "\gamma"] & y \arrow[r] \arrow[d, "\delta"] & z \arrow[d, "\epsilon"] \\ v' \arrow[r] & w' \arrow[r] & x' \arrow[r] & y' \arrow[r] & z' \end{tikzcd}$$

be a commutative diagram with exact rows. If β, δ are isomorphisms, ϵ is injective, and α is surjective then γ is an isomorphism.

Proof. Immediate consequence of Lemma 12.5.19. $\square$

12.6. Extensions

010I

010J **Definition 12.6.1.** Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. An *extension E of B by A* is a short exact sequence

$$0 \to A \to E \to B \to 0.$$

A morphism of extensions between two extensions $0 \to A \to E \to B \to 0$ and $0 \to A \to F \to B \to 0$ means a morphism $f : E \to F$ in $\mathcal{A}$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A \arrow[r] & F \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

commutative. Thus, the extensions of B by A form a category.

By abuse of language we often omit mention of the morphisms $A \to E$ and $E \to B$, although they are definitively part of the structure of an extension.

010K **Definition 12.6.2.** Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. The set of isomorphism classes of extensions of B by A is denoted

$$\mathrm{Ext}_{\mathcal{A}}(B, A).$$

This is called the *Ext-group*.

This definition works, because by our conventions $\mathrm{Ob}(\mathcal{A})$ is a set, and hence $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set. In any of the cases of "big" abelian categories listed in Categories, Remark 4.2.2 one can check by hand that $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set as well. Also, we will see later that this is always the case when $\mathcal{A}$ has either enough projectives or enough injectives. Insert future reference here.

Actually we can turn $\mathrm{Ext}_{\mathcal{A}}(-, -)$ into a functor

$$\mathcal{A} \times \mathcal{A}^{opp} \longrightarrow \textit{Sets}, \quad (A, B) \longmapsto \mathrm{Ext}_{\mathcal{A}}(B, A)$$

as follows:

(1) Given a morphism $B' \to B$ and an extension E of B by A we define $E' = E \times_B B'$ so that we have the following commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

The extension E' is called the pullback of E via $B' \to B$.

(2) Given a morphism $A \to A'$ and an extension E of B by A we define $E' = A' \amalg_A E$ so that we have the following commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

The extension E' is called the pushout of E via $A \to A'$.

To see that this defines a functor as indicated above there are several things to verify. First of all functoriality in the variable B requires that $(E \times_B B') \times_{B'} B'' = E \times_B B''$ which is a general property of fibre products. Dually one deals with functoriality in the variable A. Finally, given $A \to A'$ and $B' \to B$ we have to show that

$$A' \amalg_A (E \times_B B') \cong (A' \amalg_A E) \times_B B'$$

as extensions of B' by A'. Recall that $A' \amalg_A E$ is a quotient of $A' \oplus E$. Thus the right hand side is a quotient of $A' \oplus E \times_B B'$, and it is straightforward to see that the kernel is exactly what you need in order to get the left hand side.

Note that if E_1 and E_2 are extensions of B by A, then $E_1 \oplus E_2$ is an extension of $B \oplus B$ by $A \oplus A$. We push out by the sum map $A \oplus A \to A$ and we pull back by the diagonal map $B \to B \oplus B$ to get an extension $E_1 + E_2$ of B by A.

$$\begin{tikzcd} 0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "\Sigma"'] & E_1 \oplus E_2 \arrow[r] \arrow[d] & B \oplus B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \oplus B \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u] & E_1 + E_2 \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "\Delta"] & 0 \end{tikzcd}$$

The extension $E_1 + E_2$ is called the Baer sum of the given extensions.

010L Lemma 12.6.3. The construction $(E_1, E_2) \mapsto E_1 + E_2$ above defines a commutative group law on $\mathrm{Ext}_{\mathcal{A}}(B, A)$ which is functorial in both variables.

Proof. Omitted. ☐

05E2 Lemma 12.6.4. Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{A}$.

(1) There is a canonical six term exact sequence of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_3, N) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_2, N) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M_1, N) \arrow[dll] \\ & \mathrm{Ext}_{\mathcal{A}}(M_3, N) \arrow[r] & \mathrm{Ext}_{\mathcal{A}}(M_2, N) \arrow[r] & \mathrm{Ext}_{\mathcal{A}}(M_1, N) \end{tikzcd}$$

for all objects N of $\mathcal{A}$, and

(2) there is a canonical six term exact sequence of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_\mathcal{A}(N, M_1) \arrow[r] & \mathrm{Hom}_\mathcal{A}(N, M_2) \arrow[r] & \mathrm{Hom}_\mathcal{A}(N, M_3) \arrow[dll] \\ & \mathrm{Ext}_\mathcal{A}(N, M_1) \arrow[r] & \mathrm{Ext}_\mathcal{A}(N, M_2) \arrow[r] & \mathrm{Ext}_\mathcal{A}(N, M_3) \end{tikzcd}$$

for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: The boundary maps are defined using either the pushout or pullback of the given short exact sequence. □

12.7. Additive functors

010M First a completely silly lemma characterizing additive functors between additive categories.

0DLP Lemma 12.7.1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. The following are equivalent

1. F is additive,
2. $F(A) \oplus F(B) \to F(A \oplus B)$ is an isomorphism for all $A, B \in \mathcal{A}$, and
3. $F(A \oplus B) \to F(A) \oplus F(B)$ is an isomorphism for all $A, B \in \mathcal{A}$.

Proof. Additive functors commute with direct sums by Lemma 12.3.7 hence (1) implies (2) and (3). On the other hand (2) and (3) are equivalent because the composition $F(A) \oplus F(B) \to F(A \oplus B) \to F(A) \oplus F(B)$ is the identity map. Assume (2) and (3) hold. Let $f, g : A \to B$ be maps. Then $f + g$ is equal to the composition

$$A \to A \oplus A \xrightarrow{\mathrm{diag}(f,g)} B \oplus B \to B$$

Apply the functor F and consider the following diagram

$$\begin{tikzcd} F(A) \arrow[r] \arrow[dr] & F(A \oplus A) \arrow[rr, "F(\mathrm{diag}(f,g))"'] & & F(B \oplus B) \arrow[r] \arrow[d] & F(B) \\ & F(A) \oplus F(A) \arrow[u] \arrow[rr, "\mathrm{diag}(F(f),F(g))"] & & F(B) \oplus F(B) \arrow[ur] & \end{tikzcd}$$

We claim this is commutative. For the middle square we can verify it separately for each of the four induced maps $F(A) \to F(B)$ where it follows from the fact that F is a functor (in other words this square commutes even if F does not satisfy any properties beyond being a functor). For the triangle on the left, we use that $F(A \oplus A) \to F(A) \oplus F(A)$ is an isomorphism to see that it suffice to check after composition with this map and this check is trivial. Dually for the other triangle. Thus going around the bottom is equal to $F(f + g)$ and we conclude. □

Recall that we defined, in Categories, Definition 4.23.1 the notion of a “right exact”, “left exact” and “exact” functor in the setting of a functor between categories that have finite (co)limits. Thus this applies in particular to functors between abelian categories.

010N Lemma 12.7.2. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

1. If F is either left or right exact, then it is additive.
2. F is left exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact.

(3) F is right exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact.
(4) F is exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.

Proof. If F is left exact, i.e., F commutes with finite limits, then F sends products to products, hence F preserved direct sums, hence F is additive by Lemma 12.7.1. On the other hand, suppose that for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact. Let A, B be two objects. Then we have a short exact sequence

$$0 \to A \to A \oplus B \to B \to 0$$

see for example Lemma 12.3.10. By assumption, the lower row in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F(A) \arrow[r] \arrow[d] & F(A) \oplus F(B) \arrow[r] \arrow[d] & F(B) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F(A) \arrow[r] & F(A \oplus B) \arrow[r] & F(B) & \end{tikzcd}$$

is exact. Hence by the snake lemma (Lemma 12.5.17) we conclude that $F(A) \oplus F(B) \to F(A \oplus B)$ is an isomorphism. Hence F is additive in this case as well. Thus for the rest of the proof we may assume F is additive.

Denote $f : B \to C$ a map from B to C. Exactness of $0 \to A \to B \to C$ just means that $A = \text{Ker}(f)$. Clearly the kernel of f is the equalizer of the two maps f and 0 from B to C. Hence if F commutes with limits, then $F(\text{Ker}(f)) = \text{Ker}(F(f))$ which exactly means that $0 \to F(A) \to F(B) \to F(C)$ is exact.

Conversely, suppose that F is additive and transforms any short exact sequence $0 \to A \to B \to C \to 0$ into an exact sequence $0 \to F(A) \to F(B) \to F(C)$. Because it is additive it commutes with direct sums and hence finite products in $\mathcal{A}$. To show it commutes with finite limits it therefore suffices to show that it commutes with equalizers. But equalizers in an abelian category are the same as the kernel of the difference map, hence it suffices to show that F commutes with taking kernels. Let $f : A \to B$ be a morphism. Factor f as $A \to I \to B$ with $f' : A \to I$ surjective and $i : I \to B$ injective. (This is possible by the definition of an abelian category.) Then it is clear that $\text{Ker}(f) = \text{Ker}(f')$. Also $0 \to \text{Ker}(f') \to A \to I \to 0$ and $0 \to I \to B \to B/I \to 0$ are short exact. By the condition imposed on F we see that $0 \to F(\text{Ker}(f')) \to F(A) \to F(I)$ and $0 \to F(I) \to F(B) \to F(B/I)$ are exact. Hence it is also the case that $F(\text{Ker}(f'))$ is the kernel of the map $F(A) \to F(B)$, and we win.

The proof of (3) is similar to the proof of (2). Statement (4) is a combination of (2) and (3). □

010O Lemma 12.7.3. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. For every pair of objects A, B of $\mathcal{A}$ the functor F induces an abelian group homomorphism

$$\text{Ext}_{\mathcal{A}}(B, A) \longrightarrow \text{Ext}_{\mathcal{B}}(F(B), F(A))$$

which maps the extension E to $F(E)$.

Proof. Omitted. □

The following lemma is used in the proof that the category of abelian sheaves on a site is abelian, where the functor b is sheafification.

03A3 **Lemma 12.7.4.** Let $a : \mathcal{A} \to \mathcal{B}$ and $b : \mathcal{B} \to \mathcal{A}$ be functors. Assume that

(1) $\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b,
(2) $\mathcal{B}$ is abelian and b is left exact, and
(3) $ba \cong \mathrm{id}_{\mathcal{A}}$.

Then $\mathcal{A}$ is abelian.

Proof. As $\mathcal{B}$ is abelian we see that all finite limits and colimits exist in $\mathcal{B}$ by Lemma 12.5.5. Since b is a left adjoint we see that b is also right exact and hence exact, see Categories, Lemma 4.24.6. Let $\varphi : B_1 \to B_2$ be a morphism of $\mathcal{B}$. In particular, if $K = \mathrm{Ker}(B_1 \to B_2)$, then K is the equalizer of 0 and φ and hence bK is the equalizer of 0 and $b\varphi$, hence bK is the kernel of $b\varphi$. Similarly, if $Q = \mathrm{Coker}(B_1 \to B_2)$, then Q is the coequalizer of 0 and φ and hence bQ is the coequalizer of 0 and $b\varphi$, hence bQ is the cokernel of $b\varphi$. Thus we see that every morphism of the form $b\varphi$ in $\mathcal{A}$ has a kernel and a cokernel. However, since $ba \cong \mathrm{id}$ we see that every morphism of $\mathcal{A}$ is of this form, and we conclude that kernels and cokernels exist in $\mathcal{A}$. In fact, the argument shows that if $\psi : A_1 \to A_2$ is a morphism then

$$\mathrm{Ker}(\psi) = b\,\mathrm{Ker}(a\psi), \quad \text{and} \quad \mathrm{Coker}(\psi) = b\,\mathrm{Coker}(a\psi).$$

Now we still have to show that $\mathrm{Coim}(\psi) = \mathrm{Im}(\psi)$. We do this as follows. First note that since $\mathcal{A}$ has kernels and cokernels it has all finite limits and colimits (see proof of Lemma 12.5.5). Hence we see by Categories, Lemma 4.24.6 that a is left exact and hence transforms kernels (=equalizers) into kernels.

$$\begin{aligned}
\mathrm{Coim}(\psi) &= \mathrm{Coker}(\mathrm{Ker}(\psi) \to A_1) && \text{by definition} \\
&= b\,\mathrm{Coker}(a(\mathrm{Ker}(\psi) \to A_1)) && \text{by formula above} \\
&= b\,\mathrm{Coker}(\mathrm{Ker}(a\psi) \to aA_1)) && a \text{ preserves kernels} \\
&= b\,\mathrm{Coim}(a\psi) && \text{by definition} \\
&= b\,\mathrm{Im}(a\psi) && \mathcal{B} \text{ is abelian} \\
&= b\,\mathrm{Ker}(aA_2 \to \mathrm{Coker}(a\psi)) && \text{by definition} \\
&= \mathrm{Ker}(baA_2 \to b\,\mathrm{Coker}(a\psi)) && b \text{ preserves kernels} \\
&= \mathrm{Ker}(A_2 \to b\,\mathrm{Coker}(a\psi)) && ba = \mathrm{id}_{\mathcal{A}} \\
&= \mathrm{Ker}(A_2 \to \mathrm{Coker}(\psi)) && \text{by formula above} \\
&= \mathrm{Im}(\psi) && \text{by definition}
\end{aligned}$$

Thus the lemma holds. □

12.8. Localization

05QC In this section we note how Gabriel-Zisman localization interacts with the additive structure on a category.

05QD **Lemma 12.8.1.** Let $\mathcal{C}$ be a preadditive category. Let S be a left or right multiplicative system. There exists a canonical preadditive structure on $S^{-1}\mathcal{C}$ such that the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.

Proof. We will prove this in the case S is a left multiplicative system. The case where S is a right multiplicative system is dual. Suppose that X, Y are objects of $\mathcal{C}$ and that $\alpha, \beta : X \to Y$ are morphisms in $S^{-1}\mathcal{C}$. According to Categories, Lemma 4.27.5 we may represent these by pairs $s^{-1}f, s^{-1}g$ with common denominator s. In this case we define $\alpha + \beta$ to be the equivalence class of $s^{-1}(f + g)$. In the rest of the proof we show that this is well defined and that composition is bilinear. Once this is done it is clear that Q is an additive functor.

Let us show construction above is well defined. An abstract way of saying this is that filtered colimits of abelian groups agree with filtered colimits of sets and to use Categories, Equation (4.27.7.1). We can work this out in a bit more detail as follows. Say $s : Y \to Y_1$ and $f, g : X \to Y_1$. Suppose we have a second representation of α, β as $(s')^{-1}f', (s')^{-1}g'$ with $s' : Y \to Y_2$ and $f', g' : X \to Y_2$. By Categories, Remark 4.27.7 we can find a morphism $s_3 : Y \to Y_3$ and morphisms $a_1 : Y_1 \to Y_3$, $a_2 : Y_2 \to Y_3$ such that $a_1 \circ s = s_3 = a_2 \circ s'$ and also $a_1 \circ f = a_2 \circ f'$ and $a_1 \circ g = a_2 \circ g'$. Hence we see that $s^{-1}(f + g)$ is equivalent to

$$\begin{aligned} s_3^{-1}(a_1 \circ (f + g)) &= s_3^{-1}(a_1 \circ f + a_1 \circ g) \\ &= s_3^{-1}(a_2 \circ f' + a_2 \circ g') \\ &= s_3^{-1}(a_2 \circ (f' + g')) \end{aligned}$$

which is equivalent to $(s')^{-1}(f' + g')$.

Fix $s : Y \to Y'$ and $f, g : X \to Y'$ with $\alpha = s^{-1}f$ and $\beta = s^{-1}g$ as morphisms $X \to Y$ in $S^{-1}\mathcal{C}$. To show that composition is bilinear first consider the case of a morphism $\gamma : Y \to Z$ in $S^{-1}\mathcal{C}$. Say $\gamma = t^{-1}h$ for some $h : Y \to Z'$ and $t : Z \to Z'$ in S. Using LMS2 we choose morphisms $a : Y' \to Z''$ and $t' : Z' \to Z''$ in S such that $a \circ s = t' \circ h$. Picture

$$\begin{tikzcd} & & Z \arrow[d, "t"] \\ & Y \arrow[r, "h"] \arrow[d, "s"] & Z' \arrow[d, "t'"] \\ X \arrow[r, "{f,g}"] & Y' \arrow[r, "a"] & Z'' \end{tikzcd}$$

Then $\gamma \circ \alpha = (t' \circ t)^{-1}(a \circ f)$ and $\gamma \circ \beta = (t' \circ t)^{-1}(a \circ g)$. Hence we see that $\gamma \circ (\alpha + \beta)$ is represented by $(t' \circ t)^{-1}(a \circ (f + g)) = (t' \circ t)^{-1}(a \circ f + a \circ g)$ which represents $\gamma \circ \alpha + \gamma \circ \beta$.

Finally, assume that $\delta : W \to X$ is another morphism of $S^{-1}\mathcal{C}$. Say $\delta = r^{-1}i$ for some $i : W \to X'$ and $r : X \to X'$ in S. We claim that we can find a morphism $s' : Y' \to Y''$ in S and morphisms $a'', b'' : X' \to Y''$ such that the following diagram commutes

$$\begin{tikzcd} & & Y \arrow[d, "s"] \\ & X \arrow[r, "{f,g,f+g}"] \arrow[d, "r"] & Y' \arrow[d, "s'"] \\ W \arrow[r, "i"] & X' \arrow[r, "{a'',b'',a''+b''}"] & Y'' \end{tikzcd}$$

Namely, using LMS2 we can first choose $s_1 : Y' \to Y_1$, $s_2 : Y' \to Y_2$ in S and $a : X' \to Y_1$, $b : X' \to Y_2$ such that $a \circ r = s_1 \circ f$ and $b \circ r = s_2 \circ f$. Then using that the category Y'/S is filtered (see Categories, Remark 4.27.7), we can find a $s' : Y' \to Y''$ and morphisms $a' : Y_1 \to Y''$, $b' : Y_2 \to Y''$ such that $s' = a' \circ s_1$ and $s' = b' \circ s_2$. Setting $a'' = a' \circ a$ and $b'' = b' \circ b$ works. At this point we see that the compositions $\alpha \circ \delta$ and $\beta \circ \delta$ are represented by $(s' \circ s)^{-1}(a'' \circ i)$ and $(s' \circ s)^{-1}(b'' \circ i)$. Hence $\alpha \circ \delta + \beta \circ \delta$ is represented by $(s' \circ s)^{-1}(a'' \circ i + b'' \circ i) = (s' \circ s)^{-1}((a'' + b'') \circ i)$ which by the diagram again is a representative of $(\alpha + \beta) \circ \delta$. $\square$

05QE Lemma 12.8.2. Let $\mathcal{C}$ be an additive category. Let S be a left or right multiplicative system. Then $S^{-1}\mathcal{C}$ is an additive category and the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.

Proof. By Lemma 12.8.1 we see that $S^{-1}\mathcal{C}$ is preadditive and that Q is additive. Recall that the functor Q commutes with finite colimits (resp. finite limits), see Categories, Lemmas 4.27.9 and 4.27.17. We conclude that $S^{-1}\mathcal{C}$ has a zero object and direct sums, see Lemmas 12.3.2 and 12.3.4. $\square$

The following lemma describes the “kernel” of the localization functor in case we invert a multiplicative system.

05QF Lemma 12.8.3. Let $\mathcal{C}$ be an additive category. Let S be a multiplicative system. Let X be an object of $\mathcal{C}$. The following are equivalent

(1) $Q(X) = 0$ in $S^{-1}\mathcal{C}$,
(2) there exists $Y \in \mathrm{Ob}(\mathcal{C})$ such that $0 : X \to Y$ is an element of S, and
(3) there exists $Z \in \mathrm{Ob}(\mathcal{C})$ such that $0 : Z \to X$ is an element of S.

Proof. If (2) holds we see that $0 = Q(0) : Q(X) \to Q(Y)$ is an isomorphism. In the additive category $S^{-1}\mathcal{C}$ this implies that $Q(X) = 0$. Hence (2) $\Rightarrow$ (1). Similarly, (3) $\Rightarrow$ (1). Suppose that $Q(X) = 0$. This implies that the morphism $f : 0 \to X$ is transformed into an isomorphism in $S^{-1}\mathcal{C}$. Hence by Categories, Lemma 4.27.21 there exists a morphism $g : Z \to 0$ such that $fg \in S$. This proves (1) $\Rightarrow$ (3). Similarly, (1) $\Rightarrow$ (2). $\square$

05QG Lemma 12.8.4. Let $\mathcal{A}$ be an abelian category.

(1) If S is a left multiplicative system, then the category $S^{-1}\mathcal{A}$ has cokernels and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ commutes with them.
(2) If S is a right multiplicative system, then the category $S^{-1}\mathcal{A}$ has kernels and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ commutes with them.
(3) If S is a multiplicative system, then the category $S^{-1}\mathcal{A}$ is abelian and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ is exact.

Proof. Assume S is a left multiplicative system. Let $a : X \to Y$ be a morphism of $S^{-1}\mathcal{A}$. Then $a = s^{-1}f$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Since $Q(s)$ is an isomorphism we see that the existence of $\mathrm{Coker}(a : X \to Y)$ is equivalent to the existence of $\mathrm{Coker}(Q(f) : X \to Y')$. Since $\mathrm{Coker}(Q(f))$ is the coequalizer of 0 and $Q(f)$ we see that $\mathrm{Coker}(Q(f))$ is represented by $Q(\mathrm{Coker}(f))$ by Categories, Lemma 4.27.9. This proves (1).

Part (2) is dual to part (1).

If S is a multiplicative system, then S is both a left and a right multiplicative system. Thus we see that $S^{-1}\mathcal{A}$ has kernels and cokernels and Q commutes with kernels and cokernels. To finish the proof of (3) we have to show that Coim = Im in $S^{-1}\mathcal{A}$. Again using that any arrow in $S^{-1}\mathcal{A}$ is isomorphic to an arrow $Q(f)$ we see that the result follows from the result for $\mathcal{A}$. □

12.9. Jordan-Hölder

0FCD The Jordan-Hölder lemma is Lemma 12.9.7. First we state some definitions.

0FCE **Definition 12.9.1.** Let $\mathcal{A}$ be an abelian category. An object A of $\mathcal{A}$ is said to be *simple* if it is nonzero and the only subobjects of A are 0 and A.

0FCF **Definition 12.9.2.** Let $\mathcal{A}$ be an abelian category.

1. We say an object A of $\mathcal{A}$ is *Artinian* if and only if it satisfies the descending chain condition for subobjects.
2. We say $\mathcal{A}$ is *Artinian* if every object of $\mathcal{A}$ is Artinian.

0FCG **Definition 12.9.3.** Let $\mathcal{A}$ be an abelian category.

1. We say an object A of $\mathcal{A}$ is *Noetherian* if and only if it satisfies the ascending chain condition for subobjects.
2. We say $\mathcal{A}$ is *Noetherian* if every object of $\mathcal{A}$ is Noetherian.

0FCH **Lemma 12.9.4.** Let $\mathcal{A}$ be an abelian category. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence of $\mathcal{A}$. Then A_2 is Artinian if and only if A_1 and A_3 are Artinian.

Proof. Omitted. □

0FCI **Lemma 12.9.5.** Let $\mathcal{A}$ be an abelian category. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence of $\mathcal{A}$. Then A_2 is Noetherian if and only if A_1 and A_3 are Noetherian.

Proof. Omitted. □

0FCJ **Lemma 12.9.6.** Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$. The following are equivalent

1. A is Artinian and Noetherian, and
2. there exists a filtration $0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = A$ by subobjects such that A_i/A_{i-1} is simple for $i = 1, \ldots, n$.

Proof. Assume (1). If A is zero, then (2) holds. If A is not zero, then there exists a smallest nonzero object $A_1 \subset A$ by the Artinian property. Of course A_1 is simple. If $A_1 = A$, then we are done. If not, then we can find $A_1 \subset A_2 \subset A$ minimal with $A_2 \neq A_1$. Then A_2/A_1 is simple. Continuing in this way, we can find a sequence $0 \subset A_1 \subset A_2 \subset \ldots$ of subobjects of A such that A_i/A_{i-1} is simple. Since A is Noetherian, we conclude that the process stops. Hence (2) follows.

Assume (2). We will prove (1) by induction on n. If $n = 1$, then A is simple and clearly Noetherian and Artinian. If the result holds for $n-1$, then we use the short exact sequence $0 \to A_{n-1} \to A_n \to A_n/A_{n-1} \to 0$ and Lemmas 12.9.4 and 12.9.5 to conclude for n. □

0FCK **Lemma 12.9.7** (Jordan-Hölder)**.** Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ satisfying the equivalent conditions of Lemma 12.9.6. Given two filtrations

$$0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = A \quad \text{and} \quad 0 \subset B_1 \subset B_2 \subset \ldots \subset B_m = A$$

with $S_i = A_i/A_{i-1}$ and $T_j = B_j/B_{j-1}$ simple objects we have $n = m$ and there exists a permutation σ of $\{1, \ldots, n\}$ such that $S_i \cong T_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$.

Proof. Let j be the smallest index such that $A_1 \subset B_j$. Then the map $S_1 = A_1 \to B_j/B_{j-1} = T_j$ is an isomorphism. Moreover, the object $A/A_1 = A_n/A_1 = B_m/A_1$ has the two filtrations

$$0 \subset A_2/A_1 \subset A_3/A_1 \subset \ldots \subset A_n/A_1$$

and

$$0 \subset (B_1 + A_1)/A_1 \subset \ldots \subset (B_{j-1} + A_1)/A_1 = B_j/A_1 \subset B_{j+1}/A_1 \subset \ldots \subset B_m/A_1$$

We conclude by induction. □

12.10. Serre subcategories

02MN In [Ser53, Chapter I, Section 1] a notion of a "class" of abelian groups is defined. This notion has been extended to abelian categories by many authors (in slightly different ways). We will use the following variant which is virtually identical to Serre's original definition.

02MO Definition 12.10.1. Let $\mathcal{A}$ be an abelian category. [Ser53, Condition (I) on page 259]

(1) A Serre subcategory of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence[1]
$$A \to B \to C$$
with $A, C \in \mathrm{Ob}(\mathcal{C})$, then also $B \in \mathrm{Ob}(\mathcal{C})$.

(2) A weak Serre subcategory of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence
$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$
with A_0, A_1, A_3, A_4 in $\mathcal{C}$, then also A_2 in $\mathcal{C}$.

In some references the second notion is called a "thick" subcategory and in other references the first notion is called a "thick" subcategory. However, it seems that the notion of a Serre subcategory is universally accepted to be the one defined above. Note that in both cases the category $\mathcal{C}$ is abelian and that the inclusion functor $\mathcal{C} \to \mathcal{A}$ is a fully faithful exact functor. Let's characterize these types of subcategories in more detail.

02MP Lemma 12.10.2. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a Serre subcategory if and only if the following conditions are satisfied:

(1) $0 \in \mathrm{Ob}(\mathcal{C})$,
(2) $\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,
(3) any subobject or quotient of an object of $\mathcal{C}$ is an object of $\mathcal{C}$,
(4) if $A \in \mathrm{Ob}(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in \mathrm{Ob}(\mathcal{C})$.

Moreover, a Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

0754 Lemma 12.10.3. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a weak Serre subcategory if and only if the following conditions are satisfied:

[1] By Definition 12.5.7 this means $\mathrm{Im}(A \to B) = \mathrm{Ker}(B \to C)$.

(1) $0 \in \mathrm{Ob}(\mathcal{C})$,
(2) $\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,
(3) kernels and cokernels in $\mathcal{A}$ of morphisms between objects of $\mathcal{C}$ are in $\mathcal{C}$,
(4) if $A \in \mathrm{Ob}(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in \mathrm{Ob}(\mathcal{C})$.

Moreover, a weak Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

02MQ Lemma 12.10.4. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ forms a Serre subcategory of $\mathcal{A}$.

Proof. Omitted. □

02MR Definition 12.10.5. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ is called the kernel of the functor F, and is sometimes denoted $\mathrm{Ker}(F)$.

Any Serre subcategory of an abelian category is the kernel of an exact functor. In Examples, Section 110.76 we discuss this for Serre's original example of torsion groups.

02MS Lemma 12.10.6. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor

$$F : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

which is essentially surjective and whose kernel is $\mathcal{C}$. The category $\mathcal{A}/\mathcal{C}$ and the functor F are characterized by the following universal property: For any exact functor $G : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \subset \mathrm{Ker}(G)$ there exists a factorization $G = H \circ F$ for a unique exact functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$.

Proof. Consider the set of arrows of $\mathcal{A}$ defined by the following formula

$$S = \{f \in \mathrm{Arrows}(\mathcal{A}) \mid \mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathrm{Ob}(\mathcal{C})\}.$$

We claim that S is a multiplicative system. To prove this we have to check MS1, MS2, MS3, see Categories, Definition 4.27.1.

It is clear that identities are elements of S. Suppose that $f : A \to B$ and $g : B \to C$ are elements of S. There are exact sequences

$$\begin{gathered} 0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(gf) \to \mathrm{Ker}(g) \\ \mathrm{Coker}(f) \to \mathrm{Coker}(gf) \to \mathrm{Coker}(g) \to 0 \end{gathered}$$

Hence it follows that $gf \in S$. This proves MS1. (In fact, a similar argument will show that S is a saturated multiplicative system, see Categories, Definition 4.27.20.)

Consider a solid diagram

$$\begin{tikzcd} A \arrow[r, "g"'] \arrow[d, "t"'] & B \arrow[d, dotted, "s"] \\ C \arrow[r, dotted, "f"] & C \amalg_A B \end{tikzcd}$$

with $t \in S$. Set $W = C \amalg_A B = \mathrm{Coker}((t, -g) : A \to C \oplus B)$. Then $\mathrm{Ker}(t) \to \mathrm{Ker}(s)$ is surjective and $\mathrm{Coker}(t) \to \mathrm{Coker}(s)$ is an isomorphism. Hence s is an element of S. This proves LMS2 and the proof of RMS2 is dual.

Finally, consider morphisms $f, g : B \to C$ and a morphism $s : A \to B$ in S such that $f \circ s = g \circ s$. This means that $(f - g) \circ s = 0$. In turn this means that $I = \mathrm{Im}(f - g) \subset C$ is a quotient of $\mathrm{Coker}(s)$ hence an object of $\mathcal{C}$. Thus $t : C \to C' = C/I$ is an element of S such that $t \circ (f - g) = 0$, i.e., such that $t \circ f = t \circ g$. This proves LMS3 and the proof of RMS3 is dual.

Having proved that S is a multiplicative system we set $\mathcal{A}/\mathcal{C} = S^{-1}\mathcal{A}$, and we set F equal to the localization functor Q. By Lemma 12.8.4 the category $\mathcal{A}/\mathcal{C}$ is abelian and F is exact. If X is in the kernel of $F = Q$, then by Lemma 12.8.3 we see that $0 : X \to Z$ is an element of S and hence X is an object of $\mathcal{C}$, i.e., the kernel of F is $\mathcal{C}$. Finally, if G is as in the statement of the lemma, then G turns every element of S into an isomorphism. Hence we obtain the functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ from the universal property of localization, see Categories, Lemma 4.27.8. We still have to show the functor H is exact. To do this it suffices to show that H commutes with taking kernels and cokernels, see Lemma 12.7.2. Let $A \to B$ be a morphism in $\mathcal{A}/\mathcal{C}$. We may represent $A \to B$ as fs^{-1} where $s : A' \to A$ is in S and $f : A' \to B$ an arbitrary morphism of $\mathcal{A}$. Since $F = Q$ maps s to an isomorphism in the quotient category $\mathcal{A}/\mathcal{C}$, it suffices to show that H commutes with taking kernels and cokernels of morphisms $f : A \to B$ of $\mathcal{A}$. But here we have $H(f) = G(f)$ and the result follows from the fact that G is exact. $\square$

06XK Lemma 12.10.7. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory contained in the kernel of F. Then $\mathcal{C} = \mathrm{Ker}(F)$ if and only if the induced functor $\overline{F} : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ (Lemma 12.10.6) is faithful.

Proof. We will use the results of Lemma 12.10.6 without further mention. The "only if" direction is true because the kernel of $\overline{F}$ is zero by construction. Namely, if $f : X \to Y$ is a morphism in $\mathcal{A}/\mathcal{C}$ such that $\overline{F}(f) = 0$, then $\overline{F}(\mathrm{Im}(f)) = \mathrm{Im}(\overline{F}(f)) = 0$, hence $\mathrm{Im}(f) = 0$ by the assumption on the kernel of F. Thus $f = 0$.

For the "if" direction, let X be an object of $\mathcal{A}$ such that $F(X) = 0$. Then $\overline{F}(\mathrm{id}_X) = \mathrm{id}_{\overline{F}(X)} = 0$, thus $\mathrm{id}_X = 0$ in $\mathcal{A}/\mathcal{C}$ by faithfulness of $\overline{F}$. Hence $X = 0$ in $\mathcal{A}/\mathcal{C}$, that is $X \in \mathrm{Ob}(\mathcal{C})$. $\square$

12.11. K-groups

02MT A tiny bit about K_0 of an abelian category.

02MU Definition 12.11.1. Let $\mathcal{A}$ be an abelian category. We denote $K_0(\mathcal{A})$ the zeroth K-group of $\mathcal{A}$. It is the abelian group constructed as follows. Take the free abelian group on the objects on $\mathcal{A}$ and for every short exact sequence $0 \to A \to B \to C \to 0$ impose the relation $[B] - [A] - [C] = 0$.

Another way to say this is that there is a presentation

$$\bigoplus_{A \to B \to C \text{ ses}} \mathbf{Z}[A \to B \to C] \longrightarrow \bigoplus_{A \in \mathrm{Ob}(\mathcal{A})} \mathbf{Z}[A] \longrightarrow K_0(\mathcal{A}) \longrightarrow 0$$

with $[A \to B \to C] \mapsto [B] - [A] - [C]$ of $K_0(\mathcal{A})$. The short exact sequence $0 \to 0 \to 0 \to 0 \to 0$ leads to the relation $[0] = 0$ in $K_0(\mathcal{A})$. There are no set-theoretical issues as all of our categories are "small" if not mentioned otherwise. Some examples of K-groups for categories of modules over rings where computed in Algebra, Section 10.55.

02MV Lemma 12.11.2. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then F induces a homomorphism of K-groups $K_0(F) : K_0(\mathcal{A}) \to K_0(\mathcal{B})$ by simply setting $K_0(F)([A]) = [F(A)]$.

Proof. Proves itself. □

Suppose we are given an object M of an abelian category $\mathcal{A}$ and a complex of the form

02MW (12.11.2.1)
$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

In this situation we define

$$H^0(M, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi), \quad \text{and} \quad H^1(M, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

02MX Lemma 12.11.3. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory and set $\mathcal{B} = \mathcal{A}/\mathcal{C}$.

(1) The exact functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$ induce an exact sequence
$$K_0(\mathcal{C}) \to K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to 0$$
of K-groups, and

(2) the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is equal to the collection of elements of the form
$$[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$$
where (M, φ, ψ) is a complex as in (12.11.2.1) with the property that it becomes exact in $\mathcal{B}$; in other words that $H^0(M, \varphi, \psi)$ and $H^1(M, \varphi, \psi)$ are objects of $\mathcal{C}$.

Proof. Proof of (1). It is clear that $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ is surjective and that the composition $K_0(\mathcal{C}) \to K_0(\mathcal{A}) \to K_0(\mathcal{B})$ is zero. Let $x \in K_0(\mathcal{A})$ be an element mapping to zero in $K_0(\mathcal{B})$. We can write $x = [A] - [A']$ with A, A' in $\mathcal{A}$ (fun exercise). Denote B, B' the corresponding objects of $\mathcal{B}$. The fact that x maps to zero in $K_0(\mathcal{B})$ means that there exists a finite set $I = I^+ \amalg I^-$, for each $i \in I$ a short exact sequence
$$0 \to B_i \to B'_i \to B''_i \to 0$$
in $\mathcal{B}$ such that we have
$$[B] - [B'] = \sum\nolimits_{i \in I^+}([B'_i] - [B_i] - [B''_i]) - \sum\nolimits_{i \in I^-}([B'_i] - [B_i] - [B''_i])$$
in the free abelian group on isomorphism classes of objects of $\mathcal{B}$. We can rewrite this as
$$[B] + \sum\nolimits_{i \in I^+}([B_i] + [B''_i]) + \sum\nolimits_{i \in I^-}[B'_i] = [B'] + \sum\nolimits_{i \in I^-}([B_i] + [B''_i]) + \sum\nolimits_{i \in I^+}[B'_i].$$
Since the right and left hand side should contain the same isomorphism classes of objects of $\mathcal{B}$ counted with multiplicity, this means there should be a bijection
$$\tau : \{B\} \amalg \{B_i, B''_i; i \in I^+\} \amalg \{B'_i; i \in I^-\} \longrightarrow \{B'\} \amalg \{B_i, B''_i; i \in I^-\} \amalg \{B'_i; i \in I^+\}$$

such that N and $\tau(N)$ are isomorphic in $\mathcal{B}$. The proof of Lemmas 12.10.6 and 12.8.4 show that we choose for $i \in I$ a short exact sequence

$$0 \to A_i \to A_i' \to A_i'' \to 0$$

in $\mathcal{A}$ such that B_i, B_i', B_i'' are isomorphic to the images of A_i, A_i', A_i'' in $\mathcal{B}$. This implies that the corresponding bijection

$$\tau : \{A\} \amalg \{A_i, A_i''; i \in I^+\} \amalg \{A_i'; i \in I^-\} \longrightarrow \{A'\} \amalg \{A_i, A_i''; i \in I^-\} \amalg \{A_i'; i \in I^+\}$$

satisfies the property that M and $\tau(M)$ are objects of $\mathcal{A}$ which become isomorphic in $\mathcal{B}$. This means $[M] - [\tau(M)]$ is in the image of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$. Namely, the isomorphism in $\mathcal{B}$ is given by a diagram $M \leftarrow M' \to \tau(M)$ in $\mathcal{A}$ where both $M' \to M$ and $M' \to \tau(M)$ have kernel and cokernel in $\mathcal{C}$. Working backwards we conclude that $x = [A] - [A']$ is in the image of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ and the proof of part (1) is complete.

Proof of (2). The proof is similar to the proof of (1) but slightly more bookkeeping is involved. First we remark that any class of the type $[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$ is zero in $K_0(\mathcal{A})$ by the following calculation

$$\begin{aligned} 0 &= [M] - [M] \\ &= [\mathrm{Ker}(\varphi)] + [\mathrm{Im}(\varphi)] - [\mathrm{Ker}(\psi)] - [\mathrm{Im}(\psi)] \\ &= [\mathrm{Ker}(\varphi)/\mathrm{Im}(\psi)] - [\mathrm{Ker}(\psi)/\mathrm{Im}(\varphi)] \\ &= [H^1(M, \varphi, \psi)] - [H^0(M, \varphi, \psi)] \end{aligned}$$

as desired. Hence it suffices to show that any element in the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is of this form.

Any element x in $K_0(\mathcal{C})$ can be represented as the difference $x = [P] - [Q]$ of two objects of $\mathcal{C}$ (fun exercise). Suppose that this element maps to zero in $K_0(\mathcal{A})$. This means that there exist

(1) a finite set $I = I^+ \amalg I^-$,
(2) for $i \in I$ a short exact sequence $0 \to A_i \to B_i \to C_i \to 0$ in $\mathcal{A}$

such that

$$[P] - [Q] = \sum\nolimits_{i \in I^+} ([B_i] - [A_i] - [C_i]) - \sum\nolimits_{i \in I^-} ([B_i] - [A_i] - [C_i])$$

in the free abelian group on the objects of $\mathcal{A}$. We can rewrite this as

$$[P] + \sum\nolimits_{i \in I^+} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^-} [B_i] = [Q] + \sum\nolimits_{i \in I^-} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^+} [B_i].$$

Since the right and left hand side should contain the same objects of $\mathcal{A}$ counted with multiplicity, this means there should be a bijection τ between the terms which occur above. Set

$$T^+ = \{p\} \amalg \{a, c\} \times I^+ \amalg \{b\} \times I^-$$

and

$$T^- = \{q\} \amalg \{a, c\} \times I^- \amalg \{b\} \times I^+.$$

Set $T = T^+ \amalg T^- = \{p, q\} \amalg \{a, b, c\} \times I$. For $t \in T$ define

$$O(t) = \begin{cases} P & \text{if} \quad t = p \\ Q & \text{if} \quad t = q \\ A_i & \text{if} \quad t = (a, i) \\ B_i & \text{if} \quad t = (b, i) \\ C_i & \text{if} \quad t = (c, i) \end{cases}$$

Hence we can view $\tau : T^+ \to T^-$ as a bijection such that $O(t) = O(\tau(t))$ for all $t \in T^+$. Let $t_0^- = \tau(p)$ and let $t_0^+ \in T^+$ be the unique element such that $\tau(t_0^+) = q$. Consider the object

$$M^+ = \bigoplus\nolimits_{t \in T^+} O(t)$$

By using τ we see that it is equal to the object

$$M^- = \bigoplus\nolimits_{t \in T^-} O(t)$$

Consider the map

$$\varphi : M^+ \longrightarrow M^-$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^+$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^- and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^-$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^-. The map is zero on the summands corresponding to p and (c, i), $i \in I^+$. Similarly, consider the map

$$\psi : M^- \longrightarrow M^+$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^-$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^+ and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^+$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^+. The map is zero on the summands corresponding to q and (c, i), $i \in I^-$.

Note that the kernel of φ is equal to the direct sum of the summand P and the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b, i)) = B_i$, $i \in I^-$. The image of ψ is equal to the direct sum of the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b, i)) = B_i$, $i \in I^-$. In other words we see that

$$P \cong \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

In exactly the same way we see that

$$Q \cong \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

Since as we remarked above the existence of the bijection τ shows that $M^+ = M^-$ we see that the lemma follows. $\square$

12.12. Cohomological delta-functors

010P

010Q Definition 12.12.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A cohomological δ-functor or simply a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ is given by the following data:

(1) a collection $F^n : \mathcal{A} \to \mathcal{B}$, $n \geq 0$ of additive functors, and
(2) for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ a collection $\delta_{A \to B \to C} : F^n(C) \to F^{n+1}(A)$, $n \geq 0$ of morphisms of $\mathcal{B}$.

These data are assumed to satisfy the following axioms

(1) for every short exact sequence as above the sequence
$$\begin{tikzcd} 0 \arrow[r] & F^0(A) \arrow[r] & F^0(B) \arrow[r] & F^0(C) \arrow[dll, "\delta_{A\to B\to C}"] \\ & F^1(A) \arrow[r] & F^1(B) \arrow[r] & F^1(C) \arrow[dll, "\delta_{A\to B\to C}"] \\ & F^2(A) \arrow[r] & F^2(B) \arrow[r] & \ldots \end{tikzcd}$$
is exact, and

(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences of $\mathcal{A}$ the diagrams
$$\begin{tikzcd} F^n(C) \arrow[r, "\delta_{A\to B\to C}"'] \arrow[d] & F^{n+1}(A) \arrow[d] \\ F^n(C') \arrow[r, "\delta_{A'\to B'\to C'}"] & F^{n+1}(A') \end{tikzcd}$$
are commutative.

Note that this in particular implies that F^0 is left exact.

010R Definition 12.12.2. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let (F^n, δ_F) and (G^n, δ_G) be δ-functors from $\mathcal{A}$ to $\mathcal{B}$. A morphism of δ-functors from F to G is a collection of transformation of functors $t^n : F^n \to G^n$, $n \geq 0$ such that for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ the diagrams
$$\begin{tikzcd} F^n(C) \arrow[r, "\delta_{F,A\to B\to C}"'] \arrow[d, "t^n"'] & F^{n+1}(A) \arrow[d, "t^{n+1}"] \\ G^n(C) \arrow[r, "\delta_{G,A\to B\to C}"] & G^{n+1}(A) \end{tikzcd}$$
are commutative.

010S Definition 12.12.3. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. We say F is a universal δ-functor if and only if for every δ-functor $G = (G^n, \delta_G)$ and any morphism of functors $t : F^0 \to G^0$ there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$.

010T Lemma 12.12.4. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. Suppose that for every $n > 0$ and any $A \in \text{Ob}(\mathcal{A})$ there exists an injective morphism $u : A \to B$ (depending on A and n) such that $F^n(u) : F^n(A) \to F^n(B)$ is zero. Then F is a universal δ-functor.

Proof. Let $G = (G^n, \delta_G)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and let $t : F^0 \to G^0$ be a morphism of functors. We have to show there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$. We construct t^n by induction on n. For $n = 0$ we set $t^0 = t$. Suppose we have already constructed a unique sequence of transformation of functors t^i for $i \leq n$ compatible with the maps δ in degrees $\leq n$.

Let $A \in \text{Ob}(\mathcal{A})$. By assumption we may choose a embedding $u : A \to B$ such that $F^{n+1}(u) = 0$. Let $C = B/u(A)$. The long exact cohomology sequence for the short exact sequence $0 \to A \to B \to C \to 0$ and the δ-functor F gives that $F^{n+1}(A) = \mathrm{Coker}(F^n(B) \to F^n(C))$ by our choice of u. Since we have already defined t^n we can set

$$t_A^{n+1} : F^{n+1}(A) \to G^{n+1}(A)$$

equal to the unique map such that

$$\begin{tikzcd} \mathrm{Coker}(F^n(B) \to F^n(C)) \arrow[r, "t^n"'] \arrow[d, "\delta_{F,A\to B\to C}"'] & \mathrm{Coker}(G^n(B) \to G^n(C)) \arrow[d, "\delta_{G,A\to B\to C}"] \\ F^{n+1}(A) \arrow[r, "t_A^{n+1}"] & G^{n+1}(A) \end{tikzcd}$$

commutes. This is clearly uniquely determined by the requirements imposed. We omit the verification that this defines a transformation of functors. □

010U Lemma 12.12.5. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a universal δ-functor (F^n, δ_F) from $\mathcal{A}$ to $\mathcal{B}$ with $F^0 = F$, then it is determined up to unique isomorphism of δ-functors.

Proof. Immediate from the definitions. □

12.13. Complexes

010V Of course the notions of a chain complex and a cochain complex are dual and you only have to read one of the two parts of this section. So pick the one you like. (Actually, this doesn't quite work right since the conventions on numbering things are not adapted to an easy transition between chain and cochain complexes.)

A chain complex $A_\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A_i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d_i : A_i \to A_{i-1}$ such that $d_{i-1} \circ d_i = 0$ for all i. A morphism of chain complexes $f : A_\bullet \to B_\bullet$ is given by a family of morphisms $f_i : A_i \to B_i$ such that all the diagrams

$$\begin{tikzcd} A_i \arrow[r, "d_i"'] \arrow[d, "f_i"'] & A_{i-1} \arrow[d, "f_{i-1}"] \\ B_i \arrow[r, "d_i"] & B_{i-1} \end{tikzcd}$$

commute. The category of chain complexes of $\mathcal{A}$ is denoted $\mathrm{Ch}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to 0 \to \ldots$$

is denoted $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. In other words, a chain complex $A_\bullet$ belongs to $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ if and only if $A_i = 0$ for all $i < 0$.

Given an additive category $\mathcal{A}$ we identify $\mathcal{A}$ with the full subcategory of $\mathrm{Ch}(\mathcal{A})$ consisting of chain complexes zero except in degree 0 by the functor

$$\mathcal{A} \longrightarrow \mathrm{Ch}(\mathcal{A}), \quad A \longmapsto (\ldots \to 0 \to A \to 0 \to \ldots)$$

By abuse of notation we often denote the object on the right hand side simply A. If we want to stress that we are viewing A as a chain complex we may sometimes use the notation $A[0]$, see Section 12.14.

A homotopy h between a pair of morphisms of chain complexes $f, g : A_\bullet \to B_\bullet$ is a collection of morphisms $h_i : A_i \to B_{i+1}$ such that we have

$$f_i - g_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$$

for all i. Two morphisms $f, g : A_\bullet \to B_\bullet$ are said to be homotopic if a homotopy between f and g exists. Clearly, the notions of chain complex, morphism of chain complexes, and homotopies between morphisms of chain complexes make sense even in a preadditive category.

010W Lemma 12.13.1. Let $\mathcal{A}$ be an additive category. Let $f, g : B_\bullet \to C_\bullet$ be morphisms of chain complexes. Suppose given morphisms of chain complexes $a : A_\bullet \to B_\bullet$, and $c : C_\bullet \to D_\bullet$. If $\{h_i : B_i \to C_{i+1}\}$ defines a homotopy between f and g, then $\{c_{i+1} \circ h_i \circ a_i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.

Proof. Omitted. □

In particular this means that it makes sense to define the category of chain complexes with maps up to homotopy. We'll return to this later.

010X Definition 12.13.2. Let $\mathcal{A}$ be an additive category. We say a morphism $a : A_\bullet \to B_\bullet$ is a homotopy equivalence if there exists a morphism $b : B_\bullet \to A_\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A_\bullet$ and $B_\bullet$, then we say that $A_\bullet$ and $B_\bullet$ are homotopy equivalent.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

010Y Lemma 12.13.3. Let $\mathcal{A}$ be an abelian category.

(1) The category of chain complexes in $\mathcal{A}$ is abelian.
(2) A morphism of complexes $f : A_\bullet \to B_\bullet$ is injective if and only if each $f_n : A_n \to B_n$ is injective.
(3) A morphism of complexes $f : A_\bullet \to B_\bullet$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.
(4) A sequence of chain complexes

$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$$

is exact at $B_\bullet$ if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith homology group of a chain complex $A_\bullet$ in an abelian category is defined by the following formula

$$H_i(A_\bullet) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes of $\mathcal{A}$ then we get an induced morphism $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ because clearly $f_i(\mathrm{Ker}(d_i : A_i \to A_{i-1})) \subset \mathrm{Ker}(d_i : B_i \to B_{i-1})$, and similarly for $\mathrm{Im}(d_{i+1})$. Thus we obtain a functor

$$H_i : \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

010Z Definition 12.13.4. Let $\mathcal{A}$ be an abelian category.

(1) A morphism of chain complexes $f : A_\bullet \to B_\bullet$ is called a quasi-isomorphism if the induced map $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A chain complex $A_\bullet$ is called acyclic if all of its homology objects $H_i(A_\bullet)$ are zero.

0110 Lemma 12.13.5. Let $\mathcal{A}$ be an abelian category.

(1) If the maps $f, g : A_\bullet \to B_\bullet$ are homotopic, then the induced maps $H_i(f)$ and $H_i(g)$ are equal.
(2) If the map $f : A_\bullet \to B_\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.

Proof. Omitted. $\square$

0111 Lemma 12.13.6. Let $\mathcal{A}$ be an abelian category. Suppose that

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of chain complexes of $\mathcal{A}$. Then there is a canonical long exact homology sequence

$$\begin{tikzcd}
\ldots & \ldots & \ldots \arrow[lld] \\
H_i(A_\bullet) \arrow[r] & H_i(B_\bullet) \arrow[r] & H_i(C_\bullet) \arrow[lld] \\
H_{i-1}(A_\bullet) \arrow[r] & H_{i-1}(B_\bullet) \arrow[r] & H_{i-1}(C_\bullet) \arrow[lld] \\
\ldots & \ldots & \ldots
\end{tikzcd}$$

Proof. Omitted. The maps come from the Snake Lemma 12.5.17 applied to the diagrams

$$\begin{tikzcd}
 & A_i/\operatorname{Im}(d_{A,i+1}) \arrow[r] \arrow[d, "d_{A,i}"] & B_i/\operatorname{Im}(d_{B,i+1}) \arrow[r] \arrow[d, "d_{B,i}"] & C_i/\operatorname{Im}(d_{C,i+1}) \arrow[r] \arrow[d, "d_{C,i}"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(d_{A,i-1}) \arrow[r] & \operatorname{Ker}(d_{B,i-1}) \arrow[r] & \operatorname{Ker}(d_{C,i-1}) &
\end{tikzcd}$$

$\square$

A cochain complex $A^\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A^i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is given by a family of morphisms $f^i : A^i \to B^i$ such that all the diagrams

$$\begin{tikzcd} A^i \arrow[r, "d^i"'] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d^i"] & B^{i+1} \end{tikzcd}$$

commute. The category of cochain complexes of $\mathcal{A}$ is denoted $\text{CoCh}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to 0 \to 0 \to A^0 \to A^1 \to A^2 \to \ldots$$

is denoted $\text{CoCh}_{\geq 0}(\mathcal{A})$. In other words, a cochain complex $A^\bullet$ belongs to the subcategory $\text{CoCh}_{\geq 0}(\mathcal{A})$ if and only if $A^i = 0$ for all $i < 0$.

Given an additive category $\mathcal{A}$ we identify $\mathcal{A}$ with the full subcategory of $\text{CoCh}(\mathcal{A})$ consisting of cochain complexes zero except in degree 0 by the functor

$$\mathcal{A} \longrightarrow \text{CoCh}(\mathcal{A}), \quad A \longmapsto (\ldots \to 0 \to A \to 0 \to \ldots)$$

By abuse of notation we often denote the object on the right hand side simply A. If we want to stress that we are viewing A as a cochain complex we may sometimes use the notation $A[0]$, see Section 12.14.

A homotopy h between a pair of morphisms of cochain complexes $f, g : A^\bullet \to B^\bullet$ is a collection of morphisms $h^i : A^i \to B^{i-1}$ such that we have

$$f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$$

for all i. Two morphisms $f, g : A^\bullet \to B^\bullet$ are said to be homotopic if a homotopy between f and g exists. Clearly, the notions of cochain complex, morphism of cochain complexes, and homotopies between morphisms of cochain complexes make sense even in a preadditive category.

0112 Lemma 12.13.7. Let $\mathcal{A}$ be an additive category. Let $f, g : B^\bullet \to C^\bullet$ be morphisms of cochain complexes. Suppose given morphisms of cochain complexes $a : A^\bullet \to B^\bullet$, and $c : C^\bullet \to D^\bullet$. If $\{h^i : B^i \to C^{i-1}\}$ defines a homotopy between f and g, then $\{c^{i-1} \circ h^i \circ a^i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.

Proof. Omitted. $\square$

In particular this means that it makes sense to define the category of cochain complexes with maps up to homotopy. We'll return to this later.

0113 Definition 12.13.8. Let $\mathcal{A}$ be an additive category. We say a morphism $a : A^\bullet \to B^\bullet$ is a homotopy equivalence if there exists a morphism $b : B^\bullet \to A^\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A^\bullet$ and $B^\bullet$, then we say that $A^\bullet$ and $B^\bullet$ are homotopy equivalent.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

0114 Lemma 12.13.9. Let $\mathcal{A}$ be an abelian category.

(1) The category of cochain complexes in $\mathcal{A}$ is abelian.
(2) A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is injective if and only if each $f^n : A^n \to B^n$ is injective.

(3) A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is surjective if and only if each $f^n : A^n \to B^n$ is surjective.
(4) A sequence of cochain complexes

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$$

is exact at $B^\bullet$ if and only if each sequence

$$A^i \xrightarrow{f^i} B^i \xrightarrow{g^i} C^i$$

is exact at B^i.

Proof. Omitted. $\square$

For any $i \in \mathbf{Z}$ the ith cohomology group of a cochain complex $A^\bullet$ is defined by the following formula

$$H^i(A^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1}).$$

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes of $\mathcal{A}$ then we get an induced morphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ because clearly $f^i(\mathrm{Ker}(d^i : A^i \to A^{i+1})) \subset \mathrm{Ker}(d^i : B^i \to B^{i+1})$, and similarly for $\mathrm{Im}(d^{i-1})$. Thus we obtain a functor

$$H^i : \mathrm{CoCh}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

0115 Definition 12.13.10. Let $\mathcal{A}$ be an abelian category.

(1) A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ of $\mathcal{A}$ is called a quasi-isomorphism if the induced maps $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A cochain complex $A^\bullet$ is called acyclic if all of its cohomology objects $H^i(A^\bullet)$ are zero.

0116 Lemma 12.13.11. Let $\mathcal{A}$ be an abelian category.

(1) If the maps $f, g : A^\bullet \to B^\bullet$ are homotopic, then the induced maps $H^i(f)$ and $H^i(g)$ are equal.
(2) If $f : A^\bullet \to B^\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.

Proof. Omitted. $\square$

0117 Lemma 12.13.12. Let $\mathcal{A}$ be an abelian category. Suppose that

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

is a short exact sequence of cochain complexes of $\mathcal{A}$. Then there is a long exact cohomology sequence

$$\begin{tikzcd}
\ldots & \ldots & \ldots \arrow[dll] \\
H^i(A^\bullet) \arrow[r] & H^i(B^\bullet) \arrow[r] & H^i(C^\bullet) \arrow[dll] \\
H^{i+1}(A^\bullet) \arrow[r] & H^{i+1}(B^\bullet) \arrow[r] & H^{i+1}(C^\bullet) \arrow[dll] \\
\ldots & \ldots & \ldots
\end{tikzcd}$$

The construction produces long exact cohomology sequences which are functorial in the short exact sequence and compatible with shifts.

Proof. For the horizontal maps $H^i(A^\bullet) \to H^i(B^\bullet)$ and $H^i(B^\bullet) \to H^i(C^\bullet)$ we use the fact that H^i is a functor, see above. For the "boundary map" $H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ we use the map δ of the Snake Lemma 12.5.17 applied to the diagram

$$\begin{tikzcd} & A^i/\operatorname{Im}(d_A^{i-1}) \arrow[r] \arrow[d, "d_A^i"] & B^i/\operatorname{Im}(d_B^{i-1}) \arrow[r] \arrow[d, "d_B^i"] & C^i/\operatorname{Im}(d_C^{i-1}) \arrow[r] \arrow[d, "d_C^i"] & 0 \\ 0 \arrow[r] & \operatorname{Ker}(d_A^{i+1}) \arrow[r] & \operatorname{Ker}(d_B^{i+1}) \arrow[r] & \operatorname{Ker}(d_C^{i+1}) & \end{tikzcd}$$

This works as the kernel of the right vertical map is equal to $H^i(C^\bullet)$ and the cokernel of the left vertical map is $H^{i+1}(A^\bullet)$. We omit the verification that we obtain a long exact sequence and we omit the verification of the properties mentioned at the end of the statement of the lemma. □

12.14. Homotopy and the shift functor

0119 It is an annoying feature that signs and indices have to be part of any discussion of homological algebra[2].

011A Definition 12.14.1. Let $\mathcal{A}$ be an additive category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we define the k-shifted chain complex $A[k]_\bullet$ as follows:

1. we set $A[k]_n = A_{n+k}$, and
2. we set $d_{A[k],n} : A[k]_n \to A[k]_{n-1}$ equal to $d_{A[k],n} = (-1)^k d_{A,n+k}$.

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes, then we let $f[k] : A[k]_\bullet \to B[k]_\bullet$ be the morphism of chain complexes with $f[k]_n = f_{k+n}$.

Of course this means we have functors $[k] : \text{Ch}(\mathcal{A}) \to \text{Ch}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors), such that $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \text{id}_{\text{Ch}(\mathcal{A})}$.

Recall that we view $\mathcal{A}$ as a full subcategory of $\text{Ch}(\mathcal{A})$, see Section 12.13. Thus for any object A of $\mathcal{A}$ the notation $A[k]$ refers to the unique chain complex zero in all degrees except having A in degree $-k$.

011B Definition 12.14.2. Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we identify $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$ via the identification $A_{i+k} = A[k]_i$.

This identification is functorial in $A_\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H_{i-k}(A[k]_\bullet)$, which are further compatible with the identifications $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \text{id}_{\text{Ch}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes, $a, b : A_\bullet \to B_\bullet$ are morphisms of chain complexes, and $\{h_i : A_i \to B_{i+1}\}$ is a homotopy between a and b. Recall that this means that $a_i - b_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$. What if $a = b$? Then we obtain the formula $0 = d_{i+1} \circ h_i + h_{i-1} \circ d_i$, in other words,

[2] Please let us know if you notice sign errors or if you have improvements to our conventions.

$-d_{i+1} \circ h_i = h_{i-1} \circ d_i$. By definition above this means the collection $\{h_i\}$ above defines a morphism of chain complexes

$$A_\bullet \longrightarrow B[1]_\bullet.$$

Such a thing is the same as a morphism $A[-1]_\bullet \to B_\bullet$ by our remarks above. This proves the following lemma.

011C Lemma 12.14.3. Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes. Given any morphism of chain complexes $a : A_\bullet \to B_\bullet$ there is a bijection between the set of homotopies from a to a and $\mathrm{Mor}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B[1]_\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogeneous space under the group $\mathrm{Mor}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B[1]_\bullet)$.

Proof. See above. □

011D Lemma 12.14.4. Let $\mathcal{A}$ be an abelian category. Let

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a short exact sequence of complexes. Suppose that $\{s_n : C_n \to B_n\}$ is a family of morphisms which split the short exact sequences $0 \to A_n \to B_n \to C_n \to 0$. Let $\pi_n : B_n \to A_n$ be the associated projections, see Lemma 12.5.10. Then the family of morphisms

$$\pi_{n-1} \circ d_{B,n} \circ s_n : C_n \to A_{n-1}$$

define a morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$.

Proof. Denote $i : A_\bullet \to B_\bullet$ and $q : B_\bullet \to C_\bullet$ the maps of complexes in the short exact sequence. Then $i_{n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}$. Hence $i_{n-2} \circ d_{A,n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n-1} \circ (d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}) = -d_{B,n-1} \circ s_{n-1} \circ d_{C,n}$ as desired. □

011E Lemma 12.14.5. Notation and assumptions as in Lemma 12.14.4 above. The morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$ induces the maps

$$H_i(\delta(s)) : H_i(C_\bullet) \longrightarrow H_i(A[-1]_\bullet) = H_{i-1}(A_\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of chain complexes by Lemma 12.13.6.

Proof. Omitted. □

011F Lemma 12.14.6. Notation and assumptions as in Lemma 12.14.4 above. Suppose $\{s'_n : C_n \to B_n\}$ is a second choice of splittings. Write $s'_n = s_n + i_n \circ h_n$ for some unique morphisms $h_n : C_n \to A_n$. The family of maps $\{h_n : C_n \to A[-1]_{n+1}\}$ is a homotopy between the associated morphisms $\delta(s), \delta(s') : C_\bullet \to A[-1]_\bullet$.

Proof. Omitted. □

011G Definition 12.14.7. Let $\mathcal{A}$ be an additive category. Let $A^\bullet$ be a cochain complex with boundary maps $d_A^n : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we define the k-shifted cochain complex $A[k]^\bullet$ as follows:

(1) we set $A[k]^n = A^{n+k}$, and
(2) we set $d_{A[k]}^n : A[k]^n \to A[k]^{n+1}$ equal to $d_{A[k]}^n = (-1)^k d_A^{n+k}$.

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes, then we let $f[k] : A[k]^\bullet \to B[k]^\bullet$ be the morphism of cochain complexes with $f[k]^n = f^{k+n}$.

Of course this means we have functors $[k] : \text{CoCh}(\mathcal{A}) \to \text{CoCh}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors) and such that $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \text{id}_{\text{CoCh}(\mathcal{A})}$.

Recall that we view $\mathcal{A}$ as a full subcategory of $\text{CoCh}(\mathcal{A})$, see Section 12.13. Thus for any object A of $\mathcal{A}$ the notation $A[k]$ refers to the unique cochain complex zero in all degrees except having A in degree $-k$.

011H Definition 12.14.8. Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex with boundary maps $d_A^n : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we identify $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$ via the identification $A^{i+k} = A[k]^i$.

This identification is functorial in $A^\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H^{i-k}(A[k]^\bullet)$, which are further compatible with the identifications $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \text{id}_{\text{CoCh}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes, $a, b : A^\bullet \to B^\bullet$ are morphisms of cochain complexes, and $\{h^i : A^i \to B^{i-1}\}$ is a homotopy between a and b. Recall that this means that $a^i - b^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$. What if $a = b$? Then we obtain the formula $0 = d^{i-1} \circ h^i + h^{i+1} \circ d^i$, in other words, $-d^{i-1} \circ h^i = h^{i+1} \circ d^i$. By definition above this means the collection $\{h^i\}$ above defines a morphism of cochain complexes

$$A^\bullet \longrightarrow B[-1]^\bullet.$$

Such a thing is the same as a morphism $A[1]^\bullet \to B^\bullet$ by our remarks above. This proves the following lemma.

011I Lemma 12.14.9. Let $\mathcal{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes. Given any morphism of cochain complexes $a : A^\bullet \to B^\bullet$ there is a bijection between the set of homotopies from a to a and $\text{Mor}_{\text{CoCh}(\mathcal{A})}(A^\bullet, B[-1]^\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogeneous space under the group $\text{Mor}_{\text{CoCh}(\mathcal{A})}(A^\bullet, B[-1]^\bullet)$.

Proof. See above. $\square$

011J Lemma 12.14.10. Let $\mathcal{A}$ be an additive category. Let

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a complex (!) of complexes. Suppose that we are given splittings $B^n = A^n \oplus C^n$ compatible with the maps in the displayed sequence. Let $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ be the corresponding maps. Then the family of morphisms

$$\pi^{n+1} \circ d_B^n \circ s^n : C^n \to A^{n+1}$$

define a morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$.

Proof. Denote $i : A^\bullet \to B^\bullet$ and $q : B^\bullet \to C^\bullet$ the maps of complexes in the short exact sequence. Then $i^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^n \circ s^n - s^{n+1} \circ d_C^n$. Hence $i^{n+2} \circ d_A^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^{n+1} \circ (d_B^n \circ s^n - s^{n+1} \circ d_C^n) = -d_B^{n+1} \circ s^{n+1} \circ d_C^n$ as desired. $\square$

011K Lemma 12.14.11. Notation and assumptions as in Lemma 12.14.10 above. Assume in addition that $\mathcal{A}$ is abelian. The morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$ induces the maps

$$H^i(\delta) : H^i(C^\bullet) \longrightarrow H^i(A[1]^\bullet) = H^{i+1}(A^\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of cochain complexes by Lemma 12.13.12.

Proof. Omitted. $\square$

011L Lemma 12.14.12. Notation and assumptions as in Lemma 12.14.10. Let $\alpha : A^\bullet \to B^\bullet$, $\beta : B^\bullet \to C^\bullet$ be the given morphisms of complexes. Suppose $(s')^n : C^n \to B^n$ and $(\pi')^n : B^n \to A^n$ is a second choice of splittings. Write $(s')^n = s^n + \alpha^n \circ h^n$ and $(\pi')^n = \pi^n + g^n \circ \beta^n$ for some unique morphisms $h^n : C^n \to A^n$ and $g^n : C^n \to A^n$. Then

(1) $g^n = -h^n$, and
(2) the family of maps $\{g^n : C^n \to A[1]^{n-1}\}$ is a homotopy between $\delta, \delta' : C^\bullet \to A[1]^\bullet$, more precisely $(\delta')^n = \delta^n + g^{n+1} \circ d_C^n + d_{A[1]}^{n-1} \circ g^n$.

Proof. As $(s')^n$ and $(\pi')^n$ are splittings we have $(\pi')^n \circ (s')^n = 0$. Hence

$$0 = (\pi^n + g^n \circ \beta^n) \circ (s^n + \alpha^n \circ h^n) = g^n \circ \beta^n \circ s^n + \pi^n \circ \alpha^n \circ h^n = g^n + h^n$$

which proves (1). We compute $(\delta')^n$ as follows

$$(\pi^{n+1} + g^{n+1} \circ \beta^{n+1}) \circ d_B^n \circ (s^n + \alpha^n \circ h^n) = \delta^n + g^{n+1} \circ d_C^n + d_A^n \circ h^n$$

Since $h^n = -g^n$ and since $d_{A[1]}^{n-1} = -d_A^n$ we conclude that (2) holds. $\square$

12.15. Truncation of complexes

0118 Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex. There are several ways to truncate the complex $A_\bullet$.

(1) The "stupid" truncation $\sigma_{\leq n}$ is the subcomplex $\sigma_{\leq n} A_\bullet$ defined by the rule $(\sigma_{\leq n} A_\bullet)_i = 0$ if $i > n$ and $(\sigma_{\leq n} A_\bullet)_i = A_i$ if $i \leq n$. In a picture

$$\begin{tikzcd} \sigma_{\leq n} A_\bullet \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\ A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & A_{n-1} \arrow[r] & \ldots \end{tikzcd}$$

Note the property $\sigma_{\leq n} A_\bullet / \sigma_{\leq n-1} A_\bullet = A_n[-n]$.

(2) The "stupid" truncation $\sigma_{\geq n}$ is the quotient complex $\sigma_{\geq n} A_\bullet$ defined by the rule $(\sigma_{\geq n} A_\bullet)_i = A_i$ if $i \geq n$ and $(\sigma_{\geq n} A_\bullet)_i = 0$ if $i < n$. In a picture

$$\begin{tikzcd} A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\ \sigma_{\geq n} A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & 0 \arrow[r] & \ldots \end{tikzcd}$$

The map of complexes $\sigma_{\geq n} A_\bullet \to \sigma_{\geq n+1} A_\bullet$ is surjective with kernel $A_n[-n]$.

(3) The canonical truncation $\tau_{\geq n}A_\bullet$ is defined by the picture

$$\begin{tikzcd}
\tau_{\geq n}A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & \mathrm{Ker}(d_n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
A_\bullet & \ldots \arrow[r] & A_{n+1} \arrow[r] & A_n \arrow[r] & A_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H_i(\tau_{\geq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \geq n \\ 0 & \text{if} \quad i < n \end{cases}$$

(4) The canonical truncation $\tau_{\leq n}A_\bullet$ is defined by the picture

$$\begin{tikzcd}
A_\bullet \arrow[d] & \ldots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\
\tau_{\leq n}A_\bullet & \ldots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d_{n+1}) \arrow[r] & A_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H_i(\tau_{\leq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex. There are four ways to truncate the complex $A^\bullet$.

(1) The "stupid" truncation $\sigma_{\geq n}$ is the subcomplex $\sigma_{\geq n}A^\bullet$ defined by the rule $(\sigma_{\geq n}A^\bullet)^i = 0$ if $i < n$ and $(\sigma_{\geq n}A^\bullet)^i = A_i$ if $i \geq n$. In a picture

$$\begin{tikzcd}
\sigma_{\geq n}A^\bullet \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\
A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Note the property $\sigma_{\geq n}A^\bullet/\sigma_{\geq n+1}A^\bullet = A^n[-n]$.

(2) The "stupid" truncation $\sigma_{\leq n}$ is the quotient complex $\sigma_{\leq n}A^\bullet$ defined by the rule $(\sigma_{\leq n}A^\bullet)^i = 0$ if $i > n$ and $(\sigma_{\leq n}A^\bullet)^i = A^i$ if $i \leq n$. In a picture

$$\begin{tikzcd}
A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\
\sigma_{\leq n}A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

The map of complexes $\sigma_{\leq n}A^\bullet \to \sigma_{\leq n-1}A^\bullet$ is surjective with kernel $A^n[-n]$.

(3) The canonical truncation $\tau_{\leq n}A^\bullet$ is defined by the picture

$$\begin{tikzcd}
\tau_{\leq n}A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & \mathrm{Ker}(d^n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
A^\bullet & \ldots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Note that these complexes have the property that

$$H^i(\tau_{\leq n}A^\bullet) = \begin{cases} H^i(A^\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

(4) The canonical truncation $\tau_{\geq n}A^\bullet$ is defined by the picture

$$\begin{tikzcd} A^\bullet \arrow[d] & \ldots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \\ \tau_{\geq n}A^\bullet & \ldots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d^{n-1}) \arrow[r] & A^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Note that these complexes have the property that

$$H^i(\tau_{\geq n}A^\bullet) = \begin{cases} 0 & \text{if} \quad i < n \\ H^i(A^\bullet) & \text{if} \quad i \geq n \end{cases}$$

12.16. Graded objects

09MF We make the following definition.

0125 **Definition 12.16.1.** Let $\mathcal{A}$ be an additive category. The category of graded objects of $\mathcal{A}$, denoted $\mathrm{Gr}(\mathcal{A})$, is the category with

(1) objects $A = (A^i)$ are families of objects A^i, $i \in \mathbf{Z}$ of objects of $\mathcal{A}$, and
(2) morphisms $f : A = (A^i) \to B = (B^i)$ are families of morphisms $f^i : A^i \to B^i$ of $\mathcal{A}$.

If $\mathcal{A}$ has countable direct sums, then we can associate to an object $A = (A^i)$ of $\mathrm{Gr}(\mathcal{A})$ the object

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} A^i$$

and set $k^iA = A^i$. In this case $\mathrm{Gr}(\mathcal{A})$ is equivalent to the category of pairs (A, k) consisting of an object A of $\mathcal{A}$ and a direct sum decomposition

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} k^iA$$

by direct summands indexed by $\mathbf{Z}$ and a morphism $(A, k) \to (B, k)$ of such objects is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(k^iA) \subset k^iB$ for all $i \in \mathbf{Z}$. Whenever our additive category $\mathcal{A}$ has countable direct sums we will use this equivalence without further mention.

However, with our definitions an additive or abelian category does not necessarily have all (countable) direct sums. In this case our definition still makes sense. For example, if $\mathcal{A} = \mathrm{Vect}_k$ is the category of finite dimensional vector spaces over a field k, then $\mathrm{Gr}(\mathrm{Vect}_k)$ is the category of vector spaces with a given gradation all of whose graded pieces are finite dimensional, and not the category of finite dimensional vector spaces with a given graduation.

0126 **Lemma 12.16.2.** Let $\mathcal{A}$ be an abelian category. The category of graded objects $\mathrm{Gr}(\mathcal{A})$ is abelian.

Proof. Let $f : A = (A^i) \to B = (B^i)$ be a morphism of graded objects of $\mathcal{A}$ given by morphisms $f^i : A^i \to B^i$ of $\mathcal{A}$. Then we have $\mathrm{Ker}(f) = (\mathrm{Ker}(f^i))$ and $\mathrm{Coker}(f) = (\mathrm{Coker}(f^i))$ in the category $\mathrm{Gr}(\mathcal{A})$. Since we have $\mathrm{Im} = \mathrm{Coim}$ in $\mathcal{A}$ we see the same thing holds in $\mathrm{Gr}(\mathcal{A})$. □

0AMH **Remark 12.16.3** (Warning). There are abelian categories $\mathcal{A}$ having countable direct sums but where countable direct sums are not exact. An example is the opposite of the category of abelian sheaves on $\mathbf{R}$. Namely, the category of abelian sheaves on $\mathbf{R}$ has countable products, but countable products are not exact. For such a category the functor $\text{Gr}(\mathcal{A}) \to \mathcal{A}$, $(A^i) \mapsto \bigoplus A^i$ described above is not exact. It is still true that $\text{Gr}(\mathcal{A})$ is equivalent to the category of graded objects (A, k) of $\mathcal{A}$, but the kernel in the category of graded objects of a map $\varphi : (A, k) \to (B, k)$ is not equal to $\text{Ker}(\varphi)$ endowed with a direct sum decomposition, but rather it is the direct sum of the kernels of the maps $k^i A \to k^i B$.

09MG Definition 12.16.4. Let $\mathcal{A}$ be an additive category. If $A = (A^i)$ is a graded object, then the kth shift $A[k]$ is the graded object with $A[k]^i = A^{k+i}$.

If A and B are graded objects of $\mathcal{A}$, then we have

09MH (12.16.4.1) $$\text{Hom}_{\text{Gr}(\mathcal{A})}(A, B[k]) = \text{Hom}_{\text{Gr}(\mathcal{A})}(A[-k], B)$$

and an element of this group is sometimes called a map of graded objects homogeneous of degree k.

Given any set G we can define G-graded objects of $\mathcal{A}$ as the category whose objects are $A = (A^g)_{g \in G}$ families of objects parametrized by elements of G. Morphisms $f : A \to B$ are defined as families of maps $f^g : A^g \to B^g$ where g runs over the elements of G. If G is an abelian group, then we can (unambiguously) define shift functors $[g]$ on the category of G-graded objects by the rule $(A[g])^{g_0} = A^{g+g_0}$. A particular case of this type of construction is when $G = \mathbf{Z} \times \mathbf{Z}$. In this case the objects of the category are called bigraded objects of $\mathcal{A}$. The (p, q) component of a bigraded object A is usually denoted $A^{p,q}$. For $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ we write $A[a, b]$ in stead of $A[(a, b)]$. A morphism $A \to A[a, b]$ is sometimes called a map of bidegree (a, b).

12.17. Additive monoidal categories

0FN9 Some material about the interaction between a monoidal structure and an additive structure on a category.

0FNA Definition 12.17.1. An additive monoidal category is an additive category $\mathcal{A}$ endowed with a monoidal structure $\otimes, \phi$ (Categories, Definition 4.43.1) such that $\otimes$ is an additive functor in each variable.

0FFT Lemma 12.17.2. Let $\mathcal{A}$ be an additive monoidal category. If Y_i, $i = 1, 2$ are left duals of X_i, $i = 1, 2$, then $Y_1 \oplus Y_2$ is a left dual of $X_1 \oplus X_2$.

Proof. Follows from uniqueness of adjoints and Categories, Remark 4.43.7. $\square$

0FFU Lemma 12.17.3. In a Karoubian additive monoidal category every summand of an object which has a left dual has a left dual.

Proof. We will use Categories, Lemma 4.43.6 without further mention. Let X be an object which has a left dual Y. We have

$$\text{Hom}(X, X) = \text{Hom}(\mathbf{1}, X \otimes Y) = \text{Hom}(Y, Y)$$

If $a : X \to X$ corresponds to $b : Y \to Y$ then b is the unique endomorphism of Y such that precomposing by a on

$$\text{Hom}(Z' \otimes X, Z) = \text{Hom}(Z', Z \otimes Y)$$

is the same as postcomposing by $1 \otimes b$. Hence the bijection $\text{Hom}(X, X) \to \text{Hom}(Y, Y)$, $a \mapsto b$ is an isomorphism of the opposite of the algebra $\text{Hom}(X, X)$ with the algebra $\text{Hom}(Y, Y)$. In particular, if $X = X_1 \oplus X_2$, then the corresponding projectors e_1, e_2

are mapped to idempotents in $\mathrm{Hom}(Y, Y)$. If $Y = Y_1 \oplus Y_2$ is the corresponding direct sum decomposition of Y (Section 12.4) then we see that under the bijection $\mathrm{Hom}(Z' \otimes X, Z) = \mathrm{Hom}(Z', Z \otimes Y)$ we have $\mathrm{Hom}(Z' \otimes X_i, Z) = \mathrm{Hom}(Z', Z \otimes Y_i)$ functorially as subgroups for $i = 1, 2$. It follows that Y_i is the left dual of X_i by the discussion in Categories, Remark 4.43.7. □

0FFX Example 12.17.4. Let F be a field. Let $\mathcal{C}$ be the category of graded F-vector spaces. Given graded vector spaces V and W we let $V \otimes W$ denote the graded F-vector space whose degree n part is

$$(V \otimes W)^n = \bigoplus\nolimits_{n=p+q} V^p \otimes_F W^q$$

Given a third graded vector space U as associativity constraint $\phi : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ we use the "usual" isomorphisms

$$U^p \otimes_F (V^q \otimes_F W^r) \to (U^p \otimes_F V^q) \otimes_F W^r$$

of vectors spaces. As unit we use the graded F-vector space $\mathbf{1}$ which has F in degree 0 and is zero in other degrees. There are two commutativity constraints on $\mathcal{C}$ which turn $\mathcal{C}$ into a symmetric monoidal category: one involves the intervention of signs and the other does not. We will usually use the one that does. To be explicit, if V and W are graded F-vector spaces we will use the isomorphism $\psi : V \otimes W \to W \otimes V$ which in degree n uses

$$V^p \otimes_F W^q \to W^q \otimes_F V^p, \quad v \otimes w \mapsto (-1)^{pq} w \otimes v$$

We omit the verification that this works.

0FFV Lemma 12.17.5. Let F be a field. Let $\mathcal{C}$ be the category of graded F-vector spaces viewed as a monoidal category as in Example 12.17.4. If V in $\mathcal{C}$ has a left dual W, then $\sum_n \dim_F V^n < \infty$ and the map ϵ defines nondegenerate pairings $W^{-n} \times V^n \to F$.

Proof. As unit we take By Categories, Definition 4.43.5 we have maps

$$\eta : \mathbf{1} \to V \otimes W \quad \epsilon : W \otimes V \to \mathbf{1}$$

Since $\mathbf{1} = F$ placed in degree 0, we may think of ϵ as a sequence of pairings $W^{-n} \times V^n \to F$ as in the statement of the lemma. Choose bases $\{e_{n,i}\}_{i \in I_n}$ for V^n for all n. Write

$$\eta(1) = \sum e_{n,i} \otimes w_{-n,i}$$

for some elements $w_{-n,i} \in W^{-n}$ almost all of which are zero! The condition that $(\epsilon \otimes 1) \circ (1 \otimes \eta)$ is the identity on W means that

$$\sum\nolimits_{n,i} \epsilon(w, e_{n,i}) w_{-n,i} = w$$

Thus we see that W is generated as a graded vector space by the finitely many nonzero vectors $w_{-n,i}$. The condition that $(1 \otimes \epsilon) \circ (\eta \otimes 1)$ is the identity of V means that

$$\sum\nolimits_{n,i} e_{n,i} \ \epsilon(w_{-n,i}, v) = v$$

In particular, setting $v = e_{n,i}$ we conclude that $\epsilon(w_{-n,i}, e_{n,i'}) = \delta_{ii'}$. Thus we find that the statement of the lemma holds and that $\{w_{-n,i}\}_{i \in I_n}$ is the dual basis for W^{-n} to the chosen basis for V^n. □

12.18. Double complexes and associated total complexes

0FNB We discuss double complexes and associated total complexes.

012Y Definition 12.18.1. Let $\mathcal{A}$ be an additive category. A double complex in $\mathcal{A}$ is given by a system $(\{A^{p,q}, d_1^{p,q}, d_2^{p,q}\}_{p,q \in \mathbf{Z}})$, where each $A^{p,q}$ is an object of $\mathcal{A}$ and $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ are morphisms of $\mathcal{A}$ such that the following rules hold:

(1) $d_1^{p+1,q} \circ d_1^{p,q} = 0$
(2) $d_2^{p,q+1} \circ d_2^{p,q} = 0$
(3) $d_1^{p,q+1} \circ d_2^{p,q} = d_2^{p+1,q} \circ d_1^{p,q}$

for all $p, q \in \mathbf{Z}$.

This is just the cochain version of the definition. It says that each $A^{p,\bullet}$ is a cochain complex and that each $d_1^{p,\bullet}$ is a morphism of complexes $A^{p,\bullet} \to A^{p+1,\bullet}$ such that $d_1^{p+1,\bullet} \circ d_1^{p,\bullet} = 0$ as morphisms of complexes. In other words a double complex can be seen as a complex of complexes. So in the diagram

$$\begin{tikzcd}
\ldots & \ldots & \ldots & \ldots \\
\ldots \arrow[r] & A^{p,q+1} \arrow[u] \arrow[r, "d_1^{p,q+1}"] & A^{p+1,q+1} \arrow[u] \arrow[r] & \ldots \\
\ldots \arrow[r] & A^{p,q} \arrow[u, "d_2^{p,q}"] \arrow[r, "d_1^{p,q}"] & A^{p+1,q} \arrow[u, "d_2^{p+1,q}"'] \arrow[r] & \ldots \\
\ldots & \ldots \arrow[u] & \ldots \arrow[u] & \ldots
\end{tikzcd}$$

any square commutes. Warning: In the literature one encounters a different definition where a "bicomplex" or a "double complex" has the property that the squares in the diagram anti-commute.

0A5J Example 12.18.2. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be additive categories. Suppose that

$$\otimes : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}, \quad (X, Y) \longmapsto X \otimes Y$$

is a functor which is bilinear on morphisms, see Categories, Definition 4.2.20 for the definition of $\mathcal{A} \times \mathcal{B}$. Given complexes $X^\bullet$ of $\mathcal{A}$ and $Y^\bullet$ of $\mathcal{B}$ we obtain a double complex

$$K^{\bullet,\bullet} = X^\bullet \otimes Y^\bullet$$

in $\mathcal{C}$. Here the first differential $K^{p,q} \to K^{p+1,q}$ is the morphism $X^p \otimes Y^q \to X^{p+1} \otimes Y^q$ induced by the morphism $X^p \to X^{p+1}$ and the identity on Y^q. Similarly for the second differential.

012Z Definition 12.18.3. Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet}$ be a double complex. The associated simple complex, denoted $sA^\bullet$, also often called the associated total complex, denoted $\text{Tot}(A^{\bullet,\bullet})$, is given by

$$sA^n = \text{Tot}^n(A^{\bullet,\bullet}) = \bigoplus\nolimits_{n=p+q} A^{p,q}$$

(if it exists) with differential

$$d^n_{sA^\bullet} = d^n_{\text{Tot}(A^{\bullet,\bullet})} = \sum\nolimits_{n=p+q} (d_1^{p,q} + (-1)^p d_2^{p,q})$$

If countable direct sums exist in $\mathcal{A}$ or if for each n at most finitely many $A^{p,n-p}$ are nonzero, then $\text{Tot}(A^{\bullet,\bullet})$ exists. Note that the definition is not symmetric in the indices (p, q).

08BI Remark 12.18.4. Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet,\bullet}$ be a triple complex. The associated total complex is the complex with terms

$$\text{Tot}^n(A^{\bullet,\bullet,\bullet}) = \bigoplus\nolimits_{p+q+r=n} A^{p,q,r}$$

and differential

$$d^n_{\text{Tot}(A^{\bullet,\bullet,\bullet})} = \sum\nolimits_{p+q+r=n} d_1^{p,q,r} + (-1)^p d_2^{p,q,r} + (-1)^{p+q} d_3^{p,q,r}$$

With this definition a simple calculation shows that the associated total complex is equal to

$$\text{Tot}(A^{\bullet,\bullet,\bullet}) = \text{Tot}(\text{Tot}_{12}(A^{\bullet,\bullet,\bullet})) = \text{Tot}(\text{Tot}_{23}(A^{\bullet,\bullet,\bullet}))$$

In other words, we can either first combine the first two of the variables and then combine sum of those with the last, or we can first combine the last two variables and then combine the first with the sum of the last two.

0FLG Remark 12.18.5. Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet}$ be a double complex with differentials $d_1^{p,q}$ and $d_2^{p,q}$. Denote $A^{\bullet,\bullet}[a, b]$ the double complex with

$$(A^{\bullet,\bullet}[a, b])^{p,q} = A^{p+a,q+b}$$

and differentials

$$d^{p,q}_{A^{\bullet,\bullet}[a,b],1} = (-1)^a d_1^{p+a,q+b} \quad \text{and} \quad d^{p,q}_{A^{\bullet,\bullet}[a,b],2} = (-1)^b d_2^{p+a,q+b}$$

In this situation there is a well defined isomorphism

$$\gamma : \text{Tot}(A^{\bullet,\bullet})[a + b] \longrightarrow \text{Tot}(A^{\bullet,\bullet}[a, b])$$

which in degree n is given by the map

$$\begin{array}{c} (\text{Tot}(A^{\bullet,\bullet})[a + b])^n = \bigoplus\nolimits_{p+q=n+a+b} A^{p,q} \\ \downarrow {\scriptstyle \epsilon(p,q,a,b)\text{id}_{A^{p,q}}} \\ \text{Tot}(A^{\bullet,\bullet}[a, b])^n = \bigoplus\nolimits_{p'+q'=n} A^{p'+a,q'+b} \end{array}$$

for some sign $\epsilon(p, q, a, b)$. Of course the summand $A^{p,q}$ maps to the summand $A^{p'+a,q'+b}$ when $p = p' + a$ and $q = q' + b$. To figure out the conditions on these signs observe that on the source we have

$$d|_{A^{p,q}} = (-1)^{a+b} (d_1^{p,q} + (-1)^p d_2^{p,q})$$

whereas on the target we have

$$d|_{A^{p'+a,q'+b}} = (-1)^a d_1^{p'+a,q'+b} + (-1)^{p'} (-1)^b d_2^{p'+a,q'+b}$$

Thus our constraints are that

$$(-1)^a \epsilon(p, q, a, b) = \epsilon(p + 1, q, a, b)(-1)^{a+b} \Leftrightarrow \epsilon(p + 1, q, a, b) = (-1)^b \epsilon(p, q, a, b)$$

and

$$(-1)^{p'+b} \epsilon(p, q, a, b) = \epsilon(p, q + 1, a, b)(-1)^{a+b+p} \Leftrightarrow \epsilon(p, q, a, b) = \epsilon(p, q + 1, a, b)$$

Thus we choose $\epsilon(p, q, a, b) = (-1)^{pb}$.

0G6A Remark 12.18.6. Let $\mathcal{A}$ be an additive category with countable direct sums. Let $\text{DoubleComp}(\mathcal{A})$ denote the category of double complexes. We can consider an object $A^{\bullet,\bullet}$ of $\text{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows

$$\ldots \to A^{\bullet,-1} \to A^{\bullet,0} \to A^{\bullet,1} \to \ldots$$

For the variant where we switch the role of the indices, see Remark 12.18.7. In this remark we show that taking the associated total complex is compatible with all the structures on complexes we have studied in the chapter so far.

First, observe that the shift functor on double complexes viewed as complexes of complexes in the manner given above is the functor $[0,1]$ defined in Remark 12.18.5. By Remark 12.18.5 the functor

$$\text{Tot} : \text{DoubleComp}(\mathcal{A}) \to \text{Comp}(\mathcal{A})$$

is compatible with shift functors, in the sense that we have a functorial isomorphism $\gamma : \text{Tot}(A^{\bullet,\bullet})[1] \to \text{Tot}(A^{\bullet,\bullet}[0,1])$.

Second, if

$$f, g : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$$

are homotopic when f and g are viewed as morphisms of complexes of complexes in the manner given above, then

$$\text{Tot}(f), \text{Tot}(g) : \text{Tot}(A^{\bullet,\bullet}) \to \text{Tot}(B^{\bullet,\bullet})$$

are homotopic maps of complexes. Indeed, let $h = (h^q)$ be a homotopy between f and g. If we denote $h^{p,q} : A^{p,q} \to B^{p,q-1}$ the component in degree p of h^q, then this means that

$$f^{p,q} - g^{p,q} = d_2^{p,q-1} \circ h^{p,q} + h^{p,q+1} \circ d_2^{p,q}$$

The fact that $h^q : A^{\bullet,q} \to B^{\bullet,q-1}$ is a map of complexes means that

$$d_1^{p,q-1} \circ h^{p,q} = h^{p+1,q} \circ d_1^{p,q}$$

Let us define $h' = ((h')^n)$ the homotopy given by the maps $(h')^n : \text{Tot}^n(A^{\bullet,\bullet}) \to \text{Tot}^{n-1}(B^{\bullet,\bullet})$ using $(-1)^p h^{p,q}$ on the summand $A^{p,q}$ for $p+q=n$. Then we see that

$$d_{\text{Tot}(B^{\bullet,\bullet})} \circ h' + h' \circ d_{\text{Tot}(A^{\bullet,\bullet})}$$

restricted to the summand $A^{p,q}$ is equal to

$$d_1^{p,q-1} \circ (-1)^p h^{p,q} + (-1)^p d_2^{p,q-1} \circ (-1)^p h^{p,q} + (-1)^{p+1} h^{p+1,q} \circ d_1^{p,q} + (-1)^p h^{p,q+1} \circ (-1)^p d_2^{p,q}$$

which evaluates to $f^{p,q} - g^{p,q}$ by the equations given above. This proves the second compatibility.

Third, suppose that in the paragraph above we have $f = g$. Then the assignment $h \rightsquigarrow h'$ above is compatible with the identification of Lemma 12.14.9. More precisely, if we view h as a morphism of complexes of complexes $A^{\bullet,\bullet} \to B^{\bullet,\bullet}[0,-1]$ via this lemma then

$$\text{Tot}(A^{\bullet,\bullet}) \xrightarrow{\text{Tot}(h)} \text{Tot}(B^{\bullet,\bullet}[0,-1]) \xrightarrow{\gamma^{-1}} \text{Tot}(B^{\bullet,\bullet})[-1]$$

is equal to h' viewed as a morphism of complexes via the lemma. Here γ is the identification of Remark 12.18.5. The verification of this third point is immediate.

Fourth, let

$$0 \to A^{\bullet,\bullet} \to B^{\bullet,\bullet} \to C^{\bullet,\bullet} \to 0$$

be a complex of double complexes and suppose we are given splittings $s^q : C^{\bullet,q} \to B^{\bullet,q}$ and $\pi^q : B^{\bullet,q} \to A^{\bullet,q}$ of this as in Lemma 12.14.10 when we view double complexes as complexes of complexes in the manner given above. This on the one hand produces a map

$$\delta : C^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}[0,1]$$

by the procedure in Lemma 12.14.10. On the other hand taking Tot we obtain a complex

$$0 \to \mathrm{Tot}(A^{\bullet,\bullet}) \to \mathrm{Tot}(B^{\bullet,\bullet}) \to \mathrm{Tot}(C^{\bullet,\bullet}) \to 0$$

which is termwise split (see below) and hence comes with a morphism

$$\delta' : \mathrm{Tot}(C^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet})[1]$$

well defined up to homotopy by Lemmas 12.14.10 and 12.14.12. Claim: these maps agree in the sense that

$$\mathrm{Tot}(C^{\bullet,\bullet}) \xrightarrow{\mathrm{Tot}(\delta)} \mathrm{Tot}(A^{\bullet,\bullet}[0,1]) \xrightarrow{\gamma^{-1}} \mathrm{Tot}(A^{\bullet,\bullet})[1]$$

is equal to δ' where γ is as in Remark 12.18.5. To see this denote $s^{p,q} : C^{p,q} \to B^{\bullet,q}$ and $\pi^{p,q} : B^{p,q} \to A^{p,q}$ the components of s^q and π^q. As splittings $(s')^n : \mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^n(B^{\bullet,\bullet})$ and $(\pi')^n : \mathrm{Tot}^n(B^{\bullet,\bullet}) \to \mathrm{Tot}^n(A^{\bullet,\bullet})$ we use the maps whose components are $s^{p,q}$ and $\pi^{p,q}$ for $p+q=n$. We recall that

$$(\delta')^n = (\pi')^{n+1} \circ d^n_{\mathrm{Tot}(B^{\bullet,\bullet})} \circ (s')^n : \mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^{n+1}(A^{\bullet,\bullet})$$

The restriction of this to the summand $C^{p,q}$ is equal to

$$\pi^{p+1,q} \circ d_1^{p,q} \circ s^{p,q} + \pi^{p,q+1} \circ (-1)^p d_2^{p,q} \circ s^{p,q} = \pi^{p,q+1} \circ (-1)^p d_2^{p,q} \circ s^{p,q}$$

The equality holds because s^q is a morphism of complexes (with d_1 as differential) and because $\pi^{p+1,q} \circ s^{p+1,q} = 0$ as s and π correspond to a direct sum decomposition of B in every bidegree. On the other hand, for δ we have

$$\delta^q = \pi^q \circ d_2 \circ s^q : C^{\bullet,q} \to A^{\bullet,q+1}$$

whose restriction to the summand $C^{p,q}$ is equal to $\pi^{p,q+1} \circ d_2^{p,q} \circ s^{p,q}$. The difference in signs is exactly canceled out by the sign of $(-1)^p$ in the isomorphism γ and the fourth claim is proven.

0G6B Remark 12.18.7. Let $\mathcal{A}$ be an additive category with countable direct sums. Let $\mathrm{DoubleComp}(\mathcal{A})$ denote the category of double complexes. We can consider an object $A^{\bullet,\bullet}$ of $\mathrm{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows

$$\ldots \to A^{-1,\bullet} \to A^{0,\bullet} \to A^{1,\bullet} \to \ldots$$

For the variant where we switch the role of the indices, see Remark 12.18.6. In this remark we show that taking the associated total complex is compatible with all the structures on complexes we have studied in the chapter so far.

First, observe that the shift functor on double complexes viewed as complexes of complexes in the manner given above is the functor $[1,0]$ defined in Remark 12.18.5. By Remark 12.18.5 the functor

$$\mathrm{Tot} : \mathrm{DoubleComp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A})$$

is compatible with shift functors, in the sense that we have a functorial isomorphism $\gamma : \mathrm{Tot}(A^{\bullet,\bullet})[1] \to \mathrm{Tot}(A^{\bullet,\bullet}[1,0])$.

Second, if

$$f, g : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$$

are homotopic when f and g are viewed as morphisms of complexes of complexes in the manner given above, then

$$\mathrm{Tot}(f), \mathrm{Tot}(g) : \mathrm{Tot}(A^{\bullet,\bullet}) \to \mathrm{Tot}(B^{\bullet,\bullet})$$

are homotopic maps of complexes. Indeed, let $h = (h^p)$ be a homotopy between f and g. If we denote $h^{p,q} : A^{p,q} \to B^{p-1,q}$ the component in degree p of h^q, then this means that

$$f^{p,q} - g^{p,q} = d_1^{p-1,q} \circ h^{p,q} + h^{p+1,q} \circ d_1^{p,q}$$

The fact that $h^p : A^{p,\bullet} \to B^{p-1,\bullet}$ is a map of complexes means that

$$d_2^{p-1,q} \circ h^{p,q} = h^{p,q+1} \circ d_2^{p,q}$$

Let us define $h' = ((h')^n)$ the homotopy given by the maps $(h')^n : \mathrm{Tot}^n(A^{\bullet,\bullet}) \to \mathrm{Tot}^{n-1}(B^{\bullet,\bullet})$ using $h^{p,q}$ on the summand $A^{p,q}$ for $p + q = n$. Then we see that

$$d_{\mathrm{Tot}(B^{\bullet,\bullet})} \circ h' + h' \circ d_{\mathrm{Tot}(A^{\bullet,\bullet})}$$

restricted to the summand $A^{p,q}$ is equal to

$$d_1^{p-1,q} \circ h^{p,q} + (-1)^{p-1} d_2^{p-1,q} \circ h^{p,q} + h^{p+1,q} \circ d_1^{p,q} + h^{p,q+1} \circ (-1)^p d_2^{p,q}$$

which evaluates to $f^{p,q} - g^{p,q}$ by the equations given above. This proves the second compatibility.

Third, suppose that in the paragraph above we have $f = g$. Then the assignment $h \rightsquigarrow h'$ above is compatible with the identification of Lemma 12.14.9. More precisely, if we view h as a morphism of complexes of complexes $A^{\bullet,\bullet} \to B^{\bullet,\bullet}[-1, 0]$ via this lemma then

$$\mathrm{Tot}(A^{\bullet,\bullet}) \xrightarrow{\mathrm{Tot}(h)} \mathrm{Tot}(B^{\bullet,\bullet}[-1, 0]) \xrightarrow{\gamma^{-1}} \mathrm{Tot}(B^{\bullet,\bullet})[-1]$$

is equal to h' viewed as a morphism of complexes via the lemma. Here γ is the identification of Remark 12.18.5. The verification of this third point is immediate.

Fourth, let

$$0 \to A^{\bullet,\bullet} \to B^{\bullet,\bullet} \to C^{\bullet,\bullet} \to 0$$

be a complex of double complexes and suppose we are given splittings $s^p : C^{p,\bullet} \to B^{p,\bullet}$ and $\pi^p : B^{p,\bullet} \to A^{p,\bullet}$ of this as in Lemma 12.14.10 when we view double complexes as complexes of complexes in the manner given above. This on the one hand produces a map

$$\delta : C^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}[0, 1]$$

by the procedure in Lemma 12.14.10. On the other hand taking Tot we obtain a complex

$$0 \to \mathrm{Tot}(A^{\bullet,\bullet}) \to \mathrm{Tot}(B^{\bullet,\bullet}) \to \mathrm{Tot}(C^{\bullet,\bullet}) \to 0$$

which is termwise split (see below) and hence comes with a morphism

$$\delta' : \mathrm{Tot}(C^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet})[1]$$

well defined up to homotopy by Lemmas 12.14.10 and 12.14.12. Claim: these maps agree in the sense that

$$\mathrm{Tot}(C^{\bullet,\bullet}) \xrightarrow{\mathrm{Tot}(\delta)} \mathrm{Tot}(A^{\bullet,\bullet}[1, 0]) \xrightarrow{\gamma^{-1}} \mathrm{Tot}(A^{\bullet,\bullet})[1]$$

is equal to δ' where γ is as in Remark 12.18.5. To see this denote $s^{p,q} : C^{p,q} \to B^{\bullet,q}$ and $\pi^{p,q} : B^{p,q} \to A^{p,q}$ the components of s^q and π^q. As splittings $(s')^n$:

$\mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^n(B^{\bullet,\bullet})$ and $(\pi')^n : \mathrm{Tot}^n(B^{\bullet,\bullet}) \to \mathrm{Tot}^n(A^{\bullet,\bullet})$ we use the maps whose components are $s^{p,q}$ and $\pi^{p,q}$ for $p+q=n$. We recall that

$$(\delta')^n = (\pi')^{n+1} \circ d^n_{\mathrm{Tot}(B^{\bullet,\bullet})} \circ (s')^n : \mathrm{Tot}^n(C^{\bullet,\bullet}) \to \mathrm{Tot}^{n+1}(A^{\bullet,\bullet})$$

The restriction of this to the summand $C^{p,q}$ is equal to

$$\pi^{p+1,q} \circ d_1^{p,q} \circ s^{p,q} + \pi^{p,q+1} \circ (-1)^p d_2^{p,q} \circ s^{p,q} = \pi^{p+1,q} \circ d_1^{p,q} \circ s^{p,q}$$

The equality holds because s^p is a morphism of complexes (with d_2 as differential) and because $\pi^{p,q+1} \circ s^{p,q+1} = 0$ as s and π correspond to a direct sum decomposition of B in every bidegree. On the other hand, for δ we have

$$\delta^p = \pi^p \circ d_1 \circ s^p : C^{p,\bullet} \to A^{p+1,\bullet}$$

whose restriction to the summand $C^{p,q}$ is equal to $\pi^{p+1,q} \circ d_1^{p,q} \circ s^{p,q}$. Thus we get the same as before which matches with the fact that the isomorphism $\gamma : \mathrm{Tot}(A^{\bullet,\bullet})[1] \to \mathrm{Tot}(A^{\bullet,\bullet}[1,0])$ is defined without the intervention of signs.

12.19. Filtrations

0120 A nice reference for this material is [Del71, Section 1]. (Note that our conventions regarding abelian categories are different.)

0121 Definition 12.19.1. Let $\mathcal{A}$ be an abelian category.

(1) A decreasing filtration F on an object A is a family $(F^nA)_{n\in\mathbf{Z}}$ of subobjects of A such that
$$A \supset \ldots \supset F^nA \supset F^{n+1}A \supset \ldots \supset 0$$
(2) A filtered object of $\mathcal{A}$ is pair (A, F) consisting of an object A of $\mathcal{A}$ and a decreasing filtration F on A.
(3) A morphism $(A, F) \to (B, F)$ of filtered objects is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(F^iA) \subset F^iB$ for all $i \in \mathbf{Z}$.
(4) The category of filtered objects is denoted $\mathrm{Fil}(\mathcal{A})$.
(5) Given a filtered object (A, F) and a subobject $X \subset A$ the induced filtration on X is the filtration with $F^nX = X \cap F^nA$.
(6) Given a filtered object (A, F) and a surjection $\pi : A \to Y$ the quotient filtration is the filtration with $F^nY = \pi(F^nA)$.
(7) A filtration F on an object A is said to be finite if there exist n, m such that $F^nA = A$ and $F^mA = 0$.
(8) Given a filtered object (A, F) we say $\bigcap F^iA$ exists if there exists a biggest subobject of A contained in all F^iA. We say $\bigcup F^iA$ exists if there exists a smallest subobject of A containing all F^iA.
(9) The filtration on a filtered object (A, F) is said to be separated if $\bigcap F^iA = 0$ and exhaustive if $\bigcup F^iA = A$.

By abuse of notation we say that a morphism $f : (A, F) \to (B, F)$ of filtered objects is injective if $f : A \to B$ is injective in the abelian category $\mathcal{A}$. Similarly we say f is surjective if $f : A \to B$ is surjective in the category $\mathcal{A}$. Being injective (resp. surjective) is equivalent to being a monomorphism (resp. epimorphism) in $\mathrm{Fil}(\mathcal{A})$. By Lemma 12.19.2 this is also equivalent to having zero kernel (resp. cokernel).

0122 Lemma 12.19.2. Let $\mathcal{A}$ be an abelian category. The category of filtered objects $\mathrm{Fil}(\mathcal{A})$ has the following properties:

(1) It is an additive category.

(2) It has a zero object.
(3) It has kernels and cokernels, images and coimages.
(4) In general it is not an abelian category.

Proof. It is clear that $\text{Fil}(\mathcal{A})$ is additive with direct sum given by $(A, F) \oplus (B, F) = (A \oplus B, F)$ where $F^p(A \oplus B) = F^pA \oplus F^pB$. The kernel of a morphism $f : (A, F) \to (B, F)$ of filtered objects is the injection $\text{Ker}(f) \subset A$ where $\text{Ker}(f)$ is endowed with the induced filtration. The cokernel of a morphism $f : A \to B$ of filtered objects is the surjection $B \to \text{Coker}(f)$ where $\text{Coker}(f)$ is endowed with the quotient filtration. Since all kernels and cokernels exist, so do all coimages and images. See Example 12.3.13 for the last statement. □

0123 Definition 12.19.3. Let $\mathcal{A}$ be an abelian category. A morphism $f : A \to B$ of filtered objects of $\mathcal{A}$ is said to be strict if $f(F^iA) = f(A) \cap F^iB$ for all $i \in \mathbf{Z}$.

This also equivalent to requiring that $f^{-1}(F^iB) = F^iA + \text{Ker}(f)$ for all $i \in \mathbf{Z}$. We characterize strict morphisms as follows.

05SI Lemma 12.19.4. Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. The following are equivalent

(1) f is strict,
(2) the morphism $\text{Coim}(f) \to \text{Im}(f)$ of Lemma 12.3.12 is an isomorphism.

Proof. Note that $\text{Coim}(f) \to \text{Im}(f)$ is an isomorphism of objects of $\mathcal{A}$, and that part (2) signifies that it is an isomorphism of filtered objects. By the description of kernels and cokernels in the proof of Lemma 12.19.2 we see that the filtration on $\text{Coim}(f)$ is the quotient filtration coming from $A \to \text{Coim}(f)$. Similarly, the filtration on $\text{Im}(f)$ is the induced filtration coming from the injection $\text{Im}(f) \to B$. The definition of strict is exactly that the quotient filtration is the induced filtration. □

05SK Lemma 12.19.5. Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a strict monomorphism of filtered objects. Let $g : A \to C$ be a morphism of filtered objects. Then $f \oplus g : A \to B \oplus C$ is a strict monomorphism.

Proof. Clear from the definitions. □

05SL Lemma 12.19.6. Let $\mathcal{A}$ be an abelian category. Let $f : B \to A$ be a strict epimorphism of filtered objects. Let $g : C \to A$ be a morphism of filtered objects. Then $f \oplus g : B \oplus C \to A$ is a strict epimorphism.

Proof. Clear from the definitions. □

0124 Lemma 12.19.7. Let $\mathcal{A}$ be an abelian category. Let (A, F), (B, F) be filtered objects. Let $u : A \to B$ be a morphism of filtered objects. If u is injective then u is strict if and only if the filtration on A is the induced filtration. If u is surjective then u is strict if and only if the filtration on B is the quotient filtration.

Proof. This is immediate from the definition. □

05SJ Lemma 12.19.8. Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$, $g : B \to C$ be strict morphisms of filtered objects.

(1) In general the composition $g \circ f$ is not strict.
(2) If g is injective, then $g \circ f$ is strict.

(3) If f is surjective, then $g \circ f$ is strict.

Proof. Let B a vector space over a field k with basis e_1, e_2, with the filtration $F^nB = B$ for $n < 0$, with $F^0B = ke_1$, and $F^nB = 0$ for $n > 0$. Now take $A = k(e_1 + e_2)$ and $C = B/ke_2$ with filtrations induced by B, i.e., such that $A \to B$ and $B \to C$ are strict (Lemma 12.19.7). Then $F^n(A) = A$ for $n < 0$ and $F^n(A) = 0$ for $n \geq 0$. Also $F^n(C) = C$ for $n \leq 0$ and $F^n(C) = 0$ for $n > 0$. So the (nonzero) composition $A \to C$ is not strict.

Assume g is injective. Then

$$\begin{aligned} g(f(F^pA)) &= g(f(A) \cap F^pB) \\ &= g(f(A)) \cap g(F^p(B)) \\ &= (g \circ f)(A) \cap (g(B) \cap F^pC) \\ &= (g \circ f)(A) \cap F^pC. \end{aligned}$$

The first equality as f is strict, the second because g is injective, the third because g is strict, and the fourth because $(g \circ f)(A) \subset g(B)$.

Assume f is surjective. Then

$$\begin{aligned} (g \circ f)^{-1}(F^iC) &= f^{-1}(F^iB + \mathrm{Ker}(g)) \\ &= f^{-1}(F^iB) + f^{-1}(\mathrm{Ker}(g)) \\ &= F^iA + \mathrm{Ker}(f) + \mathrm{Ker}(g \circ f) \\ &= F^iA + \mathrm{Ker}(g \circ f) \end{aligned}$$

The first equality because g is strict, the second because f is surjective, the third because f is strict, and the last because $\mathrm{Ker}(f) \subset \mathrm{Ker}(g \circ f)$. $\square$

The following lemma says that subobjects of a filtered object have a well defined filtration independent of a choice of writing the object as a cokernel.

0129 Lemma 12.19.9. Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. Let $X \subset Y \subset A$ be subobjects of A. On the object

$$Y/X = \mathrm{Ker}(A/X \to A/Y)$$

the quotient filtration coming from the induced filtration on Y and the induced filtration coming from the quotient filtration on A/X agree. Any of the morphisms $X \to Y$, $X \to A$, $Y \to A$, $Y \to A/X$, $Y \to Y/X$, $Y/X \to A/X$ are strict (with induced/quotient filtrations).

Proof. The quotient filtration Y/X is given by $F^p(Y/X) = F^pY/(X \cap F^pY) = F^pY/F^pX$ because $F^pY = Y \cap F^pA$ and $F^pX = X \cap F^pA$. The induced filtration from the injection $Y/X \to A/X$ is given by

$$\begin{aligned} F^p(Y/X) &= Y/X \cap F^p(A/X) \\ &= Y/X \cap (F^pA + X)/X \\ &= (Y \cap F^pA)/(X \cap F^pA) \\ &= F^pY/F^pX. \end{aligned}$$

Hence the first statement of the lemma. The proof of the other cases is similar. $\square$

05SM **Lemma 12.19.10.** Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in \text{Fil}(\mathcal{A})$. Let $f : A \to B$ and $g : A \to C$ be morphisms. Then there exists a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & C \amalg_A B \end{tikzcd}$$

in $\text{Fil}(\mathcal{A})$. If f is strict, so is f'.

Proof. Set $C \amalg_A B$ equal to $\text{Coker}((1, -1) : A \to C \oplus B)$ in $\text{Fil}(\mathcal{A})$. This cokernel exists, by Lemma 12.19.2. It is a pushout, see Example 12.5.6. Note that $F^p(C \amalg_A B)$ is the image of $F^pC \oplus F^pB$. Hence

$$(f')^{-1}(F^p(C \amalg_A B)) = g(f^{-1}(F^pB))) + F^pC$$

Whence the last statement. □

05SN **Lemma 12.19.11.** Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in \text{Fil}(\mathcal{A})$. Let $f : B \to A$ and $g : C \to A$ be morphisms. Then there exists a fibre product

$$\begin{tikzcd} B \times_A C \arrow[r, "g'"] \arrow[d, "f'"] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & A \end{tikzcd}$$

in $\text{Fil}(\mathcal{A})$. If f is strict, so is f'.

Proof. This lemma is dual to Lemma 12.19.10. □

Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. We denote $\text{gr}^p_F(A) = \text{gr}^p(A)$ the object $F^pA/F^{p+1}A$ of $\mathcal{A}$. This defines an additive functor

$$\text{gr}^p : \text{Fil}(\mathcal{A}) \longrightarrow \mathcal{A}, \quad (A, F) \longmapsto \text{gr}^p(A).$$

Recall that we have defined the category $\text{Gr}(\mathcal{A})$ of graded objects of $\mathcal{A}$ in Section 12.16. For (A, F) in $\text{Fil}(\mathcal{A})$ we may set

$$\text{gr}(A) = \text{the graded object of } \mathcal{A} \text{ whose } p\text{th graded piece is } \text{gr}^p(A)$$

and if $\mathcal{A}$ has countable direct sums, then we simply have

$$\text{gr}(A) = \bigoplus \text{gr}^p(A)$$

This defines an additive functor

$$\text{gr} : \text{Fil}(\mathcal{A}) \longrightarrow \text{Gr}(\mathcal{A}), \quad (A, F) \longmapsto \text{gr}(A).$$

05SP **Lemma 12.19.12.** Let $\mathcal{A}$ be an abelian category.

(1) Let A be a filtered object and $X \subset A$. Then for each p the sequence

$$0 \to \text{gr}^p(X) \to \text{gr}^p(A) \to \text{gr}^p(A/X) \to 0$$

is exact (with induced filtration on X and quotient filtration on A/X).

(2) Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. Then for each p the sequences

$$0 \to \text{gr}^p(\text{Ker}(f)) \to \text{gr}^p(A) \to \text{gr}^p(\text{Coim}(f)) \to 0$$

and

$$0 \to \text{gr}^p(\text{Im}(f)) \to \text{gr}^p(B) \to \text{gr}^p(\text{Coker}(f)) \to 0$$

are exact.

Proof. We have $F^{p+1}X = X \cap F^{p+1}A$, hence map $\text{gr}^p(X) \to \text{gr}^p(A)$ is injective. Dually the map $\text{gr}^p(A) \to \text{gr}^p(A/X)$ is surjective. The kernel of $F^pA/F^{p+1}A \to A/X + F^{p+1}A$ is clearly $F^{p+1}A + X \cap F^pA/F^{p+1}A = F^pX/F^{p+1}X$ hence exactness in the middle. The two short exact sequence of (2) are special cases of the short exact sequence of (1). $\square$

0127 Lemma 12.19.13. Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of finite filtered objects of $\mathcal{A}$. The following are equivalent

(1) f is strict,
(2) the morphism $\text{Coim}(f) \to \text{Im}(f)$ is an isomorphism,
(3) $\text{gr}(\text{Coim}(f)) \to \text{gr}(\text{Im}(f))$ is an isomorphism,
(4) the sequence $\text{gr}(\text{Ker}(f)) \to \text{gr}(A) \to \text{gr}(B)$ is exact,
(5) the sequence $\text{gr}(A) \to \text{gr}(B) \to \text{gr}(\text{Coker}(f))$ is exact, and
(6) the sequence
$$0 \to \text{gr}(\text{Ker}(f)) \to \text{gr}(A) \to \text{gr}(B) \to \text{gr}(\text{Coker}(f)) \to 0$$
is exact.

Proof. The equivalence of (1) and (2) is Lemma 12.19.4. By Lemma 12.19.12 we see that (4), (5), (6) imply (3) and that (3) implies (4), (5), (6). Hence it suffices to show that (3) implies (2). Thus we have to show that if $f : A \to B$ is an injective and surjective map of finite filtered objects which induces and isomorphism $\text{gr}(A) \to \text{gr}(B)$, then f induces an isomorphism of filtered objects. In other words, we have to show that $f(F^pA) = F^pB$ for all p. As the filtrations are finite we may prove this by descending induction on p. Suppose that $f(F^{p+1}A) = F^{p+1}B$. Then commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}A \arrow[r] \arrow[d, "f"] & F^pA \arrow[r] \arrow[d, "f"] & \text{gr}^p(A) \arrow[r] \arrow[d, "\text{gr}^p(f)"] & 0 \\
0 \arrow[r] & F^{p+1}B \arrow[r] & F^pB \arrow[r] & \text{gr}^p(B) \arrow[r] & 0
\end{tikzcd}$$

and the five lemma imply that $f(F^pA) = F^pB$. $\square$

0128 Lemma 12.19.14. Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume $\alpha : A \to B$ and $\beta : B \to C$ are strict morphisms of filtered objects. Then $\text{gr}(\text{Ker}(\beta)/\text{Im}(\alpha)) = \text{Ker}(\text{gr}(\beta))/\text{Im}(\text{gr}(\alpha)))$.

Proof. This follows formally from Lemma 12.19.12 and the fact that $\text{Coim}(\alpha) \cong \text{Im}(\alpha)$ and $\text{Coim}(\beta) \cong \text{Im}(\beta)$ by Lemma 12.19.4. $\square$

05QH Lemma 12.19.15. Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume A, B, C have finite filtrations and that $\text{gr}(A) \to \text{gr}(B) \to \text{gr}(C)$ is exact. Then

(1) for each $p \in \mathbf{Z}$ the sequence $\text{gr}^p(A) \to \text{gr}^p(B) \to \text{gr}^p(C)$ is exact,
(2) for each $p \in \mathbf{Z}$ the sequence $F^p(A) \to F^p(B) \to F^p(C)$ is exact,
(3) for each $p \in \mathbf{Z}$ the sequence $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact,
(4) the maps $A \to B$ and $B \to C$ are strict, and
(5) $A \to B \to C$ is exact (as a sequence in $\mathcal{A}$).

Proof. Part (1) is immediate from the definitions. We will prove (3) by induction on the length of the filtrations. If each of A, B, C has only one nonzero graded part, then (3) holds as $\mathrm{gr}(A) = A$, etc. Let n be the largest integer such that at least one of F^nA, F^nB, F^nC is nonzero. Set $A' = A/F^nA$, $B' = B/F^nB$, $C' = C/F^nC$ with induced filtrations. Note that $\mathrm{gr}(A) = F^nA \oplus \mathrm{gr}(A')$ and similarly for B and C. The induction hypothesis applies to $A' \to B' \to C'$, which implies that $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact for $p \geq n$. To conclude the same for $p = n+1$, i.e., to prove that $A \to B \to C$ is exact we use the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^nA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^nB \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^nC \arrow[r] & C \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

whose rows are short exact sequences of objects of $\mathcal{A}$. The proof of (2) is dual. Of course (5) follows from (2).

To prove (4) denote $f : A \to B$ and $g : B \to C$ the given morphisms. We know that $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C))$ by (2) and $f(A) = \mathrm{Ker}(g)$ by (5). Hence $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C)) = \mathrm{Ker}(g) \cap F^p(B) = f(A) \cap F^p(B)$ which proves that f is strict. The proof that g is strict is dual to this. $\square$

12.20. Spectral sequences

011M A nice discussion of spectral sequences may be found in [Eis95]. See also [McC01], [Lan02], etc.

011N Definition 12.20.1. Let $\mathcal{A}$ be an abelian category.

(1) A spectral sequence in $\mathcal{A}$ is given by a system $(E_r, d_r)_{r\geq 1}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to E_r$ is a morphism such that $d_r \circ d_r = 0$ and $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ for $r \geq 1$.
(2) A morphism of spectral sequences $f : (E_r, d_r)_{r\geq 1} \to (E'_r, d'_r)_{r\geq 1}$ is given by a family of morphisms $f_r : E_r \to E'_r$ such that $f_r \circ d_r = d'_r \circ f_r$ and such that f_{r+1} is the morphism induced by f_r via the identifications $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ and $E'_{r+1} = \mathrm{Ker}(d'_r)/\mathrm{Im}(d'_r)$.

We will sometimes loosen this definition somewhat and allow E_{r+1} to be an object with a given isomorphism $E_{r+1} \to \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$. In addition we sometimes have a system $(E_r, d_r)_{r\geq r_0}$ for some $r_0 \in \mathbf{Z}$ satisfying the properties of the definition above for indices $\geq r_0$. We will also call this a spectral sequence since by a simple renumbering it falls under the definition anyway. In fact, the cases $r_0 = 0$ and $r_0 = -1$ can be found in the literature.

Given a spectral sequence $(E_r, d_r)_{r\geq 1}$ we define

$$0 = B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_2 \subset Z_1 = E_1$$

by the following simple procedure. Set $B_2 = \mathrm{Im}(d_1)$ and $Z_2 = \mathrm{Ker}(d_1)$. Then it is clear that $d_2 : Z_2/B_2 \to Z_2/B_2$. Hence we can define B_3 as the unique subobject of E_1 containing B_2 such that B_3/B_2 is the image of d_2. Similarly we can define

Z_3 as the unique subobject of E_1 containing B_2 such that Z_3/B_2 is the kernel of d_2. And so on and so forth. In particular we have

$$E_r = Z_r/B_r$$

for all $r \geq 1$. In case the spectral sequence starts at $r = r_0$ then we can similarly construct B_i, Z_i as subobjects in E_{r_0}. In fact, in the literature one sometimes finds the notation

$$0 = B_r(E_r) \subset B_{r+1}(E_r) \subset B_{r+2}(E_r) \subset \ldots \subset Z_{r+2}(E_r) \subset Z_{r+1}(E_r) \subset Z_r(E_r) = E_r$$

to denote the filtration described above but starting with E_r.

011O Definition 12.20.2. Let $\mathcal{A}$ be an abelian category. Let $(E_r, d_r)_{r\geq 1}$ be a spectral sequence.

(1) If the subobjects $Z_\infty = \bigcap Z_r$ and $B_\infty = \bigcup B_r$ of E_1 exist then we define the limit[3] of the spectral sequence to be the object $E_\infty = Z_\infty/B_\infty$.
(2) We say that the spectral sequence degenerates at E_r if the differentials $d_r, d_{r+1}, \ldots$ are all zero.

Note that if the spectral sequence degenerates at E_r, then we have $E_r = E_{r+1} = \ldots = E_\infty$ (and the limit exists of course). Also, almost any abelian category we will encounter has countable sums and intersections.

0AMI Remark 12.20.3 (Variant). It is often the case that the terms of a spectral sequence have additional structure, for example a grading or a bigrading. To accomodate this (and to get around certain technical issues) we introduce the following notion. Let $\mathcal{A}$ be an abelian category. Let $(T_r)_{r\geq 1}$ be a sequence of translation or shift functors, i.e., $T_r : \mathcal{A} \to \mathcal{A}$ is an isomorphism of categories. In this setting a spectral sequence is given by a system $(E_r, d_r)_{r\geq 1}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to T_r E_r$ is a morphism such that $T_r d_r \circ d_r = 0$ so that

$$\ldots \longrightarrow T_r^{-1}E_r \xrightarrow{T_r^{-1}d_r} E_r \xrightarrow{d_r} T_r E_r \xrightarrow{T_r d_r} T_r^2 E_r \longrightarrow \ldots$$

is a complex and $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(T_r^{-1}d_r)$ for $r \geq 1$. It is clear what a morphism of spectral sequences means in this setting. In this setting we can still define

$$0 = B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_2 \subset Z_1 = E_1$$

and Z_∞ and B_∞ (if they exist) as above.

12.21. Spectral sequences: exact couples

011P

011Q Definition 12.21.1. Let $\mathcal{A}$ be an abelian category.

(1) An exact couple is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$ and α, f, g are morphisms as in the following diagram

$$\begin{tikzcd} A \arrow[rr, "\alpha"'] & & A \arrow[dl, "g"] \\ & E \arrow[ul, "f"] & \end{tikzcd}$$

[3]This notation is not universally accepted. In some references an additional pair of subobjects Z_∞ and B_∞ of E_1 such that $0 = B_1 \subset B_2 \subset \ldots \subset B_\infty \subset Z_\infty \subset \ldots \subset Z_2 \subset Z_1 = E_1$ is part of the data comprising a spectral sequence!

with the property that the kernel of each arrow is the image of its predecessor. So $\mathrm{Ker}(\alpha) = \mathrm{Im}(f)$, $\mathrm{Ker}(f) = \mathrm{Im}(g)$, and $\mathrm{Ker}(g) = \mathrm{Im}(\alpha)$.
(2) A morphism of exact couples $t : (A, E, \alpha, f, g) \to (A', E', \alpha', f', g')$ is given by morphisms $t_A : A \to A'$ and $t_E : E \to E'$ such that $\alpha' \circ t_A = t_A \circ \alpha$, $f' \circ t_E = t_A \circ f$, and $g' \circ t_A = t_E \circ g$.

011R Lemma 12.21.2. Let (A, E, α, f, g) be an exact couple in an abelian category $\mathcal{A}$. Set

(1) $d = g \circ f : E \to E$ so that $d \circ d = 0$,
(2) $E' = \mathrm{Ker}(d)/\mathrm{Im}(d)$,
(3) $A' = \mathrm{Im}(\alpha)$,
(4) $\alpha' : A' \to A'$ induced by α,
(5) $f' : E' \to A'$ induced by f,
(6) $g' : A' \to E'$ induced by "$g \circ \alpha^{-1}$".

Then we have

(1) $\mathrm{Ker}(d) = f^{-1}(\mathrm{Ker}(g)) = f^{-1}(\mathrm{Im}(\alpha))$,
(2) $\mathrm{Im}(d) = g(\mathrm{Im}(f)) = g(\mathrm{Ker}(\alpha))$,
(3) $(A', E', \alpha', f', g')$ is an exact couple.

Proof. Omitted. □

Hence it is clear that given an exact couple (A, E, α, f, g) we get a spectral sequence by setting $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

011S Definition 12.21.3. Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. The spectral sequence associated to the exact couple is the spectral sequence $(E_r, d_r)_{r \geq 1}$ with $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

011T Lemma 12.21.4. Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. Let $(E_r, d_r)_{r \geq 1}$ be the spectral sequence associated to the exact couple. In this case we have

$$0 = B_1 \subset \ldots \subset B_{r+1} = g(\mathrm{Ker}(\alpha^r)) \subset \ldots \subset Z_{r+1} = f^{-1}(\mathrm{Im}(\alpha^r)) \subset \ldots \subset Z_1 = E$$

and the map $d_{r+1} : E_{r+1} \to E_{r+1}$ is described by the following rule: For any (test) object T of $\mathcal{A}$ and any elements $x : T \to Z_{r+1}$ and $y : T \to A$ such that $f \circ x = \alpha^r \circ y$ we have

$$d_{r+1} \circ \overline{x} = \overline{g \circ y}$$

where $\overline{x} : T \to E_{r+1}$ is the induced morphism.

Proof. Omitted. □

Note that in the situation of the lemma we obviously have

$$B_\infty = g\left(\bigcup\nolimits_r \mathrm{Ker}(\alpha^r)\right) \subset Z_\infty = f^{-1}\left(\bigcap\nolimits_r \mathrm{Im}(\alpha^r)\right)$$

provided $\bigcup \mathrm{Ker}(\alpha^r)$ and $\bigcap \mathrm{Im}(\alpha^r)$ exist. This produces as limit $E_\infty = Z_\infty/B_\infty$, see Definition 12.20.2.

0AMJ Remark 12.21.5 (Variant). Let $\mathcal{A}$ be an abelian category. Let $S, T : \mathcal{A} \to \mathcal{A}$ be shift functors, i.e., isomorphisms of categories. We will indicate the n-fold compositions by $S^n A$ and $T^n A$ for $A \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbf{Z}$. In this situation an exact couple is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$ and $\alpha : A \to T^{-1}A$, $f : E \to A$, $g : A \to SE$ are morphisms such that

$$TE \xrightarrow{Tf} TA \xrightarrow{T\alpha} A \xrightarrow{g} SE \xrightarrow{Sf} SA$$

is an exact complex. Let's visualize this as follows

$$\begin{tikzcd}
TA \arrow[rrr, "T\alpha"'] & & & A \arrow[rrr, "\alpha"'] \arrow[dl, "g"] & & & T^{-1}A \arrow[dl, "T^{-1}g"] \\
& TE \arrow[ul, "Tf"] \arrow[r, dotted, no head] & SE & & E \arrow[ul, "f"'] \arrow[r, dotted, no head] & T^{-1}SE &
\end{tikzcd}$$

We set $d = g \circ f : E \to SE$. Then $d \circ S^{-1}d = g \circ f \circ S^{-1}g \circ S^{-1}f = 0$ because $f \circ S^{-1}g = 0$. Set $E' = \text{Ker}(d)/\text{Im}(S^{-1}d)$. Set $A' = \text{Im}(T\alpha)$. Let $\alpha' : A' \to T^{-1}A'$ induced by α. Let $f' : E' \to A'$ be induced by f which works because $f(\text{Ker}(d)) \subset \text{Ker}(g) = \text{Im}(T\alpha)$. Finally, let $g' : A' \to TSE'$ induced by "$Tg \circ (T\alpha)^{-1}$"[4].

In exactly the same way as above we find

(1) $\text{Ker}(d) = f^{-1}(\text{Ker}(g)) = f^{-1}(\text{Im}(T\alpha))$,
(2) $\text{Im}(d) = g(\text{Im}(f)) = g(\text{Ker}(\alpha))$,
(3) $(A', E', \alpha', f', g')$ is an exact couple for the shift functors TS and T.

We obtain a spectral sequence (as in Remark 12.20.3) with $E_1 = E$, $E_2 = E'$, etc, with $d_r : E_r \to T^{r-1}SE_r$ for all $r \geq 1$. Lemma 12.21.4 tells us that

$$SB_{r+1} = g(\text{Ker}(T^{-r+1}\alpha \circ \ldots \circ T^{-1}\alpha \circ \alpha))$$

and

$$Z_{r+1} = f^{-1}(\text{Im}(T\alpha \circ T^2\alpha \circ \ldots \circ T^r\alpha))$$

in this situation. The description of the map d_{r+1} is similar to that given in the lemma. (It may be easier to use these explicit descriptions to prove one gets a spectral sequence from such an exact couple.)

12.22. Spectral sequences: differential objects

011U

011V **Definition 12.22.1.** Let $\mathcal{A}$ be an abelian category. A *differential object* of $\mathcal{A}$ is a pair (A, d) consisting of an object A of $\mathcal{A}$ endowed with a selfmap d such that $d \circ d = 0$. A *morphism of differential objects* $(A, d) \to (B, d)$ is given by a morphism $\alpha : A \to B$ such that $d \circ \alpha = \alpha \circ d$.

011W **Lemma 12.22.2.** *Let $\mathcal{A}$ be an abelian category. The category of differential objects of $\mathcal{A}$ is abelian.*

Proof. Omitted. $\square$

011X **Definition 12.22.3.** For a differential object (A, d) we denote

$$H(A, d) = \text{Ker}(d)/\text{Im}(d)$$

its *homology*.

[4]This works because $TSE' = \text{Ker}(TSd)/\text{Im}(Td)$ and $Tg(\text{Ker}(T\alpha)) = Tg(\text{Im}(Tf)) = \text{Im}(T(d))$ and $TS(d)(\text{Im}(Tg)) = \text{Im}(TSg \circ TSf \circ Tg) = 0$.

011Y **Lemma 12.22.4.** Let $\mathcal{A}$ be an abelian category. Let $0 \to (A, d) \to (B, d) \to (C, d) \to 0$ be a short exact sequence of differential objects. Then we get an exact homology sequence

$$\ldots \to H(C, d) \to H(A, d) \to H(B, d) \to H(C, d) \to \ldots$$

Proof. Apply Lemma 12.13.12 to the short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

where the vertical arrows are d. □

We come to an important example of a spectral sequence. Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : (A, d) \to (A, d)$ be an endomorphism of this differential object. If we assume α injective, then we get a short exact sequence

$$0 \to (A, d) \to (A, d) \to (A/\alpha A, d) \to 0$$

of differential objects. By the Lemma 12.22.4 we get an exact couple

$$\begin{tikzcd} H(A, d) \arrow[rr, "\overline{\alpha}"'] & & H(A, d) \arrow[dl, "g"] \\ & H(A/\alpha A, d) \arrow[ul, "f"] & \end{tikzcd}$$

where g is the canonical map and f is the map defined in the snake lemma. Thus we get an associated spectral sequence! Since in this case we have $E_1 = H(A/\alpha A, d)$ we see that it makes sense to define $E_0 = A/\alpha A$ and $d_0 = d$. In other words, we start the spectral sequence with $r = 0$. According to our conventions in Section 12.20 we define a sequence of subobjects

$$0 = B_0 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_0 = E_0$$

with the property that $E_r = Z_r/B_r$. Namely we have for $r \geq 1$ that

(1) B_r is the image of $(\alpha^{r-1})^{-1}(dA)$ under the natural map $A \to A/\alpha A$,
(2) Z_r is the image of $d^{-1}(\alpha^r A)$ under the natural map $A \to A/\alpha A$, and
(3) $d_r : E_r \to E_r$ is given as follows: given an element $z \in Z_r$ choose an element $y \in A$ such that $d(z) = \alpha^r(y)$. Then $d_r(z+B_r+\alpha A) = y+B_r+\alpha A$.

Warning: It is not necessarily the case that $\alpha A \subset (\alpha^{r-1})^{-1}(dA)$, nor $\alpha A \subset d^{-1}(\alpha^r A)$. It is true that $(\alpha^{r-1})^{-1}(dA) \subset d^{-1}(\alpha^r A)$. We have

$$E_r = \frac{d^{-1}(\alpha^r A) + \alpha A}{(\alpha^{r-1})^{-1}(dA) + \alpha A}.$$

It is not hard to verify directly that (1) – (3) give a spectral sequence.

011Z **Definition 12.22.5.** Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : A \to A$ be an injective selfmap of A which commutes with d. The spectral sequence associated to (A, d, α) is the spectral sequence $(E_r, d_r)_{r \geq 0}$ described above.

0AMK Remark 12.22.6 (Variant). Let $\mathcal{A}$ be an abelian category and let $S, T : \mathcal{A} \to \mathcal{A}$ be shift functors, i.e., isomorphisms of categories. Assume that $TS = ST$ as functors. Consider pairs (A, d) consisting of an object A of $\mathcal{A}$ and a morphism $d : A \to SA$ such that $d \circ S^{-1}d = 0$. The category of these objects is abelian. We define $H(A, d) = \mathrm{Ker}(d)/\mathrm{Im}(S^{-1}d)$ and we observe that $H(SA, Sd) = SH(A, d)$ (canonical isomorphism). Given a short exact sequence

$$0 \to (A, d) \to (B, d) \to (C, d) \to 0$$

we obtain a long exact homology sequence

$$\ldots \to S^{-1}H(C, d) \to H(A, d) \to H(B, d) \to H(C, d) \to SH(A, d) \to \ldots$$

(note the shifts in the boundary maps). Since $ST = TS$ the functor T defines a shift functor on pairs by setting $T(A, d) = (TA, Td)$. Next, let $\alpha : (A, d) \to T^{-1}(A, d)$ be injective with cokernel (Q, d). Then we get an exact couple as in Remark 12.21.5 with shift functors TS and T given by

$$(H(A, d), S^{-1}H(Q, d), \overline{\alpha}, f, g)$$

where $\overline{\alpha} : H(A, d) \to T^{-1}H(A, d)$ is induced by α, the map $f : S^{-1}H(Q, d) \to H(A, d)$ is the boundary map and $g : H(A, d) \to TH(Q, d) = TS(S^{-1}H(Q, d))$ is induced by the quotient map $A \to TQ$. Thus we get a spectral sequence as above with $E_1 = S^{-1}H(Q, d)$ and differentials $d_r : E_r \to T^rSE_r$. As above we set $E_0 = S^{-1}Q$ and $d_0 : E_0 \to SE_0$ given by $S^{-1}d : S^{-1}Q \to Q$. If according to our conventions we define $B_r \subset Z_r \subset E_0$, then we have for $r \geq 1$ that

(1) SB_r is the image of

$$(T^{-r+1}\alpha \circ \ldots \circ T^{-1}\alpha)^{-1}\,\mathrm{Im}(T^{-r}S^{-1}d)$$

under the natural map $T^{-1}A \to Q$,

(2) Z_r is the image of

$$(S^{-1}T^{-1}d)^{-1}\,\mathrm{Im}(\alpha \circ \ldots \circ T^{r-1}\alpha)$$

under the natural map $S^{-1}T^{-1}A \to S^{-1}Q$.

The differentials can be described as follows: if $x \in Z_r$, then pick $x' \in S^{-1}T^{-1}A$ mapping to x. Then $S^{-1}T^{-1}d(x')$ is $(\alpha \circ \ldots \circ T^{r-1}\alpha)(y)$ for some $y \in T^{r-1}A$. Then $d_r(x) \in T^rSE_r$ is represented by the class of the image of y in $T^rSE_0 = T^rQ$ modulo T^rSB_r.

12.23. Spectral sequences: filtered differential objects

012A We can build a spectral sequence starting with a filtered differential object.

012B Definition 12.23.1. Let $\mathcal{A}$ be an abelian category. A filtered differential object (K, F, d) is a filtered object (K, F) of $\mathcal{A}$ endowed with an endomorphism $d : (K, F) \to (K, F)$ whose square is zero: $d \circ d = 0$.

To describe the spectral sequence associated to such an object we assume, for the moment, that $\mathcal{A}$ is an abelian category which has countable direct sums and countable direct sums are exact (this is not automatic, see Remark 12.16.3). Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Note that each F^nK is a differential object by itself. Consider the object $A = \bigoplus F^nK$ and endow it with a differential d

by using d on each summand. Then (A, d) is a differential object of $\mathcal{A}$ which comes equipped with a grading. Consider the map

$$\alpha : A \to A$$

which is given by the inclusions $F^nK \to F^{n-1}K$. This is clearly an injective morphism of differential objects $\alpha : (A, d) \to (A, d)$. Hence, by Definition 12.22.5 we get a spectral sequence. We will call this the spectral sequence associated to the filtered differential object (K, F, d).

Let us figure out the terms of this spectral sequence. First, note that $A/\alpha A = \mathrm{gr}(K)$ endowed with its differential $d = \mathrm{gr}(d)$. Hence we see that

$$E_0 = \mathrm{gr}(K), \quad d_0 = \mathrm{gr}(d).$$

Hence the homology of the graded differential object $\mathrm{gr}(K)$ is the next term:

$$E_1 = H(\mathrm{gr}(K), \mathrm{gr}(d)).$$

In addition we see that E_0 is a graded object of $\mathcal{A}$ and that d_0 is compatible with the grading. Hence clearly E_1 is a graded object as well. But it turns out that the differential d_1 does not preserve this grading; instead it shifts the degree by 1.

To work this out precisely, we define

$$Z_r^p = \frac{F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K}{F^{p+1}K}$$

and

$$B_r^p = \frac{F^pK \cap d(F^{p-r+1}K) + F^{p+1}K}{F^{p+1}K}.$$

This notation, although quite natural, seems to be different from the notation in most places in the literature. Perhaps it does not matter, since the literature does not seem to have a consistent choice of notation either. With these choices we see that $B_r \subset E_0$, resp. $Z_r \subset E_0$ (as defined in Section 12.22) is equal to $\bigoplus_p B_r^p$, resp. $\bigoplus_p Z_r^p$. Hence if we define

$$E_r^p = Z_r^p / B_r^p$$

for $r \geq 0$ and $p \in \mathbf{Z}$, then we have $E_r = \bigoplus_p E_r^p$. We can define a differential $d_r^p : E_r^p \to E_r^{p+r}$ by the rule

$$z + F^{p+1}K \longmapsto dz + F^{p+r+1}K$$

where $z \in F^pK \cap d^{-1}(F^{p+r}K)$.

012C Lemma 12.23.2. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ in $\mathrm{Gr}(\mathcal{A})$ associated to (K, F, d) such that $d_r : E_r \to E_r[r]$ for all r and such that the graded pieces E_r^p and maps $d_r^p : E_r^p \to E_r^{p+r}$ are as given above. Furthermore, $E_0^p = \mathrm{gr}^pK$, $d_0^p = \mathrm{gr}^p(d)$, and $E_1^p = H(\mathrm{gr}^pK, d)$.

Proof. If $\mathcal{A}$ has countable direct sums and if countable direct sums are exact, then this follows from the discussion above. In general, we proceed as follows; we strongly suggest the reader skip this proof. Consider the object $A = (F^{p+1}K)$ of $\mathrm{Gr}(\mathcal{A})$, i.e., we put $F^{p+1}K$ in degree p (the funny shift in numbering to get numbering correct later on). We endow it with a differential d by using d on each component. Then (A, d) is a differential object of $\mathrm{Gr}(\mathcal{A})$. Consider the map

$$\alpha : A \to A[-1]$$

which is given in degree p by the inclusions $F^{p+1}A \to F^pA$. This is clearly an injective morphism of differential objects $\alpha : (A, d) \to (A, d)[-1]$. Hence, we can apply Remark 12.22.6 with $S = \mathrm{id}$ and $T = [1]$. The corresponding spectral sequence $(E_r, d_r)_{r \geq 0}$ in $\mathrm{Gr}(\mathcal{A})$ is the spectral sequence we are looking for. Let us unwind the definitions a bit. First of all we have $E_r = (E_r^p)$ is an object of $\mathrm{Gr}(\mathcal{A})$. Then, since $T^rS = [r]$ we have $d_r : E_r \to E_r[r]$ which means that $d_r^p : E_r^p \to E_r^{p+r}$.

To see that the description of the graded pieces hold, we argue as above. Namely, first we have $E_0 = \mathrm{Coker}(\alpha : A \to A[-1])$ and by our choice of numbering above this gives $E_0^p = \mathrm{gr}^pK$. The first differential is given by $d_0^p = \mathrm{gr}^pd : E_0^p \to E_0^p$. Next, the description of the boundaries B_r and the cocycles Z_r in Remark 12.22.6 translates into a straightforward manner into the formulae for Z_r^p and B_r^p given above. □

012D Lemma 12.23.3. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The spectral sequence $(E_r, d_r)_{r \geq 0}$ associated to (K, F, d) has

$$d_1^p : E_1^p = H(\mathrm{gr}^pK) \longrightarrow H(\mathrm{gr}^{p+1}K) = E_1^{p+1}$$

equal to the boundary map in homology associated to the short exact sequence of differential objects

$$0 \to \mathrm{gr}^{p+1}K \to F^pK/F^{p+2}K \to \mathrm{gr}^pK \to 0.$$

Proof. This is clear from the formula for the differential d_1^p given just above Lemma 12.23.2. □

012E Definition 12.23.4. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The induced filtration on $H(K, d)$ is the filtration defined by $F^pH(K, d) = \mathrm{Im}(H(F^pK, d) \to H(K, d))$.

Writing out what this means we see that

$$F^pH(K, d) = \frac{\mathrm{Ker}(d) \cap F^pK + \mathrm{Im}(d)}{\mathrm{Im}(d)}$$

and hence we see that

$$\mathrm{gr}^pH(K) = \frac{\mathrm{Ker}(d) \cap F^pK + \mathrm{Im}(d)}{\mathrm{Ker}(d) \cap F^{p+1}K + \mathrm{Im}(d)} = \frac{\mathrm{Ker}(d) \cap F^pK}{\mathrm{Ker}(d) \cap F^{p+1}K + \mathrm{Im}(d) \cap F^pK}$$

012F Lemma 12.23.5. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. If Z_∞^p and B_∞^p exist (see proof), then

(1) the limit E_∞ exists and is graded having $E_\infty^p = Z_\infty^p/B_\infty^p$ in degree p, and
(2) the associated graded $\mathrm{gr}(H(K))$ of the cohomology of K is a graded subquotient of the graded limit object E_∞.

Proof. The objects Z_∞, B_∞, and the limit $E_\infty = Z_\infty/B_\infty$ of Definition 12.20.2 are objects of $\mathrm{Gr}(\mathcal{A})$ by our construction of the spectral sequence in the proof of Lemma 12.23.2. Since $Z_r = \bigoplus Z_r^p$ and $B_r = \bigoplus B_r^p$, if we assume that

$$Z_\infty^p = \bigcap\nolimits_r Z_r^p = \frac{\bigcap_r (F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K)}{F^{p+1}K}$$

and

$$B_\infty^p = \bigcup\nolimits_r B_r^p = \frac{\bigcup_r (F^pK \cap d(F^{p-r+1}K) + F^{p+1}K)}{F^{p+1}K}.$$

exist, then Z_∞ and B_∞ exist with degree p parts Z_∞^p and B_∞^p (follows from an elementary argument about unions and intersections of graded subobjects). Thus

$$E_\infty^p = \frac{\bigcap_r (F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K)}{\bigcup_r (F^pK \cap d(F^{p-r+1}K) + F^{p+1}K)}.$$

where the top and bottom exist. We have

012G (12.23.5.1) $$\mathrm{Ker}(d) \cap F^pK + F^{p+1}K \subset \bigcap\nolimits_r \left(F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K\right)$$

and

012H (12.23.5.2) $$\bigcup\nolimits_r \left(F^pK \cap d(F^{p-r+1}K) + F^{p+1}K\right) \subset \mathrm{Im}(d) \cap F^pK + F^{p+1}K.$$

Thus a subquotient of E_∞^p is

$$\frac{\mathrm{Ker}(d) \cap F^pK + F^{p+1}K}{\mathrm{Im}(d) \cap F^pK + F^{p+1}K} = \frac{\mathrm{Ker}(d) \cap F^pK}{\mathrm{Im}(d) \cap F^pK + \mathrm{Ker}(d) \cap F^{p+1}K}$$

Comparing with the formula given for $\mathrm{gr}^pH(K)$ in the discussion following Definition 12.23.4 we conclude. □

012I Definition 12.23.6. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. We say the spectral sequence associated to (K, F, d)

(1) weakly converges to $H(K)$ if $\mathrm{gr}H(K) = E_\infty$ via Lemma 12.23.5,
(2) abuts to $H(K)$ if it weakly converges to $H(K)$ and we have $\bigcap F^pH(K) = 0$ and $\bigcup F^pH(K) = H(K)$,

Unfortunately, it seems hard to find a consistent terminology for these notions in the literature.

012J Lemma 12.23.7. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The associated spectral sequence

(1) weakly converges to $H(K)$ if and only if for every $p \in \mathbf{Z}$ we have equality in equations (12.23.5.2) and (12.23.5.1),
(2) abuts to $H(K)$ if and only if it weakly converges to $H(K)$ and $\bigcap_p(\mathrm{Ker}(d) \cap F^pK + \mathrm{Im}(d)) = \mathrm{Im}(d)$ and $\bigcup_p(\mathrm{Ker}(d) \cap F^pK + \mathrm{Im}(d)) = \mathrm{Ker}(d)$.

Proof. Immediate from the discussions above. □

12.24. Spectral sequences: filtered complexes

012K

012L Definition 12.24.1. Let $\mathcal{A}$ be an abelian category. A filtered complex $K^\bullet$ of $\mathcal{A}$ is a complex of $\mathrm{Fil}(\mathcal{A})$ (see Definition 12.19.1).

We will denote the filtration on the objects by F. Thus F^pK^n denotes the pth step in the filtration of the nth term of the complex. Note that each $F^pK^\bullet$ is a complex of $\mathcal{A}$. Hence we could also have defined a filtered complex as a filtered object in the (abelian) category of complexes of $\mathcal{A}$. In particular $\mathrm{gr}K^\bullet$ is a graded object of the category of complexes of $\mathcal{A}$.

To describe the spectral sequence associated to such an object we assume, for the moment, that $\mathcal{A}$ is an abelian category which has countable direct sums and countable direct sums are exact (this is not automatic, see Remark 12.16.3). Let us denote d the differential of K. Forgetting the grading we can think of $\bigoplus K^n$ as a filtered differential object of $\mathcal{A}$. Hence according to Section 12.23 we obtain a spectral sequence $(E_r, d_r)_{r \geq 0}$. In this section we work out the terms of this spectral sequence, and we endow the terms of this spectral sequence with additional structure coming from the grading of K.

First we point out that $E_0^p = \text{gr}^p K^\bullet$ is a complex and hence is graded. Thus E_0 is bigraded in a natural way. It is customary to use the bigrading

$$E_0 = \bigoplus\nolimits_{p,q} E_0^{p,q}, \quad E_0^{p,q} = \text{gr}^p K^{p+q}$$

The idea is that $p+q$ should be thought of as the total degree of the (co)homology classes. Also, p is called the filtration degree, and q is called the complementary degree. The differential d_0 is compatible with this bigrading in the following way

$$d_0 = \bigoplus d_0^{p,q}, \quad d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}.$$

Namely, d_0^p is just the differential on the complex $\text{gr}^p K^\bullet$ (which occurs as $\text{gr}^p E_0$ just shifted a bit).

To go further we identify the objects B_r^p and Z_r^p introduced in Section 12.23 as graded objects and we work out the corresponding decompositions of the differentials. We do this in a completely straightforward manner, but again we warn the reader that our notation is not the same as notation found elsewhere. We define

$$Z_r^{p,q} = \frac{F^p K^{p+q} \cap d^{-1}(F^{p+r} K^{p+q+1}) + F^{p+1} K^{p+q}}{F^{p+1} K^{p+q}}$$

and

$$B_r^{p,q} = \frac{F^p K^{p+q} \cap d(F^{p-r+1} K^{p+q-1}) + F^{p+1} K^{p+q}}{F^{p+1} K^{p+q}}$$

and of course $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$. With these definitions it is completely clear that $Z_r^p = \bigoplus_q Z_r^{p,q}$, $B_r^p = \bigoplus_q B_r^{p,q}$, and $E_r^p = \bigoplus_q E_r^{p,q}$. Moreover, we have

$$0 \subset \ldots \subset B_r^{p,q} \subset \ldots \subset Z_r^{p,q} \subset \ldots \subset E_0^{p,q}$$

Also, the map d_r^p decomposes as the direct sum of the maps

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \quad z + F^{p+1} K^{p+q} \mapsto dz + F^{p+r+1} K^{p+q+1}$$

where $z \in F^p K^{p+q} \cap d^{-1}(F^{p+r} K^{p+q+1})$.

012M Lemma 12.24.2. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ in the category of bigraded objects of $\mathcal{A}$ associated to $(K^\bullet, F)$ such that d_r has bidegree $(r, -r+1)$ and such that E_r has bigraded pieces $E_r^{p,q}$ and maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ as given above. Furthermore, we have $E_0^{p,q} = \text{gr}^p(K^{p+q})$, $d_0^{p,q} = \text{gr}^p(d^{p+q})$, and $E_1^{p,q} = H^{p+q}(\text{gr}^p(K^\bullet))$.

Proof. If $\mathcal{A}$ has countable direct sums and if countable direct sums are exact, then this follows from the discussion above. In general, we proceed as follows; we strongly suggest the reader skip this proof. Consider the bigraded object $A = (F^{p+1} K^{p+1+q})$ of $\mathcal{A}$, i.e., we put $F^{p+1} K^{p+1+q}$ in degree (p, q) (the funny shift in numbering to get numbering correct later on). We endow it with a differential $d : A \to A[0, 1]$ by using d on each component. Then (A, d) is a differential bigraded object. Consider the map

$$\alpha : A \to A[-1, 1]$$

which is given in degree (p, q) by the inclusion $F^{p+1} K^{p+1+q} \to F^p K^{p+1+q}$. This is an injective morphism of differential objects $\alpha : (A, d) \to (A, d)[-1, 1]$. Hence, we can apply Remark 12.22.6 with $S = [0, 1]$ and $T = [1, -1]$. The corresponding spectral sequence $(E_r, d_r)_{r\geq 0}$ of bigraded objects is the spectral sequence we are looking for. Let us unwind the definitions a bit. First of all we have $E_r = (E_r^{p,q})$. Then, since $T^r S = [r, -r+1]$ we have $d_r : E_r \to E_r[r, -r+1]$ which means that $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r, q-r+1}$.

To see that the description of the graded pieces hold, we argue as above. Namely, first we have

$$E_0 = \mathrm{Coker}(\alpha : A \to A[-1,1])[0,-1] = \mathrm{Coker}(\alpha[0,-1] : A[0,-1] \to A[-1,0])$$

and by our choice of numbering above this gives

$$E_0^{p,q} = \mathrm{Coker}(F^{p+1}K^{p+q} \to F^pK^{p+q}) = \mathrm{gr}^pK^{p+q}$$

The first differential is given by $d_0^{p,q} = \mathrm{gr}^p d^{p+q} : E_0^{p,q} \to E_0^{p,q+1}$. Next, the description of the boundaries B_r and the cocycles Z_r in Remark 12.22.6 translates into a straightforward manner into the formulae for $Z_r^{p,q}$ and $B_r^{p,q}$ given above. □

012N Lemma 12.24.3. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. Let $(E_r, d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$.

(1) The map

$$d_1^{p,q} : E_1^{p,q} = H^{p+q}(\mathrm{gr}^p(K^\bullet)) \longrightarrow E_1^{p+1,q} = H^{p+q+1}(\mathrm{gr}^{p+1}(K^\bullet))$$

is equal to the boundary map in cohomology associated to the short exact sequence of complexes

$$0 \to \mathrm{gr}^{p+1}(K^\bullet) \to F^pK^\bullet/F^{p+2}K^\bullet \to \mathrm{gr}^p(K^\bullet) \to 0.$$

(2) Assume that $d(F^pK) \subset F^{p+1}K$ for all $p \in \mathbf{Z}$. Then d induces the zero differential on $\mathrm{gr}^p(K^\bullet)$ and hence $E_1^{p,q} = \mathrm{gr}^p(K^\bullet)^{p+q}$. Furthermore, in this case

$$d_1^{p,q} : E_1^{p,q} = \mathrm{gr}^p(K^\bullet)^{p+q} \longrightarrow E_1^{p+1,q} = \mathrm{gr}^{p+1}(K^\bullet)^{p+q+1}$$

is the morphism induced by d.

Proof. This is clear from the formula given for the differential $d_1^{p,q}$ just above Lemma 12.24.2. □

012O Lemma 12.24.4. Let $\mathcal{A}$ be an abelian category. Let $\alpha : (K^\bullet, F) \to (L^\bullet, F)$ be a morphism of filtered complexes of $\mathcal{A}$. Let $(E_r(K), d_r)_{r\geq 0}$, resp. $(E_r(L), d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$, resp. $(L^\bullet, F)$. The morphism α induces a canonical morphism of spectral sequences $\{\alpha_r : E_r(K) \to E_r(L)\}_{r\geq 0}$ compatible with the bigradings.

Proof. Obvious from the explicit representation of the terms of the spectral sequences. □

012P Definition 12.24.5. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The induced filtration on $H^n(K^\bullet)$ is the filtration defined by $F^pH^n(K^\bullet) = \mathrm{Im}(H^n(F^pK^\bullet) \to H^n(K^\bullet))$.

Writing out what this means we see that

012R (12.24.5.1)
$$F^pH^n(K^\bullet, d) = \frac{\mathrm{Ker}(d) \cap F^pK^n + \mathrm{Im}(d) \cap K^n}{\mathrm{Im}(d) \cap K^n}$$

and hence we see that

0BDT (12.24.5.2)
$$\text{gr}^p H^n(K^\bullet) = \frac{\text{Ker}(d) \cap F^pK^n}{\text{Ker}(d) \cap F^{p+1}K^n + \text{Im}(d) \cap F^pK^n}$$

(one intermediate step omitted).

012Q Lemma 12.24.6. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. If $Z_\infty^{p,q}$ and $B_\infty^{p,q}$ exist (see proof), then

(1) the limit E_∞ exists and is a bigraded object having $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$ in bidegree (p, q),
(2) the pth graded part $\text{gr}^p H^n(K^\bullet)$ of the nth cohomology object of $K^\bullet$ is a subquotient of $E_\infty^{p,n-p}$.

Proof. The objects Z_∞, B_∞, and the limit $E_\infty = Z_\infty/B_\infty$ of Definition 12.20.2 are bigraded objects of $\mathcal{A}$ by our construction of the spectral sequence in Lemma 12.24.2. Since $Z_r = \bigoplus Z_r^{p,q}$ and $B_r = \bigoplus B_r^{p,q}$, if we assume that

$$Z_\infty^{p,q} = \bigcap\nolimits_r Z_r^{p,q} = \bigcap\nolimits_r \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_\infty^{p,q} = \bigcup\nolimits_r B_r^{p,q} = \bigcup\nolimits_r \frac{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

exist, then Z_∞ and B_∞ exist with bidegree (p, q) parts $Z_\infty^{p,q}$ and $B_\infty^{p,q}$ (follows from an elementary argument about unions and intersections of bigraded objects). Thus

$$E_\infty^{p,q} = \frac{\bigcap_r (F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q})}{\bigcup_r (F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q})}.$$

where the top and the bottom exist. With $n = p + q$ we have

(12.24.6.1)
012S
$$\text{Ker}(d) \cap F^pK^n + F^{p+1}K^n \subset \bigcap\nolimits_r \left(F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n\right)$$

and

(12.24.6.2)
012T
$$\bigcup\nolimits_r \left(F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n\right) \subset \text{Im}(d) \cap F^pK^n + F^{p+1}K^n.$$

Thus a subquotient of $E_\infty^{p,q}$ is

$$\frac{\text{Ker}(d) \cap F^pK^n + F^{p+1}K^n}{\text{Im}(d) \cap F^pK^n + F^{p+1}K^n} = \frac{\text{Ker}(d) \cap F^pK^n}{\text{Im}(d) \cap F^pK^n + \text{Ker}(d) \cap F^{p+1}K^n}$$

Comparing with (12.24.5.2) we conclude. □

0BDU Definition 12.24.7. Let $\mathcal{A}$ be an abelian category. Let $(E_r, d_r)_{r \geq r_0}$ be a spectral sequence of bigraded objects of $\mathcal{A}$ with d_r of bidegree $(r, -r + 1)$. We say such a spectral sequence is

(1) regular if for all $p, q \in \mathbf{Z}$ there is a $b = b(p, q)$ such that the maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are zero for $r \geq b$,
(2) coregular if for all $p, q \in \mathbf{Z}$ there is a $b = b(p, q)$ such that the maps $d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}$ are zero for $r \geq b$,
(3) bounded if for all n there are only a finite number of nonzero $E_{r_0}^{p,n-p}$,
(4) bounded below if for all n there is a $b = b(n)$ such that $E_{r_0}^{p,n-p} = 0$ for $p \geq b$.

(5) bounded above if for all n there is a $b = b(n)$ such that $E_{r_0}^{p,n-p} = 0$ for $p \leq b$.

Bounded below means that if we look at $E_r^{p,q}$ on the line $p + q = n$ (whose slope is -1) we obtain zeros as (p, q) moves down and to the right. As mentioned above there is no consistent terminology regarding these notions in the literature.

0BDV Lemma 12.24.8. In the situation of Definition 12.24.7. Let $Z_r^{p,q}, B_r^{p,q} \subset E_{r_0}^{p,q}$ be the (p,q)-graded parts of Z_r, B_r defined as in Section 12.20.

(1) The spectral sequence is regular if and only if for all p, q there exists an $r = r(p,q)$ such that $Z_r^{p,q} = Z_{r+1}^{p,q} = \ldots$
(2) The spectral sequence is coregular if and only if for all p, q there exists an $r = r(p,q)$ such that $B_r^{p,q} = B_{r+1}^{p,q} = \ldots$
(3) The spectral sequence is bounded if and only if it is both bounded below and bounded above.
(4) If the spectral sequence is bounded below, then it is regular.
(5) If the spectral sequence is bounded above, then it is coregular.

Proof. Omitted. Hint: If $E_r^{p,q} = 0$, then we have $E_{r'}^{p,q} = 0$ for all $r' \geq r$. □

012U Definition 12.24.9. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. We say the spectral sequence associated to $(K^\bullet, F)$

(1) weakly converges to $H^*(K^\bullet)$ if $\text{gr}^p H^n(K^\bullet) = E_\infty^{p,n-p}$ via Lemma 12.24.6 for all $p, n \in \mathbf{Z}$,
(2) abuts to $H^*(K^\bullet)$ if it weakly converges to $H^*(K^\bullet)$ and $\bigcap_p F^p H^n(K^\bullet) = 0$ and $\bigcup_p F^p H^n(K^\bullet) = H^n(K^\bullet)$ for all n,
(3) converges to $H^*(K^\bullet)$ if it is regular, abuts to $H^*(K^\bullet)$, and $H^n(K^\bullet) = \lim_p H^n(K^\bullet)/F^p H^n(K^\bullet)$.

Weak convergence, abutment, or convergence is symbolized by the notation $E_r^{p,q} \Rightarrow H^{p+q}(K^\bullet)$. As mentioned above there is no consistent terminology regarding these notions in the literature.

012V Lemma 12.24.10. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The associated spectral sequence

(1) weakly converges to $H^*(K^\bullet)$ if and only if for every $p, q \in \mathbf{Z}$ we have equality in equations (12.24.6.2) and (12.24.6.1),
(2) abuts to $H^*(K)$ if and only if it weakly converges to $H^*(K^\bullet)$ and we have $\bigcap_p(\text{Ker}(d) \cap F^p K^n + \text{Im}(d) \cap K^n) = \text{Im}(d) \cap K^n$ and $\bigcup_p(\text{Ker}(d) \cap F^p K^n + \text{Im}(d) \cap K^n) = \text{Ker}(d) \cap K^n$.

Proof. Immediate from the discussions above. □

012W Lemma 12.24.11. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume that the filtration on each K^n is finite (see Definition 12.19.1). Then

(1) the spectral sequence associated to $(K^\bullet, F)$ is bounded,
(2) the filtration on each $H^n(K^\bullet)$ is finite,
(3) the spectral sequence associated to $(K^\bullet, F)$ converges to $H^*(K^\bullet)$,
(4) if $\mathcal{C} \subset \mathcal{A}$ is a weak Serre subcategory and for some r we have $E_r^{p,q} \in \mathcal{C}$ for all $p, q \in \mathbf{Z}$, then $H^n(K^\bullet)$ is in $\mathcal{C}$.

Proof. Part (1) follows as $E_0^{p,n-p} = \text{gr}^p K^n$. Part (2) is clear from Equation (12.24.5.1). We will use Lemma 12.24.10 to prove that the spectral sequence weakly converges. Fix $p, n \in \mathbf{Z}$. The right hand side of (12.24.6.1) is equal to $F^pK^n \cap \text{Ker}(d) + F^{p+1}K^n$ because $F^{p+r}K^n = 0$ for $r \gg 0$. Thus (12.24.6.1) is an equality. The left hand side of (12.24.6.2) is equal to $F^pK^n \cap \text{Im}(d) + F^{p+1}K^n$ because $F^{p-r+1}K^{n-1} = K^{n-1}$ for $r \gg 0$. Thus (12.24.6.2) is an equality. Since the filtration on $H^n(K^\bullet)$ is finite by (2) we see that we have abutment. To prove we have convergence we have to show the spectral sequence is regular which follows as it is bounded (Lemma 12.24.8) and we have to show that $H^n(K^\bullet) = \lim_p H^n(K^\bullet)/F^pH^n(K^\bullet)$ which follows from the fact that the filtration on $H^*(K^\bullet)$ is finite proved in part (2).

Proof of (4). Assume that for some $r \geq 0$ we have $E_r^{p,q} \in \mathcal{C}$ for some weak Serre subcategory $\mathcal{C}$ of $\mathcal{A}$. Then $E_{r+1}^{p,q}$ is in $\mathcal{C}$ as well, see Lemma 12.10.3. By boundedness proved above (which implies that the spectral sequence is both regular and coregular, see Lemma 12.24.8) we can find an $r' \geq r$ such that $E_\infty^{p,q} = E_{r'}^{p,q}$ for all p, q with $p + q = n$. Thus $H^n(K^\bullet)$ is an object of $\mathcal{A}$ which has a finite filtration whose graded pieces are in $\mathcal{C}$. This implies that $H^n(K^\bullet)$ is in $\mathcal{C}$ by Lemma 12.10.3. □

0BDW Lemma 12.24.12. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume that the filtration on each K^n is finite (see Definition 12.19.1) and that for some r we have only a finite number of nonzero $E_r^{p,q}$. Then only a finite number of $H^n(K^\bullet)$ are nonzero and we have

$$\sum (-1)^n [H^n(K^\bullet)] = \sum (-1)^{p+q} [E_r^{p,q}]$$

in $K_0(\mathcal{A}')$ where $\mathcal{A}'$ is the smallest weak Serre subcategory of $\mathcal{A}$ containing the objects $E_r^{p,q}$.

Proof. Denote E_r^{even} and E_r^{odd} the even and odd part of E_r defined as the direct sum of the (p,q) components with $p+q$ even and odd. The differential d_r defines maps $\varphi : E_r^{even} \to E_r^{odd}$ and $\psi : E_r^{odd} \to E_r^{even}$ whose compositions either way give zero. Then we see that

$$\begin{aligned} [E_r^{even}] - [E_r^{odd}] &= [\text{Ker}(\varphi)] + [\text{Im}(\varphi)] - [\text{Ker}(\psi)] - [\text{Im}(\psi)] \\ &= [\text{Ker}(\varphi)/\text{Im}(\psi)] - [\text{Ker}(\psi)/\text{Im}(\varphi)] \\ &= [E_{r+1}^{even}] - [E_{r+1}^{odd}] \end{aligned}$$

Note that all the intervening objects are in the smallest Serre subcategory containing the objects $E_r^{p,q}$. Continuing in this manner we see that we can increase r at will. Since there are only a finite number of pairs (p,q) for which $E_r^{p,q}$ is nonzero, a property which is inherited by $E_{r+1}, E_{r+2}, \ldots$, we see that we may assume that $d_r = 0$. At this stage we see that $H^n(K^\bullet)$ has a finite filtration (Lemma 12.24.11) whose graded pieces are exactly the $E_r^{p,n-p}$ and the result is clear. □

The following lemma is more a kind of sanity check for our definitions. Surely, if we have a filtered complex such that for every n we have

$$H^n(F^pK^\bullet) = 0 \text{ for } p \gg 0 \quad \text{and} \quad H^n(F^pK^\bullet) = H^n(K^\bullet) \text{ for } p \ll 0,$$

then the corresponding spectral sequence should converge?

0BK5 **Lemma 12.24.13.** Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume

(1) for every n there exist $p_0(n)$ such that $H^n(F^pK^\bullet) = 0$ for $p \geq p_0(n)$,
(2) for every n there exist $p_1(n)$ such that $H^n(F^pK^\bullet) \to H^n(K^\bullet)$ is an isomorphism for $p \leq p_1(n)$.

Then

(1) the spectral sequence associated to $(K^\bullet, F)$ is bounded,
(2) the filtration on each $H^n(K^\bullet)$ is finite,
(3) the spectral sequence associated to $(K^\bullet, F)$ converges to $H^*(K^\bullet)$.

Proof. Fix n. Using the long exact cohomology sequence associated to the short exact sequence of complexes

$$0 \to F^{p+1}K^\bullet \to F^pK^\bullet \to \text{gr}^pK^\bullet \to 0$$

we find that $E_1^{p,n-p} = 0$ for $p \geq \max(p_0(n), p_0(n+1))$ and $p < \min(p_1(n), p_1(n+1))$. Hence the spectral sequence is bounded (Definition 12.24.7). This proves (1).

It is clear from the assumptions and Definition 12.24.5 that the filtration on $H^n(K^\bullet)$ is finite. This proves (2).

Next we prove that the spectral sequence weakly converges to $H^*(K^\bullet)$ using Lemma 12.24.10. Let us show that we have equality in (12.24.6.1). Namely, for $p + r > p_0(n+1)$ the map

$$d : F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) \to F^{p+r}K^{n+1}$$

ends up in the image of $d : F^{p+r}K^n \to F^{p+r}K^{n+1}$ because the complex $F^{p+r}K^\bullet$ is exact in degree $n+1$. We conclude that $F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) = d(F^{p+r}K^n) + \text{Ker}(d) \cap F^pK^n$. Hence for such r we have

$$\text{Ker}(d) \cap F^pK^n + F^{p+1}K^n = F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n$$

which proves the desired equality. To show that we have equality in (12.24.6.2) we use that for $p - r + 1 < p_1(n-1)$ we have

$$d(F^{p-r+1}K^{n-1}) = \text{Im}(d) \cap F^{p-r+1}K^n$$

because the map $F^{p-r+1}K^\bullet \to K^\bullet$ induces an isomorphism on cohomology in degree $n-1$. This shows that we have

$$F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n = \text{Im}(d) \cap F^pK^n + F^{p+1}K^n$$

for such r which proves the desired equality.

To see that the spectral sequence abuts to $H^*(K^\bullet)$ using Lemma 12.24.10 we have to show that $\bigcap_p(\text{Ker}(d) \cap F^pK^n + \text{Im}(d) \cap K^n) = \text{Im}(d) \cap K^n$ and $\bigcup_p(\text{Ker}(d) \cap F^pK^n + \text{Im}(d) \cap K^n) = \text{Ker}(d) \cap K^n$. For $p \geq p_0(n)$ we have $\text{Ker}(d) \cap F^pK^n + \text{Im}(d) \cap K^n = \text{Im}(d) \cap K^n$ and for $p \leq p_1(n)$ we have $\text{Ker}(d) \cap F^pK^n + \text{Im}(d) \cap K^n = \text{Ker}(d) \cap K^n$. Combining weak convergence, abutment, and boundedness we see that (2) and (3) are true. $\square$

12.25. Spectral sequences: double complexes

012X Let $K^{\bullet,\bullet}$ be a double complex, see Section 12.18. It is customary to denote $H_I^p(K^{\bullet,\bullet})$ the complex with terms $\text{Ker}(d_1^{p,q})/\text{Im}(d_1^{p-1,q})$ (varying q) and differential induced by d_2. Then $H_{II}^q(H_I^p(K^{\bullet,\bullet}))$ denotes its cohomology in degree q. It is also customary to denote $H_{II}^q(K^{\bullet,\bullet})$ the complex with terms $\text{Ker}(d_2^{p,q})/\text{Im}(d_2^{p,q-1})$ (varying p) and differential induced by d_1. Then $H_I^p(H_{II}^q(K^{\bullet,\bullet}))$ denotes its cohomology in degree p. It will turn out that these cohomology groups show up as the terms in the spectral sequence for a filtration on the associated total complex or simple complex, see Definition 12.18.3.

There are two natural filtrations on the total complex $\text{Tot}(K^{\bullet,\bullet})$ associated to the double complex $K^{\bullet,\bullet}$. Namely, we define

$$F_I^p(\text{Tot}^n(K^{\bullet,\bullet})) = \bigoplus\nolimits_{i+j=n,\ i\geq p} K^{i,j} \quad \text{and} \quad F_{II}^p(\text{Tot}^n(K^{\bullet,\bullet})) = \bigoplus\nolimits_{i+j=n,\ j\geq p} K^{i,j}.$$

It is immediately verified that $(\text{Tot}(K^{\bullet,\bullet}), F_I)$ and $(\text{Tot}(K^{\bullet,\bullet}), F_{II})$ are filtered complexes. By Section 12.24 we obtain two spectral sequences. It is customary to denote $({}'E_r, {}'d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_I and to denote $({}''E_r, {}''d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_{II}. Here is a description of these spectral sequences.

0130 **Lemma 12.25.1.** Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. The spectral sequences associated to $K^{\bullet,\bullet}$ have the following terms:

(1) ${}'E_0^{p,q} = K^{p,q}$ with ${}'d_0^{p,q} = (-1)^p d_2^{p,q} : K^{p,q} \to K^{p,q+1}$,
(2) ${}''E_0^{p,q} = K^{q,p}$ with ${}''d_0^{p,q} = d_1^{q,p} : K^{q,p} \to K^{q+1,p}$,
(3) ${}'E_1^{p,q} = H^q(K^{p,\bullet})$ with ${}'d_1^{p,q} = H^q(d_1^{p,\bullet})$,
(4) ${}''E_1^{p,q} = H^q(K^{\bullet,p})$ with ${}''d_1^{p,q} = (-1)^q H^q(d_2^{\bullet,p})$,
(5) ${}'E_2^{p,q} = H_I^p(H_{II}^q(K^{\bullet,\bullet}))$,
(6) ${}''E_2^{p,q} = H_{II}^p(H_I^q(K^{\bullet,\bullet}))$.

Proof. Omitted. $\square$

These spectral sequences define two filtrations on $H^n(\text{Tot}(K^{\bullet,\bullet}))$. We will denote these F_I and F_{II}.

0131 **Definition 12.25.2.** Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. We say the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ *weakly converges* to $H^n(\text{Tot}(K^{\bullet,\bullet}))$, *abuts* to $H^n(\text{Tot}(K^{\bullet,\bullet}))$, or *converges* to $H^n(\text{Tot}(K^{\bullet,\bullet}))$ if Definition 12.24.9 applies. Similarly we say the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ *weakly converges* to $H^n(\text{Tot}(K^{\bullet,\bullet}))$, *abuts* to $H^n(\text{Tot}(K^{\bullet,\bullet}))$, or *converges* to $H^n(\text{Tot}(K^{\bullet,\bullet}))$ if Definition 12.24.9 applies.

As mentioned above there is no consistent terminology regarding these notions in the literature. In the situation of the definition, we have weak convergence of the first spectral sequence if for all n

$$\text{gr}_{F_I}(H^n(\text{Tot}(K^{\bullet,\bullet}))) = \oplus_{p+q=n} {}'E_\infty^{p,q}$$

via the canonical comparison of Lemma 12.24.6. Similarly the second spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ weakly converges if for all n

$$\text{gr}_{F_{II}}(H^n(\text{Tot}(K^{\bullet,\bullet}))) = \oplus_{p+q=n} {}''E_\infty^{p,q}$$

via the canonical comparison of Lemma 12.24.6.

0132 **Lemma 12.25.3.** Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. Assume that for every $n \in \mathbf{Z}$ there are only finitely many nonzero $K^{p,q}$ with $p+q = n$. Then

(1) the two spectral sequences associated to $K^{\bullet,\bullet}$ are bounded,
(2) the filtrations F_I, F_{II} on each $H^n(\text{Tot}(K^{\bullet,\bullet}))$ are finite,
(3) the spectral sequences $({}'E_r, {}'d_r)_{r\geq 0}$ and $({}''E_r, {}''d_r)_{r\geq 0}$ converge to $H^*(\text{Tot}(K^{\bullet,\bullet}))$,
(4) if $\mathcal{C} \subset \mathcal{A}$ is a weak Serre subcategory and for some r we have ${}'E_r^{p,q} \in \mathcal{C}$ for all $p, q \in \mathbf{Z}$, then $H^n(\text{Tot}(K^{\bullet,\bullet}))$ is in $\mathcal{C}$. Similarly for $({}''E_r, {}''d_r)_{r\geq 0}$.

Proof. Follows immediately from Lemma 12.24.11. □

Here is our first application of spectral sequences.

0133 **Lemma 12.25.4.** Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex. Let $A^{\bullet,\bullet}$ be a double complex. Let $\alpha^p : K^p \to A^{p,0}$ be morphisms. Assume that

(1) For every $n \in \mathbf{Z}$ there are only finitely many nonzero $A^{p,q}$ with $p+q = n$.
(2) We have $A^{p,q} = 0$ if $q < 0$.
(3) The morphisms α^p give rise to a morphism of complexes $\alpha : K^\bullet \to A^{\bullet,0}$.
(4) The complex $A^{p,\bullet}$ is exact in all degrees $q \neq 0$ and the morphism $K^p \to A^{p,0}$ induces an isomorphism $K^p \to \text{Ker}(d_2^{p,0})$.

Then α induces a quasi-isomorphism

$$K^\bullet \longrightarrow \text{Tot}(A^{\bullet,\bullet})$$

of complexes. Moreover, there is a variant of this lemma involving the second variable q instead of p.

Proof. The map is simply the map given by the morphisms $K^n \to A^{n,0} \to \text{Tot}^n(A^{\bullet,\bullet})$, which are easily seen to define a morphism of complexes. Consider the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ associated to the double complex $A^{\bullet,\bullet}$. By Lemma 12.25.3 this spectral sequence converges and the induced filtration on $H^n(\text{Tot}(A^{\bullet,\bullet}))$ is finite for each n. By Lemma 12.25.1 and assumption (4) we have ${}'E_1^{p,q} = 0$ unless $q = 0$ and ${}'E_1^{p,0} = K^p$ with differential ${}'d_1^{p,0}$ identified with d_K^p. Hence ${}'E_2^{p,0} = H^p(K^\bullet)$ and zero otherwise. This clearly implies $d_2^{p,q} = d_3^{p,q} = \ldots = 0$ for degree reasons. Hence we conclude that $H^n(\text{Tot}(A^{\bullet,\bullet})) = H^n(K^\bullet)$. We omit the verification that this identification is given by the morphism of complexes $K^\bullet \to \text{Tot}(A^{\bullet,\bullet})$ introduced above. □

0FKH **Lemma 12.25.5.** Let $\mathcal{A}$ be an abelian category. Let $M^\bullet$ be a complex of $\mathcal{A}$. Let

$$a : M^\bullet[0] \longrightarrow \left(A^{0,\bullet} \to A^{1,\bullet} \to A^{2,\bullet} \to \ldots\right)$$

be a homotopy equivalence in the category of complexes of complexes of $\mathcal{A}$. Then the map $\alpha : M^\bullet \to \text{Tot}(A^{\bullet,\bullet})$ induced by $M^\bullet \to A^{0,\bullet}$ is a homotopy equivalence.

Proof. The statement makes sense as a complex of complexes is the same thing as a double complex. The assumption means there is a map

$$b : \left(A^{0,\bullet} \to A^{1,\bullet} \to A^{2,\bullet} \to \ldots\right) \longrightarrow M^\bullet[0]$$

such that $a \circ b$ and $b \circ a$ are homotopic to the identity in the category of complexes of complexes. This means that $b \circ a$ is the identity of $M^\bullet[0]$ (because there is only one term in degree 0). Also, observe that b is given by a map $b^0 : A^{0,\bullet} \to M^\bullet$ and zero in all other degrees. Thus b induces a map $\beta : \text{Tot}(A^{\bullet,\bullet}) \to M^\bullet$ and $\beta \circ \alpha$ is the identity on $M^\bullet$. Finally, we have to show that the map $\alpha \circ \beta$ is homotopic to the identity. For this we choose maps of complexes $h^n : A^{n,\bullet} \to A^{n-1,\bullet}$ such that $a \circ b - \text{id} = d_1 \circ h + h \circ d_1$ which exist by assumption. Here $d_1 : A^{n,\bullet} \to A^{n+1,\bullet}$ are the differentials of the complex of complexes. We will also denote d_2 the differentials of the complexes $A^{n,\bullet}$ for all n. Let $h^{n,m} : A^{n,m} \to A^{n-1,m}$ be the components of h^n. Then we can consider

$$h' : \text{Tot}(A^{\bullet,\bullet})^k = \bigoplus\nolimits_{n+m=k} A^{n,m} \to \bigoplus\nolimits_{n+m=k-1} A^{n,m} = \text{Tot}(A^{\bullet,\bullet})^{k-1}$$

given by $h^{n,m}$ on the summand $A^{n,m}$. Then we compute that the map

$$d_{\text{Tot}(A^{\bullet,\bullet})} \circ h' + h' \circ d_{\text{Tot}(A^{\bullet,\bullet})}$$

restricted to the summand $A^{n,m}$ is equal to

$$d_1^{n-1,m} \circ h^{n,m} + (-1)^{n-1} d_2^{n-1,m} \circ h^{n,m} + h^{n+1,m} \circ d_1^{n,m} + h^{n,m+1} \circ (-1)^n d_2^{n,m}$$

Since h^n is a map of complexes, the terms $(-1)^{n-1} d_2^{n-1,m} \circ h^{n,m}$ and $h^{n,m+1} \circ (-1)^n d_2^{n,m}$ cancel. The other two terms give $(\alpha \circ \beta)|_{A^{n,m}} - \text{id}_{A^{n,m}}$ because $a \circ b - \text{id} = d_1 \circ h + h \circ d_1$. This finishes the proof. $\square$

12.26. Double complexes of abelian groups

0E1P In this section we put some results on double complexes of abelian groups for which do not (yet) have the analogues results for general abelian categories. Please be careful not to use these lemmas except when the underlying abelian category is the category of abelian groups or some such (e.g., the category of modules over a ring). Some of the arguments will be difficult to follow without drawing "zig-zags" on a napkin – compare with the proof of Algebra, Lemma 10.75.3.

0E1Q **Lemma 12.26.1.** Let $M^\bullet$ be a complex of abelian groups. Let

$$0 \to M^\bullet \to A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to \ldots$$

be an exact complex of complexes of abelian groups. Set $A^{p,q} = A_p^q$ to obtain a double complex. Then the map $M^\bullet \to \text{Tot}(A^{\bullet,\bullet})$ induced by $M^\bullet \to A_0^\bullet$ is a quasi-isomorphism.

Proof. If there exists a $t \in \mathbf{Z}$ such that $A_0^q = 0$ for $q < t$, then this follows immediately from Lemma 12.25.4 (with p and q swapped as in the final statement of that lemma). OK, but for every $t \in \mathbf{Z}$ we have a complex

$$0 \to \sigma_{\geq t} M^\bullet \to \sigma_{\geq t} A_0^\bullet \to \sigma_{\geq t} A_1^\bullet \to \sigma_{\geq t} A_2^\bullet \to \ldots$$

of stupid truncations. Denote $A(t)^{\bullet,\bullet}$ the corresponding double complex. Every element ξ of $H^n(\text{Tot}(A^{\bullet,\bullet}))$ is the image of an element of $H^n(\text{Tot}(A(t)^{\bullet,\bullet}))$ for some t (look at explicit representatives of cohomology classes). Hence ξ is in the image of $H^n(\sigma_{\geq t} M^\bullet)$. Thus the map $H^n(M^\bullet) \to H^n(\text{Tot}(A^{\bullet,\bullet}))$ is surjective. It is injective because for all t the map $H^n(\sigma_{\geq t} M^\bullet) \to H^n(\text{Tot}(A(t)^{\bullet,\bullet}))$ is injective and similar arguments. $\square$

09IZ **Lemma 12.26.2.** Let $M^\bullet$ be a complex of abelian groups. Let

$$\ldots \to A_2^\bullet \to A_1^\bullet \to A_0^\bullet \to M^\bullet \to 0$$

be an exact complex of complexes of abelian groups such that for all $p \in \mathbf{Z}$ the complexes

$$\ldots \to \text{Ker}(d^p_{A_2^\bullet}) \to \text{Ker}(d^p_{A_1^\bullet}) \to \text{Ker}(d^p_{A_0^\bullet}) \to \text{Ker}(d^p_{M^\bullet}) \to 0$$

are exact as well. Set $A^{p,q} = A^q_{-p}$ to obtain a double complex. Then $\text{Tot}(A^{\bullet,\bullet}) \to M^\bullet$ induced by $A_0^\bullet \to M^\bullet$ is a quasi-isomorphism.

Proof. Using the short exact sequences $0 \to \text{Ker}(d^p_{A_n^\bullet}) \to A_n^p \to \text{Im}(d^p_{A_n^\bullet}) \to 0$ and the assumptions we see that

$$\ldots \to \text{Im}(d^p_{A_2^\bullet}) \to \text{Im}(d^p_{A_1^\bullet}) \to \text{Im}(d^p_{A_0^\bullet}) \to \text{Im}(d^p_{M^\bullet}) \to 0$$

is exact for all $p \in \mathbf{Z}$. Repeating with the exact sequences $0 \to \text{Im}(d^{p-1}_{A_n^\bullet}) \to \text{Ker}(d^p_{A_n^\bullet}) \to H^p(A_n^\bullet) \to 0$ we find that

$$\ldots \to H^p(A_2^\bullet) \to H^p(A_1^\bullet) \to H^p(A_0^\bullet) \to H^p(M^\bullet) \to 0$$

is exact for all $p \in \mathbf{Z}$.

Write $T^\bullet = \text{Tot}(A^{\bullet,\bullet})$. We will show that $H^0(T^\bullet) \to H^0(M^\bullet)$ is an isomorphism. The same argument works for other degrees. Let $x \in \text{Ker}(\text{d}^0_{T^\bullet})$ represent an element $\xi \in H^0(T^\bullet)$. Write $x = \sum_{i=n,\ldots,0} x_i$ with $x_i \in A_i^i$. Assume $n > 0$. Then x_n is in the kernel of $d^n_{A_n^\bullet}$ and maps to zero in $H^n(A_{n-1}^\bullet)$ because it maps to an element which is the boundary of x_{n-1} up to sign. By the first paragraph of the proof, we find that $x_n \bmod \text{Im}(d^{n-1}_{A_n^\bullet})$ is in the image of $H^n(A_{n+1}^\bullet) \to H^n(A_n^\bullet)$. Thus we can modify x by a boundary and reach the situation where x_n is a boundary. Modifying x once more we see that we may assume $x_n = 0$. By induction we see that every cohomology class ξ is represented by a cocycle $x = x_0$. Finally, the condition on exactness of kernels tells us two such cocycles x_0 and x_0' are cohomologous if and only if their image in $H^0(M^\bullet)$ are the same. □

09J0 Lemma 12.26.3. Let $M^\bullet$ be a complex of abelian groups. Let

$$0 \to M^\bullet \to A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to \ldots$$

be an exact complex of complexes of abelian groups such that for all $p \in \mathbf{Z}$ the complexes

$$0 \to \text{Coker}(d^p_{M^\bullet}) \to \text{Coker}(d^p_{A_0^\bullet}) \to \text{Coker}(d^p_{A_1^\bullet}) \to \text{Coker}(d^p_{A_2^\bullet}) \to \ldots$$

are exact as well. Set $A^{p,q} = A^q_p$ to obtain a double complex. Let $\text{Tot}_\pi(A^{\bullet,\bullet})$ be the product total complex associated to the double complex (see proof). Then the map $M^\bullet \to \text{Tot}_\pi(A^{\bullet,\bullet})$ induced by $M^\bullet \to A_0^\bullet$ is a quasi-isomorphism.

Proof. Abbreviating $T^\bullet = \text{Tot}_\pi(A^{\bullet,\bullet})$ we define

$$T^n = \prod\nolimits_{p+q=n} A^{p,q} = \prod\nolimits_{p+q=n} A^q_p \quad \text{with} \quad \text{d}^n_{T^\bullet} = \prod\nolimits_{n=p+q} (f_p^q + (-1)^p d^q_{A_p^\bullet})$$

where $f_p^\bullet : A_p^\bullet \to A_{p+1}^\bullet$ are the maps of complexes in the lemma.

We will show that $H^0(M^\bullet) \to H^0(T^\bullet)$ is an isomorphism. The same argument works for other degrees. Let $x \in \text{Ker}(\text{d}^0_{T^\bullet})$ represent $\xi \in H^0(T^\bullet)$. Write $x = (x_i)$ with $x_i \in A_i^{-i}$. Note that x_0 maps to zero in $\text{Coker}(A_1^{-1} \to A_1^0)$. Hence we see that $x_0 = m_0 + d^{-1}_{A_0^\bullet}(y)$ for some $m_0 \in M^0$ and $y \in A_0^{-1}$. Then $d_{M^\bullet}(m_0) = 0$ because $\text{d}_{A_0^\bullet}(x_0) = 0$ as $\text{d}_{T^\bullet}(x) = 0$. Thus, replacing ξ by something in the image of $H^0(M^\bullet) \to H^0(T^\bullet)$ we may assume that x_0 is in $\text{Im}(d^{-1}_{A_0^\bullet})$.

Assume $x_0 \in \text{Im}(d^{-1}_{A_0^\bullet})$. We claim that in this case $\xi = 0$. To prove this we find, by induction on n elements $y_0, y_1, \ldots, y_n$ with $y_i \in A_i^{-i-1}$ such that $x_0 = \text{d}^{-1}_{A_0}(y_0)$ and

$x_j = f^{-j}_{j-1}(y_{j-1}) + (-1)^j d^{-j-1}_{A^\bullet_{-j}}(y_j)$ for $j = 1, \ldots, n$. This is clear for $n = 0$. Proof of induction step: suppose we have found $y_0, \ldots, y_{n-1}$. Then $w_n = x_n - f^{-n}_{n-1}(y_{n-1})$ is in the kernel of $d^{-n}_{A^\bullet_n}$ and maps to zero in $H^n(A^\bullet_{n+1})$ (because it maps to an element which is a boundary the boundary of x_{n+1} up to sign). Exactly as in the proof of Lemma 12.26.2 the assumptions of the lemma imply that

$$0 \to H^p(M^\bullet) \to H^p(A^\bullet_0) \to H^p(A^\bullet_1) \to H^p(A^\bullet_2) \to \ldots$$

is exact for all $p \in \mathbf{Z}$. Thus after changing y_{n-1} by an element in $\mathrm{Ker}(d^{n-1}_{A^\bullet_{n-1}})$ we may assume that w_n maps to zero in $H^{-n}(A^\bullet_n)$. This means we can find y_n as desired. Observe that this procedure does not change $y_0, \ldots, y_{n-2}$. Hence continuing ad infinitum we find an element $y = (y_i)$ in T^{n-1} with $d_{T^\bullet}(y) = \xi$. This shows that $H^0(M^\bullet) \to H^0(T^\bullet)$ is surjective.

Suppose that $m_0 \in \mathrm{Ker}(d^0_{M^\bullet})$ maps to zero in $H^0(T^\bullet)$. Say it maps to the differential applied to $y = (y_i) \in T^{-1}$. Then $y_0 \in A^{-1}_0$ maps to zero in $\mathrm{Coker}(d^{-2}_{A^\bullet_1})$. By assumption this means that $y_0 \bmod \mathrm{Im}(d^{-2}_{A^\bullet_0})$ is the image of some $z \in M^{-1}$. It follows that $m_0 = d^{-1}_{M^\bullet}(z)$. This proves injectivity and the proof is complete. □

0E1R Lemma 12.26.4. Let $M^\bullet$ be a complex of abelian groups. Let

$$\ldots \to A^\bullet_2 \to A^\bullet_1 \to A^\bullet_0 \to M^\bullet \to 0$$

be an exact complex of complexes of abelian groups. Set $A^{p,q} = A^q_{-p}$ to obtain a double complex. Let $\mathrm{Tot}_\pi(A^{\bullet,\bullet})$ be the product total complex associated to the double complex (see proof). Then the map $\mathrm{Tot}_\pi(A^{\bullet,\bullet}) \to M^\bullet$ induced by $A^\bullet_0 \to M^\bullet$ is a quasi-isomorphism.

Proof. Abbreviating $T^\bullet = \mathrm{Tot}_\pi(A^{\bullet,\bullet})$ we define

$$T^n = \prod\nolimits_{p+q=n} A^{p,q} = \prod\nolimits_{p+q=n} A^q_{-p} \quad \text{with} \quad \mathrm{d}^n_{T^\bullet} = \prod\nolimits_{n=p+q} (f^q_{-p} + (-1)^p d^q_{A^\bullet_{-p}})$$

where $f^\bullet_p : A^\bullet_p \to A^\bullet_{p-1}$ are the maps of complexes in the lemma. We will show that $T^\bullet$ is acyclic when $M^\bullet$ is the zero complex. This will suffice by the following trick. Set $B^\bullet_n = A^\bullet_{n+1}$ and $B^\bullet_0 = M^\bullet$. Then we have an exact sequence

$$\ldots \to B^\bullet_2 \to B^\bullet_1 \to B^\bullet_0 \to 0 \to 0$$

as in the lemma. Let $S^\bullet = \mathrm{Tot}_\pi(B^{\bullet,\bullet})$. Then there is an obvious short exact sequence of complexes

$$0 \to M^\bullet \to S^\bullet \to T^\bullet[1] \to 0$$

and we conclude by the long exact cohomology sequence. Some details omitted.

Assume $M^\bullet = 0$. We will show $H^0(T^\bullet) = 0$. The same argument works for other degrees. Let $x = (x_n) \in \mathrm{Ker}(d_{T^\bullet})$ map to $\xi \in H^0(T^\bullet)$ with $x_n \in A^{-n,n} = A^n_n$. Since $M^0 = 0$ we find that $x_0 = f^0_1(y_0)$ for some $y_0 \in A^0_1$. Then $x_1 - d^0_{A^\bullet_1}(y_0) = f^1_2(y_1)$ because it is mapped to zero by f^1_1 as x is a cocycle. for some $y_1 \in A^1_2$. Continuing, using induction, we find $y = (y_i) \in T^{-1}$ with $d_{T^\bullet}(y) = x$ as desired. □

12.27. Injectives

0134

0135 Definition 12.27.1. Let $\mathcal{A}$ be an abelian category. An object $J \in \mathrm{Ob}(\mathcal{A})$ is called injective if for every injection $A \hookrightarrow B$ and every morphism $A \to J$ there exists a morphism $B \to J$ making the following diagram commute

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[ld, dashed] \\ J & \end{tikzcd}$$

Here is the obligatory characterization of injective objects.

0136 Lemma 12.27.2. Let $\mathcal{A}$ be an abelian category. Let I be an object of $\mathcal{A}$. The following are equivalent:

(1) The object I is injective.
(2) The functor $B \mapsto \mathrm{Hom}_{\mathcal{A}}(B, I)$ is exact.
(3) Any short exact sequence

$$0 \to I \to A \to B \to 0$$

in $\mathcal{A}$ is split.
(4) We have $\mathrm{Ext}_{\mathcal{A}}(B, I) = 0$ for all $B \in \mathrm{Ob}(\mathcal{A})$.

Proof. Omitted. $\square$

0137 Lemma 12.27.3. Let $\mathcal{A}$ be an abelian category. Suppose I_ω, $\omega \in \Omega$ is a set of injective objects of $\mathcal{A}$. If $\prod_{\omega \in \Omega} I_\omega$ exists then it is injective.

Proof. Omitted. $\square$

0138 Definition 12.27.4. Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has enough injectives if every object A has an injective morphism $A \to J$ into an injective object J.

0139 Definition 12.27.5. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has functorial injective embeddings if there exists a functor

$$J : \mathcal{A} \longrightarrow \mathrm{Arrows}(\mathcal{A})$$

such that

(1) $s \circ J = \mathrm{id}_{\mathcal{A}}$,
(2) for any object $A \in \mathrm{Ob}(\mathcal{A})$ the morphism $J(A)$ is injective, and
(3) for any object $A \in \mathrm{Ob}(\mathcal{A})$ the object $t(J(A))$ is an injective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (A \to J(A))$.

12.28. Projectives

013A

013B Definition 12.28.1. Let $\mathcal{A}$ be an abelian category. An object $P \in \mathrm{Ob}(\mathcal{A})$ is called projective if for every surjection $A \to B$ and every morphism $P \to B$ there exists a morphism $P \to A$ making the following diagram commute

$$\begin{tikzcd} A \arrow[r] & B \\ P \arrow[u, dashed] \arrow[ru] & \end{tikzcd}$$

Here is the obligatory characterization of projective objects.

013C Lemma 12.28.2. Let $\mathcal{A}$ be an abelian category. Let P be an object of $\mathcal{A}$. The following are equivalent:

(1) The object P is projective.
(2) The functor $B \mapsto \mathrm{Hom}_{\mathcal{A}}(P, B)$ is exact.
(3) Any short exact sequence
$$0 \to A \to B \to P \to 0$$
in $\mathcal{A}$ is split.
(4) We have $\mathrm{Ext}_{\mathcal{A}}(P, A) = 0$ for all $A \in \mathrm{Ob}(\mathcal{A})$.

Proof. Omitted. $\square$

013D Lemma 12.28.3. Let $\mathcal{A}$ be an abelian category. Suppose P_ω, $\omega \in \Omega$ is a set of projective objects of $\mathcal{A}$. If $\coprod_{\omega\in\Omega} P_\omega$ exists then it is projective.

Proof. Omitted. $\square$

013E Definition 12.28.4. Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has enough projectives if every object A has an surjective morphism $P \to A$ from an projective object P onto it.

013F Definition 12.28.5. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has functorial projective surjections if there exists a functor
$$P : \mathcal{A} \longrightarrow \mathrm{Arrows}(\mathcal{A})$$
such that

(1) $t \circ J = \mathrm{id}_{\mathcal{A}}$,
(2) for any object $A \in \mathrm{Ob}(\mathcal{A})$ the morphism $P(A)$ is surjective, and
(3) for any object $A \in \mathrm{Ob}(\mathcal{A})$ the object $s(P(A))$ is an projective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (P(A) \to A)$.

12.29. Injectives and adjoint functors

015Y Here are some lemmas on adjoint functors and their relationship with injectives. See also Lemma 12.7.4.

015Z Lemma 12.29.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume

(1) u is right adjoint to v, and
(2) v transforms injective maps into injective maps.

Then u transforms injectives into injectives.

Proof. Let I be an injective object of $\mathcal{A}$. Let $\varphi : N \to M$ be an injective map in $\mathcal{B}$ and let $\alpha : N \to uI$ be a morphism. By adjointness we get a morphism $\alpha : vN \to I$ and by assumption $v\varphi : vN \to vM$ is injective. Hence as I is an injective object we get a morphism $\beta : vM \to I$ extending α. By adjointness again this corresponds to a morphism $\beta : M \to uI$ as desired. $\square$

03B8 Remark 12.29.2. Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as in Lemma 12.29.1. In the presence of assumption (1) assumption (2) is equivalent to requiring that v is exact. Moreover, condition (2) is necessary. Here is an example. Let $A \to B$ be a ring map. Let $u : \text{Mod}_B \to \text{Mod}_A$ be $u(N) = N_A$ and let $v : \text{Mod}_A \to \text{Mod}_B$ be $v(M) = M \otimes_A B$. Then u is right adjoint to v, and u is exact and v is right exact, but v does not transform injective maps into injective maps in general (i.e., v is not left exact). Moreover, it is not the case that u transforms injective B-modules into injective A-modules. For example, if $A = \mathbf{Z}$ and $B = \mathbf{Z}/p\mathbf{Z}$, then the injective B-module $\mathbf{Z}/p\mathbf{Z}$ is not an injective $\mathbf{Z}$-module. In fact, the lemma applies to this example if and only if the ring map $A \to B$ is flat.

0160 Lemma 12.29.3. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume

(1) u is right adjoint to v,
(2) v transforms injective maps into injective maps,
(3) $\mathcal{A}$ has enough injectives, and
(4) $vB = 0$ implies $B = 0$ for any $B \in \text{Ob}(\mathcal{B})$.

Then $\mathcal{B}$ has enough injectives.

Proof. Pick $B \in \text{Ob}(\mathcal{B})$. Pick an injection $vB \to I$ for I an injective object of $\mathcal{A}$. According to Lemma 12.29.1 and the assumptions the corresponding map $B \to uI$ is the injection of B into an injective object. ☐

03B9 Remark 12.29.4. Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as In Lemma 12.29.3. In the presence of conditions (1) and (2) condition (4) is equivalent to v being faithful. Moreover, condition (4) is needed. An example is to consider the case where the functors u and v are both the zero functor.

0161 Lemma 12.29.5. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume

(1) u is right adjoint to v,
(2) v transforms injective maps into injective maps,
(3) $\mathcal{A}$ has enough injectives,
(4) $vB = 0$ implies $B = 0$ for any $B \in \text{Ob}(\mathcal{B})$, and
(5) $\mathcal{A}$ has functorial injective hulls.

Then $\mathcal{B}$ has functorial injective hulls.

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective hull on $\mathcal{A}$. Then $B \mapsto (B \to uJ(vB))$ is a functorial injective hull on $\mathcal{B}$. Compare with the proof of Lemma 12.29.3. ☐

0793 Lemma 12.29.6. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a subset $\mathcal{P} \subset \text{Ob}(\mathcal{B})$ such that

(1) every object of $\mathcal{B}$ is a quotient of an element of $\mathcal{P}$, and
(2) for every $P \in \mathcal{P}$ there exists an object Q of $\mathcal{A}$ such that $\text{Hom}_{\mathcal{A}}(Q, A) = \text{Hom}_{\mathcal{B}}(P, u(A))$ functorially in A,

then there exists a left adjoint v of u.

Proof. By the Yoneda lemma (Categories, Lemma 4.3.5) the object Q of $\mathcal{A}$ corresponding to P is defined up to unique isomorphism by the formula $\text{Hom}_{\mathcal{A}}(Q, A) =$

$\mathrm{Hom}_{\mathcal{B}}(P, u(A))$. Let us write $Q = v(P)$. Denote $i_P : P \to u(v(P))$ the map corresponding to $\mathrm{id}_{v(P)}$ in $\mathrm{Hom}_{\mathcal{A}}(v(P), v(P))$. Functoriality in (2) implies that the bijection is given by

$$\mathrm{Hom}_{\mathcal{A}}(v(P), A) \to \mathrm{Hom}_{\mathcal{B}}(P, u(A)), \quad \varphi \mapsto u(\varphi) \circ i_P$$

For any pair of elements $P_1, P_2 \in \mathcal{P}$ there is a canonical map

$$\mathrm{Hom}_{\mathcal{B}}(P_2, P_1) \to \mathrm{Hom}_{\mathcal{A}}(v(P_2), v(P_1)), \quad \varphi \mapsto v(\varphi)$$

which is characterized by the rule $u(v(\varphi)) \circ i_{P_2} = i_{P_1} \circ \varphi$ in $\mathrm{Hom}_{\mathcal{B}}(P_2, u(v(P_1)))$. Note that $\varphi \mapsto v(\varphi)$ is compatible with composition; this can be seen directly from the characterization. Hence $P \mapsto v(P)$ is a functor from the full subcategory of $\mathcal{B}$ whose objects are the elements of $\mathcal{P}$.

Given an arbitrary object B of $\mathcal{B}$ choose an exact sequence

$$P_2 \to P_1 \to B \to 0$$

which is possible by assumption (1). Define $v(B)$ to be the object of $\mathcal{A}$ fitting into the exact sequence

$$v(P_2) \to v(P_1) \to v(B) \to 0$$

Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(v(B), A) &= \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(v(P_1), A) \to \mathrm{Hom}_{\mathcal{A}}(v(P_2), A)) \\ &= \mathrm{Ker}(\mathrm{Hom}_{\mathcal{B}}(P_1, u(A)) \to \mathrm{Hom}_{\mathcal{B}}(P_2, u(A))) \\ &= \mathrm{Hom}_{\mathcal{B}}(B, u(A)) \end{aligned}$$

Hence we see that we may take $\mathcal{P} = \mathrm{Ob}(\mathcal{B})$, i.e., we see that v is everywhere defined. □

12.30. Essentially constant systems

0A2D In this section we discuss essentially constant systems with values in additive categories.

0A2E **Lemma 12.30.1.** Let $\mathcal{I}$ be a category, let $\mathcal{A}$ be a pre-additive Karoubian category, and let $M : \mathcal{I} \to \mathcal{A}$ be a diagram.

(1) Assume $\mathcal{I}$ is filtered. The following are equivalent
   (a) M is essentially constant,
   (b) $X = \mathrm{colim}\, M$ exists and there exists a cofinal filtered subcategory $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \mathrm{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ such that $X_{i'}$ maps isomorphically to X and $Z_{i'}$ to zero in $M_{i''}$ for some $i' \to i''$ in $\mathcal{I}'$.
(2) Assume $\mathcal{I}$ is cofiltered. The following are equivalent
   (a) M is essentially constant,
   (b) $X = \lim M$ exists and there exists an initial cofiltered subcategory $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \mathrm{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ such that X maps isomorphically to $X_{i'}$ and $M_{i''} \to Z_{i'}$ is zero for some $i'' \to i'$ in $\mathcal{I}'$.

Proof. Assume (1)(a), i.e., $\mathcal{I}$ is filtered and M is essentially constant. Let $X = \mathrm{colim}\, M_i$. Choose i and $X \to M_i$ as in Categories, Definition 4.22.1. Let $\mathcal{I}'$ be the full subcategory consisting of objects which are the target of a morphism with source i. Suppose $i' \in \mathrm{Ob}(\mathcal{I}')$ and choose a morphism $i \to i'$. Then $X \to M_i \to M_{i'}$

composed with $M_{i'} \to X$ is the identity on X. As $\mathcal{A}$ is Karoubian, we find a direct summand decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$, where $Z_{i'} = \text{Ker}(M_{i'} \to X)$ and $X_{i'}$ maps isomorphically to X. Pick $i \to k$ and $i' \to k$ such that $M_{i'} \to X \to M_i \to M_k$ equals $M_{i'} \to M_k$ as in Categories, Definition 4.22.1. Then we see that $M_{i'} \to M_k$ annihilates $Z_{i'}$. Thus (1)(b) holds.

Assume (1)(b), i.e., $\mathcal{I}$ is filtered and we have $\mathcal{I}' \subset \mathcal{I}$ and for $i' \in \text{Ob}(\mathcal{I}')$ a direct sum decomposition $M_{i'} = X_{i'} \oplus Z_{i'}$ as stated in the lemma. To see that M is essentially constant we can replace $\mathcal{I}$ by $\mathcal{I}'$, see Categories, Lemma 4.22.11. Pick any $i \in \text{Ob}(\mathcal{I})$ and denote $X \to M_i$ the inverse of the isomorphism $X_i \to X$ followed by the inclusion map $X_i \to M_i$. If j is a second object, then choose $j \to k$ such that $Z_j \to M_k$ is zero. Since $\mathcal{I}$ is filtered we may also assume there is a morphism $i \to k$ (after possibly increasing k). Then $M_j \to X \to M_i \to M_k$ and $M_j \to M_k$ both annihilate Z_j. Thus after postcomposing by a morphism $M_k \to M_l$ which annihilates the summand Z_k, we find that $M_j \to X \to M_i \to M_l$ and $M_j \to M_l$ are equal, i.e., M is essentially constant.

The proof of (2) is dual. □

0A2F Lemma 12.30.2. Let $\mathcal{I}$ be a category. Let $\mathcal{A}$ be an additive, Karoubian category. Let $F : \mathcal{I} \to \mathcal{A}$ and $G : \mathcal{I} \to \mathcal{A}$ be functors. The following are equivalent

(1) $\text{colim}_\mathcal{I} F \oplus G$ exists, and
(2) $\text{colim}_\mathcal{I} F$ and $\text{colim}_\mathcal{I} G$ exist.

In this case $\text{colim}_\mathcal{I} F \oplus G = \text{colim}_\mathcal{I} F \oplus \text{colim}_\mathcal{I} G$.

Proof. Assume (1) holds. Set $W = \text{colim}_\mathcal{I} F \oplus G$. Note that the projection onto F defines natural transformation $F \oplus G \to F \oplus G$ which is idempotent. Hence we obtain an idempotent endomorphism $W \to W$ by Categories, Lemma 4.14.8. Since $\mathcal{A}$ is Karoubian we get a corresponding direct sum decomposition $W = X \oplus Y$, see Lemma 12.4.2. A straightforward argument (omitted) shows that $X = \text{colim}_\mathcal{I} F$ and $Y = \text{colim}_\mathcal{I} G$. Thus (2) holds. We omit the proof that (2) implies (1). □

0A2G Lemma 12.30.3. Let $\mathcal{I}$ be a filtered category. Let $\mathcal{A}$ be an additive, Karoubian category. Let $F : \mathcal{I} \to \mathcal{A}$ and $G : \mathcal{I} \to \mathcal{A}$ be functors. The following are equivalent

(1) $F \oplus G : \mathcal{I} \to \mathcal{A}$ is essentially constant, and
(2) F and G are essentially constant.

Proof. Assume (1) holds. In particular $W = \text{colim}_\mathcal{I} F \oplus G$ exists and hence by Lemma 12.30.2 we have $W = X \oplus Y$ with $X = \text{colim}_\mathcal{I} F$ and $Y = \text{colim}_\mathcal{I} G$. A straightforward argument (omitted) using for example the characterization of Categories, Lemma 4.22.9 shows that F is essentially constant with value X and G is essentially constant with value Y. Thus (2) holds. The proof that (2) implies (1) is omitted. □

12.31. Inverse systems

02MY Let $\mathcal{C}$ be a category. In Categories, Section 4.21 we defined the notion of an inverse system over a preordered set (with values in the category $\mathcal{C}$). If the preordered set is $\mathbf{N} = \{1, 2, 3, \ldots\}$ with the usual ordering such an inverse system over $\mathbf{N}$ is often simply called an inverse system. It consists quite simply of a pair $(M_i, f_{ii'})$ where each M_i, $i \in \mathbf{N}$ is an object of $\mathcal{C}$, and for each $i > i'$, $i, i' \in \mathbf{N}$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that moreover $f_{i'i''} \circ f_{ii'} = f_{ii''}$ whenever this makes sense. It is clear that in fact it suffices to give the morphisms $M_2 \to M_1$, $M_3 \to M_2$, and so on. Hence an inverse system is frequently pictured as follows

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

Moreover, we often omit the transition maps φ_i from the notation and we simply say "let (M_i) be an inverse system".

The collection of all inverse systems with values in $\mathcal{C}$ forms a category with the obvious notion of morphism.

02MZ Lemma 12.31.1. Let $\mathcal{C}$ be a category.

(1) If $\mathcal{C}$ is an additive category, then the category of inverse systems with values in $\mathcal{C}$ is an additive category.
(2) If $\mathcal{C}$ is an abelian category, then the category of inverse systems with values in $\mathcal{C}$ is an abelian category. A sequence $(K_i) \to (L_i) \to (M_i)$ of inverse systems is exact if and only if each $K_i \to L_i \to N_i$ is exact.

Proof. Omitted. □

The limit (see Categories, Section 4.21) of such an inverse system is denoted $\lim M_i$, or $\lim_i M_i$. If $\mathcal{C}$ is the category of abelian groups (or sets), then the limit always exists and in fact can be described as follows

$$\lim_i M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

see Categories, Section 4.15. However, given a short exact sequence

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

of inverse systems of abelian groups it is not always the case that the associated system of limits is exact. In order to discuss this further we introduce the following notion.

02N0 Definition 12.31.2. Let $\mathcal{C}$ be an abelian category. We say the inverse system (A_i) satisfies the Mittag-Leffler condition, or for short is ML, if for every i there exists a $c = c(i) \geq i$ such that

$$\operatorname{Im}(A_k \to A_i) = \operatorname{Im}(A_c \to A_i)$$

for all $k \geq c$.

It turns out that the Mittag-Leffler condition is good enough to ensure that the lim-functor is exact, provided one works within the abelian category of abelian groups, modules over a ring, etc. It is shown in a paper by A. Neeman (see [Nee02]) that this condition is not strong enough in an abelian category having AB4* (having exact products).

02N1 Lemma 12.31.3. Let

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups.

(1) In any case the sequence

$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i$$

is exact.
(2) If (B_i) is ML, then also (C_i) is ML.

(3) If (A_i) is ML, then

$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i \to 0$$

is exact.

Proof. Nice exercise. See Algebra, Lemma 10.87.1 for part (3). □

070B Lemma 12.31.4. Let

$$(A_i) \to (B_i) \to (C_i) \to (D_i)$$

be an exact sequence of inverse systems of abelian groups. If the system (A_i) is ML, then the sequence

$$\lim_i B_i \to \lim_i C_i \to \lim_i D_i$$

is exact.

Proof. Let $Z_i = \mathrm{Ker}(C_i \to D_i)$ and $I_i = \mathrm{Im}(A_i \to B_i)$. Then $\lim Z_i = \mathrm{Ker}(\lim C_i \to \lim D_i)$ and we get a short exact sequence of systems

$$0 \to (I_i) \to (B_i) \to (Z_i) \to 0$$

Moreover, by Lemma 12.31.3 we see that (I_i) has (ML), thus another application of Lemma 12.31.3 shows that $\lim B_i \to \lim Z_i$ is surjective which proves the lemma. □

The following characterization of essentially constant inverse systems shows in particular that they have ML.

070C Lemma 12.31.5. Let $\mathcal{A}$ be an abelian category. Let (A_i) be an inverse system in $\mathcal{A}$ with limit $A = \lim A_i$. Then (A_i) is essentially constant (see Categories, Definition 4.22.1) if and only if there exists an i and for all $j \geq i$ a direct sum decomposition $A_j = A \oplus Z_j$ such that (a) the maps $A_{j'} \to A_j$ are compatible with the direct sum decompositions, (b) for all j there exists some $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero.

Proof. Assume (A_i) is essentially constant. Then there exists an i and a morphism $A_i \to A$ such that $A \to A_i \to A$ is the identity and for all $j \geq i$ there exists a $j' \geq j$ such that $A_{j'} \to A_j$ factors as $A_{j'} \to A_i \to A \to A_j$ (the last map comes from $A = \lim A_i$). Hence setting $Z_j = \mathrm{Ker}(A_j \to A)$ for all $j \geq i$ works. Proof of the converse is omitted. □

We will improve on the following lemma in More on Algebra, Lemma 15.86.13.

070D Lemma 12.31.6. Let

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be an exact sequence of inverse systems of abelian groups. If (C_i) is essentially constant, then (A_i) has ML if and only if (B_i) has ML.

Proof. After renumbering we may assume that $C_i = C \oplus Z_i$ compatible with transition maps and that for all i there exists an $i' \geq i$ such that $Z_{i'} \to Z_i$ is zero, see Lemma 12.31.5.

First, assume $C = 0$, i.e., we have $C_i = Z_i$. In this case choose $1 = n_1 < n_2 < n_3 < \ldots$ such that $Z_{n_{i+1}} \to Z_{n_i}$ is zero. Then $B_{n_{i+1}} \to B_{n_i}$ factors through $A_{n_i} \subset B_{n_i}$. It follows that for $j \geq i+1$ we have

$$\mathrm{Im}(A_{n_j} \to A_{n_i}) \subset \mathrm{Im}(B_{n_j} \to B_{n_i}) \subset \mathrm{Im}(A_{n_{j-1}} \to A_{n_i})$$

as subsets of A_{n_i}. Thus the images $\mathrm{Im}(A_{n_j} \to A_{n_i})$ stabilize for $j \geq i+1$ if and only if the same is true for the images $\mathrm{Im}(B_{n_j} \to B_{n_i})$. The equivalence follows from this (small detail omitted).

If $C \neq 0$, denote $B'_i \subset B_i$ the inverse image of C by the map $B_i \to C \oplus Z_i$. Then by the previous paragraph we see that (B'_i) has ML if and only if (B_i) has ML. Thus we may replace (B_i) by (B'_i). In this case we have exact sequences $0 \to A_i \to B_i \to C \to 0$ for all i. It follows that $0 \to \mathrm{Im}(A_j \to A_i) \to \mathrm{Im}(B_j \to B_i) \to C \to 0$ is short exact for all $j \geq i$. Hence the images $\mathrm{Im}(A_j \to A_i)$ stabilize for $j \geq i$ if and only if the same is true for $\mathrm{Im}(B_j \to B_i)$ as desired. □

The "correct" version of the following lemma is More on Algebra, Lemma 15.86.3.

070E Lemma 12.31.7. Let

$$(A_i^{-2} \to A_i^{-1} \to A_i^0 \to A_i^1)$$

be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ its limit. Denote (H_i^{-1}), (H_i^0) the inverse systems of cohomologies, and denote H^{-1}, H^0 the cohomologies of $A^{-2} \to A^{-1} \to A^0 \to A^1$. If (A_i^{-2}) and (A_i^{-1}) are ML and (H_i^{-1}) is essentially constant, then $H^0 = \lim H_i^0$.

Proof. Let $Z_i^j = \mathrm{Ker}(A_i^j \to A_i^{j+1})$ and $I_i^j = \mathrm{Im}(A_i^{j-1} \to A_i^j)$. Note that $\lim Z_i^0 = \mathrm{Ker}(\lim A_i^0 \to \lim A_i^1)$ as taking kernels commutes with limits. The systems (I_i^{-1}) and (I_i^0) have ML as quotients of the systems (A_i^{-2}) and (A_i^{-1}), see Lemma 12.31.3. Thus an exact sequence

$$0 \to (I_i^{-1}) \to (Z_i^{-1}) \to (H_i^{-1}) \to 0$$

of inverse systems where (I_i^{-1}) has ML and where (H_i^{-1}) is essentially constant by assumption. Hence (Z_i^{-1}) has ML by Lemma 12.31.6. The exact sequence

$$0 \to (Z_i^{-1}) \to (A_i^{-1}) \to (I_i^0) \to 0$$

and an application of Lemma 12.31.3 shows that $\lim A_i^{-1} \to \lim I_i^0$ is surjective. Finally, the exact sequence

$$0 \to (I_i^0) \to (Z_i^0) \to (H_i^0) \to 0$$

and Lemma 12.31.3 show that $\lim I_i^0 \to \lim Z_i^0 \to \lim H_i^0 \to 0$ is exact. Putting everything together we win. □

Sometimes we need a version of the lemma above where we take limits over big ordinals.

0AAT Lemma 12.31.8. Let α be an ordinal. Let $K_\beta^\bullet$, $\beta < \alpha$ be an inverse system of complexes of abelian groups over α. If for all $\beta < \alpha$ the complex $K_\beta^\bullet$ is acyclic and the map

$$K_\beta^n \longrightarrow \lim_{\gamma < \beta} K_\gamma^n$$

is surjective, then the complex $\lim_{\beta < \alpha} K_\beta^\bullet$ is acyclic.

Proof. By transfinite induction we prove this holds for every ordinal α and every system as in the lemma. In particular, whilst proving the result for α we may assume the complexes $\lim_{\gamma < \beta} K_\gamma^n$ are acyclic.

Let $x \in \lim_{\beta < \alpha} K_\alpha^0$ with $\mathrm{d}(x) = 0$. We will find a $y \in K_\alpha^{-1}$ with $\mathrm{d}(y) = x$. Write $x = (x_\beta)$ where $x_\beta \in K_\beta^0$ is the image of x for $\beta < \alpha$. We will construct $y = (y_\beta)$ by transfinite recursion.

For $\beta = 0$ let $y_0 \in K_0^{-1}$ be any element with $\mathrm{d}(y_0) = x_0$.

For $\beta = \gamma + 1$ a successor, we have to find an element y_β which maps both to y_γ by the transition map $f : K_\beta^\bullet \to K_\gamma^\bullet$ and to x_β under the differential. As a first approximation we choose y'_β with $\mathrm{d}(y'_\beta) = x_\beta$. Then the difference $y_\gamma - f(y'_\beta)$ is in the kernel of the differential, hence equal to $\mathrm{d}(z_\gamma)$ for some $z_\gamma \in K_\gamma^{-2}$. By assumption, the map $f^{-2} : K_\beta^{-2} \to K_\gamma^{-2}$ is surjective. Hence we write $z_\gamma = f(z_\beta)$ and change y'_β into $y_\beta = y'_\beta + \mathrm{d}(z_\beta)$ which works.

If β is a limit ordinal, then we have the element $(y_\gamma)_{\gamma<\beta}$ in $\lim_{\gamma<\beta} K_\gamma^{-1}$ whose differential is the image of x_β. Thus we can argue in exactly the same manner as above using the termwise surjective map of complexes $f : K_\beta^\bullet \to \lim_{\gamma<\beta} K_\gamma^\bullet$ and the fact (see first paragraph of proof) that we may assume $\lim_{\gamma<\beta} K_\gamma^\bullet$ is acyclic by induction. □

12.32. Exactness of products

060J

060K Lemma 12.32.1. Let I be a set. For $i \in I$ let $L_i \to M_i \to N_i$ be a complex of abelian groups. Let $H_i = \mathrm{Ker}(M_i \to N_i)/\mathrm{Im}(L_i \to M_i)$ be the cohomology. Then

$$\prod L_i \to \prod M_i \to \prod N_i$$

is a complex of abelian groups with homology $\prod H_i$.

Proof. Omitted. □

12.33. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

Topics in Scheme Theory

CHAPTER 13

Derived Categories

05QI
13.1. Introduction

05QJ We first discuss triangulated categories and localization in triangulated categories. Next, we prove that the homotopy category of complexes in an additive category is a triangulated category. Once this is done we define the derived category of an abelian category as the localization of the homotopy category with respect to quasi-isomorphisms. A good reference is Verdier's thesis [Ver96].

13.2. Triangulated categories

0143 Triangulated categories are a convenient tool to describe the type of structure inherent in the derived category of an abelian category. Some references are [Ver96], [KS06], and [Nee01].

13.3. The definition of a triangulated category

05QK In this section we collect most of the definitions concerning triangulated and pre-triangulated categories.

0144 Definition 13.3.1. Let $\mathcal{D}$ be an additive category. Let $[1] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[1]$ be an additive functor which is an auto-equivalence of $\mathcal{D}$.

(1) A triangle is a sextuple (X, Y, Z, f, g, h) where $X, Y, Z \in \mathrm{Ob}(\mathcal{D})$ and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to X[1]$ are morphisms of $\mathcal{D}$.
(2) A morphism of triangles $(X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ is given by morphisms $a : X \to X'$, $b : Y \to Y'$ and $c : Z \to Z'$ of $\mathcal{D}$ such that $b \circ f = f' \circ a$, $c \circ g = g' \circ b$ and $a[1] \circ h = h' \circ c$.

A morphism of triangles is visualized by the following commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, "c"] & X[1] \arrow[d, "a[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

In the setting of Definition 13.3.1, we write $[0] = \mathrm{id}$, for $n > 0$ we denote $[n]$ the n-fold composition of $[1]$, we choose a quasi-inverse $[-1]$ of $[1]$, and we set $[-n]$ equal to the n-fold composition of $[-1]$. Then $\{[n]\}_{n \in \mathbf{Z}}$ is a collection of additive auto-equivalences of $\mathcal{D}$ indexed by $n \in \mathbf{Z}$ such that we are given isomorphisms of functors $[n] \circ [m] \cong [n + m]$.

Here is the definition of a triangulated category as given in Verdier's thesis.

0145 Definition 13.3.2. A triangulated category consists of a triple $(\mathcal{D}, \{[n]\}_{n \in \mathbf{Z}}, \mathcal{T})$ where

(1) $\mathcal{D}$ is an additive category,
(2) $[1] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[1]$ is an additive auto-equivalence and $[n]$ for $n \in \mathbf{Z}$ is as discussed above, and
(3) $\mathcal{T}$ is a set of triangles (Definition 13.3.1) called the distinguished triangles

subject to the following conditions

TR1 Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Any triangle of the form $(X, X, 0, \text{id}, 0, 0)$ is distinguished. For any morphism $f : X \to Y$ of $\mathcal{D}$ there exists a distinguished triangle of the form (X, Y, Z, f, g, h).

TR2 The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, X[1], g, h, -f[1])$ is.

TR3 Given a solid diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, dashed] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

whose rows are distinguished triangles and which satisfies $b \circ f = f' \circ a$, there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morphism of triangles.

TR4 Given objects X, Y, Z of $\mathcal{D}$, and morphisms $f : X \to Y$, $g : Y \to Z$, and distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3), there exist morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ such that
(a) $(Q_1, Q_2, Q_3, a, b, p_1[1] \circ d_3)$ is a distinguished triangle,
(b) the triple (id_X, g, a) is a morphism of triangles $(X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$, and
(c) the triple (f, id_Z, b) is a morphism of triangles $(X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$.

We will call $(\mathcal{D}, [\ ], \mathcal{T})$ a pre-triangulated category if TR1, TR2 and TR3 hold.[1]

The explanation of TR4 is that if you think of Q_1 as Y/X, Q_2 as Z/X and Q_3 as Z/Y, then TR4(a) expresses the isomorphism $(Z/X)/(Y/X) \cong Z/Y$ and TR4(b) and TR4(c) express that we can compare the triangles $X \to Y \to Q_1 \to X[1]$ etc with morphisms of triangles. For a more precise reformulation of this idea see the proof of Lemma 13.10.2.

The sign in TR2 means that if (X, Y, Z, f, g, h) is a distinguished triangle then in the long sequence
(13.3.2.1)
05QL
$$\ldots \to Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \to \ldots$$
each four term sequence gives a distinguished triangle.

As usual we abuse notation and we simply speak of a (pre-)triangulated category $\mathcal{D}$ without explicitly introducing notation for the additional data. The notion of a pre-triangulated category is useful in finding statements equivalent to TR4.

We have the following definition of a triangulated functor.

[1]We use $[\ ]$ as an abbreviation for the family $\{[n]\}_{n \in \mathbf{Z}}$.

014V Definition 13.3.3. Let $\mathcal{D}$, $\mathcal{D}'$ be pre-triangulated categories. An exact functor, or a triangulated functor from $\mathcal{D}$ to $\mathcal{D}'$ is a functor $F : \mathcal{D} \to \mathcal{D}'$ together with given functorial isomorphisms $\xi_X : F(X[1]) \to F(X)[1]$ such that for every distinguished triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ the triangle $(F(X), F(Y), F(Z), F(f), F(g), \xi_X \circ F(h))$ is a distinguished triangle of $\mathcal{D}'$.

An exact functor is additive, see Lemma 13.4.17. When we say two triangulated categories are equivalent we mean that they are equivalent in the 2-category of triangulated categories. A 2-morphism $a : (F, \xi) \to (F', \xi')$ in this 2-category is simply a transformation of functors $a : F \to F'$ which is compatible with ξ and ξ', i.e.,

$$\begin{tikzcd} F \circ [1] \arrow[r, "\xi"'] \arrow[d, "a \star 1"'] & [1] \circ F \arrow[d, "1 \star a"] \\ F' \circ [1] \arrow[r, "\xi'"] & [1] \circ F' \end{tikzcd}$$

commutes.

05QM Definition 13.3.4. Let $(\mathcal{D}, [\ ], \mathcal{T})$ be a pre-triangulated category. A pre-triangulated subcategory[2] is a pair $(\mathcal{D}', \mathcal{T}')$ such that

(1) $\mathcal{D}'$ is an additive subcategory of $\mathcal{D}$ which is preserved under $[1]$ and such that $[1] : \mathcal{D}' \to \mathcal{D}'$ is an auto-equivalence,
(2) $\mathcal{T}' \subset \mathcal{T}$ is a subset such that for every $(X, Y, Z, f, g, h) \in \mathcal{T}'$ we have $X, Y, Z \in \mathrm{Ob}(\mathcal{D}')$ and $f, g, h \in \mathrm{Arrows}(\mathcal{D}')$, and
(3) $(\mathcal{D}', [\ ], \mathcal{T}')$ is a pre-triangulated category.

If $\mathcal{D}$ is a triangulated category, then we say $(\mathcal{D}', \mathcal{T}')$ is a triangulated subcategory if it is a pre-triangulated subcategory and $(\mathcal{D}', [\ ], \mathcal{T}')$ is a triangulated category.

In this situation the inclusion functor $\mathcal{D}' \to \mathcal{D}$ is an exact functor with $\xi_X : X[1] \to X[1]$ given by the identity on $X[1]$.

We will see in Lemma 13.4.1 that for a distinguished triangle (X, Y, Z, f, g, h) in a pre-triangulated category the composition $g \circ f : X \to Z$ is zero. Thus the sequence (13.3.2.1) is a complex. A homological functor is one that turns this complex into a long exact sequence.

0147 Definition 13.3.5. Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is called homological if for every distinguished triangle (X, Y, Z, f, g, h) the sequence

$$H(X) \to H(Y) \to H(Z)$$

is exact in the abelian category $\mathcal{A}$. An additive functor $H : \mathcal{D}^{opp} \to \mathcal{A}$ is called cohomological if the corresponding functor $\mathcal{D} \to \mathcal{A}^{opp}$ is homological.

If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor we often write $H^n(X) = H(X[n])$ so that $H(X) = H^0(X)$. Our discussion of TR2 above implies that a distinguished triangle (X, Y, Z, f, g, h) determines a long exact sequence
(13.3.5.1)

0148 $$H^{-1}(Z) \xrightarrow{H(h[-1])} H^0(X) \xrightarrow{H(f)} H^0(Y) \xrightarrow{H(g)} H^0(Z) \xrightarrow{H(h)} H^1(X)$$

[2]This definition may be nonstandard. If $\mathcal{D}'$ is a full subcategory then $\mathcal{T}'$ is the intersection of the set of triangles in $\mathcal{D}'$ with $\mathcal{T}$, see Lemma 13.4.16. In this case we drop $\mathcal{T}'$ from the notation.

This will be called the long exact sequence associated to the distinguished triangle and the homological functor. As indicated we will not use any signs for the morphisms in the long exact sequence. This has the side effect that maps in the long exact sequence associated to the rotation (TR2) of a distinguished triangle differ from the maps in the sequence above by some signs.

0150 Definition 13.3.6. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. A δ-functor from $\mathcal{A}$ to $\mathcal{D}$ is given by a functor $G : \mathcal{A} \to \mathcal{D}$ and a rule which assigns to every short exact sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

a morphism $\delta = \delta_{A\to B\to C} : G(C) \to G(A)[1]$ such that

(1) the triangle $(G(A), G(B), G(C), G(a), G(b), \delta_{A\to B\to C})$ is a distinguished triangle of $\mathcal{D}$ for any short exact sequence as above, and
(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences the diagram

$$\begin{tikzcd} G(C) \arrow[r, "\delta_{A\to B\to C}"'] \arrow[d] & G(A)[1] \arrow[d] \\ G(C') \arrow[r, "\delta_{A'\to B'\to C'}"] & G(A')[1] \end{tikzcd}$$

is commutative.

In this situation we call $(G(A), G(B), G(C), G(a), G(b), \delta_{A\to B\to C})$ the image of the short exact sequence under the given δ-functor.

Note how a δ-functor comes equipped with additional structure. Strictly speaking it does not make sense to say that a given functor $\mathcal{A} \to \mathcal{D}$ is a δ-functor, but we will often do so anyway.

13.4. Elementary results on triangulated categories

05QN Most of the results in this section are proved for pre-triangulated categories and a fortiori hold in any triangulated category.

0146 Lemma 13.4.1. Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. Then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.

Proof. By TR1 we know $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. Apply TR3 to

$$\begin{tikzcd} X \arrow[r] \arrow[d, "1"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "1[1]"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

Of course the dotted arrow is the zero map. Hence the commutativity of the diagram implies that $g \circ f = 0$. For the other cases rotate the triangle, i.e., apply TR2. □

0149 Lemma 13.4.2. Let $\mathcal{D}$ be a pre-triangulated category. For any object W of $\mathcal{D}$ the functor $\mathrm{Hom}_{\mathcal{D}}(W, -)$ is homological, and the functor $\mathrm{Hom}_{\mathcal{D}}(-, W)$ is cohomological.

Proof. Consider a distinguished triangle (X, Y, Z, f, g, h). We have already seen that $g \circ f = 0$, see Lemma 13.4.1. Suppose $a : W \to Y$ is a morphism such that $g \circ a = 0$. Then we get a commutative diagram

$$\begin{tikzcd} W \arrow[r, "1"'] \arrow[d, dotted, "b"] & W \arrow[r] \arrow[d, "a"] & 0 \arrow[r] \arrow[d, "0"] & W[1] \arrow[d, dotted, "b[1]"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

Both rows are distinguished triangles (use TR1 for the top row). Hence we can fill the dotted arrow b (first rotate using TR2, then apply TR3, and then rotate back). This proves the lemma. ☐

014A **Lemma 13.4.3.** Let $\mathcal{D}$ be a pre-triangulated category. Let

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If two among a, b, c are isomorphisms so is the third.

Proof. Assume that a and c are isomorphisms. For any object W of $\mathcal{D}$ write $H_W(-) = \mathrm{Hom}_\mathcal{D}(W, -)$. Then we get a commutative diagram of abelian groups

$$\begin{tikzcd} H_W(Z[-1]) \arrow[r] \arrow[d] & H_W(X) \arrow[r] \arrow[d] & H_W(Y) \arrow[r] \arrow[d] & H_W(Z) \arrow[r] \arrow[d] & H_W(X[1]) \arrow[d] \\ H_W(Z'[-1]) \arrow[r] & H_W(X') \arrow[r] & H_W(Y') \arrow[r] & H_W(Z') \arrow[r] & H_W(X'[1]) \end{tikzcd}$$

By assumption the right two and left two vertical arrows are bijective. As H_W is homological by Lemma 13.4.2 and the five lemma (Homology, Lemma 12.5.20) it follows that the middle vertical arrow is an isomorphism. Hence by Yoneda's lemma, see Categories, Lemma 4.3.5 we see that b is an isomorphism. This implies the other cases by rotating (using TR2). ☐

09WA **Remark 13.4.4.** Let $\mathcal{D}$ be an additive category with translation functors $[n]$ as in Definition 13.3.1. Let us call a triangle (X, Y, Z, f, g, h) *special*[3] if for every object W of $\mathcal{D}$ the long sequence of abelian groups

$$\ldots \to \mathrm{Hom}_\mathcal{D}(W, X) \to \mathrm{Hom}_\mathcal{D}(W, Y) \to \mathrm{Hom}_\mathcal{D}(W, Z) \to \mathrm{Hom}_\mathcal{D}(W, X[1]) \to \ldots$$

is exact. The proof of Lemma 13.4.3 shows that if

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

is a morphism of special triangles and if two among a, b, c are isomorphisms so is the third. There is a dual statement for *co-special* triangles, i.e., triangles which turn into long exact sequences on applying the functor $\mathrm{Hom}_\mathcal{D}(-, W)$. Thus distinguished triangles are special and co-special, but in general there are many more (co-)special triangles, than there are distinguished triangles.

05QP **Lemma 13.4.5.** Let $\mathcal{D}$ be a pre-triangulated category. Let

$$(0, b, 0), (0, b', 0) : (X, Y, Z, f, g, h) \to (X, Y, Z, f, g, h)$$

be endomorphisms of a distinguished triangle. Then $bb' = 0$.

[3]This is nonstandard notation.

Proof. Picture

$$\begin{tikzcd} X \arrow[r] \arrow[d, "0"] & Y \arrow[r] \arrow[d, "{b,b'}"] \arrow[dl, dotted, "\alpha"] & Z \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "\beta"] & X[1] \arrow[d, "0"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

Applying Lemma 13.4.2 we find dotted arrows α and β such that $b' = f \circ \alpha$ and $b = \beta \circ g$. Then $bb' = \beta \circ g \circ f \circ \alpha = 0$ as $g \circ f = 0$ by Lemma 13.4.1. $\square$

05QQ Lemma 13.4.6. Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. If

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "c"] & X[1] \arrow[d, "{a[1]}"] \\ Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

is commutative and $a^2 = a$, $c^2 = c$, then there exists a morphism $b : Y \to Y$ with $b^2 = b$ such that (a, b, c) is an endomorphism of the triangle (X, Y, Z, f, g, h).

Proof. By TR3 there exists a morphism b' such that (a, b', c) is an endomorphism of (X, Y, Z, f, g, h). Then $(0, (b')^2 - b', 0)$ is also an endomorphism. By Lemma 13.4.5 we see that $(b')^2 - b'$ has square zero. Set $b = b' - (2b' - 1)((b')^2 - b') = 3(b')^2 - 2(b')^3$. A computation shows that (a, b, c) is an endomorphism and that $b^2 - b = (4(b')^2 - 4b' - 3)((b')^2 - b')^2 = 0$. $\square$

014B Lemma 13.4.7. Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. There exists a distinguished triangle (X, Y, Z, f, g, h) which is unique up to (nonunique) isomorphism of triangles. More precisely, given a second such distinguished triangle (X, Y, Z', f, g', h') there exists an isomorphism

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Z', f, g', h')$$

Proof. Existence by TR1. Uniqueness up to isomorphism by TR3 and Lemma 13.4.3. $\square$

0FWZ Lemma 13.4.8. Let $\mathcal{D}$ be a pre-triangulated category. Let

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If one of the following conditions holds

(1) $\mathrm{Hom}(Y, X') = 0$,
(2) $\mathrm{Hom}(Z, Y') = 0$,
(3) $\mathrm{Hom}(X, X') = \mathrm{Hom}(Z, X') = 0$,
(4) $\mathrm{Hom}(Z, X') = \mathrm{Hom}(Z, Z') = 0$, or
(5) $\mathrm{Hom}(X[1], Z') = \mathrm{Hom}(Z, X') = 0$

then b is the unique morphism from $Y \to Y'$ such that (a, b, c) is a morphism of triangles.

Proof. If we have a second morphism of triangles (a, b', c) then $(0, b - b', 0)$ is a morphism of triangles. Hence we have to show: the only morphism $b : Y \to Y'$ such that $X \to Y \to Y'$ and $Y \to Y' \to Z'$ are zero is 0. We will use Lemma 13.4.2 without further mention. In particular, condition (3) implies (1). Given condition (1) if the composition $g' \circ b : Y \to Y' \to Z'$ is zero, then b lifts to a morphism $Y \to X'$ which has to be zero. This proves (1).

The proof of (2) and (4) are dual to this argument.

Assume (5). Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[d, "0"] & Y \arrow[r, "g"'] \arrow[d, "b"] & Z \arrow[r, "h"'] \arrow[d, "0"] \arrow[dl, dotted, "\epsilon"] & X[1] \arrow[d, "0"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

We may choose ϵ such that $b = \epsilon \circ g$. Then $g' \circ \epsilon \circ g = 0$ which implies that $g' \circ \epsilon = \delta \circ h$ for some $\delta \in \mathrm{Hom}(X[1], Z')$. Since $\mathrm{Hom}(X[1], Z') = 0$ we conclude that $g' \circ \epsilon = 0$. Hence $\epsilon = f' \circ \gamma$ for some $\gamma \in \mathrm{Hom}(Z, X')$. Since $\mathrm{Hom}(Z, X') = 0$ we conclude that $\epsilon = 0$ and hence $b = 0$ as desired. □

05QR Lemma 13.4.9. Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent

(1) f is an isomorphism,
(2) $(X, Y, 0, f, 0, 0)$ is a distinguished triangle, and
(3) for any distinguished triangle (X, Y, Z, f, g, h) we have $Z = 0$.

Proof. By TR1 the triangle $(X, X, 0, 1, 0, 0)$ is distinguished. Let (X, Y, Z, f, g, h) be a distinguished triangle. By TR3 there is a map of distinguished triangles $(1, f, 0) : (X, X, 0) \to (X, Y, Z)$. If f is an isomorphism, then $(1, f, 0)$ is an isomorphism of triangles by Lemma 13.4.3 and $Z = 0$. Conversely, if $Z = 0$, then $(1, f, 0)$ is an isomorphism of triangles as well, hence f is an isomorphism. □

05QS Lemma 13.4.10. Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be triangles. The following are equivalent

(1) $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$ is a distinguished triangle,
(2) both (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') are distinguished triangles.

Proof. Assume (2). By TR1 we may choose a distinguished triangle $(X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. By TR3 we can find morphisms of distinguished triangles $(X, Y, Z, f, g, h) \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$ and $(X', Y', Z', f', g', h') \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. Taking the direct sum of these morphisms we obtain a morphism of triangles

$$\begin{array}{c} (X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \\ \Big\downarrow{\scriptstyle (1,1,c)} \\ (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h''). \end{array}$$

In the terminology of Remark 13.4.4 this is a map of special triangles (because a direct sum of special triangles is special) and we conclude that c is an isomorphism. Thus (1) holds.

Assume (1). We will show that (X, Y, Z, f, g, h) is a distinguished triangle. First observe that (X, Y, Z, f, g, h) is a special triangle (terminology from Remark 13.4.4) as a direct summand of the distinguished hence special triangle $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$. Using TR1 let (X, Y, Q, f, g'', h'') be a distinguished triangle. By TR3 there exists a morphism of distinguished triangles $(X \oplus X', Y \oplus Y', Z \oplus$

$Z', f \oplus f', g \oplus g', h \oplus h') \to (X, Y, Q, f, g'', h'')$. Composing this with the inclusion map we get a morphism of triangles

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Q, f, g'', h'')$$

By Remark 13.4.4 we find that c is an isomorphism and we conclude that (2) holds. □

05QT Lemma 13.4.11. Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle.

(1) If $h = 0$, then there exists a right inverse $s : Z \to Y$ to g.
(2) For any right inverse $s : Z \to Y$ of g the map $f \oplus s : X \oplus Z \to Y$ is an isomorphism.
(3) For any objects X', Z' of $\mathcal{D}$ the triangle $(X', X' \oplus Z', Z', (1, 0), (0, 1), 0)$ is distinguished.

Proof. To see (1) use that $\mathrm{Hom}_{\mathcal{D}}(Z, Y) \to \mathrm{Hom}_{\mathcal{D}}(Z, Z) \to \mathrm{Hom}_{\mathcal{D}}(Z, X[1])$ is exact by Lemma 13.4.2. By the same token, if s is as in (2), then $h = 0$ and the sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}}(W, X) \to \mathrm{Hom}_{\mathcal{D}}(W, Y) \to \mathrm{Hom}_{\mathcal{D}}(W, Z) \to 0$$

is split exact (split by $s : Z \to Y$). Hence by Yoneda's lemma we see that $X \oplus Z \to Y$ is an isomorphism. The last assertion follows from TR1 and Lemma 13.4.10. □

05QU Lemma 13.4.12. Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent

(1) f has a kernel,
(2) f has a cokernel,
(3) f is the isomorphic to a composition $K \oplus Z \to Z \to Z \oplus Q$ of a projection and coprojection for some objects K, Z, Q of $\mathcal{D}$.

Proof. Any morphism isomorphic to a map of the form $X' \oplus Z \to Z \oplus Y'$ has both a kernel and a cokernel. Hence (3) $\Rightarrow$ (1), (2). Next we prove (1) $\Rightarrow$ (3). Suppose first that $f : X \to Y$ is a monomorphism, i.e., its kernel is zero. By TR1 there exists a distinguished triangle (X, Y, Z, f, g, h). By Lemma 13.4.1 the composition $f \circ h[-1] = 0$. As f is a monomorphism we see that $h[-1] = 0$ and hence $h = 0$. Then Lemma 13.4.11 implies that $Y = X \oplus Z$, i.e., we see that (3) holds. Next, assume f has a kernel K. As $K \to X$ is a monomorphism we conclude $X = K \oplus X'$ and $f|_{X'} : X' \to Y$ is a monomorphism. Hence $Y = X' \oplus Y'$ and we win. The implication (2) $\Rightarrow$ (3) is dual to this. □

0CRG Lemma 13.4.13. Let $\mathcal{D}$ be a pre-triangulated category. Let I be a set.

(1) Let X_i, $i \in I$ be a family of objects of $\mathcal{D}$.
  (a) If $\prod X_i$ exists, then $(\prod X_i)[1] = \prod X_i[1]$.
  (b) If $\bigoplus X_i$ exists, then $(\bigoplus X_i)[1] = \bigoplus X_i[1]$.
(2) Let $X_i \to Y_i \to Z_i \to X_i[1]$ be a family of distinguished triangles of $\mathcal{D}$.
  (a) If $\prod X_i$, $\prod Y_i$, $\prod Z_i$ exist, then $\prod X_i \to \prod Y_i \to \prod Z_i \to \prod X_i[1]$ is a distinguished triangle.
  (b) If $\bigoplus X_i$, $\bigoplus Y_i$, $\bigoplus Z_i$ exist, then $\bigoplus X_i \to \bigoplus Y_i \to \bigoplus Z_i \to \bigoplus X_i[1]$ is a distinguished triangle.

Proof. Part (1) is true because [1] is an autoequivalence of $\mathcal{D}$ and because direct sums and products are defined in terms of the category structure. Let us prove (2)(a). Choose a distinguished triangle $\prod X_i \to \prod Y_i \to Z \to \prod X_i[1]$. For each

j we can use TR3 to choose a morphism $p_j : Z \to Z_j$ fitting into a morphism of distinguished triangles with the projection maps $\prod X_i \to X_j$ and $\prod Y_i \to Y_j$. Using the definition of products we obtain a map $\prod p_i : Z \to \prod Z_i$ fitting into a morphism of triangles from the distinguished triangle to the triangle made out of the products. Observe that the "product" triangle $\prod X_i \to \prod Y_i \to \prod Z_i \to \prod X_i[1]$ is special in the terminology of Remark 13.4.4 because products of exact sequences of abelian groups are exact. Hence Remark 13.4.4 shows that the morphism of triangles is an isomorphism and we conclude by TR1. The proof of (2)(b) is dual. □

05QW Lemma 13.4.14. Let $\mathcal{D}$ be a pre-triangulated category. If $\mathcal{D}$ has countable products, then $\mathcal{D}$ is Karoubian. If $\mathcal{D}$ has countable coproducts, then $\mathcal{D}$ is Karoubian.

Proof. Assume $\mathcal{D}$ has countable products. By Homology, Lemma 12.4.3 it suffices to check that morphisms which have a right inverse have kernels. Any morphism which has a right inverse is an epimorphism, hence has a kernel by Lemma 13.4.12. The second statement is dual to the first. □

The following lemma makes it slightly easier to prove that a pre-triangulated category is triangulated.

014C Lemma 13.4.15. Let $\mathcal{D}$ be a pre-triangulated category. In order to prove TR4 it suffices to show that given any pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ there exist

(1) isomorphisms $i : X' \to X$, $j : Y' \to Y$ and $k : Z' \to Z$, and then setting $f' = j^{-1}fi : X' \to Y'$ and $g' = k^{-1}gj : Y' \to Z'$ there exist
(2) distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$, such that the assertion of TR4 holds.

Proof. The replacement of X, Y, Z by X', Y', Z' is harmless by our definition of distinguished triangles and their isomorphisms. The lemma follows from the fact that the distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$ are unique up to isomorphism by Lemma 13.4.7. □

05QX Lemma 13.4.16. Let $\mathcal{D}$ be a pre-triangulated category. Assume that $\mathcal{D}'$ is an additive full subcategory of $\mathcal{D}$. The following are equivalent

(1) there exists a set of triangles $\mathcal{T}'$ such that $(\mathcal{D}', \mathcal{T}')$ is a pre-triangulated subcategory of $\mathcal{D}$,
(2) $\mathcal{D}'$ is preserved under $[1]$ and $[1] : \mathcal{D}' \to \mathcal{D}'$ is an auto-equivalence and given any morphism $f : X \to Y$ in $\mathcal{D}'$ there exists a distinguished triangle (X, Y, Z, f, g, h) in $\mathcal{D}$ such that Z is isomorphic to an object of $\mathcal{D}'$.

In this case $\mathcal{T}'$ as in (1) is the set of distinguished triangles (X, Y, Z, f, g, h) of $\mathcal{D}$ such that $X, Y, Z \in \mathrm{Ob}(\mathcal{D}')$. Finally, if $\mathcal{D}$ is a triangulated category, then (1) and (2) are also equivalent to

(3) $\mathcal{D}'$ is a triangulated subcategory.

Proof. Omitted. □

05QY Lemma 13.4.17. An exact functor of pre-triangulated categories is additive.

Proof. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Since $(0, 0, 0, 1_0, 1_0, 0)$ is a distinguished triangle of $\mathcal{D}$ the triangle

$$(F(0), F(0), F(0), 1_{F(0)}, 1_{F(0)}, F(0))$$

is distinguished in $\mathcal{D}'$. This implies that $1_{F(0)} \circ 1_{F(0)}$ is zero, see Lemma 13.4.1. Hence $F(0)$ is the zero object of $\mathcal{D}'$. This also implies that F applied to any zero morphism is zero (since a morphism in an additive category is zero if and only if it factors through the zero object). Next, using that $(X, X \oplus Y, Y, (1,0), (0,1), 0)$ is a distinguished triangle by Lemma 13.4.11 part (3), we see that $(F(X), F(X \oplus Y), F(Y), F(1,0), F(0,1), 0)$ is one too. This implies that the map $F(X) \oplus F(Y) \to F(X \oplus Y)$ is an isomorphism by Lemma 13.4.11 part (2). To finish we apply Homology, Lemma 12.7.1. □

05SQ Lemma 13.4.18. Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor of pre-triangulated categories. Then a triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ is distinguished if and only if $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished in $\mathcal{D}'$.

Proof. The "only if" part is clear. Assume $(F(X), F(Y), F(Z))$ is distinguished in $\mathcal{D}'$. Pick a distinguished triangle (X, Y, Z', f, g', h') in $\mathcal{D}$. By Lemma 13.4.7 there exists an isomorphism of triangles

$$(1, 1, c') : (F(X), F(Y), F(Z)) \longrightarrow (F(X), F(Y), F(Z')).$$

Since F is fully faithful, there exists a morphism $c : Z \to Z'$ such that $F(c) = c'$. Then $(1, 1, c)$ is an isomorphism between (X, Y, Z) and (X, Y, Z'). Hence (X, Y, Z) is distinguished by TR1. □

014Y Lemma 13.4.19. Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be pre-triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $F' : \mathcal{D}' \to \mathcal{D}''$ be exact functors. Then $F' \circ F$ is an exact functor.

Proof. Omitted. □

05QZ Lemma 13.4.20. Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor.

(1) Let $\mathcal{D}'$ be a pre-triangulated category. Let $F : \mathcal{D}' \to \mathcal{D}$ be an exact functor. Then the composition $H \circ F$ is a homological functor as well.
(2) Let $\mathcal{A}'$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{A}'$ be an exact functor. Then $G \circ H$ is a homological functor as well.

Proof. Omitted. □

0151 Lemma 13.4.21. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor.

(1) Let $\mathcal{D}'$ be a triangulated category. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. Then the composition $F \circ G$ is a δ-functor as well.
(2) Let $\mathcal{A}'$ be an abelian category. Let $H : \mathcal{A}' \to \mathcal{A}$ be an exact functor. Then $G \circ H$ is a δ-functor as well.

Proof. Omitted. □

05SR Lemma 13.4.22. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor. Let $H : \mathcal{D} \to \mathcal{B}$ be a homological functor. Assume that $H^{-1}(G(A)) = 0$ for all A in $\mathcal{A}$. Then the collection

$$\{H^n \circ G, H^n(\delta_{A \to B \to C})\}_{n \geq 0}$$

is a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 12.12.1.

Proof. The notation signifies the following. If $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ is a short exact sequence in $\mathcal{A}$, then

$$\delta = \delta_{A \to B \to C} : G(C) \to G(A)[1]$$

is a morphism in $\mathcal{D}$ such that $(G(A), G(B), G(C), a, b, \delta)$ is a distinguished triangle, see Definition 13.3.6. Then $H^n(\delta) : H^n(G(C)) \to H^n(G(A)[1]) = H^{n+1}(G(A))$ is clearly functorial in the short exact sequence. Finally, the long exact cohomology sequence (13.3.5.1) combined with the vanishing of $H^{-1}(G(C))$ gives a long exact sequence

$$0 \to H^0(G(A)) \to H^0(G(B)) \to H^0(G(C)) \xrightarrow{H^0(\delta)} H^1(G(A)) \to \ldots$$

in $\mathcal{B}$ as desired. □

The proof of the following result uses TR4.

05R0 **Proposition 13.4.23.** Let $\mathcal{D}$ be a triangulated category. Any commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

can be extended to a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d] \\ X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & X''[1] \arrow[d] \\ X[1] \arrow[r] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2] \end{tikzcd}$$

where all the squares are commutative, except for the lower right square which is anticommutative. Moreover, each of the rows and columns are distinguished triangles. Finally, the morphisms on the bottom row (resp. right column) are obtained from the morphisms of the top row (resp. left column) by applying [1].

Proof. During this proof we avoid writing the arrows in order to make the proof legible. Choose distinguished triangles (X, Y, Z), (X', Y', Z'), (X, X', X''), (Y, Y', Y''), and (X, Y', A). Note that the morphism $X \to Y'$ is both equal to the composition $X \to Y \to Y'$ and equal to the composition $X \to X' \to Y'$. Hence, we can find morphisms

(1) $a : Z \to A$ and $b : A \to Y''$, and
(2) $a' : X'' \to A$ and $b' : A \to Z'$

as in TR4. Denote $c : Y'' \to Z[1]$ the composition $Y'' \to Y[1] \to Z[1]$ and denote $c' : Z' \to X''[1]$ the composition $Z' \to X'[1] \to X''[1]$. The conclusion of our application TR4 are that

(1) (Z, A, Y'', a, b, c), (X'', A, Z', a', b', c') are distinguished triangles,

(2) $(X, Y, Z) \to (X, Y', A)$, $(X, Y', A) \to (Y, Y', Y'')$, $(X, X', X'') \to (X, Y', A)$, $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles.

First using that $(X, X', X'') \to (X, Y', A)$ and $(X, Y', A) \to (Y, Y', Y'')$. are morphisms of triangles we see the first of the diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X'' \arrow[r, "b\circ a'"] \arrow[d] & Y'' \arrow[d] \\ X[1] \arrow[r] & Y[1] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "b'\circ a"] & X[1] \arrow[d] \\ Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

is commutative. The second is commutative too using that $(X, Y, Z) \to (X, Y', A)$ and $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles. At this point we choose a distinguished triangle (X'', Y'', Z'') starting with the map $b \circ a' : X'' \to Y''$.

Next we apply TR4 one more time to the morphisms $X'' \to A \to Y''$ and the triangles (X'', A, Z', a', b', c'), (X'', Y'', Z''), and $(A, Y'', Z[1], b, c, -a[1])$ to get morphisms $a'' : Z' \to Z''$ and $b'' : Z'' \to Z[1]$. Then $(Z', Z'', Z[1], a'', b'', -b'[1] \circ a[1])$ is a distinguished triangle, hence also $(Z, Z', Z'', -b' \circ a, a'', -b'')$ and hence also $(Z, Z', Z'', b'{\circ}a, a'', b'')$. Moreover, $(X'', A, Z') \to (X'', Y'', Z'')$ and $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ are morphisms of triangles. At this point we have defined all the distinguished triangles and all the morphisms, and all that's left is to verify some commutativity relations.

To see that the middle square in the diagram commutes, note that the arrow $Y' \to Z'$ factors as $Y' \to A \to Z'$ because $(X, Y', A) \to (X', Y', Z')$ is a morphism of triangles. Similarly, the morphism $Y' \to Y''$ factors as $Y' \to A \to Y''$ because $(X, Y', A) \to (Y, Y', Y'')$ is a morphism of triangles. Hence the middle square commutes because the square with sides (A, Z', Z'', Y'') commutes as $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles (by TR4). The square with sides $(Y'', Z'', Y[1], Z[1])$ commutes because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and $c : Y'' \to Z[1]$ is the composition $Y'' \to Y[1] \to Z[1]$. The square with sides $(Z', X'[1], X''[1], Z'')$ is commutative because $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles and $c' : Z' \to X''[1]$ is the composition $Z' \to X'[1] \to X''[1]$. Finally, we have to show that the square with sides $(Z'', X''[1], Z[1], X[2])$ anticommutes. This holds because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and we're done. □

13.5. Localization of triangulated categories

05R1 In order to construct the derived category starting from the homotopy category of complexes, we will use a localization process.

05R2 Definition 13.5.1. Let $\mathcal{D}$ be a pre-triangulated category. We say a multiplicative system S is compatible with the triangulated structure if the following two conditions hold:

MS5 For a morphism f of $\mathcal{D}$ we have $f \in S \Leftrightarrow f[1] \in S$[4].

[4]See Remark 13.5.3.

MS6 Given a solid commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"] & Y \arrow[r] \arrow[d, "s'"] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "s[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

whose rows are distinguished triangles with $s, s' \in S$ there exists a morphism $s'' : Z \to Z'$ in S such that (s, s', s'') is a morphism of triangles.

It turns out that these axioms are not independent of the axioms defining multiplicative systems.

05R3 **Lemma 13.5.2.** Let $\mathcal{D}$ be a pre-triangulated category. Let $S \subset \text{Arrows}(\mathcal{D})$.

(1) If S contains all identities and MS6 holds (Definition 13.5.1), then every isomorphism of $\mathcal{D}$ is in S.
(2) If MS1, MS5 (Categories, Definition 4.27.1) and MS6 hold, then MS2 holds.

Proof. Assume S contains all identities and MS6 holds. Let $f : X \to Y$ be an isomorphism of $\mathcal{D}$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "1"] & X \arrow[r, "1"'] \arrow[d, "1"] & X \arrow[r] \arrow[d, dashed] & 0[1] \arrow[d, "1[1]"] \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & 0[1] \end{tikzcd}$$

The rows are distinguished triangles by Lemma 13.4.9. By MS6 we see that the dotted arrow exists and is in S, so f is in S.

Assume MS1, MS5, MS6. Suppose that $f : X \to Y$ is a morphism of $\mathcal{D}$ and $t : X \to X'$ an element of S. Choose a distinguished triangle (X, Y, Z, f, g, h). Next, choose a distinguished triangle $(X', Y', Z, f', g', t[1] \circ h)$ (here we use TR1 and TR2). By MS5, MS6 (and TR2 to rotate) we can find the dotted arrow in the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "t"] & Y \arrow[r] \arrow[d, dotted, "s'"] & Z \arrow[r] \arrow[d, "1"] & X[1] \arrow[d, "t[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z \arrow[r] & X'[1] \end{tikzcd}$$

with moreover $s' \in S$. This proves LMS2. The proof of RMS2 is dual. $\square$

0H30 **Remark 13.5.3.** In the presence of MS1 and MS6, condition MS5 is equivalent to asking $s[n] \in S$ for all $s \in S$ and $n \in \mathbf{Z}$. For example, suppose MS5 holds, we have $s \in S$, and we want to show $s[-1] \in S$. This isn't immediate because $s[-1][1]$ is not equal to s, only isomorphic to s as an arrow of $\mathcal{D}$. Still, this does imply that $s[-1][1] = f \circ s \circ g$ for isomorphisms f, g. By Lemma 13.5.2 (1) we find $f, g \in S$, hence $s[-1][1] \in S$ by MS1, hence $s[-1] \in S$ by MS5. We leave a complete proof to the reader as an exercise.

05R4 **Lemma 13.5.4.** Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let

$$S = \{f \in \text{Arrows}(\mathcal{D}) \mid F(f) \text{ is an isomorphism}\}$$

Then S is a saturated (see Categories, Definition 4.27.20) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 4.27.1 and 4.27.20 and Definition 13.5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and Lemma 13.4.3. By Lemma 13.5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, d, h) using TR1. Since $a \circ t = 0$ we see by Lemma 13.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ d = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{tikzcd} Z \arrow[r, "t"'] & X \arrow[r, "d"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\ & X \arrow[r, "a"'] & Y \arrow[d, "j"] & \\ & & W & \end{tikzcd}$$

OK, and now we apply the functor F to this diagram. Since $t \in S$ we see that $F(Q) = 0$, see Lemma 13.4.9. Hence $F(j)$ is an isomorphism by the same lemma, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ d = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. $\square$

05R5 Lemma 13.5.5. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor between a pre-triangulated category and an abelian category. Let

$$S = \{f \in \text{Arrows}(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}$$

Then S is a saturated (see Categories, Definition 4.27.20) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 4.27.1 and 4.27.20 and Definition 13.5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and the long exact cohomology sequence (13.3.5.1). By Lemma 13.5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 13.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{tikzcd} Z \arrow[r, "t"'] & X \arrow[r, "g"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\ & X \arrow[r, "a"'] & Y \arrow[d, "j"] & \\ & & W & \end{tikzcd}$$

OK, and now we apply the functors H^i to this diagram. Since $t \in S$ we see that $H^i(Q) = 0$ by the long exact cohomology sequence (13.3.5.1). Hence $H^i(j)$ is an isomorphism for all i by the same argument, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ g = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. ☐

05R6 Proposition 13.5.6. Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Then there exists a unique structure of a pre-triangulated category on $S^{-1}\mathcal{D}$ such that $[1] \circ Q = Q \circ [1]$ and the localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ is exact. Moreover, if $\mathcal{D}$ is a triangulated category, so is $S^{-1}\mathcal{D}$.

Proof. We have seen that $S^{-1}\mathcal{D}$ is an additive category and that the localization functor Q is additive in Homology, Lemma 12.8.2. It follows from MS5 that there is a unique additive auto-equivalence $[1] : S^{-1}\mathcal{D} \to S^{-1}\mathcal{D}$ such that $Q \circ [1] = [1] \circ Q$ (equality of functors); we omit the details. We say a triangle of $S^{-1}\mathcal{D}$ is distinguished if it is isomorphic to the image of a distinguished triangle under the localization functor Q.

Proof of TR1. The only thing to prove here is that if $a : Q(X) \to Q(Y)$ is a morphism of $S^{-1}\mathcal{D}$, then a fits into a distinguished triangle. Write $a = Q(s)^{-1} \circ Q(f)$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Choose a distinguished triangle (X, Y', Z, f, g, h) in $\mathcal{D}$. Then we see that $(Q(X), Q(Y), Q(Z), a, Q(g) \circ Q(s), Q(h))$ is a distinguished triangle of $S^{-1}\mathcal{D}$.

Proof of TR2. This is immediate from the definitions.

Proof of TR3. Note that the existence of the dotted arrow which is required to exist may be proven after replacing the two triangles by isomorphic triangles. Hence we may assume given distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') of $\mathcal{D}$ and a commutative diagram

$$\begin{tikzcd} Q(X) \arrow[r, "Q(f)"'] \arrow[d, "a"'] & Q(Y) \arrow[d, "b"] \\ Q(X') \arrow[r, "Q(f')"] & Q(Y') \end{tikzcd}$$

in $S^{-1}\mathcal{D}$. Now we apply Categories, Lemma 4.27.10 to find a morphism $f'' : X'' \to Y''$ in $\mathcal{D}$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "k"'] \arrow[d, "f"'] & X'' \arrow[d, "f''"] & X' \arrow[l, "s"'] \arrow[d, "f'"] \\ Y \arrow[r, "l"] & Y'' & Y' \arrow[l, "t"] \end{tikzcd}$$

in $\mathcal{D}$ with $s, t \in S$ and $a = s^{-1}k$, $b = t^{-1}l$. At this point we can use TR3 for $\mathcal{D}$ and MS6 to find a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "k"] & Y \arrow[r] \arrow[d, "l"] & Z \arrow[r] \arrow[d, "m"] & X[1] \arrow[d, "g[1]"] \\
X'' \arrow[r] & Y'' \arrow[r] & Z'' \arrow[r] & X''[1] \\
X' \arrow[r] \arrow[u, "s"] & Y' \arrow[r] \arrow[u, "t"] & Z' \arrow[r] \arrow[u, "r"] & X'[1] \arrow[u, "s[1]"]
\end{tikzcd}$$

with $r \in S$. It follows that setting $c = Q(r)^{-1}Q(m)$ we obtain the desired morphism of triangles

$$\begin{tikzcd}
(Q(X), Q(Y), Q(Z), Q(f), Q(g), Q(h)) \arrow[d, "{(a,b,c)}"] \\
(Q(X'), Q(Y'), Q(Z'), Q(f'), Q(g'), Q(h'))
\end{tikzcd}$$

This proves the first statement of the lemma. If $\mathcal{D}$ is also a triangulated category, then we still have to prove TR4 in order to show that $S^{-1}\mathcal{D}$ is triangulated as well. To do this we reduce by Lemma 13.4.15 to the following statement: Given composable morphisms $a : Q(X) \to Q(Y)$ and $b : Q(Y) \to Q(Z)$ we have to produce an octahedron after possibly replacing $Q(X), Q(Y), Q(Z)$ by isomorphic objects. To do this we may first replace Y by an object such that $a = Q(f)$ for some morphism $f : X \to Y$ in $\mathcal{D}$. (More precisely, write $a = s^{-1}f$ with $s : Y \to Y'$ in S and $f : X \to Y'$. Then replace Y by Y'.) After this we similarly replace Z by an object such that $b = Q(g)$ for some morphism $g : Y \to Z$. Now we can find distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3) in $\mathcal{D}$ (by TR1), and morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ as in TR4. Then it is immediately verified that applying the functor Q to all these data gives a corresponding structure in $S^{-1}\mathcal{D}$ $\square$

The universal property of the localization of a triangulated category is as follows (we formulate this for pre-triangulated categories, hence it holds a fortiori for triangulated categories).

05R7 Lemma 13.5.7. Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ be the localization functor, see Proposition 13.5.6.

(1) If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $H(s)$ is an isomorphism for all $s \in S$, then the unique factorization $H' : S^{-1}\mathcal{D} \to \mathcal{A}$ such that $H = H' \circ Q$ (see Categories, Lemma 4.27.8) is a homological functor too.
(2) If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $F(s)$ is an isomorphism for all $s \in S$, then the unique factorization $F' : S^{-1}\mathcal{D} \to \mathcal{D}'$ such that $F = F' \circ Q$ (see Categories, Lemma 4.27.8) is an exact functor too.

Proof. This lemma proves itself. Details omitted. $\square$

0GSL Lemma 13.5.8. Let $\mathcal{D}$ be a pre-triangulated category and let $\mathcal{D}' \subset \mathcal{D}$ be a full, pre-triangulated subcategory. Let S be a saturated multiplicative system of $\mathcal{D}$ compatible with the triangulated structure. Assume that for each X in $\mathcal{D}$ there exists an $s : X' \to X$ in S such that X' is an object of $\mathcal{D}'$. Then $S' = S \cap \text{Arrows}(\mathcal{D}')$ is a saturated multiplicative system compatible with the triangulated structure and the functor

$$(S')^{-1}\mathcal{D}' \longrightarrow S^{-1}\mathcal{D}$$

is an equivalence of pre-triangulated categories.

Proof. Consider the quotient functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ of Proposition 13.5.6. Since S is saturated we have that a morphism $f : X \to Y$ is in S if and only if $Q(f)$ is invertible, see Categories, Lemma 4.27.21. Thus S' is the collection of arrows which are turned into isomorphisms by the composition $\mathcal{D}' \to \mathcal{D} \to S^{-1}\mathcal{D}$. Hence S' is is a saturated multiplicative system compatible with the triangulated structure by Lemma 13.5.4. By Lemma 13.5.7 we obtain the exact functor $(S')^{-1}\mathcal{D}' \to S^{-1}\mathcal{D}$ of pre-triangulated categories. By assumption this functor is essentially surjective. Let X', Y' be objects of $\mathcal{D}'$. By Categories, Remark 4.27.15 we have

$$\text{Mor}_{S^{-1}\mathcal{D}}(X', Y') = \text{colim}_{s : X \to X' \text{ in } S} \text{Mor}_{\mathcal{D}}(X, Y')$$

Our assumption implies that for any $s : X \to X'$ in S we can find a morphism $s' : X'' \to X$ in S with X'' in $\mathcal{D}'$. Then $s \circ s' : X'' \to X'$ is in S'. Hence the colimit above is equal to

$$\text{colim}_{s'' : X'' \to X' \text{ in } S'} \text{Mor}_{\mathcal{D}'}(X'', Y') = \text{Mor}_{(S')^{-1}\mathcal{D}'}(X', Y')$$

This proves our functor is also fully faithful and the proof is complete. □

The following lemma describes the kernel (see Definition 13.6.5) of the localization functor.

05R8 Lemma 13.5.9. Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let Z be an object of $\mathcal{D}$. The following are equivalent

(1) $Q(Z) = 0$ in $S^{-1}\mathcal{D}$,
(2) there exists $Z' \in \text{Ob}(\mathcal{D})$ such that $0 : Z \to Z'$ is an element of S,
(3) there exists $Z' \in \text{Ob}(\mathcal{D})$ such that $0 : Z' \to Z$ is an element of S, and
(4) there exists an object Z' and a distinguished triangle $(X, Y, Z \oplus Z', f, g, h)$ such that $f \in S$.

If S is saturated, then these are also equivalent to

(5) the morphism $0 \to Z$ is an element of S,
(6) the morphism $Z \to 0$ is an element of S,
(7) there exists a distinguished triangle (X, Y, Z, f, g, h) such that $f \in S$.

Proof. The equivalence of (1), (2), and (3) is Homology, Lemma 12.8.3. If (2) holds, then $(Z'[-1], Z'[-1] \oplus Z, Z, (1, 0), (0, 1), 0)$ is a distinguished triangle (see Lemma 13.4.11) with "$0 \in S$". By rotating we conclude that (4) holds. If $(X, Y, Z \oplus Z', f, g, h)$ is a distinguished triangle with $f \in S$ then $Q(f)$ is an isomorphism hence $Q(Z \oplus Z') = 0$ hence $Q(Z) = 0$. Thus (1) – (4) are all equivalent.

Next, assume that S is saturated. Note that each of (5), (6), (7) implies one of the equivalent conditions (1) – (4). Suppose that $Q(Z) = 0$. Then $0 \to Z$ is a morphism of $\mathcal{D}$ which becomes an isomorphism in $S^{-1}\mathcal{D}$. According to Categories, Lemma 4.27.21 the fact that S is saturated implies that $0 \to Z$ is in S. Hence (1) $\Rightarrow$ (5). Dually (1) $\Rightarrow$ (6). Finally, if $0 \to Z$ is in S, then the triangle $(0, Z, Z, 0, \mathrm{id}_Z, 0)$ is distinguished by TR1 and TR2 and is a triangle as in (4). □

05R9 Lemma 13.5.10. Let $\mathcal{D}$ be a triangulated category. Let S be a saturated multiplicative system in $\mathcal{D}$ that is compatible with the triangulated structure. Let (X, Y, Z, f, g, h) be a distinguished triangle in $\mathcal{D}$. Consider the category of morphisms of triangles

$$\mathcal{I} = \{(s, s', s'') : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h') \mid s, s', s'' \in S\}$$

Then $\mathcal{I}$ is a filtered category and the functors $\mathcal{I} \to X/S$, $\mathcal{I} \to Y/S$, and $\mathcal{I} \to Z/S$ are cofinal.

Proof. We strongly suggest the reader skip the proof of this lemma and instead work it out on a napkin.

The first remark is that using rotation of distinguished triangles (TR2) gives an equivalence of categories between $\mathcal{I}$ and the corresponding category for the distinguished triangle $(Y, Z, X[1], g, h, -f[1])$. Using this we see for example that if we prove the functor $\mathcal{I} \to X/S$ is cofinal, then the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

Note that if $s : X \to X'$ is a morphism of S, then using MS2 we can find $s' : Y \to Y'$ and $f' : X' \to Y'$ such that $f' \circ s = s' \circ f$, whereupon we can use MS6 to complete this into an object of $\mathcal{I}$. Hence the functor $\mathcal{I} \to X/S$ is surjective on objects. Using rotation as above this implies the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

Suppose given objects $s_1 : X \to X_1$ and $s_2 : X \to X_2$ in X/S and a morphism $a : X_1 \to X_2$ in X/S. Since S is saturated, we see that $a \in S$, see Categories, Lemma 4.27.21. By the argument of the previous paragraph we can complete $s_1 : X \to X_1$ to an object $(s_1, s_1', s_1'') : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ in $\mathcal{I}$. Then we can repeat and find $(a, b, c) : (X_1, Y_1, Z_1, f_1, g_1, h_1) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ with $a, b, c \in S$ completing the given $a : X_1 \to X_2$. But then (a, b, c) is a morphism in $\mathcal{I}$. In this way we conclude that the functor $\mathcal{I} \to X/S$ is also surjective on arrows. Using rotation as above, this implies the same thing is true for the functors $\mathcal{I} \to Y/S$ and $\mathcal{I} \to Z/S$.

The category $\mathcal{I}$ is nonempty as the identity provides an object. This proves the condition (1) of the definition of a filtered category, see Categories, Definition 4.19.1.

We check condition (2) of Categories, Definition 4.19.1 for the category $\mathcal{I}$. Suppose given objects $(s_1, s_1', s_1'') : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z, f, g, h) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ in $\mathcal{I}$. We want to find an object of $\mathcal{I}$ which is the target of an arrow from both $(X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(X_2, Y_2, Z_2, f_2, g_2, h_2)$. By Categories, Remark 4.27.7 the categories X/S, Y/S, Z/S are filtered. Thus we can find $X \to X_3$ in X/S and morphisms $s : X_2 \to X_3$ and $a : X_1 \to X_3$. By the above we can find a morphism $(s, s', s'') : (X_2, Y_2, Z_2, f_2, g_2, h_2) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ with $s', s'' \in S$. After replacing (X_2, Y_2, Z_2) by (X_3, Y_3, Z_3) we may assume that there exists a morphism $a : X_1 \to X_2$ in X/S. Repeating the argument for Y and Z (by rotating as above) we may assume there is a morphism $a : X_1 \to X_2$ in X/S, $b : Y_1 \to Y_2$ in Y/S, and $c : Z_1 \to Z_2$ in Z/S. However, these morphisms do not necessarily give rise to a morphism of distinguished triangles. On the other hand, the necessary diagrams do commute in $S^{-1}\mathcal{D}$. Hence we see (for example) that there exists a morphism $s'_2 : Y_2 \to Y_3$ in S such that $s'_2 \circ f_2 \circ a = s'_2 \circ b \circ f_1$. Another replacement of (X_2, Y_2, Z_2) as above then gets us to the situation where $f_2 \circ a = b \circ f_1$. Rotating and applying the same argument two more times we see that we may assume (a, b, c) is a morphism of triangles. This proves condition (2).

Next we check condition (3) of Categories, Definition 4.19.1. Suppose $(s_1, s'_1, s''_1) : (X, Y, Z) \to (X_1, Y_1, Z_1)$ and $(s_2, s'_2, s''_2) : (X, Y, Z) \to (X_2, Y_2, Z_2)$ are objects of $\mathcal{I}$, and suppose $(a, b, c), (a', b', c')$ are two morphisms between them. Since $a \circ s_1 = a' \circ s_1$ there exists a morphism $s_3 : X_2 \to X_3$ such that $s_3 \circ a = s_3 \circ a'$. Using the surjectivity statement we can complete this to a morphism of triangles $(s_3, s'_3, s''_3) : (X_2, Y_2, Z_2) \to (X_3, Y_3, Z_3)$ with $s_3, s'_3, s''_3 \in S$. Thus $(s_3 \circ s_2, s'_3 \circ s'_2, s''_3 \circ s''_2) : (X, Y, Z) \to (X_3, Y_3, Z_3)$ is also an object of $\mathcal{I}$ and after composing the maps $(a, b, c), (a', b', c')$ with (s_3, s'_3, s''_3) we obtain $a = a'$. By rotating we may do the same to get $b = b'$ and $c = c'$.

Finally, we check that $\mathcal{I} \to X/S$ is cofinal, see Categories, Definition 4.17.1. The first condition is true as the functor is surjective. Suppose that we have an object $s : X \to X'$ in X/S and two objects $(s_1, s'_1, s''_1) : (X, Y, Z, f, g, h) \to (X_1, Y_1, Z_1, f_1, g_1, h_1)$ and $(s_2, s'_2, s''_2) : (X, Y, Z, f, g, h) \to (X_2, Y_2, Z_2, f_2, g_2, h_2)$ in $\mathcal{I}$ as well as morphisms $t_1 : X' \to X_1$ and $t_2 : X' \to X_2$ in X/S. By property (2) of $\mathcal{I}$ proved above we can find morphisms $(s_3, s'_3, s''_3) : (X_1, Y_1, Z_1, f_1, g_1, h_1) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ and $(s_4, s'_4, s''_4) : (X_2, Y_2, Z_2, f_2, g_2, h_2) \to (X_3, Y_3, Z_3, f_3, g_3, h_3)$ in $\mathcal{I}$. We would be done if the compositions $X' \to X_1 \to X_3$ and $X' \to X_2 \to X_3$ where equal (see displayed equation in Categories, Definition 4.17.1). If not, then, because X/S is filtered, we can choose a morphism $X_3 \to X_4$ in S such that the compositions $X' \to X_1 \to X_3 \to X_4$ and $X' \to X_2 \to X_3 \to X_4$ are equal. Then we finally complete $X_3 \to X_4$ to a morphism $(X_3, Y_3, Z_3) \to (X_4, Y_4, Z_4)$ in $\mathcal{I}$ and compose with that morphism to see that the result is true. □

13.6. Quotients of triangulated categories

05RA Given a triangulated category and a triangulated subcategory we can construct another triangulated category by taking the "quotient". The construction uses a localization. This is similar to the quotient of an abelian category by a Serre subcategory, see Homology, Section 12.10. Before we do the actual construction we briefly discuss kernels of exact functors.

05RB Definition 13.6.1. Let $\mathcal{D}$ be a pre-triangulated category. We say a full pre-triangulated subcategory $\mathcal{D}'$ of $\mathcal{D}$ is saturated if whenever $X \oplus Y$ is isomorphic to an object of $\mathcal{D}'$ then both X and Y are isomorphic to objects of $\mathcal{D}'$.

A saturated triangulated subcategory is sometimes called a thick triangulated subcategory. In some references, this is only used for strictly full triangulated subcategories (and sometimes the definition is written such that it implies strictness). There is another notion, that of an épaisse triangulated subcategory. The definition is that given a commutative diagram

$$\begin{tikzcd} & S \arrow[dr] & & & \\ X \arrow[ur] \arrow[rr] & & Y \arrow[r] & T \arrow[r] & X[1] \end{tikzcd}$$

where the second line is a distinguished triangle and S and T are isomorphic to objects of $\mathcal{D}'$, then also X and Y are isomorphic to objects of $\mathcal{D}'$. It turns out that this is equivalent to being saturated (this is elementary and can be found in [Ric89a]) and the notion of a saturated category is easier to work with.

05RC Lemma 13.6.2. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let $\mathcal{D}''$ be the full subcategory of $\mathcal{D}$ with objects

$$\mathrm{Ob}(\mathcal{D}'') = \{X \in \mathrm{Ob}(\mathcal{D}) \mid F(X) = 0\}$$

Then $\mathcal{D}''$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}''$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}''$ is preserved under [1] and [−1]. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $F(X) = F(Y) = 0$, then also $F(Z) = 0$ as $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished. Hence we may apply Lemma 13.4.16 to see that $\mathcal{D}''$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The final assertion of being saturated follows from $F(X) \oplus F(Y) = 0 \Rightarrow F(X) = F(Y) = 0$. □

05RD Lemma 13.6.3. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ with objects

$$\mathrm{Ob}(\mathcal{D}') = \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \in \mathbf{Z}\}$$

Then $\mathcal{D}'$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}'$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}'$ is preserved under [1] and [−1]. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all n, then also $H(Z[n]) = 0$ for all n by the long exact sequence (13.3.5.1). Hence we may apply Lemma 13.4.16 to see that $\mathcal{D}'$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

05RE Lemma 13.6.4. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}_H^+, \mathcal{D}_H^-, \mathcal{D}_H^b$ be the full subcategory of $\mathcal{D}$ with objects

$$\begin{aligned} \mathrm{Ob}(\mathcal{D}_H^+) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \ll 0\} \\ \mathrm{Ob}(\mathcal{D}_H^-) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \gg 0\} \\ \mathrm{Ob}(\mathcal{D}_H^b) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } |n| \gg 0\} \end{aligned}$$

Each of these is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then each is a triangulated subcategory.

Proof. Let us prove this for $\mathcal{D}^+_H$. It is clear that it is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all $n \ll 0$, then also $H(Z[n]) = 0$ for all $n \ll 0$ by the long exact sequence (13.3.5.1). Hence we may apply Lemma 13.4.16 to see that $\mathcal{D}^+_H$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

05RF Definition 13.6.5. Let $\mathcal{D}$ be a (pre-)triangulated category.

(1) Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. The kernel of F is the strictly full saturated (pre-)triangulated subcategory described in Lemma 13.6.2.
(2) Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor. The kernel of H is the strictly full saturated (pre-)triangulated subcategory described in Lemma 13.6.3.

These are sometimes denoted $\text{Ker}(F)$ or $\text{Ker}(H)$.

The proof of the following lemma uses TR4.

05RG Lemma 13.6.6. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Set

05RH (13.6.6.1) $$S = \left\{ \begin{matrix} f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{D}' \end{matrix} \right\}$$

Then S is a multiplicative system compatible with the triangulated structure on $\mathcal{D}$. In this situation the following are equivalent

(1) S is a saturated multiplicative system,
(2) $\mathcal{D}'$ is a saturated triangulated subcategory.

Proof. To prove the first assertion we have to prove that MS1, MS2, MS3 and MS5, MS6 hold.

Proof of MS1. It is clear that identities are in S because $(X, X, 0, 1, 0, 0)$ is distinguished for every object X of $\mathcal{D}$ and because 0 is an object of $\mathcal{D}'$. Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms contained in S. Choose distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3). By assumption we know that Q_1 and Q_3 are isomorphic to objects of $\mathcal{D}'$. By TR4 we know there exists a distinguished triangle (Q_1, Q_2, Q_3, a, b, c). Since $\mathcal{D}'$ is a triangulated subcategory we conclude that Q_2 is isomorphic to an object of $\mathcal{D}'$. Hence $g \circ f \in S$.

Proof of MS3. Let $a : X \to Y$ be a morphism and let $t : Z \to X$ be an element of S such that $a \circ t = 0$. To prove LMS3 it suffices to find an $s \in S$ such that $s \circ a = 0$, compare with the proof of Lemma 13.5.4. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 13.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, s, k). Here is a picture

$$\begin{tikzcd}
Z \arrow[r, "t"'] & X \arrow[r, "g"'] \arrow[d, "1"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\
& X \arrow[r, "a"'] & Y \arrow[d, "s"] & \\
& & W &
\end{tikzcd}$$

Since $t \in S$ we see that Q is isomorphic to an object of $\mathcal{D}'$. Hence $s \in S$. Finally, $s \circ a = s \circ i \circ g = 0$ as $s \circ i = 0$ by Lemma 13.4.1. We conclude that LMS3 holds. The proof of RMS3 is dual.

Proof of MS5. Follows as distinguished triangles and $\mathcal{D}'$ are stable under translations

Proof of MS6. Suppose given a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "s"] & Y \arrow[d, "s'"] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

with $s, s' \in S$. By Proposition 13.4.23 we can extend this to a nine square diagram. As s, s' are elements of S we see that X'', Y'' are isomorphic to objects of $\mathcal{D}'$. Since $\mathcal{D}'$ is a full triangulated subcategory we see that Z'' is also isomorphic to an object of $\mathcal{D}'$. Whence the morphism $Z \to Z'$ is an element of S. This proves MS6.

MS2 is a formal consequence of MS1, MS5, and MS6, see Lemma 13.5.2. This finishes the proof of the first assertion of the lemma.

Let's assume that S is saturated. (In the following we will use rotation of distinguished triangles without further mention.) Let $X \oplus Y$ be an object isomorphic to an object of $\mathcal{D}'$. Consider the morphism $f : 0 \to X$. The composition $0 \to X \to X \oplus Y$ is an element of S as $(0, X \oplus Y, X \oplus Y, 0, 1, 0)$ is a distinguished triangle. The composition $Y[-1] \to 0 \to X$ is an element of S as $(X, X \oplus Y, Y, (1, 0), (0, 1), 0)$ is a distinguished triangle, see Lemma 13.4.11. Hence $0 \to X$ is an element of S (as S is saturated). Thus X is isomorphic to an object of $\mathcal{D}'$ as desired.

Finally, assume $\mathcal{D}'$ is a saturated triangulated subcategory. Let

$$W \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z$$

be composable morphisms of $\mathcal{D}$ such that $fg, gh \in S$. We will build up a picture of objects as in the diagram below.

$$\begin{tikzcd}
& & Q_{12} \arrow[dr] & & Q_{23} \arrow[dr] & & \\
& Q_1 \arrow[ur] \arrow[dl, "+1"'] & & Q_2 \arrow[ll, "+1"'] \arrow[ur] \arrow[dl, "+1"'] & & Q_3 \arrow[ll, "+1"'] \arrow[dl, "+1"'] & \\
W \arrow[rr] & & X \arrow[rr] \arrow[ul] & & Y \arrow[rr] \arrow[ul] & & Z \arrow[ul]
\end{tikzcd}$$

First choose distinguished triangles (W, X, Q_1), (X, Y, Q_2), (Y, Z, Q_3) (W, Y, Q_{12}), and (X, Z, Q_{23}). Denote $s : Q_2 \to Q_1[1]$ the composition $Q_2 \to X[1] \to Q_1[1]$. Denote $t : Q_3 \to Q_2[1]$ the composition $Q_3 \to Y[1] \to Q_2[1]$. By TR4 applied to the composition $W \to X \to Y$ and the composition $X \to Y \to Z$ there exist distinguished triangles (Q_1, Q_{12}, Q_2) and (Q_2, Q_{23}, Q_3) which use the morphisms s and t. The objects Q_{12} and Q_{23} are isomorphic to objects of $\mathcal{D}'$ as $W \to Y$ and $X \to Z$ are assumed in S. Hence also $s[1]t$ is an element of S as S is closed under compositions and shifts. Note that $s[1]t = 0$ as $Y[1] \to Q_2[1] \to X[2]$ is zero, see Lemma 13.4.1. Hence $Q_3[1] \oplus Q_1[2]$ is isomorphic to an object of $\mathcal{D}'$, see Lemma 13.4.11. By assumption on $\mathcal{D}'$ we conclude that Q_3 and Q_1 are isomorphic to objects of $\mathcal{D}'$. Looking at the distinguished triangle (Q_1, Q_{12}, Q_2) we conclude that Q_2 is also isomorphic to an object of $\mathcal{D}'$. Looking at the distinguished triangle (X, Y, Q_2) we finally conclude that $g \in S$. (It is also follows that $h, f \in S$, but we don't need this.) □

05RI Definition 13.6.7. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. We define the quotient category $\mathcal{D}/\mathcal{B}$ by the formula $\mathcal{D}/\mathcal{B} = S^{-1}\mathcal{D}$, where S is the multiplicative system of $\mathcal{D}$ associated to $\mathcal{B}$ via Lemma 13.6.6. The localization functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is called the quotient functor in this case.

Note that the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is an exact functor of triangulated categories, see Proposition 13.5.6. The universal property of this construction is the following.

05RJ Lemma 13.6.8. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. Let $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ be the quotient functor.

(1) If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $\mathcal{B} \subset \mathrm{Ker}(H)$ then there exists a unique factorization $H' : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ such that $H = H' \circ Q$ and H' is a homological functor too.
(2) If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $\mathcal{B} \subset \mathrm{Ker}(F)$ then there exists a unique factorization $F' : \mathcal{D}/\mathcal{B} \to \mathcal{D}'$ such that $F = F' \circ Q$ and F' is an exact functor too.

Proof. This lemma follows from Lemma 13.5.7. Namely, if $f : X \to Y$ is a morphism of $\mathcal{D}$ such that for some distinguished triangle (X, Y, Z, f, g, h) the object Z is isomorphic to an object of $\mathcal{B}$, then $H(f)$, resp. $F(f)$ is an isomorphism under the assumptions of (1), resp. (2). Details omitted. □

The kernel of the quotient functor can be described as follows.

05RK Lemma 13.6.9. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. The kernel of the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is the strictly full subcategory of $\mathcal{D}$ whose objects are

$$\mathrm{Ob}(\mathrm{Ker}(Q)) = \left\{ \begin{matrix} Z \in \mathrm{Ob}(\mathcal{D}) \text{ such that there exists a } Z' \in \mathrm{Ob}(\mathcal{D}) \\ \text{such that } Z \oplus Z' \text{ is isomorphic to an object of } \mathcal{B} \end{matrix} \right\}$$

In other words it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.

Proof. First note that the kernel is automatically a strictly full triangulated subcategory containing summands of any of its objects, see Lemma 13.6.2. The description of its objects follows from the definitions and Lemma 13.5.9 part (4). □

Let $\mathcal{D}$ be a triangulated category. At this point we have constructions which induce order preserving maps between

(1) the partially ordered set of multiplicative systems S in $\mathcal{D}$ compatible with the triangulated structure, and
(2) the partially ordered set of full triangulated subcategories $\mathcal{B} \subset \mathcal{D}$.

Namely, the constructions are given by $S \mapsto \mathcal{B}(S) = \text{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$ and $\mathcal{B} \mapsto S(\mathcal{B})$ where $S(\mathcal{B})$ is the multiplicative set of (13.6.6.1), i.e.,

$$S(\mathcal{B}) = \left\{ \begin{matrix} f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{B} \end{matrix} \right\}$$

Note that it is not the case that these operations are mutually inverse.

05RL Lemma 13.6.10. Let $\mathcal{D}$ be a triangulated category. The operations described above have the following properties

(1) $S(\mathcal{B}(S))$ is the "saturation" of S, i.e., it is the smallest saturated multiplicative system in $\mathcal{D}$ containing S, and
(2) $\mathcal{B}(S(\mathcal{B}))$ is the "saturation" of $\mathcal{B}$, i.e., it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.

In particular, the constructions define mutually inverse maps between the (partially ordered) set of saturated multiplicative systems in $\mathcal{D}$ compatible with the triangulated structure on $\mathcal{D}$ and the (partially ordered) set of strictly full saturated triangulated subcategories of $\mathcal{D}$.

Proof. First, let's start with a full triangulated subcategory $\mathcal{B}$. Then $\mathcal{B}(S(\mathcal{B})) = \text{Ker}(Q : \mathcal{D} \to \mathcal{D}/\mathcal{B})$ and hence (2) is the content of Lemma 13.6.9.

Next, suppose that S is multiplicative system in $\mathcal{D}$ compatible with the triangulation on $\mathcal{D}$. Then $\mathcal{B}(S) = \text{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$. Hence (using Lemma 13.4.9 in the localized category)

$$\begin{aligned} S(\mathcal{B}(S)) &= \left\{ \begin{matrix} f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished} \\ \text{triangle } (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Q(Z) = 0 \end{matrix} \right\} \\ &= \{ f \in \text{Arrows}(\mathcal{D}) \mid Q(f) \text{ is an isomorphism} \} \\ &= \hat{S} = S' \end{aligned}$$

in the notation of Categories, Lemma 4.27.21. The final statement of that lemma finishes the proof. □

05RM Lemma 13.6.11. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$, see Definition 13.3.5. The subcategory $\text{Ker}(H)$ of $\mathcal{D}$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ whose corresponding saturated multiplicative system (see Lemma 13.6.10) is the set

$$S = \{ f \in \text{Arrows}(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z} \}.$$

The functor H factors through the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\text{Ker}(H)$.

Proof. The category $\text{Ker}(H)$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ by Lemma 13.6.3. The set S is a saturated multiplicative system compatible with the triangulated structure by Lemma 13.5.5. Recall that the multiplicative system corresponding to $\text{Ker}(H)$ is the set

$$\left\{ \begin{matrix} f \in \text{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ with } H^i(Z) = 0 \text{ for all } i \end{matrix} \right\}$$

By the long exact cohomology sequence, see (13.3.5.1), it is clear that f is an element of this set if and only if f is an element of S. Finally, the factorization of H through Q is a consequence of Lemma 13.6.8. □

13.7. Adjoints for exact functors

0A8C Results on adjoint functors between triangulated categories.

0A8D Lemma 13.7.1. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. If F admits a right adjoint $G : \mathcal{D}' \to \mathcal{D}$, then G is also an exact functor.

Proof. Let X be an object of $\mathcal{D}$ and A an object of $\mathcal{D}'$. Since F is an exact functor we see that

$$\begin{aligned} \mathrm{Mor}_{\mathcal{D}}(X, G(A[1])) &= \mathrm{Mor}_{\mathcal{D}'}(F(X), A[1]) \\ &= \mathrm{Mor}_{\mathcal{D}'}(F(X)[-1], A) \\ &= \mathrm{Mor}_{\mathcal{D}'}(F(X[-1]), A) \\ &= \mathrm{Mor}_{\mathcal{D}}(X[-1], G(A)) \\ &= \mathrm{Mor}_{\mathcal{D}}(X, G(A)[1]) \end{aligned}$$

By Yoneda's lemma (Categories, Lemma 4.3.5) we obtain a canonical isomorphism $G(A)[1] = G(A[1])$. Let $A \to B \to C \to A[1]$ be a distinguished triangle in $\mathcal{D}'$. Choose a distinguished triangle

$$G(A) \to G(B) \to X \to G(A)[1]$$

in $\mathcal{D}$. Then $F(G(A)) \to F(G(B)) \to F(X) \to F(G(A))[1]$ is a distinguished triangle in $\mathcal{D}'$. By TR3 we can choose a morphism of distinguished triangles

$$\begin{tikzcd} F(G(A)) \arrow[r] \arrow[d] & F(G(B)) \arrow[r] \arrow[d] & F(X) \arrow[r] \arrow[d] & F(G(A))[1] \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

Since G is the adjoint the new morphism determines a morphism $X \to G(C)$ such that the diagram

$$\begin{tikzcd} G(A) \arrow[r] \arrow[d] & G(B) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & G(A)[1] \arrow[d] \\ G(A) \arrow[r] & G(B) \arrow[r] & G(C) \arrow[r] & G(A)[1] \end{tikzcd}$$

commutes. Applying the homological functor $\mathrm{Hom}_{\mathcal{D}'}(W, -)$ for an object W of $\mathcal{D}'$ we deduce from the 5 lemma that

$$\mathrm{Hom}_{\mathcal{D}'}(W, X) \to \mathrm{Hom}_{\mathcal{D}'}(W, G(C))$$

is a bijection and using the Yoneda lemma once more we conclude that $X \to G(C)$ is an isomorphism. Hence we conclude that $G(A) \to G(B) \to G(C) \to G(A)[1]$ is a distinguished triangle which is what we wanted to show. □

09J1 Lemma 13.7.2. Let $\mathcal{D}$, $\mathcal{D}'$ be triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be functors. Assume that

(1) F and G are exact functors,
(2) F is fully faithful,

(3) G is a right adjoint to F, and
(4) the kernel of G is zero.

Then F is an equivalence of categories.

Proof. Since F is fully faithful the adjunction map $\text{id} \to G \circ F$ is an isomorphism (Categories, Lemma 4.24.4). Let X be an object of $\mathcal{D}'$. Choose a distinguished triangle

$$F(G(X)) \to X \to Y \to F(G(X))[1]$$

in $\mathcal{D}'$. Applying G and using that $G(F(G(X))) = G(X)$ we find a distinguished triangle

$$G(X) \to G(X) \to G(Y) \to G(X)[1]$$

Hence $G(Y) = 0$. Thus $Y = 0$. Thus $F(G(X)) \to X$ is an isomorphism. $\square$

13.8. The homotopy category

05RN Let $\mathcal{A}$ be an additive category. The homotopy category $K(\mathcal{A})$ of $\mathcal{A}$ is the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Here is the formal definition.

013H Definition 13.8.1. Let $\mathcal{A}$ be an additive category.

(1) We set $\text{Comp}(\mathcal{A}) = \text{CoCh}(\mathcal{A})$ be the category of (cochain) complexes.
(2) A complex $K^\bullet$ is said to be bounded below if $K^n = 0$ for all $n \ll 0$.
(3) A complex $K^\bullet$ is said to be bounded above if $K^n = 0$ for all $n \gg 0$.
(4) A complex $K^\bullet$ is said to be bounded if $K^n = 0$ for all $|n| \gg 0$.
(5) We let $\text{Comp}^+(\mathcal{A})$, $\text{Comp}^-(\mathcal{A})$, resp. $\text{Comp}^b(\mathcal{A})$ be the full subcategory of $\text{Comp}(\mathcal{A})$ whose objects are the complexes which are bounded below, bounded above, resp. bounded.
(6) We let $K(\mathcal{A})$ be the category with the same objects as $\text{Comp}(\mathcal{A})$ but as morphisms homotopy classes of maps of complexes (see Homology, Lemma 12.13.7).
(7) We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ be the full subcategory of $K(\mathcal{A})$ whose objects are bounded below, bounded above, resp. bounded complexes of $\mathcal{A}$.

It will turn out that the categories $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. To prove this we first develop some machinery related to cones and split exact sequences.

13.9. Cones and termwise split sequences

014D Let $\mathcal{A}$ be an additive category, and let $K(\mathcal{A})$ denote the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Note that the shift functors $[n]$ on complexes, see Homology, Definition 12.14.7, give rise to functors $[n] : K(\mathcal{A}) \to K(\mathcal{A})$ such that $[n] \circ [m] = [n + m]$ and $[0] = \text{id}$.

014E Definition 13.9.1. Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. The cone of f is the complex $C(f)^\bullet$ given by $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d^n_{C(f)} = \begin{pmatrix} d^n_L & f^{n+1} \\ 0 & -d^{n+1}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L^\bullet \to C(f)^\bullet$ and $p : C(f)^\bullet \to K^\bullet[1]$ induced by the obvious maps $L^n \to C(f)^n \to K^{n+1}$.

In other words $(K, L, C(f), f, i, p)$ forms a triangle:

$$K^\bullet \to L^\bullet \to C(f)^\bullet \to K^\bullet[1]$$

The formation of this triangle is functorial in the following sense.

014F Lemma 13.9.2. Suppose that

$$\begin{tikzcd} K_1^\bullet \arrow[r, "f_1"'] \arrow[d, "a"'] & L_1^\bullet \arrow[d, "b"] \\ K_2^\bullet \arrow[r, "f_2"] & L_2^\bullet \end{tikzcd}$$

is a diagram of morphisms of complexes which is commutative up to homotopy. Then there exists a morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ which gives rise to a morphism of triangles $(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \to (K_2^\bullet, L_2^\bullet, C(f_2)^\bullet, f_2, i_2, p_2)$ of $K(\mathcal{A})$.

Proof. Let $h^n : K_1^n \to L_2^{n-1}$ be a family of morphisms such that $b \circ f_1 - f_2 \circ a = d \circ h + h \circ d$. Define c^n by the matrix

$$c^n = \begin{pmatrix} b^n & h^{n+1} \\ 0 & a^{n+1} \end{pmatrix} : L_1^n \oplus K_1^{n+1} \to L_2^n \oplus K_2^{n+1}$$

A matrix computation show that c is a morphism of complexes. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. □

Note that the morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ constructed in the proof of Lemma 13.9.2 in general depends on the chosen homotopy h between $f_2 \circ a$ and $b \circ f_1$.

08RI Lemma 13.9.3. Suppose that $f : K^\bullet \to L^\bullet$ and $g : L^\bullet \to M^\bullet$ are morphisms of complexes such that $g \circ f$ is homotopic to zero. Then

(1) g factors through a morphism $C(f)^\bullet \to M^\bullet$, and
(2) f factors through a morphism $K^\bullet \to C(g)^\bullet[-1]$.

Proof. The assumptions say that the diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "f"'] \arrow[d] & L^\bullet \arrow[d, "g"] \\ 0 \arrow[r] & M^\bullet \end{tikzcd}$$

commutes up to homotopy. Since the cone on $0 \to M^\bullet$ is $M^\bullet$ the map $C(f)^\bullet \to C(0 \to M^\bullet) = M^\bullet$ of Lemma 13.9.2 is the map in (1). The cone on $K^\bullet \to 0$ is $K^\bullet[1]$ and applying Lemma 13.9.2 gives a map $K^\bullet[1] \to C(g)^\bullet$. Applying $[-1]$ we obtain the map in (2). □

Note that the morphisms $C(f)^\bullet \to M^\bullet$ and $K^\bullet \to C(g)^\bullet[-1]$ constructed in the proof of Lemma 13.9.3 in general depend on the chosen homotopy.

014G Definition 13.9.4. Let $\mathcal{A}$ be an additive category. A termwise split injection $\alpha : A^\bullet \to B^\bullet$ is a morphism of complexes such that each $A^n \to B^n$ is isomorphic to the inclusion of a direct summand. A termwise split surjection $\beta : B^\bullet \to C^\bullet$ is a morphism of complexes such that each $B^n \to C^n$ is isomorphic to the projection onto a direct summand.

014H Lemma 13.9.5. Let $\mathcal{A}$ be an additive category. Let

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, "a"] & B^\bullet \arrow[d, "b"] \\ C^\bullet \arrow[r, "g"] & D^\bullet \end{tikzcd}$$

be a diagram of morphisms of complexes commuting up to homotopy. If f is a termwise split injection, then b is homotopic to a morphism which makes the diagram commute. If g is a termwise split surjection, then a is homotopic to a morphism which makes the diagram commute.

Proof. Let $h^n : A^n \to D^{n-1}$ be a collection of morphisms such that $bf - ga = dh + hd$. Suppose that $\pi^n : B^n \to A^n$ are morphisms splitting the morphisms f^n. Take $b' = b - dh\pi - h\pi d$. Suppose $s^n : D^n \to C^n$ are morphisms splitting the morphisms $g^n : C^n \to D^n$. Take $a' = a + dsh + shd$. Computations omitted. $\square$

The following lemma can be used to replace a morphism of complexes by a morphism where in each degree the map is the injection of a direct summand.

013N Lemma 13.9.6. Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization

$$\begin{tikzcd} K^\bullet \arrow[r, "\tilde\alpha"] \arrow[rr, bend right, "\alpha"'] & \tilde L^\bullet \arrow[r, "\pi"] & L^\bullet \end{tikzcd}$$

such that

(1) $\tilde\alpha$ is a termwise split injection (see Definition 13.9.4),
(2) there is a map of complexes $s : L^\bullet \to \tilde L^\bullet$ such that $\pi \circ s = \mathrm{id}_{L^\bullet}$ and such that $s \circ \pi$ is homotopic to $\mathrm{id}_{\tilde L^\bullet}$.

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde L^\bullet$.

Proof. We set

$$\tilde L^n = L^n \oplus K^n \oplus K^{n+1}$$

and we define

$$d^n_{\tilde L} = \begin{pmatrix} d^n_L & 0 & 0 \\ 0 & d^n_K & \mathrm{id}_{K^{n+1}} \\ 0 & 0 & -d^{n+1}_K \end{pmatrix}$$

In other words, $\tilde L^\bullet = L^\bullet \oplus C(1_{K^\bullet})$. Moreover, we set

$$\tilde\alpha = \begin{pmatrix} \alpha \\ \mathrm{id}_{K^n} \\ 0 \end{pmatrix}$$

which is clearly a split injection. It is also clear that it defines a morphism of complexes. We define

$$\pi = \begin{pmatrix} \mathrm{id}_{L^n} & 0 & 0 \end{pmatrix}$$

so that clearly $\pi \circ \tilde\alpha = \alpha$. We set

$$s = \begin{pmatrix} \mathrm{id}_{L^n} \\ 0 \\ 0 \end{pmatrix}$$

so that $\pi \circ s = \text{id}_{L^\bullet}$. Finally, let $h^n : \tilde{L}^n \to \tilde{L}^{n-1}$ be the map which maps the summand K^n of $\tilde{L}^n$ via the identity morphism to the summand K^n of $\tilde{L}^{n-1}$. Then it is a trivial matter (see computations in remark below) to prove that

$$\text{id}_{\tilde{L}^\bullet} - s \circ \pi = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

013O Remark 13.9.7. To see the last displayed equality in the proof above we can argue with elements as follows. We have $s\pi(l, k, k^+) = (l, 0, 0)$. Hence the morphism of the left hand side maps (l, k, k^+) to $(0, k, k^+)$. On the other hand $h(l, k, k^+) = (0, 0, k)$ and $d(l, k, k^+) = (dl, dk + k^+, -dk^+)$. Hence $(dh + hd)(l, k, k^+) = d(0, 0, k) + h(dl, dk + k^+, -dk^+) = (0, k, -dk) + (0, 0, dk + k^+) = (0, k, k^+)$ as desired.

0642 Lemma 13.9.8. Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization

$$\begin{tikzcd} K^\bullet \arrow[r, "i"] \arrow[rr, bend right, "\alpha"'] & \tilde{K}^\bullet \arrow[r, "\tilde{\alpha}"] & L^\bullet \end{tikzcd}$$

such that

(1) $\tilde{\alpha}$ is a termwise split surjection (see Definition 13.9.4),
(2) there is a map of complexes $s : \tilde{K}^\bullet \to K^\bullet$ such that $s \circ i = \text{id}_{K^\bullet}$ and such that $i \circ s$ is homotopic to $\text{id}_{\tilde{K}^\bullet}$.

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde{K}^\bullet$.

Proof. Dual to Lemma 13.9.6. Take

$$\tilde{K}^n = K^n \oplus L^{n-1} \oplus L^n$$

and we define

$$d^n_{\tilde{K}} = \begin{pmatrix} d^n_K & 0 & 0 \\ 0 & -d^{n-1}_L & \text{id}_{L^n} \\ 0 & 0 & d^n_L \end{pmatrix}$$

in other words $\tilde{K}^\bullet = K^\bullet \oplus C(1_{L^\bullet[-1]})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha & 0 & \text{id}_{L^n} \end{pmatrix}$$

which is clearly a split surjection. It is also clear that it defines a morphism of complexes. We define

$$i = \begin{pmatrix} \text{id}_{K^n} \\ 0 \\ 0 \end{pmatrix}$$

so that clearly $\tilde{\alpha} \circ i = \alpha$. We set

$$s = \begin{pmatrix} \text{id}_{K^n} & 0 & 0 \end{pmatrix}$$

so that $s \circ i = \text{id}_{K^\bullet}$. Finally, let $h^n : \tilde{K}^n \to \tilde{K}^{n-1}$ be the map which maps the summand L^{n-1} of $\tilde{K}^n$ via the identity morphism to the summand L^{n-1} of $\tilde{K}^{n-1}$. Then it is a trivial matter to prove that

$$\text{id}_{\tilde{K}^\bullet} - i \circ s = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

014I Definition 13.9.9. Let $\mathcal{A}$ be an additive category. A termwise split exact sequence of complexes of $\mathcal{A}$ is a complex of complexes

$$0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$$

together with given direct sum decompositions $B^n = A^n \oplus C^n$ compatible with α^n and β^n. We often write $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ for the maps induced by the direct sum decompositions. According to Homology, Lemma 12.14.10 we get an associated morphism of complexes

$$\delta : C^\bullet \longrightarrow A^\bullet[1]$$

which in degree n is the map $\pi^{n+1} \circ d_B^n \circ s^n$. In other words $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ forms a triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

This will be the triangle associated to the termwise split sequence of complexes.

05SS Lemma 13.9.10. Let $\mathcal{A}$ be an additive category. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be termwise split exact sequences as in Definition 13.9.9. Let $(\pi')^n$, $(s')^n$ be a second collection of splittings. Denote $\delta' : C^\bullet \longrightarrow A^\bullet[1]$ the morphism associated to this second set of splittings. Then

$$(1, 1, 1) : (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta) \longrightarrow (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta')$$

is an isomorphism of triangles in $K(\mathcal{A})$.

Proof. The statement simply means that δ and δ' are homotopic maps of complexes. This is Homology, Lemma 12.14.12. $\square$

014J Remark 13.9.11. Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2$ be termwise split exact sequences. Suppose that $a : A_1^\bullet \to A_2^\bullet$, $b : B_1^\bullet \to B_2^\bullet$, and $c : C_1^\bullet \to C_2^\bullet$ are morphisms of complexes such that

$$\begin{tikzcd} A_1^\bullet \arrow[r] \arrow[d, "a"] & B_1^\bullet \arrow[r] \arrow[d, "b"] & C_1^\bullet \arrow[d, "c"] \\ A_2^\bullet \arrow[r] & B_2^\bullet \arrow[r] & C_2^\bullet \end{tikzcd}$$

commutes in $K(\mathcal{A})$. In general, there does not exist a morphism $b' : B_1^\bullet \to B_2^\bullet$ which is homotopic to b such that the diagram above commutes in the category of complexes. Namely, consider Examples, Equation (110.63.0.1). If we could replace the middle map there by a homotopic one such that the diagram commutes, then we would have additivity of traces which we do not.

086L Lemma 13.9.12. Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2, 3$ be termwise split exact sequences of complexes. Let $b : B_1^\bullet \to B_2^\bullet$ and $b' : B_2^\bullet \to B_3^\bullet$ be morphisms of complexes such that

$$\begin{tikzcd} A_1^\bullet \arrow[r] \arrow[d, "0"] & B_1^\bullet \arrow[r] \arrow[d, "b"] & C_1^\bullet \arrow[d, "0"] \\ A_2^\bullet \arrow[r] & B_2^\bullet \arrow[r] & C_2^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_2^\bullet \arrow[r] \arrow[d, "0"] & B_2^\bullet \arrow[r] \arrow[d, "b'"] & C_2^\bullet \arrow[d, "0"] \\ A_3^\bullet \arrow[r] & B_3^\bullet \arrow[r] & C_3^\bullet \end{tikzcd}$$

commute in $K(\mathcal{A})$. Then $b' \circ b = 0$ in $K(\mathcal{A})$.

Proof. By Lemma 13.9.5 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\mathrm{Im}(b^n) \subset \mathrm{Im}(A_2^n \to B_2^n)$ and $\mathrm{Ker}((b')^n) \supset \mathrm{Im}(A_2^n \to B_2^n)$. Then $b' \circ b = 0$ as a map of complexes. □

014K Lemma 13.9.13. Let $\mathcal{A}$ be an additive category. Let $f_1 : K_1^\bullet \to L_1^\bullet$ and $f_2 : K_2^\bullet \to L_2^\bullet$ be morphisms of complexes. Let

$$(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \longrightarrow (K_2^\bullet, L_2^\bullet, C(f_2)^\bullet, f_2, i_2, p_2)$$

be any morphism of triangles of $K(\mathcal{A})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2^\bullet \to K_1^\bullet$ be a morphism of complexes which is inverse to a in $K(\mathcal{A})$. Let $b^{-1} : L_2^\bullet \to L_1^\bullet$ be a morphism of complexes which is inverse to b in $K(\mathcal{A})$. Let $c' : C(f_2)^\bullet \to C(f_1)^\bullet$ be the morphism from Lemma 13.9.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathcal{A})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K^\bullet, L^\bullet, C(f)^\bullet, f, i, p)$ in $K(\mathcal{A})$ the morphism c is an isomorphism in $K(\mathcal{A})$. By assumption the two squares in the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r] \arrow[d, "1"'] & C(f)^\bullet \arrow[r] \arrow[d, "c"'] & K^\bullet[1] \arrow[d, "1"'] \\
L^\bullet \arrow[r] & C(f)^\bullet \arrow[r] & K^\bullet[1]
\end{tikzcd}$$

commute up to homotopy. By construction of $C(f)^\bullet$ the rows form termwise split sequences of complexes. Thus we see that $(c-1)^2 = 0$ in $K(\mathcal{A})$ by Lemma 13.9.12. Hence c is an isomorphism in $K(\mathcal{A})$ with inverse $2 - c$. □

Hence if a and b are homotopy equivalences then the resulting morphism of triangles is an isomorphism of triangles in $K(\mathcal{A})$. It turns out that the collection of triangles of $K(\mathcal{A})$ given by cones and the collection of triangles of $K(\mathcal{A})$ given by termwise split sequences of complexes are the same up to isomorphisms, at least up to sign!

014L Lemma 13.9.14. Let $\mathcal{A}$ be an additive category.

(1) Given a termwise split sequence of complexes $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ there exists a homotopy equivalence $C(\alpha)^\bullet \to C^\bullet$ such that the diagram

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C(\alpha)^\bullet \arrow[r, "-p"'] \arrow[d] & A^\bullet[1] \arrow[d] \\
A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r, "\delta"] & A^\bullet[1]
\end{tikzcd}$$

defines an isomorphism of triangles in $K(\mathcal{A})$.

(2) Given a morphism of complexes $f : K^\bullet \to L^\bullet$ there exists an isomorphism of triangles

$$\begin{tikzcd}
K^\bullet \arrow[r] \arrow[d] & \tilde{L}^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[r, "\delta"'] \arrow[d] & K^\bullet[1] \arrow[d] \\
K^\bullet \arrow[r] & L^\bullet \arrow[r] & C(f)^\bullet \arrow[r, "-p"] & K^\bullet[1]
\end{tikzcd}$$

where the upper triangle is the triangle associated to a termwise split exact sequence $K^\bullet \to \tilde{L}^\bullet \to M^\bullet$.

Proof. Proof of (1). We have $C(\alpha)^n = B^n \oplus A^{n+1}$ and we simply define $C(\alpha)^n \to C^n$ via the projection onto B^n followed by β^n. This defines a morphism of complexes because the compositions $A^{n+1} \to B^{n+1} \to C^{n+1}$ are zero. To get a homotopy inverse we take $C^\bullet \to C(\alpha)^\bullet$ given by $(s^n, -\delta^n)$ in degree n. This is a morphism of complexes because the morphism δ^n can be characterized as the unique morphism $C^n \to A^{n+1}$ such that $d \circ s^n - s^{n+1} \circ d = \alpha \circ \delta^n$, see proof of Homology, Lemma 12.14.10. The composition $C^\bullet \to C(\alpha)^\bullet \to C^\bullet$ is the identity. The composition $C(\alpha)^\bullet \to C^\bullet \to C(\alpha)^\bullet$ is equal to the morphism

$$\begin{pmatrix} s^n \circ \beta^n & 0 \\ -\delta^n \circ \beta^n & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h^n : C(\alpha)^n \to C(\alpha)^{n-1}$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} : C(\alpha)^n = B^n \oplus A^{n+1} \to B^{n-1} \oplus A^n = C(\alpha)^{n-1}$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s^n \\ -\delta^n \end{pmatrix} \begin{pmatrix} \beta^n & 0 \end{pmatrix} = \begin{pmatrix} d & \alpha^n \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi^{n+1} & 0 \end{pmatrix} \begin{pmatrix} d & \alpha^{n+1} \\ 0 & -d \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha)^\bullet \to A^\bullet[1]$ (see Definition 13.9.1) and $C(\alpha)^\bullet \to C^\bullet \to A^\bullet[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : C^\bullet \to C(\alpha)^\bullet$ and check instead that the two maps $C^\bullet \to A^\bullet[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K^\bullet \to \tilde{L}^\bullet$, $s : L^\bullet \to \tilde{L}^\bullet$ and $\pi : \tilde{L}^\bullet \to L^\bullet$ be as in Lemma 13.9.6. By Lemmas 13.9.2 and 13.9.13 the triangles $(K^\bullet, L^\bullet, C(f), i, p)$ and $(K^\bullet, \tilde{L}^\bullet, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace $L^\bullet$ by $\tilde{L}^\bullet$ and f by $\tilde{f}$. In other words we may assume that f is a termwise split injection. In this case the result follows from part (1). □

014M Lemma 13.9.15. Let $\mathcal{A}$ be an additive category. Let $A_1^\bullet \to A_2^\bullet \to \ldots \to A_n^\bullet$ be a sequence of composable morphisms of complexes. There exists a commutative diagram

$$\begin{tikzcd} A_1^\bullet \arrow[r] & A_2^\bullet \arrow[r] & \ldots \arrow[r] & A_n^\bullet \\ B_1^\bullet \arrow[r] \arrow[u] & B_2^\bullet \arrow[r] \arrow[u] & \ldots \arrow[r] & B_n^\bullet \arrow[u] \end{tikzcd}$$

such that each morphism $B_i^\bullet \to B_{i+1}^\bullet$ is a split injection and each $B_i^\bullet \to A_i^\bullet$ is a homotopy equivalence. Moreover, if all $A_i^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so are the $B_i^\bullet$.

Proof. The case $n = 1$ is without content. Lemma 13.9.6 is the case $n = 2$. Suppose we have constructed the diagram except for $B_n^\bullet$. Apply Lemma 13.9.6 to the composition $B_{n-1}^\bullet \to A_{n-1}^\bullet \to A_n^\bullet$. The result is a factorization $B_{n-1}^\bullet \to B_n^\bullet \to A_n^\bullet$ as desired. □

014N Lemma 13.9.16. Let $\mathcal{A}$ be an additive category. Let $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ be a termwise split sequence of complexes. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the associated triangle. Then the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to the triangle $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$.

Proof. We write $B^n = A^n \oplus C^n$ and we identify α^n and β^n with the natural inclusion and projection maps. By construction of δ we have

$$d_B^n = \begin{pmatrix} d_A^n & \delta^n \\ 0 & d_C^n \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : C^\bullet[-1] \to A^\bullet$ is given as $C(\delta[-1])^n = A^n \oplus C^n$ with differential identical with the matrix above! Whence the lemma. $\square$

014O Lemma 13.9.17. Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. The triangle $(L^\bullet, C(f)^\bullet, K^\bullet[1], i, p, f[1])$ is the triangle associated to the termwise split sequence

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. $\square$

13.10. Distinguished triangles in the homotopy category

014P Since we want our boundary maps in long exact sequences of cohomology to be given by the maps in the snake lemma without signs we define distinguished triangles in the homotopy category as follows.

014Q Definition 13.10.1. Let $\mathcal{A}$ be an additive category. A triangle (X, Y, Z, f, g, h) of $K(\mathcal{A})$ is called a distinguished triangle of $K(\mathcal{A})$ if it is isomorphic to the triangle associated to a termwise split exact sequence of complexes, see Definition 13.9.9. Same definition for $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$.

Note that according to Lemma 13.9.14 a triangle of the form $(K^\bullet, L^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle. This does indeed lead to a triangulated category, see Proposition 13.10.3. Before we can prove the proposition we need one more lemma in order to be able to prove TR4.

014R Lemma 13.10.2. Let $\mathcal{A}$ be an additive category. Suppose that $\alpha : A^\bullet \to B^\bullet$ and $\beta : B^\bullet \to C^\bullet$ are split injections of complexes. Then there exist distinguished triangles $(A^\bullet, B^\bullet, Q_1^\bullet, \alpha, p_1, d_1)$, $(A^\bullet, C^\bullet, Q_2^\bullet, \beta \circ \alpha, p_2, d_2)$ and $(B^\bullet, C^\bullet, Q_3^\bullet, \beta, p_3, d_3)$ for which TR4 holds.

Proof. Say $\pi_1^n : B^n \to A^n$, and $\pi_3^n : C^n \to B^n$ are the splittings. Then also $A^\bullet \to C^\bullet$ is a split injection with splittings $\pi_2^n = \pi_1^n \circ \pi_3^n$. Let us write $Q_1^\bullet$, $Q_2^\bullet$ and $Q_3^\bullet$ for the "quotient" complexes. In other words, $Q_1^n = \text{Ker}(\pi_1^n)$, $Q_3^n = \text{Ker}(\pi_3^n)$ and $Q_2^n = \text{Ker}(\pi_2^n)$. Note that the kernels exist. Then $B^n = A^n \oplus Q_1^n$ and $C_n = B^n \oplus Q_3^n$, where we think of A^n as a subobject of B^n and so on. This implies $C^n = A^n \oplus Q_1^n \oplus Q_3^n$. Note that $\pi_2^n = \pi_1^n \circ \pi_3^n$ is zero on both Q_1^n and Q_3^n. Hence

$Q_2^n = Q_1^n \oplus Q_3^n$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A^\bullet & \to & B^\bullet & \to & Q_1^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^\bullet & \to & C^\bullet & \to & Q_2^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^\bullet & \to & C^\bullet & \to & Q_3^\bullet & \to & 0 \end{array}$$

The rows of this diagram are termwise split exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(A^\bullet \to B^\bullet \to Q_1^\bullet \to A^\bullet[1]) \longrightarrow (A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1])$$

and

$$(A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1]) \longrightarrow (B^\bullet \to C^\bullet \to Q_3^\bullet \to B^\bullet[1]).$$

Note that the splittings $Q_3^n \to C^n$ of the bottom split sequence in the diagram provides a splitting for the split sequence $0 \to Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to 0$ upon composing with $C^n \to Q_2^n$. It follows easily from this that the morphism $\delta : Q_3^\bullet \to Q_1^\bullet[1]$ in the corresponding distinguished triangle

$$(Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to Q_1^\bullet[1])$$

is equal to the composition $Q_3^\bullet \to B^\bullet[1] \to Q_1^\bullet[1]$. Hence we get a structure as in the conclusion of axiom TR4. □

014S Proposition 13.10.3. Let $\mathcal{A}$ be an additive category. The category $K(\mathcal{A})$ of complexes up to homotopy with its natural translation functors and distinguished triangles as defined above is a triangulated category.

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(A^\bullet, A^\bullet, 0, 1, 0, 0)$ is distinguished since $0 \to A^\bullet \to A^\bullet \to 0 \to 0$ is a termwise split sequence of complexes. Finally, given any morphism of complexes $f : K^\bullet \to L^\bullet$ the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 13.9.14.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists a termwise split sequence of complexes $A^\bullet \to B^\bullet \to C^\bullet$ such that the associated triangle $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, B^\bullet, -\delta[-1], \alpha, \beta)$. It follows from Lemma 13.9.16 that the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, -p)$. Hence it is distinguished by Lemma 13.9.14. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 13.9.14 this means that it is isomorphic to a triangle of the form $(K^\bullet, L^\bullet, C(f), f, i, -p)$ for some morphism of complexes f. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L^\bullet, C(f), K^\bullet[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L^\bullet, C(f), K^\bullet[1], i, p, f[1])$. By Lemma 13.9.17 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ$

f. By Lemma 13.9.14 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 13.9.2 to the commutative diagram given by f, f', a, b.

Proof of TR4. At this point we know that $K(\mathcal{A})$ is a pre-triangulated category. Hence we can use Lemma 13.4.15. Let $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ be composable morphisms of $K(\mathcal{A})$. By Lemma 13.9.15 we may assume that $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ are split injective morphisms. In this case the result follows from Lemma 13.10.2. □

05RP Remark 13.10.4. Let $\mathcal{A}$ be an additive category. Exactly the same proof as the proof of Proposition 13.10.3 shows that the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. Namely, the cone of a morphism between bounded (above, below) is bounded (above, below). But we prove below that these are triangulated subcategories of $K(\mathcal{A})$ which gives another proof.

05RQ Lemma 13.10.5. Let $\mathcal{A}$ be an additive category. The categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are full triangulated subcategories of $K(\mathcal{A})$.

Proof. Each of the categories mentioned is a full additive subcategory. We use the criterion of Lemma 13.4.16 to show that they are triangulated subcategories. It is clear that each of the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ is preserved under the shift functors $[1], [-1]$. Finally, suppose that $f : A^\bullet \to B^\bullet$ is a morphism in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$. Then $(A^\bullet, B^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle of $K(\mathcal{A})$ with $C(f)^\bullet \in K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$ as is clear from the construction of the cone. Thus the lemma is proved. (Alternatively, $K^\bullet \to L^\bullet$ is isomorphic to an termwise split injection of complexes in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, see Lemma 13.9.6 and then one can directly take the associated distinguished triangle.) □

014X Lemma 13.10.6. Let $\mathcal{A}$, $\mathcal{B}$ be additive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. The induced functors

$$\begin{aligned} F : K(\mathcal{A}) &\longrightarrow K(\mathcal{B}) \\ F : K^+(\mathcal{A}) &\longrightarrow K^+(\mathcal{B}) \\ F : K^-(\mathcal{A}) &\longrightarrow K^-(\mathcal{B}) \\ F : K^b(\mathcal{A}) &\longrightarrow K^b(\mathcal{B}) \end{aligned}$$

are exact functors of triangulated categories.

Proof. Suppose $A^\bullet \to B^\bullet \to C^\bullet$ is a termwise split sequence of complexes of $\mathcal{A}$ with splittings (s^n, π^n) and associated morphism $\delta : C^\bullet \to A^\bullet[1]$, see Definition 13.9.9. Then $F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet)$ is a termwise split sequence of complexes with splittings $(F(s^n), F(\pi^n))$ and associated morphism $F(\delta) : F(C^\bullet) \to F(A^\bullet)[1]$. Thus F transforms distinguished triangles into distinguished triangles. □

0G6C Lemma 13.10.7. Let $\mathcal{A}$ be an additive category. Let $(A^\bullet, B^\bullet, C^\bullet, a, b, c)$ be a distinguished triangle in $K(\mathcal{A})$. Then there exists an isomorphic distinguished triangle $(A^\bullet, (B')^\bullet, C^\bullet, a', b', c)$ such that $0 \to A^n \to (B')^n \to C^n \to 0$ is a split short exact sequence for all n.

Proof. We will use that $K(\mathcal{A})$ is a triangulated category by Proposition 13.10.3. Let $W^\bullet$ be the cone on $c : C^\bullet \to A^\bullet[1]$ with its maps $i : A^\bullet[1] \to W^\bullet$ and $p : W^\bullet \to C^\bullet[1]$. Then $(C^\bullet, A^\bullet[1], W^\bullet, c, i, -p)$ is a distinguished triangle by Lemma 13.9.14. Rotating backwards twice we see that $(A^\bullet, W^\bullet[-1], C^\bullet, -i[-1], p[-1], c)$

is a distinguished triangle. By TR3 there is a morphism of distinguished triangles $(\mathrm{id}, \beta, \mathrm{id}) : (A^\bullet, B^\bullet, C^\bullet, a, b, c) \to (A^\bullet, W^\bullet[-1], C^\bullet, -i[-1], p[-1], c)$ which must be an isomorphism by Lemma 13.4.3. This finishes the proof because $0 \to A^\bullet \to W^\bullet[-1] \to C^\bullet \to 0$ is a termwise split short exact sequence of complexes by the very construction of cones in Section 13.9. $\square$

0G6D Remark 13.10.8. Let $\mathcal{A}$ be an additive category with countable direct sums. Let $\text{DoubleComp}(\mathcal{A})$ denote the category of double complexes in $\mathcal{A}$, see Homology, Section 12.18. We can use this category to construct two triangulated categories.

(1) We can consider an object $A^{\bullet,\bullet}$ of $\text{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows

$$\ldots \to A^{\bullet,-1} \to A^{\bullet,0} \to A^{\bullet,1} \to \ldots$$

and take the homotopy category $K_{first}(\text{DoubleComp}(\mathcal{A}))$ with the corresponding triangulated structure given by Proposition 13.10.3. By Homology, Remark 12.18.6 the functor

$$\text{Tot} : K_{first}(\text{DoubleComp}(\mathcal{A})) \longrightarrow K(\mathcal{A})$$

is an exact functor of triangulated categories.

(2) We can consider an object $A^{\bullet,\bullet}$ of $\text{DoubleComp}(\mathcal{A})$ as a complex of complexes as follows

$$\ldots \to A^{-1,\bullet} \to A^{0,\bullet} \to A^{1,\bullet} \to \ldots$$

and take the homotopy category $K_{second}(\text{DoubleComp}(\mathcal{A}))$ with the corresponding triangulated structure given by Proposition 13.10.3. By Homology, Remark 12.18.7 the functor

$$\text{Tot} : K_{second}(\text{DoubleComp}(\mathcal{A})) \longrightarrow K(\mathcal{A})$$

is an exact functor of triangulated categories.

0G6E Remark 13.10.9. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be additive categories and assume $\mathcal{C}$ has countable direct sums. Suppose that

$$\otimes : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}, \quad (X, Y) \longmapsto X \otimes Y$$

is a functor which is bilinear on morphisms. This determines a functor

$$\text{Comp}(\mathcal{A}) \times \text{Comp}(\mathcal{B}) \longrightarrow \text{DoubleComp}(\mathcal{C}), \quad (X^\bullet, Y^\bullet) \longmapsto X^\bullet \otimes Y^\bullet$$

See Homology, Example 12.18.2.

(1) For a fixed object $X^\bullet$ of $\text{Comp}(\mathcal{A})$ the functor

$$K(\mathcal{B}) \longrightarrow K(\mathcal{C}), \quad Y^\bullet \longmapsto \text{Tot}(X^\bullet \otimes Y^\bullet)$$

is an exact functor of triangulated categories.

(2) For a fixed object $Y^\bullet$ of $\text{Comp}(\mathcal{B})$ the functor

$$K(\mathcal{A}) \longrightarrow K(\mathcal{C}), \quad X^\bullet \longmapsto \text{Tot}(X^\bullet \otimes Y^\bullet)$$

is an exact functor of triangulated categories.

This follows from Remark 13.10.8 since the functors $\text{Comp}(\mathcal{A}) \to \text{DoubleComp}(\mathcal{C})$, $Y^\bullet \mapsto X^\bullet \otimes Y^\bullet$ and $\text{Comp}(\mathcal{B}) \to \text{DoubleComp}(\mathcal{C})$, $X^\bullet \mapsto X^\bullet \otimes Y^\bullet$ are immediately seen to be compatible with homotopies and termwise split short exact sequences and hence induce exact functors of triangulated categories

$$K(\mathcal{B}) \to K_{first}(\text{DoubleComp}(\mathcal{C})) \quad \text{and} \quad K(\mathcal{A}) \to K_{second}(\text{DoubleComp}(\mathcal{C}))$$

Observe that for the first of the two the isomorphism

$$\text{Tot}(X^\bullet \otimes Y^\bullet[1]) \cong \text{Tot}(X^\bullet \otimes Y^\bullet)[1]$$

involves signs (this goes back to the signs chosen in Homology, Remark 12.18.5).

13.11. Derived categories

05RR In this section we construct the derived category of an abelian category $\mathcal{A}$ by inverting the quasi-isomorphisms in $K(\mathcal{A})$. Before we do this recall that the functors $H^i : \text{Comp}(\mathcal{A}) \to \mathcal{A}$ factor through $K(\mathcal{A})$, see Homology, Lemma 12.13.11. Moreover, in Homology, Definition 12.14.8 we have defined identifications $H^i(K^\bullet[n]) = H^{i+n}(K^\bullet)$. At this point it makes sense to redefine

$$H^i(K^\bullet) = H^0(K^\bullet[i])$$

in order to avoid confusion and possible sign errors.

05RS **Lemma 13.11.1.** Let $\mathcal{A}$ be an abelian category. The functor

$$H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$$

is homological.

Proof. Because H^0 is a functor, and by our definition of distinguished triangles it suffices to prove that given a termwise split short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ the sequence $H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet)$ is exact. This follows from Homology, Lemma 12.13.12. $\square$

In particular, this lemma implies that a distinguished triangle (X, Y, Z, f, g, h) in $K(\mathcal{A})$ gives rise to a long exact cohomology sequence
(13.11.1.1)

05ST $$\ldots \longrightarrow H^i(X) \xrightarrow{H^i(f)} H^i(Y) \xrightarrow{H^i(g)} H^i(Z) \xrightarrow{H^i(h)} H^{i+1}(X) \longrightarrow \ldots$$

see (13.3.5.1). Moreover, there is a compatibility with the long exact sequence of cohomology associated to a short exact sequence of complexes (insert future reference here). For example, if $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is the distinguished triangle associated to a termwise split exact sequence of complexes (see Definition 13.9.9), then the cohomology sequence above agrees with the one defined using the snake lemma, see Homology, Lemma 12.13.12 and for agreement of sequences, see Homology, Lemma 12.14.11.

Recall that a complex $K^\bullet$ is acyclic if $H^i(K^\bullet) = 0$ for all $i \in \mathbf{Z}$. Moreover, recall that a morphism of complexes $f : K^\bullet \to L^\bullet$ is a quasi-isomorphism if and only if $H^i(f)$ is an isomorphism for all i. See Homology, Definition 12.13.10.

05RT **Lemma 13.11.2.** Let $\mathcal{A}$ be an abelian category. The full subcategory $\text{Ac}(\mathcal{A})$ of $K(\mathcal{A})$ consisting of acyclic complexes is a strictly full saturated triangulated subcategory of $K(\mathcal{A})$. The corresponding saturated multiplicative system (see Lemma 13.6.10) of $K(\mathcal{A})$ is the set $\text{Qis}(\mathcal{A})$ of quasi-isomorphisms. In particular, the kernel of the localization functor $Q : K(\mathcal{A}) \to \text{Qis}(\mathcal{A})^{-1}K(\mathcal{A})$ is $\text{Ac}(\mathcal{A})$ and the functor H^0 factors through Q.

Proof. We know that H^0 is a homological functor by Lemma 13.11.1. Thus this lemma is a special case of Lemma 13.6.11. $\square$

05RU Definition 13.11.3. Let $\mathcal{A}$ be an abelian category. Let $\text{Ac}(\mathcal{A})$ and $\text{Qis}(\mathcal{A})$ be as in Lemma 13.11.2. The derived category of $\mathcal{A}$ is the triangulated category

$$D(\mathcal{A}) = K(\mathcal{A})/\text{Ac}(\mathcal{A}) = \text{Qis}(\mathcal{A})^{-1}K(\mathcal{A}).$$

We denote $H^0 : D(\mathcal{A}) \to \mathcal{A}$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above. Using Lemma 13.6.4 we introduce the strictly full saturated triangulated subcategories $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ whose sets of objects are

$$\begin{aligned}
\text{Ob}(D^+(\mathcal{A})) &= \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \ll 0\} \\
\text{Ob}(D^-(\mathcal{A})) &= \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \gg 0\} \\
\text{Ob}(D^b(\mathcal{A})) &= \{X \in \text{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } |n| \gg 0\}
\end{aligned}$$

The category $D^b(\mathcal{A})$ is called the bounded derived category of $\mathcal{A}$.

If $K^\bullet$ and $L^\bullet$ are complexes of $\mathcal{A}$ then we sometimes say "$K^\bullet$ is quasi-isomorphic to $L^\bullet$" to indicate that $K^\bullet$ and $L^\bullet$ are isomorphic objects of $D(\mathcal{A})$.

09PA Remark 13.11.4. In this chapter, we consistently work with "small" abelian categories (as is the convention in the Stacks project). For a "big" abelian category $\mathcal{A}$, it isn't clear that the derived category $D(\mathcal{A})$ exists, because it isn't clear that morphisms in the derived category are sets. In fact, in general they aren't, see Examples, Lemma 110.61.1. However, if $\mathcal{A}$ is a Grothendieck abelian category, and given $K^\bullet, L^\bullet$ in $K(\mathcal{A})$, then by Injectives, Theorem 19.12.6 there exists a quasi-isomorphism $L^\bullet \to I^\bullet$ to a K-injective complex $I^\bullet$ and Lemma 13.31.2 shows that

$$\text{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

Each of the variants $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ can be constructed as a localization of the corresponding homotopy category. This relies on the following simple lemma.

05RV Lemma 13.11.5. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex.

(1) If $H^n(K^\bullet) = 0$ for all $n \ll 0$, then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.
(2) If $H^n(K^\bullet) = 0$ for all $n \gg 0$, then there exists a quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^\bullet$ bounded above.
(3) If $H^n(K^\bullet) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes

$$\begin{tikzcd}
K^\bullet \arrow[r] & L^\bullet \\
M^\bullet \arrow[u] \arrow[r] & N^\bullet \arrow[u]
\end{tikzcd}$$

where all the arrows are quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Pick $a \ll 0 \ll b$ and set $M^\bullet = \tau_{\leq b}K^\bullet$, $L^\bullet = \tau_{\geq a}K^\bullet$, and $N^\bullet = \tau_{\leq b}L^\bullet = \tau_{\geq a}M^\bullet$. See Homology, Section 12.15 for the truncation functors. $\square$

To state the following lemma denote $\mathrm{Ac}^+(\mathcal{A})$, $\mathrm{Ac}^-(\mathcal{A})$, resp. $\mathrm{Ac}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\mathrm{Ac}(\mathcal{A})$. Denote $\mathrm{Qis}^+(\mathcal{A})$, $\mathrm{Qis}^-(\mathcal{A})$, resp. $\mathrm{Qis}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\mathrm{Qis}(\mathcal{A})$.

05RW Lemma 13.11.6. Let $\mathcal{A}$ be an abelian category. The subcategories $\mathrm{Ac}^+(\mathcal{A})$, $\mathrm{Ac}^-(\mathcal{A})$, resp. $\mathrm{Ac}^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 13.6.10) are the sets $\mathrm{Qis}^+(\mathcal{A})$, $\mathrm{Qis}^-(\mathcal{A})$, resp. $\mathrm{Qis}^b(\mathcal{A})$.

(1) The kernel of the functor $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is $\mathrm{Ac}^+(\mathcal{A})$ and this induces an equivalence of triangulated categories
$$K^+(\mathcal{A})/\mathrm{Ac}^+(\mathcal{A}) = \mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$$
(2) The kernel of the functor $K^-(\mathcal{A}) \to D^-(\mathcal{A})$ is $\mathrm{Ac}^-(\mathcal{A})$ and this induces an equivalence of triangulated categories
$$K^-(\mathcal{A})/\mathrm{Ac}^-(\mathcal{A}) = \mathrm{Qis}^-(\mathcal{A})^{-1}K^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A})$$
(3) The kernel of the functor $K^b(\mathcal{A}) \to D^b(\mathcal{A})$ is $\mathrm{Ac}^b(\mathcal{A})$ and this induces an equivalence of triangulated categories
$$K^b(\mathcal{A})/\mathrm{Ac}^b(\mathcal{A}) = \mathrm{Qis}^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$$

Proof. The initial statements follow from Lemma 13.6.11 by considering the restriction of the homological functor H^0. The statement on kernels in (1), (2), (3) is a consequence of the definitions in each case. Each of the functors is essentially surjective by Lemma 13.11.5. To finish the proof we have to show the functors are fully faithful. We first do this for the bounded below version.

Suppose that $K^\bullet, L^\bullet$ are bounded above complexes. A morphism between these in $D(\mathcal{A})$ is of the form $s^{-1}f$ for a pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ where s is a quasi-isomorphism. This implies that $(L')^\bullet$ has cohomology bounded below. Hence by Lemma 13.11.5 we can choose a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ with $(L'')^\bullet$ bounded below. Then the pair $(s' \circ f, s' \circ s)$ defines a morphism in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. Hence the functor is "full". Finally, suppose that the pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ defines a morphism in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$ which is zero in $D(\mathcal{A})$. This means that there exists a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ such that $s' \circ f = 0$. Using Lemma 13.11.5 once more we obtain a quasi-isomorphism $s'' : (L'')^\bullet \to (L''')^\bullet$ with $(L''')^\bullet$ bounded below. Thus we see that $s'' \circ s' \circ f = 0$ which implies that $s^{-1}f$ is zero in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. This finishes the proof that the functor in (1) is an equivalence.

The proof of (2) is dual to the proof of (1). To prove (3) we may use the result of (2). Hence it suffices to prove that the functor $\mathrm{Qis}^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \to \mathrm{Qis}^-(\mathcal{A})^{-1}K^-(\mathcal{A})$ is fully faithful. The argument given in the previous paragraph applies directly to show this where we consistently work with complexes which are already bounded above. □

13.12. The canonical delta-functor

014Z The derived category should be the receptacle for the universal cohomology functor. In order to state the result we use the notion of a δ-functor from an abelian category into a triangulated category, see Definition 13.3.6.

Consider the functor $\text{Comp}(\mathcal{A}) \to K(\mathcal{A})$. This functor is not a δ-functor in general. The easiest way to see this is to consider a nonsplit short exact sequence $0 \to A \to B \to C \to 0$ of objects of $\mathcal{A}$. Since $\text{Hom}_{K(\mathcal{A})}(C[0], A[1]) = 0$ we see that any distinguished triangle arising from this short exact sequence would look like $(A[0], B[0], C[0], a, b, 0)$. But the existence of such a distinguished triangle in $K(\mathcal{A})$ implies that the extension is split. A contradiction.

It turns out that the functor $\text{Comp}(\mathcal{A}) \to D(\mathcal{A})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet \to 0$$

of complexes in the abelian category $\mathcal{A}$. Consider the cone $C(a)^\bullet$ of the morphism a. We have $C(a)^n = B^n \oplus A^{n+1}$ and we define $q^n : C(a)^n \to C^n$ via the projection to B^n followed by b^n. Hence a morphism of complexes

$$q : C(a)^\bullet \longrightarrow C^\bullet.$$

It is clear that $q \circ i = b$ where i is as in Definition 13.9.1. Note that, as $a^\bullet$ is injective in each degree, the kernel of q is identified with the cone of $\text{id}_{A^\bullet}$ which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 13.9.14 the triangle

$$(A, B, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$. As the localization functor $K(\mathcal{A}) \to D(\mathcal{A})$ is exact we see that $(A, B, C(a), a, i, -p)$ is a distinguished triangle in $D(\mathcal{A})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(\mathcal{A})$. Hence we deduce that

$$(A, B, C, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(\mathcal{A})$. This suggests the following lemma.

0152 Lemma 13.12.1. Let $\mathcal{A}$ be an abelian category. The functor $\text{Comp}(\mathcal{A}) \to D(\mathcal{A})$ defined has the natural structure of a δ-functor, with

$$\delta_{A^\bullet \to B^\bullet \to C^\bullet} = -p \circ q^{-1}$$

with p and q as explained above. The same construction turns the functors $\text{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$, $\text{Comp}^-(\mathcal{A}) \to D^-(\mathcal{A})$, and $\text{Comp}^b(\mathcal{A}) \to D^b(\mathcal{A})$ into δ-functors.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show that given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r, "a"'] \arrow[d, "f"'] & B^\bullet \arrow[r, "b"'] \arrow[d, "g"'] & C^\bullet \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & (A')^\bullet \arrow[r, "a'"] & (B')^\bullet \arrow[r, "b'"] & (C')^\bullet \arrow[r] & 0
\end{tikzcd}$$

we get the desired commutative diagram of Definition 13.3.6 (2). By Lemma 13.9.2 the pair (f, g) induces a canonical morphism $c : C(a)^\bullet \to C(a')^\bullet$. It is a simple computation to show that $q' \circ c = h \circ q$ and $f[1] \circ p = p' \circ c$. From this the result follows directly. $\square$

0153 Lemma 13.12.2. Let $\mathcal{A}$ be an abelian category. Let

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D^\bullet \arrow[r] & E^\bullet \arrow[r] & F^\bullet \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of morphisms of complexes such that the rows are short exact sequences of complexes, and the vertical arrows are quasi-isomorphisms. The δ-functor of Lemma 13.12.1 above maps the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ and $0 \to D^\bullet \to E^\bullet \to F^\bullet \to 0$ to isomorphic distinguished triangles.

Proof. Trivial from the fact that $K(\mathcal{A}) \to D(\mathcal{A})$ transforms quasi-isomorphisms into isomorphisms and that the associated distinguished triangles are functorial. $\square$

0154 Lemma 13.12.3. Let $\mathcal{A}$ be an abelian category. Let

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a short exact sequences of complexes. Assume this short exact sequence is termwise split. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the distinguished triangle of $K(\mathcal{A})$ associated to the sequence. The δ-functor of Lemma 13.12.1 above maps the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ to a triangle isomorphic to the distinguished triangle

$$(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta).$$

Proof. Follows from Lemma 13.9.14. $\square$

08J5 Remark 13.12.4. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$. Let $a \in \mathbf{Z}$. We claim there is a canonical distinguished triangle

$$\tau_{\leq a}K^\bullet \to K^\bullet \to \tau_{\geq a+1}K^\bullet \to (\tau_{\leq a}K^\bullet)[1]$$

in $D(\mathcal{A})$. Here we have used the canonical truncation functors τ from Homology, Section 12.15. Namely, we first take the distinguished triangle associated by our δ-functor (Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to \tau_{\leq a}K^\bullet \to K^\bullet \to K^\bullet/\tau_{\leq a}K^\bullet \to 0$$

Next, we use that the map $K^\bullet \to \tau_{\geq a+1}K^\bullet$ factors through a quasi-isomorphism $K^\bullet/\tau_{\leq a}K^\bullet \to \tau_{\geq a+1}K^\bullet$ by the description of cohomology groups in Homology, Section 12.15. In a similar way we obtain canonical distinguished triangles

$$\tau_{\leq a}K^\bullet \to \tau_{\leq a+1}K^\bullet \to H^{a+1}(K^\bullet)[-a-1] \to (\tau_{\leq a}K^\bullet)[1]$$

and

$$H^a(K^\bullet)[-a] \to \tau_{\geq a}K^\bullet \to \tau_{\geq a+1}K^\bullet \to H^a(K^\bullet)[-a+1]$$

08Q2 Lemma 13.12.5. Let $\mathcal{A}$ be an abelian category. Let

$$K_0^\bullet \to K_1^\bullet \to \ldots \to K_n^\bullet$$

be maps of complexes such that

(1) $H^i(K_0^\bullet) = 0$ for $i > 0$,
(2) $H^{-j}(K_j^\bullet) \to H^{-j}(K_{j+1}^\bullet)$ is zero.

Then the composition $K_0^\bullet \to K_n^\bullet$ factors through $\tau_{\leq -n}K_n^\bullet \to K_n^\bullet$ in $D(\mathcal{A})$. Dually, given maps of complexes

$$K_n^\bullet \to K_{n-1}^\bullet \to \ldots \to K_0^\bullet$$

such that

(1) $H^i(K_0^\bullet) = 0$ for $i < 0$,
(2) $H^j(K_{j+1}^\bullet) \to H^j(K_j^\bullet)$ is zero,

then the composition $K_n^\bullet \to K_0^\bullet$ factors through $K_n^\bullet \to \tau_{\geq n}K_n^\bullet$ in $D(\mathcal{A})$.

Proof. The case $n = 1$. Since $\tau_{\leq 0}K_0^\bullet = K_0^\bullet$ in $D(\mathcal{A})$ we can replace $K_0^\bullet$ by $\tau_{\leq 0}K_0^\bullet$ and $K_1^\bullet$ by $\tau_{\leq 0}K_1^\bullet$. Consider the distinguished triangle

$$\tau_{\leq -1}K_1^\bullet \to K_1^\bullet \to H^0(K_1^\bullet)[0] \to (\tau_{\leq -1}K_1^\bullet)[1]$$

(Remark 13.12.4). The composition $K_0^\bullet \to K_1^\bullet \to H^0(K_1^\bullet)[0]$ is zero as it is equal to $K_0^\bullet \to H^0(K_0^\bullet)[0] \to H^0(K_1^\bullet)[0]$ which is zero by assumption. The fact that $\mathrm{Hom}_{D(\mathcal{A})}(K_0^\bullet, -)$ is a homological functor (Lemma 13.4.2), allows us to find the desired factorization. For $n = 2$ we get a factorization $K_0^\bullet \to \tau_{\leq -1}K_1^\bullet$ by the case $n = 1$ and we can apply the case $n = 1$ to the map of complexes $\tau_{\leq -1}K_1^\bullet \to \tau_{\leq -1}K_2^\bullet$ to get a factorization $\tau_{\leq -1}K_1^\bullet \to \tau_{\leq -2}K_2^\bullet$. The general case is proved in exactly the same manner. □

13.13. Filtered derived categories

05RX A reference for this section is [Ill72, I, Chapter V]. Let $\mathcal{A}$ be an abelian category. In this section we will define the filtered derived category $DF(\mathcal{A})$ of $\mathcal{A}$. In short, we will define it as the derived category of the exact category of objects of $\mathcal{A}$ endowed with a finite filtration. (Thus our construction is a special case of a more general construction of the derived category of an exact category, see for example [Büh10], [Kel90].) Illusie's filtered derived category is the full subcategory of ours consisting of those objects whose filtration is finite. (In our category the filtration is still finite in each degree, but may not be uniformly bounded.) The rationale for our choice is that it is not harder and it allows us to apply the discussion to the spectral sequences of Lemma 13.21.3, see also Remark 13.21.4.

We will use the notation regarding filtered objects introduced in Homology, Section 12.19. The category of filtered objects of $\mathcal{A}$ is denoted $\mathrm{Fil}(\mathcal{A})$. All filtrations will be decreasing by fiat.

05RY Definition 13.13.1. Let $\mathcal{A}$ be an abelian category. The category of finite filtered objects of $\mathcal{A}$ is the category of filtered objects (A, F) of $\mathcal{A}$ whose filtration F is finite. We denote it $\mathrm{Fil}^f(\mathcal{A})$.

Thus $\mathrm{Fil}^f(\mathcal{A})$ is a full subcategory of $\mathrm{Fil}(\mathcal{A})$. For each $p \in \mathbf{Z}$ there is a functor $\mathrm{gr}^p : \mathrm{Fil}^f(\mathcal{A}) \to \mathcal{A}$. There is a functor

$$\mathrm{gr} = \bigoplus\nolimits_{p \in \mathbf{Z}} \mathrm{gr}^p : \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A})$$

where $\mathrm{Gr}(\mathcal{A})$ is the category of graded objects of $\mathcal{A}$, see Homology, Definition 12.16.1. Finally, there is a functor

$$(\text{forget } F) : \mathrm{Fil}^f(\mathcal{A}) \longrightarrow \mathcal{A}$$

which associates to the filtered object (A, F) the underlying object of $\mathcal{A}$. The category $\text{Fil}^f(\mathcal{A})$ is an additive category, but not abelian in general, see Homology, Example 12.3.13.

Because the functors gr^p, gr, (forget F) are additive they induce exact functors of triangulated categories

$$\text{gr}^p, (\text{forget } F) : K(\text{Fil}^f(\mathcal{A})) \to K(\mathcal{A}) \quad \text{and} \quad \text{gr} : K(\text{Fil}^f(\mathcal{A})) \to K(\text{Gr}(\mathcal{A}))$$

by Lemma 13.10.6. By analogy with the case of the homotopy category of an abelian category we make the following definitions.

05RZ Definition 13.13.2. Let $\mathcal{A}$ be an abelian category.

(1) Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of $K(\text{Fil}^f(\mathcal{A}))$. We say that α is a filtered quasi-isomorphism if the morphism $\text{gr}(\alpha)$ is a quasi-isomorphism.
(2) Let $K^\bullet$ be an object of $K(\text{Fil}^f(\mathcal{A}))$. We say that $K^\bullet$ is filtered acyclic if the complex $\text{gr}(K^\bullet)$ is acyclic.

Note that $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism if and only if each $\text{gr}^p(\alpha)$ is a quasi-isomorphism. Similarly a complex $K^\bullet$ is filtered acyclic if and only if each $\text{gr}^p(K^\bullet)$ is acyclic.

05S0 Lemma 13.13.3. Let $\mathcal{A}$ be an abelian category.

(1) The functor $K(\text{Fil}^f(\mathcal{A})) \longrightarrow \text{Gr}(\mathcal{A})$, $K^\bullet \longmapsto H^0(\text{gr}(K^\bullet))$ is homological.
(2) The functor $K(\text{Fil}^f(\mathcal{A})) \to \mathcal{A}$, $K^\bullet \longmapsto H^0(\text{gr}^p(K^\bullet))$ is homological.
(3) The functor $K(\text{Fil}^f(\mathcal{A})) \longrightarrow \mathcal{A}$, $K^\bullet \longmapsto H^0((\text{forget } F)K^\bullet)$ is homological.

Proof. This follows from the fact that $H^0 : K(\mathcal{A}) \to \mathcal{A}$ is homological, see Lemma 13.11.1 and the fact that the functors $\text{gr}, \text{gr}^p, (\text{forget } F)$ are exact functors of triangulated categories. See Lemma 13.4.20. $\square$

05S1 Lemma 13.13.4. Let $\mathcal{A}$ be an abelian category. The full subcategory $\text{FAc}(\mathcal{A})$ of $K(\text{Fil}^f(\mathcal{A}))$ consisting of filtered acyclic complexes is a strictly full saturated triangulated subcategory of $K(\text{Fil}^f(\mathcal{A}))$. The corresponding saturated multiplicative system (see Lemma 13.6.10) of $K(\text{Fil}^f(\mathcal{A}))$ is the set $\text{FQis}(\mathcal{A})$ of filtered quasi-isomorphisms. In particular, the kernel of the localization functor

$$Q : K(\text{Fil}^f(\mathcal{A})) \longrightarrow \text{FQis}(\mathcal{A})^{-1}K(\text{Fil}^f(\mathcal{A}))$$

is $\text{FAc}(\mathcal{A})$ and the functor $H^0 \circ \text{gr}$ factors through Q.

Proof. We know that $H^0 \circ \text{gr}$ is a homological functor by Lemma 13.13.3. Thus this lemma is a special case of Lemma 13.6.11. $\square$

05S2 Definition 13.13.5. Let $\mathcal{A}$ be an abelian category. Let $\text{FAc}(\mathcal{A})$ and $\text{FQis}(\mathcal{A})$ be as in Lemma 13.13.4. The filtered derived category of $\mathcal{A}$ is the triangulated category

$$DF(\mathcal{A}) = K(\text{Fil}^f(\mathcal{A}))/\text{FAc}(\mathcal{A}) = \text{FQis}(\mathcal{A})^{-1}K(\text{Fil}^f(\mathcal{A})).$$

05S3 Lemma 13.13.6. The functors $\text{gr}^p, \text{gr}, (\text{forget } F)$ induce canonical exact functors

$$\text{gr}^p, \text{gr}, (\text{forget } F) : DF(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

which commute with the localization functors.

Proof. This follows from the universal property of localization, see Lemma 13.5.7, provided we can show that a filtered quasi-isomorphism is turned into a quasi-isomorphism by each of the functors $\text{gr}^p, \text{gr}, (\text{forget } F)$. This is true by definition for the first two. For the last one the statement we have to do a little bit of work. Let $f : K^\bullet \to L^\bullet$ be a filtered quasi-isomorphism in $K(\text{Fil}^f(\mathcal{A}))$. Choose a distinguished triangle $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ which contains f. Then $M^\bullet$ is filtered acyclic, see Lemma 13.13.4. Hence by the corresponding lemma for $K(\mathcal{A})$ it suffices to show that a filtered acyclic complex is an acyclic complex if we forget the filtration. This follows from Homology, Lemma 12.19.15. □

05S4 Definition 13.13.7. Let $\mathcal{A}$ be an abelian category. The bounded filtered derived category $DF^b(\mathcal{A})$ is the full subcategory of $DF(\mathcal{A})$ with objects those X such that $\text{gr}(X) \in D^b(\mathcal{A})$. Similarly for the bounded below filtered derived category $DF^+(\mathcal{A})$ and the bounded above filtered derived category $DF^-(\mathcal{A})$.

05S5 Lemma 13.13.8. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet \in K(\text{Fil}^f(\mathcal{A}))$.

(1) If $H^n(\text{gr}(K^\bullet)) = 0$ for all $n < a$, then there exists a filtered quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^n = 0$ for all $n < a$.
(2) If $H^n(\text{gr}(K^\bullet)) = 0$ for all $n > b$, then there exists a filtered quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^n = 0$ for all $n > b$.
(3) If $H^n(\text{gr}(K^\bullet)) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes

$$\begin{tikzcd} K^\bullet \arrow[r] & L^\bullet \\ M^\bullet \arrow[u] \arrow[r] & N^\bullet \arrow[u] \end{tikzcd}$$

where all the arrows are filtered quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Suppose that $H^n(\text{gr}(K^\bullet)) = 0$ for all $n < a$. By Homology, Lemma 12.19.15 the sequence

$$K^{a-1} \xrightarrow{d^{a-2}} K^{a-1} \xrightarrow{d^{a-1}} K^a$$

is an exact sequence of objects of $\mathcal{A}$ and the morphisms d^{a-2} and d^{a-1} are strict. Hence $\text{Coim}(d^{a-1}) = \text{Im}(d^{a-1})$ in $\text{Fil}^f(\mathcal{A})$ and the map $\text{gr}(\text{Im}(d^{a-1})) \to \text{gr}(K^a)$ is injective with image equal to the image of $\text{gr}(K^{a-1}) \to \text{gr}(K^a)$, see Homology, Lemma 12.19.13. This means that the map $K^\bullet \to \tau_{\geq a}K^\bullet$ into the truncation

$$\tau_{\geq a}K^\bullet = (\ldots \to 0 \to K^a/\text{Im}(d^{a-1}) \to K^{a+1} \to \ldots)$$

is a filtered quasi-isomorphism. This proves (1). The proof of (2) is dual to the proof of (1). Part (3) follows formally from (1) and (2). □

To state the following lemma denote $\text{FAc}^+(\mathcal{A})$, $\text{FAc}^-(\mathcal{A})$, resp. $\text{FAc}^b(\mathcal{A})$ the intersection of $K^+(\text{Fil}^f\mathcal{A})$, $K^-(\text{Fil}^f\mathcal{A})$, resp. $K^b(\text{Fil}^f\mathcal{A})$ with $\text{FAc}(\mathcal{A})$. Denote $\text{FQis}^+(\mathcal{A})$, $\text{FQis}^-(\mathcal{A})$, resp. $\text{FQis}^b(\mathcal{A})$ the intersection of $K^+(\text{Fil}^f\mathcal{A})$, $K^-(\text{Fil}^f\mathcal{A})$, resp. $K^b(\text{Fil}^f\mathcal{A})$ with $\text{FQis}(\mathcal{A})$.

05S6 Lemma 13.13.9. Let $\mathcal{A}$ be an abelian category. The subcategories $\text{FAc}^+(\mathcal{A})$, $\text{FAc}^-(\mathcal{A})$, resp. $\text{FAc}^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of

$K^+(\text{Fil}^f\mathcal{A})$, $K^-(\text{Fil}^f\mathcal{A})$, resp. $K^b(\text{Fil}^f\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 13.6.10) are the sets $\text{FQis}^+(\mathcal{A})$, $\text{FQis}^-(\mathcal{A})$, resp. $\text{FQis}^b(\mathcal{A})$.

(1) The kernel of the functor $K^+(\text{Fil}^f\mathcal{A}) \to DF^+(\mathcal{A})$ is $\text{FAc}^+(\mathcal{A})$ and this induces an equivalence of triangulated categories

$$K^+(\text{Fil}^f\mathcal{A})/\text{FAc}^+(\mathcal{A}) = \text{FQis}^+(\mathcal{A})^{-1}K^+(\text{Fil}^f\mathcal{A}) \longrightarrow DF^+(\mathcal{A})$$

(2) The kernel of the functor $K^-(\text{Fil}^f\mathcal{A}) \to DF^-(\mathcal{A})$ is $\text{FAc}^-(\mathcal{A})$ and this induces an equivalence of triangulated categories

$$K^-(\text{Fil}^f\mathcal{A})/\text{FAc}^-(\mathcal{A}) = \text{FQis}^-(\mathcal{A})^{-1}K^-(\text{Fil}^f\mathcal{A}) \longrightarrow DF^-(\mathcal{A})$$

(3) The kernel of the functor $K^b(\text{Fil}^f\mathcal{A}) \to DF^b(\mathcal{A})$ is $\text{FAc}^b(\mathcal{A})$ and this induces an equivalence of triangulated categories

$$K^b(\text{Fil}^f\mathcal{A})/\text{FAc}^b(\mathcal{A}) = \text{FQis}^b(\mathcal{A})^{-1}K^b(\text{Fil}^f\mathcal{A}) \longrightarrow DF^b(\mathcal{A})$$

Proof. This follows from the results above, in particular Lemma 13.13.8, by exactly the same arguments as used in the proof of Lemma 13.11.6. ☐

13.14. Derived functors in general

05S7 A reference for this section is Deligne's exposé XVII in [AGV71]. A very general notion of right and left derived functors exists where we have an exact functor between triangulated categories, a multiplicative system in the source category and we want to find the "correct" extension of the exact functor to the localized category.

05S8 Situation 13.14.1. Here $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor of triangulated categories and S is a saturated multiplicative system in $\mathcal{D}$ compatible with the structure of triangulated category on $\mathcal{D}$.

Let $X \in \text{Ob}(\mathcal{D})$. Recall from Categories, Remark 4.27.7 the filtered category X/S of arrows $s : X \to X'$ in S with source X. Dually, in Categories, Remark 4.27.15 we defined the cofiltered category S/X of arrows $s : X' \to X$ in S with target X.

05S9 Definition 13.14.2. Assumptions and notation as in Situation 13.14.1. Let $X \in \text{Ob}(\mathcal{D})$.

(1) we say the right derived functor RF is defined at X if the ind-object

$$(X/S) \longrightarrow \mathcal{D}', \quad (s : X \to X') \longmapsto F(X')$$

is essentially constant[5]; in this case the value Y in $\mathcal{D}'$ is called the value of RF at X.

(2) we say the left derived functor LF is defined at X if the pro-object

$$(S/X) \longrightarrow \mathcal{D}', \quad (s : X' \to X) \longmapsto F(X')$$

is essentially constant; in this case the value Y in $\mathcal{D}'$ is called the value of LF at X.

By abuse of notation we often denote the values simply $RF(X)$ or $LF(X)$.

It will turn out that the full subcategory of $\mathcal{D}$ consisting of objects where RF is defined is a triangulated subcategory, and RF will define a functor on this subcategory which transforms morphisms of S into isomorphisms.

[5]For a discussion of when an ind-object or pro-object of a category is essentially constant we refer to Categories, Section 4.22.

05SA **Lemma 13.14.3.** Assumptions and notation as in Situation 13.14.1. Let $f : X \to Y$ be a morphism of $\mathcal{D}$.

(1) If RF is defined at X and Y then there exists a unique morphism $RF(f) : RF(X) \to RF(Y)$ between the values such that for any commutative diagram
$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$
with $s, s' \in S$ the diagram
$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(X') \arrow[r] \arrow[d] & RF(X) \arrow[d] \\ F(Y) \arrow[r] & F(Y') \arrow[r] & RF(Y) \end{tikzcd}$$
commutes.

(2) If LF is defined at X and Y then there exists a unique morphism $LF(f) : LF(X) \to LF(Y)$ between the values such that for any commutative diagram
$$\begin{tikzcd} X' \arrow[r, "s"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "s'"] & Y \end{tikzcd}$$
with s, s' in S the diagram
$$\begin{tikzcd} LF(X) \arrow[r] \arrow[d] & F(X') \arrow[r] \arrow[d] & F(X) \arrow[d] \\ LF(Y) \arrow[r] & F(Y') \arrow[r] & F(Y) \end{tikzcd}$$
commutes.

Proof. Part (1) holds if we only assume that the colimits
$$RF(X) = \mathrm{colim}_{s:X\to X'} F(X') \quad \text{and} \quad RF(Y) = \mathrm{colim}_{s':Y\to Y'} F(Y')$$
exist. Namely, to give a morphism $RF(X) \to RF(Y)$ between the colimits is the same thing as giving for each $s : X \to X'$ in $\mathrm{Ob}(X/S)$ a morphism $F(X') \to RF(Y)$ compatible with morphisms in the category X/S. To get the morphism we choose a commutative diagram
$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$
with s, s' in S as is possible by MS2 and we set $F(X') \to RF(Y)$ equal to the composition $F(X') \to F(Y') \to RF(Y)$. To see that this is independent of the choice of the diagram above use MS3. Details omitted. The proof of (2) is dual. $\square$

05SB **Lemma 13.14.4.** Assumptions and notation as in Situation 13.14.1. Let $s : X \to Y$ be an element of S.

(1) RF is defined at X if and only if it is defined at Y. In this case the map $RF(s) : RF(X) \to RF(Y)$ between values is an isomorphism.
(2) LF is defined at X if and only if it is defined at Y. In this case the map $LF(s) : LF(X) \to LF(Y)$ between values is an isomorphism.

Proof. Omitted. □

05SU Lemma 13.14.5. Assumptions and notation as in Situation 13.14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.

(1) RF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $RF(X)[n] = RF(X[n])$ between values.
(2) LF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $LF(X)[n] \to LF(X[n])$ between values.

Proof. Omitted. □

05SC Lemma 13.14.6. Assumptions and notation as in Situation 13.14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If RF is defined at two out of three of X, Y, Z, then it is defined at the third. Moreover, in this case

$$(RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h))$$

is a distinguished triangle in $\mathcal{D}'$. Similarly for LF.

Proof. Say RF is defined at X, Y with values A, B. Let $RF(f) : A \to B$ be the induced morphism, see Lemma 13.14.3. We may choose a distinguished triangle $(A, B, C, RF(f), b, c)$ in $\mathcal{D}'$. We claim that C is a value of RF at Z.

To see this pick $s : X \to X'$ in S such that there exists a morphism $\alpha : A \to F(X')$ as in Categories, Definition 4.22.1. We may choose a commutative diagram

$$\begin{tikzcd} X \arrow[r, "s"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "s'"] & Y' \end{tikzcd}$$

with $s' \in S$ by MS2. Using that Y/S is filtered we can (after replacing s' by some $s'' : Y \to Y''$ in S) assume that there exists a morphism $\beta : B \to F(Y')$ as in Categories, Definition 4.22.1. Picture

$$\begin{tikzcd} A \arrow[r, "\alpha"'] \arrow[d, "RF(f)"'] & F(X') \arrow[r] \arrow[d, "F(f')"] & A \arrow[d, "RF(f)"] \\ B \arrow[r, "\beta"] & F(Y') \arrow[r] & B \end{tikzcd}$$

It may not be true that the left square commutes, but the outer and right squares commute. The assumption that the ind-object $\{F(Y')\}_{s' : Y' \to Y}$ is essentially constant means that there exists a $s'' : Y \to Y''$ in S and a morphism $h : Y' \to Y''$ such that $s'' = h \circ s'$ and such that $F(h)$ equal to $F(Y') \to B \to F(Y') \to F(Y'')$. Hence after replacing Y' by Y'' and β by $F(h) \circ \beta$ the diagram will commute (by direct computation with arrows).

Using MS6 choose a morphism of triangles

$$(s, s', s'') : (X, Y, Z, f, g, h) \longrightarrow (X', Y', Z', f', g', h')$$

with $s'' \in S$. By TR3 choose a morphism of triangles

$$(\alpha, \beta, \gamma) : (A, B, C, RF(f), b, c) \longrightarrow (F(X'), F(Y'), F(Z'), F(f'), F(g'), F(h'))$$

By Lemma 13.14.4 it suffices to prove that $RF(Z')$ is defined and has value C. Consider the category $\mathcal{I}$ of Lemma 13.5.10 of triangles

$$\mathcal{I} = \{(t, t', t'') : (X', Y', Z', f', g', h') \to (X'', Y'', Z'', f'', g'', h'') \mid (t, t', t'') \in S\}$$

To show that the system $F(Z'')$ is essentially constant over the category Z'/S is equivalent to showing that the system of $F(Z'')$ is essentially constant over $\mathcal{I}$ because $\mathcal{I} \to Z'/S$ is cofinal, see Categories, Lemma 4.22.11 (cofinality is proven in Lemma 13.5.10). For any object W in $\mathcal{D}'$ we consider the diagram

$$\begin{tikzcd}
\operatorname{colim}_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(W, F(X'')) & \operatorname{Mor}_{\mathcal{D}'}(W, A) \arrow[l] \\
\operatorname{colim}_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(W, F(Y'')) \arrow[u] & \operatorname{Mor}_{\mathcal{D}'}(W, B) \arrow[l] \arrow[u] \\
\operatorname{colim}_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(W, F(Z'')) \arrow[u] & \operatorname{Mor}_{\mathcal{D}'}(W, C) \arrow[l] \arrow[u] \\
\operatorname{colim}_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(W, F(X''[1])) \arrow[u] & \operatorname{Mor}_{\mathcal{D}'}(W, A[1]) \arrow[l] \arrow[u] \\
\operatorname{colim}_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(W, F(Y''[1])) \arrow[u] & \operatorname{Mor}_{\mathcal{D}'}(W, B[1]) \arrow[l] \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are given by composing with (α, β, γ). Since filtered colimits are exact (Algebra, Lemma 10.8.8) the left column is an exact sequence. Thus the 5 lemma (Homology, Lemma 12.5.20) tells us the

$$\operatorname{colim}_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(W, F(Z'')) \longrightarrow \operatorname{Mor}_{\mathcal{D}'}(W, C)$$

is bijective. Choose an object $(t, t', t'') : (X', Y', Z') \to (X'', Y'', Z'')$ of $\mathcal{I}$. Applying what we just showed to $W = F(Z'')$ and the element $\mathrm{id}_{F(X'')}$ of the colimit we find a unique morphism $c_{(X'', Y'', Z'')} : F(Z'') \to C$ such that for some $(X'', Y'', Z'') \to (X''', Y''', Z'')$ in $\mathcal{I}$

$$F(Z'') \xrightarrow{c_{(X'',Y'',Z'')}} C \xrightarrow{\gamma} F(Z') \to F(Z'') \to F(Z''') \quad \text{equals} \quad F(Z'') \to F(Z''')$$

The family of morphisms $c_{(X'', Y'', Z'')}$ form an element c of $\lim_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(F(Z''), C)$ by uniqueness (computation omitted). Finally, we show that $\operatorname{colim}_\mathcal{I} F(Z'') = C$ via the morphisms $c_{(X'', Y'', Z'')}$ which will finish the proof by Categories, Lemma 4.22.9. Namely, let W be an object of $\mathcal{D}'$ and let $d_{(X'', Y'', Z'')} : F(Z'') \to W$ be a family of maps corresponding to an element of $\lim_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(F(Z''), W)$. If $d_{(X', Y', Z')} \circ \gamma = 0$, then for every object (X'', Y'', Z'') of $\mathcal{I}$ the morphism $d_{(X'', Y'', Z'')}$ is zero by the existence of $c_{(X'', Y'', Z'')}$ and the morphism $(X'', Y'', Z'') \to (X''', Y''', Z'')$ in $\mathcal{I}$ satisfying the displayed equality above. Hence the map

$$\lim_\mathcal{I} \operatorname{Mor}_{\mathcal{D}'}(F(Z''), W) \longrightarrow \operatorname{Mor}_{\mathcal{D}'}(C, W)$$

(coming from precomposing by γ) is injective. However, it is also surjective because the element c gives a left inverse. We conclude that C is the colimit by Categories, Remark 4.14.4. $\square$

05SD **Lemma 13.14.7.** Assumptions and notation as in Situation 13.14.1. Let X, Y be objects of $\mathcal{D}$.

(1) If RF is defined at X and Y, then RF is defined at $X \oplus Y$.
(2) If $\mathcal{D}'$ is Karoubian and RF is defined at $X \oplus Y$, then RF is defined at both X and Y.

In either case we have $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. Similarly for LF.

Proof. If RF is defined at X and Y, then the distinguished triangle $X \to X \oplus Y \to Y \to X[1]$ (Lemma 13.4.11) and Lemma 13.14.6 shows that RF is defined at $X \oplus Y$ and that we have a distinguished triangle $RF(X) \to RF(X \oplus Y) \to RF(Y) \to RF(X)[1]$. Applying Lemma 13.4.11 to this once more we find that $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. This proves (1) and the final assertion.

Conversely, assume that RF is defined at $X \oplus Y$ and that $\mathcal{D}'$ is Karoubian. Since S is a saturated system S is the set of arrows which become invertible under the additive localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$, see Categories, Lemma 4.27.21. Thus for any $s : X \to X'$ and $s' : Y \to Y'$ in S the morphism $s \oplus s' : X \oplus Y \to X' \oplus Y'$ is an element of S. In this way we obtain a functor

$$X/S \times Y/S \longrightarrow (X \oplus Y)/S$$

Recall that the categories $X/S, Y/S, (X \oplus Y)/S$ are filtered (Categories, Remark 4.27.7). By Categories, Lemma 4.22.12 $X/S \times Y/S$ is filtered and $F|_{X/S} : X/S \to \mathcal{D}'$ (resp. $G|_{Y/S} : Y/S \to \mathcal{D}'$) is essentially constant if and only if $F|_{X/S} \circ \text{pr}_1 : X/S \times Y/S \to \mathcal{D}'$ (resp. $G|_{Y/S} \circ \text{pr}_2 : X/S \times Y/S \to \mathcal{D}'$) is essentially constant. Below we will show that the displayed functor is cofinal, hence by Categories, Lemma 4.22.11. we see that $F|_{(X \oplus Y)/S}$ is essentially constant implies that $F|_{X/S} \circ \text{pr}_1 \oplus F|_{Y/S} \circ \text{pr}_2 : X/S \times Y/S \to \mathcal{D}'$ is essentially constant. By Homology, Lemma 12.30.3 (and this is where we use that $\mathcal{D}'$ is Karoubian) we see that $F|_{X/S} \circ \text{pr}_1 \oplus F|_{Y/S} \circ \text{pr}_2$ being essentially constant implies $F|_{X/S} \circ \text{pr}_1$ and $F|_{Y/S} \circ \text{pr}_2$ are essentially constant proving that RF is defined at X and Y.

Proof that the displayed functor is cofinal. To do this pick any $t : X \oplus Y \to Z$ in S. Using MS2 we can find morphisms $Z \to X'$, $Z \to Y'$ and $s : X \to X'$, $s' : Y \to Y'$ in S such that

$$\begin{tikzcd} X \arrow[d, "s"] & X \oplus Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d, "s'"'] \\ X' & Z \arrow[l] \arrow[r] & Y' \end{tikzcd}$$

commutes. This proves there is a map $Z \to X' \oplus Y'$ in $(X \oplus Y)/S$, i.e., we get part (1) of Categories, Definition 4.17.1. To prove part (2) it suffices to prove that given $t : X \oplus Y \to Z$ and morphisms $s_i \oplus s'_i : Z \to X'_i \oplus Y'_i$, $i = 1, 2$ in $(X \oplus Y)/S$ we can find morphisms $a : X'_1 \to X'$, $b : X'_2 \to X'$, $c : Y'_1 \to Y'$, $d : Y'_2 \to Y'$ in S such that $a \circ s_1 = b \circ s_2$ and $c \circ s'_1 = d \circ s'_2$. To do this we first choose any X' and Y' and maps a, b, c, d in S; this is possible as X/S and Y/S are filtered. Then the two maps $a \circ s_1, b \circ s_2 : Z \to X'$ become equal in $S^{-1}\mathcal{D}$. Hence we can find a morphism $X' \to X''$ in S equalizing them. Similarly we find $Y' \to Y''$ in S

equalizing $c \circ s_1'$ and $d \circ s_2'$. Replacing X' by X'' and Y' by Y'' we get $a \circ s_1 = b \circ s_2$ and $c \circ s_1' = d \circ s_2'$.

The proof of the corresponding statements for LF are dual. $\square$

05SE Proposition 13.14.8. Assumptions and notation as in Situation 13.14.1.

(1) The full subcategory $\mathcal{E}$ of $\mathcal{D}$ consisting of objects at which RF is defined is a strictly full triangulated subcategory of $\mathcal{D}$.
(2) We obtain an exact functor $RF : \mathcal{E} \longrightarrow \mathcal{D}'$ of triangulated categories.
(3) Elements of S with either source or target in $\mathcal{E}$ are morphisms of $\mathcal{E}$.
(4) The functor $S_\mathcal{E}^{-1}\mathcal{E} \to S^{-1}\mathcal{D}$ is a fully faithful exact functor of triangulated categories.
(5) Any element of $S_\mathcal{E} = \text{Arrows}(\mathcal{E}) \cap S$ is mapped to an isomorphism by RF.
(6) We obtain an exact functor
$$RF : S_\mathcal{E}^{-1}\mathcal{E} \longrightarrow \mathcal{D}'.$$
(7) If $\mathcal{D}'$ is Karoubian, then $\mathcal{E}$ is a saturated triangulated subcategory of $\mathcal{D}$.

A similar result holds for LF.

Proof. Since S is saturated it contains all isomorphisms (see remark following Categories, Definition 4.27.20). Hence (1) follows from Lemmas 13.14.4, 13.14.6, and 13.14.5. We get (2) from Lemmas 13.14.3, 13.14.5, and 13.14.6. We get (3) from Lemma 13.14.4. The fully faithfulness in (4) follows from (3) and the definitions. The fact that $S_\mathcal{E}^{-1}\mathcal{E} \to S^{-1}\mathcal{D}$ is exact follows from the fact that a triangle in $S_\mathcal{E}^{-1}\mathcal{E}$ is distinguished if and only if it is isomorphic to the image of a distinguished triangle in $\mathcal{E}$, see proof of Proposition 13.5.6. Part (5) follows from Lemma 13.14.4. The factorization of $RF : \mathcal{E} \to \mathcal{D}'$ through an exact functor $S_\mathcal{E}^{-1}\mathcal{E} \to \mathcal{D}'$ follows from Lemma 13.5.7. Part (7) follows from Lemma 13.14.7. $\square$

Proposition 13.14.8 tells us that RF lives on a maximal strictly full triangulated subcategory of $S^{-1}\mathcal{D}$ and is an exact functor on this triangulated category. Picture:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "F"'] \arrow[d, "Q"'] & & \mathcal{D}' \\
S^{-1}\mathcal{D} & & S_\mathcal{E}^{-1}\mathcal{E} \arrow[ll, "\text{fully faithful}"', "\text{exact}"] \arrow[u, "RF"']
\end{tikzcd}$$

05SV Definition 13.14.9. In Situation 13.14.1. We say F is right derivable, or that RF everywhere defined if RF is defined at every object of $\mathcal{D}$. We say F is left derivable, or that LF everywhere defined if LF is defined at every object of $\mathcal{D}$.

In this case we obtain a right (resp. left) derived functor

05SW (13.14.9.1) $$RF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}', \quad (\text{resp. } LF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}'),$$

see Proposition 13.14.8. In most interesting situations it is not the case that $RF \circ Q$ is equal to F. In fact, it might happen that the canonical map $F(X) \to RF(X)$ is never an isomorphism. In practice this does not happen, because in practice we only know how to prove F is right derivable by showing that RF can be computed by evaluating F at judiciously chosen objects of the triangulated category $\mathcal{D}$. This warrants a definition.

05SX Definition 13.14.10. In Situation 13.14.1.

(1) An object X of $\mathcal{D}$ computes RF if RF is defined at X and the canonical map $F(X) \to RF(X)$ is an isomorphism.
(2) An object X of $\mathcal{D}$ computes LF if LF is defined at X and the canonical map $LF(X) \to F(X)$ is an isomorphism.

05SY Lemma 13.14.11. Assumptions and notation as in Situation 13.14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.

(1) X computes RF if and only if $X[n]$ computes RF.
(2) X computes LF if and only if $X[n]$ computes LF.

Proof. Omitted. $\square$

05SZ Lemma 13.14.12. Assumptions and notation as in Situation 13.14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If X, Y compute RF then so does Z. Similar for LF.

Proof. By Lemma 13.14.6 we know that RF is defined at Z and that RF applied to the triangle produces a distinguished triangle. Consider the morphism of distinguished triangles

$$\begin{array}{c} (F(X), F(Y), F(Z), F(f), F(g), F(h)) \\ \downarrow \\ (RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h)) \end{array}$$

Two out of three maps are isomorphisms, hence so is the third. $\square$

05T0 Lemma 13.14.13. Assumptions and notation as in Situation 13.14.1. Let X, Y be objects of $\mathcal{D}$. If $X \oplus Y$ computes RF, then X and Y compute RF. Similarly for LF.

Proof. If $X \oplus Y$ computes RF, then $RF(X \oplus Y) = F(X) \oplus F(Y)$. In the proof of Lemma 13.14.7 we have seen that the functor $X/S \times Y/S \to (X \oplus Y)/S$, $(s, s') \mapsto s \oplus s'$ is cofinal. We will use this without further mention. Let $s : X \to X'$ be an element of S. Then $F(X) \to F(X')$ has a section, namely,

$$F(X') \to F(X' \oplus Y) \to RF(X' \oplus Y) = RF(X \oplus Y) = F(X) \oplus F(Y) \to F(X).$$

where we have used Lemma 13.14.4. Hence $F(X') = F(X) \oplus E$ for some object E of $\mathcal{D}'$ such that $E \to F(X' \oplus Y) \to RF(X' \oplus Y) = RF(X \oplus Y)$ is zero (Lemma 13.4.12). Because RF is defined at $X' \oplus Y$ with value $F(X) \oplus F(Y)$ we can find a morphism $t : X' \oplus Y \to Z$ of S such that $F(t)$ annihilates E. We may assume $Z = X'' \oplus Y''$ and $t = t' \oplus t''$ with $t', t'' \in S$. Then $F(t')$ annihilates E. It follows that F is essentially constant on X/S with value $F(X)$ as desired. $\square$

05T1 Lemma 13.14.14. Assumptions and notation as in Situation 13.14.1.

(1) If for every object $X \in \text{Ob}(\mathcal{D})$ there exists an arrow $s : X \to X'$ in S such that X' computes RF, then RF is everywhere defined.
(2) If for every object $X \in \text{Ob}(\mathcal{D})$ there exists an arrow $s : X' \to X$ in S such that X' computes LF, then LF is everywhere defined.

Proof. This is clear from the definitions. $\square$

06XN Lemma 13.14.15. Assumptions and notation as in Situation 13.14.1. If there exists a subset $\mathcal{I} \subset \text{Ob}(\mathcal{D})$ such that

(1) for all $X \in \mathrm{Ob}(\mathcal{D})$ there exists $s : X \to X'$ in S with $X' \in \mathcal{I}$, and
(2) for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{I}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,

then RF is everywhere defined and every $X \in \mathcal{I}$ computes RF. Dually, if there exists a subset $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ such that

(1) for all $X \in \mathrm{Ob}(\mathcal{D})$ there exists $s : X' \to X$ in S with $X' \in \mathcal{P}$, and
(2) for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{P}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,

then LF is everywhere defined and every $X \in \mathcal{P}$ computes LF.

Proof. Let X be an object of $\mathcal{D}$. Assumption (1) implies that the arrows $s : X \to X'$ in S with $X' \in \mathcal{I}$ are cofinal in the category X/S. Assumption (2) implies that F is constant on this cofinal subcategory. Clearly this implies that $F : (X/S) \to \mathcal{D}'$ is essentially constant with value $F(X')$ for any $s : X \to X'$ in S with $X' \in \mathcal{I}$. □

05T2 Lemma 13.14.16. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be triangulated categories. Let S, resp. S' be a saturated multiplicative system in $\mathcal{A}$, resp. $\mathcal{B}$ compatible with the triangulated structure. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be exact functors. Denote $F' : \mathcal{A} \to (S')^{-1}\mathcal{B}$ the composition of F with the localization functor.

(1) If RF', RG, $R(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : R(G \circ F) \longrightarrow RG \circ RF'$.
(2) If LF', LG, $L(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : LG \circ LF' \to L(G \circ F)$.

Proof. In this proof we try to be careful. Hence let us think of the derived functors as the functors

$$RF' : S^{-1}\mathcal{A} \to (S')^{-1}\mathcal{B}, \quad R(G \circ F) : S^{-1}\mathcal{A} \to \mathcal{C}, \quad RG : (S')^{-1}\mathcal{B} \to \mathcal{C}.$$

Let us denote $Q_A : \mathcal{A} \to S^{-1}\mathcal{A}$ and $Q_B : \mathcal{B} \to (S')^{-1}\mathcal{B}$ the localization functors. Then $F' = Q_B \circ F$. Note that for every object Y of $\mathcal{B}$ there is a canonical map

$$G(Y) \longrightarrow RG(Q_B(Y))$$

in other words, there is a transformation of functors $t' : G \to RG \circ Q_B$. Let X be an object of $\mathcal{A}$. We have

$$\begin{aligned} R(G \circ F)(Q_A(X)) &= \mathrm{colim}_{s : X \to X' \in S}\, G(F(X')) \\ &\xrightarrow{t'} \mathrm{colim}_{s : X \to X' \in S}\, RG(Q_B(F(X'))) \\ &= \mathrm{colim}_{s : X \to X' \in S}\, RG(F'(X')) \\ &= RG(\mathrm{colim}_{s : X \to X' \in S}\, F'(X')) \\ &= RG(RF'(X)). \end{aligned}$$

The system $F'(X')$ is essentially constant in the category $(S')^{-1}\mathcal{B}$. Hence we may pull the colimit inside the functor RG in the third equality of the diagram above, see Categories, Lemma 4.22.8 and its proof. We omit the proof this defines a transformation of functors. The case of left derived functors is similar. □

13.15. Derived functors on derived categories

05T3 In practice derived functors come about most often when given an additive functor between abelian categories.

05T4 **Situation 13.15.1.** Here $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. This induces exact functors

$$F : K(\mathcal{A}) \to K(\mathcal{B}), \quad K^+(\mathcal{A}) \to K^+(\mathcal{B}), \quad K^-(\mathcal{A}) \to K^-(\mathcal{B}).$$

See Lemma 13.10.6. We also denote F the composition $K(\mathcal{A}) \to D(\mathcal{B})$, $K^+(\mathcal{A}) \to D^+(\mathcal{B})$, and $K^-(\mathcal{A}) \to D^-(\mathcal{B})$ of F with the localization functor $K(\mathcal{B}) \to D(\mathcal{B})$, etc. This situation leads to four derived functors we will consider in the following.

(1) The right derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\text{Qis}(\mathcal{A})$.
(2) The right derived functor of $F : K^+(\mathcal{A}) \to D^+(\mathcal{B})$ relative to the multiplicative system $\text{Qis}^+(\mathcal{A})$.
(3) The left derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\text{Qis}(\mathcal{A})$.
(4) The left derived functor of $F : K^-(\mathcal{A}) \to D^-(\mathcal{B})$ relative to the multiplicative system $\text{Qis}^-(\mathcal{A})$.

Each of these cases is an example of Situation 13.14.1.

Some of the ambiguity that may arise is alleviated by the following.

05T5 **Lemma 13.15.2.** *In Situation 13.15.1.*

(1) *Let X be an object of $K^+(\mathcal{A})$. The right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(2) *Let X be an object of $K^+(\mathcal{A})$. Then X computes the right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *Let X be an object of $K^-(\mathcal{A})$. The left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(4) *Let X be an object of $K^-(\mathcal{A})$. Then X computes the left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$.*

Proof. Let X be an object of $K^+(\mathcal{A})$. Consider a quasi-isomorphism $s : X \to X'$ in $K(\mathcal{A})$. By Lemma 13.11.5 there exists quasi-isomorphism $X' \to X''$ with X'' bounded below. Hence we see that $X/\text{Qis}^+(\mathcal{A})$ is cofinal in $X/\text{Qis}(\mathcal{A})$. Thus it is clear that (1) holds. Part (2) follows directly from part (1). Parts (3) and (4) are dual to parts (1) and (2). □

Given an object A of an abelian category $\mathcal{A}$ we get a complex

$$A[0] = (\ldots \to 0 \to A \to 0 \to \ldots)$$

where A is placed in degree zero. Hence a functor $\mathcal{A} \to K(\mathcal{A})$, $A \mapsto A[0]$. Let us temporarily say that a partial functor is one that is defined on a subcategory.

0157 **Definition 13.15.3.** In Situation 13.15.1.

(1) The right derived functors of F are the partial functors RF associated to cases (1) and (2) of Situation 13.15.1.
(2) The left derived functors of F are the partial functors LF associated to cases (3) and (4) of Situation 13.15.1.
(3) An object A of $\mathcal{A}$ is said to be right acyclic for F, or acyclic for RF if $A[0]$ computes RF.
(4) An object A of $\mathcal{A}$ is said to be left acyclic for F, or acyclic for LF if $A[0]$ computes LF.

The following few lemmas give some criteria for the existence of enough acyclics.

05T7 Lemma 13.15.4. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset containing 0 such that every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Let $a \in \mathbf{Z}$.

(1) Given $K^\bullet$ with $K^n = 0$ for $n > a$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^n \in \mathcal{P}$ and $P^n \to K^n$ surjective for all n and $P^n = 0$ for $n > a$.
(2) Given $K^\bullet$ with $H^n(K^\bullet) = 0$ for $n > a$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^n \in \mathcal{P}$ for all n and $P^n = 0$ for $n > a$.

Proof. Proof of part (1). Consider the following induction hypothesis IH_n: There are $P^j \in \mathcal{P}$, $j \geq n$, with $P^j = 0$ for $j > a$, maps $d^j : P^j \to P^{j+1}$ for $j \geq n$, and surjective maps $\alpha^j : P^j \to K^j$ for $j \geq n$ such that the diagram

$$\begin{tikzcd}
 & P^n \arrow[r] \arrow[d, "\alpha"] & P^{n+1} \arrow[r] \arrow[d, "\alpha"] & P^{n+2} \arrow[r] \arrow[d, "\alpha"] & \ldots \\
\ldots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & K^{n+2} \arrow[r] & \ldots
\end{tikzcd}$$

is commutative, such that $d^{j+1} \circ d^j = 0$ for $j \geq n$, such that α induces isomorphisms $H^j(K^\bullet) \to \mathrm{Ker}(d^j)/\mathrm{Im}(d^{j-1})$ for $j > n$, and such that $\alpha : \mathrm{Ker}(d^n) \to \mathrm{Ker}(d^n_K)$ is surjective. Then we choose a surjection

$$P^{n-1} \longrightarrow K^{n-1} \times_{K^n} \mathrm{Ker}(d^n) = K^{n-1} \times_{\mathrm{Ker}(d^n_K)} \mathrm{Ker}(d^n)$$

with P^{n-1} in $\mathcal{P}$. This allows us to extend the diagram above to

$$\begin{tikzcd}
 & P^{n-1} \arrow[r] \arrow[d, "\alpha"] & P^n \arrow[r] \arrow[d, "\alpha"] & P^{n+1} \arrow[r] \arrow[d, "\alpha"] & P^{n+2} \arrow[r] \arrow[d, "\alpha"] & \ldots \\
\ldots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & K^{n+2} \arrow[r] & \ldots
\end{tikzcd}$$

The reader easily checks that IH_{n-1} holds with this choice.

We finish the proof of (1) as follows. First we note that IH_n is true for $n = a + 1$ since we can just take $P^j = 0$ for $j > a$. Hence we see that proceeding by descending induction we produce a complex $P^\bullet$ with $P^n = 0$ for $n > a$ consisting of objects from $\mathcal{P}$, and a termwise surjective quasi-isomorphism $\alpha : P^\bullet \to K^\bullet$ as desired.

Proof of part (2). The assumption implies that the morphism $\tau_{\leq a} K^\bullet \to K^\bullet$ (Homology, Section 12.15) is a quasi-isomorphism. Apply part (1) to find $P^\bullet \to \tau_{\leq a} K^\bullet$. The composition $P^\bullet \to K^\bullet$ is the desired quasi-isomorphism. □

05T6 Lemma 13.15.5. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset containing 0 such that every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. Let $a \in \mathbf{Z}$.

(1) Given $K^\bullet$ with $K^n = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \to I^\bullet$ with $K^n \to I^n$ injective and $I^n \in \mathcal{I}$ for all n and $I^n = 0$ for $n < a$,
(2) Given $K^\bullet$ with $H^n(K^\bullet) = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^n \in \mathcal{I}$ and $I^n = 0$ for $n < a$.

Proof. This lemma is dual to Lemma 13.15.4. $\square$

05T8 Lemma 13.15.6. In Situation 13.15.1. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset with the following properties:

(1) every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$,
(2) for any short exact sequence $0 \to P \to Q \to R \to 0$ of $\mathcal{A}$ with $P, Q \in \mathcal{I}$, then $R \in \mathcal{I}$, and $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.

Then every object of $\mathcal{I}$ is acyclic for RF.

Proof. We may add 0 to $\mathcal{I}$ if necessary. Pick $A \in \mathcal{I}$. Let $A[0] \to K^\bullet$ be a quasi-isomorphism with $K^\bullet$ bounded below. Then we can find a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and each $I^n \in \mathcal{I}$, see Lemma 13.15.5. Hence we see that these resolutions are cofinal in the category $A[0]/\mathrm{Qis}^+(\mathcal{A})$. To finish the proof it therefore suffices to show that for any quasi-isomorphism $A[0] \to I^\bullet$ with $I^\bullet$ bounded below and $I^n \in \mathcal{I}$ we have $F(A)[0] \to F(I^\bullet)$ is a quasi-isomorphism. To see this suppose that $I^n = 0$ for $n < n_0$. Of course we may assume that $n_0 < 0$. Starting with $n = n_0$ we prove inductively that $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n)$ and $\mathrm{Im}(d^{-1})$ are elements of $\mathcal{I}$ using property (2) and the exact sequences

$$0 \to \mathrm{Ker}(d^n) \to I^n \to \mathrm{Im}(d^n) \to 0.$$

Moreover, property (2) also guarantees that the complex

$$0 \to F(I^{n_0}) \to F(I^{n_0+1}) \to \ldots \to F(I^{-1}) \to F(\mathrm{Im}(d^{-1})) \to 0$$

is exact. The exact sequence $0 \to \mathrm{Im}(d^{-1}) \to I^0 \to I^0/\mathrm{Im}(d^{-1}) \to 0$ implies that $I^0/\mathrm{Im}(d^{-1})$ is an element of $\mathcal{I}$. The exact sequence $0 \to A \to I^0/\mathrm{Im}(d^{-1}) \to \mathrm{Im}(d^0) \to 0$ then implies that $\mathrm{Im}(d^0) = \mathrm{Ker}(d^1)$ is an elements of $\mathcal{I}$ and from then on one continues as before to show that $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n)$ is an element of $\mathcal{I}$ for all $n > 0$. Applying F to each of the short exact sequences mentioned above and using (2) we observe that $F(A)[0] \to F(I^\bullet)$ is an isomorphism as desired. $\square$

05T9 Lemma 13.15.7. In Situation 13.15.1. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset with the following properties:

(1) every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,
(2) for any short exact sequence $0 \to P \to Q \to R \to 0$ of $\mathcal{A}$ with $Q, R \in \mathcal{P}$, then $P \in \mathcal{P}$, and $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.

Then every object of $\mathcal{P}$ is acyclic for LF.

Proof. Dual to the proof of Lemma 13.15.6. $\square$

13.16. Higher derived functors

05TB The following simple lemma shows that right derived functors "move to the right".

05TC Lemma 13.16.1. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $K^\bullet$ be a complex of $\mathcal{A}$ and $a \in \mathbf{Z}$.

(1) If $H^i(K^\bullet) = 0$ for all $i < a$ and RF is defined at $K^\bullet$, then $H^i(RF(K^\bullet)) = 0$ for all $i < a$.

(2) If RF is defined at $K^\bullet$ and $\tau_{\leq a}K^\bullet$, then $H^i(RF(\tau_{\leq a}K^\bullet)) = H^i(RF(K^\bullet))$ for all $i \leq a$.

Proof. Assume $K^\bullet$ satisfies the assumptions of (1). Let $K^\bullet \to L^\bullet$ be any quasi-isomorphism. Then it is also true that $K^\bullet \to \tau_{\geq a}L^\bullet$ is a quasi-isomorphism by our assumption on $K^\bullet$. Hence in the category $K^\bullet/\text{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : K^\bullet \to L^\bullet$ with $L^n = 0$ for $n < a$ are cofinal. Thus RF is the value of the essentially constant ind-object $F(L^\bullet)$ for these s it follows that $H^i(RF(K^\bullet)) = 0$ for $i < a$.

To prove (2) we use the distinguished triangle

$$\tau_{\leq a}K^\bullet \to K^\bullet \to \tau_{\geq a+1}K^\bullet \to (\tau_{\leq a}K^\bullet)[1]$$

of Remark 13.12.4 to conclude via Lemma 13.14.6 that RF is defined at $\tau_{\geq a+1}K^\bullet$ as well and that we have a distinguished triangle

$$RF(\tau_{\leq a}K^\bullet) \to RF(K^\bullet) \to RF(\tau_{\geq a+1}K^\bullet) \to RF(\tau_{\leq a}K^\bullet)[1]$$

in $D(\mathcal{B})$. By part (1) we see that $RF(\tau_{\geq a+1}K^\bullet)$ has vanishing cohomology in degrees $< a+1$. The long exact cohomology sequence of this distinguished triangle then shows what we want. □

015A Definition 13.16.2. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $i \in \mathbf{Z}$. The ith right derived functor R^iF of F is the functor

$$R^iF = H^i \circ RF : \mathcal{A} \longrightarrow \mathcal{B}$$

The following lemma shows that it really does not make a lot of sense to take the right derived functor unless the functor is left exact.

05TD Lemma 13.16.3. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.

(1) We have $R^iF = 0$ for $i < 0$,
(2) R^0F is left exact,
(3) the map $F \to R^0F$ is an isomorphism if and only if F is left exact.

Proof. Let A be an object of $\mathcal{A}$. Let $A[0] \to K^\bullet$ be any quasi-isomorphism. Then it is also true that $A[0] \to \tau_{\geq 0}K^\bullet$ is a quasi-isomorphism. Hence in the category $A[0]/\text{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : A[0] \to K^\bullet$ with $K^n = 0$ for $n < 0$ are cofinal. Thus it is clear that $H^i(RF(A[0])) = 0$ for $i < 0$. Moreover, for such an s the sequence

$$0 \to A \to K^0 \to K^1$$

is exact. Hence if F is left exact, then $0 \to F(A) \to F(K^0) \to F(K^1)$ is exact as well, and we see that $F(A) \to H^0(F(K^\bullet))$ is an isomorphism for every $s : A[0] \to K^\bullet$ as above which implies that $H^0(RF(A[0])) = F(A)$.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$. By Lemma 13.12.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $D^+(\mathcal{A})$. From the long exact cohomology sequence (and the vanishing for $i < 0$ proved above) we deduce that $0 \to R^0F(A) \to R^0F(B) \to R^0F(C)$ is exact. Hence R^0F is left exact. Of course this also proves that if $F \to R^0F$ is an isomorphism, then F is left exact. □

015C Lemma 13.16.4. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let A be an object of $\mathcal{A}$.

(1) A is right acyclic for F if and only if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$,
(2) if F is left exact, then A is right acyclic for F if and only if $R^iF(A) = 0$ for all $i > 0$.

Proof. If A is right acyclic for F, then $RF(A[0]) = F(A)[0]$ and in particular $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for $i \neq 0$. Conversely, if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$ then $F(A[0]) \to RF(A[0])$ is a quasi-isomorphism by Lemma 13.16.3 part (1) and hence A is acyclic. If F is left exact then $F = R^0F$, see Lemma 13.16.3. □

015D Lemma 13.16.5. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$.

(1) If A and C are right acyclic for F then so is B.
(2) If A and B are right acyclic for F then so is C.
(3) If B and C are right acyclic for F and $F(B) \to F(C)$ is surjective then A is right acyclic for F.

In each of the three cases

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a short exact sequence of $\mathcal{B}$.

Proof. By Lemma 13.12.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $K^+(\mathcal{A})$. As RF is an exact functor and since $R^iF = 0$ for $i < 0$ and $R^0F = F$ (Lemma 13.16.3) we obtain an exact cohomology sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to \ldots$$

in the abelian category $\mathcal{B}$. Thus the lemma follows from the characterization of acyclic objects in Lemma 13.16.4. □

05TE Lemma 13.16.6. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.

(1) The functors R^iF, $i \geq 0$ come equipped with a canonical structure of a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 12.12.1.
(2) If every object of $\mathcal{A}$ is a subobject of a right acyclic object for F, then $\{R^iF, \delta\}_{i \geq 0}$ is a universal δ-functor, see Homology, Definition 12.12.3.

Proof. The functor $\mathcal{A} \to \text{Comp}^+(\mathcal{A})$, $A \mapsto A[0]$ is exact. The functor $\text{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 13.12.1. The functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is exact. Finally, the functor $H^0 : D^+(\mathcal{B}) \to \mathcal{B}$ is a homological functor, see Definition 13.11.3. Hence we get the structure of a δ-functor from Lemma 13.4.22 and Lemma 13.4.21. Part (2) follows from Homology, Lemma 12.12.4 and the description of acyclics in Lemma 13.16.4. □

015E Lemma 13.16.7 (Leray's acyclicity lemma). Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $A^\bullet$ be a bounded below complex of right F-acyclic objects such that RF is defined at $A^\bullet$[6]. The canonical map

$$F(A^\bullet) \longrightarrow RF(A^\bullet)$$

is an isomorphism in $D^+(\mathcal{B})$, i.e., $A^\bullet$ computes RF.

Proof. Let $A^\bullet$ be a bounded complex of right F-acyclic objects. We claim that RF is defined at $A^\bullet$ and that $F(A^\bullet) \to RF(A^\bullet)$ is an isomorphism in $D^+(\mathcal{B})$. Namely, it holds for complexes with at most one nonzero right F-acyclic object for example by Lemma 13.16.4. Next, suppose that $A^n = 0$ for $n \not\in [a, b]$. Using the "stupid" truncations we obtain a termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1}A^\bullet \to A^\bullet \to \sigma_{\leq a}A^\bullet \to 0$$

see Homology, Section 12.15. Thus a distinguished triangle $(\sigma_{\geq a+1}A^\bullet, A^\bullet, \sigma_{\leq a}A^\bullet)$. By induction hypothesis RF is defined for the two outer complexes and these complexes compute RF. Then the same is true for the middle one by Lemma 13.14.12.

Suppose that $A^\bullet$ is a bounded below complex of acyclic objects such that RF is defined at $A^\bullet$. To show that $F(A^\bullet) \to RF(A^\bullet)$ is an isomorphism in $D^+(\mathcal{B})$ it suffices to show that $H^i(F(A^\bullet)) \to H^i(RF(A^\bullet))$ is an isomorphism for all i. Pick i. Consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq i+2}A^\bullet \to A^\bullet \to \sigma_{\leq i+1}A^\bullet \to 0.$$

Note that this induces a termwise split short exact sequence

$$0 \to \sigma_{\geq i+2}F(A^\bullet) \to F(A^\bullet) \to \sigma_{\leq i+1}F(A^\bullet) \to 0.$$

Hence we get distinguished triangles

$$(\sigma_{\geq i+2}A^\bullet, A^\bullet, \sigma_{\leq i+1}A^\bullet) \quad \text{and} \quad (\sigma_{\geq i+2}F(A^\bullet), F(A^\bullet), \sigma_{\leq i+1}F(A^\bullet))$$

Since RF is defined at $A^\bullet$ (by assumption) and at $\sigma_{\leq i+1}A^\bullet$ (by the first paragraph) we see that RF is defined at $\sigma_{\geq i+1}A^\bullet$ and we get a distinghuished triangle

$$(RF(\sigma_{\geq i+2}A^\bullet), RF(A^\bullet), RF(\sigma_{\leq i+1}A^\bullet))$$

See Lemma 13.14.6. Using these distinguished triangles we obtain a map of exact sequences

$$\begin{tikzcd}
H^i(\sigma_{\geq i+2}F(A^\bullet)) \arrow[r] \arrow[d] & H^i(F(A^\bullet)) \arrow[r] \arrow[d, "\alpha"] & H^i(\sigma_{\leq i+1}F(A^\bullet)) \arrow[r] \arrow[d, "\beta"] & H^{i+1}(\sigma_{\geq i+2}F(A^\bullet)) \arrow[d] \\
H^i(RF(\sigma_{\geq i+2}A^\bullet)) \arrow[r] & H^i(RF(A^\bullet)) \arrow[r] & H^i(RF(\sigma_{\leq i+1}A^\bullet)) \arrow[r] & H^{i+1}(RF(\sigma_{\geq i+2}A^\bullet))
\end{tikzcd}$$

By the results of the first paragraph the map β is an isomorphism. By inspection the objects on the upper left and the upper right are zero. Hence to finish the proof it suffices to show that $H^i(RF(\sigma_{\geq i+2}A^\bullet)) = 0$ and $H^{i+1}(RF(\sigma_{\geq i+2}A^\bullet)) = 0$. This follows immediately from Lemma 13.16.1. $\square$

05TA Proposition 13.16.8. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories.

[6]For example this holds if $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.

(1) If every object of $\mathcal{A}$ injects into an object acyclic for RF, then RF is defined on all of $K^+(\mathcal{A})$ and we obtain an exact functor
$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$
see (13.14.9.1). Moreover, any bounded below complex $A^\bullet$ whose terms are acyclic for RF computes RF.
(2) If every object of $\mathcal{A}$ is quotient of an object acyclic for LF, then LF is defined on all of $K^-(\mathcal{A})$ and we obtain an exact functor
$$LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})$$
see (13.14.9.1). Moreover, any bounded above complex $A^\bullet$ whose terms are acyclic for LF computes LF.

Proof. Assume every object of $\mathcal{A}$ injects into an object acyclic for RF. Let $\mathcal{I}$ be the set of objects acyclic for RF. Let $K^\bullet$ be a bounded below complex in $\mathcal{A}$. By Lemma 13.15.5 there exists a quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and $I^n \in \mathcal{I}$. Hence in order to prove (1) it suffices to show that $F(I^\bullet) \to F((I')^\bullet)$ is a quasi-isomorphism when $s : I^\bullet \to (I')^\bullet$ is a quasi-isomorphism of bounded below complexes of objects from $\mathcal{I}$, see Lemma 13.14.15. Note that the cone $C(s)^\bullet$ is an acyclic bounded below complex all of whose terms are in $\mathcal{I}$. Hence it suffices to show: given an acyclic bounded below complex $I^\bullet$ all of whose terms are in $\mathcal{I}$ the complex $F(I^\bullet)$ is acyclic.

Say $I^n = 0$ for $n < n_0$. Setting $J^n = \mathrm{Im}(d^n)$ we break $I^\bullet$ into short exact sequences $0 \to J^n \to I^{n+1} \to J^{n+1} \to 0$ for $n \geq n_0$. These sequences induce distinguished triangles (J^n, I^{n+1}, J^{n+1}) in $D^+(\mathcal{A})$ by Lemma 13.12.1. For each $k \in \mathbf{Z}$ denote H_k the assertion: For all $n \leq k$ the right derived functor RF is defined at J^n and $R^iF(J^n) = 0$ for $i \neq 0$. Then H_k holds trivially for $k \leq n_0$. If H_n holds, then, using Proposition 13.14.8, we see that RF is defined at J^{n+1} and $(RF(J^n), RF(I^{n+1}), RF(J^{n+1}))$ is a distinguished triangle of $D^+(\mathcal{B})$. Thus the long exact cohomology sequence (13.11.1.1) associated to this triangle gives an exact sequence
$$0 \to R^{-1}F(J^{n+1}) \to R^0F(J^n) \to F(I^{n+1}) \to R^0F(J^{n+1}) \to 0$$
and gives that $R^iF(J^{n+1}) = 0$ for $i \notin \{-1, 0\}$. By Lemma 13.16.1 we see that $R^{-1}F(J^{n+1}) = 0$. This proves that H_{n+1} is true hence H_k holds for all k. We also conclude that
$$0 \to R^0F(J^n) \to F(I^{n+1}) \to R^0F(J^{n+1}) \to 0$$
is short exact for all n. This in turn proves that $F(I^\bullet)$ is exact.

The proof in the case of LF is dual. □

015F Lemma 13.16.9. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories. Then

(1) every object of $\mathcal{A}$ is right acyclic for F,
(2) $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined,
(3) $RF : D(\mathcal{A}) \to D(\mathcal{B})$ is everywhere defined,
(4) every complex computes RF, in other words, the canonical map $F(K^\bullet) \to RF(K^\bullet)$ is an isomorphism for all complexes, and
(5) $R^iF = 0$ for $i \neq 0$.

Proof. This is true because F transforms acyclic complexes into acyclic complexes and quasi-isomorphisms into quasi-isomorphisms. Details omitted. □

13.17. Triangulated subcategories of the derived category

06UP Let $\mathcal{A}$ be an abelian category. In this section we look at certain strictly full saturated triangulated subcategories $\mathcal{D}' \subset D(\mathcal{A})$.

Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory, see Homology, Definition 12.10.1 and Lemma 12.10.3. We let $D_\mathcal{B}(\mathcal{A})$ the full subcategory of $D(\mathcal{A})$ whose objects are

$$\mathrm{Ob}(D_\mathcal{B}(\mathcal{A})) = \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) \text{ is an object of } \mathcal{B} \text{ for all } n\}$$

We also define $D^+_\mathcal{B}(\mathcal{A}) = D^+(\mathcal{A}) \cap D_\mathcal{B}(\mathcal{A})$ and similarly for the other bounded versions.

06UQ **Lemma 13.17.1.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory. The category $D_\mathcal{B}(\mathcal{A})$ is a strictly full saturated triangulated subcategory of $D(\mathcal{A})$. Similarly for the bounded versions.*

Proof. It is clear that $D_\mathcal{B}(\mathcal{A})$ is an additive subcategory preserved under the translation functors. If $X \oplus Y$ is in $D_\mathcal{B}(\mathcal{A})$, then both $H^n(X)$ and $H^n(Y)$ are kernels of maps between maps of objects of $\mathcal{B}$ as $H^n(X \oplus Y) = H^n(X) \oplus H^n(Y)$. Hence both X and Y are in $D_\mathcal{B}(\mathcal{A})$. By Lemma 13.4.16 it therefore suffices to show that given a distinguished triangle (X, Y, Z, f, g, h) such that X and Y are in $D_\mathcal{B}(\mathcal{A})$ then Z is an object of $D_\mathcal{B}(\mathcal{A})$. The long exact cohomology sequence (13.11.1.1) and the definition of a weak Serre subcategory (see Homology, Definition 12.10.1) show that $H^n(Z)$ is an object of $\mathcal{B}$ for all n. Thus Z is an object of $D_\mathcal{B}(\mathcal{A})$. $\square$

We continue to assume that $\mathcal{B}$ is a weak Serre subcategory of the abelian category $\mathcal{A}$. Then $\mathcal{B}$ is an abelian category and the inclusion functor $\mathcal{B} \to \mathcal{A}$ is exact. Hence we obtain a derived functor $D(\mathcal{B}) \to D(\mathcal{A})$, see Lemma 13.16.9. Clearly the functor $D(\mathcal{B}) \to D(\mathcal{A})$ factors through a canonical exact functor

06UR (13.17.1.1)
$$D(\mathcal{B}) \longrightarrow D_\mathcal{B}(\mathcal{A})$$

After all a complex made from objects of $\mathcal{B}$ certainly gives rise to an object of $D_\mathcal{B}(\mathcal{A})$ and as distinguished triangles in $D_\mathcal{B}(\mathcal{A})$ are exactly the distinguished triangles of $D(\mathcal{A})$ whose vertices are in $D_\mathcal{B}(\mathcal{A})$ we see that the functor is exact since $D(\mathcal{B}) \to D(\mathcal{A})$ is exact. Similarly we obtain functors $D^+(\mathcal{B}) \to D^+_\mathcal{B}(\mathcal{A})$, $D^-(\mathcal{B}) \to D^-_\mathcal{B}(\mathcal{A})$, and $D^b(\mathcal{B}) \to D^b_\mathcal{B}(\mathcal{A})$ for the bounded versions. A key question in many cases is whether the displayed functor is an equivalence.

Now, suppose that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$. In this case we have the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$, see Homology, Lemma 12.10.6. In this case $D_\mathcal{B}(\mathcal{A})$ is the kernel of the functor $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$. Thus we obtain a canonical functor

$$D(\mathcal{A})/D_\mathcal{B}(\mathcal{A}) \longrightarrow D(\mathcal{A}/\mathcal{B})$$

by Lemma 13.6.8. Similarly for the bounded versions.

06XL **Lemma 13.17.2.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Then $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective.*

Proof. We will use the description of the category $\mathcal{A}/\mathcal{B}$ in the proof of Homology, Lemma 12.10.6. Let $(X^\bullet, d^\bullet)$ be a complex of $\mathcal{A}/\mathcal{B}$. This means that X^i is an object of $\mathcal{A}$ and $d^i : X^i \to X^{i+1}$ is a morphism in $\mathcal{A}/\mathcal{B}$ such that $d^i \circ d^{i-1} = 0$ in $\mathcal{A}/\mathcal{B}$.

For $i \geq 0$ we may write $d^i = (s^i, f^i)$ where $s^i : Y^i \to X^i$ is a morphism of $\mathcal{A}$ whose kernel and cokernel are in $\mathcal{B}$ (equivalently s^i becomes an isomorphism in the quotient category) and $f^i : Y^i \to X^{i+1}$ is a morphism of $\mathcal{A}$. By induction we will construct a commutative diagram

$$\begin{tikzcd}
 & (X')^1 \arrow[rr, dotted] & & (X')^2 \arrow[rr, dotted] & & \ldots \\
X^0 \arrow[ur, dotted] & & X^1 \arrow[ul, dotted] & & X^2 \arrow[ul, dotted] & & \ldots \\
Y^0 \arrow[u, "s^0"] \arrow[urr, "f^0"'] & & Y^1 \arrow[u, "s^1"] \arrow[urr, "f^1"'] & & Y^2 \arrow[u, "s^2"] \arrow[urr, "f^2"'] & & \ldots
\end{tikzcd}$$

where the vertical arrows $X^i \to (X')^i$ become isomorphisms in the quotient category. Namely, we first let $(X')^1 = \mathrm{Coker}(Y^0 \to X^0 \oplus X^1)$ (or rather the pushout of the diagram with arrows s^0 and f^0) which gives the first commutative diagram. Next, we take $(X')^2 = \mathrm{Coker}(Y^1 \to (X')^1 \oplus X^2)$. And so on. Setting additionally $(X')^n = X^n$ for $n \leq 0$ we see that the map $(X^\bullet, d^\bullet) \to ((X')^\bullet, (d')^\bullet)$ is an isomorphism of complexes in $\mathcal{A}/\mathcal{B}$. Hence we may assume $d^n : X^n \to X^{n+1}$ is given by a map $X^n \to X^{n+1}$ in $\mathcal{A}$ for $n \geq 0$.

Dually, for $i < 0$ we may write $d^i = (g^i, t^{i+1})$ where $t^{i+1} : X^{i+1} \to Z^{i+1}$ is an isomorphism in the quotient category and $g^i : X^i \to Z^{i+1}$ is a morphism. By induction we will construct a commutative diagram

$$\begin{tikzcd}
\ldots & Z^{-2} & & Z^{-1} & & Z^0 \\
\ldots & X^{-2} \arrow[u, "t_{-2}"] \arrow[urr, "g_{-2}"'] & & X^{-1} \arrow[u, "t_{-1}"] \arrow[urr, "g_{-1}"'] & & X^0 \arrow[u, "t^0"] \\
\ldots & (X')^{-2} \arrow[u, dotted] \arrow[rr, dotted] & & (X')^{-1} \arrow[u, dotted] \arrow[urr, dotted] & &
\end{tikzcd}$$

where the vertical arrows $(X')^i \to X^i$ become isomorphisms in the quotient category. Namely, we take $(X')^{-1} = X^{-1} \times_{Z^0} X^0$. Then we take $(X')^{-2} = X^{-2} \times_{Z^{-1}} (X')^{-1}$. And so on. Setting additionally $(X')^n = X^n$ for $n \geq 0$ we see that the map $((X')^\bullet, (d')^\bullet) \to (X^\bullet, d^\bullet)$ is an isomorphism of complexes in $\mathcal{A}/\mathcal{B}$. Hence we may assume $d^n : X^n \to X^{n+1}$ is given by a map $d^n : X^n \to X^{n+1}$ in $\mathcal{A}$ for all $n \in \mathbf{Z}$.

In this case we know the compositions $d^n \circ d^{n-1}$ are zero in $\mathcal{A}/\mathcal{B}$. If for $n > 0$ we replace X^n by

$$(X')^n = X^n / \sum\nolimits_{0 < k \leq n} \mathrm{Im}(\mathrm{Im}(X^{k-2} \to X^k) \to X^n)$$

then the compositions $d^n \circ d^{n-1}$ are zero for $n \geq 0$. (Similarly to the second paragraph above we obtain an isomorphism of complexes $(X^\bullet, d^\bullet) \to ((X')^\bullet, (d')^\bullet)$.) Finally, for $n < 0$ we replace X^n by

$$(X')^n = \bigcap\nolimits_{n \leq k < 0} (X^n \to X^k)^{-1} \mathrm{Ker}(X^k \to X^{k+2})$$

and we argue in the same manner to get a complex in $\mathcal{A}$ whose image in $\mathcal{A}/\mathcal{B}$ is isomorphic to the given one. $\square$

06XM Lemma 13.17.3. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Suppose that the functor $v : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ has a left adjoint $u : \mathcal{A}/\mathcal{B} \to \mathcal{A}$ such that $vu \cong \text{id}$. Then

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) = D(\mathcal{A}/\mathcal{B})$$

and similarly for the bounded versions.

Proof. The functor $D(v) : D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective by Lemma 13.17.2. For an object X of $D(\mathcal{A})$ the adjunction mapping $c_X : uvX \to X$ maps to an isomorphism in $D(\mathcal{A}/\mathcal{B})$ because $vuv \cong v$ by the assumption that $vu \cong \text{id}$. Thus in a distinguished triangle (uvX, X, Z, c_X, g, h) the object Z is an object of $D_{\mathcal{B}}(\mathcal{A})$ as we see by looking at the long exact cohomology sequence. Hence c_X is an element of the multiplicative system used to define the quotient category $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. Thus $uvX \cong X$ in $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. For $X, Y \in \text{Ob}(\mathcal{A}))$ the map

$$\text{Hom}_{D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})}(X, Y) \longrightarrow \text{Hom}_{D(\mathcal{A}/\mathcal{B})}(vX, vY)$$

is bijective because u gives an inverse (by the remarks above). $\square$

For certain Serre subcategories $\mathcal{B} \subset \mathcal{A}$ we can prove that the functor $D(\mathcal{B}) \to D_{\mathcal{B}}(\mathcal{A})$ is fully faithful.

0FCL Lemma 13.17.4. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Assume that for every surjection $X \to Y$ with $X \in \text{Ob}(\mathcal{A})$ and $Y \in \text{Ob}(\mathcal{B})$ there exists $X' \subset X$, $X' \in \text{Ob}(\mathcal{B})$ which surjects onto Y. Then the functor $D^-(\mathcal{B}) \to D^-_{\mathcal{B}}(\mathcal{A})$ of (13.17.1.1) is an equivalence.

Proof. Let $X^\bullet$ be a bounded above complex of $\mathcal{A}$ such that $H^i(X^\bullet) \in \text{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$. Moreover, suppose we are given $B^i \subset X^i$, $B^i \in \text{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$. Claim: there exists a subcomplex $Y^\bullet \subset X^\bullet$ such that

(1) $Y^\bullet \to X^\bullet$ is a quasi-isomorphism,
(2) $Y^i \in \text{Ob}(\mathcal{B})$ for all $i \in \mathbf{Z}$, and
(3) $B^i \subset Y^i$ for all $i \in \mathbf{Z}$.

To prove the claim, using the assumption of the lemma we first choose $C^i \subset \text{Ker}(d^i : X^i \to X^{i+1})$, $C^i \in \text{Ob}(\mathcal{B})$ surjecting onto $H^i(X^\bullet)$. Setting $D^i = C^i + d^{i-1}(B^{i-1}) + B^i$ we find a subcomplex $D^\bullet$ satisfying (2) and (3) such that $H^i(D^\bullet) \to H^i(X^\bullet)$ is surjective for all $i \in \mathbf{Z}$. For any choice of $E^i \subset X^i$ with $E^i \in \text{Ob}(\mathcal{B})$ and $d^i(E^i) \subset D^{i+1} + E^{i+1}$ we see that setting $Y^i = D^i + E^i$ gives a subcomplex whose terms are in $\mathcal{B}$ and whose cohomology surjects onto the cohomology of $X^\bullet$. Clearly, if $d^i(E^i) = (D^{i+1} + E^{i+1}) \cap \text{Im}(d^i)$ then we see that the map on cohomology is also injective. For $n \gg 0$ we can take E^n equal to 0. By descending induction we can choose E^i for all i with the desired property. Namely, given $E^{i+1}, E^{i+2}, \ldots$ we choose $E^i \subset X^i$ such that $d^i(E^i) = (D^{i+1} + E^{i+1}) \cap \text{Im}(d^i)$. This is possible by our assumption in the lemma combined with the fact that $(D^{i+1} + E^{i+1}) \cap \text{Im}(d^i)$ is in $\mathcal{B}$ as $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$.

The claim above implies the lemma. Essential surjectivity is immediate from the claim. Let us prove faithfulness. Namely, suppose we have a morphism $f : U^\bullet \to V^\bullet$ of bounded above complexes of $\mathcal{B}$ whose image in $D(\mathcal{A})$ is zero. Then there exists a quasi-isomorphism $s : V^\bullet \to X^\bullet$ into a bounded above complex of $\mathcal{A}$ such that

$s \circ f$ is homotopic to zero. Choose a homotopy $h^i : U^i \to X^{i-1}$ between 0 and $s \circ f$. Apply the claim with $B^i = h^{i+1}(U^{i+1}) + s^i(V^i)$. The resulting map $s' : V^\bullet \to Y^\bullet$ is a quasi-isomorphism as well and $s' \circ f$ is homotopic to zero as is clear from the fact that h^i factors through Y^{i-1}. This proves faithfulness. Fully faithfulness is proved in the exact same manner. □

13.18. Injective resolutions

013G In this section we prove some lemmas regarding the existence of injective resolutions in abelian categories having enough injectives.

013I Definition 13.18.1. Let $\mathcal{A}$ be an abelian category. Let $A \in \mathrm{Ob}(\mathcal{A})$. An injective resolution of A is a complex $I^\bullet$ together with a map $A \to I^0$ such that:

(1) We have $I^n = 0$ for $n < 0$.
(2) Each I^n is an injective object of $\mathcal{A}$.
(3) The map $A \to I^0$ is an isomorphism onto $\mathrm{Ker}(d^0)$.
(4) We have $H^i(I^\bullet) = 0$ for $i > 0$.

Hence $A[0] \to I^\bullet$ is a quasi-isomorphism. In other words the complex

$$\ldots \to 0 \to A \to I^0 \to I^1 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An injective resolution of $K^\bullet$ is a complex $I^\bullet$ together with a map $\alpha : K^\bullet \to I^\bullet$ of complexes such that

(1) We have $I^n = 0$ for $n \ll 0$, i.e., $I^\bullet$ is bounded below.
(2) Each I^n is an injective object of $\mathcal{A}$.
(3) The map $\alpha : K^\bullet \to I^\bullet$ is a quasi-isomorphism.

In other words an injective resolution $K^\bullet \to I^\bullet$ gives rise to a diagram

$$\begin{tikzcd}
\ldots \arrow[r] & K^{n-1} \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

which induces an isomorphism on cohomology objects in each degree. An injective resolution of an object A of $\mathcal{A}$ is almost the same thing as an injective resolution of the complex $A[0]$.

013J Lemma 13.18.2. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.

(1) If $K^\bullet$ has an injective resolution then $H^n(K^\bullet) = 0$ for $n \ll 0$.
(2) If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.

Proof. Omitted. For the second statement use $L^\bullet = \tau_{\geq n} K^\bullet$ for some $n \ll 0$. See Homology, Section 12.15 for the definition of the truncation $\tau_{\geq n}$. □

013K Lemma 13.18.3. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives.

(1) Any object of $\mathcal{A}$ has an injective resolution.
(2) If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then $K^\bullet$ has an injective resolution.
(3) If $K^\bullet$ is a complex with $K^n = 0$ for $n < a$, then there exists an injective resolution $\alpha : K^\bullet \to I^\bullet$ with $I^n = 0$ for $n < a$ such that each $\alpha^n : K^n \to I^n$ is injective.

Proof. Proof of (1). First choose an injection $A \to I^0$ of A into an injective object of $\mathcal{A}$. Next, choose an injection $I_0/A \to I^1$ into an injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose an injection $I^1/\operatorname{Im}(d^0) \to I^2$ into an injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. By Lemma 13.18.2 part (2) follows from part (3). Part (3) is a special case of Lemma 13.15.5. □

013R Lemma 13.18.4. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $I^\bullet$ be bounded below and consisting of injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n|_{d^{n-1}(K^{n-1})} = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is acyclic we have $d^{n-1}(K^{n-1}) = \operatorname{Ker}(K^n \to K^{n+1})$. Hence we can think of α^n as a map into I^n defined on the subobject $\operatorname{Im}(K^n \to K^{n+1})$ of K^{n+1}. By injectivity of the object I^n we can extend this to a map $h : K^{n+1} \to I^n$ as desired. □

05TF Remark 13.18.5. Let $\mathcal{A}$ be an abelian category. Using the fact that $K(\mathcal{A})$ is a triangulated category we may use Lemma 13.18.4 to obtain proofs of some of the lemmas below which are usually proved by chasing through diagrams. Namely, suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Then

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$ (Lemma 13.9.14) and $C(\alpha)^\bullet$ is an acyclic complex (Lemma 13.11.2). Next, let $I^\bullet$ be a bounded below complex of injective objects. Then

$$\begin{tikzcd} \operatorname{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet, I^\bullet) \arrow[r] & \operatorname{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) \arrow[r] & \operatorname{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) \arrow[dll] \\ \operatorname{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet[-1], I^\bullet) & & \end{tikzcd}$$

is an exact sequence of abelian groups, see Lemma 13.4.2. At this point Lemma 13.18.4 guarantees that the outer two groups are zero and hence $\operatorname{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \operatorname{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

013P Lemma 13.18.6. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"] & L^\bullet \arrow[dl, dashed, "\beta"] \\ I^\bullet & \end{tikzcd}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism.

(1) There exists a map of complexes β making the diagram commute up to homotopy.
(2) If α is injective in every degree then we can find a β which makes the diagram commute.

Proof. The "correct" proof of part (1) is explained in Remark 13.18.5. We also give a direct proof here.

We first show that (2) implies (1). Namely, let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 13.9.6. Since $\tilde{\alpha}$ is injective by (2) there exists a morphism $\tilde{\beta} : \tilde{L}^\bullet \to I^\bullet$ such that $\gamma = \tilde{\beta} \circ \tilde{\alpha}$. Set $\beta = \tilde{\beta} \circ s$. Then we have

$$\beta \circ \alpha = \tilde{\beta} \circ s \circ \pi \circ \tilde{\alpha} \sim \tilde{\beta} \circ \tilde{\alpha} = \gamma$$

as desired.

Assume that $\alpha : K^\bullet \to L^\bullet$ is injective. Suppose we have already defined β in all degrees $\leq n - 1$ compatible with differentials and such that $\gamma^j = \beta^j \circ \alpha^j$ for all $j \leq n - 1$. Consider the commutative solid diagram

$$\begin{tikzcd}
K^{n-1} \arrow[r] \arrow[d, "\alpha"] \arrow[dd, bend right=60, "\gamma"'] & K^n \arrow[d, "\alpha"] \arrow[dd, bend left=60, "\gamma"] \\
L^{n-1} \arrow[r] \arrow[d, "\beta"] & L^n \arrow[d, dashed] \\
I^{n-1} \arrow[r] & I^n
\end{tikzcd}$$

Thus we see that the dotted arrow is prescribed on the subobjects $\alpha(K^n)$ and $d^{n-1}(L^{n-1})$. Moreover, these two arrows agree on $\alpha(d^{n-1}(K^{n-1}))$. Hence if

013Q (13.18.6.1) $$\alpha(d^{n-1}(K^{n-1})) = \alpha(K^n) \cap d^{n-1}(L^{n-1})$$

then these morphisms glue to a morphism $\alpha(K^n) + d^{n-1}(L^{n-1}) \to I^n$ and, using the injectivity of I^n, we can extend this to a morphism from all of L^n into I^n. After this by induction we get the morphism β for all n simultaneously (note that we can set $\beta^n = 0$ for all $n \ll 0$ since $I^\bullet$ is bounded below – in this way starting the induction).

It remains to prove the equality (13.18.6.1). The reader is encouraged to argue this for themselves with a suitable diagram chase. Nonetheless here is our argument. Note that the inclusion $\alpha(d^{n-1}(K^{n-1})) \subset \alpha(K^n) \cap d^{n-1}(L^{n-1})$ is obvious. Take an object T of $\mathcal{A}$ and a morphism $x : T \to L^n$ whose image is contained in the subobject $\alpha(K^n) \cap d^{n-1}(L^{n-1})$. Since α is injective we see that $x = \alpha \circ x'$ for some $x' : T \to K^n$. Moreover, since x lies in $d^{n-1}(L^{n-1})$ we see that $d^n \circ x = 0$. Hence using injectivity of α again we see that $d^n \circ x' = 0$. Thus x' gives a morphism $[x'] : T \to H^n(K^\bullet)$. On the other hand the corresponding map $[x] : T \to H^n(L^\bullet)$ induced by x is zero by assumption. Since α is a quasi-isomorphism we conclude that $[x'] = 0$. This of course means exactly that the image of x' is contained in $d^{n-1}(K^{n-1})$ and we win. $\square$

013S Lemma 13.18.7. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & L^\bullet \arrow[dl, dashed, "\beta_i"] \\ I^\bullet & \end{tikzcd}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. This follows from Remark 13.18.5. We also give a direct argument here.

Let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 13.9.6. If we can show that $\beta_1 \circ \pi$ is homotopic to $\beta_2 \circ \pi$, then we deduce that $\beta_1 \sim \beta_2$ because $\pi \circ s$ is the identity. Hence we may assume $\alpha^n : K^n \to L^n$ is the inclusion of a direct summand for all n. Thus we get a short exact sequence of complexes

$$0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$$

which is termwise split and such that $M^\bullet$ is acyclic. We choose splittings $L^n = K^n \oplus M^n$, so we have $\beta_i^n : K^n \oplus M^n \to I^n$ and $\gamma^n : K^n \to I^n$. In this case the condition on β_i is that there are morphisms $h_i^n : K^n \to I^{n-1}$ such that

$$\gamma^n - \beta_i^n|_{K^n} = d \circ h_i^n + h_i^{n+1} \circ d$$

Thus we see that

$$\beta_1^n|_{K^n} - \beta_2^n|_{K^n} = d \circ (h_1^n - h_2^n) + (h_1^{n+1} - h_2^{n+1}) \circ d$$

Consider the map $h^n : K^n \oplus M^n \to I^{n-1}$ which equals $h_1^n - h_2^n$ on the first summand and zero on the second. Then we see that

$$\beta_1^n - \beta_2^n - (d \circ h^n + h^{n+1}) \circ d$$

is a morphism of complexes $L^\bullet \to I^\bullet$ which is identically zero on the subcomplex $K^\bullet$. Hence it factors as $L^\bullet \to M^\bullet \to I^\bullet$. Thus the result of the lemma follows from Lemma 13.18.4. □

05TG Lemma 13.18.8. Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be bounded below complex consisting of injective objects. Let $L^\bullet \in K(\mathcal{A})$. Then

$$\mathrm{Mor}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Mor}_{D(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By Lemma 13.18.6 we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then Lemma 13.18.7 shows that β is homotopic to zero also, i.e., $b = 0$. □

013T Lemma 13.18.9. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of $\mathrm{Comp}^+(\mathcal{A})$ there exists a commutative diagram in $\text{Comp}^+(\mathcal{A})$

$$\begin{array}{ccccccccc} 0 & \to & A^\bullet & \to & B^\bullet & \to & C^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I_1^\bullet & \to & I_2^\bullet & \to & I_3^\bullet & \to & 0 \end{array}$$

where the vertical arrows are injective resolutions and the rows are short exact sequences of complexes. In fact, given any injective resolution $A^\bullet \to I^\bullet$ we may assume $I_1^\bullet = I^\bullet$.

Proof. Step 1. Choose an injective resolution $A^\bullet \to I^\bullet$ (see Lemma 13.18.3) or use the given one. Recall that $\text{Comp}^+(\mathcal{A})$ is an abelian category, see Homology, Lemma 12.13.9. Hence we may form the pushout along the map $A^\bullet \to I^\bullet$ to get

$$\begin{array}{ccccccccc} 0 & \to & A^\bullet & \to & B^\bullet & \to & C^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I^\bullet & \to & E^\bullet & \to & C^\bullet & \to & 0 \end{array}$$

Because of the 5-lemma and the last assertion of Homology, Lemma 12.13.12 the map $B^\bullet \to A^\bullet$ is a quasi-isomorphism. Note that the lower short exact sequence is termwise split, see Homology, Lemma 12.27.2. Hence it suffices to prove the lemma when $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is termwise split.

Step 2. Choose splittings. In other words, write $B^n = A^n \oplus C^n$. Denote $\delta : C^\bullet \to A^\bullet[1]$ the morphism as in Homology, Lemma 12.14.10. Choose injective resolutions $f_1 : A^\bullet \to I_1^\bullet$ and $f_3 : C^\bullet \to I_3^\bullet$. (If $A^\bullet$ is a complex of injectives, then use $I_1^\bullet = A^\bullet$.) We may assume f_3 is injective in every degree. By Lemma 13.18.6 we may find a morphism $\delta' : I_3^\bullet \to I_1^\bullet[1]$ such that $\delta' \circ f_3 = f_1[1] \circ \delta$ (equality of morphisms of complexes). Set $I_2^n = I_1^n \oplus I_3^n$. Define

$$d_{I_2}^n = \begin{pmatrix} d_{I_1}^n & (\delta')^n \\ 0 & d_{I_3}^n \end{pmatrix}$$

and define the maps $B^n \to I_2^n$ to be given as the sum of the maps $A^n \to I_1^n$ and $C^n \to I_3^n$. Everything is clear. $\square$

13.19. Projective resolutions

0643 This section is dual to Section 13.18. We give definitions and state results, but we do not reprove the lemmas.

0644 Definition 13.19.1. Let $\mathcal{A}$ be an abelian category. Let $A \in \text{Ob}(\mathcal{A})$. An projective resolution of A is a complex $P^\bullet$ together with a map $P^0 \to A$ such that:

(1) We have $P^n = 0$ for $n > 0$.
(2) Each P^n is an projective object of $\mathcal{A}$.
(3) The map $P^0 \to A$ induces an isomorphism $\text{Coker}(d^{-1}) \to A$.
(4) We have $H^i(P^\bullet) = 0$ for $i < 0$.

Hence $P^\bullet \to A[0]$ is a quasi-isomorphism. In other words the complex

$$\ldots \to P^{-1} \to P^0 \to A \to 0 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An projective resolution of $K^\bullet$ is a complex $P^\bullet$ together with a map $\alpha : P^\bullet \to K^\bullet$ of complexes such that

(1) We have $P^n = 0$ for $n \gg 0$, i.e., $P^\bullet$ is bounded above.
(2) Each P^n is an projective object of $\mathcal{A}$.
(3) The map $\alpha : P^\bullet \to K^\bullet$ is a quasi-isomorphism.

0645 Lemma 13.19.2. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.

(1) If $K^\bullet$ has a projective resolution then $H^n(K^\bullet) = 0$ for $n \gg 0$.
(2) If $H^n(K^\bullet) = 0$ for $n \gg 0$ then there exists a quasi-isomorphism $L^\bullet \to K^\bullet$ with $L^\bullet$ bounded above.

Proof. Dual to Lemma 13.18.2. □

0646 Lemma 13.19.3. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives.

(1) Any object of $\mathcal{A}$ has a projective resolution.
(2) If $H^n(K^\bullet) = 0$ for all $n \gg 0$ then $K^\bullet$ has a projective resolution.
(3) If $K^\bullet$ is a complex with $K^n = 0$ for $n > a$, then there exists a projective resolution $\alpha : P^\bullet \to K^\bullet$ with $P^n = 0$ for $n > a$ such that each $\alpha^n : P^n \to K^n$ is surjective.

Proof. Dual to Lemma 13.18.3. □

0647 Lemma 13.19.4. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $P^\bullet$ be bounded above and consisting of projective objects. Any morphism $P^\bullet \to K^\bullet$ is homotopic to zero.

Proof. Dual to Lemma 13.18.4. □

0648 Remark 13.19.5. Let $\mathcal{A}$ be an abelian category. Suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Let $P^\bullet$ be a bounded above complex of projectives. Then

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, L^\bullet)$$

is an isomorphism. This is dual to Remark 13.18.5.

0649 Lemma 13.19.6. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet & L^\bullet \arrow[l, "\alpha"'] \\ P^\bullet \arrow[u] \arrow[ru, dashed, "\beta"'] & \end{tikzcd}$$

where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism.

(1) There exists a map of complexes β making the diagram commute up to homotopy.
(2) If α is surjective in every degree then we can find a β which makes the diagram commute.

Proof. Dual to Lemma 13.18.6. □

064A Lemma 13.19.7. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{tikzcd} K^\bullet & L^\bullet \arrow[l, "\alpha"'] \\ P^\bullet \arrow[u] \arrow[ru, dashed, "\beta_i"'] & \end{tikzcd}$$

where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. Dual to Lemma 13.18.7. □

064B Lemma 13.19.8. Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$ be bounded above complex consisting of projective objects. Let $L^\bullet \in K(\mathcal{A})$. Then

$$\mathrm{Mor}_{K(\mathcal{A})}(P^\bullet, L^\bullet) = \mathrm{Mor}_{D(\mathcal{A})}(P^\bullet, L^\bullet).$$

Proof. Dual to Lemma 13.18.8. □

064C Lemma 13.19.9. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of $\mathrm{Comp}^+(\mathcal{A})$ there exists a commutative diagram in $\mathrm{Comp}^+(\mathcal{A})$

$$\begin{tikzcd}
0 \arrow[r] & P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & P_3^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are projective resolutions and the rows are short exact sequences of complexes. In fact, given any projective resolution $P^\bullet \to C^\bullet$ we may assume $P_3^\bullet = P^\bullet$.

Proof. Dual to Lemma 13.18.9. □

064D Lemma 13.19.10. Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$, $K^\bullet$ be complexes. Let $n \in \mathbf{Z}$. Assume that

(1) $P^\bullet$ is a bounded complex consisting of projective objects,
(2) $P^i = 0$ for $i < n$, and
(3) $H^i(K^\bullet) = 0$ for $i \geq n$.

Then $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = 0$.

Proof. The first equality follows from Lemma 13.19.8. Note that there is a distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, \tau_{\geq n}K^\bullet, f, g, h)$$

by Remark 13.12.4. Hence, by Lemma 13.4.2 it suffices to prove $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\leq n-1}K^\bullet) = 0$ and $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\geq n}K^\bullet) = 0$. The first vanishing is trivial and the second is Lemma 13.19.4. □

064E Lemma 13.19.11. Let $\mathcal{A}$ be an abelian category. Let $\beta : P^\bullet \to L^\bullet$ and $\alpha : E^\bullet \to L^\bullet$ be maps of complexes. Let $n \in \mathbf{Z}$. Assume

(1) $P^\bullet$ is a bounded complex of projectives and $P^i = 0$ for $i < n$,
(2) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$.

Then there exists a map of complexes $\gamma : P^\bullet \to E^\bullet$ such that $\alpha \circ \gamma$ and β are homotopic.

Proof. Consider the cone $C^\bullet = C(\alpha)^\bullet$ with map $i : L^\bullet \to C^\bullet$. Note that $i \circ \beta$ is zero by Lemma 13.19.10. Hence we can lift β to $E^\bullet$ by Lemma 13.4.2. □

13.20. Right derived functors and injective resolutions

0156 At this point we can use the material above to define the right derived functors of an additive functor between an abelian category having enough injectives and a general abelian category.

05TH **Lemma 13.20.1.** *Let $\mathcal{A}$ be an abelian category. Let $I \in \mathrm{Ob}(\mathcal{A})$ be an injective object. Let $I^\bullet$ be a bounded below complex of injectives in $\mathcal{A}$.*

1. *$I^\bullet$ computes RF relative to $\mathrm{Qis}^+(\mathcal{A})$ for any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into any triangulated category $\mathcal{D}$.*
2. *I is right acyclic for any additive functor $F : \mathcal{A} \to \mathcal{B}$ into any abelian category $\mathcal{B}$.*

Proof. Part (2) is a direct consequences of part (1) and Definition 13.15.3. To prove (1) let $\alpha : I^\bullet \to K^\bullet$ be a quasi-isomorphism into a complex. By Lemma 13.18.6 we see that α has a left inverse. Hence the category $I^\bullet/\mathrm{Qis}^+(\mathcal{A})$ is essentially constant with value $\mathrm{id} : I^\bullet \to I^\bullet$. Thus also the ind-object

$$I^\bullet/\mathrm{Qis}^+(\mathcal{A}) \longrightarrow \mathcal{D}, \quad (I^\bullet \to K^\bullet) \longmapsto F(K^\bullet)$$

is essentially constant with value $F(I^\bullet)$. This proves (1), see Definitions 13.14.2 and 13.14.10. $\square$

05TI **Lemma 13.20.2.** *Let $\mathcal{A}$ be an abelian category with enough injectives.*

1. *For any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into a triangulated category $\mathcal{D}$ the right derived functor*
$$RF : D^+(\mathcal{A}) \longrightarrow \mathcal{D}$$
is everywhere defined.
2. *For any additive functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ the right derived functor*
$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$
is everywhere defined.

Proof. Combine Lemma 13.20.1 and Proposition 13.16.8 for the second assertion. To see the first assertion combine Lemma 13.18.3, Lemma 13.20.1, Lemma 13.14.14, and Equation (13.14.9.1). $\square$

0159 **Lemma 13.20.3.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor.*

1. *The functor RF is an exact functor $D^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
2. *The functor RF induces an exact functor $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
3. *The functor RF induces a δ-functor $\mathrm{Comp}^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
4. *The functor RF induces a δ-functor $\mathcal{A} \to D^+(\mathcal{B})$.*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 13.20.2 RF is everywhere defined. Here are some references:

1. The derived functor is exact: This boils down to Lemma 13.14.6.
2. This is true because $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is exact and compositions of exact functors are exact.
3. This is true because $\mathrm{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 13.12.1.

(4) This is true because $\mathcal{A} \to \text{Comp}^+(\mathcal{A})$ is exact and precomposing a δ-functor by an exact functor gives a δ-functor.

$\square$

015B Lemma 13.20.4. Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor.

(1) For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of complexes in $\text{Comp}^+(\mathcal{A})$ there is an associated long exact sequence

$$\ldots \to H^i(RF(A^\bullet)) \to H^i(RF(B^\bullet)) \to H^i(RF(C^\bullet)) \to H^{i+1}(RF(A^\bullet)) \to \ldots$$

(2) The functors $R^iF : \mathcal{A} \to \mathcal{B}$ are zero for $i < 0$. Also $R^0F = F : \mathcal{A} \to \mathcal{B}$.
(3) We have $R^iF(I) = 0$ for $i > 0$ and I injective.
(4) The sequence (R^iF, δ) forms a universal δ-functor (see Homology, Definition 12.12.3) from $\mathcal{A}$ to $\mathcal{B}$.

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 13.20.2 RF is everywhere defined. Here are some references:

(1) This follows from Lemma 13.20.3 part (3) combined with the long exact cohomology sequence (13.11.1.1) for $D^+(\mathcal{B})$.
(2) This is Lemma 13.16.3.
(3) This is the fact that injective objects are acyclic.
(4) This is Lemma 13.16.6.

$\square$

13.21. Cartan-Eilenberg resolutions

015G This section can be expanded. The material can be generalized and applied in more cases. Resolutions need not use injectives and the method also works in the unbounded case in some situations.

015H Definition 13.21.1. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a bounded below complex. A Cartan-Eilenberg resolution of $K^\bullet$ is given by a double complex $I^{\bullet,\bullet}$ and a morphism of complexes $\epsilon : K^\bullet \to I^{\bullet,0}$ with the following properties:

(1) There exists a $i \ll 0$ such that $I^{p,q} = 0$ for all $p < i$ and all q.
(2) We have $I^{p,q} = 0$ if $q < 0$.
(3) The complex $I^{p,\bullet}$ is an injective resolution of K^p.
(4) The complex $\text{Ker}(d_1^{p,\bullet})$ is an injective resolution of $\text{Ker}(d_K^p)$.
(5) The complex $\text{Im}(d_1^{p,\bullet})$ is an injective resolution of $\text{Im}(d_K^p)$.
(6) The complex $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$.

015I Lemma 13.21.2. Let $\mathcal{A}$ be an abelian category with enough injectives. Let $K^\bullet$ be a bounded below complex. There exists a Cartan-Eilenberg resolution of $K^\bullet$.

Proof. Suppose that $K^p = 0$ for $p < n$. Decompose $K^\bullet$ into short exact sequences as follows: Set $Z^p = \text{Ker}(d^p)$, $B^p = \text{Im}(d^{p-1})$, $H^p = Z^p/B^p$, and consider

$$\begin{gathered} 0 \to Z^n \to K^n \to B^{n+1} \to 0 \\ 0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0 \\ 0 \to Z^{n+1} \to K^{n+1} \to B^{n+2} \to 0 \\ 0 \to B^{n+2} \to Z^{n+2} \to H^{n+2} \to 0 \\ \ldots \end{gathered}$$

Set $I^{p,q} = 0$ for $p < n$. Inductively we choose injective resolutions as follows:

(1) Choose an injective resolution $Z^n \to J_Z^{n,\bullet}$.
(2) Using Lemma 13.18.9 choose injective resolutions $K^n \to I^{n,\bullet}$, $B^{n+1} \to J_B^{n+1,\bullet}$, and an exact sequence of complexes $0 \to J_Z^{n,\bullet} \to I^{n,\bullet} \to J_B^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to Z^n \to K^n \to B^{n+1} \to 0$.
(3) Using Lemma 13.18.9 choose injective resolutions $Z^{n+1} \to J_Z^{n+1,\bullet}$, $H^{n+1} \to J_H^{n+1,\bullet}$, and an exact sequence of complexes $0 \to J_B^{n+1,\bullet} \to J_Z^{n+1,\bullet} \to J_H^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0$.
(4) Etc.

Taking as maps $d_1^\bullet : I^{p,\bullet} \to I^{p+1,\bullet}$ the compositions $I^{p,\bullet} \to J_B^{p+1,\bullet} \to J_Z^{p+1,\bullet} \to I^{p+1,\bullet}$ everything is clear. □

015J Lemma 13.21.3. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution for $K^\bullet$. The spectral sequences $({}'E_r, {}'d_r)_{r\geq 0}$ and $({}''E_r, {}''d_r)_{r\geq 0}$ associated to the double complex $F(I^{\bullet,\bullet})$ satisfy the relations

$$ {}'E_1^{p,q} = R^qF(K^p) \quad \text{and} \quad {}''E_2^{p,q} = R^pF(H^q(K^\bullet)) $$

Moreover, these spectral sequences are bounded, converge to $H^*(RF(K^\bullet))$, and the associated induced filtrations on $H^n(RF(K^\bullet))$ are finite.

Proof. We will use the following remarks without further mention:
(1) As $I^{p,\bullet}$ is an injective resolution of K^p we see that RF is defined at $K^p[0]$ with value $F(I^{p,\bullet})$.
(2) As $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$ the derived functor RF is defined at $H^p(K^\bullet)[0]$ with value $F(H_I^p(I^{\bullet,\bullet}))$.
(3) By Homology, Lemma 12.25.4 the total complex $\text{Tot}(I^{\bullet,\bullet})$ is an injective resolution of $K^\bullet$. Hence RF is defined at $K^\bullet$ with value $F(\text{Tot}(I^{\bullet,\bullet}))$.

Consider the two spectral sequences associated to the double complex $L^{\bullet,\bullet} = F(I^{\bullet,\bullet})$, see Homology, Lemma 12.25.1. These are both bounded, converge to $H^*(\text{Tot}(L^{\bullet,\bullet}))$, and induce finite filtrations on $H^n(\text{Tot}(L^{\bullet,\bullet}))$, see Homology, Lemma 12.25.3. Since $\text{Tot}(L^{\bullet,\bullet}) = \text{Tot}(F(I^{\bullet,\bullet})) = F(\text{Tot}(I^{\bullet,\bullet}))$ computes $H^n(RF(K^\bullet))$ we find the final assertion of the lemma holds true.

Computation of the first spectral sequence. We have ${}'E_1^{p,q} = H^q(L^{p,\bullet})$ in other words

$$ {}'E_1^{p,q} = H^q(F(I^{p,\bullet})) = R^qF(K^p) $$

as desired. Observe for later use that the maps ${}'d_1^{p,q} : {}'E_1^{p,q} \to {}'E_1^{p+1,q}$ are the maps $R^qF(K^p) \to R^qF(K^{p+1})$ induced by $K^p \to K^{p+1}$ and the fact that R^qF is a functor.

Computation of the second spectral sequence. We have ${}''E_1^{p,q} = H^q(L^{\bullet,p}) = H^q(F(I^{\bullet,p}))$. Note that the complex $I^{\bullet,p}$ is bounded below, consists of injectives, and moreover each kernel, image, and cohomology group of the differentials is an injective object of $\mathcal{A}$. Hence we can split the differentials, i.e., each differential is a split surjection onto a direct summand. It follows that the same is true after applying F. Hence ${}''E_1^{p,q} = F(H^q(I^{\bullet,p})) = F(H_I^q(I^{\bullet,p}))$. The differentials on this are $(-1)^q$ times F applied to the differential of the complex $H_I^p(I^{\bullet,\bullet})$ which is an injective resolution of $H^p(K^\bullet)$. Hence the description of the E_2 terms. □

015K Remark 13.21.4. The spectral sequences of Lemma 13.21.3 are functorial in the complex $K^\bullet$. This follows from functoriality properties of Cartan-Eilenberg resolutions. On the other hand, they are both examples of a more general spectral sequence which may be associated to a filtered complex of $\mathcal{A}$. The functoriality will follow from its construction. We will return to this in the section on the filtered derived category, see Remark 13.26.15.

13.22. Composition of right derived functors

015L Sometimes we can compute the right derived functor of a composition. Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that the right derived functors $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, $RG : D^+(\mathcal{B}) \to D^+(\mathcal{C})$, and $R(G \circ F) : D^+(\mathcal{A}) \to D^+(\mathcal{C})$ are everywhere defined. Then there exists a canonical transformation

$$t : R(G \circ F) \longrightarrow RG \circ RF$$

of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$, see Lemma 13.14.16. This transformation need not always be an isomorphism.

015M Lemma 13.22.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume $\mathcal{A}$, $\mathcal{B}$ have enough injectives. The following are equivalent

(1) $F(I)$ is right acyclic for G for each injective object I of $\mathcal{A}$, and
(2) the canonical map

$$t : R(G \circ F) \longrightarrow RG \circ RF.$$

is isomorphism of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$.

Proof. If (2) holds, then (1) follows by evaluating the isomorphism t on $RF(I) = F(I)$. Conversely, assume (1) holds. Let $A^\bullet$ be a bounded below complex of $\mathcal{A}$. Choose an injective resolution $A^\bullet \to I^\bullet$. The map t is given (see proof of Lemma 13.14.16) by the maps

$$R(G \circ F)(A^\bullet) = (G \circ F)(I^\bullet) = G(F(I^\bullet))) \to RG(F(I^\bullet)) = RG(RF(A^\bullet))$$

where the arrow is an isomorphism by Lemma 13.16.7. □

015N Lemma 13.22.2 (Grothendieck spectral sequence). With assumptions as in Lemma 13.22.1 and assuming the equivalent conditions (1) and (2) hold. Let X be an object of $D^+(\mathcal{A})$. There exists a spectral sequence $(E_r, d_r)_{r\geq 0}$ consisting of bigraded objects E_r of $\mathcal{C}$ with d_r of bidegree $(r, -r+1)$ and with

$$E_2^{p,q} = R^pG(H^q(RF(X)))$$

Moreover, this spectral sequence is bounded, converges to $H^*(R(G \circ F)(X))$, and induces a finite filtration on each $H^n(R(G \circ F)(X))$.

For an object A of $\mathcal{A}$ we get $E_2^{p,q} = R^pG(R^qF(A))$ converging to $R^{p+q}(G \circ F)(A)$.

Proof. We may represent X by a bounded below complex $A^\bullet$. Choose an injective resolution $A^\bullet \to I^\bullet$. Choose a Cartan-Eilenberg resolution $F(I^\bullet) \to I^{\bullet,\bullet}$ using Lemma 13.21.2. Apply the second spectral sequence of Lemma 13.21.3. □

13.23. Resolution functors

013U Let $\mathcal{A}$ be an abelian category with enough injectives. Denote $\mathcal{I}$ the full additive subcategory of $\mathcal{A}$ whose objects are the injective objects of $\mathcal{A}$. It turns out that $K^+(\mathcal{I})$ and $D^+(\mathcal{A})$ are equivalent in this case (see Proposition 13.23.1). For many purposes it therefore makes sense to think of $D^+(\mathcal{A})$ as the (easier to grok) category $K^+(\mathcal{I})$ in this case.

013V Proposition 13.23.1. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Denote $\mathcal{I} \subset \mathcal{A}$ the strictly full additive subcategory whose objects are the injective objects of $\mathcal{A}$. The functor

$$K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories.

Proof. It is clear that the functor is exact. It is essentially surjective by Lemma 13.18.3. Fully faithfulness is a consequence of Lemma 13.18.8. $\square$

Proposition 13.23.1 implies that we can find resolution functors. It turns out that we can prove resolution functors exist even in some cases where the abelian category $\mathcal{A}$ is a "big" category, i.e., has a class of objects.

013W Definition 13.23.2. Let $\mathcal{A}$ be an abelian category with enough injectives. A resolution functor[7] for $\mathcal{A}$ is given by the following data:

(1) for all $K^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ a bounded below complex of injectives $j(K^\bullet)$, and
(2) for all $K^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ a quasi-isomorphism $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$.

05TJ Lemma 13.23.3. Let $\mathcal{A}$ be an abelian category with enough injectives. Given a resolution functor (j, i) there is a unique way to turn j into a functor and i into a 2-isomorphism producing a 2-commutative diagram

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[rr, "j"'] \arrow[dr] & & K^+(\mathcal{I}) \arrow[dl] \\ & D^+(\mathcal{A}) & \end{tikzcd}$$

where $\mathcal{I}$ is the full additive subcategory of $\mathcal{A}$ consisting of injective objects.

Proof. For every morphism $\alpha : K^\bullet \to L^\bullet$ of $K^+(\mathcal{A})$ there is a unique morphism $j(\alpha) : j(K^\bullet) \to j(L^\bullet)$ in $K^+(\mathcal{I})$ such that

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"'] \arrow[d, "i_{K^\bullet}"'] & L^\bullet \arrow[d, "i_{L^\bullet}"] \\ j(K^\bullet) \arrow[r, "j(\alpha)"] & j(L^\bullet) \end{tikzcd}$$

is commutative in $K^+(\mathcal{A})$. To see this either use Lemmas 13.18.6 and 13.18.7 or the equivalent Lemma 13.18.8. The uniqueness implies that j is a functor, and the commutativity of the diagram implies that i gives a 2-morphism which witnesses the 2-commutativity of the diagram of categories in the statement of the lemma. $\square$

[7]This is likely nonstandard terminology.

013X Lemma 13.23.4. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Then a resolution functor j exists and is unique up to unique isomorphism of functors.

Proof. Consider the set of all objects $K^\bullet$ of $K^+(\mathcal{A})$. (Recall that by our conventions any category has a set of objects unless mentioned otherwise.) By Lemma 13.18.3 every object has an injective resolution. By the axiom of choice we can choose for each $K^\bullet$ an injective resolution $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$. □

014W Lemma 13.23.5. Let $\mathcal{A}$ be an abelian category with enough injectives. Any resolution functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is exact.

Proof. Denote $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$ the canonical maps of Definition 13.23.2. First we discuss the existence of the functorial isomorphism $j(K^\bullet[1]) \to j(K^\bullet)[1]$. Consider the diagram

$$\begin{tikzcd} K^\bullet[1] \arrow[r, equal] \arrow[d, "i_{K^\bullet[1]}"] & K^\bullet[1] \arrow[d, "i_{K^\bullet}[1]"] \\ j(K^\bullet[1]) \arrow[r, dotted, "\xi_{K^\bullet}"] & j(K^\bullet)[1] \end{tikzcd}$$

By Lemmas 13.18.6 and 13.18.7 there exists a unique dotted arrow $\xi_{K^\bullet}$ in $K^+(\mathcal{I})$ making the diagram commute in $K^+(\mathcal{A})$. We omit the verification that this gives a functorial isomorphism. (Hint: use Lemma 13.18.7 again.)

Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle of $K^+(\mathcal{A})$. We have to show that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle of $K^+(\mathcal{I})$. Note that we have a commutative diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "f"'] \arrow[d] & L^\bullet \arrow[r, "g"'] \arrow[d] & M^\bullet \arrow[r, "h"'] \arrow[d] & K^\bullet[1] \arrow[d] \\ j(K^\bullet) \arrow[r, "j(f)"] & j(L^\bullet) \arrow[r, "j(g)"] & j(M^\bullet) \arrow[r, "\xi_{K^\bullet} \circ j(h)"] & j(K^\bullet)[1] \end{tikzcd}$$

in $K^+(\mathcal{A})$ whose vertical arrows are the quasi-isomorphisms i_K, i_L, i_M. Hence we see that the image of $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ in $D^+(\mathcal{A})$ is isomorphic to a distinguished triangle and hence a distinguished triangle by TR1. Thus we see from Lemma 13.4.18 that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle in $K^+(\mathcal{I})$. □

05TK Lemma 13.23.6. Let $\mathcal{A}$ be an abelian category which has enough injectives. Let j be a resolution functor. Write $Q : K^+(\mathcal{A}) \to D^+(\mathcal{A})$ for the natural functor. Then $j = j' \circ Q$ for a unique functor $j' : D^+(\mathcal{A}) \to K^+(\mathcal{I})$ which is quasi-inverse to the canonical functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$.

Proof. By Lemma 13.11.6 Q is a localization functor. To prove the existence of j' it suffices to show that any element of $\text{Qis}^+(\mathcal{A})$ is mapped to an isomorphism under the functor j, see Lemma 13.5.7. This is true by the remarks following Definition 13.23.2. □

013Y Remark 13.23.7. Suppose that $\mathcal{A}$ is a "big" abelian category with enough injectives such as the category of abelian groups. In this case we have to be slightly more careful in constructing our resolution functor since we cannot use the axiom of choice with a quantifier ranging over a class. But note that the proof of the lemma does show that any two localization functors are canonically isomorphic. Namely, given quasi-isomorphisms $i : K^\bullet \to I^\bullet$ and $i' : K^\bullet \to J^\bullet$ of a bounded below complex $K^\bullet$ into bounded below complexes of injectives there exists a unique(!) morphism $a : I^\bullet \to J^\bullet$ in $K^+(\mathcal{I})$ such that $i' = i \circ a$ as morphisms in $K^+(\mathcal{I})$. Hence the only issue is existence, and we will see how to deal with this in the next section.

13.24. Functorial injective embeddings and resolution functors

0140 In this section we redo the construction of a resolution functor $K^+(\mathcal{A}) \to K^+(\mathcal{I})$ in case the category $\mathcal{A}$ has functorial injective embeddings. There are two reasons for this: (1) the proof is easier and (2) the construction also works if $\mathcal{A}$ is a "big" abelian category. See Remark 13.24.3 below.

Let $\mathcal{A}$ be an abelian category. As before denote $\mathcal{I}$ the additive full subcategory of $\mathcal{A}$ consisting of injective objects. Consider the category $\text{InjRes}(\mathcal{A})$ of arrows $\alpha : K^\bullet \to I^\bullet$ where $K^\bullet$ is a bounded below complex of $\mathcal{A}$, $I^\bullet$ is a bounded below complex of injectives of $\mathcal{A}$ and α is a quasi-isomorphism. In other words, α is an injective resolution and $K^\bullet$ is bounded below. There is an obvious functor

$$s : \text{InjRes}(\mathcal{A}) \longrightarrow \text{Comp}^+(\mathcal{A})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto K^\bullet$. There is also a functor

$$t : \text{InjRes}(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto I^\bullet$.

0141 **Lemma 13.24.1.** Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has functorial injective embeddings, see Homology, Definition 12.27.5.

(1) There exists a functor $inj : \text{Comp}^+(\mathcal{A}) \to \text{InjRes}(\mathcal{A})$ such that $s \circ inj = \text{id}$.
(2) For any functor $inj : \text{Comp}^+(\mathcal{A}) \to \text{InjRes}(\mathcal{A})$ such that $s \circ inj = \text{id}$ we obtain a resolution functor, see Definition 13.23.2.

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective embedding, see Homology, Definition 12.27.5. We first note that we may assume $J(0) = 0$. Namely, if not then for any object A we have $0 \to A \to 0$ which gives a direct sum decomposition $J(A) = J(0) \oplus \text{Ker}(J(A) \to J(0))$. Note that the functorial morphism $A \to J(A)$ has to map into the second summand. Hence we can replace our functor by $J'(A) = \text{Ker}(J(A) \to J(0))$ if needed.

Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Say $K^p = 0$ if $p < B$. We are going to construct a double complex $I^{\bullet,\bullet}$ of injectives, together with a map $\alpha : K^\bullet \to I^{\bullet,0}$ such that α induces a quasi-isomorphism of $K^\bullet$ with the associated total complex of $I^{\bullet,\bullet}$. First we set $I^{p,q} = 0$ whenever $q < 0$. Next, we set $I^{p,0} = J(K^p)$ and $\alpha^p : K^p \to I^{p,0}$ the functorial embedding. Since J is a functor we see that $I^{\bullet,0}$ is a complex and that α is a morphism of complexes. Each α^p is injective. And $I^{p,0} = 0$ for $p < B$ because $J(0) = 0$. Next, we set $I^{p,1} = J(\text{Coker}(K^p \to I^{p,0}))$. Again by functoriality we see that $I^{\bullet,1}$ is a complex. And again we get that $I^{p,1} = 0$ for $p < B$. It is also clear that K^p maps isomorphically onto $\text{Ker}(I^{p,0} \to I^{p,1})$. As our third step we take $I^{p,2} = J(\text{Coker}(I^{p,0} \to I^{p,1}))$. And so on and so forth.

At this point we can apply Homology, Lemma 12.25.4 to get that the map

$$\alpha : K^\bullet \longrightarrow \text{Tot}(I^{\bullet,\bullet})$$

is a quasi-isomorphism. To prove we get a functor inj it rests to show that the construction above is functorial. This verification is omitted.

Suppose we have a functor inj such that $s \circ inj = \text{id}$. For every object $K^\bullet$ of $\text{Comp}^+(\mathcal{A})$ we can write

$$inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$$

This provides us with a resolution functor as in Definition 13.23.2. □

05TL Remark 13.24.2. Suppose inj is a functor such that $s \circ inj = \text{id}$ as in part (2) of Lemma 13.24.1. Write $inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$ as in the proof of that lemma. Suppose $\alpha : K^\bullet \to L^\bullet$ is a map of bounded below complexes. Consider the map $inj(\alpha)$ in the category $\text{InjRes}(\mathcal{A})$. It induces a commutative diagram

$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d, "i_K"'] & L^\bullet \arrow[d, "i_L"] \\ j(K)^\bullet \arrow[r, "inj(\alpha)"] & j(L)^\bullet \end{tikzcd}$$

of morphisms of complexes. Hence, looking at the proof of Lemma 13.23.3 we see that the functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is given by the rule

$$j(\alpha \text{ up to homotopy}) = inj(\alpha) \text{ up to homotopy} \in \text{Hom}_{K^+(\mathcal{I})}(j(K^\bullet), j(L^\bullet))$$

Hence we see that j matches $t \circ inj$ in this case, i.e., the diagram

$$\begin{tikzcd} \text{Comp}^+(\mathcal{A}) \arrow[rr, "t \circ inj"'] \arrow[dr] & & K^+(\mathcal{I}) \\ & K^+(\mathcal{A}) \arrow[ur, "j"'] & \end{tikzcd}$$

is commutative.

0142 Remark 13.24.3. Let $\text{Mod}(\mathcal{O}_X)$ be the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ (or more generally on a ringed site). We will see later that $\text{Mod}(\mathcal{O}_X)$ has enough injectives and in fact functorial injective embeddings, see Injectives, Theorem 19.8.4. Note that the proof of Lemma 13.23.4 does not apply to $\text{Mod}(\mathcal{O}_X)$. But the proof of Lemma 13.24.1 does apply to $\text{Mod}(\mathcal{O}_X)$. Thus we obtain

$$j : K^+(\text{Mod}(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

which is a resolution functor where $\mathcal{I}$ is the additive category of injective $\mathcal{O}_X$-modules. This argument also works in the following cases:

(1) The category Mod_R of R-modules over a ring R.
(2) The category $\text{PMod}(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules on a site endowed with a presheaf of rings.
(3) The category $\text{Mod}(\mathcal{O})$ of sheaves of $\mathcal{O}$-modules on a ringed site.
(4) Add more here as needed.

13.25. Right derived functors via resolution functors

05TM The content of the following lemma is that we can simply define $RF(K^\bullet) = F(j(K^\bullet))$ if we are given a resolution functor j.

05TN Lemma 13.25.1. Let $\mathcal{A}$ be an abelian category with enough injectives Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor into an abelian category. Let (i, j) be a resolution functor, see Definition 13.23.2. The right derived functor RF of F fits into the following 2-commutative diagram

$$\begin{tikzcd} D^+(\mathcal{A}) \arrow[rr, "j'"] \arrow[dr, "RF"'] & & K^+(\mathcal{I}) \arrow[dl, "F"] \\ & D^+(\mathcal{B}) & \end{tikzcd}$$

where j' is the functor from Lemma 13.23.6.

Proof. By Lemma 13.20.1 we have $RF(K^\bullet) = F(j(K^\bullet))$. □

0158 Remark 13.25.2. In the situation of Lemma 13.25.1 we see that we have actually lifted the right derived functor to an exact functor $F \circ j' : D^+(\mathcal{A}) \to K^+(\mathcal{B})$. It is occasionally useful to use such a factorization.

13.26. Filtered derived category and injective resolutions

015O Let $\mathcal{A}$ be an abelian category. In this section we will show that if $\mathcal{A}$ has enough injectives, then so does the category $\text{Fil}^f(\mathcal{A})$ in some sense. One can use this observation to compute in the filtered derived category of $\mathcal{A}$.

The category $\text{Fil}^f(\mathcal{A})$ is an example of an exact category, see Injectives, Remark 19.9.6. A special role is played by the strict morphisms, see Homology, Definition 12.19.3, i.e., the morphisms f such that $\text{Coim}(f) = \text{Im}(f)$. We will say that a complex $A \to B \to C$ in $\text{Fil}^f(\mathcal{A})$ is exact if the sequence $\text{gr}(A) \to \text{gr}(B) \to \text{gr}(C)$ is exact in $\mathcal{A}$. This implies that $A \to B$ and $B \to C$ are strict morphisms, see Homology, Lemma 12.19.15.

015P Definition 13.26.1. Let $\mathcal{A}$ be an abelian category. We say an object I of $\text{Fil}^f(\mathcal{A})$ is filtered injective if each $\text{gr}^p(I)$ is an injective object of $\mathcal{A}$.

05TP Lemma 13.26.2. Let $\mathcal{A}$ be an abelian category. An object I of $\text{Fil}^f(\mathcal{A})$ is filtered injective if and only if there exist $a \leq b$, injective objects I_n, $a \leq n \leq b$ of $\mathcal{A}$ and an isomorphism $I \cong \bigoplus_{a \leq n \leq b} I_n$ such that $F^pI = \bigoplus_{n \geq p} I_n$.

Proof. Follows from the fact that any injection $J \to M$ of $\mathcal{A}$ is split if J is an injective object. Details omitted. □

05TQ Lemma 13.26.3. Let $\mathcal{A}$ be an abelian category. Any strict monomorphism $u : I \to A$ of $\text{Fil}^f(\mathcal{A})$ where I is a filtered injective object is a split injection.

Proof. Let p be the largest integer such that $F^pI \neq 0$. In particular $\text{gr}^p(I) = F^pI$. Let I' be the object of $\text{Fil}^f(\mathcal{A})$ whose underlying object of $\mathcal{A}$ is F^pI and with filtration given by $F^nI' = 0$ for $n > p$ and $F^nI' = I' = F^pI$ for $n \leq p$. Note that $I' \to I$ is a strict monomorphism too. The fact that u is a strict monomorphism implies that $F^pI \to A/F^{p+1}(A)$ is injective, see Homology, Lemma 12.19.13. Choose a splitting $s : A/F^{p+1}A \to F^pI$ in $\mathcal{A}$. The induced morphism $s' : A \to I'$ is a strict morphism of filtered objects splitting the composition $I' \to I \to A$. Hence we can write $A = I' \oplus \text{Ker}(s')$ and $I = I' \oplus \text{Ker}(s'|_I)$. Note that $\text{Ker}(s'|_I) \to \text{Ker}(s')$ is a strict monomorphism and that $\text{Ker}(s'|_I)$ is a filtered injective object.

By induction on the length of the filtration on I the map $\text{Ker}(s'|_I) \to \text{Ker}(s')$ is a split injection. Thus we win. $\square$

05TR Lemma 13.26.4. Let $\mathcal{A}$ be an abelian category. Let $u : A \to B$ be a strict monomorphism of $\text{Fil}^f(\mathcal{A})$ and $f : A \to I$ a morphism from A into a filtered injective object in $\text{Fil}^f(\mathcal{A})$. Then there exists a morphism $g : B \to I$ such that $f = g \circ u$.

Proof. The pushout $f' : I \to I \amalg_A B$ of f by u is a strict monomorphism, see Homology, Lemma 12.19.10. Hence the result follows formally from Lemma 13.26.3. $\square$

05TS Lemma 13.26.5. Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $\text{Fil}^f(\mathcal{A})$ there exists a strict monomorphism $A \to I$ where I is a filtered injective object.

Proof. Pick $a \leq b$ such that $\text{gr}^p(A) = 0$ unless $p \in \{a, a+1, \ldots, b\}$. For each $n \in \{a, a+1, \ldots, b\}$ choose an injection $u_n : A/F^{n+1}A \to I_n$ with I_n an injective object. Set $I = \bigoplus_{a \leq n \leq b} I_n$ with filtration $F^pI = \bigoplus_{n \geq p} I_n$ and set $u : A \to I$ equal to the direct sum of the maps u_n. $\square$

05TT Lemma 13.26.6. Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $\text{Fil}^f(\mathcal{A})$ there exists a filtered quasi-isomorphism $A[0] \to I^\bullet$ where $I^\bullet$ is a complex of filtered injective objects with $I^n = 0$ for $n < 0$.

Proof. First choose a strict monomorphism $u_0 : A \to I^0$ of A into a filtered injective object, see Lemma 13.26.5. Next, choose a strict monomorphism $u_1 : \text{Coker}(u_0) \to I^1$ into a filtered injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose a strict monomorphism $u_2 : \text{Coker}(u_1) \to I^2$ into a filtered injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. This works because each of the sequences

$$0 \to \text{Coker}(u_n) \to I^{n+1} \to \text{Coker}(u_{n+1}) \to 0$$

is short exact, i.e., induces a short exact sequence on applying gr. To see this use Homology, Lemma 12.19.13. $\square$

05TU Lemma 13.26.7. Let $\mathcal{A}$ be an abelian category with enough injectives. Let $f : A \to B$ be a morphism of $\text{Fil}^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $B[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram

$$\begin{tikzcd} A[0] \arrow[r] \arrow[d] & B[0] \arrow[d] \\ I^\bullet \arrow[r] & J^\bullet \end{tikzcd}$$

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $b : B \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 12.19.15. We will inductively construct the maps f^n in the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"'] \arrow[d, "f"'] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[r] \arrow[d, "f^1"] & I^2 \arrow[r] \arrow[d, "f^2"] & \ldots \\ B \arrow[r, "b"] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \ldots \end{tikzcd}$$

Because $A \to I^0$ is a strict monomorphism and because J^0 is filtered injective, we can find a morphism $f^0 : I^0 \to J^0$ such that $f^0 \circ a = b \circ f$, see Lemma 13.26.4. The composition $d_J^0 \circ b \circ f$ is zero, hence $d_J^0 \circ f^0 \circ a = 0$, hence $d_J^0 \circ f^0$ factors through a unique morphism

$$\mathrm{Coker}(a) = \mathrm{Coim}(d_I^0) = \mathrm{Im}(d_I^0) \longrightarrow J^1.$$

As $\mathrm{Im}(d_I^0) \to I^1$ is a strict monomorphism we can extend the displayed arrow to a morphism $f^1 : I^1 \to J^1$ by Lemma 13.26.4 again. And so on. $\square$

05TV Lemma 13.26.8. Let $\mathcal{A}$ be an abelian category with enough injectives. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathrm{Fil}^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A[0] \arrow[r] \arrow[d] & B[0] \arrow[r] \arrow[d] & C[0] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^\bullet \arrow[r] & M^\bullet \arrow[r] & J^\bullet \arrow[r] & 0 \end{tikzcd}$$

where the lower row is a termwise split sequence of complexes.

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $c : C \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 13.13.4. We are going to step by step construct the south-east and the south arrows in the following commutative diagram

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[dr, "b"] & C \arrow[r, "c"] \arrow[dr, "\overline{b}"] & J^0 \arrow[r] \arrow[d, "\delta^0"] & J^1 \arrow[r] \arrow[d, "\delta^1"] & \ldots \\ A \arrow[u, "\alpha"] \arrow[r, "a"] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \ldots \end{tikzcd}$$

As $A \to B$ is a strict monomorphism, we can find a morphism $b : B \to I^0$ such that $b \circ \alpha = a$, see Lemma 13.26.4. As A is the kernel of the strict morphism $I^0 \to I^1$ and $\beta = \mathrm{Coker}(\alpha)$ we obtain a unique morphism $\overline{b} : C \to I^1$ fitting into the diagram. As c is a strict monomorphism and I^1 is filtered injective we can find $\delta^0 : J^0 \to I^1$, see Lemma 13.26.4. Because $B \to C$ is a strict epimorphism and because $B \to I^0 \to I^1 \to I^2$ is zero, we see that $C \to I^1 \to I^2$ is zero. Hence $d_I^1 \circ \delta^0$ is zero on $C \cong \mathrm{Im}(c)$. Hence $d_I^1 \circ \delta^0$ factors through a unique morphism

$$\mathrm{Coker}(c) = \mathrm{Coim}(d_J^0) = \mathrm{Im}(d_J^0) \longrightarrow I^2.$$

As I^2 is filtered injective and $\mathrm{Im}(d_J^0) \to J^1$ is a strict monomorphism we can extend the displayed morphism to a morphism $\delta^1 : J^1 \to I^2$, see Lemma 13.26.4. And so on. We set $M^\bullet = I^\bullet \oplus J^\bullet$ with differential

$$d_M^n = \begin{pmatrix} d_I^n & (-1)^{n+1}\delta^n \\ 0 & d_J^n \end{pmatrix}$$

Finally, the map $B[0] \to M^\bullet$ is given by $b \oplus c \circ \beta : M \to I^0 \oplus J^0$. $\square$

05TW Lemma 13.26.9. Let $\mathcal{A}$ be an abelian category with enough injectives. For every $K^\bullet \in K^+(\mathrm{Fil}^f(\mathcal{A}))$ there exists a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below, each I^n a filtered injective object, and each $K^n \to I^n$ a strict monomorphism.

Proof. After replacing $K^\bullet$ by a shift (which is harmless for the proof) we may assume that $K^n = 0$ for $n < 0$. Consider the short exact sequences

$$\begin{matrix} 0 \to \text{Ker}(d_K^0) \to K^0 \to \text{Coim}(d_K^0) \to 0 \\ 0 \to \text{Ker}(d_K^1) \to K^1 \to \text{Coim}(d_K^1) \to 0 \\ 0 \to \text{Ker}(d_K^2) \to K^2 \to \text{Coim}(d_K^2) \to 0 \\ \ldots \end{matrix}$$

of the exact category $\text{Fil}^f(\mathcal{A})$ and the maps $u_i : \text{Coim}(d_K^i) \to \text{Ker}(d_K^{i+1})$. For each $i \geq 0$ we may choose filtered quasi-isomorphisms

$$\begin{matrix} \text{Ker}(d_K^i)[0] \to I^\bullet_{ker,i} \\ \text{Coim}(d_K^i)[0] \to I^\bullet_{coim,i} \end{matrix}$$

with $I^n_{ker,i}, I^n_{coim,i}$ filtered injective and zero for $n < 0$, see Lemma 13.26.6. By Lemma 13.26.7 we may lift u_i to a morphism of complexes $u_i^\bullet : I^\bullet_{coim,i} \to I^\bullet_{ker,i+1}$. Finally, for each $i \geq 0$ we may complete the diagrams

$$\begin{tikzcd} 0 \arrow[r] & \text{Ker}(d_K^i)[0] \arrow[r] \arrow[d] & K^i[0] \arrow[r] \arrow[d] & \text{Coim}(d_K^i)[0] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^\bullet_{ker,i} \arrow[r, "\alpha_i"] & I_i^\bullet \arrow[r, "\beta_i"] & I^\bullet_{coim,i} \arrow[r] & 0 \end{tikzcd}$$

with the lower sequence a termwise split exact sequence, see Lemma 13.26.8. For $i \geq 0$ set $d_i : I_i^\bullet \to I_{i+1}^\bullet$ equal to $d_i = \alpha_{i+1} \circ u_i^\bullet \circ \beta_i$. Note that $d_i \circ d_{i-1} = 0$ because $\beta_i \circ \alpha_i = 0$. Hence we have constructed a commutative diagram

$$\begin{tikzcd} I_0^\bullet \arrow[r] & I_1^\bullet \arrow[r] & I_2^\bullet \arrow[r] & \ldots \\ K^0[0] \arrow[r] \arrow[u] & K^1[0] \arrow[r] \arrow[u] & K^2[0] \arrow[r] \arrow[u] & \ldots \end{tikzcd}$$

Here the vertical arrows are filtered quasi-isomorphisms. The upper row is a complex of complexes and each complex consists of filtered injective objects with no nonzero objects in degree < 0. Thus we obtain a double complex by setting $I^{a,b} = I_a^b$ and using

$$d_1^{a,b} : I^{a,b} = I_a^b \to I_{a+1}^b = I^{a+1,b}$$

the map d_a^b and using for

$$d_2^{a,b} : I^{a,b} = I_a^b \to I_a^{b+1} = I^{a,b+1}$$

the map $d_{I_a}^b$. Denote $\text{Tot}(I^{\bullet,\bullet})$ the total complex associated to this double complex, see Homology, Definition 12.18.3. Observe that the maps $K^n[0] \to I_n^\bullet$ come from maps $K^n \to I^{n,0}$ which give rise to a map of complexes

$$K^\bullet \longrightarrow \text{Tot}(I^{\bullet,\bullet})$$

We claim this is a filtered quasi-isomorphism. As $\text{gr}(-)$ is an additive functor, we see that $\text{gr}(\text{Tot}(I^{\bullet,\bullet})) = \text{Tot}(\text{gr}(I^{\bullet,\bullet}))$. Thus we can use Homology, Lemma 12.25.4 to conclude that $\text{gr}(K^\bullet) \to \text{gr}(\text{Tot}(I^{\bullet,\bullet}))$ is a quasi-isomorphism as desired. □

05TX Lemma 13.26.10. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet, I^\bullet \in K(\text{Fil}^f(\mathcal{A}))$. Assume $K^\bullet$ is filtered acyclic and $I^\bullet$ bounded below and consisting of filtered injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero: $\text{Hom}_{K(\text{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet) = 0$.

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n \circ d_K^{n-1} = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is filtered acyclic we see that d_K^{n-1} and d_K^n are strict and that

$$0 \to \mathrm{Im}(d_K^{n-1}) \to K^n \to \mathrm{Im}(d_K^n) \to 0$$

is an exact sequence of the exact category $\mathrm{Fil}^f(\mathcal{A})$, see Homology, Lemma 12.19.15. Hence we can think of α^n as a map into I^n defined on $\mathrm{Im}(d_K^n)$. Using that $\mathrm{Im}(d_K^n) \to K^{n+1}$ is a strict monomorphism and that I^n is filtered injective we may lift this map to a map $h : K^{n+1} \to I^n$ as desired, see Lemma 13.26.4. $\square$

05TY Lemma 13.26.11. Let $\mathcal{A}$ be an abelian category. Let $I^\bullet \in K(\mathrm{Fil}^f(\mathcal{A}))$ be a bounded below complex consisting of filtered injective objects.

(1) Let $\alpha : K^\bullet \to L^\bullet$ in $K(\mathrm{Fil}^f(\mathcal{A}))$ be a filtered quasi-isomorphism. Then the map
$$\mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet)$$
is bijective.

(2) Let $L^\bullet \in K(\mathrm{Fil}^f(\mathcal{A}))$. Then
$$\mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) = \mathrm{Hom}_{DF(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Proof of (1). Note that

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathrm{Fil}^f(\mathcal{A}))$ (Lemma 13.9.14) and $C(\alpha)^\bullet$ is a filtered acyclic complex (Lemma 13.13.4). Then

$$\begin{tikzcd} \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet, I^\bullet) \arrow[r] & \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) \arrow[r] & \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet) \arrow[dll] \\ \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet[-1], I^\bullet) & & \end{tikzcd}$$

is an exact sequence of abelian groups, see Lemma 13.4.2. At this point Lemma 13.26.10 guarantees that the outer two groups are zero and hence $\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Proof of (2). Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By part (1) we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a filtered quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then part (1) shows that β is homotopic to zero also, i.e., $b = 0$. $\square$

015Q Lemma 13.26.12. Let $\mathcal{A}$ be an abelian category with enough injectives. Let $\mathcal{I}^f \subset \text{Fil}^f(\mathcal{A})$ denote the strictly full additive subcategory whose objects are the filtered injective objects. The canonical functor

$$K^+(\mathcal{I}^f) \longrightarrow DF^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories. Furthermore the diagrams

$$\begin{tikzcd} K^+(\mathcal{I}^f) \arrow[r] \arrow[d, "\text{gr}^p"'] & DF^+(\mathcal{A}) \arrow[d, "\text{gr}^p"'] \\ K^+(\mathcal{I}) \arrow[r] & D^+(\mathcal{A}) \end{tikzcd} \quad \begin{tikzcd} K^+(\mathcal{I}^f) \arrow[r] \arrow[d, "\text{forget } F"] & DF^+(\mathcal{A}) \arrow[d, "\text{forget } F"] \\ K^+(\mathcal{I}) \arrow[r] & D^+(\mathcal{A}) \end{tikzcd}$$

are commutative, where $\mathcal{I} \subset \mathcal{A}$ is the strictly full additive subcategory whose objects are the injective objects.

Proof. The functor $K^+(\mathcal{I}^f) \to DF^+(\mathcal{A})$ is essentially surjective by Lemma 13.26.9. It is fully faithful by Lemma 13.26.11. It is an exact functor by our definitions regarding distinguished triangles. The commutativity of the squares is immediate. $\square$

015R Remark 13.26.13. We can invert the arrow of the lemma only if $\mathcal{A}$ is a category in our sense, namely if it has a set of objects. However, suppose given a big abelian category $\mathcal{A}$ with enough injectives, such as $\text{Mod}(\mathcal{O}_X)$ for example. Then for any given set of objects $\{A_i\}_{i\in I}$ there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ containing all of them and having enough injectives, see Sets, Lemma 3.12.1. Thus we may use the lemma above for $\mathcal{A}'$. This essentially means that if we use a set worth of diagrams, etc then we will never run into trouble using the lemma.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. (We cannot use the letter F for the functor since this would conflict too much with our use of the letter F to indicate filtrations.) Note that T induces an additive functor

$$T : \text{Fil}^f(\mathcal{A}) \to \text{Fil}^f(\mathcal{B})$$

by the rule $T(A, F) = (T(A), F)$ where $F^pT(A) = T(F^pA)$ which makes sense as T is left exact. (Warning: It may not be the case that $\text{gr}(T(A)) = T(\text{gr}(A))$.) This induces functors of triangulated categories

05TZ (13.26.13.1) $$T : K^+(\text{Fil}^f(\mathcal{A})) \longrightarrow K^+(\text{Fil}^f(\mathcal{B}))$$

The filtered right derived functor of T is the right derived functor of Definition 13.14.2 for this exact functor composed with the exact functor $K^+(\text{Fil}^f(\mathcal{B})) \to DF^+(\mathcal{B})$ and the multiplicative set $\text{FQis}^+(\mathcal{A})$. Assume $\mathcal{A}$ has enough injectives. At this point we can redo the discussion of Section 13.20 to define the filtered right derived functors

015S (13.26.13.2) $$RT : DF^+(\mathcal{A}) \longrightarrow DF^+(\mathcal{B})$$

of our functor T.

However, instead we will proceed as in Section 13.25, and it will turn out that we can define RT even if T is just additive. Namely, we first choose a quasi-inverse $j' : DF^+(\mathcal{A}) \to K^+(\mathcal{I}^f)$ of the equivalence of Lemma 13.26.12. By Lemma 13.4.18

we see that j' is an exact functor of triangulated categories. Next, we note that for a filtered injective object I we have a (noncanonical) decomposition

015T (13.26.13.3)
$$I \cong \bigoplus\nolimits_{p \in \mathbf{Z}} I_p, \quad \text{with} \quad F^pI = \bigoplus\nolimits_{q \geq p} I_q$$

by Lemma 13.26.2. Hence if T is any additive functor $T : \mathcal{A} \to \mathcal{B}$ then we get an additive functor

05U0 (13.26.13.4)
$$T_{ext} : \mathcal{I}^f \to \text{Fil}^f(\mathcal{B})$$

by setting $T_{ext}(I) = \bigoplus T(I_p)$ with $F^pT_{ext}(I) = \bigoplus_{q \geq p} T(I_q)$. Note that we have the property $\text{gr}(T_{ext}(I)) = T(\text{gr}(I))$ by construction. Hence we obtain a functor

05U1 (13.26.13.5)
$$T_{ext} : K^+(\mathcal{I}^f) \to K^+(\text{Fil}^f(\mathcal{B}))$$

which commutes with gr. Then we define (13.26.13.2) by the composition

05U2 (13.26.13.6)
$$RT = T_{ext} \circ j'.$$

Since $RT : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is computed by injective resolutions as well, see Lemmas 13.20.1, the commutation of T with gr, and the commutative diagrams of Lemma 13.26.12 imply that

015U (13.26.13.7)
$$\text{gr}^p \circ RT \cong RT \circ \text{gr}^p$$

and

015V (13.26.13.8)
$$(\text{forget } F) \circ RT \cong RT \circ (\text{forget } F)$$

as functors $DF^+(\mathcal{A}) \to D^+(\mathcal{B})$.

The filtered derived functor RT (13.26.13.2) induces functors

$$\begin{gathered} RT : \text{Fil}^f(\mathcal{A}) \to DF^+(\mathcal{B}), \\ RT : \text{Comp}^+(\text{Fil}^f(\mathcal{A})) \to DF^+(\mathcal{B}), \\ RT : KF^+(\mathcal{A}) \to DF^+(\mathcal{B}). \end{gathered}$$

Note that since $\text{Fil}^f(\mathcal{A})$, and $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$ are no longer abelian it does not make sense to say that RT restricts to a δ-functor on them. (This can be repaired by thinking of these categories as exact categories and formulating the notion of a δ-functor from an exact category into a triangulated category.) But it does make sense, and it is true by construction, that RT is an exact functor on the triangulated category $KF^+(\mathcal{A})$.

015W **Lemma 13.26.14.** Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume $\mathcal{A}$ has enough injectives. Let $(K^\bullet, F)$ be an object of $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$. There exists a spectral sequence $(E_r, d_r)_{r \geq 0}$ consisting of bigraded objects E_r of $\mathcal{B}$ and d_r of bidegree $(r, -r+1)$ and with

$$E_1^{p,q} = R^{p+q}T(\text{gr}^p(K^\bullet))$$

Moreover, this spectral sequence is bounded, converges to $R^*T(K^\bullet)$, and induces a finite filtration on each $R^nT(K^\bullet)$. The construction of this spectral sequence is functorial in the object $K^\bullet$ of $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$ and the terms (E_r, d_r) for $r \geq 1$ do not depend on any choices.

Proof. Choose a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ a bounded below complex of filtered injective objects, see Lemma 13.26.9. Consider the complex $RT(K^\bullet) = T_{ext}(I^\bullet)$, see (13.26.13.6). Thus we can consider the spectral sequence $(E_r, d_r)_{r \geq 0}$ associated to this as a filtered complex in $\mathcal{B}$, see Homology, Section 12.24. By Homology, Lemma 12.24.2 we have $E_1^{p,q} = H^{p+q}(\text{gr}^p(T(I^\bullet)))$. By Equation (13.26.13.3) we have $E_1^{p,q} = H^{p+q}(T(\text{gr}^p(I^\bullet)))$, and by definition of a filtered injective resolution the map $\text{gr}^p(K^\bullet) \to \text{gr}^p(I^\bullet)$ is an injective resolution. Hence $E_1^{p,q} = R^{p+q}T(\text{gr}^p(K^\bullet))$.

On the other hand, each I^n has a finite filtration and hence each $T(I^n)$ has a finite filtration. Thus we may apply Homology, Lemma 12.24.11 to conclude that the spectral sequence is bounded, converges to $H^n(T(I^\bullet)) = R^nT(K^\bullet)$ moreover inducing finite filtrations on each of the terms.

Suppose that $K^\bullet \to L^\bullet$ is a morphism of $\text{Comp}^+(\text{Fil}^f(\mathcal{A}))$. Choose a filtered quasi-isomorphism $L^\bullet \to J^\bullet$ with $J^\bullet$ a bounded below complex of filtered injective objects, see Lemma 13.26.9. By our results above, for example Lemma 13.26.11, there exists a diagram

$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d] & L^\bullet \arrow[d] \\ I^\bullet \arrow[r] & J^\bullet \end{tikzcd}$$

which commutes up to homotopy. Hence we get a morphism of filtered complexes $T(I^\bullet) \to T(J^\bullet)$ which gives rise to the morphism of spectral sequences, see Homology, Lemma 12.24.4. The last statement follows from this. $\square$

015X Remark 13.26.15. As promised in Remark 13.21.4 we discuss the connection of the lemma above with the constructions using Cartan-Eilenberg resolutions. Namely, let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, assume $\mathcal{A}$ has enough injectives, and let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. We give an alternative construction of the spectral sequences $'E$ and $''E$ of Lemma 13.21.3.

First spectral sequence. Consider the "stupid" filtration on $K^\bullet$ obtained by setting $F^p(K^\bullet) = \sigma_{\geq p}(K^\bullet)$, see Homology, Section 12.15. Note that this stupid in the sense that $d(F^p(K^\bullet)) \subset F^{p+1}(K^\bullet)$, compare Homology, Lemma 12.24.3. Note that $\text{gr}^p(K^\bullet) = K^p[-p]$ with this filtration. According to Lemma 13.26.14 there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(K^p[-p]) = R^qT(K^p)$$

as in the spectral sequence $'E_r$. Observe moreover that the differentials $E_1^{p,q} \to E_1^{p+1,q}$ agree with the differentials in $'E_1$, see Homology, Lemma 12.24.3 part (2) and the description of $'d_1$ in the proof of Lemma 13.21.3.

Second spectral sequence. Consider the filtration on the complex $K^\bullet$ obtained by setting $F^p(K^\bullet) = \tau_{\leq -p}(K^\bullet)$, see Homology, Section 12.15. The minus sign is necessary to get a decreasing filtration. Note that $\text{gr}^p(K^\bullet)$ is quasi-isomorphic to $H^{-p}(K^\bullet)[p]$ with this filtration. According to Lemma 13.26.14 there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(H^{-p}(K^\bullet)[p]) = R^{2p+q}T(H^{-p}(K^\bullet)) = {''E}_2^{i,j}$$

with $i = 2p+q$ and $j = -p$. (This looks unnatural, but note that we could just have well developed the whole theory of filtered complexes using increasing filtrations, with the end result that this then looks natural, but the other one doesn't.) We leave it to the reader to see that the differentials match up.

Actually, given a Cartan-Eilenberg resolution $K^\bullet \to I^{\bullet,\bullet}$ the induced morphism $K^\bullet \to \text{Tot}(I^{\bullet,\bullet})$ into the associated total complex will be a filtered injective resolution for either filtration using suitable filtrations on $\text{Tot}(I^{\bullet,\bullet})$. This can be used to match up the spectral sequences exactly.

13.27. Ext groups

06XP In this section we start describing the Ext groups of objects of an abelian category. First we have the following very general definition.

06XQ **Definition 13.27.1.** Let $\mathcal{A}$ be an abelian category. Let $i \in \mathbf{Z}$. Let X, Y be objects of $D(\mathcal{A})$. The *ith extension group* of X by Y is the group

$$\text{Ext}^i_\mathcal{A}(X, Y) = \text{Hom}_{D(\mathcal{A})}(X, Y[i]) = \text{Hom}_{D(\mathcal{A})}(X[-i], Y).$$

If $A, B \in \text{Ob}(\mathcal{A})$ we set $\text{Ext}^i_\mathcal{A}(A, B) = \text{Ext}^i_\mathcal{A}(A[0], B[0])$.

Since $\text{Hom}_{D(\mathcal{A})}(X, -)$, resp. $\text{Hom}_{D(\mathcal{A})}(-, Y)$ is a homological, resp. cohomological functor, see Lemma 13.4.2, we see that a distinguished triangle (Y, Y', Y''), resp. (X, X', X'') leads to a long exact sequence

$$\ldots \to \text{Ext}^i_\mathcal{A}(X, Y) \to \text{Ext}^i_\mathcal{A}(X, Y') \to \text{Ext}^i_\mathcal{A}(X, Y'') \to \text{Ext}^{i+1}_\mathcal{A}(X, Y) \to \ldots$$

respectively

$$\ldots \to \text{Ext}^i_\mathcal{A}(X'', Y) \to \text{Ext}^i_\mathcal{A}(X', Y) \to \text{Ext}^i_\mathcal{A}(X, Y) \to \text{Ext}^{i+1}_\mathcal{A}(X'', Y) \to \ldots$$

Note that since $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ are full subcategories we may compute the Ext groups by Hom groups in these categories provided X, Y are contained in them.

In case the category $\mathcal{A}$ has enough injectives or enough projectives we can compute the Ext groups using injective or projective resolutions. To avoid confusion, recall that having an injective (resp. projective) resolution implies vanishing of homology in all low (resp. high) degrees, see Lemmas 13.18.2 and 13.19.2.

06XR **Lemma 13.27.2.** Let $\mathcal{A}$ be an abelian category. Let $X^\bullet, Y^\bullet \in \text{Ob}(K(\mathcal{A}))$.

(1) Let $Y^\bullet \to I^\bullet$ be an injective resolution (Definition 13.18.1). Then
$$\text{Ext}^i_\mathcal{A}(X^\bullet, Y^\bullet) = \text{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet[i]).$$
(2) Let $P^\bullet \to X^\bullet$ be a projective resolution (Definition 13.19.1). Then
$$\text{Ext}^i_\mathcal{A}(X^\bullet, Y^\bullet) = \text{Hom}_{K(\mathcal{A})}(P^\bullet[-i], Y^\bullet).$$

Proof. Follows immediately from Lemma 13.18.8 and Lemma 13.19.8. $\square$

In the rest of this section we discuss extensions of objects of the abelian category itself. First we observe the following.

06XS **Lemma 13.27.3.** Let $\mathcal{A}$ be an abelian category.

(1) Let X, Y be objects of $D(\mathcal{A})$. Given $a, b \in \mathbf{Z}$ such that $H^i(X) = 0$ for $i > a$ and $H^j(Y) = 0$ for $j < b$, we have $\text{Ext}^n_\mathcal{A}(X, Y) = 0$ for $n < b - a$ and
$$\text{Ext}^{b-a}_\mathcal{A}(X, Y) = \text{Hom}_\mathcal{A}(H^a(X), H^b(Y))$$

(2) Let $A, B \in \text{Ob}(\mathcal{A})$. For $i < 0$ we have $\text{Ext}^i_{\mathcal{A}}(B, A) = 0$. We have $\text{Ext}^0_{\mathcal{A}}(B, A) = \text{Hom}_{\mathcal{A}}(B, A)$.

Proof. Choose complexes $X^\bullet$ and $Y^\bullet$ representing X and Y. Since $Y^\bullet \to \tau_{\geq b}Y^\bullet$ is a quasi-isomorphism, we may assume that $Y^j = 0$ for $j < b$. Let $L^\bullet \to X^\bullet$ be any quasi-isomorphism. Then $\tau_{\leq a}L^\bullet \to X^\bullet$ is a quasi-isomorphism. Hence a morphism $X \to Y[n]$ in $D(\mathcal{A})$ can be represented as fs^{-1} where $s : L^\bullet \to X^\bullet$ is a quasi-isomorphism, $f : L^\bullet \to Y^\bullet[n]$ a morphism, and $L^i = 0$ for $i < a$. Note that f maps L^i to Y^{i+n}. Thus $f = 0$ if $n < b - a$ because always either L^i or Y^{i+n} is zero. If $n = b - a$, then f corresponds exactly to a morphism $H^a(X) \to H^b(Y)$. Part (2) is a special case of (1). $\square$

Let $\mathcal{A}$ be an abelian category. Suppose that $0 \to A \to A' \to A'' \to 0$ is a short exact sequence of objects of $\mathcal{A}$. Then $0 \to A[0] \to A'[0] \to A''[0] \to 0$ leads to a distinguished triangle in $D(\mathcal{A})$ (see Lemma 13.12.1) hence a long exact sequence of Ext groups

$$0 \to \text{Ext}^0_{\mathcal{A}}(B, A) \to \text{Ext}^0_{\mathcal{A}}(B, A') \to \text{Ext}^0_{\mathcal{A}}(B, A'') \to \text{Ext}^1_{\mathcal{A}}(B, A) \to \ldots$$

Similarly, given a short exact sequence $0 \to B \to B' \to B'' \to 0$ we obtain a long exact sequence of Ext groups

$$0 \to \text{Ext}^0_{\mathcal{A}}(B'', A) \to \text{Ext}^0_{\mathcal{A}}(B', A) \to \text{Ext}^0_{\mathcal{A}}(B, A) \to \text{Ext}^1_{\mathcal{A}}(B'', A) \to \ldots$$

We may view these Ext groups as an application of the construction of the derived category. It shows one can define Ext groups and construct the long exact sequence of Ext groups without needing the existence of enough injectives or projectives. There is an alternative construction of the Ext groups due to Yoneda which avoids the use of the derived category, see [Yon60].

06XT Definition 13.27.4. Let $\mathcal{A}$ be an abelian category. Let $A, B \in \text{Ob}(\mathcal{A})$. A degree i Yoneda extension of B by A is an exact sequence

$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$

in $\mathcal{A}$. We say two Yoneda extensions E and E' of the same degree are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & Z_{i-1} \arrow[r] & \ldots \arrow[r] & Z_0 \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\text{id}"] \arrow[d, "\text{id}"'] & Z''_{i-1} \arrow[r] \arrow[u] \arrow[d] & \ldots \arrow[r] & Z''_0 \arrow[r] \arrow[u] \arrow[d] & B \arrow[r] \arrow[u, "\text{id}"'] \arrow[d, "\text{id}"] & 0 \\
0 \arrow[r] & A \arrow[r] & Z'_{i-1} \arrow[r] & \ldots \arrow[r] & Z'_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

where the middle row is a Yoneda extension as well.

It is not immediately clear that the equivalence of the definition is an equivalence relation. Although it is instructive to prove this directly this will also follow from Lemma 13.27.5 below.

Let $\mathcal{A}$ be an abelian category with objects A, B. Given a Yoneda extension $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ we define an associated element

$\delta(E) \in \text{Ext}^i(B, A)$ as the morphism $\delta(E) = fs^{-1} : B[0] \to A[i]$ where s is the quasi-isomorphism

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow B[0]$$

and f is the morphism of complexes

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow A[i]$$

We call $\delta(E) = fs^{-1}$ the class of the Yoneda extension. It turns out that this class characterizes the equivalence class of the Yoneda extension.

06XU Lemma 13.27.5. Let $\mathcal{A}$ be an abelian category with objects A, B. Any element in $\text{Ext}^i_\mathcal{A}(B, A)$ is $\delta(E)$ for some degree i Yoneda extension of B by A. Given two Yoneda extensions E, E' of the same degree then E is equivalent to E' if and only if $\delta(E) = \delta(E')$.

Proof. Let $\xi : B[0] \to A[i]$ be an element of $\text{Ext}^i_\mathcal{A}(B, A)$. We may write $\xi = fs^{-1}$ for some quasi-isomorphism $s : L^\bullet \to B[0]$ and map $f : L^\bullet \to A[i]$. After replacing $L^\bullet$ by $\tau_{\leq 0}L^\bullet$ we may assume that $L^j = 0$ for $j > 0$. Picture

$$\begin{tikzcd} L^{-i-1} \arrow[r] & L^{-i} \arrow[r] \arrow[d] & \ldots \arrow[r] & L^0 \arrow[r] & B \arrow[r] & 0 \\ & A & & & & \end{tikzcd}$$

Then setting $Z_{i-1} = (L^{-i+1} \oplus A)/L^{-i}$ and $Z_j = L^{-j}$ for $j = i-2, \ldots, 0$ we see that we obtain a degree i extension E of B by A whose class $\delta(E)$ equals ξ.

It is immediate from the definitions that equivalent Yoneda extensions have the same class. Suppose that $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ and $E' : 0 \to A \to Z'_{i-1} \to Z'_{i-2} \to \ldots \to Z'_0 \to B \to 0$ are Yoneda extensions with the same class. By construction of $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ at the set of quasi-isomorphisms, this means there exists a complex $L^\bullet$ and quasi-isomorphisms

$$t : L^\bullet \to (\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots)$$

and

$$t' : L^\bullet \to (\ldots \to 0 \to A \to Z'_{i-1} \to \ldots \to Z'_0 \to 0 \to \ldots)$$

such that $s \circ t = s' \circ t'$ and $f \circ t = f' \circ t'$, see Categories, Section 4.27. Let E'' be the degree i extension of B by A constructed from the pair $L^\bullet \to B[0]$ and $L^\bullet \to A[i]$ in the first paragraph of the proof. Then the reader sees readily that there exists "morphisms" of degree i Yoneda extensions $E'' \to E$ and $E'' \to E'$ as in the definition of equivalent Yoneda extensions (details omitted). This finishes the proof. $\square$

06XV Lemma 13.27.6. Let $\mathcal{A}$ be an abelian category. Let A, B be objects of $\mathcal{A}$. Then $\text{Ext}^1_\mathcal{A}(B, A)$ is the group $\text{Ext}_\mathcal{A}(B, A)$ constructed in Homology, Definition 12.6.2.

Proof. This is the case $i = 1$ of Lemma 13.27.5. $\square$

0GSM Lemma 13.27.7. Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to Z \to B \to 0$ and $0 \to B \to Z' \to C \to 0$ be short exact sequences in $\mathcal{A}$. Denote $[Z] \in \text{Ext}^1(B, A)$

and $[Z'] \in \text{Ext}^1(C, B)$ their classes. Then $[Z] \circ [Z'] \in \text{Ext}^2_{\mathcal{A}}(C, A)$ is 0 if and only if there exists a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "1"] & Z \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & W \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, "1"] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns in $\mathcal{A}$.

Proof. Omitted. Hints: You can argue this using the result of Lemma 13.27.5 and working out what it means for a 2-extension class to be zero. Or you can use that if $[Z] \circ [Z'] \in \text{Ext}^2_{\mathcal{A}}(C, A)$ is zero, then by the long exact cohomology sequence of Ext the element $[Z] \in \text{Ext}^1(B, A)$ is the image of some element in $\text{Ext}^1(W', A)$. $\square$

0EWW Lemma 13.27.8. Let $\mathcal{A}$ be an abelian category and let $p \geq 0$. If $\text{Ext}^p_{\mathcal{A}}(B, A) = 0$ for any pair of objects A, B of $\mathcal{A}$, then $\text{Ext}^i_{\mathcal{A}}(B, A) = 0$ for $i \geq p$ and any pair of objects A, B of $\mathcal{A}$.

Proof. For $i > p$ write any class ξ as $\delta(E)$ where E is a Yoneda extension

$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$

This is possible by Lemma 13.27.5. Set $C = \text{Ker}(Z_{p-1} \to Z_p) = \text{Im}(Z_p \to Z_{p-1})$. Then $\delta(E)$ is the composition of $\delta(E')$ and $\delta(E'')$ where

$$E' : 0 \to C \to Z_{p-1} \to \ldots \to Z_0 \to B \to 0$$

and

$$E'' : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_p \to C \to 0$$

Since $\delta(E') \in \text{Ext}^p_{\mathcal{A}}(B, C) = 0$ we conclude. $\square$

0GM4 Lemma 13.27.9. Let $\mathcal{A}$ be an abelian category. Let K be an object of $D^b(\mathcal{A})$ such that $\text{Ext}^p_{\mathcal{A}}(H^i(K), H^j(K)) = 0$ for all $p \geq 2$ and $i > j$. Then K is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$.

Proof. Choose a, b such that $H^i(K) = 0$ for $i \notin [a, b]$. We will prove the lemma by induction on $b - a$. If $b - a \leq 0$, then the result is clear. If $b - a > 0$, then we look at the distinguished triangle of truncations

$$\tau_{\leq b-1}K \to K \to H^b(K)[-b] \to (\tau_{\leq b-1}K)[1]$$

see Remark 13.12.4. By Lemma 13.4.11 if the last arrow is zero, then $K \cong \tau_{\leq b-1}K \oplus H^b(K)[-b]$ and we win by induction. Again using induction we see that

$$\text{Hom}_{D(\mathcal{A})}(H^b(K)[-b], (\tau_{\leq b-1}K)[1]) = \bigoplus\nolimits_{i<b} \text{Ext}^{b-i+1}_{\mathcal{A}}(H^b(K), H^i(K))$$

By assumption the direct sum is zero and the proof is complete. $\square$

0EWX Lemma 13.27.10. Let $\mathcal{A}$ be an abelian category. Assume $\mathrm{Ext}^2_{\mathcal{A}}(B, A) = 0$ for any pair of objects A, B of $\mathcal{A}$. Then any object K of $D^b(\mathcal{A})$ is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$.

Proof. The assumption implies that $\mathrm{Ext}^i_{\mathcal{A}}(B, A) = 0$ for $i \geq 2$ and any pair of objects A, B of $\mathcal{A}$ by Lemma 13.27.8. Hence this lemma is a special case of Lemma 13.27.9. □

13.28. K-groups

0FCM A tiny bit about K_0 of a triangulated category.

0FCN Definition 13.28.1. Let $\mathcal{D}$ be a triangulated category. We denote $K_0(\mathcal{D})$ the zeroth K-group of $\mathcal{D}$. It is the abelian group constructed as follows. Take the free abelian group on the objects on $\mathcal{D}$ and for every distinguished triangle $X \to Y \to Z$ impose the relation $[Y] - [X] - [Z] = 0$.

Observe that this implies that $[X[n]] = (-1)^n[X]$ because we have the distinguished triangle $(X, 0, X[1], 0, 0, -\mathrm{id}[1])$.

0FCP Lemma 13.28.2. Let $\mathcal{A}$ be an abelian category. Then there is a canonical identification $K_0(D^b(\mathcal{A})) = K_0(\mathcal{A})$ of zeroth K-groups.

Proof. Given an object A of $\mathcal{A}$ denote $A[0]$ the object A viewed as a complex sitting in degree 0. If $0 \to A \to A' \to A'' \to 0$ is a short exact sequence, then we get a distinguished triangle $A[0] \to A'[0] \to A''[0] \to A[1]$, see Section 13.12. This shows that we obtain a map $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$ by sending $[A]$ to $[A[0]]$ with apologies for the horrendous notation.

On the other hand, given an object X of $D^b(\mathcal{A})$ we can consider the element

$$c(X) = \sum (-1)^i [H^i(X)] \in K_0(\mathcal{A})$$

Given a distinguished triangle $X \to Y \to Z$ the long exact sequence of cohomology (13.11.1.1) and the relations in $K_0(\mathcal{A})$ show that $c(Y) = c(X) + c(Z)$. Thus c factors through a map $c : K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A})$.

We want to show that the two maps above are mutually inverse. It is clear that the composition $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A})$ is the identity. Suppose that $X^\bullet$ is a bounded complex of $\mathcal{A}$. The existence of the distinguished triangles of "stupid truncations" (see Homology, Section 12.15)

$$\sigma_{\geq n} X^\bullet \to \sigma_{\geq n-1} X^\bullet \to X^{n-1}[-n+1] \to (\sigma_{\geq n} X^\bullet)[1]$$

and induction show that

$$[X^\bullet] = \sum (-1)^i [X^i[0]]$$

in $K_0(D^b(\mathcal{A}))$ (with again apologies for the notation). It follows that the composition $K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$ is surjective which finishes the proof. □

0FCQ Lemma 13.28.3. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories. Then F induces a group homomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{D}')$.

Proof. Omitted. □

0FCR Lemma 13.28.4. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor from a triangulated category to an abelian category. Assume that for any X in $\mathcal{D}$ only a finite number of the objects $H(X[i])$ are nonzero in $\mathcal{A}$. Then H induces a group homomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{A})$ sending $[X]$ to $\sum(-1)^i[H(X[i])]$.

Proof. Omitted. $\square$

0FCS Lemma 13.28.5. Let $\mathcal{B}$ be a weak Serre subcategory of the abelian category $\mathcal{A}$. There is a canonical isomorphism

$$K_0(\mathcal{B}) \longrightarrow K_0(D^b_{\mathcal{B}}(\mathcal{A})), \quad [B] \longmapsto [B[0]]$$

The inverse sends the class $[X]$ of X to the element $\sum(-1)^i[H^i(X)]$.

Proof. We omit the verification that the rule for the inverse gives a well defined map $K_0(D^b_{\mathcal{B}}(\mathcal{A})) \to K_0(\mathcal{B})$. It is immediate that the composition $K_0(\mathcal{B}) \to K_0(D^b_{\mathcal{B}}(\mathcal{A})) \to K_0(\mathcal{B})$ is the identity. On the other hand, using the distinguished triangles of Remark 13.12.4 and an induction argument the reader may show that the displayed arrow in the statement of the lemma is surjective (details omitted). The lemma follows. $\square$

0FCT Lemma 13.28.6. Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$ be triangulated categories. Let

$$\otimes : \mathcal{D} \times \mathcal{D}' \longrightarrow \mathcal{D}''$$

be a functor such that for fixed X in $\mathcal{D}$ the functor $X \otimes - : \mathcal{D}' \to \mathcal{D}''$ is an exact functor and for fixed X' in $\mathcal{D}'$ the functor $- \otimes X' : \mathcal{D} \to \mathcal{D}''$ is an exact functor. Then $\otimes$ induces a bilinear map $K_0(\mathcal{D}) \times K_0(\mathcal{D}') \to K_0(\mathcal{D}'')$ which sends $([X], [X'])$ to $[X \otimes X']$.

Proof. Omitted. $\square$

13.29. Unbounded complexes

06XW A reference for the material in this section is [Spa88]. The following lemma is useful to find "good" left resolutions of unbounded complexes.

06XX Lemma 13.29.1. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume $\mathcal{P}$ contains 0, is closed under (finite) direct sums, and every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Let $K^\bullet$ be a complex. There exists a commutative diagram

$$\begin{tikzcd} P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}K^\bullet \arrow[r] & \tau_{\leq 2}K^\bullet \arrow[r] & \ldots \end{tikzcd}$$

in the category of complexes such that

1. the vertical arrows are quasi-isomorphisms and termwise surjective,
2. $P_n^\bullet$ is a bounded above complex with terms in $\mathcal{P}$,
3. the arrows $P_n^\bullet \to P_{n+1}^\bullet$ are termwise split injections and each cokernel P^i_{n+1}/P^i_n is an element of $\mathcal{P}$.

Proof. We are going to use that the homotopy category $K(\mathcal{A})$ is a triangulated category, see Proposition 13.10.3. By Lemma 13.15.4 we can find a termwise surjective map of complexes $P_1^\bullet \to \tau_{\leq 1}K^\bullet$ which is a quasi-isomorphism such that the terms of $P_1^\bullet$ are in $\mathcal{P}$. By induction it suffices, given $P_1^\bullet, \ldots, P_n^\bullet$ to construct $P_{n+1}^\bullet$ and the maps $P_n^\bullet \to P_{n+1}^\bullet$ and $P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$.

Choose a distinguished triangle $P_n^\bullet \to \tau_{\leq n+1}K^\bullet \to C^\bullet \to P_n^\bullet[1]$ in $K(\mathcal{A})$. Applying Lemma 13.15.4 we choose a map of complexes $Q^\bullet \to C^\bullet$ which is a quasi-isomorphism such that the terms of $Q^\bullet$ are in $\mathcal{P}$. By the axioms of triangulated categories we may fit the composition $Q^\bullet \to C^\bullet \to P_n^\bullet[1]$ into a distinguished triangle $P_n^\bullet \to P_{n+1}^\bullet \to Q^\bullet \to P_n^\bullet[1]$ in $K(\mathcal{A})$. By Lemma 13.10.7 we may and do assume $0 \to P_n^\bullet \to P_{n+1}^\bullet \to Q^\bullet \to 0$ is a termwise split short exact sequence. This implies that the terms of $P_{n+1}^\bullet$ are in $\mathcal{P}$ and that $P_n^\bullet \to P_{n+1}^\bullet$ is a termwise split injection whose cokernels are in $\mathcal{P}$. By the axioms of triangulated categories we obtain a map of distinguished triangles

$$\begin{tikzcd} P_n^\bullet \arrow[r] \arrow[d] & P_{n+1}^\bullet \arrow[r] \arrow[d] & Q^\bullet \arrow[r] \arrow[d] & P_n^\bullet[1] \arrow[d] \\ P_n^\bullet \arrow[r] & \tau_{\leq n+1}K^\bullet \arrow[r] & C^\bullet \arrow[r] & P_n^\bullet[1] \end{tikzcd}$$

in the triangulated category $K(\mathcal{A})$. Choose an actual morphism of complexes $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$. The left square of the diagram above commutes up to homotopy, but as $P_n^\bullet \to P_{n+1}^\bullet$ is a termwise split injection we can lift the homotopy and modify our choice of f to make it commute. Finally, f is a quasi-isomorphism, because both $P_n^\bullet \to P_n^\bullet$ and $Q^\bullet \to C^\bullet$ are.

At this point we have all the properties we want, except we don't know that the map $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$ is termwise surjective. Since we have the commutative diagram

$$\begin{tikzcd} P_n^\bullet \arrow[r] \arrow[d] & P_{n+1}^\bullet \arrow[d] \\ \tau_{\leq n}K^\bullet \arrow[r] & \tau_{\leq n+1}K^\bullet \end{tikzcd}$$

of complexes, by induction hypothesis we see that f is surjective on terms in all degrees except possibly n and $n+1$. Choose an object $P \in \mathcal{P}$ and a surjection $q : P \to K^n$. Consider the map

$$g : P^\bullet = (\ldots \to 0 \to P \xrightarrow{1} P \to 0 \to \ldots) \longrightarrow \tau_{\leq n+1}K^\bullet$$

with first copy of P in degree n and maps given by q in degree n and $d_K \circ q$ in degree $n+1$. This is a surjection in degree n and the cokernel in degree $n+1$ is $H^{n+1}(\tau_{\leq n+1}K^\bullet)$; to see this recall that $\tau_{\leq n+1}K^\bullet$ has $\text{Ker}(d_K^{n+1})$ in degree $n+1$. However, since f is a quasi-isomorphism we know that $H^{n+1}(f)$ is surjective. Hence after replacing $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$ by $f \oplus g : P_{n+1}^\bullet \oplus P^\bullet \to \tau_{\leq n+1}K^\bullet$ we win. $\square$

In some cases we can use the lemma above to show that a left derived functor is everywhere defined.

0794 Proposition 13.29.2. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. Let $\mathcal{P} \subset \text{Ob}(\mathcal{A})$ be a subset. Assume

(1) $\mathcal{P}$ contains 0, is closed under (finite) direct sums, and every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,
(2) for any bounded above acyclic complex $P^\bullet$ of $\mathcal{A}$ with $P^n \in \mathcal{P}$ for all n the complex $F(P^\bullet)$ is exact,

(3) $\mathcal{A}$ and $\mathcal{B}$ have colimits of systems over $\mathbf{N}$,
(4) colimits over $\mathbf{N}$ are exact in both $\mathcal{A}$ and $\mathcal{B}$, and
(5) F commutes with colimits over $\mathbf{N}$.

Then LF is defined on all of $D(\mathcal{A})$.

Proof. By (1) and Lemma 13.15.4 for any bounded above complex $K^\bullet$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^\bullet$ bounded above and $P^n \in \mathcal{P}$ for all n. Suppose that $s : P^\bullet \to (P')^\bullet$ is a quasi-isomorphism of bounded above complexes consisting of objects of $\mathcal{P}$. Then $F(P^\bullet) \to F((P')^\bullet)$ is a quasi-isomorphism because $F(C(s)^\bullet)$ is acyclic by assumption (2). This already shows that LF is defined on $D^-(\mathcal{A})$ and that a bounded above complex consisting of objects of $\mathcal{P}$ computes LF, see Lemma 13.14.15.

Next, let $K^\bullet$ be an arbitrary complex of $\mathcal{A}$. Choose a diagram

$$\begin{tikzcd} P_1^\bullet \arrow[r] \arrow[d] & P_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}K^\bullet \arrow[r] & \tau_{\leq 2}K^\bullet \arrow[r] & \ldots \end{tikzcd}$$

as in Lemma 13.29.1. Note that the map $\operatorname{colim} P_n^\bullet \to K^\bullet$ is a quasi-isomorphism because colimits over $\mathbf{N}$ in $\mathcal{A}$ are exact and $H^i(P_n^\bullet) = H^i(K^\bullet)$ for $n > i$. We claim that

$$F(\operatorname{colim} P_n^\bullet) = \operatorname{colim} F(P_n^\bullet)$$

(termwise colimits) is $LF(K^\bullet)$, i.e., that $\operatorname{colim} P_n^\bullet$ computes LF. To see this, by Lemma 13.14.15, it suffices to prove the following claim. Suppose that

$$\operatorname{colim} Q_n^\bullet = Q^\bullet \xrightarrow{\alpha} P^\bullet = \operatorname{colim} P_n^\bullet$$

is a quasi-isomorphism of complexes, such that each $P_n^\bullet$, $Q_n^\bullet$ is a bounded above complex whose terms are in $\mathcal{P}$ and the maps $P_n^\bullet \to \tau_{\leq n}P^\bullet$ and $Q_n^\bullet \to \tau_{\leq n}Q^\bullet$ are quasi-isomorphisms. Claim: $F(\alpha)$ is a quasi-isomorphism.

The problem is that we do not assume that α is given as a colimit of maps between the complexes $P_n^\bullet$ and $Q_n^\bullet$. However, for each n we know that the solid arrows in the diagram

$$\begin{tikzcd} & R^\bullet \arrow[d, dotted] & \\ P_n^\bullet \arrow[d] & L^\bullet \arrow[l, dotted] \arrow[r, dotted] & Q_n^\bullet \arrow[d] \\ \tau_{\leq n}P^\bullet \arrow[rr, "\tau_{\leq n}\alpha"] & & \tau_{\leq n}Q^\bullet \end{tikzcd}$$

are quasi-isomorphisms. Because quasi-isomorphisms form a multiplicative system in $K(\mathcal{A})$ (see Lemma 13.11.2) we can find a quasi-isomorphism $L^\bullet \to P_n^\bullet$ and map of complexes $L^\bullet \to Q_n^\bullet$ such that the diagram above commutes up to homotopy. Then $\tau_{\leq n}L^\bullet \to L^\bullet$ is a quasi-isomorphism. Hence (by the first part of the proof) we can find a bounded above complex $R^\bullet$ whose terms are in $\mathcal{P}$ and a quasi-isomorphism $R^\bullet \to L^\bullet$ (as indicated in the diagram). Using the result of the first paragraph of the proof we see that $F(R^\bullet) \to F(P_n^\bullet)$ and $F(R^\bullet) \to F(Q_n^\bullet)$ are quasi-isomorphisms. Thus we obtain a isomorphisms $H^i(F(P_n^\bullet)) \to H^i(F(Q_n^\bullet))$ fitting into the commutative diagram

$$\begin{tikzcd} H^i(F(P_n^\bullet)) \arrow[r] \arrow[d] & H^i(F(Q_n^\bullet)) \arrow[d] \\ H^i(F(P^\bullet)) \arrow[r] & H^i(F(Q^\bullet)) \end{tikzcd}$$

The exact same argument shows that these maps are also compatible as n varies. Since by (4) and (5) we have

$$H^i(F(P^\bullet)) = H^i(F(\operatorname{colim} P_n^\bullet)) = H^i(\operatorname{colim} F(P_n^\bullet)) = \operatorname{colim} H^i(F(P_n^\bullet))$$

and similarly for $Q^\bullet$ we conclude that $H^i(\alpha) : H^i(F(P^\bullet) \to H^i(F(Q^\bullet)$ is an isomorphism and the claim follows. $\square$

070F Lemma 13.29.3. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset \text{Ob}(\mathcal{A})$ be a subset. Assume $\mathcal{I}$ contains 0, is closed under (finite) products, and every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. Let $K^\bullet$ be a complex. There exists a commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & \tau_{\geq -2}K^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}K^\bullet \arrow[d] \\ \ldots \arrow[r] & I_2^\bullet \arrow[r] & I_1^\bullet \end{tikzcd}$$

in the category of complexes such that

(1) the vertical arrows are quasi-isomorphisms and termwise injective,
(2) $I_n^\bullet$ is a bounded below complex with terms in $\mathcal{I}$,
(3) the arrows $I_{n+1}^\bullet \to I_n^\bullet$ are termwise split surjections and $\text{Ker}(I_{n+1}^i \to I_n^i)$ is an element of $\mathcal{I}$.

Proof. This lemma is dual to Lemma 13.29.1. $\square$

13.30. Deriving adjoints

0FNC Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be exact functors of triangulated categories. Let S, resp. S' be a multiplicative system for $\mathcal{D}$, resp. $\mathcal{D}'$ compatible with the triangulated structure. Denote $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ and $Q' : \mathcal{D}' \to (S')^{-1}\mathcal{D}'$ the localization functors. In this situation, by abuse of notation, one often denotes RF the partially defined right derived functor corresponding to $Q' \circ F : \mathcal{D} \to (S')^{-1}\mathcal{D}'$ and the multiplicative system S. Similarly one denotes LG the partially defined left derived functor corresponding to $Q \circ G : \mathcal{D}' \to S^{-1}\mathcal{D}$ and the multiplicative system S'. Picture

$$\begin{tikzcd} \mathcal{D} \arrow[r, "F"'] \arrow[d, "Q"'] & \mathcal{D}' \arrow[d, "Q'"] \\ S^{-1}\mathcal{D} \arrow[r, dotted, "RF"] & (S')^{-1}\mathcal{D}' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{D}' \arrow[r, "G"'] \arrow[d, "Q'"'] & \mathcal{D} \arrow[d, "Q"] \\ (S')^{-1}\mathcal{D}' \arrow[r, dotted, "LG"] & S^{-1}\mathcal{D} \end{tikzcd}$$

0FND **Lemma 13.30.1.** In the situation above assume F is right adjoint to G. Let $K \in \text{Ob}(\mathcal{D})$ and $M \in \text{Ob}(\mathcal{D}')$. If RF is defined at K and LG is defined at M, then there is a canonical isomorphism

$$\text{Hom}_{(S')^{-1}\mathcal{D}'}(M, RF(K)) = \text{Hom}_{S^{-1}\mathcal{D}}(LG(M), K)$$

This isomorphism is functorial in both variables on the triangulated subcategories of $S^{-1}\mathcal{D}$ and $(S')^{-1}\mathcal{D}'$ where RF and LG are defined.

Proof. Since RF is defined at K, we see that the rule which assigns to an $s : K \to I$ in S the object $F(I)$ is essentially constant as an ind-object of $(S')^{-1}\mathcal{D}'$ with value $RF(K)$. Similarly, the rule which assigns to a $t : P \to M$ in S' the object $G(P)$ is essentially constant as a pro-object of $S^{-1}\mathcal{D}$ with value $LG(M)$. Thus we have

$$\begin{aligned} \text{Hom}_{(S')^{-1}\mathcal{D}'}(M, RF(K)) &= \text{colim}_{s:K\to I} \text{Hom}_{(S')^{-1}\mathcal{D}'}(M, F(I)) \\ &= \text{colim}_{s:K\to I} \text{colim}_{t:P\to M} \text{Hom}_{\mathcal{D}'}(P, F(I)) \\ &= \text{colim}_{t:P\to M} \text{colim}_{s:K\to I} \text{Hom}_{\mathcal{D}'}(P, F(I)) \\ &= \text{colim}_{t:P\to M} \text{colim}_{s:K\to I} \text{Hom}_{\mathcal{D}}(G(P), I) \\ &= \text{colim}_{t:P\to M} \text{Hom}_{S^{-1}\mathcal{D}}(G(P), K) \\ &= \text{Hom}_{S^{-1}\mathcal{D}}(LG(M), K) \end{aligned}$$

The first equality holds by Categories, Lemma 4.22.9. The second equality holds by the definition of morphisms in $D(\mathcal{B})$, see Categories, Remark 4.27.15. The third equality holds by Categories, Lemma 4.14.10. The fourth equality holds because F and G are adjoint. The fifth equality holds by definition of morphism in $D(\mathcal{A})$, see Categories, Remark 4.27.7. The sixth equality holds by Categories, Lemma 4.22.10. We omit the proof of functoriality. □

0DVC **Lemma 13.30.2.** Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors of abelian categories such that F is a right adjoint to G. Let $K^\bullet$ be a complex of $\mathcal{A}$ and let $M^\bullet$ be a complex of $\mathcal{B}$. If RF is defined at $K^\bullet$ and LG is defined at $M^\bullet$, then there is a canonical isomorphism

$$\text{Hom}_{D(\mathcal{B})}(M^\bullet, RF(K^\bullet)) = \text{Hom}_{D(\mathcal{A})}(LG(M^\bullet), K^\bullet)$$

This isomorphism is functorial in both variables on the triangulated subcategories of $D(\mathcal{A})$ and $D(\mathcal{B})$ where RF and LG are defined.

Proof. This is a special case of the very general Lemma 13.30.1. □

The following lemma is an example of why it is easier to work with unbounded derived categories. Namely, without having the unbounded derived functors, the lemma could not even be stated.

09T5 **Lemma 13.30.3.** Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors of abelian categories such that F is a right adjoint to G. If the derived functors $RF : D(\mathcal{A}) \to D(\mathcal{B})$ and $LG : D(\mathcal{B}) \to D(\mathcal{A})$ exist, then RF is a right adjoint to LG.

Proof. Immediate from Lemma 13.30.2. □

13.31. K-injective complexes

070G The following types of complexes can be used to compute right derived functors on the unbounded derived category.

070H Definition 13.31.1. Let $\mathcal{A}$ be an abelian category. A complex $I^\bullet$ is K-injective if for every acyclic complex $M^\bullet$ we have $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.

In the situation of the definition we have in fact $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet[i], I^\bullet) = 0$ for all i as the translate of an acyclic complex is acyclic.

070I Lemma 13.31.2. Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be a complex. The following are equivalent

(1) $I^\bullet$ is K-injective,
(2) for every quasi-isomorphism $M^\bullet \to N^\bullet$ the map
$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$$
is bijective, and
(3) for every complex $N^\bullet$ the map
$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(N^\bullet, I^\bullet)$$
is an isomorphism.

Proof. Assume (1). Then (2) holds because the functor $\mathrm{Hom}_{K(\mathcal{A})}(-, I^\bullet)$ is cohomological and the cone on a quasi-isomorphism is acyclic.

Assume (2). A morphism $N^\bullet \to I^\bullet$ in $D(\mathcal{A})$ is of the form $fs^{-1} : N^\bullet \to I^\bullet$ where $s : M^\bullet \to N^\bullet$ is a quasi-isomorphism and $f : M^\bullet \to I^\bullet$ is a map. By (2) this corresponds to a unique morphism $N^\bullet \to I^\bullet$ in $K(\mathcal{A})$, i.e., (3) holds.

Assume (3). If $M^\bullet$ is acyclic then $M^\bullet$ is isomorphic to the zero complex in $D(\mathcal{A})$ hence $\mathrm{Hom}_{D(\mathcal{A})}(M^\bullet, I^\bullet) = 0$, whence $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$ by (3), i.e., (1) holds. □

090X Lemma 13.31.3. Let $\mathcal{A}$ be an abelian category. Let (K, L, M, f, g, h) be a distinguished triangle of $K(\mathcal{A})$. If two out of K, L, M are K-injective complexes, then the third is too.

Proof. Follows from the definition, Lemma 13.4.2, and the fact that $K(\mathcal{A})$ is a triangulated category (Proposition 13.10.3). □

070J Lemma 13.31.4. Let $\mathcal{A}$ be an abelian category. A bounded below complex of injectives is K-injective.

Proof. Follows from Lemmas 13.31.2 and 13.18.8. □

0BK6 Lemma 13.31.5. Let $\mathcal{A}$ be an abelian category. Let T be a set and for each $t \in T$ let $I_t^\bullet$ be a K-injective complex. If $I^n = \prod_t I_t^n$ exists for all n, then $I^\bullet$ is a K-injective complex. Moreover, $I^\bullet$ represents the product of the objects $I_t^\bullet$ in $D(\mathcal{A})$.

Proof. Let $K^\bullet$ be an complex. Observe that the complex
$$C : \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^{b-1}) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^b) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I^{b+1})$$
has cohomology $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$ in the middle. Similarly, the complex
$$C_t : \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^{b-1}) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^b) \to \prod\nolimits_b \mathrm{Hom}(K^{-b}, I_t^{b+1})$$

computes $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I_t^\bullet)$. Next, observe that we have

$$C = \prod\nolimits_{t \in T} C_t$$

as complexes of abelian groups by our choice of I. Taking products is an exact functor on the category of abelian groups. Hence if $K^\bullet$ is acyclic, then $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I_t^\bullet) = 0$, hence C_t is acyclic, hence C is acyclic, hence we get $\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) = 0$. Thus we find that $I^\bullet$ is K-injective. Having said this, we can use Lemma 13.31.2 to conclude that

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I^\bullet) = \prod\nolimits_{t \in T} \mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I_t^\bullet)$$

and indeed $I^\bullet$ represents the product in the derived category. □

070Y Lemma 13.31.6. Let $\mathcal{A}$ be an abelian category. Let $F : K(\mathcal{A}) \to \mathcal{D}'$ be an exact functor of triangulated categories. Then RF is defined at every complex in $K(\mathcal{A})$ which is quasi-isomorphic to a K-injective complex. In fact, every K-injective complex computes RF.

Proof. By Lemma 13.14.4 it suffices to show that RF is defined at a K-injective complex, i.e., it suffices to show a K-injective complex $I^\bullet$ computes RF. Any quasi-isomorphism $I^\bullet \to N^\bullet$ is a homotopy equivalence as it has an inverse by Lemma 13.31.2. Thus $I^\bullet \to I^\bullet$ is a final object of $I^\bullet/\mathrm{Qis}(\mathcal{A})$ and we win. □

070K Lemma 13.31.7. Let $\mathcal{A}$ be an abelian category. Assume every complex has a quasi-isomorphism towards a K-injective complex. Then any exact functor $F : K(\mathcal{A}) \to \mathcal{D}'$ of triangulated categories has a right derived functor

$$RF : D(\mathcal{A}) \longrightarrow \mathcal{D}'$$

and $RF(I^\bullet) = F(I^\bullet)$ for K-injective complexes $I^\bullet$.

Proof. To see this we apply Lemma 13.14.15 with $\mathcal{I}$ the collection of K-injective complexes. Since (1) holds by assumption, it suffices to prove that if $I^\bullet \to J^\bullet$ is a quasi-isomorphism of K-injective complexes, then $F(I^\bullet) \to F(J^\bullet)$ is an isomorphism. This is clear because $I^\bullet \to J^\bullet$ is a homotopy equivalence, i.e., an isomorphism in $K(\mathcal{A})$, by Lemma 13.31.2. □

The following lemma can be generalized to limits over bigger ordinals.

070L Lemma 13.31.8. Let $\mathcal{A}$ be an abelian category. Let

$$\ldots \to I_3^\bullet \to I_2^\bullet \to I_1^\bullet$$

be an inverse system of complexes. Assume

(1) each $I_n^\bullet$ is K-injective,
(2) each map $I_{n+1}^m \to I_n^m$ is a split surjection,
(3) the limits $I^m = \lim I_n^m$ exist.

Then the complex $I^\bullet$ is K-injective.

Proof. We urge the reader to skip the proof of this lemma. Let $M^\bullet$ be an acyclic complex. Let us abbreviate $H_n(a, b) = \mathrm{Hom}_\mathcal{A}(M^a, I_n^b)$. With this notation $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ is the cohomology of the complex

$$\prod_m \lim_n H_n(m, m-2) \to \prod_m \lim_n H_n(m, m-1) \to \prod_m \lim_n H_n(m, m) \to \prod_m \lim_n H_n(m, m+1)$$

in the third spot from the left. We may exchange the order of $\prod$ and lim and each of the complexes

$$\prod_m H_n(m, m-2) \to \prod_m H_n(m, m-1) \to \prod_m H_n(m, m) \to \prod_m H_n(m, m+1)$$

is exact by assumption (1). By assumption (2) the maps in the systems

$$\ldots \to \prod_m H_3(m, m-2) \to \prod_m H_2(m, m-2) \to \prod_m H_1(m, m-2)$$

are surjective. Thus the lemma follows from Homology, Lemma 12.31.4. □

It appears that a combination of Lemmas 13.29.3, 13.31.4, and 13.31.8 produces "enough K-injectives" for any abelian category with enough injectives and countable products. Actually, this may not work! See Lemma 13.34.4 for an explanation.

08BJ Lemma 13.31.9. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume

(1) u is right adjoint to v, and
(2) v is exact.

Then u transforms K-injective complexes into K-injective complexes.

Proof. Let $I^\bullet$ be a K-injective complex of $\mathcal{A}$. Let $M^\bullet$ be a acyclic complex of $\mathcal{B}$. As v is exact we see that $v(M^\bullet)$ is an acyclic complex. By adjointness we get

$$0 = \mathrm{Hom}_{K(\mathcal{A})}(v(M^\bullet), I^\bullet) = \mathrm{Hom}_{K(\mathcal{B})}(M^\bullet, u(I^\bullet))$$

hence the lemma follows. □

13.32. Bounded cohomological dimension

07K5 There is another case where the unbounded derived functor exists. Namely, when the functor has bounded cohomological dimension.

07K6 Lemma 13.32.1. Let $\mathcal{A}$ be an abelian category. Let $d : \mathrm{Ob}(\mathcal{A}) \to \{0, 1, 2, \ldots, \infty\}$ be a function. Assume that

(1) every object of $\mathcal{A}$ is a subobject of an object A with $d(A) = 0$,
(2) $d(A \oplus B) \leq \max\{d(A), d(B)\}$ for $A, B \in \mathcal{A}$, and
(3) if $0 \to A \to B \to C \to 0$ is short exact, then $d(C) \leq \max\{d(A) - 1, d(B)\}$.

Let $K^\bullet$ be a complex such that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$. Then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $d(L^n) = 0$ for all $n \in \mathbf{Z}$.

Proof. By Lemma 13.15.5 we can find a quasi-isomorphism $\sigma_{\geq 0}K^\bullet \to M^\bullet$ with $M^n = 0$ for $n < 0$ and $d(M^n) = 0$ for $n \geq 0$. Then $K^\bullet$ is quasi-isomorphic to the complex

$$\ldots \to K^{-2} \to K^{-1} \to M^0 \to M^1 \to \ldots$$

Hence we may assume that $d(K^n) = 0$ for $n \gg 0$. Note that the condition $n + d(K^n) \to -\infty$ as $n \to -\infty$ is not violated by this replacement.

We are going to improve $K^\bullet$ by an (infinite) sequence of elementary replacements. An elementary replacement is the following. Choose an index n such that $d(K^n) >$

0. Choose an injection $K^n \to M$ where $d(M) = 0$. Set $M' = \text{Coker}(K^n \to M \oplus K^{n+1})$. Consider the map of complexes

$$\begin{array}{ccccccccc} K^\bullet : & & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & K^{n+2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (K')^\bullet : & & K^{n-1} & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & K^{n+2} \end{array}$$

It is clear that $K^\bullet \to (K')^\bullet$ is a quasi-isomorphism. Moreover, it is clear that $d((K')^n) = 0$ and

$$d((K')^{n+1}) \leq \max\{d(K^n) - 1, d(M \oplus K^{n+1})\} \leq \max\{d(K^n) - 1, d(K^{n+1})\}$$

and the other values are unchanged.

To finish the proof we carefuly choose the order in which to do the elementary replacements so that for every integer m the complex $\sigma_{\geq m} K^\bullet$ is changed only a finite number of times. To do this set

$$\xi(K^\bullet) = \max\{n + d(K^n) \mid d(K^n) > 0\}$$

and

$$I = \{n \in \mathbf{Z} \mid \xi(K^\bullet) = n + d(K^n) \text{ and } d(K^n) > 0\}$$

Our assumption that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$ and the fact that $d(K^n) = 0$ for $n >> 0$ implies $\xi(K^\bullet) < +\infty$ and that I is a finite set. It is clear that $\xi((K')^\bullet) \leq \xi(K^\bullet)$ for an elementary transformation as above. An elementary transformation changes the complex in degrees $\leq \xi(K^\bullet) + 1$. Hence if we can find finite sequence of elementary transformations which decrease $\xi(K^\bullet)$, then we win. However, note that if we do an elementary transformation starting with the smallest element $n \in I$, then we either decrease the size of I, or we increase $\min I$. Since every element of I is $\leq \xi(K^\bullet)$ we see that we win after a finite number of steps. $\square$

07K7 **Lemma 13.32.2.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Assume

(1) every object of $\mathcal{A}$ is a subobject of an object which is right acyclic for F,
(2) there exists an integer $n \geq 0$ such that $R^n F = 0$,

Then

(1) $RF : D(\mathcal{A}) \to D(\mathcal{B})$ exists,
(2) any complex consisting of right acyclic objects for F computes RF,
(3) any complex is the source of a quasi-isomorphism into a complex consisting of right acyclic objects for F,
(4) for $E \in D(\mathcal{A})$
  (a) $H^i(RF(\tau_{\leq a} E) \to H^i(RF(E))$ is an isomorphism for $i \leq a$,
  (b) $H^i(RF(E)) \to H^i(RF(\tau_{\geq b-n+1} E))$ is an isomorphism for $i \geq b$,
  (c) if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(RF(E)) = 0$ for $i \notin [a, b + n - 1]$.

Proof. Note that the first assumption implies that $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists, see Proposition 13.16.8. Let A be an object of $\mathcal{A}$. Choose an injection $A \to A'$ with A' acyclic. Then we see that $R^{n+1}F(A) = R^n F(A'/A) = 0$ by the long exact cohomology sequence. Hence we conclude that $R^{n+1}F = 0$. Continuing like this using induction we find that $R^m F = 0$ for all $m \geq n$.

We are going to use Lemma 13.32.1 with the function $d : \text{Ob}(\mathcal{A}) \to \{0, 1, 2, \ldots\}$ given by $d(A) = \max\{0\} \cup \{i \mid R^iF(A) \neq 0\}$. The first assumption of Lemma 13.32.1 is our assumption (1). The second assumption of Lemma 13.32.1 follows from the fact that $RF(A \oplus B) = RF(A) \oplus RF(B)$. The third assumption of Lemma 13.32.1 follows from the long exact cohomology sequence. Hence for every complex $K^\bullet$ there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ into a complex of objects right acyclic for F. This proves statement (3).

We claim that if $L^\bullet \to M^\bullet$ is a quasi-isomorphism of complexes of right acyclic objects for F, then $F(L^\bullet) \to F(M^\bullet)$ is a quasi-isomorphism. If we prove this claim then we get statements (1) and (2) of the lemma by Lemma 13.14.15. To prove the claim pick an integer $i \in \mathbf{Z}$. Consider the distinguished triangle

$$\sigma_{\geq i-n-1}L^\bullet \to \sigma_{\geq i-n-1}M^\bullet \to Q^\bullet,$$

i.e., let $Q^\bullet$ be the cone of the first map. Note that $Q^\bullet$ is bounded below and that $H^j(Q^\bullet)$ is zero except possibly for $j = i - n - 1$ or $j = i - n - 2$. We may apply RF to $Q^\bullet$. Using the second spectral sequence of Lemma 13.21.3 and the assumed vanishing of cohomology (2) we conclude that $H^j(RF(Q^\bullet))$ is zero except possibly for $j \in \{i-n-2, \ldots, i-1\}$. Hence we see that $RF(\sigma_{\geq i-n-1}L^\bullet) \to RF(\sigma_{\geq i-n-1}M^\bullet)$ induces an isomorphism of cohomology objects in degrees $\geq i$. By Proposition 13.16.8 we know that $RF(\sigma_{\geq i-n-1}L^\bullet) = \sigma_{\geq i-n-1}F(L^\bullet)$ and $RF(\sigma_{\geq i-n-1}M^\bullet) = \sigma_{\geq i-n-1}F(M^\bullet)$. We conclude that $F(L^\bullet) \to F(M^\bullet)$ is an isomorphism in degree i as desired.

Part (4)(a) follows from Lemma 13.16.1.

For part (4)(b) let E be represented by the complex $L^\bullet$ of objects right acyclic for F. By part (2) $RF(E)$ is represented by the complex $F(L^\bullet)$ and $RF(\sigma_{\geq c}L^\bullet)$ is represented by $\sigma_{\geq c}F(L^\bullet)$. Consider the distinguished triangle

$$H^{b-n}(L^\bullet)[n-b] \to \tau_{\geq b-n}L^\bullet \to \tau_{\geq b-n+1}L^\bullet$$

of Remark 13.12.4. The vanishing established above gives that $H^i(RF(\tau_{\geq b-n}L^\bullet))$ agrees with $H^i(RF(\tau_{\geq b-n+1}L^\bullet))$ for $i \geq b$. Consider the short exact sequence of complexes

$$0 \to \text{Im}(L^{b-n-1} \to L^{b-n})[n-b] \to \sigma_{\geq b-n}L^\bullet \to \tau_{\geq b-n}L^\bullet \to 0$$

Using the distinguished triangle associated to this (see Section 13.12) and the vanishing as before we conclude that $H^i(RF(\tau_{\geq b-n}L^\bullet))$ agrees with $H^i(RF(\sigma_{\geq b-n}L^\bullet))$ for $i \geq b$. Since the map $RF(\sigma_{\geq b-n}L^\bullet) \to RF(L^\bullet)$ is represented by $\sigma_{\geq b-n}F(L^\bullet) \to F(L^\bullet)$ we conclude that this in turn agrees with $H^i(RF(L^\bullet))$ for $i \geq b$ as desired.

Proof of (4)(c). Under the assumption on E we have $\tau_{\leq a-1}E = 0$ and we get the vanishing of $H^i(RF(E))$ for $i \leq a-1$ from part (4)(a). Similarly, we have $\tau_{\geq b+1}E = 0$ and hence we get the vanishing of $H^i(RF(E))$ for $i \geq b+n$ from part (4)(b). □

07K8 **Lemma 13.32.3.** Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. If

(1) every object of $\mathcal{A}$ is a quotient of an object which is left acyclic for F,
(2) there exists an integer $n \geq 0$ such that $L^nF = 0$,

Then

(1) $LF : D(\mathcal{A}) \to D(\mathcal{B})$ exists,
(2) any complex consisting of left acyclic objects for F computes LF,

(3) any complex is the target of a quasi-isomorphism from a complex consisting of left acyclic objects for F,
(4) for $E \in D(\mathcal{A})$
   (a) $H^i(LF(\tau_{\leq a+n-1}E) \to H^i(LF(E))$ is an isomorphism for $i \leq a$,
   (b) $H^i(LF(E)) \to H^i(LF(\tau_{\geq b}E))$ is an isomorphism for $i \geq b$,
   (c) if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(LF(E)) = 0$ for $i \notin [a - n + 1, b]$.

Proof. This is dual to Lemma 13.32.2. □

13.33. Derived colimits

0A5K In a triangulated category there is a notion of derived colimit.

090Z **Definition 13.33.1.** Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. We say an object K is a *derived colimit*, or a *homotopy colimit* of the system (K_n) if the direct sum $\bigoplus K_n$ exists and there is a distinguished triangle

$$\bigoplus K_n \to \bigoplus K_n \to K \to \bigoplus K_n[1]$$

where the map $\bigoplus K_n \to \bigoplus K_n$ is given by $1 - f_n$ in degree n. If this is the case, then we sometimes indicate this by the notation $K = \text{hocolim} K_n$.

By TR3 a derived colimit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived colimit of K_n exists as soon as $\bigoplus K_n$ exists. The derived category $D(\textit{Ab})$ of the category of abelian groups is an example of a triangulated category where all homotopy colimits exist.

The nonuniqueness makes it hard to pin down the derived colimit. In More on Algebra, Lemma 15.86.5 the reader finds an exact sequence

$$0 \to R^1 \lim \text{Hom}(K_n, L[-1]) \to \text{Hom}(\text{hocolim} K_n, L) \to \lim \text{Hom}(K_n, L) \to 0$$

describing the Homs out of a homotopy colimit in terms of the usual Homs.

0CRH **Remark 13.33.2.** Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. We may think of a derived colimit as an object K of $\mathcal{D}$ endowed with morphisms $i_n : K_n \to K$ such that $i_{n+1} \circ f_n = i_n$ and such that there exists a morphism $c : K \to \bigoplus K_n$ with the property that

$$\bigoplus K_n \xrightarrow{1-f_n} \bigoplus K_n \xrightarrow{i_n} K \xrightarrow{c} \bigoplus K_n[1]$$

is a distinguished triangle. If (K', i'_n, c') is a second derived colimit, then there exists an isomorphism $\varphi : K \to K'$ such that $\varphi \circ i_n = i'_n$ and $c' \circ \varphi = c$. The existence of φ is TR3 and the fact that φ is an isomorphism is Lemma 13.4.3.

0CRI **Remark 13.33.3.** Let $\mathcal{D}$ be a triangulated category. Let $(a_n) : (K_n, f_n) \to (L_n, g_n)$ be a morphism of systems of objects of $\mathcal{D}$. Let (K, i_n, c) be a derived colimit of the first system and let (L, j_n, d) be a derived colimit of the second system with notation as in Remark 13.33.2. Then there exists a morphism $a : K \to L$ such that $a \circ i_n = j_n$ and $d \circ a = (a_n[1]) \circ c$. This follows from TR3 applied to the defining distinguished triangles.

0CRJ **Lemma 13.33.4.** *Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. Let $n_1 < n_2 < n_3 < \ldots$ be a sequence of integers. Assume $\bigoplus K_n$ and*

$\bigoplus K_{n_i}$ exist. Then there exists an isomorphism $\text{hocolim} K_{n_i} \to \text{hocolim} K_n$ such that

$$\begin{tikzcd} K_{n_i} \arrow[r] \arrow[d, "\text{id}"'] & \text{hocolim} K_{n_i} \arrow[d] \\ K_{n_i} \arrow[r] & \text{hocolim} K_n \end{tikzcd}$$

commutes for all i.

Proof. Let $g_i : K_{n_i} \to K_{n_{i+1}}$ be the composition $f_{n_{i+1}-1} \circ \ldots \circ f_{n_i}$. We construct commutative diagrams

$$\begin{tikzcd} \bigoplus_i K_{n_i} \arrow[r, "1-g_i"'] \arrow[d, "b"'] & \bigoplus_i K_{n_i} \arrow[d, "a"] \\ \bigoplus_n K_n \arrow[r, "1-f_n"] & \bigoplus_n K_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bigoplus_n K_n \arrow[r, "1-f_n"'] \arrow[d, "d"'] & \bigoplus_n K_n \arrow[d, "c"] \\ \bigoplus_i K_{n_i} \arrow[r, "1-g_i"] & \bigoplus_i K_{n_i} \end{tikzcd}$$

as follows. Let $a_i = a|_{K_{n_i}}$ be the inclusion of K_{n_i} into the direct sum. In other words, a is the natural inclusion. Let $b_i = b|_{K_{n_i}}$ be the map

$$K_{n_i} \xrightarrow{1,\ f_{n_i},\ f_{n_i+1} \circ f_{n_i},\ \ldots,\ f_{n_{i+1}-2} \circ \ldots \circ f_{n_i}} K_{n_i} \oplus K_{n_i+1} \oplus \ldots \oplus K_{n_{i+1}-1}$$

If $n_{i-1} < j \leq n_i$, then we let $c_j = c|_{K_j}$ be the map

$$K_j \xrightarrow{f_{n_i-1} \circ \ldots \circ f_j} K_{n_i}$$

We let $d_j = d|_{K_j}$ be zero if $j \neq n_i$ for any i and we let d_{n_i} be the natural inclusion of K_{n_i} into the direct sum. In other words, d is the natural projection. By TR3 these diagrams define morphisms

$$\varphi : \text{hocolim} K_{n_i} \to \text{hocolim} K_n \quad \text{and} \quad \psi : \text{hocolim} K_n \to \text{hocolim} K_{n_i}$$

Since $c \circ a$ and $d \circ b$ are the identity maps we see that $\varphi \circ \psi$ is an isomorphism by Lemma 13.4.3. The other way around we get the morphisms $a \circ c$ and $b \circ d$. Consider the morphism $h = (h_j) : \bigoplus K_n \to \bigoplus K_n$ given by the rule: for $n_{i-1} < j < n_i$ we set

$$h_j : K_j \xrightarrow{1,\ f_j,\ f_{j+1} \circ f_j,\ \ldots,\ f_{n_i-1} \circ \ldots \circ f_j} K_j \oplus \ldots \oplus K_{n_i}$$

Then the reader verifies that $(1-f) \circ h = \text{id} - a \circ c$ and $h \circ (1-f) = \text{id} - b \circ d$. This means that $\text{id} - \psi \circ \varphi$ has square zero by Lemma 13.4.5 (small argument omitted). In other words, $\psi \circ \varphi$ differs from the identity by a nilpotent endomorphism, hence is an isomorphism. Thus φ and ψ are isomorphisms as desired. $\square$

0A5L **Lemma 13.33.5.** Let $\mathcal{A}$ be an abelian category. If $\mathcal{A}$ has exact countable direct sums, then $D(\mathcal{A})$ has countable direct sums. In fact given a collection of complexes $K_i^\bullet$ indexed by a countable index set I the termwise direct sum $\bigoplus K_i^\bullet$ is the direct sum of $K_i^\bullet$ in $D(\mathcal{A})$.

Proof. Let $L^\bullet$ be a complex. Suppose given maps $\alpha_i : K_i^\bullet \to L^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : M_i^\bullet \to K_i^\bullet$ of complexes and maps of complexes $f_i : M_i^\bullet \to L^\bullet$ such that $\alpha_i = f_i s_i^{-1}$. By assumption the map of complexes

$$s : \bigoplus M_i^\bullet \longrightarrow \bigoplus K_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \bigoplus f_i$ we see that $\alpha = fs^{-1}$ is a map in $D(\mathcal{A})$ whose composition with the coprojection $K_i^\bullet \to \bigoplus K_i^\bullet$ is α_i. We omit the verification that α is unique. □

093W Lemma 13.33.6. Let $\mathcal{A}$ be an abelian category. Assume colimits over $\mathbf{N}$ exist and are exact. Then countable direct sums exists and are exact. Moreover, if (A_n, f_n) is a system over $\mathbf{N}$, then there is a short exact sequence

$$0 \to \bigoplus A_n \to \bigoplus A_n \to \operatorname{colim} A_n \to 0$$

where the first map in degree n is given by $1 - f_n$.

Proof. The first statement follows from $\bigoplus A_n = \operatorname{colim}(A_1 \oplus \ldots \oplus A_n)$. For the second, note that for each n we have the short exact sequence

$$0 \to A_1 \oplus \ldots \oplus A_{n-1} \to A_1 \oplus \ldots \oplus A_n \to A_n \to 0$$

where the first map is given by the maps $1 - f_i$ and the second map is the sum of the transition maps. Take the colimit to get the sequence of the lemma. □

0949 Lemma 13.33.7. Let $\mathcal{A}$ be an abelian category. Let $L_n^\bullet$ be a system of complexes of $\mathcal{A}$. Assume colimits over $\mathbf{N}$ exist and are exact in $\mathcal{A}$. Then the termwise colimit $L^\bullet = \operatorname{colim} L_n^\bullet$ is a homotopy colimit of the system in $D(\mathcal{A})$.

Proof. We have an exact sequence of complexes

$$0 \to \bigoplus L_n^\bullet \to \bigoplus L_n^\bullet \to L^\bullet \to 0$$

by Lemma 13.33.6. The direct sums are direct sums in $D(\mathcal{A})$ by Lemma 13.33.5. Thus the result follows from the definition of derived colimits in Definition 13.33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 13.12.1). □

0CRK Lemma 13.33.8. Let $\mathcal{D}$ be a triangulated category having countable direct sums. Let $\mathcal{A}$ be an abelian category with exact colimits over $\mathbf{N}$. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor commuting with countable direct sums. Then $H(\text{hocolim} K_n) = \operatorname{colim} H(K_n)$ for any system of objects of $\mathcal{D}$.

Proof. Write $K = \text{hocolim} K_n$. Apply H to the defining distinguished triangle to get

$$\bigoplus H(K_n) \to \bigoplus H(K_n) \to H(K) \to \bigoplus H(K_n[1]) \to \bigoplus H(K_n[1])$$

where the first map is given by $1 - H(f_n)$ and the last map is given by $1 - H(f_n[1])$. Apply Lemma 13.33.6 to see that this proves the lemma. □

The following lemma tells us that taking maps out of a compact object (to be defined later) commutes with derived colimits.

094A Lemma 13.33.9. Let $\mathcal{D}$ be a triangulated category with countable direct sums. Let $K \in \mathcal{D}$ be an object such that for every countable set of objects $E_n \in \mathcal{D}$ the canonical map

$$\bigoplus \operatorname{Hom}_{\mathcal{D}}(K, E_n) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(K, \bigoplus E_n)$$

is a bijection. Then, given any system L_n of $\mathcal{D}$ over $\mathbf{N}$ whose derived colimit $L = \text{hocolim} L_n$ exists we have that

$$\operatorname{colim} \operatorname{Hom}_{\mathcal{D}}(K, L_n) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(K, L)$$

is a bijection.

Proof. Consider the defining distinguished triangle

$$\bigoplus L_n \to \bigoplus L_n \to L \to \bigoplus L_n[1]$$

Apply the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(K, -)$ (see Lemma 13.4.2). By elementary considerations concerning colimits of abelian groups we get the result. □

13.34. Derived limits

08TB In a triangulated category there is a notion of derived limit.

08TC Definition 13.34.1. Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be an inverse system of objects of $\mathcal{D}$. We say an object K is a derived limit, or a homotopy limit of the system (K_n) if the product $\prod K_n$ exists and there is a distinguished triangle

$$K \to \prod K_n \to \prod K_n \to K[1]$$

where the map $\prod K_n \to \prod K_n$ is given by $(k_n) \mapsto (k_n - f_{n+1}(k_{n+1}))$. If this is the case, then we sometimes indicate this by the notation $K = R\lim K_n$.

By TR3 a derived limit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived limit $R\lim K_n$ exists as soon as $\prod K_n$ exists. The derived category $D(\mathrm{Ab})$ of the category of abelian groups is an example of a triangulated category where all derived limits exist.

The nonuniqueness makes it hard to pin down the derived limit. In More on Algebra, Lemma 15.86.4 the reader finds an exact sequence

$$0 \to R^1 \lim \mathrm{Hom}(L, K_n[-1]) \to \mathrm{Hom}(L, R\lim K_n) \to \lim \mathrm{Hom}(L, K_n) \to 0$$

describing the Homs into a derived limit in terms of the usual Homs.

07KC Lemma 13.34.2. Let $\mathcal{A}$ be an abelian category with exact countable products. Then

(1) $D(\mathcal{A})$ has countable products,
(2) countable products $\prod K_i$ in $D(\mathcal{A})$ are obtained by taking termwise products of any complexes representing the K_i, and
(3) $H^p(\prod K_i) = \prod H^p(K_i)$.

Proof. Let $K_i^\bullet$ be a complex representing K_i in $D(\mathcal{A})$. Let $L^\bullet$ be a complex. Suppose given maps $\alpha_i : L^\bullet \to K_i^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : K_i^\bullet \to M_i^\bullet$ of complexes and maps of complexes $f_i : L^\bullet \to M_i^\bullet$ such that $\alpha_i = s_i^{-1} f_i$. By assumption the map of complexes

$$s : \prod K_i^\bullet \longrightarrow \prod M_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \prod f_i$ we see that $\alpha = s^{-1} f$ is a map in $D(\mathcal{A})$ whose composition with the projection $\prod K_i^\bullet \to K_i^\bullet$ is α_i. We omit the verification that α is unique. □

The duals of Lemmas 13.33.6, 13.33.7, and 13.33.9 should be stated here and proved. However, we do not know any applications of these lemmas for now.

0BK7 Lemma 13.34.3. Let $\mathcal{A}$ be an abelian category with countable products and enough injectives. Let (K_n) be an inverse system of $D^+(\mathcal{A})$. Then $R\lim K_n$ exists.

Proof. It suffices to show that $\prod K_n$ exists in $D(\mathcal{A})$. For every n we can represent K_n by a bounded below complex $I_n^\bullet$ of injectives (Lemma 13.18.3). Then $\prod K_n$ is represented by $\prod I_n^\bullet$, see Lemma 13.31.5. $\square$

070M **Lemma 13.34.4.** Let $\mathcal{A}$ be an abelian category with countable products and enough injectives. Let $K^\bullet$ be a complex. Let $I_n^\bullet$ be the inverse system of bounded below complexes of injectives produced by Lemma 13.29.3. Then $I^\bullet = \lim I_n^\bullet$ exists, is K-injective, and the following are equivalent

(1) the map $K^\bullet \to I^\bullet$ is a quasi-isomorphism,
(2) the canonical map $K^\bullet \to R\lim \tau_{\geq -n}K^\bullet$ is an isomorphism in $D(\mathcal{A})$.

Proof. The statement of the lemma makes sense as $R\lim \tau_{\geq -n}K^\bullet$ exists by Lemma 13.34.3. Each complex $I_n^\bullet$ is K-injective by Lemma 13.31.4. Choose direct sum decompositions $I_{n+1}^p = C_{n+1}^p \oplus I_n^p$ for all $n \geq 1$. Set $C_1^p = I_1^p$. The complex $I^\bullet = \lim I_n^\bullet$ exists because we can take $I^p = \prod_{n \geq 1} C_n^p$. Fix $p \in \mathbf{Z}$. We claim there is a split short exact sequence

$$0 \to I^p \to \prod I_n^p \to \prod I_n^p \to 0$$

of objects of $\mathcal{A}$. Here the first map is given by the projection maps $I^p \to I_n^p$ and the second map by $(x_n) \mapsto (x_n - f_{n+1}^p(x_{n+1}))$ where $f_n^p : I_n^p \to I_{n-1}^p$ are the transition maps. The splitting comes from the map $\prod I_n^p \to \prod C_n^p = I^p$. We obtain a termwise split short exact sequence of complexes

$$0 \to I^\bullet \to \prod I_n^\bullet \to \prod I_n^\bullet \to 0$$

Hence a corresponding distinguished triangle in $K(\mathcal{A})$ and $D(\mathcal{A})$. By Lemma 13.31.5 the products are K-injective and represent the corresponding products in $D(\mathcal{A})$. It follows that $I^\bullet$ represents $R\lim I_n^\bullet$ (Definition 13.34.1). Moreover, it follows that $I^\bullet$ is K-injective by Lemma 13.31.3. By the commutative diagram of Lemma 13.29.3 we obtain a corresponding commutative diagram

$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d] & R\lim \tau_{\geq -n}K^\bullet \arrow[d] \\ I^\bullet \arrow[r] & R\lim I_n^\bullet \end{tikzcd}$$

in $D(\mathcal{A})$. Since the right vertical arrow is an isomorphism (as derived limits are defined on the level of the derived category and since $\tau_{\geq -n}K^\bullet \to I_n^\bullet$ is a quasi-isomorphism), the lemma follows. $\square$

090Y **Lemma 13.34.5.** Let $\mathcal{A}$ be an abelian category having enough injectives and exact countable products. Then for every complex there is a quasi-isomorphism to a K-injective complex.

Proof. By Lemma 13.34.4 it suffices to show that $K \to R\lim \tau_{\geq -n}K$ is an isomorphism for all K in $D(\mathcal{A})$. Consider the defining distinguished triangle

$$R\lim \tau_{\geq -n}K \to \prod \tau_{\geq -n}K \to \prod \tau_{\geq -n}K \to (R\lim \tau_{\geq -n}K)[1]$$

By Lemma 13.34.2 we have

$$H^p(\prod \tau_{\geq -n}K) = \prod\nolimits_{p \geq -n} H^p(K)$$

It follows in a straightforward manner from the long exact cohomology sequence of the displayed distinguished triangle that $H^p(R\lim \tau_{\geq -n}K) = H^p(K)$. $\square$

13.35. Operations on full subcategories

0FX0 Let $\mathcal{T}$ be a triangulated category. We will identify full subcategories of $\mathcal{T}$ with subsets of $\text{Ob}(\mathcal{T})$. Given full subcategories $\mathcal{A}, \mathcal{B}, \ldots$ we let

(1) $\mathcal{A}[a, b]$ for $-\infty \leq a \leq b \leq \infty$ be the full subcategory of $\mathcal{T}$ consisting of all objects $A[-i]$ with $i \in [a, b] \cap \mathbf{Z}$ with $A \in \text{Ob}(\mathcal{A})$ (note the minus sign!),
(2) $smd(\mathcal{A})$ be the full subcategory of $\mathcal{T}$ consisting of all objects which are isomorphic to direct summands of objects of $\mathcal{A}$,
(3) $add(\mathcal{A})$ be the full subcategory of $\mathcal{T}$ consisting of all objects which are isomorphic to finite direct sums of objects of $\mathcal{A}$,
(4) $\mathcal{A} \star \mathcal{B}$ be the full subcategory of $\mathcal{T}$ consisting of all objects X of $\mathcal{T}$ which fit into a distinguished triangle $A \to X \to B$ with $A \in \text{Ob}(\mathcal{A})$ and $B \in \text{Ob}(\mathcal{B})$,
(5) $\mathcal{A}^{\star n} = \mathcal{A} \star \ldots \star \mathcal{A}$ with $n \geq 1$ factors (we will see $\star$ is associative below),
(6) $smd(add(\mathcal{A})^{\star n}) = smd(add(\mathcal{A}) \star \ldots \star add(\mathcal{A}))$ with $n \geq 1$ factors.

If E is an object of $\mathcal{T}$, then we think of E sometimes also as the full subcategory of $\mathcal{T}$ whose single object is E. Then we can consider things like $add(E[-1, 2])$ and so on and so forth. We warn the reader that this notation is not universally accepted.

0FX1 **Lemma 13.35.1.** Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ we have $(\mathcal{A} \star \mathcal{B}) \star \mathcal{C} = \mathcal{A} \star (\mathcal{B} \star \mathcal{C})$.

Proof. If we have distinguished triangles $A \to X \to B$ and $X \to Y \to C$ then by Axiom TR4 we have distinguished triangles $A \to Y \to Z$ and $B \to Z \to C$. $\square$

0FX2 **Lemma 13.35.2.** Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$ we have $smd(\mathcal{A}) \star smd(\mathcal{B}) \subset smd(\mathcal{A} \star \mathcal{B})$ and $smd(smd(\mathcal{A}) \star smd(\mathcal{B})) = smd(\mathcal{A} \star \mathcal{B})$.

Proof. Suppose we have a distinguished triangle $A_1 \to X \to B_1$ where $A_1 \oplus A_2 \in \text{Ob}(\mathcal{A})$ and $B_1 \oplus B_2 \in \text{Ob}(\mathcal{B})$. Then we obtain a distinguished triangle $A_1 \oplus A_2 \to A_2 \oplus X \oplus B_2 \to B_1 \oplus B_2$ which proves that X is in $smd(\mathcal{A} \star \mathcal{B})$. This proves the inclusion. The equality follows trivially from this. $\square$

0FX3 **Lemma 13.35.3.** Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}$, $\mathcal{B}$ the full subcategories $add(\mathcal{A}) \star add(\mathcal{B})$ and $smd(add(\mathcal{A}))$ are closed under direct sums.

Proof. Namely, if $A \to X \to B$ and $A' \to X' \to B'$ are distinguished triangles and $A, A' \in add(\mathcal{A})$ and $B, B' \in add(\mathcal{B})$ then $A \oplus A' \to X \oplus X' \to B \oplus B'$ is a distinguished triangle with $A \oplus A' \in add(\mathcal{A})$ and $B \oplus B' \in add(\mathcal{B})$. The result for $smd(add(\mathcal{A}))$ is trivial. $\square$

0FX4 **Lemma 13.35.4.** Let $\mathcal{T}$ be a triangulated category. Given a full subcategory $\mathcal{A}$ for $n \geq 1$ the subcategory

$$\mathcal{C}_n = smd(add(\mathcal{A})^{\star n}) = smd(add(\mathcal{A}) \star \ldots \star add(\mathcal{A}))$$

defined above is a strictly full subcategory of $\mathcal{T}$ closed under direct sums and direct summands and $\mathcal{C}_{n+m} = smd(\mathcal{C}_n \star \mathcal{C}_m)$ for all $n, m \geq 1$.

Proof. Immediate from Lemmas 13.35.1, 13.35.2, and 13.35.3. $\square$

0FX5 Remark 13.35.5. Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor of triangulated categories. Given a full subcategory $\mathcal{A}$ of $\mathcal{T}$ we denote $F(\mathcal{A})$ the full subcategory of $\mathcal{T}'$ whose objects consists of all objects $F(A)$ with $A \in \mathrm{Ob}(\mathcal{A})$. We have

$$\begin{gathered} F(\mathcal{A}[a,b]) = F(\mathcal{A})[a,b] \\ F(smd(\mathcal{A})) \subset smd(F(\mathcal{A})), \\ F(add(\mathcal{A})) \subset add(F(\mathcal{A})), \\ F(\mathcal{A} \star \mathcal{B}) \subset F(\mathcal{A}) \star F(\mathcal{B}), \\ F(\mathcal{A}^{\star n}) \subset F(\mathcal{A})^{\star n}. \end{gathered}$$

We omit the trivial verifications.

0FX6 Remark 13.35.6. Let $\mathcal{T}$ be a triangulated category. Given full subcategories $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \ldots$ and $\mathcal{B}$ of $\mathcal{T}$ we have

$$\begin{gathered} \left(\bigcup \mathcal{A}_i\right)[a,b] = \bigcup \mathcal{A}_i[a,b] \\ smd\left(\bigcup \mathcal{A}_i\right) = \bigcup smd(\mathcal{A}_i), \\ add\left(\bigcup \mathcal{A}_i\right) = \bigcup add(\mathcal{A}_i), \\ \left(\bigcup \mathcal{A}_i\right) \star \mathcal{B} = \bigcup \mathcal{A}_i \star \mathcal{B}, \\ \mathcal{B} \star \left(\bigcup \mathcal{A}_i\right) = \bigcup \mathcal{B} \star \mathcal{A}_i, \\ \left(\bigcup \mathcal{A}_i\right)^{\star n} = \bigcup \mathcal{A}_i^{\star n}. \end{gathered}$$

We omit the trivial verifications.

0FX7 Lemma 13.35.7. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D} = D(\mathcal{A})$. Let $\mathcal{E} \subset \mathrm{Ob}(\mathcal{A})$ be a subset which we view as a subset of $\mathrm{Ob}(\mathcal{D})$ also. Let K be an object of $\mathcal{D}$.

(1) Let $b \geq a$ and assume $H^i(K)$ is zero for $i \notin [a,b]$ and $H^i(K) \in \mathcal{E}$ if $i \in [a,b]$. Then K is in $smd(add(\mathcal{E}[a,b])^{\star(b-a+1)})$.
(2) Let $b \geq a$ and assume $H^i(K)$ is zero for $i \notin [a,b]$ and $H^i(K) \in smd(add(\mathcal{E}))$ if $i \in [a,b]$. Then K is in $smd(add(\mathcal{E}[a,b])^{\star(b-a+1)})$.
(3) Let $b \geq a$ and assume K can be represented by a complex $K^\bullet$ with $K^i = 0$ for $i \notin [a,b]$ and $K^i \in \mathcal{E}$ for $i \in [a,b]$. Then K is in $smd(add(\mathcal{E}[a,b])^{\star(b-a+1)})$.
(4) Let $b \geq a$ and assume K can be represented by a complex $K^\bullet$ with $K^i = 0$ for $i \notin [a,b]$ and $K^i \in smd(add(\mathcal{E}))$ for $i \in [a,b]$. Then K is in $smd(add(\mathcal{E}[a,b])^{\star(b-a+1)})$.

Proof. We will use Lemma 13.35.4 without further mention. We will prove (2) which trivially implies (1). We use induction on $b - a$. If $b - a = 0$, then K is isomorphic to $H^i(K)[-a]$ in $\mathcal{D}$ and the result is immediate. If $b - a > 0$, then we consider the distinguished triangle

$$\tau_{\leq b-1}K^\bullet \to K^\bullet \to K^b[-b]$$

and we conclude by induction on $b - a$. We omit the proof of (3) and (4). $\square$

0FX8 Lemma 13.35.8. Let $\mathcal{T}$ be a triangulated category. Let $H : \mathcal{T} \to \mathcal{A}$ be a homological functor to an abelian category $\mathcal{A}$. Let $a \leq b$ and $\mathcal{E} \subset \mathrm{Ob}(\mathcal{T})$ be a subset such that $H^i(E) = 0$ for $E \in \mathcal{E}$ and $i \notin [a,b]$. Then for $X \in smd(add(\mathcal{E}[-m,m])^{\star n})$ we have $H^i(X) = 0$ for $i \notin [-m + na, m + nb]$.

Proof. Omitted. Pleasant exercise in the definitions. $\square$

13.36. Generators of triangulated categories

09SI In this section we briefly introduce a few of the different notions of a generator for a triangulated category. Our terminology is taken from [BV03] (except that we use "saturated" for what they call "épaisse", see Definition 13.6.1, and our definition of $add(\mathcal{A})$ is different).

Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. Denote $\langle E\rangle_1$ the strictly full subcategory of $\mathcal{D}$ consisting of objects in $\mathcal{D}$ isomorphic to direct summands of finite direct sums

$$\bigoplus\nolimits_{i=1,\ldots,r} E[n_i]$$

of shifts of E. It is clear that in the notation of Section 13.35 we have

$$\langle E\rangle_1 = smd(add(E[-\infty,\infty]))$$

For $n > 1$ let $\langle E\rangle_n$ denote the full subcategory of $\mathcal{D}$ consisting of objects of $\mathcal{D}$ isomorphic to direct summands of objects X which fit into a distinguished triangle

$$A \to X \to B \to A[1]$$

where A is an object of $\langle E\rangle_1$ and B an object of $\langle E\rangle_{n-1}$. In the notation of Section 13.35 we have

$$\langle E\rangle_n = smd(\langle E\rangle_1 \star \langle E\rangle_{n-1})$$

Each of the categories $\langle E\rangle_n$ is a strictly full additive (by Lemma 13.35.3) subcategory of $\mathcal{D}$ preserved under shifts and under taking summands. But, $\langle E\rangle_n$ is not necessarily closed under "taking cones" or "extensions", hence not necessarily a triangulated subcategory. This will be true for the subcategory

$$\langle E\rangle = \bigcup\nolimits_n \langle E\rangle_n$$

as will be shown in the lemmas below.

0FX9 **Lemma 13.36.1.** Let $\mathcal{T}$ be a triangulated category. Let E be an object of $\mathcal{T}$. For $n \geq 1$ we have

$$\langle E\rangle_n = smd(\langle E\rangle_1 \star \ldots \star \langle E\rangle_1) = smd({\langle E\rangle_1}^{\star n}) = \bigcup\nolimits_{m\geq 1} smd(add(E[-m,m])^{\star n})$$

For $n, n' \geq 1$ we have $\langle E\rangle_{n+n'} = smd(\langle E\rangle_n \star \langle E\rangle_{n'})$.

Proof. The left equality in the displayed formula follows from Lemmas 13.35.1 and 13.35.2 and induction. The middle equality is a matter of notation. Since $\langle E\rangle_1 = smd(add(E[-\infty,\infty]))$ and since $E[-\infty,\infty] = \bigcup_{m\geq 1} E[-m,m]$ we see from Remark 13.35.6 and Lemma 13.35.2 that we get the equality on the right. Then the final statement follows from the remark and the corresponding statement of Lemma 13.35.4. $\square$

0ATG **Lemma 13.36.2.** Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. The subcategory

$$\langle E\rangle = \bigcup\nolimits_n \langle E\rangle_n = \bigcup\nolimits_{n,m\geq 1} smd(add(E[-m,m])^{\star n})$$

is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ and it is the smallest such subcategory of $\mathcal{D}$ containing the object E.

Proof. The equality on the right follows from Lemma 13.36.1. It is clear that $\langle E \rangle = \bigcup \langle E \rangle_n$ contains E, is preserved under shifts, direct sums, direct summands. If $A \in \langle E \rangle_a$ and $B \in \langle E \rangle_b$ and if $A \to X \to B \to A[1]$ is a distinguished triangle, then $X \in \langle E \rangle_{a+b}$ by Lemma 13.36.1. Hence $\bigcup \langle E \rangle_n$ is also preserved under extensions and it follows that it is a triangulated subcategory.

Finally, let $\mathcal{D}' \subset \mathcal{D}$ be a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ containing E. Then $\mathcal{D}'[-\infty, \infty] \subset \mathcal{D}'$, $add(\mathcal{D}) \subset \mathcal{D}'$, $smd(\mathcal{D}') \subset \mathcal{D}'$, and $\mathcal{D}' \star \mathcal{D}' \subset \mathcal{D}'$. In other words, all the operations we used to construct $\langle E \rangle$ out of E preserve $\mathcal{D}'$. Hence $\langle E \rangle \subset \mathcal{D}'$ and this finishes the proof. $\square$

09SJ **Definition 13.36.3.** Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$.

(1) We say E is a *classical generator* of $\mathcal{D}$ if the smallest strictly full, saturated, triangulated subcategory of $\mathcal{D}$ containing E is equal to $\mathcal{D}$, in other words, if $\langle E \rangle = \mathcal{D}$.
(2) We say E is a *strong generator* of $\mathcal{D}$ if $\langle E \rangle_n = \mathcal{D}$ for some $n \geq 1$.
(3) We say E is a *weak generator* or a *generator* of $\mathcal{D}$ if for any nonzero object K of $\mathcal{D}$ there exists an integer n and a nonzero map $E \to K[n]$.

This definition can be generalized to the case of a family of objects.

09SK **Lemma 13.36.4.** Let $\mathcal{D}$ be a triangulated category. Let E, K be objects of $\mathcal{D}$. The following are equivalent

(1) $\mathrm{Hom}(E, K[i]) = 0$ for all $i \in \mathbf{Z}$,
(2) $\mathrm{Hom}(E', K) = 0$ for all $E' \in \langle E \rangle$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Conversely, assume (1). Then $\mathrm{Hom}(X, K) = 0$ for all X in $\langle E \rangle_1$. Arguing by induction on n and using Lemma 13.4.2 we see that $\mathrm{Hom}(X, K) = 0$ for all X in $\langle E \rangle_n$. $\square$

09SL **Lemma 13.36.5.** Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. If E is a classical generator of $\mathcal{D}$, then E is a generator.

Proof. Assume E is a classical generator. Let K be an object of $\mathcal{D}$ such that $\mathrm{Hom}(E, K[i]) = 0$ for all $i \in \mathbf{Z}$. By Lemma 13.36.4 $\mathrm{Hom}(E', K) = 0$ for all E' in $\langle E \rangle$. However, since $\mathcal{D} = \langle E \rangle$ we conclude that $\mathrm{id}_K = 0$, i.e., $K = 0$. $\square$

0FXA **Lemma 13.36.6.** Let $\mathcal{D}$ be a triangulated category which has a strong generator. Let E be an object of $\mathcal{D}$. If E is a classical generator of $\mathcal{D}$, then E is a strong generator.

Proof. Let E' be an object of $\mathcal{D}$ such that $\mathcal{D} = \langle E' \rangle_n$. Since $\mathcal{D} = \langle E \rangle$ we see that $E' \in \langle E \rangle_m$ for some $m \geq 1$ by Lemma 13.36.2. Then $\langle E' \rangle_1 \subset \langle E \rangle_m$ hence

$$\mathcal{D} = \langle E' \rangle_n = smd(\langle E' \rangle_1 \star \ldots \star \langle E' \rangle_1) \subset smd(\langle E \rangle_m \star \ldots \star \langle E \rangle_m) = \langle E \rangle_{nm}$$

as desired. Here we used Lemma 13.36.1. $\square$

0ATH **Remark 13.36.7.** Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. Let T be a property of objects of $\mathcal{D}$. Suppose that

(1) if $K_i \in D(A)$, $i = 1, \ldots, r$ with $T(K_i)$ for $i = 1, \ldots, r$, then $T(\bigoplus K_i)$,
(2) if $K \to L \to M \to K[1]$ is a distinguished triangle and T holds for two, then T holds for the third object,
(3) if $T(K \oplus L)$ then $T(K)$ and $T(L)$, and

(4) $T(E[n])$ holds for all n.

Then T holds for all objects of $\langle E\rangle$.

13.37. Compact objects

09SM Here is the definition.

07LS Definition 13.37.1. Let $\mathcal{D}$ be an additive category with arbitrary direct sums. A compact object of $\mathcal{D}$ is an object K such that the map

$$\bigoplus\nolimits_{i\in I} \mathrm{Hom}_\mathcal{D}(K, E_i) \longrightarrow \mathrm{Hom}_\mathcal{D}(K, \bigoplus\nolimits_{i\in I} E_i)$$

is bijective for any set I and objects $E_i \in \mathrm{Ob}(\mathcal{D})$ parametrized by $i \in I$.

This notion turns out to be very useful in algebraic geometry. It is an intrinsic condition on objects that forces the objects to be, well, compact.

09QH Lemma 13.37.2. Let $\mathcal{D}$ be a (pre-)triangulated category with direct sums. Then the compact objects of $\mathcal{D}$ form the objects of a Karoubian, saturated, strictly full, (pre-)triangulated subcategory $\mathcal{D}_c$ of $\mathcal{D}$.

Proof. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$ with X and Y compact. Then it follows from Lemma 13.4.2 and the five lemma (Homology, Lemma 12.5.20) that Z is a compact object too. It is clear that if $X \oplus Y$ is compact, then X, Y are compact objects too. Hence $\mathcal{D}_c$ is a saturated triangulated subcategory. Since $\mathcal{D}$ is Karoubian by Lemma 13.4.14 we conclude that the same is true for $\mathcal{D}_c$. □

09SN Lemma 13.37.3. Let $\mathcal{D}$ be a triangulated category with direct sums. Let E_i, $i \in I$ be a family of compact objects of $\mathcal{D}$ such that $\bigoplus E_i$ generates $\mathcal{D}$. Then every object X of $\mathcal{D}$ can be written as

$$X = \mathrm{hocolim} X_n$$

where X_1 is a direct sum of shifts of the E_i and each transition morphism fits into a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ where Y_n is a direct sum of shifts of the E_i.

Proof. Set $X_1 = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$ and $\varphi : E_i[m] \to X$. Then X_1 comes equipped with a canonical morphism $X_1 \to X$. Given $X_n \to X$ we set $Y_n = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$, and $\varphi : E_i[m] \to X_n$ is a morphism such that $E_i[m] \to X_n \to X$ is zero. Choose a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ and let $X_{n+1} \to X$ be any morphism such that $X_n \to X_{n+1} \to X$ is the given one; such a morphism exists by our choice of Y_n. We obtain a morphism $\mathrm{hocolim} X_n \to X$ by the construction of our maps $X_n \to X$. Choose a distinguished triangle

$$C \to \mathrm{hocolim} X_n \to X \to C[1]$$

Let $E_i[m] \to C$ be a morphism. Since E_i is compact, the composition $E_i[m] \to C \to \mathrm{hocolim} X_n$ factors through X_n for some n, say by $E_i[m] \to X_n$. Then the construction of Y_n shows that the composition $E_i[m] \to X_n \to X_{n+1}$ is zero. In other words, the composition $E_i[m] \to C \to \mathrm{hocolim} X_n$ is zero. This means that our morphism $E_i[m] \to C$ comes from a morphism $E_i[m] \to X[-1]$. The construction of X_1 then shows that such morphism lifts to $\mathrm{hocolim} X_n$ and we conclude that our morphism $E_i[m] \to C$ is zero. The assumption that $\bigoplus E_i$ generates $\mathcal{D}$ implies that C is zero and the proof is done. □

09SP Lemma 13.37.4. With assumptions and notation as in Lemma 13.37.3. If C is a compact object and $C \to X_n$ is a morphism, then there is a factorization $C \to E \to X_n$ where E is an object of $\langle E_{i_1} \oplus \ldots \oplus E_{i_t} \rangle$ for some $i_1, \ldots, i_t \in I$.

Proof. We prove this by induction on n. The base case $n = 1$ is clear. If $n > 1$ consider the composition $C \to X_n \to Y_{n-1}[1]$. This can be factored through some $E'[1] \to Y_{n-1}[1]$ where E' is a finite direct sum of shifts of the E_i. Let $I' \subset I$ be the finite set of indices that occur in this direct sum. Thus we obtain

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & E'[1] \arrow[d] \\ Y_{n-1} \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[r] & Y_{n-1}[1] \end{tikzcd}$$

By induction the morphism $C' \to X_{n-1}$ factors through $E'' \to X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I''} E_i \rangle$ for some finite subset $I'' \subset I$. Choose a distinguished triangle

$$E' \to E'' \to E \to E'[1]$$

then E is an object of $\langle \bigoplus_{i \in I' \cup I''} E_i \rangle$. By construction and the axioms of a triangulated category we can choose morphisms $C \to E$ and a morphism $E \to X_n$ fitting into morphisms of triangles $(E', C', C) \to (E', E'', E)$ and $(E', E'', E) \to (Y_{n-1}, X_{n-1}, X_n)$. The composition $C \to E \to X_n$ may not equal the given morphism $C \to X_n$, but the compositions into Y_{n-1} are equal. Let $C \to X_{n-1}$ be a morphism that lifts the difference. By induction assumption we can factor this through a morphism $E''' \to X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I'''} E_i \rangle$ for some finite subset $I' \subset I$. Thus we see that we get a solution on considering $E \oplus E''' \to X_n$ because $E \oplus E'''$ is an object of $\langle \bigoplus_{i \in I' \cup I'' \cup I'''} E_i \rangle$. $\square$

09SQ Definition 13.37.5. Let $\mathcal{D}$ be a triangulated category with arbitrary direct sums. We say $\mathcal{D}$ is compactly generated if there exists a set E_i, $i \in I$ of compact objects such that $\bigoplus E_i$ generates $\mathcal{D}$.

The following proposition clarifies the relationship between classical generators and weak generators.

09SR Proposition 13.37.6. Let $\mathcal{D}$ be a triangulated category with direct sums. Let E be a compact object of $\mathcal{D}$. The following are equivalent

(1) E is a classical generator for $\mathcal{D}_c$ and $\mathcal{D}$ is compactly generated, and
(2) E is a generator for $\mathcal{D}$.

Proof. If E is a classical generator for $\mathcal{D}_c$, then $\mathcal{D}_c = \langle E \rangle$. It follows formally from the assumption that $\mathcal{D}$ is compactly generated and Lemma 13.36.4 that E is a generator for $\mathcal{D}$.

The converse is more interesting. Assume that E is a generator for $\mathcal{D}$. Let X be a compact object of $\mathcal{D}$. Apply Lemma 13.37.3 with $I = \{1\}$ and $E_1 = E$ to write

$$X = \text{hocolim} X_n$$

as in the lemma. Since X is compact we find that $X \to \text{hocolim} X_n$ factors through X_n for some n (Lemma 13.33.9). Thus X is a direct summand of X_n. By Lemma 13.37.4 we see that X is an object of $\langle E \rangle$ and the lemma is proven. $\square$

13.38. Brown representability

0A8E A reference for the material in this section is [Nee96].

0A8F **Lemma 13.38.1.** Let $\mathcal{D}$ be a triangulated category with direct sums which is compactly generated. Let $H : \mathcal{D} \to \textit{Ab}$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable. [Nee96, Theorem 3.1].

Proof. Let E_i, $i \in I$ be a set of compact objects such that $\bigoplus_{i \in I} E_i$ generates $\mathcal{D}$. We may and do assume that the set of objects $\{E_i\}$ is preserved under shifts. Consider pairs (i, a) where $i \in I$ and $a \in H(E_i)$ and set

$$X_1 = \bigoplus\nolimits_{(i,a)} E_i$$

Since $H(X_1) = \prod_{(i,a)} H(E_i)$ we see that $(a)_{(i,a)}$ defines an element $a_1 \in H(X_1)$. Set $H_1 = \mathrm{Hom}_\mathcal{D}(-, X_1)$. By Yoneda's lemma (Categories, Lemma 4.3.5) the element a_1 defines a natural transformation $H_1 \to H$.

We are going to inductively construct X_n and transformations $a_n : H_n \to H$ where $H_n = \mathrm{Hom}_\mathcal{D}(-, X_n)$. Namely, we apply the procedure above to the functor $\mathrm{Ker}(H_n \to H)$ to get an object

$$K_{n+1} = \bigoplus\nolimits_{(i,k),\ k \in \mathrm{Ker}(H_n(E_i) \to H(E_i))} E_i$$

and a transformation $\mathrm{Hom}_\mathcal{D}(-, K_{n+1}) \to \mathrm{Ker}(H_n \to H)$. By Yoneda's lemma the composition $\mathrm{Hom}_\mathcal{D}(-, K_{n+1}) \to H_n$ gives a morphism $K_{n+1} \to X_n$. We choose a distinguished triangle

$$K_{n+1} \to X_n \to X_{n+1} \to K_{n+1}[1]$$

in $\mathcal{D}$. The element $a_n \in H(X_n)$ maps to zero in $H(K_{n+1})$ by construction. Since H is cohomological we can lift it to an element $a_{n+1} \in H(X_{n+1})$.

We claim that $X = \mathrm{hocolim} X_n$ represents H. Applying H to the defining distinguished triangle

$$\bigoplus X_n \to \bigoplus X_n \to X \to \bigoplus X_n[1]$$

we obtain an exact sequence

$$\prod H(X_n) \leftarrow \prod H(X_n) \leftarrow H(X)$$

Thus there exists an element $a \in H(X)$ mapping to (a_n) in $\prod H(X_n)$. Hence a natural transformation $\mathrm{Hom}_\mathcal{D}(-, X) \to H$ such that

$$\mathrm{Hom}_\mathcal{D}(-, X_1) \to \mathrm{Hom}_\mathcal{D}(-, X_2) \to \mathrm{Hom}_\mathcal{D}(-, X_3) \to \ldots \to \mathrm{Hom}_\mathcal{D}(-, X) \to H$$

commutes. For each i the map $\mathrm{Hom}_\mathcal{D}(E_i, X) \to H(E_i)$ is surjective, by construction of X_1. On the other hand, by construction of $X_n \to X_{n+1}$ the kernel of $\mathrm{Hom}_\mathcal{D}(E_i, X_n) \to H(E_i)$ is killed by the map $\mathrm{Hom}_\mathcal{D}(E_i, X_n) \to \mathrm{Hom}_\mathcal{D}(E_i, X_{n+1})$. Since

$$\mathrm{Hom}_\mathcal{D}(E_i, X) = \mathrm{colim}\, \mathrm{Hom}_\mathcal{D}(E_i, X_n)$$

by Lemma 13.33.9 we see that $\mathrm{Hom}_\mathcal{D}(E_i, X) \to H(E_i)$ is injective.

To finish the proof, consider the subcategory

$$\mathcal{D}' = \{Y \in \mathrm{Ob}(\mathcal{D}) \mid \mathrm{Hom}_\mathcal{D}(Y[n], X) \to H(Y[n]) \text{ is an isomorphism for all } n\}$$

As $\mathrm{Hom}_\mathcal{D}(-, X) \to H$ is a transformation between cohomological functors, the subcategory $\mathcal{D}'$ is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ (details omitted; see proof of Lemma 13.6.3). Moreover, as both H and $\text{Hom}_\mathcal{D}(-, X)$ transform direct sums into products, we see that direct sums of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Thus derived colimits of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Since $\{E_i\}$ is preserved under shifts, we see that E_i is an object of $\mathcal{D}'$ for all i. It follows from Lemma 13.37.3 that $\mathcal{D}' = \mathcal{D}$ and the proof is complete. □

0A8G Proposition 13.38.2. Let $\mathcal{D}$ be a triangulated category with direct sums which is compactly generated. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint. [Nee96, Theorem 4.1].

Proof. For an object Y of $\mathcal{D}'$ consider the contravariant functor

$$\mathcal{D} \to \text{Ab}, \quad W \mapsto \text{Hom}_{\mathcal{D}'}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 13.38.1 we find an object X of $\mathcal{D}$ such that $\text{Hom}_\mathcal{D}(W, X) = \text{Hom}_{\mathcal{D}'}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 4.24.2. Exactness follows from Lemma 13.7.1. □

13.39. Brown representability, bis

0GYF In this section we explain a version of Brown representability for triangulated categories which have a suitable set of generators; for other versions, please see [Fra01], [Nee01], and [Kra02].

0GYG Lemma 13.39.1. Let $\mathcal{D}$ be a triangulated category with direct sums. Suppose given a set $\mathcal{E}$ of objects of $\mathcal{D}$ such that Weak version of [Kra02, Theorem A]

(1) if X is a nonzero object of $\mathcal{D}$, then there exists an $E \in \mathcal{E}$ and a nonzero map $E \to X$, and
(2) given objects X_n, $n \in \mathbf{N}$ of $\mathcal{D}$, $E \in \mathcal{E}$, and $\alpha : E \to \bigoplus X_n$, there exist $E_n \in \mathcal{E}$ and $\beta_n : E_n \to X_n$ and a morphism $\gamma : E \to \bigoplus E_n$ such that $\alpha = (\bigoplus \beta_n) \circ \gamma$.

Let $H : \mathcal{D} \to \text{Ab}$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.

Proof. This proof is very similar to the proof of Lemma 13.38.1. We may replace $\mathcal{E}$ by $\bigcup_{i \in \mathbf{Z}} \mathcal{E}[i]$ and assume that $\mathcal{E}$ is preserved by shifts. Consider pairs (E, a) where $E \in \mathcal{E}$ and $a \in H(E)$ and set

$$X_1 = \bigoplus\nolimits_{(E,a)} E$$

Since $H(X_1) = \prod_{(E,a)} H(E)$ we see that $(a)_{(E,a)}$ defines an element $a_1 \in H(X_1)$. Set $H_1 = \text{Hom}_\mathcal{D}(-, X_1)$. By Yoneda's lemma (Categories, Lemma 4.3.5) the element a_1 defines a natural transformation $H_1 \to H$.

We are going to inductively construct X_n and transformations $a_n : H_n \to H$ where $H_n = \text{Hom}_\mathcal{D}(-, X_n)$. Namely, we apply the procedure above to the functor $\text{Ker}(H_n \to H)$ to get an object

$$K_{n+1} = \bigoplus\nolimits_{(E,k),\ k \in \text{Ker}(H_n(E) \to H(E))} E$$

and a transformation $\mathrm{Hom}_\mathcal{D}(-, K_{n+1}) \to \mathrm{Ker}(H_n \to H)$. By Yoneda's lemma the composition $\mathrm{Hom}_\mathcal{D}(-, K_{n+1}) \to H_n$ gives a morphism $K_{n+1} \to X_n$. We choose a distinguished triangle

$$K_{n+1} \to X_n \to X_{n+1} \to K_{n+1}[1]$$

in $\mathcal{D}$. The element $a_n \in H(X_n)$ maps to zero in $H(K_{n+1})$ by construction. Since H is cohomological we can lift it to an element $a_{n+1} \in H(X_{n+1})$.

Set $X = \mathrm{hocolim} X_n$. Applying H to the defining distinguished triangle

$$\bigoplus X_n \to \bigoplus X_n \to X \to \bigoplus X_n[1]$$

we obtain an exact sequence

$$\prod H(X_n) \leftarrow \prod H(X_n) \leftarrow H(X)$$

Thus there exists an element $a \in H(X)$ mapping to (a_n) in $\prod H(X_n)$. Hence a natural transformation $\mathrm{Hom}_\mathcal{D}(-, X) \to H$ such that

$$\mathrm{Hom}_\mathcal{D}(-, X_1) \to \mathrm{Hom}_\mathcal{D}(-, X_2) \to \mathrm{Hom}_\mathcal{D}(-, X_3) \to \ldots \to \mathrm{Hom}_\mathcal{D}(-, X) \to H$$

commutes. We claim that $\mathrm{Hom}_\mathcal{D}(-, X) \to H(-)$ is an isomorphism.

Let $E \in \mathcal{E}$. Let us show that

$$\mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n) \to \mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n)$$

is injective. Namely, let $\alpha : E \to \bigoplus X_n$. Then by assumption (2) we obtain a factorization $\alpha = (\bigoplus \beta_n) \circ \gamma$. Since $E_n \to X_n \to X_{n+1}$ is zero by construction, we see that the composition $\bigoplus E_n \to \bigoplus X_n \to \bigoplus X_n$ is equal to $\bigoplus \beta_n$. Hence also the composition $E \to \bigoplus X_n \to \bigoplus X_n$ is equal to α. This proves the stated injectivity and hence also

$$\mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n[1]) \to \mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n[1])$$

is injective. It follows that we have an exact sequence

$$\mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n) \to \mathrm{Hom}_\mathcal{D}(E, \bigoplus X_n) \to \mathrm{Hom}_\mathcal{D}(E, X) \to 0$$

for all $E \in \mathcal{E}$.

Let $E \in \mathcal{E}$ and let $f : E \to X$ be a morphism. By the previous paragraph, we may choose $\alpha : E \to \bigoplus X_n$ lifting f. Then by assumption (2) we obtain a factorization $\alpha = (\bigoplus \beta_n) \circ \gamma$. For each n there is a morphism $\delta_n : E_n \to X_1$ such that δ_n and β_n map to the same element of $H(E_n)$. Then the compositions

$$E_n \to X_n \to X_{n+1} \quad \text{and} \quad E_n \to X_1 \to X_{n+1}$$

are equal by construction of $X_n \to X_{n+1}$. It follows that

$$\bigoplus E_n \to \bigoplus X_n \to X \quad \text{and} \quad \bigoplus E_n \to \bigoplus X_1 \to X$$

are the same too. Observing that $\bigoplus X_1 \to X$ factors as $\bigoplus X_1 \to X_1 \to X$, we conclude that

$$\mathrm{Hom}_\mathcal{D}(E, X_1) \to \mathrm{Hom}_\mathcal{D}(E, X)$$

is surjective. Since by construction the map $\mathrm{Hom}_\mathcal{D}(E, X_1) \to H(E)$ is surjective and by construction the kernel of this map is annihilated by $\mathrm{Hom}_\mathcal{D}(E, X_1) \to \mathrm{Hom}_\mathcal{D}(E, X)$ we conclude that $\mathrm{Hom}_\mathcal{D}(E, X) \to H(E)$ is a bijection for all $E \in \mathcal{E}$.

To finish the proof, consider the subcategory

$$\mathcal{D}' = \{Y \in \text{Ob}(\mathcal{D}) \mid \text{Hom}_\mathcal{D}(Y[n], X) \to H(Y[n]) \text{ is an isomorphism for all } n\}$$

As $\text{Hom}_\mathcal{D}(-, X) \to H$ is a transformation between cohomological functors, the subcategory $\mathcal{D}'$ is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ (details omitted; see proof of Lemma 13.6.3). Moreover, as both H and $\text{Hom}_\mathcal{D}(-, X)$ transform direct sums into products, we see that direct sums of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Thus derived colimits of objects of $\mathcal{D}'$ are in $\mathcal{D}'$. Since $\mathcal{E}$ is preserved by shifts, we conclude that $\mathcal{E} \subset \text{Ob}(\mathcal{D}')$ by the result of the previous paragraph. To finish the proof we have to show that $\mathcal{D}' = \mathcal{D}$.

Let Y be an object of $\mathcal{D}$ and set $H(-) = \text{Hom}_\mathcal{D}(-, Y)$. Then H is a cohomological functor which transforms direct sums into products. By the construction in the first part of the proof we obtain a morphism $\text{colim}\, X_n = X \to Y$ such that $\text{Hom}_\mathcal{D}(E, X) \to \text{Hom}_\mathcal{D}(E, Y)$ is bijective for all $E \in \mathcal{E}$. Then assumption (1) tells us that $X \to Y$ is an isomorphism! On the other hand, by construction $X_1, X_2, \ldots$ are in $\mathcal{D}'$ and so is X. Thus $Y \in \mathcal{D}'$ and the proof is complete. □

0GYH Proposition 13.39.2. Let $\mathcal{D}$ be a triangulated category with direct sums. Assume there exists a set $\mathcal{E}$ of objects of $\mathcal{D}$ satisfying conditions (1) and (2) of Lemma 13.39.1. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint.

Proof. For an object Y of $\mathcal{D}'$ consider the contravariant functor

$$\mathcal{D} \to \text{Ab}, \quad W \mapsto \text{Hom}_{\mathcal{D}'}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 13.39.1 we find an object X of $\mathcal{D}$ such that $\text{Hom}_\mathcal{D}(W, X) = \text{Hom}_{\mathcal{D}'}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 4.24.2. Exactness follows from Lemma 13.7.1. □

13.40. Admissible subcategories

0CQP A reference for this section is [BK89, Section 1].

0FXB Definition 13.40.1. Let $\mathcal{D}$ be an additive category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory. The right orthogonal $\mathcal{A}^\perp$ of $\mathcal{A}$ is the full subcategory consisting of the objects X of $\mathcal{D}$ such that $\text{Hom}(A, X) = 0$ for all $A \in \text{Ob}(\mathcal{A})$. The left orthogonal ${}^\perp\mathcal{A}$ of $\mathcal{A}$ is the full subcategory consisting of the objects X of $\mathcal{D}$ such that $\text{Hom}(X, A) = 0$ for all $A \in \text{Ob}(\mathcal{A})$.

0CQQ Lemma 13.40.2. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Consider a distinguished triangle

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. The following are equivalent

(1) Z is in $\mathcal{A}^\perp$, and
(2) $\text{Hom}(A, X) = \text{Hom}(A, Y)$ for all $A \in \text{Ob}(\mathcal{A})$.

Proof. By Lemma 13.4.2 the functor $\text{Hom}(A, -)$ is homological and hence we get a long exact sequence as in (13.3.5.1). Assume (1) and let $A \in \text{Ob}(\mathcal{A})$. Then we consider the exact sequence

$$\text{Hom}(A[1], Z) \to \text{Hom}(A, X) \to \text{Hom}(A, Y) \to \text{Hom}(A, Z)$$

Since $A[1] \in \text{Ob}(\mathcal{A})$ we see that the first and last groups are zero. Thus we get (2). Assume (2) and let $A \in \text{Ob}(\mathcal{A})$. Then we consider the exact sequence

$$\text{Hom}(A, X) \to \text{Hom}(A, Y) \to \text{Hom}(A, Z) \to \text{Hom}(A[-1], X) \to \text{Hom}(A[-1], Y)$$

and we conclude that $\text{Hom}(A, Z) = 0$ as desired. □

0H0M Lemma 13.40.3. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Consider a distinguished triangle

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. The following are equivalent

(1) X is in ${}^\perp\mathcal{B}$, and
(2) $\text{Hom}(Y, B) = \text{Hom}(Z, B)$ for all $B \in \text{Ob}(\mathcal{B})$.

Proof. Dual to Lemma 13.40.2. □

0FXC Lemma 13.40.4. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full subcategory invariant under all shifts. Then both the right orthogonal $\mathcal{A}^\perp$ and the left orthogonal ${}^\perp\mathcal{A}$ of $\mathcal{A}$ are strictly full, saturated[8], triangulated subcagories of $\mathcal{D}$.

Proof. It is immediate from the definitions that the orthogonals are preserved under taking shifts, direct sums, and direct summands. Consider a distinguished triangle

$$X \to Y \to Z \to X[1]$$

of $\mathcal{D}$. By Lemma 13.4.16 it suffices to show that if X and Y are in $\mathcal{A}^\perp$, then Z is in $\mathcal{A}^\perp$. This is immediate from Lemma 13.40.2. □

0CQR Lemma 13.40.5. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be a full triangulated subcategory of $\mathcal{D}$. For an object X of $\mathcal{D}$ consider the property $P(X)$: there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with A in $\mathcal{A}$ and B in $\mathcal{A}^\perp$.

(1) If $X_1 \to X_2 \to X_3 \to X_1[1]$ is a distinguished triangle and P holds for two out of three, then it holds for the third.
(2) If P holds for X_1 and X_2, then it holds for $X_1 \oplus X_2$.

Proof. Let $X_1 \to X_2 \to X_3 \to X_1[1]$ be a distinguished triangle and assume P holds for X_1 and X_2. Choose distinguished triangles

$$A_1 \to X_1 \to B_1 \to A_1[1] \quad \text{and} \quad A_2 \to X_2 \to B_2 \to A_2[1]$$

as in condition P. Since $\text{Hom}(A_1, A_2) = \text{Hom}(A_1, X_2)$ by Lemma 13.40.2 there is a unique morphism $A_1 \to A_2$ such that the diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ A_2 \arrow[r] & X_2 \end{tikzcd}$$

[8] Definition 13.6.1.

commutes. Choose an extension of this to a diagram

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & X_1 & \longrightarrow & Q_1 & \longrightarrow & A_1[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_2 & \longrightarrow & X_2 & \longrightarrow & Q_2 & \longrightarrow & A_2[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_3 & \longrightarrow & X_3 & \longrightarrow & Q_3 & \longrightarrow & A_3[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_1[1] & \longrightarrow & X_1[1] & \longrightarrow & Q_1[1] & \longrightarrow & A_1[2]
\end{array}$$

as in Proposition 13.4.23. By TR3 we see that $Q_1 \cong B_1$ and $Q_2 \cong B_2$ and hence $Q_1, Q_2 \in \text{Ob}(\mathcal{A}^\perp)$. As $Q_1 \to Q_2 \to Q_3 \to Q_1[1]$ is a distinguished triangle we see that $Q_3 \in \text{Ob}(\mathcal{A}^\perp)$ by Lemma 13.40.4. Since $\mathcal{A}$ is a full triangulated subcategory, we see that A_3 is isomorphic to an object of $\mathcal{A}$. Thus X_3 satisfies P. The other cases of (1) follow from this case by translation. Part (2) is a special case of (1) via Lemma 13.4.11. □

0H0N Lemma 13.40.6. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. For an object X of $\mathcal{D}$ consider the property $P(X)$: there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with B in $\mathcal{B}$ and A in ${}^\perp\mathcal{B}$.

(1) If $X_1 \to X_2 \to X_3 \to X_1[1]$ is a distinguished triangle and P holds for two out of three, then it holds for the third.
(2) If P holds for X_1 and X_2, then it holds for $X_1 \oplus X_2$.

Proof. Dual to Lemma 13.40.5. □

0CQS Lemma 13.40.7. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ be a full triangulated subcategory. The following are equivalent

(1) the inclusion functor $\mathcal{A} \to \mathcal{D}$ has a right adjoint, and
(2) for every X in $\mathcal{D}$ there exists a distinguished triangle
$$A \to X \to B \to A[1]$$
in $\mathcal{D}$ with $A \in \text{Ob}(\mathcal{A})$ and $B \in \text{Ob}(\mathcal{A}^\perp)$.

If this holds, then $\mathcal{A}$ is saturated (Definition 13.6.1) and if $\mathcal{A}$ is strictly full in $\mathcal{D}$, then $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$.

Proof. Assume (1) and denote $v : \mathcal{D} \to \mathcal{A}$ the right adjoint. Let $X \in \text{Ob}(\mathcal{D})$. Set $A = v(X)$. We may extend the adjunction mapping $A \to X$ to a distinguished triangle $A \to X \to B \to A[1]$. Since

$$\text{Hom}_\mathcal{A}(A', A) = \text{Hom}_\mathcal{A}(A', v(X)) = \text{Hom}_\mathcal{D}(A', X)$$

for $A' \in \text{Ob}(\mathcal{A})$, we conclude that $B \in \text{Ob}(\mathcal{A}^\perp)$ by Lemma 13.40.2.

Assume (2). We will contruct the adjoint v explictly. Let $X \in \text{Ob}(\mathcal{D})$. Choose $A \to X \to B \to A[1]$ as in (2). Set $v(X) = A$. Let $f : X \to Y$ be a morphism in

$\mathcal{D}$. Choose $A' \to Y \to B' \to A'[1]$ as in (2). Since $\mathrm{Hom}(A, A') = \mathrm{Hom}(A, Y)$ by Lemma 13.40.2 there is a unique morphism $f' : A \to A'$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ A' \arrow[r] & Y \end{tikzcd}$$

commutes. Hence we can set $v(f) = f'$ to get a functor. To see that v is adjoint to the inclusion morphism use Lemma 13.40.2 again.

Proof of the final statement. In order to prove that $\mathcal{A}$ is saturated we may replace $\mathcal{A}$ by the strictly full subcategory having the same isomorphism classes as $\mathcal{A}$; details omitted. Assume $\mathcal{A}$ is strictly full. If we show that $\mathcal{A} = {}^\perp(\mathcal{A}^\perp)$, then $\mathcal{A}$ will be saturated by Lemma 13.40.4. Since the incusion $\mathcal{A} \subset {}^\perp(\mathcal{A}^\perp)$ is clear it suffices to prove the other inclusion. Let X be an object of ${}^\perp(\mathcal{A}^\perp)$. Choose a distinguished triangle $A \to X \to B \to A[1]$ as in (2). As $\mathrm{Hom}(X, B) = 0$ by assumption we see that $A \cong X \oplus B[-1]$ by Lemma 13.4.11. Since $\mathrm{Hom}(A, B[-1]) = 0$ as $B \in \mathcal{A}^\perp$ this implies $B[-1] = 0$ and $A \cong X$ as desired. $\square$

0CQT Lemma 13.40.8. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B} \subset \mathcal{D}$ be a full triangulated subcategory. The following are equivalent

(1) the inclusion functor $\mathcal{B} \to \mathcal{D}$ has a left adjoint, and
(2) for every X in $\mathcal{D}$ there exists a distinguished triangle
$$A \to X \to B \to A[1]$$
in $\mathcal{D}$ with $B \in \mathrm{Ob}(\mathcal{B})$ and $A \in \mathrm{Ob}({}^\perp\mathcal{B})$.

If this holds, then $\mathcal{B}$ is saturated (Definition 13.6.1) and if $\mathcal{B}$ is strictly full in $\mathcal{D}$, then $\mathcal{B} = ({}^\perp\mathcal{B})^\perp$.

Proof. Dual to Lemma 13.40.7. $\square$

0FXD Definition 13.40.9. Let $\mathcal{D}$ be a triangulated category. A right admissible subcategory of $\mathcal{D}$ is a strictly full triangulated subcategory satisfying the equivalent conditions of Lemma 13.40.7. A left admissible subcategory of $\mathcal{D}$ is a strictly full triangulated subcategory satisfying the equivalent conditions of Lemma 13.40.8. A two-sided admissible subcategory is one which is both right and left admissible.

Let $\mathcal{A}$ be a right admissible subcategory of the triangulated category $\mathcal{D}$. Then we observe that for $X \in \mathcal{D}$ the distinguished triangle

$$A \to X \to B \to A[1]$$

with $A \in \mathcal{A}$ and $B \in \mathcal{A}^\perp$ is canonical in the following sense: for any other distinguished triangle $A' \to X \to B' \to A'[1]$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{A}^\perp$ there is an isomorphism $(\alpha, \mathrm{id}_X, \beta) : (A, X, B) \to (A', X, B')$ of triangles. The following proposition summarizes what was said above.

0H0P Proposition 13.40.10. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A} \subset \mathcal{D}$ and $\mathcal{B} \subset \mathcal{D}$ be subcategories. The following are equivalent

(1) $\mathcal{A}$ is right admissible and $\mathcal{B} = \mathcal{A}^\perp$,
(2) $\mathcal{B}$ is left admissible and $\mathcal{A} = {}^\perp\mathcal{B}$,

(3) $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and for every X in $\mathcal{D}$ there exists a distinguished triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

If this is true, then $\mathcal{A} \to \mathcal{D}/\mathcal{B}$ and $\mathcal{B} \to \mathcal{D}/\mathcal{A}$ are equivalences of triangulated categories, the right adjoint to the inclusion functor $\mathcal{A} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{B} \to \mathcal{A}$, and the left adjoint to the inclusion functor $\mathcal{B} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{A} \to \mathcal{B}$.

Proof. The equivalence between (1), (2), and (3) follows in a straighforward manner from Lemmas 13.40.7 and 13.40.8 (small detail omitted). Denote $v : \mathcal{D} \to \mathcal{A}$ the right adjoint of the inclusion functor $i : \mathcal{A} \to \mathcal{D}$. It is immediate that $\mathrm{Ker}(v) = \mathcal{A}^\perp = \mathcal{B}$. Thus v factors over a functor $\overline{v} : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ by the universal property of the quotient. Since $v \circ i = \mathrm{id}_\mathcal{A}$ by Categories, Lemma 4.24.4 we see that $\overline{v}$ is a left quasi-inverse to $\overline{i} : \mathcal{A} \to \mathcal{D}/\mathcal{B}$. We claim also the composition $\overline{i} \circ \overline{v}$ is isomorphic to $\mathrm{id}_{\mathcal{D}/\mathcal{B}}$. Namely, suppose we have X fitting into a distinguished triangle $A \to X \to B \to A[1]$ as in (3). Then $v(X) = A$ as was seen in the proof of Lemma 13.40.7. Viewing X as an object of $\mathcal{D}/\mathcal{B}$ we have $\overline{i}(\overline{v}(X)) = A$ and there is a functorial isomorphism $\overline{i}(\overline{v}(X)) = A \to X$ in $\mathcal{D}/\mathcal{B}$. Thus we find that indeed $\overline{v} : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ is an equivalence. To show that $\mathcal{B} \to \mathcal{D}/\mathcal{A}$ is an equivalence and the left adjoint to the inclusion functor $\mathcal{B} \to \mathcal{D}$ is $\mathcal{D} \to \mathcal{D}/\mathcal{A} \to \mathcal{B}$ is dual to what we just said. □

13.41. Postnikov systems

0D7Y A reference for this section is [Orl97]. Let $\mathcal{D}$ be a triangulated category. Let

$$X_n \to X_{n-1} \to \ldots \to X_0$$

be a complex in $\mathcal{D}$. In this section we consider the problem of constructing a "totalization" of this complex.

0D7Z Definition 13.41.1. Let $\mathcal{D}$ be a triangulated category. Let

$$X_n \to X_{n-1} \to \ldots \to X_0$$

be a complex in $\mathcal{D}$. A Postnikov system is defined inductively as follows.

(1) If $n = 0$, then it is an isomorphism $Y_0 \to X_0$.
(2) If $n = 1$, then it is a choice of an isomorphism $Y_0 \to X_0$ and a choice of a distinguished triangle

$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$

where $X_1 \to Y_0$ composed with $Y_0 \to X_0$ is the given morphism $X_1 \to X_0$.
(3) If $n > 1$, then it is a choice of a Postnikov system for $X_{n-1} \to \ldots \to X_0$ and a choice of a distinguished triangle

$$Y_n \to X_n \to Y_{n-1} \to Y_n[1]$$

where the morphism $X_n \to Y_{n-1}$ composed with $Y_{n-1} \to X_{n-1}$ is the given morphism $X_n \to X_{n-1}$.

Given a morphism

0D80 (13.41.1.1)
$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & X_0 \arrow[d] \\ X'_n \arrow[r] & X'_{n-1} \arrow[r] & \ldots \arrow[r] & X'_0 \end{tikzcd}$$

between complexes of the same length in $\mathcal{D}$ there is an obvious notion of a morphism of Postnikov systems.

Here is a key example.

0D8Z Example 13.41.2. Let $\mathcal{A}$ be an abelian category. Let $\ldots \to A_2 \to A_1 \to A_0$ be a chain complex in $\mathcal{A}$. Then we can consider the objects

$$X_n = A_n \quad \text{and} \quad Y_n = (A_n \to A_{n-1} \to \ldots \to A_0)[-n]$$

of $D(\mathcal{A})$. With the evident canonical maps $Y_n \to X_n$ and $Y_0 \to Y_1[1] \to Y_2[2] \to \ldots$ the distinguished triangles $Y_n \to X_n \to Y_{n-1} \to Y_n[1]$ define a Postnikov system as in Definition 13.41.1 for $\ldots \to X_2 \to X_1 \to X_0$. Here we are using the obvious extension of Postnikov systems for an infinite complex of $D(\mathcal{A})$. Finally, if colimits over $\mathbf{N}$ exist and are exact in $\mathcal{A}$ then

$$\text{hocolim} Y_n[n] = (\ldots \to A_2 \to A_1 \to A_0 \to 0 \to \ldots)$$

in $D(\mathcal{A})$. This follows immediately from Lemma 13.33.7.

Given a complex $X_n \to X_{n-1} \to \ldots \to X_0$ and a Postnikov system as in Definition 13.41.1 we can consider the maps

$$Y_0 \to Y_1[1] \to \ldots \to Y_n[n]$$

These maps fit together in certain distinguished triangles and fit with the given maps between the X_i. Here is a picture for $n = 3$:

$$\begin{tikzcd}
Y_0 \arrow[rr] & & Y_1[1] \arrow[rr] \arrow[dl] & & Y_2[2] \arrow[rr] \arrow[dl] & & Y_3[3] \arrow[dl] \\
& X_1[1] \arrow[ul, "+1"] & & X_2[2] \arrow[ll, "+1"'] \arrow[ul, "+1"] & & X_3[3] \arrow[ll, "+1"'] \arrow[ul, "+1"] &
\end{tikzcd}$$

We encourage the reader to think of $Y_n[n]$ as obtained from $X_0, X_1[1], \ldots, X_n[n]$; for example if the maps $X_i \to X_{i-1}$ are zero, then we can take $Y_n[n] = \bigoplus_{i=0,\ldots,n} X_i[i]$. Postnikov systems do not always exist. Here is a simple lemma for low n.

0D81 Lemma 13.41.3. Let $\mathcal{D}$ be a triangulated category. Consider Postnikov systems for complexes of length n.

(1) For $n = 0$ Postnikov systems always exist and any morphism (13.41.1.1) of complexes extends to a unique morphism of Postnikov systems.
(2) For $n = 1$ Postnikov systems always exist and any morphism (13.41.1.1) of complexes extends to a (nonunique) morphism of Postnikov systems.
(3) For $n = 2$ Postnikov systems always exist but morphisms (13.41.1.1) of complexes in general do not extend to morphisms of Postnikov systems.
(4) For $n > 2$ Postnikov systems do not always exist.

Proof. The case $n = 0$ is immediate as isomorphisms are invertible. The case $n = 1$ follows immediately from TR1 (existence of triangles) and TR3 (extending morphisms to triangles). For the case $n = 2$ we argue as follows. Set $Y_0 = X_0$. By the case $n = 1$ we can choose a Postnikov system

$$Y_1 \to X_1 \to Y_0 \to Y_1[1]$$

Since the composition $X_2 \to X_1 \to X_0$ is zero, we can factor $X_2 \to X_1$ (nonuniquely) as $X_2 \to Y_1 \to X_1$ by Lemma 13.4.2. Then we simply fit the morphism $X_2 \to Y_1$ into a distinguished triangle

$$Y_2 \to X_2 \to Y_1 \to Y_2[1]$$

to get the Postnikov system for $n = 2$. For $n > 2$ we cannot argue similarly, as we do not know whether the composition $X_n \to X_{n-1} \to Y_{n-1}$ is zero in $\mathcal{D}$. □

0D82 **Lemma 13.41.4.** Let $\mathcal{D}$ be a triangulated category. Given a map (13.41.1.1) consider the condition

0DW1 (13.41.4.1) $$\mathrm{Hom}(X_i[i-j-1], X'_j) = 0 \text{ for } i > j+1$$

Then

(1) If we have a Postnikov system for $X'_n \to X'_{n-1} \to \ldots \to X'_0$ then property (13.41.4.1) implies that
$$\mathrm{Hom}(X_i[i-j-1], Y'_j) = 0 \text{ for } i > j+1$$
(2) If we are given Postnikov systems for both complexes and we have (13.41.4.1), then the map extends to a (nonunique) map of Postnikov systems.

Proof. We first prove (1) by induction on j. For the base case $j = 0$ there is nothing to prove as $Y'_0 \to X'_0$ is an isomorphism. Say the result holds for $j-1$. We consider the distinguished triangle

$$Y'_j \to X'_j \to Y'_{j-1} \to Y'_j[1]$$

The long exact sequence of Lemma 13.4.2 gives an exact sequence

$$\mathrm{Hom}(X_i[i-j-1], Y'_{j-1}[-1]) \to \mathrm{Hom}(X_i[i-j-1], Y'_j) \to \mathrm{Hom}(X_i[i-j-1], X'_j)$$

From the induction hypothesis and (13.41.4.1) we conclude the outer groups are zero and we win.

Proof of (2). For $n = 1$ the existence of morphisms has been established in Lemma 13.41.3. For $n > 1$ by induction, we may assume given the map of Postnikov systems of length $n-1$. The problem is that we do not know whether the diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ X'_n \arrow[r] & Y'_{n-1} \end{tikzcd}$$

is commutative. Denote $\alpha : X_n \to Y'_{n-1}$ the difference. Then we do know that the composition of α with $Y'_{n-1} \to X'_{n-1}$ is zero (because of what it means to be a map of Postnikov systems of length $n-1$). By the distinguished triangle $Y'_{n-1} \to X'_{n-1} \to Y'_{n-2} \to Y'_{n-1}[1]$, this means that α is the composition of $Y'_{n-2}[-1] \to Y'_{n-1}$ with a map $\alpha' : X_n \to Y'_{n-2}[-1]$. Then (13.41.4.1) guarantees α' is zero by part (1) of the lemma. Thus α is zero. To finish the proof of existence, the commutativity guarantees we can choose the dotted arrow fitting into the diagram

$$\begin{tikzcd} Y_{n-1}[-1] \arrow[r] \arrow[d] & Y_n \arrow[r] \arrow[d, dotted] & X_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ Y'_{n-1}[-1] \arrow[r] & Y'_n \arrow[r] & X'_n \arrow[r] & Y'_{n-1} \end{tikzcd}$$

by TR3. ☐

0FXE **Lemma 13.41.5.** Let $\mathcal{D}$ be a triangulated category. Given a map (13.41.1.1) assume we are given Postnikov systems for both complexes. If

1. $\mathrm{Hom}(X_i[i], Y'_n[n]) = 0$ for $i = 1, \ldots, n$, or
2. $\mathrm{Hom}(Y_n[n], X'_{n-i}[n-i]) = 0$ for $i = 1, \ldots, n$, or
3. $\mathrm{Hom}(X_{j-i}[-i+1], X'_j) = 0$ and $\mathrm{Hom}(X_j, X'_{j-i}[-i]) = 0$ for $j \geq i > 0$,

then there exists at most one morphism between these Postnikov systems.

Proof. Proof of (1). Look at the following diagram

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & Y_1[1] \arrow[r] \arrow[dl] & Y_2[2] \arrow[r] \arrow[dll] & \ldots \arrow[r] & Y_n[n] \arrow[dllll] \\ Y'_n[n] \end{tikzcd}$$

The arrows are the composition of the morphism $Y_n[n] \to Y'_n[n]$ and the morphism $Y_i[i] \to Y_n[n]$. The arrow $Y_0 \to Y'_n[n]$ is determined as it is the composition $Y_0 = X_0 \to X'_0 = Y'_0 \to Y'_n[n]$. Since we have the distinguished triangle $Y_0 \to Y_1[1] \to X_1[1]$ we see that $\mathrm{Hom}(X_1[1], Y'_n[n]) = 0$ guarantees that the second vertical arrow is unique. Since we have the distinguished triangle $Y_1[1] \to Y_2[2] \to X_2[2]$ we see that $\mathrm{Hom}(X_2[2], Y'_n[n]) = 0$ guarantees that the third vertical arrow is unique. And so on.

Proof of (2). The composition $Y_n[n] \to Y'_n[n] \to X_n[n]$ is the same as the composition $Y_n[n] \to X_n[n] \to X'_n[n]$ and hence is unique. Then using the distinguished triangle $Y'_{n-1}[n-1] \to Y'_n[n] \to X'_n[n]$ we see that it suffices to show $\mathrm{Hom}(Y_n[n], Y'_{n-1}[n-1]) = 0$. Using the distinguished triangles

$$Y'_{n-i-1}[n-i-1] \to Y'_{n-i}[n-i] \to X'_{n-i}[n-i]$$

we get this vanishing from our assumption. Small details omitted.

Proof of (3). Looking at the proof of Lemma 13.41.4 and arguing by induction on n it suffices to show that the dotted arrow in the morphism of triangles

$$\begin{tikzcd} Y_{n-1}[-1] \arrow[r] \arrow[d] & Y_n \arrow[r] \arrow[d, dotted] & X_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ Y'_{n-1}[-1] \arrow[r] & Y'_n \arrow[r] & X'_n \arrow[r] & Y'_{n-1} \end{tikzcd}$$

is unique. By Lemma 13.4.8 part (5) it suffices to show that $\mathrm{Hom}(Y_{n-1}, X'_n) = 0$ and $\mathrm{Hom}(X_n, Y'_{n-1}[-1]) = 0$. To prove the first vanishing we use the distinguished triangles $Y_{n-i-1}[-i] \to Y_{n-i}[-(i-1)] \to X_{n-i}[-(i-1)]$ for $i > 0$ and induction on i to see that the assumed vanishing of $\mathrm{Hom}(X_{n-i}[-i+1], X'_n)$ is enough. For the second we similarly use the distinguished triangles $Y'_{n-i-1}[-i-1] \to Y'_{n-i}[-i] \to X'_{n-i}[-i]$ to see that the assumed vanishing of $\mathrm{Hom}(X_n, X'_{n-i}[-i])$ is enough as well. ☐

0D83 **Lemma 13.41.6.** Let $\mathcal{D}$ be a triangulated category. Let $X_n \to X_{n-1} \to \ldots \to X_0$ be a complex in $\mathcal{D}$. If

$$\mathrm{Hom}(X_i[i-j-2], X_j) = 0 \text{ for } i > j+2$$

then there exists a Postnikov system. If we have

$$\mathrm{Hom}(X_i[i-j-1], X_j) = 0 \text{ for } i > j+1$$

then any two Postnikov systems are isomorphic.

Proof. We argue by induction on n. The cases $n = 0, 1, 2$ follow from Lemma 13.41.3. Assume $n > 2$. Suppose given a Postnikov system for the complex $X_{n-1} \to X_{n-2} \to \ldots \to X_0$. The only obstruction to extending this to a Postnikov system of length n is that we have to find a morphism $X_n \to Y_{n-1}$ such that the composition $X_n \to Y_{n-1} \to X_{n-1}$ is equal to the given map $X_n \to X_{n-1}$. Considering the distinguished triangle

$$Y_{n-1} \to X_{n-1} \to Y_{n-2} \to Y_{n-1}[1]$$

and the associated long exact sequence coming from this and the functor $\mathrm{Hom}(X_n, -)$ (see Lemma 13.4.2) we find that it suffices to show that the composition $X_n \to X_{n-1} \to Y_{n-2}$ is zero. Since we know that $X_n \to X_{n-1} \to X_{n-2}$ is zero we can apply the distinguished triangle

$$Y_{n-2} \to X_{n-2} \to Y_{n-3} \to Y_{n-2}[1]$$

to see that it suffices if $\mathrm{Hom}(X_n, Y_{n-3}[-1]) = 0$. Arguing exactly as in the proof of Lemma 13.41.4 part (1) the reader easily sees this follows from the condition stated in the lemma.

The statement on isomorphisms follows from the existence of a map between the Postnikov systems extending the identity on the complex proven in Lemma 13.41.4 part (2) and Lemma 13.4.3 to show all the maps are isomorphisms. $\square$

13.42. Essentially constant systems

0G38 Some preliminary lemmas on essentially constant systems in triangulated categories.

0G39 Lemma 13.42.1. Let $\mathcal{D}$ be a triangulated category. Let (A_i) be an inverse system in $\mathcal{D}$. Then (A_i) is essentially constant (see Categories, Definition 4.22.1) if and only if there exists an i and for all $j \geq i$ a direct sum decomposition $A_j = A \oplus Z_j$ such that (a) the maps $A_{j'} \to A_j$ are compatible with the direct sum decompositions and identity on A, (b) for all $j \geq i$ there exists some $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero.

Proof. Assume (A_i) is essentially constant with value A. Then $A = \lim A_i$ and there exists an i and a morphism $A_i \to A$ such that (1) the composition $A \to A_i \to A$ is the identity on A and (2) for all $j \geq i$ there exists a $j' \geq j$ such that $A_{j'} \to A_j$ factors as $A_{j'} \to A_i \to A \to A_j$. From (1) we conclude that for $j \geq i$ the maps $A \to A_j$ and $A_j \to A_i \to A$ compose to the identity on A. It follows that $A_j \to A$ has a kernel Z_j and that the map $A \oplus Z_j \to A_j$ is an isomorphism, see Lemmas 13.4.12 and 13.4.11. These direct sum decompositions clearly satisfy (a). From (2) we conclude that for all j there is a $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero, so (b) holds. Proof of the converse is omitted. $\square$

0G3A Lemma 13.42.2. Let $\mathcal{D}$ be a triangulated category. Let

$$A_n \to B_n \to C_n \to A_n[1]$$

be an inverse system of distinguished triangles in $\mathcal{D}$. If (A_n) and (C_n) are essentially constant, then (B_n) is essentially constant and their values fit into a distinguished triangle $A \to B \to C \to A[1]$ such that for some $n \geq 1$ there is a map

$$\begin{tikzcd} A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & A_n[1] \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

of distinguished triangles which induces an isomorphism $\lim_{n' \geq n} A_{n'} \to A$ and similarly for B and C.

Proof. After renumbering we may assume that $A_n = A \oplus A'_n$ and $C_n = C \oplus C'_n$ for inverse systems (A'_n) and (C'_n) which are essentially zero, see Lemma 13.42.1. In particular, the morphism

$$C \oplus C'_n \to (A \oplus A'_n)[1]$$

maps the summand C into the summand $A[1]$ for all n by a map $\delta : C \to A[1]$ which is independent of n. Choose a distinguished triangle

$$A \to B \to C \xrightarrow{\delta} A[1]$$

Next, choose a morphism of distingished triangles

$$(A_1 \to B_1 \to C_1 \to A_1[1]) \to (A \to B \to C \to A[1])$$

which is possible by TR3. For any object D of $\mathcal{D}$ this induces a commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & \mathrm{Hom}_\mathcal{D}(C, D) \arrow[r] \arrow[d] & \mathrm{Hom}_\mathcal{D}(B, D) \arrow[r] \arrow[d] & \mathrm{Hom}_\mathcal{D}(A, D) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & \mathrm{colim}\, \mathrm{Hom}_\mathcal{D}(C_n, D) \arrow[r] & \mathrm{colim}\, \mathrm{Hom}_\mathcal{D}(B_n, D) \arrow[r] & \mathrm{colim}\, \mathrm{Hom}_\mathcal{D}(A_n, D) \arrow[r] & \ldots \end{tikzcd}$$

The left and right vertical arrows are isomorphisms and so are the ones to the left and right of those. Thus by the 5-lemma we conclude that the middle arrow is an isomorphism. It follows that (B_n) is isomorphic to the constant inverse system with value B by the discussion in Categories, Remark 4.22.7. Since this is equivalent to (B_n) being essentially constant with value B by Categories, Remark 4.22.5 the proof is complete. □

0G3B Lemma 13.42.3. Let $\mathcal{A}$ be an abelian category. Let A_n be an inverse system of objects of $D(\mathcal{A})$. Assume

(1) there exist integers $a \leq b$ such that $H^i(A_n) = 0$ for $i \not\in [a, b]$, and
(2) the inverse systems $H^i(A_n)$ of $\mathcal{A}$ are essentially constant for all $i \in \mathbf{Z}$.

Then A_n is an essentially constant system of $D(\mathcal{A})$ whose value A satisfies that $H^i(A)$ is the value of the constant system $H^i(A_n)$ for each $i \in \mathbf{Z}$.

Proof. By Remark 13.12.4 we obtain an inverse system of distinguished triangles

$$\tau_{\leq a} A_n \to A_n \to \tau_{\geq a+1} A_n \to (\tau_{\leq a} A_n)[1]$$

Of course we have $\tau_{\leq a} A_n = H^a(A_n)[-a]$ in $D(\mathcal{A})$. Thus by assumption these form an essentially constant system. By induction on $b - a$ we find that the inverse system $\tau_{\geq a+1} A_n$ is essentially constant, say with value A'. By Lemma 13.42.2 we find that A_n is an essentially constant system. We omit the proof of the statement on cohomologies (hint: use the final part of Lemma 13.42.2). □

0G3C Lemma 13.42.4. Let $\mathcal{D}$ be a triangulated category. Let

$$A_n \to B_n \to C_n \to A_n[1]$$

be an inverse system of distinguished triangles. If the system C_n is pro-zero (essentially constant with value 0), then the maps $A_n \to B_n$ determine a pro-isomorphism between the pro-object (A_n) and the pro-object (B_n).

Proof. For any object X of $\mathcal{D}$ consider the exact sequence

$$\mathrm{colim}\,\mathrm{Hom}(C_n, X) \to \mathrm{colim}\,\mathrm{Hom}(B_n, X) \to \mathrm{colim}\,\mathrm{Hom}(A_n, X) \to \mathrm{colim}\,\mathrm{Hom}(C_n[-1], X) \to$$

Exactness follows from Lemma 13.4.2 combined with Algebra, Lemma 10.8.8. By assumption the first and last term are zero. Hence the map $\mathrm{colim}\,\mathrm{Hom}(B_n, X) \to \mathrm{colim}\,\mathrm{Hom}(A_n, X)$ is an isomorphism for all X. The lemma follows from this and Categories, Remark 4.22.7. □

0G3D Lemma 13.42.5. Let $\mathcal{A}$ be an abelian category.

$$A_n \to B_n$$

be an inverse system of maps of $D(\mathcal{A})$. Assume

(1) there exist integers $a \leq b$ such that $H^i(A_n) = 0$ and $H^i(B_n) = 0$ for $i \notin [a, b]$, and
(2) the inverse system of maps $H^i(A_n) \to H^i(B_n)$ of $\mathcal{A}$ define an isomorphism of pro-objects of $\mathcal{A}$ for all $i \in \mathbf{Z}$.

Then the maps $A_n \to B_n$ determine a pro-isomorphism between the pro-object (A_n) and the pro-object (B_n).

Proof. We can inductively extend the maps $A_n \to B_n$ to an inverse system of distinguished triangles $A_n \to B_n \to C_n \to A_n[1]$ by axiom TR3. By Lemma 13.42.4 it suffices to prove that C_n is pro-zero. By Lemma 13.42.3 it suffices to show that $H^p(C_n)$ is pro-zero for each p. This follows from assumption (2) and the long exact sequences

$$H^p(A_n) \xrightarrow{\alpha_n} H^p(B_n) \xrightarrow{\beta_n} H^p(C_n) \xrightarrow{\delta_n} H^{p+1}(A_n) \xrightarrow{\epsilon_n} H^{p+1}(B_n)$$

Namely, for every n we can find an $m > n$ such that $\mathrm{Im}(\beta_m)$ maps to zero in $H^p(C_n)$ because we may choose m such that $H^p(B_m) \to H^p(B_n)$ factors through $\alpha_n : H^p(A_n) \to H^p(B_n)$. For a similar reason we may then choose $k > m$ such that $\mathrm{Im}(\delta_k)$ maps to zero in $H^{p+1}(A_m)$. Then $H^p(C_k) \to H^p(C_n)$ is zero because $H^p(C_k) \to H^p(C_m)$ maps into $\mathrm{Ker}(\delta_m)$ and $H^p(C_m) \to H^p(C_n)$ annihilates $\mathrm{Ker}(\delta_m) = \mathrm{Im}(\beta_m)$. □

13.43. Other chapters

Preliminaries

CHAPTER 14

Simplicial Methods

0162 14.1. Introduction

0163 This is a minimal introduction to simplicial methods. We just add here whenever something is needed later on. A general reference to this material is perhaps [GJ99]. An example of the things you can do is the paper by Quillen on Homotopical Algebra, see [Qui67] or the paper on Étale Homotopy by Artin and Mazur, see [AM69].

14.2. The category of finite ordered sets

0164 The category Δ is the category with

(1) objects $[0], [1], [2], \ldots$ with $[n] = \{0, 1, 2, \ldots, n\}$ and
(2) a morphism $[n] \to [m]$ is a nondecreasing map $\{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, m\}$ between the corresponding sets.

Here nondecreasing for a map $\varphi : [n] \to [m]$ means by definition that $\varphi(i) \geq \varphi(j)$ if $i \geq j$. In other words, Δ is a category equivalent to the "big" category of nonempty finite totally ordered sets and nondecreasing maps. There are exactly $n+1$ morphisms $[0] \to [n]$ and there is exactly 1 morphism $[n] \to [0]$. There are exactly $(n+1)(n+2)/2$ morphisms $[1] \to [n]$ and there are exactly $n+2$ morphisms $[n] \to [1]$. And so on and so forth.

0165 Definition 14.2.1. For any integer $n \geq 1$, and any $0 \leq j \leq n$ we let $\delta^n_j : [n-1] \to [n]$ denote the injective order preserving map skipping j. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $\sigma^n_j : [n+1] \to [n]$ the surjective order preserving map with $(\sigma^n_j)^{-1}(\{j\}) = \{j, j+1\}$.

0166 Lemma 14.2.2. Any morphism in Δ can be written as a composition of the morphisms δ^n_j and σ^n_j.

Proof. Let $\varphi : [n] \to [m]$ be a morphism of Δ. If $j \notin \mathrm{Im}(\varphi)$, then we can write φ as $\delta^m_j \circ \psi$ for some morphism $\psi : [n] \to [m-1]$. If $\varphi(j) = \varphi(j+1)$ then we can write φ as $\psi \circ \sigma^{n-1}_j$ for some morphism $\psi : [n-1] \to [m]$. The result follows because each replacement as above lowers $n+m$ and hence at some point φ is both injective and surjective, hence an identity morphism. $\square$

0167 Lemma 14.2.3. The morphisms δ^n_j and σ^n_j satisfy the following relations.

(1) If $0 \leq i < j \leq n+1$, then $\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$. In other words the diagram

$$\begin{tikzcd} & {[n]} \arrow[dr, "\delta_j^{n+1}"] & \\ {[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\delta_{j-1}^n"'] & & {[n+1]} \\ & {[n]} \arrow[ur, "\delta_i^{n+1}"'] & \end{tikzcd}$$

commutes.

(2) If $0 \leq i < j \leq n-1$, then $\sigma_j^{n-1} \circ \delta_i^n = \delta_i^{n-1} \circ \sigma_{j-1}^{n-2}$. In other words the diagram

$$\begin{tikzcd} & {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\ {[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\sigma_{j-1}^{n-2}"'] & & {[n-1]} \\ & {[n-2]} \arrow[ur, "\delta_i^{n-1}"'] & \end{tikzcd}$$

commutes.

(3) If $0 \leq j \leq n-1$, then $\sigma_j^{n-1} \circ \delta_j^n = \mathrm{id}_{[n-1]}$ and $\sigma_j^{n-1} \circ \delta_{j+1}^n = \mathrm{id}_{[n-1]}$. In other words the diagram

$$\begin{tikzcd} & {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\ {[n-1]} \arrow[ur, "\delta_j^n"] \arrow[rr, "\mathrm{id}_{[n-1]}"] \arrow[dr, "\delta_{j+1}^n"'] & & {[n-1]} \\ & {[n]} \arrow[ur, "\sigma_j^{n-1}"'] & \end{tikzcd}$$

commutes.

(4) If $0 < j+1 < i \leq n$, then $\sigma_j^{n-1} \circ \delta_i^n = \delta_{i-1}^{n-1} \circ \sigma_j^{n-2}$. In other words the diagram

$$\begin{tikzcd} & {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\ {[n-1]} \arrow[ur, "\delta_i^n"] \arrow[dr, "\sigma_j^{n-2}"'] & & {[n-1]} \\ & {[n-2]} \arrow[ur, "\delta_{i-1}^{n-1}"'] & \end{tikzcd}$$

commutes.

(5) If $0 \leq i \leq j \leq n-1$, then $\sigma_j^{n-1} \circ \sigma_i^n = \sigma_i^{n-1} \circ \sigma_{j+1}^n$. In other words the diagram

$$\begin{tikzcd} & {[n]} \arrow[dr, "\sigma_j^{n-1}"] & \\ {[n+1]} \arrow[ur, "\sigma_i^n"] \arrow[dr, "\sigma_{j+1}^n"'] & & {[n-1]} \\ & {[n]} \arrow[ur, "\sigma_i^{n-1}"'] & \end{tikzcd}$$

commutes.

Proof. Omitted. □

0168 Lemma 14.2.4. The category Δ is the universal category with objects $[n]$, $n \geq 0$ and morphisms δ_j^n and σ_j^n such that (a) every morphism is a composition of these morphisms, (b) the relations listed in Lemma 14.2.3 are satisfied, and (c) any relation among the morphisms is a consequence of those relations.

Proof. Omitted. □

14.3. Simplicial objects

0169

016A Definition 14.3.1. Let $\mathcal{C}$ be a category.

(1) A simplicial object U of $\mathcal{C}$ is a contravariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta^{opp} \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a simplicial set.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a simplicial abelian group.
(4) A morphism of simplicial objects $U \to U'$ is a transformation of functors.
(5) The category of simplicial objects of $\mathcal{C}$ is denoted $\text{Simp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([n]) \to U([m])$, satisfying $U(\varphi \circ \psi) = U(\psi) \circ U(\varphi)$.

In particular there is a unique morphism $U([0]) \to U([n])$ and there are exactly $n+1$ morphisms $U([n]) \to U([0])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 14.2 above.

016B Lemma 14.3.2. Let $\mathcal{C}$ be a category.

(1) Given a simplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $d_j^n = U(\delta_j^n) : U_n \to U_{n-1}$ and $s_j^n = U(\sigma_j^n) : U_n \to U_{n+1}$. These morphisms satisfy the opposites of the relations displayed in Lemma 14.2.3, namely
  (a) If $0 \leq i < j \leq n+1$, then $d_i^n \circ d_j^{n+1} = d_{j-1}^n \circ d_i^{n+1}$.
  (b) If $0 \leq i < j \leq n-1$, then $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$.

   (c) If $0 \leq j \leq n-1$, then $\text{id} = d^n_j \circ s^{n-1}_j = d^n_{j+1} \circ s^{n-1}_j$.
   (d) If $0 < j+1 < i \leq n$, then $d^n_i \circ s^{n-1}_j = s^{n-2}_j \circ d^{n-1}_{i-1}$.
   (e) If $0 \leq i \leq j \leq n-1$, then $s^n_i \circ s^{n-1}_j = s^n_{j+1} \circ s^{n-1}_i$.
(2) Conversely, given a sequence of objects U_n and morphisms d^n_j, s^n_j satisfying (1)(a) – (e) there exists a unique simplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $d^n_j = U(\delta^n_j)$, and $s^n_j = U(\sigma^n_j)$.
(3) A morphism between simplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms d^n_j and s^n_j.

Proof. This follows from Lemma 14.2.4. □

016C Remark 14.3.3. By abuse of notation we sometimes write $d_i : U_n \to U_{n-1}$ instead of d^n_i, and similarly for $s_i : U_n \to U_{n+1}$. The relations among the morphisms d^n_i and s^n_i may be expressed as follows:

(1) If $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.
(2) If $i < j$, then $d_i \circ s_j = s_{j-1} \circ d_i$.
(3) We have $\text{id} = d_j \circ s_j = d_{j+1} \circ s_j$.
(4) If $i > j+1$, then $d_i \circ s_j = s_j \circ d_{i-1}$.
(5) If $i \leq j$, then $s_i \circ s_j = s_{j+1} \circ s_i$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $s^0_0 = U(\sigma^0_0) : U_0 \to U_1$ and two morphisms $d^1_0 = U(\delta^1_0)$, and $d^1_1 = U(\delta^1_1)$ which are morphisms $U_1 \to U_0$. There are two morphisms $s^1_0 = U(\sigma^1_0)$, $s^1_1 = U(\sigma^1_1)$ which are morphisms $U_1 \to U_2$. Three morphisms $d^2_0 = U(\delta^2_0)$, $d^2_1 = U(\delta^2_1)$, $d^2_2 = U(\delta^2_2)$ which are morphisms $U_3 \to U_2$. And so on.

Pictorially we think of U as follows:

$$U_2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; U_1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; U_0$$

Here the d-morphisms are the arrows pointing right and the s-morphisms are the arrows pointing left.

016D Example 14.3.4. The simplest example is the constant simplicial object with value $X \in \text{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

016E Example 14.3.5. Suppose that $Y \to X$ is a morphism of $\mathcal{C}$ such that all the fibred products $Y \times_X Y \times_X \ldots \times_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold fibre product, and we let $\varphi : [n] \to [m]$ correspond to the map (on "coordinates") $(y_0, \ldots, y_m) \mapsto (y_{\varphi(0)}, \ldots, y_{\varphi(n)})$. In other words, the map $U_0 = Y \to U_1 = Y \times_X Y$ is the diagonal map. The two maps $U_1 = Y \times_X Y \to U_0 = Y$ are the projection maps.

Geometrically Example 14.3.5 above is an important example. It tells us that it is a good idea to think of the maps $d^n_j : U_n \to U_{n-1}$ as projection maps (forgetting the jth component), and to think of the maps $s^n_j : U_n \to U_{n+1}$ as diagonal maps (repeating the jth coordinate). We will return to this in the sections below.

016F Lemma 14.3.6. Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. Each of the morphisms $s^n_i : U_n \to U_{n+1}$ has a left inverse. In particular s^n_i is a monomorphism.

Proof. This is true because $d^{n+1}_i \circ s^n_i = \text{id}_{U_n}$. □

14.4. Simplicial objects as presheaves

016G Another observation is that we may think of a simplicial object of $\mathcal{C}$ as a presheaf with values in $\mathcal{C}$ over Δ. See Sites, Definition 7.2.2. And in fact, if U, U' are simplicial objects of $\mathcal{C}$, then we have

016H (14.4.0.1)
$$\mathrm{Mor}(U, U') = \mathrm{Mor}_{\mathrm{PSh}(\Delta)}(U, U').$$

Some of the material below could be replaced by the more general constructions in the chapter on sites. However, it seems a clearer picture arises from the arguments specific to simplicial objects.

14.5. Cosimplicial objects

016I A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{opp}$. This is not really how the human brain works, so we introduce them separately here and point out some simple properties.

016J Definition 14.5.1. Let $\mathcal{C}$ be a category.

(1) A cosimplicial object U of $\mathcal{C}$ is a covariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a cosimplicial set.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a cosimplicial abelian group.
(4) A morphism of cosimplicial objects $U \to U'$ is a transformation of functors.
(5) The category of cosimplicial objects of $\mathcal{C}$ is denoted $\mathrm{CoSimp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([m]) \to U([n])$, satisfying $U(\varphi \circ \psi) = U(\varphi) \circ U(\psi)$.

In particular there is a unique morphism $U([n]) \to U([0])$ and there are exactly $n+1$ morphisms $U([0]) \to U([n])$ corresponding to the $n + 1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 14.2 above.

016K Lemma 14.5.2. Let $\mathcal{C}$ be a category.

(1) Given a cosimplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $\delta^n_j = U(\delta^n_j) : U_{n-1} \to U_n$ and $\sigma^n_j = U(\sigma^n_j) : U_{n+1} \to U_n$. These morphisms satisfy the relations displayed in Lemma 14.2.3.
(2) Conversely, given a sequence of objects U_n and morphisms δ^n_j, σ^n_j satisfying these relations there exists a unique cosimplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $\delta^n_j = U(\delta^n_j)$, and $\sigma^n_j = U(\sigma^n_j)$.
(3) A morphism between cosimplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms δ^n_j and σ^n_j.

Proof. This follows from Lemma 14.2.4. $\square$

016L Remark 14.5.3. By abuse of notation we sometimes write $\delta_i : U_{n-1} \to U_n$ instead of δ_i^n, and similarly for $\sigma_i : U_{n+1} \to U_n$. The relations among the morphisms δ_i^n and σ_i^n may be expressed as follows:

(1) If $i < j$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$.
(2) If $i < j$, then $\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1}$.
(3) We have $\mathrm{id} = \sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1}$.
(4) If $i > j + 1$, then $\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j$.
(5) If $i \leq j$, then $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $\sigma_0^0 = U(\sigma_0^0) : U_1 \to U_0$ and two morphisms $\delta_0^1 = U(\delta_0^1)$, and $\delta_1^1 = U(\delta_1^1)$ which are morphisms $U_0 \to U_1$. There are two morphisms $\sigma_0^1 = U(\sigma_0^1)$, $\sigma_1^1 = U(\sigma_1^1)$ which are morphisms $U_2 \to U_1$. Three morphisms $\delta_0^2 = U(\delta_0^2)$, $\delta_1^2 = U(\delta_1^2)$, $\delta_2^2 = U(\delta_2^2)$ which are morphisms $U_2 \to U_3$. And so on.

Pictorially we think of U as follows:

$$\begin{tikzcd} U_0 \arrow[r, shift left=2] \arrow[r, shift right=2] & U_1 \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & U_2 \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

Here the δ-morphisms are the arrows pointing right and the σ-morphisms are the arrows pointing left.

016M Example 14.5.4. The simplest example is the constant cosimplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

016N Example 14.5.5. Suppose that $X \to Y$ is a morphism of C such that all the pushouts $Y \amalg_X Y \amalg_X \ldots \amalg_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold pushout, and we let $\varphi : [n] \to [m]$ correspond to the map

$$(y \text{ in } i\text{th component}) \mapsto (y \text{ in } \varphi(i)\text{th component})$$

on "coordinates". In other words, the map $U_1 = Y \amalg_X Y \to U_0 = Y$ is the identity on each component. The two maps $U_0 = Y \to U_1 = Y \amalg_X Y$ are the two coprojections.

0B13 Example 14.5.6. For every $n \geq 0$ we denote $C[n]$ the cosimplicial set

$$\Delta \longrightarrow \text{Sets}, \quad [k] \longmapsto \mathrm{Mor}_\Delta([n], [k])$$

This example is dual to Example 14.11.2.

016O Lemma 14.5.7. Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$. Each of the morphisms $\delta_i^n : U_{n-1} \to U_n$ has a left inverse. In particular δ_i^n is a monomorphism.

Proof. This is true because $\sigma_i^{n-1} \circ \delta_i^n = \mathrm{id}_{U_n}$ for $j < n$. $\square$

14.6. Products of simplicial objects

016P Of course we should define the product of simplicial objects as the product in the category of simplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

016Q Definition 14.6.1. Let $\mathcal{C}$ be a category. Let U and V be simplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The product of U and V is the simplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $U \times V$ is the product of the presheaves U and V on Δ.

016R Lemma 14.6.2. If U and V are simplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third simplicial object W of $\mathcal{C}$.

Proof. Omitted. □

14.7. Fibre products of simplicial objects

016S Of course we should define the fibre product of simplicial objects as the fibre product in the category of simplicial objects. This may lead to the potentially confusing situation where the fibre product exists but is not described as below. To avoid this we define the fibre product directly as follows.

016T Definition 14.7.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : V \to U$, $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The fibre product of V and W over U is the simplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \times_U W$ is the fibre product of the presheaves V and W over the presheaf U on Δ.

016U Lemma 14.7.2. If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.8. Pushouts of simplicial objects

016V Of course we should define the pushout of simplicial objects as the pushout in the category of simplicial objects. This may lead to the potentially confusing situation where the pushouts exist but are not as described below. To avoid this we define the pushout directly as follows.

016W Definition 14.8.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : U \to V$, $b : U \to W$ be morphisms. Assume the pushouts $V_n \amalg_{U_n} W_n$ exist in $\mathcal{C}$. The pushout of V and W over U is the simplicial object $V \amalg_U W$ defined as follows:

(1) $(V \amalg_U W)_n = V_n \amalg_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \amalg_U W$ is the pushout of the presheaves V and W over the presheaf U on Δ.

016X Lemma 14.8.2. If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : U \to V$, $b : U \to W$ are morphisms and if $V \amalg_U W$ exists, then we have

$$\mathrm{Mor}(V \amalg_U W, T) = \mathrm{Mor}(V, T) \times_{\mathrm{Mor}(U,T)} \mathrm{Mor}(W, T)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. $\square$

14.9. Products of cosimplicial objects

016Y Of course we should define the product of cosimplicial objects as the product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

016Z Definition 14.9.1. Let $\mathcal{C}$ be a category. Let U and V be cosimplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The product of U and V is the cosimplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(U \times V)(\varphi) : U_n \times V_n \to U_m \times V_m$ is the product $U(\varphi) \times V(\varphi)$.

0170 Lemma 14.9.2. If U and V are cosimplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third cosimplicial object W of $\mathcal{C}$.

Proof. Omitted. $\square$

14.10. Fibre products of cosimplicial objects

0171 Of course we should define the fibre product of cosimplicial objects as the fibre product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the fibre product directly as follows.

0172 Definition 14.10.1. Let $\mathcal{C}$ be a category. Let U, V, W be cosimplicial objects of $\mathcal{C}$. Let $a : V \to U$ and $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The fibre product of V and W over U is the cosimplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(V \times_U W)(\varphi) : V_n \times_{U_n} W_n \to V_m \times_{U_m} W_m$ is the product $V(\varphi) \times_{U(\varphi)} W(\varphi)$.

0173 Lemma 14.10.2. If U, V, W are cosimplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth cosimplicial object T of $\mathcal{C}$.

Proof. Omitted. $\square$

14.11. Simplicial sets

0174 Let U be a simplicial set. It is a good idea to think of U_0 as the 0-simplices, the set U_1 as the 1-simplices, the set U_2 as the 2-simplices, and so on.

We think of the maps $s^n_j : U_n \to U_{n+1}$ as the map that associates to an n-simplex A the degenerate $(n+1)$-simplex B whose $(j, j+1)$-edge is collapsed to the vertex j of A. We think of the map $d^n_j : U_n \to U_{n-1}$ as the map that associates to an n-simplex A one of the faces, namely the face that omits the vertex j. In this way it become possible to visualize the relations among the maps s^n_j and d^n_j geometrically.

0175 Definition 14.11.1. Let U be a simplicial set. We say x is an n-simplex of U to signify that x is an element of U_n. We say that y is the $j$the face of x to signify that $d^n_j x = y$. We say that z is the jth degeneracy of x if $z = s^n_j x$. A simplex is called degenerate if it is the degeneracy of another simplex.

Here are a few fundamental examples.

0176 Example 14.11.2. For every $n \geq 0$ we denote $\Delta[n]$ the simplicial set

$$\Delta^{opp} \longrightarrow \text{Sets}, \quad [k] \longmapsto \text{Mor}_\Delta([k],[n])$$

We leave it to the reader to verify the following statements. Every m-simplex of $\Delta[n]$ with $m > n$ is degenerate. There is a unique nondegenerate n-simplex of $\Delta[n]$, namely $\text{id}_{[n]}$.

0177 Lemma 14.11.3. Let U be a simplicial set. Let $n \geq 0$ be an integer. There is a canonical bijection

$$\text{Mor}(\Delta[n], U) \longrightarrow U_n$$

which maps a morphism φ to the value of φ on the unique nondegenerate n-simplex of $\Delta[n]$.

Proof. Omitted. □

0178 Example 14.11.4. Consider the category $\Delta/[n]$ of objects over $[n]$ in Δ, see Categories, Example 4.2.13. There is a functor $p : \Delta/[n] \to \Delta$. The fibre category of p over $[k]$, see Categories, Section 4.35, has as objects the set $\Delta[n]_k$ of k-simplices in $\Delta[n]$, and as morphisms only identities. For every morphism $\varphi : [k] \to [l]$ of Δ, and every object $\psi : [l] \to [n]$ in the fibre category over $[l]$ there is a unique object over $[k]$ with a morphism covering φ, namely $\psi \circ \varphi : [k] \to [n]$. Thus $\Delta/[n]$ is fibred in sets over Δ. In other words, we may think of $\Delta/[n]$ as a presheaf of sets over Δ. See also, Categories, Example 4.38.7. And this presheaf of sets agrees with the simplicial set $\Delta[n]$. In particular, from Equation (14.4.0.1) and Lemma 14.11.3 above we get the formula

$$\text{Mor}_{\text{PSh}(\Delta)}(\Delta/[n], U) = U_n$$

for any simplicial set U.

0179 Lemma 14.11.5. Let U, V be simplicial sets. Let $a, b \geq 0$ be integers. Assume every n-simplex of U is degenerate if $n > a$. Assume every n-simplex of V is degenerate if $n > b$. Then every n-simplex of $U \times V$ is degenerate if $n > a + b$.

Proof. Suppose $n > a + b$. Let $(u, v) \in (U \times V)_n = U_n \times V_n$. By assumption, there exists a $\alpha : [n] \to [a]$ and a $u' \in U_a$ and a $\beta : [n] \to [b]$ and a $v' \in V_b$ such that $u = U(\alpha)(u')$ and $v = V(\beta)(v')$. Because $n > a + b$, there exists an $0 \leq i \leq a + b$

such that $\alpha(i) = \alpha(i+1)$ and $\beta(i) = \beta(i+1)$. It follows immediately that (u, v) is in the image of s_i^{n-1}. $\square$

14.12. Truncated simplicial objects and skeleton functors

017Z Let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], [1], [2], \ldots, [n]$. Let $\mathcal{C}$ be a category.

0180 Definition 14.12.1. An n-truncated simplicial object of $\mathcal{C}$ is a contravariant functor from $\Delta_{\leq n}$ to $\mathcal{C}$. A morphism of n-truncated simplicial objects is a transformation of functors. We denote the category of n-truncated simplicial objects of $\mathcal{C}$ by the symbol $\text{Simp}_n(\mathcal{C})$.

Given a simplicial object U of $\mathcal{C}$ the truncation $\text{sk}_n U$ is the restriction of U to the subcategory $\Delta_{\leq n}$. This defines a skeleton functor

$$\text{sk}_n : \text{Simp}(\mathcal{C}) \longrightarrow \text{Simp}_n(\mathcal{C})$$

from the category of simplicial objects of $\mathcal{C}$ to the category of n-truncated simplicial objects of $\mathcal{C}$. See Remark 14.21.6 to avoid possible confusion with other functors in the literature.

14.13. Products with simplicial sets

017A Let $\mathcal{C}$ be a category. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. We can consider the covariant functor which associates to a simplicial object W of $\mathcal{C}$ the set

(14.13.0.1)

017B
$$\left\{ (f_{n,u} : V_n \to W_n)_{n \geq 0, u \in U_n} \text{ such that } \begin{matrix} \forall \varphi : [m] \to [n] \\ f_{m, U(\varphi)(u)} \circ V(\varphi) = W(\varphi) \circ f_{n,u} \end{matrix} \right\}$$

If this functor is of the form $\text{Mor}_{\text{Simp}(\mathcal{C})}(Q, -)$ then we can think of Q as the product of U with V. Instead of formalizing this in this way we just directly define the product as follows.

017C Definition 14.13.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. In this case we define the product $U \times V$ of U and V to be the simplicial object of $\mathcal{C}$ whose nth term is the object

$$(U \times V)_n = \coprod\nolimits_{u \in U_n} V_n$$

with maps for $\varphi : [m] \to [n]$ given by the morphism

$$\coprod\nolimits_{u \in U_n} V_n \longrightarrow \coprod\nolimits_{u' \in U_m} V_m$$

which maps the component V_n corresponding to u to the component V_m corresponding to $u' = U(\varphi)(u)$ via the morphism $V(\varphi)$. More loosely, if all of the coproducts displayed above exist (without assuming anything about $\mathcal{C}$) we will say that the product $U \times V$ exists.

017D Lemma 14.13.2. Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. The functor $W \mapsto \text{Mor}_{\text{Simp}(\mathcal{C})}(U \times V, W)$ is canonically isomorphic to the functor which maps W to the set in Equation (14.13.0.1).

Proof. Omitted. □

017E Lemma 14.13.3. Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let us temporarily denote FSSets the category of simplicial sets all of whose components are finite nonempty.

(1) The rule $(U, V) \mapsto U \times V$ defines a functor $\text{FSSets} \times \text{Simp}(\mathcal{C}) \to \text{Simp}(\mathcal{C})$.
(2) For every U, V as above there is a canonical map of simplicial objects
$$U \times V \longrightarrow V$$
defined by taking the identity on each component of $(U \times V)_n = \coprod_u V_n$.

Proof. Omitted. □

We briefly study a special case of the construction above. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $k \geq 0$ be an integer. If all coproducts $X \amalg \ldots \amalg X$ exist then according to the definition above the product
$$X \times \Delta[k]$$
exists, where we think of X as the corresponding constant simplicial object.

017F Lemma 14.13.4. With X and k as above. For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection
$$\text{Mor}_{\text{Simp}(\mathcal{C})}(X \times \Delta[k], V) \longrightarrow \text{Mor}_{\mathcal{C}}(X, V_k).$$
which maps γ to the restriction of the morphism γ_k to the component corresponding to $\text{id}_{[k]}$. Similarly, for any $n \geq k$, if W is an n-truncated simplicial object of $\mathcal{C}$, then we have
$$\text{Mor}_{\text{Simp}_n(\mathcal{C})}(\text{sk}_n(X \times \Delta[k]), W) = \text{Mor}_{\mathcal{C}}(X, W_k).$$

Proof. A morphism $\gamma : X \times \Delta[k] \to V$ is given by a family of morphisms $\gamma_\alpha : X \to V_n$ where $\alpha : [n] \to [k]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams
$$\begin{tikzcd} X \arrow[r, "\gamma_\alpha"] \arrow[d, "\text{id}_X"] & V_n \arrow[d, "V(\varphi)"] \\ X \arrow[r, "\gamma_{\alpha \circ \varphi}"] & V_m \end{tikzcd}$$
commute. Taking $\alpha = \text{id}_{[k]}$, we see that for any $\varphi : [m] \to [k]$ we have $\gamma_\varphi = V(\varphi) \circ \gamma_{\text{id}_{[k]}}$. Thus the morphism γ is determined by the value of γ on the component corresponding to $\text{id}_{[k]}$. Conversely, given such a morphism $f : X \to V_k$ we easily construct a morphism γ by putting $\gamma_\alpha = V(\alpha) \circ f$.

The truncated case is similar, and left to the reader. □

A particular example of this is the case $k = 0$. In this case the formula of the lemma just says that
$$\text{Mor}_{\mathcal{C}}(X, V_0) = \text{Mor}_{\text{Simp}(\mathcal{C})}(X, V)$$
where on the right hand side X indicates the constant simplicial object with value X. We will use this formula without further mention in the following.

14.14. Hom from simplicial sets into cosimplicial objects

07K9 Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$, and let V be a cosimplicial object of $\mathcal{C}$. Then we get a cosimplicial set $\mathrm{Hom}_\mathcal{C}(U, V)$ as follows:

(1) we set $\mathrm{Hom}_\mathcal{C}(U, V)_n = \mathrm{Mor}_\mathcal{C}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}_\mathcal{C}(U, V)_m \to \mathrm{Hom}_\mathcal{C}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

019V Definition 14.14.1. Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object of $\mathcal{C}$. Let U be a simplicial set such that each U_n is finite nonempty. We define $\mathrm{Hom}(U, V)$ to be the cosimplicial object of $\mathcal{C}$ defined as follows:

(1) we set $\mathrm{Hom}(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy
$$\mathrm{Mor}_\mathcal{C}(X, \mathrm{Hom}(U, V)_n) = \mathrm{Map}(U_n, \mathrm{Mor}_\mathcal{C}(X, V_n))$$
and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}(U, V)_m \to \mathrm{Hom}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 14.13.3 in the case of products of simplicial sets with simplicial objects.

14.15. Hom from cosimplicial sets into simplicial objects

0B14 Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$, and let V be a simplicial object of $\mathcal{C}$. Then we get a simplicial set $\mathrm{Hom}_\mathcal{C}(U, V)$ as follows:

(1) we set $\mathrm{Hom}_\mathcal{C}(U, V)_n = \mathrm{Mor}_\mathcal{C}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}_\mathcal{C}(U, V)_n \to \mathrm{Hom}_\mathcal{C}(U, V)_m$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

0B15 Definition 14.15.1. Let $\mathcal{C}$ be a category with finite products. Let V be a simplicial object of $\mathcal{C}$. Let U be a cosimplicial set such that each U_n is finite nonempty. We define $\mathrm{Hom}(U, V)$ to be the simplicial object of $\mathcal{C}$ defined as follows:

(1) we set $\mathrm{Hom}(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy
$$\mathrm{Mor}_\mathcal{C}(X, \mathrm{Hom}(U, V)_n) = \mathrm{Map}(U_n, \mathrm{Mor}_\mathcal{C}(X, V_n))$$
and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}(U, V)_n \to \mathrm{Hom}(U, V)_m$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 14.13.3 in the case of products of simplicial sets with simplicial objects.

We spell out the construction above in a special case. Let X be an object of a category $\mathcal{C}$. Assume that self products $X \times \ldots \times X$ exist. Let k be an integer. Consider the simplicial object U with terms

$$U_n = \prod\nolimits_{\alpha \in \mathrm{Mor}([k],[n])} X$$

and maps given $\varphi : [m] \to [n]$

$$U(\varphi) : \prod\nolimits_{\alpha \in \mathrm{Mor}([k],[n])} X \longrightarrow \prod\nolimits_{\alpha' \in \mathrm{Mor}([k],[m])} X, \quad (f_\alpha)_\alpha \longmapsto (f_{\varphi \circ \alpha'})_{\alpha'}$$

In terms of "coordinates", the element $(x_\alpha)_\alpha$ is mapped to the element $(x_{\varphi\circ\alpha'})_{\alpha'}$. We claim this object is equal to $\mathrm{Hom}(C[k], X)$ where we think of X as the constant simplicial object X and where $C[k]$ is the cosimplicial set from Example 14.5.6.

017M Lemma 14.15.2. With X, k and U as above.

(1) For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(V, U) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(V_k, X).$$

wich maps γ to the morphism γ_k composed with the projection onto the factor corresponding to $\mathrm{id}_{[k]}$.

(2) Similarly, if W is an k-truncated simplicial object of $\mathcal{C}$, then we have

$$\mathrm{Mor}_{\mathrm{Simp}_k(\mathcal{C})}(W, \mathrm{sk}_k U) = \mathrm{Mor}_{\mathcal{C}}(W_k, X).$$

(3) The object U constructed above is an incarnation of $\mathrm{Hom}(C[k], X)$ where $C[k]$ is the cosimplicial set from Example 14.5.6.

Proof. We first prove (1). Suppose that $\gamma : V \to U$ is a morphism. This is given by a family of morphisms $\gamma_\alpha : V_n \to X$ for $\alpha : [k] \to [n]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$\begin{tikzcd} X \arrow[d, "\mathrm{id}_X"'] & V_n \arrow[l, "\gamma_{\varphi\circ\alpha'}"'] \arrow[d, "V(\varphi)"] \\ X & V_m \arrow[l, "\gamma_{\alpha'}"] \end{tikzcd}$$

commute for all $\alpha' : [k] \to [m]$. Taking $\alpha' = \mathrm{id}_{[k]}$, we see that for any $\varphi : [k] \to [n]$ we have $\gamma_\varphi = \gamma_{\mathrm{id}_{[k]}} \circ V(\varphi)$. Thus the morphism γ is determined by the component of γ_k corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : V_k \to X$ we easily construct a morphism γ by putting $\gamma_\alpha = f \circ V(\alpha)$.

The truncated case is similar, and left to the reader.

Part (3) is immediate from the construction of U and the fact that $C[k]_n = \mathrm{Mor}([k],[n])$ which are the index sets used in the construction of U_n. $\square$

14.16. Internal Hom

017G Let $\mathcal{C}$ be a category with finite nonempty products. Let U, V be simplicial objects $\mathcal{C}$. In some cases the functor

$$\mathrm{Simp}(\mathcal{C})^{opp} \longrightarrow \mathrm{Sets}, \quad W \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times V, U)$$

is representable. In this case we denote $\mathcal{H}om(V, U)$ the resulting simplicial object of $\mathcal{C}$, and we say that the internal hom of V into U exists. Moreover, in this case, given X in $\mathcal{C}$, we would have

$$\begin{aligned}\mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(V, U)_n) &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n], \mathcal{H}om(V, U)) \\ &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n] \times V, U) \\ &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, \mathcal{H}om(\Delta[n] \times V, U)) \\ &= \mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(\Delta[n] \times V, U)_0)\end{aligned}$$

provided that $\mathcal{H}om(\Delta[n] \times V, U)$ exists also. The first and last equalities follow from Lemma 14.13.4.

The lesson we learn from this is that, given U and V, if we want to construct the internal hom then we should try to construct the objects

$$\mathcal{H}om(\Delta[n] \times V, U)_0$$

because these should be the nth term of $\mathcal{H}om(V, U)$. In the next section we study a construction of simplicial objects "$\mathrm{Hom}(\Delta[n], U)$".

14.17. Hom from simplicial sets into simplicial objects

017H Motivated by the discussion on internal hom we define what should be the simplicial object classifying morphisms from a simplicial set into a given simplicial object of the category $\mathcal{C}$.

017I Definition 14.17.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects exists. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. We denote $\mathrm{Hom}(U, V)$ any simplicial object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W, \mathrm{Hom}(U, V)) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times U, V)$$

functorially in the simplicial object W of $\mathcal{C}$.

Of course $\mathrm{Hom}(U, V)$ need not exist. Also, by the discussion in Section 14.16 we expect that if it does exist, then $\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0$. We do not use the italic notation for these Hom objects since $\mathrm{Hom}(U, V)$ is not an internal hom.

017J Lemma 14.17.2. Assume the category $\mathcal{C}$ has coproducts of any two objects and countable limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor

$$\begin{array}{ccc}\mathcal{C}^{opp} & \longrightarrow & \mathrm{Sets} \\ X & \longmapsto & \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times U, V)\end{array}$$

is representable.

Proof. A morphism from $X \times U$ into V is given by a collection of morphisms $f_u : X \to V_n$ with $n \geq 0$ and $u \in U_n$. And such a collection actually defines a morphism if and only if for all $\varphi : [m] \to [n]$ all the diagrams

$$\begin{tikzcd} X \arrow[r, "f_u"] \arrow[d, "\mathrm{id}_X"'] & V_n \arrow[d, "V(\varphi)"] \\ X \arrow[r, "f_{U(\varphi)(u)}"] & V_m \end{tikzcd}$$

commute. Thus it is natural to introduce a category $\mathcal{U}$ and a functor $\mathcal{V} : \mathcal{U}^{opp} \to \mathcal{C}$ as follows:

(1) The set of objects of $\mathcal{U}$ is $\coprod_{n \geq 0} U_n$,
(2) a morphism from $u' \in U_m$ to $u \in U_n$ is a $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$
(3) for $u \in U_n$ we set $\mathcal{V}(u) = V_n$, and
(4) for $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$ we set $\mathcal{V}(\varphi) = V(\varphi) : V_n \to V_m$.

At this point it is clear that our functor is nothing but the functor defining

$$\lim_{\mathcal{U}^{opp}} \mathcal{V}$$

Thus if $\mathcal{C}$ has countable limits then this limit and hence an object representing the functor of the lemma exist. $\square$

017K Lemma 14.17.3. Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor

$$\begin{array}{ccc} \mathcal{C}^{opp} & \longrightarrow & \textit{Sets} \\ X & \longmapsto & \mathrm{Mor}_{\text{Simp}(\mathcal{C})}(X \times U, V) \end{array}$$

is representable.

Proof. We have to show that the category $\mathcal{U}$ described in the proof of Lemma 14.17.2 has a finite subcategory $\mathcal{U}'$ such that the limit of $\mathcal{V}$ over $\mathcal{U}'$ is the same as the limit of $\mathcal{V}$ over $\mathcal{U}$. We will use Categories, Lemma 4.17.4. For $m > 0$ let $\mathcal{U}_{\leq m}$ denote the full subcategory with objects $\coprod_{0 \leq n \leq m} U_m$. Let m_0 be an integer such that every n-simplex of the simplicial set U is degenerate if $n > m_0$. For any $m \geq m_0$ large enough, the subcategory $\mathcal{U}_{\leq m}$ satisfies property (1) of Categories, Definition 4.17.3.

Suppose that $u \in U_n$ and $u' \in U_{n'}$ with $n, n' \leq m_0$ and suppose that $\varphi : [k] \to [n]$, $\varphi' : [k] \to [n']$ are morphisms such that $U(\varphi)(u) = U(\varphi')(u')$. A simple combinatorial argument shows that if $k > 2m_0$, then there exists an index $0 \leq i \leq 2m_0$ such that $\varphi(i) = \varphi(i+1)$ and $\varphi'(i) = \varphi'(i+1)$. (The pigeon hole principle would tell you this works if $k > m_0^2$ which is good enough for the argument below anyways.) Hence, if $k > 2m_0$, we may write $\varphi = \psi \circ \sigma^{k-1}_i$ and $\varphi' = \psi' \circ \sigma^{k-1}_i$ for some $\psi : [k-1] \to [n]$ and some $\psi' : [k-1] \to [n']$. Since $s^{k-1}_i : U_{k-1} \to U_k$ is injective, see Lemma 14.3.6, we conclude that $U(\psi)(u) = U(\psi')(u')$ also. Continuing in this fashion we conclude that given morphisms $u \to z$ and $u' \to z$ of $\mathcal{U}$ with $u, u' \in \mathcal{U}_{\leq m_0}$, there exists a commutative diagram

$$\begin{tikzcd} u \arrow[dr] \arrow[drr] & & \\ & a \arrow[r] & z \\ u' \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

with $a \in \mathcal{U}_{\leq 2m_0}$.

It is easy to deduce from this that the finite subcategory $\mathcal{U}_{\leq 2m_0}$ works. Namely, suppose given $x' \in U_n$ and $x'' \in U_{n'}$ with $n, n' \leq 2m_0$ as well as morphisms $x' \to x$ and $x'' \to x$ of $\mathcal{U}$ with the same target. By our choice of m_0 we can find objects u, u' of $\mathcal{U}_{\leq m_0}$ and morphisms $u \to x'$, $u' \to x''$. By the above we can find $a \in \mathcal{U}_{\leq 2m_0}$ and morphisms $u \to a$, $u' \to a$ such that

$$\begin{tikzcd} u \arrow[r] \arrow[dr] \arrow[drr] & x' \arrow[dr] & \\ & a \arrow[r] & x \\ u' \arrow[ur] \arrow[urr] \arrow[r] & x'' \arrow[ur] & \end{tikzcd}$$

is commutative. Turning this diagram 90 degrees clockwise we get the desired diagram as in (2) of Categories, Definition 4.17.3. $\square$

017L Lemma 14.17.4. Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then $\text{Hom}(U, V)$ exists, moreover we have the expected equalities

$$\text{Hom}(U, V)_n = \text{Hom}(U \times \Delta[n], V)_0.$$

Proof. We construct this simplicial object as follows. For $n \geq 0$ let $\text{Hom}(U, V)_n$ denote the object of $\mathcal{C}$ representing the functor

$$X \longmapsto \text{Mor}_{\text{Simp}(\mathcal{C})}(X \times U \times \Delta[n], V)$$

This exists by Lemma 14.17.3 because $U \times \Delta[n]$ is a simplicial set with finite sets of simplices and no nondegenerate simplices in high enough degree, see Lemma 14.11.5. For $\varphi : [m] \to [n]$ we obtain an induced map of simplicial sets $\varphi : \Delta[m] \to \Delta[n]$. Hence we obtain a morphism $X \times U \times \Delta[m] \to X \times U \times \Delta[n]$ functorial in X, and hence a transformation of functors, which in turn gives

$$\text{Hom}(U, V)(\varphi) : \text{Hom}(U, V)_n \longrightarrow \text{Hom}(U, V)_m.$$

Clearly this defines a contravariant functor $\text{Hom}(U, V)$ from Δ into the category $\mathcal{C}$. In other words, we have a simplicial object of $\mathcal{C}$.

We have to show that $\text{Hom}(U, V)$ satisfies the desired universal property

$$\text{Mor}_{\text{Simp}(\mathcal{C})}(W, \text{Hom}(U, V)) = \text{Mor}_{\text{Simp}(\mathcal{C})}(W \times U, V)$$

To see this, let $f : W \to \text{Hom}(U, V)$ be given. We want to construct the element $f' : W \times U \to V$ of the right hand side. By construction, each $f_n : W_n \to \text{Hom}(U, V)_n$ corresponds to a morphism $f_n : W_n \times U \times \Delta[n] \to V$. Further, for every morphism $\varphi : [m] \to [n]$ the diagram

$$\begin{tikzcd} W_n \times U \times \Delta[m] \arrow[r, "W(\varphi) \times \text{id} \times \text{id}"'] \arrow[d, "\text{id} \times \text{id} \times \varphi"'] & W_m \times U \times \Delta[m] \arrow[d, "f_m"] \\ W_n \times U \times \Delta[n] \arrow[r, "f_n"] & V \end{tikzcd}$$

is commutative. For $\psi : [n] \to [k]$ in $(\Delta[n])_k$ we denote $(f_n)_{k,\psi} : W_n \times U_k \to V_k$ the component of $(f_n)_k$ corresponding to the element ψ. We define $f'_n : W_n \times U_n \to V_n$

as $f'_n = (f_n)_{n,\mathrm{id}}$, in other words, as the restriction of $(f_n)_n : W_n \times U_n \times (\Delta[n])_n \to V_n$ to $W_n \times U_n \times \mathrm{id}_{[n]}$. To see that the collection (f'_n) defines a morphism of simplicial objects, we have to show for any $\varphi : [m] \to [n]$ that $V(\varphi) \circ f'_n = f'_m \circ W(\varphi) \times U(\varphi)$. The commutative diagram above says that $(f_n)_{m,\varphi} : W_n \times U_m \to V_m$ is equal to $(f_m)_{m,\mathrm{id}} \circ W(\varphi) : W_n \times U_m \to V_m$. But then the fact that f_n is a morphism of simplicial objects implies that the diagram

$$\begin{tikzcd} W_n \times U_n \times (\Delta[n])_n \arrow[r, "(f_n)_n"'] \arrow[d, "\mathrm{id} \times U(\varphi) \times \varphi"'] & V_n \arrow[d, "V(\varphi)"] \\ W_n \times U_m \times (\Delta[n])_m \arrow[r, "(f_n)_m"] & V_m \end{tikzcd}$$

is commutative. And this implies that $(f_n)_{m,\varphi} \circ U(\varphi)$ is equal to $V(\varphi) \circ (f_n)_{n,\mathrm{id}}$. Altogether we obtain $V(\varphi) \circ (f_n)_{n,\mathrm{id}} = (f_n)_{m,\varphi} \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \times U(\varphi)$ as desired.

On the other hand, given a morphism $f' : W \times U \to V$ we define a morphism $f : W \to \mathrm{Hom}(U, V)$ as follows. By Lemma 14.13.4 the morphisms $\mathrm{id} : W_n \to W_n$ corresponds to a unique morphism $c_n : W_n \times \Delta[n] \to W$. Hence we can consider the composition

$$W_n \times \Delta[n] \times U \xrightarrow{c_n} W \times U \xrightarrow{f'} V.$$

By construction this corresponds to a unique morphism $f_n : W_n \to \mathrm{Hom}(U, V)_n$. We leave it to the reader to see that these define a morphism of simplicial sets as desired.

We also leave it to the reader to see that $f \mapsto f'$ and $f' \mapsto f$ are mutually inverse operations. $\square$

017N Lemma 14.17.5. Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let $a : U \to V$, $b : U \to W$ be morphisms of simplicial sets. Assume U_n, V_n, W_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U, V, W are degenerate for all $n \gg 0$. Let T be a simplicial object of $\mathcal{C}$. Then

$$\mathrm{Hom}(V, T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W, T) = \mathrm{Hom}(V \amalg_U W, T)$$

In other words, the fibre product on the left hand side is represented by the Hom object on the right hand side.

Proof. By Lemma 14.17.4 all the required Hom objects exist and satisfy the correct functorial properties. Now we can identify the nth term on the left hand side as the object representing the functor that associates to X the first set of the following sequence of functorial equalities

$$\begin{aligned} & \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V, T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W, T)) \\ & = \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V, T)) \times_{\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(U,T))} \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(W, T)) \\ & = \mathrm{Mor}(X \times \Delta[n] \times V, T) \times_{\mathrm{Mor}(X \times \Delta[n] \times U, T)} \mathrm{Mor}(X \times \Delta[n] \times W, T) \\ & = \mathrm{Mor}(X \times \Delta[n] \times (V \amalg_U W), T)) \end{aligned}$$

Here we have used the fact that

$$(X \times \Delta[n] \times V) \times_{X \times \Delta[n] \times U} (X \times \Delta[n] \times W) = X \times \Delta[n] \times (V \amalg_U W)$$

which is easy to verify term by term. The result of the lemma follows as the last term in the displayed sequence of equalities corresponds to $\mathrm{Hom}(V \amalg_U W, T)_n$. $\square$

14.18. Splitting simplicial objects

017O A subobject N of an object X of the category $\mathcal{C}$ is an object N of $\mathcal{C}$ together with a monomorphism $N \to X$. Of course we say (by abuse of notation) that the subobjects N, N' are equal if there exists an isomorphism $N \to N'$ compatible with the morphisms to X. The collection of subobjects forms a partially ordered set. (Because of our conventions on categories; not true for category of spaces up to homotopy for example.)

017P **Definition 14.18.1.** Let $\mathcal{C}$ be a category which admits finite nonempty coproducts. We say a simplicial object U of $\mathcal{C}$ is split if there exist subobjects $N(U_m)$ of U_m, $m \geq 0$ with the property that

017Q (14.18.1.1)
$$\coprod\nolimits_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) \longrightarrow U_n$$

is an isomorphism for all $n \geq 0$. If U is an r-truncated simplicial object of $\mathcal{C}$ then we say U is split if there exist subobjects $N(U_m)$ of U_m, $r \geq m \geq 0$ with the property that (14.18.1.1) is an isomorphism for $r \geq n \geq 0$.

If this is the case, then $N(U_0) = U_0$. Next, we have $U_1 = U_0 \amalg N(U_1)$. Second we have

$$U_2 = U_0 \amalg N(U_1) \amalg N(U_1) \amalg N(U_2).$$

It turns out that in many categories $\mathcal{C}$ every simplicial object is split.

017R **Lemma 14.18.2.** Let U be a simplicial set. Then U has a unique splitting with $N(U_m)$ equal to the set of nondegenerate m-simplices.

Proof. From the definition it follows immediately, that if there is a splitting then $N(U_m)$ has to be the set of nondegenerate simplices. Let $x \in U_n$. Suppose that there are surjections $\varphi : [n] \to [k]$ and $\psi : [n] \to [l]$ and nondegenerate simplices $y \in U_k$, $z \in U_l$ such that $x = U(\varphi)(y)$ and $x = U(\psi)(z)$. Choose a right inverse $\xi : [l] \to [n]$ of ψ, i.e., $\psi \circ \xi = \text{id}_{[l]}$. Then $z = U(\xi)(x)$. Hence $z = U(\xi)(x) = U(\varphi \circ \xi)(y)$. Since z is nondegenerate we conclude that $\varphi \circ \xi : [l] \to [k]$ is surjective, and hence $l \geq k$. Similarly $k \geq l$. Hence we see that $\varphi \circ \xi : [l] \to [k]$ has to be the identity map for any choice of right inverse ξ of ψ. This easily implies that $\psi = \varphi$. □

Of course it can happen that a map of simplicial sets maps a nondegenerate n-simplex to a degenerate n-simplex. Thus the splitting of Lemma 14.18.2 is not functorial. Here is a case where it is functorial.

017S **Lemma 14.18.3.** Let $f : U \to V$ be a morphism of simplicial sets. Suppose that (a) the image of every nondegenerate simplex of U is a nondegenerate simplex of V and (b) the restriction of f to a map from the set of nondegenerate simplices of U to the set of nondegenerate simplices of V is injective. Then f_n is injective for all n. Same holds with "injective" replaced by "surjective" or "bijective".

Proof. Under hypothesis (a) we see that the map f preserves the disjoint union decompositions of the splitting of Lemma 14.18.2, in other words that we get commutative diagrams

$$\begin{array}{ccc} \coprod_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) & \longrightarrow & U_n \\ \downarrow & & \downarrow \\ \coprod_{\varphi:[n]\to[m] \text{ surjective}} N(V_m) & \longrightarrow & V_n. \end{array}$$

And then (b) clearly shows that the left vertical arrow is injective (resp. surjective, resp. bijective). □

017T Lemma 14.18.4. Let U be a simplicial set. Let $n \geq 0$ be an integer. The rule

$$U'_m = \bigcup\nolimits_{\varphi:[m]\to[i],\ i\leq n} \mathrm{Im}(U(\varphi))$$

defines a sub simplicial set $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, all m-simplices of U' are degenerate for all $m > n$.

Proof. If $x \in U_m$ and $x = U(\varphi)(y)$ for some $y \in U_i$, $i \leq n$ and some $\varphi : [m] \to [i]$ then any image $U(\psi)(x)$ for any $\psi : [m'] \to [m]$ is equal to $U(\varphi \circ \psi)(y)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial set. By construction all simplices in dimension $n+1$ and higher are degenerate. □

017U Lemma 14.18.5. Let U be a simplicial abelian group. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking

$$N(U_m) = \bigcap\nolimits_{i=0}^{m-1} \mathrm{Ker}(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial abelian groups.

Proof. By induction on n we will show that the choice of $N(U_m)$ in the lemma guarantees that (14.18.1.1) is an isomorphism for $m \leq n$. This is clear for $n = 0$. In the rest of this proof we are going to drop the superscripts from the maps d_i and s_i in order to improve readability. We will also repeatedly use the relations from Remark 14.3.3.

First we make a general remark. For $0 \leq i \leq m$ and $z \in U_m$ we have $d_i(s_i(z)) = z$. Hence we can write any $x \in U_{m+1}$ uniquely as $x = x' + x''$ with $d_i(x') = 0$ and $x'' \in \mathrm{Im}(s_i)$ by taking $x' = (x - s_i(d_i(x)))$ and $x'' = s_i(d_i(x))$. Moreover, the element $z \in U_m$ such that $x'' = s_i(z)$ is unique because s_i is injective.

Here is a procedure for decomposing any $x \in U_{n+1}$. First, write $x = x_0 + s_0(z_0)$ with $d_0(x_0) = 0$. Next, write $x_0 = x_1 + s_1(z_1)$ with $d_n(x_1) = 0$. Continue like this to get

$$\begin{array}{rcl} x & = & x_0 + s_0(z_0), \\ x_0 & = & x_1 + s_1(z_1), \\ x_1 & = & x_2 + s_2(z_2), \\ \ldots & \ldots & \ldots \\ x_{n-1} & = & x_n + s_n(z_n) \end{array}$$

where $d_i(x_i) = 0$ for all $i = n, \ldots, 0$. By our general remark above all of the x_i and z_i are determined uniquely by x. We claim that $x_i \in \mathrm{Ker}(d_0) \cap \mathrm{Ker}(d_1) \cap \ldots \cap \mathrm{Ker}(d_i)$

and $z_i \in \text{Ker}(d_0) \cap \ldots \cap \text{Ker}(d_{i-1})$ for $i = n, \ldots, 0$. Here and in the following an empty intersection of kernels indicates the whole space; i.e., the notation $z_0 \in \text{Ker}(d_0) \cap \ldots \cap \text{Ker}(d_{i-1})$ when $i = 0$ means $z_0 \in U_n$ with no restriction.

We prove this by ascending induction on i. It is clear for $i = 0$ by construction of x_0 and z_0. Let us prove it for $0 < i \leq n$ assuming the result for $i-1$. First of all we have $d_i(x_i) = 0$ by construction. So pick a j with $0 \leq j < i$. We have $d_j(x_{i-1}) = 0$ by induction. Hence

$$0 = d_j(x_{i-1}) = d_j(x_i) + d_j(s_i(z_i)) = d_j(x_i) + s_{i-1}(d_j(z_i)).$$

The last equality by the relations of Remark 14.3.3. These relations also imply that $d_{i-1}(d_j(x_i)) = d_j(d_i(x_i)) = 0$ because $d_i(x_i) = 0$ by construction. Then the uniqueness in the general remark above shows the equality $0 = x' + x'' = d_j(x_i) + s_{i-1}(d_j(z_i))$ can only hold if both terms are zero. We conclude that $d_j(x_i) = 0$ and by injectivity of s_{i-1} we also conclude that $d_j(z_i) = 0$. This proves the claim.

The claim implies we can uniquely write

$$x = s_0(z_0) + s_1(z_1) + \ldots + s_n(z_n) + x_0$$

with $x_0 \in N(U_{n+1})$ and $z_i \in \text{Ker}(d_0) \cap \ldots \cap \text{Ker}(d_{i-1})$. We can reformulate this as saying that we have found a direct sum decomposition

$$U_{n+1} = N(U_{n+1}) \oplus \bigoplus\nolimits_{i=0}^{i=n} s_i\Big(\text{Ker}(d_0) \cap \ldots \cap \text{Ker}(d_{i-1})\Big)$$

with the property that

$$\text{Ker}(d_0) \cap \ldots \cap \text{Ker}(d_j) = N(U_{n+1}) \oplus \bigoplus\nolimits_{i=j+1}^{i=n} s_i\Big(\text{Ker}(d_n) \cap \ldots \cap \text{Ker}(d_{i-1})\Big)$$

for $j = 0, \ldots, n$. The result follows from this statement as follows. Each of the z_i in the expression for x can be written uniquely as

$$z_i = s_i(z'_{i,i}) + \ldots + s_{n-1}(z'_{i,n-1}) + z_{i,0}$$

with $z_{i,0} \in N(U_n)$ and $z'_{i,j} \in \text{Ker}(d_0) \cap \ldots \cap \text{Ker}(d_{j-1})$. The first few steps in the decomposition of z_i are zero because z_i already is in the kernel of $d_0, \ldots, d_i$. This in turn uniquely gives

$$x = x_0 + s_0(z_{0,0}) + s_1(z_{1,0}) + \ldots + s_n(z_{n,0}) + \sum\nolimits_{0 \leq i \leq j \leq n-1} s_i(s_j(z'_{i,j})).$$

Continuing in this fashion we see that we in the end obtain a decomposition of x as a sum of terms of the form

$$s_{i_1} s_{i_2} \ldots s_{i_k}(z)$$

with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n - k + 1$ and $z \in N(U_{n+1-k})$. This is exactly the required decomposition, because any surjective map $[n+1] \to [n+1-k]$ can be uniquely expressed in the form

$$\sigma^{n-k}_{i_k} \ldots \sigma^{n-1}_{i_2} \sigma^n_{i_1}$$

with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n - k + 1$. $\square$

017V Lemma 14.18.6. Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking

$$N(U_m) = \bigcap\nolimits_{i=0}^{m-1} \text{Ker}(d^m_i).$$

Moreover, this splitting is functorial on the category of simplicial objects of $\mathcal{A}$.

Proof. For any object A of $\mathcal{A}$ we obtain a simplicial abelian group $\text{Mor}_{\mathcal{A}}(A, U)$. Each of these are canonically split by Lemma 14.18.5. Moreover,

$$N(\text{Mor}_{\mathcal{A}}(A, U_m)) = \bigcap\nolimits_{i=0}^{m-1} \text{Ker}(d_i^m) = \text{Mor}_{\mathcal{A}}(A, N(U_m)).$$

Hence we see that the morphism (14.18.1.1) becomes an isomorphism after applying the functor $\text{Mor}_{\mathcal{A}}(A, -)$ for any object of $\mathcal{A}$. Hence it is an isomorphism by the Yoneda lemma. $\square$

017W Lemma 14.18.7. Let $\mathcal{A}$ be an abelian category. Let $f : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If the induced morphisms $N(f)_i : N(U)_i \to N(V)_i$ are injective for all i, then f_i is injective for all i. Same holds with "injective" replaced with "surjective", or "isomorphism".

Proof. This is clear from Lemma 14.18.6 and the definition of a splitting. $\square$

017X Lemma 14.18.8. Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Let $N(U_m)$ as in Lemma 14.18.6 above. Then $d_m^m(N(U_m)) \subset N(U_{m-1})$.

Proof. For $j = 0, \ldots, m-2$ we have $d_j^{m-1} d_m^m = d_{m-1}^{m-1} d_j^m$ by the relations in Remark 14.3.3. The result follows. $\square$

017Y Lemma 14.18.9. Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The rule

$$U'_m = \sum\nolimits_{\varphi : [m] \to [i],\ i \leq n} \text{Im}(U(\varphi))$$

defines a sub simplicial object $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, $N(U'_m) = 0$ for all $m > n$.

Proof. Pick m, $i \leq n$ and some $\varphi : [m] \to [i]$. The image under $U(\psi)$ of $\text{Im}(U(\varphi))$ for any $\psi : [m'] \to [m]$ is equal to the image of $U(\varphi \circ \psi)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial object. Pick $m > n$. We have to show $N(U'_m) = 0$. By definition of $N(U_m)$ and $N(U'_m)$ we have $N(U'_m) = U'_m \cap N(U_m)$ (intersection of subobjects). Since U is split by Lemma 14.18.6, it suffices to show that U'_m is contained in the sum

$$\sum\nolimits_{\varphi : [m] \to [m'] \text{ surjective},\ m' < m} \text{Im}(U(\varphi)|_{N(U_{m'})}).$$

By the splitting each $U_{m'}$ is the sum of images of $N(U_{m''})$ via $U(\psi)$ for surjective maps $\psi : [m'] \to [m'']$. Hence the displayed sum above is the same as

$$\sum\nolimits_{\varphi : [m] \to [m'] \text{ surjective},\ m' < m} \text{Im}(U(\varphi)).$$

Clearly U'_m is contained in this by the simple fact that any $\varphi : [m] \to [i]$, $i \leq n$ occurring in the definition of U'_m may be factored as $[m] \to [m'] \to [i]$ with $[m] \to [m']$ surjective and $m' < m$ as in the last displayed sum above. $\square$

14.19. Coskeleton functors

0AMA Let $\mathcal{C}$ be a category. The coskeleton functor (if it exists) is a functor

$$\text{cosk}_n : \text{Simp}_n(\mathcal{C}) \longrightarrow \text{Simp}(\mathcal{C})$$

which is right adjoint to the skeleton functor. In a formula

0181 (14.19.0.1) $$\text{Mor}_{\text{Simp}(\mathcal{C})}(U, \text{cosk}_n V) = \text{Mor}_{\text{Simp}_n(\mathcal{C})}(\text{sk}_n U, V)$$

Given a n-truncated simplicial object V we say that $\text{cosk}_n V$ exists if there exists a $\text{cosk}_n V \in \text{Ob}(\text{Simp}(\mathcal{C}))$ and a morphism $\text{sk}_n \text{cosk}_n V \to V$ such that the displayed formula holds, in other words if the functor $U \mapsto \text{Mor}_{\text{Simp}_n(\mathcal{C})}(\text{sk}_n U, V)$ is representable. If it exists it is unique up to unique isomorphism by the Yoneda lemma. See Categories, Section 4.3.

0182 Example 14.19.1. Suppose the category $\mathcal{C}$ has finite nonempty self products. A 0-truncated simplicial object of $\mathcal{C}$ is the same as an object X of $\mathcal{C}$. In this case we claim that $\text{cosk}_0(X)$ is the simplicial object U with $U_n = X^{n+1}$ the $(n+1)$-fold self product of X, and structure of simplicial object as in Example 14.3.5. Namely, a morphism $V \to U$ where V is a simplicial object is given by morphisms $V_n \to X^{n+1}$, such that all the diagrams

$$\begin{tikzcd} V_n \arrow[r] \arrow[d, "{V([0]\to[n], 0\mapsto i)}"'] & X^{n+1} \arrow[d, "\text{pr}_i"] \\ V_0 \arrow[r] & X \end{tikzcd}$$

commute. Clearly this means that the map determines and is determined by a unique morphism $V_0 \to X$. This proves that formula (14.19.0.1) holds.

Recall the category $\Delta/[n]$, see Example 14.11.4. We let $(\Delta/[n])_{\leq m}$ denote the full subcategory of $\Delta/[n]$ consisting of objects $[k] \to [n]$ of $\Delta/[n]$ with $k \leq m$. In other words we have the following commutative diagram of categories and functors

$$\begin{tikzcd} (\Delta/[n])_{\leq m} \arrow[r] \arrow[d] & \Delta/[n] \arrow[d] \\ \Delta_{\leq m} \arrow[r] & \Delta \end{tikzcd}$$

Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : (\Delta/[n])_{\leq m}^{opp} \longrightarrow \mathcal{C}$$

by the rules

$$\begin{matrix} ([k] \to [n]) & \longmapsto & U_k \\ \psi : ([k'] \to [n]) \to ([k] \to [n]) & \longmapsto & U(\psi) : U_k \to U_{k'} \end{matrix}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\overline{\varphi} : (\Delta/[n])_{\leq m} \longrightarrow (\Delta/[n'])_{\leq m}$$

which maps $\alpha : [k] \to [n]$ to $\varphi \circ \alpha : [k] \to [n']$. The composition $U(n') \circ \overline{\varphi}$ is equal to the functor $U(n)$.

0183 Lemma 14.19.2. If the category $\mathcal{C}$ has finite limits, then cosk_m functors exist for all m. Moreover, for any m-truncated simplicial object U the simplicial object $\text{cosk}_m U$ is described by the formula

$$(\text{cosk}_m U)_n = \lim_{(\Delta/[n])_{\leq m}^{opp}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $\text{cosk}_m U(\varphi)$ comes from the identification $U(n') \circ \overline{\varphi} = U(n)$ above via Categories, Lemma 4.14.9.

Proof. During the proof of this lemma we denote $\text{cosk}_m U$ the simplicial object with $(\text{cosk}_m U)_n$ equal to $\lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$. We will conclude at the end of the proof that it does satisfy the required mapping property.

Suppose that V is a simplicial object. A morphism $\gamma : V \to \text{cosk}_m U$ is given by a sequence of morphisms $\gamma_n : V_n \to (\text{cosk}_m U)_n$. By definition of a limit, this is given by a collection of morphisms $\gamma(\alpha) : V_n \to U_k$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$. These morphisms then also satisfy the rules that

$$\begin{tikzcd} V_n \arrow[r, "\gamma(\alpha)"'] & U_k \\ V_{n'} \arrow[u, "V(\varphi)"] \arrow[r, "\gamma(\alpha')"] & U_{k'} \arrow[u, "U(\psi)"'] \end{tikzcd}$$

are commutative, given any $0 \leq k, k' \leq m$, $0 \leq n, n'$ and any $\psi : [k] \to [k']$, $\varphi : [n] \to [n']$, $\alpha : [k] \to [n]$ and $\alpha' : [k'] \to [n']$ in Δ such that $\varphi \circ \alpha = \alpha' \circ \psi$. Taking $n = k = k'$, $\varphi = \alpha'$, and $\alpha = \psi = \text{id}_{[k]}$ we deduce that $\gamma(\alpha') = \gamma(\text{id}_{[k]}) \circ V(\alpha')$. In other words, the morphisms $\gamma(\text{id}_{[k]})$, $k \leq m$ determine the morphism γ. And it is easy to see that these morphisms form a morphism $\text{sk}_m V \to U$.

Conversely, given a morphism $\gamma : \text{sk}_m V \to U$, we obtain a family of morphisms $\gamma(\alpha)$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$ by setting $\gamma(\alpha) = \gamma(\text{id}_{[k]}) \circ V(\alpha)$. These morphisms satisfy all the displayed commutativity restraints pictured above, and hence give rise to a morphism $V \to \text{cosk}_m U$. $\square$

0184 Lemma 14.19.3. Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For $n \leq m$ the limit $\lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$ exists and is canonically isomorphic to U_n.

Proof. This is true because the category $(\Delta/[n])_{\leq m}$ has an final object in this case, namely the identity map $[n] \to [n]$. $\square$

0185 Lemma 14.19.4. Let $\mathcal{C}$ be a category with finite limits. Let U be an n-truncated simplicial object of $\mathcal{C}$. The morphism $\text{sk}_n \text{cosk}_n U \to U$ is an isomorphism.

Proof. Combine Lemmas 14.19.2 and 14.19.3. $\square$

Let us describe a particular instance of the coskeleton functor in more detail. By abuse of notation we will denote sk_n also the restriction functor $\text{Simp}_{n'}(\mathcal{C}) \to \text{Simp}_n(\mathcal{C})$ for any $n' \geq n$. We are going to describe a right adjoint of the functor $\text{sk}_n : \text{Simp}_{n+1}(\mathcal{C}) \to \text{Simp}_n(\mathcal{C})$. For $n \geq 1$, $0 \leq i < j \leq n+1$ define $\delta^{n+1}_{i,j} : [n-1] \to [n+1]$ to be the increasing map omitting i and j. Note that $\delta^{n+1}_{i,j} = \delta^{n+1}_j \circ \delta^n_i = \delta^{n+1}_i \circ \delta^n_{j-1}$, see Lemma 14.2.3. This motivates the following lemma.

0186 Lemma 14.19.5. Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set

$$\{(f_0, \ldots, f_{n+1}) \in \text{Mor}_{\mathcal{C}}(T, U_n) \mid d^n_{j-1} \circ f_i = d^n_i \circ f_j \ \forall\ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then

$$U_{n+1} = \lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n)$$

Proof. The limit, if it exists, represents the functor that associates to an object T the set

$$\{(f_\alpha)_{\alpha:[k]\to[n+1],k\le n} \mid f_{\alpha\circ\psi} = U(\psi)\circ f_\alpha \ \forall\ \psi:[k']\to[k], \alpha:[k]\to[n+1]\}.$$

In fact we will show this functor is isomorphic to the one displayed in the lemma. The map in one direction is given by the rule

$$(f_\alpha)_\alpha \longmapsto (f_{\delta_0^{n+1}}, \ldots, f_{\delta_{n+1}^{n+1}}).$$

This satisfies the conditions of the lemma because

$$d_{j-1}^n \circ f_{\delta_i^{n+1}} = f_{\delta_i^{n+1}\circ\delta_{j-1}^n} = f_{\delta_j^{n+1}\circ\delta_i^n} = d_i^n \circ f_{\delta_j^{n+1}}$$

by the relations we recalled above the lemma. To construct a map in the other direction we have to associate to a system $(f_0, \ldots, f_{n+1})$ as in the displayed formula of the lemma a system of maps f_α. Let $\alpha:[k]\to[n+1]$ be given. Since $k\le n$ the map α is not surjective. Hence we can write $\alpha = \delta_i^{n+1}\circ\psi$ for some $0\le i\le n+1$ and some $\psi:[k]\to[n]$. We have no choice but to define

$$f_\alpha = U(\psi)\circ f_i.$$

Of course we have to check that this is independent of the choice of the pair (i,ψ). First, observe that given i there is a unique ψ which works. Second, suppose that (j,ϕ) is another pair. Then $i\ne j$ and we may assume $i<j$. Since both i,j are not in the image of α we may actually write $\alpha = \delta_{i,j}^{n+1}\circ\xi$ and then we see that $\psi = \delta_{j-1}^n\circ\xi$ and $\phi = \delta_i^n\circ\xi$. Thus

$$\begin{aligned} U(\psi)\circ f_i &= U(\delta_{j-1}^n\circ\xi)\circ f_i \\ &= U(\xi)\circ d_{j-1}^n\circ f_i \\ &= U(\xi)\circ d_i^n\circ f_j \\ &= U(\delta_i^n\circ\xi)\circ f_j \\ &= U(\phi)\circ f_j \end{aligned}$$

as desired. We still have to verify that the maps f_α so defined satisfy the rules of a system of maps $(f_\alpha)_\alpha$. To see this suppose that $\psi:[k']\to[k]$, $\alpha:[k]\to[n+1]$ with $k,k'\le n$. Set $\alpha' = \alpha\circ\psi$. Choose i not in the image of α. Then clearly i is not in the image of α' also. Write $\alpha = \delta_i^{n+1}\circ\phi$ (we cannot use the letter ψ here because we've already used it). Then obviously $\alpha' = \delta_i^{n+1}\circ\phi\circ\psi$. By construction above we then have

$$U(\psi)\circ f_\alpha = U(\psi)\circ U(\phi)\circ f_i = U(\phi\circ\psi)\circ f_i = f_{\alpha\circ\psi} = f_{\alpha'}$$

as desired. We leave to the reader the pleasant task of verifying that our constructions are mutually inverse bijections, and are functorial in T. □

0187 **Lemma 14.19.6.** Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set

$$\{(f_0,\ldots,f_{n+1})\in \mathrm{Mor}_\mathcal{C}(T,U_n) \mid d_{j-1}^n\circ f_i = d_i^n\circ f_j \ \forall\ 0\le i<j\le n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then there exists an $(n+1)$-truncated simplicial object $\tilde U$, with $\mathrm{sk}_n\tilde U = U$ and $\tilde U_{n+1} = U_{n+1}$ such that the following adjointness holds

$$\mathrm{Mor}_{\mathrm{Simp}_{n+1}(\mathcal{C})}(V,\tilde U) = \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n V, U)$$

Proof. By Lemma 14.19.3 there are identifications

$$U_i = \lim_{(\Delta/[i])^{opp}_{\leq n}} U(i)$$

for $0 \leq i \leq n$. By Lemma 14.19.5 we have

$$U_{n+1} = \lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n).$$

Thus we may define for any $\varphi : [i] \to [j]$ with $i, j \leq n+1$ the corresponding map $\tilde{U}(\varphi) : \tilde{U}_j \to \tilde{U}_i$ exactly as in Lemma 14.19.2. This defines an $(n+1)$-truncated simplicial object $\tilde{U}$ with $\text{sk}_n\tilde{U} = U$.

To see the adjointness we argue as follows. Given any element $\gamma : \text{sk}_n V \to U$ of the right hand side of the formula consider the morphisms $f_i = \gamma_n \circ d^{n+1}_i : V_{n+1} \to V_n \to U_n$. These clearly satisfy the relations $d^n_{j-1} \circ f_i = d^n_i \circ f_j$ and hence define a unique morphism $V_{n+1} \to U_{n+1}$ by our choice of U_{n+1}. Conversely, given a morphism $\gamma' : V \to \tilde{U}$ of the left hand side we can simply restrict to $\Delta_{\leq n}$ to get an element of the right hand side. We leave it to the reader to show these are mutually inverse constructions. $\square$

0188 Remark 14.19.7. Let U, and U_{n+1} be as in Lemma 14.19.6. On T-valued points we can easily describe the face and degeneracy maps of $\tilde{U}$. Explicitly, the maps $d^{n+1}_i : U_{n+1} \to U_n$ are given by

$$(f_0, \ldots, f_{n+1}) \longmapsto f_i.$$

And the maps $s^n_j : U_n \to U_{n+1}$ are given by

$$\begin{array}{rcl} f & \longmapsto & (s^{n-1}_{j-1} \circ d^{n-1}_0 \circ f, \\ & & s^{n-1}_{j-1} \circ d^{n-1}_1 \circ f, \\ & & \ldots \\ & & s^{n-1}_{j-1} \circ d^{n-1}_{j-1} \circ f, \\ & & f, \\ & & f, \\ & & s^{n-1}_j \circ d^{n-1}_{j+1} \circ f, \\ & & s^{n-1}_j \circ d^{n-1}_{j+2} \circ f, \\ & & \ldots \\ & & s^{n-1}_j \circ d^{n-1}_n \circ f) \end{array}$$

where we leave it to the reader to verify that the RHS is an element of the displayed set of Lemma 14.19.6. For $n = 0$ there is one map, namely $f \mapsto (f, f)$. For $n = 1$ there are two maps, namely $f \mapsto (f, f, s_0 d_1 f)$ and $f \mapsto (s_0 d_0 f, f, f)$. For $n = 2$ there are three maps, namely $f \mapsto (f, f, s_0 d_1 f, s_0 d_2 f)$, $f \mapsto (s_0 d_0 f, f, f, s_1 d_2 f)$, and $f \mapsto (s_1 d_0 f, s_1 d_1 f, f, f)$. And so on and so forth.

0189 Remark 14.19.8. The construction of Lemma 14.19.6 above in the case of simplicial sets is the following. Given an n-truncated simplicial set U, we make a canonical $(n+1)$-truncated simplicial set $\tilde{U}$ as follows. We add a set of $(n+1)$-simplices U_{n+1} by the formula of the lemma. Namely, an element of U_{n+1} is a numbered collection of $(f_0, \ldots, f_{n+1})$ of n-simplices, with the property that they glue as they would in a $(n+1)$-simplex. In other words, the ith face of f_j is the $(j-1)$st face of f_i for $i < j$. Geometrically it is obvious how to define the face and degeneracy maps for $\tilde{U}$. If V is an $(n+1)$-truncated simplicial set, then its $(n+1)$-simplices give rise to compatible collections of n-simplices $(f_0, \ldots, f_{n+1})$ with $f_i \in V_n$. Hence there is a natural map $\text{Mor}(\text{sk}_n V, U) \to \text{Mor}(V, \tilde{U})$ which is inverse to the canonical restriction mapping the other way.

Also, it is enough to do the combinatorics of the construction in the case of truncated simplicial sets. Namely, for any object T of the category $\mathcal{C}$, and any n-truncated simplicial object U of $\mathcal{C}$ we can consider the n-truncated simplicial set $\text{Mor}(T, U)$. We may apply the construction to this, and take its set of $(n+1)$-simplices, and require this to be representable. This is a good way to think about the result of Lemma 14.19.6.

018A Remark 14.19.9. Inductive construction of coskeleta. Suppose that $\mathcal{C}$ is a category with finite limits. Suppose that U is an m-truncated simplicial object in $\mathcal{C}$. Then we can inductively construct n-truncated objects U^n as follows:

(1) To start, set $U^m = U$.
(2) Given U^n for $n \geq m$ set $U^{n+1} = \tilde{U}^n$, where $\tilde{U}^n$ is constructed from U^n as in Lemma 14.19.6.

Since the construction of Lemma 14.19.6 has the property that it leaves the n-skeleton of U^n unchanged, we can then define $\text{cosk}_m U$ to be the simplicial object with $(\text{cosk}_m U)_n = U^n_n = U^{n+1}_n = \ldots$. And it follows formally from Lemma 14.19.6 that U^n satisfies the formula

$$\text{Mor}_{\text{Simp}_n(\mathcal{C})}(V, U^n) = \text{Mor}_{\text{Simp}_m(\mathcal{C})}(\text{sk}_m V, U)$$

for all $n \geq m$. It also then follows formally from this that

$$\text{Mor}_{\text{Simp}(\mathcal{C})}(V, \text{cosk}_m U) = \text{Mor}_{\text{Simp}_m(\mathcal{C})}(\text{sk}_m V, U)$$

with $\text{cosk}_m U$ chosen as above.

018B Lemma 14.19.10. Let $\mathcal{C}$ be a category which has finite limits.

(1) For every n the functor $\text{sk}_n : \text{Simp}(\mathcal{C}) \to \text{Simp}_n(\mathcal{C})$ has a right adjoint cosk_n.
(2) For every $n' \geq n$ the functor $\text{sk}_n : \text{Simp}_{n'}(\mathcal{C}) \to \text{Simp}_n(\mathcal{C})$ has a right adjoint, namely $\text{sk}_{n'}\text{cosk}_n$.
(3) For every $m \geq n \geq 0$ and every n-truncated simplicial object U of $\mathcal{C}$ we have $\text{cosk}_m \text{sk}_m \text{cosk}_n U = \text{cosk}_n U$.
(4) If U is a simplicial object of $\mathcal{C}$ such that the canonical map $U \to \text{cosk}_n \text{sk}_n U$ is an isomorphism for some $n \geq 0$, then the canonical map $U \to \text{cosk}_m \text{sk}_m U$ is an isomorphism for all $m \geq n$.

Proof. The existence in (1) follows from Lemma 14.19.2 above. Parts (2) and (3) follow from the discussion in Remark 14.19.9. After this (4) is obvious. □

09VS Remark 14.19.11. We do not need all finite limits in order to be able to define the coskeleton functors. Here are some remarks

(1) We have seen in Example 14.19.1 that if $\mathcal{C}$ has products of pairs of objects then cosk_0 exists.
(2) For $k > 0$ the functor cosk_k exists if $\mathcal{C}$ has finite connected limits.

This is clear from the inductive procedure of constructing coskeleta (Remarks 14.19.8 and 14.19.9) but it also follows from the fact that the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ used in Lemma 14.19.2 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 14.19.3 or Lemma 14.19.4. (As k gets higher the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ are more and more connected in a topological sense.)

018C Lemma 14.19.12. Let U, V be n-truncated simplicial objects of a category $\mathcal{C}$. Then

$$\text{cosk}_n(U \times V) = \text{cosk}_n U \times \text{cosk}_n V$$

whenever the left and right hand sides exist.

Proof. Let W be a simplicial object. We have

$$\begin{aligned} \text{Mor}(W, \text{cosk}_n(U \times V)) &= \text{Mor}(\text{sk}_n W, U \times V) \\ &= \text{Mor}(\text{sk}_n W, U) \times \text{Mor}(\text{sk}_n W, V) \\ &= \text{Mor}(W, \text{cosk}_n U) \times \text{Mor}(W, \text{cosk}_n V) \\ &= \text{Mor}(W, \text{cosk}_n U \times \text{cosk}_n V) \end{aligned}$$

The lemma follows. □

018D Lemma 14.19.13. Assume $\mathcal{C}$ has fibre products. Let $U \to V$ and $W \to V$ be morphisms of n-truncated simplicial objects of the category $\mathcal{C}$. Then

$$\text{cosk}_n(U \times_V W) = \text{cosk}_n U \times_{\text{cosk}_n V} \text{cosk}_n W$$

whenever the left and right hand side exist.

Proof. Omitted, but very similar to the proof of Lemma 14.19.12 above. □

08NJ Lemma 14.19.14. Let $\mathcal{C}$ be a category with finite limits. Let $X \in \text{Ob}(\mathcal{C})$. The functor $\mathcal{C}/X \to \mathcal{C}$ commutes with the coskeleton functors cosk_k for $k \geq 1$.

Proof. The statement means that if U is a simplicial object of $\mathcal{C}/X$ which we can think of as a simplicial object of $\mathcal{C}$ with a morphism towards the constant simplicial object X, then $\text{cosk}_k U$ computed in $\mathcal{C}/X$ is the same as computed in $\mathcal{C}$. This follows for example from Categories, Lemma 4.16.2 because the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ used in Lemma 14.19.2 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 14.19.3 or Lemma 14.19.4. □

018E Lemma 14.19.15. The canonical map $\Delta[n] \to \text{cosk}_1 \text{sk}_1 \Delta[n]$ is an isomorphism.

Proof. Consider a simplicial set U and a morphism $f : U \to \Delta[n]$. This is a rule that associates to each $u \in U_i$ a map $f_u : [i] \to [n]$ in Δ. Furthermore, these maps should have the property that $f_u \circ \varphi = f_{U(\varphi)(u)}$ for any $\varphi : [j] \to [i]$. Denote $\epsilon^i_j : [0] \to [i]$ the map which maps 0 to j. Denote $F : U_0 \to [n]$ the map $u \mapsto f_u(0)$. Then we see that

$$f_u(j) = F(\epsilon^i_j(u))$$

for all $0 \leq j \leq i$ and $u \in U_i$. In particular, if we know the function F then we know the maps f_u for all $u \in U_i$ all i. Conversely, given a map $F : U_0 \to [n]$, we can set for any i, and any $u \in U_i$ and any $0 \leq j \leq i$

$$f_u(j) = F(\epsilon^i_j(u))$$

This does not in general define a morphism f of simplicial sets as above. Namely, the condition is that all the maps f_u are nondecreasing. This clearly is equivalent to the condition that $F(\epsilon^i_j(u)) \leq F(\epsilon^i_{j'}(u))$ whenever $0 \leq j \leq j' \leq i$ and $u \in U_i$. But in this case the morphisms

$$\epsilon^i_j, \epsilon^i_{j'} : [0] \to [i]$$

both factor through the map $\epsilon^i_{j,j'} : [1] \to [i]$ defined by the rules $0 \mapsto j$, $1 \mapsto j'$. In other words, it is enough to check the inequalities for $i = 1$ and $u \in X_1$. In other words, we have

$$\mathrm{Mor}(U, \Delta[n]) = \mathrm{Mor}(\mathrm{sk}_1 U, \mathrm{sk}_1 \Delta[n])$$

as desired. □

14.20. Augmentations

018F

018G Definition 14.20.1. Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. An augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$ is a morphism from U into the constant simplicial object X.

018H Lemma 14.20.2. Let $\mathcal{C}$ be a category. Let $X \in \mathrm{Ob}(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. To give an augmentation of U towards X is the same as giving a morphism $\epsilon_0 : U_0 \to X$ such that $\epsilon_0 \circ d^1_0 = \epsilon_0 \circ d^1_1$.

Proof. Given a morphism $\epsilon : U \to X$ we certainly obtain an ϵ_0 as in the lemma. Conversely, given ϵ_0 as in the lemma, define $\epsilon_n : U_n \to X$ by choosing any morphism $\alpha : [0] \to [n]$ and taking $\epsilon_n = \epsilon_0 \circ U(\alpha)$. Namely, if $\beta : [0] \to [n]$ is another choice, then there exists a morphism $\gamma : [1] \to [n]$ such that α and β both factor as $[0] \to [1] \to [n]$. Hence the condition on ϵ_0 shows that ϵ_n is well defined. Then it is easy to show that $(\epsilon_n) : U \to X$ is a morphism of simplicial objects. □

018I Lemma 14.20.3. Let $\mathcal{C}$ be a category with fibred products. Let $f : Y \to X$ be a morphism of $\mathcal{C}$. Let U be the simplicial object of $\mathcal{C}$ whose nth term is the $(n+1)$fold fibred product $Y \times_X Y \times_X \ldots \times_X Y$. See Example 14.3.5. For any simplicial object V of $\mathcal{C}$ we have

$$\begin{aligned} \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(V, U) &= \mathrm{Mor}_{\mathrm{Simp}_1(\mathcal{C})}(\mathrm{sk}_1 V, \mathrm{sk}_1 U) \\ &= \{g_0 : V_0 \to Y \mid f \circ g_0 \circ d^1_0 = f \circ g_0 \circ d^1_1\} \end{aligned}$$

In particular we have $U = \mathrm{cosk}_1 \mathrm{sk}_1 U$.

Proof. Suppose that $g : \mathrm{sk}_1 V \to \mathrm{sk}_1 U$ is a morphism of 1-truncated simplicial objects. Then the diagram

$$\begin{tikzcd} V_1 \arrow[r, shift left, "d^1_0"] \arrow[r, shift right, "d^1_1"'] \arrow[d, "g_1"'] & V_0 \arrow[d, "g_0"] & \\ Y \times_X Y \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_0"'] & Y \arrow[r] & X \end{tikzcd}$$

is commutative, which proves that the relation shown in the lemma holds. We have to show that, conversely, given a morphism g_0 satisfying the relation $f \circ g_0 \circ d^1_0 = f \circ g_0 \circ d^1_1$ we get a unique morphism of simplicial objects $g : V \to U$. This is done as follows. For any $n \geq 1$ let $g_{n,i} = g_0 \circ V([0] \to [n], 0 \mapsto i) : V_n \to Y$. The equality above implies that $f \circ g_{n,i} = f \circ g_{n,i+1}$ because of the commutative diagram

$$\begin{tikzcd}
{[0]} \arrow[dr, "0\mapsto 0"'] \arrow[drr, "0\mapsto i"] & & \\
& {[1]} \arrow[r, "{0\mapsto i, 1\mapsto i+1}"] & {[n]} \\
{[0]} \arrow[ur, "0\mapsto 1"] \arrow[urr, "0\mapsto i+1"'] & &
\end{tikzcd}$$

Hence we get $(g_{n,0}, \ldots, g_{n,n}) : V_n \to Y \times_X \ldots \times_X Y = U_n$. We leave it to the reader to see that this is a morphism of simplicial objects. The last assertion of the lemma is equivalent to the first equality in the displayed formula of the lemma. □

018J Remark 14.20.4. Let $\mathcal{C}$ be a category with fibre products. Let V be a simplicial object. Let $\epsilon : V \to X$ be an augmentation. Let U be the simplicial object whose nth term is the $(n+1)$st fibred product of V_0 over X. By a simple combination of Lemmas 14.20.2 and 14.20.3 we obtain a canonical morphism $V \to U$.

14.21. Left adjoints to the skeleton functors

018K In this section we construct a left adjoint $i_{m!}$ of the skeleton functor sk_m in certain cases. The adjointness formula is

$$\text{Mor}_{\text{Simp}_m(\mathcal{C})}(U, \text{sk}_m V) = \text{Mor}_{\text{Simp}(\mathcal{C})}(i_{m!}U, V).$$

It turns out that this left adjoint exists when the category $\mathcal{C}$ has finite colimits.

We use a similar construction as in Section 14.12. Recall the category $[n]/\Delta$ of objects under $[n]$, see Categories, Example 4.2.14. Its objects are morphisms $\alpha : [n] \to [k]$ and its morphisms are commutative triangles. We let $([n]/\Delta)_{\leq m}$ denote the full subcategory of $[n]/\Delta$ consisting of objects $[n] \to [k]$ with $k \leq m$. Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : ([n]/\Delta)_{\leq m}^{opp} \longrightarrow \mathcal{C}$$

by the rules

$$\begin{array}{rcl} ([n] \to [k]) & \longmapsto & U_k \\ \psi : ([n] \to [k']) \to ([n] \to [k]) & \longmapsto & U(\psi) : U_k \to U_{k'} \end{array}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\underline{\varphi} : ([n']/\Delta)_{\leq m} \longrightarrow ([n]/\Delta)_{\leq m}$$

which maps $\alpha : [n'] \to [k]$ to $\varphi \circ \alpha : [n] \to [k]$. The composition $U(n) \circ \underline{\varphi}$ is equal to the functor $U(n')$.

018L Lemma 14.21.1. Let $\mathcal{C}$ be a category which has finite colimits. The functors $i_{m!}$ exist for all m. Let U be an m-truncated simplicial object of $\mathcal{C}$. The simplicial object $i_{m!}U$ is described by the formula

$$(i_{m!}U)_n = \text{colim}_{([n]/\Delta)_{\leq m}^{opp}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $i_{m!}U(\varphi)$ comes from the identification $U(n) \circ \underline{\varphi} = U(n')$ above via Categories, Lemma 4.14.8.

Proof. In this proof we denote $i_{m!}U$ the simplicial object whose nth term is given by the displayed formula of the lemma. We will show it satisfies the adjointness property.

Let V be a simplicial object of $\mathcal{C}$. Let $\gamma : U \to \text{sk}_m V$ be given. A morphism

$$\text{colim}_{([n]/\Delta)_{\leq m}^{opp}} U(n) \to T$$

is given by a compatible system of morphisms $f_\alpha : U_k \to T$ where $\alpha : [n] \to [k]$ with $k \leq m$. Certainly, we have such a system of morphisms by taking the compositions

$$U_k \xrightarrow{\gamma_k} V_k \xrightarrow{V(\alpha)} V_n.$$

Hence we get an induced morphism $(i_{m!}U)_n \to V_n$. We leave it to the reader to see that these form a morphism of simplicial objects $\gamma' : i_{m!}U \to V$.

Conversely, given a morphism $\gamma' : i_{m!}U \to V$ we obtain a morphism $\gamma : U \to \text{sk}_m V$ by setting $\gamma_i : U_i \to V_i$ equal to the composition

$$U_i \xrightarrow{\text{id}_{[i]}} \text{colim}_{([i]/\Delta)_{\leq m}^{opp}} U(i) \xrightarrow{\gamma'_i} V_i$$

for $0 \leq i \leq n$. We leave it to the reader to see that this is the inverse of the construction above. □

018M Lemma 14.21.2. Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For any $n \leq m$ the colimit

$$\text{colim}_{([n]/\Delta)_{\leq m}^{opp}} U(n)$$

exists and is equal to U_n.

Proof. This is so because the category $([n]/\Delta)_{\leq m}$ has an initial object, namely $\text{id} : [n] \to [n]$. □

018N Lemma 14.21.3. Let $\mathcal{C}$ be a category which has finite colimits. Let U be an m-truncated simplicial object of $\mathcal{C}$. The map $U \to \text{sk}_m i_{m!}U$ is an isomorphism.

Proof. Combine Lemmas 14.21.1 and 14.21.2. □

018O Lemma 14.21.4. If U is an m-truncated simplicial set and $n > m$ then all n-simplices of $i_{m!}U$ are degenerate.

Proof. This can be seen from the construction of $i_{m!}U$ in Lemma 14.21.1, but we can also argue directly as follows. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subset with $V'_i = V_i$ for $i \leq m$ and all i simplices degenerate for $i > m$, see Lemma 14.18.4. By the adjunction formula, since $\text{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. □

018P Lemma 14.21.5. Let U be a simplicial set. Let $n \geq 0$ be an integer. The morphism $i_{n!}\text{sk}_n U \to U$ identifies $i_{n!}\text{sk}_n U$ with the simplicial set $U' \subset U$ defined in Lemma 14.18.4.

Proof. By Lemma 14.21.4 the only nondegenerate simplices of $i_{n!}\text{sk}_n U$ are in degrees $\leq n$. The map $i_{n!}\text{sk}_n U \to U$ is an isomorphism in degrees $\leq n$. Combined we conclude that the map $i_{n!}\text{sk}_n U \to U$ maps nondegenerate simplices to nondegenerate simplices and no two nondegenerate simplices have the same image. Hence Lemma 14.18.3 applies. Thus $i_{n!}\text{sk}_n U \to U$ is injective. The result follows easily from this. □

018Q Remark 14.21.6. In some texts the composite functor

$$\text{Simp}(\mathcal{C}) \xrightarrow{\text{sk}_m} \text{Simp}_m(\mathcal{C}) \xrightarrow{i_{m!}} \text{Simp}(\mathcal{C})$$

is denoted sk_m. This makes sense for simplicial sets, because then Lemma 14.21.5 says that $i_{m!}\text{sk}_m V$ is just the sub simplicial set of V consisting of all i-simplices of V, $i \leq m$ and their degeneracies. In those texts it is also customary to denote the composition

$$\text{Simp}(\mathcal{C}) \xrightarrow{\text{sk}_m} \text{Simp}_m(\mathcal{C}) \xrightarrow{\text{cosk}_m} \text{Simp}(\mathcal{C})$$

by cosk_m.

018R Lemma 14.21.7. Let $U \subset V$ be simplicial sets. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that

(1) $V_i = U_i$ for $i < n$,
(2) $V_n = U_n \cup \{x\}$,
(3) any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.

Let $\Delta[n] \to V$ be the unique morphism mapping the nondegenerate n-simplex of $\Delta[n]$ to x. In this case the diagram

$$\begin{tikzcd} \Delta[n] \arrow[r] & V \\ i_{(n-1)!}\text{sk}_{n-1}\Delta[n] \arrow[r] \arrow[u] & U \arrow[u] \end{tikzcd}$$

is a pushout diagram.

Proof. Let us denote $\partial\Delta[n] = i_{(n-1)!}\text{sk}_{n-1}\Delta[n]$ for convenience. There is a natural map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$. We have to show that it is bijective in degree j for all j. This is clear for $j \leq n$. Let $j > n$. The third condition means that any $z \in V_j$, $z \notin U_j$ is a degenerate simplex, say $z = s^{j-1}_i(z')$. Of course $z' \notin U_{j-1}$. By induction it follows that z' is a degeneracy of x. Thus we conclude that all j-simplices of V are either in U or degeneracies of x. This implies that the map $U\amalg_{\partial\Delta[n]}\Delta[n] \to V$ is surjective. Note that a nondegenerate simplex of $U\amalg_{\partial\Delta[n]}\Delta[n]$ is either the image of a nondegenerate simplex of U, or the image of the (unique) nondegenerate n-simplex of $\Delta[n]$. Since clearly x is nondegenerate we deduce that $U\amalg_{\partial\Delta[n]}\Delta[n] \to V$ maps nondegenerate simplices to nondegenerate simplices and is injective on nondegenerate simplices. Hence it is injective, by Lemma 14.18.3. $\square$

018S Lemma 14.21.8. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then there exists a sequence of sub simplicial sets

$$U = W^0 \subset W^1 \subset W^2 \subset \ldots W^r = V$$

such that Lemma 14.21.7 applies to each of the inclusions $W^i \subset W^{i+1}$.

Proof. Let n be the smallest integer such that V has a nondegenerate simplex that does not belong to U. Let $x \in V_n$, $x \notin U_n$ be such a nondegenerate simplex. Let $W \subset V$ be the set of elements which are either in U, or are a (repeated) degeneracy of x (in other words, are of the form $V(\varphi)(x)$ with $\varphi : [m] \to [n]$ surjective). It is easy to see that W is a simplicial set. The inclusion $U \subset W$ satisfies the conditions of Lemma 14.21.7. Moreover the number of nondegenerate simplices of V which are not contained in W is exactly one less than the number of nondegenerate simplices of V which are not contained in U. Hence we win by induction on this number. ☐

018T Lemma 14.21.9. Let $\mathcal{A}$ be an abelian category Let U be an m-truncated simplicial object of $\mathcal{A}$. For $n > m$ we have $N(i_{m!}U)_n = 0$.

Proof. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subobject of V with $V'_i = V_i$ for $i \leq m$ and $N(V'_i) = 0$ for $i > m$, see Lemma 14.18.9. By the adjunction formula, since $\text{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. ☐

018U Lemma 14.21.10. Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The morphism $i_{n!}\text{sk}_n U \to U$ identifies $i_{n!}\text{sk}_n U$ with the simplicial subobject $U' \subset U$ defined in Lemma 14.18.9.

Proof. By Lemma 14.21.9 we have $N(i_{n!}\text{sk}_n U)_i = 0$ for $i > n$. The map $i_{n!}\text{sk}_n U \to U$ is an isomorphism in degrees $\leq n$, see Lemma 14.21.3. Combined we conclude that the map $i_{n!}\text{sk}_n U \to U$ induces injective maps $N(i_{n!}\text{sk}_n U)_i \to N(U)_i$ for all i. Hence Lemma 14.18.7 applies. Thus $i_{n!}\text{sk}_n U \to U$ is injective. The result follows easily from this. ☐

Here is another way to think about the coskeleton functor using the material above.

018V Lemma 14.21.11. Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let V be a simplicial object of $\mathcal{C}$. In this case

$$(\text{cosk}_n \text{sk}_n V)_{n+1} = \text{Hom}(i_{n!}\text{sk}_n \Delta[n+1], V)_0.$$

Proof. By Lemma 14.13.4 the object on the left represents the functor which assigns to X the first set of the following equalities

$$\begin{aligned} \text{Mor}(X \times \Delta[n+1], \text{cosk}_n \text{sk}_n V) &= \text{Mor}(X \times \text{sk}_n \Delta[n+1], \text{sk}_n V) \\ &= \text{Mor}(X \times i_{n!}\text{sk}_n \Delta[n+1], V). \end{aligned}$$

The object on the right in the formula of the lemma is represented by the functor which assigns to X the last set in the sequence of equalities. This proves the result.

In the sequence of equalities we have used that $\text{sk}_n(X \times \Delta[n+1]) = X \times \text{sk}_n \Delta[n+1]$ and that $i_{n!}(X \times \text{sk}_n \Delta[n+1]) = X \times i_{n!}\text{sk}_n \Delta[n+1]$. The first equality is obvious. For any (possibly truncated) simplicial object W of $\mathcal{C}$ and any object X of $\mathcal{C}$ denote temporarily $\text{Mor}_{\mathcal{C}}(X, W)$ the (possibly truncated) simplicial set $[n] \mapsto \text{Mor}_{\mathcal{C}}(X, W_n)$. From the definitions it follows that $\text{Mor}(U \times X, W) = \text{Mor}(U, \text{Mor}_{\mathcal{C}}(X, W))$ for any (possibly truncated) simplicial set U. Hence

$$\begin{aligned} \text{Mor}(X \times i_{n!}\text{sk}_n \Delta[n+1], W) &= \text{Mor}(i_{n!}\text{sk}_n \Delta[n+1], \text{Mor}_{\mathcal{C}}(X, W)) \\ &= \text{Mor}(\text{sk}_n \Delta[n+1], \text{sk}_n \text{Mor}_{\mathcal{C}}(X, W)) \\ &= \text{Mor}(X \times \text{sk}_n \Delta[n+1], \text{sk}_n W) \\ &= \text{Mor}(i_{n!}(X \times \text{sk}_n \Delta[n+1]), W). \end{aligned}$$

This proves the second equality used, and ends the proof of the lemma. ☐

14.22. Simplicial objects in abelian categories

018Y Recall that an abelian category is defined in Homology, Section 12.5.

018Z Lemma 14.22.1. Let $\mathcal{A}$ be an abelian category.

(1) The categories $\text{Simp}(\mathcal{A})$ and $\text{CoSimp}(\mathcal{A})$ are abelian.

(2) A morphism of (co)simplicial objects $f : A \to B$ is injective if and only if each $f_n : A_n \to B_n$ is injective.
(3) A morphism of (co)simplicial objects $f : A \to B$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.
(4) A sequence of (co)simplicial objects

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Pre-additivity is easy. A final object is given by $U_n = 0$ in all degrees. Existence of direct products we saw in Lemmas 14.6.2 and 14.9.2. Kernels and cokernels are obtained by taking termwise kernels and cokernels. $\square$

For an object A of $\mathcal{A}$ and an integer k consider the k-truncated simplicial object U with

(1) $U_i = 0$ for $i < k$,
(2) $U_k = A$,
(3) all morphisms $U(\varphi)$ equal to zero, except $U(\text{id}_{[k]}) = \text{id}_A$.

Since $\mathcal{A}$ has both finite limits and finite colimits we see that both $\text{cosk}_k U$ and $i_{k!}U$ exist. We will describe both of these and the canonical map $i_{k!}U \to \text{cosk}_k U$.

0190 Lemma 14.22.2. With A, k and U as above, so $U_i = 0$, $i < k$ and $U_k = A$.

(1) Given a k-truncated simplicial object V we have

$$\text{Mor}(U, V) = \{f : A \to V_k \mid d^k_i \circ f = 0,\ i = 0, \ldots, k\}$$

and

$$\text{Mor}(V, U) = \{f : V_k \to A \mid f \circ s^{k-1}_i = 0,\ i = 0, \ldots, k-1\}.$$

(2) The object $i_{k!}U$ has nth term equal to $\bigoplus_\alpha A$ where α runs over all surjective morphisms $\alpha : [n] \to [k]$.
(3) For any $\varphi : [m] \to [n]$ the map $i_{k!}U(\varphi)$ is described as the mapping $\bigoplus_\alpha A \to \bigoplus_{\alpha'} A$ which maps to component corresponding to $\alpha : [n] \to [k]$ to zero if $\alpha \circ \varphi$ is not surjective and by the identity to the component corresponding to $\alpha \circ \varphi$ if it is surjective.
(4) The object $\text{cosk}_k U$ has nth term equal to $\bigoplus_\beta A$, where β runs over all injective morphisms $\beta : [k] \to [n]$.
(5) For any $\varphi : [m] \to [n]$ the map $\text{cosk}_k U(\varphi)$ is described as the mapping $\bigoplus_\beta A \to \bigoplus_{\beta'} A$ which maps to component corresponding to $\beta : [k] \to [n]$ to zero if β does not factor through φ and by the identity to each of the components corresponding to β' such that $\beta = \varphi \circ \beta'$ if it does.
(6) The canonical map $c : i_{k!}U \to \text{cosk}_k U$ in degree n has (α, β) coefficient $A \to A$ equal to zero if $\alpha \circ \beta$ is not the identity and equal to id_A if it is.
(7) The canonical map $c : i_{k!}U \to \text{cosk}_k U$ is injective.

Proof. The proof of (1) is left to the reader.

Let us take the rules of (2) and (3) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $i_{k!}U$. This will prove (2), (3) at the same time. We have to show that given a morphism $f : U \to \text{sk}_k V$ there exists a unique morphism $\tilde{f} : \tilde{U} \to V$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : A \to V_k$ which maps into the kernel of d^k_i for all i. For any surjective $\alpha : [n] \to [k]$ we set $\tilde{f}_\alpha : A \to V_n$ equal to the composition $\tilde{f}_\alpha = V(\alpha) \circ f_k : A \to V_n$. We define $\tilde{f}_n : \tilde{U}_n \to V_n$ as the sum of the $\tilde{f}_\alpha$ over $\alpha : [n] \to [k]$ surjective. Such a collection of $\tilde{f}_\alpha$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$\begin{tikzcd}
\bigoplus_{\alpha:[n]\to[k] \text{ surjective}} A \arrow[r, "\tilde{f}_n"'] \arrow[d, "(3)"'] & V_n \arrow[d, "V(\varphi)"] \\
\bigoplus_{\alpha':[m]\to[k] \text{ surjective}} A \arrow[r, "\tilde{f}_m"] & V_m
\end{tikzcd}$$

is commutative. Choosing $\varphi = \alpha$ shows our choice of $\tilde{f}_\alpha$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the left upper corner separately. It is clear for the summands corresponding to α where $\alpha\circ\varphi$ is surjective, because those get mapped by id_A to the summand with $\alpha' = \alpha \circ \varphi$, and we have $\tilde{f}_{\alpha'} = V(\alpha') \circ f_k = V(\alpha \circ \varphi) \circ f_k = V(\varphi) \circ \tilde{f}_\alpha$. For those where $\alpha \circ \varphi$ is not surjective, we have to show that $V(\varphi) \circ \tilde{f}_\alpha = 0$. By definition this is equal to $V(\varphi) \circ V(\alpha) \circ f_k = V(\alpha \circ \varphi) \circ f_k$. Since $\alpha \circ \varphi$ is not surjective we can write it as $\delta^k_i \circ \psi$, and we deduce that $V(\varphi) \circ V(\alpha) \circ f_k = V(\psi) \circ d^k_i \circ f_k = 0$ see above.

Let us take the rules of (4) and (5) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $\text{cosk}_k U$. This will prove (4), (5) at the same time. The argument is completely dual to the proof of (2), (3) above, but we give it anyway. We have to show that given a morphism $f : \text{sk}_k V \to U$ there exists a unique morphism $\tilde{f} : V \to \tilde{U}$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : V_k \to A$ which is zero on the image of s^{k-1}_i for all i. For any injective $\beta : [k] \to [n]$ we set $\tilde{f}_\beta : V_n \to A$ equal to the composition $\tilde{f}_\beta = f_k \circ V(\beta) : V_n \to A$. We define $\tilde{f}_n : V_n \to \tilde{U}_n$ as the sum of the $\tilde{f}_\beta$ over $\beta : [k] \to [n]$ injective. Such a collection of $\tilde{f}_\beta$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$\begin{tikzcd}
V_n \arrow[r, "\tilde{f}_n"'] \arrow[d, "V(\varphi)"'] & \bigoplus_{\beta:[k]\to[n] \text{ injective}} A \arrow[d, "(5)"] \\
V_m \arrow[r, "\tilde{f}_m"] & \bigoplus_{\beta':[k]\to[m] \text{ injective}} A
\end{tikzcd}$$

is commutative. Choosing $\varphi = \beta$ shows our choice of $\tilde{f}_\beta$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the right lower corner separately. It is clear for the summands corresponding to β' where $\varphi\circ\beta'$ is injective, because these summands get mapped into by exactly the summand with $\beta = \varphi \circ \beta'$ and we have in that case $\tilde{f}_{\beta'} \circ V(\varphi) = f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\beta) = \tilde{f}_\beta$. For those where $\varphi \circ \beta'$ is not injective, we have to show that $\tilde{f}_{\beta'} \circ V(\varphi) = 0$. By definition this is equal to $f_k\circ V(\beta')\circ V(\varphi) = f_k\circ V(\varphi\circ\beta')$. Since $\varphi\circ\beta'$ is not injective we can write it as $\psi\circ\sigma^{k-1}_i$, and we deduce that $f_k\circ V(\beta')\circ V(\varphi) = f_k\circ s^{k-1}_i\circ V(\psi) = 0$ see above.

The composition $i_{k!}U \to \text{cosk}_k U$ is the unique map of simplicial objects which is the identity on $A = U_k = (i_{k!}U)_k = (\text{cosk}_k U)_k$. Hence it suffices to check that the proposed rule defines a morphism of simplicial objects. To see this we have to show that for any $\varphi : [m] \to [n]$ the diagram

$$\begin{tikzcd} \bigoplus_{\alpha:[n]\to[k] \text{ surjective}} A \arrow[r, "(6)"'] \arrow[d, "(3)"'] & \bigoplus_{\beta:[k]\to[n] \text{ injective}} A \arrow[d, "(5)"] \\ \bigoplus_{\alpha':[m]\to[k] \text{ surjective}} A \arrow[r, "(6)"] & \bigoplus_{\beta':[k]\to[m] \text{ injective}} A \end{tikzcd}$$

is commutative. Now we can think of this in terms of matrices filled with only 0's and 1's as follows: The matrix of (3) has a nonzero (α', α) entry if and only if $\alpha' = \alpha \circ \varphi$. Likewise the matrix of (5) has a nonzero (β', β) entry if and only if $\beta = \varphi \circ \beta'$. The upper matrix of (6) has a nonzero (α, β) entry if and only if $\alpha \circ \beta = \text{id}_{[k]}$. Similarly for the lower matrix of (6). The commutativity of the diagram then comes down to computing the (α, β') entry for both compositions and seeing they are equal. This comes down to the following equality

$$\# \left\{\beta \mid \beta = \varphi \circ \beta' \text{ and } \alpha \circ \beta = \text{id}_{[k]}\right\} = \# \left\{\alpha' \mid \alpha' = \alpha \circ \varphi \text{ and } \alpha' \circ \beta' = \text{id}_{[k]}\right\}$$

whose proof may safely be left to the reader.

Finally, we prove (7). This follows directly from Lemmas 14.18.7, 14.19.4, 14.21.3 and 14.21.9. □

0191 Definition 14.22.3. Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. The Eilenberg-Maclane object $K(A, k)$ is given by the object $K(A, k) = i_{k!}U$ which is described in Lemma 14.22.2 above.

0192 Lemma 14.22.4. Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. Consider the simplicial object E defined by the following rules

1. $E_n = \bigoplus_\alpha A$, where the sum is over $\alpha : [n] \to [k+1]$ whose image is either $[k]$ or $[k+1]$.
2. Given $\varphi : [m] \to [n]$ the map $E_n \to E_m$ maps the summand corresponding to α via id_A to the summand corresponding to $\alpha \circ \varphi$, provided $\text{Im}(\alpha \circ \varphi)$ is equal to $[k]$ or $[k+1]$.

Then there exists a short exact sequence

$$0 \to K(A, k) \to E \to K(A, k+1) \to 0$$

which is term by term split exact.

Proof. The maps $K(A, k)_n \to E_n$ resp. $E_n \to K(A, k+1)_n$ are given by the inclusion of direct sums, resp. projection of direct sums which is obvious from the inclusions of index sets. It is clear that these are maps of simplicial objects. □

0193 Lemma 14.22.5. Let $\mathcal{A}$ be an abelian category. For any simplicial object V of $\mathcal{A}$ we have

$$V = \text{colim}_n\, i_{n!}\text{sk}_n V$$

where all the transition maps are injections.

Proof. This is true simply because each V_m is equal to $(i_{n!}\text{sk}_n V)_m$ as soon as $n \geq m$. See also Lemma 14.21.10 for the transition maps. □

14.23. Simplicial objects and chain complexes

0194 Let $\mathcal{A}$ be an abelian category. See Homology, Section 12.13 for conventions and notation regarding chain complexes. Let U be a simplicial object of $\mathcal{A}$. The associated chain complex $s(U)$ of U, sometimes called the Moore complex, is the chain complex

$$\ldots \to U_2 \to U_1 \to U_0 \to 0 \to 0 \to \ldots$$

with boundary maps $d_n : U_n \to U_{n-1}$ given by the formula

$$d_n = \sum\nolimits_{i=0}^{n} (-1)^i d^n_i.$$

This is a complex because, by the relations listed in Remark 14.3.3, we have

$$\begin{aligned} d_n \circ d_{n+1} &= (\sum\nolimits_{i=0}^{n} (-1)^i d^n_i) \circ (\sum\nolimits_{j=0}^{n+1} (-1)^j d^{n+1}_j) \\ &= \sum\nolimits_{0 \le i < j \le n+1} (-1)^{i+j} d^n_{j-1} \circ d^{n+1}_i + \sum\nolimits_{n \ge i \ge j \ge 0} (-1)^{i+j} d^n_i \circ d^{n+1}_j \\ &= 0. \end{aligned}$$

The signs cancel! We denote the associated chain complex $s(U)$. Clearly, the construction is functorial and hence defines a functor

$$s : \text{Simp}(\mathcal{A}) \longrightarrow \text{Ch}_{\ge 0}(\mathcal{A}).$$

Thus we have the confusing but correct formula $s(U)_n = U_n$.

0195 **Lemma 14.23.1.** The functor s is exact.

Proof. Clear from Lemma 14.22.1. $\square$

0196 **Lemma 14.23.2.** Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. Let E be the object described in Lemma 14.22.4. Then the complex $s(E)$ is acyclic.

Proof. For a morphism $\alpha : [n] \to [k+1]$ we define $\alpha' : [n+1] \to [k+1]$ to be the map such that $\alpha'|_{[n]} = \alpha$ and $\alpha'(n+1) = k+1$. Note that if the image of α is $[k]$ or $[k+1]$, then the image of α' is $[k+1]$. Consider the family of maps $h_n : E_n \to E_{n+1}$ which maps the summand corresponding to α to the summand corresponding to α' via the identity on A. Let us compute $d_{n+1} \circ h_n - h_{n-1} \circ d_n$. We will first do this in case the category $\mathcal{A}$ is the category of abelian groups. Let us use the notation x_α to indicate the element $x \in A$ in the summand of E_n corresponding to the map α occurring in the index set. Let us also adopt the convention that x_α designates the zero element of E_n whenever $\text{Im}(\alpha)$ is not $[k]$ or $[k+1]$. With these conventions we see that

$$d_{n+1}(h_n(x_\alpha)) = \sum\nolimits_{i=0}^{n+1} (-1)^i x_{\alpha' \circ \delta^{n+1}_i}$$

and

$$h_{n-1}(d_n(x_\alpha)) = \sum\nolimits_{i=0}^{n} (-1)^i x_{(\alpha \circ \delta^n_i)'}$$

It is easy to see that $\alpha' \circ \delta^{n+1}_i = (\alpha \circ \delta^n_i)'$ for $i = 0, \ldots, n$. It is also easy to see that $\alpha' \circ \delta^{n+1}_{n+1} = \alpha$. Thus we see that

$$(d_{n+1} \circ h_n - h_{n-1} \circ d_n)(x_\alpha) = (-1)^{n+1} x_\alpha$$

These identities continue to hold if $\mathcal{A}$ is any abelian category because they hold in the simplicial abelian group $[n] \mapsto \text{Hom}(A, E_n)$; details left to the reader. We conclude that the identity map on E is homotopic to zero, with homotopy given by the system of maps $h'_n = (-1)^{n+1} h_n : E_n \to E_{n+1}$. Hence we see that E is acyclic, for example by Homology, Lemma 12.13.5. $\square$

0197 **Lemma 14.23.3.** Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $H_i(s(K(A,k))) = A$ if $i = k$ and 0 else.

Proof. First, let us prove this if $k = 0$. In this case we have $K(A,0)_n = A$ for all n. Furthermore, all the maps in this simplicial abelian group are id_A, in other words $K(A,0)$ is the constant simplicial object with value A. The boundary maps $d_n = \sum_{i=0}^{n} (-1)^i \mathrm{id}_A = 0$ if n odd and $= \mathrm{id}_A$ if n is even. Thus $s(K(A,0))$ looks like this

$$\ldots \to A \xrightarrow{0} A \xrightarrow{1} A \xrightarrow{0} A \to 0$$

and the result is clear.

Next, we prove the result for all k by induction. Given the result for k consider the short exact sequence

$$0 \to K(A,k) \to E \to K(A,k+1) \to 0$$

from Lemma 14.22.4. By Lemma 14.22.1 the associated sequence of chain complexes is exact. By Lemma 14.23.2 we see that $s(E)$ is acyclic. Hence the result for $k+1$ follows from the long exact sequence of homology, see Homology, Lemma 12.13.6. $\square$

There is a second chain complex we can associate to a simplicial object of $\mathcal{A}$. Recall that by Lemma 14.18.6 any simplicial object U of $\mathcal{A}$ is canonically split with $N(U_m) = \bigcap_{i=0}^{m-1} \mathrm{Ker}(d_i^m)$. We define the normalized chain complex $N(U)$ to be the chain complex

$$\ldots \to N(U_2) \to N(U_1) \to N(U_0) \to 0 \to 0 \to \ldots$$

with boundary map $d_n : N(U_n) \to N(U_{n-1})$ given by the restriction of $(-1)^n d_n^n$ to the direct summand $N(U_n)$ of U_n. Note that Lemma 14.18.8 implies that $d_n^n(N(U_n)) \subset N(U_{n-1})$. It is a complex because $d_n^n \circ d_{n+1}^{n+1} = d_n^n \circ d_n^{n+1}$ and d_n^{n+1} is zero on $N(U_{n+1})$ by definition. Thus we obtain a second functor

$$N : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

Here is the reason for the sign in the differential.

0198 **Lemma 14.23.4.** Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical map $N(U_n) \to U_n$ gives rise to a morphism of complexes $N(U) \to s(U)$.

Proof. This is clear because the differential on $s(U)_n = U_n$ is $\sum (-1)^i d_i^n$ and the maps d_i^n, $i < n$ are zero on $N(U_n)$, whereas the restriction of $(-1)^n d_n^n$ is the boundary map of $N(U)$ by definition. $\square$

0199 **Lemma 14.23.5.** Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $N(K(A,k))_i = A$ if $i = k$ and 0 else.

Proof. It is clear that $N(K(A,k))_i = 0$ when $i < k$ because $K(A,k)_i = 0$ in that case. It is clear that $N(K(A,k))_k = A$ since $K(A,k)_{k-1} = 0$ and $K(A,k)_k = A$. For $i > k$ we have $N(K(A,k))_i = 0$ by Lemma 14.21.9 and the definition of $K(A,k)$, see Definition 14.22.3. $\square$

019A Lemma 14.23.6. Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(U) \to s(U)$ is split. In fact,

$$s(U) = N(U) \oplus D(U)$$

for some complex $D(U)$. The construction $U \mapsto D(U)$ is functorial.

Proof. Define $D(U)_n$ to be the image of

$$\bigoplus\nolimits_{\varphi:[n]\to[m] \text{ surjective}, \ m<n} N(U_m) \xrightarrow{\bigoplus U(\varphi)} U_n$$

which is a subobject of U_n complementary to $N(U_n)$ according to Lemma 14.18.6 and Definition 14.18.1. We show that $D(U)$ is a subcomplex. Pick a surjective map $\varphi : [n] \to [m]$ with $m < n$ and consider the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

This composition is the sum of the maps

$$N(U_m) \xrightarrow{U(\varphi\circ\delta^n_i)} U_{n-1}$$

with sign $(-1)^i$, $i = 0, \ldots, n$.

First we will prove by ascending induction on m, $0 \leq m < n-1$ that all the maps $U(\varphi \circ \delta^n_i)$ map $N(U_m)$ into $D(U)_{n-1}$. (The case $m = n-1$ is treated below.) Whenever the map $\varphi \circ \delta^n_i : [n-1] \to [m]$ is surjective then the image of $N(U_m)$ under $U(\varphi \circ \delta^n_i)$ is contained in $D(U)_{n-1}$ by definition. If $\varphi \circ \delta^n_i : [n-1] \to [m]$ is not surjective, set $j = \varphi(i)$ and observe that i is the unique index whose image under φ is j. We may write $\varphi \circ \delta^n_i = \delta^m_j \circ \psi \circ \delta^n_i$ for some $\psi : [n-1] \to [m-1]$. Hence $U(\varphi \circ \delta^n_i) = U(\psi \circ \delta^n_i) \circ d^m_j$ which is zero on $N(U_m)$ unless $j = m$. If $j = m$, then $d^m_m(N(U_m)) \subset N(U_{m-1})$ and hence $U(\varphi\circ\delta^n_i)(N(U_m)) \subset U(\psi\circ\delta^n_i)(N(U_{m-1}))$ and we win by induction hypothesis.

To finish proving that $D(U)$ is a subcomplex we still have to deal with the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

in case $m = n-1$. In this case $\varphi = \sigma^{n-1}_j$ for some $0 \leq j \leq n-1$ and $U(\varphi) = s^{n-1}_j$. Thus the composition is given by the sum

$$\sum (-1)^i d^n_i \circ s^{n-1}_j$$

Recall from Remark 14.3.3 that $d^n_j \circ s^{n-1}_j = d^n_{j+1} \circ s^{n-1}_j = \text{id}$ and these drop out because the corresponding terms have opposite signs. The map $d^n_n \circ s^{n-1}_j$, if $j < n-1$, is equal to $s^{n-2}_j \circ d^{n-1}_{n-1}$. Since d^{n-1}_{n-1} maps $N(U_{n-1})$ into $N(U_{n-2})$, we see that the image $d^n_n(s^{n-1}_j(N(U_{n-1}))$ is contained in $s^{n-2}_j(N(U_{n-2}))$ which is contained in $D(U_{n-1})$ by definition. For all other combinations of (i,j) we have either $d^n_i \circ s^{n-1}_j = s^{n-2}_{j-1} \circ d^{n-1}_i$ (if $i < j$), or $d^n_i \circ s^{n-1}_j = s^{n-2}_j \circ d^{n-1}_{i-1}$ (if $n > i > j+1$) and in these cases the map is zero because of the definition of $N(U_{n-1})$. □

0FKI Remark 14.23.7. In the situation of Lemma 14.23.6 the subcomplex $D(U) \subset s(U)$ can also be defined as the subcomplex with terms

$$D(U)_n = \text{Im}\left(\bigoplus\nolimits_{\varphi:[n]\to[m] \text{ surjective}, \ m<n} U_m \xrightarrow{\bigoplus U(\varphi)} U_n\right)$$

Namely, since U_m is the direct sum of the subobject $N(U_m)$ and the images of $N(U_k)$ for surjections $[m] \to [k]$ with $k < m$ this is clearly the same as the definition of $D(U)_n$ given in the proof of Lemma 14.23.6. Thus we see that if U is a simplicial abelian group, then elements of $D(U)_n$ are exactly the sums of degenerate n-simplices.

019B Lemma 14.23.8. The functor N is exact.

Proof. By Lemma 14.23.1 and the functorial decomposition of Lemma 14.23.6. $\square$

019C Lemma 14.23.9. Let $\mathcal{A}$ be an abelian category. Let V be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(V) \to s(V)$ is a quasi-isomorphism. In other words, the complex $D(V)$ of Lemma 14.23.6 is acyclic.

Proof. Note that the result holds for $K(A, k)$ for any object A and any $k \geq 0$, by Lemmas 14.23.3 and 14.23.5. Consider the hypothesis $IH_{n,m}$: for all V such that $V_j = 0$ for $j \leq m$ and all $i \leq n$ the map $N(V) \to s(V)$ induces an isomorphism $H_i(N(V)) \to H_i(s(V))$.

To start of the induction, note that $IH_{n,n}$ is trivially true, because in that case $N(V)_n = 0$ and $s(V)_n = 0$.

Assume $IH_{n,m}$, with $m \leq n$. Pick a simplicial object V such that $V_j = 0$ for $j < m$. By Lemma 14.22.2 and Definition 14.22.3 we have $K(V_m, m) = i_{m!}\text{sk}_m V$. By Lemma 14.21.10 the natural morphism

$$K(V_m, m) = i_{m!}\text{sk}_m V \to V$$

is injective. Thus we get a short exact sequence

$$0 \to K(V_m, m) \to V \to W \to 0$$

for some W with $W_i = 0$ for $i = 0, \ldots, m$. This short exact sequence induces a morphism of short exact sequence of associated complexes

$$\begin{tikzcd} 0 \arrow[r] & N(K(V_m, m)) \arrow[r] \arrow[d] & N(V) \arrow[r] \arrow[d] & N(W) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & s(K(V_m, m)) \arrow[r] & s(V) \arrow[r] & s(W) \arrow[r] & 0 \end{tikzcd}$$

see Lemmas 14.23.1 and 14.23.8. Hence we deduce the result for V from the result on the ends. $\square$

14.24. Dold-Kan

019D In this section we prove the Dold-Kan theorem relating simplicial objects in an abelian category with chain complexes.

019E Lemma 14.24.1. Let $\mathcal{A}$ be an abelian category. The functor N is faithful, and reflects isomorphisms, injections and surjections.

Proof. The faithfulness is immediate from the canonical splitting of Lemma 14.18.6. The statement on reflecting injections, surjections, and isomorphisms follows from Lemma 14.18.7. $\square$

019F Lemma 14.24.2. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $N : \mathcal{A} \to \mathcal{B}$, and $S : \mathcal{B} \to \mathcal{A}$ be functors. Suppose that

(1) the functors S and N are exact,
(2) there is an isomorphism $g : N \circ S \to \mathrm{id}_{\mathcal{B}}$ to the identity functor of $\mathcal{B}$,
(3) N is faithful, and
(4) S is essentially surjective.

Then S and N are quasi-inverse equivalences of categories.

Proof. It suffices to construct a functorial isomorphism $S(N(A)) \cong A$. To do this choose B and an isomorphism $f : A \to S(B)$. Consider the map

$$f^{-1} \circ g_{S(B)} \circ S(N(f)) : S(N(A)) \to S(N(S(B))) \to S(B) \to A.$$

It is easy to show this does not depend on the choice of f, B and gives the desired isomorphism $S \circ N \to \mathrm{id}_{\mathcal{A}}$. $\square$

019G Theorem 14.24.3. Let $\mathcal{A}$ be an abelian category. The functor N induces an equivalence of categories

$$N : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$$

Proof. We will describe a functor in the reverse direction inspired by the construction of Lemma 14.22.4 (except that we throw in a sign to get the boundaries right). Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For each $n \geq 0$ denote

$$I_n = \Big\{\alpha : [n] \to \{0, 1, 2, \ldots\} \mid \mathrm{Im}(\alpha) = [k] \text{ for some } k\Big\}.$$

For $\alpha \in I_n$ we denote $k(\alpha)$ the unique integer such that $\mathrm{Im}(\alpha) = [k]$. We define a simplicial object $S(A_\bullet)$ as follows:

(1) $S(A_\bullet)_n = \bigoplus_{\alpha \in I_n} A_{k(\alpha)}$, which we will write as $\bigoplus_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$ to suggest thinking of "α" as a basis vector for the summand corresponding to it,
(2) given $\varphi : [m] \to [n]$ we define $S(A_\bullet)(\varphi)$ by its restriction to the direct summand $A_{k(\alpha)} \cdot \alpha$ of $S(A_\bullet)_n$ as follows
  (a) $\alpha \circ \varphi \notin I_m$ then we set it equal to zero,
  (b) $\alpha \circ \varphi \in I_m$ but $k(\alpha \circ \varphi)$ not equal to either $k(\alpha)$ or $k(\alpha) - 1$ then we set it equal to zero as well,
  (c) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha)$ then we use the identity map to the summand $A_{k(\alpha\circ\varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$, and
  (d) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha) - 1$ then we use $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ to the summand $A_{k(\alpha\circ\varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$.

Let us show that $S(A_\bullet)$ is a simplicial object of $\mathcal{A}$. To do this, assume we have maps $\varphi : [m] \to [n]$ and $\psi : [n] \to [p]$. We will show that $S(A_\bullet)(\varphi) \circ S(A_\bullet)(\psi) = S(A_\bullet)(\psi \circ \varphi)$. Choose $\beta \in I_p$ and set $\alpha = \beta \circ \psi$ and $\gamma = \alpha \circ \varphi$ viewed as maps $\alpha : [n] \to \{0, 1, 2, \ldots\}$ and $\gamma : [m] \to \{0, 1, 2, \ldots\}$. Picture

$$\begin{tikzcd}
{[m]} \arrow[r, "\varphi"'] \arrow[d, "\gamma"'] & {[n]} \arrow[r, "\psi"'] \arrow[d, "\alpha"'] & {[p]} \arrow[d, "\beta"] \\
\mathrm{Im}(\gamma) \arrow[r] & \mathrm{Im}(\alpha) \arrow[r] & {[k(\beta)]}
\end{tikzcd}$$

We will show that the restriction of the maps $S(A_\bullet)(\varphi) \circ S(A_\bullet)(\psi)$ and $S(A_\bullet)(\psi \circ \varphi)$. to the summand $A_{k(\beta)} \cdot \beta$ agree. There are several cases to consider

(1) Say $\alpha \notin I_n$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is zero. Then either $\gamma \notin I_m$ or we have $[k(\gamma)] = \mathrm{Im}(\gamma) \subset \mathrm{Im}(\alpha) \subset [k(\beta)]$ and the subset $\mathrm{Im}(\alpha)$

of $[k(\beta)]$ has a gap so $k(\gamma) < k(\beta) - 1$. In both cases we see that the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to $A_{k(\beta)} \cdot \beta$ is zero as well.

(2) Say $\alpha \in I_n$ and $k(\alpha) < k(\beta) - 1$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is zero. Then either $\gamma \notin I_m$ or we have $[k(\gamma)] \subset [k(\alpha)] \subset [k(\beta)]$ and it follows that $k(\gamma) < k(\beta) - 1$. In both cases we see that the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to $A_{k(\beta)} \cdot \beta$ is zero as well.

(3) Say $\alpha \in I_n$ and $k(\alpha) = k(\beta)$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is the identity map from $A_{k(\beta)} \cdot \beta$ to $A_{k(\alpha)} \cdot \alpha$. In this case because $\mathrm{Im}(\alpha) = [k(\beta)]$ the rule describing the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ is exactly the same as the rule describing the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ and hence agreement holds.

(4) Say $\alpha \in I_n$ and $k(\alpha) = k(\beta) - 1$ so the restriction of $S(A_\bullet)(\psi)$ to $A_{k(\beta)} \cdot \beta$ is given by $(-1)^{k(\beta)} d_{A,k(\beta)}$ to $A_{k(\alpha)} \cdot \alpha$. Subcases
   (a) If $\gamma \notin I_m$, then both the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ and the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ are zero and we get agreement.
   (b) If $\gamma \in I_m$ but $k(\gamma) < k(\alpha) - 1$, then again both restrictions are zero and we get agreement.
   (c) If $\gamma \in I_m$ and $k(\gamma) = k(\alpha)$ then $\mathrm{Im}(\gamma) = \mathrm{Im}(\alpha)$. In this case the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ is given by $(-1)^{k(\beta)} d_{A,k(\beta)}$ to $A_{k(\gamma)} \cdot \gamma$ and the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ is the identity map $A_{k(\alpha)} \cdot \alpha \to A_{k(\gamma)} \cdot \gamma$. Hence agreement holds.
   (d) Finally, if $\gamma \in I_m$ and $k(\gamma) = k(\alpha) - 1$ then the restriction of $S(A_\bullet)(\varphi)$ to the summand $A_{k(\alpha)} \cdot \alpha$ is given by $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ as a map $A_{k(\alpha)} \cdot \alpha \to A_{k(\beta)} \cdot \beta$. Since $A_\bullet$ is a complex we see that the composition $A_{k(\beta)} \cdot \beta \to A_{k(\alpha)} \cdot \alpha \to A_{k(\gamma)} \cdot \gamma$ is zero which matches what we get for the restriction of $S(A_\bullet)(\psi \circ \varphi)$ to the summand $A_{k(\beta)} \cdot \beta$ because $k(\gamma) = k(\beta) - 2 < k(\beta) - 1$.

Thus $S(A_\bullet)$ is a simplicial object of $\mathcal{A}$.

Let us construct an isomorphism $A_\bullet \to N(S(A_\bullet))$ functorial in $A_\bullet$. Recall that

$$S(A_\bullet) = N(S(A_\bullet)) \oplus D(S(A_\bullet))$$

as chain complexes by Lemma 14.23.6. On the other hand it follows from Remark 14.23.7 and the construction of $S(A_\bullet)$ that

$$D(S(A_\bullet))_n = \bigoplus\nolimits_{\alpha \in I_n,\ k(\alpha) < n} A_{k(\alpha)} \cdot \alpha \subset \bigoplus\nolimits_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$$

However, if $\alpha \in I_n$ then we have $k(\alpha) \geq n \Leftrightarrow \alpha = \mathrm{id}_{[n]} : [n] \to [n]$. Thus the summand $A_n \cdot \mathrm{id}_{[n]}$ of $S(A_\bullet)_n$ is a complement to the summand $D(S(A_\bullet))_n$. All the maps $d^n_i : S(A_\bullet)_n \to S(A_\bullet)_n$ restrict to zero on the summand $A_n \cdot \mathrm{id}_{[n]}$ except for d^n_n which produces $(-1)^n d_{A,n}$ from $A_n \cdot \mathrm{id}_{[n]}$ to $A_{n-1} \cdot \mathrm{id}_{[n-1]}$. We conclude that $A_n \cdot \mathrm{id}_{[n]}$ must be equal to the summand $N(S(A_\bullet))_n$ and moreover the restriction of the differential $d_n = \sum (-1)^i d^n_i : S(A_\bullet)_n \to S(A_\bullet)_{n-1}$ to the summand $A_n \cdot \mathrm{id}_{[n]}$ gives what we want!

Finally, we have to show that $S \circ N$ is isomorphic to the identity functor. Let U be a simplicial object of $\mathcal{A}$. Then we can define an obvious map

$$S(N(U))_n = \bigoplus\nolimits_{\alpha \in I_n} N(U)_{k(\alpha)} \cdot \alpha \longrightarrow U_n$$

by using $U(\alpha) : N(U)_{k(\alpha)} \to U_n$ on the summand corresponding to α. By Definition 14.18.1 this is an isomorphism. To finish the proof we have to show that this is compatible with the maps in the simplicial objects. Thus let $\varphi : [m] \to [n]$ and let $\alpha \in I_n$. Set $\beta = \alpha \circ \varphi$. Picture

$$\begin{tikzcd} {[m]} \arrow[r, "\varphi"'] \arrow[d, "\beta"'] & {[n]} \arrow[d, "\alpha"'] \\ \mathrm{Im}(\beta) \arrow[r] & {[k(\alpha)]} \end{tikzcd}$$

There are several cases to consider

(1) Say $\beta \not\in I_m$. Then there exists an index $0 \leq j < k(\alpha)$ with $j \not\in \mathrm{Im}(\alpha \circ \varphi)$ and hence we can choose a factorization $\alpha \circ \varphi = \delta^{k(\alpha)}_j \circ \psi$ for some $\psi : [m] \to [k(\alpha) - 1]$. It follows that $U(\varphi)$ is zero on the image of the summand $N(U)_{k(\alpha)} \cdot \alpha$ because $U(\varphi) \circ U(\alpha) = U(\alpha \circ \varphi) = U(\psi) \circ d^{k(\alpha)}_j$ is zero on $N(U)_{k(\alpha)}$ by construction of N. This matches our rule for $S(N(U))$ given above.
(2) Say $\beta \in I_m$ and $k(\beta) < k(\alpha) - 1$. Here we argue exactly as in case (1) with $j = k(\alpha) - 1$.
(3) Say $\beta \in I_m$ and $k(\beta) = k(\alpha)$. Here the summand $N(U)_{k(\alpha)} \cdot \alpha$ is mapped by the identity to the summand $N(U)_{k(\beta)} \cdot \beta$. This is the same as the effect of $U(\varphi)$ since in this case $U(\varphi) \circ U(\alpha) = U(\beta)$.
(4) Say $\beta \in I_m$ and $k(\beta) = k(\alpha) - 1$. Here we use the differential $(-1)^{k(\alpha)} d_{N(U), k(\alpha)}$ to map the summand $N(U)_{k(\alpha)} \cdot \alpha$ to the summand $N(U)_{k(\beta)} \cdot \beta$. On the other hand, since $\mathrm{Im}(\beta) = [k(\beta)]$ in this case we get $\alpha \circ \varphi = \delta^{k(\alpha)}_{k(\alpha)} \circ \beta$. Thus we see that $U(\varphi)$ composed with the restriction of $U(\alpha)$ to $N(U)_{k(\alpha)}$ is equal to $U(\beta)$ precomposed with $d^{k(\alpha)}_{k(\alpha)}$ restricted to $N(U)_{k(\alpha)}$. Since $d_{N(U), k(\alpha)} = \sum (-1)^i d^{k(\alpha)}_i$ and since $d^{k(\alpha)}_i$ restricts to zero on $N(U)_{k(\alpha)}$ for $i < k(\alpha)$ we see that equality holds.

This finishes the proof of the theorem. $\square$

14.25. Dold-Kan for cosimplicial objects

019H Let $\mathcal{A}$ be an abelian category. According to Homology, Lemma 12.5.2 also $\mathcal{A}^{opp}$ is abelian. It follows formally from the definitions that

$$\mathrm{CoSimp}(\mathcal{A}) = \mathrm{Simp}(\mathcal{A}^{opp})^{opp}.$$

Thus Dold-Kan (Theorem 14.24.3) implies that $\mathrm{CoSimp}(\mathcal{A})$ is equivalent to the category $\mathrm{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}$. And it follows formally from the definitions that

$$\mathrm{CoCh}_{\geq 0}(\mathcal{A}) = \mathrm{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}.$$

Putting these arrows together we obtain an equivalence

$$Q : \mathrm{CoSimp}(\mathcal{A}) \longrightarrow \mathrm{CoCh}_{\geq 0}(\mathcal{A}).$$

In this section we describe Q.

First we define the cochain complex $s(U)$ associated to a cosimplicial object U. It is the cochain complex with terms zero in negative degrees, and $s(U)^n = U_n$ for $n \geq 0$. As differentials we use the maps $d^n : s(U)^n \to s(U)^{n+1}$ defined by $d^n = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}$. In other words the complex $s(U)$ looks like

$$0 \longrightarrow U_0 \xrightarrow{\delta_0^1 - \delta_1^1} U_1 \xrightarrow{\delta_0^2 - \delta_1^2 + \delta_2^2} U_2 \longrightarrow \ldots$$

This is sometimes also called the Moore complex associated to U.

On the other hand, given a cosimplicial object U of $\mathcal{A}$ set $Q(U)^0 = U_0$ and

$$Q(U)^n = \text{Coker}(\ \bigoplus\nolimits_{i=0}^{n-1} U_{n-1} \xrightarrow{\delta_i^n} U_n\ ).$$

The differential $d^n : Q(U)^n \to Q(U)^{n+1}$ is induced by $(-1)^{n+1}\delta_{n+1}^{n+1}$, i.e., by fitting the morphism $(-1)^{n+1}\delta_{n+1}^{n+1}$ into a commutative diagram

$$\begin{tikzcd}
U_n \arrow[r, "(-1)^{n+1}\delta_{n+1}^{n+1}"'] \arrow[d] & U_{n+1} \arrow[d] \\
Q(U)^n \arrow[r, "d_n"] & Q(U)^{n+1}.
\end{tikzcd}$$

We leave it to the reader to show that this diagram makes sense, i.e., that the image of δ_i^n maps into the kernel of the right vertical arrow for $i = 0, \ldots, n-1$. (This is dual to Lemma 14.18.8.) Thus our cochain complex $Q(U)$ looks like this

$$0 \to Q(U)^0 \to Q(U)^1 \to Q(U)^2 \to \ldots$$

This is called the normalized cochain complex associated to U. The dual to the Dold-Kan Theorem 14.24.3 is the following.

019I Lemma 14.25.1. Let $\mathcal{A}$ be an abelian category.

(1) The functor $s : \text{CoSimp}(\mathcal{A}) \to \text{CoCh}_{\geq 0}(\mathcal{A})$ is exact.
(2) The maps $s(U)^n \to Q(U)^n$ define a morphism of cochain complexes.
(3) There exists a functorial direct sum decomposition $s(U) = D(U) \oplus Q(U)$ in $\text{CoCh}_{\geq 0}(\mathcal{A})$.
(4) The functor Q is exact.
(5) The morphism of complexes $s(U) \to Q(U)$ is a quasi-isomorphism.
(6) The functor $U \mapsto Q(U)^\bullet$ defines an equivalence of categories $\text{CoSimp}(\mathcal{A}) \to \text{CoCh}_{\geq 0}(\mathcal{A})$.

Proof. Omitted. But the results are the exact dual statements to Lemmas 14.23.1, 14.23.4, 14.23.6, 14.23.8, 14.23.9, and Theorem 14.24.3. □

14.26. Homotopies

019J Consider the simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \longrightarrow \Delta[1],$$

coming from the morphisms $[0] \to [1]$ mapping 0 to an element of $[1] = \{0, 1\}$. Recall also that each set $\Delta[1]_k$ is finite. Hence, if the category $\mathcal{C}$ has finite coproducts, then we can form the product

$$U \times \Delta[1]$$

for any simplicial object U of $\mathcal{C}$, see Definition 14.13.1. Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \to U \times \Delta[1].$$

019K Definition 14.26.1. Let $\mathcal{C}$ be a category having finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms.

(1) We say a morphism
$$h : U \times \Delta[1] \longrightarrow V$$
is a homotopy from a to b if $a = h \circ e_0$ and $b = h \circ e_1$.
(2) We say the morphisms a and b are homotopic or are in the same homotopy class if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}.

The relation "there is a homotopy from a to b" is in general not transitive or symmetric; we will see it is reflexive in Example 14.26.3. Of course, "being homotopic" is an equivalence relation on the set $\mathrm{Mor}(U, V)$ and it is the equivalence relation generated by the relation "there is a homotopy from a to b" . It turns out we can define homotopies between pairs of maps of simplicial objects in any category. We will do this in Remark 14.26.4 after we work out in some detail what it means to have a morphism $h : U \times \Delta[1] \to V$.

Let $\mathcal{C}$ be a category with finite coproducts. Let U, V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy from a to b. For every $n \geq 0$ let us write

$$\Delta[1]_n = \{\alpha^n_0, \ldots, \alpha^n_{n+1}\}$$

where $\alpha^n_i : [n] \to [1]$ is the map such that

$$\alpha^n_i(j) = \begin{cases} 0 & \text{if} \quad j < i \\ 1 & \text{if} \quad j \geq i \end{cases}$$

Thus

$$h_n : (U \times \Delta[1])_n = \coprod U_n \cdot \alpha^n_i \longrightarrow V_n$$

has a component $h_{n,i} : U_n \to V_n$ which is the restriction to the summand corresponding to α^n_i for all $i = 0, \ldots, n + 1$.

019L Lemma 14.26.2. In the situation above, we have the following relations:

(1) We have $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$.
(2) We have $d^n_j \circ h_{n,i} = h_{n-1,i-1} \circ d^n_j$ for $i > j$.
(3) We have $d^n_j \circ h_{n,i} = h_{n-1,i} \circ d^n_j$ for $i \leq j$.
(4) We have $s^n_j \circ h_{n,i} = h_{n+1,i+1} \circ s^n_j$ for $i > j$.
(5) We have $s^n_j \circ h_{n,i} = h_{n+1,i} \circ s^n_j$ for $i \leq j$.

Conversely, given a system of maps $h_{n,i}$ satisfying the properties listed above, then these define a morphism h which is a homotopy from a to b.

Proof. Omitted. You can prove the last statement using the fact, see Lemma 14.2.4 that to give a morphism of simplicial objects is the same as giving a sequence of morphisms h_n commuting with all d^n_j and s^n_j. $\square$

07KA Example 14.26.3. Suppose in the situation above $a = b$. Then there is a trivial homotopy from a to b, namely the one with $h_{n,i} = a_n = b_n$.

019M Remark 14.26.4. Let $\mathcal{C}$ be any category (no assumptions whatsoever). Let U and V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{C}$. A homotopy from a to b is given by morphisms[1] $h_{n,i} : U_n \to V_n$, for $n \geq 0$, $i = 0, \ldots, n+1$ satisfying the relations of Lemma 14.26.2. As in Definition 14.26.1 we say the morphisms a and b are homotopic if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}. Clearly, if $F : \mathcal{C} \to \mathcal{C}'$ is any functor and $\{h_{n,i}\}$ is a homotopy from a to b, then $\{F(h_{n,i})\}$ is a homotopy from $F(a)$ to $F(b)$. Similarly, if a and b are homotopic, then $F(a)$ and $F(b)$ are homotopic. Since the lemma says that the newer notion is the same as the old one in case finite coproduct exist, we deduce in particular that functors preserve the original notion whenever both categories have finite coproducts.

08RJ Remark 14.26.5. Let $\mathcal{C}$ be any category. Suppose two morphisms $a, a' : U \to V$ of simplicial objects are homotopic. Then for any morphism $b : V \to W$ the two maps $b \circ a, b \circ a' : U \to W$ are homotopic. Similarly, for any morphism $c : X \to U$ the two maps $a \circ c, a' \circ c : X \to V$ are homotopic. In fact the maps $b \circ a \circ c, b \circ a' \circ c : X \to W$ are homotopic. Namely, if the maps $h_{n,i} : U_n \to V_n$ define a homotopy from a to a' then the maps $b \circ h_{n,i} \circ c$ define a homotopy from $b \circ a \circ c$ to $b \circ a' \circ c$. In this way we see that we obtain a new category $\text{hSimp}(\mathcal{C})$ with the same objects as $\text{Simp}(\mathcal{C})$ but whose morphisms are homotopy classes of of morphisms of $\text{Simp}(\mathcal{C})$. Thus there is a canonical functor

$$\text{Simp}(\mathcal{C}) \longrightarrow \text{hSimp}(\mathcal{C})$$

which is essentially surjective and surjective on sets of morphisms.

019N Definition 14.26.6. Let U and V be two simplicial objects of a category $\mathcal{C}$. We say a morphism $a : U \to V$ is a homotopy equivalence if there exists a morphism $b : V \to U$ such that $a \circ b$ is homotopic to id_V and $b \circ a$ is homotopic to id_U. We say U and V are homotopy equivalent if there exists a homotopy equivalence $a : U \to V$.

08Q3 Example 14.26.7. The simplicial set $\Delta[m]$ is homotopy equivalent to $\Delta[0]$. Namely, consider the unique morphism $f : \Delta[m] \to \Delta[0]$ and the morphism $g : \Delta[0] \to \Delta[m]$ given by the inclusion of the last 0-simplex of $\Delta[m]$. We have $f \circ g = \text{id}$. We will give a homotopy $h : \Delta[m] \times \Delta[1] \to \Delta[m]$ from $\text{id}_{\Delta[m]}$ to $g \circ f$. Namely h is given by the maps

$$\text{Mor}_\Delta([n], [m]) \times \text{Mor}_\Delta([n], [1]) \to \text{Mor}_\Delta([n], [m])$$

which send (φ, α) to

$$k \mapsto \begin{cases} \varphi(k) & \text{if} \quad \alpha(k) = 0 \\ m & \text{if} \quad \alpha(k) = 1 \end{cases}$$

Note that this only works because we took g to be the inclusion of the last 0-simplex. If we took g to be the inclusion of the first 0-simplex we could find a homotopy from $g \circ f$ to $\text{id}_{\Delta[m]}$. This is an illustration of the asymmetry inherent in homotopies in the category of simplicial sets.

[1]In the literature, often the maps $h_{n+1,i} \circ s_i : U_n \to V_{n+1}$ are used instead of the maps $h_{n,i}$. Of course the relations these maps satisfy are different from the ones in Lemma 14.26.2.

The following lemma says that $U \times \Delta[1]$ is homotopy equivalent to U.

019O **Lemma 14.26.8.** Let $\mathcal{C}$ be a category with finite coproducts. Let U be a simplicial object of $\mathcal{C}$. Consider the maps $e_1, e_0 : U \to U \times \Delta[1]$, and $\pi : U \times \Delta[1] \to U$, see Lemma 14.13.3.

(1) We have $\pi \circ e_1 = \pi \circ e_0 = \mathrm{id}_U$, and
(2) The morphisms $\mathrm{id}_{U \times \Delta[1]}$, and $e_0 \circ \pi$ are homotopic.
(3) The morphisms $\mathrm{id}_{U \times \Delta[1]}$, and $e_1 \circ \pi$ are homotopic.

Proof. The first assertion is trivial. For the second, consider the map of simplicial sets $\Delta[1] \times \Delta[1] \longrightarrow \Delta[1]$ which in degree n assigns to a pair (β_1, β_2), $\beta_i : [n] \to [1]$ the morphism $\beta : [n] \to [1]$ defined by the rule

$$\beta(i) = \max\{\beta_1(i), \beta_2(i)\}.$$

It is a morphism of simplicial sets, because the action $\Delta[1](\varphi) : \Delta[1]_n \to \Delta[1]_m$ of $\varphi : [m] \to [n]$ is by precomposing. Clearly, using notation from Section 14.26, we have $\beta = \beta_1$ if $\beta_2 = \alpha^n_0$ and $\beta = \alpha^n_{n+1}$ if $\beta_2 = \alpha^n_{n+1}$. This implies easily that the induced morphism

$$U \times \Delta[1] \times \Delta[1] \longrightarrow U \times \Delta[1]$$

of Lemma 14.13.3 is a homotopy from $\mathrm{id}_{U \times \Delta[1]}$ to $e_0 \circ \pi$. Similarly for $e_1 \circ \pi$ (use minimum instead of maximum). $\square$

019P **Lemma 14.26.9.** Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with fibre products. Assume f has a section s. Consider the simplicial object U constructed in Example 14.3.5 starting with f. The morphism $U \to U$ which in each degree is the self map $(s \circ f)^{n+1}$ of $Y \times_X \ldots \times_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant simplicial object X.

Proof. Set $g^0 = \mathrm{id}_Y$ and $g^1 = s \circ f$. We use the morphisms

$$\begin{array}{rcl} Y \times_X \ldots \times_X Y \times \mathrm{Mor}([n], [1]) & \to & Y \times_X \ldots \times_X Y \\ (y_0, \ldots, y_n) \times \alpha & \mapsto & (g^{\alpha(0)}(y_0), \ldots, g^{\alpha(n)}(y_n)) \end{array}$$

where we use the functor of points point of view to define the maps. Another way to say this is to say that $h_{n,0} = \mathrm{id}$, $h_{n,n+1} = (s \circ f)^{n+1}$ and $h_{n,i} = \mathrm{id}_Y^{i+1} \times (s \circ f)^{n+1-i}$. We leave it to the reader to show that these satisfy the relations of Lemma 14.26.2. Hence they define the desired homotopy. See also Remark 14.26.4 which shows that we do not need to assume anything else on the category $\mathcal{C}$. $\square$

08Q4 **Lemma 14.26.10.** Let $\mathcal{C}$ be a category. Let T be a set. For $t \in T$ let X_t, Y_t be simplicial objects of $\mathcal{C}$. Assume $X = \prod_{t \in T} X_t$ and $Y = \prod_{t \in T} Y_t$ exist.

(1) If X_t and Y_t are homotopy equivalent for all $t \in T$ and T is finite, then X and Y are homotopy equivalent.

For $t \in T$ let $a_t, b_t : X_t \to Y_t$ be morphisms. Set $a = \prod a_t : X \to Y$ and $b = \prod b_t : X \to Y$.

(2) If there exists a homotopy from a_t to b_t for all $t \in T$, then there exists a homotopy from a to b.
(3) If T is finite and $a_t, b_t : X_t \to Y_t$ for $t \in T$ are homotopic, then a and b are homotopic.

Proof. If $h_t = (h_{t,n,i})$ is a homotopy from a_t to b_t (see Remark 14.26.4), then $h = (\prod_t h_{t,n,i})$ is a homotopy from $\prod a_t$ to $\prod b_t$. This proves (2).

Proof of (3). Choose $t \in T$. There exists an integer $n \geq 0$ and a chain $a_t = a_{t,0}, a_{t,1}, \ldots, a_{t,n} = b_t$ such that for every $1 \leq i \leq n$ either there is a homotopy from $a_{t,i-1}$ to $a_{t,i}$ or there is a homotopy from $a_{t,i}$ to $a_{t,i-1}$. If $n = 0$, then we pick another t. (We're done if $a_t = b_t$ for all $t \in T$.) So assume $n > 0$. By Example 14.26.3 there are is a homotopy from $b_{t'}$ to $b_{t'}$ for all $t' \in T \setminus \{t\}$. Thus by (2) there is a homotopy from $a_{t,n-1} \times \prod_{t'} b_{t'}$ to b or there is a homotopy from b to $a_{t,n-1} \times \prod_{t'} b_{t'}$. In this way we can decrease n by 1. This proves (3).

Part (1) follows from part (3) and the definitions. □

14.27. Homotopies in abelian categories

019Q Let $\mathcal{A}$ be an additive category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy from a to b. Let us prove the two morphisms of chain complexes $s(a), s(b) : s(U) \longrightarrow s(V)$ are homotopic in the sense of Homology, Section 12.13. Using the notation introduced in Section 14.26 we define

$$s(h)_n : U_n \longrightarrow V_{n+1}$$

by the formula

019R (14.27.0.1)
$$s(h)_n = \sum\nolimits_{i=0}^{n} (-1)^{i+1} h_{n+1,i+1} \circ s_i^n.$$

Let us compute $d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n$. We first compute

$$\begin{aligned}
d_{n+1} \circ s(h)_n &= \sum\nolimits_{j=0}^{n+1} \sum\nolimits_{i=0}^{n} (-1)^{j+i+1} d_j^{n+1} \circ h_{n+1,i+1} \circ s_i^n \\
&= \sum\nolimits_{1 \leq i+1 \leq j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \circ d_j^{n+1} \circ s_i^n \\
&\quad + \sum\nolimits_{n \geq i \geq j \geq 0} (-1)^{i+j+1} h_{n,i} \circ d_j^{n+1} \circ s_i^n \\
&= \sum\nolimits_{1 \leq i+1 < j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \circ s_i^{n-1} \circ d_{j-1}^n \\
&\quad + \sum\nolimits_{1 \leq i+1 = j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \\
&\quad + \sum\nolimits_{n \geq i = j \geq 0} (-1)^{i+j+1} h_{n,i} \\
&\quad + \sum\nolimits_{n \geq i > j \geq 0} (-1)^{i+j+1} h_{n,i} \circ s_{i-1}^{n-1} \circ d_j^n
\end{aligned}$$

We leave it to the reader to see that the first and the last of the four sums cancel exactly against all the terms of

$$s(h)_{n-1} \circ d_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+1+j} h_{n,i+1} \circ s_i^{n-1} \circ d_j^n.$$

Hence we obtain

$$\begin{aligned}
d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n &= \sum_{j=1}^{n+1} (-1)^{2j} h_{n,j} + \sum_{i=0}^{n} (-1)^{2i+1} h_{n,i} \\
&= h_{n,n+1} - h_{n,0} \\
&= a_n - b_n
\end{aligned}$$

as desired.

019S Lemma 14.27.1. Let $\mathcal{A}$ be an additive category. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$ are homotopic maps of chain complexes. If $\mathcal{A}$ is abelian, then also $N(a), N(b) : N(U) \to N(V)$ are homotopic maps of chain complexes.

Proof. We may choose a sequence $a = a_0, a_1, \ldots, a_n = b$ of morphisms from U to V such that for each $i = 1, \ldots, n$ either there is a homotopy from a_i to a_{i-1} or there is a homotopy from a_{i-1} to a_i. The calculation above shows that in this case either $s(a_i)$ is homotopic to $s(a_{i-1})$ as a map of chain complexes or $s(a_{i-1})$ is homotopic to $s(a_i)$ as a map of chain complexes. Of course, these things are equivalent and moreover being homotopic is an equivalence relation on the set of maps of chain complexes, see Homology, Section 12.13. This proves that $s(a)$ and $s(b)$ are homotopic as maps of chain complexes.

Next, we turn to $N(a)$ and $N(b)$. It follows from Lemma 14.23.6 that $N(a)$, $N(b)$ are compositions

$$N(U) \to s(U) \to s(V) \to N(V)$$

where we use $s(a)$, $s(b)$ in the middle. Hence the assertion follows from Homology, Lemma 12.13.1. □

019T Lemma 14.27.2. Let $\mathcal{A}$ be an additive category. Let $a : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$ is a homotopy equivalence of chain complexes. If in addition $\mathcal{A}$ is abelian, then also $N(a) : N(U) \to N(V)$ is a homotopy equivalence of chain complexes.

Proof. Omitted. See Lemma 14.27.1 above. □

14.28. Homotopies and cosimplicial objects

019U Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object and consider $\mathrm{Hom}(\Delta[1], V)$, see Section 14.14. The morphisms $e_0, e_1 : \Delta[0] \to \Delta[1]$ produce two morphisms $e_0, e_1 : \mathrm{Hom}(\Delta[1], V) \to V$.

019W Definition 14.28.1. Let $\mathcal{C}$ be a category having finite products. Let U and V be two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms of cosimplicial objects of $\mathcal{C}$.

(1) We say a morphism

$$h : U \longrightarrow \mathrm{Hom}(\Delta[1], V)$$

such that $a = e_0 \circ h$ and $b = e_1 \circ h$ is a homotopy from a to b.

(2) We say a and b are homotopic or are in the same homotopy class if there exists a sequence $a = a_0, a_1, \ldots, a_n = b$ of morphisms from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_i to a_{i-1} or there exists a homotopy from a_{i-1} to a_i.

This is dual to the notion we introduced for simplicial objects in Section 14.26. To explain this, consider a homotopy $h : U \to \mathrm{Hom}(\Delta[1], V)$ from a to b as in the definition. Recall that $\Delta[1]_n$ is a finite set. The degree n component of h is a morphism

$$h_n = (h_{n,\alpha}) : U \longrightarrow \mathrm{Hom}(\Delta[1], V)_n = \prod\nolimits_{\alpha \in \Delta[1]_n} V_n$$

The morphisms $h_{n,\alpha} : U_n \to V_n$ of $\mathcal{C}$ have the property that for every morphism $f : [n] \to [m]$ of Δ we have

07KB (14.28.1.1)
$$h_{m,\alpha} \circ U(f) = V(f) \circ h_{n,\alpha \circ f}$$

Moreover, the condition that $a = e_0 \circ h$ means that $a_n = h_{n,0:[n]\to[1]}$ where $0 : [n] \to [1]$ is the constant map with value 0. Similarly, the condition that $b = e_1 \circ h$ means that $b_n = h_{n,1:[n]\to[1]}$ where $1 : [n] \to [1]$ is the constant map with value 1. Conversly, given a family of morphisms $\{h_{n,\alpha}\}$ such that (14.28.1.1) holds for all morphisms f of Δ and such that $a_n = h_{n,0:[n]\to[1]}$ and $b_n = h_{n,1:[n]\to[1]}$ for all $n \geq 0$, then we obtain a homotopy h from a to b by setting $h = \prod_{\alpha \in \Delta[1]_n} h_{n,\alpha}$.

0FKJ Remark 14.28.2. Let $\mathcal{C}$ be any category (no assumptions whatsoever). Let U and V be cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. A homotopy from a to b is given by morphisms $h_{n,\alpha} : U_n \to V_n$, for $n \geq 0$, $\alpha \in \Delta[1]_n$ satisfying (14.28.1.1) for all morphisms f of Δ and such that $a_n = h_{n,0:[n]\to[1]}$ and $b_n = h_{n,1:[n]\to[1]}$ for all $n \geq 0$. As in Definition 14.28.1 we say the morphisms a and b are homotopic if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each $i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}. Clearly, if $F : \mathcal{C} \to \mathcal{C}'$ is any functor and $\{h_{n,i}\}$ is a homotopy from a to b, then $\{F(h_{n,i})\}$ is a homotopy from $F(a)$ to $F(b)$. Similarly, if a and b are homotopic, then $F(a)$ and $F(b)$ are homotopic. This new notion is the same as the old one in case finite products exist. We deduce in particular that functors preserve the original notion whenever both categories have finite products.

019X Lemma 14.28.3. Let $\mathcal{C}$ be a category. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects. Recall that U, V correspond to simplicial objects U', V' of $\mathcal{C}^{opp}$. Moreover a, b correspond to morphisms $a', b' : V' \to U'$. The following are equivalent

(1) There exists a homotopy $h = \{h_{n,\alpha}\}$ from a to b as in Remark 14.28.2.
(2) There exists a homotopy $h = \{h_{n,i}\}$ from a' to b' as in Remark 14.26.4.

Thus a is homotopic to b as in Remark 14.28.2 if and only if a' is homotopic to b' as in Remark 14.26.4.

Proof. In case $\mathcal{C}$ has finite products, then $\mathcal{C}^{opp}$ has finite coproducts and we may use Definitions 14.28.1 and 14.26.1 instead of Remarks 14.28.2 and 14.26.4. In this case $h : U \to \text{Hom}(\Delta[1], V)$ is the same as a morphism $h' : \text{Hom}(\Delta[1], V)' \to U'$. Since products and coproducts get switched too, it is immediate that $(\text{Hom}(\Delta[1], V))' = V' \times \Delta[1]$. Moreover, the "primed" version of the morphisms $e_0, e_1 : \text{Hom}(\Delta[1], V) \to V$ are the morphisms $e_0, e_1 : V' \to \Delta[1] \times V$. Thus $e_0 \circ h = a$ translates into $h' \circ e_0 = a'$ and similarly $e_1 \circ h = b$ translates into $h' \circ e_1 = b'$. This proves the lemma in this case.

In the general case, one needs to translate the relations given by (14.28.1.1) into the relations given in Lemma 14.26.2. We omit the details.

The final assertion is formal from the equivalence of (1) and (2). $\square$

019Y Lemma 14.28.4. Let $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}'$ be categories. With terminology as in Remarks 14.28.2 and 14.26.4.

(1) Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{D}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of simplicial objects.
(2) Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of cosimplicial objects.
(3) Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{C}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of cosimplicial objects.
(4) Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of simplicial objects.

Proof. By Lemma 14.28.3 above, we can turn F into a covariant functor between a pair of categories, and we have to show that the functor preserves homotopic pairs of maps. This is explained in Remark 14.26.4. □

019Z Lemma 14.28.5. Let $f : X \to Y$ be a morphism of a category $\mathcal{C}$ with pushouts. Assume there is a morphism $s : Y \to X$ with $s \circ f = \text{id}_X$. Consider the cosimplicial object U constructed in Example 14.5.5 starting with f. The morphism $U \to U$ which in each degree is the self map of $Y \amalg_X \ldots \amalg_X Y$ given by $f \circ s$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant cosimplicial object X.

Proof. This lemma is dual to Lemma 14.26.9. Hence this lemma follows on applying Lemma 14.28.3. □

01A0 Lemma 14.28.6. Let $\mathcal{A}$ be an additive category. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$ are homotopic maps of cochain complexes. If in addition $\mathcal{A}$ is abelian, then $Q(a), Q(b) : Q(U) \to Q(V)$ are homotopic maps of cochain complexes.

Proof. Let $(-)' : \mathcal{A} \to \mathcal{A}^{opp}$ be the contravariant functor $A \mapsto A$. By Lemma 14.28.5 the maps a' and b' are homotopic. By Lemma 14.27.1 we see that $s(a')$ and $s(b')$ are homotopic maps of chain complexes. Since $s(a') = (s(a))'$ and $s(b') = (s(b))'$ we conclude that also $s(a)$ and $s(b)$ are homotopic by applying the additive contravariant functor $(-)'' : \mathcal{A}^{opp} \to \mathcal{A}$. The result for the Q-complexes follows in the same manner using that $Q(U)' = N(U')$. □

0FKK Lemma 14.28.7. Let $\mathcal{A}$ be an additive category. Let $a : U \to V$ be a morphism of cosimplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$ is a homotopy equivalence of chain complexes. If in addition $\mathcal{A}$ is abelian, then also $Q(a) : Q(U) \to Q(V)$ is a homotopy equivalence of chain complexes.

Proof. Omitted. See Lemma 14.28.6 above. □

14.29. More homotopies in abelian categories

01A1 Let $\mathcal{A}$ be an abelian category. In this section we show that a homotopy between morphisms in $\text{Ch}_{\geq 0}(\mathcal{A})$ always comes from a morphism $U \times \Delta[1] \to V$ in the category of simplicial objects. In some sense this will provide a converse to Lemma 14.27.1. We first develop some material on homotopies between morphisms of chain complexes.

01A2 **Lemma 14.29.1.** Let $\mathcal{A}$ be an abelian category. Let A be a chain complex. Consider the covariant functor

$$B \longmapsto \{(a, b, h) \mid a, b : A \to B \text{ and } h \text{ a homotopy between } a, b\}$$

There exists a chain complex $\diamond A$ such that $\mathrm{Mor}_{\mathrm{Ch}(\mathcal{A})}(\diamond A, -)$ is isomorphic to the displayed functor. The construction $A \mapsto \diamond A$ is functorial.

Proof. We set $\diamond A_n = A_n \oplus A_n \oplus A_{n-1}$, and we define $d_{\diamond A,n}$ by the matrix

$$d_{\diamond A,n} = \begin{pmatrix} d_{A,n} & 0 & \mathrm{id}_{A_{n-1}} \\ 0 & d_{A,n} & -\mathrm{id}_{A_{n-1}} \\ 0 & 0 & -d_{A,n-1} \end{pmatrix} : A_n \oplus A_n \oplus A_{n-1} \to A_{n-1} \oplus A_{n-1} \oplus A_{n-2}$$

If $\mathcal{A}$ is the category of abelian groups, and $(x, y, z) \in A_n \oplus A_n \oplus A_{n-1}$ then $d_{\diamond A,n}(x, y, z) = (d_n(x) + z, d_n(y) - z, -d_{n-1}(z))$. It is easy to verify that $d^2 = 0$. Clearly, there are two maps $\diamond a, \diamond b : A \to \diamond A$ (first summand and second summand), and a map $\diamond A \to A[-1]$ which give a short exact sequence

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

which is termwise split. Moreover, there is a sequence of maps $\diamond h_n : A_n \to \diamond A_{n+1}$, namely the identity from A_n to the summand A_n of $\diamond A_{n+1}$, such that $\diamond h$ is a homotopy between $\diamond a$ and $\diamond b$.

We conclude that any morphism $f : \diamond A \to B$ gives rise to a triple (a, b, h) by setting $a = f \circ \diamond a$, $b = f \circ \diamond b$ and $h_n = f_{n+1} \circ \diamond h_n$. Conversely, given a triple (a, b, h) we get a morphism $f : \diamond A \to B$ by taking

$$f_n = (a_n, b_n, h_{n-1}).$$

To see that this is a morphism of chain complexes you have to do a calculation. We only do this in case $\mathcal{A}$ is the category of abelian groups: Say $(x, y, z) \in \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$. Then

$$\begin{aligned} f_{n-1}(d_n(x, y, z)) &= f_{n-1}(d_n(x) + z, d_n(y) - z, -d_{n-1}(z)) \\ &= a_n(d_n(x)) + a_n(z) + b_n(d_n(y)) - b_n(z) - h_{n-2}(d_{n-1}(z)) \end{aligned}$$

and

$$\begin{aligned} d_n(f_n(x, y, z) &= d_n(a_n(x) + b_n(y) + h_{n-1}(z)) \\ &= d_n(a_n(x)) + d_n(b_n(y)) + d_n(h_{n-1}(z)) \end{aligned}$$

which are the same by definition of a homotopy. $\square$

Note that the extension

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

comes with sections of the morphisms $\diamond A_n \to A[-1]_n$ with the property that the associated morphism $\delta : A[-1] \to (A \oplus A)[-1]$, see Homology, Lemma 12.14.4 equals the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$.

01A3 **Lemma 14.29.2.** Let $\mathcal{A}$ be an abelian category. Let

$$0 \to A \oplus A \to B \to C \to 0$$

be a short exact sequence of chain complexes of $\mathcal{A}$. Suppose given in addition morphisms $s_n : C_n \to B_n$ splitting the associated short exact sequence in degree n. Let $\delta(s) : C \to (A \oplus A)[-1] = A[-1] \oplus A[-1]$ be the associated morphism of complexes, see Homology, Lemma 12.14.4. If $\delta(s)$ factors through the morphism $(1,-1) : A[-1] \to A[-1] \oplus A[-1]$, then there is a unique morphism $B \to \diamond A$ fitting into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \oplus A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \oplus A \arrow[r] & \diamond A \arrow[r] & A[-1] \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are compatible with the splittings s_n and the splittings of $\diamond A_n \to A[-1]_n$ as well.

Proof. Denote $(p_n, q_n) : B_n \to A_n \oplus A_n$ the morphism π_n of Homology, Lemma 12.14.4. Also write $(a, b) : A \oplus A \to B$, and $r : B \to C$ for the maps in the short exact sequence. Write the factorization of $\delta(s)$ as $\delta(s) = (1,-1) \circ f$. This means that $p_{n-1} \circ d_{B,n} \circ s_n = f_n$, and $q_{n-1} \circ d_{B,n} \circ s_n = -f_n$, and Set $B_n \to \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$ equal to $(p_n, q_n, f_n \circ r_n)$.

Now we have to check that this actually defines a morphism of complexes. We will only do this in the case of abelian groups. Pick $x \in B_n$. Then $x = a_n(x_1) + b_n(x_2) + s_n(x_3)$ and it suffices to show that our definition commutes with differential for each term separately. For the term $a_n(x_1)$ we have $(p_n, q_n, f_n \circ r_n)(a_n(x_1)) = (x_1, 0, 0)$ and the result is obvious. Similarly for the term $b_n(x_2)$. For the term $s_n(x_3)$ we have

$$\begin{aligned} (p_n, q_n, f_n \circ r_n)(d_n(s_n(x_3))) &= (p_n, q_n, f_n \circ r_n)(\\ &\qquad a_n(f_n(x_3)) - b_n(f_n(x_3)) + s_n(d_n(x_3))) \\ &= (f_n(x_3), -f_n(x_3), f_n(d_n(x_3))) \end{aligned}$$

by definition of f_n. And

$$\begin{aligned} d_n(p_n, q_n, f_n \circ r_n)(s_n(x_3)) &= d_n(0, 0, f_n(x_3)) \\ &= (f_n(x_3), -f_n(x_3), d_{A[-1],n}(f_n(x_3))) \end{aligned}$$

The result follows as f is a morphism of complexes. □

01A4 Lemma 14.29.3. Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be a pair of morphisms. Assume the corresponding maps of chain complexes $N(a), N(b) : N(U) \to N(V)$ are homotopic by a homotopy $\{N_n : N(U)_n \to N(V)_{n+1}\}$. Then there exists a homotopy from a to b as in Definition 14.26.1. Moreover, one can choose the homotopy $h : U \times \Delta[1] \to V$ such that $N_n = N(h)_n$ where $N(h)$ is the homotopy coming from h as in Section 14.27.

Proof. Let $(\diamond N(U), \diamond a, \diamond b, \diamond h)$ be as in Lemma 14.29.1 and its proof. By that lemma there exists a morphism $\diamond N(U) \to N(V)$ representing the triple $(N(a), N(b), \{N_n\})$. We will show there exists a morphism $\psi : N(U \times \Delta[1]) \to \diamond N(U)$ such that $\diamond a = \psi \circ N(e_0)$, and $\diamond b = \psi \circ N(e_1)$. Moreover, we will show that the homotopy between $N(e_0), N(e_1) : N(U) \to N(U \times \Delta[1])$ coming from (14.27.0.1) and Lemma 14.27.1 with $h = \text{id}_{U \times \Delta[1]}$ is mapped via ψ to the canonical homotopy $\diamond h$ between the two maps $\diamond a, \diamond b : N(U) \to \diamond N(U)$. Certainly this will imply the lemma.

Note that $N : \text{Simp}(\mathcal{A}) \to \text{Ch}_{\geq 0}(\mathcal{A})$ as a functor is a direct summand of the functor $s : \text{Simp}(\mathcal{A}) \to \text{Ch}_{\geq 0}(\mathcal{A})$. Also, the functor $\diamond$ is compatible with direct sums. Thus it suffices instead to construct a morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$ with the corresponding properties. This is what we do below.

By Definition 14.26.1 the morphisms $e_0 : U \to U \times \Delta[1]$ and $e_1 : U \to U \times \Delta[1]$ are homotopic with homotopy $\text{id}_{U\times\Delta[1]}$. By Lemma 14.27.1 we get an explicit homotopy $\{h_n : s(U)_n \to s(U \times \Delta[1])_{n+1}\}$ between the morphisms of chain complexes $s(e_0) : s(U) \to s(U \times \Delta[1])$ and $s(e_1) : s(U) \to s(U \times \Delta[1])$. By Lemma 14.29.2 above we get a corresponding morphism

$$\Phi : \diamond s(U) \to s(U \times \Delta[1])$$

According to the construction, Φ_n restricted to the summand $s(U)[-1]_n = s(U)_{n-1}$ of $\diamond s(U)_n$ is equal to h_{n-1}. And

$$h_{n-1} = \sum\nolimits_{i=0}^{n-1} (-1)^{i+1} s_i^n \cdot \alpha_{i+1}^n : U_{n-1} \to \bigoplus\nolimits_j U_n \cdot \alpha_j^n.$$

with obvious notation.

On the other hand, the morphisms $e_i : U \to U \times \Delta[1]$ induce a morphism $(e_0, e_1) : U \oplus U \to U \times \Delta[1]$. Denote W the cokernel. Note that, if we write $(U \times \Delta[1])_n = \bigoplus_{\alpha:[n]\to[1]} U_n \cdot \alpha$, then we may identify $W_n = \bigoplus_{i=1}^n U_n \cdot \alpha_i^n$ with α_i^n as in Section 14.26. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \oplus U \arrow[r] \arrow[dr, "{(1,1)}"'] & U \times \Delta[1] \arrow[r] \arrow[d, "\pi"] & W \arrow[r] & 0 \\ & & U & & \end{tikzcd}$$

This implies we have a similar commutative diagram after applying the functor s. Next, we choose the splittings $\sigma_n : s(W)_n \to s(U\times\Delta[1])_n$ by mapping the summand $U_n \cdot \alpha_i^n \subset W_n$ via $(-1, 1)$ to the summands $U_n \cdot \alpha_0^n \oplus U_n \cdot \alpha_i^n \subset (U \times \Delta[1])_n$. Note that $s(\pi)_n \circ \sigma_n = 0$. It follows that $(1, 1) \circ \delta(\sigma)_n = 0$. Hence $\delta(\sigma)$ factors as in Lemma 14.29.2. By that lemma we obtain a canonical morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$.

To compute Ψ we first compute the morphism $\delta(\sigma) : s(W) \to s(U)[-1] \oplus s(U)[-1]$. According to Homology, Lemma 12.14.4 and its proof, to do this we have compute

$$d_{s(U\times\delta[1]),n} \circ \sigma_n - \sigma_{n-1} \circ d_{s(W),n}$$

and write it as a morphism into $U_{n-1} \cdot \alpha_0^{n-1} \oplus U_{n-1} \cdot \alpha_n^{n-1}$. We only do this in case $\mathcal{A}$ is the category of abelian groups. We use the short hand notation x_α for $x \in U_n$ to denote the element x in the summand $U_n \cdot \alpha$ of $(U \times \Delta[1])_n$. Recall that

$$d_{s(U\times\delta[1]),n} = \sum\nolimits_{i=0}^n (-1)^i d_i^n$$

where d_i^n maps the summand $U_n \cdot \alpha$ to the summand $U_{n-1} \cdot (\alpha \circ \delta_i^n)$ via the morphism d_i^n of the simplicial object U. In terms of the notation above this means

$$d_{s(U\times\delta[1]),n}(x_\alpha) = \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n}$$

Starting with $x_\alpha \in W_n$, in other words $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$, we see that $\sigma_n(x_\alpha) = x_\alpha - x_{\alpha_0^n}$ and hence

$$(d_{s(U\times\delta[1]),n} \circ \sigma_n)(x_\alpha) = \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - \sum\nolimits_{i=0}^n (-1)^i (d_i^n(x))_{\alpha_0^n\circ\delta_i^n}$$

To compute $d_{s(W),n}(x_\alpha)$, we have to omit all terms where $\alpha \circ \delta^n_i = \alpha^{n-1}_0, \alpha^{n-1}_n$. Hence we get

$$(\sigma_{n-1} \circ d_{s(W),n})(x_\alpha) = \\ \sum\nolimits_{i=0,\ldots,n \text{ and } \alpha\circ\delta^n_i \neq \alpha^{n-1}_0 \text{ or } \alpha^{n-1}_n} \left((-1)^i (d^n_i(x))_{\alpha\circ\delta^n_i} - (-1)^i (d^n_i(x))_{\alpha^{n-1}_0}\right)$$

Clearly the difference of the two terms is the sum

$$\sum\nolimits_{i=0,\ldots,n \text{ and } \alpha\circ\delta^n_i = \alpha^{n-1}_0 \text{ or } \alpha^{n-1}_n} \left((-1)^i (d^n_i(x))_{\alpha\circ\delta^n_i} - (-1)^i (d^n_i(x))_{\alpha^{n-1}_0}\right)$$

Of course, if $\alpha \circ \delta^n_i = \alpha^{n-1}_0$ then the term drops out. Recall that $\alpha = \alpha^n_j$ for some $j \in \{1, \ldots, n\}$. The only way $\alpha^n_j \circ \delta^n_i = \alpha^{n-1}_n$ is if $j = n$ and $i = n$. Thus we actually get 0 unless $j = n$ and in that case we get $(-1)^n (d^n_n(x))_{\alpha^{n-1}_n} - (-1)^n (d^n_n(x))_{\alpha^{n-1}_0}$. In other words, we conclude the morphism

$$\delta(\sigma)_n : W_n \to (s(U)[-1] \oplus s(U)[-1])_n = U_{n-1} \oplus U_{n-1}$$

is zero on all summands except $U_n \cdot \alpha^n_n$ and on that summand it is equal to $((-1)^n d^n_n, -(-1)^n d^n_n)$. (Namely, the first summand of the two corresponds to the factor with α^{n-1}_n because that is the map $[n-1] \to [1]$ which maps everybody to 0, and hence corresponds to e_0.)

We obtain a canonical diagram

$$\begin{tikzcd}
0 \arrow[r] & s(U) \oplus s(U) \arrow[r] \arrow[d] & \diamond s(U) \arrow[r] \arrow[d, "\Phi"] & s(U)[-1] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & s(U) \oplus s(U) \arrow[r] \arrow[d] & s(U \times \Delta[1]) \arrow[r] \arrow[d, "\Psi"] & s(W) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & s(U) \oplus s(U) \arrow[r] & \diamond s(U) \arrow[r] & s(U)[-1] \arrow[r] & 0
\end{tikzcd}$$

We claim that $\Phi \circ \Psi$ is the identity. To see this it is enough to prove that the composition of Φ and $\delta(\sigma)$ as a map $s(U)[-1] \to s(W) \to s(U)[-1] \oplus s(U)[-1]$ is the identity in the first factor and minus identity in the second. By the computations above it is $((-1)^n d^n_0, -(-1)^n d^n_0) \circ (-1)^n s^n_n = (1, -1)$ as desired. $\square$

14.30. Trivial Kan fibrations

08NK Recall that for $n \geq 0$ the simplicial set $\Delta[n]$ is given by the rule $[k] \mapsto \mathrm{Mor}_\Delta([k], [n])$, see Example 14.11.2. Recall that $\Delta[n]$ has a unique nondegenerate n-simplex and all nondegenerate simplices are faces of this n-simplex. In fact, the nondegenerate simplices of $\Delta[n]$ correspond exactly to injective morphisms $[k] \to [n]$, which we may identify with subsets of $[n]$. Moreover, recall that $\mathrm{Mor}(\Delta[n], X) = X_n$ for any simplicial set X (Lemma 14.11.3). We set

$$\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$$

and we call it the boundary of $\Delta[n]$. From Lemma 14.21.5 we see that $\partial\Delta[n] \subset \Delta[n]$ is the simplicial subset having the same nondegenerate simplices in degrees $\leq n-1$ but not containing the nondegenerate n-simplex.

08NL Definition 14.30.1. A map $X \to Y$ of simplicial sets is called a trivial Kan fibration if $X_0 \to Y_0$ is surjective and for all $n \geq 1$ and any commutative solid diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

a dotted arrow exists making the diagram commute.

A trivial Kan fibration satisfies a very general lifting property.

08NM Lemma 14.30.2. Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. For any solid commutative diagram

$$\begin{tikzcd} Z \arrow[r, "b"'] \arrow[d] & X \arrow[d] \\ W \arrow[r, "a"] \arrow[ur, dashed] & Y \end{tikzcd}$$

of simplicial sets with $Z \to W$ (termwise) injective a dotted arrow exists making the diagram commute.

Proof. Suppose that $Z \neq W$. Let n be the smallest integer such that $Z_n \neq W_n$. Let $x \in W_n$, $x \notin Z_n$. Denote $Z' \subset W$ the simplicial subset containing Z, x, and all degeneracies of x. Let $\varphi : \Delta[n] \to Z'$ be the morphism corresponding to x (Lemma 14.11.3). Then $\varphi|_{\partial\Delta[n]}$ maps into Z as all the nondegenerate simplices of $\partial\Delta[n]$ end up in Z. By assumption we can extend $b \circ \varphi|_{\partial\Delta[n]}$ to $\beta : \Delta[n] \to X$. By Lemma 14.21.7 the simplicial set Z' is the pushout of $\Delta[n]$ and Z along $\partial\Delta[n]$. Hence b and β define a morphism $b' : Z' \to X$. In other words, we have extended the morphism b to a bigger simplicial subset of Z.

The proof is finished by an application of Zorn's lemma (omitted). $\square$

08NN Lemma 14.30.3. Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a trivial Kan fibration.

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 14.7.2) and the definitions. $\square$

08NP Lemma 14.30.4. The composition of two trivial Kan fibrations is a trivial Kan fibration.

Proof. Omitted. $\square$

08NQ Lemma 14.30.5. Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of trivial Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U_n^t$. Then $U \to U^0$ is a trivial Kan fibration.

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. $\square$

08NR Lemma 14.30.6. Let $X_i \to Y_i$ be a set of trivial Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a trivial Kan fibration.

Proof. Omitted. $\square$

08Q5 **Lemma 14.30.7.** A filtered colimit of trivial Kan fibrations is a trivial Kan fibration.

Proof. Omitted. Hint: See description of filtered colimits of sets in Categories, Section 4.19. ☐

08NS **Lemma 14.30.8.** Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Then f is a homotopy equivalence.

Proof. By Lemma 14.30.2 we can choose an right inverse $g : Y \to X$ to f. Consider the diagram

$$\begin{tikzcd} \partial\Delta[1] \times X \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[1] \times X \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

Here the top horizontal arrow is given by id_X and $g \circ f$ where we use that $(\partial\Delta[1] \times X)_n = X_n \amalg X_n$ for all $n \geq 0$. The bottom horizontal arrow is given by the map $\Delta[1] \to \Delta[0]$ and $f : X \to Y$. The diagram commutes as $f \circ g \circ f = f$. By Lemma 14.30.2 we can fill in the dotted arrow and we win. ☐

14.31. Kan fibrations

08NT Let n, k be integers with $0 \leq k \leq n$ and $1 \leq n$. Let $\sigma_0, \ldots, \sigma_n$ be the $n+1$ faces of the unique nondegenerate n-simplex σ of $\Delta[n]$, i.e., $\sigma_i = d_i\sigma$. We let

$$\Lambda_k[n] \subset \Delta[n]$$

be the kth horn of the n-simplex $\Delta[n]$. It is the simplicial subset of $\Delta[n]$ generated by $\sigma_0, \ldots, \hat{\sigma}_k, \ldots, \sigma_n$. In other words, the image of the displayed inclusion contains all the nondegenerate simplices of $\Delta[n]$ except for σ and σ_k.

08NU **Definition 14.31.1.** A map $X \to Y$ of simplicial sets is called a *Kan fibration* if for all k, n with $1 \leq n$, $0 \leq k \leq n$ and any commutative solid diagram

$$\begin{tikzcd} \Lambda_k[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

a dotted arrow exists making the diagram commute. A *Kan complex* is a simplicial set X such that $X \to *$ is a Kan fibration, where $*$ is the constant simplicial set on a singleton.

Note that $\Lambda_k[n]$ is always nonempty. Thus a morphism from the empty simplicial set to any simplicial set is always a Kan fibration. It follows from Lemma 14.30.2 that a trivial Kan fibration is a Kan fibration.

08NV **Lemma 14.31.2.** Let $f : X \to Y$ be a Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a Kan fibration.

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 14.7.2) and the definitions. ☐

08NW **Lemma 14.31.3.** The composition of two Kan fibrations is a Kan fibration.

Proof. Omitted. ☐

08NX Lemma 14.31.4. Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U^t_n$. Then $U \to U^0$ is a Kan fibration.

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. $\square$

08NY Lemma 14.31.5. Let $X_i \to Y_i$ be a set of Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a Kan fibration.

Proof. Omitted. $\square$

The following lemma is due to J.C. Moore, see [Moo55].

08NZ Lemma 14.31.6. Let X be a simplicial group. Then X is a Kan complex.

Proof. The following proof is basically just a translation into English of the proof in the reference mentioned above. Using the terminology as explained in the introduction to this section, suppose $f : \Lambda_k[n] \to X$ is a morphism from a horn. Set $x_i = f(\sigma_i) \in X_{n-1}$ for $i = 0, \ldots, \hat{k}, \ldots, n$. This means that for $i < j$ we have $d_i x_j = d_{j-1} x_i$ whenever $i, j \neq k$. We have to find an $x \in X_n$ such that $x_i = d_i x$ for $i = 0, \ldots, \hat{k}, \ldots, n$.

We first prove there exists a $u \in X_n$ such that $d_i u = x_i$ for $i < k$. This is trivial for $k = 0$. If $k > 0$, one defines by induction an element $u^r \in X_n$ such that $d_i u^r = x_i$ for $0 \leq i \leq r$. Start with $u^0 = s_0 x_0$. If $r < k - 1$, we set

$$y^r = s_{r+1}((d_{r+1} u^r)^{-1} x_{r+1}), \quad u^{r+1} = u^r y^r.$$

An easy calculation shows that $d_i y^r = 1$ (unit element of the group X_{n-1}) for $i \leq r$ and $d_{r+1} y^r = (d_{r+1} u^r)^{-1} x_{r+1}$. It follows that $d_i u^{r+1} = x_i$ for $i \leq r + 1$. Finally, take $u = u^{k-1}$ to get u as promised.

Next we prove, by induction on the integer r, $0 \leq r \leq n - k$, there exists a $x^r \in X_n$ such that

$$d_i x^r = x_i \quad \text{for } i < k \text{ and } i > n - r.$$

Start with $x^0 = u$ for $r = 0$. Having defined x^r for $r \leq n - k - 1$ we set

$$z^r = s_{n-r-1}((d_{n-r} x^r)^{-1} x_{n-r}), \quad x^{r+1} = x^r z^r$$

A simple calculation, using the given relations, shows that $d_i z^r = 1$ for $i < k$ and $i > n - r$ and that $d_{n-r}(z^r) = (d_{n-r} x^r)^{-1} x_{n-r}$. It follows that $d_i x^{r+1} = x_i$ for $i < k$ and $i > n - r - 1$. Finally, we take $x = x^{n-k}$ which finishes the proof. $\square$

08P0 Lemma 14.31.7. Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective. Then f is a Kan fibration of simplicial sets.

Proof. Consider a commutative solid diagram

$$\begin{tikzcd} \Lambda_k[n] \arrow[r, "a"] \arrow[d] & X \arrow[d] \\ \Delta[n] \arrow[r, "b"] \arrow[ur, dashed] & Y \end{tikzcd}$$

as in Definition 14.31.1. The map a corresponds to $x_0, \ldots, \hat{x}_k, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$, $i, j \neq k$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$ for $i \neq k$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ for $i \neq k$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$ for $i \neq k$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\text{Ker}(f) \subset X$ and Y by 0. In this case the statement become Lemma 14.31.6. □

08P1 Lemma 14.31.8. Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective and induces a quasi-isomorphism on associated chain complexes. Then f is a trivial Kan fibration of simplicial sets.

Proof. Consider a commutative solid diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "a"] \arrow[d] & X \arrow[d] \\ \Delta[n] \arrow[r, "b"] \arrow[ru, dashed] & Y \end{tikzcd}$$

as in Definition 14.30.1. The map a corresponds to $x_0, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\text{Ker}(f) \subset X$ and Y by 0. This works, because by Homology, Lemma 12.13.6 the homology of the chain complex associated to $\text{Ker}(f)$ is zero and hence $\text{Ker}(f) \to 0$ induces a quasi-isomorphism on associated chain complexes.

Since X is a Kan complex (Lemma 14.31.6) we can find $x \in X_n$ with $d_i x = x_i$ for $i = 0, \ldots, n-1$. After replacing x_i by $x_i - d_i x$ for $i = 0, \ldots, n$ we may assume that $x_0 = x_1 = \ldots = x_{n-1} = 0$. In this case we see that $d_i x_n = 0$ for $i = 0, \ldots, n-1$. Thus $x_n \in N(X)_{n-1}$ and lies in the kernel of the differential $N(X)_{n-1} \to N(X)_{n-2}$. Here $N(X)$ is the normalized chain complex associated to X, see Section 14.23. Since $N(X)$ is quasi-isomorphic to $s(X)$ (Lemma 14.23.9) and thus acyclic we find $x \in N(X_n)$ whose differential is x_n. This x answers the question posed by the lemma and we are done. □

08P2 Lemma 14.31.9. Let $f : X \to Y$ be a map of simplicial abelian groups. If f is a homotopy equivalence of simplicial sets, then f induces a quasi-isomorphism of associated chain complexes.

Proof. In this proof we will write $H_n(Z) = H_n(s(Z)) = H_n(N(Z))$ when Z is a simplicial abelian group, with s and N as in Section 14.23. Let $\mathbf{Z}[X]$ denote the free abelian group on X viewed as a simplicial set and similarly for $\mathbf{Z}[Y]$. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{Z}[X] \arrow[r, "g"] \arrow[d] & \mathbf{Z}[Y] \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of simplicial abelian groups. Since taking the free abelian group on a set is a functor, we see that the horizontal arrow is a homotopy equivalence of simplicial abelian groups, see Lemma 14.28.4. By Lemma 14.27.2 we see that $H_n(g) : H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective for all $n \geq 0$.

Let $\xi \in H_n(Y)$. By definition of $N(Y)$ we can represent ξ by an element $y \in N(Y_n)$ whose boundary is zero. This means $y \in Y_n$ with $d^n_0(y) = \ldots = d^n_{n-1}(y) = 0$ because $y \in N(Y_n)$ and $d^n_n(y) = 0$ because the boundary of y is zero. Denote $0_n \in Y_n$ the zero element. Then we see that

$$\tilde{y} = [y] - [0_n] \in (\mathbf{Z}[Y])_n$$

is an element with $d^n_0(\tilde{y}) = \ldots = d^n_{n-1}(\tilde{y}) = 0$ and $d^n_n(\tilde{y}) = 0$. Thus $\tilde{y}$ is in $N(\mathbf{Z}[Y])_n$ has boundary 0, i.e., $\tilde{y}$ determines a class $\tilde{\xi} \in H_n(\mathbf{Z}[Y])$ mapping to ξ. Because $H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective we can lift $\tilde{\xi}$ to a class in $H_n(\mathbf{Z}[X])$. Looking at the commutative diagram above we see that ξ is in the image of $H_n(X) \to H_n(Y)$.

Let $\xi \in H_n(X)$ be an element mapping to zero in $H_n(Y)$. Exactly as in the previous parapgraph we can represent ξ by an element $x \in N(X_n)$ whose boundary is zero, i.e., $d^n_0(x) = \ldots = d^n_{n-1}(x) = d^n_n(x) = 0$. In particular, we see that $[x] - [0_n]$ is an element of $N(\mathbf{Z}[X])_n$ whose boundary is zero, whence defines a lift $\tilde{\xi} \in H_n(\mathbf{Z}[x])$ of ξ. The fact that ξ maps to zero in $H_n(Y)$ means there exists a $y \in N(Y_{n+1})$ whose boundary is $f_n(x)$. This means $d^{n+1}_0(y) = \ldots = d^{n+1}_n(y) = 0$ and $d^{n+1}_{n+1}(y) = f(x)$. However, this means exactly that $z = [y] - [0_{n+1}]$ is in $N(\mathbf{Z}[y])_{n+1}$ and

$$g([x] - [0_n]) = [f(x)] - [0_n] = \text{boundary of } z$$

This proves that $\tilde{\xi}$ maps to zero in $H_n(\mathbf{Z}[y])$. As $H_n(\mathbf{Z}[X]) \to H_n(\mathbf{Z}[Y])$ is bijective we conclude $\tilde{\xi} = 0$ and hence $\xi = 0$. □

14.32. A homotopy equivalence

01A5 Suppose that A, B are sets, and that $f : A \to B$ is a map. Consider the associated map of simplicial sets

$$\begin{tikzcd}
\mathrm{cosk}_0(A) \arrow[r, equal] & \Big(\ldots A \times A \times A \arrow[d] \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[r, shift right=5] & A \times A \arrow[d] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & A \Big) \arrow[d] \\
\mathrm{cosk}_0(B) \arrow[r, equal] & \Big(\ldots B \times B \times B \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[r, shift right=5] & B \times B \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & B \Big)
\end{tikzcd}$$

See Example 14.19.1. The case $n = 0$ of the following lemma says that this map of simplicial sets is a trivial Kan fibration if f is surjective.

01A6 Lemma 14.32.1. Let $f : V \to U$ be a morphism of simplicial sets. Let $n \geq 0$ be an integer. Assume

(1) The map $f_i : V_i \to U_i$ is a bijection for $i < n$.
(2) The map $f_n : V_n \to U_n$ is a surjection.
(3) The canonical morphism $U \to \mathrm{cosk}_n \mathrm{sk}_n U$ is an isomorphism.
(4) The canonical morphism $V \to \mathrm{cosk}_n \mathrm{sk}_n V$ is an isomorphism.

Then f is a trivial Kan fibration.

Proof. Consider a solid diagram

$$\begin{tikzcd}
\partial\Delta[k] \arrow[r] \arrow[d] & V \arrow[d] \\
\Delta[k] \arrow[r] \arrow[ru, dashed] & U
\end{tikzcd}$$

as in Definition 14.30.1. Let $x \in U_k$ be the k-simplex corresponding to the lower horizontal arrow. If $k \leq n$ then the dotted arrow is the one corresponding to a lift $y \in V_k$ of x; the diagram will commute as the other nondegenerate simplices of $\Delta[k]$ are in degrees $< k$ where f is an isomorphism. If $k > n$, then by conditions (3) and (4) we have (using adjointness of skeleton and coskeleton functors)

$$\mathrm{Mor}(\Delta[k], U) = \mathrm{Mor}(\mathrm{sk}_n\Delta[k], \mathrm{sk}_n U) = \mathrm{Mor}(\mathrm{sk}_n\partial\Delta[k], \mathrm{sk}_n U) = \mathrm{Mor}(\partial\Delta[k], U)$$

and similarly for V because $\mathrm{sk}_n\Delta[k] = \mathrm{sk}_n\partial\Delta[k]$ for $k > n$. Thus we obtain a unique dotted arrow fitting into the diagram in this case also. □

Let A, B be sets. Let $f^0, f^1 : A \to B$ be maps of sets. Consider the induced maps $f^0, f^1 : \mathrm{cosk}_0(A) \to \mathrm{cosk}_0(B)$ abusively denoted by the same symbols. The following lemma for $n = 0$ says that f^0 is homotopic to f^1. In fact, there is a homotopy $h : \mathrm{cosk}_0(A) \times \Delta[1] \to \mathrm{cosk}_0(A)$ from f^0 to f^1 with components

$$\begin{array}{rcl} h_m : A \times \ldots \times A \times \mathrm{Mor}_\Delta([m], [1]) & \longrightarrow & B \times \ldots \times B, \\ (a_0, \ldots, a_m, \alpha) & \longmapsto & (f^{\alpha(0)}(a_0), \ldots, f^{\alpha(m)}(a_m)) \end{array}$$

To check that this works, note that for a map $\varphi : [k] \to [m]$ the induced maps are $(a_0, \ldots, a_m) \mapsto (a_{\varphi(0)}, \ldots, a_{\varphi(k)})$ and $\alpha \mapsto \alpha \circ \varphi$. Thus $h = (h_m)_{m \geq 0}$ is clearly a map of simplicial sets as desired.

01A9 Lemma 14.32.2. Let $f^0, f^1 : V \to U$ be maps of simplicial sets. Let $n \geq 0$ be an integer. Assume

(1) The maps $f_i^j : V_i \to U_i$, $j = 0, 1$ are equal for $i < n$.
(2) The canonical morphism $U \to \mathrm{cosk}_n\mathrm{sk}_n U$ is an isomorphism.
(3) The canonical morphism $V \to \mathrm{cosk}_n\mathrm{sk}_n V$ is an isomorphism.

Then f^0 is homotopic to f^1.

First proof. Let W be the n-truncated simplicial set with $W_i = U_i$ for $i < n$ and $W_n = U_n/\sim$ where $\sim$ is the equivalence relation generated by $f^0(y) \sim f^1(y)$ for $y \in V_n$. This makes sense as the morphisms $U(\varphi) : U_n \to U_i$ corresponding to $\varphi : [i] \to [n]$ for $i < n$ factor through the quotient map $U_n \to W_n$ because f^0 and f^1 are morphisms of simplicial sets and equal in degrees $< n$. Next, we upgrade W to a simplicial set by taking $\mathrm{cosk}_n W$. By Lemma 14.32.1 the morphism $g : U \to W$ is a trivial Kan fibration. Observe that $g \circ f^0 = g \circ f^1$ by construction and denote this morphism $f : V \to W$. Consider the diagram

$$\begin{tikzcd} \partial\Delta[1] \times V \arrow[r, "f^0, f^1"'] \arrow[d] & U \arrow[d] \\ \Delta[1] \times V \arrow[r, "f"] \arrow[ur, dashed] & W \end{tikzcd}$$

By Lemma 14.30.2 the dotted arrow exists and the proof is done. □

Second proof. We have to construct a morphism of simplicial sets $h : V \times \Delta[1] \to U$ which recovers f^i on composing with e_i. The case $n = 0$ was dealt with above the lemma. Thus we may assume that $n \geq 1$. The map $\Delta[1] \to \mathrm{cosk}_1\mathrm{sk}_1\Delta[1]$ is an isomorphism, see Lemma 14.19.15. Thus we see that $\Delta[1] \to \mathrm{cosk}_n\mathrm{sk}_n\Delta[1]$ is an isomorphism as $n \geq 1$, see Lemma 14.19.10. And hence $V \times \Delta[1] \to \mathrm{cosk}_n\mathrm{sk}_n(V \times \Delta[1])$ is an isomorphism too, see Lemma 14.19.12. In other words, in order to construct the homotopy it suffices to construct a suitable morphism of n-truncated simplicial sets $h : \mathrm{sk}_n V \times \mathrm{sk}_n \Delta[1] \to \mathrm{sk}_n U$.

For $k = 0, \ldots, n-1$ we define h_k by the formula $h_k(v, \alpha) = f^0(v) = f^1(v)$. The map $h_n : V_n \times \mathrm{Mor}_\Delta([k], [1]) \to U_n$ is defined as follows. Pick $v \in V_n$ and $\alpha : [n] \to [1]$:

(1) If $\mathrm{Im}(\alpha) = \{0\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(2) If $\mathrm{Im}(\alpha) = \{0, 1\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(3) If $\mathrm{Im}(\alpha) = \{1\}$, then we set $h_n(v, \alpha) = f^1(v)$.

Let $\varphi : [k] \to [l]$ be a morphism of $\Delta_{\leq n}$. We will show that the diagram

$$\begin{tikzcd} V_l \times \mathrm{Mor}([l], [1]) \arrow[r] \arrow[d] & U_l \arrow[d] \\ V_k \times \mathrm{Mor}([k], [1]) \arrow[r] & U_k \end{tikzcd}$$

commutes. Pick $v \in V_l$ and $\alpha : [l] \to [1]$. The commutativity means that

$$h_k(V(\varphi)(v), \alpha \circ \varphi) = U(\varphi)(h_l(v, \alpha)).$$

In almost every case this holds because $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^0(V(\varphi)(v))$ and $U(\varphi)(h_l(v, \alpha)) = U(\varphi)(f^0(v))$, combined with the fact that f^0 is a morphism of simplicial sets. The only cases where this does not hold is when either (A) $\mathrm{Im}(\alpha) = \{1\}$ and $l = n$ or (B) $\mathrm{Im}(\alpha \circ \varphi) = \{1\}$ and $k = n$. Observe moreover that necessarily $f^0(v) = f^1(v)$ for any degenerate n-simplex of V. Thus we can narrow the cases above down even further to the cases (A) $\mathrm{Im}(\alpha) = \{1\}$, $l = n$ and v nondegenerate, and (B) $\mathrm{Im}(\alpha \circ \varphi) = \{1\}$, $k = n$ and $V(\varphi)(v)$ nondegenerate.

In case (A), we see that also $\mathrm{Im}(\alpha \circ \varphi) = \{1\}$. Hence we see that not only $h_l(v, \alpha) = f^1(v)$ but also $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^1(V(\varphi)(v))$. Thus we see that the relation holds because f^1 is a morphism of simplicial sets.

In case (B) we conclude that $l = k = n$ and φ is bijective, since otherwise $V(\varphi)(v)$ is degenerate. Thus $\varphi = \mathrm{id}_{[n]}$, which is a trivial case. ☐

01AB Lemma 14.32.3. Let A, B be sets, and that $f : A \to B$ is a map. Consider the simplicial set U with n-simplices

$$A \times_B A \times_B \ldots \times_B A \ (n + 1 \text{ factors}).$$

see Example 14.3.5. If f is surjective, the morphism $U \to B$ where B indicates the constant simplicial set with value B is a trivial Kan fibration.

Proof. Observe that U fits into a cartesian square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathrm{cosk}_0(A) \arrow[d] \\ B \arrow[r] & \mathrm{cosk}_0(B) \end{tikzcd}$$

Since the right vertical arrow is a trivial Kan fibration by Lemma 14.32.1, so is the left by Lemma 14.30.3. ☐

14.33. Preparation for standard resolutions

0G5L The material in this section can be found in [God73, Appendix 1]

0G5M **Example 14.33.1.** Let $Y : \mathcal{C} \to \mathcal{C}$ be a functor from a category to itself and suppose given transformations of functors

$$d : Y \longrightarrow \text{id}_\mathcal{C} \quad \text{and} \quad s : Y \longrightarrow Y \circ Y$$

Using these transformations we can construct something that looks like a simplicial object. Namely, for $n \geq 0$ we define

$$X_n = Y \circ \ldots \circ Y \quad (n+1 \text{ compositions})$$

Observe that $X_{n+m+1} = X_n \circ X_m$ for $n, m \geq 0$. Next, for $n \geq 0$ and $0 \leq j \leq n$ we define using notation as in Categories, Section 4.28

$$d^n_j = 1_{X_{j-1}} \star d \star 1_{X_{n-j-1}} : X_n \to X_{n-1} \quad \text{and} \quad s^n_j = 1_{X_{j-1}} \star s \star 1_{X_{n-j-1}} : X_n \to X_{n+1}$$

So d^n_j, resp. s^n_j is the natural transformation using d, resp. s on the jth Y (counted from the left) in the composition defining X_n.

0G5N **Lemma 14.33.2.** In Example 14.33.1 if

$$1_Y = (d \star 1_Y) \circ s = (1_Y \star d) \circ s \quad \text{and} \quad (s \star 1) \circ s = (1 \star s) \circ s$$

then $X = (X_n, d^n_j, s^n_j)$ is a simplicial object in the category of endofunctors of $\mathcal{C}$ and $d : X_0 = Y \to \text{id}_\mathcal{C}$ defines an augmentation.

Proof. To see that we obtain a simplicial object we have to check that the relations (1)(a) – (e) of Lemma 14.3.2 are satisfied. We will use the short hand notation

$$1_a = 1_{X_{a-1}} = 1_Y \star \ldots \star 1_Y \quad (a \text{ factors})$$

for $a \geq 0$. With this notation we have

$$d^n_j = 1_j \star d \star 1_{n-j} \quad \text{and} \quad s^n_j = 1_j \star s \star 1_{n-j}$$

We are repeatedly going to use the rule that for transformations of funtors a, a', b, b' we have $(a' \circ a) \star (b' \circ b) = (a' \star b') \circ (a \star b)$ provided that the $\star$ and $\circ$ compositions in this formula make sense, see Categories, Lemma 4.28.2.

Condition (1)(a) always holds (no conditions needed on d and s). Namely, let $0 \leq i < j \leq n+1$. We have to show that $d^n_i \circ d^{n+1}_j = d^n_{j-1} \circ d^{n+1}_i$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star d \star 1_{n+1-j}) = (1_{j-1} \star d \star 1_{n+1-j}) \circ (1_i \star d \star 1_{n+1-i})$$

We can rewrite the left hand side as

$$\begin{aligned}
&(1_i \star d \star 1_{j-i-1} \star 1_{n+1-j}) \circ (1_i \star 1_1 \star 1_{j-i-1} \star d \star 1_{n+1-j}) \\
&= 1_i \star ((d \star 1_{j-i-1}) \circ (1_1 \star 1_{j-i-1} \star d)) \star 1_{n+1-j} \\
&= 1_i \star d \star 1_{j-i-1} \star d \star 1_{n+1-j}
\end{aligned}$$

The second equality is true because $d \circ 1_1 = d$ and $1_{j-i} \circ (1_{j-i-1} \star d) = 1_{j-i-1} \star d$. A similar computation gives the same result for the right hand side.

We check condition (1)(b). Let $0 \leq i < j \leq n-1$. We have to show that $d^n_i \circ s^{n-1}_j = s^{n-2}_{j-1} \circ d^{n-1}_i$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j-1} \star s \star 1_{n-1-j}) \circ (1_i \star d \star 1_{n-1-i})$$

By the same kind of calculus as in case (1)(a) both sides simplify to $1_i \star d \star 1_{j-i-1} \star s \star 1_{n-j-1}$.

We check condition (1)(c). Let $0 \leq j \leq n-1$. We have to show $\text{id} = d^n_j \circ s^{n-1}_j = d^n_{j+1} \circ s^{n-1}_j$, i.e.,

$$1_n = (1_j \star d \star 1_{n-j}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j+1} \star d \star 1_{n-j-1}) \circ (1_j \star s \star 1_{n-1-j})$$

This is easily seen to be implied by the first assumption of the lemma.

We check condition (1)(d). Let $0 < j+1 < i \leq n$. We have to show $d^n_i \circ s^{n-1}_j = s^{n-2}_j \circ d^{n-1}_{i-1}$, i.e.,

$$(1_i \star d \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_j \star s \star 1_{n-2-j}) \circ (1_{i-1} \star d \star 1_{n-i})$$

By the same kind of calculus as in case (1)(a) both sides simplify to $1_j \star s \star 1_{i-j-2} \star d \star 1_{n-i}$.

We check condition (1)(e). Let $0 \leq i \leq j \leq n-1$. We have to show that $s^n_i \circ s^{n-1}_j = s^n_{j+1} \circ s^{n-1}_i$, i.e.,

$$(1_i \star s \star 1_{n-i}) \circ (1_j \star s \star 1_{n-1-j}) = (1_{j+1} \star s \star 1_{n-1-j}) \circ (1_i \star s \star 1_{n-1-i})$$

By the same kind of calculus as in case (1)(a) this reduces to

$$(s \star 1_{j-i+1}) \circ (1_{j-i} \star s) = (1_{j-i+1} \star s) \circ (s \star 1_{j-i})$$

If $j = i$ this is exactly one of the two assumptions of the lemma. For $j > i$ left and right hand side both reduce to the equality $s \star 1_{j-i-1} \star s$ by calculations similar to those we did in case (1)(a).

Finally, in order to show that d defines an augmentation we have to show that $d \circ (1_1 \star d) = d \circ (d \star 1_1)$ which is true because both sides are equal to $d \star d$. □

0G5P Example 14.33.3. Let $\mathcal{C}$, Y, d, s be as in Example 14.33.1 satisfying the equations of Lemma 14.33.2. Given functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{B}$ we obtain a simplicial object $G \circ X \circ F$ in the category of functors from $\mathcal{A}$ to $\mathcal{B}$ which comes with an augmentation to $G \circ F$.

0G5Q Lemma 14.33.4. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, Y, d, s, F, G be as in Example 14.33.3. Given a transformation of functors $h_0 : G \circ F \to G \circ Y \circ F$ such that

$$1_{G \circ F} = (1_G \star d \star 1_F) \circ h_0$$

Then there is a morphism $h : G \circ F \to G \circ X \circ F$ of simplicial objects such that $\epsilon \circ h = \text{id}$ where $\epsilon : G \circ X \circ F \to G \circ F$ is the augmentation.

Proof. Denote $u_n : Y = X_0 \to X_n$ the map of the simplicial object X corresponding to the unique morphism $[n] \to [0]$ in Δ. Set $h_n : G \circ F \to G \circ X_n \circ F$ equal to $(1_G \star u_n \star 1_F) \circ h_0$.

For any simplicial object $X = (X_n)$ in any category $u = (u_n) : X_0 \to X$ is a morphism from the constant simplicial object on X_0 to X. Hence h is a morphism of simplicial objects because it is the composition of $1_G \star u \star 1_F$ and h_0.

Let us check that $\epsilon \circ h = \text{id}$. We compute

$$\epsilon_n \circ (1_G \star u_n \star 1_F) \circ h_0 = \epsilon_0 \circ h_0 = \text{id}$$

The first equality because ϵ is a morphism of simplicial objects and the second equality because $\epsilon_0 = (1_G \star d \star 1_F)$ and we can apply the assumption in the statement of the lemma. □

0G5R **Lemma 14.33.5.** Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, Y, d, s, F, G be as in Example 14.33.3. Let $F' : \mathcal{A} \to \mathcal{C}$ and $G' : \mathcal{C} \to \mathcal{B}$ be two functors. Let $(a_n) : G \circ X \to G' \circ X$ be a morphism of simplicial objects compatible via augmentations with $a : G \to G'$. Let $(b_n) : X \circ F \to X \circ F'$ be a morphism of simplicial objects compatible via augmentations with $b : F \to F'$. Then the two maps

$$a \star (b_n), (a_n) \star b : G \circ X \circ F \to G' \circ X \circ F'$$

are homotopic.

Proof. To show the morphisms are homotopic we construct morphisms

$$h_{n,i} : G \circ X_n \circ F \to G' \circ X_n \circ F'$$

for $n \geq 0$ and $0 \leq i \leq n+1$ satisfying the relations described in Lemma 14.26.2. See also Remark 14.26.4. To satisfy condition (1) of Lemma 14.26.2 we are forced to set $h_{n,0} = a \star b_n$ and $h_{n,n+1} = a_n \star b$. Thus a logical choice is

$$h_{n,i} = a_{i-1} \star b_{n-i}$$

for $1 \leq i \leq n$. Setting $a = a_{-1}$ and $b = b_{-1}$ we see the displayed formular holds for $0 \leq i \leq n+1$.

Recall that

$$d^n_j = 1_G \star 1_j \star d \star 1_{n-j} \star 1_F$$

on $G \circ X \circ F$ where we use the notation $1_a = 1_{Y \circ \ldots \circ Y}$ introduced in the proof of Lemma 14.33.2. We are going to use below that we can rewrite this as

$$\begin{aligned} d^n_j &= d^j_j \star 1_{n-j} = d^{j+1}_j \star 1_{n-j} = \ldots = d^{n-1}_j \star 1_1 \\ &= 1_j \star d^{n-j}_0 = 1_{j-1} \star d^{n-j+1}_1 = \ldots = 1_1 \star d^{n-1}_{j-1} \end{aligned}$$

Of course we have the analogous formulae for d^n_j on $G' \circ X \circ F'$.

We check condition (2) of Lemma 14.26.2. Let $i > j$. We have to show

$$d^n_j \circ (a_{i-1} \star b_{n-i}) = (a_{i-2} \star b_{n-i}) \circ d^n_j$$

Since $i - 1 \geq j$ we can use one of the possible descriptions of d^n_j to rewrite the left hand side as

$$(d^{i-1}_j \star 1_{n-i+1}) \circ (a_{i-1} \star b_{n-i}) = (d^{i-1}_j \circ a_{i-1}) \star b_{n-i} = (a_{i-2} \circ d^{i-1}_j) \star b_{n-i}$$

Similarly the right hand side becomes

$$(a_{i-2} \star b_{n-i}) \circ (d^{i-1}_j \star 1_{n-i+1}) = (a_{i-2} \circ d^{i-1}_j) \star b_{n-i}$$

Thus we obtain the same result and (2) is checked.

We check condition (3) of Lemma 14.26.2. Let $i \leq j$. We have to show

$$d^n_j \circ (a_{i-1} \star b_{n-i}) = (a_{i-1} \star b_{n-1-i}) \circ d^n_j$$

Since $j \geq i$ we may rewrite the left hand side as

$$(1_i \star d^{n-i}_{j-i}) \circ (a_{i-1} \star b_{n-i}) = a_{i-1} \star (b_{n-1-i} \circ d^{n-i}_{j-i})$$

A similar manipulation shows this agrees with the right hand side.

Recall that

$$s^n_j = 1_G \star 1_j \star s \star 1_{n-j} \star 1_F$$

on $G \circ X \circ F$. We are going to use below that we can rewrite this as

$$\begin{aligned} s^n_j &= s^j_j \star 1_{n-j} = s^{j+1}_j \star 1_{n-j-1} = \ldots = s^{n-1}_j \star 1_1 \\ &= 1_j \star s^{n-j}_0 = 1_{j-1} \star s^{n-j+1}_1 = \ldots = 1_1 \star s^{n-1}_{j-1} \end{aligned}$$

Of course we have the analogous formulae for s^n_j on $G' \circ X \circ F'$.

We check condition (4) of Lemma 14.26.2. Let $i > j$. We have to show

$$s^n_j \circ (a_{i-1} \star b_{n-i}) = (a_i \star b_{n-i}) \circ s^n_j$$

Since $i - 1 \geq j$ we can rewrite the left hand side as

$$(s^{i-1}_j \star 1_{n-i+1}) \circ (a_{i-1} \star b_{n-i}) = (s^{i-1}_j \circ a_{i-1}) \star b_{n-i} = (a_i \circ s^{i-1}_j) \star b_{n-i}$$

Similarly the right hand side becomes

$$(a_i \star b_{n-i}) \circ (s^{i-1}_j \star 1_{n-i+1}) = (a_i \circ s^{i-1}_j) \star b_{n-i}$$

as desired.

We check condition (5) of Lemma 14.26.2. Let $i \leq j$. We have to show

$$s^n_j \circ (a_{i-1} \star b_{n-i}) = (a_{i-1} \star b_{n+1-i}) \circ s^n_j$$

This equality holds because both sides evaluate to $a_{i-1} \star (s^{n-i}_{j-i} \circ b_{n-i}) = a_{i-1} \star (b_{n+1-i} \circ s^{n-i}_{j-i})$ by exactly the same arguments as above. □

0G5S Lemma 14.33.6. Let $\mathcal{C}$, Y, d, s be as in Example 14.33.1 satisfying the equations of Lemma 14.33.2. Let $f : \text{id}_\mathcal{C} \to \text{id}_\mathcal{C}$ be an endomorphism of the identity functor. Then $f \star 1_X, 1_X \star f : X \to X$ are maps of simplicial objects compatible with f via the augmentation $\epsilon : X \to \text{id}_\mathcal{C}$. Moreover, $f \star 1_X$ and $1_X \star f$ are homotopic.

Proof. The map $f \star 1_X$ is the map with components

$$X_n = \text{id}_\mathcal{C} \circ X_n \xrightarrow{f \star 1_{X_n}} \text{id}_\mathcal{C} \circ X_n = X_n$$

For a transformation $a : F \to G$ of endofunctors of $\mathcal{C}$ we have $a \circ (f \star 1_F) = f \star a = (f \star 1_G) \circ a$. Thus $f \star 1_X$ is indeed a morphism of simplicial objects. Similarly for $1_X \star f$.

To show the morphisms are homotopic we construct morphisms $h_{n,i} : X_n \to X_n$ for $n \geq 0$ and $0 \leq i \leq n + 1$ satisfying the relations described in Lemma 14.26.2. See also Remark 14.26.4. It turns out we can take

$$h_{n,i} = 1_i \star f \star 1_{n+1-i}$$

where 1_i is the identity transformation on $Y \circ \ldots \circ Y$ as in the proof of Lemma 14.33.2. We have $h_{n,0} = f \star 1_{X_n}$ and $h_{n,n+1} = 1_{X_n} \star f$ which checks the first condition. In checking the other conditions we use the comments made in the proof of Lemma 14.33.5 about the maps d^n_j and s^n_j.

We check condition (2) of Lemma 14.26.2. Let $i > j$. We have to show

$$d^n_j \circ (1_i \star f \star 1_{n+1-i}) = (1_{i-1} \star f \star 1_{n+1-i}) \circ d^n_j$$

Since $i - 1 \geq j$ we can use one of the possible descriptions of d^n_j to rewrite the left hand side as

$$(d^{i-1}_j \star 1_{n-i+1}) \circ (1_i \star f \star 1_{n+1-i}) = d^{i-1}_j \star f \star 1_{n+1-i}$$

Similarly the right hand side becomes

$$(1_{i-1} \star f \star 1_{n+1-i}) \circ (d_j^{i-1} \star 1_{n-i+1}) = d_j^{i-1} \star f \star 1_{n+1-i}$$

Thus we obtain the same result and (2) is checked.

The conditions (3), (4), and (5) of Lemma 14.26.2 are checked in exactly the same manner using the strategy of the proof of Lemma 14.33.5. We omit the details[2]. □

14.34. Standard resolutions

08N8 Some of the material in this section can be found in [God73, Appendix 1] and [Ill72, I 1.5].

08N9 Situation 14.34.1. Let $\mathcal{A}$, $\mathcal{S}$ be categories and let $V : \mathcal{A} \to \mathcal{S}$ be a functor with a left adjoint $U : \mathcal{S} \to \mathcal{A}$.

In this very general situation we will construct a simplicial object X in the category of functors from $\mathcal{A}$ to $\mathcal{A}$. We suggest looking at the examples presented later on before reading the text of this section.

For the construction we will use the horizontal composition as defined in Categories, Section 4.28. The definition of the adjunction morphisms[3]

$$d : U \circ V \to \mathrm{id}_\mathcal{A} \quad \text{(counit)} \quad \text{and} \quad \eta : \mathrm{id}_\mathcal{S} \to V \circ U \quad \text{(unit)}$$

in Categories, Section 4.24 shows that the compositions

08NB (14.34.1.1) $$V \xrightarrow{\eta \star 1_V} V \circ U \circ V \xrightarrow{1_V \star d} V \quad \text{and} \quad U \xrightarrow{1_U \star \eta} U \circ V \circ U \xrightarrow{d \star 1_U} U$$

are the identity morphisms. Here to define the morphism $\eta \star 1_V$ we silently identify V with $\mathrm{id}_\mathcal{S} \circ V$ and 1_V stands for $\mathrm{id}_V : V \to V$. We will use this notation and these relations repeatedly in what follows. For $n \geq 0$ we set

$$X_n = (U \circ V)^{\circ(n+1)} = U \circ V \circ U \circ \ldots \circ U \circ V$$

In other words, X_n is the $(n+1)$-fold composition of $U \circ V$ with itself. We also set $X_{-1} = \mathrm{id}_\mathcal{A}$. We have $X_{n+m+1} = X_n \circ X_m$ for all $n, m \geq -1$. We will endow this sequence of functors with the structure of a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ by constructing the morphisms of functors

$$d_j^n : X_n \to X_{n-1}, \quad s_j^n : X_n \to X_{n+1}$$

satisfying the relations displayed in Lemma 14.2.3. Namely, we set

$$d_j^n = 1_{X_{j-1}} \star d \star 1_{X_{n-j-1}} \quad \text{and} \quad s_j^n = 1_{X_{j-1} \circ U} \star \eta \star 1_{V \circ X_{n-j-1}}$$

Finally, write $\epsilon_0 = d : X_0 \to X_{-1}$.

08NC Lemma 14.34.2. In Situation 14.34.1 the system $X = (X_n, d_j^n, s_j^n)$ is a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ and ϵ_0 defines an augmentation ϵ from X to the constant simplicial object with value $X_{-1} = \mathrm{id}_\mathcal{A}$.

[2] When f is invertible it suffices to prove that $(a_n) = 1_X$ and $(b_n) = f^{-1} \star 1_X \star f$ are homotopic. But this follows from Lemma 14.33.5 because in this case $a = b = 1_{\mathrm{id}_\mathcal{C}}$.

[3] We can't use ϵ for the counit of the adjunction because we want to use ϵ for the augmentation of our simplicial obejct.

Proof. Consider $Y = U \circ V : \mathcal{A} \to \mathcal{A}$. We already have the transformation $d : Y = U \circ V \to \mathrm{id}_\mathcal{A}$. Let us denote

$$s = 1_U \star \eta \star 1_V : Y = U \circ \mathrm{id}_\mathcal{S} \circ V \longrightarrow U \circ V \circ U \circ V = Y \circ Y$$

This places us in the sitation of Example 14.33.1. It is immediate from the formulas that the X, d^n_i, s^n_i constructed above and the X, s^n_i, s^n_i constructed from Y, d, s in Example 14.33.1 agree. Thus, according to Lemma 14.33.2 it suffices to prove that

$$1_Y = (d \star 1_Y) \circ s = (1_Y \star d) \circ s \quad \text{and} \quad (s \star 1) \circ s = (1 \star s) \circ s$$

The first equal sign translates into the equality

$$1_U \star 1_V = (d \star 1_U \star 1_V) \circ (1_U \star \eta \star 1_V)$$

which holds if we have $1_U = (d \star 1_U) \circ (1_U \star \eta)$ which in turn holds by (14.34.1.1). Similarly for the second equal sign. For the last equation we need to prove

$$(1_U \star \eta \star 1_V \star 1_U \star 1_V) \circ (1_U \star \eta \star 1_V) = (1_U \star 1_V \star 1_U \star \eta \star 1_V) \circ (1_U \star \eta \star 1_V)$$

For this it suffices to prove $(\eta \star 1_V \star 1_U) \circ \eta = (1_V \star 1_U \star \eta) \circ \eta$ which is true because both sides are the same as $\eta \star \eta$. $\square$

Before reading the proof of the following lemma, we advise the reader to look at the example discussed in Example 14.34.8 in order to understand the purpose of the lemma.

08ND **Lemma 14.34.3.** In Situation 14.34.1 the maps

$$1_V \star \epsilon : V \circ X \to V, \quad \text{and} \quad \epsilon \star 1_U : X \circ U \to U$$

are homotopy equivalences.

Proof. As in the proof of Lemma 14.34.2 we set $Y = U \circ V$ so that we are in the sitation of Example 14.33.1.

Proof of the first homotopy equivalence. By Lemma 14.33.4 to construct a map $h : V \to V \circ X$ right inverse to $1_V \star \epsilon$ it suffices to construct a map $h_0 : V \to V \circ Y = V \circ U \circ V$ such that $1_V = (1_V \star d) \circ h_0$. Of course we take $h_0 = \eta \star 1_V$ and the equality holds by (14.34.1.1). To finish the proof we need to show the two maps

$$(1_V \star \epsilon) \circ h, 1_V \star \mathrm{id}_X : V \circ X \longrightarrow V \circ X$$

are homotopic. This follows immediately from Lemma 14.33.5 (with $G = G' = V$ and $F = F' = \mathrm{id}_\mathcal{S}$).

The proof of the second homotopy equivalence. By Lemma 14.33.4 to construct a map $h : U \to X \circ U$ right inverse to $\epsilon \star 1_U$ it suffices to construct a map $h_0 : U \to Y \circ U = U \circ V \circ U$ such that $1_U = (d \star 1_U) \circ h_0$. Of course we take $h_0 = 1_U \star \eta$ and the equality holds by (14.34.1.1). To finish the proof we need to show the two maps

$$(\epsilon \star 1_U) \circ h, \mathrm{id}_X \star 1_U : X \circ U \longrightarrow X \circ U$$

are homotopic. This follows immediately from Lemma 14.33.5 (with $G = G' = \mathrm{id}_\mathcal{A}$ and $F = F' = U$). $\square$

0G5T Example 14.34.4. Let R be a ring. As an example of the above we can take $i : \text{Mod}_R \to \text{Sets}$ to be the forgetful functor and $F : \text{Sets} \to \text{Mod}_R$ to be the functor that associates to a set E the free R-module $R[E]$ on E. For an R-module M the simplicial R-module $X(M)$ will have the following shape

$$X(M) = \left(\ldots \ R[R[R[M]]] \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} R[R[M]] \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} R[M] \ \right)$$

which comes with an augmentation towards M. We will also show this augmentation is a homotopy equivalence of sets. By Lemmas 14.30.8, 14.31.9, and 14.31.8 this is equivalent to asking M to be the only nonzero cohomology group of the chain complex associated to the simplicial module $X(M)$.

08NA Example 14.34.5. Let A be a ring. Let Alg_A be the category of commutative A-algebras. As an example of the above we can take $i : \text{Alg}_A \to \text{Sets}$ to be the forgetful functor and $F : \text{Sets} \to \text{Alg}_A$ to be the functor that associates to a set E the polynomial algebra $A[E]$ on E over A. (We apologize for the overlap in notation between this example and Example 14.34.4.) For an A-algebra B the simplicial A-algebra $X(B)$ will have the following shape

$$X(B) = \left(\ldots \ A[A[A[B]]] \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A[A[B]] \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} A[B] \ \right)$$

which comes with an augmentation towards B. We will also show this augmentation is a homotopy equivalence of sets. By Lemmas 14.30.8, 14.31.9, and 14.31.8 this is equivalent to asking B to be the only nonzero cohomology group of the chain complex of A-modules associated to $X(B)$ viewed as a simplicial A-module.

0G5U Example 14.34.6. In Example 14.34.4 we have $X_n(M) = R[R[\ldots[M]\ldots]]$ with $n+1$ brackets. We describe the maps constructed above using a typical element

$$\xi = \sum\nolimits_i r_i \left[\sum\nolimits_j r_{ij}[m_{ij}]\right]$$

of $X_1(M)$. The maps $d_0, d_1 : R[R[M]] \to R[M]$ are given by

$$d_0(\xi) = \sum\nolimits_{i,j} r_i r_{ij}[m_{ij}] \quad \text{and} \quad d_1(\xi) = \sum\nolimits_i r_i \left[\sum\nolimits_j r_{ij} m_{ij}\right].$$

The maps $s_0, s_1 : R[R[M]] \to R[R[R[M]]]$ are given by

$$s_0(\xi) = \sum\nolimits_i r_i \left[\left[\sum\nolimits_j r_{ij}[m_{ij}]\right]\right] \quad \text{and} \quad s_1(\xi) = \sum\nolimits_i r_i \left[\sum\nolimits_j r_{ij}[[m_{ij}]]\right].$$

09CB Example 14.34.7. In Example 14.34.5 we have $X_n(B) = A[A[\ldots[B]\ldots]]$ with $n+1$ brackets. We describe the maps constructed above using a typical element

$$\xi = \sum\nolimits_i a_i[x_{i,1}]\ldots[x_{i,m_i}] \in A[A[B]] = X_1(B)$$

where for each i, j we can write

$$x_{i,j} = \sum a_{i,j,k}[b_{i,j,k,1}]\ldots[b_{i,j,k,n_{i,j,k}}] \in A[B]$$

Obviously this is horrendous! To ease the notation, to see what the A-algebra maps $d_0, d_1 : A[A[B]] \to A[B]$ are doing it suffices to see what happens to the variables $[x]$ where

$$x = \sum a_k[b_{k,1}]\ldots[b_{k,n_k}] \in A[B]$$

is a general element. For these we get

$$d_0([x]) = x = \sum a_k[b_{k,1}]\ldots[b_{k,n_k}] \quad \text{and} \quad d_1([x]) = \left[\sum a_k b_{k,1}\ldots b_{k,n_k}\right]$$

The maps $s_0, s_1 : A[A[B]] \to A[A[A[B]]]$ are given by

$$s_0([x]) = \left[\left[\sum a_k[b_{k,1}] \ldots [b_{k,n_k}]\right]\right] \quad \text{and} \quad s_1([x]) = \left[\sum a_k[[b_{k,1}]] \ldots [[b_{k,n_k}]]\right]$$

08NE Example 14.34.8. Going back to the example discussed in Example 14.34.5 our Lemma 14.34.3 signifies that for any ring map $A \to B$ the map of simplicial rings

$$\begin{tikzcd}
A[A[A[B]]] \arrow[r, shift left=4] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=4] \arrow[d] & A[A[B]] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d] & A[B] \arrow[d] \\
B \arrow[r, shift left=4] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=4] & B \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & B
\end{tikzcd}$$

is a homotopy equivalence on underlying simplicial sets. Moreover, the inverse map constructed in Lemma 14.34.3 is in degree n given by

$$b \longmapsto [\ldots [b] \ldots]$$

with obvious notation. In the other direction the lemma tells us that for every set E there is a homotopy equivalence

$$\begin{tikzcd}
A[A[A[A[E]]]] \arrow[r, shift left=4] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=4] \arrow[d] & A[A[A[E]]] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d] & A[A[E]] \arrow[d] \\
A[E] \arrow[r, shift left=4] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=4] & A[E] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & A[E]
\end{tikzcd}$$

of rings. The inverse map constructed in the lemma is in degree n given by the ring map

$$\sum a_{e_1,\ldots,e_p}[e_1][e_2]\ldots[e_p] \longmapsto \sum a_{e_1,\ldots,e_p}[\ldots[e_1]\ldots][\ldots[e_2]\ldots]\ldots[\ldots[e_p]\ldots]$$

(with obvious notation).

14.35. Other chapters

CHAPTER 15

More on Algebra

05E3 ## 15.1. Introduction

05E4 In this chapter we prove some results in commutative algebra which are less elementary than those in the first chapter on commutative algebra, see Algebra, Section 10.1. A reference is [Mat70a].

15.2. Advice for the reader

0910 More than in the chapter on commutative algebra, each of the sections in this chapter stands on its own. Starting with Section 15.56 we freely use the (unbounded) derived category of modules over rings and all the machinery that comes with it.

15.3. Stably free modules

0BC2 Here is what seems to be the generally accepted definition.

0BC3 Definition 15.3.1. Let R be a ring.

(1) Two modules M, N over R are said to be stably isomorphic if there exist $n, m \geq 0$ such that $M \oplus R^{\oplus m} \cong N \oplus R^{\oplus n}$ as R-modules.
(2) A module M is stably free if it is stably isomorphic to a free module.

Observe that a stably free module is projective.

0BC4 Lemma 15.3.2. Let R be a ring. Let $0 \to P' \to P \to P'' \to 0$ be a short exact sequence of finite projective R-modules. If 2 out of 3 of these modules are stably free, then so is the third.

Proof. Since the modules are projective, the sequence is split. Thus we can choose an isomorphism $P = P' \oplus P''$. If $P' \oplus R^{\oplus n}$ and $P'' \oplus R^{\oplus m}$ are free, then we see that $P \oplus R^{\oplus n+m}$ is free. Suppose that P' and P are stably free, say $P \oplus R^{\oplus n}$ is free and $P' \oplus R^{\oplus m}$ is free. Then

$$P'' \oplus (P' \oplus R^{\oplus m}) \oplus R^{\oplus n} = (P'' \oplus P') \oplus R^{\oplus m} \oplus R^{\oplus n} = (P \oplus R^{\oplus n}) \oplus R^{\oplus m}$$

is free. Thus P'' is stably free. By symmetry we get the last of the three cases. □

0BC5 Lemma 15.3.3. Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). For every finite stably free R/I-module E there exists a finite stably free R-module M such that $M/IM \cong E$.

Proof. Choose a n and m and an isomorphism $E \oplus (R/I)^{\oplus n} \cong (R/I)^{\oplus m}$. Choose R-linear maps $\varphi : R^{\oplus m} \to R^{\oplus n}$ and $\psi : R^{\oplus n} \to R^{\oplus m}$ lifting the projection $(R/I)^{\oplus m} \to (R/I)^{\oplus n}$ and injection $(R/I)^{\oplus n} \to (R/I)^{\oplus m}$. Then $\varphi \circ \psi : R^{\oplus n} \to R^{\oplus n}$ reduces to the identity modulo I. Thus the determinant of this map is invertible by our assumption on I. Hence $P = \mathrm{Ker}(\varphi)$ is stably free and lifts E. □

0D48 **Lemma 15.3.4.** Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). Let M be a finite flat R-module such that M/IM is a projective R/I-module. Then M is a finite projective R-module.

Proof. By Algebra, Lemma 10.78.5 we see that $M_{\mathfrak p}$ is finite free for all prime ideals $\mathfrak p \subset R$. By Algebra, Lemma 10.78.2 it suffices to show that the function $\rho_M : \mathfrak p \mapsto \dim_{\kappa(\mathfrak p)} M \otimes_R \kappa(\mathfrak p)$ is locally constant on $\mathrm{Spec}(R)$. Because M/IM is finite projective, this is true on $V(I) \subset \mathrm{Spec}(R)$. Since every closed point of $\mathrm{Spec}(R)$ is in $V(I)$ and since $\rho_M(\mathfrak p) = \rho_M(\mathfrak q)$ whenever $\mathfrak p \subset \mathfrak q \subset R$ are prime ideals, we conclude by an elementary argument on topological spaces which we omit. $\square$

The lift of Lemma 15.3.3 is unique up to isomorphism by the following lemma.

0BC6 **Lemma 15.3.5.** Let R be a ring. Let $I \subset R$ be an ideal. Assume that every element of $1 + I$ is a unit (in other words I is contained in the Jacobson radical of R). If P and P' are finite projective R-modules, then

(1) if $\varphi : P \to P'$ is an R-module map inducing an isomorphism $\overline{\varphi} : P/IP \to P'/IP'$, then φ is an isomorphism,
(2) if $P/IP \cong P'/IP'$, then $P \cong P'$.

Proof. Proof of (1). As P' is projective as an R-module we may choose a lift $\psi : P' \to P$ of the map $P' \to P'/IP' \xrightarrow{\overline{\varphi}^{-1}} P/IP$. By Nakayama's lemma (Algebra, Lemma 10.20.1) $\psi \circ \varphi$ and $\varphi \circ \psi$ are surjective. Hence these maps are isomorphisms (Algebra, Lemma 10.16.4). Thus φ is an isomorphism.

Proof of (2). Choose an isomorphism $P/IP \cong P'/IP'$. Since P is projective we can choose a lift $\varphi : P \to P'$ of the map $P \to P/IP \to P'/IP'$. Then φ is an isomorphism by (1). $\square$

15.4. A comment on the Artin-Rees property

07VD Some of this material is taken from [CdJ02]. A general discussion with additional references can be found in [EH05, Section 1].

Let A be a Noetherian ring and let $I \subset A$ be an ideal. Given a homomorphism $f : M \to N$ of finite A-modules there exists a $c \geq 0$ such that

$$f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$, see Algebra, Lemma 10.51.3. In this situation we will say c works for f in the Artin-Rees lemma.

07VE **Lemma 15.4.1.** Let A be a Noetherian ring. Let $I \subset A$ be an ideal contained in the Jacobson radical of A. Let

$$S : L \xrightarrow{f} M \xrightarrow{g} N \quad \text{and} \quad S' : L \xrightarrow{f'} M \xrightarrow{g'} N$$

be two complexes of finite A-modules as shown. Assume that

(1) c works in the Artin-Rees lemma for f and g,
(2) the complex S is exact, and
(3) $f' = f \bmod I^{c+1}M$ and $g' = g \bmod I^{c+1}N$.

Then c works in the Artin-Rees lemma for g' and the complex S' is exact.

Proof. We first show that $g'(M) \cap I^nN \subset g'(I^{n-c}M)$ for $n \geq c$. Let a be an element of M such that $g'(a) \in I^nN$. We want to adjust a by an element of $f'(L)$, i.e, without changing $g'(a)$, so that $a \in I^{n-c}M$. Assume that $a \in I^rM$, where $r < n - c$. Then

$$g(a) = g'(a) + (g - g')(a) \in I^nN + I^{r+c+1}N = I^{r+c+1}N.$$

By Artin-Rees for g we have $g(a) \in g(I^{r+1}M)$. Say $g(a) = g(a_1)$ with $a_1 \in I^{r+1}M$. Since the sequence S is exact, $a - a_1 \in f(L)$. Accordingly, we write $a = f(b) + a_1$ for some $b \in L$. Then $f(b) = a - a_1 \in I^rM$. Artin-Rees for f shows that if $r \geq c$, we may replace b by an element of $I^{r-c}L$. Then in all cases, $a = f'(b) + a_2$, where $a_2 = (f - f')(b) + a_1 \in I^{r+1}M$. (Namely, either $c \geq r$ and $(f - f')(b) \in I^{r+1}M$ by assumption, or $c < r$ and $b \in I^{r-c}$, whence again $(f - f')(b) \in I^{c+1}I^{r-c}M = I^{r+1}M$.) So we can adjust a by the element $f'(b) \in f'(L)$ to increase r by 1.

In fact, the argument above shows that $(g')^{-1}(I^nN) \subset f'(L) + I^{n-c}M$ for all $n \geq c$. Hence S' is exact because

$$(g')^{-1}(0) = (g')^{-1}(\bigcap I^nN) \subset \bigcap f'(L) + I^{n-c}M = f'(L)$$

as I is contained in the Jacobson radical of A, see Algebra, Lemma 10.51.5. □

Given an ideal $I \subset A$ of a ring A and an A-module M we set

$$\mathrm{Gr}_I(M) = \bigoplus I^nM/I^{n+1}M.$$

We think of this as a graded $\mathrm{Gr}_I(A)$-module.

07VF Lemma 15.4.2. Assumptions as in Lemma 15.4.1. Let $Q = \mathrm{Coker}(g)$ and $Q' = \mathrm{Coker}(g')$. Then $\mathrm{Gr}_I(Q) \cong \mathrm{Gr}_I(Q')$ as graded $\mathrm{Gr}_I(A)$-modules.

Proof. In degree n we have $\mathrm{Gr}_I(Q)_n = I^nN/(I^{n+1}N + g(M) \cap I^nN)$ and similarly for Q'. We claim that

$$g(M) \cap I^nN \subset I^{n+1}N + g'(M) \cap I^nN.$$

By symmetry (the proof of the claim will only use that c works for g which also holds for g' by the lemma) this will imply that

$$I^{n+1}N + g(M) \cap I^nN = I^{n+1}N + g'(M) \cap I^nN$$

whence $\mathrm{Gr}_I(Q)_n$ and $\mathrm{Gr}_I(Q')_n$ agree as subquotients of N, implying the lemma. Observe that the claim is clear for $n \leq c$ as $f = f' \bmod I^{c+1}N$. If $n > c$, then suppose $b \in g(M) \cap I^nN$. Write $b = g(a)$ for $a \in I^{n-c}M$. Set $b' = g'(a)$. We have $b - b' = (g - g')(a) \in I^{n+1}N$ as desired. □

07VG Lemma 15.4.3. Let $A \to B$ be a flat map of Noetherian rings. Let $I \subset A$ be an ideal. Let $f : M \to N$ be a homomorphism of finite A-modules. Assume that c works for f in the Artin-Rees lemma. Then c works for $f \otimes 1 : M \otimes_A B \to N \otimes_A B$ in the Artin-Rees lemma for the ideal IB.

Proof. Note that

$$(f \otimes 1)(M) \cap I^nN \otimes_A B = (f \otimes 1)\left((f \otimes 1)^{-1}(I^nN \otimes_A B)\right)$$

On the other hand,

$$\begin{aligned}(f \otimes 1)^{-1}(I^nN \otimes_A B) &= \mathrm{Ker}(M \otimes_A B \to N \otimes_A B/(I^nN \otimes_A B)) \\ &= \mathrm{Ker}(M \otimes_A B \to (N/I^nN) \otimes_A B)\end{aligned}$$

As $A \to B$ is flat taking kernels and cokernels commutes with tensoring with B, whence this is equal to $f^{-1}(I^n N) \otimes_A B$. By assumption $f^{-1}(I^n N)$ is contained in $\mathrm{Ker}(f) + I^{n-c}M$. Thus the lemma holds. $\square$

15.5. Fibre products of rings, I

08KG Fibre products of rings have to do with pushouts of schemes. Some cases of pushouts of schemes are discussed in More on Morphisms, Section 37.14.

00IT Lemma 15.5.1. Let R be a ring. Let $A \to B$ and $C \to B$ be R-algebra maps. Assume

(1) R is Noetherian,
(2) A, B, C are of finite type over R,
(3) $A \to B$ is surjective, and
(4) B is finite over C.

Then $A \times_B C$ is of finite type over R.

Proof. Set $D = A \times_B C$. There is a commutative diagram

$$\begin{tikzcd} 0 & B \arrow[l] & A \arrow[l] & I \arrow[l] & 0 \arrow[l] \\ 0 & C \arrow[l] \arrow[u] & D \arrow[l] \arrow[u] & I \arrow[l] \arrow[u] & 0 \arrow[l] \end{tikzcd}$$

with exact rows. Choose $y_1, \ldots, y_n \in B$ which are generators for B as a C-module. Choose $x_i \in A$ mapping to y_i. Then $1, x_1, \ldots, x_n$ are generators for A as a D-module. The map $D \to A \times C$ is injective, and the ring $A \times C$ is finite as a D-module (because it is the direct sum of the finite D-modules A and C). Hence the lemma follows from the Artin-Tate lemma (Algebra, Lemma 10.51.7). $\square$

08NI Lemma 15.5.2. Let R be a Noetherian ring. Let I be a finite set. Suppose given a cartesian diagram

$$\begin{tikzcd} \prod B_i & \prod A_i \arrow[l, "\prod \varphi_i"'] \\ Q \arrow[u, "\prod \psi_i"] & P \arrow[l] \arrow[u] \end{tikzcd}$$

with ψ_i and φ_i surjective, and Q, A_i, B_i of finite type over R. Then P is of finite type over R.

Proof. Follows from Lemma 15.5.1 and induction on the size of I. Namely, let $I = I' \amalg \{i_0\}$. Let P' be the ring defined by the diagram of the lemma using I'. Then P' is of finite type by induction hypothesis. Finally, P sits in a fibre product diagram

$$\begin{tikzcd} B_{i_0} & A_{i_0} \arrow[l] \\ P' \arrow[u] & P \arrow[l] \arrow[u] \end{tikzcd}$$

to which the lemma applies. $\square$

01Z8 **Lemma 15.5.3.** Suppose given a cartesian diagram of rings

$$\begin{tikzcd} R & R' \arrow[l, "t"] \\ B \arrow[u, "s"] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

i.e., $B' = B \times_R R'$. If $h \in B'$ corresponds to $g \in B$ and $f \in R'$ such that $s(g) = t(f)$, then the diagram

$$\begin{tikzcd} R_{s(g)} = R_{t(f)} & (R')_f \arrow[l, "t"] \\ B_g \arrow[u, "s"] & (B')_h \arrow[l] \arrow[u] \end{tikzcd}$$

is cartesian too.

Proof. The equality $B' = B \times_R R'$ tells us that

$$0 \to B' \to B \oplus R' \xrightarrow{s,-t} R$$

is an exact sequence of B'-modules. We have $B_g = B_h$, $R'_f = R'_h$, and $R_{s(g)} = R_{t(f)} = R_h$ as B'-modules. By exactness of localization (Algebra, Proposition 10.9.12) we find that

$$0 \to B'_h \to B_g \oplus R'_f \xrightarrow{s,-t} R_{s(g)} = R_{t(f)}$$

is an exact sequence. This proves the lemma. $\square$

Consider a commutative diagram of rings

$$\begin{tikzcd} R & R' \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

Consider the functor (where the fibre product of categories is as constructed in Categories, Example 4.31.3)

0D2E (15.5.3.1) $$\mathrm{Mod}_{B'} \longrightarrow \mathrm{Mod}_B \times_{\mathrm{Mod}_R} \mathrm{Mod}_{R'}, \quad L' \longmapsto (L' \otimes_{B'} B, L' \otimes_{B'} R', can)$$

where *can* is the canonical identification $L' \otimes_{B'} B \otimes_B R = L' \otimes_{B'} R' \otimes_{R'} R$. In the following we will write (N, M', φ) for an object of the right hand side, i.e., N is a B-module, M' is an R'-module and $\varphi : N \otimes_B R \to M' \otimes_{R'} R$ is an isomorphism.

0D2F **Lemma 15.5.4.** Given a commutative diagram of rings

$$\begin{tikzcd} R & R' \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

the functor (15.5.3.1) has a right adjoint, namely the functor

$$F : (N, M', \varphi) \longmapsto N \times_\varphi M'$$

(see proof for elucidation).

Proof. Given an object (N, M', φ) of the category $\text{Mod}_B \times_{\text{Mod}_R} \text{Mod}_{R'}$ we set

$$N \times_\varphi M' = \{(n, m') \in N \times M' \mid \varphi(n \otimes 1) = m' \otimes 1 \text{ in } M' \otimes_{R'} R\}$$

viewed as a B'-module. The adjointness statement is that for a B'-module L' and a triple (N, M', φ) we have

$$\text{Hom}_{B'}(L', N \times_\varphi M') = \text{Hom}_B(L' \otimes_{B'} B, N) \times_{\text{Hom}_R(L' \otimes_{B'} R, M' \otimes_{R'} R)} \text{Hom}_{R'}(L' \otimes_{B'} R', M')$$

By Algebra, Lemma 10.14.3 the right hand side is equal to

$$\text{Hom}_{B'}(L', N) \times_{\text{Hom}_{B'}(L', M' \otimes_{R'} R)} \text{Hom}_{B'}(L', M')$$

Thus it is clear that for a pair (g, f') of elements of this fibre product we get an B'-linear map $L' \to N \times_\varphi M'$, $l' \mapsto (g(l'), f'(l'))$. Conversely, given a B' linear map $g' : L' \to N \times_\varphi M'$ we can set g equal to the composition $L' \to N \times_\varphi M' \to N$ and f' equal to the composition $L' \to N \times_\varphi M' \to M'$. These constructions are mutually inverse to each other and define the desired isomorphism. $\square$

15.6. Fibre products of rings, II

0D2G In this section we discuss fibre products in the following situation.

08KH **Situation 15.6.1.** In the following we will consider ring maps

$$B \longrightarrow A \longleftarrow A'$$

where we assume $A' \to A$ is surjective with kernel I. In this situation we set $B' = B \times_A A'$ to obtain a cartesian square

$$\begin{tikzcd} A & A' \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

0B7J **Lemma 15.6.2.** *In Situation 15.6.1 we have*

$$\text{Spec}(B') = \text{Spec}(B) \amalg_{\text{Spec}(A)} \text{Spec}(A')$$

as topological spaces.

Proof. Since $B' = B \times_A A'$ we obtain a commutative square of spectra, which induces a continuous map

$$can : \text{Spec}(B) \amalg_{\text{Spec}(A)} \text{Spec}(A') \longrightarrow \text{Spec}(B')$$

as the source is a pushout in the category of topological spaces (which exists by Topology, Section 5.29).

To show the map can is surjective, let $\mathfrak{q}' \subset B'$ be a prime ideal. If $I \subset \mathfrak{q}'$ (here and below we take the liberty of considering I as an ideal of B' as well as an ideal of A'), then $\mathfrak{q}'$ corresponds to a prime ideal of B and is in the image. If not, then pick $h \in I$, $h \notin \mathfrak{q}'$. In this case $B_h = A_h = 0$ and the ring map $B'_h \to A'_h$ is an isomorphism, see Lemma 15.5.3. Thus we see that $\mathfrak{q}'$ corresponds to a unique prime ideal $\mathfrak{p}' \subset A'$ which does not contain I.

Since $B' \to B$ is surjective, we see that can is injective on the summand $\text{Spec}(B)$. We have seen above that $\text{Spec}(A') \to \text{Spec}(B')$ is injective on the complement of $V(I) \subset \text{Spec}(A')$. Since $V(I) \subset \text{Spec}(A')$ is exactly the image of $\text{Spec}(A) \to \text{Spec}(A')$ a trivial set theoretic argument shows that can is injective.

To finish the proof we have to show that *can* is open. To do this, observe that an open of the pushout is of the form $V \amalg U'$ where $V \subset \mathrm{Spec}(B)$ and $U' \subset \mathrm{Spec}(A')$ are opens whose inverse images in $\mathrm{Spec}(A)$ agree. Let $v \in V$. We can find a $g \in B$ such that $v \in D(g) \subset V$. Let $f \in A$ be the image. Pick $f' \in A'$ mapping to f. Then $D(f') \cap U' \cap V(I) = D(f') \cap V(I)$. Hence $V(I) \cap D(f')$ and $D(f') \cap (U')^c$ are disjoint closed subsets of $D(f') = \mathrm{Spec}(A'_{f'})$. Write $(U')^c = V(J)$ for some ideal $J \subset A'$. Since $A'_{f'} \to A'_{f'}/IA'_{f'} \times A'_{f'}/JA'_{f'}$ is surjective by the disjointness just shown, we can find an $a'' \in A'_{f'}$ mapping to 1 in $A'_{f'}/IA'_{f'}$ and mapping to zero in $A'_{f'}/JA'_{f'}$. Clearing denominators, we find an element $a' \in J$ mapping to f^n in A. Then $D(a'f') \subset U'$. Let $h' = (g^{n+1}, a'f') \in B'$. Since $B'_{h'} = B_{g^{n+1}} \times_{A_{f^{n+1}}} A'_{a'f'}$ by a previously cited lemma, we see that $D(h')$ pulls back to an open neighbourhood of v in the pushout, i.e., the image of $V \amalg U'$ contains an open neighbourhood of the image of v. We omit the (easier) proof that the same thing is true for $u' \in U'$ with $u' \notin V(I)$. $\square$

0E1S **Lemma 15.6.3.** In Situation 15.6.1 if $B \to A$ is integral, then $B' \to A'$ is integral.

Proof. Let $a' \in A'$ with image $a \in A$. Let $x^d + b_1 x^{d-1} + \ldots + b_d$ be a monical polynomial with coefficients in B satisfied by a. Choose $b'_i \in B'$ mapping to $b_i \in B$ (possible). Then $(a')^d + b'_1(a')^{d-1} + \ldots + b'_d$ is in the kernel of $A' \to A$. Since $\mathrm{Ker}(B' \to B) = \mathrm{Ker}(A' \to A)$ we can modify our choice of b'_d to get $(a')^d + b'_1(a')^{d-1} + \ldots + b'_d = 0$ as desired. $\square$

In Situation 15.6.1 we'd like to understand B'-modules in terms of modules over A', A, and B. In order to do this we consider the functor (where the fibre product of categories as constructed in Categories, Example 4.31.3)

08KI (15.6.3.1) $$\mathrm{Mod}_{B'} \longrightarrow \mathrm{Mod}_B \times_{\mathrm{Mod}_A} \mathrm{Mod}_{A'}, \quad L' \longmapsto (L' \otimes_{B'} B, L' \otimes_{B'} A', can)$$

where *can* is the canonical identification $L' \otimes_{B'} B \otimes_B A = L' \otimes_{B'} A' \otimes_{A'} A$. In the following we will write (N, M', φ) for an object of the right hand side, i.e., N is a B-module, M' is an A'-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism. However, it is often more convenient think of φ as a B-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

07RU **Lemma 15.6.4.** In Situation 15.6.1 the functor (15.6.3.1) has a right adjoint, namely the functor

$$F : (N, M', \varphi) \longmapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. Moreover, the composition of F with (15.6.3.1) is the identity functor on $\mathrm{Mod}_B \times_{\mathrm{Mod}_A} \mathrm{Mod}_{A'}$. In other words, setting $N' = N \times_{\varphi, M} M'$ we have $N' \otimes_{B'} B = N$ and $N' \otimes_{B'} A' = M'$.

Proof. The adjointness statement follows from the more general Lemma 15.5.4. To prove the final assertion, recall that $B' = B \times_A A'$ and $N' = N \times_{\varphi, M} M'$ and extend these equalities to

$$\begin{tikzcd} A & A' \arrow[l] & I \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] & J \arrow[l] \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M & M' \arrow[l] & K \arrow[l] \\ N \arrow[u, "\varphi"] & N' \arrow[l] \arrow[u] & L \arrow[l] \arrow[u] \end{tikzcd}$$

where I, J, K, L are the kernels of the horizontal maps of the original diagrams. We present the proof as a sequence of observations:

(1) $K = IM'$ (see statement lemma),
(2) $B' \to B$ is surjective with kernel J and $J \to I$ is bijective,
(3) $N' \to N$ is surjective with kernel L and $L \to K$ is bijective,
(4) $JN' \subset L$,
(5) $\text{Im}(N \to M)$ generates M as an A-module (because $N \otimes_B A = M$),
(6) $\text{Im}(N' \to M')$ generates M' as an A'-module (because it holds modulo K and L maps isomorphically to K),
(7) $JN' = L$ (because $L \cong K = IM'$ is generated by images of elements xn' with $x \in I$ and $n' \in N'$ by the previous statement),
(8) $N' \otimes_{B'} B = N$ (because $N = N'/L$, $B = B'/J$, and the previous statement),
(9) there is a map $\gamma : N' \otimes_{B'} A' \to M'$,
(10) γ is surjective (see above),
(11) the kernel of the composition $N' \otimes_{B'} A' \to M' \to M$ is generated by elements $l \otimes 1$ and $n' \otimes x$ with $l \in K$, $n' \in N'$, $x \in I$ (because $M = N \otimes_B A$ by assumption and because $N' \to N$ and $A' \to A$ are surjective with kernels L and I),
(12) any element of $N' \otimes_{B'} A'$ in the submodule generated by the elements $l \otimes 1$ and $n' \otimes x$ with $l \in L$, $n' \in N'$, $x \in I$ can be written as $l \otimes 1$ for some $l \in L$ (because J maps isomorphically to I we see that $n' \otimes x = n'x \otimes 1$ in $N' \otimes_{B'} A'$; similarly $xn' \otimes a' = n' \otimes xa' = n'(xa') \otimes 1$ in $N' \otimes_{B'} A'$ when $n' \in N'$, $x \in J$ and $a' \in A'$; since we have seen that $JN' = L$ this proves the assertion),
(13) the kernel of γ is zero (because by (10) and (11) any element of the kernel is of the form $l \otimes 1$ with $l \in L$ which is mapped to $l \in K \subset M'$ by γ).

This finishes the proof. $\square$

08IG Lemma 15.6.5. In the situation of Lemma 15.6.4 for a B'-module L' the adjunction map

$$L' \longrightarrow (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A)} (L' \otimes_{B'} A')$$

is surjective but in general not injective.

Proof. As in the proof of Lemma 15.6.4 let $J \subset B'$ be the kernel of the map $B' \to B$. Then $L' \otimes_{B'} B = L'/JL'$. Hence to prove surjectivity it suffices to show that elements of the form $(0, z)$ of the fibre product are in the image of the map of the lemma. The kernel of the map $L' \otimes_{B'} A' \to L' \otimes_{B'} A$ is the image of $L' \otimes_{B'} I \to L' \otimes_{B'} A'$. Since the map $J \to I$ induced by $B' \to A'$ is an isomorphism the composition

$$L' \otimes_{B'} J \to L' \to (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A)} (L' \otimes_{B'} A')$$

induces a surjection of $L' \otimes_{B'} J$ onto the set of elements of the form $(0, z)$. To see the map is not injective in general we present a simple example. Namely, take a field k, set $B' = k[x, y]/(xy)$, $A' = B'/(x)$, $B = B'/(y)$, $A = B'/(x, y)$ and $L' = B'/(x - y)$. In that case the class of x in L' is nonzero but is mapped to zero under the displayed arrow. $\square$

08KJ Lemma 15.6.6. In Situation 15.6.1 let $(N_1, M_1', \varphi_1) \to (N_2, M_2', \varphi_2)$ be a morphism of $\text{Mod}_B \times_{\text{Mod}_A} \text{Mod}_{A'}$ with $N_1 \to N_2$ and $M_1' \to M_2'$ surjective. Then

$$N_1 \times_{\varphi_1, M_1} M_1' \to N_2 \times_{\varphi_2, M_2} M_2'$$

where $M_1 = M'_1/IM'_1$ and $M_2 = M'_2/IM'_2$ is surjective.

Proof. Pick $(x_2, y_2) \in N_2 \times_{\varphi_2, M_2} M'_2$. Choose $x_1 \in N_1$ mapping to x_2. Since $M'_1 \to M_1$ is surjective we can find $y_1 \in M'_1$ mapping to $\varphi_1(x_1)$. Then (x_1, y_1) maps to (x_2, y'_2) in $N_2 \times_{\varphi_2, M_2} M'_2$. Thus it suffices to show that elements of the form $(0, y_2)$ are in the image of the map. Here we see that $y_2 \in IM'_2$. Write $y_2 = \sum t_i y_{2,i}$ with $t_i \in I$. Choose $y_{1,i} \in M'_1$ mapping to $y_{2,i}$. Then $y_1 = \sum t_i y_{1,i} \in IM'_1$ and the element $(0, y_1)$ does the job. $\square$

0D2H Lemma 15.6.7. Let $A, A', B, B', I, M, M', N, \varphi$ be as in Lemma 15.6.4. If N finite over B and M' finite over A', then $N' = N \times_{\varphi, M} M'$ is finite over B'.

Proof. We will use the results of Lemma 15.6.4 without further mention. Choose generators $y_1, \ldots, y_r$ of N over B and generators $x_1, \ldots, x_s$ of M' over A'. Using that $N = N' \otimes_{B'} B$ and $B' \to B$ is surjective we can find $u_1, \ldots, u_r \in N'$ mapping to $y_1, \ldots, y_r$ in N. Using that $M' = N' \otimes_{B'} A'$ we can find $v_1, \ldots, v_t \in N'$ such that $x_i = \sum v_j \otimes a'_{ij}$ for some $a'_{ij} \in A'$. In particular we see that the images $\overline{v}_j \in M'$ of the v_j generate M' over A'. We claim that $u_1, \ldots, u_r, v_1, \ldots, v_t$ generate N' as a B'-module. Namely, pick $\xi \in N'$. We first choose $b'_1, \ldots, b'_r \in B'$ such that ξ and $\sum b'_i u_i$ map to the same element of N. This is possible because $B' \to B$ is surjective and $y_1, \ldots, y_r$ generate N over B. The difference $\xi - \sum b'_i u_i$ is of the form $(0, \theta)$ for some θ in IM'. Say θ is $\sum t_j \overline{v}_j$ with $t_j \in I$. As $J = \mathrm{Ker}(B' \to B)$ maps isomorphically to I we can choose $s_j \in J \subset B'$ mapping to t_j. Because $N' = N \times_{\varphi, M} M'$ it follows that $\xi = \sum b'_i u_i + \sum s_j v_j$ as desired. $\square$

0D2I Lemma 15.6.8. With A, A', B, B', I as in Situation 15.6.1.

(1) Let (N, M', φ) be an object of $\mathrm{Mod}_B \times_{\mathrm{Mod}_A} \mathrm{Mod}_{A'}$. If M' is flat over A' and N is flat over B, then $N' = N \times_{\varphi, M} M'$ is flat over B'.
(2) If L' is a flat B'-module, then $L' = (L \otimes_{B'} B) \times_{(L \otimes_{B'} A)} (L \otimes_{B'} A')$.
(3) The category of flat B'-modules is equivalent to the full subcategory of $\mathrm{Mod}_B \times_{\mathrm{Mod}_A} \mathrm{Mod}_{A'}$ consisting of triples (N, M', φ) with N flat over B and M' flat over A'.

Proof. In the proof we will use Lemma 15.6.4 without further mention.

Proof of (1). Set $J = \mathrm{Ker}(B' \to B)$. This is an ideal of B' mapping isomorphically to $I = \mathrm{Ker}(A' \to A)$. Let $\mathfrak{b}' \subset B'$ be an ideal. We have to show that $\mathfrak{b}' \otimes_{B'} N' \to N'$ is injective, see Algebra, Lemma 10.39.5. We know that

$$\mathfrak{b}'/(\mathfrak{b}' \cap J) \otimes_{B'} N' = \mathfrak{b}'/(\mathfrak{b}' \cap J) \otimes_B N \to N$$

is injective as N is flat over B. As $\mathfrak{b}' \cap J \to \mathfrak{b}' \to \mathfrak{b}'/(\mathfrak{b}' \cap J) \to 0$ is exact, we conclude that it suffices to show that $(\mathfrak{b}' \cap J) \otimes_{B'} N' \to N'$ is injective. Thus we may assume that $\mathfrak{b}' \subset J$. Next, since $J \to I$ is an isomorphism we have

$$J \otimes_{B'} N' = I \otimes_{A'} A' \otimes_{B'} N' = I \otimes_{A'} M'$$

which maps injectively into M' as M' is a flat A'-module. Hence $J \otimes_{B'} N' \to N'$ is injective and we conclude that $\mathrm{Tor}_1^{B'}(B'/J, N') = 0$, see Algebra, Remark 10.75.9. Thus we may apply Algebra, Lemma 10.99.8 to N' over B' and the ideal J. Going back to our ideal $\mathfrak{b}' \subset J$, let $\mathfrak{b}' \subset \mathfrak{b}'' \subset J$ be the smallest ideal whose image in I is an A'-submodule of I. In other words, we have $\mathfrak{b}'' = A'\mathfrak{b}'$ if we view $J = I$ as A'-module. Then $\mathfrak{b}''/\mathfrak{b}'$ is killed by J and we get a short exact sequence

$$0 \to \mathfrak{b}' \otimes_{B'} N' \to \mathfrak{b}'' \otimes_{B'} N' \to \mathfrak{b}''/\mathfrak{b}' \otimes_{B'} N' \to 0$$

by the vanishing of $\text{Tor}_1^{B'}(\mathfrak b''/\mathfrak b', N')$ we get from the application of the lemma. Thus we may replace $\mathfrak b'$ by $\mathfrak b''$. In particular we may assume $\mathfrak b'$ is an A'-module and maps to an ideal of A'. Then

$$\mathfrak b' \otimes_{B'} N' = \mathfrak b' \otimes_{A'} A' \otimes_{B'} N' = \mathfrak b' \otimes_{A'} M'$$

This tensor product maps injectively into M' by our assumption that M' is flat over A'. We conclude that $\mathfrak b' \otimes_{B'} N' \to N' \to M'$ is injective and hence the first map is injective as desired.

Proof of (2). This follows by tensoring the short exact sequence $0 \to B' \to B \oplus A' \to A \to 0$ with L' over B'.

Proof of (3). Immediate consequence of (1) and (2). □

0D2J Lemma 15.6.9. Let A, A', B, B', I be as in Situation 15.6.1. The category of finite projective B'-modules is equivalent to the full subcategory of $\text{Mod}_B \times_{\text{Mod}_A} \text{Mod}_{A'}$ consisting of triples (N, M', φ) with N finite projective over B and M' finite projective over A'.

Proof. Recall that a module is finite projective if and only if it is finitely presented and flat, see Algebra, Lemma 10.78.2. Using Lemmas 15.6.8 and 15.6.7 we reduce to showing that $N' = N \times_{\varphi, M} M'$ is a B'-module of finite presentation if N finite projective over B and M' finite projective over A'.

By Lemma 15.6.7 the module N' is finite over B'. Choose a surjection $(B')^{\oplus n} \to N'$ with kernel K'. By base change we obtain maps $B^{\oplus n} \to N$, $(A')^{\oplus n} \to M'$, and $A^{\oplus n} \to M$ with kernels K_B, $K_{A'}$, and K_A. There is a canonical map

$$K' \longrightarrow K_B \times_{K_A} K_{A'}$$

On the other hand, since $N' = N \times_{\varphi, M} M'$ and $B' = B \times_A A'$ there is also a canonical map $K_B \times_{K_A} K_{A'} \to K'$ inverse to the displayed arrow. Hence the displayed map is an isomorphism. By Algebra, Lemma 10.5.3 the modules K_B and $K_{A'}$ are finite. We conclude from Lemma 15.6.7 that K' is a finite B'-module provided that $K_B \to K_A$ and $K_{A'} \to K_A$ induce isomorphisms $K_B \otimes_B A = K_A = K_{A'} \otimes_{A'} A$. This is true because the flatness assumptions implies the sequences

$$0 \to K_B \to B^{\oplus n} \to N \to 0 \quad \text{and} \quad 0 \to K_{A'} \to (A')^{\oplus n} \to M' \to 0$$

stay exact upon tensoring, see Algebra, Lemma 10.39.12. □

15.7. Fibre products of rings, III

0D2K In this section we discuss fibre products in the following situation.

08KK Situation 15.7.1. Let A, A', B, B', I be as in Situation 15.6.1. Let $B' \to D'$ be a ring map. Set $D = D' \otimes_{B'} B$, $C' = D' \otimes_{B'} A'$, and $C = D' \otimes_{B'} A$. This leads to a big commutative diagram

$$\begin{tikzcd}
C & & & C' \arrow[lll] \\
& A \arrow[ul] & A' \arrow[l] \arrow[ur] & \\
& B \arrow[u] \arrow[dl] & B' \arrow[l] \arrow[u] \arrow[dr] & \\
D \arrow[uuu] & & & D' \arrow[lll] \arrow[uuu]
\end{tikzcd}$$

of rings. Observe that we do not assume that the map $D' \to D \times_C C'$ is an isomorphism[1]. In this situation we have the functor

08KL (15.7.1.1) $$\text{Mod}_{D'} \longrightarrow \text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}, \quad L' \longmapsto (L' \otimes_{D'} D, L' \otimes_{D'} C', can)$$

analogous to (15.6.3.1). Note that $L' \otimes_{D'} D = L \otimes_{D'} (D' \otimes_{B'} B) = L \otimes_{B'} B$ and similarly $L' \otimes_{D'} C' = L \otimes_{D'} (D' \otimes_{B'} A') = L \otimes_{B'} A'$ hence the diagram

$$\begin{tikzcd}
\text{Mod}_{D'} \arrow[r] \arrow[d] & \text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'} \arrow[d] \\
\text{Mod}_{B'} \arrow[r] & \text{Mod}_B \times_{\text{Mod}_A} \text{Mod}_{A'}
\end{tikzcd}$$

is commutative. In the following we will write (N, M', φ) for an object of $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$, i.e., N is a D-module, M' is an C'-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism of C-modules. However, it is often more convenient think of φ as a D-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

08KM **Lemma 15.7.2.** In Situation 15.7.1 the functor (15.7.1.1) has a right adjoint, namely the functor

$$F : (N, M', \varphi) \longmapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. Moreover, the composition of F with (15.7.1.1) is the identity functor on $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$. In other words, setting $N' = N \times_{\varphi, M} M'$ we have $N' \otimes_{D'} D = N$ and $N' \otimes_{D'} C' = M'$.

Proof. The adjointness statement follows from the more general Lemma 15.5.4. The final assertion follows from the corresponding assertion of Lemma 15.6.4 because $N' \otimes_{D'} D = N' \otimes_{D'} D' \otimes_{B'} B = N' \otimes_{B'} B$ and $N' \otimes_{D'} C' = N' \otimes_{D'} D' \otimes_{B'} A' = N' \otimes_{B'} A'$. $\square$

08KN **Lemma 15.7.3.** In Situation 15.7.1 the map $JD' \to IC'$ is surjective where $J = \text{Ker}(B' \to B)$.

Proof. Since $C' = D' \otimes_{B'} A'$ we have that IC' is the image of $D' \otimes_{B'} I = C' \otimes_{A'} I \to C'$. As the ring map $B' \to A'$ induces an isomorphism $J \to I$ the lemma follows. $\square$

08IH **Lemma 15.7.4.** Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 15.7.2. If N finite over D and M' finite over C', then $N' = N \times_{\varphi, M} M'$ is finite over D'.

[1] But $D' \to D \times_C C'$ is surjective by Lemma 15.6.5.

Proof. Recall that $D' \to D \times_C C'$ is surjective by Lemma 15.6.5. Observe that $N' = N \times_{\varphi,M} M'$ is a module over $D \times_C C'$. We can apply Lemma 15.6.7 to the data $C, C', D, D', IC', M', M, N, \varphi$ to see that $N' = N \times_{\varphi,M} M'$ is finite over $D \times_C C'$. Thus it is finite over D'. $\square$

07RW Lemma 15.7.5. With $A, A', B, B', C, C', D, D', I$ as in Situation 15.7.1.

(1) Let (N, M', φ) be an object of $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$. If M' is flat over A' and N is flat over B, then $N' = N \times_{\varphi,M} M'$ is flat over B'.
(2) If L' is a D'-module flat over B', then $L' = (L \otimes_{D'} D) \times_{(L \otimes_{D'} C)} (L \otimes_{D'} C')$.
(3) The category of D'-modules flat over B' is equivalent to the categories of objects (N, M', φ) of $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$ with N flat over B and M' flat over A'.

Proof. Part (1) follows from part (1) of Lemma 15.6.8.

Part (2) follows from part (2) of Lemma 15.6.8 using that $L' \otimes_{D'} D = L' \otimes_{B'} B$, $L' \otimes_{D'} C' = L' \otimes_{B'} A'$, and $L' \otimes_{D'} C = L' \otimes_{B'} A$, see discussion in Situation 15.7.1.

Part (3) is an immediate consequence of (1) and (2). $\square$

The following lemma is a good deal more interesting than its counter part in the absolute case (Lemma 15.6.9), although the proof is essentially the same.

08KP Lemma 15.7.6. Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 15.7.2. If

(1) N is finitely presented over D and flat over B,
(2) M' finitely presented over C' and flat over A', and
(3) the ring map $B' \to D'$ factors as $B' \to D'' \to D'$ with $B' \to D''$ flat and $D'' \to D'$ of finite presentation,

then $N' = N \times_M M'$ is finitely presented over D'.

Proof. Choose a surjection $D''' = D''[x_1, \ldots, x_n] \to D'$ with finitely generated kernel J. By Algebra, Lemma 10.36.23 it suffices to show that N' is finitely presented as a D'''-module. Moreover, $D''' \otimes_{B'} B \to D' \otimes_{B'} B = D$ and $D''' \otimes_{B'} A' \to D' \otimes_{B'} A' = C'$ are surjections whose kernels are generated by the image of J, hence N is a finitely presented $D''' \otimes_{B'} B$-module and M' is a finitely presented $D''' \otimes_{B'} A'$-module by Algebra, Lemma 10.36.23 again. Thus we may replace D' by D''' and D by $D''' \otimes_{B'} B$, etc. Since D''' is flat over B', it follows that we may assume that $B' \to D'$ is flat.

Assume $B' \to D'$ is flat. By Lemma 15.7.4 the module N' is finite over D'. Choose a surjection $(D')^{\oplus n} \to N'$ with kernel K'. By base change we obtain maps $D^{\oplus n} \to N$, $(C')^{\oplus n} \to M'$, and $C^{\oplus n} \to M$ with kernels K_D, $K_{C'}$, and K_C. There is a canonical map

$$K' \longrightarrow K_D \times_{K_C} K_{C'}$$

On the other hand, since $N' = N \times_M M'$ and $D' = D \times_C C'$ (by Lemma 15.6.8; applied to the flat B'-module D') there is also a canonical map $K_D \times_{K_C} K_{C'} \to K'$ inverse to the displayed arrow. Hence the displayed map is an isomorphism. By Algebra, Lemma 10.5.3 the modules K_D and $K_{C'}$ are finite. We conclude from Lemma 15.7.4 that K' is a finite D'-module provided that $K_D \to K_C$ and $K_{C'} \to$

K_C induce isomorphisms $K_D \otimes_B A = K_C = K_{C'} \otimes_{A'} A$. This is true because the flatness assumptions implies the sequences

$$0 \to K_D \to D^{\oplus n} \to N \to 0 \quad \text{and} \quad 0 \to K_{C'} \to (C')^{\oplus n} \to M' \to 0$$

stay exact upon tensoring, see Algebra, Lemma 10.39.12. □

08KQ **Lemma 15.7.7.** Let A, A', B, B', I be as in Situation 15.6.1. Let (D, C', φ) be a system consisting of an B-algebra D, a A'-algebra C' and an isomorphism $D \otimes_B A \to C'/IC' = C$. Set $D' = D \times_C C'$ (as in Lemma 15.6.4). Then

(1) $B' \to D'$ is finite type if and only if $B \to D$ and $A' \to C'$ are finite type,
(2) $B' \to D'$ is flat if and only if $B \to D$ and $A' \to C'$ are flat,
(3) $B' \to D'$ is flat and of finite presentation if and only if $B \to D$ and $A' \to C'$ are flat and of finite presentation,
(4) $B' \to D'$ is smooth if and only if $B \to D$ and $A' \to C'$ are smooth,
(5) $B' \to D'$ is étale if and only if $B \to D$ and $A' \to C'$ are étale.

Moreover, if D' is a flat B'-algebra, then $D' \to (D' \otimes_{B'} B) \times_{(D' \otimes_{B'} A)} (D' \otimes_{B'} A')$ is an isomorphism. In this way the category of flat B'-algebras is equivalent to the categories of systems (D, C', φ) as above with D flat over B and C' flat over A'.

Proof. The implication "⇒" follows from Algebra, Lemmas 10.14.2, 10.39.7, 10.137.4, and 10.143.3 because we have $D' \otimes_{B'} B = D$ and $D' \otimes_{B'} A' = C'$ by Lemma 15.6.4. Thus it suffices to prove the implications in the other direction.

Ad (1). Assume D of finite type over B and C' of finite type over A'. We will use the results of Lemma 15.6.4 without further mention. Choose generators $x_1, \ldots, x_r$ of D over B and generators $y_1, \ldots, y_s$ of C' over A'. Using that $D = D' \otimes_{B'} B$ and $B' \to B$ is surjective we can find $u_1, \ldots, u_r \in D'$ mapping to $x_1, \ldots, x_r$ in D. Using that $C' = D' \otimes_{B'} A'$ we can find $v_1, \ldots, v_t \in D'$ such that $y_i = \sum v_j \otimes a'_{ij}$ for some $a'_{ij} \in A'$. In particular, the images of v_j in C' generate C' as an A'-algebra. Set $N = r + t$ and consider the cube of rings

$$\begin{tikzcd}
A[x_1, \ldots, x_N] & & A'[x_1, \ldots, x_N] \arrow[ll] & \\
& A \arrow[ul] & & A' \arrow[ll] \arrow[ul] \\
B[x_1, \ldots, x_N] \arrow[uu] & & B'[x_1, \ldots, x_N] \arrow[ll] \arrow[uu] & \\
& B \arrow[uu] \arrow[ul] & & B' \arrow[ll] \arrow[uu] \arrow[ul]
\end{tikzcd}$$

Observe that the back square is cartesian as well. Consider the ring map

$$B'[x_1, \ldots, x_N] \to D', \quad x_i \mapsto u_i \quad \text{and} \quad x_{r+j} \mapsto v_j.$$

Then we see that the induced maps $B[x_1, \ldots, x_N] \to D$ and $A'[x_1, \ldots, x_N] \to C'$ are surjective, in particular finite. We conclude from Lemma 15.7.4 that $B'[x_1, \ldots, x_N] \to D'$ is finite, which implies that D' is of finite type over B' for example by Algebra, Lemma 10.6.2.

Ad (2). The implication "⇐" follows from Lemma 15.7.5. Moreover, the final statement follows from the final statement of Lemma 15.7.5.

Ad (3). Assume $B \to D$ and $A' \to C'$ are flat and of finite presentation. The flatness of $B' \to D'$ we've seen in (2). We know $B' \to D'$ is of finite type by (1). Choose a surjection $B'[x_1, \ldots, x_N] \to D'$. By Algebra, Lemma 10.6.3 the ring D is of finite presentation as a $B[x_1, \ldots, x_N]$-module and the ring C' is of finite presentation as a $A'[x_1, \ldots, x_N]$-module. By Lemma 15.7.6 we see that D' is of finite presentation as a $B'[x_1, \ldots, x_N]$-module, i.e., $B' \to D'$ is of finite presentation.

Ad (4). Assume $B \to D$ and $A' \to C'$ smooth. By (3) we see that $B' \to D'$ is flat and of finite presentation. By Algebra, Lemma 10.137.17 it suffices to check that $D' \otimes_{B'} k$ is smooth for any field k over B'. If the composition $J \to B' \to k$ is zero, then $B' \to k$ factors as $B' \to B \to k$ and we see that

$$D' \otimes_{B'} k = D' \otimes_{B'} B \otimes_B k = D \otimes_B k$$

is smooth as $B \to D$ is smooth. If the composition $J \to B' \to k$ is nonzero, then there exists an $h \in J$ which does not map to zero in k. Then $B' \to k$ factors as $B' \to B'_h \to k$. Observe that h maps to zero in B, hence $B_h = 0$. Thus by Lemma 15.5.3 we have $B'_h = A'_h$ and we get

$$D' \otimes_{B'} k = D' \otimes_{B'} B'_h \otimes_{B'_h} k = C'_h \otimes_{A'_h} k$$

is smooth as $A' \to C'$ is smooth.

Ad (5). Assume $B \to D$ and $A' \to C'$ are étale. By (4) we see that $B' \to D'$ is smooth. As we can read off whether or not a smooth map is étale from the dimension of fibres we see that (5) holds (argue as in the proof of (4) to identify fibres – some details omitted). □

08KR Remark 15.7.8. In Situation 15.7.1. Assume $B' \to D'$ is of finite presentation and suppose we are given a D'-module L'. We claim there is a bijective correspondence between

(1) surjections of D'-modules $L' \to Q'$ with Q' of finite presentation over D' and flat over B', and
(2) pairs of surjections of modules $(L' \otimes_{D'} D \to Q_1, L' \otimes_{D'} C' \to Q_2)$ with
    (a) Q_1 of finite presentation over D and flat over B,
    (b) Q_2 of finite presentation over C' and flat over A',
    (c) $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ as quotients of $L' \otimes_{D'} C$.

The correspondence between these is given by $Q \mapsto (Q_1, Q_2)$ with $Q_1 = Q \otimes_{D'} D$ and $Q_2 = Q \otimes_{D'} C'$. And for the converse we use $Q = Q_1 \times_{Q_{12}} Q_2$ where Q_{12} the common quotient $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ of $L' \otimes_{D'} C$. As quotient map we use

$$L' \longrightarrow (L' \otimes_{D'} D) \times_{(L' \otimes_{D'} C)} (L' \otimes_{D'} C') \longrightarrow Q_1 \times_{Q_{12}} Q_2 = Q$$

where the first arrow is surjective by Lemma 15.6.5 and the second by Lemma 15.6.6. The claim follows by Lemmas 15.7.5 and 15.7.6.

15.8. Fitting ideals

07Z6 The Fitting ideals of a finite module are the ideals determined by the construction of Lemma 15.8.2.

07Z7 Lemma 15.8.1. Let R be a ring. Let A be an $n \times m$ matrix with coefficients in R. Let $I_r(A)$ be the ideal generated by the $r \times r$-minors of A with the convention that $I_0(A) = R$ and $I_r(A) = 0$ if $r > \min(n, m)$. Then

(1) $I_0(A) \supset I_1(A) \supset I_2(A) \supset \ldots$,

(2) if B is an $(n + n') \times m$ matrix, and A is the first n rows of B, then $I_{r+n'}(B) \subset I_r(A)$,
(3) if C is an $n \times n$ matrix then $I_r(CA) \subset I_r(A)$.
(4) If A is a block matrix
$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$
then $I_r(A) = \sum_{r_1+r_2=r} I_{r_1}(A_1)I_{r_2}(A_2)$.
(5) Add more here.

Proof. Omitted. (Hint: Use that a determinant can be computed by expanding along a column or a row.) ☐

07Z8 Lemma 15.8.2. Let R be a ring. Let M be a finite R-module. Choose a presentation
$$\bigoplus\nolimits_{j\in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$
of M. Let $A = (a_{ij})_{i=1,\ldots,n,j\in J}$ be the matrix of the map $\bigoplus_{j\in J} R \to R^{\oplus n}$. The ideal $\text{Fit}_k(M)$ generated by the $(n-k) \times (n-k)$ minors of A is independent of the choice of the presentation.

Proof. Let $K \subset R^{\oplus n}$ be the kernel of the surjection $R^{\oplus n} \to M$. Pick $z_1, \ldots, z_{n-k} \in K$ and write $z_j = (z_{1j}, \ldots, z_{nj})$. Another description of the ideal $\text{Fit}_k(M)$ is that it is the ideal generated by the $(n-k) \times (n-k)$ minors of all the matrices (z_{ij}) we obtain in this way.

Suppose we change the surjection into the surjection $R^{\oplus n+n'} \to M$ with kernel K' where we use the original map on the first n standard basis elements of $R^{\oplus n+n'}$ and 0 on the last n' basis vectors. Then the corresponding ideals are the same. Namely, if $z_1, \ldots, z_{n-k} \in K$ as above, let $z'_j = (z_{1j}, \ldots, z_{nj}, 0, \ldots, 0) \in K'$ for $j = 1, \ldots, n-k$ and $z'_{n+j'} = (0, \ldots, 0, 1, 0, \ldots, 0) \in K'$. Then we see that the ideal of $(n-k)\times(n-k)$ minors of (z_{ij}) agrees with the ideal of $(n+n'-k)\times(n+n'-k)$ minors of (z'_{ij}). This gives one of the inclusions. Conversely, given $z'_1, \ldots, z'_{n+n'-k}$ in K' we can project these to $R^{\oplus n}$ to get $z_1, \ldots, z_{n+n'-k}$ in K. By Lemma 15.8.1 we see that the ideal generated by the $(n + n' - k) \times (n + n' - k)$ minors of (z'_{ij}) is contained in the ideal generated by the $(n - k) \times (n - k)$ minors of (z_{ij}). This gives the other inclusion.

Let $R^{\oplus m} \to M$ be another surjection with kernel L. By Schanuel's lemma (Algebra, Lemma 10.109.1) and the results of the previous paragraph, we may assume $m = n$ and that there is an isomorphism $R^{\oplus n} \to R^{\oplus m}$ commuting with the surjections to M. Let $C = (c_{li})$ be the (invertible) matrix of this map (it is a square matrix as $n = m$). Then given $z'_1, \ldots, z'_{n-k} \in L$ as above we can find $z_1, \ldots, z_{n-k} \in K$ with $z'_1 = Cz_1, \ldots, z'_{n-k} = Cz_{n-k}$. By Lemma 15.8.1 we get one of the inclusions. By symmetry we get the other. ☐

07Z9 Definition 15.8.3. Let R be a ring. Let M be a finite R-module. Let $k \geq 0$. The kth Fitting ideal of M is the ideal $\text{Fit}_k(M)$ constructed in Lemma 15.8.2. Set $\text{Fit}_{-1}(M) = 0$.

Since the Fitting ideals are the ideals of minors of a big matrix (numbered in reverse ordering from the ordering in Lemma 15.8.1) we see that
$$0 = \text{Fit}_{-1}(M) \subset \text{Fit}_0(M) \subset \text{Fit}_1(M) \subset \ldots \subset \text{Fit}_t(M) = R$$

for some $t \gg 0$. Here are some basic properties of Fitting ideals.

07ZA Lemma 15.8.4. Let R be a ring. Let M be a finite R-module.

(1) If M can be generated by n elements, then $\text{Fit}_n(M) = R$.
(2) Given a second finite R-module M' we have
$$\text{Fit}_l(M \oplus M') = \sum\nolimits_{k+k'=l} \text{Fit}_k(M)\text{Fit}_{k'}(M')$$
(3) If $R \to R'$ is a ring map, then $\text{Fit}_k(M \otimes_R R')$ is the ideal of R' generated by the image of $\text{Fit}_k(M)$.
(4) If M is of finite presentation, then $\text{Fit}_k(M)$ is a finitely generated ideal.
(5) If $M \to M'$ is a surjection, then $\text{Fit}_k(M) \subset \text{Fit}_k(M')$.
(6) We have $\text{Fit}_0(M) \subset \text{Ann}_R(M)$.
(7) We have $V(\text{Fit}_0(M)) = \text{Supp}(M)$.
(8) Add more here.

Proof. Part (1) follows from the fact that $I_0(A) = R$ in Lemma 15.8.1.

Part (2) follows form the corresponding statement in Lemma 15.8.1.

Part (3) follows from the fact that $\otimes_R R'$ is right exact, so the base change of a presentation of M is a presentation of $M \otimes_R R'$.

Proof of (4). Let $R^{\oplus m} \xrightarrow{A} R^{\oplus n} \to M \to 0$ be a presentation. Then $\text{Fit}_k(M)$ is the ideal generated by the $n-k \times n-k$ minors of the matrix A.

Part (5) is immediate from the definition.

Proof of (6). Choose a presentation of M with matrix A as in Lemma 15.8.2. Let $J' \subset J$ be a subset of cardinality n. It suffices to show that $f = \det(a_{ij})_{i=1,\ldots,n,j \in J'}$ annihilates M. This is clear because the cokernel of
$$R^{\oplus n} \xrightarrow{A' = (a_{ij})_{i=1,\ldots,n,j\in J'}} R^{\oplus n} \to M \to 0$$
is killed by f as there is a matrix B with $A'B = f1_{n\times n}$.

Proof of (7). Choose a presentation of M with matrix A as in Lemma 15.8.2. By Nakayama's lemma (Algebra, Lemma 10.20.1) we have
$$M_\mathfrak{p} \neq 0 \Leftrightarrow M \otimes_R \kappa(\mathfrak{p}) \neq 0 \Leftrightarrow \text{rank}(\text{image } A \text{ in } \kappa(\mathfrak{p})) < n$$
Clearly $\text{Fit}_0(M)$ exactly cuts out the set of primes with this property. □

07ZB Example 15.8.5. Let R be a ring. The Fitting ideals of the finite free module $M = R^{\oplus n}$ are $\text{Fit}_k(M) = 0$ for $k < n$ and $\text{Fit}_k(M) = R$ for $k \geq n$.

07ZC Lemma 15.8.6. Let R be a ring. Let M be a finite R-module. Let $k \geq 0$. Let $\mathfrak{p} \subset R$ be a prime ideal. The following are equivalent

(1) $\text{Fit}_k(M) \not\subset \mathfrak{p}$,
(2) $\dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p}) \leq k$,
(3) $M_\mathfrak{p}$ can be generated by k elements over $R_\mathfrak{p}$, and
(4) M_f can be generated by k elements over R_f for some $f \in R$, $f \notin \mathfrak{p}$.

Proof. By Nakayama's lemma (Algebra, Lemma 10.20.1) we see that M_f can be generated by k elements over R_f for some $f \in R$, $f \notin \mathfrak{p}$ if $M \otimes_R \kappa(\mathfrak{p})$ can be generated by k elements. Hence (2), (3), and (4) are equivalent. Using Lemma 15.8.4 part (3) this reduces the problem to the case where R is a field and $\mathfrak{p} = (0)$. In this case the result follows from Example 15.8.5. □

07ZD Lemma 15.8.7. Let R be a ring. Let M be a finite R-module. Let $r \geq 0$. The following are equivalent

1. M is finite locally free of rank r (Algebra, Definition 10.78.1),
2. $\text{Fit}_{r-1}(M) = 0$ and $\text{Fit}_r(M) = R$, and
3. $\text{Fit}_k(M) = 0$ for $k < r$ and $\text{Fit}_k(M) = R$ for $k \geq r$.

Proof. It is immediate that (2) is equivalent to (3) because the Fitting ideals form an increasing sequence of ideals. Since the formation of $\text{Fit}_k(M)$ commutes with base change (Lemma 15.8.4) we see that (1) implies (2) by Example 15.8.5 and glueing results (Algebra, Section 10.23). Conversely, assume (2). By Lemma 15.8.6 we may assume that M is generated by r elements. Thus a presentation $\bigoplus_{j \in J} R \to R^{\oplus r} \to M \to 0$. But now the assumption that $\text{Fit}_{r-1}(M) = 0$ implies that all entries of the matrix of the map $\bigoplus_{j \in J} R \to R^{\oplus r}$ are zero. Thus M is free. $\square$

080Z Lemma 15.8.8. Let R be a local ring. Let M be a finite R-module. Let $k \geq 0$. Assume that $\text{Fit}_k(M) = (f)$ for some $f \in R$. Let M' be the quotient of M by $\{x \in M \mid fx = 0\}$. Then M' can be generated by k elements.

Proof. Choose generators $x_1, \ldots, x_n \in M$ corresponding to the surjection $R^{\oplus n} \to M$. Since R is local if a set of elements $E \subset (f)$ generates (f), then some $e \in E$ generates (f), see Algebra, Lemma 10.20.1. Hence we may pick $z_1, \ldots, z_{n-k}$ in the kernel of $R^{\oplus n} \to M$ such that some $(n-k) \times (n-k)$ minor of the $n \times (n-k)$ matrix $A = (z_{ij})$ generates (f). After renumbering the x_i we may assume the first minor $\det(z_{ij})_{1 \leq i,j \leq n-k}$ generates (f), i.e., $\det(z_{ij})_{1 \leq i,j \leq n-k} = uf$ for some unit $u \in R$. Every other minor is a multiple of f. By Algebra, Lemma 10.15.6 there exists a $n-k \times n-k$ matrix B such that

$$AB = f \begin{pmatrix} u1_{n-k \times n-k} \\ C \end{pmatrix}$$

for some matrix C with coefficients in R. This implies that for every $i \leq n-k$ the element $y_i = ux_i + \sum_j c_{ji} x_j$ is annihilated by f. Since $M/\sum Ry_i$ is generated by the images of $x_{n-k+1}, \ldots, x_n$ we win. $\square$

0F7M Lemma 15.8.9. Let R be a ring. Let M be a finitely presented R-module. Let $k \geq 0$. Assume that $\text{Fit}_k(M) = (f)$ for some nonzerodivisor $f \in R$ and $\text{Fit}_{k-1}(M) = 0$. Then

1. M has projective dimension ≤ 1,
2. $M' = \text{Ker}(f : M \to M)$ is the f-power torsion submodule of M,
3. M' has projective dimension ≤ 1,
4. M/M' is finite locally free of rank k, and
5. $M \cong M/M' \oplus M'$.

Proof. Choose a presentation

$$R^{\oplus m} \xrightarrow{A} R^{\oplus n} \to M \to 0$$

for some matrix A with coefficients in R.

We first prove the lemma when R is local. Set $M' = \{x \in M \mid fx = 0\}$ as in the statement. By Lemma 15.8.8 we can choose $x_1, \ldots, x_k \in M$ which generate M/M'. Then $x_1, \ldots, x_k$ generate $M_f = (M/M')_f$. Hence, if there is a relation $\sum a_i x_i = 0$ in M, then we see that $a_1, \ldots, a_k$ map to zero in R_f since otherwise $\text{Fit}_{k-1}(M)R_f = \text{Fit}_{k-1}(M_f)$ would be nonzero. Since f is a nonzerodivisor, we conclude $a_1 = \ldots = a_k = 0$. Thus $M \cong R^{\oplus k} \oplus M'$. After a change of basis in our presentation above, we may assume the first $n-k$ basis vectors of $R^{\oplus n}$ map into the summand M' of M and the last k-basis vectors of $R^{\oplus n}$ map to basis elements of the summand $R^{\oplus k}$ of M. Having done so, the last k rows of the matrix A vanish. In this way we see that, replacing M by M', k by 0, n by $n-k$, and A by the submatrix where we delete the last k rows, we reduce to the case discussed in the next paragraph.

Assume R is local, $k = 0$, and M annihilated by f. Now the 0th Fitting ideal of M is (f) and is generated by the $n \times n$ minors of the matrix A of size $n \times m$. (This in particular implies $m \geq n$.) Since R is local, some $n \times n$ minor of A is uf for a unit $u \in R$. After renumbering we may assume this minor is the first one. Moreover, we know all other $n \times n$ minors of A are divisible by f. Write $A = (A_1 A_2)$ in block form where A_1 is an $n \times n$ matrix and A_2 is an $n \times (m-n)$ matrix. By Algebra, Lemma 10.15.6 applied to the transpose of A (!) we find there exists an $n \times n$ matrix B such that

$$BA = B(A_1 A_2) = f \begin{pmatrix} u1_{n\times n} & C \end{pmatrix}$$

for some $n \times (m-n)$ matrix C with coefficients in R. Then we first conclude $BA_1 = fu1_{n\times n}$. Thus

$$BA_2 = fC = u^{-1}fuC = u^{-1}BA_1C$$

Since the determinant of B is a nonzerodivisor we conclude that $A_2 = u^{-1}A_1C$. Therefore the image of A is equal to the image of A_1 which is isomorphic to $R^{\oplus n}$ because the determinant of A_1 is a nonzerodivisor. Hence M has projective dimension ≤ 1.

We return to the case of a general ring R. By the local case we see that M/M' is a finite locally free module of rank k, see Algebra, Lemma 10.78.2. Hence the extension $0 \to M' \to M \to M/M' \to 0$ splits. It follows that M' is a finitely presented module. Choose a short exact sequence $0 \to K \to R^{\oplus a} \to M' \to 0$. Then K is a finite R-module, see Algebra, Lemma 10.5.3. By the local case we see that $K_{\mathfrak p} \cong R_{\mathfrak p}^{\oplus a}$ for all primes. Hence by Algebra, Lemma 10.78.2 again we see that K is finite locally free of rank a. It follows that M' has projective dimension ≤ 1 and the lemma is proved. □

15.9. Lifting

07LW In this section we collection some lemmas concerning lifting statements of the following kind: If A is a ring and $I \subset A$ is an ideal, and $\overline{\xi}$ is some kind of structure over A/I, then we can lift $\overline{\xi}$ to a similar kind of structure ξ over A or over some étale extension of A. Here are some types of structure for which we have already proved some results:

(1) idempotents, see Algebra, Lemmas 10.32.6 and 10.32.7,
(2) projective modules, see Algebra, Lemmas 10.77.5 and 10.77.6,
(3) finite stably free modules, see Lemma 15.3.3,
(4) basis elements, see Algebra, Lemmas 10.101.1 and 10.101.3,
(5) ring maps, i.e., proving certain algebras are formally smooth, see Algebra, Lemma 10.138.4, Proposition 10.138.13, and Lemma 10.138.17,
(6) syntomic ring maps, see Algebra, Lemma 10.136.18,
(7) smooth ring maps, see Algebra, Lemma 10.137.20,

(8) étale ring maps, see Algebra, Lemma 10.143.10,
(9) factoring polynomials, see Algebra, Lemma 10.143.13, and
(10) Algebra, Section 10.153 discusses henselian local rings.

The interested reader will find more results of this nature in Smoothing Ring Maps, Section 16.3 in particular Smoothing Ring Maps, Proposition 16.3.2.

Let A be a ring and let $I \subset A$ be an ideal. Let $\overline{\xi}$ be some kind of structure over A/I. In the following lemmas we look for étale ring maps $A \to A'$ which induce isomorphisms $A/I \to A'/IA'$ and objects ξ' over A' lifting $\overline{\xi}$. A general remark is that given étale ring maps $A \to A' \to A''$ such that $A/I \cong A'/IA'$ and $A'/IA' \cong A''/IA''$ the composition $A \to A''$ is also étale (Algebra, Lemma 10.143.3) and also satisfies $A/I \cong A''/IA''$. We will frequently use this in the following lemmas without further mention. Here is a trivial example of the type of result we are looking for.

07LX Lemma 15.9.1. Let A be a ring, let $I \subset A$ be an ideal, let $\overline{u} \in A/I$ be an invertible element. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an invertible element $u' \in A'$ lifting $\overline{u}$.

Proof. Choose any lift $f \in A$ of $\overline{u}$ and set $A' = A_f$ and u the image of f in A'. $\square$

07LY Lemma 15.9.2. Let A be a ring, let $I \subset A$ be an ideal, let $\overline{e} \in A/I$ be an idempotent. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in A'$ lifting $\overline{e}$.

Proof. Choose any lift $x \in A$ of $\overline{e}$. Set

$$A' = A[t]/(t^2 - t)\left[\frac{1}{t - 1 + x}\right].$$

The ring map $A \to A'$ is étale because $(2t-1)\mathrm{d}t = 0$ and $(2t-1)(2t-1) = 1$ which is invertible. We have $A'/IA' = A/I[t]/(t^2-t)[\frac{1}{t-1+\overline{e}}] \cong A/I$ the last map sending t to $\overline{e}$ which works as $\overline{e}$ is a root of $t^2 - t$. This also shows that setting e' equal to the class of t in A' works. $\square$

07LZ Lemma 15.9.3. Let A be a ring, let $I \subset A$ be an ideal. Let $\mathrm{Spec}(A/I) = \coprod_{j \in J} \overline{U}_j$ be a finite disjoint open covering. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite disjoint open covering $\mathrm{Spec}(A') = \coprod_{j \in J} U'_j$ lifting the given covering.

Proof. This follows from Lemma 15.9.2 and the fact that open and closed subsets of Spectra correspond to idempotents, see Algebra, Lemma 10.21.3. $\square$

07M0 Lemma 15.9.4. Let $A \to B$ be a ring map and $J \subset B$ an ideal. If $A \to B$ is étale at every prime of $V(J)$, then there exists a $g \in B$ mapping to an invertible element of B/J such that $A' = B_g$ is étale over A.

Proof. The set of points of $\mathrm{Spec}(B)$ where $A \to B$ is not étale is a closed subset of $\mathrm{Spec}(B)$, see Algebra, Definition 10.143.1. Write this as $V(J')$ for some ideal $J' \subset B$. Then $V(J') \cap V(J) = \emptyset$ hence $J + J' = B$ by Algebra, Lemma 10.17.2. Write $1 = f + g$ with $f \in J$ and $g \in J'$. Then g works. $\square$

Next we have three lemmas saying we can lift factorizations of polynomials.

0ALH Lemma 15.9.5. Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ such that $\overline{g}$ and $\overline{h}$ are monic and generate the unit ideal in $A/I[x]$. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ with g', h' monic lifting the given factorization over A/I.

Proof. We will deduce this from results on the universal factorization proved earlier; however, we encourage the reader to find their own proof not using this trick. Say $\deg(\overline{g}) = n$ and $\deg(\overline{h}) = m$ so that $\deg(f) = n+m$. Write $f = x^{n+m} + \sum \alpha_i x^{n+m-i}$ for some $\alpha_1, \ldots, \alpha_{n+m} \in A$. Consider the ring map

$$R = \mathbf{Z}[a_1, \ldots, a_{n+m}] \longrightarrow S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

of Algebra, Example 10.143.12. Let $R \to A$ be the ring map which sends a_i to α_i. Set

$$B = A \otimes_R S$$

By construction the image f_B of f in $B[x]$ factors, say $f_B = g_B h_B$ with $g_B = x^n + \sum (1 \otimes b_i)x^{n-i}$ and similarly for h_B. Write $\overline{g} = x^n + \sum \overline{\beta}_i x^{n-i}$ and $\overline{h} = x^m + \sum \overline{\gamma}_i x^{m-i}$. The A-algebra map

$$B \longrightarrow A/I, \quad 1 \otimes b_i \mapsto \overline{\beta}_i, \quad 1 \otimes c_i \mapsto \overline{\gamma}_i$$

maps g_B and h_B to $\overline{g}$ and $\overline{h}$ in $A/I[x]$. The displayed map is surjective; denote $J \subset B$ its kernel. From the discussion in Algebra, Example 10.143.12 it is clear that $A \to B$ is etale at all points of $V(J) \subset \text{Spec}(B)$. Choose $g \in B$ as in Lemma 15.9.4 and consider the A-algebra B_g. Since g maps to a unit in $B/J = A/I$ we obtain also a map $B_g/IB_g \to A/I$ of A/I-algebras. Since $A/I \to B_g/IB_g$ is étale, also $B_g/IB_g \to A/I$ is étale (Algebra, Lemma 10.143.8). Hence there exists an idempotent $e \in B_g/IB_g$ such that $A/I = (B_g/IB_g)_e$ (Algebra, Lemma 10.143.9). Choose a lift $h \in B_g$ of e. Then $A \to A' = (B_g)_h$ with factorization given by the image of the factorization $f_B = g_B h_B$ in A' is a solution to the problem posed by the lemma. ☐

The assumption on the leading coefficient in the following lemma will be removed in Lemma 15.9.7.

07M1 Lemma 15.9.6. Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume

(1) the leading coefficient of $\overline{g}$ is an invertible element of A/I, and
(2) $\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$.

Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.

Proof. Applying Lemma 15.9.1 we may assume that the leading coefficient of $\overline{g}$ is the reduction of an invertible element $u \in A$. Then we may replace $\overline{g}$ by $\overline{u}^{-1}\overline{g}$ and $\overline{h}$ by $\overline{u}\overline{h}$. Thus we may assume that $\overline{g}$ is monic. Since f is monic we conclude that $\overline{h}$ is monic too. In this case the result follows from Lemma 15.9.5. ☐

07M2 Lemma 15.9.7. Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume that $\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.

Proof. Say $f = x^d + a_1x^{d-1} + \ldots + a_d$ has degree d. Write $\overline{g} = \sum \overline{b}_j x^j$ and $\overline{h} = \sum \overline{c}_j x^j$. Then we see that $1 = \sum \overline{b}_j \overline{c}_{d-j}$. It follows that $\text{Spec}(A/I)$ is covered by the standard opens $D(\overline{b}_j \overline{c}_{d-j})$. However, each point $\mathfrak{p}$ of $\text{Spec}(A/I)$ is contained in at most one of these as by looking at the induced factorization of f over the field $\kappa(\mathfrak{p})$ we see that $\deg(\overline{g} \bmod \mathfrak{p}) + \deg(\overline{h} \bmod \mathfrak{p}) = d$. Hence our open covering is a disjoint open covering. Applying Lemma 15.9.3 (and replacing A by A') we see that we may assume there is a corresponding disjoint open covering of $\text{Spec}(A)$. This disjoint open covering corresponds to a product decomposition of A, see Algebra, Lemma 10.24.3. It follows that

$$A = A_0 \times \ldots \times A_d, \quad I = I_0 \times \ldots \times I_d,$$

where the image of $\overline{g}$, resp. $\overline{h}$ in A_j/I_j has degree j, resp. $d-j$ with invertible leading coefficient. Clearly, it suffices to prove the result for each factor A_j separatedly. Hence the lemma follows from Lemma 15.9.6. □

07M3 Lemma 15.9.8. Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal of R and let $J \subset S$ be an ideal of S. If the closure of the image of $V(J)$ in $\text{Spec}(R)$ is disjoint from $V(I)$, then there exists an element $f \in R$ which maps to 1 in R/I and to an element of J in S.

Proof. Let $I' \subset R$ be an ideal such that $V(I')$ is the closure of the image of $V(J)$. Then $V(I) \cap V(I') = \emptyset$ by assumption and hence $I + I' = R$ by Algebra, Lemma 10.17.2. Write $1 = g + f$ with $g \in I$ and $f \in I'$. We have $V(f') \supset V(J)$ where f' is the image of f in S. Hence $(f')^n \in J$ for some n, see Algebra, Lemma 10.17.2. Replacing f by f^n we win. □

09XG Lemma 15.9.9. Let I be an ideal of a ring A. Let $A \to B$ be an integral ring map. Let $b \in B$ map to an idempotent in B/IB. Then there exists a monic $f \in A[x]$ with $f(b) = 0$ and $f \bmod I = x^d(x-1)^d$ for some $d \geq 1$.

Proof. Observe that $z = b^2 - b$ is an element of IB. By Algebra, Lemma 10.38.4 there exist a monic polynomial $g(x) = x^d + \sum a_j x^j$ of degree d with $a_j \in I$ such that $g(z) = 0$ in B. Hence $f(x) = g(x^2 - x) \in A[x]$ is a monic polynomial such that $f(x) \equiv x^d(x-1)^d \bmod I$ and such that $f(b) = 0$ in B. □

07M4 Lemma 15.9.10. Let A be a ring, let $I \subset A$ be an ideal. Let $A \to B$ be an integral ring map. Let $\overline{e} \in B/IB$ be an idempotent. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in B \otimes_A A'$ lifting $\overline{e}$.

Proof. Choose an element $y \in B$ lifting $\overline{e}$. Choose $f \in A[x]$ as in Lemma 15.9.9 for y. By Lemma 15.9.6 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and such that $f = gh$ in $A[x]$ with $g(x) = x^d \bmod IA'$ and $h(x) = (x-1)^d \bmod IA'$. After replacing A by A' we may assume that the factorization is defined over A. In that case we see that $b_1 = g(y) \in B$ is a lift of $\overline{e}^d = \overline{e}$ and $b_2 = h(y) \in B$ is a lift of $(\overline{e}-1)^d = (-1)^d(1-\overline{e})^d = (-1)^d(1-\overline{e})$ and moreover $b_1b_2 = 0$. Thus $(b_1, b_2)B/IB = B/IB$ and $V(b_1, b_2) \subset \text{Spec}(B)$ is disjoint from $V(IB)$. Since $\text{Spec}(B) \to \text{Spec}(A)$ is closed (see Algebra, Lemmas 10.36.22 and 10.41.6) we can find an $a \in A$ which maps to an invertible element of A/I whose image in B lies in (b_1, b_2), see Lemma 15.9.8. After replacing A by the localization A_a we get that $(b_1, b_2) = B$. Then $\text{Spec}(B) = D(b_1) \amalg D(b_2)$; disjoint union because $b_1b_2 = 0$ and covers $\text{Spec}(B)$ because $(b_1, b_2) = B$. Let $e \in B$ be the idempotent corresponding to the open and closed subset $D(b_1)$, see Algebra, Lemma 10.21.3. Since b_1 is a lift of $\overline{e}$ and b_2 is a lift of $\pm(1-\overline{e})$ we conclude that e is a lift of $\overline{e}$ by the uniqueness statement in Algebra, Lemma 10.21.3. $\square$

07M5 **Lemma 15.9.11.** Let A be a ring, let $I \subset A$ be an ideal. Let $\overline{P}$ be a finite projective A/I-module. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A'-module P' lifting $\overline{P}$.

Proof. We can choose an integer n and a direct sum decomposition $(A/I)^{\oplus n} = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Choose a lift $\varphi : A^{\oplus n} \to A^{\oplus n}$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$. Let $f \in A[x]$ be the characteristic polynomial of φ. Set $B = A[x]/(f)$. By Cayley-Hamilton (Algebra, Lemma 10.16.1) there is a map $B \to \text{End}_A(A^{\oplus n})$ mapping x to φ. For every prime $\mathfrak{p} \supset I$ the image of f in $\kappa(\mathfrak{p})$ is $(x-1)^r x^{n-r}$ where r is the dimension of $\overline{P} \otimes_{A/I} \kappa(\mathfrak{p})$. Hence $(x-1)^n x^n$ maps to zero in $B \otimes_A \kappa(\mathfrak{p})$ for all $\mathfrak{p} \supset I$. Thus $x(1-x)$ is contained in every prime ideal of B/IB. Hence $x^N(1-x)^N$ is contained in IB for some $N \geq 1$. It follows that $x^N + (1-x)^N$ is a unit in B/IB and that

$$\overline{e} = \text{image of } \frac{x^N}{x^N + (1-x)^N} \text{ in } B/IB$$

is an idempotent as both assertions hold in $\mathbf{Z}[x]/(x^N(x-1)^N)$. The image of $\overline{e}$ in $\text{End}_{A/I}((A/I)^{\oplus n})$ is

$$\frac{\overline{p}^N}{\overline{p}^N + (1-\overline{p})^N} = \overline{p}$$

as $\overline{p}$ is an idempotent. After replacing A by an étale extension A' as in the lemma, we may assume there exists an idempotent $e \in B$ which maps to $\overline{e}$ in B/IB, see Lemma 15.9.10. Then the image of e under the map

$$B = A[x]/(f) \longrightarrow \text{End}_A(A^{\oplus n}).$$

is an idempotent element p which lifts $\overline{p}$. Setting $P = \text{Im}(p)$ we win. $\square$

07EV **Lemma 15.9.12.** Let A be a ring. Let $0 \to K \to A^{\oplus m} \to M \to 0$ be a sequence of A-modules. Consider the A-algebra $C = \text{Sym}^*_A(M)$ with its presentation $\alpha : A[y_1, \ldots, y_m] \to C$ coming from the surjection $A^{\oplus m} \to M$. Then

$$NL(\alpha) = (K \otimes_A C \to \bigoplus\nolimits_{j=1,\ldots,m} C \text{d}y_j)$$

(see Algebra, Section 10.134) in particular $\Omega_{C/A} = M \otimes_A C$.

Proof. Let $J = \text{Ker}(\alpha)$. The lemma asserts that $J/J^2 \cong K \otimes_A C$. Note that α is a homomorphism of graded algebras. We will prove that in degree d we have $(J/J^2)_d = K \otimes_A C_{d-1}$. Note that

$$J_d = \text{Ker}(\text{Sym}^d_A(A^{\oplus m}) \to \text{Sym}^d_A(M)) = \text{Im}(K \otimes_A \text{Sym}^{d-1}_A(A^{\oplus m}) \to \text{Sym}^d_A(A^{\oplus m})),$$

see Algebra, Lemma 10.13.2. It follows that $(J^2)_d = \sum_{a+b=d} J_a \cdot J_b$ is the image of

$$K \otimes_A K \otimes_A \text{Sym}^{d-2}_A(A^{\otimes m}) \to \text{Sym}^d_A(A^{\oplus m}).$$

The cokernel of the map $K \otimes_A \text{Sym}^{d-2}_A(A^{\otimes m}) \to \text{Sym}^{d-1}_A(A^{\oplus m})$ is $\text{Sym}^{d-1}_A(M)$ by the lemma referenced above. Hence it is clear that $(J/J^2)_d = J_d/(J^2)_d$ is equal to

$$\begin{aligned} \text{Coker}(K \otimes_A K \otimes_A \text{Sym}^{d-2}_A(A^{\otimes m}) \to K \otimes_A \text{Sym}^{d-1}_A(A^{\otimes m})) &= K \otimes_A \text{Sym}^{d-1}_A(M) \\ &= K \otimes_A C_{d-1} \end{aligned}$$

as desired. $\square$

07M6 Lemma 15.9.13. Let A be a ring. Let M be an A-module. Then $C = \text{Sym}_A^*(M)$ is smooth over A if and only if M is a finite projective A-module.

Proof. Let $\sigma : C \to A$ be the projection onto the degree 0 part of C. Then $J = \text{Ker}(\sigma)$ is the part of degree > 0 and we see that $J/J^2 = M$ as an A-module. Hence if $A \to C$ is smooth then M is a finite projective A-module by Algebra, Lemma 10.139.4.

Conversely, assume that M is finite projective and choose a surjection $A^{\oplus n} \to M$ with kernel K. Of course the sequence $0 \to K \to A^{\oplus n} \to M \to 0$ is split as M is projective. In particular we see that K is a finite A-module and hence C is of finite presentation over A as C is a quotient of $A[x_1, \ldots, x_n]$ by the ideal generated by $K \subset \bigoplus Ax_i$. The computation of Lemma 15.9.12 shows that $NL_{C/A}$ is homotopy equivalent to $(K \to M) \otimes_A C$. Hence $NL_{C/A}$ is quasi-isomorphic to $C \otimes_A M$ placed in degree 0 which means that C is smooth over A by Algebra, Definition 10.137.1. $\square$

07M7 Lemma 15.9.14. Let A be a ring, let $I \subset A$ be an ideal. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[dr] & \\ A \arrow[u] \arrow[r] & A/I \end{tikzcd}$$

where B is a smooth A-algebra. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an A-algebra map $B \to A'$ lifting the ring map $B \to A/I$.

Proof. Let $J \subset B$ be the kernel of $B \to A/I$ so that $B/J = A/I$. By Algebra, Lemma 10.139.3 the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is split exact. Thus $\overline{P} = J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a finite projective A/I-module. Choose an integer n and a direct sum decomposition $A/I^{\oplus n} = \overline{P} \oplus \overline{K}$. By Lemma 15.9.11 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A-module K which lifts $\overline{K}$. We may and do replace A by A'. Set $B' = B \otimes_A \text{Sym}_A^*(K)$. Since $A \to \text{Sym}_A^*(K)$ is smooth by Lemma 15.9.13 we see that $B \to B'$ is smooth which in turn implies that $A \to B'$ is smooth (see Algebra, Lemmas 10.137.4 and 10.137.13). Moreover the section $\text{Sym}_A^*(K) \to A$ determines a section $B' \to B$ and we let $B' \to A/I$ be the composition $B' \to B \to A/I$. Let $J' \subset B'$ be the kernel of $B' \to A/I$. We have $JB' \subset J'$ and $B \otimes_A K \subset J'$. These maps combine to give an isomorphism

$$(A/I)^{\oplus n} \cong J/J^2 \oplus \overline{K} \longrightarrow J'/((J')^2 + IB')$$

Thus, after replacing B by B' we may assume that $J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a free A/I-module of rank n.

In this case, choose $f_1, \ldots, f_n \in J$ which map to a basis of $J/(J^2 + IB)$. Consider the finitely presented A-algebra $C = B/(f_1, \ldots, f_n)$. Note that we have an exact sequence

$$0 \to H_1(L_{C/A}) \to (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

see Algebra, Lemma 10.134.4 (note that $H_1(L_{B/A}) = 0$ and that $\Omega_{B/A}$ is finite projective, in particular flat so the Tor group vanishes). For any prime $\mathfrak{q} \supset J$ of B the module $\Omega_{B/A,\mathfrak{q}}$ is free of rank n because $\Omega_{B/A}$ is finite projective and because $\Omega_{B/A} \otimes_B B/J$ is free of rank n (see Algebra, Lemma 10.78.2). By our choice of $f_1, \ldots, f_n$ the map

$$\left((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2\right)_{\mathfrak{q}} \to \Omega_{B/A,\mathfrak{q}}$$

is surjective modulo J. Hence we see that this map of modules over the local ring $C_{\mathfrak{q}}$ has to be an isomorphism (this is because by Nakayama's Algebra, Lemma 10.20.1 the map is surjective and then for example by Algebra, Lemma 10.16.4 because $((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2)_{\mathfrak{q}}$ is generated by n elements the map is injective). Thus $H_1(L_{C/A})_{\mathfrak{q}} = 0$ and $\Omega_{C/A,\mathfrak{q}} = 0$. By Algebra, Lemma 10.137.12 we see that $A \to C$ is smooth at the prime $\overline{\mathfrak{q}}$ of C corresponding to $\mathfrak{q}$. Since $\Omega_{C/A,\mathfrak{q}} = 0$ it is actually étale at $\overline{\mathfrak{q}}$. Thus $A \to C$ is étale at all primes of C containing JC. By Lemma 15.9.4 we can find an $f \in C$ mapping to an invertible element of C/JC such that $A \to C_f$ is étale. By our choice of f it is still true that $C_f/JC_f = A/I$. The map $C_f/IC_f \to A/I$ is surjective and étale by Algebra, Lemma 10.143.8. Hence A/I is isomorphic to the localization of C_f/IC_f at some element $g \in C$, see Algebra, Lemma 10.143.9. Set $A' = C_{fg}$ to conclude the proof. □

15.10. Zariski pairs

0ELX In this section and the next a pair is a pair (A, I) where A is a ring and $I \subset A$ is an ideal. A morphism of pairs $(A, I) \to (B, J)$ is a ring map $\varphi : A \to B$ with $\varphi(I) \subset J$.

0ELY Definition 15.10.1. A Zariski pair is a pair (A, I) such that I is contained in the Jacobson radical of A.

09XF Lemma 15.10.2. Let (A, I) be a Zariski pair. Then the map from idempotents of A to idempotents of A/I is injective.

Proof. An idempotent of a local ring is either 0 or 1. Thus an idempotent is determined by the set of maximal ideals where it vanishes, by Algebra, Lemma 10.23.1. □

0ELZ Lemma 15.10.3. Let (A, I) be a Zariski pair. Let $A \to B$ be a flat, integral, finitely presented ring map such that $A/I \to B/IB$ is an isomorphism. Then $A \to B$ is an isomorphism.

Proof. The ring map $A \to B$ is finite by Algebra, Lemma 10.36.5. Hence B is finitely presented as an A-module by Algebra, Lemma 10.36.23. Hence B is a finite locally free A-module by Algebra, Lemma 10.78.2. Since the module B has rank 1 along $V(I)$ (see rank function described in Algebra, Lemma 10.78.2), and as (A, I) is a Zariski pair, we conclude that the rank is 1 everywhere. It follows that $A \to B$ is an isomorphism: it is a pleasant exercise to show that a ring map $R \to S$ such that S is a locally free R-module of rank 1 is an isomorphism (hint: look at local rings). □

0EM0 Lemma 15.10.4. Let (A, I) be a Zariski pair. Let $A \to B$ be a finite ring map. Assume

(1) $B/IB = B_1 \times B_2$ is a product of A/I-algebras
(2) $A/I \to B_1/IB_1$ is surjective,
(3) $b \in B$ maps to $(1, 0)$ in the product.

Then there exists a monic $f \in A[x]$ with $f(b) = 0$ and $f \bmod I = (x-1)x^d$ for some $d \geq 1$.

Proof. By Lemma 15.9.10 we can find an étale ring map $A \to A'$ inducing an isomorphism $A/I \to A'/IA'$ such that $B' = B \otimes_A A'$ contains an idempotent e' lifting the image of b in B'/IB'. Consider the corresponding A'-algebra decomposition

$$B' = B'_1 \times B'_2$$

which is compatible with the one given in the lemma upon reduction modulo I. The map $A' \to B'_1$ is surjective modulo IA'. By Nakayama's lemma (Algebra, Lemma 10.20.1) we can find $i \in IA'$ such that after replacing A' by A'_{1+i} the map $A' \to B'_1$ is surjective. Observe that the image $b'_1 \in B'_1$ of b satisfies $b'_1 - 1 \in IB'_1$. Thus we may pick $a' \in IA'$ mapping to $b'_1 - 1$. On the other hand, the image $b'_2 \in B'_2$ of b is in IB'_2. By Algebra, Lemma 10.38.4 there exist a monic polynomial $g(x) = x^d + \sum a'_j x^j$ of degree d with $a'_j \in IA'$ such that $g(b'_2) = 0$ in B'_2. Thus the image $b' = (b'_1, b'_2) \in B'$ of b is a root of the polynomial $(x - 1 - a')g(x)$. We conclude that

$$(b' - 1)(b')^d \in \sum\nolimits_{j=0,\ldots,d} IA' \cdot (b')^j$$

We claim that this implies

$$(b-1)b^d \in \sum\nolimits_{j=0,\ldots,d} I \cdot b^j$$

in B. For this it is enough to see that the ring map $A \to A'$ is faithfully flat, because the condition is that the image of $(b-1)b^d$ is zero in $B/\sum_{j=0,\ldots,d} Ib^j$ (use Algebra, Lemma 10.82.11). The map $A \to A'$ flat because it is étale (Algebra, Lemma 10.143.3). On the other hand, the induced map on spectra is open (see Algebra, Proposition 10.41.8 and use previous lemma referenced) and the image contains $V(I)$. Since I is contained in the Jacobson radical of A we conclude. □

0GED Lemma 15.10.5. Let (A, I) be a Zariski pair with A Noetherian. Let $f \in I$. Then A_f is a Jacobson ring.

Proof. We will use the criterion of Algebra, Lemma 10.61.4. Let $\mathfrak{p} \subset A$ be a prime ideal such that $\mathfrak{p}_f = \mathfrak{p}A_f$ is prime and not maximal. We have to show that $A_f/\mathfrak{p}_f = (A/\mathfrak{p})_f$ has infinitely many prime ideals. After replacing A by $A/\mathfrak{p}$ we may assume A is a domain, $\dim A_f > 0$, and our goal is to show that $\mathrm{Spec}(A_f)$ is infinite. Since $\dim A_f > 0$ we can find a nonzero prime ideal $\mathfrak{q} \subset A$ not containing f. Choose a maximal ideal $\mathfrak{m} \subset A$ containing $\mathfrak{q}$. Since (A, I) is a Zariski pair, we see $I \subset \mathfrak{m}$. Hence $\mathfrak{m} \neq \mathfrak{q}$ and $\dim(A_\mathfrak{m}) > 1$. Hence $\mathrm{Spec}((A_\mathfrak{m})_f) \subset \mathrm{Spec}(A_f)$ is infinite by Algebra, Lemma 10.61.1 and we win. □

15.11. Henselian pairs

09XD Some of the results of Section 15.9 may be viewed as results about henselian pairs. In this section a pair is a pair (A, I) where A is a ring and $I \subset A$ is an ideal. A morphism of pairs $(A, I) \to (B, J)$ is a ring map $\varphi : A \to B$ with $\varphi(I) \subset J$. As in Section 15.9 given an object ξ over A we denote $\overline{\xi}$ the "base change" of ξ to an object over A/I (provided this makes sense).

09XE Definition 15.11.1. A henselian pair is a pair (A, I) satisfying

(1) I is contained in the Jacobson radical of A, and
(2) for any monic polynomial $f \in A[T]$ and factorization $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$, there exists a factorization $f = gh$ in $A[T]$ with g, h monic and $g_0 = \overline{g}$ and $h_0 = \overline{h}$.

Observe that if A is a local ring and $I = \mathfrak{m}$ is the maximal ideal, then (A, I) is a henselian pair if and only if A is a henselian local ring, see Algebra, Lemma 10.153.3. In Lemma 15.11.6 we give a number of equivalent characterizations of henselian pairs (and we will add more as time goes on).

0ALI Lemma 15.11.2. Let (A, I) be a pair with I locally nilpotent. Then the functor $B \mapsto B/IB$ induces an equivalence between the category of étale algebras over A and the category of étale algebras over A/I. Moreover, the pair is henselian.

Proof. Essential surjectivity holds by Algebra, Lemma 10.143.10. If B, B' are étale over A and $B/IB \to B'/IB'$ is a morphism of A/I-algebras, then we can lift this by Algebra, Lemma 10.138.17. Finally, suppose that $f, g : B \to B'$ are two A-algebra maps with $f \bmod I = g \bmod I$. Choose an idempotent $e \in B \otimes_A B$ generating the kernel of the multiplication map $B \otimes_A B \to B$, see Algebra, Lemmas 10.151.4 and 10.151.3 (to see that étale is unramified). Then $(f \otimes g)(e) \in IB'$. Since IB' is locally nilpotent (Algebra, Lemma 10.32.3) this implies $(f \otimes g)(e) = 0$ by Algebra, Lemma 10.32.6. Thus $f = g$.

It is clear that I is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0 h_0$ be a factorization of $\overline{f} = f \bmod I$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. By Lemma 15.9.5 there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ such that the factorization lifts to a factorization into monic polynomials over A'. By the above we have $A = A'$ and the factorization is over A. $\square$

0CT7 Lemma 15.11.3. Let $A = \lim A_n$ where (A_n) is an inverse system of rings whose transition maps are surjective and have locally nilpotent kernels. Then (A, I_n) is a henselian pair, where $I_n = \mathrm{Ker}(A \to A_n)$.

Proof. Fix n. Let $a \in A$ be an element which maps to 1 in A_n. By Algebra, Lemma 10.32.4 we see that a maps to a unit in A_m for all $m \geq n$. Hence a is a unit in A. Thus by Algebra, Lemma 10.19.1 the ideal I_n is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_n h_n$ be a factorization of $\overline{f} = f \bmod I_n$ with $g_n, h_n \in A_n[T]$ monic generating the unit ideal in $A_n[T]$. By Lemma 15.11.2 we can successively lift this factorization to $f \bmod I_m = g_m h_m$ with g_m, h_m monic in $A_m[T]$ for all $m \geq n$. At each step we have to verify that our lifts g_m, h_m generate the unit ideal in $A_n[T]$; this follows from the corresponding fact for g_n, h_n and the fact that $\mathrm{Spec}(A_n[T]) = \mathrm{Spec}(A_m[T])$ because the kernel of $A_m \to A_n$ is locally nilpotent. As $A = \lim A_m$ this finishes the proof. $\square$

0ALJ Lemma 15.11.4. Let (A, I) be a pair. If A is I-adically complete, then the pair is henselian.

Proof. By Algebra, Lemma 10.96.6 the ideal I is contained in the Jacobson radical of A. Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0 h_0$ be a factorization of $\overline{f} = f \bmod I$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. By Lemma 15.11.2 we can successively lift this factorization to $f \bmod I^n = g_n h_n$ with g_n, h_n monic in $A/I^n[T]$ for all $n \geq 1$. As $A = \lim A/I^n$ this finishes the proof. $\square$

09XH Lemma 15.11.5. Let (A, I) be a pair. Let $A \to B$ be a finite type ring map such that $B/IB = C_1 \times C_2$ with $A/I \to C_1$ finite. Let B' be the integral closure of A in B. Then we can write $B'/IB' = C_1 \times C_2'$ such that the map $B'/IB' \to B/IB$ preserves product decompositions and there exists a $g \in B'$ mapping to $(1, 0)$ in $C_1 \times C_2'$ with $B'_g \to B_g$ an isomorphism.

Proof. Observe that $A \to B$ is quasi-finite at every prime of the closed subset $T = \mathrm{Spec}(C_1) \subset \mathrm{Spec}(B)$ (this follows by looking at fibre rings, see Algebra, Definition 10.122.3). Consider the diagram of topological spaces

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & \mathrm{Spec}(B') \arrow[dl, "\psi'"] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

By Algebra, Theorem 10.123.12 for every $\mathfrak{p} \in T$ there is a $h_\mathfrak{p} \in B'$, $h_\mathfrak{p} \notin \mathfrak{p}$ such that $B'_h \to B_h$ is an isomorphism. The union $U = \bigcup D(h_\mathfrak{p})$ gives an open $U \subset \mathrm{Spec}(B')$ such that $\phi^{-1}(U) \to U$ is a homeomorphism and $T \subset \phi^{-1}(U)$. Since T is open in $\psi^{-1}(V(I))$ we conclude that $\phi(T)$ is open in $U \cap (\psi')^{-1}(V(I))$. Thus $\phi(T)$ is open in $(\psi')^{-1}(V(I))$. On the other hand, since C_1 is finite over A/I it is finite over B'. Hence $\phi(T)$ is a closed subset of $\mathrm{Spec}(B')$ by Algebra, Lemmas 10.41.6 and 10.36.22. We conclude that $\mathrm{Spec}(B'/IB') \supset \phi(T)$ is open and closed. By Algebra, Lemma 10.24.3 we get a corresponding product decomposition $B'/IB' = C_1' \times C_2'$. The map $B'/IB' \to B/IB$ maps C_1' into C_1 and C_2' into C_2 as one sees by looking at what happens on spectra (hint: the inverse image of $\phi(T)$ is exactly T; some details omitted). Pick a $g \in B'$ mapping to $(1, 0)$ in $C_1' \times C_2'$ such that $D(g) \subset U$; this is possible because $\mathrm{Spec}(C_1')$ and $\mathrm{Spec}(C_2')$ are disjoint and closed in $\mathrm{Spec}(B')$ and $\mathrm{Spec}(C_1')$ is contained in U. Then $B'_g \to B_g$ defines a homeomorphism on spectra and an isomorphism on local rings (by our choice of U above). Hence it is an isomorphism, as follows for example from Algebra, Lemma 10.23.1. Finally, it follows that $C_1' = C_1$ and the proof is complete. $\square$

09XI Lemma 15.11.6. Let (A, I) be a pair. The following are equivalent [Ray70, Chapter XI] and [Gab92, Proposition 1]

(1) (A, I) is a henselian pair,
(2) given an étale ring map $A \to A'$ and an A-algebra map $\sigma : A' \to A/I$, there exists an A-algebra map $A' \to A$ lifting σ,
(3) for any finite A-algebra B the map $B \to B/IB$ induces a bijection on idempotents,
(4) for any integral A-algebra B the map $B \to B/IB$ induces a bijection on idempotents, and
(5) (Gabber) I is contained in the Jacobson radical of A and every monic polynomial $f(T) \in A[T]$ of the form
$$f(T) = T^n(T - 1) + a_n T^n + \ldots + a_1 T + a_0$$
with $a_n, \ldots, a_0 \in I$ and $n \geq 1$ has a root $\alpha \in 1 + I$.

Moreover, in part (5) the root is unique.

Proof. Assume (2) holds. Then I is contained in the Jacobson radical of A, since otherwise there would be a nonunit $f \in A$ congruent to 1 modulo I and the map $A \to A_f$ would contradict (2). Hence $IB \subset B$ is contained in the Jacobson radical of B for B integral over A because $\text{Spec}(B) \to \text{Spec}(A)$ is closed by Algebra, Lemmas 10.41.6 and 10.36.22. Thus the map from idempotents of B to idempotents of B/IB is injective by Lemma 15.10.2. On the other hand, since (2) holds, every idempotent of B/IB lifts to an idempotent of B by Lemma 15.9.10. In this way we see that (2) implies (4).

The implication (4) $\Rightarrow$ (3) is trivial.

Assume (3). Let $\mathfrak m$ be a maximal ideal and consider the finite map $A \to B = A/(I \cap \mathfrak m)$. The condition that $B \to B/IB$ induces a bijection on idempotents implies that $I \subset \mathfrak m$ (if not, then $B = A/I \times A/\mathfrak m$ and $B/IB = A/I$). Thus we see that I is contained in the Jacobson radical of A. Let $f \in A[T]$ be monic and suppose given a factorization $\overline{f} = g_0h_0$ with $g_0, h_0 \in A/I[T]$ monic. Set $B = A[T]/(f)$. Let $\overline{e}$ be the idempotent of B/IB corresponding to the decomposition

$$B/IB = A/I[T]/(g_0) \times A[T]/(h_0)$$

of A-algebras. Let $e \in B$ be an idempotent lifting $\overline{e}$ which exists as we assumed (3). This gives a product decomposition

$$B = eB \times (1 - e)B$$

Note that B is free of rank $\deg(f)$ as an A-module. Hence eB and $(1 - e)B$ are finite locally free A-modules. However, since eB and $(1 - e)B$ have constant rank $\deg(g_0)$ and $\deg(h_0)$ over A/I we find that the same is true over $\text{Spec}(A)$. We conclude that

$$\begin{aligned} f &= \text{CharPol}_A(T : B \to B) \\ &= \text{CharPol}_A(T : eB \to eB)\text{CharPol}_A(T : (1 - e)B \to (1 - e)B) \end{aligned}$$

is a factorization into monic polynomials reducing to the given factorization modulo I. Here CharPol_A denotes the characteristic polynomial of an endomorphism of a finite locally free module over A. If the module is free the CharPol_A is defined as the characteristic polynomial of the corresponding matrix and in general one uses Algebra, Lemma 10.24.2 to glue. Details omitted. Thus (3) implies (1).

Assume (1). Let f be as in (5). The factorization of f mod I as T^n times $T - 1$ lifts to a factorization $f = gh$ with g and h monic by Definition 15.11.1. Then h has to have degree 1 and we see that f has a root reducing to 1 modulo 1. Finally, I is contained in the Jacobson radical by the definition of a henselian pair. Thus (1) implies (5).

Before we give the proof of the last step, let us show that the root α in (5), if it exists, is unique. Namely, Due to the explicit shape of $f(T)$, we have $f'(\alpha) \in 1 + I$ where f' is the derivative of f with respect to T. An elementary argument shows that

$$f(T) = f(\alpha + T - \alpha) = f(\alpha) + f'(\alpha) \cdot (T - \alpha) \bmod (T - \alpha)^2 A[T]$$

This shows that any other root $\alpha' \in 1 + I$ of $f(T)$ satisfies $0 = f(\alpha') - f(\alpha) = (\alpha' - \alpha)(1 + i)$ for some $i \in I$, so that, since $1 + i$ is a unit in A, we have $\alpha = \alpha'$.

Assume (5). We will show that (2) holds, in other words, that for every étale map $A \to A'$, every section $\sigma : A' \to A/I$ modulo I lifts to a section $A' \to A$. Since $A \to A'$ is étale, the section σ determines a decomposition

0EM1 (15.11.6.1) $$A'/IA' \cong A/I \times C$$

of A/I-algebras. Namely, the surjective ring map $A'/IA' \to A/I$ is étale by Algebra, Lemma 10.143.8 and then we get the desired idempotent by Algebra, Lemma 10.143.9. We will show that this decomposition lifts to a decomposition

0EM2 (15.11.6.2) $$A' \cong A'_1 \times A'_2$$

of A-algebras with A'_1 integral over A. Then $A \to A'_1$ is integral and étale and $A/I \to A'_1/IA'_1$ is an isomorphism, thus $A \to A'_1$ is an isomorphism by Lemma 15.10.3 (here we also use that an étale ring map is flat and of finite presentation, see Algebra, Lemma 10.143.3).

Let B' be the integral closure of A in A'. By Lemma 15.11.5 we may decompose

0EM3 (15.11.6.3) $$B'/IB' \cong A/I \times C'$$

as A/I-algebras compatibly with (15.11.6.1) and we may find $b \in B'$ that lifts $(1, 0)$ such that $B'_b \to A'_b$ is an isomorphism. If the decomposition (15.11.6.3) lifts to a decomposition

0EM4 (15.11.6.4) $$B' \cong B'_1 \times B'_2$$

of A-algebras, then the induced decomposition $A' = A'_1 \times A'_2$ will give the desired (15.11.6.2): indeed, since b is a unit in B'_1 (details omitted), we will have $B'_1 \cong A'_1$, so that A'_1 will be integral over A.

Choose a finite A-subalgebra $B'' \subset B'$ containing b (observe that any finitely generated A-subalgebra of B' is finite over A). After enlarging B'' we may assume b maps to an idempotent in B''/IB'' producing

0EM5 (15.11.6.5) $$B''/IB'' \cong C''_1 \times C''_2$$

Since $B'_b \cong A'_b$ we see that B'_b is of finite type over A. Say B'_b is generated by $b_1/b^n, \ldots, b_t/b^n$ over A and enlarge B'' so that $b_1, \ldots, b_t \in B''$. Then $B''_b \to B'_b$ is surjective as well as injective, hence an isomorphism. In particular, we see that $C''_1 = A/I$! Therefore $A/I \to C''_1$ is an isomorphism, in particular surjective. By Lemma 15.10.4 we can find an $f(T) \in A[T]$ of the form

$$f(T) = T^n(T - 1) + a_nT^n + \ldots + a_1T + a_0$$

with $a_n, \ldots, a_0 \in I$ and $n \geq 1$ such that $f(b) = 0$. In particular, we find that B' is a $A[T]/(f)$-algebra. By (5) we deduce there is a root $a \in 1 + I$ of f. This produces a product decomposition $A[T]/(f) = A[T]/(T - a) \times D$ compatible with the splitting (15.11.6.3) of B'/IB'. The induced splitting of B' is then a desired (15.11.6.4). □

09XJ Lemma 15.11.7. Let A be a ring. Let $I, J \subset A$ be ideals with $V(I) = V(J)$. Then (A, I) is henselian if and only if (A, J) is henselian.

Proof. For any integral ring map $A \to B$ we see that $V(IB) = V(JB)$. Hence idempotents of B/IB and B/JB are in bijective correspondence (Algebra, Lemma 10.21.3). It follows that $B \to B/IB$ induces a bijection on sets of idempotents if and only if $B \to B/JB$ induces a bijection on sets of idempotents. Thus we conclude by Lemma 15.11.6. □

09XK **Lemma 15.11.8.** Let (A, I) be a henselian pair and let $A \to B$ be an integral ring map. Then (B, IB) is a henselian pair.

Proof. Immediate from the fourth characterization of henselian pairs in Lemma 15.11.6 and the fact that the composition of integral ring maps is integral. $\square$

0DYD **Lemma 15.11.9.** Let $I \subset J \subset A$ be ideals of a ring A. The following are equivalent

(1) (A, I) and $(A/I, J/I)$ are henselian pairs, and
(2) (A, J) is an henselian pair.

Proof. Assume (1). Let B be an integral A-algebra. Consider the ring maps

$$B \to B/IB \to B/JB$$

By Lemma 15.11.6 we find that both arrows induce bijections on idempotents. Hence so does the composition. Whence (A, J) is a henselian pair by Lemma 15.11.6.

Conversely, assume (2) holds. Then $(A/I, J/I)$ is a henselian pair by Lemma 15.11.8. Let B be an integral A-algebra. Consider the ring maps

$$B \to B/IB \to B/JB$$

By Lemma 15.11.6 we find that the composition and the second arrow induce bijections on idempotents. Hence so does the first arrow. It follows that (A, I) is a henselian pair (by the lemma again). $\square$

0G1R **Lemma 15.11.10.** Let A be a ring and let (A, I) and (A, I') be henselian pairs. Then $(A, I + I')$ is an henselian pair.

Proof. By Lemma 15.11.8 the pair $(A/I, (I' + I)/I)$ is henselian. Thus we get the conclusion from Lemma 15.11.9. $\square$

0ATD **Lemma 15.11.11.** Let J be a set and let $\{(A_j, I_j)\}_{j \in J}$ be a collection of pairs. Then $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is Henselian if and only if so is each (A_j, I_j).

Proof. For every $j \in J$, the projection $\prod_{j\in J} A_j \to A_j$ is an integral ring map, so Lemma 15.11.8 proves that each (A_j, I_j) is Henselian if $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is Henselian.

Conversely, suppose that each (A_j, I_j) is a Henselian pair. Then every $1 + x$ with $x \in \prod_{j\in J} I_j$ is a unit in $\prod_{j\in J} A_j$ because it is so componentwise by Algebra, Lemma 10.19.1 and Definition 15.11.1. Thus, by Algebra, Lemma 10.19.1 again, $\prod_{j\in J} I_j$ is contained in the Jacobson radical of $\prod_{j\in J} A_j$. Continuing to work componentwise, it likewise follows that for every monic $f \in (\prod_{j\in J} A_j)[T]$ and every factorization $\overline{f} = g_0h_0$ with monic $g_0, h_0 \in (\prod_{j\in J} A_j/\prod_{j\in J} I_j)[T] = (\prod_{j\in J} A_j/I_j)[T]$ that generate the unit ideal in $(\prod_{j\in J} A_j/\prod_{j\in J} I_j)[T]$, there exists a factorization $f = gh$ in $(\prod_{j\in J} A_j)[T]$ with g, h monic and reducing to g_0, h_0. In conclusion, according to Definition 15.11.1 $(\prod_{j\in J} A_j, \prod_{j\in J} I_j)$ is a Henselian pair. $\square$

0EM6 **Lemma 15.11.12.** The property of being Henselian is preserved under limits of pairs. More precisely, let J be a preordered set and let (A_j, I_j) be an inverse system of henselian pairs over J. Then $A = \lim A_j$ equipped with the ideal $I = \lim I_j$ is a henselian pair (A, I).

Proof. By Categories, Lemma 4.14.11, we only need to consider products and equalizers. For products, the claim follows from Lemma 15.11.11. Thus, consider an equalizer diagram

$$(A, I) \longrightarrow (A', I') \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (A'', I'')$$

in which the pairs (A', I') and (A'', I'') are henselian. To check that the pair (A, I) is also henselian, we will use the Gabber's criterion in Lemma 15.11.6. Every element of $1 + I$ is a unit in A because, due to the uniqueness of the inverses of units, this may be checked in (A', I'). Thus I is contained in the Jacobson radical of A, see Algebra, Lemma 10.19.1. Thus, let

$$f(T) = T^{N-1}(T - 1) + a_{N-1}T^{N-1} + \cdots + a_1T + a_0$$

be a polynomial in $A[T]$ with $a_{N-1}, \ldots, a_0 \in I$ and $N \geq 1$. The image of $f(T)$ in $A'[T]$ has a unique root $\alpha' \in 1 + I'$ and likewise for the further image in $A''[T]$. Thus, due to the uniqueness, $\varphi(\alpha') = \psi(\alpha')$, to the effect that α' defines a root of $f(T)$ in $1 + I$, as desired. □

0FWT Lemma 15.11.13. The property of being Henselian is preserved under filtered colimits of pairs. More precisely, let J be a directed set and let (A_j, I_j) be a system of henselian pairs over J. Then $A = \operatorname{colim} A_j$ equipped with the ideal $I = \operatorname{colim} I_j$ is a henselian pair (A, I).

Proof. If $u \in 1+I$ then for some $j \in J$ we see that u is the image of some $u_j \in 1+I_j$. Then u_j is invertible in A_j by Algebra, Lemma 10.19.1 and the assumption that I_j is contained in the Jacobson radical of A_j. Hence u is invertible in A. Thus I is contained in the Jacobson radical of A (by the lemma).

Let $f \in A[T]$ be a monic polynomial and let $\overline{f} = g_0h_0$ be a factorization with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$. Write $1 = g_0g_0' + h_0h_0'$ for some $g_0', h_0' \in A/I[T]$. Since $A = \operatorname{colim} A_j$ and $A/I = \operatorname{colim} A_j/I_j$ are filterd colimits we can find a $j \in J$ and $f_j \in A_j$ and a factorization $\overline{f}_j = g_{j,0}h_{j,0}$ with $g_{j,0}, h_{j,0} \in A_j/I_j[T]$ monic and $1 = g_{j,0}g_{j,0}' + h_{j,0}h_{j,0}'$ for some $g_{j,0}', h_{j,0}' \in A_j/I_j[T]$ with $f_j, g_{j,0}, h_{j,0}, g_{j,0}', h_{j,0}'$ mapping to f, g_0, h_0, g_0', h_0'. Since (A_j, I_j) is a henselian pair, we can lift $\overline{f}_j = g_{j,0}h_{j,0}$ to a factorization over A_j and taking the image in A we obtain a corresponding factorization in A. Hence (A, I) is henselian. □

0FWU Example 15.11.14 (Moret-Bailly). Lemma 15.11.13 is wrong if the colimit isn't filtered. For example, if we take the coproduct of the henselian pairs $(\mathbf{Z}_p, (p))$ and $(\mathbf{Z}_p, (p))$, then we obtain (A, pA) with $A = \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathbf{Z}_p$. This isn't a henselian pair: $A/pA = \mathbf{F}_p$ hence if (A, pA) where henselian, then A would have to be local. However, $\operatorname{Spec}(A)$ is disconnected; for example for odd primes p we have the nontrivial idempotent

$$(1/2 \otimes 1)\left(1 \otimes 1 - (1 + p)^{-1}u \otimes u\right)$$

where $u \in \mathbf{Z}_p$ is a square root of $1 + p$. Some details omitted.

0G1S Lemma 15.11.15. Let A be a ring. There exists a largest ideal $I \subset A$ such that (A, I) is a henselian pair.

Proof. Combine Lemmas 15.11.9, 15.11.10, and 15.11.13. □

09Y6 Lemma 15.11.16. Let (A, I) be a henselian pair. Let $\mathfrak{p} \subset A$ be a prime ideal. Then $V(\mathfrak{p} + I)$ is connected.

Proof. By Lemma 15.11.8 we see that $(A/\mathfrak{p}, I + \mathfrak{p}/\mathfrak{p})$ is a henselian pair. Thus it suffices to prove: If (A, I) is a henselian pair and A is a domain, then $\mathrm{Spec}(A/I) = V(I)$ is connected. If not, then A/I has a nontrivial idempotent by Algebra, Lemma 10.21.4. By Lemma 15.11.6 this would imply A has a nontrivial idempotent. This is a contradiction. $\square$

15.12. Henselization of pairs

0EM7 We continue the discussion started in Section 15.11.

0A02 Lemma 15.12.1. The inclusion functor

$$\text{category of henselian pairs} \longrightarrow \text{category of pairs}$$

has a left adjoint $(A, I) \mapsto (A^h, I^h)$.

Proof. Let (A, I) be a pair. Consider the category $\mathcal{C}$ consisting of étale ring maps $A \to B$ such that $A/I \to B/IB$ is an isomorphism. We will show that the category $\mathcal{C}$ is directed and that $A^h = \mathrm{colim}_{B \in \mathcal{C}} B$ with ideal $I^h = IA^h$ gives the desired adjoint.

We first prove that $\mathcal{C}$ is directed (Categories, Definition 4.19.1). It is nonempty because $\mathrm{id} : A \to A$ is an object. If B and B' are two objects of $\mathcal{C}$, then $B'' = B \otimes_A B'$ is an object of $\mathcal{C}$ (use Algebra, Lemma 10.143.3) and there are morphisms $B \to B''$ and $B' \to B''$. Suppose that $f, g : B \to B'$ are two maps between objects of $\mathcal{C}$. Then a coequalizer is

$$(B' \otimes_{f,B,g} B') \otimes_{(B' \otimes_A B')} B'$$

which is étale over A by Algebra, Lemmas 10.143.3 and 10.143.8. Thus the category $\mathcal{C}$ is directed.

Since $B/IB = A/I$ for all objects B of $\mathcal{C}$ we see that $A^h/I^h = A^h/IA^h = \mathrm{colim}\, B/IB = \mathrm{colim}\, A/I = A/I$.

Next, we show that $A^h = \mathrm{colim}_{B \in \mathcal{C}} B$ with $I^h = IA^h$ is a henselian pair. To do this we will verify condition (2) of Lemma 15.11.6. Namely, suppose given an étale ring map $A^h \to A'$ and A^h-algebra map $\sigma : A' \to A^h/I^h$. Then there exists a $B \in \mathcal{C}$ and an étale ring map $B \to B'$ such that $A' = B' \otimes_B A^h$. See Algebra, Lemma 10.143.3. Since $A^h/I^h = A/IB$, the map σ induces an A-algebra map $s : B' \to A/I$. Then $B'/IB' = A/I \times C$ as A/I-algebra, where C is the kernel of the map $B'/IB' \to A/I$ induced by s. Let $g \in B'$ map to $(1, 0) \in A/I \times C$. Then $B \to B'_g$ is étale and $A/I \to B'_g/IB'_g$ is an isomorphism, i.e., B'_g is an object of $\mathcal{C}$. Thus we obtain a canonical map $B'_g \to A^h$ such that

$$\begin{tikzcd} B'_g \arrow[r] & A^h \\ B \arrow[u] \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B' \arrow[r] \arrow[drr, "s"'] & B'_g \arrow[r] & A^h \arrow[d] \\ & & A/I \end{tikzcd}$$

commute. This induces a map $A' = B' \otimes_B A^h \to A^h$ compatible with σ as desired.

Let $(A, I) \to (A', I')$ be a morphism of pairs with (A', I') henselian. We will show there is a unique factorization $A \to A^h \to A'$ which will finish the proof. Namely, for each $A \to B$ in $\mathcal{C}$ the ring map $A' \to B' = A' \otimes_A B$ is étale and induces an isomorphism $A'/I' \to B'/I'B'$. Hence there is a section $\sigma_B : B' \to A'$ by Lemma 15.11.6. Given a morphism $B_1 \to B_2$ in $\mathcal{C}$ we claim the diagram

$$\begin{tikzcd} B'_1 \arrow[rr] \arrow[dr, "\sigma_{B_1}"'] & & B'_2 \arrow[dl, "\sigma_{B_2}"] \\ & A' & \end{tikzcd}$$

commutes. This follows once we prove that for every B in $\mathcal{C}$ the section σ_B is the unique A'-algebra map $B' \to A'$. We have $B' \otimes_{A'} B' = B' \times R$ for some ring R, see Algebra, Lemma 10.151.4. In our case $R/I'R = 0$ as $B'/I'B' = A'/I'$. Thus given two A'-algebra maps $\sigma_B, \sigma'_B : B' \to A'$ then $e = (\sigma_B \otimes \sigma'_B)(0, 1) \in A'$ is an idempotent contained in I'. We conclude that $e = 0$ by Lemma 15.10.2. Hence $\sigma_B = \sigma'_B$ as desired. Using the commutativity we obtain

$$A^h = \text{colim}_{B \in \mathcal{C}} B \to \text{colim}_{B \in \mathcal{C}} A' \otimes_A B \xrightarrow{\text{colim}\, \sigma_B} A'$$

as desired. The uniqueness of the maps σ_B also guarantees that this map is unique. Hence $(A, I) \mapsto (A^h, I^h)$ is the desired adjoint. $\square$

0AGU Lemma 15.12.2. Let (A, I) be a pair. Let (A^h, I^h) be as in Lemma 15.12.1. Then $A \to A^h$ is flat, $I^h = IA^h$ and $A/I^n \to A^h/I^nA^h$ is an isomorphism for all n.

Proof. In the proof of Lemma 15.12.1 we have seen that A^h is a filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism and we have seen that $I^h = IA^h$. As an étale ring map is flat (Algebra, Lemma 10.143.3) we conclude that $A \to A^h$ is flat by Algebra, Lemma 10.39.3. Since each $A \to B$ is flat we find that the maps $A/I^n \to B/I^nB$ are isomorphisms as well (for example by Algebra, Lemma 10.101.3). Taking the colimit we find that $A/I^n = A^h/I^nA^h$ as desired. $\square$

0A03 Lemma 15.12.3. The functor of Lemma 15.12.1 associates to a local ring $(A, \mathfrak{m})$ its henselization.

Proof. Let $(A^h, \mathfrak{m}^h)$ be the henselization of the pair $(A, \mathfrak{m})$ constructed in Lemma 15.12.1. Then $\mathfrak{m}^h = \mathfrak{m}A^h$ is a maximal ideal by Lemma 15.12.2 and since it is contained in the Jacobson radical, we conclude A^h is local with maximal ideal $\mathfrak{m}^h$. Having said this there are two ways to finish the proof.

First proof: observe that the construction in the proof of Algebra, Lemma 10.155.1 as a colimit is the same as the colimit used to construct A^h in Lemma 15.12.1. Second proof: Both the henselization $A \to S$ and $A \to A^h$ of Lemma 15.12.1 are local ring homomorphisms, both S and A^h are filtered colimits of étale A-algebras, both S and A^h are henselian local rings, and both S and A^h have residue fields equal to $\kappa(\mathfrak{m})$ (by Lemma 15.12.2 for the second case). Hence they are canonically isomorphic by Algebra, Lemma 10.154.7. $\square$

0AGV Lemma 15.12.4. Let (A, I) be a pair with A Noetherian. Let (A^h, I^h) be as in Lemma 15.12.1. Then the map of I-adic completions

$$A^\wedge \to (A^h)^\wedge$$

is an isomorphism. Moreover, A^h is Noetherian, the maps $A \to A^h \to A^\wedge$ are flat, and $A^h \to A^\wedge$ is faithfully flat.

Proof. The first statement is an immediate consequence of Lemma 15.12.2 and in fact holds without assuming A is Noetherian. In the proof of Lemma 15.12.1 we have seen that A^h is a filtered colimit of étale A-algebras B such that $A/I \to B/IB$ is an isomorphism. For each such $A \to B$ the induced map $A^\wedge \to B^\wedge$ is an isomorphism (see proof of Lemma 15.12.2). By Algebra, Lemma 10.97.2 the ring map $B \to A^\wedge = B^\wedge = (A^h)^\wedge$ is flat for each B. Thus $A^h \to A^\wedge = (A^h)^\wedge$ is flat by Algebra, Lemma 10.39.6. Since $I^h = IA^h$ is contained in the Jacobson radical of A^h and since $A^h \to A^\wedge$ induces an isomorphism $A^h/I^h \to A/I$ we see that $A^h \to A^\wedge$ is faithfully flat by Algebra, Lemma 10.39.15. By Algebra, Lemma 10.97.6 the ring $A^\wedge$ is Noetherian. Hence we conclude that A^h is Noetherian by Algebra, Lemma 10.164.1. $\square$

0A04 Lemma 15.12.5. Let $(A, I) = \text{colim}(A_i, I_i)$ be a filtered colimit of pairs. The functor of Lemma 15.12.1 gives $A^h = \text{colim}\, A_i^h$ and $I^h = \text{colim}\, I_i^h$.

This lemma is false for non-filtered colimits, see Example 15.11.14.

Proof. By Categories, Lemma 4.24.5 we see that (A^h, I^h) is the colimit of the system (A_i^h, I_i^h) in the category of henselian pairs. Thus for a henselian pair (B, J) we have

$$\text{Mor}((A^h, I^h), (B, J)) = \lim \text{Mor}((A_i^h, I_i^h), (B, J)) = \text{Mor}(\text{colim}(A_i^h, I_i^h), (B, J))$$

Here the colimit is in the category of pairs. Since the colimit is filtered we obtain $\text{colim}(A_i^h, I_i^h) = (\text{colim}\, A_i^h, \text{colim}\, I_i^h)$ in the category of pairs; details omitted. Again using the colimit is filtered, this is a henselian pair (Lemma 15.11.13). Hence by the Yoneda lemma we find $(A^h, I^h) = (\text{colim}\, A_i^h, \text{colim}\, I_i^h)$. $\square$

0F0L Lemma 15.12.6. Let A be a ring with ideals I and J. If $V(I) = V(J)$ then the functor of Lemma 15.12.1 produces the same ring for the pair (A, I) as for the pair (A, J).

Proof. Let (A', IA') be the pair produced by Lemma 15.12.1 starting with the pair (A, I), see Lemma 15.12.2. Let (A'', JA'') be the pair produced by Lemma 15.12.1 starting with the pair (A, J). By Lemma 15.11.7 we see that (A', JA') is a henselian pair and (A'', IA'') is a henselian pair. By the universal property of the construction we obtain unique A-algebra maps $A'' \to A'$ and $A' \to A''$. The uniqueness shows that these are mutually inverse. $\square$

0DYE Lemma 15.12.7. Let $(A, I) \to (B, J)$ be a map of pairs such that $V(J) = V(IB)$. Let $(A^h, I^h) \to (B^h, J^h)$ be the induced map on henselizations (Lemma 15.12.1). If $A \to B$ is integral, then the induced map $A^h \otimes_A B \to B^h$ is an isomorphism.

Proof. By Lemma 15.12.6 we may assume $J = IB$. By Lemma 15.11.8 the pair $(A^h \otimes_A B, I^h(A^h \otimes_A B))$ is henselian. By the universal property of (B^h, IB^h) we obtain a map $B^h \to A^h \otimes_A B$. We omit the proof that this map is the inverse of the map in the lemma. $\square$

15.13. Lifting and henselian pairs

0D49 In this section we mostly combine results from Sections 15.9 and 15.11.

0D4A **Lemma 15.13.1.** Let (R, I) be a henselian pair. The map

$$P \longrightarrow P/IP$$

induces a bijection between the sets of isomorphism classes of finite projective R-modules and finite projective R/I-modules. In particular, any finite projective R/I-module is isomorphic to P/IP for some finite projective R-module P.

Proof. We first prove the final statement. Let $\overline{P}$ be a finite projective R/I-module. We can find a finite projective module P' over some R' étale over R with $R/I = R'/IR'$ such that P'/IP' is isomorphic to $\overline{P}$, see Lemma 15.9.11. Then, since (R, I) is a henselian pair, the étale ring map $R \to R'$ has a section $\tau : R' \to R$ (Lemma 15.11.6). Setting $P = P' \otimes_{R',\tau} R$ we conclude that P/IP is isomorphic to $\overline{P}$. Of course, this tells us that the map in the statement of the lemma is surjective.

Injectivity. Suppose that P_1 and P_2 are finite projective R-modules such that $P_1/IP_1 \cong P_2/IP_2$ as R/I-modules. Since P_1 is projective, we can find an R-module map $u : P_1 \to P_2$ lifting the given isomorphism. Then u is surjective by Nakayama's lemma (Algebra, Lemma 10.20.1). We similarly find a surjection $v : P_2 \to P_1$. By Algebra, Lemma 10.16.4 the map $v \circ u$ is an isomorphism and we conclude u is an isomorphism. $\square$

09ZL **Lemma 15.13.2.** Let (A, I) be a henselian pair. The functor $B \to B/IB$ determines an equivalence between finite étale A-algebras and finite étale A/I-algebras.

Proof. Let B, B' be two A-algebras finite étale over A. Then $B' \to B'' = B \otimes_A B'$ is finite étale as well (Algebra, Lemmas 10.143.3 and 10.36.13). Now we have 1-to-1 correspondences between

(1) A-algebra maps $B \to B'$,
(2) sections of $B' \to B''$, and
(3) idempotents e of B'' such that $B' \to B'' \to eB''$ is an isomorphism.

The bijection between (2) and (3) sends $\sigma : B'' \to B'$ to e such that $(1 - e)$ is the idempotent that generates the kernel of σ which exists by Algebra, Lemmas 10.143.8 and 10.143.9. There is a similar correspondence between A/I-algebra maps $B/IB \to B'/IB'$ and idempotents $\overline{e}$ of B''/IB'' such that $B'/IB' \to B''/IB'' \to \overline{e}(B''/IB'')$ is an isomorphism. However every idempotent $\overline{e}$ of B''/IB'' lifts uniquely to an idempotent e of B'' (Lemma 15.11.6). Moreover, if $B'/IB' \to \overline{e}(B''/IB'')$ is an isomorphism, then $B' \to eB''$ is an isomorphism too by Nakayama's lemma (Algebra, Lemma 10.20.1). In this way we see that the functor is fully faithful.

Essential surjectivity. Let $A/I \to C$ be a finite étale map. By Algebra, Lemma 10.143.10 there exists an étale map $A \to B$ such that $B/IB \cong C$. Let B' be the integral closure of A in B. By Lemma 15.11.5 we have $B'/IB' = C \times C'$ for some ring C' and $B'_g \cong B_g$ for some $g \in B'$ mapping to $(1, 0) \in C \times C'$. Since idempotents lift (Lemma 15.11.6) we get $B' = B'_1 \times B'_2$ with $C = B'_1/IB'_1$ and $C' = B'_2/IB'_2$. The image of g in B'_1 is invertible. Then $B_g = B'_g = B'_1 \times (B_2)_g$ and this implies that $A \to B'_1$ is étale. We conclude that B'_1 is finite étale over A (integral étale implies finite étale by Algebra, Lemma 10.36.5 for example) and the proof is done. $\square$

0D4B Lemma 15.13.3. Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given A_n-modules M_n and A_{n+1}-module maps $M_{n+1} \to M_n$. Assume

(1) the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,
(2) M_1 is a finite projective A_1-module,
(3) M_n is a finite flat A_n-module, and
(4) the maps induce isomorphisms $M_{n+1} \otimes_{A_{n+1}} A_n \to M_n$.

Then $M = \lim M_n$ is a finite projective A-module and $M \otimes_A A_n \to M_n$ is an isomorphism for all n.

Proof. By Lemma 15.11.3 the pair $(A, \text{Ker}(A \to A_1))$ is henselian. By Lemma 15.13.1 we can choose a finite projective A-module P and an isomorphism $P \otimes_A A_1 \to M_1$. Since P is projective, we can successively lift the A-module map $P \to M_1$ to A-module maps $P \to M_2$, $P \to M_3$, and so on. Thus we obtain a map

$$P \longrightarrow M$$

Since P is finite projective, we can write $A^{\oplus m} = P \oplus Q$ for some $m \geq 0$ and A-module Q. Since $A = \lim A_n$ we conclude that $P = \lim P \otimes_A A_n$. Hence, in order to show that the displayed A-module map is an isomorphism, it suffices to show that the maps $P \otimes_A A_n \to M_n$ are isomorphisms. From Lemma 15.3.4 we see that M_n is a finite projective module. By Lemma 15.3.5 the maps $P \otimes_A A_n \to M_n$ are isomorphisms. $\square$

15.14. Absolute integral closure

0DCK Here is our definition.

0DCL Definition 15.14.1. A ring A is absolutely integrally closed if every monic $f \in A[T]$ is a product of linear factors.

Be careful: it may be possible to write f as a product of linear factors in many different ways.

0DCM Lemma 15.14.2. Let A be a ring. The following are equivalent

(1) A is absolutely integrally closed, and
(2) any monic $f \in A[T]$ has a root in A.

Proof. Omitted. $\square$

0DCN Lemma 15.14.3. Let A be absolutely integrally closed.

(1) Any quotient ring A/I of A is absolutely integrally closed.
(2) Any localization $S^{-1}A$ is absolutely integrally closed.

Proof. Omitted. $\square$

0DCP Lemma 15.14.4. Let A be a ring. Let $S \subset A$ be a multiplicative subset consisting of nonzerodivisors. If $S^{-1}A$ is absolutely integrally closed and $A \subset S^{-1}A$ is integrally closed in $S^{-1}A$, then A is absolutely integrally closed.

Proof. Omitted. $\square$

0DCQ Lemma 15.14.5. Let A be a normal domain. Then A is absolutely integrally closed if and only if its fraction field is algebraically closed.

Proof. Observe that a field is algebraically closed if and only if it is absolutely integrally closed as a ring. Hence the lemma follows from Lemmas 15.14.3 and 15.14.4. □

0DCR Lemma 15.14.6. For any ring A there exists an extension $A \subset B$ such that

(1) B is a filtered colimit of finite free A-algebras,
(2) B is free as an A-module, and
(3) B is absolutely integrally closed.

Proof. Let I be the set of monic polynomials over A. For $i \in I$ denote x_i a variable and P_i the corresponding monic polynomial in the variable x_i. Then we set

$$F(A) = A[x_i; i \in I]/(P_i; i \in I)$$

As the notation suggests F is a functor from the category of rings to itself. Note that $A \subset F(A)$, that $F(A)$ is free as an A-module, and that $F(A)$ is a filtered colimit of finite free A-algebras. Then we take

$$B = \operatorname{colim} F^n(A)$$

where the transition maps are the inclusions $F^n(A) \subset F(F^n(A)) = F^{n+1}(A)$. Any monic polynomial with coefficients in B actually has coefficients in $F^n(A)$ for some n and hence has a solution in $F^{n+1}(A)$ by construction. This implies that B is absolutely integrally closed by Lemma 15.14.2. We omit the proof of the other properties. □

0DCS Lemma 15.14.7. Let A be absolutely integrally closed. Let $\mathfrak{p} \subset A$ be a prime. Then the local ring $A_{\mathfrak{p}}$ is strictly henselian.

Proof. By Lemma 15.14.3 we may assume A is a local ring and $\mathfrak{p}$ is its maximal ideal. The residue field is algebraically closed by Lemma 15.14.3. Every monic polynomial decomposes completely into linear factors hence Algebra, Definition 10.153.1 applies directly. □

0DCT Lemma 15.14.8. Let A be absolutely integrally closed. Let $I \subset A$ be an ideal. Then (A, I) is a henselian pair if (and only if) the following conditions hold

(1) I is contained in the Jacobson radical of A,
(2) $A \to A/I$ induces a bijection on idempotents.

Proof. Let $f \in A[T]$ be a monic polynomial and let $f \bmod I = g_0h_0$ be a factorization over A/I with g_0, h_0 monic such that g_0 and h_0 generate the unit ideal of $A/I[T]$. This means that

$$A/I[T]/(f) = A/I[T]/(g_0) \times A/I[T]/(h_0)$$

Denote $e \in A/I[T]/(f)$ the element correspoing to the idempotent $(1, 0)$ in the ring on the right. Write $f = (T - a_1) \ldots (T - a_d)$ with $a_i \in A$. For each $i \in \{1, \ldots, d\}$ we obtain an A-algebra map $\varphi_i : A[T]/(f) \to A$, $T \mapsto a_i$ which induces a similar A/I-algebra map $\overline{\varphi}_i : A/I[T]/(f) \to A/I$. Denote $e_i = \overline{\varphi}_i(e) \in A/I$. These are idempotents. By our assumption (2) we can lift e_i to an idempotent in A. This means we can write $A = \prod A_j$ as a finite product of rings such that in A_j/IA_j each e_i is either 0 or 1. Some details omitted. Observe that A_j is absolutely integrally closed as a factor ring of A. It suffices to lift the factorization of f over A_j/IA_j to A_j. This reduces us to the situation discussed in the next paragraph.

Assume $e_i = 1$ for $i = 1, \ldots, r$ and $e_i = 0$ for $i = r+1, \ldots, d$. From $(g_0, h_0) = A/I[T]$ we have that there are $k_0, l_0 \in A/I[T]$ such that $g_0k_0 + h_0l_0 = 1$. We see that $e = h_0l_0$ and $e_i = h_0(a_i)l_0(a_i)$. We conclude that $h_0(a_i)$ is a unit for $i = 1, \ldots, r$. Since $f(a_i) = 0$ we find $0 = h_0(a_i)g_0(a_i)$ and we conclude that $g_0(a_i) = 0$ for $i = 1, \ldots, r$. Thus $(T - a_1)$ divides g_0 in $A/I[T]$, say $g_0 = (T - a_1)g_0'$. Set $f' = (T - a_2) \ldots (T - a_d)$ and $h_0' = h_0$. By induction on d we can lift the factorization $f' \bmod I = g_0'h_0'$ to a factorization of $f' = g'h'$ over over A which gives the factorization $f = (T - a_1)g'h'$ lifting the factorization $f \bmod I = g_0h_0$ as desired. $\square$

15.15. Auto-associated rings

05GL Some of this material is in [Laz69].

05GM **Definition 15.15.1.** A ring R is said to be *auto-associated* if R is local and its maximal ideal $\mathfrak{m}$ is weakly associated to R.

05GN **Lemma 15.15.2.** *An auto-associated ring R has the following property: (P) Every proper finitely generated ideal $I \subset R$ has a nonzero annihilator.*

Proof. By assumption there exists a nonzero element $x \in R$ such that for every $f \in \mathfrak{m}$ we have $f^n x = 0$. Say $I = (f_1, \ldots, f_r)$. Then x is in the kernel of $R \to \bigoplus R_{f_i}$. Hence we see that there exists a nonzero $y \in R$ such that $f_i y = 0$ for all i, see Algebra, Lemma 10.24.4. As $y \in \text{Ann}_R(I)$ we win. $\square$

05GP **Lemma 15.15.3.** *Let R be a ring having property (P) of Lemma 15.15.2. Let $u : N \to M$ be a homomorphism of projective R-modules. Then u is universally injective if and only if u is injective.*

Proof. Assume u is injective. Our goal is to show u is universally injective. First we choose a module Q such that $N \oplus Q$ is free. On considering the map $N \oplus Q \to M \oplus Q$ we see that it suffices to prove the lemma in case N is free. In this case N is a directed colimit of finite free R-modules. Thus we reduce to the case that N is a finite free R-module, say $N = R^{\oplus n}$. We prove the lemma by induction on n. The case $n = 0$ is trivial.

Let $u : R^{\oplus n} \to M$ be an injective module map with M projective. Choose an R-module Q such that $M \oplus Q$ is free. After replacing u by the composition $R^{\oplus n} \to M \to M \oplus Q$ we see that we may assume that M is free. Then we can find a direct summand $R^{\oplus m} \subset M$ such that $u(R^{\oplus n}) \subset R^{\oplus m}$. Hence we may assume that $M = R^{\oplus m}$. In this case u is given by a matrix $A = (a_{ij})$ so that $u(x_1, \ldots, x_n) = (\sum x_i a_{i1}, \ldots, \sum x_i a_{im})$. As u is injective, in particular $u(x, 0, \ldots, 0) = (xa_{11}, xa_{12}, \ldots, xa_{1m}) \neq 0$ if $x \neq 0$, and as R has property (P) we see that $a_{11}R + a_{12}R + \ldots + a_{1m}R = R$. Hence see that $R(a_{11}, \ldots, a_{1m}) \subset R^{\oplus m}$ is a direct summand of $R^{\oplus m}$, in particular $R^{\oplus m}/R(a_{11}, \ldots, a_{1m})$ is a projective R-module. We get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, "1"] & R^{\oplus n} \arrow[r] \arrow[d, "u"] & R^{\oplus n-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R \arrow[r, "(a_{11}, \ldots, a_{1m})"] & R^{\oplus m} \arrow[r] & R^{\oplus m}/R(a_{11}, \ldots, a_{1m}) \arrow[r] & 0 \end{tikzcd}$$

with split exact rows. Thus the right vertical arrow is injective and we may apply the induction hypothesis to conclude that the right vertical arrow is universally injective. It follows that the middle vertical arrow is universally injective. □

05GQ Lemma 15.15.4. Let R be a ring. The following are equivalent

(1) R has property (P) of Lemma 15.15.2,
(2) any injective map of projective R-modules is universally injective,
(3) if $u : N \to M$ is injective and N, M are finite projective R-modules then $\text{Coker}(u)$ is a finite projective R-module,
(4) if $N \subset M$ and N, M are finite projective as R-modules, then N is a direct summand of M, and
(5) any injective map $R \to R^{\oplus n}$ is a split injection.

Proof. The implication (1) ⇒ (2) is Lemma 15.15.3. It is clear that (3) and (4) are equivalent. We have (2) ⇒ (3), (4) by Algebra, Lemma 10.82.4. Part (5) is a special case of (4). Assume (5). Let $I = (a_1, \ldots, a_n)$ be a proper finitely generated ideal of R. As $I \neq R$ we see that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is not a split injection. Hence it has a nonzero kernel and we conclude that $\text{Ann}_R(I) \neq 0$. Thus (1) holds. □

05GR Example 15.15.5. If the equivalent conditions of Lemma 15.15.4 hold, then it is not always the case that every injective map of free R-modules is a split injection. For example suppose that $R = k[x_1, x_2, x_3, \ldots]/(x_i^2)$. This is an auto-associated ring. Consider the map of free R-modules

$$u : \bigoplus\nolimits_{i \geq 1} Re_i \longrightarrow \bigoplus\nolimits_{i \geq 1} Rf_i, \quad e_i \longmapsto f_i - x_i f_{i+1}.$$

For any integer n the restriction of u to $\bigoplus_{i=1,\ldots,n} Re_i$ is injective as the images $u(e_1), \ldots, u(e_n)$ are R-linearly independent. Hence u is injective and hence universally injective by the lemma. Since $u \otimes \text{id}_k$ is bijective we see that if u were a split injection then u would be surjective. But u is not surjective because the inverse image of f_1 would be the element

$$\sum\nolimits_{i \geq 0} x_1 \ldots x_i e_{i+1} = e_1 + x_1 e_2 + x_1 x_2 e_3 + \ldots$$

which is not an element of the direct sum. A side remark is that $\text{Coker}(u)$ is a flat (because u is universally injective), countably generated R-module which is not projective (as u is not split), hence not Mittag-Leffler (see Algebra, Lemma 10.93.1).

The following lemma is a special case of Algebra, Proposition 10.102.9 in case the local ring is Noetherian.

00MX Lemma 15.15.6. Let $(R, \mathfrak{m})$ be a local ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules. The following are equivalent

(1) φ is injective,
(2) the rank of φ is m and the annihilator of $I(\varphi)$ in R is zero.

If R is Noetherian these are also equivalent to

(3) the rank of φ is m and either $I(\varphi) = R$ or it contains a nonzerodivisor.

Here the rank of φ and $I(\varphi)$ are defined as in Algebra, Definition 10.102.5.

Proof. If any matrix coefficient of φ is not in $\mathfrak{m}$, then we apply Algebra, Lemma 10.102.2 to write φ as the sum of $1 : R \to R$ and a map $\varphi' : R^{m-1} \to R^{n-1}$. It is easy to see that the lemma for φ' implies the lemma for φ. Thus we may assume from the outset that all the matrix coefficients of φ are in $\mathfrak{m}$.

Suppose φ is injective. We may assume $m > 0$. Let $\mathfrak{q} \in \text{WeakAss}(R)$ so that $R_\mathfrak{q}$ is an auto-associated ring. Then φ induces a injective map $R_\mathfrak{q}^m \to R_\mathfrak{q}^n$ which is universally injective by Lemmas 15.15.2 and 15.15.3. Thus $\varphi : \kappa(\mathfrak{q})^m \to \kappa(\mathfrak{q})^n$ is injective. Hence the rank of φ mod $\mathfrak{q}$ is m and $I(\varphi \otimes \kappa(\mathfrak{q}))$ is not the zero ideal. Since m is the maximum rank φ can have, we conclude that φ has rank m as well (ranks of matrices can only drop after base change). Hence $I(\varphi) \cdot \kappa(\mathfrak{q}) = I(\varphi \otimes \kappa(\mathfrak{q}))$ is not zero. Thus $I(\varphi)$ is not contained in $\mathfrak{q}$. Thus none of the weakly associated primes of R are weakly associated primes of the R-module $\text{Ann}_R I(\varphi)$. Thus $\text{Ann}_R I(\varphi)$ has no weakly associated primes, see Algebra, Lemma 10.66.4. It follows from Algebra, Lemma 10.66.5 that $\text{Ann}_R I(\varphi)$ is zero.

Conversely, assume (2). The rank being m implies $n \geq m$. Write $I(\varphi) = (f_1, \ldots, f_r)$ which is possible as $I(\varphi)$ is finitely generated. By Algebra, Lemma 10.15.5 we can find maps $\psi_i : R^n \to R^m$ such that $\psi \circ \varphi = f_i \text{id}_{R^m}$. Thus $\varphi(x) = 0$ implies $f_i x = 0$ for $i = 1, \ldots, r$. This implies $x = 0$ and hence φ is injective.

For the equivalence of (1) and (3) in the Noetherian local case we refer to Algebra, Proposition 10.102.9. If the ring R is Noetherian but not local, then the reader can deduce it from the local case; details omitted. Another option is to redo the argument above using associated primes, using that there are finitely many of these, using prime avoidance, and using the characterization of nonzerodivisors as elements of a Noetherian ring not contained in any associated prime. □

0EWY Lemma 15.15.7. Let R be a ring. Suppose that $\varphi : R^n \to R^n$ be an injective map of finite free modules of the same rank. Then $\text{Hom}_R(\text{Coker}(\varphi), R) = 0$.

Proof. Let $\varphi^t : R^n \to R^n$ be the transpose of φ. The lemma claims that φ^t is injective. With notation as in Lemma 15.15.6 we see that the rank of φ^t is n and that $I(\varphi) = I(\varphi^t)$. Thus we conclude by the equivalence of (1) and (2) of the lemma. □

15.16. Flattening stratification

0521 Let $R \to S$ be a ring map and let M be an S-module. For any R-algebra R' we can consider the base changes $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We say $R \to R'$ flattens M if the module M' is flat over R'. We would like to understand the structure of the collection of ring maps $R \to R'$ which flatten M. In particular we would like to know if there exists a universal flattening $R \to R_{univ}$ of M, i.e., a ring map $R \to R_{univ}$ which flattens M and has the property that any ring map $R \to R'$ which flattens M factors through $R \to R_{univ}$. It turns out that such a universal solution usually does not exist.

We will discuss universal flattenings and flattening stratifications in a scheme theoretic setting $\mathcal{F}/X/S$ in More on Flatness, Section 38.21. If the universal flattening $R \to R_{univ}$ exists then the morphism of schemes $\text{Spec}(R_{univ}) \to \text{Spec}(R)$ is the universal flattening of the quasi-coherent module $\widetilde{M}$ on $\text{Spec}(S)$.

In this and the next few sections we prove some basic algebra facts related to this. The most basic result is perhaps the following.

0522 Lemma 15.16.1. Let R be a ring. Let M be an R-module. Let I_1, I_2 be ideals of R. If M/I_1M is flat over R/I_1 and M/I_2M is flat over R/I_2, then $M/(I_1 \cap I_2)M$ is flat over $R/(I_1 \cap I_2)$.

Proof. By replacing R with $R/(I_1 \cap I_2)$ and M by $M/(I_1 \cap I_2)M$ we may assume that $I_1 \cap I_2 = 0$. Let $J \subset R$ be an ideal. To prove that M is flat over R we have to show that $J \otimes_R M \to M$ is injective, see Algebra, Lemma 10.39.5. By flatness of M/I_1M over R/I_1 the map

$$J/(J \cap I_1) \otimes_R M = (J + I_1)/I_1 \otimes_{R/I_1} M/I_1M \longrightarrow M/I_1M$$

is injective. As $0 \to (J \cap I_1) \to J \to J/(J \cap I_1) \to 0$ is exact we obtain a diagram

$$\begin{tikzcd}
(J \cap I_1) \otimes_R M \arrow[r] \arrow[d] & J \otimes_R M \arrow[r] \arrow[d] & J/(J \cap I_1) \otimes_R M \arrow[r] \arrow[d] & 0 \\
M \arrow[r, equal] & M \arrow[r] & M/I_1M &
\end{tikzcd}$$

hence it suffices to show that $(J \cap I_1) \otimes_R M \to M$ is injective. Since $I_1 \cap I_2 = 0$ the ideal $J \cap I_1$ maps isomorphically to an ideal $J' \subset R/I_2$ and we see that $(J \cap I_1) \otimes_R M = J' \otimes_{R/I_2} M/I_2M$. By flatness of M/I_2M over R/I_2 the map $J' \otimes_{R/I_2} M/I_2M \to M/I_2M$ is injective, which clearly implies that $(J \cap I_1) \otimes_R M \to M$ is injective. $\square$

15.17. Flattening over an Artinian ring

05LJ A universal flattening exists when the base ring is an Artinian local ring. It exists for an arbitrary module. Hence, as we will see later, a flatting stratification exists when the base scheme is the spectrum of an Artinian local ring.

0524 Lemma 15.17.1. Let R be an Artinian ring. Let M be an R-module. Then there exists a smallest ideal $I \subset R$ such that M/IM is flat over R/I.

Proof. This follows directly from Lemma 15.16.1 and the Artinian property. $\square$

This ideal has the following universal property.

0525 Lemma 15.17.2. Let R be an Artinian ring. Let M be an R-module. Let $I \subset R$ be the smallest ideal $I \subset R$ such that M/IM is flat over R/I. Then I has the following universal property: For every ring map $\varphi : R \to R'$ we have

$$R' \otimes_R M \text{ is flat over } R' \Leftrightarrow \text{we have } \varphi(I) = 0.$$

Proof. Note that I exists by Lemma 15.17.1. The implication $\Rightarrow$ follows from Algebra, Lemma 10.39.7. Let $\varphi : R \to R'$ be such that $M \otimes_R R'$ is flat over R'. Let $J = \mathrm{Ker}(\varphi)$. By Algebra, Lemma 10.101.7 and as $R' \otimes_R M = R' \otimes_{R/J} M/JM$ is flat over R' we conclude that M/JM is flat over R/J. Hence $I \subset J$ as desired. $\square$

15.18. Flattening over a closed subset of the base

05LK Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. In the following we will consider the following condition

052W (15.18.0.1)
$$\forall \mathfrak q \in V(IS) \subset \operatorname{Spec}(S) : M_{\mathfrak q} \text{ is flat over } R.$$

Geometrically, this means that M is flat over R along the inverse image of $V(I)$ in $\operatorname{Spec}(S)$. If R and S are Noetherian rings and M is a finite S-module, then (15.18.0.1) is equivalent to the condition that M/I^nM is flat over R/I^n for all $n \geq 1$, see Algebra, Lemma 10.99.11.

052X **Lemma 15.18.1.** Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal. If (15.18.0.1) holds for $(R \to S, I, M)$, then (15.18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$.

Proof. Assume (15.18.0.1) holds for $(R \to S, I \subset R, M)$. Let $I'(S \otimes_R R') \subset \mathfrak q'$ be a prime of $S \otimes_R R'$. Let $\mathfrak q \subset S$ be the corresponding prime of S. Then $IS \subset \mathfrak q$. Note that $(M \otimes_R R')_{\mathfrak q'}$ is a localization of the base change $M_{\mathfrak q} \otimes_R R'$. Hence $(M \otimes_R R')_{\mathfrak q'}$ is flat over R' as a localization of a flat module, see Algebra, Lemmas 10.39.7 and 10.39.18. $\square$

05LL **Lemma 15.18.2.** Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal such that

1. the map $V(I') \to V(I)$ induced by $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is surjective, and
2. $R'_{\mathfrak p'}$ is flat over R for all primes $\mathfrak p' \in V(I')$.

If (15.18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then (15.18.0.1) holds for $(R \to S, I, M)$.

Proof. Assume (15.18.0.1) holds for $(R' \to S \otimes_R R', IR', M \otimes_R R')$. Pick a prime $IS \subset \mathfrak q \subset S$. Let $I \subset \mathfrak p \subset R$ be the corresponding prime of R. By assumption there exists a prime $\mathfrak p' \in V(I')$ of R' lying over $\mathfrak p$ and $R_{\mathfrak p} \to R'_{\mathfrak p'}$ is flat. Choose a prime $\overline{\mathfrak q}' \subset \kappa(\mathfrak q) \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak p')$ which corresponds to a prime $\mathfrak q' \subset S \otimes_R R'$ which lies over $\mathfrak q$ and over $\mathfrak p'$. Note that $(S \otimes_R R')_{\mathfrak q'}$ is a localization of $S_{\mathfrak q} \otimes_{R_{\mathfrak p}} R'_{\mathfrak p'}$. By assumption the module $(M \otimes_R R')_{\mathfrak q'}$ is flat over $R'_{\mathfrak p'}$. Hence Algebra, Lemma 10.100.1 implies that $M_{\mathfrak q}$ is flat over $R_{\mathfrak p}$ which is what we wanted to prove. $\square$

05LM **Lemma 15.18.3.** Let $R \to S$ be a ring map of finite presentation. Let M be an S-module of finite presentation. Let $R' = \operatorname{colim}_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \operatorname{colim}_\lambda I_\lambda$. If (15.18.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then there exists a $\lambda \in \Lambda$ such that (15.18.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, I_\lambda, M \otimes_R R_\lambda)$.

Proof. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 10.14.2. By Algebra, Theorem 10.129.4 the set

$$U' = \{\mathfrak q' \in \operatorname{Spec}(S') \mid M'_{\mathfrak q'} \text{ is flat over } R'\}$$

is open in $\operatorname{Spec}(S')$. Note that $V(I'S')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S') \subset \bigcup D(g'_j)$ means that $I'S' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as $1 = \sum z_s h_s + \sum f_j g'_j$ for some $z_s \in I'$, $h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 10.168.1 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes lying over the ideal I_λ. □

15.19. Flattening over a closed subsets of source and base

05LN In this section we slightly generalize the discussion in Section 15.18. We strongly suggest the reader first read and understand that section.

05LP Situation 15.19.1. Let $R \to S$ be a ring map. Let $J \subset S$ be an ideal. Let M be an S-module.

In this situation, given an R-algebra R' and an ideal $I' \subset R'$ we set $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We will consider the condition

05LQ (15.19.1.1) $$\forall \mathfrak q' \in V(I'S' + JS') \subset \text{Spec}(S') : M'_{\mathfrak q'} \text{ is flat over } R'.$$

Geometrically, this means that M' is flat over R' along the intersection of the inverse image of $V(I')$ with the inverse image of $V(J)$. Since $(R \to S, J, M)$ are fixed, condition (15.19.1.1) only depends on the pair (R', I') where R' is viewed as an R-algebra.

05LR Lemma 15.19.2. In Situation 15.19.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. If (15.19.1.1) holds for (R', I'), then (15.19.1.1) holds for (R'', I'').

Proof. Assume (15.19.1.1) holds for (R', I'). Let $I''S'' + JS'' \subset \mathfrak q''$ be a prime of S''. Let $\mathfrak q' \subset S'$ be the corresponding prime of S'. Then both $I'S' \subset \mathfrak q'$ and $JS' \subset \mathfrak q'$ because the corresponding conditions hold for $\mathfrak q''$. Note that $(M'')_{\mathfrak q''}$ is a localization of the base change $M'_{\mathfrak q'} \otimes_R R''$. Hence $(M'')_{\mathfrak q''}$ is flat over R'' as a localization of a flat module, see Algebra, Lemmas 10.39.7 and 10.39.18. □

05LS Lemma 15.19.3. In Situation 15.19.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. Assume

(1) the map $V(I'') \to V(I')$ induced by $\text{Spec}(R'') \to \text{Spec}(R')$ is surjective, and
(2) $R''_{\mathfrak p''}$ is flat over R' for all primes $\mathfrak p'' \in V(I'')$.

If (15.19.1.1) holds for (R'', I''), then (15.19.1.1) holds for (R', I').

Proof. Assume (15.19.1.1) holds for (R'', I''). Pick a prime $I'S' + JS' \subset \mathfrak q' \subset S'$. Let $I' \subset \mathfrak p' \subset R'$ be the corresponding prime of R'. By assumption there exists a prime $\mathfrak p'' \in V(I'')$ of R'' lying over $\mathfrak p'$ and $R'_{\mathfrak p'} \to R''_{\mathfrak p''}$ is flat. Choose a prime $\overline{\mathfrak q}'' \subset \kappa(\mathfrak q') \otimes_{\kappa(\mathfrak p')} \kappa(\mathfrak p'')$. This corresponds to a prime $\mathfrak q'' \subset S'' = S' \otimes_{R'} R''$ which lies over $\mathfrak q'$ and over $\mathfrak p''$. In particular we see that $I''S'' \subset \mathfrak q''$ and that $JS'' \subset \mathfrak q''$. Note that $(S' \otimes_{R'} R'')_{\mathfrak q''}$ is a localization of $S'_{\mathfrak q'} \otimes_{R'_{\mathfrak p'}} R''_{\mathfrak p''}$. By assumption the module $(M' \otimes_{R'} R'')_{\mathfrak q''}$ is flat over $R''_{\mathfrak p''}$. Hence Algebra, Lemma 10.100.1 implies that $M'_{\mathfrak q'}$ is flat over $R'_{\mathfrak p'}$ which is what we wanted to prove. □

05LT Lemma 15.19.4. In Situation 15.19.1 assume $R \to S$ is essentially of finite presentation and M is an S-module of finite presentation. Let $R' = \mathrm{colim}_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \mathrm{colim}_\lambda I_\lambda$. If (15.19.1.1) holds for (R', I'), then there exists a $\lambda \in \Lambda$ such that (15.19.1.1) holds for (R_λ, I_λ).

Proof. We first prove the lemma in case $R \to S$ is of finite presentation and then we explain what needs to be changed in the general case. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 10.14.2. By Algebra, Theorem 10.129.4 the set

$$U' = \{\mathfrak q' \in \mathrm{Spec}(S') \mid M'_{\mathfrak q'} \text{ is flat over } R'\}$$

is open in $\mathrm{Spec}(S')$. Note that $V(I'S' + JS')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S' + JS') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S'+JS') \subset \bigcup D(g'_j)$ means that $I'S' + JS' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as

$$1 = \sum y_t k_t + \sum z_s h_s + \sum f_j g'_j$$

for some $z_s \in I'$, $y_t \in J$, $k_t, h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda + JS_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 10.168.1 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes corresponding to points of $V(I_\lambda S_\lambda + JS_\lambda)$.

In the case that S is essentially of finite presentation, we can write $S = \Sigma^{-1}C$ where $R \to C$ is of finite presentation and $\Sigma \subset C$ is a multiplicative subset. We can also write $M = \Sigma^{-1}N$ for some finitely presented C-module N, see Algebra, Lemma 10.126.3. At this point we introduce C_λ, C', N_λ, N'. Then in the discussion above we obtain an open $U' \subset \mathrm{Spec}(C')$ over which N' is flat over R'. The assumption that (15.19.1.1) is true means that $V(I'S' + JS')$ maps into U', because for a prime $\mathfrak q' \subset S'$, corresponding to a prime $\mathfrak r' \subset C'$ we have $M'_{\mathfrak q'} = N'_{\mathfrak r'}$. Thus we can find $g'_j \in C'$ such that $\bigcup D(g'_j)$ contains the image of $V(I'S' + JS')$. The rest of the proof is exactly the same as before. □

05LU Lemma 15.19.5. In Situation 15.19.1. Let $I \subset R$ be an ideal. Assume

(1) R is a Noetherian ring,
(2) S is a Noetherian ring,
(3) M is a finite S-module, and
(4) for each $n \geq 1$ and any prime $\mathfrak q \in V(J + IS)$ the module $(M/I^nM)_{\mathfrak q}$ is flat over R/I^n.

Then (15.19.1.1) holds for (R, I), i.e., for every prime $\mathfrak q \in V(J+IS)$ the localization $M_{\mathfrak q}$ is flat over R.

Proof. Let $\mathfrak q \in V(J + IS)$. Then Algebra, Lemma 10.99.11 applied to $R \to S_{\mathfrak q}$ and $M_{\mathfrak q}$ implies that $M_{\mathfrak q}$ is flat over R. □

15.20. Flattening over a Noetherian complete local ring

05LV The following three lemmas give a completely algebraic proof of the existence of the "local" flattening stratification when the base is a complete local Noetherian ring R and the given module is finite over a finite type R-algebra S.

0526 Lemma 15.20.1. Let $R \to S$ be a ring map. Let M be an S-module. Assume

(1) $(R, \mathfrak{m})$ is a complete local Noetherian ring,
(2) S is a Noetherian ring, and
(3) M is finite over S.

Then there exists an ideal $I \subset \mathfrak{m}$ such that

(1) $(M/IM)_{\mathfrak{q}}$ is flat over R/I for all primes $\mathfrak{q}$ of S/IS lying over $\mathfrak{m}$, and
(2) if $J \subset R$ is an ideal such that $(M/JM)_{\mathfrak{q}}$ is flat over R/J for all primes $\mathfrak{q}$ lying over $\mathfrak{m}$, then $I \subset J$.

In other words, I is the smallest ideal of R such that (15.18.0.1) holds for $(\overline{R} \to \overline{S}, \overline{\mathfrak{m}}, \overline{M})$ where $\overline{R} = R/I$, $\overline{S} = S/IS$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$ and $\overline{M} = M/IM$.

Proof. Let $J \subset R$ be an ideal. Apply Algebra, Lemma 10.99.11 to the module M/JM over the ring R/J. Then we see that $(M/JM)_{\mathfrak{q}}$ is flat over R/J for all primes $\mathfrak{q}$ of S/JS if and only if $M/(J + \mathfrak{m}^n)M$ is flat over $R/(J + \mathfrak{m}^n)$ for all $n \geq 1$. We will use this remark below.

For every $n \geq 1$ the local ring $R/\mathfrak{m}^n$ is Artinian. Hence, by Lemma 15.17.1 there exists a smallest ideal $I_n \supset \mathfrak{m}^n$ such that M/I_nM is flat over R/I_n. It is clear that $I_{n+1} + \mathfrak{m}^n$ is contains I_n and applying Lemma 15.16.1 we see that $I_n = I_{n+1} + \mathfrak{m}^n$. Since $R = \lim_n \ R/\mathfrak{m}^n$ we see that $I = \lim_n \ I_n/\mathfrak{m}^n$ is an ideal in R such that $I_n = I + \mathfrak{m}^n$ for all $n \geq 1$. By the initial remarks of the proof we see that I verifies (1) and (2). Some details omitted. $\square$

0527 Lemma 15.20.2. With notation $R \to S$, M, and I and assumptions as in Lemma 15.20.1. Consider a local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that R' is Noetherian. Then the following are equivalent

(1) condition (15.18.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and
(2) $\varphi(I) = 0$.

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 15.18.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). We have to show that $\varphi(I) = 0$. This is equivalent to the condition that $\varphi(I)R' = 0$. By Artin-Rees in the Noetherian local ring R' (see Algebra, Lemma 10.51.4) this is equivalent to the condition that $\varphi(I)R' + (\mathfrak{m}')^n = (\mathfrak{m}')^n$ for all $n > 0$. Hence this is equivalent to the condition that the composition $\varphi_n : R \to R' \to R'/(\mathfrak{m}')^n$ annihilates I for each n. Now assumption (1) for φ implies assumption (1) for φ_n by Lemma 15.18.1. This reduces us to the case where R' is Artinian local.

Assume R' Artinian. Let $J = \mathrm{Ker}(\varphi)$. We have to show that $I \subset J$. By the construction of I in Lemma 15.20.1 it suffices to show that $(M/JM)_{\mathfrak{q}}$ is flat over R/J for every prime $\mathfrak{q}$ of S/JS lying over $\mathfrak{m}$. As R' is Artinian, condition (1) signifies that $M \otimes_R R'$ is flat over R'. As R' is Artinian and $R/J \to R'$ is a local injective ring map, it follows that R/J is Artinian too. Hence the flatness of $M \otimes_R R' = M/JM \otimes_{R/J} R'$ over R' implies that M/JM is flat over R/J by Algebra, Lemma 10.101.7. This concludes the proof. $\square$

0528 **Lemma 15.20.3.** With notation $R \to S$, M, and I and assumptions as in Lemma 15.20.1. In addition assume that $R \to S$ is of finite type. Then for any local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ the following are equivalent

(1) condition (15.18.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and
(2) $\varphi(I) = 0$.

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 15.18.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). As R is Noetherian we see that $R \to S$ is of finite presentation and M is an S-module of finite presentation. Write $R' = \operatorname{colim}_\lambda R_\lambda$ as a directed colimit of local R-subalgebras $R_\lambda \subset R'$, with maximal ideals $\mathfrak{m}_\lambda = R_\lambda \cap \mathfrak{m}'$ such that each R_λ is essentially of finite type over R. By Lemma 15.18.3 we see that condition (15.18.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, \mathfrak{m}_\lambda, M \otimes_R R_\lambda)$ for some λ. Hence Lemma 15.20.2 applies to the ring map $R \to R_\lambda$ and we see that I maps to zero in R_λ, a fortiori it maps to zero in R'. $\square$

15.21. Descent of flatness along integral maps

052Y First a few simple lemmas.

052Z **Lemma 15.21.1.** Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. Let $\alpha \in R$ be such that $P(\alpha) = 0$. Then $P(T) = (T - \alpha)Q(T)$ for some monic polynomial $Q(T) \in R[T]$.

Proof. By induction on the degree of P. If $\deg(P) = 1$, then $P(T) = T - \alpha$ and the result is true. If $\deg(P) > 1$, then we can write $P(T) = (T - \alpha)Q(T) + r$ for some polynomial $Q \in R[T]$ of degree $< \deg(P)$ and some $r \in R$ by long division. By assumption $0 = P(\alpha) = (\alpha - \alpha)Q(\alpha) + r = r$ and we conclude that $r = 0$ as desired. $\square$

0530 **Lemma 15.21.2.** Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. There exists a finite free ring map $R \to R'$ such that $P(T) = (T - \alpha)Q(T)$ for some $\alpha \in R'$ and some monic polynomial $Q(T) \in R'[T]$.

Proof. Write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_0$. Set $R' = R[x]/(x^d + a_1 x^{d-1} + \ldots + a_0)$. Set α equal to the congruence class of x. Then it is clear that $P(\alpha) = 0$. Thus we win by Lemma 15.21.1. $\square$

0531 **Lemma 15.21.3.** Let $R \to S$ be a finite ring map. There exists a finite free ring extension $R \subset R'$ such that $S \otimes_R R'$ is a quotient of a ring of the form

$$R'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$.

Proof. Let $x_1, \ldots, x_n \in S$ be generators of S over R. For each i we can choose a monic polynomial $P_i(T) \in R[T]$ such that $P_i(x_i) = 0$ in S, see Algebra, Lemma 10.36.3. Say $\deg(P_i) = d_i$. By Lemma 15.21.2 (applied $\sum d_i$ times) there exists a finite free ring extension $R \subset R'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{j=1,\ldots,d_i}(T - \alpha_{ij})$$

for certain $\alpha_{ik} \in R'$. Let $R'[T_1, \ldots, T_n] \to S \otimes_R R'$ be the R'-algebra map which maps T_i to $x_i \otimes 1$. As this maps $P_i(T_i)$ to zero, this induces the desired surjection. $\square$

0532 Lemma 15.21.4. Let R be a ring. Let $S = R[T_1, \ldots, T_n]/J$. Assume J contains elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R$. For $\underline{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i$ consider the ring map

$$\Phi_{\underline{k}} : R[T_1, \ldots, T_n] \to R, \quad T_i \longmapsto \alpha_{ik_i}$$

Set $J_{\underline{k}} = \Phi_{\underline{k}}(J)$. Then the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is equal to $V(\bigcap J_{\underline{k}})$.

Proof. This lemma proves itself. Hint: $V(\bigcap J_{\underline{k}}) = \bigcup V(J_{\underline{k}})$. □

The following result is due to Ferrand, see [Fer69].

0533 Lemma 15.21.5. Let $R \to S$ be a finite injective homomorphism of Noetherian rings. Let M be an R-module. If $M \otimes_R S$ is a flat S-module, then M is a flat R-module.

Proof. Let M be an R-module such that $M \otimes_R S$ is flat over S. By Algebra, Lemma 10.39.8 in order to prove that M is flat we may replace R by any faithfully flat ring extension. By Lemma 15.21.3 we can find a finite locally free ring extension $R \subset R'$ such that $S' = S \otimes_R R' = R'[T_1, \ldots, T_n]/J$ for some ideal $J \subset R'[T_1, \ldots, T_n]$ which contains the elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$. Note that R' is Noetherian and that $R' \subset S'$ is a finite extension of rings. Hence we may replace R by R' and assume that S has a presentation as in Lemma 15.21.4. Note that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, see Algebra, Lemma 10.36.17. Thus, using Lemma 15.21.4 we conclude that $I = \bigcap J_{\underline{k}}$ is an ideal such that $V(I) = \mathrm{Spec}(R)$. This means that $I \subset \sqrt{(0)}$, and since R is Noetherian that I is nilpotent. The maps $\Phi_{\underline{k}}$ induce commutative diagrams

$$\begin{tikzcd} S \arrow[rr] & & R/J_{\underline{k}} \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

from which we conclude that $M/J_{\underline{k}}M$ is flat over $R/J_{\underline{k}}$. By Lemma 15.16.1 we see that M/IM is flat over R/I. Finally, applying Algebra, Lemma 10.101.5 we conclude that M is flat over R. □

0534 Lemma 15.21.6. Let $R \to S$ be an injective integral ring map. Let M be a finitely presented module over $R[x_1, \ldots, x_n]$. If $M \otimes_R S$ is flat over S, then M is flat over R.

Proof. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus t} \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Let's say that the first map is given by the $r \times t$-matrix $T = (f_{ij})$ with $f_{ij} \in R[x_1, \ldots, x_n]$. Write $f_{ij} = \sum f_{ij,I} x^I$ with $f_{ij,I} \in R$ (multi-index notation). Consider diagrams

$$\begin{tikzcd} R \arrow[r] & S \\ R_\lambda \arrow[u] \arrow[r] & S_\lambda \arrow[u] \end{tikzcd}$$

where R_λ is a finitely generated $\mathbf{Z}$-subalgebra of R containing all $f_{ij,I}$ and S_λ is a finite R_λ-subalgebra of S. Let M_λ be the finite $R_\lambda[x_1, \ldots, x_n]$-module defined by a presentation as above, using the same matrix T but now viewed as a matrix over $R_\lambda[x_1, \ldots, x_n]$. Note that S is the directed colimit of the S_λ (details omitted). By Algebra, Lemma 10.168.1 we see that for some λ the module $M_\lambda \otimes_{R_\lambda} S_\lambda$ is flat over S_λ. By Lemma 15.21.5 we conclude that M_λ is flat over R_λ. Since $M = M_\lambda \otimes_{R_\lambda} R$ we win by Algebra, Lemma 10.39.7. □

15.22. Torsion free modules

0549 In this section we discuss torsion free modules and the relationship with flatness (especially over dimension 1 rings).

0536 Definition 15.22.1. Let R be a domain. Let M be an R-module.

(1) We say an element $x \in M$ is torsion if there exists a nonzero $f \in R$ such that $fx = 0$.
(2) We say M is torsion free if the only torsion element of M is 0.

Let R be a domain and let $S = R \setminus \{0\}$ be the multiplicative set of nonzero elements of R. Then an R-module M is torsion free if and only if $M \to S^{-1}M$ is injective. In other words, if and only if the map $M \to M \otimes_R K$ is injective where $K = S^{-1}R$ is the fraction field of R.

0537 Lemma 15.22.2. Let R be a domain. Let M be an R-module. The set of torsion elements of M forms a submodule $M_{tors} \subset M$. The quotient module M/M_{tors} is torsion free.

Proof. Omitted. □

0AUR Lemma 15.22.3. Let R be a domain. Let M be a torsion free R-module. For any multiplicative set $S \subset R$ the module $S^{-1}M$ is a torsion free $S^{-1}R$-module.

Proof. Omitted. □

0AXM Lemma 15.22.4. Let $R \to R'$ be a flat homomorphism of domains. If M is a torsion free R-module, then $M \otimes_R R'$ is a torsion free R'-module.

Proof. If M is torsion free, then $M \subset M \otimes_R K$ is injective where K is the fraction field of R. Since R' is flat over R we see that $M \otimes_R R' \to (M \otimes_R K) \otimes_R R'$ is injective. Since $M \otimes_R K$ is isomorphic to a direct sum of copies of K, it suffices to see that $K \otimes_R R'$ is torsion free. This is true because it is a localization of R'. □

0AUS Lemma 15.22.5. Let R be a domain. Let $0 \to M \to M' \to M'' \to 0$ be a short exact sequence of R-modules. If M and M'' are torsion free, then M' is torsion free.

Proof. Omitted. □

0AUT Lemma 15.22.6. Let R be a domain. Let M be an R-module. Then M is torsion free if and only if $M_\mathfrak{m}$ is a torsion free $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.

Proof. Omitted. Hint: Use Lemma 15.22.3 and Algebra, Lemma 10.23.1. □

0AUU Lemma 15.22.7. Let R be a domain. Let M be a finite R-module. Then M is torsion free if and only if M is a submodule of a finite free module.

Proof. If M is a submodule of $R^{\oplus n}$, then M is torsion free. For the converse, assume M is torsion free. Let K be the fraction field of R. Then $M \otimes_R K$ is a finite dimensional K-vector space. Choose a basis $e_1, \ldots, e_r$ for this vector space. Let $x_1, \ldots, x_n$ be generators of M. Write $x_i = \sum (a_{ij}/b_{ij})e_j$ for some $a_{ij}, b_{ij} \in R$ with $b_{ij} \neq 0$. Set $b = \prod_{i,j} b_{ij}$. Since M is torsion free the map $M \to M \otimes_R K$ is injective and the image is contained in $R^{\oplus r} = Re_1/b \oplus \ldots \oplus Re_r/b$. $\square$

0AUV Lemma 15.22.8. Let R be a Noetherian domain. Let M be a nonzero finite R-module. The following are equivalent

(1) M is torsion free,
(2) M is a submodule of a finite free module,
(3) (0) is the only associated prime of M,
(4) (0) is in the support of M and M has property (S_1), and
(5) (0) is in the support of M and M has no embedded associated prime.

Proof. We have seen the equivalence of (1) and (2) in Lemma 15.22.7. We have seen the equivalence of (4) and (5) in Algebra, Lemma 10.157.2. The equivalence between (3) and (5) is immediate from the definition. A localization of a torsion free module is torsion free (Lemma 15.22.3), hence it is clear that a M has no associated primes different from (0). Thus (1) implies (5). Conversely, assume (5). If M has torsion, then there exists an embedding $R/I \subset M$ for some nonzero ideal I of R. Hence M has an associated prime different from (0) (see Algebra, Lemmas 10.63.3 and 10.63.7). This is an embedded associated prime which contradicts the assumption. $\square$

0538 Lemma 15.22.9. Let R be a domain. Any flat R-module is torsion free.

Proof. If $x \in R$ is nonzero, then $x : R \to R$ is injective, and hence if M is flat over R, then $x : M \to M$ is injective. Thus if M is flat over R, then M is torsion free. $\square$

0539 Lemma 15.22.10. Let A be a valuation ring. An A-module M is flat over A if and only if M is torsion free.

Proof. The implication "flat $\Rightarrow$ torsion free" is Lemma 15.22.9. For the converse, assume M is torsion free. By the equational criterion of flatness (see Algebra, Lemma 10.39.11) we have to show that every relation in M is trivial. To do this assume that $\sum_{i=1,\ldots,n} a_i x_i = 0$ with $x_i \in M$ and $a_i \in A$. After renumbering we may assume that $v(a_1) \leq v(a_i)$ for all i. Hence we can write $a_i = a'_i a_1$ for some $a'_i \in A$. Note that $a'_1 = 1$. As M is torsion free we see that $x_1 = -\sum_{i \geq 2} a'_i x_i$. Thus, if we choose $y_i = x_i$, $i = 2, \ldots, n$ then

$$x_1 = \sum\nolimits_{j \geq 2} -a'_j y_j, \quad x_i = y_i, (i \geq 2) \quad 0 = a_1 \cdot (-a'_j) + a_j \cdot 1 (j \geq 2)$$

shows that the relation was trivial (to be explicit the elements a_{ij} are defined by setting $a_{11} = 0$, $a_{1j} = -a'_j$ for $j > 1$, and $a_{ij} = \delta_{ij}$ for $i, j \geq 2$). $\square$

0AUW Lemma 15.22.11. Let A be a Dedekind domain (for example a discrete valuation ring or more generally a PID).

(1) An A-module is flat if and only if it is torsion free.
(2) A finite torsion free A-module is finite locally free.
(3) A finite torsion free A-module is finite free if A is a PID.

Proof. (For the parenthetical remark in the statement of the lemma, see Algebra, Lemma 10.120.15.) Proof of (1). By Lemma 15.22.6 and Algebra, Lemma 10.39.18 it suffices to check the statement over $A_\mathfrak{m}$ for $\mathfrak{m} \subset A$ maximal. Since $A_\mathfrak{m}$ is a discrete valuation ring (Algebra, Lemma 10.120.17) we win by Lemma 15.22.10.

Proof of (2). Follows from Algebra, Lemma 10.78.2 and (1).

Proof of (3). Let A be a PID and let M be a finite torsion free module. By Lemma 15.22.7 we see that $M \subset A^{\oplus n}$ for some n. We argue that M is free by induction on n. The case $n = 1$ expresses exactly the fact that A is a PID. If $n > 1$ let $M' \subset R^{\oplus n-1}$ be the image of the projection onto the last $n-1$ summands of $R^{\oplus n}$. Then we obtain a short exact sequence $0 \to I \to M \to M' \to 0$ where I is the intersection of M with the first summand R of $R^{\oplus n}$. By induction we see that M is an extension of finite free R-modules, whence finite free. □

0AUX Lemma 15.22.12. Let R be a domain. Let M, N be R-modules. If N is torsion free, so is $\text{Hom}_R(M, N)$.

Proof. Choose a surjection $\bigoplus_{i \in I} R \to M$. Then $\text{Hom}_R(M, N) \subset \prod_{i \in I} N$. □

15.23. Reflexive modules

0AUY Here is our definition.

0AUZ Definition 15.23.1. Let R be a domain. We say an R-module M is reflexive if the natural map

$$j : M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)$$

which sends $m \in M$ to the map sending $\varphi \in \text{Hom}_R(M, R)$ to $\varphi(m) \in R$ is an isomorphism.

We can make this definition for more general rings, but already the definition above has drawbacks. It would be wise to restrict to Noetherian domains and finite torsion free modules and (perhaps) impose some regularity conditions on R (e.g., R is normal).

0AV0 Lemma 15.23.2. Let R be a domain and let M be an R-module.

(1) If M is reflexive, then M is torsion free.
(2) If M is finite, then $j : M \to \text{Hom}_R(\text{Hom}_R(M, R), R)$ is injective if and only if M is torsion free

Proof. Follows immediately from Lemmas 15.22.12 and 15.22.7. □

0B36 Lemma 15.23.3. Let R be a discrete valuation ring and let M be a finite R-module. Then the map $j : M \to \text{Hom}_R(\text{Hom}_R(M, R), R)$ is surjective.

Proof. Let $M_{tors} \subset M$ be the torsion submodule. Then we have $\text{Hom}_R(M, R) = \text{Hom}_R(M/M_{tors}, R)$ (holds over any domain). Hence we may assume that M is torsion free. Then M is free by Lemma 15.22.11 and the lemma is clear. □

0AV1 Lemma 15.23.4. Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent:

(1) M is reflexive,
(2) $M_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module for all primes $\mathfrak{p} \subset R$, and
(3) $M_\mathfrak{m}$ is a reflexive $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.

Proof. The localization of $j : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ at a prime $\mathfrak{p}$ is the corresponding map for the module $M_\mathfrak{p}$ over the Noetherian local domain $R_\mathfrak{p}$. See Algebra, Lemma 10.10.2. Thus the lemma holds by Algebra, Lemma 10.23.1. $\square$

0EB8 **Lemma 15.23.5.** *Let R be a Noetherian domain. Let $0 \to M \to M' \to M''$ an exact sequence of finite R-modules. If M' is reflexive and M'' is torsion free, then M is reflexive.*

Proof. We will use without further mention that $\mathrm{Hom}_R(N, N')$ is a finite R-module for any finite R-modules N and N', see Algebra, Lemma 10.71.9. We take duals to get a sequence

$$\mathrm{Hom}_R(M, R) \leftarrow \mathrm{Hom}_R(M', R) \leftarrow \mathrm{Hom}_R(M'', R)$$

Dualizing again we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \arrow[r, "j"'] & \mathrm{Hom}_R(\mathrm{Hom}_R(M', R), R) \arrow[r] & \mathrm{Hom}_R(\mathrm{Hom}_R(M'', R), R) \\
M \arrow[u] \arrow[r] & M' \arrow[u] \arrow[r] & M'' \arrow[u]
\end{tikzcd}$$

We do not know the top row is exact. But, by assumption the middle vertical arrow is an isomorphism and the right vertical arrow is injective (Lemma 15.23.2). We claim j is injective. Assuming the claim a diagram chase shows that the left vertical arrow is an isomorphism, i.e., M is reflexive.

Proof of the claim. Consider the exact sequence $\mathrm{Hom}_R(M', R) \to \mathrm{Hom}_R(M, R) \to Q \to 0$ defining Q. One applies Algebra, Lemma 10.10.2 to obtain

$$\mathrm{Hom}_K(M' \otimes_R K, K) \to \mathrm{Hom}_K(M \otimes_R K, K) \to Q \otimes_R K \to 0$$

But $M \otimes_R K \to M' \otimes_R K$ is an injective map of vector spaces, hence split injective, so $Q \otimes_R K = 0$, that is, Q is torsion. Then one gets the exact sequence

$$0 \to \mathrm{Hom}_R(Q, R) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M', R), R)$$

and $\mathrm{Hom}_R(Q, R) = 0$ because Q is torsion. $\square$

0AV2 **Lemma 15.23.6.** *Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent*

1. *M is reflexive,*
2. *there exists a short exact sequence $0 \to M \to F \to N \to 0$ with F finite free and N torsion free.*

Proof. Observe that a finite free module is reflexive. By Lemma 15.23.5 we see that (2) implies (1). Assume M is reflexive. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to \mathrm{Hom}_R(M, R) \to 0$. Dualizing we get an exact sequence

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) \to R^{\oplus n} \to N \to 0$$

with $N = \mathrm{Im}(R^{\oplus n} \to R^{\oplus m})$ a torsion free module. As $M = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ we get an exact sequence as in (2). $\square$

0EB9 **Lemma 15.23.7.** *Let $R \to R'$ be a flat homomorphism of Noetherian domains. If M is a finite reflexive R-module, then $M \otimes_R R'$ is a finite reflexive R'-module.*

Proof. Choose a short exact sequence $0 \to M \to F \to N \to 0$ with F finite free and N torsion free, see Lemma 15.23.6. Since $R \to R'$ is flat we obtain a short exact sequence $0 \to M \otimes_R R' \to F \otimes_R R' \to N \otimes_R R' \to 0$ with $F \otimes_R R'$ finite free and $N \otimes_R R'$ torsion free (Lemma 15.22.4). Thus $M \otimes_R R'$ is reflexive by Lemma 15.23.6. □

0AV3 **Lemma 15.23.8.** Let R be a Noetherian domain. Let M be a finite R-module. Let N be a finite reflexive R-module. Then $\mathrm{Hom}_R(M, N)$ is reflexive.

Proof. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Then we obtain

$$0 \to \mathrm{Hom}_R(M, N) \to N^{\oplus n} \to N' \to 0$$

with $N' = \mathrm{Im}(N^{\oplus n} \to N^{\oplus m})$ torsion free. We conclude by Lemma 15.23.5. □

0AV4 **Definition 15.23.9.** Let R be a Noetherian domain. Let M be a finite R-module. The module $M^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is called the *reflexive hull* of M.

This makes sense because the reflexive hull is reflexive by Lemma 15.23.8. The assignment $M \mapsto M^{**}$ is a functor. If $\varphi : M \to N$ is an R-module map into a reflexive R-module N, then φ factors $M \to M^{**} \to N$ through the reflexive hull of M. Another way to say this is that taking the reflexive hull is the left adjoint to the inclusion functor

$$\text{finite reflexive modules} \subset \text{finite modules}$$

over a Noetherian domain R.

0AV5 **Lemma 15.23.10.** Let R be a Noetherian local ring. Let M, N be finite R-modules.

(1) If N has depth ≥ 1, then $\mathrm{Hom}_R(M, N)$ has depth ≥ 1.
(2) If N has depth ≥ 2, then $\mathrm{Hom}_R(M, N)$ has depth ≥ 2.

Proof. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Dualizing we get an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to N^{\oplus n} \to N' \to 0$$

with $N' = \mathrm{Im}(N^{\oplus n} \to N^{\oplus m})$. A submodule of a module with depth ≥ 1 has depth ≥ 1; this follows immediately from the definition. Thus part (1) is clear. For (2) note that here the assumption and the previous remark implies N' has depth ≥ 1. The module $N^{\oplus n}$ has depth ≥ 2. From Algebra, Lemma 10.72.6 we conclude $\mathrm{Hom}_R(M, N)$ has depth ≥ 2. □

0AV6 **Lemma 15.23.11.** Let R be a Noetherian ring. Let M, N be finite R-modules.

(1) If N has property (S_1), then $\mathrm{Hom}_R(M, N)$ has property (S_1).
(2) If N has property (S_2), then $\mathrm{Hom}_R(M, N)$ has property (S_2).
(3) If R is a domain, N is torsion free and (S_2), then $\mathrm{Hom}_R(M, N)$ is torsion free and has property (S_2).

Proof. Since localizing at primes commutes with taking Hom_R for finite R-modules (Algebra, Lemma 10.71.9) parts (1) and (2) follow immediately from Lemma 15.23.10. Part (3) follows from (2) and Lemma 15.22.12. □

0AV7 **Lemma 15.23.12.** Let R be a Noetherian ring. Let $\varphi : M \to N$ be a map of R-modules. Assume that for every prime $\mathfrak{p}$ of R at least one of the following happens

(1) $M_\mathfrak{p} \to N_\mathfrak{p}$ is injective, or
(2) $\mathfrak{p} \notin \mathrm{Ass}(M)$.

Then φ is injective.

Proof. Let $\mathfrak{p}$ be an associated prime of $\text{Ker}(\varphi)$. Then there exists an element $x \in M_\mathfrak{p}$ which is in the kernel of $M_\mathfrak{p} \to N_\mathfrak{p}$ and is annihilated by $\mathfrak{p}R_\mathfrak{p}$ (Algebra, Lemma 10.63.15). This is impossible in both cases. Hence $\text{Ass}(\text{Ker}(\varphi)) = \emptyset$ and we conclude $\text{Ker}(\varphi) = 0$ by Algebra, Lemma 10.63.7. □

0AV8 Lemma 15.23.13. Let R be a Noetherian ring. Let $\varphi : M \to N$ be a map of R-modules. Assume M is finite and that for every prime $\mathfrak{p}$ of R one of the following happens

(1) $M_\mathfrak{p} \to N_\mathfrak{p}$ is an isomorphism, or
(2) $\text{depth}(M_\mathfrak{p}) \geq 2$ and $\mathfrak{p} \notin \text{Ass}(N)$.

Then φ is an isomorphism.

Proof. By Lemma 15.23.12 we see that φ is injective. Let $N' \subset N$ be an finitely generated R-module containing the image of M. Then $\text{Ass}(N_\mathfrak{p}) = \emptyset$ implies $\text{Ass}(N'_\mathfrak{p}) = \emptyset$. Hence the assumptions of the lemma hold for $M \to N'$. In order to prove that φ is an isomorphism, it suffices to prove the same thing for every such $N' \subset N$. Thus we may assume N is a finite R-module. In this case, $\mathfrak{p} \notin \text{Ass}(N) \Rightarrow \text{depth}(N_\mathfrak{p}) \geq 1$, see Algebra, Lemma 10.63.18. Consider the short exact sequence

$$0 \to M \to N \to Q \to 0$$

defining Q. Looking at the conditions we see that either $Q_\mathfrak{p} = 0$ in case (1) or $\text{depth}(Q_\mathfrak{p}) \geq 1$ in case (2) by Algebra, Lemma 10.72.6. This implies that Q does not have any associated primes, hence $Q = 0$ by Algebra, Lemma 10.63.7. □

0AV9 Lemma 15.23.14. Let R be a Noetherian domain. Let $\varphi : M \to N$ be a map of R-modules. Assume M is finite, N is torsion free, and that for every prime $\mathfrak{p}$ of R one of the following happens

(1) $M_\mathfrak{p} \to N_\mathfrak{p}$ is an isomorphism, or
(2) $\text{depth}(M_\mathfrak{p}) \geq 2$.

Then φ is an isomorphism.

Proof. This is a special case of Lemma 15.23.13. □

0AVA Lemma 15.23.15. Let R be a Noetherian domain. Let M be a finite R-module. The following are equivalent

(1) M is reflexive,
(2) for every prime $\mathfrak{p}$ of R one of the following happens
    (a) $M_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module, or
    (b) $\text{depth}(M_\mathfrak{p}) \geq 2$.

Proof. If (1) is true, then $M_\mathfrak{p}$ is a reflexive module for all primes of $\mathfrak{p}$ by Lemma 15.23.4. Thus (1) $\Rightarrow$ (2). Assume (2). Set $N = \text{Hom}_R(\text{Hom}_R(M, R), R)$ so that

$$N_\mathfrak{p} = \text{Hom}_{R_\mathfrak{p}}(\text{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}), R_\mathfrak{p})$$

for every prime $\mathfrak{p}$ of R. See Algebra, Lemma 10.10.2. We apply Lemma 15.23.14 to the map $j : M \to N$. This is allowed because M is finite and N is torsion free by Lemma 15.22.12. In case (2)(a) the map $M_\mathfrak{p} \to N_\mathfrak{p}$ is an isomorphism and in case (2)(b) we have $\text{depth}(M_\mathfrak{p}) \geq 2$. □

0EBA Lemma 15.23.16. Let R be a Noetherian domain. Let M be a finite reflexive R-module. Let $\mathfrak{p} \subset R$ be a prime ideal.

(1) If $\text{depth}(R_\mathfrak{p}) \geq 2$, then $\text{depth}(M_\mathfrak{p}) \geq 2$.
(2) If R is (S_2), then M is (S_2).

Proof. Since formation of reflexive hull $\text{Hom}_R(\text{Hom}_R(M, R), R)$ commutes with localization (Algebra, Lemma 10.10.2) part (1) follows from Lemma 15.23.10. Part (2) is immediate from Lemma 15.23.11. $\square$

0EBB Example 15.23.17. The results above and below suggest reflexivity is related to the (S_2) condition; here is an example to prevent too optimistic conjectures. Let k be a field. Let R be the k-subalgebra of $k[x, y]$ generated by $1, y, x^2, xy, x^3$. Then R is not (S_2). So R as an R-module is an example of a reflexive R-module which is not (S_2). Let $M = k[x, y]$ viewed as an R-module. Then M is a reflexive R-module because

$$\text{Hom}_R(M, R) = \mathfrak{m} = (y, x^2, xy, x^3) \quad \text{and} \quad \text{Hom}_R(\mathfrak{m}, R) = M$$

and M is (S_2) as an R-module (computations omitted). Thus R is a Noetherian domain possessing a reflexive (S_2) module but R is not (S_2) itself.

0AVB Lemma 15.23.18. Let R be a Noetherian normal domain with fraction field K. Let M be a finite R-module. The following are equivalent

(1) M is reflexive,
(2) M is torsion free and has property (S_2),
(3) M is torsion free and $M = \bigcap_{\text{height}(\mathfrak{p})=1} M_\mathfrak{p}$ where the intersection happens in $M_K = M \otimes_R K$.

Proof. By Algebra, Lemma 10.157.4 we see that R satisfies (R_1) and (S_2).

Assume (1). Then M is torsion free by Lemma 15.23.2 and satisfies (S_2) by Lemma 15.23.16. Thus (2) holds.

Assume (2). By definition $M' = \bigcap_{\text{height}(\mathfrak{p})=1} M_\mathfrak{p}$ is the kernel of the map

$$M_K \longrightarrow \bigoplus\nolimits_{\text{height}(\mathfrak{p})=1} M_K/M_\mathfrak{p} \subset \prod\nolimits_{\text{height}(\mathfrak{p})=1} M_K/M_\mathfrak{p}$$

Observe that our map indeed factors through the direct sum as indicated since given $a/b \in K$ there are at most finitely many height 1 primes $\mathfrak{p}$ with $b \in \mathfrak{p}$. Let $\mathfrak{p}_0$ be a prime of height 1. Then $(M_K/M_\mathfrak{p})_{\mathfrak{p}_0} = 0$ unless $\mathfrak{p} = \mathfrak{p}_0$ in which case we get $(M_K/M_\mathfrak{p})_{\mathfrak{p}_0} = M_K/M_{\mathfrak{p}_0}$. Thus by exactness of localization and the fact that localization commutes with direct sums, we see that $M'_{\mathfrak{p}_0} = M_{\mathfrak{p}_0}$. Since M has depth ≥ 2 at primes of height > 1, we see that $M \to M'$ is an isomorphism by Lemma 15.23.14. Hence (3) holds.

Assume (3). Let $\mathfrak{p}$ be a prime of height 1. Then $R_\mathfrak{p}$ is a discrete valuation ring by (R_1). By Lemma 15.22.11 we see that $M_\mathfrak{p}$ is finite free, in particular reflexive. Hence the map $M \to M^{**}$ induces an isomorphism at all the primes $\mathfrak{p}$ of height 1. Thus the condition $M = \bigcap_{\text{height}(\mathfrak{p})=1} M_\mathfrak{p}$ implies that $M = M^{**}$ and (1) holds. $\square$

0AVC Lemma 15.23.19. Let R be a Noetherian normal domain. Let M be a finite R-module. Then the reflexive hull of M is the intersection

$$M^{**} = \bigcap\nolimits_{\text{height}(\mathfrak{p})=1} M_\mathfrak{p}/(M_\mathfrak{p})_{tors} = \bigcap\nolimits_{\text{height}(\mathfrak{p})=1} (M/M_{tors})_\mathfrak{p}$$

taken in $M \otimes_R K$.

Proof. Let $\mathfrak p$ be a prime of height 1. The kernel of $M_\mathfrak p \to M \otimes_R K$ is the torsion submodule $(M_\mathfrak p)_{tors}$ of $M_\mathfrak p$. Moreover, we have $(M/M_{tors})_\mathfrak p = M_\mathfrak p/(M_\mathfrak p)_{tors}$ and this is a finite free module over the discrete valuation ring $R_\mathfrak p$ (Lemma 15.22.11). Then $M_\mathfrak p/(M_\mathfrak p)_{tors} \to (M_\mathfrak p)^{**} = (M^{**})_\mathfrak p$ is an isomorphism, hence the lemma is a consequence of Lemma 15.23.18. ☐

0BM4 Lemma 15.23.20. Let A be a Noetherian normal domain with fraction field K. Let L be a finite extension of K. If the integral closure B of A in L is finite over A, then B is reflexive as an A-module.

Proof. It suffices to show that $B = \bigcap B_\mathfrak p$ where the intersection is over height 1 primes $\mathfrak p \subset A$, see Lemma 15.23.18. Let $b \in \bigcap B_\mathfrak p$. Let $x^d + a_1x^{d-1} + \ldots + a_d$ be the minimal polynomial of b over K. We want to show $a_i \in A$. By Algebra, Lemma 10.38.6 we see that $a_i \in A_\mathfrak p$ for all i and all height one primes $\mathfrak p$. Hence we get what we want from Algebra, Lemma 10.157.6 (or the lemma already cited as A is a reflexive module over itself). ☐

15.24. Content ideals

0AS9 The definition may not be what you expect.

0ASA Definition 15.24.1. Let A be a ring. Let M be a flat A-module. Let $x \in M$. If the set of ideals I in A such that $x \in IM$ has a smallest element, we call it the content ideal of x.

Note that since M is flat over A, for a pair of ideals I, I' of A we have $IM \cap I'M = (I \cap I')M$ as can be seen by tensoring the exact sequence $0 \to I \cap I' \to I \oplus I' \to I + I' \to 0$ by M.

0ASB Lemma 15.24.2. Let A be a ring. Let M be a flat A-module. Let $x \in M$. The content ideal of x, if it exists, is finitely generated.

Proof. Say $x \in IM$. Then we can write $x = \sum_{i=1,\ldots,n} f_ix_i$ with $f_i \in I$ and $x_i \in M$. Hence $x \in I'M$ with $I' = (f_1, \ldots, f_n)$. ☐

0ASC Lemma 15.24.3. Let $(A, \mathfrak m)$ be a local ring. Let $u : M \to N$ be a map of flat A-modules such that $\overline{u} : M/\mathfrak mM \to N/\mathfrak mN$ is injective. If $x \in M$ has content ideal I, then $u(x)$ has content ideal I as well.

Proof. It is clear that $u(x) \in IN$. If $u(x) \in I'N$, then $u(x) \in (I' \cap I)N$, see discussion following Definition 15.24.1. Hence it suffices to show: if $x \in I'N$ and $I' \subset I$, $I' \neq I$, then $u(x) \notin I'N$. Since I/I' is a nonzero finite A-module (Lemma 15.24.2) there is a nonzero map $\chi : I/I' \to A/\mathfrak m$ of A-modules by Nakayama's lemma (Algebra, Lemma 10.20.1). Since I is the content ideal of x we see that $x \notin I''M$ where $I'' = \text{Ker}(\chi)$. Hence x is not in the kernel of the map

$$IM = I \otimes_A M \xrightarrow{\chi\otimes 1} A/\mathfrak m \otimes M \cong M/\mathfrak mM$$

Applying our hypothesis on $\overline{u}$ we conclude that $u(x)$ does not map to zero under the map

$$IN = I \otimes_A N \xrightarrow{\chi\otimes 1} A/\mathfrak m \otimes N \cong N/\mathfrak mN$$

and we conclude. ☐

0ASD Lemma 15.24.4. Let A be a ring. Let M be a flat Mittag-Leffler module. Then every element of M has a content ideal.

Proof. This is a special case of Algebra, Lemma 10.91.2. □

15.25. Flatness and finiteness conditions

054A In this section we discuss some implications of the type "flat + finite type ⇒ finite presentation". We will revisit this result in the chapter on flatness, see More on Flatness, Section 38.1. A first result of this type was proved in Algebra, Lemma 10.108.6.

053A **Lemma 15.25.1.** Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be an S-module. Assume

(1) there exist finitely many primes $\mathfrak p_1, \ldots, \mathfrak p_m$ of R such that the map $R \to \prod R_{\mathfrak p_j}$ is injective,
(2) M is a finite S-module,
(3) M flat over R, and
(4) for every prime $\mathfrak p$ of R the module $M_{\mathfrak p}$ is of finite presentation over $S_{\mathfrak p}$.

Then M is of finite presentation over S.

Proof. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0$$

of M as an S-module. Let $\mathfrak q$ be a prime ideal of S lying over a prime $\mathfrak p$ of R. By assumption there exist finitely many elements $k_1, \ldots, k_t \in K$ such that if we set $K' = \sum S k_j \subset K$ then $K'_{\mathfrak p} = K_{\mathfrak p}$ and $K'_{\mathfrak p_j} = K_{\mathfrak p_j}$ for $j = 1, \ldots, m$. Setting $M' = S^{\oplus r}/K'$ we deduce that in particular $M'_{\mathfrak q} = M_{\mathfrak q}$. By openness of flatness, see Algebra, Theorem 10.129.4 we conclude that there exists a $g \in S$, $g \notin \mathfrak q$ such that M'_g is flat over R. Thus $M'_g \to M_g$ is a surjective map of flat R-modules. Consider the commutative diagram

$$\begin{tikzcd} M'_g \arrow[r] \arrow[d] & M_g \arrow[d] \\ \prod (M'_g)_{\mathfrak p_j} \arrow[r] & \prod (M_g)_{\mathfrak p_j} \end{tikzcd}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $R \to \prod R_{\mathfrak p_j}$ is injective and M'_g is flat over R. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that M_g is of finite presentation over S_g. We conclude by applying Algebra, Lemma 10.23.2. □

053B **Lemma 15.25.2.** Let $R \to S$ be a ring homomorphism. Assume

(1) there exist finitely many primes $\mathfrak p_1, \ldots, \mathfrak p_m$ of R such that the map $R \to \prod R_{\mathfrak p_j}$ is injective,
(2) $R \to S$ is of finite type,
(3) S flat over R, and
(4) for every prime $\mathfrak p$ of R the ring $S_{\mathfrak p}$ is of finite presentation over $R_{\mathfrak p}$.

Then S is of finite presentation over R.

Proof. By assumption S is a quotient of a polynomial ring over R. Thus the result follows directly from Lemma 15.25.1. □

053C Lemma 15.25.3. Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a graded polynomial algebra over R, i.e., $\deg(x_i) > 0$ but not necessarily equal to 1. Let M be a graded S-module. Assume

(1) R is a local ring,
(2) M is a finite S-module, and
(3) M is flat over R.

Then M is finitely presented as an S-module.

Proof. Let $M = \bigoplus M_d$ be the grading on M. Pick homogeneous generators $m_1, \ldots, m_r \in M$ of M. Say $\deg(m_i) = d_i \in \mathbf{Z}$. This gives us a presentation

$$0 \to K \to \bigoplus\nolimits_{i=1,\ldots,r} S(-d_i) \to M \to 0$$

which in each degree d leads to the short exact sequence

$$0 \to K_d \to \bigoplus\nolimits_{i=1,\ldots,r} S_{d-d_i} \to M_d \to 0.$$

By assumption each M_d is a finite flat R-module. By Algebra, Lemma 10.78.5 this implies each M_d is a finite free R-module. Hence we see each K_d is a finite R-module. Also each K_d is flat over R by Algebra, Lemma 10.39.13. Hence we conclude that each K_d is finite free by Algebra, Lemma 10.78.5 again.

Let $\mathfrak m$ be the maximal ideal of R. By the flatness of M over R the short exact sequences above remain short exact after tensoring with $\kappa = \kappa(\mathfrak m)$. As the ring $S \otimes_R \kappa$ is Noetherian we see that there exist homogeneous elements $k_1, \ldots, k_t \in K$ such that the images $\overline{k}_j$ generate $K \otimes_R \kappa$ over $S \otimes_R \kappa$. Say $\deg(k_j) = e_j$. Thus for any d the map

$$\bigoplus\nolimits_{j=1,\ldots,t} S_{d-e_j} \longrightarrow K_d$$

becomes surjective after tensoring with κ. By Nakayama's lemma (Algebra, Lemma 10.20.1) this implies the map is surjective over R. Hence K is generated by $k_1, \ldots, k_t$ over S and we win. $\square$

053D Lemma 15.25.4. Let R be a ring. Let $S = \bigoplus_{n \geq 0} S_n$ be a graded R-algebra. Let $M = \bigoplus_{d \in \mathbf{Z}} M_d$ be a graded S-module. Assume S is finitely generated as an R-algebra, assume S_0 is a finite R-algebra, and assume there exist finitely many primes $\mathfrak p_j$, $i = 1, \ldots, m$ such that $R \to \prod R_{\mathfrak p_j}$ is injective.

(1) If S is flat over R, then S is a finitely presented R-algebra.
(2) If M is flat as an R-module and finite as an S-module, then M is finitely presented as an S-module.

Proof. As S is finitely generated as an R-algebra, it is finitely generated as an S_0 algebra, say by homogeneous elements $t_1, \ldots, t_n \in S$ of degrees $d_1, \ldots, d_n > 0$. Set $P = R[x_1, \ldots, x_n]$ with $\deg(x_i) = d_i$. The ring map $P \to S$, $x_i \to t_i$ is finite as S_0 is a finite R-module. To prove (1) it suffices to prove that S is a finitely presented P-module. To prove (2) it suffices to prove that M is a finitely presented P-module. Thus it suffices to prove that if $S = P$ is a graded polynomial ring and M is a finite S-module flat over R, then M is finitely presented as an S-module. By Lemma 15.25.3 we see $M_{\mathfrak p}$ is a finitely presented $S_{\mathfrak p}$-module for every prime $\mathfrak p$ of R. Thus the result follows from Lemma 15.25.1. $\square$

05GS Remark 15.25.5. Let R be a ring. When does R satisfy the condition mentioned in Lemmas 15.25.1, 15.25.2, and 15.25.4? This holds if

(1) R is local,
(2) R is Noetherian,
(3) R is a domain,
(4) R is a reduced ring with finitely many minimal primes, or
(5) R has finitely many weakly associated primes, see Algebra, Lemma 10.66.17.

Thus these lemmas hold in all cases listed above.

The following lemma will be improved on in More on Flatness, Proposition 38.13.10.

053E Lemma 15.25.6. Let A be a valuation ring. Let $A \to B$ be a ring map of finite type. Let M be a finite B-module. [Nag66, Theorem 3]

(1) If B is flat over A, then B is a finitely presented A-algebra.
(2) If M is flat as an A-module, then M is finitely presented as a B-module.

Proof. We are going to use that an A-module is flat if and only if it is torsion free, see Lemma 15.22.10. By Algebra, Lemma 10.57.10 we can find a graded A-algebra S with $S_0 = A$ and generated by finitely many elements in degree 1, an element $f \in S_1$ and a finite graded S-module N such that $B \cong S_{(f)}$ and $M \cong N_{(f)}$. If M is torsion free, then we can take N torsion free by replacing it by N/N_{tors}, see Lemma 15.22.2. Similarly, if B is torsion free, then we can take S torsion free by replacing it by S/S_{tors}. Hence in case (1), we may apply Lemma 15.25.4 to see that S is a finitely presented A-algebra, which implies that $B = S_{(f)}$ is a finitely presented A-algebra. To see (2) we may first replace S by a graded polynomial ring, and then we may apply Lemma 15.25.3 to conclude. $\square$

0GSE Lemma 15.25.7. Let A be a valuation ring. Let $A \to B$ be a local homomorphism which is essentially of finite type. Let M be a finite B-module.

(1) If B is flat over A, then B is essentially of finite presentation over A.
(2) If M is flat as an A-module, then M is finitely presented as a B-module.

Proof. By assumption we can write B as a quotient of the localization of a polynomial algebra $P = A[x_1, \ldots, x_n]$ at a prime ideal $\mathfrak q$. In case (1) we consider $M = B$ as a finite module over $P_\mathfrak q$ and in case (2) we consider M as a finite module over $P_\mathfrak q$. In both cases, we have to show that this is a finitely presented $P_\mathfrak q$-module, see Algebra, Lemma 10.6.4 for case (2).

Choose a presentation $0 \to K \to P_\mathfrak q^{\oplus r} \to M \to 0$ which is possible because M is finite over $P_\mathfrak q$. Let $L = P^{\oplus r} \cap K$. Then $K = L_\mathfrak q$, see Algebra, Lemma 10.9.15. Then $N = P^{\oplus r}/L$ is a submodule of M and hence flat by Lemma 15.22.10. Since also N is a finite P-module, we see that N is finitely presented as a P-module by Lemma 15.25.6. Since localization is exact (Algebra, Proposition 10.9.12) we see that $N_\mathfrak q = M$ and we conclude. $\square$

15.26. Blowing up and flatness

0535 In this section we begin our discussion of results of the form: "After a blowup the strict transform becomes flat". More results of this type may be found in Divisors, Section 31.35 and More on Flatness, Section 38.30.

053H Definition 15.26.1. Let R be a ring. Let $I \subset R$ be an ideal and $a \in I$. Let $R[\frac{I}{a}]$ be the affine blowup algebra, see Algebra, Definition 10.70.1. Let M be an R-module. The strict transform of M along $R \to R[\frac{I}{a}]$ is the $R[\frac{I}{a}]$-module

$$M' = \left(M \otimes_R R[\tfrac{I}{a}]\right)/a\text{-power torsion}$$

The following is a very weak version of flattening by blowing up, but it is already sometimes a useful result.

053I Lemma 15.26.2. Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let S be a finite type R-algebra. Let M be a finite S-module. For every valuation ring $A \subset K$ dominating R there exists an ideal $I \subset \mathfrak{m}$ and a nonzero element $a \in I$ such that

(1) I is finitely generated,
(2) A has center on $R[\frac{I}{a}]$,
(3) the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero, and
(4) the strict transform $S_{I,a}$ of S along $R \to R[\frac{I}{a}]$ is flat and of finite presentation over R, and the strict transform $M_{I,a}$ of M along $R \to R[\frac{I}{a}]$ is flat over R and finitely presented over $S_{I,a}$.

Proof. Write $S = R[x_1, \ldots, x_n]/J$ and denote $N = S \oplus M$ viewed as a module over $P = R[x_1, \ldots, x_n]$. If we can prove the lemma in case S is a polynomial algebra over R, then we can find I, a satisfying (1), (2), (3) such that the strict transform $N_{I,a}$ of N along $R \to R[\frac{I}{a}]$ is flat over R and finitely presented as a module over the strict transform $P_{I,a}]$ of P. Since $P_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$ (small detail omitted) we find that the summand $S_{I,a} \subset N_{I,a}$ is flat over R and finitely presented as a module over $R[\frac{I}{a}][x_1, \ldots, x_n]$. Hence $S_{I,a}$ is finitely presented as an $R[\frac{I}{a}]$-algebra. Moreover, the summand $M_{I,a} \subset N_{I,a}$ is flat over R and finitely presented as a module over $P_{I,a}$ hence also finitely presented as a module over $S_{I,a}$, see Algebra, Lemma 10.6.4. This reduces us to the case discussed in the next paragraph.

Assume $S = R[x_1, \ldots, x_n]$. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0.$$

Let M_A be the quotient of $M \otimes_R A$ by its torsion submodule, see Lemma 15.22.2. Then M_A is a finite module over $S_A = A[x_1, \ldots, x_n]$. By Lemma 15.22.10 we see that M_A is flat over A. By Lemma 15.25.6 we see that M_A is finitely presented. Hence there exist finitely many elements $k_1, \ldots, k_t \in S_A^{\oplus r}$ which generate the kernel of the presentation $S_A^{\oplus r} \to M_A$ as an S_A-module. For any choice of $a \in I \subset \mathfrak{m}$ satisfying (1), (2), and (3) we denote $M_{I,a}$ the strict transform of M along $R \to R[\frac{I}{a}]$. It is a finite module over $S_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$. By Algebra, Lemma 10.70.12 we have $A = \text{colim}_{I,a} R[\frac{I}{a}]$. This implies that $S_A = \text{colim}\, S_{I,a}$ and

$$\text{colim}\, M \otimes_R R[\tfrac{I}{a}] = M \otimes_R A$$

Choose I, a and lifts $k_1, \ldots, k_t \in S_{I,a}^{\oplus r}$. Since M_A is the quotient of $M \otimes_R A$ by torsion, we see that the images of $k_1, \ldots, k_t$ in $M \otimes_R A$ are annihilated by a nonzero element $\alpha \in A$. After replacing I, a by a different pair (recall that the colimit is filtered), we may assume $\alpha = x/a^n$ for some $x \in I^n$ nonzero. Then we find that $xk_1, \ldots, xk_t$ map to zero in $M \otimes_R A$. Hence after replacing I, a by a different pair we may assume $xk_1, \ldots, xk_t$ map to zero in $M \otimes_R R[\frac{I}{a}]$ for some nonzero $x \in R$. Then finally replacing I, a by xI, xa we find that we may assume $k_1, \ldots, k_t$ map to a-power torsion elements of $M \otimes_R R[\frac{I}{a}]$. For any such pair (I, a) we set

$$M'_{I,a} = S_{I,a}^{\oplus r} / \sum S_{I,a} k_j.$$

Since $M_A = S_A^{\oplus r} / \sum S_A k_j$ we see that $M_A = \text{colim}_{I,a} M'_{I,a}$. At this point we finally apply Algebra, Lemma 10.168.1 (3) to conclude that $M'_{I,a}$ is flat for some pair (I, a)

as above. This lemma does not apply a priori to the system of strict transforms

$$M_{I,a} = (M \otimes_R R[\tfrac{I}{a}])/a\text{-power torsion}$$

as the transition maps may not satisfy the assumptions of the lemma. But now, since flatness implies torsion free (Lemma 15.22.9) and since $M_{I,a}$ is the quotient of $M'_{I,a}$ (because we arranged it so the elements $k_1, \ldots, k_t$ map to zero in $M_{I,a}$) by the a-power torsion submodule we also conclude that $M'_{I,a} = M_{I,a}$ for such a pair and we win. ☐

0CZM Lemma 15.26.3. Let R be a ring. Let M be a finite R-module. Let $k \geq 0$ and $I = \text{Fit}_k(M)$. For every $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

has $\text{Fit}_k(M') = R'$.

Proof. First observe that $\text{Fit}_k(M \otimes_R R') = IR' = aR'$. The first equality by Lemma 15.8.4 part (3) and the second equality by Algebra, Lemma 10.70.2. From Lemma 15.8.8 and exactness of localization we see that $M'_{\mathfrak{p}'}$ can be generated by $\leq k$ elements for every prime $\mathfrak{p}'$ of R'. Then $\text{Fit}_k(M') = R'$ for example by Lemma 15.8.6. ☐

0CZN Lemma 15.26.4. Let R be a ring. Let M be a finite R-module. Let $k \geq 0$ and $I = \text{Fit}_k(M)$. Asssume that $M_{\mathfrak{p}}$ is free of rank k for every $\mathfrak{p} \notin V(I)$. Then for every $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

is locally free of rank k.

Proof. By Lemma 15.26.3 we have $\text{Fit}_k(M') = R'$. By Lemma 15.8.7 it suffices to show that $\text{Fit}_{k-1}(M') = 0$. Recall that $R' \subset R'_a = R_a$, see Algebra, Lemma 10.70.2. Hence it suffices to prove that $\text{Fit}_{k-1}(M')$ maps to zero in $R'_a = R_a$. Since clearly $(M')_a = M_a$ this reduces us to showing that $\text{Fit}_{k-1}(M_a) = 0$ because formation of Fitting ideals commutes with base change according to Lemma 15.8.4 part (3). This is true by our assumption that M_a is finite locally free of rank k (see Algebra, Lemma 10.78.2) and the already cited Lemma 15.8.7. ☐

0BBJ Lemma 15.26.5. Let R be a ring. Let M be a finite R-module. Let $f \in R$ be an element such that M_f is finite locally free of rank r. Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the strict transform

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

is locally free of rank r.

Proof. Choose a surjection $R^{\oplus n} \to M$. Choose a finite submodule $K \subset \text{Ker}(R^{\oplus n} \to M)$ such that $R^{\oplus n}/K \to M$ becomes an isomorphism after inverting f. This is possible because M_f is of finite presentation for example by Algebra, Lemma 10.78.2. Set $M_1 = R^{\oplus n}/K$ and suppose we can prove the lemma for M_1. Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M'_1 = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 10.70.7. But a is a nonzerodivisor on M'_1, whence the displayed map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented R-module.

Assume M is a finitely presented R-module. Then $J = \text{Fit}_r(M) \subset S$ is a finitely generated ideal. We claim that $I = fJ$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \notin \mathfrak{p}$, then $\mathfrak{p}$ is not an element of $V(J)$ by Lemma 15.8.6. Thus $\mathfrak{p} \notin V(fJ) = V(I)$.

Let $a \in I$ and set $R' = R[\frac{I}{a}]$. We may write $a = fb$ for some $b \in J$. By Algebra, Lemmas 10.70.2 and 10.70.8 we see that $JR' = bR'$ and b is a nonzerodivisor in R'. Let $\mathfrak{p}' \subset R' = R[\frac{I}{a}]$ be a prime ideal. Then $JR'_{\mathfrak{p}'}$ is generated by b. It follows from Lemma 15.8.8 that $M'_{\mathfrak{p}'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \ldots, m_r \in M'$ and $g \in R'$, $g \notin \mathfrak{p}'$ such that the corresponding map $(R')^{\oplus r} \to M'$ becomes surjective after inverting g.

Finally, consider the ideal $J' = \text{Fit}_{k-1}(M')$. Note that $J'R'_g$ is generated by the coefficients of relations between $m_1, \ldots, m_r$ (compatibility of Fitting ideal with base change). Thus it suffices to show that $J' = 0$, see Lemma 15.8.7. Since $R'_a = R_f$ (Algebra, Lemma 10.70.7) and $M'_a = M_f$ is free of rank r we see that $J'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. $\square$

15.27. Completion and flatness

06LD In this section we discuss when the completion of a “big” flat module is flat.

05BC Lemma 15.27.1. Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. There is a canonical map

$$\left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow \prod\nolimits_{\alpha \in A} R$$

from the I-adic completion of the direct sum into the product which is universally injective.

Proof. By definition an element x of the left hand side is $x = (x_n)$ where $x_n = (x_{n,\alpha}) \in \bigoplus_{\alpha \in A} R/I^n$ such that $x_{n,\alpha} = x_{n+1,\alpha} \bmod I^n$. As $R = R^\wedge$ we see that for any α there exists a $y_\alpha \in R$ such that $x_{n,\alpha} = y_\alpha \bmod I^n$. Note that for each n there are only finitely many α such that the elements $x_{n,\alpha}$ are nonzero. Conversely, given $(y_\alpha) \in \prod_\alpha R$ such that for each n there are only finitely many α such that $y_\alpha \bmod I^n$ is nonzero, then this defines an element of the left hand side. Hence we can think of an element of the left hand side as infinite “convergent sums” $\sum_\alpha y_\alpha$ with $y_\alpha \in R$ such that for each n there are only finitely many y_α which are nonzero modulo I^n. The displayed map maps this element to the element to (y_α) in the product. In particular the map is injective.

Let Q be a finite R-module. We have to show that the map

$$Q \otimes_R \left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow Q \otimes_R \left(\prod\nolimits_{\alpha \in A} R\right)$$

is injective, see Algebra, Theorem 10.82.3. Choose a presentation $R^{\oplus k} \to R^{\oplus m} \to Q \to 0$ and denote $q_1, \ldots, q_m \in Q$ the corresponding generators for Q. By Artin-Rees (Algebra, Lemma 10.51.2) there exists a constant c such that $\text{Im}(R^{\oplus k} \to$

$R^{\oplus m}) \cap (I^N)^{\oplus m} \subset \mathrm{Im}((I^{N-c})^{\oplus k} \to R^{\oplus m})$. Let us contemplate the diagram

$$\begin{tikzcd} \bigoplus_{l=1}^k \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & \bigoplus_{j=1}^m \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & Q \otimes_R \left(\bigoplus_{\alpha \in A} R\right)^\wedge \arrow[r] \arrow[d] & 0 \\ \bigoplus_{l=1}^k \left(\prod_{\alpha \in A} R\right) \arrow[r] & \bigoplus_{j=1}^m \left(\prod_{\alpha \in A} R\right) \arrow[r] & Q \otimes_R \left(\prod_{\alpha \in A} R\right) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Pick an element $\sum_j \sum_\alpha y_{j,\alpha}$ of $\bigoplus_{j=1,\ldots,m} \left(\bigoplus_{\alpha \in A} R\right)^\wedge$. If this element maps to zero in the module $Q \otimes_R \left(\prod_{\alpha \in A} R\right)$, then we see in particular that $\sum_j q_j \otimes y_{j,\alpha} = 0$ in Q for each α. Thus we can find an element $(z_{1,\alpha}, \ldots, z_{k,\alpha}) \in \bigoplus_{l=1,\ldots,k} R$ which maps to $(y_{1,\alpha}, \ldots, y_{m,\alpha}) \in \bigoplus_{j=1,\ldots,m} R$. Moreover, if $y_{j,\alpha} \in I^{N_\alpha}$ for $j = 1, \ldots, m$, then we may assume that $z_{l,\alpha} \in I^{N_\alpha - c}$ for $l = 1, \ldots, k$. Hence the sum $\sum_l \sum_\alpha z_{l,\alpha}$ is "convergent" and defines an element of $\bigoplus_{l=1,\ldots,k} \left(\bigoplus_{\alpha \in A} R\right)^\wedge$ which maps to the element $\sum_j \sum_\alpha y_{j,\alpha}$ we started out with. Thus the right vertical arrow is injective and we win. $\square$

The following lemma can also be deduced from Lemma 15.27.4 below.

06LE Lemma 15.27.2. Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian. The completion $(\bigoplus_{\alpha \in A} R)^\wedge$ is a flat R-module.

Proof. Denote $R^\wedge$ the completion of R with respect to I. As $R \to R^\wedge$ is flat by Algebra, Lemma 10.97.2 it suffices to prove that $(\bigoplus_{\alpha \in A} R)^\wedge$ is a flat $R^\wedge$-module (use Algebra, Lemma 10.39.4). Since

$$(\bigoplus\nolimits_{\alpha \in A} R)^\wedge = (\bigoplus\nolimits_{\alpha \in A} R^\wedge)^\wedge$$

we may replace R by $R^\wedge$ and assume that R is complete with respect to I (see Algebra, Lemma 10.97.4). In this case Lemma 15.27.1 tells us the map $(\bigoplus_{\alpha \in A} R)^\wedge \to \prod_{\alpha \in A} R$ is universally injective. Thus, by Algebra, Lemma 10.82.7 it suffices to show that $\prod_{\alpha \in A} R$ is flat. By Algebra, Proposition 10.90.6 (and Algebra, Lemma 10.90.5) we see that $\prod_{\alpha \in A} R$ is flat. $\square$

0911 Lemma 15.27.3. Let A be a Noetherian ring. Let I be an ideal of A. Let M be a finite A-module. For every $p > 0$ there exists a $c > 0$ such that $\mathrm{Tor}_p^A(M, A/I^n) \to \mathrm{Tor}_p^A(M, A/I^{n-c})$ is zero for all $n \geq c$.

This is [Qui, Lemma 9.9]; note that the author forgot the word "strict" in the statement although it was clearly intended.

Proof. Proof for $p = 1$. Choose a short exact sequence $0 \to K \to A^{\oplus t} \to M \to 0$. Then $\mathrm{Tor}_1^A(M, A/I^n) = K \cap (I^n)^{\oplus t}/I^n K$. By Artin-Rees (Algebra, Lemma 10.51.2) there is a constant $c \geq 0$ such that $K \cap (I^n)^{\oplus t} \subset I^{n-c}K$ for $n \geq c$. Thus the result for $p = 1$. For $p > 1$ we have $\mathrm{Tor}_p^A(M, A/I^n) = \mathrm{Tor}_{p-1}^A(K, A/I^n)$. Thus the lemma follows by induction. $\square$

0912 Lemma 15.27.4. Let A be a Noetherian ring. Let I be an ideal of A. Let (M_n) be an inverse system of A-modules such that

(1) M_n is a flat A/I^n-module,
(2) $M_{n+1} \to M_n$ is surjective.

Then $M = \lim M_n$ is a flat A-module and $Q \otimes_A M = \lim Q \otimes_A M_n$ for every finite A-module Q.

Proof. We first show that $Q \otimes_A M = \lim Q \otimes_A M_n$ for every finite A-module Q. Choose a resolution $F_2 \to F_1 \to F_0 \to Q \to 0$ by finite free A-modules F_i. Then

$$F_2 \otimes_A M_n \to F_1 \otimes_A M_n \to F_0 \otimes_A M_n$$

is a chain complex whose homology in degree 0 is $Q \otimes_A M_n$ and whose homology in degree 1 is

$$\text{Tor}_1^A(Q, M_n) = \text{Tor}_1^A(Q, A/I^n) \otimes_{A/I^n} M_n$$

as M_n is flat over A/I^n. By Lemma 15.27.3 we see that this system is essentially constant (with value 0). It follows from Homology, Lemma 12.31.7 that $\lim Q \otimes_A A/I^n = \text{Coker}(\lim F_1 \otimes_A M_n \to \lim F_0 \otimes_A M_n)$. Since F_i is finite free this equals $\text{Coker}(F_1 \otimes_A M \to F_0 \otimes_A M) = Q \otimes_A M$.

Next, let $Q \to Q'$ be an injective map of finite A-modules. We have to show that $Q \otimes_A M \to Q' \otimes_A M$ is injective (Algebra, Lemma 10.39.5). By the above we see

$$\text{Ker}(Q \otimes_A M \to Q' \otimes_A M) = \text{Ker}(\lim Q \otimes_A M_n \to \lim Q' \otimes_A M_n).$$

For each n we have an exact sequence

$$\text{Tor}_1^A(Q', M_n) \to \text{Tor}_1^A(Q'', M_n) \to Q \otimes_A M_n \to Q' \otimes_A M_n$$

where $Q'' = \text{Coker}(Q \to Q')$. Above we have seen that the inverse systems of Tor's are essentially constant with value 0. It follows from Homology, Lemma 12.31.7 that the inverse limit of the right most maps is injective. $\square$

0AGW **Lemma 15.27.5.** Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume

1. I is finitely generated,
2. R/I is Noetherian,
3. M/IM is flat over R/I,
4. $\text{Tor}_1^R(M, R/I) = 0$.

Then the I-adic completion $R^\wedge$ is a Noetherian ring and $M^\wedge$ is flat over $R^\wedge$.

Proof. By Algebra, Lemma 10.99.8 the modules M/I^nM are flat over R/I^n for all n. By Algebra, Lemma 10.96.3 we have (a) $R^\wedge$ and $M^\wedge$ are I-adically complete and (b) $R/I^n = R^\wedge/I^nR^\wedge$ for all n. By Algebra, Lemma 10.97.5 the ring $R^\wedge$ is Noetherian. Applying Lemma 15.27.4 we conclude that $M^\wedge = \lim M/I^nM$ is flat as an $R^\wedge$-module. $\square$

15.28. The Koszul complex

0621 We define the Koszul complex as follows.

0622 **Definition 15.28.1.** Let R be a ring. Let $\varphi : E \to R$ be an R-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the commutative differential graded algebra defined as follows:

1. the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(E)$,
2. the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all $e \in E = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is one of the generators of degree n in $K_\bullet(\varphi)$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \otimes e$ and hence factors through the exterior algebra.

We often assume that E is a finite free module, say $E = R^{\oplus n}$. In this case the map φ is given by a sequence of elements $f_1, \ldots, f_n \in R$.

0623 Definition 15.28.2. Let R be a ring and let $f_1, \ldots, f_r \in R$. The Koszul complex on $f_1, \ldots, f_r$ is the Koszul complex associated to the map $(f_1, \ldots, f_r) : R^{\oplus r} \to R$. Notation $K_\bullet(f_\bullet)$, $K_\bullet(f_1, \ldots, f_r)$, $K_\bullet(R, f_1, \ldots, f_r)$, or $K_\bullet(R, f_\bullet)$.

Of course, if E is finite locally free, then $K_\bullet(\varphi)$ is locally on $\text{Spec}(R)$ isomorphic to a Koszul complex $K_\bullet(f_1, \ldots, f_r)$. This complex has many interesting formal properties.

0624 Lemma 15.28.3. Let $\varphi : E \to R$ and $\varphi' : E' \to R$ be R-module maps. Let $\psi : E \to E'$ be an R-module map such that $\varphi' \circ \psi = \varphi$. Then ψ induces a homomorphism of differential graded algebras $K_\bullet(\varphi) \to K_\bullet(\varphi')$.

Proof. This is immediate from the definitions. $\square$

0625 Lemma 15.28.4. Let $f_1, \ldots, f_r \in R$ be a sequence. Let (x_{ij}) be an invertible $r \times r$-matrix with coefficients in R. Then the complexes $K_\bullet(f_\bullet)$ and

$$K_\bullet(\sum x_{1j} f_j, \sum x_{2j} f_j, \ldots, \sum x_{rj} f_j)$$

are isomorphic.

Proof. Set $g_i = \sum x_{ij} f_j$. The matrix (x_{ji}) gives an isomorphism $x : R^{\oplus r} \to R^{\oplus r}$ such that $(g_1, \ldots, g_r) = (f_1, \ldots, f_r) \circ x$. Hence this follows from the functoriality of the Koszul complex described in Lemma 15.28.3. $\square$

0626 Lemma 15.28.5. Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $e \in E$ with image $f = \varphi(e)$ in R. Then

$$f = de + ed$$

as endomorphisms of $K_\bullet(\varphi)$.

Proof. This is true because $d(ea) = d(e)a - ed(a) = fa - ed(a)$. $\square$

0663 Lemma 15.28.6. Let R be a ring. Let $f_1, \ldots, f_r \in R$ be a sequence. Multiplication by f_i on $K_\bullet(f_\bullet)$ is homotopic to zero, and in particular the cohomology modules $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_r)$.

Proof. Special case of Lemma 15.28.5. $\square$

In Derived Categories, Section 13.9 we defined the cone of a morphism of cochain complexes. The cone $C(f)_\bullet$ of a morphism of chain complexes $f : A_\bullet \to B_\bullet$ is the complex $C(f)_\bullet$ given by $C(f)_n = B_n \oplus A_{n-1}$ and differential

0627 (15.28.6.1)
$$d_{C(f),n} = \begin{pmatrix} d_{B,n} & f_{n-1} \\ 0 & -d_{A,n-1} \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : B_\bullet \to C(f)_\bullet$ and $p : C(f)_\bullet \to A_\bullet[-1]$ induced by the obvious maps $B_n \to C(f)_n \to A_{n-1}$.

0628 Lemma 15.28.7. Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f \in R$. Set $E' = E \oplus R$ and define $\varphi' : E' \to R$ by φ on E and multiplication by f on R. The complex $K_\bullet(\varphi')$ is isomorphic to the cone of the map of complexes

$$f : K_\bullet(\varphi) \longrightarrow K_\bullet(\varphi).$$

Proof. Denote $e_0 \in E'$ the element $1 \in R \subset R \oplus E$. By our definition of the cone above we see that

$$C(f)_n = K_n(\varphi) \oplus K_{n-1}(\varphi) = \wedge^n(E) \oplus \wedge^{n-1}(E) = \wedge^n(E')$$

where in the last $=$ we map $(0, e_1 \wedge \ldots \wedge e_{n-1})$ to $e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}$ in $\wedge^n(E')$. A computation shows that this isomorphism is compatible with differentials. Namely, this is clear for elements of the first summand as $\varphi'|_E = \varphi$ and $d_{C(f)}$ restricted to the first summand is just $d_{K_\bullet(\varphi)}$. On the other hand, if $e_1 \wedge \ldots \wedge e_{n-1}$ is in the second summand, then

$$d_{C(f)}(0, e_1 \wedge \ldots \wedge e_{n-1}) = fe_1 \wedge \ldots \wedge e_{n-1} - d_{K_\bullet(\varphi)}(e_1 \wedge \ldots \wedge e_{n-1})$$

and on the other hand

$$\begin{aligned} & d_{K_\bullet(\varphi')}(0, e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}) \\ & = \sum\nolimits_{i=0,\ldots,n-1} (-1)^i \varphi'(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\ & = fe_1 \wedge \ldots \wedge e_{n-1} + \sum\nolimits_{i=1,\ldots,n-1} (-1)^i \varphi(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\ & = fe_1 \wedge \ldots \wedge e_{n-1} - e_0 \left(\sum\nolimits_{i=1,\ldots,n-1} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1}\right) \end{aligned}$$

which is the image of the result of the previous computation. □

0629 Lemma 15.28.8. Let R be a ring. Let $f_1, \ldots, f_r$ be a sequence of elements of R. The complex $K_\bullet(f_1, \ldots, f_r)$ is isomorphic to the cone of the map of complexes

$$f_r : K_\bullet(f_1, \ldots, f_{r-1}) \longrightarrow K_\bullet(f_1, \ldots, f_{r-1}).$$

Proof. Special case of Lemma 15.28.7. □

062A Lemma 15.28.9. Let R be a ring. Let $A_\bullet$ be a complex of R-modules. Let $f, g \in R$. Let $C(f)_\bullet$ be the cone of $f : A_\bullet \to A_\bullet$. Define similarly $C(g)_\bullet$ and $C(fg)_\bullet$. Then $C(fg)_\bullet$ is homotopy equivalent to the cone of a map

$$C(f)_\bullet[1] \longrightarrow C(g)_\bullet$$

Proof. We first prove this if $A_\bullet$ is the complex consisting of R placed in degree 0. In this case the complex $C(f)_\bullet$ is the complex

$$\ldots \to 0 \to R \xrightarrow{f} R \to 0 \to \ldots$$

with R placed in (homological) degrees 1 and 0. The map of complexes we use is

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R \arrow[r, "f"] \arrow[d, "1"] & R \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & R \arrow[r, "g"] & R \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

The cone of this is the chain complex consisting of $R^{\oplus 2}$ placed in degrees 1 and 0 and differential (15.28.6.1)

$$\begin{pmatrix} g & 1 \\ 0 & -f \end{pmatrix} : R^{\oplus 2} \longrightarrow R^{\oplus 2}$$

To see this chain complex is homotopic to $C(fg)_\bullet$, i.e., to $R \xrightarrow{fg} R$, consider the maps of complexes

$$\begin{tikzcd} R \arrow[r, "fg"'] \arrow[d, "{(1,-g)}"'] & R \arrow[d, "{(0,1)}"] \\ R^{\oplus 2} \arrow[r] & R^{\oplus 2} \end{tikzcd} \qquad \begin{tikzcd} R^{\oplus 2} \arrow[r] \arrow[d, "{(1,0)}"'] & R^{\oplus 2} \arrow[d, "{(f,1)}"] \\ R \arrow[r, "fg"] & R \end{tikzcd}$$

with obvious notation. The composition of these two maps in one direction is the identity on $C(fg)_\bullet$, but in the other direction it isn't the identity. We omit writing out the required homotopy.

To see the result holds in general, we use that we have a functor $K_\bullet \mapsto \text{Tot}(A_\bullet \otimes_R K_\bullet)$ on the category of complexes which is compatible with homotopies and cones. Then we write $C(f)_\bullet$ and $C(g)_\bullet$ as the total complex of the double complexes

$$(R \xrightarrow{f} R) \otimes_R A_\bullet \quad \text{and} \quad (R \xrightarrow{g} R) \otimes_R A_\bullet$$

and in this way we deduce the result from the special case discussed above. Some details omitted. □

062B Lemma 15.28.10. Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f, g \in R$. Set $E' = E \oplus R$ and define $\varphi'_f, \varphi'_g, \varphi'_{fg} : E' \to R$ by φ on E and multiplication by f, g, fg on R. The complex $K_\bullet(\varphi'_{fg})$ is homotopy equivalent to the cone of a map of complexes

$$K_\bullet(\varphi'_f)[1] \longrightarrow K_\bullet(\varphi'_g).$$

Proof. By Lemma 15.28.7 the complex $K_\bullet(\varphi'_f)$ is isomorphic to the cone of multiplication by f on $K_\bullet(\varphi)$ and similarly for the other two cases. Hence the lemma follows from Lemma 15.28.9. □

062C Lemma 15.28.11. Let R be a ring. Let $f_1, \ldots, f_{r-1}$ be a sequence of elements of R. Let $f, g \in R$. The complex $K_\bullet(f_1, \ldots, f_{r-1}, fg)$ is homotopy equivalent to the cone of a map of complexes

$$K_\bullet(f_1, \ldots, f_{r-1}, f)[1] \longrightarrow K_\bullet(f_1, \ldots, f_{r-1}, g)$$

Proof. Special case of Lemma 15.28.10. □

0664 Lemma 15.28.12. Let R be a ring. Let $f_1, \ldots, f_r, g_1, \ldots, g_s$ be elements of R. Then there is an isomorphism of Koszul complexes

$$K_\bullet(R, f_1, \ldots, f_r, g_1, \ldots, g_s) = \text{Tot}(K_\bullet(R, f_1, \ldots, f_r) \otimes_R K_\bullet(R, g_1, \ldots, g_s)).$$

Proof. Omitted. Hint: If $K_\bullet(R, f_1, \ldots, f_r)$ is generated as a differential graded algebra by $x_1, \ldots, x_r$ with $\text{d}(x_i) = f_i$ and $K_\bullet(R, g_1, \ldots, g_s)$ is generated as a differential graded algebra by $y_1, \ldots, y_s$ with $\text{d}(y_j) = g_j$, then we can think of $K_\bullet(R, f_1, \ldots, f_r, g_1, \ldots, g_s)$ as the differential graded algebra generated by the sequence of elements $x_1, \ldots, x_r, y_1, \ldots, y_s$ with $\text{d}(x_i) = f_i$ and $\text{d}(y_j) = g_j$. □

15.29. The extended alternating Čech complex

0G6F Let R be a ring. Let $f_1, \ldots, f_r \in R$. The extended alternating Čech complex of R is the cochain complex

$$R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$$

where R is in degree 0, the term $\bigoplus_{i_0} R_{f_{i_0}}$ is in degre 1, and so on. The maps are defined as follows

(1) The map $R \to \bigoplus_{i_0} R_{f_{i_0}}$ is given by the canonical maps $R \to R_{f_{i_0}}$.
(2) Given $1 \leq i_0 < \ldots < i_{p+1} \leq r$ and $0 \leq j \leq p+1$ we have the canonical localization map

$$R_{f_{i_0} \ldots \hat{f}_{i_j} \ldots f_{i_{p+1}}} \to R_{f_{i_0} \ldots f_{i_{p+1}}}$$

(3) The differentials use the canonical maps of (2) with sign $(-1)^j$.

If M is any R-module, the extended alternating Čech complex of M is the similarly constructed cochain complex

$$M \to \bigoplus\nolimits_{i_0} M_{f_{i_0}} \to \bigoplus\nolimits_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \to \ldots \to M_{f_1 \ldots f_r}$$

where M is in degree 0 as before.

0G6G **Lemma 15.29.1.** The extended alternating Čech complexes defined above are complexes of R-modules.

Proof. Omitted. □

0G6H **Lemma 15.29.2.** Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. The extended alternating Čech complex of M is the tensor product over R of M with the extended alternating Čech complex of R.

Proof. Omitted. □

0G6I **Lemma 15.29.3.** Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. Let $R \to S$ be a ring map, denote $g_1, \ldots, g_r \in S$ the images of $f_1, \ldots, f_r$, and set $N = M \otimes_R S$. The extended alternating Čech complex constructed using S, $g_1, \ldots, g_r$, and N is the tensor product of the extended alternating Čech complex of M with S over R.

Proof. Omitted. □

0G6J **Lemma 15.29.4.** Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. If there exists an $i \in \{1, \ldots, r\}$ such that f_i is a unit, then the extended alternating Čech complex of M is homotopy equivalent to 0.

Proof. We will use the following notation: a cochain x of degree $p+1$ in the extended alternating Čech complex of M is $x = (x_{i_0 \ldots i_p})$ where $x_{i_0 \ldots i_p}$ is in $M_{f_{i_0} \ldots f_{i_p}}$. With this notation we have

$$d(x)_{i_0 \ldots i_{p+1}} = \sum\nolimits_j (-1)^j x_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$$

As homotopy we use the maps

$$h : \text{cochains of degree } p+2 \to \text{cochains of degree } p+1$$

given by the rule

$$h(x)_{i_0\ldots i_p} = 0 \text{ if } i \in \{i_0, \ldots, i_p\} \text{ and } h(x)_{i_0\ldots i_p} = (-1)^j x_{i_0\ldots i_j i i_{j+1}\ldots i_p} \text{ if not}$$

Here j is the unique index such that $i_j < i < i_{j+1}$ in the second case; also, since f_i is a unit we have the equality

$$M_{f_{i_0}\ldots f_{i_p}} = M_{f_{i_0}\ldots f_{i_j} f_i f_{i_{j+1}}\ldots f_{i_p}}$$

which we can use to make sense of thinking of $(-1)^j x_{i_0\ldots i_j i i_{j+1}\ldots i_p}$ as an element of $M_{f_{i_0}\ldots f_{i_p}}$. We will show by a computation that $dh + hd$ equals the negative of the identity map which finishes the proof. To do this fix x a cochain of degree $p+1$ and let $1 \leq i_0 < \ldots < i_p \leq r$.

Case I: $i \in \{i_0, \ldots, i_p\}$. Say $i = i_t$. Then we have $h(d(x))_{i_0\ldots i_p} = 0$. On the other hand we have

$$d(h(x))_{i_0\ldots i_p} = \sum (-1)^j h(x)_{i_0\ldots \hat{i}_j \ldots i_p} = (-1)^t h(x)_{i_0\ldots \hat{i} \ldots i_p} = (-1)^t(-1)^{t-1} x_{i_0\ldots i_p}$$

Thus $(dh + hd)(x)_{i_0\ldots i_p} = -x_{i_0\ldots i_p}$ as desired.

Case II: $i \notin \{i_0, \ldots, i_p\}$. Let j be such that $i_j < i < i_{j+1}$. Then we see that

$$\begin{aligned} h(d(x))_{i_0\ldots i_p} &= (-1)^j d(x)_{i_0\ldots i_j i i_{j+1}\ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j+j'} x_{i_0\ldots \hat{i}_{j'} \ldots i_j i i_{j+1}\ldots i_p} - x_{i_0\ldots i_p} \\ &+ \sum\nolimits_{j' > j} (-1)^{j+j'+1} x_{i_0\ldots i_j i i_{j+1}\ldots \hat{i}_{j'} \ldots i_p} \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(h(x))_{i_0\ldots i_p} &= \sum\nolimits_{j'} (-1)^{j'} h(x)_{i_0\ldots \hat{i}_{j'} \ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j'+j-1} x_{i_0\ldots \hat{i}_{j'} \ldots i_j i i_{j+1}\ldots i_p} \\ &+ \sum\nolimits_{j' > j} (-1)^{j'+j} x_{i_0\ldots i_j i i_{j+1}\ldots \hat{i}_{j'} \ldots i_p} \end{aligned}$$

Adding these up we obtain $(dh + hd)(x)_{i_0\ldots i_p} = -x_{i_0\ldots i_p}$ as desired. □

0G6K Lemma 15.29.5. Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let M be an R-module. Let H^q be the qth cohomology module of the extended alternation Čech complex of M. Then

1. $H^q = 0$ if $q \notin [0, r]$,
2. for $x \in H^i$ there exists an $n \geq 1$ such that $f_i^n x = 0$ for $i = 1, \ldots, r$,
3. the support of H^q is contained in $V(f_1, \ldots, f_r)$,
4. if there is an $f \in (f_1, \ldots, f_r)$ which acts invertibly on M, then $H^q = 0$.

Proof. Part (1) follows from the fact that the extended alternating Čech complex is zero in degrees < 0 and $> r$. To prove (2) it suffices to show that for each i there exists an $n \geq 1$ such that $f_i^n x = 0$. To see this it suffices to show that $(H^q)_{f_i} = 0$. Since localization is exact, $(H^q)_{f_i}$ is the qth cohomology module of the localization of the extended alternating complex of M at f_i. By Lemma 15.29.3 this localization is the extended alternating Čech complex of M_{f_i} over R_{f_i} with respect to the images of $f_1, \ldots, f_r$ in R_{f_i}. Thus we reduce to showing that H^q is zero if f_i is invertible, which follows from Lemma 15.29.4. Part (3) follows from the observation that $(H^q)_{f_i} = 0$ for all i that we just proved. To see part (4) note that in this case f acts invertibly on H^q and H^q is supported on $V(f)$ by (3). This forces H^q to be zero (small detail omitted). $\square$

0913 Lemma 15.29.6. Let R be a ring. Let $f_1, \ldots, f_r \in R$. The extended alternating Čech complex

$$R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0<i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1\ldots f_r}$$

is a colimit of the Koszul complexes $K(R, f_1^n, \ldots, f_r^n)$; see proof for a precise statement.

Proof. We urge the reader to prove this for themselves. Denote $K(R, f_1^n, \ldots, f_r^n)$ the Koszul complex of Definition 15.28.2 viewed as a cochain complex sitting in degrees $0, \ldots, r$. Thus we have

$$K(R, f_1^n, \ldots, f_r^n) : 0 \to \wedge^r(R^{\oplus r}) \to \wedge^{r-1}(R^{\oplus r}) \to \ldots \to R^{\oplus r} \to R \to 0$$

with the term $\wedge^r(R^{\oplus r})$ sitting in degree 0. Let $e_1^n, \ldots, e_r^n$ be the standard basis of $R^{\oplus r}$. Then the elements $e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n$ for $1 \leq j_1 < \ldots < j_{r-p} \leq r$ form a basis for the term in degree p of the Koszul complex. Further, observe that

$$d(e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n) = \sum (-1)^{a+1} f_{j_a}^n e_{j_1}^n \wedge \ldots \wedge \hat{e}_{j_a}^n \wedge \ldots \wedge e_{j_{r-p}}^n$$

by our construction of the Koszul complex in Section 15.28. The transition maps of our system

$$K(R, f_1^n, \ldots, f_r^n) \to K(R, f_1^{n+1}, \ldots, f_r^{n+1})$$

are given by the rule

$$e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n \longmapsto f_{i_0} \ldots f_{i_{p-1}} e_{j_1}^{n+1} \wedge \ldots \wedge e_{j_{r-p}}^{n+1}$$

where the indices $1 \leq i_0 < \ldots < i_{p-1} \leq r$ are such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. We omit the short computation that shows this is compatible with differentials. Observe that the transition maps are always 1 in degree 0 and equal to $f_1 \ldots f_r$ in degree r.

Denote $K^p(R, f_1^n, \ldots, f_r^n)$ the term of degree p in the Koszul complex. Observe that for any $f \in R$ we have

$$R_f = \text{colim}(R \xrightarrow{f} R \xrightarrow{f} R \to \ldots)$$

Hence we see that in degree p we obtain

$$\text{colim}\, K^p(R, f_1^n, \ldots f_r^n) = \bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0}\ldots f_{i_{p-1}}}$$

Here the element $e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n$ of the Koszul complex above maps in the colimit to the element $(f_{i_0} \ldots f_{i_{p-1}})^{-n}$ in the summand $R_{f_{i_0}\ldots f_{i_{p-1}}}$ where the indices are chosen such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. Thus the differential on this complex is given by

$$d(1 \text{ in } R_{f_{i_0}\ldots f_{i_{p-1}}}) = \sum\nolimits_{i \not\in \{i_0, \ldots, i_{p-1}\}} (-1)^{i-t} \text{ in } R_{f_{i_0}\ldots f_{i_t} f_i f_{i_{t+1}} \ldots f_{i_{p-1}}}$$

Thus if we consider the map of complexes given in degree p by the map

$$\bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0}\ldots f_{i_{p-1}}} \longrightarrow \bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0}\ldots f_{i_{p-1}}}$$

determined by the rule

$$1 \text{ in } R_{f_{i_0}\ldots f_{i_{p-1}}} \longmapsto (-1)^{i_0 + \ldots + i_{p-1} + p} \text{ in } R_{f_{i_0}\ldots f_{i_{p-1}}}$$

then we get an isomorphism of complexes from $\mathrm{colim}\, K(R, f_1^n, \ldots, f_r^n)$ to the extended alternating Čech complex defined in this section. We omit the verification that the signs work out. $\square$

15.30. Koszul regular sequences

062D Please take a look at Algebra, Sections 10.68, 10.69, and 10.72 before looking at this one.

062E Definition 15.30.1. Let R be a ring. Let $r \geq 0$ and let $f_1, \ldots, f_r \in R$ be a sequence of elements. Let M be an R-module. The sequence $f_1, \ldots, f_r$ is called

(1) M-Koszul-regular if $H_i(K_\bullet(f_1, \ldots, f_r) \otimes_R M) = 0$ for all $i \neq 0$,
(2) M-H_1-regular if $H_1(K_\bullet(f_1, \ldots, f_r) \otimes_R M) = 0$,
(3) Koszul-regular if $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for all $i \neq 0$, and
(4) H_1-regular if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$.

We will see in Lemmas 15.30.2, 15.30.3, and 15.30.6 that for elements $f_1, \ldots, f_r$ of a ring R we have the following implications

$$\begin{aligned} f_1, \ldots, f_r \text{ is a regular sequence} &\Rightarrow f_1, \ldots, f_r \text{ is a Koszul-regular sequence} \\ &\Rightarrow f_1, \ldots, f_r \text{ is an } H_1\text{-regular sequence} \\ &\Rightarrow f_1, \ldots, f_r \text{ is a quasi-regular sequence.} \end{aligned}$$

In general none of these implications can be reversed, but if R is a Noetherian local ring and $f_1, \ldots, f_r \in \mathfrak{m}_R$, then the four conditions are all equivalent (Lemma 15.30.7). If $f = f_1 \in R$ is a length 1 sequence and f is not a unit of R then it is clear that the following are all equivalent

(1) f is a regular sequence of length one,
(2) f is a Koszul-regular sequence of length one, and
(3) f is a H_1-regular sequence of length one.

It is also clear that these imply that f is a quasi-regular sequence of length one. But there do exist quasi-regular sequences of length 1 which are not regular sequences. Namely, let

$$R = k[x, y_0, y_1, \ldots]/(xy_0, xy_1 - y_0, xy_2 - y_1, \ldots)$$

and let f be the image of x in R. Then f is a zerodivisor, but $\bigoplus_{n \geq 0}(f^n)/(f^{n+1}) \cong k[x]$ is a polynomial ring.

062F Lemma 15.30.2. An M-regular sequence is M-Koszul-regular. A regular sequence is Koszul-regular.

Proof. Let R be a ring and let M be an R-module. It is immediate that an M-regular sequence of length 1 is M-Koszul-regular. Let $f_1, \ldots, f_r$ be an M-regular sequence. Then f_1 is a nonzerodivisor on M. Hence

$$0 \to K_\bullet(f_2, \ldots, f_r) \otimes M \xrightarrow{f_1} K_\bullet(f_2, \ldots, f_r) \otimes M \to K_\bullet(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M \to 0$$

is a short exact sequence of complexes where $\overline{f}_i$ is the image of f_i in $R/(f_1)$. By Lemma 15.28.8 the complex $K_\bullet(R, f_1, \ldots, f_r)$ is isomorphic to the cone of multiplication by f_1 on $K_\bullet(f_2, \ldots, f_r)$. Thus $K_\bullet(R, f_1, \ldots, f_r) \otimes M$ is isomorphic to the cone on the first map. Hence $K_\bullet(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M$ is quasi-isomorphic to $K_\bullet(f_1, \ldots, f_r) \otimes M$. As $\overline{f}_2, \ldots, \overline{f}_r$ is an M/f_1M-regular sequence in $R/(f_1)$ the result follows from the case $r = 1$ and induction. $\square$

0CEM Lemma 15.30.3. A M-Koszul-regular sequence is M-H_1-regular. A Koszul-regular sequence is H_1-regular.

Proof. This is immediate from the definition. $\square$

062G Lemma 15.30.4. Let $f_1, \ldots, f_{r-1} \in R$ be a sequence and $f, g \in R$. Let M be an R-module.

(1) If $f_1, \ldots, f_{r-1}, f$ and $f_1, \ldots, f_{r-1}, g$ are M-H_1-regular then $f_1, \ldots, f_{r-1}, fg$ is M-H_1-regular too.
(2) If $f_1, \ldots, f_{r-1}, f$ and $f_1, \ldots, f_{r-1}, g$ are M-Koszul-regular then $f_1, \ldots, f_{r-1}, fg$ is M-Koszul-regular too.

Proof. By Lemma 15.28.11 we have exact sequences

$$H_i(K_\bullet(f_1, \ldots, f_{r-1}, f)\otimes M) \to H_i(K_\bullet(f_1, \ldots, f_{r-1}, fg)\otimes M) \to H_i(K_\bullet(f_1, \ldots, f_{r-1}, g)\otimes M)$$

for all i. $\square$

062H Lemma 15.30.5. Let $\varphi : R \to S$ be a flat ring map. Let $f_1, \ldots, f_r \in R$. Let M be an R-module and set $N = M \otimes_R S$.

(1) If $f_1, \ldots, f_r$ in R is an M-H_1-regular sequence, then $\varphi(f_1), \ldots, \varphi(f_r)$ is an N-H_1-regular sequence in S.
(2) If $f_1, \ldots, f_r$ is an M-Koszul-regular sequence in R, then $\varphi(f_1), \ldots, \varphi(f_r)$ is an N-Koszul-regular sequence in S.

Proof. This is true because $K_\bullet(f_1, \ldots, f_r) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r))$ and therefore $(K_\bullet(f_1, \ldots, f_r) \otimes_R M) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r)) \otimes_S N$. $\square$

062I Lemma 15.30.6. An M-H_1-regular sequence is M-quasi-regular.

Proof. Let R be a ring and let M be an R-module. Let $f_1, \ldots, f_r$ be an M-H_1-regular sequence. Denote $J = (f_1, \ldots, f_r)$. The assumption means that we have an exact sequence

$$\wedge^2(R^r) \otimes M \to R^{\oplus r} \otimes M \to JM \to 0$$

where the first arrow is given by $e_i \wedge e_j \otimes m \mapsto (f_i e_j - f_j e_i) \otimes m$. Tensoring the sequence with R/J we see that

$$JM/J^2M = (R/J)^{\oplus r} \otimes_R M = (M/JM)^{\oplus r}$$

is a finite free module. To finish the proof we have to prove for every $n \geq 2$ the following: if

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_r)} m_I f_1^{i_1} \ldots f_r^{i_r} \in J^{n+1}M$$

then $m_I \in JM$ for all I. In the next paragraph, we prove $m_I \in JM$ for $I = (0, \ldots, 0, n)$ and in the last paragraph we deduce the general case from this special case.

Let $I = (0, \ldots, 0, n)$. Let ξ be as above. We can write $\xi = m_1 f_1 + \ldots + m_{r-1} f_{r-1} + m_I f_r^n$. As we have assumed $\xi \in J^{n+1}M$, we can also write $\xi = \sum_{1 \leq i \leq j \leq r-1} m_{ij} f_i f_j + \sum_{1 \leq i \leq r-1} m'_i f_i f_r^n + m'' f_r^{n+1}$. Then we see that

$$\begin{matrix} (m_1 - m_{11} f_1 - m'_1 f_r^n) f_1 + \\ (m_2 - m_{12} f_1 - m_{22} f_2 - m'_2 f_r^n) f_2 + \\ \ldots + \\ (m_{r-1} - m_{1r-1} f_1 - \ldots - m_{r-1r-1} f_{r-1} - m'_{r-1} f_r^n) f_{r-1} + \\ (m_I - m'' f_r) f_r^n = 0 \end{matrix}$$

Since $f_1, \ldots, f_{r-1}, f_r^n$ is M-H_1-regular by Lemma 15.30.4 we see that $m_I - m'' f_r$ is in the submodule $f_1M + \ldots + f_{r-1}M + f_r^nM$. Thus $m_I \in f_1M + \ldots + f_rM$.

Let $S = R[x_1, x_2, \ldots, x_r, 1/x_r]$. The ring map $R \to S$ is faithfully flat, hence $f_1, \ldots, f_r$ is an M-H_1-regular sequence in S, see Lemma 15.30.5. By Lemma 15.28.4 we see that

$$g_1 = f_1 - \frac{x_1}{x_r} f_r, \ \ldots, \ g_{r-1} = f_{r-1} - \frac{x_{r-1}}{x_r} f_r, \ g_r = \frac{1}{x_r} f_r$$

is an M-H_1-regular sequence in S. Finally, note that our element ξ can be rewritten

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_r)} m_I (g_1 + x_i g_r)^{i_1} \ldots (g_{r-1} + x_i g_r)^{i_{r-1}} (x_r g_r)^{i_r}$$

and the coefficient of g_r^n in this expression is

$$\sum m_I x_1^{i_1} \ldots x_r^{i_r}$$

By the case discussed in the previous paragraph this sum is in $J(M \otimes_R S)$. Since the monomials $x_1^{i_1} \ldots x_r^{i_r}$ form part of an R-basis of S over R we conclude that $m_I \in J$ for all I as desired. □

For nonzero finite modules over Noetherian local rings all of the types of regular sequences introduced so far are equivalent.

09CC Lemma 15.30.7. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_r \in \mathfrak{m}$. The following are equivalent

(1) $f_1, \ldots, f_r$ is an M-regular sequence,
(2) $f_1, \ldots, f_r$ is a M-Koszul-regular sequence,
(3) $f_1, \ldots, f_r$ is an M-H_1-regular sequence,
(4) $f_1, \ldots, f_r$ is an M-quasi-regular sequence.

In particular the sequence $f_1, \ldots, f_r$ is a regular sequence in R if and only if it is a Koszul regular sequence, if and only if it is a H_1-regular sequence, if and only if it is a quasi-regular sequence.

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 15.30.2. The implication (2) $\Rightarrow$ (3) is Lemma 15.30.3. The implication (3) $\Rightarrow$ (4) is Lemma 15.30.6. The implication (4) $\Rightarrow$ (1) is Algebra, Lemma 10.69.6. □

0665 Lemma 15.30.8. Let A be a ring. Let $I \subset A$ be an ideal. Let $g_1, \ldots, g_m$ be a sequence in A whose image in A/I is H_1-regular. Then $I \cap (g_1, \ldots, g_m) = I(g_1, \ldots, g_m)$.

Proof. Consider the exact sequence of complexes

$$0 \to I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A/I, g_1, \ldots, g_m) \to 0$$

Since the complex on the right has $H_1 = 0$ by assumption we see that

$$\text{Coker}(I^{\oplus m} \to I) \longrightarrow \text{Coker}(A^{\oplus m} \to A)$$

is injective. This is equivalent to the assertion of the lemma. □

0666 Lemma 15.30.9. Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Then $I \cap J^2 = IJ$.

Proof. To prove this choose $g_1, \ldots, g_m \in J$ whose images in A/I form a H_1-regular sequence which generates J/I. In particular $J = I + (g_1, \ldots, g_m)$. Suppose that $x \in I \cap J^2$. Because $x \in J^2$ can write

$$x = \sum a_{ij} g_i g_j + \sum a_j g_j + a$$

with $a_{ij} \in A$, $a_j \in I$ and $a \in I^2$. Then $\sum a_{ij} g_i g_j \in I \cap (g_1, \ldots, g_m)$ hence by Lemma 15.30.8 we see that $\sum a_{ij} g_i g_j \in I(g_1, \ldots, g_m)$. Thus $x \in IJ$ as desired. $\square$

0667 Lemma 15.30.10. Let A be a ring. Let I be an ideal generated by a quasi-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A.

Proof. We claim that $g_1, \ldots, g_m$ forms an H_1-regular sequence in A/I^d for every d. By induction assume that this holds in A/I^{d-1}. We have a short exact sequence of complexes

$$0 \to K_\bullet(A, g_\bullet) \otimes_A I^{d-1}/I^d \to K_\bullet(A/I^d, g_\bullet) \to K_\bullet(A/I^{d-1}, g_\bullet) \to 0$$

Since $f_1, \ldots, f_n$ is quasi-regular we see that the first complex is a direct sum of copies of $K_\bullet(A/I, g_1, \ldots, g_m)$ hence acyclic in degree 1. By induction hypothesis the last complex is acyclic in degree 1. Hence also the middle complex is. In particular, the sequence $g_1, \ldots, g_m$ forms a quasi-regular sequence in A/I^d for every $d \geq 1$, see Lemma 15.30.6. Now we are ready to prove that $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A. Namely, set $J = (f_1, \ldots, f_n, g_1, \ldots, g_m)$ and suppose that (with multinomial notation)

$$\sum\nolimits_{|N|+|M|=d} a_{N,M} f^N g^M \in J^{d+1}$$

for some $a_{N,M} \in A$. We have to show that $a_{N,M} \in J$ for all N, M. Let $e \in \{0, 1, \ldots, d\}$. Then

$$\sum\nolimits_{|N|=d-e,\ |M|=e} a_{N,M} f^N g^M \in (g_1, \ldots, g_m)^{e+1} + I^{d-e+1}$$

Because $g_1, \ldots, g_m$ is a quasi-regular sequence in A/I^{d-e+1} we deduce

$$\sum\nolimits_{|N|=d-e} a_{N,M} f^N \in (g_1, \ldots, g_m) + I^{d-e+1}$$

for each M with $|M| = e$. By Lemma 15.30.8 applied to I^{d-e}/I^{d-e+1} in the ring A/I^{d-e+1} this implies $\sum_{|N|=d-e} a_{N,M} f^N \in I^{d-e}(g_1, \ldots, g_m)$. Since $f_1, \ldots, f_n$ is quasi-regular in A this implies that $a_{N,M} \in J$ for each N, M with $|N| = d - e$ and $|M| = e$. This proves the lemma. $\square$

0668 Lemma 15.30.11. Let A be a ring. Let I be an ideal generated by an H_1-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is an H_1-regular sequence in A.

Proof. We have to show that $H_1(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = 0$. To do this consider the commutative diagram

$$\begin{tikzcd} \wedge^2(A^{\oplus n+m}) \arrow[r] \arrow[d] & A^{\oplus n+m} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ \wedge^2(A/I^{\oplus m}) \arrow[r] & A/I^{\oplus m} \arrow[r] & A/I \arrow[r] & 0 \end{tikzcd}$$

Consider an element $(a_1, \ldots, a_{n+m}) \in A^{\oplus n+m}$ which maps to zero in A. Because $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I we see that $(\overline{a}_{n+1}, \ldots, \overline{a}_{n+m})$ is the image of some element $\overline{\alpha}$ of $\wedge^2(A/I^{\oplus m})$. We can lift $\overline{\alpha}$ to an element $\alpha \in \wedge^2(A^{\oplus n+m})$ and substract the image of it in $A^{\oplus n+m}$ from our element $(a_1, \ldots, a_{n+m})$. Thus we may assume that $a_{n+1}, \ldots, a_{n+m} \in I$. Since $I = (f_1, \ldots, f_n)$ we can modify our element $(a_1, \ldots, a_{n+m})$ by linear combinations of the elements

$$(0, \ldots, g_j, 0, \ldots, 0, f_i, 0, \ldots, 0)$$

in the image of the top left horizontal arrow to reduce to the case that $a_{n+1}, \ldots, a_{n+m}$ are zero. In this case $(a_1, \ldots, a_n, 0, \ldots, 0)$ defines an element of $H_1(A, f_1, \ldots, f_n)$ which we assumed to be zero. $\square$

068L **Lemma 15.30.12.** Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$ be an H_1-regular sequence. Then the images $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form an H_1-regular sequence.

Proof. Set $I = (f_1, \ldots, f_n)$. We have to show that any relation $\sum_{j=1,\ldots,m} \overline{a}_j\overline{g}_j$ in A/I is a linear combination of trivial relations. Because $I = (f_1, \ldots, f_n)$ we can lift this relation to a relation

$$\sum\nolimits_{j=1,\ldots,m} a_j g_j + \sum\nolimits_{i=1,\ldots,n} b_i f_i = 0$$

in A. By assumption this relation in A is a linear combination of trivial relations. Taking the image in A/I we obtain what we want. $\square$

0669 **Lemma 15.30.13.** Let A be a ring. Let I be an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a Koszul-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a Koszul-regular sequence in A.

Proof. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A/I. Then

$$\begin{aligned} K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) &= \text{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \\ &\cong A/(f_i, g_j) \end{aligned}$$

The first equality by Lemma 15.28.12. The first quasi-isomorphism $\cong$ by (the dual of) Homology, Lemma 12.25.4 as the qth row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. The last quasi-isomorphism by assumption. Hence we win. $\square$

To conclude in the following lemma it is necessary to assume that both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular. A counter example to dropping the assumption that $f_1, \ldots, f_n$ is Koszul-regular is Examples, Lemma 110.14.1.

068M Lemma 15.30.14. Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$. If both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular sequences in A, then $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form a Koszul-regular sequence.

Proof. Set $I = (f_1, \ldots, f_n)$. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A. Then

$$\begin{aligned} A/(f_i, g_j) &\cong K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) \\ &= \mathrm{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \end{aligned}$$

The first quasi-isomorphism $\cong$ by assumption. The first equality by Lemma 15.28.12. The second quasi-isomorphism by (the dual of) Homology, Lemma 12.25.4 as the qth row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. Hence we win. $\square$

066A Lemma 15.30.15. Let R be a ring. Let I be an ideal generated by $f_1, \ldots, f_r \in R$.

(1) If I can be generated by a quasi-regular sequence of length r, then $f_1, \ldots, f_r$ is a quasi-regular sequence.
(2) If I can be generated by an H_1-regular sequence of length r, then $f_1, \ldots, f_r$ is an H_1-regular sequence.
(3) If I can be generated by a Koszul-regular sequence of length r, then $f_1, \ldots, f_r$ is a Koszul-regular sequence.

Proof. If I can be generated by a quasi-regular sequence of length r, then I/I^2 is free of rank r over R/I. Since $f_1, \ldots, f_r$ generate by assumption we see that the images $\overline{f}_i$ form a basis of I/I^2 over R/I. It follows that $f_1, \ldots, f_r$ is a quasi-regular sequence as all this means, besides the freeness of I/I^2, is that the maps $\mathrm{Sym}^n_{R/I}(I/I^2) \to I^n/I^{n+1}$ are isomorphisms.

We continue to assume that I can be generated by a quasi-regular sequence, say $g_1, \ldots, g_r$. Write $g_j = \sum a_{ij} f_i$. As $f_1, \ldots, f_r$ is quasi-regular according to the previous paragraph, we see that $\det(a_{ij})$ is invertible mod I. The matrix a_{ij} gives a map $R^{\oplus r} \to R^{\oplus r}$ which induces a map of Koszul complexes $\alpha : K_\bullet(R, f_1, \ldots, f_r) \to K_\bullet(R, g_1, \ldots, g_r)$, see Lemma 15.28.3. This map becomes an isomorphism on inverting $\det(a_{ij})$. Since the cohomology modules of both $K_\bullet(R, f_1, \ldots, f_r)$ and $K_\bullet(R, g_1, \ldots, g_r)$ are annihilated by I, see Lemma 15.28.6, we see that α is a quasi-isomorphism.

Now assume that $g_1, \ldots, g_r$ is a H_1-regular sequence generating I. Then $g_1, \ldots, g_r$ is a quasi-regular sequence by Lemma 15.30.6. By the previous paragraph we conclude that $f_1, \ldots, f_r$ is a H_1-regular sequence. Similarly for Koszul-regular sequences. $\square$

068P Lemma 15.30.16. Let R be a ring. Let $a_1, \ldots, a_n \in R$ be elements such that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is injective. Then the element $\sum a_i t_i$ of the polynomial ring $R[t_1, \ldots, t_n]$ is a nonzerodivisor.

This is a particular case of [McC57, Corollary]

Proof. If one of the a_i is a unit this is just the statement that any element of the form $t_1 + a_2t_2 + \ldots + a_nt_n$ is a nonzerodivisor in the polynomial ring over R.

Case I: R is Noetherian. Let $\mathfrak q_j$, $j = 1, \ldots, m$ be the associated primes of R. We have to show that each of the maps

$$\sum a_it_i : \text{Sym}^d(R^{\oplus n}) \longrightarrow \text{Sym}^{d+1}(R^{\oplus n})$$

is injective. As $\text{Sym}^d(R^{\oplus n})$ is a free R-module its associated primes are $\mathfrak q_j$, $j = 1, \ldots, m$. For each j there exists an $i = i(j)$ such that $a_i \not\in \mathfrak q_j$ because there exists an $x \in R$ with $\mathfrak q_jx = 0$ but $a_ix \neq 0$ for some i by assumption. Hence a_i is a unit in $R_{\mathfrak q_j}$ and the map is injective after localizing at $\mathfrak q_j$. Thus the map is injective, see Algebra, Lemma 10.63.19.

Case II: R general. We can write R as the union of Noetherian rings R_λ with $a_1, \ldots, a_n \in R_\lambda$. For each R_λ the result holds, hence the result holds for R. □

068Q Lemma 15.30.17. Let R be a ring. Let $f_1, \ldots, f_n$ be a Koszul-regular sequence in R such that $(f_1, \ldots, f_n) \neq R$. Consider the faithfully flat, smooth ring map

$$R \longrightarrow S = R[\{t_{ij}\}_{i \leq j}, t_{11}^{-1}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$$

For $1 \leq i \leq n$ set

$$g_i = \sum\nolimits_{i \leq j} t_{ij}f_j \in S.$$

Then $g_1, \ldots, g_n$ is a regular sequence in S and $(f_1, \ldots, f_n)S = (g_1, \ldots, g_n)$.

Proof. The equality of ideals is obvious as the matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & \ldots \\ 0 & t_{22} & t_{23} & \ldots \\ 0 & 0 & t_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

is invertible in S. Because $f_1, \ldots, f_n$ is a Koszul-regular sequence we see that the kernel of $R \to R^{\oplus n}$, $x \mapsto (xf_1, \ldots, xf_n)$ is zero (as it computes the $n$the Koszul homology of R w.r.t. $f_1, \ldots, f_n$). Hence by Lemma 15.30.16 we see that $g_1 = f_1t_{11} + \ldots + f_nt_{1n}$ is a nonzerodivisor in $S' = R[t_{11}, t_{12}, \ldots, t_{1n}, t_{11}^{-1}]$. We see that $g_1, f_2, \ldots, f_n$ is a Koszul-sequence in S' by Lemma 15.30.5 and 15.30.15. We conclude that $\overline{f}_2, \ldots, \overline{f}_n$ is a Koszul-regular sequence in $S'/(g_1)$ by Lemma 15.30.14. Hence by induction on n we see that the images $\overline{g}_2, \ldots, \overline{g}_n$ of $g_2, \ldots, g_n$ in $S'/(g_1)[\{t_{ij}\}_{2 \leq i \leq j}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$ form a regular sequence. This in turn means that $g_1, \ldots, g_n$ forms a regular sequence in S. □

15.31. More on Koszul regular sequences

0CEN We continue the discussion from Section 15.30.

0G6L Lemma 15.31.1. Let R be a ring. Let $f_1, \ldots, f_r \in R$ be an Koszul-regular sequence. Then the extended alternating Čech complex $R \to \bigoplus_{i_0} R_{f_{i_0}} \to \bigoplus_{i_0 < i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$ from Section 15.29 only has cohomology in degree r.

Proof. By Lemma 15.30.4 and induction the sequence $f_1, \ldots, f_{r-1}, f_r^n$ is Koszul regular for all $n \geq 1$. By Lemma 15.28.4 any permutation of a Koszul regular sequence is a Koszul regular sequence. Hence we see that we may replace any (or all) f_i by its nth power and still have a Koszul regular sequence. Thus $K_\bullet(R, f_1^n, \ldots, f_r^n)$

has nonzero cohomology only in homological degree 0. This implies what we want by Lemma 15.29.6. $\square$

0BIQ Lemma 15.31.2. Let $a, a_2, \ldots, a_r$ be an H_1-regular sequence in a ring R (for example a Koszul regular sequence or a regular sequence, see Lemmas 15.30.2 and 15.30.3). With $I = (a, a_2, \ldots, a_r)$ the blowup algebra $R' = R[\frac{I}{a}]$ is isomorphic to $R'' = R[y_2, \ldots, y_r]/(ay_i - a_i)$.

Proof. By Algebra, Lemma 10.70.6 it suffices to show that R'' is a-torsion free.

We claim $a, ay_2 - a_2, \ldots, ay_n - a_r$ is a H_1-regular sequence in $R[y_2, \ldots, y_r]$. Namely, the map

$$(a, ay_2 - a_2, \ldots, ay_n - a_r) : R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]$$

used to define the Koszul complex on $a, ay_2 - a_2, \ldots, ay_n - a_r$ is isomorphic to the map

$$(a, a_2, \ldots, a_r) : R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]$$

used to the define the Koszul complex on $a, a_2, \ldots, a_r$ via the isomorphism

$$R[y_2, \ldots, y_r]^{\oplus r} \longrightarrow R[y_2, \ldots, y_r]^{\oplus r}$$

sending $(b_1, \ldots, b_r)$ to $(b_1 - b_2 y_2 \ldots - b_r y_r, -b_2, \ldots, -b_r)$. By Lemma 15.28.3 these Koszul complexes are isomorphic. By Lemma 15.30.5 applied to the flat ring map $R \to R[y_2, \ldots, y_r]$ we conclude our claim is true. By Lemma 15.28.8 we see that the Koszul complex K on $a, ay_2 - a_2, \ldots, ay_n - a_r$ is the cone on $a : L \to L$ where L is the Koszul complex on $ay_2 - a_2, \ldots, ay_n - a_r$. Since $H_1(K) = 0$ by the claim, we conclude that $a : H_0(L) \to H_0(L)$ is injective, in other words that $R'' = R[y_2, \ldots, y_r]/(ay_i - a_i)$ has no nonzero a-torsion elements as desired. $\square$

063Q Lemma 15.31.3. Let $A \to B$ be a ring map. Let $f_1, \ldots, f_r$ be a sequence in B such that $B/(f_1, \ldots, f_r)$ is A-flat. Let $A \to A'$ be a ring map. Then the canonical map

$$H_1(K_\bullet(B, f_1, \ldots, f_r)) \otimes_A A' \longrightarrow H_1(K_\bullet(B', f_1', \ldots, f_r'))$$

is surjective. Here $B' = B \otimes_A A'$ and $f_i' \in B'$ is the image of f_i.

Proof. The sequence

$$\wedge^2(B^{\oplus r}) \to B^{\oplus r} \to B \to B/J \to 0$$

is a complex of A-modules with B/J flat over A and cohomology group $H_1 = H_1(K_\bullet(B, f_1, \ldots, f_r))$ in the spot $B^{\oplus r}$. If we tensor this with A' we obtain a complex

$$\wedge^2((B')^{\oplus r}) \to (B')^{\oplus r} \to B' \to B'/J' \to 0$$

which is exact at B' and B'/J'. In order to compute its cohomology group $H_1' = H_1(K_\bullet(B', f_1', \ldots, f_r'))$ at $(B')^{\oplus r}$ we split the first sequence above into the exact sequences $0 \to J \to B \to B/J \to 0$, $0 \to K \to B^{\oplus r} \to J \to 0$, and $\wedge^2(B^{\oplus r}) \to K \to H_1 \to 0$. Tensoring over A with A' we obtain the exact sequences

$$\begin{gathered} 0 \to J \otimes_A A' \to B \otimes_A A' \to (B/J) \otimes_A A' \to 0 \\ K \otimes_A A' \to B^{\oplus r} \otimes_A A' \to J \otimes_A A' \to 0 \\ \wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A' \to H_1 \otimes_A A' \to 0 \end{gathered}$$

where the first one is exact as B/J is flat over A, see Algebra, Lemma 10.39.12. We conclude that $J' = J \otimes_A A' \subset B'$ and that $K \otimes_A A' \to \text{Ker}((B')^{\oplus r} \to B')$ is surjective. Thus

$$\begin{aligned} H_1 \otimes_A A' &= \text{Coker}\left(\wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A'\right) \\ &\to \text{Coker}\left(\wedge^2((B')^{\oplus r}) \to \text{Ker}((B')^{\oplus r} \to B')\right) = H_1' \end{aligned}$$

is surjective too. □

0CEP Lemma 15.31.4. Let $A \to B$ and $A \to A'$ be ring maps. Set $B' = B \otimes_A A'$. Let $f_1, \ldots, f_r \in B$. Assume $B/(f_1, \ldots, f_r)B$ is flat over A

(1) If $f_1, \ldots, f_r$ is a quasi-regular sequence, then the image in B' is a quasi-regular sequence.
(2) If $f_1, \ldots, f_r$ is a H_1-regular sequence, then the image in B' is a H_1-regular sequence.

Proof. Assume $f_1, \ldots, f_r$ is quasi-regular. Set $J = (f_1, \ldots, f_r)$. By assumption J^n/J^{n+1} is isomorphic to a direct sum of copies of B/J hence flat over A. By induction and Algebra, Lemma 10.39.13 we conclude that B/J^n is flat over A. The ideal $(J')^n$ is equal to $J^n \otimes_A A'$, see Algebra, Lemma 10.39.12. Hence $(J')^n/(J')^{n+1} = J^n/J^{n+1} \otimes_A A'$ which clearly implies that $f_1, \ldots, f_r$ is a quasi-regular sequence in B'.

Assume $f_1, \ldots, f_r$ is H_1-regular. By Lemma 15.31.3 the vanishing of the Koszul homology group $H_1(K_\bullet(B, f_1, \ldots, f_r))$ implies the vanishing of $H_1(K_\bullet(B', f_1', \ldots, f_r'))$ and we win. □

0CEQ Lemma 15.31.5. Let $A' \to B'$ be a ring map. Let $I \subset A'$ be an ideal. Set $A = A'/I$ and $B = B'/IB'$. Let $f_1', \ldots, f_r' \in B'$. Assume

(1) $A' \to B'$ is flat and of finite presentation,
(2) I is locally nilpotent,
(3) the images $f_1, \ldots, f_r \in B$ form a quasi-regular sequence,
(4) $B/(f_1, \ldots, f_r)$ is flat over A.

Then $B'/(f_1', \ldots, f_r')$ is flat over A'.

Proof. Set $C' = B'/(f_1', \ldots, f_r')$. We have to show $A' \to C'$ is flat. Let $\mathfrak{r}' \subset C'$ be a prime ideal lying over $\mathfrak{p}' \subset A'$. We let $\mathfrak{q}' \subset B'$ be the inverse image of $\mathfrak{r}'$. By Algebra, Lemma 10.39.18 it suffices to show that $A'_{\mathfrak{p}'} \to C'_{\mathfrak{q}'}$ is flat. Algebra, Lemma 10.128.6 tells us it suffices to show that $f_1', \ldots, f_r'$ map to a regular sequence in

$$B'_{\mathfrak{q}'}/\mathfrak{p}'B'_{\mathfrak{q}'} = B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = (B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{q}}$$

with obvious notation. What we know is that $f_1, \ldots, f_r$ is a quasi-regular sequence in B and that $B/(f_1, \ldots, f_r)$ is flat over A. By Lemma 15.31.4 the images $\overline{f}_1, \ldots, \overline{f}_r$ of $f_1', \ldots, f_r'$ in $B \otimes_A \kappa(\mathfrak{p})$ form a quasi-regular sequence. Since $(B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{q}}$ is a Noetherian local ring, we conclude by Lemma 15.30.7. □

0CER Lemma 15.31.6. Let $A' \to B'$ be a ring map. Let $I \subset A'$ be an ideal. Set $A = A'/I$ and $B = B'/IB'$. Let $f_1', \ldots, f_r' \in B'$. Assume

(1) $A' \to B'$ is flat and of finite presentation (for example smooth),
(2) I is locally nilpotent,
(3) the images $f_1, \ldots, f_r \in B$ form a quasi-regular sequence,
(4) $B/(f_1, \ldots, f_r)$ is smooth over A.

Then $B'/(f'_1, \ldots, f'_r)$ is smooth over A'.

Proof. Set $C' = B'/(f'_1, \ldots, f'_r)$ and $C = B/(f_1, \ldots, f_r)$. Then $A' \to C'$ is of finite presentation. By Lemma 15.31.5 we see that $A' \to C'$ is flat. The fibre rings of $A' \to C'$ are equal to the fibre rings of $A \to C$ and hence smooth by assumption (4). It follows that $A' \to C'$ is smooth by Algebra, Lemma 10.137.17. □

15.32. Regular ideals

07CU We will discuss the notion of a regular ideal sheaf in great generality in Divisors, Section 31.20. Here we define the corresponding notion in the affine case, i.e., in the case of an ideal in a ring.

07CV Definition 15.32.1. Let R be a ring and let $I \subset R$ be an ideal.

(1) We say I is a regular ideal if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \not\in \mathfrak{p}$ and a regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(2) We say I is a Koszul-regular ideal if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \not\in \mathfrak{p}$ and a Koszul-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(3) We say I is a H_1-regular ideal if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \not\in \mathfrak{p}$ and an H_1-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(4) We say I is a quasi-regular ideal if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \not\in \mathfrak{p}$ and a quasi-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.

It is clear that given $I \subset R$ we have the implications

$$\begin{aligned} I \text{ is a regular ideal} &\Rightarrow I \text{ is a Koszul-regular ideal} \\ &\Rightarrow I \text{ is a } H_1\text{-regular ideal} \\ &\Rightarrow I \text{ is a quasi-regular ideal} \end{aligned}$$

see Lemmas 15.30.2, 15.30.3, and 15.30.6. Such an ideal is always finitely generated.

07CW Lemma 15.32.2. A quasi-regular ideal is finitely generated.

Proof. Let $I \subset R$ be a quasi-regular ideal. Since $V(I)$ is quasi-compact, there exist $g_1, \ldots, g_m \in R$ such that $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$ and such that I_{g_j} is generated by a quasi-regular sequence $g_{j1}, \ldots, g_{jr_j} \in R_{g_j}$. Write $g_{ji} = g'_{ji}/g_j^{e_{ij}}$ for some $g'_{ij} \in I$. Write $1 + x = \sum g_j h_j$ for some $x \in I$ which is possible as $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$. Note that $\text{Spec}(R) = D(g_1) \cup \ldots \cup D(g_m) \bigcup D(x)$ Then I is generated by the elements g'_{ij} and x as these generate on each of the pieces of the cover, see Algebra, Lemma 10.23.2. □

08RK Lemma 15.32.3. Let $I \subset R$ be a quasi-regular ideal of a ring. Then I/I^2 is a finite projective R/I-module.

Proof. This follows from Algebra, Lemma 10.78.2 and the definitions. □

We prove flat descent for Koszul-regular, H_1-regular, quasi-regular ideals.

068N Lemma 15.32.4. Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in B, then I is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in A.

Proof. We fix the prime $\mathfrak{p} \supset I$ throughout the proof. Assume IB is quasi-regular. By Lemma 15.32.2 IB is a finite module, hence I is a finite A-module by Algebra, Lemma 10.83.2. As $A \to B$ is flat we see that

$$I/I^2 \otimes_{A/I} B/IB = I/I^2 \otimes_A B = IB/(IB)^2.$$

As IB is quasi-regular, the B/IB-module $IB/(IB)^2$ is finite locally free. Hence I/I^2 is finite projective, see Algebra, Proposition 10.83.3. In particular, after replacing A by A_f for some $f \in A$, $f \notin \mathfrak{p}$ we may assume that I/I^2 is free of rank r. Pick $f_1, \ldots, f_r \in I$ which give a basis of I/I^2. By Nakayama's lemma (see Algebra, Lemma 10.20.1) we see that, after another replacement $A \rightsquigarrow A_f$ as above, I is generated by $f_1, \ldots, f_r$.

Proof of the "quasi-regular" case. Above we have seen that I/I^2 is free on the r-generators $f_1, \ldots, f_r$. To finish the proof in this case we have to show that the maps $\text{Sym}^d(I/I^2) \to I^d/I^{d+1}$ are isomorphisms for each $d \geq 2$. This is clear as the faithfully flat base changes $\text{Sym}^d(IB/(IB)^2) \to (IB)^d/(IB)^{d+1}$ are isomorphisms locally on B by assumption. Details omitted.

Proof of the "H_1-regular" and "Koszul-regular" case. Consider the sequence of elements $f_1, \ldots, f_r$ generating I we constructed above. By Lemma 15.30.15 we see that $f_1, \ldots, f_r$ map to a H_1-regular or Koszul-regular sequence in B_g for any $g \in B$ such that IB is generated by an H_1-regular or Koszul-regular sequence. Hence $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B_g$ has vanishing H_1 or H_i, $i > 0$. Since the homology of $K_\bullet(B, f_1, \ldots, f_r) = K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ is annihilated by IB (see Lemma 15.28.6) and since $V(IB) \subset \bigcup_{g \text{ as above}} D(g)$ we conclude that $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ has vanishing homology in degree 1 or all positive degrees. Using that $A \to B$ is faithfully flat we conclude that the same is true for $K_\bullet(A, f_1, \ldots, f_r)$. $\square$

07CX Lemma 15.32.5. Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is a H_1-regular ideal. Then $I \cap J^2 = IJ$.

Proof. Follows immediately from Lemma 15.30.9 by localizing. $\square$

15.33. Local complete intersection maps

07CY We can use the material above to define a local complete intersection map between rings using presentations by (finite) polynomial algebras.

07CZ Lemma 15.33.1. Let $A \to B$ be a finite type ring map. If for some presentation $\alpha : A[x_1, \ldots, x_n] \to B$ the kernel I is a Koszul-regular ideal then for any presentation $\beta : A[y_1, \ldots, y_m] \to B$ the kernel J is a Koszul-regular ideal.

Proof. Choose $f_j \in A[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in A[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} A[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & A[x_1, \ldots, x_n] \arrow[d] \\ A[y_1, \ldots, y_m] \arrow[r] & B \end{tikzcd}$$

Note that the kernel K of $A[x_i, y_j] \to B$ is equal to $K = (I, y_j - f_j) = (J, x_i - f_i)$. In particular, as I is finitely generated by Lemma 15.32.2 we see that $J = K/(x_i - f_i)$ is finitely generated too.

Pick a prime $\mathfrak{q} \subset B$. Since $I/I^2 \oplus B^{\oplus m} = J/J^2 \oplus B^{\oplus n}$ (Algebra, Lemma 10.134.15) we see that

$$\dim J/J^2 \otimes_B \kappa(\mathfrak{q}) + n = \dim I/I^2 \otimes_B \kappa(\mathfrak{q}) + m.$$

Pick $p_1, \ldots, p_t \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q}) = I \otimes_{A[x_i]} \kappa(\mathfrak{q})$. Pick $q_1, \ldots, q_s \in J$ which map to a basis of $J/J^2 \otimes \kappa(\mathfrak{q}) = J \otimes_{A[y_j]} \kappa(\mathfrak{q})$. So $s+n = t+m$. By Nakayama's lemma there exist $h \in A[x_i]$ and $h' \in A[y_j]$ both mapping to a nonzero element of $\kappa(\mathfrak{q})$ such that $I_h = (p_1, \ldots, p_t)$ in $A[x_i, 1/h]$ and $J_{h'} = (q_1, \ldots, q_s)$ in $A[y_j, 1/h']$. As I is Koszul-regular we may also assume that I_h is generated by a Koszul regular sequence. This sequence must necessarily have length $t = \dim I/I^2 \otimes_B \kappa(\mathfrak{q})$, hence we see that $p_1, \ldots, p_t$ is a Koszul-regular sequence by Lemma 15.30.15. As also $y_1 - f_1, \ldots, y_m - f_m$ is a regular sequence we conclude

$$y_1 - f_1, \ldots, y_m - f_m, p_1, \ldots, p_t$$

is a Koszul-regular sequence in $A[x_i, y_j, 1/h]$ (see Lemma 15.30.13). This sequence generates the ideal K_h. Hence the ideal $K_{hh'}$ is generated by a Koszul-regular sequence of length $m + t = n + s$. But it is also generated by the sequence

$$x_1 - g_1, \ldots, x_n - g_n, q_1, \ldots, q_s$$

of the same length which is thus a Koszul-regular sequence by Lemma 15.30.15. Finally, by Lemma 15.30.14 we conclude that the images of $q_1, \ldots, q_s$ in

$$A[x_i, y_j, 1/hh']/(x_1 - g_1, \ldots, x_n - g_n) \cong A[y_j, 1/h'']$$

form a Koszul-regular sequence generating $J_{h''}$. Since h'' is the image of hh' it doesn't map to zero in $\kappa(\mathfrak{q})$ and we win. □

This lemma allows us to make the following definition.

07D0 Definition 15.33.2. A ring map $A \to B$ is called a local complete intersection if it is of finite type and for some (equivalently any) presentation $B = A[x_1, \ldots, x_n]/I$ the ideal I is Koszul-regular.

This notion is local.

07D1 Lemma 15.33.3. Let $R \to S$ be a ring map. Let $g_1, \ldots, g_m \in S$ generate the unit ideal. If each $R \to S_{g_j}$ is a local complete intersection so is $R \to S$.

Proof. Let $S = R[x_1, \ldots, x_n]/I$ be a presentation. Pick $h_j \in R[x_1, \ldots, x_n]$ mapping to g_j in S. Then $R[x_1, \ldots, x_n, x_{n+1}]/(I, x_{n+1}h_j - 1)$ is a presentation of S_{g_j}. Hence $I_j = (I, x_{n+1}h_j - 1)$ is a Koszul-regular ideal in $R[x_1, \ldots, x_n, x_{n+1}]$. Pick a prime $I \subset \mathfrak{q} \subset R[x_1, \ldots, x_n]$. Then $h_j \notin \mathfrak{q}$ for some j and $\mathfrak{q}_j = (\mathfrak{q}, x_{n+1}h_j - 1)$ is a prime ideal of $V(I_j)$ lying over $\mathfrak{q}$. Pick $f_1, \ldots, f_r \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q})$. Then $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a sequence of elements of I_j which map to a basis of $I_j \otimes \kappa(\mathfrak{q}_j)$. By Nakayama's lemma there exists an $h \in R[x_1, \ldots, x_n, x_{n+1}]$ such that $(I_j)_h$ is generated by $x_{n+1}h_j - 1, f_1, \ldots, f_r$. We may also assume that $(I_j)_h$ is generated by a Koszul regular sequence of some length e. Looking at the dimension of $I_j \otimes \kappa(\mathfrak{q}_j)$ we see that $e = r + 1$. Hence by Lemma 15.30.15 we see that $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a Koszul-regular sequence generating $(I_j)_h$ for some $h \in R[x_1, \ldots, x_n, x_{n+1}]$, $h \notin \mathfrak{q}_j$. By Lemma 15.30.14 we see that $I_{h'}$ is generated by a Koszul-regular sequence for some $h' \in R[x_1, \ldots, x_n]$, $h' \notin \mathfrak{q}$ as desired. □

07D2 Lemma 15.33.4. Let R be a ring. If $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection, then $f_1, \ldots, f_c$ is a Koszul regular sequence.

Proof. Recall that the homology groups $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_c)$. Hence it suffices to show that $H_i(K_\bullet(f_\bullet))_\mathfrak{q}$ is zero for all primes $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ containing $(f_1, \ldots, f_c)$. This follows from Algebra, Lemma 10.136.12 and the fact that a regular sequence is Koszul regular (Lemma 15.30.2). $\square$

07D3 **Lemma 15.33.5.** Let $R \to S$ be a ring map. The following are equivalent

(1) $R \to S$ is syntomic (Algebra, Definition 10.136.1), and
(2) $R \to S$ is flat and a local complete intersection.

Proof. Assume (1). Then $R \to S$ is flat by definition. By Algebra, Lemma 10.136.15 and Lemma 15.33.3 we see that it suffices to show a relative global complete intersection is a local complete intersection homomorphism which is Lemma 15.33.4.

Assume (2). A local complete intersection is of finite presentation because a Koszul-regular ideal is finitely generated. Let $R \to k$ be a map to a field. It suffices to show that $S' = S \otimes_R k$ is a local complete intersection over k, see Algebra, Definition 10.135.1. Choose a prime $\mathfrak{q}' \subset S'$. Write $S = R[x_1, \ldots, x_n]/I$. Then $S' = k[x_1, \ldots, x_n]/I'$ where $I' \subset k[x_1, \ldots, x_n]$ is the image of I. Let $\mathfrak{p}' \subset k[x_1, \ldots, x_n]$, $\mathfrak{q} \subset S$, and $\mathfrak{p} \subset R[x_1, \ldots, x_n]$ be the corresponding primes. By Definition 15.32.1 exists an $g \in R[x_1, \ldots, x_n]$, $g \notin \mathfrak{p}$ and $f_1, \ldots, f_r \in R[x_1, \ldots, x_n]_g$ which form a Koszul-regular sequence generating I_g. Since S and hence S_g is flat over R we see that the images $f_1', \ldots, f_r'$ in $k[x_1, \ldots, x_n]_g$ form a H_1-regular sequence generating I'_g, see Lemma 15.31.4. Thus $f_1', \ldots, f_r'$ map to a regular sequence in $k[x_1, \ldots, x_n]_{\mathfrak{p}'}$ generating $I'_{\mathfrak{p}'}$ by Lemma 15.30.7. Applying Algebra, Lemma 10.135.4 we conclude $S'_{gg'}$ for some $g' \in S$, $g' \notin \mathfrak{q}'$ is a global complete intersection over k as desired. $\square$

For a local complete intersection $R \to S$ we have $H_n(L_{S/R}) = 0$ for $n \geq 2$. Since we haven't (yet) defined the full cotangent complex we can't state and prove this, but we can deduce one of the consequences.

07D4 **Lemma 15.33.6.** Let $A \to B \to C$ be ring maps. Assume $B \to C$ is a local complete intersection homomorphism. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_1, \ldots, y_m] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{A[x_s]/A} \otimes C \arrow[r] & \Omega_{A[x_s,y_t]/A} \otimes C \arrow[r] & \Omega_{B[y_t]/B} \otimes C \arrow[r] & 0 \\ 0 \arrow[r] & I/I^2 \otimes C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J/J^2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. In particular, the six term exact sequence of Algebra, Lemma 10.134.4 can be completed with a zero on the left, i.e., the sequence

$$0 \to H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is exact.

Proof. The only thing to prove is the injectivity of the map $I/I^2 \otimes C \to K/K^2$. By assumption the ideal J is Koszul-regular. Hence we have $IA[x_s, y_j] \cap K^2 = IK$ by Lemma 15.32.5. This means that the kernel of $K/K^2 \to J/J^2$ is isomorphic to $IA[x_s, y_j]/IK$. Since $I/I^2 \otimes_A C = IA[x_s, y_j]/IK$ by right exactness of tensor product, this provides us with the desired injectivity of $I/I^2 \otimes_A C \to K/K^2$. $\square$

07D5 Lemma 15.33.7. Let $A \to B \to C$ be ring maps. If $B \to C$ is a filtered colimit of local complete intersection homomorphisms then the conclusion of Lemma 15.33.6 remains valid.

Proof. Follows from Lemma 15.33.6 and Algebra, Lemma 10.134.9. □

0D08 Lemma 15.33.8. Let $A \to B$ be a local homomorphism of local rings. Let $A^h \to B^h$, resp. $A^{sh} \to B^{sh}$ be the induced map on henselizations, resp. strict henselizations (Algebra, Lemma 10.155.6, resp. Lemma 10.155.10). Then $NL_{B/A} \otimes_B B^h \to NL_{B^h/A^h}$ and $NL_{B/A} \otimes_B B^{sh} \to NL_{B^{sh}/A^{sh}}$ induce isomorphisms on cohomology groups.

Proof. Since A^h is a filtered colimit of étale algebras over A we see that $NL_{A^h/A}$ is an acyclic complex by Algebra, Lemma 10.134.9 and Algebra, Definition 10.143.1. The same is true for B^h/B. Using the Jacobi-Zariski sequence (Algebra, Lemma 10.134.4) for $A \to A^h \to B^h$ we find that $NL_{B^h/A} \to NL_{B^h/A^h}$ induces isomorphisms on cohomology groups. Moreover, an étale ring map is a local complete intersection as it is even a global complete intersection, see Algebra, Lemma 10.143.2. By Lemma 15.33.7 we get a six term exact Jacobi-Zariski sequence associated to $A \to B \to B^h$ which proves that $NL_{B/A} \otimes_B B^h \to NL_{B^h/A}$ induces isomorphisms on cohomology groups. This finishes the proof in the case of the map on henselizations. The case of strict henselization is proved in exactly the same manner. □

15.34. Cartier's equality and geometric regularity

07E0 A reference for this section and the next is [Mat70a, Section 39]. In order to comfortably read this section the reader should be familiar with the naive cotangent complex and its properties, see Algebra, Section 10.134.

07E1 Lemma 15.34.1 (Cartier equality). Let K/k be a finitely generated field extension. Then $\Omega_{K/k}$ and $H_1(L_{K/k})$ are finite dimensional and $\text{trdeg}_k(K) = \dim_K \Omega_{K/k} - \dim_K H_1(L_{K/k})$.

Proof. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 10.158.11 and its proof. In this case we see that $NL_{K/k}$ is homotopy equivalent to the complex

$$\bigoplus\nolimits_{j=1,\ldots,c} K \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} K \mathrm{d}x_i$$

by Algebra, Lemmas 10.134.2 and 10.134.13. The transcendence degree of K over k is the dimension of A (by Algebra, Lemma 10.116.1) which is $n-c$ and we win. □

07E2 Lemma 15.34.2. Let $M/L/K$ be field extensions. Then the Jacobi-Zariski sequence

$$0 \to H_1(L_{L/K}) \otimes_L M \to H_1(L_{M/K}) \to H_1(L_{M/L}) \to \Omega_{L/K} \otimes_L M \to \Omega_{M/K} \to \Omega_{M/L} \to 0$$

is exact.

Proof. Combine Lemma 15.33.7 with Algebra, Lemma 10.158.11. □

07E3 Lemma 15.34.3. Given a commutative diagram of fields

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

with k'/k and K'/K finitely generated field extensions the kernel and cokernel of the maps

$$\alpha : \Omega_{K/k} \otimes_K K' \to \Omega_{K'/k'} \quad \text{and} \quad \beta : H_1(L_{K/k}) \otimes_K K' \to H_1(L_{K'/k'})$$

are finite dimensional and

$$\dim \mathrm{Ker}(\alpha) - \dim \mathrm{Coker}(\alpha) - \dim \mathrm{Ker}(\beta) + \dim \mathrm{Coker}(\beta) = \mathrm{trdeg}_k(k') - \mathrm{trdeg}_K(K')$$

Proof. The Jacobi-Zariski sequences for $k \subset k' \subset K'$ and $k \subset K \subset K'$ are

$$0 \to H_1(L_{k'/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/k'}) \to \Omega_{k'/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/k'} \to 0$$

and

$$0 \to H_1(L_{K/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/K}) \to \Omega_{K/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/K} \to 0$$

By Lemma 15.34.1 the vector spaces $\Omega_{k'/k}$, $\Omega_{K'/K}$, $H_1(L_{K'/K})$, and $H_1(L_{k'/k})$ are finite dimensional and the alternating sum of their dimensions is $\mathrm{trdeg}_k(k') - \mathrm{trdeg}_K(K')$. The lemma follows. □

15.35. Geometric regularity

07E4 Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The Jacobi-Zariski sequence (Algebra, Lemma 10.134.4) is a canonical exact sequence

$$H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A K \to \Omega_{K/k} \to 0$$

because $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ by Algebra, Lemma 10.134.6. We will show that exactness on the left of this sequence characterizes whether or not a regular local ring A is geometrically regular over k. We will link this to the notion of formal smoothness in Section 15.40.

07E5 Proposition 15.35.1. Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The following are equivalent

(1) A is geometrically regular over k,
(2) for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,
(3) A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and
(4) A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.

Proof. Proof of (3) ⇒ (1). Assume (3). Let k'/k be a finite purely inseparable extension. Set $A' = A \otimes_k k'$. This is a local ring with maximal ideal $\mathfrak{m}'$. Set $K' = A'/\mathfrak{m}'$. We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_1(L_{K/k}) \otimes K' \arrow[r] \arrow[d, "\beta"] & \mathfrak{m}/\mathfrak{m}^2 \otimes K' \arrow[r] \arrow[d] & \Omega_{A/k} \otimes_A K' \arrow[r] \arrow[d, "\cong"] & \Omega_{K/k} \otimes K' \arrow[r] \arrow[d, "\alpha"] & 0 \\
 & H_1(L_{K'/k'}) \arrow[r] & \mathfrak{m}'/(\mathfrak{m}')^2 \arrow[r] & \Omega_{A'/k'} \otimes_{A'} K' \arrow[r] & \Omega_{K'/k'} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The third vertical arrow is an isomorphism by base change for modules of differentials (Algebra, Lemma 10.131.12). Thus α is surjective. By Lemma 15.34.3 we have

$$\dim \mathrm{Ker}(\alpha) - \dim \mathrm{Ker}(\beta) + \dim \mathrm{Coker}(\beta) = 0$$

(and these dimensions are all finite). A diagram chase shows that $\dim \mathfrak{m}'/(\mathfrak{m}')^2 \leq \dim \mathfrak{m}/\mathfrak{m}^2$. However, since $A \to A'$ is finite flat we see that $\dim(A) = \dim(A')$, see Algebra, Lemma 10.112.6. Hence A' is regular by definition.

Equivalence of (3) and (4). Consider the Jacobi-Zariski sequences for rows of the commutative diagram

$$\begin{tikzcd} \mathbf{F}_p \arrow[r] & A \arrow[r] & K \\ \mathbf{F}_p \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & K \arrow[u] \end{tikzcd}$$

to get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] & \Omega_{A/\mathbf{F}_p} \otimes_A K \arrow[r] & \Omega_{K/\mathbf{F}_p} \arrow[r] & 0 \\ 0 \arrow[r] & H_1(L_{K/k}) \arrow[r] \arrow[u] & \Omega_{k/\mathbf{F}_p} \otimes_k K \arrow[r] \arrow[u] & \Omega_{K/\mathbf{F}_p} \arrow[r] \arrow[u] & \Omega_{K/k} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. We have used that $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ and that $H_1(L_{K/\mathbf{F}_p}) = 0$ as $K/\mathbf{F}_p$ is separable, see Algebra, Proposition 10.158.9. Thus it is clear that the kernels of $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ and $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ have the same dimension.

Proof of (2) $\Rightarrow$ (4) following Faltings, see [Fal78a]. Let $a_1, \ldots, a_n \in k$ be elements such that $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Consider the field extension $k' = k(a_1^{1/p}, \ldots, a_n^{1/p})$. By Algebra, Lemma 10.158.3 we see that $k' = k[x_1, \ldots, x_n]/(x_1^p - a_1, \ldots, x_n^p - a_n)$. In particular we see that the naive cotangent complex of k'/k is homotopic to the complex $\bigoplus_{j=1,\ldots,n} k' \to \bigoplus_{i=1,\ldots,n} k'$ with the zero differential as $\mathrm{d}(x_j^p - a_j) = 0$ in $\Omega_{k[x_1,\ldots,x_n]/k}$. Set $A' = A \otimes_k k'$ and $K' = A'/\mathfrak{m}'$ as above. By Algebra, Lemma 10.134.8 we see that $NL_{A'/A}$ is homotopy equivalent to the complex $\bigoplus_{j=1,\ldots,n} A' \to \bigoplus_{i=1,\ldots,n} A'$ with the zero differential, i.e., $H_1(L_{A'/A})$ and $\Omega_{A'/A}$ are free of rank n. The Jacobi-Zariski sequence for $\mathbf{F}_p \to A \to A'$ is

$$H_1(L_{A'/A}) \to \Omega_{A/\mathbf{F}_p} \otimes_A A' \to \Omega_{A'/\mathbf{F}_p} \to \Omega_{A'/A} \to 0$$

Using the presentation $A[x_1, \ldots, x_n] \to A'$ with kernel $(x_j^p - a_j)$ we see, unwinding the maps in Algebra, Lemma 10.134.4, that the jth basis vector of $H_1(L_{A'/A})$ maps to $\mathrm{d}a_j \otimes 1$ in $\Omega_{A/\mathbf{F}_p} \otimes A'$. As $\Omega_{A'/A}$ is free (hence flat) we get on tensoring with K' an exact sequence

$$K'^{\oplus n} \to \Omega_{A/\mathbf{F}_p} \otimes_A K' \xrightarrow{\beta} \Omega_{A'/\mathbf{F}_p} \otimes_{A'} K' \to K'^{\oplus n} \to 0$$

We conclude that the elements $\mathrm{d}a_j \otimes 1$ generate $\mathrm{Ker}(\beta)$ and we have to show that are linearly independent, i.e., we have to show $\dim(\mathrm{Ker}(\beta)) = n$. Consider the following big diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}'/(\mathfrak{m}')^2 \arrow[r] & \Omega_{A'/\mathbf{F}_p} \otimes K' \arrow[r] & \Omega_{K'/\mathbf{F}_p} \arrow[r] & 0 \\ 0 \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \otimes K' \arrow[r] \arrow[u, "\alpha"] & \Omega_{A/\mathbf{F}_p} \otimes K' \arrow[r] \arrow[u, "\beta"] & \Omega_{K/\mathbf{F}_p} \otimes K' \arrow[r] \arrow[u, "\gamma"] & 0 \end{tikzcd}$$

By Lemma 15.34.1 and the Jacobi-Zariski sequence for $\mathbf{F}_p \to K \to K'$ we see that the kernel and cokernel of γ have the same finite dimension. By assumption A' is regular (and of the same dimension as A, see above) hence the kernel and cokernel of α have the same dimension. It follows that the kernel and cokernel of β have the same dimension which is what we wanted to show.

The implication (1) $\Rightarrow$ (2) is trivial. This finishes the proof of the proposition. $\square$

07E6 Lemma 15.35.2. Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. Assume A is geometrically regular over k. Let $K/F/k$ be a finitely generated subextension. Let $\varphi : k[y_1, \ldots, y_m] \to A$ be a k-algebra map such that y_i maps to an element of F in K and such that $\text{d}y_1, \ldots, \text{d}y_m$ map to a basis of $\Omega_{F/k}$. Set $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$. Then

$$k[y_1, \ldots, y_m]_{\mathfrak{p}} \to A$$

is flat and $A/\mathfrak{p}A$ is regular.

Proof. Set $A_0 = k[y_1, \ldots, y_m]_{\mathfrak{p}}$ with maximal ideal $\mathfrak{m}_0$ and residue field K_0. Note that $\Omega_{A_0/k}$ is free of rank m and $\Omega_{A_0/k} \otimes K_0 \to \Omega_{K_0/k}$ is an isomorphism. It is clear that A_0 is geometrically regular over k. Hence $H_1(L_{K_0/k}) \to \mathfrak{m}_0/\mathfrak{m}_0^2$ is an isomorphism, see Proposition 15.35.1. Now consider

$$\begin{tikzcd} H_1(L_{K_0/k}) \otimes K \arrow[r] \arrow[d] & \mathfrak{m}_0/\mathfrak{m}_0^2 \otimes K \arrow[d] \\ H_1(L_{K/k}) \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \end{tikzcd}$$

Since the left vertical arrow is injective by Lemma 15.34.2 and the lower horizontal by Proposition 15.35.1 we conclude that the right vertical one is too. Hence a regular system of parameters in A_0 maps to part of a regular system of parameters in A. We win by Algebra, Lemmas 10.128.2 and 10.106.3. $\square$

15.36. Topological rings and modules

07E7 Let's quickly discuss some properties of topological abelian groups. An abelian group M is a topological abelian group if M is endowed with a topology such that addition $M \times M \to M$, $(x, y) \mapsto x + y$ and inverse $M \to M$, $x \mapsto -x$ are continuous. A homomorphism of topological abelian groups is just a homomorphism of abelian groups which is continuous. The category of commutative topological groups is additive and has kernels and cokernels, but is not abelian (as the axiom $\text{Im} = \text{Coim}$ doesn't hold). If $N \subset M$ is a subgroup, then we think of N and M/N as topological groups also, namely using the induced topology on N and the quotient topology on M/N (i.e., such that $M \to M/N$ is submersive). Note that if $N \subset M$ is an open subgroup, then the topology on M/N is discrete.

We say the topology on M is linear if there exists a fundamental system of neighbourhoods of 0 consisting of subgroups. If so then these subgroups are also open. An example is the following. Let I be a directed set and let G_i be an inverse system of (discrete) abelian groups over I. Then

$$G = \lim_{i \in I} G_i$$

with the inverse limit topology is linearly topologized with a fundamental system of neighbourhoods of 0 given by $\text{Ker}(G \to G_i)$. Conversely, let M be a linearly topologized abelian group. Choose any fundamental system of open subgroups $U_i \subset M$, $i \in I$ (i.e., the U_i form a fundamental system of open neighbourhoods and each U_i is a subgroup of M). Setting $i \geq i' \Leftrightarrow U_i \subset U_{i'}$ we see that I is a directed set. We obtain a homomorphism of linearly topologized abelian groups

$$c : M \longrightarrow \lim_{i \in I} M/U_i.$$

It is clear that M is separated (as a topological space) if and only if c is injective. We say that M is complete if c is an isomorphism[2]. We leave it to the reader to check that this condition is independent of the choice of fundamental system of open subgroups $\{U_i\}_{i \in I}$ chosen above. In fact the topological abelian group $M^\wedge = \lim_{i \in I} M/U_i$ is independent of this choice and is sometimes called the completion of M. Any $G = \lim G_i$ as above is complete, in particular, the completion $M^\wedge$ is always complete.

07E8 Definition 15.36.1 (Topological rings). Let R be a ring and let M be an R-module. [GD60, Chapter 0, Sections 7.1 and 7.2]

(1) We say R is a topological ring if R is endowed with a topology such that both addition and multiplication are continuous as maps $R \times R \to R$ where $R \times R$ has the product topology. In this case we say M is a topological module if M is endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous.
(2) A homomorphism of topological modules is just a continuous R-module map. A homomorphism of topological rings is a ring homomorphism which is continuous for the given topologies.
(3) We say M is linearly topologized if 0 has a fundamental system of neighbourhoods consisting of submodules. We say R is linearly topologized if 0 has a fundamental system of neighbourhoods consisting of ideals.
(4) If R is linearly topologized, we say that $I \subset R$ is an ideal of definition if I is open and if every neighbourhood of 0 contains I^n for some n.
(5) If R is linearly topologized, we say that R is pre-admissible if R has an ideal of definition.
(6) If R is linearly topologized, we say that R is admissible if it is pre-admissible and complete[3].
(7) If R is linearly topologized, we say that R is pre-adic if there exists an ideal of definition I such that $\{I^n\}_{n \geq 0}$ forms a fundamental system of neighbourhoods of 0.
(8) If R is linearly topologized, we say that R is adic if R is pre-adic and complete.

Note that a (pre)adic topological ring is the same thing as a (pre)admissible topological ring which has an ideal of definition I such that I^n is open for all $n \geq 1$.

Let R be a ring and let M be an R-module. Let $I \subset R$ be an ideal. Then we can consider the linear topology on R which has $\{I^n\}_{n \geq 0}$ as a fundamental system of neighbourhoods of 0. This topology is called the I-adic topology; R is a pre-adic topological ring in the I-adic topology[4]. Moreover, the linear topology on M which has $\{I^n M\}_{n \geq 0}$ as a fundamental system of open neighbourhoods of 0 turns M into a topological R-module. This is called the I-adic topology on M. We see that M is I-adically complete (as defined in Algebra, Definition 10.96.2) if and only if

[2]We include being separated as part of being complete as we'd like to have a unique limits in complete groups. There is a definition of completeness for any topological group, agreeing, modulo the separation issue, with this one in our special case.

[3]By our conventions this includes separated.

[4]Thus the I-adic topology is sometimes called the I-pre-adic topology.

M is complete in the I-adic topology[5]. In particular, we see that R is I-adically complete if and only if R is an adic topological ring in the I-adic topology.

As a special case, note that the discrete topology is the 0-adic topology and that any ring in the discrete topology is adic.

07E9 Lemma 15.36.2. Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ and $J \subset S$ be ideals and endow R with the I-adic topology and S with the J-adic topology. Then φ is a homomorphism of topological rings if and only if $\varphi(I^n) \subset J$ for some $n \geq 1$.

Proof. Omitted. $\square$

0CQU Lemma 15.36.3 (Baire category theorem). Let M be a topological abelian group. Assume M is linearly topologized, complete, and has a countable fundamental system of neighbourhoods of 0. If $U_n \subset M$, $n \geq 1$ are open dense subsets, then $\bigcap_{n \geq 1} U_n$ is dense.

Proof. Let U_n be as in the statement of the lemma. After replacing U_n by $U_1 \cap \ldots \cap U_n$, we may assume that $U_1 \supset U_2 \supset \ldots$. Let M_n, $n \in \mathbf{N}$ be a fundamental system of neighbourhoods of 0. We may assume that $M_{n+1} \subset M_n$. Pick $x \in M$. We will show that for every $k \geq 1$ there exists a $y \in \bigcap_{n \geq 1} U_n$ with $x - y \in M_k$.

To construct y we argue as follows. First, we pick a $y_1 \in U_1$ with $y_1 \in x + M_k$. This is possible because U_1 is dense and $x + M_k$ is open. Then we pick a $k_1 > k$ such that $y_1 + M_{k_1} \subset U_1$. This is possible because U_1 is open. Next, we pick a $y_2 \in U_2$ with $y_2 \in y_1 + M_{k_1}$. This is possible because U_2 is dense and $y_2 + M_{k_1}$ is open. Then we pick a $k_2 > k_1$ such that $y_2 + M_{k_2} \subset U_2$. This is possible because U_2 is open.

Continuing in this fashion we get a converging sequence y_i of elements of M with limit y. By construction $x - y \in M_k$. Since

$$y - y_i = (y_{i+1} - y_i) + (y_{i+2} - y_{i+1}) + \ldots$$

is in M_{k_i} we see that $y \in y_i + M_{k_i} \subset U_i$ for all i as desired. $\square$

0CQV Lemma 15.36.4. With same assumptions as Lemma 15.36.3 if $M = \bigcup_{n \geq 1} N_n$ for some closed subgroups N_n, then N_n is open for some n.

Proof. If not, then $U_n = M \setminus N_n$ is dense for all n and we get a contradiction with Lemma 15.36.3. $\square$

0CQW Lemma 15.36.5 (Open mapping lemma). Let $u : N \to M$ be a continuous map of linearly topologized abelian groups. Assume that N is complete, M separated, and N has a countable fundamental system of neighbourhoods of 0. Then exactly one of the following holds

(1) u is open, or
(2) for some open subgroup $N' \subset N$ the image $u(N')$ is nowhere dense in M.

Proof. Let N_n, $n \in \mathbf{N}$ be a fundamental system of neighbourhoods of 0. We may assume that $N_{n+1} \subset N_n$. If (2) does not hold, then the closure M_n of $u(N_n)$ is an open subgroup for $n = 1, 2, 3, \ldots$. Since u is continuous, we see that M_n, $n \in \mathbf{N}$

[5] It may happen that the I-adic completion $M^\wedge$ is not I-adically complete, even though $M^\wedge$ is always complete with respect to the limit topology. If I is finitely generated then the I-adic topology and the limit topology on $M^\wedge$ agree, see Algebra, Lemma 10.96.3 and its proof.

must be a fundamental system of open neighbourhoods of 0 in M. Also, since M_n is the closure of $u(N_n)$ we see that

$$u(N_n) + M_{n+1} = M_n$$

for all $n \geq 1$. Pick $x_1 \in M_1$. Then we can inductively choose $y_i \in N_i$ and $x_{i+1} \in M_{i+1}$ such that

$$u(y_i) + x_{i+1} = x_i$$

The element $y = y_1 + y_2 + y_3 + \ldots$ of N exists because N is complete. Whereupon we see that $x = u(y)$ because M is separated. Thus $M_1 = u(N_1)$. In exactly the same way the reader shows that $M_i = u(N_i)$ for all $i \geq 2$ and we see that u is open. □

15.37. Formally smooth maps of topological rings

07EA There is a version of formal smoothness which applies to homomorphisms of topological rings.

07EB Definition 15.37.1. Let $R \to S$ be a homomorphism of topological rings with R and S linearly topologized. We say S is formally smooth over R if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of homomorphisms of topological rings where A is a discrete ring and $J \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

We will mostly use this notion when given ideals $\mathfrak m \subset R$ and $\mathfrak n \subset S$ and we endow R with the $\mathfrak m$-adic topology and S with the $\mathfrak n$-adic topology. Continuity of $\varphi : R \to S$ holds if and only if $\varphi(\mathfrak m^m) \subset \mathfrak n$ for some $m \geq 1$, see Lemma 15.36.2. It turns out that in this case only the topology on S is relevant.

07EC Lemma 15.37.2. Let $\varphi : R \to S$ be a ring map.

(1) If $R \to S$ is formally smooth in the sense of Algebra, Definition 10.138.1, then $R \to S$ is formally smooth for any linear topology on R and any pre-adic topology on S such that $R \to S$ is continuous.
(2) Let $\mathfrak n \subset S$ and $\mathfrak m \subset R$ ideals such that φ is continuous for the $\mathfrak m$-adic topology on R and the $\mathfrak n$-adic topology on S. Then the following are equivalent
    (a) φ is formally smooth for the $\mathfrak m$-adic topology on R and the $\mathfrak n$-adic topology on S, and
    (b) φ is formally smooth for the discrete topology on R and the $\mathfrak n$-adic topology on S.

Proof. Assume $R \to S$ is formally smooth in the sense of Algebra, Definition 10.138.1. If S has a pre-adic topology, then there exists an ideal $\mathfrak n \subset S$ such that S has the $\mathfrak n$-adic topology. Suppose given a solid commutative diagram as in Definition 15.37.1. Continuity of $S \to A/J$ means that $\mathfrak n^k$ maps to zero in A/J for some $k \geq 1$, see Lemma 15.36.2. We obtain a ring map $\psi : S \to A$ from the assumed formal smoothness of S over R. Then $\psi(\mathfrak n^k) \subset J$ hence $\psi(\mathfrak n^{2k}) = 0$ as $J^2 = 0$. Hence ψ is continuous by Lemma 15.36.2. This proves (1).

The proof of (2)(b) $\Rightarrow$ (2)(a) is the same as the proof of (1). Assume (2)(a). Suppose given a solid commutative diagram as in Definition 15.37.1 where we use the discrete topology on R. Since φ is continuous we see that $\varphi(\mathfrak{m}^n) \subset \mathfrak{n}$ for some $n \geq 1$. As $S \to A/J$ is continuous we see that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$. Hence $\mathfrak{m}^{nk}$ maps into J under the map $R \to A$. Thus $\mathfrak{m}^{2nk}$ maps to zero in A and we see that $R \to A$ is continuous in the $\mathfrak{m}$-adic topology. Thus (2)(a) gives a dotted arrow as desired. $\square$

07NI Definition 15.37.3. Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset S$ be an ideal. If the equivalent conditions (2)(a) and (2)(b) of Lemma 15.37.2 hold, then we say $R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology.

This property is inherited by the completions.

07ED Lemma 15.37.4. Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be rings endowed with finitely generated ideals. Endow R and S with the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies. Let $R \to S$ be a homomorphism of topological rings. The following are equivalent

(1) $R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology,
(2) $R \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology,
(3) $R^\wedge \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology.

Here $R^\wedge$ and $S^\wedge$ are the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic completions of R and S.

Proof. The assumption that $\mathfrak{m}$ is finitely generated implies that $R^\wedge$ is $\mathfrak{m}R^\wedge$-adically complete, that $\mathfrak{m}R^\wedge = \mathfrak{m}^\wedge$ and that $R^\wedge/\mathfrak{m}^n R^\wedge = R/\mathfrak{m}^n$, see Algebra, Lemma 10.96.3 and its proof. Similarly for $(S, \mathfrak{n})$. Thus it is clear that diagrams as in Definition 15.37.1 for the cases (1), (2), and (3) are in 1-to-1 correspondence. $\square$

The advantage of working with adic rings is that one gets a stronger lifting property.

07NJ Lemma 15.37.5. Let $R \to S$ be a ring map. Let $\mathfrak{n}$ be an ideal of S. Assume that $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology. Consider a solid commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed, "\psi"] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of homomorphisms of topological rings where A is adic and A/J is the quotient (as topological ring) of A by a closed ideal $J \subset A$ such that J^t is contained in an ideal of definition of A for some $t \geq 1$. Then there exists a dotted arrow in the category of topological rings which makes the diagram commute.

Proof. Let $I \subset A$ be an ideal of definition so that $I \supset J^t$ for some n. Then $A = \lim A/I^n$ and $A/J = \lim A/J + I^n$ because J is assumed closed. Consider the following diagram of discrete R algebras $A_{n,m} = A/J^n + I^m$:

$$\begin{tikzcd} A/J^3 + I^3 \arrow[r] \arrow[d] & A/J^2 + I^3 \arrow[r] \arrow[d] & A/J + I^3 \arrow[d] \\ A/J^3 + I^2 \arrow[r] \arrow[d] & A/J^2 + I^2 \arrow[r] \arrow[d] & A/J + I^2 \arrow[d] \\ A/J^3 + I \arrow[r] & A/J^2 + I \arrow[r] & A/J + I \end{tikzcd}$$

Note that each of the commutative squares defines a surjection

$$A_{n+1,m+1} \longrightarrow A_{n+1,m} \times_{A_{n,m}} A_{n,m+1}$$

of R-algebras whose kernel has square zero. We will inductively construct R-algebra maps $\varphi_{n,m} : S \to A_{n,m}$. Namely, we have the maps $\varphi_{1,m} = \psi \bmod J + I^m$. Note that each of these maps is continuous as ψ is. We can inductively choose the maps $\varphi_{n,1}$ by starting with our choice of $\varphi_{1,1}$ and lifting up, using the formal smoothness of S over R, along the right column of the diagram above. We construct the remaining maps $\varphi_{n,m}$ by induction on $n + m$. Namely, we choose $\varphi_{n+1,m+1}$ by lifting the pair $(\varphi_{n+1,m}, \varphi_{n,m+1})$ along the displayed surjection above (again using the formal smoothness of S over R). In this way all of the maps $\varphi_{n,m}$ are compatible with the transition maps of the system. As $J^t \subset I$ we see that for example $\varphi_n = \varphi_{nt,n} \bmod I^n$ induces a map $S \to A/I^n$. Taking the limit $\varphi = \lim \varphi_n$ we obtain a map $S \to A = \lim A/I^n$. The composition into A/J agrees with ψ as we have seen that $A/J = \lim A/J + I^n$. Finally we show that φ is continuous. Namely, we know that $\psi(\mathfrak{n}^r) \subset J + I/J$ for some $r \geq 1$ by our assumption that ψ is a morphism of topological rings, see Lemma 15.36.2. Hence $\varphi(\mathfrak{n}^r) \subset J + I$ hence $\varphi(\mathfrak{n}^{rt}) \subset I$ as desired. □

07EE Lemma 15.37.6. Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset \mathfrak{n}' \subset S$ be ideals. If $R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology, then $R \to S$ is formally smooth for the $\mathfrak{n}'$-adic topology.

Proof. Omitted. □

07EF Lemma 15.37.7. A composition of formally smooth continuous homomorphisms of linearly topologized rings is formally smooth.

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) □

07EG Lemma 15.37.8. Let R, S be rings. Let $\mathfrak{n} \subset S$ be an ideal. Let $R \to S$ be formally smooth for the $\mathfrak{n}$-adic topology. Let $R \to R'$ be any ring map. Then $R' \to S' = S \otimes_R R'$ is formally smooth in the $\mathfrak{n}' = \mathfrak{n}S'$-adic topology.

Proof. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rrd, dashed] & S' \arrow[r] \arrow[rd, dashed] & A/J \\ R \arrow[r] \arrow[u] & R' \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

as in Definition 15.37.1 be given. Then the composition $S \to S' \to A/J$ is continuous. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

We have seen descent for formal smoothness along faithfully flat ring maps in Algebra, Lemma 10.138.16. Something similar holds in the current setting of topological rings. However, here we just prove the following very simple and easy to prove version which is already quite useful.

07EH Lemma 15.37.9. Let R, S be rings. Let $\mathfrak{n} \subset S$ be an ideal. Let $R \to R'$ be a ring map. Set $S' = S \otimes_R R'$ and $\mathfrak{n}' = \mathfrak{n}S$. If

(1) the map $R \to R'$ embeds R as a direct summand of R' as an R-module, and
(2) $R' \to S'$ is formally smooth for the $\mathfrak{n}'$-adic topology,

then $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology.

Proof. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 15.37.1 be given. Set $A' = A \otimes_R R'$ and $J' = \mathrm{Im}(J \otimes_R R' \to A')$. The base change of the diagram above is the diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[rd, dashed, "\psi'"] & A'/J' \\ R' \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

with continuous arrows. By condition (2) we obtain the dotted arrow $\psi' : S' \to A'$. Using condition (1) choose a direct summand decomposition $R' = R \oplus C$ as R-modules. (Warning: C isn't an ideal in R'.) Then $A' = A \oplus A \otimes_R C$. Set

$$J'' = \mathrm{Im}(J \otimes_R C \to A \otimes_R C) \subset J' \subset A'.$$

Then $J' = J \oplus J''$ as A-modules. The image of the composition $\psi : S \to A'$ of ψ' with $S \to S'$ is contained in $A + J' = A \oplus J''$. However, in the ring $A + J' = A \oplus J''$ the A-submodule J'' is an ideal! (Use that $J^2 = 0$.) Hence the composition $S \to A + J' \to (A + J')/J'' = A$ is the arrow we were looking for. $\square$

15.38. Formally smooth maps of local rings

0DYF In the case of a local homomorphism of local rings one can limit the diagrams for which the lifting property has to be checked. Please compare with Algebra, Lemma 10.141.2.

0DYG Lemma 15.38.1. Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local rings. The following are equivalent

(1) $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology,
(2) for every solid commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of local homomorphisms of local rings where $J \subset A$ is an ideal of square zero, $\mathfrak{m}_A^n = 0$ for some $n > 0$, and $S \to A/J$ induces an isomorphism on residue fields, a dotted arrow exists which makes the diagram commute.

If S is Noetherian these conditions are also equivalent to

(3) same as in (2) but only for diagrams where in addition $A \to A/J$ is a small extension (Algebra, Definition 10.141.1).

Proof. The implication (1) $\Rightarrow$ (2) follows from the definitions. Consider a diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 15.37.1 for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S. Pick $m > 0$ with $\mathfrak{n}^m(A/J) = 0$ (possible by continuity of maps in diagram). Consider the subring A' of A which is the inverse image of the image of S in A/J. Set $J' = J$ viewed as an ideal in A'. Then J' is an ideal of square zero in A' and A'/J' is a quotient of $S/\mathfrak{n}^m$. Hence A' is local and $\mathfrak{m}_{A'}^{2m} = 0$. Thus we get a diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A'/J' \\ R \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

as in (2). If we can construct the dotted arrow in this diagram, then we obtain the dotted arrow in the original one by composing with $A' \to A$. In this way we see that (2) implies (1).

Assume S Noetherian. The implication (1) $\Rightarrow$ (3) is immediate. Assume (3) and suppose a diagram as in (2) is given. Then $\mathfrak{m}_A^n J = 0$ for some $n > 0$. Considering the maps

$$A \to A/\mathfrak{m}_A^{n-1}J \to \ldots \to A/\mathfrak{m}J \to A/J$$

we see that it suffices to produce the lifting if $\mathfrak{m}_A J = 0$. Assume $\mathfrak{m}_A J = 0$ and let $A' \subset A$ be the ring constructed above. Then A'/J' is Artinian as a quotient of the Artinian local ring $S/\mathfrak{n}^m$. Thus it suffices to show that given property (3) we can find the dotted arrow in diagrams as in (2) with A/J Artinian and $\mathfrak{m}_A J = 0$. Let κ be the common residue field of A, A/J, and S. By (3), if $J_0 \subset J$ is an ideal with $\dim_\kappa(J/J_0) = 1$, then we can produce a dotted arrow $S \to A/J_0$. Taking the product we obtain

$$S \longrightarrow \prod\nolimits_{J_0 \text{ as above}} A/J_0$$

Clearly the image of this arrow is contained in the sub R-algebra A' of elements which map into the small diagonal $A/J \subset \prod_{J_0} A/J$. Let $J' \subset A'$ be the elements mapping to zero in A/J. Then J' is an ideal of square zero and as κ-vector space equal to

$$J' = \prod\nolimits_{J_0 \text{ as above}} J/J_0$$

Thus the map $J \to J'$ is injective. By the theory of vector spaces we can choose a splitting $J' = J \oplus M$. It follows that

$$A' = A \oplus M$$

as an R-algebra. Hence the map $S \to A'$ can be composed with the projection $A' \to A$ to give the desired dotted arrow thereby finishing the proof of the lemma. $\square$

The following lemma will be improved on in Section 15.40.

07EI Lemma 15.38.2. Let k be a field and let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology, then A is a regular local ring.

Proof. Let $k_0 \subset k$ be the prime field. Then k_0 is perfect, hence k/k_0 is separable, hence formally smooth by Algebra, Lemma 10.158.7. By Lemmas 15.37.2 and 15.37.7 we see that $k_0 \to A$ is formally smooth for the $\mathfrak{m}$-adic topology on A. Hence we may assume $k = \mathbf{Q}$ or $k = \mathbf{F}_p$.

By Algebra, Lemmas 10.97.3 and 10.110.9 it suffices to prove the completion $A^\wedge$ is regular. By Lemma 15.37.4 we may replace A by $A^\wedge$. Thus we may assume that A is a Noetherian complete local ring. By the Cohen structure theorem (Algebra, Theorem 10.160.8) there exist a map $K \to A$. As k is the prime field we see that $K \to A$ is a k-algebra map.

Let $x_1, \ldots, x_n \in \mathfrak{m}$ be elements whose images form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Set $T = K[[X_1, \ldots, X_n]]$. Note that

$$A/\mathfrak{m}^2 \cong K[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/\mathfrak{m}_T^2 \cong K[X_1, \ldots, X_n]/(X_i X_j).$$

Let $A/\mathfrak{m}^2 \to T/m_T^2$ be the local K-algebra isomorphism given by mapping the class of x_i to the class of X_i. Denote $f_1 : A \to T/\mathfrak{m}_T^2$ the composition of this isomorphism with the quotient map $A \to A/\mathfrak{m}^2$. The assumption that $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology means we can lift f_1 to a map $f_2 : A \to T/\mathfrak{m}_T^3$, then to a map $f_3 : A \to T/\mathfrak{m}_T^4$, and so on, for all $n \geq 1$. Warning: the maps f_n are continuous k-algebra maps and may not be K-algebra maps. We get an induced map $f : A \to T = \lim T/\mathfrak{m}_T^n$ of local k-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_T/\mathfrak{m}_T^2$ hence each f_n is surjective and we conclude f is surjective as A is complete. This implies $\dim(A) \geq \dim(T) = n$. Hence A is regular by definition. (It also follows that f is an isomorphism.) □

0C34 Lemma 15.38.3. Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a complete local k-algebra. If κ/k is separable, then there exists a k-algebra map $\kappa \to A$ such that $\kappa \to A \to \kappa$ is id_κ.

Proof. By Algebra, Proposition 10.158.9 the extension κ/k is formally smooth. By Lemma 15.37.2 $k \to \kappa$ is formally smooth in the sense of Definition 15.37.1. Then we get $\kappa \to A$ from Lemma 15.37.5. □

0C35 Lemma 15.38.4. Let k be a field. Let $(A, \mathfrak{m}, \kappa)$ be a complete local k-algebra. If κ/k is separable and A regular, then there exists an isomorphism of $A \cong \kappa[[t_1, \ldots, t_d]]$ as k-algebras.

Proof. Choose $\kappa \to A$ as in Lemma 15.38.3 and apply Algebra, Lemma 10.160.10. □

The following result will be improved on in Section 15.40

07EJ Lemma 15.38.5. Let k be a field. Let $(A, \mathfrak{m}, K)$ be a regular local k-algebra such that K/k is separable. Then $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.

Proof. It suffices to prove that the completion of A is formally smooth over k, see Lemma 15.37.4. Hence we may assume that A is a complete local regular k-algebra with residue field K separable over k. By Lemma 15.38.4 we see that $A = K[[x_1, \ldots, x_n]]$.

The power series ring $K[[x_1, \ldots, x_n]]$ is formally smooth over k. Namely, K is formally smooth over k and $K[x_1, \ldots, x_n]$ is formally smooth over K as a polynomial algebra. Hence $K[x_1, \ldots, x_n]$ is formally smooth over k by Algebra, Lemma 10.138.3. It follows that $k \to K[x_1, \ldots, x_n]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 15.37.2. Finally, it follows that $k \to K[[x_1, \ldots, x_n]]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 15.37.4. □

07VH Lemma 15.38.6. Let $A \to B$ be a finite type ring map with A Noetherian. Let $\mathfrak{q} \subset B$ be a prime ideal lying over $\mathfrak{p} \subset A$. The following are equivalent

(1) $A \to B$ is smooth at $\mathfrak{q}$, and
(2) $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology.

Proof. The implication (2) $\Rightarrow$ (1) follows from Algebra, Lemma 10.141.2. Conversely, if $A \to B$ is smooth at $\mathfrak{q}$, then $A \to B_g$ is smooth for some $g \in B$, $g \notin \mathfrak{q}$. Then $A \to B_g$ is formally smooth by Algebra, Proposition 10.138.13. Hence $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth as localization preserves formal smoothness (for example by the criterion of Algebra, Proposition 10.138.8 and the fact that the cotangent complex behaves well with respect to localization, see Algebra, Lemmas 10.134.11 and 10.134.13). Finally, Lemma 15.37.2 implies that $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology. □

15.39. Some results on power series rings

07NK Questions on formally smooth maps between Noetherian local rings can often be reduced to questions on maps between power series rings. In this section we prove some helper lemmas to facilitate this kind of argument.

07NL Lemma 15.39.1. Let K be a field of characteristic 0 and $A = K[[x_1, \ldots, x_n]]$. Let L be a field of characteristic $p > 0$ and $B = L[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring. Let $C = \Lambda[[x_1, \ldots, x_n]]$.

(1) $\mathbf{Q} \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.
(2) $\mathbf{F}_p \to B$ is formally smooth in the $\mathfrak{m}$-adic topology.
(3) $\mathbf{Z} \to C$ is formally smooth in the $\mathfrak{m}$-adic topology.

Proof. By the universal property of power series rings it suffices to prove:

(1) $\mathbf{Q} \to K$ is formally smooth.
(2) $\mathbf{F}_p \to L$ is formally smooth.
(3) $\mathbf{Z} \to \Lambda$ is formally smooth in the $\mathfrak{m}$-adic topology.

The first two are Algebra, Proposition 10.158.9. The third follows from Algebra, Lemma 10.160.7 since for any test diagram as in Definition 15.37.1 some power of p will be zero in A/J and hence some power of p will be zero in A. □

07NM Lemma 15.39.2. Let K be a field and $A = K[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring and let $B = \Lambda[[x_1, \ldots, x_n]]$.

(1) If $y_1, \ldots, y_n \in A$ is a regular system of parameters then $K[[y_1, \ldots, y_n]] \to A$ is an isomorphism.
(2) If $z_1, \ldots, z_r \in A$ form part of a regular system of parameters for A, then $r \leq n$ and $A/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$.
(3) If $p, y_1, \ldots, y_n \in B$ is a regular system of parameters then $\Lambda[[y_1, \ldots, y_n]] \to B$ is an isomorphism.

(4) If $p, z_1, \ldots, z_r \in B$ form part of a regular system of parameters for B, then $r \leq n$ and $B/(z_1, \ldots, z_r) \cong \Lambda[[y_1, \ldots, y_{n-r}]]$.

Proof. Proof of (1). Set $A' = K[[y_1, \ldots, y_n]]$. It is clear that the map $A' \to A$ induces an isomorphism $A'/\mathfrak{m}_{A'}^n \to A/\mathfrak{m}_A^n$ for all $n \geq 1$. Since A and A' are both complete we deduce that $A' \to A$ is an isomorphism. Proof of (2). Extend $z_1, \ldots, z_r$ to a regular system of parameters $z_1, \ldots, z_r, y_1, \ldots, y_{n-r}$ of A. Consider the map $A' = K[[z_1, \ldots, z_r, y_1, \ldots, y_{n-r}]] \to A$. This is an isomorphism by (1). Hence (2) follows as it is clear that $A'/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$. The proofs of (3) and (4) are exactly the same as the proofs of (1) and (2). □

07NN Lemma 15.39.3. Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Then there exists a commutative diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with the following properties:

(1) the horizontal arrows are surjective,
(2) if the characteristic of $A/\mathfrak{m}_A$ is zero, then S and R are power series rings over fields,
(3) if the characteristic of $A/\mathfrak{m}_A$ is $p > 0$, then S and R are power series rings over Cohen rings, and
(4) $R \to S$ maps a regular system of parameters of R to part of a regular system of parameters of S.

In particular $R \to S$ is flat (see Algebra, Lemma 10.128.2) with regular fibre $S/\mathfrak{m}_R S$ (see Algebra, Lemma 10.106.3).

Proof. Use the Cohen structure theorem (Algebra, Theorem 10.160.8) to choose a surjection $S \to B$ as in the statement of the lemma where we choose S to be a power series over a Cohen ring if the residue characteristic is $p > 0$ and a power series over a field else. Let $J \subset S$ be the kernel of $S \to B$. Next, choose a surjection $R = \Lambda[[x_1, \ldots, x_n]] \to A$ where we choose Λ to be a Cohen ring if the residue characteristic of A is $p > 0$ and Λ equal to the residue field of A otherwise. We lift the composition $\Lambda[[x_1, \ldots, x_n]] \to A \to B$ to a map $\varphi : R \to S$. This is possible because $\Lambda[[x_1, \ldots, x_n]]$ is formally smooth over $\mathbf{Z}$ in the $\mathfrak{m}$-adic topology (see Lemma 15.39.1) by an application of Lemma 15.37.5. Finally, we replace φ by the map $\varphi' : R = \Lambda[[x_1, \ldots, x_n]] \to S' = S[[y_1, \ldots, y_n]]$ with $\varphi'|_\Lambda = \varphi|_\Lambda$ and $\varphi'(x_i) = \varphi(x_i) + y_i$. We also replace $S \to B$ by the map $S' \to B$ which maps y_i to zero. After this replacement it is clear that a regular system of parameters of R maps to part of a regular sequence in S' and we win. □

There should be an elementary proof of the following lemma.

09Q8 Lemma 15.39.4. Let $S \to R$ and $S' \to R$ be surjective maps of complete Noetherian local rings. Then $S \times_R S'$ is a complete Noetherian local ring.

Proof. Let k be the residue field of R. If the characteristic of k is $p > 0$, then we denote Λ a Cohen ring (Algebra, Definition 10.160.5) with residue field k (Algebra, Lemma 10.160.6). If the characteristic of k is 0 we set $\Lambda = k$. Choose a surjection

$\Lambda[[x_1, \ldots, x_n]] \to R$ (as in the Cohen structure theorem, see Algebra, Theorem 10.160.8) and lift this to maps $\Lambda[[x_1, \ldots, x_n]] \to S$ and $\varphi : \Lambda[[x_1, \ldots, x_n]] \to S$ and $\varphi' : \Lambda[[x_1, \ldots, x_n]] \to S'$ using Lemmas 15.39.1 and 15.37.5. Next, choose $f_1, \ldots, f_m \in S$ generating the kernel of $S \to R$ and $f'_1, \ldots, f'_{m'} \in S'$ generating the kernel of $S' \to R$. Then the map

$$\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_{m'}]] \longrightarrow S \times_R S,$$

which sends x_i to $(\varphi(x_i), \varphi'(x_i))$ and y_j to $(f_j, 0)$ and $z_{j'}$ to $(0, f'_j)$ is surjective. Thus $S \times_R S'$ is a quotient of a complete local ring, whence complete. □

15.40. Geometric regularity and formal smoothness

07EK In this section we combine the results of the previous sections to prove the following characterization of geometrically regular local rings over fields. We then recycle some of our arguments to prove a characterization of formally smooth maps in the $\mathfrak{m}$-adic topology between Noetherian local rings.

07EL **Theorem 15.40.1.** Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If the characteristic of k is zero then the following are equivalent

(1) A is a regular local ring, and
(2) $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.

If the characteristic of k is $p > 0$ then the following are equivalent

(1) A is geometrically regular over k,
(2) $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.
(3) for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,
(4) A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and
(5) A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.

Proof. If the characteristic of k is zero, then the equivalence of (1) and (2) follows from Lemmas 15.38.2 and 15.38.5.

If the characteristic of k is $p > 0$, then it follows from Proposition 15.35.1 that (1), (3), (4), and (5) are equivalent. Assume (2) holds. By Lemma 15.37.8 we see that $k' \to A' = A \otimes_k k'$ is formally smooth for the $\mathfrak{m}' = \mathfrak{m}A'$-adic topology. Hence if $k \subset k'$ is finite purely inseparable, then A' is a regular local ring by Lemma 15.38.2. Thus we see that (1) holds.

Finally, we will prove that (5) implies (2). Choose a solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed, "\bar\psi"] & B/J \\ k \arrow[u, "i"] \arrow[r, "\varphi"] & B \arrow[u, "\pi"'] \end{tikzcd}$$

as in Definition 15.37.1. As $J^2 = 0$ we see that J has a canonical B/J module structure and via $\bar\psi$ an A-module structure. As $\bar\psi$ is continuous for the $\mathfrak{m}$-adic topology we see that $\mathfrak{m}^n J = 0$ for some n. Hence we can filter J by B/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}$. Considering the sequence of ring maps $B \to B/J_1 \to B/J_2 \to \ldots \to B/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}$. By Lemma 15.38.5 we see that $\mathbf{F}_p \to A$ is formally smooth in the $\mathfrak{m}$-adic topology. Hence we can find a ring map $\psi : A \to B$ such that $\pi \circ \psi = \bar{\psi}$. Then $\psi \circ i, \varphi : k \to B$ are two maps whose compositions with π are equal. Hence $D = \psi \circ i - \varphi : k \to J$ is a derivation. By Algebra, Lemma 10.131.3 we can write $D = \xi \circ \mathrm{d}$ for some k-linear map $\xi : \Omega_{k/\mathbf{F}_p} \to J$. Using the K-vector space structure on J we extend ξ to a K-linear map $\xi' : \Omega_{k/\mathbf{F}_p} \otimes_k K \to J$. Using (5) we can find a K-linear map $\xi'' : \Omega_{A/\mathbf{F}_p} \otimes_A K$ whose restriction to $\Omega_{k/\mathbf{F}_p} \otimes_k K$ is ξ'. Write

$$D' : A \xrightarrow{\mathrm{d}} \Omega_{A/\mathbf{F}_p} \to \Omega_{A/\mathbf{F}_p} \otimes_A K \xrightarrow{\xi''} J.$$

Finally, set $\psi' = \psi - D' : A \to B$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar{\psi}$ and such that $\psi' \circ i = \varphi$ as desired. □

07EM Example 15.40.2. Let k be a field of characteristic $p > 0$. Suppose that $a \in k$ is an element which is not a pth power. A standard example of a geometrically regular local k-algebra whose residue field is purely inseparable over k is the ring

$$A = k[x, y]_{(x, y^p - a)}/(y^p - a - x)$$

Namely, A is a localization of a smooth algebra over k hence $k \to A$ is formally smooth, hence $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology. A closely related example is the following. Let $k = \mathbf{F}_p(s)$ and $K = \mathbf{F}_p(t)^{perf}$. We claim the ring map

$$k \longrightarrow A = K[[x]], \quad s \longmapsto t + x$$

is formally smooth for the (x)-adic topology on A. Namely, $\Omega_{k/\mathbf{F}_p}$ is 1-dimensional with basis $\mathrm{d}s$. It maps to the element $\mathrm{d}x + \mathrm{d}t = \mathrm{d}x$ in $\Omega_{A/\mathbf{F}_p}$. We leave it to the reader to show that $\Omega_{A/\mathbf{F}_p}$ is free on $\mathrm{d}x$ as an A-module. Hence we see that condition (5) of Theorem 15.40.1 holds and we conclude that $k \to A$ is formally smooth in the (x)-adic topology.

07NP Lemma 15.40.3. Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then $A \to B$ is flat.

Proof. We may assume that A and B a Noetherian complete local rings by Lemma 15.37.4 and Algebra, Lemma 10.97.6 (this also uses Algebra, Lemma 10.39.9 and 10.97.3 to see that flatness of the map on completions implies flatness of $A \to B$). Choose a commutative diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 15.39.3 with $R \to S$ flat. Let $I \subset R$ be the kernel of $R \to A$. Because B is formally smooth over A we see that the A-algebra map

$$S/IS \longrightarrow B$$

has a section, see Lemma 15.37.5. Hence B is a direct summand of the flat A-module S/IS (by base change of flatness, see Algebra, Lemma 10.39.7), whence flat. □

0DYH Lemma 15.40.4. Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Let K be the residue field of B. Then the Jacobi-Zariski sequence for $A \to B \to K$ gives an exact sequence

$$0 \to H_1(NL_{K/A}) \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to \Omega_{B/A} \otimes_B K \to \Omega_{K/A} \to 0$$

Proof. Observe that $\mathfrak{m}_B/\mathfrak{m}_B^2 = H_1(NL_{K/B})$ by Algebra, Lemma 10.134.6. By Algebra, Lemma 10.134.4 it remains to show injectivity of $H_1(NL_{K/A}) \to \mathfrak{m}_B/\mathfrak{m}_B^2$. With k the residue field of A, the Jacobi-Zariski sequence for $A \to k \to K$ gives $\Omega_{K/A} = \Omega_{K/k}$ and an exact sequence

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \to H_1(NL_{K/A}) \to H_1(NL_{K/k}) \to 0$$

Set $\overline{B} = B \otimes_A k$. Since $\overline{B}$ is regular the ideal $\mathfrak{m}_{\overline{B}}$ is generated by a regular sequence. Applying Lemmas 15.30.9 and 15.30.7 to $\mathfrak{m}_A B \subset \mathfrak{m}_B$ we find $\mathfrak{m}_A B/(\mathfrak{m}_A B \cap \mathfrak{m}_B^2) = \mathfrak{m}_A B/\mathfrak{m}_A \mathfrak{m}_B$ which is equal to $\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K$ as $A \to B$ is flat by Lemma 15.40.3. Thus we obtain a short exact sequence

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 \to 0$$

Functoriality of the Jacobi-Zariski sequences shows that we obtain a commutative diagram

$$\begin{tikzcd} & \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \arrow[r] \arrow[d] & H_1(NL_{K/A}) \arrow[r] \arrow[d] & H_1(NL_{K/k}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_k K \arrow[r] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r] & \mathfrak{m}_{\overline{B}}/\mathfrak{m}_{\overline{B}}^2 \arrow[r] & 0 \end{tikzcd}$$

The left vertical arrow is injective by Theorem 15.40.1 as $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology by Lemma 15.37.8. This finishes the proof by the snake lemma. $\square$

07NQ Proposition 15.40.5. Let $A \to B$ be a local homomorphism of Noetherian local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent

(1) $A \to B$ is flat and $\overline{B}$ is geometrically regular over k,
(2) $A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and
(3) $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.

Proof. The equivalence of (1) and (2) follows from Theorem 15.40.1.

Assume (3). By Lemma 15.40.3 we see that $A \to B$ is flat. By Lemma 15.37.8 we see that $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology. Thus (2) holds.

Assume (2). Lemma 15.37.4 tells us formal smoothness is preserved under completion. The same is true for flatness by Algebra, Lemma 10.97.3. Hence we may replace A and B by their respective completions and assume that A and B are Noetherian complete local rings. In this case choose a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

as in Lemma 15.39.3. We will use all of the properties of this diagram without further mention. Fix a regular system of parameters $t_1, \ldots, t_d$ of R with $t_1 = p$ in case the characteristic of k is $p > 0$. Set $\overline{S} = S \otimes_R k$. Consider the short exact sequence

$$0 \to J \to S \to B \to 0$$

As $\overline{B}$ and $\overline{S}$ are regular, the kernel of $\overline{S} \to \overline{B}$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 10.106.4. Lift these elements to $x_1, \ldots, x_r \in J$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 15.39.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 15.39.3 with moreover $\overline{S} = \overline{B}$. In this case the map

$$S \otimes_R A \longrightarrow B$$

is an isomorphism as it is surjective, an isomorphism on special fibres, and source and target are flat over A (for example use Algebra, Lemma 10.99.1 or use that tensoring the short exact sequence $0 \to I \to S \otimes_R A \to B \to 0$ over A with k we find $I \otimes_A k = 0$ hence $I = 0$ by Nakayama). Thus by Lemma 15.37.8 it suffices to show that $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology. Of course, since $\overline{S} = \overline{B}$, we have that $\overline{S}$ is formally smooth over $k = R/\mathfrak{m}_R$.

Choose elements $y_1, \ldots, y_m \in S$ such that $t_1, \ldots, t_d, y_1, \ldots, y_m$ is a regular system of parameters for S. If the characteristic of k is zero, choose a coefficient field $K \subset S$ and if the characteristic of k is $p > 0$ choose a Cohen ring $\Lambda \subset S$ with residue field K. At this point the map $K[[t_1, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic zero case) or $\Lambda[[t_2, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic $p > 0$ case) is an isomorphism, see Lemma 15.39.2. From now on we think of S as the above power series ring.

The rest of the proof is analogous to the argument in the proof of Theorem 15.40.1. Choose a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed, "\bar{\psi}"] & N/J \\ R \arrow[u, "i"] \arrow[r, "\varphi"] & N \arrow[u, "\pi"'] \end{tikzcd}$$

as in Definition 15.37.1. As $J^2 = 0$ we see that J has a canonical N/J module structure and via $\bar{\psi}$ a S-module structure. As $\bar{\psi}$ is continuous for the $\mathfrak{m}_S$-adic topology we see that $\mathfrak{m}_S^n J = 0$ for some n. Hence we can filter J by N/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}_S$. Considering the sequence of ring maps $N \to N/J_1 \to N/J_2 \to \ldots \to N/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}_S$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}_S$. As $\mathbf{Q} \to S$ (characteristic zero case) or $\mathbf{Z} \to S$ (characteristic $p > 0$ case) is formally smooth in the $\mathfrak{m}_S$-adic topology (see Lemma 15.39.1), we can find a ring map $\psi : S \to N$ such that $\pi \circ \psi = \bar{\psi}$. Since S is a power series ring in $t_1, \ldots, t_d$ (characteristic zero) or $t_2, \ldots, t_d$ (characteristic $p > 0$) over a subring, it follows from the universal property of power series rings that we can change our choice of ψ so that $\psi(t_i)$ equals $\varphi(t_i)$ (automatic for $t_1 = p$ in the characteristic p case). Then $\psi \circ i$ and $\varphi : R \to N$ are two maps whose compositions with π are equal and which agree on $t_1, \ldots, t_d$. Hence $D = \psi \circ i - \varphi : R \to J$ is a derivation which annihilates $t_1, \ldots, t_d$. By Algebra, Lemma 10.131.3 we can write $D = \xi \circ \mathrm{d}$ for some R-linear map $\xi : \Omega_{R/\mathbf{Z}} \to J$ which annihilates $\mathrm{d}t_1, \ldots, \mathrm{d}t_d$ (by construction) and $\mathfrak{m}_R \Omega_{R/\mathbf{Z}}$ (as J is annihilated by $\mathfrak{m}_R$). Hence ξ factors as a composition

$$\Omega_{R/\mathbf{Z}} \to \Omega_{k/\mathbf{Z}} \xrightarrow{\xi'} J$$

where ξ' is k-linear. Using the K-vector space structure on J we extend ξ' to a K-linear map

$$\xi'' : \Omega_{k/\mathbf{Z}} \otimes_k K \longrightarrow J.$$

Using that $\overline{S}/k$ is formally smooth we see that

$$\Omega_{k/\mathbf{Z}} \otimes_k K \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K$$

is injective by Theorem 15.40.1 (this is true also in the characteristic zero case as it is even true that $\Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective in characteristic zero, see Algebra, Proposition 10.158.9). Hence we can find a K-linear map $\xi''' : \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \to J$ whose restriction to $\Omega_{k/\mathbf{Z}} \otimes_k K$ is ξ''. Write

$$D' : S \xrightarrow{\mathrm{d}} \Omega_{S/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \xrightarrow{\xi'''} J.$$

Finally, set $\psi' = \psi - D' : S \to N$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar{\psi}$ and such that $\psi' \circ i = \varphi$ as desired. $\square$

As an application of the result above we prove that deformations of formally smooth algebras are unobstructed.

07NR Lemma 15.40.6. Let A be a Noetherian complete local ring with residue field k. Let B be a Noetherian complete local k-algebra. Assume $k \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists a Noetherian complete local ring C and a local homomorphism $A \to C$ which is formally smooth in the $\mathfrak{m}_C$-adic topology such that $C \otimes_A k \cong B$.

Proof. Choose a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 15.39.3. Let $t_1, \ldots, t_d$ be a regular system of parameters for R with $t_1 = p$ in case the characteristic of k is $p > 0$. As B and $\overline{S} = S \otimes_R k$ are regular we see that $\mathrm{Ker}(\overline{S} \to B)$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 10.106.4. Lift these elements to $x_1, \ldots, x_r \in S$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 15.39.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Lemma 15.39.3 with moreover $\overline{S} = B$. In this case $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology by Proposition 15.40.5. Hence the base change $C = S \otimes_R A$ is formally smooth over A in the $\mathfrak{m}_C$-adic topology by Lemma 15.37.8. $\square$

07NS Remark 15.40.7. The assertion of Lemma 15.40.6 is quite strong. Namely, suppose that we have a diagram

$$\begin{tikzcd} & B \\ A \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of local homomorphisms of Noetherian complete local rings where $A \to A'$ induces an isomorphism of residue fields $k = A/\mathfrak{m}_A = A'/\mathfrak{m}_{A'}$ and with $B \otimes_{A'} k$ formally smooth over k. Then we can extend this to a commutative diagram

$$\begin{tikzcd} C \arrow[r] & B \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of local homomorphisms of Noetherian complete local rings where $A \to C$ is formally smooth in the $\mathfrak{m}_C$-adic topology and where $C \otimes_A k \cong B \otimes_{A'} k$. Namely, pick $A \to C$ as in Lemma 15.40.6 lifting $B \otimes_{A'} k$ over k. By formal smoothness we can find the arrow $C \to B$, see Lemma 15.37.5. Denote $C \otimes_A^\wedge A'$ the completion of $C \otimes_A A'$ with respect to the ideal $C \otimes_A \mathfrak{m}_{A'}$. Note that $C \otimes_A^\wedge A'$ is a Noetherian complete local ring (see Algebra, Lemma 10.97.5) which is flat over A' (see Algebra, Lemma 10.99.11). We have moreover

(1) $C \otimes_A^\wedge A' \to B$ is surjective,
(2) if $A \to A'$ is surjective, then $C \to B$ is surjective,
(3) if $A \to A'$ is finite, then $C \to B$ is finite, and
(4) if $A' \to B$ is flat, then $C \otimes_A^\wedge A' \cong B$.

Namely, by Nakayama's lemma for nilpotent ideals (see Algebra, Lemma 10.20.1) we see that $C \otimes_A k \cong B \otimes_{A'} k$ implies that $C \otimes_A A'/\mathfrak{m}_{A'}^n \to B/\mathfrak{m}_{A'}^n B$ is surjective for all n. This proves (1). Parts (2) and (3) follow from part (1). Part (4) follows from Algebra, Lemma 10.99.1.

15.41. Regular ring maps

07BY Let k be a field. Recall that a Noetherian k-algebra A is said to be geometrically regular over k if and only if $A \otimes_k k'$ is regular for all finite purely inseparable extensions k' of k, see Algebra, Definition 10.166.2. Moreover, if this is the case then $A \otimes_k k'$ is regular for every finitely generated field extension k'/k, see Algebra, Lemma 10.166.1. We use this notion in the following definition.

07BZ Definition 15.41.1. A ring map $R \to \Lambda$ is regular if it is flat and for every prime $\mathfrak{p} \subset R$ the fibre ring

$$\Lambda \otimes_R \kappa(\mathfrak{p}) = \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}$$

is Noetherian and geometrically regular over $\kappa(\mathfrak{p})$.

If $R \to \Lambda$ is a ring map with Λ Noetherian, then the fibre rings are always Noetherian.

07C0 Lemma 15.41.2 (Regular is a local property). Let $R \to \Lambda$ be a ring map with Λ Noetherian. The following are equivalent

(1) $R \to \Lambda$ is regular,
(2) $R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is regular for all $\mathfrak{q} \subset \Lambda$ lying over $\mathfrak{p} \subset R$, and
(3) $R_\mathfrak{m} \to \Lambda_{\mathfrak{m}'}$ is regular for all maximal ideals $\mathfrak{m}' \subset \Lambda$ lying over $\mathfrak{m}$ in R.

Proof. This is true because a Noetherian ring is regular if and only if all the local rings are regular local rings, see Algebra, Definition 10.110.7 and a ring map is flat if and only if all the induced maps of local rings are flat, see Algebra, Lemma 10.39.18. $\square$

07C1 Lemma 15.41.3 (Regular maps and base change). Let $R \to \Lambda$ be a regular ring map. For any finite type ring map $R \to R'$ the base change $R' \to \Lambda \otimes_R R'$ is regular too.

Proof. Flatness is preserved under any base change, see Algebra, Lemma 10.39.7. Consider a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$. The residue field extension $\kappa(\mathfrak{p}')/\kappa(\mathfrak{p})$ is finitely generated as R' is of finite type over R. Hence the fibre ring

$$(\Lambda \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = \Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is Noetherian by Algebra, Lemma 10.31.8 and the assumption on the fibre rings of $R \to \Lambda$. Geometric regularity of the fibres is preserved by Algebra, Lemma 10.166.1. $\square$

07QI Lemma 15.41.4 (Composition of regular maps). Let $A \to B$ and $B \to C$ be regular ring maps. If the fibre rings of $A \to C$ are Noetherian, then $A \to C$ is regular.

Proof. Let $\mathfrak{p} \subset A$ be a prime. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. We have to show that $C \otimes_A k$ is regular. By Lemma 15.41.3 we may assume that $A = k$ and we reduce to proving that C is regular. The assumption is that B is regular and that $B \to C$ is flat with regular fibres. Then C is regular by Algebra, Lemma 10.112.8. Some details omitted. $\square$

07EP Lemma 15.41.5. Let R be a ring. Let $(A_i, \varphi_{ii'})$ be a directed system of smooth R-algebras. Set $\Lambda = \mathrm{colim}\, A_i$. If the fibre rings $\Lambda \otimes_R \kappa(\mathfrak{p})$ are Noetherian for all $\mathfrak{p} \subset R$, then $R \to \Lambda$ is regular.

Proof. Note that Λ is flat over R by Algebra, Lemmas 10.39.3 and 10.137.10. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. Note that

$$\Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} k = \Lambda \otimes_R k = \mathrm{colim}\, A_i \otimes_R k$$

is a colimit of smooth k-algebras, see Algebra, Lemma 10.137.4. Since each local ring of a smooth k-algebra is regular by Algebra, Lemma 10.140.3 we conclude that all local rings of $\Lambda \otimes_R k$ are regular by Algebra, Lemma 10.106.8. This proves the lemma. □

Let's see when a field extension defines a regular ring map.

07EQ Lemma 15.41.6. Let K/k be a field extension. Then $k \to K$ is a regular ring map if and only if K is a separable field extension of k.

Proof. If $k \to K$ is regular, then K is geometrically reduced over k, hence K is separable over k by Algebra, Proposition 10.158.9. Conversely, if K/k is separable, then K is a colimit of smooth k-algebras, see Algebra, Lemma 10.158.11 hence is regular by Lemma 15.41.5. □

07NT Lemma 15.41.7. Let $A \to B \to C$ be ring maps. If $A \to C$ is regular and $B \to C$ is flat and surjective on spectra, then $A \to B$ is regular.

Proof. By Algebra, Lemma 10.39.10 we see that $A \to B$ is flat. Let $\mathfrak p \subset A$ be a prime. The ring map $B \otimes_A \kappa(\mathfrak p) \to C \otimes_A \kappa(\mathfrak p)$ is flat and surjective on spectra. Hence $B \otimes_A \kappa(\mathfrak p)$ is geometrically regular by Algebra, Lemma 10.166.3. □

15.42. Ascending properties along regular ring maps

07QJ This section is the analogue of Algebra, Section 10.163 but where the ring map $R \to S$ is regular.

07QK Lemma 15.42.1. Let $\varphi : R \to S$ be a ring map. Assume

(1) φ is regular,
(2) S is Noetherian, and
(3) R is Noetherian and reduced.

Then S is reduced.

Proof. For Noetherian rings being reduced is the same as having properties (S_1) and (R_0), see Algebra, Lemma 10.157.3. Hence we may apply Algebra, Lemmas 10.163.4 and 10.163.5. □

0BFK Lemma 15.42.2. Let $\varphi : R \to S$ be a ring map. Assume

(1) φ is regular,
(2) S is Noetherian, and
(3) R is Noetherian and normal.

Then S is normal.

Proof. For Noetherian rings being normal is the same as having properties (S_2) and (R_1), see Algebra, Lemma 10.157.4. Hence we may apply Algebra, Lemmas 10.163.4 and 10.163.5. □

15.43. Permanence of properties under completion

07NU Given a Noetherian local ring $(A, \mathfrak m)$ we denote $A^\wedge$ the completion of A with respect to $\mathfrak m$. We will use without further mention that $A^\wedge$ is a Noetherian complete local ring with maximal ideal $\mathfrak m^\wedge = \mathfrak m A^\wedge$ and that $A \to A^\wedge$ is faithfully flat. See Algebra, Lemmas 10.97.6, 10.97.4, and 10.97.3.

07NV Lemma 15.43.1. Let A be a Noetherian local ring. Then $\dim(A) = \dim(A^\wedge)$.

Proof. By Algebra, Lemma 10.97.4 the map $A \to A^\wedge$ induces isomorphisms $A/\mathfrak{m}^n = A^\wedge/(\mathfrak{m}^\wedge)^n$ for $n \geq 1$. By Algebra, Lemma 10.52.12 this implies that

$$\text{length}_A(A/\mathfrak{m}^n) = \text{length}_{A^\wedge}(A^\wedge/(\mathfrak{m}^\wedge)^n)$$

for all $n \geq 1$. Thus $d(A) = d(A^\wedge)$ and we conclude by Algebra, Proposition 10.60.9. An alternative proof is to use Algebra, Lemma 10.112.7. □

07NW **Lemma 15.43.2.** *Let A be a Noetherian local ring. Then $\text{depth}(A) = \text{depth}(A^\wedge)$.*

Proof. See Algebra, Lemma 10.163.2. □

07NX **Lemma 15.43.3.** *Let A be a Noetherian local ring. Then A is Cohen-Macaulay if and only if $A^\wedge$ is so.*

Proof. A local ring A is Cohen-Macaulay if and only if $\dim(A) = \text{depth}(A)$. As both of these invariants are preserved under completion (Lemmas 15.43.1 and 15.43.2) the claim follows. □

07NY **Lemma 15.43.4.** *Let A be a Noetherian local ring. Then A is regular if and only if $A^\wedge$ is so.*

Proof. If $A^\wedge$ is regular, then A is regular by Algebra, Lemma 10.110.9. Assume A is regular. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim(A) = \dim(A^\wedge)$ (Lemma 15.43.1). On the other hand, $\mathfrak{m}A^\wedge$ is the maximal ideal of $A^\wedge$ and hence $\mathfrak{m}_{A^\wedge}$ is generated by at most $\dim(A^\wedge)$ elements. Thus $A^\wedge$ is regular. (You can also use Algebra, Lemma 10.112.8.) □

0AP1 **Lemma 15.43.5.** *Let A be a Noetherian local ring. Then A is a discrete valuation ring if and only if $A^\wedge$ is so.*

Proof. This follows from Lemmas 15.43.1 and 15.43.4 and Algebra, Lemma 10.119.7. □

07NZ **Lemma 15.43.6.** *Let A be a Noetherian local ring.*

(1) *If $A^\wedge$ is reduced, then so is A.*
(2) *In general A reduced does not imply $A^\wedge$ is reduced.*
(3) *If A is Nagata, then A is reduced if and only if $A^\wedge$ is reduced.*

Proof. As $A \to A^\wedge$ is faithfully flat we have (1) by Algebra, Lemma 10.164.2. For (2) see Algebra, Example 10.119.5 (there are also examples in characteristic zero, see Algebra, Remark 10.119.6). For (3) see Algebra, Lemmas 10.162.13 and 10.162.10. □

0FIZ **Lemma 15.43.7.** *Let A be a Noetherian local ring. If $A^\wedge$ is normal, then so is A.*

Proof. As $A \to A^\wedge$ is faithfully flat this follows from Algebra, Lemma 10.164.3. □

0C4G **Lemma 15.43.8.** *Let $A \to B$ be a local homomorphism of Noetherian local rings. Then the induced map of completions $A^\wedge \to B^\wedge$ is flat if and only if $A \to B$ is flat.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} A^\wedge \arrow[r] & B^\wedge \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

The vertical arrows are faithfully flat. Assume that $A^\wedge \to B^\wedge$ is flat. Then $A \to B^\wedge$ is flat. Hence B is flat over A by Algebra, Lemma 10.39.9.

Assume that $A \to B$ is flat. Then $A \to B^\wedge$ is flat. Hence $B^\wedge/\mathfrak m_A^n B^\wedge$ is flat over $A/\mathfrak m_A^n$ for all $n \geq 1$. Note that $\mathfrak m_A^n A^\wedge$ is the nth power of the maximal ideal $\mathfrak m_A^\wedge$ of $A^\wedge$ and $A/\mathfrak m_A^n = A^\wedge/(\mathfrak m_A^\wedge)^n$. Thus we see that $B^\wedge$ is flat over $A^\wedge$ by applying Algebra, Lemma 10.99.11 (with $R = A^\wedge$, $I = \mathfrak m_A^\wedge$, $S = B^\wedge$, $M = S$). $\square$

0AGX Lemma 15.43.9. Let $A \to B$ be a flat local homomorphism of Noetherian local rings such that $\mathfrak m_A B = \mathfrak m_B$ and $\kappa(\mathfrak m_A) = \kappa(\mathfrak m_B)$. Then $A \to B$ induces an isomorphism $A^\wedge \to B^\wedge$ of completions.

Proof. By Algebra, Lemma 10.97.7 we see that $B^\wedge$ is the $\mathfrak m_A$-adic completion of B and that $A^\wedge \to B^\wedge$ is finite. Since $A \to B$ is flat we have $\text{Tor}_1^A(B, \kappa(\mathfrak m_A)) = 0$. Hence we see that $B^\wedge$ is flat over $A^\wedge$ by Lemma 15.27.5. Thus $B^\wedge$ is a free $A^\wedge$-module by Algebra, Lemma 10.78.5. Since $A^\wedge \to B^\wedge$ induces an isomorphism $\kappa(\mathfrak m_A) = A^\wedge/\mathfrak m_A A^\wedge \to B^\wedge/\mathfrak m_A B^\wedge = B^\wedge/\mathfrak m_B B^\wedge = \kappa(\mathfrak m_B)$ by our assumptions (and Algebra, Lemma 10.96.3), we see that $B^\wedge$ is free of rank 1. Thus $A^\wedge \to B^\wedge$ is an isomorphism. $\square$

15.44. Permanence of properties under étale maps

0AGY In this section we consider an étale ring map $\varphi : A \to B$ and we study which properties of A are inherited by B and which properties of the local ring of B at $\mathfrak q$ are inherited by the local ring of A at $\mathfrak p = \varphi^{-1}(\mathfrak q)$. Basically, this section reviews and collects earlier results and does not add any new material.

We will use without further mention that an étale ring map is flat (Algebra, Lemma 10.143.3) and that a flat local homomorphism of local rings is faithfully flat (Algebra, Lemma 10.39.17).

0AGZ Lemma 15.44.1. If $A \to B$ is an étale ring map and $\mathfrak q$ is a prime of B lying over $\mathfrak p \subset A$, then $A_\mathfrak p$ is Noetherian if and only if $B_\mathfrak q$ is Noetherian.

Proof. Since $A_\mathfrak p \to B_\mathfrak q$ is faithfully flat we see that $B_\mathfrak q$ Noetherian implies that $A_\mathfrak p$ is Noetherian, see Algebra, Lemma 10.164.1. Conversely, if $A_\mathfrak p$ is Noetherian, then $B_\mathfrak q$ is Noetherian as it is a localization of a finite type $A_\mathfrak p$-algebra. $\square$

07QP Lemma 15.44.2. If $A \to B$ is an étale ring map and $\mathfrak q$ is a prime of B lying over $\mathfrak p \subset A$, then $\dim(A_\mathfrak p) = \dim(B_\mathfrak q)$.

Proof. Namely, because $A_\mathfrak p \to B_\mathfrak q$ is flat we have going down, and hence the inequality $\dim(A_\mathfrak p) \leq \dim(B_\mathfrak q)$, see Algebra, Lemma 10.112.1. On the other hand, suppose that $\mathfrak q_0 \subset \mathfrak q_1 \subset \ldots \subset \mathfrak q_n$ is a chain of primes in $B_\mathfrak q$. Then the corresponding sequence of primes $\mathfrak p_0 \subset \mathfrak p_1 \subset \ldots \subset \mathfrak p_n$ (with $\mathfrak p_i = \mathfrak q_i \cap A_\mathfrak p$) is chain also (i.e., no equalities in the sequence) as an étale ring map is quasi-finite (see Algebra, Lemma 10.143.6) and a quasi-finite ring map induces a map of spectra with discrete fibres (by definition). This means that $\dim(A_\mathfrak p) \geq \dim(B_\mathfrak q)$ as desired. $\square$

0AH0 Lemma 15.44.3. If $A \to B$ is an étale ring map and $\mathfrak q$ is a prime of B lying over $\mathfrak p \subset A$, then $A_\mathfrak p$ is regular if and only if $B_\mathfrak q$ is regular.

Proof. By Lemma 15.44.1 we may assume both $A_\mathfrak p$ and $B_\mathfrak q$ are Noetherian in order to prove the equivalence. Let $x_1, \ldots, x_t \in \mathfrak p A_\mathfrak p$ be a minimal set of generators. As $A_\mathfrak p \to B_\mathfrak q$ is faithfully flat we see that the images $y_1, \ldots, y_t$ in $B_\mathfrak q$ form a minimal system of generators for $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$ (Algebra, Lemma 10.143.5). Regularity of $A_\mathfrak{p}$ by definition means $t = \dim(A_\mathfrak{p})$ and similarly for $B_\mathfrak{q}$. Hence the lemma follows from the equality $\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q})$ of Lemma 15.44.2. □

0AP2 Lemma 15.44.4. If $A \to B$ is an étale ring map and A is a Dedekind domain, then B is a finite product of Dedekind domains. In particular, the localizations $B_\mathfrak{q}$ for $\mathfrak{q} \subset B$ maximal are discrete valuation rings.

Proof. The statement on the local rings follows from Lemmas 15.44.2 and 15.44.3 and Algebra, Lemma 10.119.7. It follows that B is a Noetherian normal ring of dimension 1. By Algebra, Lemma 10.37.16 we conclude that B is a finite product of normal domains of dimension 1. These are Dedekind domains by Algebra, Lemma 10.120.17. □

15.45. Permanence of properties under henselization

07QL Given a local ring R we denote R^h, resp. R^{sh} the henselization, resp. strict henselization of R, see Algebra, Definition 10.155.3. Many of the properties of R are reflected in R^h and R^{sh} as we will show in this section.

07QM Lemma 15.45.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then we have the following

(1) $R \to R^h \to R^{sh}$ are faithfully flat ring maps,
(2) $\mathfrak{m}R^h = \mathfrak{m}^h$ and $\mathfrak{m}R^{sh} = \mathfrak{m}^hR^{sh} = \mathfrak{m}^{sh}$,
(3) $R/\mathfrak{m}^n = R^h/\mathfrak{m}^nR^h$ for all n,
(4) there exist elements $x_i \in R^{sh}$ such that $R^{sh}/\mathfrak{m}^nR^{sh}$ is a free $R/\mathfrak{m}^n$-module on $x_i \bmod \mathfrak{m}^nR^{sh}$.

Proof. By construction R^h is a colimit of étale R-algebras, see Algebra, Lemma 10.155.1. Since étale ring maps are flat (Algebra, Lemma 10.143.3) we see that R^h is flat over R by Algebra, Lemma 10.39.3. As a flat local ring homomorphism is faithfully flat (Algebra, Lemma 10.39.17) we see that $R \to R^h$ is faithfully flat. The ring map $R^h \to R^{sh}$ is a colimit of finite étale ring maps, see proof of Algebra, Lemma 10.155.2. Hence the same arguments as above show that $R^h \to R^{sh}$ is faithfully flat.

Part (2) follows from Algebra, Lemmas 10.155.1 and 10.155.2. Part (3) follows from Algebra, Lemma 10.101.1 because $R/\mathfrak{m} \to R^h/\mathfrak{m}R^h$ is an isomorphism and $R/\mathfrak{m}^n \to R^h/\mathfrak{m}^nR^h$ is flat as a base change of the flat ring map $R \to R^h$ (Algebra, Lemma 10.39.7). Let κ^{sep} be the residue field of R^{sh} (it is a separable algebraic closure of κ). Choose $x_i \in R^{sh}$ mapping to a basis of κ^{sep} as a κ-vector space. Then (4) follows from Algebra, Lemma 10.101.1 in exactly the same way as above. □

07QN Lemma 15.45.2. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then

(1) $R \to R^h$, $R^h \to R^{sh}$, and $R \to R^{sh}$ are formally étale,
(2) $R \to R^h$, $R^h \to R^{sh}$, resp. $R \to R^{sh}$ are formally smooth in the $\mathfrak{m}^h$, $\mathfrak{m}^{sh}$, resp. $\mathfrak{m}^{sh}$-topology.

Proof. Part (1) follows from the fact that R^h and R^{sh} are directed colimits of étale algebras (by construction), that étale algebras are formally étale (Algebra, Lemma 10.150.2), and that colimits of formally étale algebras are formally étale (Algebra, Lemma 10.150.3). Part (2) follows from the fact that a formally étale ring map is formally smooth and Lemma 15.37.2. □

06LJ **Lemma 15.45.3.** Let R be a local ring. The following are equivalent [DG67, IV, Theorem 18.6.6 and Proposition 18.8.8]

(1) R is Noetherian,
(2) R^h is Noetherian, and
(3) R^{sh} is Noetherian.

In this case we have

(a) $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian complete local rings,
(b) $R^\wedge \to (R^h)^\wedge$ is an isomorphism,
(c) $R^h \to (R^h)^\wedge$ and $R^{sh} \to (R^{sh})^\wedge$ are flat,
(d) $R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology,
(e) $(R^\wedge)^{sh} = R^\wedge \otimes_{R^h} R^{sh}$, and
(f) $((R^\wedge)^{sh})^\wedge = (R^{sh})^\wedge$.

Proof. Since $R \to R^h \to R^{sh}$ are faithfully flat (Lemma 15.45.1), we see that R^h or R^{sh} being Noetherian implies that R is Noetherian, see Algebra, Lemma 10.164.1. In the rest of the proof we assume R is Noetherian.

As $\mathfrak{m} \subset R$ is finitely generated it follows that $\mathfrak{m}^h = \mathfrak{m}R^h$ and $\mathfrak{m}^{sh} = \mathfrak{m}R^{sh}$ are finitely generated, see Lemma 15.45.1. Hence $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian by Algebra, Lemma 10.160.3. This proves (a).

Note that (b) is immediate from Lemma 15.45.1. In particular we see that $(R^h)^\wedge$ is flat over R, see Algebra, Lemma 10.97.3.

Next, we show that $R^h \to (R^h)^\wedge$ is flat. Write $R^h = \mathrm{colim}_i R_i$ as a directed colimit of localizations of étale R-algebras. By Algebra, Lemma 10.39.6 if $(R^h)^\wedge$ is flat over each R_i, then $R^h \to (R^h)^\wedge$ is flat. Note that $R^h = R_i^h$ (by construction). Hence $R_i^\wedge = (R^h)^\wedge$ by part (b) is flat over R_i as desired. To finish the proof of (c) we show that $R^{sh} \to (R^{sh})^\wedge$ is flat. To do this, by a limit argument as above, it suffices to show that $(R^{sh})^\wedge$ is flat over R. Note that it follows from Lemma 15.45.1 that $(R^{sh})^\wedge$ is the completion of a free R-module. By Lemma 15.27.2 we see this is flat over R as desired. This finishes the proof of (c).

At this point we know (c) is true and that $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian. It follows from Algebra, Lemma 10.164.1 that R^h and R^{sh} are Noetherian.

Part (d) follows from Lemma 15.45.2 and Lemma 15.37.4.

Part (e) follows from Algebra, Lemma 10.155.13 and the fact that $R^\wedge$ is henselian by Algebra, Lemma 10.153.9.

Proof of (f). Using (e) there is a map $R^{sh} \to (R^\wedge)^{sh}$ which induces a map $(R^{sh})^\wedge \to ((R^\wedge)^{sh})^\wedge$ upon completion. Using (e) there is a map $R^\wedge \to (R^{sh})^\wedge$. Since $(R^{sh})^\wedge$ is strictly henselian (see above) this map induces a map $(R^\wedge)^{sh} \to (R^{sh})^\wedge$ by Algebra, Lemma 10.155.10. Completing we obtain a map $((R^\wedge)^{sh})^\wedge \to (R^{sh})^\wedge$. We omit the verification that these two maps are mutually inverse. □

06DH **Lemma 15.45.4.** Let R be a local ring. The following are equivalent: R is reduced, the henselization R^h of R is reduced, and the strict henselization R^{sh} of R is reduced.

Proof. The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 10.164.2. Conversely, assume R is reduced. Since R^h and R^{sh} are filtered colimits of étale, hence smooth R-algebras, the result follows from Algebra, Lemma 10.163.7. □

0ASE Lemma 15.45.5. Let R be a local ring. Let $nil(R)$ denote the ideal of nilpotent elements of R. Then $nil(R)R^h = nil(R^h)$ and $nil(R)R^{sh} = nil(R^{sh})$.

Proof. Note that $nil(R)$ is the biggest ideal consisting of nilpotent elements such that the quotient $R/nil(R)$ is reduced. Note that $nil(R)R^h$ consists of nilpotent elements by Algebra, Lemma 10.32.3. Also, note that $R^h/nil(R)R^h$ is the henselization of $R/nil(R)$ by Algebra, Lemma 10.156.2. Hence $R^h/nil(R)R^h$ is reduced by Lemma 15.45.4. We conclude that $nil(R)R^h = nil(R^h)$ as desired. Similarly for the strict henselization but using Algebra, Lemma 10.156.4. □

06DI Lemma 15.45.6. Let R be a local ring. The following are equivalent: R is a normal domain, the henselization R^h of R is a normal domain, and the strict henselization R^{sh} of R is a normal domain.

Proof. A preliminary remark is that a local ring is normal if and only if it is a normal domain (see Algebra, Definition 10.37.11). The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 10.164.3. Conversely, assume R is normal. Since R^h and R^{sh} are filtered colimits of étale hence smooth R-algebras, the result follows from Algebra, Lemmas 10.163.9 and 10.37.17. □

06LK Lemma 15.45.7. Given any local ring R we have $\dim(R) = \dim(R^h) = \dim(R^{sh})$.

Proof. Since $R \to R^{sh}$ is faithfully flat (Lemma 15.45.1) we see that $\dim(R^{sh}) \geq \dim(R)$ by going down, see Algebra, Lemma 10.112.1. For the converse, we write $R^{sh} = \operatorname{colim} R_i$ as a directed colimit of local rings R_i each of which is a localization of an étale R-algebra. Now if $\mathfrak q_0 \subset \mathfrak q_1 \subset \ldots \subset \mathfrak q_n$ is a chain of prime ideals in R^{sh}, then for some sufficiently large i the sequence

$$R_i \cap \mathfrak q_0 \subset R_i \cap \mathfrak q_1 \subset \ldots \subset R_i \cap \mathfrak q_n$$

is a chain of primes in R_i. Thus we see that $\dim(R^{sh}) \leq \sup_i \dim(R_i)$. But by the result of Lemma 15.44.2 we have $\dim(R_i) = \dim(R)$ for each i and we win. □

06LL Lemma 15.45.8. Given a Noetherian local ring R we have $\text{depth}(R) = \text{depth}(R^h) = \text{depth}(R^{sh})$.

Proof. By Lemma 15.45.3 we know that R^h and R^{sh} are Noetherian. Hence the lemma follows from Algebra, Lemma 10.163.2. □

06LM Lemma 15.45.9. Let R be a Noetherian local ring. The following are equivalent: R is Cohen-Macaulay, the henselization R^h of R is Cohen-Macaulay, and the strict henselization R^{sh} of R is Cohen-Macaulay.

Proof. By Lemma 15.45.3 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Since we have $\text{depth}(R) = \text{depth}(R^h) = \text{depth}(R^{sh})$ and $\dim(R) = \dim(R^h) = \dim(R^{sh})$ by Lemmas 15.45.8 and 15.45.7 we conclude. □

06LN Lemma 15.45.10. Let R be a Noetherian local ring. The following are equivalent: R is a regular local ring, the henselization R^h of R is a regular local ring, and the strict henselization R^{sh} of R is a regular local ring.

Proof. By Lemma 15.45.3 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Let $\mathfrak m$ be the maximal ideal of R. Let $x_1, \ldots, x_t \in \mathfrak m$ be a minimal system of generators of $\mathfrak m$, i.e., such that the images in $\mathfrak m/\mathfrak m^2$ form a basis over $\kappa = R/\mathfrak{m}$. Because $R \to R^h$ and $R \to R^{sh}$ are faithfully flat, it follows that the images $x_1^h, \ldots, x_t^h$ in R^h, resp. $x_1^{sh}, \ldots, x_t^{sh}$ in R^{sh} are a minimal system of generators for $\mathfrak{m}^h = \mathfrak{m}R^h$, resp. $\mathfrak{m}^{sh} = \mathfrak{m}R^{sh}$. Regularity of R by definition means $t = \dim(R)$ and similarly for R^h and R^{sh}. Hence the lemma follows from the equality of dimensions $\dim(R) = \dim(R^h) = \dim(R^{sh})$ of Lemma 15.45.7 $\square$

0AP3 Lemma 15.45.11. Let R be a Noetherian local ring. Then R is a discrete valuation ring if and only if R^h is a discrete valuation ring if and only if R^{sh} is a discrete valuation ring.

Proof. This follows from Lemmas 15.45.7 and 15.45.10 and Algebra, Lemma 10.119.7. $\square$

0AH1 Lemma 15.45.12. Let A be a ring. Let B be a filtered colimit of étale A-algebras. Let $\mathfrak{p}$ be a prime of A. If B is Noetherian, then there are finitely many primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ lying over $\mathfrak{p}$, we have $B \otimes_A \kappa(\mathfrak{p}) = \prod \kappa(\mathfrak{q}_i)$, and each of the field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ is separable algebraic.

Proof. Write B as a filtered colimit $B = \operatorname{colim} B_i$ with $A \to B_i$ étale. Then on the one hand $B \otimes_A \kappa(\mathfrak{p}) = \operatorname{colim} B_i \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of étale $\kappa(\mathfrak{p})$-algebras, and on the other hand it is Noetherian. An étale $\kappa(\mathfrak{p})$-algebra is a finite product of finite separable field extensions (Algebra, Lemma 10.143.4). Hence there are no nontrivial specializations between the primes (which are all maximal and minimal primes) of the algebras $B_i \otimes_A \kappa(\mathfrak{p})$ and hence there are no nontrivial specializations between the primes of $B \otimes_A \kappa(\mathfrak{p})$. Thus $B \otimes_A \kappa(\mathfrak{p})$ is reduced and has finitely many primes which all minimal. Thus it is a finite product of fields (use Algebra, Lemma 10.25.4 or Algebra, Proposition 10.60.7). Each of these fields is a colimit of finite separable extensions and hence the final statement of the lemma follows. $\square$

07QQ Lemma 15.45.13. Let R be a Noetherian local ring. Let $\mathfrak{p} \subset R$ be a prime. Then

$$R^h \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1,\ldots,t} \kappa(\mathfrak{q}_i) \quad \text{resp.} \quad R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1,\ldots,s} \kappa(\mathfrak{r}_i)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, resp. $\mathfrak{r}_1, \ldots, \mathfrak{r}_s$ are the prime of R^h, resp. R^{sh} lying over $\mathfrak{p}$. Moreover, the field extensions $\kappa(\mathfrak{q}_i)/\kappa(\mathfrak{p})$ resp. $\kappa(\mathfrak{r}_i)/\kappa(\mathfrak{p})$ are separable algebraic.

Proof. This can be deduced from the more general Lemma 15.45.12 using that the henselization and strict henselization are Noetherian (as we've seen above). But we also give a direct proof as follows.

We will use without further mention the results of Lemmas 15.45.1 and 15.45.3. Note that $R^h/\mathfrak{p}R^h$, resp. $R^{sh}/\mathfrak{p}R^{sh}$ is the henselization, resp. strict henselization of $R/\mathfrak{p}$, see Algebra, Lemma 10.156.2 resp. Algebra, Lemma 10.156.4. Hence we may replace R by $R/\mathfrak{p}$ and assume that R is a Noetherian local domain and that $\mathfrak{p} = (0)$. Since R^h, resp. R^{sh} is Noetherian, it has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, resp. $\mathfrak{r}_1, \ldots, \mathfrak{r}_s$. Since $R \to R^h$, resp. $R \to R^{sh}$ is flat these are exactly the primes lying over $\mathfrak{p} = (0)$ (by going down). Finally, as R is a domain, we see that R^h, resp. R^{sh} is reduced, see Lemma 15.45.4. Thus we see that $R^h \otimes_R \kappa(\mathfrak{p})$ resp. $R^{sh} \otimes_R \kappa(\mathfrak{p})$ is a reduced Noetherian ring with finitely many primes, all of which are minimal (and hence maximal). Thus these rings are Artinian and are products of their localizations at maximal ideals, each necessarily a field (see Algebra, Proposition 10.60.7 and Algebra, Lemma 10.25.1).

The final statement follows from the fact that $R \to R^h$, resp. $R \to R^{sh}$ is a colimit of étale ring maps and hence the induced residue field extensions are colimits of finite separable extensions, see Algebra, Lemma 10.143.5. $\square$

15.46. Field extensions, revisited

07P0 In this section we study some peculiarities of field extensions in characteristic $p > 0$.

07P1 Definition 15.46.1. Let p be a prime number. Let $k \to K$ be an extension of fields of characteristic p. Denote kK^p the compositum of k and K^p in K.

(1) A subset $\{x_i\} \subset K$ is called p-independent over k if the elements $x^E = \prod x_i^{e_i}$ where $0 \leq e_i < p$ are linearly independent over kK^p.
(2) A subset $\{x_i\}$ of K is called a p-basis of K over k if the elements x^E form a basis of K over kK^p.

This is related to the notion of a p-basis of a $\mathbf{F}_p$-algebra which we will discuss later (insert future reference here).

07P2 Lemma 15.46.2. Let K/k be a field extension. Assume k has characteristic $p > 0$. Let $\{x_i\}$ be a subset of K. The following are equivalent

(1) the elements $\{x_i\}$ are p-independent over k, and
(2) the elements $\mathrm{d}x_i$ are K-linearly independent in $\Omega_{K/k}$.

Any p-independent collection can be extended to a p-basis of K over k. In particular, the field K has a p-basis over k. Moreover, the following are equivalent:

(a) $\{x_i\}$ is a p-basis of K over k, and
(b) $\mathrm{d}x_i$ is a basis of the K-vector space $\Omega_{K/k}$.

Proof. Assume (2) and suppose that $\sum a_E x^E = 0$ is a linear relation with $a_E \in kK^p$. Let $\theta_i : K \to K$ be a k-derivation such that $\theta_i(x_j) = \delta_{ij}$ (Kronecker delta). Note that any k-derivation of K annihilates kK^p. Applying θ_i to the given relation we obtain new relations

$$\sum\nolimits_{E, e_i > 0} e_i a_E x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} = 0$$

Hence if we pick $\sum a_E x^E$ as the relation with minimal total degree $|E| = \sum e_i$ for some $a_E \neq 0$, then we get a contradiction. Hence (1) holds.

If $\{x_i\}$ is a p-basis for K over k, then $K \cong kK^p[X_i]/(X_i^p - x_i^p)$. Hence we see that $\mathrm{d}x_i$ forms a basis for $\Omega_{K/k}$ over K. Thus (a) implies (b).

Let $\{x_i\}$ be a p-independent subset of K over k. An application of Zorn's lemma shows that we can enlarge this to a maximal p-independent subset of K over k. We claim that any maximal p-independent subset $\{x_i\}$ of K is a p-basis of K over k. The claim will imply that (1) implies (2) and establish the existence of p-bases. To prove the claim let L be the subfield of K generated by kK^p and the x_i. We have to show that $L = K$. If $x \in K$ but $x \notin L$, then $x^p \in L$ and $L(x) \cong L[z]/(z^p - x)$. Hence $\{x_i\} \cup \{x\}$ is p-independent over k, a contradiction.

Finally, we have to show that (b) implies (a). By the equivalence of (1) and (2) we see that $\{x_i\}$ is a maximal p-independent subset of K over k. Hence by the claim above it is a p-basis. $\square$

07P3 Lemma 15.46.3. Let K/k be a field extension. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties

(1) $k \subset K_\alpha$ for all $\alpha \in A$,
(2) $k = \bigcap_{\alpha \in A} K_\alpha$,
(3) for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.

Then for $n \geq 1$ and $V \subset K^{\oplus n}$ a K-vector space we have $V \cap k^{\oplus n} \neq 0$ if and only if $V \cap K_\alpha^{\oplus n} \neq 0$ for all $\alpha \in A$.

Proof. By induction on n. The case $n = 1$ follows from the assumptions. Assume the result proven for subspaces of $K^{\oplus n-1}$. Assume that $V \subset K^{\oplus n}$ has nonzero intersection with $K_\alpha^{\oplus n}$ for all $\alpha \in A$. If $V \cap 0 \oplus k^{\oplus n-1}$ is nonzero then we win. Hence we may assume this is not the case. By induction hypothesis we can find an α such that $V \cap 0 \oplus K_\alpha^{\oplus n-1}$ is zero. Let $v = (x_1, \ldots, x_n) \in V \cap K_\alpha^{\oplus n}$ be a nonzero element. By our choice of α we see that x_1 is not zero. Replace v by $x_1^{-1}v$ so that $v = (1, x_2, \ldots, x_n)$. Note that if $v' = (x'_1, \ldots, x'_n) \in V \cap K_\alpha$, then $v' - x'_1 v = 0$ by our choice of α. Hence we see that $V \cap K_\alpha^{\oplus n} = K_\alpha v$. If we choose some α' such that $K_{\alpha'} \subset K_\alpha$, then we see that necessarily $v \in V \cap K_{\alpha'}^{\oplus n}$ (by the same arguments applied to α'). Hence

$$x_2, \ldots, x_n \in \bigcap\nolimits_{\alpha' \in A, K_{\alpha'} \subset K_\alpha} K_{\alpha'}$$

which equals k by (2) and (3). $\square$

07P4 Lemma 15.46.4. Let K be a field of characteristic p. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties

(1) $K^p \subset K_\alpha$ for all $\alpha \in A$,
(2) $K^p = \bigcap_{\alpha \in A} K_\alpha$,
(3) for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.

Then

(1) the intersection of the kernels of the maps $\Omega_{K/\mathbf{F}_p} \to \Omega_{K/K_\alpha}$ is zero,
(2) for any finite extension L/K we have $L^p = \bigcap_{\alpha \in A} L^p K_\alpha$.

Proof. Proof of (1). Choose a p-basis $\{x_i\}$ for K over $\mathbf{F}_p$. Suppose that $\eta = \sum_{i \in I'} y_i \text{d}x_i$ maps to zero in Ω_{K/K_α} for every $\alpha \in A$. Here the index set I' is finite. By Lemma 15.46.2 this means that for every α there exists a relation

$$\sum\nolimits_E a_{E,\alpha} x^E, \quad a_{E,\alpha} \in K_\alpha$$

where E runs over multi-indices $E = (e_i)_{i \in I'}$ with $0 \leq e_i < p$. On the other hand, Lemma 15.46.2 guarantees there is no such relation $\sum a_E x^E = 0$ with $a_E \in K^p$. This is a contradiction by Lemma 15.46.3.

Proof of (2). Suppose that we have a tower $L/M/K$ of finite extensions of fields. Set $M_\alpha = M^p K_\alpha$ and $L_\alpha = L^p K_\alpha = L^p M_\alpha$. Then we can first prove that $M^p = \bigcap_{\alpha \in A} M_\alpha$, and after that prove that $L^p = \bigcap_{\alpha \in A} L_\alpha$. Hence it suffices to prove (2) for primitive field extensions having no nontrivial subfields. First, assume that $L = K(\theta)$ is separable over K. Then L is generated by θ^p over K, hence we may assume that $\theta \in L^p$. In this case we see that

$$L^p = K^p \oplus K^p\theta \oplus \ldots K^p\theta^{d-1} \quad \text{and} \quad L^p K_\alpha = K_\alpha \oplus K_\alpha\theta \oplus \ldots K_\alpha\theta^{d-1}$$

where $d = [L : K]$. Thus the conclusion is clear in this case. The other case is where $L = K(\theta)$ with $\theta^p = t \in K$, $t \notin K^p$. In this case we have

$$L^p = K^p \oplus K^p t \oplus \ldots K^p t^{p-1} \quad \text{and} \quad L^p K_\alpha = K_\alpha \oplus K_\alpha t \oplus \ldots K_\alpha t^{p-1}$$

Again the result is clear. □

07P5 Lemma 15.46.5. Let k be a field of characteristic $p > 0$. Let $n, m \geq 0$. Let K be the fraction field of $k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$. As k' ranges through all subfields $k/k'/k^p$ with $[k : k'] < \infty$ the subfields

$$\text{fraction field of } k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p] \subset K$$

form a family of subfields as in Lemma 15.46.4. Moreover, each of the ring extensions $k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p] \subset k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$ is finite.

Proof. Write $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$ and $A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$. We also denote K' the fraction field of A'. The ring extension $k'[[x_1^p, \ldots, x_d^p]] \subset k[[x_1, \ldots, x_d]]$ is finite by Algebra, Lemma 10.97.7 which implies that $A' \to A$ is finite. For $f \in A$ we see that $f^p \in A'$. Hence $K^p \subset K'$. Any element of K' can be written as a/b^p with $a \in A'$ and $b \in A$ nonzero. Suppose that $f/g^p \in K$, $f, g \in A$, $g \neq 0$ is contained in K' for every choice of k'. Fix a choice of k' for the moment. By the above we see $f/g^p = a/b^p$ for some $a \in A'$ and some nonzero $b \in A$. Hence $b^p f \in A'$. For any A'-derivation $D : A \to A$ we see that $0 = D(b^p f) = b^p D(f)$ hence $D(f) = 0$ as A is a domain. Taking $D = \partial_{x_i}$ and $D = \partial_{y_j}$ we conclude that $f \in k[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_d^p]$. Applying a k'-derivation $\theta : k \to k$ we similarly conclude that all coefficients of f are in k', i.e., $f \in A'$. Since it is clear that $A^p = \bigcap_{k'} A'$ where k' ranges over all subfields as in the lemma we win. □

15.47. The singular locus

07P6 Let R be a Noetherian ring. The regular locus $\text{Reg}(X)$ of $X = \text{Spec}(R)$ is the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is a regular local ring. The singular locus $\text{Sing}(X)$ of $X = \text{Spec}(R)$ is the complement $X \setminus \text{Reg}(X)$, i.e., the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is not a regular local ring. By the discussion preceding Algebra, Definition 10.110.7 we see that $\text{Reg}(X)$ is stable under generalization. In this section we study conditions that guarantee that $\text{Reg}(X)$ is open.

07P7 Definition 15.47.1. Let R be a Noetherian ring. Let $X = \text{Spec}(R)$. [Mat70a, (32.B)]

1. We say R is J-0 if $\text{Reg}(X)$ contains a nonempty open.
2. We say R is J-1 if $\text{Reg}(X)$ is open.
3. We say R is J-2 if any finite type R-algebra is J-1.

The ring $\mathbf{Q}[x]/(x^2)$ does not satisfy J-0, but it does satisfy J-1. On the other hand, J-1 implies J-0 for Noetherian domains and more generally nonzero reduced Noetherian rings as such a ring is regular at the minimal primes. Here is a characterization of the J-1 property.

07P8 Lemma 15.47.2. Let R be a Noetherian ring. Let $X = \text{Spec}(R)$. The ring R is J-1 if and only if $V(\mathfrak{p}) \cap \text{Reg}(X)$ contains a nonempty open subset of $V(\mathfrak{p})$ for all $\mathfrak{p} \in \text{Reg}(X)$.

Proof. This follows from Topology, Lemma 5.16.5 and the fact that $\text{Reg}(X)$ is stable under generalization by Algebra, Lemma 10.110.6. □

07P9 Lemma 15.47.3. Let R be a Noetherian ring. Let $X = \text{Spec}(R)$. Assume that for all primes $\mathfrak{p} \subset R$ the ring $R/\mathfrak{p}$ is J-0. Then R is J-1.

Proof. We will show that the criterion of Lemma 15.47.2 applies. Let $\mathfrak p \in \text{Reg}(X)$ be a prime of height r. Pick $f_1, \ldots, f_r \in \mathfrak p$ which map to generators of $\mathfrak pR_{\mathfrak p}$. Since $\mathfrak p \in \text{Reg}(X)$ we see that $f_1, \ldots, f_r$ maps to a regular sequence in $R_{\mathfrak p}$, see Algebra, Lemma 10.106.3. Thus by Algebra, Lemma 10.68.6 we see that after replacing R by R_g for some $g \in R$, $g \notin \mathfrak p$ the sequence $f_1, \ldots, f_r$ is a regular sequence in R. After another replacement we may also assume $f_1, \ldots, f_r$ generate $\mathfrak p$. Next, let $\mathfrak p \subset \mathfrak q$ be a prime ideal such that $(R/\mathfrak p)_{\mathfrak q}$ is a regular local ring. By the assumption of the lemma there exists a non-empty open subset of $V(\mathfrak p)$ consisting of such primes, hence it suffices to prove $R_{\mathfrak q}$ is regular. Note that $f_1, \ldots, f_r$ is a regular sequence in $R_{\mathfrak q}$ such that $R_{\mathfrak q}/(f_1, \ldots, f_r)R_{\mathfrak q}$ is regular. Hence $R_{\mathfrak q}$ is regular by Algebra, Lemma 10.106.7. □

07PA **Lemma 15.47.4.** *Let $R \to S$ be a ring map. Assume that*

1. *R is a Noetherian domain,*
2. *$R \to S$ is injective and of finite type, and*
3. *S is a domain and J-0.*

Then R is J-0.

Proof. After replacing S by S_g for some nonzero $g \in S$ we may assume that S is a regular ring. By generic flatness we may assume that also $R \to S$ is faithfully flat, see Algebra, Lemma 10.118.1. Then R is regular by Algebra, Lemma 10.164.4. □

07PB **Lemma 15.47.5.** *Let $R \to S$ be a ring map. Assume that*

1. *R is a Noetherian domain and J-0,*
2. *$R \to S$ is injective and of finite type, and*
3. *S is a domain, and*
4. *the induced extension of fraction fields is separable.*

Then S is J-0.

Proof. We may replace R by a principal localization and assume R is a regular ring. By Algebra, Lemma 10.140.9 the ring map $R \to S$ is smooth at (0). Hence after replacing S by a principal localization we may assume that S is smooth over R. Then S is regular too, see Algebra, Lemma 10.163.10. □

07PC **Lemma 15.47.6.** *Let R be a Noetherian ring. The following are equivalent*

1. *R is J-2,*
2. *every finite type R-algebra which is a domain is J-0,*
3. *every finite R-algebra is J-1,*
4. *for every prime $\mathfrak p$ and every finite purely inseparable extension $L/\kappa(\mathfrak p)$ there exists a finite R-algebra R' which is a domain, which is J-0, and whose field of fractions is L.*

Proof. It is clear that we have the implications (1) ⇒ (2) and (2) ⇒ (4). Recall that a domain which is J-1 is J-0. Hence we also have the implications (1) ⇒ (3) and (3) ⇒ (4).

Let $R \to S$ be a finite type ring map and let's try to show S is J-1. By Lemma 15.47.3 it suffices to prove that $S/\mathfrak q$ is J-0 for every prime $\mathfrak q$ of S. In this way we see (2) ⇒ (1).

Assume (4). We will show that (2) holds which will finish the proof. Let $R \to S$ be a finite type ring map with S a domain. Let $\mathfrak{p} = \mathrm{Ker}(R \to S)$. Let K be the fraction field of S. There exists a diagram of fields

$$\begin{tikzcd} K \arrow[r] & K' \\ \kappa(\mathfrak{p}) \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

where the horizontal arrows are finite purely inseparable field extensions and where K'/L is separable, see Algebra, Lemma 10.42.4. Choose $R' \subset L$ as in (4) and let S' be the image of the map $S \otimes_R R' \to K'$. Then S' is a domain whose fraction field is K', hence S' is J-0 by Lemma 15.47.5 and our choice of R'. Then we apply Lemma 15.47.4 to see that S is J-0 as desired. $\square$

15.48. Regularity and derivations

07PD Let $R \to S$ be a ring map. Let $D : R \to R$ be a derivation. We say that D extends to S if there exists a derivation $D' : S \to S$ such that

$$\begin{tikzcd} S \arrow[r, "D'"'] & S \\ R \arrow[u] \arrow[r, "D"] & R \arrow[u] \end{tikzcd}$$

is commutative.

07PE **Lemma 15.48.1.** Let R be a ring. Let $D : R \to R$ be a derivation.

(1) For any ideal $I \subset R$ the derivation D extends canonically to a derivation $D^\wedge : R^\wedge \to R^\wedge$ on the I-adic completion.
(2) For any multiplicative subset $S \subset R$ the derivation D extends uniquely to the localization $S^{-1}R$ of R.

If $R \subset R'$ is a finite type extension of rings such that $R_g \cong R'_g$ for some $g \in R$ which is a nonzerodivisor in R', then $g^N D$ extends to R' for some $N \geq 0$.

Proof. Proof of (1). For $n \geq 2$ we have $D(I^n) \subset I^{n-1}$ by the Leibniz rule. Hence D induces maps $D_n : R/I^n \to R/I^{n-1}$. Taking the limit we obtain $D^\wedge$. We omit the verification that $D^\wedge$ is a derivation.

Proof of (2). To extend D to $S^{-1}R$ just set $D(r/s) = D(r)/s - rD(s)/s^2$ and check the axioms.

Proof of the final statement. Let $x_1, \ldots, x_n \in R'$ be generators of R' over R. Choose an N such that $g^N x_i \in R$. Consider $g^{N+1}D$. By (2) this extends to R_g. Moreover, by the Leibniz rule and our construction of the extension above we have

$$g^{N+1}D(x_i) = g^{N+1}D(g^{-N}g^N x_i) = -Ng^N x_i D(g) + gD(g^N x_i)$$

and both terms are in R. This implies that

$$g^{N+1}D(x_1^{e_1} \ldots x_n^{e_n}) = \sum e_i x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} g^{N+1} D(x_i)$$

is an element of R'. Hence every element of R' (which can be written as a sum of monomials in the x_i with coefficients in R) is mapped to an element of R' by $g^{N+1}D$ and we win. $\square$

07PF **Lemma 15.48.2.** Let R be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of $R/(f)$. Then $R/(f)$ is regular.

Proof. It suffices to prove this when R is a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case it suffices to prove that $f \not\in \mathfrak{m}^2$, see Algebra, Lemma 10.106.3. However, if $f \in \mathfrak{m}^2$ then $D(f) \in \mathfrak{m}$ by the Leibniz rule, a contradiction. $\square$

0GEE **Lemma 15.48.3.** Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring. Let $m \geq 1$. Let $f_1, \ldots, f_m \in \mathfrak{m}$. Assume there exist derivations $D_1, \ldots, D_m : R \to R$ such that $\det_{1 \leq i,j \leq m}(D_i(f_j))$ is a unit of R. Then $R/(f_1, \ldots, f_m)$ is regular and $f_1, \ldots, f_m$ is a regular sequence.

Proof. It suffices to prove that $f_1, \ldots, f_m$ are κ-linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, see Algebra, Lemma 10.106.3. However, if there is a nontrivial linear relation the we get $\sum a_i f_i \in \mathfrak{m}^2$ for some $a_i \in R$ but not all $a_i \in \mathfrak{m}$. Observe that $D_i(\mathfrak{m}^2) \subset \mathfrak{m}$ and $D_i(a_j f_j) \equiv a_j D_i(f_j) \bmod \mathfrak{m}$ by the Leibniz rule for derivations. Hence this would imply

$$\sum a_j D_i(f_j) \in \mathfrak{m}$$

which would contradict the assumption on the determinant. $\square$

07PG **Lemma 15.48.4.** Let R be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of R. Then $R[z]/(z^n - f)$ is regular for any integer $n \geq 1$. More generally, $R[z]/(p(z) - f)$ is regular for any $p \in \mathbf{Z}[z]$.

Proof. By Algebra, Lemma 10.163.10 we see that $R[z]$ is a regular ring. Apply Lemma 15.48.2 to the extension of D to $R[z]$ which maps z to zero. This works because D annihilates any polynomial with integer coefficients and sends f to a unit. $\square$

07PH **Lemma 15.48.5.** Let p be a prime number. Let B be a domain with $p = 0$ in B. Let $f \in B$ be an element which is not a pth power in the fraction field of B. If B is of finite type over a Noetherian complete local ring, then there exists a derivation $D : B \to B$ such that $D(f)$ is not zero.

Proof. Let R be a Noetherian complete local ring such that there exists a finite type ring map $R \to B$. Of course we may replace R by its image in B, hence we may assume R is a domain of characteristic $p > 0$ (as well as Noetherian complete local). By Algebra, Lemma 10.160.11 we can write R as a finite extension of $k[[x_1, \ldots, x_n]]$ for some field k and integer n. Hence we may replace R by $k[[x_1, \ldots, x_n]]$. Next, we use Algebra, Lemma 10.115.7 to factor $R \to B$ as

$$R \subset R[y_1, \ldots, y_d] \subset B' \subset B$$

with B' finite over $R[y_1, \ldots, y_d]$ and $B'_g \cong B_g$ for some nonzero $g \in R$. Note that $f' = g^{pN} f \in B'$ for some large integer N. It is clear that f' is not a pth power in the fraction field of B'. If we can find a derivation $D' : B' \to B'$ with $D'(f') \neq 0$, then Lemma 15.48.1 guarantees that $D = g^M D'$ extends to B for some $M > 0$. Then $D(f) = g^N D'(f) = g^M D'(g^{-pN} f') = g^{M-pN} D'(f')$ is nonzero. Thus it suffices to prove the lemma in case B is a finite extension of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$.

Assume B is a finite extension of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$. Denote L the fraction field of B. Note that $\mathrm{d}f$ is not zero in $\Omega_{L/\mathbf{F}_p}$, see Algebra, Lemma 10.158.2. We apply Lemma 15.46.5 to find a subfield $k' \subset k$ of finite index such that with

$A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$ the element $\mathrm{d}f$ does not map to zero in $\Omega_{L/K'}$ where K' is the fraction field of A'. Thus we can choose a K'-derivation $D' : L \to L$ with $D'(f) \neq 0$. Since $A' \subset A$ and $A \subset B$ are finite by construction we see that $A' \subset B$ is finite. Choose $b_1, \ldots, b_t \in B$ which generate B as an A'-module. Then $D'(b_i) = f_i/g_i$ for some $f_i, g_i \in B$ with $g_i \neq 0$. Setting $D = g_1 \ldots g_t D'$ we win. □

07PI Lemma 15.48.6. Let A be a Noetherian complete local domain. Then A is J-0.

Proof. By Algebra, Lemma 10.160.11 we can find a regular subring $A_0 \subset A$ with A finite over A_0. The induced extension K/K_0 of fraction fields is finite. If K/K_0 is separable, then we are done by Lemma 15.47.5. If not, then A_0 and A have characteristic $p > 0$. For any subextension $K/M/K_0$ there exists a finite subextension $A_0 \subset B \subset A$ whose fraction field is M. Hence, arguing by induction on $[K : K_0]$ we may assume there exists $A_0 \subset B \subset A$ such that B is J-0 and K/M has no nontrivial subextensions. In this case, if K/M is separable, then we see that A is J-0 by Lemma 15.47.5. If not, then $K = M[z]/(z^p - b_1/b_2)$ for some $b_1, b_2 \in B$ with $b_2 \neq 0$ and b_1/b_2 not a pth power in M. Choose $a \in A$ nonzero such that $az \in A$. After replacing z by $b_2 a^p z$ we obtain $K = M[z]/(z^p - b)$ with $z \in A$ and $b \in B$ not a pth power in M. By Lemma 15.48.5 we can find a derivation $D : B \to B$ with $D(b) \neq 0$. Applying Lemma 15.48.4 we see that $A_\mathfrak{p}$ is regular for any prime $\mathfrak{p}$ of A lying over a regular prime of B and not containing $D(b)$. As B is J-0 we conclude A is too. □

07PJ Proposition 15.48.7. The following types of rings are J-2:

(1) fields,
(2) Noetherian complete local rings,
(3) $\mathbf{Z}$,
(4) Noetherian local rings of dimension 1,
(5) Nagata rings of dimension 1,
(6) Dedekind domains with fraction field of characteristic zero,
(7) finite type ring extensions of any of the above.

Proof. For cases (1), (3), (5), and (6) this is proved by checking condition (4) of Lemma 15.47.6. We will only do this in case R is a Nagata ring of dimension 1. Let $\mathfrak{p} \subset R$ be a prime ideal and let $L/\kappa(\mathfrak{p})$ be a finite purely inseparable extension. If $\mathfrak{p} \subset R$ is a maximal ideal, then $R \to L$ is finite and L is a regular ring and we've checked the condition. If $\mathfrak{p} \subset R$ is a minimal prime, then the Nagata condition insures that the integral closure $R' \subset L$ of R in L is finite over R. Then R' is a normal domain of dimension 1 (Algebra, Lemma 10.112.3) hence regular (Algebra, Lemma 10.157.4) and we've checked the condition in this case as well.

For case (2), we will use condition (3) of Lemma 15.47.6. Let R be a Noetherian complete local ring. Note that if $R \to R'$ is finite, then R' is a product of Noetherian complete local rings, see Algebra, Lemma 10.160.2. Hence it suffices to prove that a Noetherian complete local ring which is a domain is J-0, which is Lemma 15.48.6.

For case (4), we also use condition (3) of Lemma 15.47.6. Namely, if R is a local Noetherian ring of dimension 1 and $R \to R'$ is finite, then $\mathrm{Spec}(R')$ is finite. Since the regular locus is stable under generalization, we see that R' is J-1. □

15.49. Formal smoothness and regularity

07PK The title of this section refers to Proposition 15.49.2.

07PL **Lemma 15.49.1.** Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $D : A \to A$ be a derivation. Assume that B is complete and $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists an extension $D' : B \to B$ of D.

Proof. Denote $B[\epsilon] = B[x]/(x^2)$ the ring of dual numbers over B. Consider the ring map $\psi : A \to B[\epsilon]$, $a \mapsto a + \epsilon D(a)$. Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "1"'] & B \\ A \arrow[u] \arrow[r, "\psi"] & B[\epsilon] \arrow[u] \end{tikzcd}$$

By Lemma 15.37.5 and the assumption of formal smoothness of B/A we find a map $\varphi : B \to B[\epsilon]$ fitting into the diagram. Write $\varphi(b) = b + \epsilon D'(b)$. Then $D' : B \to B$ is the desired extension. $\square$

07PM **Proposition 15.49.2.** Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent

(1) $A \to B$ is regular,
(2) $A \to B$ is flat and $\overline{B}$ is geometrically regular over k,
(3) $A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and
(4) $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.

Proof. We have seen the equivalence of (2), (3), and (4) in Proposition 15.40.5. It is clear that (1) implies (2). Thus we assume the equivalent conditions (2), (3), and (4) hold and we prove (1).

Let $\mathfrak{p}$ be a prime of A. We will show that $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$. By Lemma 15.37.8 we may replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$. Thus we may assume that A is a domain and that $\mathfrak{p} = (0)$.

Choose $A_0 \subset A$ as in Algebra, Lemma 10.160.11. We will use all the properties stated in that lemma without further mention. As $A_0 \to A$ induces an isomorphism on residue fields, and as $B/\mathfrak{m}_A B$ is geometrically regular over $A/\mathfrak{m}_A$ we can find a diagram

$$\begin{tikzcd} C \arrow[r] & B \\ A_0 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with $A_0 \to C$ formally smooth in the $\mathfrak{m}_C$-adic topology such that $B = C \otimes_{A_0} A$, see Remark 15.40.7. (Completion in the tensor product is not needed as $A_0 \to A$ is finite, see Algebra, Lemma 10.97.1.) Hence it suffices to show that $C \otimes_{A_0} K_0$ is a geometrically regular algebra over the fraction field K_0 of A_0.

The upshot of the preceding paragraph is that we may assume that $A = k[[x_1, \ldots, x_n]]$ where k is a field or $A = \Lambda[[x_1, \ldots, x_n]]$ where Λ is a Cohen ring. In this case B is a regular ring, see Algebra, Lemma 10.112.8. Hence $B \otimes_A K$ is a regular ring too (where K is the fraction field of A) and we win if the characteristic of K is zero.

Thus we are left with the case where $A = k[[x_1, \ldots, x_n]]$ and k is a field of characteristic $p > 0$. Let L/K be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $B \otimes_A L$ is regular. The base case is $L = K$ which we've seen above. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let A' be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in A'$. Set $A'' = A'[z]/(z^p - f)$ and note that $A' \subset A''$ is finite and that the fraction field of A'' is L. By induction we know that $B \otimes_A M$ ring is regular. We have

$$B \otimes_A L = B \otimes_A M[z]/(z^p - f)$$

By Lemma 15.48.5 we know there exists a derivation $D : A' \to A'$ such that $D(f) \neq 0$. As $A' \to B \otimes_A A'$ is formally smooth in the $\mathfrak{m}$-adic topology by Lemma 15.37.9 we can use Lemma 15.49.1 to extend D to a derivation $D' : B \otimes_A A' \to B \otimes_A A'$. Note that $D'(f) = D(f)$ is a unit in $B \otimes_A M$ as $D(f)$ is not zero in $A' \subset M$. Hence $B \otimes_A L$ is regular by Lemma 15.48.4 and we win. □

15.50. G-rings

07GG Let A be a Noetherian local ring A. In Section 15.43 we have seen that some but not all properties of A are reflected in the completion $A^\wedge$ of A. To study this further we introduce some terminology. For a prime $\mathfrak{q}$ of A the fibre ring

$$A^\wedge \otimes_A \kappa(\mathfrak{q}) = (A^\wedge)_\mathfrak{q}/\mathfrak{q}(A^\wedge)_\mathfrak{q} = (A/\mathfrak{q})^\wedge \otimes_{A/q} \kappa(\mathfrak{q})$$

is called a formal fibre of A. We think of the formal fibre as an algebra over $\kappa(\mathfrak{q})$. Thus $A \to A^\wedge$ is a regular ring homomorphism if and only if all the formal fibres are geometrically regular algebras.

07GH **Definition 15.50.1.** A ring R is called a G-ring if R is Noetherian and for every prime $\mathfrak{p}$ of R the ring map $R_\mathfrak{p} \to (R_\mathfrak{p})^\wedge$ is regular.

By the discussion above we see that R is a G-ring if and only if every local ring $R_\mathfrak{p}$ has geometrically regular formal fibres. Note that if $\mathbf{Q} \subset R$, then it suffices to check the formal fibres are regular. Another way to express the G-ring condition is described in the following lemma.

07PN **Lemma 15.50.2.** Let R be a Noetherian ring. Then R is a G-ring if and only if for every pair of primes $\mathfrak{q} \subset \mathfrak{p} \subset R$ the algebra

$$(R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

is geometrically regular over $\kappa(\mathfrak{q})$.

Proof. This follows from the fact that

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) = (R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

as algebras over $\kappa(\mathfrak{q})$. □

07PP Lemma 15.50.3. Let $R \to R'$ be a finite type map of Noetherian rings and let

$$\begin{tikzcd} \mathfrak{q}' \arrow[r] \arrow[d, dash] & \mathfrak{p}' \arrow[r] \arrow[d, dash] & R' \\ \mathfrak{q} \arrow[r] & \mathfrak{p} \arrow[r] & R \arrow[u] \end{tikzcd}$$

be primes. Assume $R \to R'$ is quasi-finite at $\mathfrak{p}'$.

(1) If the formal fibre $R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$, then the formal fibre $R'_{\mathfrak{p}'} \otimes_{R'} \kappa(\mathfrak{q}')$ is geometrically regular over $\kappa(\mathfrak{q}')$.
(2) If the formal fibres of $R_\mathfrak{p}$ are geometrically regular, then the formal fibres of $R'_{\mathfrak{p}'}$ are geometrically regular.
(3) If $R \to R'$ is quasi-finite and R is a G-ring, then R' is a G-ring.

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume $R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$. By Algebra, Lemma 10.124.3 we see that

$$R_\mathfrak{p}^\wedge \otimes_R R' = (R'_{\mathfrak{p}'})^\wedge \times B$$

for some $R_\mathfrak{p}^\wedge$-algebra B. Hence $R'_{\mathfrak{p}'} \to (R'_{\mathfrak{p}'})^\wedge$ is a factor of a base change of the map $R_\mathfrak{p} \to R_\mathfrak{p}^\wedge$. It follows that $(R'_{\mathfrak{p}'})^\wedge \otimes_{R'} \kappa(\mathfrak{q}')$ is a factor of

$$R_\mathfrak{p}^\wedge \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}').$$

Thus the result follows as extension of base field preserves geometric regularity, see Algebra, Lemma 10.166.1. ☐

07PQ Lemma 15.50.4. Let R be a Noetherian ring. Then R is a G-ring if and only if for every finite free ring map $R \to S$ the formal fibres of S are regular rings.

Proof. Assume that for any finite free ring map $R \to S$ the ring S has regular formal fibres. Let $\mathfrak{q} \subset \mathfrak{p} \subset R$ be primes and let $\kappa(\mathfrak{q}) \subset L$ be a finite purely inseparable extension. To show that R is a G-ring it suffices to show that

$$R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L$$

is a regular ring. Choose a finite free extension $R \to R'$ such that $\mathfrak{q}' = \mathfrak{q}R'$ is a prime and such that $\kappa(\mathfrak{q}')$ is isomorphic to L over $\kappa(\mathfrak{q})$, see Algebra, Lemma 10.159.3. By Algebra, Lemma 10.97.8 we have

$$R_\mathfrak{p}^\wedge \otimes_R R' = \prod (R'_{\mathfrak{p}'_i})^\wedge$$

where $\mathfrak{p}'_i$ are the primes of R' lying over $\mathfrak{p}$. Thus we have

$$R_\mathfrak{p}^\wedge \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L = R_\mathfrak{p}^\wedge \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = \prod (R'_{\mathfrak{p}'_i})^\wedge \otimes_{R'_{\mathfrak{p}'_i}} \kappa(\mathfrak{q}')$$

Our assumption is that the rings on the right are regular, hence the ring on the left is regular too. Thus R is a G-ring. The converse follows from Lemma 15.50.3. ☐

07PR Lemma 15.50.5. Let k be a field of characteristic p. Let $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_n]$ and denote K the fraction field of A. Let $\mathfrak{p} \subset A$ be a prime. Then $A_\mathfrak{p}^\wedge \otimes_A K$ is geometrically regular over K.

Proof. Let L/K be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $A_\mathfrak{p}^\wedge \otimes L$ is regular. The base case is $L = K$: as A is regular, $A_\mathfrak{p}^\wedge$ is regular (Lemma 15.43.4), hence the localization $A_\mathfrak{p}^\wedge \otimes K$ is regular. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let B be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in B$. Set $C = B[z]/(z^p - f)$ and note that $B \subset C$ is finite and that the fraction field of C is L. Since $A \subset B \subset C$ are finite and $L/M/K$ are purely inseparable we see that for every element of B or C some power of it lies in A. Hence there is a unique prime $\mathfrak{r} \subset B$, resp. $\mathfrak{q} \subset C$ lying over $\mathfrak{p}$. Note that

$$A_\mathfrak{p}^\wedge \otimes_A M = B_\mathfrak{r}^\wedge \otimes_B M$$

see Algebra, Lemma 10.97.8. By induction we know that this ring is regular. In the same manner we have

$$A_\mathfrak{p}^\wedge \otimes_A L = C_\mathfrak{r}^\wedge \otimes_C L = B_\mathfrak{r}^\wedge \otimes_B M[z]/(z^p - f)$$

the last equality because the completion of $C = B[z]/(z^p - f)$ equals $B_\mathfrak{r}^\wedge[z]/(z^p - f)$. By Lemma 15.48.5 we know there exists a derivation $D : B \to B$ such that $D(f) \neq 0$. In other words, $g = D(f)$ is a unit in M! By Lemma 15.48.1 D extends to a derivation of $B_\mathfrak{r}$, $B_\mathfrak{r}^\wedge$ and $B_\mathfrak{r}^\wedge \otimes_B M$ (successively extending through a localization, a completion, and a localization). Since it is an extension we end up with a derivation of $B_\mathfrak{r}^\wedge \otimes_B M$ which maps f to g and g is a unit of the ring $B_\mathfrak{r}^\wedge \otimes_B M$. Hence $A_\mathfrak{p}^\wedge \otimes_A L$ is regular by Lemma 15.48.4 and we win. □

07PS Proposition 15.50.6. A Noetherian complete local ring is a G-ring.

Proof. Let A be a Noetherian complete local ring. By Lemma 15.50.2 it suffices to check that $B = A/\mathfrak{q}$ has geometrically regular formal fibres over the minimal prime (0) of B. Thus we may assume that A is a domain and it suffices to check the condition for the formal fibres over the minimal prime (0) of A. Let K be the fraction field of A.

We can choose a subring $A_0 \subset A$ which is a regular complete local ring such that A is finite over A_0, see Algebra, Lemma 10.160.11. Moreover, we may assume that A_0 is a power series ring over a field or a Cohen ring. By Lemma 15.50.3 we see that it suffices to prove the result for A_0.

Assume that A is a power series ring over a field or a Cohen ring. Since A is regular the localizations $A_\mathfrak{p}$ are regular (see Algebra, Definition 10.110.7 and the discussion preceding it). Hence the completions $A_\mathfrak{p}^\wedge$ are regular, see Lemma 15.43.4. Hence the fibre $A_\mathfrak{p}^\wedge \otimes_A K$ is, as a localization of $A_\mathfrak{p}^\wedge$, also regular. Thus we are done if the characteristic of K is 0. The positive characteristic case is the case $A = k[[x_1, \ldots, x_d]]$ which is a special case of Lemma 15.50.5. □

07PT Lemma 15.50.7. Let R be a Noetherian ring. Then R is a G-ring if and only if $R_\mathfrak{m}$ has geometrically regular formal fibres for every maximal ideal $\mathfrak{m}$ of R.

Proof. Assume $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ is regular for every maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{p}$ be a prime of R and choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$. Since $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ is faithfully flat we can choose a prime $\mathfrak{p}'$ if $R_\mathfrak{m}^\wedge$ lying over $\mathfrak{p}R_\mathfrak{m}$. Consider the commutative diagram

$$\begin{tikzcd} R_\mathfrak{m}^\wedge \arrow[r] & (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'} \arrow[r] & (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge \\ R_\mathfrak{m} \arrow[r] \arrow[u] & R_\mathfrak{p} \arrow[r] \arrow[u] & R_\mathfrak{p}^\wedge \arrow[u] \end{tikzcd}$$

By assumption the ring map $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ is regular. By Proposition 15.50.6 $(R_\mathfrak{m}^\wedge)_{\mathfrak{p}'} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular. The localization $R_\mathfrak{m}^\wedge \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}$ is regular. Hence $R_\mathfrak{m} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular by Lemma 15.41.4. Since it factors through the localization $R_\mathfrak{p}$, also the ring map $R_\mathfrak{p} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular. Thus we may apply Lemma 15.41.7 to see that $R_\mathfrak{p} \to R_\mathfrak{p}^\wedge$ is regular. □

07QR Lemma 15.50.8. Let R be a Noetherian local ring which is a G-ring. Then the henselization R^h and the strict henselization R^{sh} are G-rings.

Proof. We will use the criterion of Lemma 15.50.7. Let $\mathfrak{q} \subset R^h$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{q}$. Set $\mathfrak{q}_1 = \mathfrak{q}$ and let $\mathfrak{q}_2, \ldots, \mathfrak{q}_t$ be the other primes of R^h lying over $\mathfrak{p}$, so that $R^h \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 15.45.13. Using that $(R^h)^\wedge = R^\wedge$ (Lemma 15.45.3) we see

$$\prod\nolimits_{i=1,\ldots,t} (R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i) = (R^h)^\wedge \otimes_{R^h} (R^h \otimes_R \kappa(\mathfrak{p})) = R^\wedge \otimes_R \kappa(\mathfrak{p})$$

Hence $(R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i)$ is geometrically regular over $\kappa(\mathfrak{p})$ by assumption. Since $\kappa(\mathfrak{q}_i)$ is separable algebraic over $\kappa(\mathfrak{p})$ it follows from Algebra, Lemma 10.166.6 that $(R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i)$ is geometrically regular over $\kappa(\mathfrak{q}_i)$.

Let $\mathfrak{r} \subset R^{sh}$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{r}$. Set $\mathfrak{r}_1 = \mathfrak{r}$ and let $\mathfrak{r}_2, \ldots, \mathfrak{r}_s$ be the other primes of R^{sh} lying over $\mathfrak{p}$, so that $R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,s} \kappa(\mathfrak{r}_i)$, see Lemma 15.45.13. Then we see that

$$\prod\nolimits_{i=1,\ldots,s} (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i) = (R^{sh})^\wedge \otimes_{R^{sh}} (R^{sh} \otimes_R \kappa(\mathfrak{p})) = (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$$

Note that $R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology, see Lemma 15.45.3. Hence $R^\wedge \to (R^{sh})^\wedge$ is regular by Proposition 15.49.2. We conclude that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i)$ is regular over $\kappa(\mathfrak{p})$ by Lemma 15.41.4 as $R^\wedge \otimes_R \kappa(\mathfrak{p})$ is regular over $\kappa(\mathfrak{p})$ by assumption. Since $\kappa(\mathfrak{r}_i)$ is separable algebraic over $\kappa(\mathfrak{p})$ it follows from Algebra, Lemma 10.166.6 that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i)$ is geometrically regular over $\kappa(\mathfrak{r}_i)$. □

07PU Lemma 15.50.9. Let p be a prime number. Let A be a Noetherian complete local domain with fraction field K of characteristic p. Let $\mathfrak{q} \subset A[x]$ be a maximal ideal lying over the maximal ideal of A and let $(0) \neq \mathfrak{r} \subset \mathfrak{q}$ be a prime lying over $(0) \subset A$. Then $A[x]_\mathfrak{q}^\wedge \otimes_{A[x]} \kappa(\mathfrak{r})$ is geometrically regular over $\kappa(\mathfrak{r})$.

Proof. Note that $K \subset \kappa(\mathfrak{r})$ is finite. Hence, given a finite purely inseparable extension $L/\kappa(\mathfrak{r})$ there exists a finite extension of Noetherian complete local domains $A \subset B$ such that $\kappa(\mathfrak{r}) \otimes_A B$ surjects onto L. Namely, you take $B \subset L$ a finite A-subalgebra whose field of fractions is L. Denote $\mathfrak{r}' \subset B[x]$ the kernel of the map $B[x] = A[x] \otimes_A B \to \kappa(\mathfrak{r}) \otimes_A B \to L$ so that $\kappa(\mathfrak{r}') = L$. Then

$$A[x]_\mathfrak{q}^\wedge \otimes_{A[x]} L = A[x]_\mathfrak{q}^\wedge \otimes_{A[x]} B[x] \otimes_{B[x]} \kappa(\mathfrak{r}') = \prod B[x]_{\mathfrak{q}_i}^\wedge \otimes_{B[x]} \kappa(\mathfrak{r}')$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the primes of $B[x]$ lying over $\mathfrak{q}$, see Algebra, Lemma 10.97.8. Thus we see that it suffices to prove the rings $B[x]_{\mathfrak{q}_i}^\wedge \otimes_{B[x]} \kappa(\mathfrak{r}')$ are regular. This reduces us to showing that $A[x]_\mathfrak{q}^\wedge \otimes_{A[x]} \kappa(\mathfrak{r})$ is regular in the special case that $K = \kappa(\mathfrak{r})$.

Assume $K = \kappa(\mathfrak{r})$. In this case we see that $\mathfrak{r}K[x]$ is generated by $x - f$ for some $f \in K$ and

$$A[x]_\mathfrak{q}^\wedge \otimes_{A[x]} \kappa(\mathfrak{r}) = (A[x]_\mathfrak{q}^\wedge \otimes_A K)/(x - f)$$

The derivation $D = \mathrm{d}/\mathrm{d}x$ of $A[x]$ extends to $K[x]$ and maps $x - f$ to a unit of $K[x]$. Moreover D extends to $A[x]^\wedge_\mathfrak{q} \otimes_A K$ by Lemma 15.48.1. As $A \to A[x]^\wedge_\mathfrak{q}$ is formally smooth (see Lemmas 15.37.2 and 15.37.4) the ring $A[x]^\wedge_\mathfrak{q} \otimes_A K$ is regular by Proposition 15.49.2 (the arguments of the proof of that proposition simplify significantly in this particular case). We conclude by Lemma 15.48.2. □

07PV Proposition 15.50.10. Let R be a G-ring. If $R \to S$ is essentially of finite type then S is a G-ring.

Proof. Since being a G-ring is a property of the local rings it is clear that a localization of a G-ring is a G-ring. Conversely, if every localization at a prime is a G-ring, then the ring is a G-ring. Thus it suffices to show that $S_\mathfrak{q}$ is a G-ring for every finite type R-algebra S and every prime $\mathfrak{q}$ of S. Writing S as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 15.50.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a G-ring. By induction on n it suffices to prove that $R[x]$ is a G-ring. Let $\mathfrak{q} \subset R[x]$ be a maximal ideal. By Lemma 15.50.7 it suffices to show that

$$R[x]_\mathfrak{q} \longrightarrow R[x]^\wedge_\mathfrak{q}$$

is regular. If $\mathfrak{q}$ lies over $\mathfrak{p} \subset R$, then we may replace R by $R_\mathfrak{p}$. Hence we may assume that R is a Noetherian local G-ring with maximal ideal $\mathfrak{m}$ and that $\mathfrak{q} \subset R[x]$ lies over $\mathfrak{m}$. Note that there is a unique prime $\mathfrak{q}' \subset R^\wedge[x]$ lying over $\mathfrak{q}$. Consider the diagram

$$\begin{tikzcd} R[x]^\wedge_\mathfrak{q} \arrow[r] & (R^\wedge[x]_{\mathfrak{q}'})^\wedge \\ R[x]_\mathfrak{q} \arrow[r] \arrow[u] & R^\wedge[x]_{\mathfrak{q}'} \arrow[u] \end{tikzcd}$$

Since R is a G-ring the lower horizontal arrow is regular (as a localization of a base change of the regular ring map $R \to R^\wedge$). Suppose we can prove the right vertical arrow is regular. Then it follows that the composition $R[x]_\mathfrak{q} \to (R^\wedge[x]_{\mathfrak{q}'})^\wedge$ is regular, and hence the left vertical arrow is regular by Lemma 15.41.7. Hence we see that we may assume R is a Noetherian complete local ring and $\mathfrak{q}$ a prime lying over the maximal ideal of R.

Let R be a Noetherian complete local ring and let $\mathfrak{q} \subset R[x]$ be a maximal ideal lying over the maximal ideal of R. Let $\mathfrak{r} \subset \mathfrak{q}$ be a prime ideal. We want to show that $R[x]^\wedge_\mathfrak{q} \otimes_{R[x]} \kappa(\mathfrak{r})$ is a geometrically regular algebra over $\kappa(\mathfrak{r})$. Set $\mathfrak{p} = R \cap \mathfrak{r}$. Then we can replace R by $R/\mathfrak{p}$ and $\mathfrak{q}$ and $\mathfrak{r}$ by their images in $R/\mathfrak{p}[x]$, see Lemma 15.50.2. Hence we may assume that R is a domain and that $\mathfrak{r} \cap R = (0)$.

By Algebra, Lemma 10.160.11 we can find $R_0 \subset R$ which is regular and such that R is finite over R_0. Applying Lemma 15.50.3 we see that it suffices to prove $R[x]^\wedge_\mathfrak{q} \otimes_{R[x]} \kappa(\mathfrak{r})$ is geometrically regular over $\kappa(r)$ when, in addition to the above, R is a regular complete local ring.

Now R is a regular complete local ring, we have $\mathfrak{q} \subset \mathfrak{r} \subset R[x]$, we have $(0) = R \cap \mathfrak{r}$ and $\mathfrak{q}$ is a maximal ideal lying over the maximal ideal of R. Since R is regular the ring $R[x]$ is regular (Algebra, Lemma 10.163.10). Hence the localization $R[x]_\mathfrak{q}$ is regular. Hence the completions $R[x]^\wedge_\mathfrak{q}$ are regular, see Lemma 15.43.4. Hence the fibre $R[x]^\wedge_\mathfrak{q} \otimes_{R[x]} \kappa(\mathfrak{r})$ is, as a localization of $R[x]^\wedge_\mathfrak{q}$, also regular. Thus we are done if the characteristic of the fraction field of R is 0.

If the characteristic of R is positive, then $R = k[[x_1, \ldots, x_n]]$. In this case we split the argument in two subcases:

(1) The case $\mathfrak{r} = (0)$. The result is a direct consequence of Lemma 15.50.5.
(2) The case $\mathfrak{r} \neq (0)$. This is Lemma 15.50.9.

□

07PW Remark 15.50.11. Let R be a G-ring and let $I \subset R$ be an ideal. In general it is not the case that the I-adic completion $R^\wedge$ is a G-ring. An example was given by Nishimura in [Nis81]. A generalization and, in some sense, clarification of this example can be found in the last section of [Dum00].

07PX Proposition 15.50.12. The following types of rings are G-rings:

(1) fields,
(2) Noetherian complete local rings,
(3) $\mathbf{Z}$,
(4) Dedekind domains with fraction field of characteristic zero,
(5) finite type ring extensions of any of the above.

Proof. For fields, $\mathbf{Z}$ and Dedekind domains of characteristic zero this follows immediately from the definition and the fact that the completion of a discrete valuation ring is a discrete valuation ring. A Noetherian complete local ring is a G-ring by Proposition 15.50.6. The statement on finite type overrings is Proposition 15.50.10. □

0A41 Lemma 15.50.13. Let $(A, \mathfrak{m})$ be a henselian local ring. Then A is a filtered colimit of a system of henselian local G-rings with local transition maps.

Proof. Write $A = \text{colim}\, A_i$ as a filtered colimit of finite type $\mathbf{Z}$-algebras. Let $\mathfrak{p}_i$ be the prime ideal of A_i lying under $\mathfrak{m}$. We may replace A_i by the localization of A_i at $\mathfrak{p}_i$. Then A_i is a Noetherian local G-ring (Proposition 15.50.12). By Lemma 15.12.5 we see that $A = \text{colim}\, A_i^h$. By Lemma 15.50.8 the rings A_i^h are G-rings. □

0AH2 Lemma 15.50.14. Let A be a G-ring. Let $I \subset A$ be an ideal and let $A^\wedge$ be the completion of A with respect to I. Then $A \to A^\wedge$ is regular. [Mat70a, Theorem 79]

Proof. The ring map $A \to A^\wedge$ is flat by Algebra, Lemma 10.97.2. The ring $A^\wedge$ is Noetherian by Algebra, Lemma 10.97.6. Thus it suffices to check the third condition of Lemma 15.41.2. Let $\mathfrak{m}' \subset A^\wedge$ be a maximal ideal lying over $\mathfrak{m} \subset A$. By Algebra, Lemma 10.96.6 we have $IA^\wedge \subset \mathfrak{m}'$. Since $A^\wedge/IA^\wedge = A/I$ we see that $I \subset \mathfrak{m}$, $\mathfrak{m}/I = \mathfrak{m}'/IA^\wedge$, and $A/\mathfrak{m} = A^\wedge/\mathfrak{m}'$. Since $A^\wedge/\mathfrak{m}'$ is a field, we conclude that $\mathfrak{m}$ is a maximal ideal as well. Then $A_\mathfrak{m} \to A^\wedge_{\mathfrak{m}'}$ is a flat local ring homomorphism of Noetherian local rings which identifies residue fields and such that $\mathfrak{m}A^\wedge_{\mathfrak{m}'} = \mathfrak{m}'A^\wedge_{\mathfrak{m}'}$. Thus it induces an isomorphism on complete local rings, see Lemma 15.43.9. Let $(A_\mathfrak{m})^\wedge$ be the completion of $A_\mathfrak{m}$ with respect to its maximal ideal. The ring map

$$(A^\wedge)_{\mathfrak{m}'} \to ((A^\wedge)_{\mathfrak{m}'})^\wedge = (A_\mathfrak{m})^\wedge$$

is faithfully flat (Algebra, Lemma 10.97.3). Thus we can apply Lemma 15.41.7 to the ring maps

$$A_\mathfrak{m} \to (A^\wedge)_{\mathfrak{m}'} \to (A_\mathfrak{m})^\wedge$$

to conclude because $A_\mathfrak{m} \to (A_\mathfrak{m})^\wedge$ is regular as A is a G-ring. □

0AH3 **Lemma 15.50.15.** Let A be a G-ring. Let $I \subset A$ be an ideal. Let (A^h, I^h) be the henselization of the pair (A, I), see Lemma 15.12.1. Then A^h is a G-ring. [Gre76, Theorem 5.3 i)]

Proof. Let $\mathfrak{m}^h \subset A^h$ be a maximal ideal. We have to show that the map from $A^h_{\mathfrak{m}^h}$ to its completion has geometrically regular fibres, see Lemma 15.50.7. Let $\mathfrak{m}$ be the inverse image of $\mathfrak{m}^h$ in A. Note that $I^h \subset \mathfrak{m}^h$ and hence $I \subset \mathfrak{m}$ as (A^h, I^h) is a henselian pair. Recall that A^h is Noetherian, $I^h = IA^h$, and that $A \to A^h$ induces an isomorphism on I-adic completions, see Lemma 15.12.4. Then the local homomorphism of Noetherian local rings

$$A_{\mathfrak{m}} \to A^h_{\mathfrak{m}^h}$$

induces an isomorphism on completions at maximal ideals by Lemma 15.43.9 (details omitted). Let $\mathfrak{q}^h$ be a prime of $A^h_{\mathfrak{m}^h}$ lying over $\mathfrak{q} \subset A_{\mathfrak{m}}$. Set $\mathfrak{q}_1 = \mathfrak{q}^h$ and let $\mathfrak{q}_2, \ldots, \mathfrak{q}_t$ be the other primes of A^h lying over $\mathfrak{q}$, so that $A^h \otimes_A \kappa(\mathfrak{q}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 15.45.12. Using that $(A^h)^\wedge_{\mathfrak{m}^h} = (A_{\mathfrak{m}})^\wedge$ as discussed above we see

$$\prod\nolimits_{i=1,\ldots,t} (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}_i) = (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} (A^h_{\mathfrak{m}^h} \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})) = (A_{\mathfrak{m}})^\wedge \otimes_{A_{\mathfrak{m}}} \kappa(\mathfrak{q})$$

Hence, as one of the components, the ring

$$(A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$$

is geometrically regular over $\kappa(\mathfrak{q})$ by assumption on A. Since $\kappa(\mathfrak{q}^h)$ is separable algebraic over $\kappa(\mathfrak{q})$ it follows from Algebra, Lemma 10.166.6 that

$$(A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$$

is geometrically regular over $\kappa(\mathfrak{q}^h)$ as desired. $\square$

15.51. Properties of formal fibres

0BIR In this section we redo some of the arguments of Section 15.50 for to be able to talk intelligently about properties of the formal fibres of Noetherian rings.

Let P be a property of ring maps $k \to R$ where k is a field and R is Noetherian. We say P holds for the fibres of a ring homomorphism $A \to B$ with B Noetherian if P holds for $\kappa(\mathfrak{q}) \to B \otimes_A \kappa(\mathfrak{q})$ for all primes $\mathfrak{q}$ of A. In the following we will use the following assertions

(A) $P(k \to R) \Rightarrow P(k' \to R \otimes_k k')$ for finitely generated field extensions k'/k,
(B) $P(k \to R_{\mathfrak{p}})$, $\forall \mathfrak{p} \in \mathrm{Spec}(R) \Leftrightarrow P(k \to R)$,
(C) given flat maps $A \to B \to C$ of Noetherian rings, if the fibres of $A \to B$ have P and $B \to C$ is regular, then the fibres of $A \to C$ have P,
(D) given flat maps $A \to B \to C$ of Noetherian rings if the fibres of $A \to C$ have P and $B \to C$ is faithfully flat, then the fibres of $A \to B$ have P,
(E) given $k \to k' \to R$ with R Noetherian if k'/k is separable algebraic and $P(k \to R)$, then $P(k' \to R)$, and
(F) add more here.

Given a Noetherian local ring A we say "the formal fibres of A have P" if P holds for the fibres of $A \to A^\wedge$. We say that R is a P-ring if R is Noetherian and for all primes $\mathfrak{p}$ of R the formal fibres of $R_{\mathfrak{p}}$ have P.

0BIS Lemma 15.51.1. Let R be a Noetherian ring. Let P be a property as above. Then R is a P-ring if and only if for every pair of primes $\mathfrak{q} \subset \mathfrak{p} \subset R$ the $\kappa(\mathfrak{q})$-algebra

$$(R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

has property P.

Proof. This follows from the fact that

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) = (R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

as algebras over $\kappa(\mathfrak{q})$. □

0BK8 Lemma 15.51.2. Let $R \to \Lambda$ be a homomorphism of Noetherian rings. Assume P has property (B). The following are equivalent

(1) the fibres of $R \to \Lambda$ have P,
(2) the fibres of $R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ have P for all $\mathfrak{q} \subset \Lambda$ lying over $\mathfrak{p} \subset R$, and
(3) the fibres of $R_\mathfrak{m} \to \Lambda_{\mathfrak{m}'}$ have P for all maximal ideals $\mathfrak{m}' \subset \Lambda$ lying over $\mathfrak{m}$ in R.

Proof. Let $\mathfrak{p} \subset R$ be a prime. Then the fibre over $\mathfrak{p}$ is the ring $\Lambda \otimes_R \kappa(\mathfrak{p})$ whose spectrum maps bijectively onto the subset of $\text{Spec}(\Lambda)$ consisting of primes $\mathfrak{q}$ lying over $\mathfrak{p}$, see Algebra, Remark 10.17.8. For such a prime $\mathfrak{q}$ choose a maximal ideal $\mathfrak{q} \subset \mathfrak{m}'$ and set $\mathfrak{m} = R \cap \mathfrak{m}'$. Then $\mathfrak{p} \subset \mathfrak{m}$ and we have

$$(\Lambda \otimes_R \kappa(\mathfrak{p}))_\mathfrak{q} \cong (\Lambda_{\mathfrak{m}'} \otimes_{R_\mathfrak{m}} \kappa(\mathfrak{p}))_\mathfrak{q}$$

as $\kappa(\mathfrak{q})$-algebras. Thus (1), (2), and (3) are equivalent because by (B) we can check property P on local rings. □

0BIT Lemma 15.51.3. Let $R \to R'$ be a finite type map of Noetherian rings and let

$$\begin{tikzcd} \mathfrak{q}' \arrow[r] \arrow[d, dash] & \mathfrak{p}' \arrow[r] \arrow[d, dash] & R' \\ \mathfrak{q} \arrow[r] & \mathfrak{p} \arrow[r] & R \arrow[u] \end{tikzcd}$$

be primes. Assume $R \to R'$ is quasi-finite at $\mathfrak{p}'$. Assume P satisfies (A) and (B).

(1) If $\kappa(\mathfrak{q}) \to R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$ has P, then $\kappa(\mathfrak{q}') \to R'^\wedge_{\mathfrak{p}'} \otimes_{R'} \kappa(\mathfrak{q}')$ has P.
(2) If the formal fibres of $R_\mathfrak{p}$ have P, then the formal fibres of $R'_{\mathfrak{p}'}$ have P.
(3) If $R \to R'$ is quasi-finite and R is a P-ring, then R' is a P-ring.

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume P holds for $\kappa(\mathfrak{q}) \to R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$. By Algebra, Lemma 10.124.3 we see that

$$R^\wedge_\mathfrak{p} \otimes_R R' = (R'_{\mathfrak{p}'})^\wedge \times B$$

for some $R^\wedge_\mathfrak{p}$-algebra B. Hence $R'_{\mathfrak{p}'} \to (R'_{\mathfrak{p}'})^\wedge$ is a factor of a base change of the map $R_\mathfrak{p} \to R^\wedge_\mathfrak{p}$. It follows that $(R'_{\mathfrak{p}'})^\wedge \otimes_{R'} \kappa(\mathfrak{q}')$ is a factor of

$$R^\wedge_\mathfrak{p} \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}').$$

Thus the result follows from the assumptions on P. □

0BIU Lemma 15.51.4. Let R be a Noetherian ring. Assume P satisfies (C) and (D). Then R is a P-ring if and only if the formal fibres of $R_\mathfrak{m}$ have P for every maximal ideal $\mathfrak{m}$ of R.

Proof. Assume the formal fibres of $R_\mathfrak{m}$ have P for all maximal ideals $\mathfrak{m}$ of R. Let $\mathfrak{p}$ be a prime of R and choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$. Since $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ is faithfully flat we can choose a prime $\mathfrak{p}'$ if $R_\mathfrak{m}^\wedge$ lying over $\mathfrak{p}R_\mathfrak{m}$. Consider the commutative diagram

$$\begin{tikzcd} R_\mathfrak{m}^\wedge \arrow[r] & (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'} \arrow[r] & (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge \\ R_\mathfrak{m} \arrow[r] \arrow[u] & R_\mathfrak{p} \arrow[r] \arrow[u] & R_\mathfrak{p}^\wedge \arrow[u] \end{tikzcd}$$

By assumption the fibres of the ring map $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ have P. By Proposition 15.50.6 $(R_\mathfrak{m}^\wedge)_{\mathfrak{p}'} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular. The localization $R_\mathfrak{m}^\wedge \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}$ is regular. Hence $R_\mathfrak{m}^\wedge \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ is regular by Lemma 15.41.4. Hence the fibres of $R_\mathfrak{m} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ have P by (C). Since $R_\mathfrak{m} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ factors through the localization $R_\mathfrak{p}$, also the fibres of $R_\mathfrak{p} \to (R_\mathfrak{m}^\wedge)_{\mathfrak{p}'}^\wedge$ have P. Thus we may apply (D) to see that the fibres of $R_\mathfrak{p} \to R_\mathfrak{p}^\wedge$ have P. $\square$

0BIV Proposition 15.51.5. Let R be a P-ring where P satisfies (A), (B), (C), and (D). If $R \to S$ is essentially of finite type then S is a P-ring.

Proof. Since being a P-ring is a property of the local rings it is clear that a localization of a P-ring is a P-ring. Conversely, if every localization at a prime is a P-ring, then the ring is a P-ring. Thus it suffices to show that $S_\mathfrak{q}$ is a P-ring for every finite type R-algebra S and every prime $\mathfrak{q}$ of S. Writing S as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 15.51.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a P-ring. By induction on n it suffices to prove that $R[x]$ is a P-ring. Let $\mathfrak{q} \subset R[x]$ be a maximal ideal. By Lemma 15.51.4 it suffices to show that the fibres of

$$R[x]_\mathfrak{q} \longrightarrow R[x]_\mathfrak{q}^\wedge$$

have P. If $\mathfrak{q}$ lies over $\mathfrak{p} \subset R$, then we may replace R by $R_\mathfrak{p}$. Hence we may assume that R is a Noetherian local P-ring with maximal ideal $\mathfrak{m}$ and that $\mathfrak{q} \subset R[x]$ lies over $\mathfrak{m}$. Note that there is a unique prime $\mathfrak{q}' \subset R^\wedge[x]$ lying over $\mathfrak{q}$. Consider the diagram

$$\begin{tikzcd} R[x]_\mathfrak{q}^\wedge \arrow[r] & (R^\wedge[x]_{\mathfrak{q}'})^\wedge \\ R[x]_\mathfrak{q} \arrow[r] \arrow[u] & R^\wedge[x]_{\mathfrak{q}'} \arrow[u] \end{tikzcd}$$

Since R is a P-ring the fibres of $R[x] \to R^\wedge[x]$ have P because they are base changes of the fibres of $R \to R^\wedge$ by a finitely generated field extension so (A) applies. Hence the fibres of the lower horizontal arrow have P for example by Lemma 15.51.2. The right vertical arrow is regular because $R^\wedge$ is a G-ring (Propositions 15.50.6 and 15.50.10). It follows that the fibres of the composition $R[x]_\mathfrak{q} \to (R^\wedge[x]_{\mathfrak{q}'})^\wedge$ have P by (C). Hence the fibres of the left vertical arrow have P by (D) and the proof is complete. $\square$

0BK9 Lemma 15.51.6. Let A be a P-ring where P satisfies (B) and (D). Let $I \subset A$ be an ideal and let $A^\wedge$ be the completion of A with respect to I. Then the fibres of $A \to A^\wedge$ have P.

Proof. The ring map $A \to A^\wedge$ is flat by Algebra, Lemma 10.97.2. The ring $A^\wedge$ is Noetherian by Algebra, Lemma 10.97.6. Thus it suffices to check the third condition of Lemma 15.51.2. Let $\mathfrak{m}' \subset A^\wedge$ be a maximal ideal lying over $\mathfrak{m} \subset A$. By Algebra, Lemma 10.96.6 we have $IA^\wedge \subset \mathfrak{m}'$. Since $A^\wedge/IA^\wedge = A/I$ we see that $I \subset \mathfrak{m}$, $\mathfrak{m}/I = \mathfrak{m}'/IA^\wedge$, and $A/\mathfrak{m} = A^\wedge/\mathfrak{m}'$. Since $A^\wedge/\mathfrak{m}'$ is a field, we conclude that $\mathfrak{m}$ is a maximal ideal as well. Then $A_\mathfrak{m} \to A^\wedge_{\mathfrak{m}'}$ is a flat local ring homomorphism of Noetherian local rings which identifies residue fields and such that $\mathfrak{m}A^\wedge_{\mathfrak{m}'} = \mathfrak{m}'A^\wedge_{\mathfrak{m}'}$. Thus it induces an isomorphism on complete local rings, see Lemma 15.43.9. Let $(A_\mathfrak{m})^\wedge$ be the completion of $A_\mathfrak{m}$ with respect to its maximal ideal. The ring map

$$(A^\wedge)_{\mathfrak{m}'} \to ((A^\wedge)_{\mathfrak{m}'})^\wedge = (A_\mathfrak{m})^\wedge$$

is faithfully flat (Algebra, Lemma 10.97.3). Thus we can apply (D) to the ring maps

$$A_\mathfrak{m} \to (A^\wedge)_{\mathfrak{m}'} \to (A_\mathfrak{m})^\wedge$$

to conclude because the fibres of $A_\mathfrak{m} \to (A_\mathfrak{m})^\wedge$ have P as A is a P-ring. □

0BKA Lemma 15.51.7. Let A be a P-ring where P satisfies (B), (C), (D), and (E). Let $I \subset A$ be an ideal. Let (A^h, I^h) be the henselization of the pair (A, I), see Lemma 15.12.1. Then A^h is a P-ring.

Proof. Let $\mathfrak{m}^h \subset A^h$ be a maximal ideal. We have to show that the fibres of $A^h_{\mathfrak{m}^h} \to (A^h_{\mathfrak{m}^h})^\wedge$ have P, see Lemma 15.51.4. Let $\mathfrak{m}$ be the inverse image of $\mathfrak{m}^h$ in A. Note that $I^h \subset \mathfrak{m}^h$ and hence $I \subset \mathfrak{m}$ as (A^h, I^h) is a henselian pair. Recall that A^h is Noetherian, $I^h = IA^h$, and that $A \to A^h$ induces an isomorphism on I-adic completions, see Lemma 15.12.4. Then the local homomorphism of Noetherian local rings

$$A_\mathfrak{m} \to A^h_{\mathfrak{m}^h}$$

induces an isomorphism on completions at maximal ideals by Lemma 15.43.9 (details omitted). Let $\mathfrak{q}^h$ be a prime of $A^h_{\mathfrak{m}^h}$ lying over $\mathfrak{q} \subset A_\mathfrak{m}$. Set $\mathfrak{q}_1 = \mathfrak{q}^h$ and let $\mathfrak{q}_2, \ldots, \mathfrak{q}_t$ be the other primes of A^h lying over $\mathfrak{q}$, so that $A^h \otimes_A \kappa(\mathfrak{q}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 15.45.12. Using that $(A^h)^\wedge_{\mathfrak{m}^h} = (A_\mathfrak{m})^\wedge$ as discussed above we see

$$\prod\nolimits_{i=1,\ldots,t} (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}_i) = (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} (A^h_{\mathfrak{m}^h} \otimes_{A_\mathfrak{m}} \kappa(\mathfrak{q})) = (A_\mathfrak{m})^\wedge \otimes_{A_\mathfrak{m}} \kappa(\mathfrak{q})$$

Hence, looking at local rings and using (B), we see that

$$\kappa(\mathfrak{q}) \longrightarrow (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$$

has P as $\kappa(\mathfrak{q}) \to (A_\mathfrak{m})^\wedge \otimes_{A_\mathfrak{m}} \kappa(\mathfrak{q})$ does by assumption on A. Since $\kappa(\mathfrak{q}^h)/\kappa(\mathfrak{q})$ is separable algebraic, by (E) we find that $\kappa(\mathfrak{q}^h) \to (A^h_{\mathfrak{m}^h})^\wedge \otimes_{A^h_{\mathfrak{m}^h}} \kappa(\mathfrak{q}^h)$ has P as desired. □

0C36 Lemma 15.51.8. Let R be a Noetherian local ring which is a P-ring where P satisfies (B), (C), (D), and (E). Then the henselization R^h and the strict henselization R^{sh} are P-rings.

Proof. We have seen this for the henselization in Lemma 15.51.7. To prove it for the strict henselization, it suffices to show that the formal fibres of R^{sh} have P, see Lemma 15.51.4. Let $\mathfrak{r} \subset R^{sh}$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{r}$. Set $\mathfrak{r}_1 = \mathfrak{r}$

and let $\mathfrak{r}_2, \ldots, \mathfrak{r}_s$ be the other primes of R^{sh} lying over $\mathfrak{p}$, so that $R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,s} \kappa(\mathfrak{r}_i)$, see Lemma 15.45.13. Then we see that

$$\prod\nolimits_{i=1,\ldots,t} (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i) = (R^{sh})^\wedge \otimes_{R^{sh}} (R^{sh} \otimes_R \kappa(\mathfrak{p})) = (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$$

Note that $R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology, see Lemma 15.45.3. Hence $R^\wedge \to (R^{sh})^\wedge$ is regular by Proposition 15.49.2. We conclude that property P holds for $\kappa(\mathfrak{p}) \to (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$ by (C) and our assumption on R. Using property (B), using the decomposition above, and looking at local rings we conclude that property P holds for $\kappa(\mathfrak{p}) \to (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r})$. Since $\kappa(\mathfrak{r})/\kappa(\mathfrak{p})$ is separable algebraic, it follows from (E) that P holds for $\kappa(\mathfrak{r}) \to (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r})$. $\square$

0BIW Lemma 15.51.9. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is geometrically reduced over k".

Proof. Part (A) follows from the definition of geometrically reduced algebras (Algebra, Definition 10.43.1). Part (B) follows too: a ring is reduced if and only if all local rings are reduced. Part (C). This follows from Lemma 15.42.1. Part (D). This follows from Algebra, Lemma 10.164.2. Part (E). This follows from Algebra, Lemma 10.43.9. $\square$

0BIX Lemma 15.51.10. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is geometrically normal over k".

Proof. Part (A) follows from the definition of geometrically normal algebras (Algebra, Definition 10.165.2). Part (B) follows too: a ring is normal if and only if all of its local rings are normal. Part (C). This follows from Lemma 15.42.2. Part (D). This follows from Algebra, Lemma 10.164.3. Part (E). This follows from Algebra, Lemma 10.165.6. $\square$

0BIY Lemma 15.51.11. Fix $n \geq 1$. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R has (S_n)".

Proof. Let $k \to R$ be a ring map where k is a field and R a Noetherian ring. Let k'/k be a finitely generated field extension. Then the fibres of the ring map $R \to R \otimes_k k'$ are Cohen-Macaulay by Algebra, Lemma 10.167.1. Hence we may apply Algebra, Lemma 10.163.4 to the ring map $R \to R \otimes_k k'$ to see that if R has (S_n) so does $R \otimes_k k'$. This proves (A). Part (B) follows too: a Noetherian rings has (S_n) if and only if all of its local rings have (S_n). Part (C). This follows from Algebra, Lemma 10.163.4 as the fibres of a regular homomorphism are regular and in particular Cohen-Macaulay. Part (D). This follows from Algebra, Lemma 10.164.5. Part (E). This is immediate as the condition does not refer to the ground field. $\square$

0BJ9 Lemma 15.51.12. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"R is Cohen-Macaulay".

Proof. Follows immediately from Lemma 15.51.11 and the fact that a Noetherian ring is Cohen-Macaulay if and only if it satisfies conditions (S_n) for all n. $\square$

0BIZ Lemma 15.51.13. Fix $n \geq 0$. Properties (A), (B), (C), (D), and (E) hold for $P(k \to R) =$"$R \otimes_k k'$ has (R_n) for all finite extensions k'/k".

Proof. Let $k \to R$ be a ring map where k is a field and R a Noetherian ring. Assume $P(k \to R)$ is true. Let K/k be a finitely generated field extension. By Algebra, Lemma 10.45.3 we can find a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Algebra, Lemma 10.158.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $R \otimes_k k'$ satisfies (S_n) because we assumed P for $k \to R$. Step 2: $R \otimes_k k' \to R \otimes_k k' \otimes_{k'} B$ is a smooth ring map (Algebra, Lemma 10.137.4) and we conclude $R \otimes_k k' \otimes_{k'} B$ satisfies (S_n) by Algebra, Lemma 10.163.5 (and using Algebra, Lemma 10.140.3 to see that the hypotheses are satisfied). Step 3. $R \otimes_k k' \otimes_{k'} K' = R \otimes_k K'$ satisfies (R_n) as it is a localization of a ring having (R_n). Step 4. Finally $R \otimes_k K$ satisfies (R_n) by descent of (R_n) along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Algebra, Lemma 10.164.6). This proves (A). Part (B) follows too: a Noetherian ring has (R_n) if and only if all of its local rings have (R_n). Part (C). This follows from Algebra, Lemma 10.163.5 as the fibres of a regular homomorphism are regular (small detail omitted). Part (D). This follows from Algebra, Lemma 10.164.6 (small detail omitted).

Part (E). Let l/k be a separable algebraic extension of fields and let $l \to R$ be a ring map with R Noetherian. Assume that $k \to R$ has P. We have to show that $l \to R$ has P. Let l'/l be a finite extension. First observe that there exists a finite subextension $l/m/k$ and a finite extension m'/m such that $l' = l \otimes_m m'$. Then $R \otimes_l l' = R \otimes_m m'$. Hence it suffices to prove that $m \to R$ has property P, i.e., we may assume that l/k is finite. If l/k is finite, then l'/k is finite and we see that

$$l' \otimes_l R = (l' \otimes_k R) \otimes_{l \otimes_k l} l$$

is a localization (by Algebra, Lemma 10.43.8) of the Noetherian ring $l' \otimes_k R$ which has property (R_n) by assumption P for $k \to R$. This proves that $l' \otimes_l R$ has property (R_n) as desired. $\square$

15.52. Excellent rings

07QS In this section we discuss Grothendieck's notion of excellent rings. For the definitions of G-rings, J-2 rings, and universally catenary rings we refer to Definition 15.50.1, Definition 15.47.1, and Algebra, Definition 10.105.3.

07QT Definition 15.52.1. Let R be a ring.

(1) We say R is quasi-excellent if R is Noetherian, a G-ring, and J-2.
(2) We say R is excellent if R is quasi-excellent and universally catenary.

Thus a Noetherian ring is quasi-excellent if it has geometrically regular formal fibres and if any finite type algebra over it has closed singular set. For such a ring to be excellent we require in addition that there exists (locally) a good dimension function. We will see later (Section 15.109) that to be universally catenary can be formulated as a condition on the maps $R_\mathfrak{m} \to R_\mathfrak{m}^\wedge$ for maximal ideals $\mathfrak{m}$ of R.

07QU Lemma 15.52.2. Any localization of a finite type ring over a (quasi-)excellent ring is (quasi-)excellent.

Proof. For finite type algebras this follows from the definitions for the properties J-2 and universally catenary. For G-rings, see Proposition 15.50.10. We omit the proof that localization preserves (quasi-)excellency. □

07QW Proposition 15.52.3. The following types of rings are excellent:

(1) fields,
(2) Noetherian complete local rings,
(3) $\mathbf{Z}$,
(4) Dedekind domains with fraction field of characteristic zero,
(5) finite type ring extensions of any of the above.

Proof. See Propositions 15.50.12 and 15.48.7 to see that these rings are G-rings and have J-2. Any Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 10.105.9. In particular fields, Dedekind rings, and more generally regular rings are universally catenary. Via the Cohen structure theorem we see that complete local rings are universally catenary, see Algebra, Remark 10.160.9. □

The material developed above has some consequences for Nagata rings.

0BJ0 Lemma 15.52.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring. The following are equivalent

(1) A is Nagata, and
(2) the formal fibres of A are geometrically reduced.

Proof. Assume (2). By Algebra, Lemma 10.162.14 we have to show that if $A \to B$ is finite, B is a domain, and $\mathfrak{m}' \subset B$ is a maximal ideal, then $B_{\mathfrak{m}'}$ is analytically unramified. Combining Lemmas 15.51.9 and 15.51.4 and Proposition 15.51.5 we see that the formal fibres of $B_{\mathfrak{m}'}$ are geometrically reduced. In particular $B_{\mathfrak{m}'}^\wedge \otimes_B L$ is reduced where L is the fraction field of B. It follows that $B_{\mathfrak{m}'}^\wedge$ is reduced, i.e., $B_{\mathfrak{m}'}$ is analytically unramified.

Assume (1). Let $\mathfrak{q} \subset A$ be a prime ideal and let $K/\kappa(\mathfrak{q})$ be a finite extension. We have to show that $A^\wedge \otimes_A K$ is reduced. Let $A/\mathfrak{q} \subset B \subset K$ be a local subring finite over A whose fraction field is K. To construct B choose $x_1, \ldots, x_n \in K$ which generate K over $\kappa(\mathfrak{q})$ and which satisfy monic polynomials $P_i(T) = T^{d_i} + a_{i,1}T^{d_i-1} + \ldots + a_{i,d_i} = 0$ with $a_{i,j} \in \mathfrak{m}$. Then let B be the A-subalgebra of K generated by $x_1, \ldots, x_n$. (For more details see the proof of Algebra, Lemma 10.162.14.) Then

$$A^\wedge \otimes_A K = (A^\wedge \otimes_A B)_\mathfrak{q} = B^\wedge_\mathfrak{q}$$

Since $B^\wedge$ is reduced by Algebra, Lemma 10.162.14 the proof is complete. □

07QV Lemma 15.52.5. A quasi-excellent ring is Nagata.

Proof. Let R be quasi-excellent. Using that a finite type algebra over R is quasi-excellent (Lemma 15.52.2) we see that it suffices to show that any quasi-excellent domain is N-1, see Algebra, Lemma 10.162.3. Applying Algebra, Lemma 10.161.15 (and using that a quasi-excellent ring is J-2) we reduce to showing that a quasi-excellent local domain R is N-1. As $R \to R^\wedge$ is regular we see that $R^\wedge$ is reduced by Lemma 15.42.1. In other words, R is analytically unramified. Hence R is N-1 by Algebra, Lemma 10.162.10. □

0C23 Lemma 15.52.6. Let $(A, \mathfrak{m})$ be a Noetherian local ring. If A is normal and the formal fibres of A are normal (for example if A is excellent or quasi-excellent), then $A^\wedge$ is normal.

Proof. Follows immediately from Algebra, Lemma 10.163.8. □

15.53. Abelian categories of modules

0AZ5 Let R be a ring. The category Mod_R of R-modules is an abelian category. Here are some examples of subcategories of Mod_R which are abelian (we use the terminology introduced in Homology, Definition 12.10.1 as well as Homology, Lemmas 12.10.2 and 12.10.3):

(1) The category of coherent R-modules is a weak Serre subcategory of Mod_R. This follows from Algebra, Lemma 10.90.3.
(2) Let $S \subset R$ be a multiplicative subset. The full subcategory consisting of R-modules M such that multiplication by $s \in S$ is an isomorphism on M is a Serre subcategory of Mod_R. This follows from Algebra, Lemma 10.9.5.
(3) Let $I \subset R$ be a finitely generated ideal. The full subcategory of I-power torsion modules is a Serre subcategory of Mod_R. See Lemma 15.88.5.
(4) In some texts a torsion module is defined as a module M such that for all $x \in M$ there exists a nonzerodivisor $f \in R$ such that $fx = 0$. The full subcategory of torsion modules is a Serre subcategory of Mod_R.
(5) If R is not Noetherian, then the category Mod_R^{fg} of finitely generated R-modules is not abelian. Namely, if $I \subset R$ is a non-finitely generated ideal, then the map $R \to R/I$ does not have a kernel in Mod_R^{fg}.
(6) If R is Noetherian, then coherent R-modules agree with finitely generated (i.e., finite) R-modules, see Algebra, Lemmas 10.90.5, 10.90.4, and 10.31.4. Hence Mod_R^{fg} is abelian by (1) above, but in fact,in this case the category Mod_R^{fg} is a (strong) Serre subcategory of Mod_R.

15.54. Injective abelian groups

01D6 In this section we show the category of abelian groups has enough injectives. Recall that an abelian group M is divisible if and only if for every $x \in M$ and every $n \in \mathbf{N}$ there exists a $y \in M$ such that $ny = x$.

01D7 Lemma 15.54.1. An abelian group J is an injective object in the category of abelian groups if and only if J is divisible.

Proof. Suppose that J is not divisible. Then there exists an $x \in J$ and $n \in \mathbf{N}$ such that there is no $y \in J$ with $ny = x$. Then the morphism $\mathbf{Z} \to J$, $m \mapsto mx$ does not extend to $\frac{1}{n}\mathbf{Z} \supset \mathbf{Z}$. Hence J is not injective.

Let $A \subset B$ be abelian groups. Assume that J is a divisible abelian group. Let $\varphi : A \to J$ be a morphism. Consider the set of homomorphisms $\varphi' : A' \to J$ with $A \subset A' \subset B$ and $\varphi'|_A = \varphi$. Define $(A', \varphi') \geq (A'', \varphi'')$ if and only if $A' \supset A''$ and $\varphi'|_{A''} = \varphi''$. If $(A_i, \varphi_i)_{i \in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i \in I} A_i \to J$ defined by $a \in A_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (A', φ') is maximal then $A' = B$. In other words, it suffices to show, given any subgroup $A \subset B$, $A \neq B$ and any $\varphi : A \to J$, then we can find $\varphi' : A' \to J$ with $A \subset A' \subset B$ such that (a) the inclusion $A \subset A'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in B$, $x \notin A$. If there exists no $n \in \mathbf{N}$ such that $nx \in A$, then $A \oplus \mathbf{Z} \cong A + \mathbf{Z}x$. Hence we can extend φ to $A' = A + \mathbf{Z}x$ by using φ on A and mapping x to zero for example. If there does exist an $n \in \mathbf{N}$ such that $nx \in A$, then let n be the minimal such integer. Let $z \in J$ be an element such that $nz = \varphi(nx)$. Define a morphism $\tilde{\varphi} : A \oplus \mathbf{Z} \to J$ by $(a, m) \mapsto \varphi(a) + mz$. By our choice of z the kernel of $\tilde{\varphi}$ contains the kernel of the map $A \oplus \mathbf{Z} \to B$, $(a, m) \mapsto a + mx$. Hence $\tilde{\varphi}$ factors through the image $A' = A + \mathbf{Z}x$, and this extends the morphism φ. $\square$

We can use this lemma to show that every abelian group can be embedded in a injective abelian group. But this is a special case of the result of the following section.

15.55. Injective modules

01D8 Some lemmas on injective modules.

0AVD Definition 15.55.1. Let R be a ring. An R-module J is injective if and only if the functor $\mathrm{Hom}_R(-, J) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor.

The functor $\mathrm{Hom}_R(-, M)$ is left exact for any R-module M, see Algebra, Lemma 10.10.1. Hence the condition for J to be injective really signifies that given an injection of R-modules $M \to M'$ the map $\mathrm{Hom}_R(M', J) \to \mathrm{Hom}_R(M, J)$ is surjective.

Before we reformulate this in terms of *Ext*-modules we discuss the relationship between $\mathrm{Ext}^1_R(M, N)$ and extensions as in Homology, Section 12.6.

0AUL Lemma 15.55.2. Let R be a ring. Let $\mathcal{A}$ be the abelian category of R-modules. There is a canonical isomorphism $\mathrm{Ext}_{\mathcal{A}}(M, N) = \mathrm{Ext}^1_R(M, N)$ compatible with the long exact sequences of Algebra, Lemmas 10.71.6 and 10.71.7 and the 6-term exact sequences of Homology, Lemma 12.6.4.

Proof. Omitted. $\square$

0AVE Lemma 15.55.3. Let R be a ring. Let J be an R-module. The following are equivalent

(1) J is injective,
(2) $\mathrm{Ext}^1_R(M, J) = 0$ for every R-module M.

Proof. Let $0 \to M'' \to M' \to M \to 0$ be a short exact sequence of R-modules. Consider the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M, J) \to \mathrm{Hom}_R(M', J) \to \mathrm{Hom}_R(M'', J) \\ \to \mathrm{Ext}^1_R(M, J) \to \mathrm{Ext}^1_R(M', J) \to \mathrm{Ext}^1_R(M'', J) \to \ldots \end{aligned}$$

of Algebra, Lemma 10.71.7. Thus we see that (2) implies (1). Conversely, if J is injective then the Ext-group is zero by Homology, Lemma 12.27.2 and Lemma 15.55.2. $\square$

0AVF Lemma 15.55.4. Let R be a ring. Let J be an R-module. The following are equivalent

(1) J is injective,
(2) $\mathrm{Ext}^1_R(R/I, J) = 0$ for every ideal $I \subset R$, and

(3) for an ideal $I \subset R$ and module map $I \to J$ there exists an extension $R \to J$.

Proof. If $I \subset R$ is an ideal, then the short exact sequence $0 \to I \to R \to R/I \to 0$ gives an exact sequence

$$\mathrm{Hom}_R(R, J) \to \mathrm{Hom}_R(I, J) \to \mathrm{Ext}^1_R(R/I, J) \to 0$$

by Algebra, Lemma 10.71.7 and the fact that $\mathrm{Ext}^1_R(R, J) = 0$ as R is projective (Algebra, Lemma 10.77.2). Thus (2) and (3) are equivalent. In this proof we will show that (1) $\Leftrightarrow$ (3) which is known as Baer's criterion.

Assume (1). Given a module map $I \to J$ as in (3) we find the extension $R \to J$ because the map $\mathrm{Hom}_R(R, J) \to \mathrm{Hom}_R(I, J)$ is surjective by definition.

Assume (3). Let $M \subset N$ be an inclusion of R-modules. Let $\varphi : M \to J$ be a homomorphism. We will show that φ extends to N which finishes the proof of the lemma. Consider the set of homomorphisms $\varphi' : M' \to J$ with $M \subset M' \subset N$ and $\varphi'|_M = \varphi$. Define $(M', \varphi') \geq (M'', \varphi'')$ if and only if $M' \supset M''$ and $\varphi'|_{M''} = \varphi''$. If $(M_i, \varphi_i)_{i \in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i \in I} M_i \to J$ defined by $a \in M_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (M', φ') is maximal then $M' = N$. In other words, it suffices to show, given any subgroup $M \subset N$, $M \neq N$ and any $\varphi : M \to J$, then we can find $\varphi' : M' \to J$ with $M \subset M' \subset N$ such that (a) the inclusion $M \subset M'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in N$, $x \notin M$. Let $I = \{f \in R \mid fx \in M\}$. This is an ideal of R. Define a homomorphism $\psi : I \to J$ by $f \mapsto \varphi(fx)$. Extend to a map $\tilde{\psi} : R \to J$ which is possible by assumption (3). By our choice of I the kernel of $M \oplus R \to J$, $(y, f) \mapsto y - \tilde{\psi}(f)$ contains the kernel of the map $M \oplus R \to N$, $(y, f) \mapsto y + fx$. Hence this homomorphism factors through the image $M' = M + Rx$ and this extends the given homomorphism as desired. $\square$

In the rest of this section we prove that there are enough injective modules over a ring R. We start with the fact that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. This follows from Lemma 15.54.1.

01D9 Definition 15.55.5. Let R be a ring.

(1) For any R-module M over R we denote $M^\vee = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ with its natural R-module structure. We think of $M \mapsto M^\vee$ as a contravariant functor from the category of R-modules to itself.
(2) For any R-module M we denote

$$F(M) = \bigoplus\nolimits_{m \in M} R[m]$$

the free module with basis given by the elements $[m]$ with $m \in M$. We let $F(M) \to M$, $\sum f_i[m_i] \mapsto \sum f_i m_i$ be the natural surjection of R-modules. We think of $M \mapsto (F(M) \to M)$ as a functor from the category of R-modules to the category of arrows in R-modules.

01DA Lemma 15.55.6. Let R be a ring. The functor $M \mapsto M^\vee$ is exact.

Proof. This because $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group by Lemma 15.54.1. $\square$

There is a canonical map $ev : M \to (M^\vee)^\vee$ given by evaluation: given $x \in M$ we let $ev(x) \in (M^\vee)^\vee = \mathrm{Hom}(M^\vee, \mathbf{Q}/\mathbf{Z})$ be the map $\varphi \mapsto \varphi(x)$.

01DB **Lemma 15.55.7.** For any R-module M the evaluation map $ev : M \to (M^\vee)^\vee$ is injective.

Proof. You can check this using that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. Namely, if $x \in M$ is not zero, then let $M' \subset M$ be the cyclic group it generates. There exists a nonzero map $M' \to \mathbf{Q}/\mathbf{Z}$ which necessarily does not annihilate x. This extends to a map $\varphi : M \to \mathbf{Q}/\mathbf{Z}$ and then $ev(x)(\varphi) = \varphi(x) \neq 0$. $\square$

The canonical surjection $F(M) \to M$ of R-modules turns into a canonical injection, see above, of R-modules

$$(M^\vee)^\vee \longrightarrow (F(M^\vee))^\vee.$$

Set $J(M) = (F(M^\vee))^\vee$. The composition of ev with this the displayed map gives $M \to J(M)$ functorially in M.

01DC **Lemma 15.55.8.** Let R be a ring. For every R-module M the R-module $J(M)$ is injective.

Proof. Note that $J(M) \cong \prod_{\varphi \in M^\vee} R^\vee$ as an R-module. As the product of injective modules is injective, it suffices to show that $R^\vee$ is injective. For this we use that

$$\mathrm{Hom}_R(N, R^\vee) = \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})) = N^\vee$$

and the fact that $(-)^\vee$ is an exact functor by Lemma 15.55.6. $\square$

01DD **Lemma 15.55.9.** Let R be a ring. The construction above defines a covariant functor $M \mapsto (M \to J(M))$ from the category of R-modules to the category of arrows of R-modules such that for every module M the output $M \to J(M)$ is an injective map of M into an injective R-module $J(M)$.

Proof. Follows from the above. $\square$

In particular, for any map of R-modules $M \to N$ there is an associated morphism $J(M) \to J(N)$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ J(M) \arrow[r] & J(N) \end{tikzcd}$$

This is the kind of construction we would like to have in general. In Homology, Section 12.27 we introduced terminology to express this. Namely, we say this means that the category of R-modules has functorial injective embeddings.

15.56. Derived categories of modules

0914 In this section we put some generalities concerning the derived category of modules over a ring.

Let A be a ring. The category of A-modules is denoted Mod_A. We will use the symbol $K(A)$ to denote the homotopy category of complexes of A-modules, i.e., we set $K(A) = K(\mathrm{Mod}_A)$ as a category, see Derived Categories, Section 13.8. The bounded versions are $K^+(A)$, $K^-(A)$, and $K^b(A)$. We view $K(A)$ as a triangulated category as in Derived Categories, Section 13.10. The derived category of A, denoted $D(A)$, is the category obtained from $K(A)$ by inverting quasi-isomorphisms, i.e., we set $D(A) = D(\text{Mod}_A)$, see Derived Categories, Section 13.11[6]. The bounded versions are $D^+(A)$, $D^-(A)$, and $D^b(A)$.

Let A be a ring. The category of A-modules has products and products are exact. The category of A-modules has enough injectives by Lemma 15.55.9. Hence every complex of A-modules is quasi-isomorphic to a K-injective complex (Derived Categories, Lemma 13.34.5). It follows that $D(A)$ has countable products (Derived Categories, Lemma 13.34.2) and in fact arbitrary products (Injectives, Lemma 19.13.4). This implies that every inverse system of objects of $D(A)$ has a derived limit (well defined up to isomorphism), see Derived Categories, Section 13.34.

0915 Lemma 15.56.1. Let $R \to S$ be a flat ring map. If $I^\bullet$ is a K-injective complex of S-modules, then $I^\bullet$ is K-injective as a complex of R-modules.

Proof. This is true because $\text{Hom}_{K(R)}(M^\bullet, I^\bullet) = \text{Hom}_{K(S)}(M^\bullet \otimes_R S, I^\bullet)$ by Algebra, Lemma 10.14.3 and the fact that tensoring with S is exact. $\square$

0916 Lemma 15.56.2. Let $R \to S$ be an epimorphism of rings. Let $I^\bullet$ be a complex of S-modules. If $I^\bullet$ is K-injective as a complex of R-modules, then $I^\bullet$ is a K-injective complex of S-modules.

Proof. This is true because $\text{Hom}_{K(R)}(N^\bullet, I^\bullet) = \text{Hom}_{K(S)}(N^\bullet, I^\bullet)$ for any complex of S-modules $N^\bullet$, see Algebra, Lemma 10.107.14. $\square$

0917 Lemma 15.56.3. Let $A \to B$ be a ring map. If $I^\bullet$ is a K-injective complex of A-modules, then $\text{Hom}_A(B, I^\bullet)$ is a K-injective complex of B-modules.

Proof. This is true because $\text{Hom}_{K(B)}(N^\bullet, \text{Hom}_A(B, I^\bullet)) = \text{Hom}_{K(A)}(N^\bullet, I^\bullet)$ by Algebra, Lemma 10.14.4. $\square$

15.57. Computing Tor

064F Let R be a ring. We denote $D(R)$ the derived category of the abelian category Mod_R of R-modules. Note that Mod_R has enough projectives as every free R-module is projective. Thus we can define the left derived functors of any additive functor from Mod_R to any abelian category.

This applies in particular to the functor $-\otimes_R M : \text{Mod}_R \to \text{Mod}_R$ whose left derived functors are the Tor functors $\text{Tor}_i^R(-, M)$, see Algebra, Section 10.75. There is also a total left derived functor

064G (15.57.0.1)
$$- \otimes_R^{\mathbf{L}} M : D^-(R) \longrightarrow D^-(R)$$

which is denoted $- \otimes_R^{\mathbf{L}} M$. Its satellites are the Tor modules, i.e., we have

$$H^{-p}(N \otimes_R^{\mathbf{L}} M) = \text{Tor}_p^R(N, M).$$

A special situation occurs when we consider the tensor product with an R-algebra A. In this case we think of $- \otimes_R A$ as a functor from Mod_R to Mod_A. Hence the total right derived functor

064H (15.57.0.2)
$$- \otimes_R^{\mathbf{L}} A : D^-(R) \longrightarrow D^-(A)$$

which is denoted $- \otimes_R^{\mathbf{L}} A$. Its satellites are the tor groups, i.e., we have

$$H^{-p}(N \otimes_R^{\mathbf{L}} A) = \text{Tor}_p^R(N, A).$$

[6]See also Injectives, Remark 19.13.3.

In particular these Tor groups naturally have the structure of A-modules.

We will generalize the material in this section to unbounded complexes in the next few sections.

15.58. Tensor products of complexes

0GWN Let R be a ring. The category $\text{Comp}(R)$ of complexes of R-modules has a symmetric monoidal structure. Namely, suppose that we have two complexes of R-modules $L^\bullet$ and $M^\bullet$. Using Homology, Example 12.18.2 and Homology, Definition 12.18.3 we obtain a third complex of R-modules, namely

$$\text{Tot}(L^\bullet \otimes_R M^\bullet)$$

Clearly this construction is functorial in both $L^\bullet$ and $M^\bullet$. The associativity constraint will be the canonical isomorphism of complexes

$$\text{Tot}(\text{Tot}(K^\bullet \otimes_R L^\bullet) \otimes_R M^\bullet) \longrightarrow \text{Tot}(K^\bullet \otimes_R \text{Tot}(L^\bullet \otimes_R M^\bullet))$$

constructed in Homology, Remark 12.18.4 from the triple complex $K^\bullet \otimes_R L^\bullet \otimes_R M^\bullet$. The commutativity constraint is the canonical isomorphism

$$\text{Tot}(L^\bullet \otimes_R M^\bullet) \to \text{Tot}(M^\bullet \otimes_R L^\bullet)$$

which uses the sign $(-1)^{pq}$ on the summand $L^p \otimes_R M^q$. To see that it is a map of complexes we compute for $x \in L^p$ and $y \in M^q$ that

$$\text{d}(x \otimes y) = \text{d}_L(x) \otimes y + (-1)^p x \otimes \text{d}_M(y)$$

Our rule says the right hand side is mapped to

$$(-1)^{(p+1)q} y \otimes \text{d}_L(x) + (-1)^{p+p(q+1)} \text{d}_M(y) \otimes x$$

On the other hand, we see that

$$\text{d}((-1)^{pq} y \otimes x) = (-1)^{pq} \text{d}_M(y) \otimes x + (-1)^{pq+q} y \otimes \text{d}_L(x)$$

These two expressions agree by inspection as desired.

0FNI **Lemma 15.58.1.** Let R be a ring. The category $\text{Comp}(R)$ of complexes of R-modules endowed with the functor $(L^\bullet, M^\bullet) \mapsto \text{Tot}(L^\bullet \otimes_R M^\bullet)$ and associativity and commutativity constraints as above is a symmetric monoidal category.

Proof. Omitted. Hints: as unit $\mathbf{1}$ we take the complex having R in degree 0 and zero in other degrees with obvious isomorphisms $\text{Tot}(\mathbf{1} \otimes_R M^\bullet) = M^\bullet$ and $\text{Tot}(K^\bullet \otimes_R \mathbf{1}) = K^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 4.43.1 and 4.43.9. The verifications are straightforward in each case. □

064I **Lemma 15.58.2.** Let R be a ring. Let $P^\bullet$ be a complex of R-modules. Let $\alpha, \beta : L^\bullet \to M^\bullet$ be homotopic maps of complexes. Then α and β induce homotopic maps

$$\text{Tot}(\alpha \otimes \text{id}_P), \text{Tot}(\beta \otimes \text{id}_P) : \text{Tot}(L^\bullet \otimes_R P^\bullet) \longrightarrow \text{Tot}(M^\bullet \otimes_R P^\bullet).$$

In particular the construction $L^\bullet \mapsto \text{Tot}(L^\bullet \otimes_R P^\bullet)$ defines an endo-functor of the homotopy category of complexes.

Proof. Say $\alpha = \beta + dh + hd$ for some homotopy h defined by $h^n : L^n \to M^{n-1}$. Set

$$H^n = \bigoplus\nolimits_{a+b=n} h^a \otimes \mathrm{id}_{P^b} : \bigoplus\nolimits_{a+b=n} L^a \otimes_R P^b \longrightarrow \bigoplus\nolimits_{a+b=n} M^{a-1} \otimes_R P^b$$

Then a straightforward computation shows that

$$\mathrm{Tot}(\alpha \otimes \mathrm{id}_P) = \mathrm{Tot}(\beta \otimes \mathrm{id}_P) + dH + Hd$$

as maps $\mathrm{Tot}(L^\bullet \otimes_R P^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R P^\bullet)$. $\square$

0GWP **Lemma 15.58.3.** Let R be a ring. The homotopy category $K(R)$ of complexes of R-modules endowed with the functor $(L^\bullet, M^\bullet) \mapsto \mathrm{Tot}(L^\bullet \otimes_R M^\bullet)$ and associativity and commutativity constraints as above is a symmetric monoidal category.

Proof. This follows from Lemmas 15.58.1 and 15.58.2. Details omitted. $\square$

064J **Lemma 15.58.4.** Let R be a ring. Let $P^\bullet$ be a complex of R-modules. The functors

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto \mathrm{Tot}(P^\bullet \otimes_R L^\bullet)$$

and

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto \mathrm{Tot}(L^\bullet \otimes_R P^\bullet)$$

are exact functors of triangulated categories.

Proof. This follows from Derived Categories, Remark 13.10.9. $\square$

15.59. Derived tensor product

06XY We can construct the derived tensor product in greater generality. In fact, it turns out that the boundedness assumptions are not necessary, provided we choose K-flat resolutions.

06XZ **Definition 15.59.1.** Let R be a ring. A complex $K^\bullet$ is called *K-flat* if for every acyclic complex $M^\bullet$ the total complex $\mathrm{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic.

06Y0 **Lemma 15.59.2.** Let R be a ring. Let $K^\bullet$ be a K-flat complex. Then the functor

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto \mathrm{Tot}(L^\bullet \otimes_R K^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 15.58.4 and the fact that quasi-isomorphisms in $K(R)$ are characterized by having acyclic cones. $\square$

06Y1 **Lemma 15.59.3.** Let $R \to R'$ be a ring map. If $K^\bullet$ is a K-flat complex of R-modules, then $K^\bullet \otimes_R R'$ is a K-flat complex of R'-modules.

Proof. Follows from the definitions and the fact that $(K^\bullet \otimes_R R') \otimes_{R'} L^\bullet = K^\bullet \otimes_R L^\bullet$ for any complex $L^\bullet$ of R'-modules. $\square$

0795 **Lemma 15.59.4.** Let R be a ring. If $K^\bullet$, $L^\bullet$ are K-flat complexes of R-modules, then $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$ is a K-flat complex of R-modules.

Proof. Follows from the isomorphism

$$\mathrm{Tot}(M^\bullet \otimes_R \mathrm{Tot}(K^\bullet \otimes_R L^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(M^\bullet \otimes_R K^\bullet) \otimes_R L^\bullet)$$

and the definition. $\square$

06Y2 **Lemma 15.59.5.** Let R be a ring. Let $(K_1^\bullet, K_2^\bullet, K_3^\bullet)$ be a distinguished triangle in $K(R)$. If two out of three of $K_i^\bullet$ are K-flat, so is the third.

Proof. Follows from Lemma 15.58.4 and the fact that in a distinguished triangle in $K(R)$ if two out of three are acyclic, so is the third. $\square$

0BYH Lemma 15.59.6. Let R be a ring. Let $0 \to K_1^\bullet \to K_2^\bullet \to K_3^\bullet \to 0$ be a short exact sequence of complexes. If K_3^n is flat for all $n \in \mathbf{Z}$ and two out of three of $K_i^\bullet$ are K-flat, so is the third.

Proof. Let $L^\bullet$ be a complex of R-modules. Then

$$0 \to \text{Tot}(L^\bullet \otimes_R K_1^\bullet) \to \text{Tot}(L^\bullet \otimes_R K_2^\bullet) \to \text{Tot}(L^\bullet \otimes_R K_3^\bullet) \to 0$$

is a short exact sequence of complexes. Namely, for each n, m the sequence of modules $0 \to L^n \otimes_R K_1^m \to L^n \otimes_R K_2^m \to L^n \otimes_R K_3^m \to 0$ is exact by Algebra, Lemma 10.39.12 and the sequence of complexes is a direct sum of these. Thus the lemma follows from this and the fact that in a short exact sequence of complexes if two out of three are acyclic, so is the third. $\square$

064K Lemma 15.59.7. Let R be a ring. Let $P^\bullet$ be a bounded above complex of flat R-modules. Then $P^\bullet$ is K-flat.

Proof. Let $L^\bullet$ be an acyclic complex of R-modules. Let $\xi \in H^n(\text{Tot}(L^\bullet \otimes_R P^\bullet))$. We have to show that $\xi = 0$. Since $\text{Tot}^n(L^\bullet \otimes_R P^\bullet)$ is a direct sum with terms $L^a \otimes_R P^b$ we see that ξ comes from an element in $H^n(\text{Tot}(\tau_{\leq m} L^\bullet \otimes_R P^\bullet))$ for some $m \in \mathbf{Z}$. Since $\tau_{\leq m} L^\bullet$ is also acyclic we may replace $L^\bullet$ by $\tau_{\leq m} L^\bullet$. Hence we may assume that $L^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 12.25.3 has

$${}'E_1^{p,q} = H^p(L^\bullet \otimes_R P^q)$$

which is zero as P^q is flat and $L^\bullet$ acyclic. Hence $H^*(\text{Tot}(L^\bullet \otimes_R P^\bullet)) = 0$. $\square$

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

06Y3 Lemma 15.59.8. Let R be a ring. Let $K_1^\bullet \to K_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\text{colim}_i K_i^\bullet$ is K-flat. More generally any filtered colimit of K-flat complexes is K-flat.

Proof. Because we are taking termwise colimits we have

$$\text{colim}_i \text{Tot}(M^\bullet \otimes_R K_i^\bullet) = \text{Tot}(M^\bullet \otimes_R \text{colim}_i K_i^\bullet)$$

by Algebra, Lemma 10.12.9. Hence the lemma follows from the fact that filtered colimits are exact, see Algebra, Lemma 10.8.8. $\square$

0E8F Lemma 15.59.9. Let R be a ring. Let $K^\bullet$ be a complex of R-modules. If $K^\bullet \otimes_R M$ is acyclic for all finitely presented R-modules M, then $K^\bullet$ is K-flat.

Proof. We will use repeatedly that tensor product commute with colimits (Algebra, Lemma 10.12.9). Thus we see that $K^\bullet \otimes_R M$ is acyclic for any R-module M, because any R-module is a filtered colimit of finitely presented R-modules M, see Algebra, Lemma 10.11.3. Let $M^\bullet$ be an acyclic complex of R-modules. We have to show that $\text{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic. Since $M^\bullet = \text{colim}\, \tau_{\leq n} M^\bullet$ (termwise colimit) we have

$$\text{Tot}(M^\bullet \otimes_R K^\bullet) = \text{colim}\, \text{Tot}(\tau_{\leq n} M^\bullet \otimes_R K^\bullet)$$

with truncations as in Homology, Section 12.15. As filtered colimits are exact (Algebra, Lemma 10.8.8) we may replace $M^\bullet$ by $\tau_{\leq n} M^\bullet$ and assume that $M^\bullet$ is bounded above. In the bounded above case, we can write $M^\bullet = \mathop{\mathrm{colim}} \sigma_{\geq -n} M^\bullet$ where the complexes $\sigma_{\geq -n} M^\bullet$ are bounded but possibly no longer acyclic. Arguing as above we reduce to the case where $M^\bullet$ is a bounded complex. Finally, for a bounded complex $M^a \to \ldots \to M^b$ we can argue by induction on the length $b - a$ of the complex. The case $b - a = 1$ we have seen above. For $b - a > 1$ we consider the split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1} M^\bullet \to M^\bullet \to M^a[-a] \to 0$$

and we apply Lemma 15.58.4 to do the induction step. Some details omitted. □

06Y4 Lemma 15.59.10. Let R be a ring. For any complex $M^\bullet$ there exists a K-flat complex $K^\bullet$ whose terms are flat R-modules and a quasi-isomorphism $K^\bullet \to M^\bullet$ which is termwise surjective.

Proof. Let $\mathcal{P} \subset \mathrm{Ob}(\mathrm{Mod}_R)$ be the class of flat R-modules. By Derived Categories, Lemma 13.29.1 there exists a system $K_1^\bullet \to K_2^\bullet \to \ldots$ and a diagram

$$\begin{tikzcd} K_1^\bullet \arrow[r] \arrow[d] & K_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1} M^\bullet \arrow[r] & \tau_{\leq 2} M^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the properties (1), (2), (3) listed in that lemma. These properties imply each complex $K_i^\bullet$ is a bounded above complex of flat modules. Hence $K_i^\bullet$ is K-flat by Lemma 15.59.7. The induced map $\mathop{\mathrm{colim}}_i K_i^\bullet \to M^\bullet$ is a quasi-isomorphism and termwise surjective by construction. The complex $\mathop{\mathrm{colim}}_i K_i^\bullet$ is K-flat by Lemma 15.59.8. The terms $\mathop{\mathrm{colim}} K_i^n$ are flat because filtered colimits of flat modules are flat, see Algebra, Lemma 10.39.3. □

09PB Remark 15.59.11. In fact, we can do better than Lemma 15.59.10. Namely, we can find a quasi-isomorphism $P^\bullet \to M^\bullet$ where $P^\bullet$ is a complex of R-modules endowed with a filtration

$$0 = F_{-1}P^\bullet \subset F_0 P^\bullet \subset F_1 P^\bullet \subset \ldots \subset P^\bullet$$

by subcomplexes such that

(1) $P^\bullet = \bigcup F_p P^\bullet$,
(2) the inclusions $F_i P^\bullet \to F_{i+1} P^\bullet$ are termwise split injections,
(3) the quotients $F_{i+1} P^\bullet / F_i P^\bullet$ are isomorphic to direct sums of shifts $R[k]$ (as complexes, so differentials are zero).

This will be shown in Differential Graded Algebra, Lemma 22.20.4. Moreover, given such a complex we obtain a distinguished triangle

$$\bigoplus F_i P^\bullet \to \bigoplus F_i P^\bullet \to M^\bullet \to \bigoplus F_i P^\bullet[1]$$

in $D(R)$. Using this we can sometimes reduce statements about general complexes to statements about $R[k]$ (this of course only works if the statement is preserved under taking direct sums). More precisely, let T be a property of objects of $D(R)$. Suppose that

(1) if $K_i \in D(R)$, $i \in I$ is a family of objects with $T(K_i)$ for all $i \in I$, then $T(\bigoplus K_i)$,
(2) if $K \to L \to M \to K[1]$ is a distinguished triangle and T holds for two, then T holds for the third object,

(3) $T(R[k])$ holds for all k.

Then T holds for all objects of $D(R)$.

064L Lemma 15.59.12. Let R be a ring. Let $\alpha : P^\bullet \to Q^\bullet$ be a quasi-isomorphism of K-flat complexes of R-modules. For every complex $L^\bullet$ of R-modules the induced map

$$\text{Tot}(\text{id}_L \otimes \alpha) : \text{Tot}(L^\bullet \otimes_R P^\bullet) \longrightarrow \text{Tot}(L^\bullet \otimes_R Q^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $K^\bullet \to L^\bullet$ with $K^\bullet$ a K-flat complex, see Lemma 15.59.10. Consider the commutative diagram

$$\begin{tikzcd} \text{Tot}(K^\bullet \otimes_R P^\bullet) \arrow[r] \arrow[d] & \text{Tot}(K^\bullet \otimes_R Q^\bullet) \arrow[d] \\ \text{Tot}(L^\bullet \otimes_R P^\bullet) \arrow[r] & \text{Tot}(L^\bullet \otimes_R Q^\bullet) \end{tikzcd}$$

The result follows as by Lemma 15.59.2 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. Choose a K-flat resolution $K^\bullet \to M^\bullet$, see Lemma 15.59.10. By Lemmas 15.58.2 and 15.58.4 we obtain an exact functor of triangulated categories

$$K(R) \longrightarrow K(R), \quad L^\bullet \longmapsto \text{Tot}(L^\bullet \otimes_R K^\bullet)$$

By Lemma 15.59.2 this functor induces a functor $D(R) \to D(R)$ simply because $D(R)$ is the localization of $K(R)$ at quasi-isomorphism. By Lemma 15.59.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

064M Definition 15.59.13. Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. The derived tensor product

$$- \otimes_R^{\mathbf{L}} M^\bullet : D(R) \longrightarrow D(R)$$

is the exact functor of triangulated categories described above.

This functor extends the functor (15.57.0.1). It is clear from our explicit constructions that there is an isomorphism (involving a choice of signs, see below)

$$M^\bullet \otimes_R^{\mathbf{L}} L^\bullet \cong L^\bullet \otimes_R^{\mathbf{L}} M^\bullet$$

whenever both $L^\bullet$ and $M^\bullet$ are in $D(R)$. Hence when we write $M^\bullet \otimes_R^{\mathbf{L}} L^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

0BYI Lemma 15.59.14. Let R be a ring. Let $K^\bullet, L^\bullet$ be complexes of R-modules. There is a canonical isomorphism

$$K^\bullet \otimes_R^{\mathbf{L}} L^\bullet \longrightarrow L^\bullet \otimes_R^{\mathbf{L}} K^\bullet$$

functorial in both complexes which uses a sign of $(-1)^{pq}$ for the map $K^p \otimes_R L^q \to L^q \otimes_R K^p$ (see proof for explanation).

Proof. We may and do replace the complexes by K-flat complexes $K^\bullet$ and $L^\bullet$ and then we use the commutativity constraint discussed in Section 15.58. □

0BYJ **Lemma 15.59.15.** Let R be a ring. Let $K^\bullet, L^\bullet, M^\bullet$ be complexes of R-modules. There is a canonical isomorphism

$$(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet) \otimes_R^{\mathbf{L}} M^\bullet = K^\bullet \otimes_R^{\mathbf{L}} (L^\bullet \otimes_R^{\mathbf{L}} M^\bullet)$$

functorial in all three complexes.

Proof. Replace the complexes by K-flat complexes and use the associativity constraint in Section 15.58. $\square$

0G6M **Lemma 15.59.16.** Let R be a ring. Let $a : K^\bullet \to L^\bullet$ be a map of complexes of R-modules. If $K^\bullet$ is K-flat, then there exist a complex $N^\bullet$ and maps of complexes $b : K^\bullet \to N^\bullet$ and $c : N^\bullet \to L^\bullet$ such that

(1) $N^\bullet$ is K-flat,
(2) c is a quasi-isomorphism,
(3) a is homotopic to $c \circ b$.

If the terms of $K^\bullet$ are flat, then we may choose $N^\bullet$, b, and c such that the same is true for $N^\bullet$.

Proof. We will use that the homotopy category $K(R)$ is a triangulated category, see Derived Categories, Proposition 13.10.3. Choose a distinguished triangle $K^\bullet \to L^\bullet \to C^\bullet \to K^\bullet[1]$. Choose a quasi-isomorphism $M^\bullet \to C^\bullet$ with $M^\bullet$ K-flat with flat terms, see Lemma 15.59.10. By the axioms of triangulated categories, we may fit the composition $M^\bullet \to C^\bullet \to K^\bullet[1]$ into a distinguished triangle $K^\bullet \to N^\bullet \to M^\bullet \to K^\bullet[1]$. By Lemma 15.59.5 we see that $N^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $N^\bullet \to L^\bullet$ fitting into the following morphism of distinghuised triangles

$$\begin{tikzcd}
K^\bullet \arrow[r] \arrow[d] & N^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[r] \arrow[d] & K^\bullet[1] \arrow[d] \\
K^\bullet \arrow[r] & L^\bullet \arrow[r] & C^\bullet \arrow[r] & K^\bullet[1]
\end{tikzcd}$$

Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $N^\bullet \to L^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(R)$ and use Derived Categories, Lemma 13.4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $N^\bullet$ such that $N^n \cong M^n \oplus K^n$, see Derived Categories, Lemma 13.10.7. Hence we get the desired flatness if the terms of $K^\bullet$ are flat. $\square$

15.60. Derived change of rings

06Y5 Let $R \to A$ be a ring map. Let $N^\bullet$ be a complex of A-modules. We can also use K-flat resolutions to define a functor

$$- \otimes_R^{\mathbf{L}} N^\bullet : D(R) \to D(A)$$

as the left derived functor of the functor $K(R) \to K(A)$, $M^\bullet \mapsto \text{Tot}(M^\bullet \otimes_R N^\bullet)$. In particular, taking $N^\bullet = A[0]$ we obtain a derived base change functor

$$- \otimes_R^{\mathbf{L}} A : D(R) \to D(A)$$

extending the functor (15.57.0.2). Namely, for every complex of R-modules $M^\bullet$ we can choose a K-flat resolution $K^\bullet \to M^\bullet$ and set

$$M^\bullet \otimes_R^{\mathbf{L}} N^\bullet = \text{Tot}(K^\bullet \otimes_R N^\bullet).$$

You can use Lemmas 15.59.10 and 15.59.12 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

06Y6 **Lemma 15.60.1.** The construction above is independent of choices and defines an exact functor of triangulated categories $- \otimes_R^{\mathbf{L}} N^\bullet : D(R) \to D(A)$. There is a functorial isomorphism

$$E^\bullet \otimes_R^{\mathbf{L}} N^\bullet = (E^\bullet \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} N^\bullet$$

for $E^\bullet$ in $D(R)$.

Proof. To prove the existence of the derived functor $- \otimes_R^{\mathbf{L}} N^\bullet$ we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(R)$ and $\mathcal{D}' = D(A)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(M^\bullet) = \text{Tot}(M^\bullet \otimes_R N^\bullet)$. To prove the stated properties of F use Lemmas 15.58.2 and 15.58.4. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(R)$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \text{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 15.59.10 and (2) follows from Lemma 15.59.12. Thus we obtain a derived functor

$$LF : D(R) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(A)$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 guarantees that $LF(K^\bullet) = F(K^\bullet) = \text{Tot}(K^\bullet \otimes_R N^\bullet)$ when $K^\bullet$ is K-flat, i.e., LF is indeed computed in the way described above. Moreover, by Lemma 15.59.3 the complex $K^\bullet \otimes_R A$ is a K-flat complex of A-modules. Hence

$$(K^\bullet \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} N^\bullet = \text{Tot}((K^\bullet \otimes_R A) \otimes_A N^\bullet) = \text{Tot}(K^\bullet \otimes_A N^\bullet) = K^\bullet \otimes_A^{\mathbf{L}} N^\bullet$$

which proves the final statement of the lemma. $\square$

0BYK **Lemma 15.60.2.** Let $R \to A$ be a ring map. Let $f : L^\bullet \to N^\bullet$ be a map of complexes of A-modules. Then f induces a transformation of functors

$$1 \otimes f : - \otimes_A^{\mathbf{L}} L^\bullet \longrightarrow - \otimes_A^{\mathbf{L}} N^\bullet$$

If f is a quasi-isomorphism, then $1 \otimes f$ is an isomorphism of functors.

Proof. Since the functors are computing by evaluating on K-flat complexes $K^\bullet$ we can simply use the functoriality

$$\text{Tot}(K^\bullet \otimes_R L^\bullet) \to \text{Tot}(K^\bullet \otimes_R N^\bullet)$$

to define the transformation. The last statement follows from Lemma 15.59.2. $\square$

0GMT **Lemma 15.60.3.** Let $R \to A$ be a ring map. The functor $D(R) \to D(A)$, $E \mapsto E \otimes_R^{\mathbf{L}} A$ of Lemma 15.60.1 is left adjoint to the restriction functor $D(A) \to D(R)$.

Proof. This follows from Derived Categories, Lemma 13.30.1 and the fact that $- \otimes_R A$ and restriction are adjoint by Algebra, Lemma 10.14.3. $\square$

08YT Remark 15.60.4 (Warning). Let $R \to A$ be a ring map, and let N and N' be A-modules. Denote N_R and N'_R the restriction of N and N' to R-modules, see Algebra, Section 10.14. In this situation, the objects $N_R \otimes_R^{\mathbf{L}} N'$ and $N \otimes_R^{\mathbf{L}} N'_R$ of $D(A)$ are in general not isomorphic! In other words, one has to pay careful attention as to which of the two sides is being used to provide the A-module structure.

For a specific example, set $R = k[x, y]$, $A = R/(xy)$, $N = R/(x)$ and $N' = A = R/(xy)$. The resolution $0 \to R \xrightarrow{xy} R \to N'_R \to 0$ shows that $N \otimes_R^{\mathbf{L}} N'_R = N[1] \oplus N$ in $D(A)$. The resolution $0 \to R \xrightarrow{x} R \to N_R \to 0$ shows that $N_R \otimes_R^{\mathbf{L}} N'$ is represented by the complex $A \xrightarrow{x} A$. To see these two complexes are not isomorphic, one can show that the second complex is not isomorphic in $D(A)$ to the direct sum of its cohomology groups, or one can show that the first complex is not a perfect object of $D(A)$ whereas the second one is. Some details omitted.

08YU Lemma 15.60.5. Let $A \to B \to C$ be ring maps. Let $N^\bullet$ be a complex of B-modules and $K^\bullet$ a complex of C-modules. The compositions of the functors

$$D(A) \xrightarrow{- \otimes_A^{\mathbf{L}} N^\bullet} D(B) \xrightarrow{- \otimes_B^{\mathbf{L}} K^\bullet} D(C)$$

is the functor $- \otimes_A^{\mathbf{L}} (N^\bullet \otimes_B^{\mathbf{L}} K^\bullet) : D(A) \to D(C)$. If M, N, K are modules over A, B, C, then we have

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} K = M \otimes_A^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} K) = (M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} K)$$

in $D(C)$. We also have a canonical isomorphism

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} K \longrightarrow (M \otimes_A^{\mathbf{L}} K) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} C)$$

using signs. Similar results holds for complexes.

Proof. Choose a K-flat complex $P^\bullet$ of B-modules and a quasi-isomorphism $P^\bullet \to N^\bullet$ (Lemma 15.59.10). Let $M^\bullet$ be a K-flat complex of A-modules representing an arbitrary object of $D(A)$. Then we see that

$$(M^\bullet \otimes_A^{\mathbf{L}} P^\bullet) \otimes_B^{\mathbf{L}} K^\bullet \longrightarrow (M^\bullet \otimes_A^{\mathbf{L}} N^\bullet) \otimes_B^{\mathbf{L}} K^\bullet$$

is an isomorphism by Lemma 15.60.2 applied to the material inside the brackets. By Lemmas 15.59.3 and 15.59.4 the complex

$$\mathrm{Tot}(M^\bullet \otimes_A P^\bullet) = \mathrm{Tot}((M^\bullet \otimes_R A) \otimes_A P^\bullet$$

is K-flat as a complex of B-modules and it represents the derived tensor product in $D(B)$ by construction. Hence we see that $(M^\bullet \otimes_A^{\mathbf{L}} P^\bullet) \otimes_B^{\mathbf{L}} K^\bullet$ is represented by the complex

$$\mathrm{Tot}(\mathrm{Tot}(M^\bullet \otimes_A P^\bullet) \otimes_B K^\bullet) = \mathrm{Tot}(M^\bullet \otimes_A \mathrm{Tot}(P^\bullet \otimes_B K^\bullet))$$

of C-modules. Equality by Homology, Remark 12.18.4. Going back the way we came we see that this is equal to

$$M^\bullet \otimes_A^{\mathbf{L}} (P^\bullet \otimes_B^{\mathbf{L}} K^\bullet) \longleftarrow M^\bullet \otimes_A^{\mathbf{L}} (N^\bullet \otimes_B^{\mathbf{L}} K^\bullet)$$

The arrow is an isomorphism by definition of the functor $- \otimes_B^{\mathbf{L}} K^\bullet$. All of these constructions are functorial in the complex $M^\bullet$ and hence we obtain our isomorphism of functors.

By the above we have the first equality in

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} K = M \otimes_A^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} K) = (M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} K)$$

The second equality follows from the final statement of Lemma 15.60.1. The same thing allows us to write $N \otimes_B^{\mathbf{L}} K = (N \otimes_B^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} K$ and substituting we get

$$\begin{aligned}(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} K &= (M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} ((N \otimes_B^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} K) \\ &= (M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} (K \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} C)) \\ &= ((M \otimes_A^{\mathbf{L}} C) \otimes_C^{\mathbf{L}} K) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} C)) \\ &= (M \otimes_C^{\mathbf{L}} K) \otimes_C^{\mathbf{L}} (N \otimes_B^{\mathbf{L}} C)\end{aligned}$$

by Lemmas 15.59.14 and 15.59.15 as well as the previously mentioned lemma. □

15.61. Tor independence

065Y Consider a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

of rings. Given an object K of $D(A)$ we can consider its derived base change $K \otimes_A^{\mathbf{L}} A'$ to an object of $D(A')$. Or we can take the restriction of K to an object of $D(R)$ and consider the derived base change of this to an object of $D(R')$, denoted $K \otimes_R^{\mathbf{L}} R'$ We claim there is a functorial comparison map

065Z (15.61.0.1) $$K \otimes_R^{\mathbf{L}} R' \longrightarrow K \otimes_A^{\mathbf{L}} A'$$

in $D(R')$. To construct this comparison map choose a K-flat complex $K^\bullet$ of A-modules representing K. Next, choose a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a K-flat complex of R-modules. The map above is the map

$$K \otimes_R^{\mathbf{L}} R' = E^\bullet \otimes_R R' \longrightarrow K^\bullet \otimes_A A' = K \otimes_A^{\mathbf{L}} A'$$

In general there is no chance that this map is an isomorphism.

However, we often encounter the situation where the diagram above is a "base change" diagram of rings, i.e., $A' = A \otimes_R R'$. In this situation, for any A-module M we have $M \otimes_A A' = M \otimes_R R'$. Thus $- \otimes_R R'$ is equal to $- \otimes_A A'$ as a functor $\text{Mod}_A \to \text{Mod}_{A'}$. In general this equality does not extend to derived tensor products. In other words, the comparison map is not an isomorphism. A simple example is to take $R = k[x]$, $A = R' = A' = k[x]/(x) = k$ and $K^\bullet = A[0]$. Clearly, a necessary condition is that $\text{Tor}_p^R(A, R') = 0$ for all $p > 0$.

0660 **Definition 15.61.1.** Let R be a ring. Let A, B be R-algebras. We say A and B are *Tor independent over R* if $\text{Tor}_p^R(A, B) = 0$ for all $p > 0$.

0661 **Lemma 15.61.2.** *The comparison map (15.61.0.1) is an isomorphism if $A' = A \otimes_R R'$ and A and R' are Tor independent over R.*

Proof. To prove this we choose a free resolution $F^\bullet \to R'$ of R' as an R-module. Because A and R' are Tor independent over R we see that $F^\bullet \otimes_R A$ is a free A-module resolution of A' over A. By our general construction of the derived tensor product above we see that

$$K^\bullet \otimes_A A' \cong \text{Tot}(K^\bullet \otimes_A (F^\bullet \otimes_R A)) = \text{Tot}(K^\bullet \otimes_R F^\bullet) \cong \text{Tot}(E^\bullet \otimes_R F^\bullet) \cong E^\bullet \otimes_R R'$$

as desired. □

08HW **Lemma 15.61.3.** Consider a commutative diagram of rings

$$\begin{tikzcd} A' & R' \arrow[l] \arrow[r] & B' \\ A \arrow[u] & R \arrow[l] \arrow[r] \arrow[u] & B \arrow[u] \end{tikzcd}$$

Assume that R' is flat over R and A' is flat over $A \otimes_R R'$ and B' is flat over $R' \otimes_R B$. Then

$$\mathrm{Tor}_i^R(A, B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B') = \mathrm{Tor}_i^{R'}(A', B')$$

Proof. By Algebra, Section 10.76 there are canonical maps

$$\mathrm{Tor}_i^R(A, B) \longrightarrow \mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R') \longrightarrow \mathrm{Tor}_i^{R'}(A', B')$$

These induce a map from left to right in the formula of the lemma.

Take a free resolution $F_\bullet \to A$ of A as an R-module. Then we see that $F_\bullet \otimes_R R'$ is a resolution of $A \otimes_R R'$. Hence $\mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R')$ is computed by $F_\bullet \otimes_R B \otimes_R R'$. By our assumption that R' is flat over R, this computes $\mathrm{Tor}_i^R(A, B) \otimes_R R'$. Thus $\mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R') = \mathrm{Tor}_i^R(A, B) \otimes_R R'$ (uses only flatness of R' over R).

By Lazard's theorem (Algebra, Theorem 10.81.4) we can write A', resp. B' as a filtered colimit of finite free $A \otimes_R R'$, resp. $B \otimes_R R'$-modules. Say $A' = \mathrm{colim}\, M_i$ and $B' = \mathrm{colim}\, N_j$. The result above gives

$$\mathrm{Tor}_i^{R'}(M_i, N_j) = \mathrm{Tor}_i^R(A, B) \otimes_{A \otimes_R B} (M_i \otimes_{R'} N_j)$$

as one can see by writing everything out in terms of bases. Taking the colimit we get the result of the lemma. □

0FXF **Lemma 15.61.4.** Let $R \to A$ and $R \to B$ be ring maps. Let $R \to R'$ be a ring map and set $A' = A \otimes_R R'$ and $B' = B \otimes_R R'$. If A and B are tor independent over R and $R \to R'$ is flat, then A' and B' are tor independent over R'.

Proof. Follows immediately from Lemma 15.61.3 and Definition 15.61.1. □

0DJD **Lemma 15.61.5.** Assumptions as in Lemma 15.61.3. For $M \in D(A)$ there are canonical isomorphisms

$$H^i((M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B') = H^i(M \otimes_R^{\mathbf{L}} B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B')$$

of $A' \otimes_{R'} B'$-modules.

Proof. Let us elucidate the two sides of the equation. On the left hand side we have the composition of the functors $D(A) \to D(A') \to D(R') \to D(B')$ with the functor $H^i : D(B') \to \mathrm{Mod}_{B'}$. Since there is a map from A' to the endomorphisms of the object $(M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B'$ in $D(B')$, we see that the left hand side is indeed an $A' \otimes_{R'} B'$-module. By the same arguments we see that $H^i(M \otimes_R^{\mathbf{L}} B)$ has an $A \otimes_R B$-module structure.

We first prove the result in case $B' = R' \otimes_R B$. In this case we choose a resolution $F^\bullet \to B$ by free R-modules. We also choose a K-flat complex $M^\bullet$ of A-modules representing M. Then the left hand side is represented by

$$\begin{aligned} H^i(\mathrm{Tot}((M^\bullet \otimes_A A') \otimes_{R'} (R' \otimes_R F^\bullet))) &= H^i(\mathrm{Tot}(M^\bullet \otimes_A A' \otimes_R F^\bullet)) \\ &= H^i(\mathrm{Tot}(M^\bullet \otimes_R F^\bullet) \otimes_A A') \\ &= H^i(M \otimes_R^{\mathbf{L}} B) \otimes_A A' \end{aligned}$$

The final equality because $A \to A'$ is flat. The final module is the desired module because $A' \otimes_{R'} B' = A' \otimes_R B$ since we've assumed $B' = R' \otimes_R B$ in this paragraph.

General case. Suppose that $B' \to B''$ is a flat ring map. Then it is easy to see that

$$H^i((M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B'') = H^i((M \otimes_A^{\mathbf{L}} A') \otimes_{R'}^{\mathbf{L}} B') \otimes_{B'} B''$$

and

$$H^i(M \otimes_R^{\mathbf{L}} B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B'') = \left(H^i(M \otimes_R^{\mathbf{L}} B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B')\right) \otimes_{B'} B''$$

Thus the result for B' implies the result for B''. Since we've proven the result for $R' \otimes_R B$ in the previous paragraph, this implies the result in general. □

08HX Lemma 15.61.6. Let R be a ring. Let A, B be R-algebras. The following are equivalent

(1) A and B are Tor independent over R,
(2) for every pair of primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ lying over the same prime $\mathfrak{r} \subset R$ the rings $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are Tor independent over $R_\mathfrak{r}$, and
(3) For every prime $\mathfrak{s}$ of $A \otimes_R B$ the module

$$\mathrm{Tor}_i^R(A, B)_\mathfrak{s} = \mathrm{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q})_\mathfrak{s}$$

(where $\mathfrak{p} = A \cap \mathfrak{s}$, $\mathfrak{q} = B \cap \mathfrak{s}$ and $\mathfrak{r} = R \cap \mathfrak{s}$) is zero.

Proof. Let $\mathfrak{s}$ be a prime of $A \otimes_R B$ as in (3). The equality

$$\mathrm{Tor}_i^R(A, B)_\mathfrak{s} = \mathrm{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q})_\mathfrak{s}$$

where $\mathfrak{p} = A \cap \mathfrak{s}$, $\mathfrak{q} = B \cap \mathfrak{s}$ and $\mathfrak{r} = R \cap \mathfrak{s}$ follows from Lemma 15.61.3. Hence (2) implies (3). Since we can test the vanishing of modules by localizing at primes (Algebra, Lemma 10.23.1) we conclude that (3) implies (1). For (1) ⇒ (2) we use that

$$\mathrm{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q}) = \mathrm{Tor}_i^R(A, B) \otimes_{(A \otimes_R B)} (A_\mathfrak{p} \otimes_{R_\mathfrak{r}} B_\mathfrak{q})$$

again by Lemma 15.61.3. □

15.62. Spectral sequences for Tor

061Y In this section we collect various spectral sequences that come up when considering the Tor functors.

061Z Example 15.62.1. Let R be a ring. Let $K_\bullet$ be a chain complex of R-modules with $K_n = 0$ for $n \ll 0$. Let M be an R-module. Choose a resolution $P_\bullet \to M$ of M by free R-modules. We obtain a double chain complex $K_\bullet \otimes_R P_\bullet$. Applying the material in Homology, Section 12.25 (especially Homology, Lemma 12.25.3) translated into the language of chain complexes we find two spectral sequences converging to $H_*(K_\bullet \otimes_R^{\mathbf{L}} M)$. Namely, on the one hand a spectral sequence with E_2-page

$$(E_2)_{i,j} = \mathrm{Tor}_j^R(H_i(K_\bullet), M) \Rightarrow H_{i+j}(K_\bullet \otimes_R^{\mathbf{L}} M)$$

and differential d_2 given by maps $\text{Tor}_j^R(H_i(K_\bullet), M) \to \text{Tor}_{j-2}^R(H_{i+1}(K_\bullet), M)$. Another spectral sequence with E_1-page

$$(E_1)_{i,j} = \text{Tor}_j^R(K_i, M) \Rightarrow H_{i+j}(K_\bullet \otimes_R^{\mathbf{L}} M)$$

with differential d_1 given by maps $\text{Tor}_j^R(K_i, M) \to \text{Tor}_j^R(K_{i-1}, M)$ induced by $K_i \to K_{i-1}$.

068F Example 15.62.2. Let $R \to S$ be a ring map. Let M be an R-module and let N be an S-module. Then there is a spectral sequence

$$\text{Tor}_n^S(\text{Tor}_m^R(M, S), N) \Rightarrow \text{Tor}_{n+m}^R(M, N).$$

To construct it choose a R-free resolution $P_\bullet$ of M. Then we have

$$M \otimes_R^{\mathbf{L}} N = P^\bullet \otimes_R N = (P^\bullet \otimes_R S) \otimes_S N$$

and then apply the first spectral sequence of Example 15.62.1.

0620 Example 15.62.3. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B' = B \otimes_A A' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

and B-modules M, N. Set $M' = M \otimes_A A' = M \otimes_B B'$ and $N' = N \otimes_A A' = N \otimes_B B'$. Assume that $A \to B$ is flat and that M and N are A-flat. Then there is a spectral sequence

$$\text{Tor}_i^A(\text{Tor}_j^B(M, N), A') \Rightarrow \text{Tor}_{i+j}^{B'}(M', N')$$

The reason is as follows. Choose free resolution $F_\bullet \to M$ as a B-module. As B and M are A-flat we see that $F_\bullet \otimes_A A'$ is a free B'-resolution of M'. Hence we see that the groups $\text{Tor}_n^{B'}(M', N')$ are computed by the complex

$$(F_\bullet \otimes_A A') \otimes_{B'} N' = (F_\bullet \otimes_B N) \otimes_A A' = (F_\bullet \otimes_B N) \otimes_A^{\mathbf{L}} A'$$

the last equality because $F_\bullet \otimes_B N$ is a complex of flat A-modules as N is flat over A. Hence we obtain the spectral sequence by applying the spectral sequence of Example 15.62.1.

0662 Example 15.62.4. Let $K^\bullet, L^\bullet$ be objects of $D^-(R)$. Then there are spectral sequences

$$E_2^{p,q} = H^p(K^\bullet \otimes_R^{\mathbf{L}} H^q(L^\bullet)) \Rightarrow H^{p+q}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

with $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ and

$$H^q(H^p(K^\bullet) \otimes_R^{\mathbf{L}} L^\bullet) \Rightarrow H^{p+q}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

After replacing $K^\bullet$ and $L^\bullet$ by bounded above complexes of projectives, these spectral sequences are simply the two spectral sequences for computing the cohomology of $\text{Tot}(K^\bullet \otimes L^\bullet)$ discussed in Homology, Section 12.25.

15.63. Products and Tor

068G The simplest example of the product maps comes from the following situation. Suppose that $K^\bullet, L^\bullet \in D(R)$. Then there are maps

068H (15.63.0.1)
$$H^i(K^\bullet) \otimes_R H^j(L^\bullet) \longrightarrow H^{i+j}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

Namely, to define these maps we may assume that one of $K^\bullet, L^\bullet$ is a K-flat complex of R-modules (for example a bounded above complex of free or projective R-modules). In that case $K^\bullet \otimes_R^{\mathbf{L}} L^\bullet$ is represented by the complex $\text{Tot}(K^\bullet \otimes_R L^\bullet)$, see Section 15.59 (or Section 15.57). Next, suppose that $\xi \in H^i(K^\bullet)$ and $\zeta \in H^j(L^\bullet)$. Choose $k \in \text{Ker}(K^i \to K^{i+1})$ and $l \in \text{Ker}(L^j \to L^{j+1})$ representing ξ and ζ. Then we set

$$\xi \cup \zeta = \text{class of } k \otimes l \text{ in } H^{i+j}(\text{Tot}(K^\bullet \otimes_R L^\bullet)).$$

This make sense because the formula (see Homology, Definition 12.18.3) for the differential d on the total complex shows that $k \otimes l$ is a cocycle. Moreover, if $k' = d_K(k'')$ for some $k'' \in K^{i-1}$, then $k' \otimes l = \text{d}(k'' \otimes l)$ because l is a cocycle. Similarly, altering the choice of l representing ζ does not change the class of $k \otimes l$. It is equally clear that $\cup$ is bilinear, and hence to a general element of $H^i(K^\bullet) \otimes_R H^j(L^\bullet)$ we assign

$$\sum \xi_i \otimes \zeta_i \longmapsto \sum \xi_i \cup \zeta_i$$

in $H^{i+j}(\text{Tot}(K^\bullet \otimes_R L^\bullet))$.

Let $R \to A$ be a ring map. Let $K^\bullet, L^\bullet \in D(R)$. Then we have a canonical identification

068I (15.63.0.2)
$$(K^\bullet \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (L^\bullet \otimes_R^{\mathbf{L}} A) = (K^\bullet \otimes_R^{\mathbf{L}} L^\bullet) \otimes_R^{\mathbf{L}} A$$

in $D(A)$. It is constructed as follows. First, choose K-flat resolutions $P^\bullet \to K^\bullet$ and $Q^\bullet \to L^\bullet$ over R. Then the left hand side is represented by the complex $\text{Tot}((P^\bullet \otimes_R A) \otimes_A (Q^\bullet \otimes_R A))$ and the right hand side by the complex $\text{Tot}(P^\bullet \otimes_R Q^\bullet) \otimes_R A$. These complexes are canonically isomorphic. Thus the construction above induces products

$$\text{Tor}_n^R(K^\bullet, A) \otimes_A \text{Tor}_m^R(L^\bullet, A) \longrightarrow \text{Tor}_{n+m}^R(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet, A)$$

which are occasionally useful.

Let M, N be R-modules. Using the general construction above, the canonical map $M \otimes_R^{\mathbf{L}} N \to M \otimes_R N$ and functoriality of Tor we obtain canonical maps

068J (15.63.0.3)
$$\text{Tor}_n^R(M, A) \otimes_A \text{Tor}_m^R(N, A) \longrightarrow \text{Tor}_{n+m}^R(M \otimes_R N, A)$$

Here is a direct construction using projective resolutions. First, choose projective resolutions

$$P_\bullet \to M, \quad Q_\bullet \to N, \quad T_\bullet \to M \otimes_R N$$

over R. We have $H_0(\text{Tot}(P_\bullet \otimes_R Q_\bullet)) = M \otimes_R N$ by right exactness of $\otimes_R$. Hence Derived Categories, Lemmas 13.19.6 and 13.19.7 guarantee the existence and uniqueness of a map of complexes $\mu : \text{Tot}(P_\bullet \otimes_R Q_\bullet) \to T_\bullet$ such that $H_0(\mu) = \text{id}_{M \otimes_R N}$.

This induces a canonical map

$$\begin{aligned}(M \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (N \otimes_R^{\mathbf{L}} A) &= \mathrm{Tot}((P_\bullet \otimes_R A) \otimes_A (Q_\bullet \otimes_R A)) \\ &= \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \otimes_R A \\ &\to T_\bullet \otimes_R A \\ &= (M \otimes_R N) \otimes_R^{\mathbf{L}} A\end{aligned}$$

in $D(A)$. Hence the products (15.63.0.3) above are constructed using (15.63.0.1) over A to construct

$$\mathrm{Tor}_n^R(M, A) \otimes_A \mathrm{Tor}_m^R(N, A) \to H^{-n-m}((M \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (N \otimes_R^{\mathbf{L}} A))$$

and then composing by the displayed map above to end up in $\mathrm{Tor}_{n+m}^R(M \otimes_R N, A)$.

An interesting special case of the above occurs when $M = N = B$ where B is an R-algebra. In this case we obtain maps

$$\mathrm{Tor}_n^R(B, A) \otimes_A \mathrm{Tor}_m^R(B, A) \longrightarrow \mathrm{Tor}_{n+m}^R(B \otimes_R B, A) \longrightarrow \mathrm{Tor}_{n+m}^R(B, A)$$

the second arrow being induced by the multiplication map $B \otimes_R B \to B$ via functoriality for Tor. In other words we obtain an A-algebra structure on $\mathrm{Tor}_\star^R(B, A)$. This algebra structure has many intriguing properties (associativity, graded commutative, B-algebra structure, divided powers in some case, etc) which we will discuss elsewhere (insert future reference here).

068K Lemma 15.63.1. Let R be a ring. Let A, B, C be R-algebras and let $B \to C$ be an R-algebra map. Then the induced map

$$\mathrm{Tor}_\star^R(B, A) \longrightarrow \mathrm{Tor}_\star^R(C, A)$$

is an A-algebra homomorphism.

Proof. Omitted. Hint: You can prove this by working through the definitions, writing all the complexes explicitly. □

15.64. Pseudo-coherent modules, I

064N Suppose that R is a ring. Recall that an R-module M is of finite type if there exists a surjection $R^{\oplus a} \to M$ and of finite presentation if there exists a presentation $R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$. Similarly, we can consider those R-modules for which there exists a length n resolution

064P (15.64.0.1)
$$R^{\oplus a_n} \to R^{\oplus a_{n-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0$$

by finite free R-modules. A module is called pseudo-coherent if we can find such a resolution for every n. Here is the formal definition.

064Q Definition 15.64.1. Let R be a ring. Denote $D(R)$ its derived category. Let $m \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ is m-pseudo-coherent if there exists a bounded complex $E^\bullet$ of finite free R-modules and a morphism $\alpha : E^\bullet \to K^\bullet$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and $H^m(\alpha)$ is surjective.
(2) An object $K^\bullet$ of $D(R)$ is pseudo-coherent if it is quasi-isomorphic to a bounded above complex of finite free R-modules.
(3) An R-module M is called m-pseudo-coherent if $M[0]$ is an m-pseudo-coherent object of $D(R)$.

(4) An R-module M is called pseudo-coherent[7] if $M[0]$ is a pseudo-coherent object of $D(R)$.

As usual we apply this terminology also to complexes of R-modules. Since any morphism $E^\bullet \to K^\bullet$ in $D(R)$ is represented by an actual map of complexes, see Derived Categories, Lemma 13.19.8, there is no ambiguity. It turns out that $K^\bullet$ is pseudo-coherent if and only if $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 15.64.5. Also, if the ring is Noetherian the condition can be understood as a finite generation condition on the cohomology, see Lemma 15.64.17. Let us first relate this to the informal discussion above.

064R Lemma 15.64.2. Let R be a ring and $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$.

(1) If $K^\bullet$ is $(m+1)$-pseudo-coherent and $L^\bullet$ is m-pseudo-coherent then $M^\bullet$ is m-pseudo-coherent.
(2) If $K^\bullet, M^\bullet$ are m-pseudo-coherent, then $L^\bullet$ is m-pseudo-coherent.
(3) If $L^\bullet$ is $(m+1)$-pseudo-coherent and $M^\bullet$ is m-pseudo-coherent, then $K^\bullet$ is $(m+1)$-pseudo-coherent.

Proof. Proof of (1). Choose $\alpha : P^\bullet \to K^\bullet$ with $P^\bullet$ a bounded complex of finite free modules such that $H^i(\alpha)$ is an isomorphism for $i > m+1$ and surjective for $i = m+1$. We may replace $P^\bullet$ by $\sigma_{\geq m+1} P^\bullet$ and hence we may assume that $P^i = 0$ for $i < m+1$. Choose $\beta : E^\bullet \to L^\bullet$ with $E^\bullet$ a bounded complex of finite free modules such that $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$. By Derived Categories, Lemma 13.19.11 we can find a map $\gamma : P^\bullet \to E^\bullet$ such that the diagram

$$\begin{tikzcd} K^\bullet \arrow[r] & L^\bullet \\ P^\bullet \arrow[u] \arrow[r, "\gamma"] & E^\bullet \arrow[u, "\beta"'] \end{tikzcd}$$

is commutative in $D(R)$. The cone $C(\gamma)^\bullet$ is a bounded complex of finite free R-modules, and the commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(P^\bullet, E^\bullet, C(\gamma)^\bullet) \longrightarrow (K^\bullet, L^\bullet, M^\bullet).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\gamma)^\bullet \to M^\bullet$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence $M^\bullet$ is m-pseudo-coherent.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. $\square$

064S Lemma 15.64.3. Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.

(1) If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m$, then $H^m(K^\bullet)$ is a finite type R-module.
(2) If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m+1$, then $H^{m+1}(K^\bullet)$ is a finitely presented R-module.

[7] This clashes with what is meant by a pseudo-coherent module in [Bou61].

Proof. Proof of (1). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It is clear that it suffices to prove the result for $E^\bullet$. Let n be the largest integer such that $E^n \neq 0$. If $n = m$, then the result is clear. If $n > m$, then $E^{n-1} \to E^n$ is surjective as $H^n(E^\bullet) = 0$. As E^n is finite projective we see that $E^{n-1} = E' \oplus E^n$. Hence it suffices to prove the result for the complex $(E')^\bullet$ which is the same as $E^\bullet$ except has E' in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $E^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K^\bullet) \cong H^{m+1}(E^\bullet) = \text{Coker}(E^m \to E^{m+1})$ which is of finite presentation. $\square$

064T **Lemma 15.64.4.** Let R be a ring. Let M be an R-module. Then

(1) M is 0-pseudo-coherent if and only if M is a finite R-module,
(2) M is (-1)-pseudo-coherent if and only if M is a finitely presented R-module,
(3) M is $(-d)$-pseudo-coherent if and only if there exists a resolution
$$R^{\oplus a_d} \to R^{\oplus a_{d-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0$$
of length d, and
(4) M is pseudo-coherent if and only if there exists an infinite resolution
$$\ldots \to R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$$
by finite free R-modules.

Proof. If M is of finite type (resp. of finite presentation), then M is 0-pseudo-coherent (resp. (-1)-pseudo-coherent) as follows from the discussion preceding Definition 15.64.1. Conversely, if M is 0-pseudo-coherent, then $M = H^0(M[0])$ is of finite type by Lemma 15.64.3. If M is (-1)-pseudo-coherent, then it is 0-pseudo-coherent hence of finite type. Choose a surjection $R^{\oplus a} \to M$ and denote $K = \text{Ker}(R^{\oplus a} \to M)$. By Lemma 15.64.2 we see that K is 0-pseudo-coherent, hence of finite type, whence M is of finite presentation.

To prove the third and fourth statement use induction and an argument similar to the above (details omitted). $\square$

064U **Lemma 15.64.5.** Let R be a ring. Let $K^\bullet$ be a complex of R-modules. The following are equivalent

(1) $K^\bullet$ is pseudo-coherent,
(2) $K^\bullet$ is m-pseudo-coherent for every $m \in \mathbf{Z}$, and
(3) $K^\bullet$ is quasi-isomorphic to a bounded above complex of finite projective R-modules.

If (1), (2), and (3) hold and $H^i(K^\bullet) = 0$ for $i > b$, then we can find a quasi-isomorphism $F^\bullet \to K^\bullet$ with F^i finite free R-modules and $F^i = 0$ for $i > b$.

Proof. We see that (1) $\Rightarrow$ (3) as a finite free module is a finite projective R-module. Conversely, suppose $P^\bullet$ is a bounded above complex of finite projective R-modules.

Say $P^i = 0$ for $i > n_0$. We choose a direct sum decompositions $F^{n_0} = P^{n_0} \oplus C^{n_0}$ with F^{n_0} a finite free R-module, and inductively

$$F^{n-1} = P^{n-1} \oplus C^n \oplus C^{n-1}$$

for $n \leq n_0$ with F^{n_0} a finite free R-module. As a complex $F^\bullet$ has maps $F^{n-1} \to F^n$ which agree with $P^{n-1} \to P^n$, induce the identity $C^n \to C^n$, and are zero on C^{n-1}. The map $F^\bullet \to P^\bullet$ is a quasi-isomorphism (even a homotopy equivalence) and hence (3) implies (1).

Assume (1). Let $E^\bullet$ be a bounded above complex of finite free R-modules and let $E^\bullet \to K^\bullet$ be a quasi-isomorphism. Then the induced maps $\sigma_{\geq m}E^\bullet \to K^\bullet$ from the stupid truncation of $E^\bullet$ to $K^\bullet$ show that $K^\bullet$ is m-pseudo-coherent. Hence (1) implies (2).

Assume (2). Since $K^\bullet$ is 0-pseudo-coherent we see in particular that $K^\bullet$ is bounded above. Let b be an integer such that $H^i(K^\bullet) = 0$ for $i > b$. By descending induction on $n \in \mathbf{Z}$ we are going to construct finite free R-modules F^i for $i \geq n$, differentials $d^i : F^i \to F^{i+1}$ for $i \geq n$, maps $\alpha : F^i \to K^i$ compatible with differentials, such that (1) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, and (2) $F^i = 0$ for $i > b$. Picture

$$\begin{tikzcd}
 & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\
K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

The base case is $n = b+1$ where we can take $F^i = 0$ for all i. Induction step. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 13.9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet) = 0$ for $i \geq n$. By Lemma 15.64.2 we see that $C^\bullet$ is $(n-1)$-pseudo-coherent. By Lemma 15.64.3 we see that $H^{n-1}(C^\bullet)$ is a finite R-module. Choose a finite free R-module F^{n-1} and a map $\beta : F^{n-1} \to C^{n-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that F^{n-1} surjects onto $H^{n-1}(C^\bullet)$. Since $C^{n-1} = K^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{tikzcd}
 & F^{n-1} \arrow[r, "d^{n-1}"'] \arrow[d, "\alpha^{n-1}"] & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\
\ldots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{tikzcd}
(F^n \to \ldots) \arrow[r] \arrow[d] & (F^{n-1} \to \ldots) \arrow[r] \arrow[d] & F^{n-1} \arrow[r] \arrow[d, "\beta"'] & (F^n \to \ldots)[1] \arrow[d] \\
(F^n \to \ldots) \arrow[r] & K^\bullet \arrow[r] & C^\bullet \arrow[r] & (F^n \to \ldots)[1]
\end{tikzcd}$$

Hence our choice of β implies that the map of complexes $(F^{n-1} \to \ldots) \to K^\bullet$ induces an isomorphism on cohomology in degrees $\geq n$ and a surjection in degree $n-1$. This finishes the proof of the lemma. $\square$

064V **Lemma 15.64.6.** Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent then the third is also pseudo-coherent.

Proof. Combine Lemmas 15.64.2 and 15.64.5. $\square$

064W **Lemma 15.64.7.** Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.

(1) If $H^i(K^\bullet) = 0$ for all $i \geq m$, then $K^\bullet$ is m-pseudo-coherent.
(2) If $H^i(K^\bullet) = 0$ for $i > m$ and $H^m(K^\bullet)$ is a finite R-module, then $K^\bullet$ is m-pseudo-coherent.
(3) If $H^i(K^\bullet) = 0$ for $i > m + 1$, the module $H^{m+1}(K^\bullet)$ is of finite presentation, and $H^m(K^\bullet)$ is of finite type, then $K^\bullet$ is m-pseudo-coherent.

Proof. It suffices to prove (3). Set $M = H^{m+1}(K^\bullet)$. Note that $\tau_{\geq m+1}K^\bullet$ is quasi-isomorphic to $M[-m-1]$. By Lemma 15.64.4 we see that $M[-m-1]$ is m-pseudo-coherent. Since we have the distinguished triangle

$$(\tau_{\leq m}K^\bullet, K^\bullet, \tau_{\geq m+1}K^\bullet)$$

(Derived Categories, Remark 13.12.4) by Lemma 15.64.2 it suffices to prove that $\tau_{\leq m}K^\bullet$ is pseudo-coherent. By assumption $H^m(\tau_{\leq m}K^\bullet)$ is a finite type R-module. Hence we can find a finite free R-module E and a map $E \to \mathrm{Ker}(d_K^m)$ such that the composition $E \to \mathrm{Ker}(d_K^m) \to H^m(\tau_{\leq m}K^\bullet)$ is surjective. Then $E[-m] \to \tau_{\leq m}K^\bullet$ witnesses the fact that $\tau_{\leq m}K^\bullet$ is m-pseudo-coherent. $\square$

064X **Lemma 15.64.8.** Let R be a ring. Let $m \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) so are $K^\bullet$ and $L^\bullet$.

Proof. In this proof we drop the superscript $^\bullet$. Assume that $K \oplus L$ is m-pseudo-coherent. It is clear that $K, L \in D^-(R)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.10. By Lemma 15.64.2 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. By Lemma 15.64.7 we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n], L[n-1], \ldots, L$ are m-pseudo-coherent as desired. The pseudo-coherent case follows from this and Lemma 15.64.5. $\square$

064Y **Lemma 15.64.9.** Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of R-modules such that K^i is $(m-i)$-pseudo-coherent for all i. Then $K^\bullet$ is m-pseudo-coherent. In particular, if $K^\bullet$ is a bounded above complex of pseudo-coherent R-modules, then $K^\bullet$ is pseudo-coherent.

Proof. We may replace $K^\bullet$ by $\sigma_{\geq m-1}K^\bullet$ (for example) and hence assume that $K^\bullet$ is bounded. Then the complex $K^\bullet$ is m-pseudo-coherent as each $K^i[-i]$ is m-pseudo-coherent by induction on the length of the complex: use Lemma 15.64.2 and the stupid truncations. For the final statement, it suffices to prove that $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 15.64.5. This follows from the first part. $\square$

066B Lemma 15.64.10. Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(R)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).

Proof. Assume $K^\bullet$ is an object of $D^-(R)$ such that each $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent. Let n be the largest integer such that $H^n(K^\bullet)$ is nonzero. We will prove the lemma by induction on n. If $n < m$, then $K^\bullet$ is m-pseudo-coherent by Lemma 15.64.7. If $n \geq m$, then we have the distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, H^n(K^\bullet)[-n])$$

(Derived Categories, Remark 13.12.4) Since $H^n(K^\bullet)[-n]$ is m-pseudo-coherent by assumption, we can use Lemma 15.64.2 to see that it suffices to prove that $\tau_{\leq n-1}K^\bullet$ is m-pseudo-coherent. By induction on n we win. (The pseudo-coherent case follows from this and Lemma 15.64.5.) $\square$

064Z Lemma 15.64.11. Let $A \to B$ be a ring map. Assume that B is pseudo-coherent as an A-module. Let $K^\bullet$ be a complex of B-modules. The following are equivalent

(1) $K^\bullet$ is m-pseudo-coherent as a complex of B-modules, and
(2) $K^\bullet$ is m-pseudo-coherent as a complex of A-modules.

The same equivalence holds for pseudo-coherence.

Proof. Assume (1). Choose a bounded complex of finite free B-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the distinguished triangle $(E^\bullet, K^\bullet, C(\alpha)^\bullet)$. By Lemma 15.64.7 $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of A-modules. Hence it suffices to prove that $E^\bullet$ is pseudo-coherent as a complex of A-modules, which follows from Lemma 15.64.9. The pseudo-coherent case of (1) $\Rightarrow$ (2) follows from this and Lemma 15.64.5.

Assume (2). Let n be the largest integer such that $H^n(K^\bullet) \neq 0$. We will prove that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules by induction on $n - m$. The case $n < m$ follows from Lemma 15.64.7. Choose a bounded complex of finite free A-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the induced map of complexes

$$\alpha \otimes 1 : E^\bullet \otimes_A B \to K^\bullet.$$

Note that $C(\alpha \otimes 1)^\bullet$ is acyclic in degrees $\geq n$ as $H^n(E) \to H^n(E^\bullet \otimes_A B) \to H^n(K^\bullet)$ is surjective by construction and since $H^i(E^\bullet \otimes_A B) = 0$ for $i > n$ by the spectral sequence of Example 15.62.4. On the other hand, $C(\alpha \otimes 1)^\bullet$ is m-pseudo-coherent as a complex of A-modules because both $K^\bullet$ and $E^\bullet \otimes_A B$ (see Lemma 15.64.9) are so, see Lemma 15.64.2. Hence by induction we see that $C(\alpha \otimes 1)^\bullet$ is m-pseudo-coherent as a complex of B-modules. Finally another application of Lemma 15.64.2 shows that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules (as clearly $E^\bullet \otimes_A B$ is pseudo-coherent as a complex of B-modules). The pseudo-coherent case of (2) $\Rightarrow$ (1) follows from this and Lemma 15.64.5. $\square$

0650 Lemma 15.64.12. Let $A \to B$ be a ring map. Let $K^\bullet$ be an m-pseudo-coherent (resp. pseudo-coherent) complex of A-modules. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ is an m-pseudo-coherent (resp. pseudo-coherent) complex of B-modules.

Proof. First we note that the statement of the lemma makes sense as $K^\bullet$ is bounded above and hence $K^\bullet \otimes_A^{\mathbf{L}} B$ is defined by Equation (15.57.0.2). Having said this, choose a bounded complex $E^\bullet$ of finite free A-modules and $\alpha : E^\bullet \to K^\bullet$ with $H^i(\alpha)$ an isomorphism for $i > m$ and surjective for $i = m$. Then the cone $C(\alpha)^\bullet$ is acyclic in degrees $\geq m$. Since $- \otimes_A^{\mathbf{L}} B$ is an exact functor we get a distinguished triangle

$$(E^\bullet \otimes_A^{\mathbf{L}} B, K^\bullet \otimes_A^{\mathbf{L}} B, C(\alpha)^\bullet \otimes_A^{\mathbf{L}} B)$$

of complexes of B-modules. By the dual to Derived Categories, Lemma 13.16.1 we see that $H^i(C(\alpha)^\bullet \otimes_A^{\mathbf{L}} B) = 0$ for $i \geq m$. Since $E^\bullet$ is a complex of projective R-modules we see that $E^\bullet \otimes_A^{\mathbf{L}} B = E^\bullet \otimes_A B$ and hence

$$E^\bullet \otimes_A B \longrightarrow K^\bullet \otimes_A^{\mathbf{L}} B$$

is a morphism of complexes of B-modules that witnesses the fact that $K^\bullet \otimes_A^{\mathbf{L}} B$ is m-pseudo-coherent. The case of pseudo-coherent complexes follows from the case of m-pseudo-coherent complexes via Lemma 15.64.5. □

066C Lemma 15.64.13. Let $A \to B$ be a flat ring map. Let M be an m-pseudo-coherent (resp. pseudo-coherent) A-module. Then $M \otimes_A B$ is an m-pseudo-coherent (resp. pseudo-coherent) B-module.

Proof. Immediate consequence of Lemma 15.64.12 and the fact that $M \otimes_A^{\mathbf{L}} B = M \otimes_A B$ because B is flat over A. □

The following lemma also follows from the stronger Lemma 15.64.15.

066D Lemma 15.64.14. Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).

Proof. We will use without further mention that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

Assume $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent for $i = 1, \ldots, r$. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet \otimes_R R_{f_i})$ is nonzero for some i. This implies in particular that $H^i(K^\bullet) = 0$ for $i > n$ (and that $H^n(K^\bullet) \neq 0$) see Algebra, Lemma 10.23.2. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is true by Lemma 15.64.7. If $n \geq m$, then $H^n(K^\bullet)_{f_i}$ is a finite R_{f_i}-module for each i, see Lemma 15.64.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 10.23.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complexes $C(\alpha)^\bullet \otimes_R R_{f_i}$ are m-pseudo-coherent for each i, see Lemma 15.64.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 15.64.2 once more we conclude. □

068R Lemma 15.64.15. Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).

Proof. We will use without further mention that $- \otimes_R R'$ is an exact functor and that therefore

$$H^i(K^\bullet) \otimes_R R' = H^i(K^\bullet \otimes_R R').$$

Assume $K^\bullet \otimes_R R'$ is m-pseudo-coherent. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet)$ is nonzero; then n is also the largest integer such that $H^n(K^\bullet \otimes_R R')$ is nonzero. We will prove the lemma by induction on $n-m$. If $n < m$, then the lemma is true by Lemma 15.64.7. If $n \geq m$, then $H^n(K^\bullet) \otimes_R R'$ is a finite R'-module, see Lemma 15.64.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 10.83.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complex $C(\alpha)^\bullet \otimes_R R'$ is m-pseudo-coherent, see Lemma 15.64.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 15.64.2 once more we conclude. □

0DJE **Lemma 15.64.16.** Let R be a ring. Let K, L be objects of $D(R)$.

1. If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes_R^{\mathbf{L}} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.
2. If K and L are pseudo-coherent, then $K \otimes_R^{\mathbf{L}} L$ is pseudo-coherent.

Proof. Proof of (1). We may assume there exist bounded complexes $K^\bullet$ and $L^\bullet$ of finite free R-modules and maps $\alpha : K^\bullet \to K$ and $\beta : L^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(K^\bullet \otimes_R L^\bullet) \to K \otimes_R^{\mathbf{L}} L$$

induces isomorphisms on cohomology in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted). Part (2) follows from part (1) and Lemma 15.64.5. □

066E **Lemma 15.64.17.** Let R be a Noetherian ring. Then

1. A complex of R-modules $K^\bullet$ is m-pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for $i \geq m$.
2. A complex of R-modules $K^\bullet$ is pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for all i.
3. An R-module is pseudo-coherent if and only if it is finite.

Proof. In Algebra, Lemma 10.71.1 we have seen that any finite R-module is pseudo-coherent. On the other hand, a pseudo-coherent module is finite, see Lemma 15.64.4. Hence (3) holds. Suppose that $K^\bullet$ is an m-pseudo-coherent complex. Then there exists a bounded complex of finite free R-modules $E^\bullet$ such that $H^i(K^\bullet)$ is isomorphic to $H^i(E^\bullet)$ for $i > m$ and such that $H^m(K^\bullet)$ is a quotient of $H^m(E^\bullet)$. Thus it is clear that each $H^i(K^\bullet)$, $i \geq m$ is a finite module. The converse implication in (1) follows from Lemma 15.64.10 and part (3). Part (2) follows from (1) and Lemma 15.64.5. □

0EWZ **Lemma 15.64.18.** Let R be a coherent ring (Algebra, Definition 10.90.1). Let $K \in D^-(R)$. The following are equivalent

1. K is m-pseudo-coherent,

(2) $H^m(K)$ is a finite R-module and $H^i(K)$ is coherent for $i > m$, and
(3) $H^m(K)$ is a finite R-module and $H^i(K)$ is finitely presented for $i > m$.

Thus K is pseudo-coherent if and only if $H^i(K)$ is a coherent module for all i.

Proof. Recall that an R-module M is coherent if and only if it is of finite presentation (Algebra, Lemma 10.90.4). This explains the equivalence of (2) and (3). If so and if we choose an exact sequence $0 \to N \to R^{\oplus m} \to M \to 0$, then N is coherent by Algebra, Lemma 10.90.3. Thus in this case, repeating this procedure with N we find a resolution

$$\ldots \to R^{\oplus n} \to R^{\oplus m} \to M \to 0$$

by finite free R-modules. In other words, M is pseudo-coherent. The equivalence of (1) and (2) follows from this and Lemmas 15.64.10 and 15.64.4. The final assertion follows from the equivalence of (1) and (2) combined with Lemma 15.64.5. □

15.65. Pseudo-coherent modules, II

0G8V We continue the discussion started in Section 15.64.

0G8W Lemma 15.65.1. Let R be a ring. Let $M = \operatorname{colim} M_i$ be a filtered colimit of R-modules. Let $K \in D(R)$ be m-pseudo-coherent. Then $\operatorname{colim} \operatorname{Ext}^n_R(K, M_i) = \operatorname{Ext}^n_R(K, M)$ for $n < -m$ and $\operatorname{colim} \operatorname{Ext}^{-m}_R(K, M_i) \to \operatorname{Ext}^{-m}_R(K, M)$ is injective.

Proof. By definition we can find a distinguished triangle

$$E \to K \to L \to E[1]$$

in $D(R)$ such that E is represented by a bounded complex of finite free R-modules and such that $H^i(L) = 0$ for $i \geq m$. Then $\operatorname{Ext}^n_R(L, N) = 0$ for any R-module N and $n \leq -m$, see Derived Categories, Lemma 13.27.3. By the long exact sequence of Ext associated to the distinguished triangle we see that $\operatorname{Ext}^n_R(K, N) \to \operatorname{Ext}^n_R(E, N)$ is an isomorphism for $n < -m$ and injective for $n = -m$. Thus it suffices to prove that $M \mapsto \operatorname{Ext}^n_R(E, M)$ commutes with filtered colimits when E can be represented by a bounded complex of finite free R-modules $E^\bullet$. The modules $\operatorname{Ext}^n_R(E, M)$ are computed by the complex $\operatorname{Hom}_R(E^\bullet, M)$, see Derived Categories, Lemma 13.19.8. The functor $M \mapsto \operatorname{Hom}_R(E^p, M)$ commutes with filtered colimits as E^p is finite free. Thus $\operatorname{Hom}_R(E^\bullet, M) = \operatorname{colim} \operatorname{Hom}_R(E^\bullet, M_i)$ as complexes. Since filtered colimits are exact (Algebra, Lemma 10.8.8) we conclude. □

0G8X Lemma 15.65.2. Let R be a ring. Let $K \in D^-(R)$. Let $m \in \mathbf{Z}$. Then K is m-pseudo-coherent if and only if for any filtered colimit $M = \operatorname{colim} M_i$ of R-modules we have $\operatorname{colim} \operatorname{Ext}^n_R(K, M_i) = \operatorname{Ext}^n_R(K, M)$ for $n < -m$ and $\operatorname{colim} \operatorname{Ext}^{-m}_R(K, M_i) \to \operatorname{Ext}^{-m}_R(K, M)$ is injective.

Proof. One implication was shown in Lemma 15.65.1. Assume for any filtered colimit $M = \operatorname{colim} M_i$ of R-modules we have $\operatorname{colim} \operatorname{Ext}^n_R(K, M_i) = \operatorname{Ext}^n_R(K, M)$ for $n < -m$ and $\operatorname{colim} \operatorname{Ext}^{-m}_R(K, M_i) \to \operatorname{Ext}^{-m}_R(K, M)$ is injective. We will show K is m-pseudo-coherent.

Let t be the maximal integer such that $H^t(K)$ is nonzero. We will use induction on t. If $t < m$, then K is m-pseudo-coherent by Lemma 15.64.7. If $t \geq m$, then since $\operatorname{Hom}_R(H^t(K), M) = \operatorname{Ext}^{-t}_R(K, M)$ we conclude that $\operatorname{colim} \operatorname{Hom}_R(H^t(K), M_i) \to \operatorname{Hom}_R(H^t(K), M)$ is injective for any filtered colimit $M = \operatorname{colim} M_i$. This implies that $H^t(K)$ is a finite R-module by Algebra, Lemma 10.11.1. Choose a finite free R-module F and a surjection $F \to H^t(K)$. We can lift this to a morphism $F[-t] \to K$ in $D(R)$ and choose a distinguished triangle

$$F[-t] \to K \to L \to F[-t+1]$$

in $D(R)$. Then $H^i(L) = 0$ for $i \geq t$. Moreover, the long exact sequence of Ext associated to this distinguished triangle shows that L inherts the assumption we made on K by a small argument we omit. By induction on t we conclude that L is m-pseudo-coherent. Hence K is m-pseudo-coherent by Lemma 15.64.2. □

087Q Lemma 15.65.3. Let R be a ring. Let L, M, N be R-modules.

(1) If M is finitely presented and L is flat, then the canonical map $\mathrm{Hom}_R(M, N) \otimes_R L \to \mathrm{Hom}_R(M, N \otimes_R L)$ is an isomorphism.
(2) If M is $(-m)$-pseudo-coherent and L is flat, then the canonical map $\mathrm{Ext}^i_R(M, N) \otimes_R L \to \mathrm{Ext}^i_R(M, N \otimes_R L)$ is an isomorphism for $i < m$.

Proof. Choose a resolution $F_\bullet \to M$ whose terms are free R-modules, see Algebra, Lemma 10.71.1. The complex $\mathrm{Hom}_R(F_\bullet, N)$ computes $\mathrm{Ext}^i_R(M, N)$ and the complex $\mathrm{Hom}_R(F_\bullet, N \otimes_R L)$ computes $\mathrm{Ext}^i_R(M, N \otimes_R L)$. There always is a map of cochain complexes

$$\mathrm{Hom}_R(F_\bullet, N) \otimes_R L \longrightarrow \mathrm{Hom}_R(F_\bullet, N \otimes_R L)$$

which induces canonical maps $\mathrm{Ext}^i_R(M, N) \otimes_R L \to \mathrm{Ext}^i_R(M, N \otimes_R L)$ for all $i \geq 0$ (canonical for example in the sense that these maps do not depend on the choice of the resolution $F_\bullet$). If L is flat, then the complex $\mathrm{Hom}_R(F_\bullet, N) \otimes_R L$ computes $\mathrm{Ext}^i_R(M, N) \otimes_R L$ since taking cohomology commutes with tensoring by L.

Having said all of the above, if M is $(-m)$-pseudo-coherent, then we may choose $F_\bullet$ such that F_i is finite free for $i = 0, \ldots, m$. Then the map of cochain complexes displayed above is an isomorphism in degrees $\leq m$ and hence an isomorphism on cohomology groups in degrees $< m$. This proves (2). If M is finitely presented, then M is (-1)-pseudo-coherent by Lemma 15.64.4 and we get the result because $\mathrm{Hom} = \mathrm{Ext}^0$. □

087R Lemma 15.65.4. Let $R \to R'$ be a flat ring map. Let M, N be R-modules.

(1) If M is a finitely presented R-module, then $\mathrm{Hom}_R(M, N) \otimes_R R' = \mathrm{Hom}_{R'}(M \otimes_R R', N \otimes_R R')$.
(2) If M is $(-m)$-pseudo-coherent, then $\mathrm{Ext}^i_R(M, N) \otimes_R R' = \mathrm{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R')$ for $i < m$.

In particular if R is Noetherian and M is a finite module this holds for all i.

Proof. By Algebra, Lemma 10.73.1 we have $\mathrm{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R') = \mathrm{Ext}^i_R(M, N \otimes_R R')$. Combined with Lemma 15.65.3 we conclude (1) and (2) holds. The final statement follows from this and Lemma 15.64.17. □

0CYB Lemma 15.65.5. Let R be a ring. Let $K \in D^-(R)$. The following are equivalent:

(1) K is pseudo-coherent,
(2) for every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map

$$\alpha : K \otimes_R^{\mathbf{L}} \left(\prod\nolimits_\alpha Q_\alpha\right) \longrightarrow \prod\nolimits_\alpha (K \otimes_R^{\mathbf{L}} Q_\alpha)$$

is an isomorphism in $D(R)$,

(3) for every R-module Q and every set A, the canonical map

$$\beta : K \otimes_R^{\mathbf{L}} Q^A \longrightarrow (K \otimes_R^{\mathbf{L}} Q)^A$$

is an isomorphism in $D(R)$, and

(4) for every set A, the canonical map

$$\gamma : K \otimes_R^{\mathbf{L}} R^A \longrightarrow K^A$$

is an isomorphism in $D(R)$.

Given $m \in \mathbf{Z}$ the following are equivalent

(a) K is m-pseudo-coherent,
(b) for every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, with α as above $H^i(\alpha)$ is an isomorphism for $i > m$ and surjective for $i = m$,
(c) for every R-module Q and every set A, with β as above $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$,
(d) for every set A, with γ as above $H^i(\gamma)$ is an isomorphism for $i > m$ and surjective for $i = m$.

Proof. If K is pseudo-coherent, then K can be represented by a bounded above complex of finite free R-modules. Then the derived tensor products are computed by tensoring with this complex. Also, products in $D(R)$ are given by taking products of any choices of representative complexes. Hence (1) implies (2), (3), (4) by the corresponding fact for modules, see Algebra, Proposition 10.89.3.

In the same way (using the tensor product is right exact) the reader shows that (a) implies (b), (c), and (d).

Assume (4) holds. To show that K is pseudo-coherent it suffices to show that K is m-pseudo-coherent for all m (Lemma 15.64.5). Hence to finish then proof it suffices to prove that (d) implies (a).

Assume (d). Let i be the largest integer such that $H^i(K)$ is nonzero. If $i < m$, then we are done. If not, then from (d) and the description of products in $D(R)$ given above we find that $H^i(K) \otimes_R R^A \to H^i(K)^A$ is surjective. Hence $H^i(K)$ is a finitely generated R-module by Algebra, Proposition 10.89.2. Thus we may choose a complex L consisting of a single finite free module sitting in degree i and a map of complexes $L \to K$ such that $H^i(L) \to H^i(K)$ is surjective. In particular L satisfies (1), (2), (3), and (4). Choose a distinguished triangle

$$L \to K \to M \to L[1]$$

Then we see that $H^j(M) = 0$ for $j \geq i$. On the other hand, M still has property (d) by a small argument which we omit. By induction on i we find that M is m-pseudo-coherent. Hence K is m-pseudo-coherent by Lemma 15.64.2. □

0G8Y Lemma 15.65.6. Let R be a ring. Let $K \in D(R)$ be pseudo-coherent. Let $i \in \mathbf{Z}$. There exists a finitely presented R-module M and a map $K \to M[-i]$ in $D(R)$ which induces an injection $H^i(K) \to M$.

Proof. By Definition 15.64.1 we may represent K by a complex $P^\bullet$ of finite free R-modules. Set $M = \mathrm{Coker}(P^{i-1} \to P^i)$. □

0A7D Lemma 15.65.7. Let A be a Noetherian ring. Let $K \in D(A)$ be pseudo-coherent, i.e., $K \in D^-(A)$ with finite cohomology modules. Let $\mathfrak{m}$ be a maximal ideal of

A. If $H^i(K)/\mathfrak{m}H^i(K) \neq 0$, then there exists a finite A-module E annihilated by a power of $\mathfrak{m}$ and a map $K \to E[-i]$ which is nonzero on $H^i(K)$.

Proof. (The equivalent formulation of pseudo-coherence in the statement of the lemma is Lemma 15.64.17.) Choose $K \to M[-i]$ as in Lemma 15.65.6. By Artin-Rees (Algebra, Lemma 10.51.2) we can find an n such that $H^i(K) \cap \mathfrak{m}^n M \subset \mathfrak{m}H^i(K)$. Take $E = M/\mathfrak{m}^n M$. $\square$

15.66. Tor dimension

0651 Instead of resolving by projective modules we can look at resolutions by flat modules. This leads to the following concept.

0652 Definition 15.66.1. Let R be a ring. Denote $D(R)$ its derived category. Let $a, b \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ has tor-amplitude in $[a, b]$ if $H^i(K^\bullet \otimes_R^{\mathbf{L}} M) = 0$ for all R-modules M and all $i \notin [a, b]$.
(2) An object $K^\bullet$ of $D(R)$ has finite tor dimension if it has tor-amplitude in $[a, b]$ for some a, b.
(3) An R-module M has tor dimension $\leq d$ if $M[0]$ as an object of $D(R)$ has tor-amplitude in $[-d, 0]$.
(4) An R-module M has finite tor dimension if $M[0]$ as an object of $D(R)$ has finite tor dimension.

We observe that if $K^\bullet$ has finite tor dimension, then $K^\bullet \in D^b(R)$.

0653 Lemma 15.66.2. Let R be a ring. Let $K^\bullet$ be a bounded above complex of flat R-modules with tor-amplitude in $[a, b]$. Then $\mathrm{Coker}(d_K^{a-1})$ is a flat R-module.

Proof. As $K^\bullet$ is a bounded above complex of flat modules we see that $K^\bullet \otimes_R M = K^\bullet \otimes_R^{\mathbf{L}} M$. Hence for every R-module M the sequence

$$K^{a-2} \otimes_R M \to K^{a-1} \otimes_R M \to K^a \otimes_R M$$

is exact in the middle. Since $K^{a-2} \to K^{a-1} \to K^a \to \mathrm{Coker}(d_K^{a-1}) \to 0$ is a flat resolution this implies that $\mathrm{Tor}_1^R(\mathrm{Coker}(d_K^{a-1}), M) = 0$ for all R-modules M. This means that $\mathrm{Coker}(d_K^{a-1})$ is flat, see Algebra, Lemma 10.75.8. $\square$

0654 Lemma 15.66.3. Let R be a ring. Let $K^\bullet$ be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent

(1) $K^\bullet$ has tor-amplitude in $[a, b]$.
(2) $K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of flat R-modules with $E^i = 0$ for $i \notin [a, b]$.

Proof. If (2) holds, then we may compute $K^\bullet \otimes_R^{\mathbf{L}} M = E^\bullet \otimes_R M$ and it is clear that (1) holds. Assume that (1) holds. We may replace $K^\bullet$ by a projective resolution with $K^i = 0$ for $i > b$. See Derived Categories, Lemma 13.19.3. Set $E^\bullet = \tau_{\geq a} K^\bullet$. Everything is clear except that E^a is flat which follows immediately from Lemma 15.66.2 and the definitions. $\square$

0BYL Lemma 15.66.4. Let R be a ring. Let $a \in \mathbf{Z}$ and let K be an object of $D(R)$. The following are equivalent

(1) K has tor-amplitude in $[a, \infty]$, and
(2) K is quasi-isomorphic to a K-flat complex $E^\bullet$ whose terms are flat R-modules with $E^i = 0$ for $i \notin [a, \infty]$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1) holds. First we choose a K-flat complex $K^\bullet$ with flat terms representing K, see Lemma 15.59.10. For any R-module M the cohomology of

$$K^{n-1} \otimes_R M \to K^n \otimes_R M \to K^{n+1} \otimes_R M$$

computes $H^n(K \otimes_R^{\mathbf{L}} M)$. This is always zero for $n < a$. Hence if we apply Lemma 15.66.2 to the complex $\ldots \to K^{a-1} \to K^a \to K^{a+1}$ we conclude that $N = \text{Coker}(K^{a-1} \to K^a)$ is a flat R-module. We set

$$E^\bullet = \tau_{\geq a} K^\bullet = (\ldots \to 0 \to N \to K^{a+1} \to \ldots)$$

The kernel $L^\bullet$ of $K^\bullet \to E^\bullet$ is the complex

$$L^\bullet = (\ldots \to K^{a-1} \to I \to 0 \to \ldots)$$

where $I \subset K^a$ is the image of $K^{a-1} \to K^a$. Since we have the short exact sequence $0 \to I \to K^a \to N \to 0$ we see that I is a flat R-module. Thus $L^\bullet$ is a bounded above complex of flat modules, hence K-flat by Lemma 15.59.7. It follows that $E^\bullet$ is K-flat by Lemma 15.59.6. $\square$

0655 Lemma 15.66.5. Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. Let $a, b \in \mathbf{Z}$.

(1) If $K^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $L^\bullet$ has tor-amplitude in $[a, b]$ then $M^\bullet$ has tor-amplitude in $[a, b]$.
(2) If $K^\bullet, M^\bullet$ have tor-amplitude in $[a, b]$, then $L^\bullet$ has tor-amplitude in $[a, b]$.
(3) If $L^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $M^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ has tor-amplitude in $[a+1, b+1]$.

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes_R^{\mathbf{L}} M$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. $\square$

066F Lemma 15.66.6. Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent

(1) M has tor dimension $\leq d$, and
(2) there exists a resolution

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with F_i a flat R-module.

In particular an R-module has tor dimension 0 if and only if it is a flat R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to M. Hence the Tor dimension of M is at most d by Lemma 15.66.3. Conversely, assume (1). Let $P^\bullet \to M$ be a projective resolution of M. By Lemma 15.66.2 we see that $\tau_{\geq -d} P^\bullet$ is a flat resolution of M of length d, i.e., (2) holds. $\square$

066G Lemma 15.66.7. Let R be a ring. Let $a, b \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ has tor amplitude in $[a, b]$ so do $K^\bullet$ and $L^\bullet$.

Proof. Clear from the fact that the Tor functors are additive. $\square$

066H Lemma 15.66.8. Let R be a ring. Let $K^\bullet$ be a bounded complex of R-modules such that K^i has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet$ is a finite complex of R-modules of finite tor dimension, then $K^\bullet$ has finite tor dimension.

Proof. Follows by induction on the length of the finite complex: use Lemma 15.66.5 and the stupid truncations. ☐

066I Lemma 15.66.9. Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet \in D^b(R)$ such that $H^i(K^\bullet)$ has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet \in D^b(R)$ and all its cohomology groups have finite tor dimension then $K^\bullet$ has finite tor dimension.

Proof. Follows by induction on the length of the finite complex: use Lemma 15.66.5 and the canonical truncations. ☐

0B66 Lemma 15.66.10. Let $A \to B$ be a ring map. Let $K^\bullet$ and $L^\bullet$ be complexes of B-modules. Let $a, b, c, d \in \mathbf{Z}$. If

(1) $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$,
(2) $L^\bullet$ as a complex of A-modules has tor amplitude in $[c, d]$,

then $K^\bullet \otimes_B^{\mathbf{L}} L^\bullet$ as a complex of A-modules has tor amplitude in $[a+c, b+d]$.

Proof. We may assume that $K^\bullet$ is a complex of flat B-modules with $K^i = 0$ for $i \notin [a, b]$, see Lemma 15.66.3. Let M be an A-module. Choose a free resolution $F^\bullet \to M$. Then

$$(K^\bullet \otimes_B^{\mathbf{L}} L^\bullet) \otimes_A^{\mathbf{L}} M = \text{Tot}(\text{Tot}(K^\bullet \otimes_B L^\bullet) \otimes_A F^\bullet) = \text{Tot}(K^\bullet \otimes_B \text{Tot}(L^\bullet \otimes_A F^\bullet))$$

see Homology, Remark 12.18.4 for the second equality. By assumption (2) the complex $\text{Tot}(L^\bullet \otimes_A F^\bullet)$ has nonzero cohomology only in degrees $[c, d]$. Hence the spectral sequence of Homology, Lemma 12.25.1 for the double complex $K^\bullet \otimes_B \text{Tot}(L^\bullet \otimes_A F^\bullet)$ proves that $(K^\bullet \otimes_B^{\mathbf{L}} L^\bullet) \otimes_A^{\mathbf{L}} M$ has nonzero cohomology only in degrees $[a+c, b+d]$. ☐

066J Lemma 15.66.11. Let $A \to B$ be a ring map. Assume that B is flat as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a, b]$.

Proof. This is a special case of Lemma 15.66.10, but can also be seen directly as follows. We have $K^\bullet \otimes_A^{\mathbf{L}} M = K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A B)$ since any projective resolution of $K^\bullet$ as a complex of B-modules is a flat resolution of $K^\bullet$ as a complex of A-modules and can be used to compute $K^\bullet \otimes_A^{\mathbf{L}} M$. ☐

066K Lemma 15.66.12. Let $A \to B$ be a ring map. Assume that B has tor dimension $\leq d$ as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a-d, b]$.

Proof. This is a special case of Lemma 15.66.10, but can also be seen directly as follows. Let M be an A-module. Choose a free resolution $F^\bullet \to M$. Then

$$K^\bullet \otimes_A^{\mathbf{L}} M = \text{Tot}(K^\bullet \otimes_A F^\bullet) = \text{Tot}(K^\bullet \otimes_B (F^\bullet \otimes_A B)) = K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A^{\mathbf{L}} B).$$

By our assumption on B as an A-module we see that $M \otimes_A^{\mathbf{L}} B$ has cohomology only in degrees $-d, -d+1, \ldots, 0$. Because $K^\bullet$ has tor amplitude in $[a, b]$ we see from the spectral sequence in Example 15.62.4 that $K^\bullet \otimes_B^{\mathbf{L}} (M \otimes_A^{\mathbf{L}} B)$ has cohomology only in degrees $[-d+a, b]$ as desired. □

066L Lemma 15.66.13. Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules with tor amplitude in $[a, b]$. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ as a complex of B-modules has tor amplitude in $[a, b]$.

Proof. By Lemma 15.66.3 we can find a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a complex of flat A-modules with $E^i = 0$ for $i \notin [a, b]$. Then $E^\bullet \otimes_A B$ computes $K^\bullet \otimes_A^{\mathbf{L}} B$ by construction and each $E^i \otimes_A B$ is a flat B-module by Algebra, Lemma 10.39.7. Hence we conclude by Lemma 15.66.3. □

066M Lemma 15.66.14. Let $A \to B$ be a flat ring map. Let $d \geq 0$. Let M be an A-module of tor dimension $\leq d$. Then $M \otimes_A B$ is a B-module of tor dimension $\leq d$.

Proof. Immediate consequence of Lemma 15.66.13 and the fact that $M \otimes_A^{\mathbf{L}} B = M \otimes_A B$ because B is flat over A. □

0B67 Lemma 15.66.15. Let $A \to B$ be a ring map. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. The following are equivalent

(1) $K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules,
(2) $K^\bullet_\mathfrak{q}$ has tor amplitude in $[a, b]$ as a complex of $A_\mathfrak{p}$-modules for every prime $\mathfrak{q} \subset B$ with $\mathfrak{p} = A \cap \mathfrak{q}$,
(3) $K^\bullet_\mathfrak{m}$ has tor amplitude in $[a, b]$ as a complex of $A_\mathfrak{p}$-modules for every maximal ideal $\mathfrak{m} \subset B$ with $\mathfrak{p} = A \cap \mathfrak{m}$.

Proof. Assume (3) and let M be an A-module. Then $H^i = H^i(K^\bullet \otimes_A^{\mathbf{L}} M)$ is a B-module and $(H^i)_\mathfrak{m} = H^i(K^\bullet_\mathfrak{m} \otimes_{A_\mathfrak{p}}^{\mathbf{L}} M_\mathfrak{p})$. Hence $H^i = 0$ for $i \notin [a, b]$ by Algebra, Lemma 10.23.1. Thus (3) $\Rightarrow$ (1). We omit the proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). □

066N Lemma 15.66.16. Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.

Proof. This follows immediately from Lemma 15.66.15 but can also be seen directly as follows. Note that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

and similarly for every R-module M we have

$$H^i(K^\bullet \otimes_R^{\mathbf{L}} M)_{f_i} = H^i(K^\bullet \otimes_R^{\mathbf{L}} M) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i} \otimes_{R_{f_i}}^{\mathbf{L}} M_{f_i}).$$

Hence the result follows from the fact that an R-module N is zero if and only if N_{f_i} is zero for each i, see Algebra, Lemma 10.23.2. □

068S Lemma 15.66.17. Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.

Proof. Let M be an R-module. Since $R \to R'$ is flat we see that

$$(M \otimes_R^{\mathbf{L}} K^\bullet) \otimes_R R' = ((M \otimes_R R') \otimes_{R'}^{\mathbf{L}} (K^\bullet \otimes_R R')$$

and taking cohomology commutes with tensoring with R'. Hence $\text{Tor}_i^R(M, K^\bullet) \otimes_R R' = \text{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$. Since $R \to R'$ is faithfully flat, the vanishing of $\text{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$ for $i \not\in [a, b]$ implies the same thing for $\text{Tor}_i^R(M, K^\bullet)$. $\square$

0DJF Lemma 15.66.18. Given ring maps $R \to A \to B$ with $A \to B$ faithfully flat and $K \in D(A)$ the tor amplitude of K over R is the same as the tor amplitude of $K \otimes_A^{\mathbf{L}} B$ over R.

Proof. This is true because for an R-module M we have $H^i(K \otimes_R^{\mathbf{L}} M) \otimes_A B = H^i((K \otimes_A^{\mathbf{L}} B) \otimes_R^{\mathbf{L}} M)$ for all i. Namely, represent K by a complex $K^\bullet$ of A-modules and choose a free resolution $F^\bullet \to M$. Then we have the equality

$$\text{Tot}(K^\bullet \otimes_A B \otimes_R F^\bullet) = \text{Tot}(K^\bullet \otimes_R F^\bullet) \otimes_A B$$

The cohomology groups of the left hand side are $H^i((K \otimes_A^{\mathbf{L}} B) \otimes_R^{\mathbf{L}} M)$ and on the right hand side we obtain $H^i(K \otimes_R^{\mathbf{L}} M) \otimes_A B$. $\square$

066P Lemma 15.66.19. Let R be a ring of finite global dimension d. Then

(1) every module has tor dimension $\leq d$,
(2) a complex of R-modules $K^\bullet$ with $H^i(K^\bullet) \neq 0$ only if $i \in [a, b]$ has tor amplitude in $[a - d, b]$, and
(3) a complex of R-modules $K^\bullet$ has finite tor dimension if and only if $K^\bullet \in D^b(R)$.

Proof. The assumption on R means that every module has a finite projective resolution of length at most d, in particular every module has tor dimension $\leq d$. The second statement follows from Lemma 15.66.9 and the definitions. The third statement is a rephrasing of the second. $\square$

15.67. Spectral sequences for Ext

0AVG In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of objects L, K of the derived category $D(R)$ of a ring R we denote

$$\text{Ext}_R^n(L, K) = \text{Hom}_{D(R)}(L, K[n])$$

according to our general conventions in Derived Categories, Section 13.27.

For M an R-module and $K \in D^+(R)$ there is a spectral sequence

0AVH (15.67.0.1) $$E_2^{i,j} = \text{Ext}_R^i(M, H^j(K)) \Rightarrow \text{Ext}_R^{i+j}(M, K)$$

and if K is represented by the bounded below complex $K^\bullet$ of R-modules there is a spectral sequence

0AVI (15.67.0.2) $$E_1^{i,j} = \text{Ext}_R^j(M, K^i) \Rightarrow \text{Ext}_R^{i+j}(M, K)$$

These spectral sequences come from applying Derived Categories, Lemma 13.21.3 to the functor $\text{Hom}_R(M, -)$.

15.68. Projective dimension

0A5M We defined the projective dimension of a module in Algebra, Definition 10.109.2.

0A5N **Definition 15.68.1.** Let R be a ring. Let K be an object of $D(R)$. We say K has *finite projective dimension* if K can be represented by a bounded complex of projective modules. We say K has *projective-amplitude in* $[a, b]$ if K is quasi-isomorphic to a complex

$$\ldots \to 0 \to P^a \to P^{a+1} \to \ldots \to P^{b-1} \to P^b \to 0 \to \ldots$$

where P^i is a projective R-module for all $i \in \mathbf{Z}$.

Clearly, K has finite projective dimension if and only if K has projective-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Furthermore, if K has finite projective dimension, then K is bounded. Here is a lemma to detect such objects of $D(R)$.

0A5P **Lemma 15.68.2.** Let R be a ring. Let K be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent

(1) K has projective-amplitude in $[a, b]$,
(2) $\text{Ext}^i_R(K, N) = 0$ for all R-modules N and all $i \not\in [-b, -a]$,
(3) $H^n(K) = 0$ for $n > b$ and $\text{Ext}^i_R(K, N) = 0$ for all R-modules N and all $i > -a$, and
(4) $H^n(K) = 0$ for $n \not\in [a - 1, b]$ and $\text{Ext}^{-a+1}_R(K, N) = 0$ for all R-modules N.

Proof. Assume (1). We may assume K is the complex

$$\ldots \to 0 \to P^a \to P^{a+1} \to \ldots \to P^{b-1} \to P^b \to 0 \to \ldots$$

where P^i is a projective R-module for all $i \in \mathbf{Z}$. In this case we can compute the ext groups by the complex

$$\ldots \to 0 \to \text{Hom}_R(P^b, N) \to \ldots \to \text{Hom}_R(P^a, N) \to 0 \to \ldots$$

and we obtain (2).

Assume (2) holds. Choose an injection $H^n(K) \to I$ where I is an injective R-module. Since $\text{Hom}_R(-, I)$ is an exact functor, we see that $\text{Ext}^{-n}(K, I) = \text{Hom}_R(H^n(K), I)$. We conclude in particular that $H^n(K)$ is zero for $n > b$. Thus (2) implies (3).

By the same argument as in (2) implies (3) gives that (3) implies (4).

Assume (4). The same argument as in (2) implies (3) shows that $H^{a-1}(K) = 0$, i.e., we have $H^i(K) = 0$ unless $i \in [a, b]$. In particular, K is bounded above and we can choose a a complex $P^\bullet$ representing K with P^i projective (for example free) for all $i \in \mathbf{Z}$ and $P^i = 0$ for $i > b$. See Derived Categories, Lemma 13.15.4. Let $Q = \text{Coker}(P^{a-1} \to P^a)$. Then K is quasi-isomorphic to the complex

$$\ldots \to 0 \to Q \to P^{a+1} \to \ldots \to P^b \to 0 \to \ldots$$

as $H^i(K) = 0$ for $i < a$. Denote $K' = (P^{a+1} \to \ldots \to P^b)$ the corresponding object of $D(R)$. We obtain a distinguished triangle

$$K' \to K \to Q[-a] \to K'[1]$$

in $D(R)$. Thus for every R-module N an exact sequence

$$\text{Ext}^{-a}(K', N) \to \text{Ext}^1(Q, N) \to \text{Ext}^{1-a}(K, N)$$

By assumption the term on the right vanishes. By the implication (1) $\Rightarrow$ (2) the term on the left vanishes. Thus Q is a projective R-module by Algebra, Lemma 10.77.2. Hence (1) holds and the proof is complete. □

0A5Q Example 15.68.3. Let k be a field and let R be the ring of dual numbers over k, i.e., $R = k[x]/(x^2)$. Denote $\epsilon \in R$ the class of x. Let $M = R/(\epsilon)$. Then M is quasi-isomorphic to the complex

$$R \xrightarrow{\epsilon} R \xrightarrow{\epsilon} R \to \ldots$$

but M does not have finite projective dimension as defined in Algebra, Definition 10.109.2. This explains why we consider bounded (in both directions) complexes of projective modules in our definition of finite projective dimension of objects of $D(R)$.

15.69. Injective dimension

0A5R This section is the dual of the section on projective dimension.

0A5S Definition 15.69.1. Let R be a ring. Let K be an object of $D(R)$. We say K has finite injective dimension if K can be represented by a finite complex of injective R-modules. We say K has injective-amplitude in $[a, b]$ if K is isomorphic to a complex

$$\ldots \to 0 \to I^a \to I^{a+1} \to \ldots \to I^{b-1} \to I^b \to 0 \to \ldots$$

with I^i an injective R-module for all $i \in \mathbf{Z}$.

Clearly, K has bounded injective dimension if and only if K has injective-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Furthermore, if K has bounded injective dimension, then K is bounded. Here is the obligatory lemma.

0A5T Lemma 15.69.2. Let R be a ring. Let K be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent

(1) K has injective-amplitude in $[a, b]$,
(2) $\mathrm{Ext}^i_R(N, K) = 0$ for all R-modules N and all $i \not\in [a, b]$,
(3) $\mathrm{Ext}^i(R/I, K) = 0$ for all ideals $I \subset R$ and all $i \not\in [a, b]$.

Proof. Assume (1). We may assume K is the complex

$$\ldots \to 0 \to I^a \to I^{a+1} \to \ldots \to I^{b-1} \to I^b \to 0 \to \ldots$$

where I^i is a injective R-module for all $i \in \mathbf{Z}$. In this case we can compute the ext groups by the complex

$$\ldots \to 0 \to \mathrm{Hom}_R(N, I^a) \to \ldots \to \mathrm{Hom}_R(N, I^b) \to 0 \to \ldots$$

and we obtain (2). It is clear that (2) implies (3).

Assume (3) holds. Choose a nonzero map $R \to H^n(K)$. Since $\mathrm{Hom}_R(R, -)$ is an exact functor, we see that $\mathrm{Ext}^n_R(R, K) = \mathrm{Hom}_R(R, H^n(K)) = H^n(K)$. We conclude that $H^n(K)$ is zero for $n \not\in [a, b]$. In particular, K is bounded below and we can choose a quasi-isomorphism

$$K \to I^\bullet$$

with I^i injective for all $i \in \mathbf{Z}$ and $I^i = 0$ for $i < a$. See Derived Categories, Lemma 13.15.5. Let $J = \mathrm{Ker}(I^b \to I^{b+1})$. Then K is quasi-isomorphic to the complex

$$\ldots \to 0 \to I^a \to \ldots \to I^{b-1} \to J \to 0 \to \ldots$$

Denote $K' = (I^a \to \ldots \to I^{b-1})$ the corresponding object of $D(R)$. We obtain a distinguished triangle

$$J[-b] \to K \to K' \to J[1-b]$$

in $D(R)$. Thus for every ideal $I \subset R$ an exact sequence

$$\mathrm{Ext}^b(R/I, K') \to \mathrm{Ext}^1(R/I, J) \to \mathrm{Ext}^{1+b}(R/I, K)$$

By assumption the term on the right vanishes. By the implication (1) $\Rightarrow$ (2) the term on the left vanishes. Thus J is a injective R-module by Lemma 15.55.4. $\square$

0EX0 Example 15.69.3. Let R be a Dedekind domain. Then every nonzero ideal I is a finite projective module, see Lemma 15.22.11. Thus R/I has projective dimension 1. Hence every R-module M has injective dimension ≤ 1 by Lemma 15.69.2. Thus $\mathrm{Ext}^i_R(M, N) = 0$ for $i \geq 2$ and any pair of R-modules M, N. It follows that any object K in $D^b(R)$ is isomorphic to the direct sum of its cohomologies: $K \cong \bigoplus H^i(K)[-i]$, see Derived Categories, Lemma 13.27.10.

0A5U Example 15.69.4. Let k be a field and let R be the ring of dual numbers over k, i.e., $R = k[x]/(x^2)$. Denote $\epsilon \in R$ the class of x. Let $M = R/(\epsilon)$. Then M is quasi-isomorphic to the complex

$$\ldots \to R \xrightarrow{\epsilon} R \xrightarrow{\epsilon} R$$

and R is an injective R-module. However one usually does not consider M to have finite injective dimension in this situation. This explains why we consider bounded (in both directions) complexes of injective modules in our definition of bounded injective dimension of objects of $D(R)$.

0A5V Lemma 15.69.5. Let R be a ring. Let $K \in D(R)$.

(1) If K is in $D^b(R)$ and $H^i(K)$ has finite injective dimension for all i, then K has finite injective dimension.
(2) If $K^\bullet$ represents K, is a bounded complex of R-modules, and K^i has finite injective dimension for all i, then K has finite injective dimension.

Proof. Omitted. Hint: Apply the spectral sequences of Derived Categories, Lemma 13.21.3 to the functor $F = \mathrm{Hom}_R(N, -)$ to get a computation of $\mathrm{Ext}^i_A(N, K)$ and use the criterion of Lemma 15.69.2. $\square$

0DW2 Lemma 15.69.6. Let R be a Noetherian ring. Let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let $K \in D^+(R)$ have finite cohomology modules. Then the following are equivalent

(1) K has finite injective dimension, and
(2) there exists a b such that $\mathrm{Ext}^i_R(R/J, K) = 0$ for $i > b$ and any ideal $J \supset I$.

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). Say $H^i(K) = 0$ for $i < a$. Then $\mathrm{Ext}^i(M, K) = 0$ for $i < a$ and all R-modules M. Thus it suffices to show that $\mathrm{Ext}^i(M, K) = 0$ for $i > b$ any finite R-module M, see Lemma 15.69.2. By Algebra, Lemma 10.62.1 the module M has a finite filtration whose successive quotients are of the form $R/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. If $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence and $\mathrm{Ext}^i(M_j, K) = 0$ for $i > b$ and $j = 1, 2$, then $\mathrm{Ext}^i(M, K) = 0$ for $i > b$. Thus we may assume $M = R/\mathfrak{p}$. If $I \subset \mathfrak{p}$, then the vanishing follows from the assumption. If not, then choose $f \in I$, $f \not\in \mathfrak p$. Consider the short exact sequence

$$0 \to R/\mathfrak p \xrightarrow{f} R/\mathfrak p \to R/(\mathfrak p, f) \to 0$$

The R-module $R/(\mathfrak p, f)$ has a filtration whose successive quotients are $R/\mathfrak q$ with $(\mathfrak p, f) \subset \mathfrak q$. Thus by Noetherian induction and the argument above we may assume the vanishing holds for $R/(\mathfrak p, f)$. On the other hand, the modules $E^i = \text{Ext}^i(R/\mathfrak p, K)$ are finite by our assumption on K (bounded below with finite cohomology modules), the spectral sequence (15.67.0.1), and Algebra, Lemma 10.71.9. Thus E^i for $i > b$ is a finite R-module such that $E^i/fE^i = 0$. We conclude by Nakayama's lemma (Algebra, Lemma 10.20.1) that E^i is zero. □

0AVJ Lemma 15.69.7. Let $(R, \mathfrak m, \kappa)$ be a local Noetherian ring. Let $K \in D^+(R)$ have finite cohomology modules. Then the following are equivalent

(1) K has finite injective dimension, and
(2) $\text{Ext}^i_R(\kappa, K) = 0$ for $i \gg 0$.

Proof. This is a special case of Lemma 15.69.6. □

15.70. Modules which are close to being projective

0G8Z There seem to be many different of definitions in the literature of "almost projective modules". In this section we discuss just one of the many possibilities.

0G90 Lemma 15.70.1. Let R be a ring. Let M, N be R-modules.

(1) Given an R-module map $\varphi : M \to N$ the following are equivalent: (a) φ factors through a projective R-module, and (b) φ factors through a free R-module.
(2) The set of $\varphi : M \to N$ satisfying the equivalent conditions of (1) is an R-submodule of $\text{Hom}_R(M, N)$.
(3) Given maps $\psi : M' \to M$ and $\xi : N \to N'$, if $\varphi : M \to N$ satisfies the equivalent conditions of (1), then $\xi \circ \varphi \circ \psi : M' \to N'$ does too.

Proof. The equivalence of (1)(a) and (1)(b) follows from Algebra, Lemma 10.77.2. If $\varphi : M \to N$ and $\varphi' : M \to N$ factor through the modules P and P' then $\varphi + \varphi'$ factors through $P \oplus P'$ and $\lambda\varphi$ factors through P for all $\lambda \in R$. This proves (2). If $\varphi : M \to N$ factors through the module P and ψ and ξ are as in (3), then $\xi \circ \varphi \circ \psi$ factors through P. This proves (3). □

0G91 Lemma 15.70.2. Let R be a ring. Let $\varphi : M \to N$ be an R-module map. If φ factors through a projective module and M is a finite R-module, then φ factors through a finite projective module.

Proof. By Lemma 15.70.1 we can factor $\varphi = \tau \circ \sigma$ where the target of σ is $\bigoplus_{i \in I} R$ for some set I. Choose generators $x_1, \ldots, x_n$ for M. Write $\sigma(x_j) = (a_{ji})_{i \in I}$. For each j only a finite number of a_{ij} are nonzero. Hence σ has image contained in a finite free R-module and we conclude. □

Let R be a ring. Observe that an R-module is projective if and only if the identity on R factors through a projective module.

0G92 Lemma 15.70.3. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. The following conditions are equivalent

(1) for every $a \in I$ the map $a : M \to M$ factors through a projective R-module,
(2) for every $a \in I$ the map $a : M \to M$ factors through a free R-module, and
(3) $\text{Ext}^1_R(M, N)$ is annihilated by I for every R-module N.

Proof. The equivalence of (1) and (2) follows from Lemma 15.70.1. If (1) holds, then (3) holds because $\text{Ext}^1_R(P, N)$ for any N and any projective module P. Conversely, assume (3) holds. Choose a short exact sequence $0 \to N \to P \to M \to 0$ with P projective (or even free). By assumption the corresponding element of $\text{Ext}^1_R(M, N)$ is annihilated by I. Hence for every $a \in I$ the map $a : M \to M$ can be factored through the surjection $P \to M$ and we conclude (1) holds. □

In order to comfortably talk about modules satisfying the equivalent conditions of Lemma 15.70.3 we give the property a name.

0G93 Definition 15.70.4. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. We say M is I-projective[8] if the equivalent conditions of Lemma 15.70.3 hold.

Modules annihilated by I are I-projective.

0G94 Lemma 15.70.5. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. If M is annihilated by I, then M is I-projective.

Proof. Immediate from the definition and the fact that the zero module is projective. □

0G95 Lemma 15.70.6. Let R be a ring. Let $I \subset R$ be an ideal. Let

$$0 \to K \to P \to M \to 0$$

be a short exact sequence of R-modules. If M is I-projective and P is projective, then K is I-projective.

Proof. The element $\text{id}_K \in \text{Hom}_R(K, K)$ maps to the class of the given extension in $\text{Ext}^1_R(M, K)$. Since by assumption this class is annihilated by any $a \in I$ we see that $a : K \to K$ factors through $K \to P$ and we conclude. □

0G96 Lemma 15.70.7. Let R be a ring. Let $I \subset R$ be an ideal. If M is a finite, I-projective R-module, then $M^\vee = \text{Hom}_R(M, R)$ is I-projective.

Proof. Assume M is finite and I-projective. Choose a short exact sequence $0 \to K \to R^{\oplus r} \to M \to 0$. This produces an injection $M^\vee \to R^{\oplus r} = (R^{\oplus r})^\vee$. Since the extension class in $\text{Ext}^1_R(M, K)$ corresponding to the short exact sequence is annihilated by I, we see that for any $a \in I$ we can find a map $M \to R^{\oplus r}$ such that the composition with the given map $R^{\oplus r} \to M$ is equal to $a : M \to M$. Taking duals we find that $a : M^\vee \to M^\vee$ factors through the map $M^\vee \to R^{\oplus r}$ given above and we conclude. □

[8] This is nonstandard notation.

15.71. Hom complexes

0A8H Let R be a ring. Let $L^\bullet$ and $M^\bullet$ be two complexes of R-modules. We construct a complex $\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)$. Namely, for each n we set

$$\mathrm{Hom}^n(L^\bullet, M^\bullet) = \prod\nolimits_{n=p+q} \mathrm{Hom}_R(L^{-q}, M^p)$$

It is a good idea to think of Hom^n as the R-module of all R-linear maps from $L^\bullet$ to $M^\bullet$ (viewed as graded modules) which are homogenous of degree n. In this terminology, we define the differential by the rule

$$\mathrm{d}(f) = \mathrm{d}_M \circ f - (-1)^n f \circ \mathrm{d}_L$$

for $f \in \mathrm{Hom}^n(L^\bullet, M^\bullet)$. We omit the verification that $\mathrm{d}^2 = 0$. See Section 15.72 for sign rules. This construction is a special case of Differential Graded Algebra, Example 22.26.6. It follows immediately from the construction that we have

0A5X (15.71.0.1) $$H^n(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Hom}_{K(R)}(L^\bullet, M^\bullet[n])$$

for all $n \in \mathbf{Z}$.

0A5Y **Lemma 15.71.1.** *Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical isomorphism*

$$\mathrm{Hom}^\bullet(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Hom}^\bullet(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

of complexes of R-modules.

Proof. Let α be an element of degree n on the left hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(K^{-q}, \mathrm{Hom}^p(L^\bullet, M^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \prod\nolimits_{r+s+q=n} \mathrm{Hom}_R(K^{-q}, \mathrm{Hom}_R(L^{-s}, M^r))$$

If we make the identifications

0A5Z (15.71.1.1) $$\mathrm{Hom}_R(K^{-q}, \mathrm{Hom}_R(L^{-s}, M^r)) = \mathrm{Hom}_R(K^{-q} \otimes_R L^{-s}, M^r)$$

then by our sign rules we get

$$\begin{aligned} \mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_K \\ &= \mathrm{d}_M \circ \alpha^{r,s,q} - (-1)^{r+s} \alpha^{r,s,q} \circ \mathrm{d}_L - (-1)^{r+s+q} \alpha^{r,s,q} \circ \mathrm{d}_K \end{aligned}$$

On the other hand, if β is an element of degree n of the right hand side, then

$$\beta = (\beta^{r,s,q}) \in \prod\nolimits_{r+s+q=n} \mathrm{Hom}_R(K^{-q} \otimes_R L^{-s}, M^r)$$

and by our sign rule (Homology, Definition 12.18.3) we get

$$\begin{aligned} \mathrm{d}(\beta^{r,s,q}) &= \mathrm{d}_M \circ \beta^{r,s,q} - (-1)^n \beta^{r,s,q} \circ \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes L^\bullet)} \\ &= \mathrm{d}_M \circ \beta^{r,s,q} - (-1)^{r+s+q} \left(\beta^{r,s,q} \circ \mathrm{d}_K + (-1)^{-q} \beta^{r,s,q} \circ \mathrm{d}_L\right) \end{aligned}$$

Thus we see that the map induced by the identifications (15.71.1.1) indeed is a morphism of complexes. $\square$

0GWQ Remark 15.71.2. Let R be a ring. The category $\text{Comp}(R)$ of complexes of R-modules is a symmetric monoidal category with tensor product given by $\text{Tot}(- \otimes_R -)$, see Lemma 15.58.1. Given $L^\bullet$ and $M^\bullet$ in $\text{Comp}(R)$ an element $f \in \text{Hom}^0(L^\bullet, M^\bullet)$ defines a map of complexes $f : L^\bullet \to M^\bullet$ if and only if $\text{d}(f) = 0$. Hence Lemma 15.71.1 also tells us that

$$\text{Mor}_{\text{Comp}(R)}(K^\bullet, \text{Hom}^\bullet(L^\bullet, M^\bullet)) = \text{Mor}_{\text{Comp}(R)}(\text{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

functorially in $K^\bullet, L^\bullet, M^\bullet$ in $\text{Comp}(R)$. This means that $\text{Hom}^\bullet(-,-)$ is an internal hom for the symmetric monoidal category $\text{Comp}(R)$ as discussed in Categories, Remark 4.43.12.

0A8I Lemma 15.71.3. Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism

$$\text{Tot}\left(\text{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R \text{Hom}^\bullet(K^\bullet, L^\bullet)\right) \longrightarrow \text{Hom}^\bullet(K^\bullet, M^\bullet)$$

of complexes of R-modules.

Proof. Via the discussion in Remark 15.71.2 the existence of such a canonical map follows from Categories, Remark 4.43.12. We also give a direct construction.

An element α of degree n of the left hand side is

$$\alpha = (\alpha^{p,q}) \in \bigoplus\nolimits_{p+q=n} \text{Hom}^p(L^\bullet, M^\bullet) \otimes_R \text{Hom}^q(K^\bullet, L^\bullet)$$

The element $\alpha^{p,q}$ is a finite sum $\alpha^{p,q} = \sum \beta_i^p \otimes \gamma_i^q$ with

$$\beta_i^p = (\beta_i^{r,s}) \in \prod\nolimits_{r+s=p} \text{Hom}_R(L^{-s}, M^r)$$

and

$$\gamma_i^q = (\gamma_i^{u,v}) \in \prod\nolimits_{u+v=q} \text{Hom}_R(K^{-v}, L^u)$$

The map is given by sending α to $\delta = (\delta^{r,v})$ with

$$\delta^{r,v} = \sum\nolimits_{i,s} \beta_i^{r,s} \circ \gamma_i^{-s,v} \in \text{Hom}_R(K^{-v}, M^r)$$

For given $r + v = n$ this sum is finite as there are only finitely many nonzero $\alpha^{p,q}$, hence only finitely many nonzero β_i^p and γ_i^q. By our sign rules we have

$$\begin{aligned} \text{d}(\alpha^{p,q}) &= \text{d}_{\text{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q}) + (-1)^p \text{d}_{\text{Hom}^\bullet(K^\bullet, L^\bullet)}(\alpha^{p,q}) \\ &= \sum \left(\text{d}_M \circ \beta_i^p \circ \gamma_i^q - (-1)^p \beta_i^p \circ \text{d}_L \circ \gamma_i^q\right) \\ &\quad + (-1)^p \sum \left(\beta_i^p \circ \text{d}_L \circ \gamma_i^q - (-1)^q \beta_i^p \circ \gamma_i^q \circ \text{d}_K\right) \\ &= \sum \left(\text{d}_M \circ \beta_i^p \circ \gamma_i^q - (-1)^n \beta_i^p \circ \gamma_i^q \circ \text{d}_K\right) \end{aligned}$$

It follows that the rules $\alpha \mapsto \delta$ is compatible with differentials and the lemma is proved. □

0BYM Lemma 15.71.4. Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism

$$\text{Tot}(K^\bullet \otimes_R \text{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \text{Hom}^\bullet(M^\bullet, \text{Tot}(K^\bullet \otimes_R L^\bullet))$$

of complexes of R-modules functorial in all three complexes.

Proof. Via the discussion in Remark 15.71.2 the existence of such a canonical map follows from Categories, Remark 4.43.12. We also give a direct construction.

Let α be an element of degree n of the right hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(M^{-q}, \mathrm{Tot}^p(K^\bullet \otimes_R L^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \mathrm{Hom}_R(M^{-q}, \bigoplus\nolimits_{r+s+q=n} K^r \otimes_R L^s)$$

where we think of $\alpha^{r,s,q}$ as a family of maps such that for every $x \in M^{-q}$ only a finite number of $\alpha^{r,s,q}(x)$ are nonzero. By our sign rules we get

$$\begin{aligned} \mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_M \\ &= \mathrm{d}_K \circ \alpha^{r,s,q} + (-1)^r \mathrm{d}_L \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_M \end{aligned}$$

On the other hand, if β is an element of degree n of the left hand side, then

$$\beta = (\beta^{p,q}) \in \bigoplus\nolimits_{p+q=n} K^p \otimes_R \mathrm{Hom}^q(M^\bullet, L^\bullet)$$

and we can write $\beta^{p,q} = \sum \gamma_i^p \otimes \delta_i^q$ with $\gamma_i^p \in K^p$ and

$$\delta_i^q = (\delta_i^{r,s}) \in \prod\nolimits_{r+s=q} \mathrm{Hom}_R(M^{-s}, L^r)$$

By our sign rules we have

$$\begin{aligned} \mathrm{d}(\beta^{p,q}) &= \mathrm{d}_K(\beta^{p,q}) + (-1)^p \mathrm{d}_{\mathrm{Hom}^\bullet(M^\bullet, L^\bullet)}(\beta^{p,q}) \\ &= \sum \mathrm{d}_K(\gamma_i^p) \otimes \delta_i^q + (-1)^p \sum \gamma_i^p \otimes (\mathrm{d}_L \circ \delta_i^q - (-1)^q \delta_i^q \circ \mathrm{d}_M) \end{aligned}$$

We send the element β to α with

$$\alpha^{r,s,q} = c^{r,s,q}(\sum \gamma_i^r \otimes \delta_i^{s,q})$$

where $c^{r,s,q} : K^r \otimes_R \mathrm{Hom}_R(M^{-q}, L^s) \to \mathrm{Hom}_R(M^{-q}, K^r \otimes_R L^s)$ is the canonical map. For a given β and r there are only finitely many nonzero γ_i^r hence only finitely many nonzero $\alpha^{r,s,q}$ are nonzero (for a given r). Thus this family of maps satisfies the conditions above and the map is well defined. Comparing signs we see that this is compatible with differentials. ☐

0A62 Lemma 15.71.5. Let R be a ring. Given complexes $K^\bullet, L^\bullet$ of R-modules there is a canonical morphism

$$K^\bullet \longrightarrow \mathrm{Hom}^\bullet(L^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

of complexes of R-modules functorial in both complexes.

Proof. Via the discussion in Remark 15.71.2 the existence of such a canonical map follows from Categories, Remark 4.43.12. We also give a direct construction.

Let α be an element of degree n of the right hand side. Thus

$$\alpha = (\alpha^{p,q}) \in \prod\nolimits_{p+q=n} \mathrm{Hom}_R(L^{-q}, \mathrm{Tot}^p(K^\bullet \otimes_R L^\bullet))$$

Each $\alpha^{p,q}$ is an element

$$\alpha^{p,q} = (\alpha^{r,s,q}) \in \mathrm{Hom}_R(L^{-q}, \bigoplus\nolimits_{r+s+q=n} K^r \otimes_R L^s)$$

where we think of $\alpha^{r,s,q}$ as a family of maps such that for every $x \in L^{-q}$ only a finite number of $\alpha^{r,s,q}(x)$ are nonzero. By our sign rules we get

$$\begin{aligned}
\mathrm{d}(\alpha^{r,s,q}) &= \mathrm{d}_{\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)} \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_L \\
&= \mathrm{d}_K \circ \alpha^{r,s,q} + (-1)^r \mathrm{d}_L \circ \alpha^{r,s,q} - (-1)^n \alpha^{r,s,q} \circ \mathrm{d}_L
\end{aligned}$$

Now an element $\beta \in K^n$ we send to α with $\alpha^{n,-q,q} = \beta \otimes \mathrm{id}_{L^{-q}}$ and $\alpha^{r,s,q} = 0$ if $r \neq n$. This is indeed an element as above, as for fixed q there is only one nonzero $\alpha^{r,s,q}$. The description of the differential shows this is compatible with differentials. $\square$

0A60 Lemma 15.71.6. Let R be a ring. Given complexes $K^\bullet, L^\bullet, M^\bullet$ of R-modules there is a canonical morphism

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R K^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, L^\bullet), M^\bullet)$$

of complexes of R-modules functorial in all three complexes.

Proof. Via the discussion in Remark 15.71.2 the existence of such a canonical map follows from Categories, Remark 4.43.12. We also give a direct construction.

Consider an element β of degree n of the right hand side. Then

$$\beta = (\beta^{p,s}) \in \prod\nolimits_{p+s=n} \mathrm{Hom}_R(\mathrm{Hom}^{-s}(K^\bullet, L^\bullet), M^p)$$

Our sign rules tell us that

$$\mathrm{d}(\beta^{p,s}) = \mathrm{d}_M \circ \beta^{p,s} - (-1)^n \beta^{p,s} \circ \mathrm{d}_{\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)}$$

We can describe the last term as follows

$$(\beta^{p,s} \circ \mathrm{d}_{\mathrm{Hom}^\bullet(K^\bullet, L^\bullet)})(f) = \beta^{p,s}(\mathrm{d}_L \circ f - (-1)^{s+1} f \circ \mathrm{d}_K)$$

if $f \in \mathrm{Hom}^{-s-1}(K^\bullet, L^\bullet)$. We conclude that in some unspecified sense $\mathrm{d}(\beta^{p,s})$ is a sum of three terms with signs as follows

0FNE (15.71.6.1) $$\mathrm{d}(\beta^{p,s}) = \mathrm{d}_M(\beta^{p,s}) - (-1)^n \mathrm{d}_L(\beta^{p,s}) + (-1)^{p+1} \mathrm{d}_K(\beta^{p,s})$$

Next, we consider an element α of degree n of the left hand side. We can write it like so

$$\alpha = (\alpha^{t,r}) \in \bigoplus\nolimits_{t+r=n} \mathrm{Hom}^t(L^\bullet, M^\bullet) \otimes K^r$$

Each $\alpha^{t,r}$ maps to an element

$$\alpha^{t,r} \mapsto (\alpha^{p,q,r}) \in \prod\nolimits_{p+q=t} \mathrm{Hom}_R(L^{-q}, M^p) \otimes_R K^r$$

Our sign rules tell us that

$$\mathrm{d}(\alpha^{p,q,r}) = \mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q,r}) + (-1)^{p+q} \mathrm{d}_K(\alpha^{p,q,r})$$

where if we further write $\alpha^{p,q,r} = \sum g_i^{p,q} \otimes k_i^r$ then we have

$$\mathrm{d}_{\mathrm{Hom}^\bullet(L^\bullet, M^\bullet)}(\alpha^{p,q,r}) = \sum (\mathrm{d}_M \circ g_i^{p,q}) \otimes k_i^r - (-1)^{p+q} \sum (g_i^{p,q} \circ \mathrm{d}_L) \otimes k_i^r$$

We conclude that in some unspecified sense $\mathrm{d}(\alpha^{p,q,r})$ is a sum of three terms with signs as follows

0FNF (15.71.6.2) $$\mathrm{d}(\alpha^{p,q,r}) = \mathrm{d}_M(\alpha^{p,q,r}) - (-1)^{p+q} \mathrm{d}_L(\alpha^{p,q,r}) + (-1)^{p+q} \mathrm{d}_K(\alpha^{p,q,r})$$

To define our map we will use the canonical maps

$$c_{p,q,r} : \mathrm{Hom}_R(L^{-q}, M^p) \otimes_R K^r \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(K^r, L^{-q}), M^p)$$

which sends $\varphi \otimes k$ to the map $\psi \mapsto \varphi(\psi(k))$. This is functorial in all three variables. With $s = q + r$ there is an inclusion

$$\mathrm{Hom}_R(\mathrm{Hom}_R(K^r, L^{-q}), M^p) \subset \mathrm{Hom}_R(\mathrm{Hom}^{-s}(K^\bullet, L^\bullet), M^p)$$

coming from the projection $\mathrm{Hom}^{-s}(K^\bullet, L^\bullet) \to \mathrm{Hom}_R(K^r, L^{-q})$. Since $\alpha^{p,q,r}$ is nonzero only for a finite number of r we see that for a given s there is only a finite number of q, r with $q + r = s$. Thus we can send α to the element β with

$$\beta^{p,s} = \sum\nolimits_{q+r=s} \epsilon_{p,q,r} c_{p,q,r}(\alpha^{p,q,r})$$

where where the sum uses the inclusions given above and where $\epsilon_{p,q,r} \in \{\pm 1\}$. Comparing signs in the equations (15.71.6.1) and (15.71.6.2) we see that

(1) $\epsilon_{p,q,r} = \epsilon_{p+1,q,r}$
(2) $-(-1)^n \epsilon_{p,q,r} = -(-1)^{p+q}\epsilon_{p,q-1,r}$ or equivalently $\epsilon_{p,q,r} = (-1)^r \epsilon_{p,q-1,r}$
(3) $(-1)^{p+1}\epsilon_{p,q,r} = (-1)^{p+q}\epsilon_{p,q,r+1}$ or equivalently $(-1)^{q+1}\epsilon_{p,q,r} = \epsilon_{p,q,r+1}$.

A good solution is to take

$$\epsilon_{p,r,s} = (-1)^{r+qr}$$

The choice of this sign is explained in the remark following the proof. ☐

0A61 Remark 15.71.7. Let us explain why the sign used in the direct construction in the proof of Lemma 15.71.6 agrees with the sign we get from the construction using the discussion in Remark 15.71.2 and Categories, Remark 4.43.12. Denote $- \otimes - = \mathrm{Tot}(- \otimes_R -)$ and $hom(-,-) = \mathrm{Hom}^\bullet(-,-)$. The construction using monoidal category language tells us to use the arrow

$$hom(L^\bullet, M^\bullet) \otimes K^\bullet \longrightarrow hom(hom(K^\bullet, L^\bullet), M^\bullet)$$

in $\mathrm{Comp}(R)$ corresponding to the arrow

$$hom(L^\bullet, M^\bullet) \otimes K^\bullet \otimes hom(K^\bullet, L^\bullet) \longrightarrow M^\bullet$$

gotten by swapping the order of the last two tensor products and then using the evaluation maps $hom(K^\bullet, L^\bullet) \otimes K^\bullet \to L^\bullet$ and $hom(L^\bullet, K^\bullet) \otimes L^\bullet \to M^\bullet$. Only in swapping does a sign intervene. Namely, in the isomorphism

$$K^\bullet \otimes hom(K^\bullet, L^\bullet) \to hom(K^\bullet, L^\bullet) \otimes K^\bullet$$

there is a sign $(-1)^{r(q+r')}$ on $K^r \otimes_R \mathrm{Hom}_R(K^{-r'}, L^q)$, see Section 15.72 item (9). The reader can convince themselves that, because of the correspondence we are using to describe maps into an internal hom, this sign only matters if $r = r'$ and in this case we obtain $(-1)^{r(q+r)} = (-1)^{r+qr}$ as in the direct proof.

15.72. Sign rules

0FNG In this section we review the sign rules used so far and we discuss some of their ramifications. It also seems appropriate to discuss these issues in the setting of the category of complexes of modules over a ring, as most interesting phenomena already occur in this case. We sincerely hope the reader will not need to use the more esoteric aspects of this section.

For the rest of this section, we fix a ring R and we denote $M^\bullet$ a complex of R-modules with differentials $d_M^n : M^n \to M^{n+1}$.

(1) The kth shifted complex $M^\bullet[k]$ has terms $(M^\bullet[k])^n = M^{n+k}$ and differentials $d_{M[k]}^n = (-1)^k d_M^{n+k}$, see Homology, Definition 12.14.7.

(2) Given a map $f : M^\bullet \to N^\bullet$ of complexes, we define $f[k] : M^\bullet[k] \to N^\bullet[k]$ without the intervention of signs, see Homology, Definition 12.14.7.
(3) We identify $H^n(M^\bullet[k])$ with $H^{n+k}(M^\bullet)$ without the intervention of signs, see Homology, Definition 12.14.8.
(4) The boundary map of a short exact sequence of complexes is defined as in the snake lemma without the intervention of signs, see Homology, Lemma 12.13.12.
(5) The distinguished triangle associated to a termwise split short exact sequence $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ of complexes is given by
$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$$
where $M^n \to K^{n+1}$ is the map $\pi^{n+1} \circ d_L^n \circ s^n$ if s and π are compatible termwise splittings. In other words, without the intervention of signs. See Derived Categories, Definitions 13.10.1 and 13.9.9.
(6) The total complex $\text{Tot}(M^\bullet \otimes_R N^\bullet)$ has differential d satisfying the Leibniz rule $d(x \otimes y) = d(x) \otimes y + (-1)^{\deg(x)} x \otimes d(y)$. See Homology, Example 12.18.2 and Homology, Definition 12.18.3.

0FNH (7) There is a canonical isomorphism
$$\text{Tot}(M^\bullet \otimes_R N^\bullet)[a+b] \to \text{Tot}(M^\bullet[a] \otimes_R N^\bullet[b])$$
which uses the sign $(-1)^{pb}$ on the summand $M^p \otimes_R N^q$, see Homology, Remark 12.18.5. It is often more convenient to consider the corresponding shifted map $\text{Tot}(M^\bullet \otimes_R N^\bullet) \to \text{Tot}(M^\bullet[a] \otimes_R N^\bullet[b])[-a-b]$.
(8) There is a canonical isomorphism of complexes
$$\text{Tot}(\text{Tot}(K^\bullet \otimes_R L^\bullet) \otimes_R M^\bullet) \to \text{Tot}(K^\bullet \otimes_R \text{Tot}(L^\bullet \otimes_R M^\bullet))$$
defined without the intervention of signs. See Section 15.58.

0GWR (9) There is a canonical isomorphism
$$\text{Tot}(L^\bullet \otimes_R M^\bullet) \to \text{Tot}(M^\bullet \otimes_R L^\bullet)$$
which uses the sign $(-1)^{pq}$ on the summand $L^p \otimes_R M^q$. See Section 15.58.

Before we get into a discussion of the sign conventions regarding Hom-complexes, we construct the dual of a complex with respect to the conventions above.

0FNJ **Lemma 15.72.1.** Let R be a ring. Let M be an R-module. Let N, η, ϵ be a left dual of M in the monoidal category of R-modules, see Categories, Definition 4.43.5. Then

(1) M and N are finite projective R-modules,
(2) the map $e : \text{Hom}_R(M, R) \to N$, $\lambda \mapsto (\lambda \otimes 1)(\eta)$ is an isomorphism,
(3) we have $\epsilon(n, m) = e^{-1}(n)(m)$ for $n \in N$ and $m \in M$.

Proof. The assumptions mean that
$$M \xrightarrow{\eta\otimes 1} M \otimes_R N \otimes_R M \xrightarrow{1\otimes\epsilon} M \quad \text{and} \quad N \xrightarrow{1\otimes\eta} N \otimes_R M \otimes_R N \xrightarrow{\epsilon\otimes 1} N$$
are the identity map. We can choose a finite free module F, an R-module map $F \to M$, and a lift $\tilde{\eta} : R \to F \otimes_R N$ of η. We obtain a commutative diagram
$$\begin{tikzcd}
M \arrow[r, "\eta\otimes 1"'] \arrow[dr, "\tilde{\eta}\otimes 1"'] & M \otimes_R N \otimes_R M \arrow[r, "1\otimes\epsilon"'] & M \\
 & F \otimes_R N \otimes_R M \arrow[u] \arrow[r, "1\otimes\epsilon"] & F \arrow[u]
\end{tikzcd}$$

This shows that the identity on M factors through a finite free module and hence M is finite projective. By symmetry we see that N is finite projective. This proves part (1). Part (2) follows from Categories, Lemma 4.43.6 and its proof. Part (3) follows from the first equality of the proof. □

0FNK Lemma 15.72.2. Let R be a ring. Let $M^\bullet$ be a complex of R-modules. Let $N^\bullet, \eta, \epsilon$ be a left dual of $M^\bullet$ in the monoidal category of complexes of R-modules. Then

(1) $M^\bullet$ and $N^\bullet$ are bounded,
(2) M^n and N^n are finite projective R-modules,
(3) writing $\epsilon = \sum \epsilon_n$ with $\epsilon_n : N^{-n} \otimes_R M^n \to R$ and $\eta = \sum \eta_n$ with $\eta_n : R \to M^n \otimes_R N^{-n}$ then $(N^{-n}, \eta_n, \epsilon_n)$ is the left dual of M^n as in Lemma 15.72.1,
(4) the differential $d_N^n : N^n \to N^{n+1}$ is equal to $-(-1)^n$ times the map

$$N^n = \mathrm{Hom}_R(M^{-n}, R) \xrightarrow{d_M^{-n-1}} \mathrm{Hom}_R(M^{-n-1}, R) = N^{n+1}$$

where the equality signs are the identifications from Lemma 15.72.1 part (2).

Conversely, given a bounded complex $M^\bullet$ of finite projective R-modules, setting $N^n = \mathrm{Hom}_R(M^{-n}, R)$ with differentials as above, setting $\epsilon = \sum \epsilon_n$ with $\epsilon_n : N^{-n} \otimes_R M^n \to R$ given by evaluation, and setting $\eta = \sum \eta_n$ with $\eta_n : R \to M^n \otimes_R N^{-n}$ mapping 1 to id_{M_n} we obtain a left dual of $M^\bullet$ in the monoidal category of complexes of R-modules.

Proof. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{M^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{N^\bullet}$ by Categories, Definition 4.43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \mathrm{id}_{M^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \mathrm{id}_{N^{-n}}$ which proves (3). By Lemma 15.72.1 we have (2). Since the sum $\eta = \sum \eta_n$ is finite, we get (1). Since $\eta = \sum \eta_n$ is a map of complexes $R \to \mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$ we see that

$$(d_M^{-n-1} \otimes 1) \circ \eta_{-n-1} + (-1)^n (1 \otimes d_N^{-n}) \circ \eta_{-n} = 0$$

by our choice of signs for the differential on $\mathrm{Tot}(M^\bullet \otimes_R N^\bullet)$. Unwinding definitions, this proves (4). To see the final statement of the lemma one reads the above backwards. □

We will use the description of the left dual of a complex in Lemma 15.72.2 as a motivation for our sign rule on the Hom-complex. Namely, we choose the signs such that (11) holds. We continue with the discussion of various sign rules as above

(10) Given complexes $K^\bullet$, $M^\bullet$ we let $\mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$ be the complex with terms

$$\mathrm{Hom}^n(M^\bullet, K^\bullet) = \prod\nolimits_{n=p+q} \mathrm{Hom}_R(M^{-q}, K^p)$$

and differential given by the rule

$$d(f) = d_K \circ f - (-1)^n f \circ d_M$$

0FNL (11) The choice above is such that if $M^\bullet$ has a left dual $N^\bullet$ as in Lemma 15.72.2, then we have a canonical isomorphism

$$\mathrm{Tot}(K^\bullet \otimes_R N^\bullet) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$$

defined without the intervention of signs sending the summand $K^p \otimes_R N^q$ to the summand $\mathrm{Hom}_R(M^{-q}, K^p)$ via $N^q = \mathrm{Hom}_R(M^{-q}, R)$ and the canonical map $K^p \otimes_R \mathrm{Hom}_R(M^{-q}, R) \to \mathrm{Hom}_R(M^{-q}, K^p)$.

(12) There is a composition

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(L^\bullet, K^\bullet) \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, K^\bullet)$$

defined without the intervention of signs, see Lemma 15.71.3.

(13) There is a canonical isomorphism

$$\mathrm{Hom}^\bullet(K^\bullet, \mathrm{Hom}^\bullet(L^\bullet, M^\bullet)) = \mathrm{Hom}^\bullet(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet), M^\bullet)$$

defined without the intervention of signs, see Lemma 15.71.1.

(14) There is a canonical map

$$\mathrm{Tot}(K^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet)) \longrightarrow \mathrm{Hom}^\bullet(M^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

defined without the intervention of signs, see Lemma 15.71.4.

(15) There is a canonical map

$$K^\bullet \longrightarrow \mathrm{Hom}^\bullet(L^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

defined without the intervention of signs, see Lemma 15.71.5.

(16) By Lemma 15.71.6 is a canonical map

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(L^\bullet, M^\bullet) \otimes_R K^\bullet) \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, L^\bullet), M^\bullet)$$

which uses a sign $(-1)^{r+qr}$ on the module $\mathrm{Hom}_R(L^{-q}, M^p) \otimes_R K^r$ whose reason is explained in Remark 15.71.7.

0FNM (17) Taking $L^\bullet = M^\bullet$ and using $R \to \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$ the map from the previous item becomes the evaluation map

$$ev : K^\bullet \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(K^\bullet, M^\bullet), M^\bullet)$$

It sends $x \in K^n$ to the map which sends $f \in \mathrm{Hom}^m(K^\bullet, M^\bullet)$ to $(-1)^{nm} f(x)$.

0FNN (18) There is a canonical identification

$$\mathrm{Hom}^\bullet(M^\bullet, K^\bullet)[a-b] \to \mathrm{Hom}^\bullet(M^\bullet[b], K^\bullet[a])$$

which uses signs. It is defined as the map whose corresponding shifted map

$$\mathrm{Hom}^\bullet(M^\bullet, K^\bullet) \to \mathrm{Hom}^\bullet(M^\bullet[b], K^\bullet[a])[b-a]$$

uses the sign $(-1)^{nb}$ on the module $\mathrm{Hom}_R(M^{-q}, K^p)$ with $p + q = n$. Namely, if $f \in \mathrm{Hom}^n(M^\bullet, K^\bullet)$ then

$$d(f) = d_K \circ f - (-1)^n f \circ d_M$$

on the source, whereas on the target f lies in $(\mathrm{Hom}^\bullet(M^\bullet[b], K^\bullet[a])[b-a])^n = \mathrm{Hom}^{n+b-a}(M^\bullet[b], K^\bullet[a])$ and hence we get

$$\begin{aligned} d(f) &= (-1)^{b-a} \left(d_{K[a]} \circ f - (-1)^{n+b-a} f \circ d_{M[b]}\right) \\ &= (-1)^{b-a} \left((-1)^a d_K \circ f - (-1)^{n+b-a} f \circ (-1)^b d_M\right) \\ &= (-1)^b d_K \circ f - (-1)^{n+b} f \circ d_M \end{aligned}$$

and one sees that the chosen sign of $(-1)^{nb}$ in degree n produces a map of complexes for these differentials.

15.73. Derived hom

0A5W Let R be a ring. The derived hom we will define in this section is a functor

$$D(R)^{opp} \times D(R) \longrightarrow D(R), \quad (K, L) \longmapsto R\operatorname{Hom}_R(K, L)$$

This is an internal hom in the derived category of R-modules in the sense that it is characterized by the formula

0A63 (15.73.0.1)
$$\operatorname{Hom}_{D(R)}(K, R\operatorname{Hom}_R(L, M)) = \operatorname{Hom}_{D(R)}(K \otimes_R^{\mathbf{L}} L, M)$$

for objects K, L, M of $D(R)$. Note that this formula characterizes the objects up to unique isomorphism by the Yoneda lemma. A construction can be given as follows. Choose a K-injective complex $I^\bullet$ of R-modules representing M, choose a complex $L^\bullet$ representing L, and set

$$R\operatorname{Hom}_R(L, M) = \operatorname{Hom}^\bullet(L^\bullet, I^\bullet)$$

with notation as in Section 15.71. A generalization of this construction is discussed in Differential Graded Algebra, Section 22.31. From (15.71.0.1) and Derived Categories, Lemma 13.31.2 that we have

0A64 (15.73.0.2)
$$H^n(R\operatorname{Hom}_R(L, M)) = \operatorname{Hom}_{D(R)}(L, M[n])$$

for all $n \in \mathbf{Z}$. In particular, the object $R\operatorname{Hom}_R(L, M)$ of $D(R)$ is well defined, i.e., independent of the choice of the K-injective complex $I^\bullet$.

0A65 **Lemma 15.73.1.** Let R be a ring. Let K, L, M be objects of $D(R)$. There is a canonical isomorphism

$$R\operatorname{Hom}_R(K, R\operatorname{Hom}_R(L, M)) = R\operatorname{Hom}_R(K \otimes_R^{\mathbf{L}} L, M)$$

in $D(R)$ functorial in K, L, M which recovers (15.73.0.1) by taking H^0.

Proof. Choose a K-injective complex $I^\bullet$ representing M and a K-flat complex of R-modules $L^\bullet$ representing L. For any complex of R-modules $K^\bullet$ we have

$$\operatorname{Hom}^\bullet(K^\bullet, \operatorname{Hom}^\bullet(L^\bullet, I^\bullet)) = \operatorname{Hom}^\bullet(\operatorname{Tot}(K^\bullet \otimes_R L^\bullet), I^\bullet)$$

by Lemma 15.71.1. The lemma follows by the definition of $R\operatorname{Hom}$ and because $\operatorname{Tot}(K^\bullet \otimes_R L^\bullet)$ represents the derived tensor product. $\square$

0A66 **Lemma 15.73.2.** Let R be a ring. Let $P^\bullet$ be a bounded above complex of projective R-modules. Let $L^\bullet$ be a complex of R-modules. Then $R\operatorname{Hom}_R(P^\bullet, L^\bullet)$ is represented by the complex $\operatorname{Hom}^\bullet(P^\bullet, L^\bullet)$.

Proof. By (15.71.0.1) and Derived Categories, Lemma 13.19.8 the cohomology groups of the complex are "correct". Hence if we choose a quasi-isomorphism $L^\bullet \to I^\bullet$ with $I^\bullet$ a K-injective complex of R-modules then the induced map

$$\operatorname{Hom}^\bullet(P^\bullet, L^\bullet) \longrightarrow \operatorname{Hom}^\bullet(P^\bullet, I^\bullet)$$

is a quasi-isomorphism. As the right hand side is our definition of $R\operatorname{Hom}_R(P^\bullet, L^\bullet)$ we win. $\square$

0A67 **Lemma 15.73.3.** Let R be a ring. Let K, L, M be objects of $D(R)$. There is a canonical morphism

$$R\operatorname{Hom}_R(L, M) \otimes_R^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}_R(R\operatorname{Hom}_R(K, L), M)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and a K-flat complex $K^\bullet$ representing K. The map is defined using the map

$$\text{Tot}(\text{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R K^\bullet) \longrightarrow \text{Hom}^\bullet(\text{Hom}^\bullet(K^\bullet, J^\bullet), I^\bullet)$$

of Lemma 15.71.6. We omit the proof that this is functorial in all three objects of $D(R)$. ☐

0A8J Lemma 15.73.4. Let R be a ring. Given K, L, M in $D(R)$ there is a canonical morphism

$$R\,\text{Hom}_R(L, M) \otimes_R^{\mathbf{L}} R\,\text{Hom}_R(K, L) \longrightarrow R\,\text{Hom}_R(K, M)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and any complex of R-modules $K^\bullet$ representing K. By Lemma 15.71.3 there is a map of complexes

$$\text{Tot}\left(\text{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R \text{Hom}^\bullet(K^\bullet, J^\bullet)\right) \longrightarrow \text{Hom}^\bullet(K^\bullet, I^\bullet)$$

The complexes of R-modules $\text{Hom}^\bullet(J^\bullet, I^\bullet)$, $\text{Hom}^\bullet(K^\bullet, J^\bullet)$, and $\text{Hom}^\bullet(K^\bullet, I^\bullet)$ represent $R\,\text{Hom}_R(L, M)$, $R\,\text{Hom}_R(K, L)$, and $R\,\text{Hom}_R(K, M)$. If we choose a K-flat complex $H^\bullet$ and a quasi-isomorphism $H^\bullet \to \text{Hom}^\bullet(K^\bullet, J^\bullet)$, then there is a map

$$\text{Tot}\left(\text{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R H^\bullet\right) \longrightarrow \text{Tot}\left(\text{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R \text{Hom}^\bullet(K^\bullet, J^\bullet)\right)$$

whose source represents $R\,\text{Hom}_R(L, M) \otimes_R^{\mathbf{L}} R\,\text{Hom}_R(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. ☐

0BYN Lemma 15.73.5. Let R be a ring. Given complexes K, L, M in $D(R)$ there is a canonical morphism

$$K \otimes_R^{\mathbf{L}} R\,\text{Hom}_R(M, L) \longrightarrow R\,\text{Hom}_R(M, K \otimes_R^{\mathbf{L}} L)$$

in $D(R)$ functorial in K, L, M.

Proof. Choose a K-flat complex $K^\bullet$ representing K, and a K-injective complex $I^\bullet$ representing L, and choose any complex $M^\bullet$ representing M. Choose a quasi-isomorphism $\text{Tot}(K^\bullet \otimes_R I^\bullet) \to J^\bullet$ where $J^\bullet$ is K-injective. Then we use the map

$$\text{Tot}\left(K^\bullet \otimes_R \text{Hom}^\bullet(M^\bullet, I^\bullet)\right) \to \text{Hom}^\bullet(M^\bullet, \text{Tot}(K^\bullet \otimes_R I^\bullet)) \to \text{Hom}^\bullet(M^\bullet, J^\bullet)$$

where the first map is the map from Lemma 15.71.4. ☐

0A6B Lemma 15.73.6. Let R be a ring. Given complexes K, L in $D(R)$ there is a canonical morphism

$$K \longrightarrow R\,\text{Hom}_R(L, K \otimes_R^{\mathbf{L}} L)$$

in $D(R)$ functorial in both K and L.

Proof. This is a special case of Lemma 15.73.5 but we will also prove it directly. Choose a K-flat complex $K^\bullet$ representing K and any complex $L^\bullet$ representing L. Choose a quasi-isomorphism $\text{Tot}(K^\bullet \otimes_R L^\bullet) \to J^\bullet$ where $J^\bullet$ is K-injective. Then we use the map

$$K^\bullet \to \text{Hom}^\bullet(L^\bullet, \text{Tot}(K^\bullet \otimes_R L^\bullet)) \to \text{Hom}^\bullet(L^\bullet, J^\bullet)$$

where the first map is the map from Lemma 15.71.5. ☐

15.74. Perfect complexes

0656 A perfect complex is a pseudo-coherent complex of finite tor dimension. We will not use this as the definition, but define perfect complexes over a ring directly as follows.

0657 Definition 15.74.1. Let R be a ring. Denote $D(R)$ the derived category of the abelian category of R-modules.

(1) An object K of $D(R)$ is perfect if it is quasi-isomorphic to a bounded complex of finite projective R-modules.
(2) An R-module M is perfect if $M[0]$ is a perfect object in $D(R)$.

For example, over a Noetherian ring a finite module is perfect if and only if it has finite projective dimension, see Lemma 15.74.3 and Algebra, Definition 10.109.2.

0658 Lemma 15.74.2. Let $K^\bullet$ be an object of $D(R)$. The following are equivalent

(1) $K^\bullet$ is perfect, and
(2) $K^\bullet$ is pseudo-coherent and has finite tor dimension.

If (1) and (2) hold and $K^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of finite projective R-modules with $E^i = 0$ for $i \not\in [a, b]$.

Proof. It is clear that (1) implies (2), see Lemmas 15.64.5 and 15.66.3. Assume (2) holds and that $K^\bullet$ has tor-amplitude in $[a, b]$. In particular, $H^i(K^\bullet) = 0$ for $i > b$. Choose a complex $F^\bullet$ of finite free R-modules with $F^i = 0$ for $i > b$ and a quasi-isomorphism $F^\bullet \to K^\bullet$ (Lemma 15.64.5). Set $E^\bullet = \tau_{\geq a} F^\bullet$. Note that E^i is finite free except E^a which is a finitely presented R-module. By Lemma 15.66.2 E^a is flat. Hence by Algebra, Lemma 10.78.2 we see that E^a is finite projective. □

066Q Lemma 15.74.3. Let M be a module over a ring R. The following are equivalent

(1) M is a perfect module, and
(2) there exists a resolution
$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$
with each F_i a finite projective R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to $M[0]$. Hence M is perfect. Conversely, assume (1). By Lemmas 15.74.2 and 15.64.4 we can find resolution $E^\bullet \to M$ with E^{-i} a finite free R-module. By Lemma 15.66.2 we see that $F_d = \text{Coker}(E^{d-1} \to E^d)$ is flat for some d sufficiently large. By Algebra, Lemma 10.78.2 we see that F_d is finite projective. Hence
$$0 \to F_d \to E^{-d+1} \to \ldots \to E^0 \to M \to 0$$
is the desired resolution. □

066R Lemma 15.74.4. Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are perfect then the third is also perfect.

Proof. Combine Lemmas 15.74.2, 15.64.6, and 15.66.5. □

066S Lemma 15.74.5. Let R be a ring. If $K^\bullet \oplus L^\bullet$ is perfect, then so are $K^\bullet$ and $L^\bullet$.

Proof. Follows from Lemmas 15.74.2, 15.64.8, and 15.66.7. □

066T Lemma 15.74.6. Let R be a ring. Let $K^\bullet$ be a bounded complex of perfect R-modules. Then $K^\bullet$ is a perfect complex.

Proof. Follows by induction on the length of the finite complex: use Lemma 15.74.4 and the stupid truncations. $\square$

066U Lemma 15.74.7. Let R be a ring. If $K^\bullet \in D^b(R)$ and all its cohomology modules are perfect, then $K^\bullet$ is perfect.

Proof. Follows by induction on the length of the finite complex: use Lemma 15.74.4 and the canonical truncations. $\square$

066V Lemma 15.74.8. Let $A \to B$ be a ring map. Assume that B is perfect as an A-module. Let $K^\bullet$ be a perfect complex of B-modules. Then $K^\bullet$ is perfect as a complex of A-modules.

Proof. Using Lemma 15.74.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.66.12 and Lemma 15.64.11 for those results. $\square$

066W Lemma 15.74.9. Let $A \to B$ be a ring map. Let $K^\bullet$ be a perfect complex of A-modules. Then $K^\bullet \otimes_A^{\mathbf{L}} B$ is a perfect complex of B-modules.

Proof. Using Lemma 15.74.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.66.13 and Lemma 15.64.12 for those results. $\square$

066X Lemma 15.74.10. Let $A \to B$ be a flat ring map. Let M be a perfect A-module. Then $M \otimes_A B$ is a perfect B-module.

Proof. By Lemma 15.74.3 the assumption implies that M has a finite resolution $F_\bullet$ by finite projective R-modules. As $A \to B$ is flat the complex $F_\bullet \otimes_A B$ is a finite length resolution of $M \otimes_A B$ by finite projective modules over B. Hence $M \otimes_A B$ is perfect. $\square$

0GM0 Lemma 15.74.11. Let R be a ring. If K and L are perfect objects of $D(R)$, then $K \otimes_R^{\mathbf{L}} L$ is a perfect object too.

Proof. We can prove this using the definition as follows. We may represent K, resp. L by a bounded complex $K^\bullet$, resp. $L^\bullet$ of finite projective R-modules. Then $K \otimes_R^{\mathbf{L}} L$ is represented by the bounded complex $\text{Tot}(K^\bullet \otimes_R L^\bullet)$. The terms of this complex are direct sums of the modules $M^a \otimes_R L^b$. Since M^a and L^b are direct summands of finite free R-modules, so is $M^a \otimes_R L^b$. Hence we conclude the terms of the complex $\text{Tot}(K^\bullet \otimes_R L^\bullet)$ are finite projective.

Another proof can be given using the characterization of perfect complexes in Lemma 15.74.2 and the corresponding lemmas for pseudo-coherent complexes (Lemma 15.64.16) and for tor amplitude (Lemma 15.66.10 used with $A = B = R$). $\square$

066Y Lemma 15.74.12. Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is perfect, then $K^\bullet$ is perfect.

Proof. Using Lemma 15.74.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.66.16 and Lemma 15.64.14 for those results. $\square$

068T Lemma 15.74.13. Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ is perfect, then $K^\bullet$ is perfect.

Proof. Using Lemma 15.74.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.66.17 and Lemma 15.64.15 for those results. $\square$

066Z Lemma 15.74.14. Let R be a regular ring. Then

(1) an R-module is perfect if and only if it is a finite R-module, and
(2) a complex of R-modules $K^\bullet$ is perfect if and only if $K^\bullet \in D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module.

Proof. Any perfect R-module is finite by definition. Conversely, let M be a finite R-module. Choose a resolution

$$\ldots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to M \to 0$$

with F_i finite free R-modules (Algebra, Lemma 10.71.1). Set $M_i = \text{Ker}(d_i)$. Denote $U_i \subset \text{Spec}(R)$ the set of primes $\mathfrak p$ such that $M_{i,\mathfrak p}$ is free; U_i is open by Algebra, Lemma 10.79.3. We have a exact sequence $0 \to M_{i+1} \to F_{i+1} \to M_i \to 0$. If $\mathfrak p \in U_i$, then $0 \to M_{i+1,\mathfrak p} \to F_{i+1,\mathfrak p} \to M_{i,\mathfrak p} \to 0$ splits. Thus $M_{i+1,\mathfrak p}$ is finite projective, hence free (Algebra, Lemma 10.78.2). This shows that $U_i \subset U_{i+1}$. We claim that $\text{Spec}(R) = \bigcup U_i$. Namely, for every prime ideal $\mathfrak p$ the regular local ring $R_\mathfrak p$ has finite global dimension by Algebra, Proposition 10.110.1. It follows that $M_{i,\mathfrak p}$ is finite projective (hence free) for $i \gg 0$ for example by Algebra, Lemma 10.109.3. Since the spectrum of R is Noetherian (Algebra, Lemma 10.31.5) we conclude that $U_n = \text{Spec}(R)$ for some n. Then M_n is a projective R-module by Algebra, Lemma 10.78.2. Thus

$$0 \to M_n \to F_n \to \ldots \to F_1 \to M \to 0$$

is a bounded resolution by finite projective modules and hence M is perfect. This proves part (1).

Let $K^\bullet$ be a complex of R-modules. If $K^\bullet$ is perfect, then it is in $D^b(R)$ and it is quasi-isomorphic to a finite complex of finite projective R-modules so certainly each $H^i(K^\bullet)$ is a finite R-module (as R is Noetherian). Conversely, suppose that $K^\bullet$ is in $D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module. Then by (1) each $H^i(K^\bullet)$ is a perfect R-module, whence $K^\bullet$ is perfect by Lemma 15.74.7 $\square$

07VI Lemma 15.74.15. Let A be a ring. Let $K \in D(A)$ be perfect. Then $K^\vee = R\,\text{Hom}_A(K, A)$ is a perfect complex and $K \cong (K^\vee)^\vee$. There are functorial isomorphisms

$$L \otimes_A^{\mathbf{L}} K^\vee = R\,\text{Hom}_A(K, L) \quad \text{and} \quad H^0(L \otimes_A^{\mathbf{L}} K^\vee) = \text{Ext}^0_A(K, L)$$

for $L \in D(A)$.

Proof. We can represent K by a complex $K^\bullet$ of finite projective A-modules. By Lemma 15.73.2 the object $K^\vee$ is represented by the complex $E^\bullet = \text{Hom}^\bullet(K^\bullet, A)$. Note that $E^n = \text{Hom}_A(K^{-n}, A)$ and the differentials of $E^\bullet$ are the transpose of the differentials of $K^\bullet$ up to sign. Observe that $E^\bullet$ is the left dual of $K^\bullet$ in the symmetric monoidal category of complexes of R-modules, see Lemma 15.72.2. There is a canonical map

$$K^\bullet = \text{Tot}(\text{Hom}^\bullet(A, A) \otimes_A K^\bullet) \longrightarrow \text{Hom}^\bullet(\text{Hom}^\bullet(K^\bullet, A), A)$$

which up to sign uses the evaluation map in each degree, see Lemma 15.71.6. (For sign rules see Section 15.72.) Thus this map defines a canonical isomorphism $(K^\vee)^\vee \cong K$ as the double dual of a finite projective module is itself.

The second equality follows from the first by Lemma 15.73.1 and Derived Categories, Lemma 13.19.8 as well as the definition of Ext groups, see Derived Categories, Section 13.27. Let $L^\bullet$ be a complex of A-modules representing L. By Section 15.72 item (11) there is a canonical isomorphism

$$\text{Tot}(L^\bullet \otimes_A E^\bullet) \longrightarrow \text{Hom}^\bullet(K^\bullet, L^\bullet)$$

of complexes of A-modules. This proves the first displayed equality and the proof is complete. □

0BKB Lemma 15.74.16. Let A be a ring. Let $(K_n)_{n\in\mathbf{N}}$ be a system of perfect objects of $D(A)$. Let $K = \text{hocolim}K_n$ be the derived colimit (Derived Categories, Definition 13.33.1). Then for any object E of $D(A)$ we have

$$R\,\text{Hom}_A(K, E) = R\lim E \otimes^{\mathbf{L}}_A K_n^\vee$$

where $(K_n^\vee)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 15.74.15 we have $R\lim E\otimes^{\mathbf{L}}_A K_n^\vee = R\lim R\,\text{Hom}_A(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\,\text{Hom}_A(K_n, E) \to \prod R\,\text{Hom}_A(K_n, E) \to \prod R\,\text{Hom}_A(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\,\text{Hom}_A(K_n, E) = R\,\text{Hom}_A(\bigoplus K_n, E)$. This is a formal consequence of (15.73.0.1) and the fact that derived tensor product commutes with direct sums. □

0BC7 Lemma 15.74.17. Let $R = \text{colim}_{i\in I} R_i$ be a filtered colimit of rings.

(1) Given a perfect K in $D(R)$ there exists an $i \in I$ and a perfect K_i in $D(R_i)$ such that $K \cong K_i \otimes^{\mathbf{L}}_{R_i} R$ in $D(R)$.
(2) Given $0 \in I$ and $K_0, L_0 \in D(R_0)$ with K_0 perfect, we have

$$\text{Hom}_{D(R)}(K_0 \otimes^{\mathbf{L}}_{R_0} R, L_0 \otimes^{\mathbf{L}}_{R_0} R) = \text{colim}_{i\geq 0} \text{Hom}_{D(R_i)}(K_0 \otimes^{\mathbf{L}}_{R_0} R_i, L_0 \otimes^{\mathbf{L}}_{R_0} R_i)$$

In other words, the triangulated category of perfect complexes over R is the colimit of the triangulated categories of perfect complexes over R_i.

Proof. We will use the results of Algebra, Lemmas 10.127.5 and 10.127.6 without further mention. These lemmas in particular say that the category of finitely presented R-modules is the colimit of the categories of finitely presented R_i-modules. Since finite projective modules can be characterized as summands of finite free modules (Algebra, Lemma 10.78.2) we see that the same is true for the category of finite projective modules. This proves (1) by our definition of perfect objects of $D(R)$.

To prove (2) we may represent K_0 by a bounded complex $K_0^\bullet$ of finite projective R_0-modules. We may represent L_0 by a K-flat complex $L_0^\bullet$ (Lemma 15.59.10). Then we have

$$\text{Hom}_{D(R)}(K_0 \otimes^{\mathbf{L}}_{R_0} R, L_0 \otimes^{\mathbf{L}}_{R_0} R) = \text{Hom}_{K(R)}(K_0^\bullet \otimes_{R_0} R, L_0^\bullet \otimes_{R_0} R)$$

by Derived Categories, Lemma 13.19.8. Similarly for the Hom with R replaced by R_i. Since in the right hand side only a finite number of terms are involved, since

$$\mathrm{Hom}_R(K_0^p \otimes_{R_0} R, L_0^q \otimes_{R_0} R) = \mathrm{colim}_{i \geq 0} \mathrm{Hom}_{R_i}(K_0^p \otimes_{R_0} R_i, L_0^q \otimes_{R_0} R_i)$$

by the lemmas cited at the beginning of the proof, and since filtered colimits are exact (Algebra, Lemma 10.8.8) we conclude that (2) holds as well. □

15.75. Lifting complexes

0BC8 Let R be a ring. Let $I \subset R$ be an ideal. The lifting problem we will consider is the following. Suppose given an object K of $D(R)$ and a complex $E^\bullet$ of R/I-modules such that $E^\bullet$ represents $K \otimes_R^{\mathbf{L}} R/I$ in $D(R)$. Question: Does there exist a complex of R-modules $P^\bullet$ lifting $E^\bullet$ representing K in $D(R)$? In general the answer to this question is no, but in good cases something can be done. We first discuss lifting acyclic complexes.

0BC9 **Lemma 15.75.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $\mathcal{P}$ be a class of R-modules. Assume*

1. *each $P \in \mathcal{P}$ is a projective R-module,*
2. *if $P_1 \in \mathcal{P}$ and $P_1 \oplus P_2 \in \mathcal{P}$, then $P_2 \in \mathcal{P}$, and*
3. *if $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ is surjective modulo I, then f is surjective.*

Then given any bounded above acyclic complex $E^\bullet$ whose terms are of the form P/IP for $P \in \mathcal{P}$ there exists a bounded above acyclic complex $P^\bullet$ whose terms are in $\mathcal{P}$ lifting $E^\bullet$.

Proof. Say $E^i = 0$ for $i > b$. Assume given n and a morphism of complexes

$$\begin{tikzcd} & & P^n \arrow[r] \arrow[d] & P^{n+1} \arrow[r] \arrow[d] & \ldots \arrow[r] & P^b \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & E^{n-1} \arrow[r] & E^n \arrow[r] & E^{n+1} \arrow[r] & \ldots \arrow[r] & E^b \arrow[r] & 0 \arrow[r] & \ldots \end{tikzcd}$$

with $P^i \in \mathcal{P}$, with $P^n \to P^{n+1} \to \ldots \to P^b$ acyclic in degrees $\geq n+1$, and with vertical maps inducing isomorphisms $P^i/IP^i \to E^i$. In this situation one can inductively choose isomorphisms $P^i = Z^i \oplus Z^{i+1}$ such that the maps $P^i \to P^{i+1}$ are given by $Z^i \oplus Z^{i+1} \to Z^{i+1} \to Z^{i+1} \oplus Z^{i+2}$. By property (2) and arguing inductively we see that $Z^i \in \mathcal{P}$. Choose $P^{n-1} \in \mathcal{P}$ and an isomorphism $P^{n-1}/IP^{n-1} \to E^{n-1}$. Since P^{n-1} is projective and since $Z^n/IZ^n = \mathrm{Im}(E^{n-1} \to E^n)$, we can lift the map $P^{n-1} \to E^{n-1} \to E^n$ to a map $P^{n-1} \to Z^n$. By property (3) the map $P^{n-1} \to Z^n$ is surjective. Thus we obtain an extension of the diagram by adding P^{n-1} and the maps just constructed to the left of P^n. Since a diagram of the desired form exists for $n > b$ we conclude by induction on n. □

0BCA **Lemma 15.75.2.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $\mathcal{P}$ be a class of R-modules. Let $K \in D(R)$ and let $E^\bullet$ be a complex of R/I-modules representing $K \otimes_R^{\mathbf{L}} R/I$. Assume*

1. *each $P \in \mathcal{P}$ is a projective R-module,*
2. *$P_1 \in \mathcal{P}$ and $P_1 \oplus P_2 \in \mathcal{P}$ if and only if $P_1, P_2 \in \mathcal{P}$,*
3. *if $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ is surjective modulo I, then f is surjective,*
4. *$E^\bullet$ is bounded above and E^i is of the form P/IP for $P \in \mathcal{P}$, and*
5. *K can be represented by a bounded above complex whose terms are in $\mathcal{P}$.*

Then there exists a bounded above complex $P^\bullet$ whose terms are in $\mathcal{P}$ with $P^\bullet/IP^\bullet$ isomorphic to $E^\bullet$ and representing K in $D(R)$.

Proof. By assumption (5) we can represent K by a bounded above complex $K^\bullet$ whose terms are in $\mathcal{P}$. Then $K \otimes_R^{\mathbf{L}} R/I$ is represented by $K^\bullet/IK^\bullet$. Since $E^\bullet$ is a bounded above complex of projective R/I-modules by (4), we can choose a quasi-isomorphism $\delta : E^\bullet \to K^\bullet/IK^\bullet$ (Derived Categories, Lemma 13.19.8). Let $C^\bullet$ be cone on δ (Derived Categories, Definition 13.9.1). The module C^i is the direct sum $K^i/IK^i \oplus E^{i+1}$ hence is of the form P/IP for some $P \in \mathcal{P}$ as (2) says in particular that $\mathcal{P}$ is preserved under taking sums. Since $C^\bullet$ is acyclic, we can apply Lemma 15.75.1 and find a acyclic lift $A^\bullet$ of $C^\bullet$. The complex $A^\bullet$ is bounded above and has terms in $\mathcal{P}$. In

$$\begin{tikzcd} K^\bullet \arrow[r, dotted] \arrow[d] & A^\bullet \arrow[d] & \\ K^\bullet/IK^\bullet \arrow[r] & C^\bullet \arrow[r] & E^\bullet[1] \end{tikzcd}$$

we can find the dotted arrow making the diagram commute by Derived Categories, Lemma 13.19.6. We will show below that it follows from (1), (2), (3) that $K^i \to A^i$ is the inclusion of a direct summand for every i. By property (2) we see that $P^i = \mathrm{Coker}(K^i \to A^i)$ is in $\mathcal{P}$. Thus we can take $P^\bullet = \mathrm{Coker}(K^\bullet \to A^\bullet)[-1]$ to conclude.

To finish the proof we have to show the following: Let $f : P_1 \to P_2$, $P_1, P_2 \in \mathcal{P}$ and $P_1/IP_1 \to P_2/IP_2$ is split injective with cokernel of the form P_3/IP_3 for some $P_3 \in \mathcal{P}$, then f is split injective. Write $E_i = P_i/IP_i$. Then $E_2 = E_1 \oplus E_3$. Since P_2 is projective we can choose a map $g : P_2 \to P_3$ lifting the map $E_2 \to E_3$. By condition (3) the map g is surjective, hence split as P_3 is projective. Set $P_1' = \mathrm{Ker}(g)$ and choose a splitting $P_2 = P_1' \oplus P_3$. Then $P_1' \in \mathcal{P}$ by (2). We do not know that $g \circ f = 0$, but we can consider the map

$$P_1 \xrightarrow{f} P_2 \xrightarrow{projection} P_1'$$

The composition modulo I is an isomorphism. Since P_1' is projective we can split $P_1 = T \oplus P_1'$. If $T = 0$, then we are done, because then $P_2 \to P_1'$ is a splitting of f. We see that $T \in \mathcal{P}$ by (2). Calculating modulo I we see that $T/IT = 0$. Since $0 \in \mathcal{P}$ (as the summand of any P in $\mathcal{P}$) we see the map $0 \to T$ is surjective and we conclude that $T = 0$ as desired. □

09AR Lemma 15.75.3. Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that

(1) $E^\bullet$ is a bounded above complex of projective R/I-modules,
(2) $K \otimes_R^{\mathbf{L}} R/I$ is represented by $E^\bullet$ in $D(R/I)$, and
(3) I is a nilpotent ideal.

Then there exists a bounded above complex $P^\bullet$ of projective R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $E^\bullet$.

Proof. We apply Lemma 15.75.2 using the class $\mathcal{P}$ of all projective R-modules. Properties (1) and (2) of the lemma are immediate. Property (3) follows from Nakayama's lemma (Algebra, Lemma 10.20.1). Property (4) follows from the fact that we can lift projective R/I-modules to projective R-modules, see Algebra, Lemma 10.77.5. To see that (5) holds it suffices to show that K is in $D^-(R)$.

We are given that $K \otimes_R^{\mathbf{L}} R/I$ is in $D^-(R/I)$ (because $E^\bullet$ is bounded above). We will show by induction on n that $K \otimes_R^{\mathbf{L}} R/I^n$ is in $D^-(R/I^n)$. This will finish the proof because I being nilpotent exactly means that $I^n = 0$ for some n. We may represent K by a K-flat complex $K^\bullet$ with flat terms (Lemma 15.59.10). Then derived tensor products are represented by usual tensor products. Thus we consider the exact sequence

$$0 \to K^\bullet \otimes_R I^n/I^{n+1} \to K^\bullet \otimes_R R/I^{n+1} \to K^\bullet \otimes_R R/I^n \to 0$$

Thus the cohomology of $K \otimes_R^{\mathbf{L}} R/I^{n+1}$ sits in a long exact sequence with the cohomology of $K \otimes_R^{\mathbf{L}} R/I^n$ and the cohomology of

$$K \otimes_R^{\mathbf{L}} I^n/I^{n+1} = K \otimes_R^{\mathbf{L}} R/I \otimes_{R/I}^{\mathbf{L}} I^n/I^{n+1}$$

The first cohomologies vanish above a certain degree by induction assumption and the second cohomologies vanish above a certain degree because $K^\bullet \otimes_R^{\mathbf{L}} R/I$ is bounded above and I^n/I^{n+1} is in degree 0. $\square$

0BCB Lemma 15.75.4. Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that

(1) $E^\bullet$ is a bounded above complex of finite stably free R/I-modules,
(2) $K \otimes_R^{\mathbf{L}} R/I$ is represented by $E^\bullet$ in $D(R/I)$,
(3) $K^\bullet$ is pseudo-coherent, and
(4) every element of $1 + I$ is invertible.

Then there exists a bounded above complex $P^\bullet$ of finite stably free R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $E^\bullet$. Moreover, if E^i is free, then P^i is free.

Proof. We apply Lemma 15.75.2 using the class $\mathcal{P}$ of all finite stably free R-modules. Property (1) of the lemma is immediate. Property (2) follows from Lemma 15.3.2. Property (3) follows from Nakayama's lemma (Algebra, Lemma 10.20.1). Property (4) follows from the fact that we can lift finite stably free R/I-modules to finite stably free R-modules, see Lemma 15.3.3. Part (5) holds because a pseudo-coherent complex can be represented by a bounded above complex of finite free R-modules. The final assertion of the lemma follows from Lemma 15.3.5. $\square$

0BCC Lemma 15.75.5. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $K \in D(R)$ be pseudo-coherent. Set $d_i = \dim_\kappa H^i(K \otimes_R^{\mathbf{L}} \kappa)$. Then $d_i < \infty$ and for some $b \in \mathbf{Z}$ we have $d_i = 0$ for $i > b$. Then there exists a complex

$$\ldots \to R^{\oplus d_{b-2}} \to R^{\oplus d_{b-1}} \to R^{\oplus d_b} \to 0 \to \ldots$$

representing K in $D(R)$. Moreover, this complex is unique up to isomorphism(!).

Proof. Observe that $K \otimes_R^{\mathbf{L}} \kappa$ is pseudo-coherent as an object of $D(\kappa)$, see Lemma 15.64.12. Hence the cohomology spaces are finite dimensional and vanish above some cutoff. Every object of $D(\kappa)$ is isomorphic in $D(\kappa)$ to a complex $E^\bullet$ with zero differentials. In particular $E^i \cong \kappa^{\oplus d_i}$ is finite free. Applying Lemma 15.75.4 we obtain the existence.

If we have two complexes $F^\bullet$ and $G^\bullet$ with F^i and G^i free of rank d_i representing K. Then we may choose a map of complexes $\beta : F^\bullet \to G^\bullet$ representing the isomorphism $F^\bullet \cong K \cong G^\bullet$, see Derived Categories, Lemma 13.19.8. The induced map of complexes $\beta \otimes 1 : F^\bullet \otimes_R^{\mathbf{L}} \kappa \to G^\bullet \otimes_R^{\mathbf{L}} \kappa$ must be an isomorphism of complexes as the differentials in $F^\bullet \otimes_R^{\mathbf{L}} \kappa$ and $G^\bullet \otimes_R^{\mathbf{L}} \kappa$ are zero. Thus $\beta^i : F^i \to G^i$ is a map of finite free R-modules whose reduction modulo $\mathfrak m$ is an isomorphism. Hence β^i is an isomorphism and we win. $\square$

0BCD Lemma 15.75.6. Let R be a ring. Let $\mathfrak p \subset R$ be a prime. Let $K \in D(R)$ be perfect. Set $d_i = \dim_{\kappa(\mathfrak p)} H^i(K \otimes_R^{\mathbf{L}} \kappa(\mathfrak p))$. Then $d_i < \infty$ and only a finite number are nonzero. Then there exists an $f \in R$, $f \not\in \mathfrak p$ and a complex

$$\ldots \to 0 \to R_f^{\oplus d_a} \to R_f^{\oplus d_{a+1}} \to \ldots \to R_f^{\oplus d_{b-1}} \to R_f^{\oplus d_b} \to 0 \to \ldots$$

representing $K \otimes_R^{\mathbf{L}} R_f$ in $D(R_f)$.

Proof. Observe that $K \otimes_R^{\mathbf{L}} \kappa(\mathfrak p)$ is perfect as an object of $D(\kappa(\mathfrak p))$, see Lemma 15.74.9. Hence only a finite number of d_i are nonzero and they are all finite. Applying Lemma 15.75.5 we get a complex representing K having the desired shape over the local ring $R_{\mathfrak p}$. We have $R_{\mathfrak p} = \operatorname{colim} R_f$ for $f \in R$, $f \not\in \mathfrak p$ (Algebra, Lemma 10.9.9). We conclude by Lemma 15.74.17. Some details omitted. $\square$

0F9V Lemma 15.75.7. Let R be a ring. Let $\mathfrak p \subset R$ be a prime. Let $M^\bullet$ and $N^\bullet$ be bounded complexes of finite projective R-modules representing the same object of $D(R)$. Then there exists an $f \in R$, $f \not\in \mathfrak p$ such that there is an isomorphism (!) of complexes

$$M_f^\bullet \oplus P^\bullet \cong N_f^\bullet \oplus Q^\bullet$$

where $P^\bullet$ and $Q^\bullet$ are finite direct sums of trivial complexes, i.e., complexes of the form the form $\ldots \to 0 \to R_f \xrightarrow{1} R_f \to 0 \to \ldots$ (placed in arbitrary degrees).

Proof. If we have an isomorphism of the type described over the localization $R_{\mathfrak p}$, then using that $R_{\mathfrak p} = \operatorname{colim} R_f$ (Algebra, Lemma 10.9.9) we can descend the isomorphism to an isomorphism over R_f for some f. Thus we may assume R is local and $\mathfrak p$ is the maximal ideal. In this case the result follows from the uniqueness of a "minimal" complex representing a perfect object, see Lemma 15.75.5, and the fact that any complex is a direct sum of a trivial complex and a minimal one (Algebra, Lemma 10.102.2). $\square$

0BCE Lemma 15.75.8. Let R be a ring. Let $I \subset R$ be an ideal. Let $E^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that

(1) $E^\bullet$ is a bounded above complex of finite projective R/I-modules,
(2) $K \otimes_R^{\mathbf{L}} R/I$ is represented by $E^\bullet$ in $D(R/I)$,
(3) K is pseudo-coherent, and
(4) (R, I) is a henselian pair.

Then there exists a bounded above complex $P^\bullet$ of finite projective R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $E^\bullet$. Moreover, if E^i is free, then P^i is free.

Proof. We apply Lemma 15.75.2 using the class $\mathcal P$ of all finite projective R-modules. Properties (1) and (2) of the lemma are immediate. Property (3) follows from Nakayama's lemma (Algebra, Lemma 10.20.1). Property (4) follows from the fact that we can lift finite projective R/I-modules to finite projective R-modules, see Lemma 15.13.1. Property (5) holds because a pseudo-coherent complex can be represented by a bounded above complex of finite free R-modules. Thus Lemma 15.75.2 applies and we find $P^\bullet$ as desired. The final assertion of the lemma follows from Lemma 15.3.5. $\square$

15.76. Splitting complexes

0BCF In this section we discuss conditions which imply an object of the derived category of a ring is a direct sum of its truncations. Our method is to use the following lemma (under suitable hypotheses) to split the canonical distinguished triangles

$$\tau_{\leq i}K \to K \to \tau_{\geq i+1}K \to (\tau_{\leq i}K)[1]$$

in $D(R)$, see Derived Categories, Remark 13.12.4.

0BCG **Lemma 15.76.1.** Let R be a ring. Let K and L be objects of $D(R)$. Assume L has projective-amplitude in $[a, b]$, for example if L is perfect of tor-amplitude in $[a, b]$.

(1) If $H^i(K) = 0$ for $i \geq a$, then $\mathrm{Hom}_{D(R)}(L, K) = 0$.
(2) If $H^i(K) = 0$ for $i \geq a + 1$, then given any distinguished triangle $K \to M \to L \to K[1]$ there is an isomorphism $M \cong K \oplus L$ in $D(R)$ compatible with the maps in the distinguished triangle.
(3) If $H^i(K) = 0$ for $i \geq a$, then the isomorphism in (2) exists and is unique.

Proof. The assumption that L has projective-amplitude in $[a, b]$ means we can represent L by a complex $L^\bullet$ of projective R-modules with $L^i = 0$ for $i \notin [a, b]$, see Definition 15.68.1. If L is perfect of tor-amplitude in $[a, b]$, then we can represent L by a complex $L^\bullet$ of finite projective R-modules with $L^i = 0$ for $i \notin [a, b]$, see Lemma 15.74.2. If $H^i(K) = 0$ for $i \geq a$, then K is quasi-isomorphic to $\tau_{\leq a-1}K$. Hence we can represent K by a complex $K^\bullet$ of R-modules with $K^i = 0$ for $i \geq a$. Then we obtain

$$\mathrm{Hom}_{D(R)}(L, K) = \mathrm{Hom}_{K(R)}(L^\bullet, K^\bullet) = 0$$

by Derived Categories, Lemma 13.19.8. This proves (1). Under the hypotheses of (2) we see that $\mathrm{Hom}_{D(R)}(L, K[1]) = 0$ by (1), hence the distinguished triangle is split by Derived Categories, Lemma 13.4.11. The uniqueness of (3) follows from (1). ☐

0A1U **Lemma 15.76.2.** Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Assume that for some $i \in \mathbf{Z}$ the map

$$H^i(K^\bullet) \otimes_R \kappa(\mathfrak{p}) \longrightarrow H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p}))$$

is surjective. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ is a perfect object of $D(R_f)$ with tor amplitude in $[i+1, \infty]$ and a canonical isomorphism

$$K^\bullet \otimes_R R_f \cong \tau_{\leq i}(K^\bullet \otimes_R R_f) \oplus \tau_{\geq i+1}(K^\bullet \otimes_R R_f)$$

in $D(R_f)$.

Proof. In this proof all tensor products are over R and we write $\kappa = \kappa(\mathfrak{p})$. We may assume that $K^\bullet$ is a bounded above complex of finite free R-modules. Let us inspect what is happening in degree i:

$$\ldots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \ldots$$

Let $0 \subset V \subset W \subset K^i \otimes \kappa$ be defined by the formulas

$$V = \mathrm{Im}\left(K^{i-1} \otimes \kappa \to K^i \otimes \kappa\right) \quad \text{and} \quad W = \mathrm{Ker}\left(K^i \otimes \kappa \to K^{i+1} \otimes \kappa\right)$$

Set $\dim(V) = r$, $\dim(W/V) = s$, and $\dim(K^i \otimes \kappa/W) = t$. We can pick $x_1, \ldots, x_r \in K^{i-1}$ which map by d^{i-1} to a basis of V. By our assumption we can pick $y_1, \ldots, y_s \in \mathrm{Ker}(d^i)$ mapping to a basis of W/V. Finally, choose $z_1, \ldots, z_t \in K^i$ mapping to a basis of $K^i \otimes \kappa/W$. Then we see that the elements $d^i(z_1), \ldots, d^i(z_t) \in K^{i+1}$

are linearly independent in $K^{i+1} \otimes_R \kappa$. By Algebra, Lemma 10.79.4 we may after replacing R by R_f for some $f \in R$, $f \not\in \mathfrak p$ assume that

1. $d^i(x_a), y_b, z_c$ is an R-basis of K^i,
2. $d^i(z_1), \ldots, d^i(z_t)$ are R-linearly independent in K^{i+1}, and
3. the quotient $E^{i+1} = K^{i+1}/\sum Rd^i(z_c)$ is finite projective.

Since d^i annihilates $d^{i-1}(x_a)$ and y_b, we deduce from condition (2) that $E^{i+1} = \text{Coker}(d^i : K^i \to K^{i+1})$. Thus we see that

$$\tau_{\geq i+1}K^\bullet = (\ldots \to 0 \to E^{i+1} \to K^{i+2} \to \ldots)$$

is a bounded complex of finite projective modules sitting in degrees $[i+1, b]$ for some b. Thus $\tau_{\geq i+1}K^\bullet$ is perfect of amplitude $[i+1, b]$. Since $\tau_{\leq i}K^\bullet$ has no cohomology in degrees $> i$, we may apply Lemma 15.76.1 to the distinguished triangle

$$\tau_{\leq i}K^\bullet \to K^\bullet \to \tau_{\geq i+1}K^\bullet \to (\tau_{\leq i}K^\bullet)[1]$$

(Derived Categories, Remark 13.12.4) to conclude. □

0A1V Lemma 15.76.3. Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Assume that for some $i \in \mathbf{Z}$ the maps

$$H^i(K^\bullet)\otimes_R\kappa(\mathfrak p) \longrightarrow H^i(K^\bullet\otimes_R^{\mathbf{L}}\kappa(\mathfrak p)) \quad \text{and} \quad H^{i-1}(K^\bullet)\otimes_R\kappa(\mathfrak p) \longrightarrow H^{i-1}(K^\bullet\otimes_R^{\mathbf{L}}\kappa(\mathfrak p))$$

are surjective. Then there exists an $f \in R$, $f \not\in \mathfrak p$ such that

1. $\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ is a perfect object of $D(R_f)$ with tor amplitude in $[i+1, \infty]$,
2. $H^i(K^\bullet)_f$ is a finite free R_f-module, and
3. there is a canonical direct sum decomposition
$$K^\bullet \otimes_R R_f \cong \tau_{\leq i-1}(K^\bullet \otimes_R R_f) \oplus H^i(K^\bullet)_f[-i] \oplus \tau_{\geq i+1}(K^\bullet \otimes_R R_f)$$
in $D(R_f)$.

Proof. We get (1) from Lemma 15.76.2 as well as a splitting $K^\bullet \otimes_R R_f = \tau_{\leq i}K^\bullet \otimes_R R_f \oplus \tau_{\geq i+1}K^\bullet \otimes_R R_f$ in $D(R_f)$. Applying Lemma 15.76.2 once more to $\tau_{\leq i}K^\bullet \otimes_R R_f$ we obtain (after suitably choosing f) a splitting $\tau_{\leq i}K^\bullet \otimes_R R_f = \tau_{\leq i-1}K^\bullet \otimes_R R_f \oplus H^i(K^\bullet)_f$ in $D(R_f)$ as well as the conclusion that $H^i(K)_f$ is a flat perfect module, i.e., finite projective. □

068U Lemma 15.76.4. Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Let $i \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules such that $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak p)) = 0$. Then there exists an $f \in R$, $f \not\in \mathfrak p$ and a canonical direct sum decomposition

$$K^\bullet \otimes_R R_f = \tau_{\geq i+1}(K^\bullet \otimes_R R_f) \oplus \tau_{\leq i-1}(K^\bullet \otimes_R R_f)$$

in $D(R_f)$ with $\tau_{\geq i+1}(K^\bullet \otimes_R R_f)$ a perfect complex with tor-amplitude in $[i+1, \infty]$.

Proof. This is an often used special case of Lemma 15.76.2. A direct proof is as follows. We may assume that $K^\bullet$ is a bounded above complex of finite free R-modules. Let us inspect what is happening in degree i:

$$\ldots \to K^{i-2} \to R^{\oplus l} \to R^{\oplus m} \to R^{\oplus n} \to K^{i+2} \to \ldots$$

Let A be the $m \times l$ matrix corresponding to $K^{i-1} \to K^i$ and let B be the $n \times m$ matrix corresponding to $K^i \to K^{i+1}$. The assumption is that $A \bmod \mathfrak p$ has rank r and that $B \bmod \mathfrak p$ has rank $m - r$. In other words, there is some $r \times r$ minor a of A which is not in $\mathfrak p$ and there is some $(m-r) \times (m-r)$-minor b of B which is not in $\mathfrak p$. Set $f = ab$. Then after inverting f we can find direct sum decompositions

$K^{i-1} = R^{\oplus l-r} \oplus R^{\oplus r}$, $K^i = R^{\oplus r} \oplus R^{\oplus m-r}$, $K^{i+1} = R^{\oplus m-r} \oplus R^{\oplus n-m+r}$ such that the module map $K^{i-1} \to K^i$ kills of $R^{\oplus l-r}$ and induces an isomorphism of $R^{\oplus r}$ onto the corresponding summand of K^i and such that the module map $K^i \to K^{i+1}$ kills of $R^{\oplus r}$ and induces an isomorphism of $R^{\oplus m-r}$ onto the corresponding summand of K^{i+1}. Thus $K^\bullet$ becomes quasi-isomorphic to

$$\ldots \to K^{i-2} \to R^{\oplus l-r} \to 0 \to R^{\oplus n-m+r} \to K^{i+2} \to \ldots$$

and everything is clear. □

0G97 Lemma 15.76.5. Let R be a ring. Let $K \in D^-(R)$. Let $a \in \mathbf{Z}$. Assume that for any injective R-module map $M \to M'$ the map $\text{Ext}_R^{-a}(K, M) \to \text{Ext}_R^{-a}(K, M')$ is injective. Then there is a unique direct sum decomposition $K \cong \tau_{\leq a}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b.

Proof. Consider the distinguished triangle

$$\tau_{\leq a}K \to K \to \tau_{\geq a+1}K \to (\tau_{\leq a}K)[1]$$

in $D(R)$, see Derived Categories, Remark 13.12.4. Observe that $\text{Ext}_R^{-a}(\tau_{\leq a}K, M) = \text{Hom}_R(H^a(K), M)$ and $\text{Ext}_R^{-a-1}(\tau_{\leq a}K, M) = 0$, see Derived Categories, Lemma 13.27.3. Thus the long exact sequence of Ext gives an exact sequence

$$0 \to \text{Ext}_R^{-a}(\tau_{\geq a+1}K, M) \to \text{Ext}_R^{-a}(K, M) \to \text{Hom}_R(H^a(K), M)$$

functorial in the R-module M. Now if I is an injective R-module, then $\text{Ext}_R^{-a}(\tau_{\geq a+1}K, I) = 0$ for example by Derived Categories, Lemma 13.27.2. Since every module injects into an injective module, we conclude that $\text{Ext}_R^{-a}(\tau_{\geq a+1}K, M) = 0$ for every R-module M. By Lemma 15.68.2 we conclude that $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b (this is where we use that K is bounded above). We obtain the splitting by Lemma 15.76.1. □

0G98 Lemma 15.76.6. Let R be a ring. Let $K \in D^-(R)$. Let $a \in \mathbf{Z}$. Assume $\text{Ext}_R^{-a}(K, M) = 0$ for any R-module M. Then there is a unique direct sum decomposition $K \cong \tau_{\leq a-1}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b.

Proof. By Lemma 15.76.5 we have a direct sum decomposition $K \cong \tau_{\leq a}K \oplus \tau_{\geq a+1}K$ and $\tau_{\geq a+1}K$ has projective-amplitude in $[a+1, b]$ for some b. Clearly, we must have $H^a(K) = 0$ and we conclude that $\tau_{\leq a}K = \tau_{\leq a-1}K$ in $D(R)$. □

15.77. Recognizing perfect complexes

0G99 Some lemmas that allow us to prove certain complexes are perfect.

0BYP Lemma 15.77.1. Let R be a ring and let $\mathfrak{p} \subset R$ be a prime. Let K be pseudo-coherent and bounded below. Set $d_i = \dim_{\kappa(\mathfrak{p})} H^i(K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p}))$. If there exists an $a \in \mathbf{Z}$ such that $d_i = 0$ for $i < a$, then there exists an $f \in R$, $f \not\in \mathfrak{p}$ and a complex

$$\ldots \to 0 \to R_f^{\oplus d_a} \to R_f^{\oplus d_{a+1}} \to \ldots \to R_f^{\oplus d_{b-1}} \to R_f^{\oplus d_b} \to 0 \to \ldots$$

representing $K \otimes_R^{\mathbf{L}} R_f$ in $D(R_f)$. In particular $K \otimes_R^{\mathbf{L}} R_f$ is perfect.

Proof. After decreasing a we may assume that also $H^i(K^\bullet) = 0$ for $i < a$. By Lemma 15.76.4 after replacing R by R_f for some $f \in R$, $f \not\in \mathfrak{p}$ we can write $K^\bullet = \tau_{\leq a-1}K^\bullet \oplus \tau_{\geq a}K^\bullet$ in $D(R)$ with $\tau_{\geq a}K^\bullet$ perfect. Since $H^i(K^\bullet) = 0$ for $i < a$

we see that $\tau_{\leq a-1}K^\bullet = 0$ in $D(R)$. Hence $K^\bullet$ is perfect. Then we can conclude using Lemma 15.75.6. □

068V Lemma 15.77.2. Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. The following are equivalent

(1) $K^\bullet$ is perfect with tor amplitude in $[a, b]$,
(2) for every prime $\mathfrak p$ we have $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak p)) = 0$ for all $i \not\in [a, b]$, and
(3) for every maximal ideal $\mathfrak m$ we have $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak m)) = 0$ for all $i \not\in [a, b]$.

Proof. We omit the proof of the implications (1) ⇒ (2) ⇒ (3). Assume (3). Let $i \in \mathbf{Z}$ with $i \not\in [a, b]$. By Lemma 15.76.4 we see that the assumption implies that $H^i(K^\bullet)_{\mathfrak m} = 0$ for all maximal ideals of R. Hence $H^i(K^\bullet) = 0$, see Algebra, Lemma 10.23.1. Moreover, Lemma 15.76.4 now also implies that for every maximal ideal $\mathfrak m$ there exists an element $f \in R$, $f \not\in \mathfrak m$ such that $K^\bullet \otimes_R R_f$ is perfect with tor amplitude in $[a, b]$. Hence we conclude by appealing to Lemmas 15.74.12 and 15.66.16. □

068W Lemma 15.77.3. Let R be a ring. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. Consider the following conditions

(1) $K^\bullet$ is perfect,
(2) for every prime ideal $\mathfrak p$ the complex $K^\bullet \otimes_R R_{\mathfrak p}$ is perfect,
(3) for every maximal ideal $\mathfrak m$ the complex $K^\bullet \otimes_R R_{\mathfrak m}$ is perfect,
(4) for every prime $\mathfrak p$ we have $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak p)) = 0$ for all $i \ll 0$,
(5) for every maximal ideal $\mathfrak m$ we have $H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa(\mathfrak m)) = 0$ for all $i \ll 0$.

We always have the implications

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$$

If $K^\bullet$ is bounded below, then all conditions are equivalent.

Proof. By Lemma 15.74.9 we see that (1) implies (2). It is immediate that (2) ⇒ (3). Since every prime $\mathfrak p$ is contained in a maximal ideal $\mathfrak m$, we can apply Lemma 15.74.9 to the map $R_{\mathfrak m} \to R_{\mathfrak p}$ to see that (3) implies (2). Applying Lemma 15.74.9 to the residue maps $R_{\mathfrak p} \to \kappa(\mathfrak p)$ and $R_{\mathfrak m} \to \kappa(\mathfrak m)$ we see that (2) implies (4) and (3) implies (5).

Assume R is local with maximal ideal $\mathfrak m$ and residue field κ. We will show that if $H^i(K^\bullet \otimes^{\mathbf{L}} \kappa) = 0$ for $i < a$ for some a, then K is perfect. This will show that (4) implies (2) and (5) implies (3) whence the first part of the lemma. First we apply Lemma 15.76.4 with $i = a - 1$ to see that $K^\bullet = \tau_{\leq a-1}K^\bullet \oplus \tau_{\geq a}K^\bullet$ in $D(R)$ with $\tau_{\geq a}K^\bullet$ perfect of tor-amplitude contained in $[a, \infty]$. To finish we need to show that $\tau_{\leq a-1}K$ is zero, i.e., that its cohomology groups are zero. If not let i be the largest index such that $M = H^i(\tau_{\leq a-1}K)$ is not zero. Then M is a finite R-module because $\tau_{\leq a-1}K^\bullet$ is pseudo-coherent (Lemmas 15.64.3 and 15.64.8). Thus by Nakayama's lemma (Algebra, Lemma 10.20.1) we find that $M \otimes_R \kappa$ is nonzero. This implies that

$$H^i((\tau_{\leq a-1}K^\bullet) \otimes_R^{\mathbf{L}} \kappa) = H^i(K^\bullet \otimes_R^{\mathbf{L}} \kappa)$$

is nonzero which is a contradiction.

Assume the equivalent conditions (2) – (5) hold and that $K^\bullet$ is bounded below. Say $H^i(K^\bullet) = 0$ for $i < a$. Pick a maximal ideal $\mathfrak m$ of R. It suffices to show there exists an $f \in R$, $f \not\in \mathfrak m$ such that $K^\bullet \otimes_R^{\mathbf{L}} R_f$ is perfect (Lemma 15.74.12 and Algebra, Lemma 10.17.10). This follows from Lemma 15.77.1. □

0G9A Lemma 15.77.4. Let R be a ring. Let K be a pseudo-coherent object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent

(1) K has projective-amplitude in $[a, b]$,
(2) K is perfect of tor-amplitude in $[a, b]$,
(3) $\text{Ext}^i_R(K, N) = 0$ for all finitely presented R-modules N and all $i \not\in [-b, -a]$,
(4) $H^n(K) = 0$ for $n > b$ and $\text{Ext}^i_R(K, N) = 0$ for all finitely presented R-modules N and all $i > -a$, and
(5) $H^n(K) = 0$ for $n \not\in [a-1, b]$ and $\text{Ext}^{-a+1}_R(K, N) = 0$ for all finitely presented R-modules N.

Proof. From the final statement of Lemma 15.74.2 we see that (2) implies (1). If (1) holds, then K can be represented by a complex of projective modules P^i with $P^i = 0$ for $i \not\in [a, b]$. Since projective modules are flat (as summands of free modules), we see that K has tor-amplitude in $[a, b]$, see Lemma 15.66.3. Thus by Lemma 15.74.2 we see that (2) holds.

In conditions (3), (4), (5) the assumed vanishing of ext groups $\text{Ext}^i_R(K, M)$ for M of finite presentation is equivalent to the vanishing for all R-modules M by Lemma 15.65.1 and Algebra, Lemma 10.11.3. Thus the equivalence of (1), (3), (4), and (5) follows from Lemma 15.68.2. □

The following lemma useful in order to find perfect complexes over a polynomial ring $B = A[x_1, \ldots, x_d]$.

068X Lemma 15.77.5. Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume

(1) the ring map $A \to B$ is flat,
(2) for every prime $\mathfrak{p} \subset A$ the ring $B \otimes_A \kappa(\mathfrak{p})$ has finite global dimension $\leq d$,
(3) $K^\bullet$ is pseudo-coherent as a complex of B-modules, and
(4) $K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules.

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a-d, b]$.

Proof. We may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. In particular, $K^\bullet$ is flat as a complex of A-modules and $K^\bullet \otimes_A M = K^\bullet \otimes_A^{\mathbf{L}} M$ for any A-module M. For every prime $\mathfrak{p}$ of A the complex

$$K^\bullet \otimes_A \kappa(\mathfrak{p})$$

is a bounded above complex of finite free modules over $B \otimes_A \kappa(\mathfrak{p})$ with vanishing H^i except for $i \in [a, b]$. As $B \otimes_A \kappa(\mathfrak{p})$ has global dimension d we see from Lemma 15.66.19 that $K^\bullet \otimes_A \kappa(\mathfrak{p})$ has tor amplitude in $[a-d, b]$. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p}$. Since $K^\bullet \otimes_A \kappa(\mathfrak{p})$ is a bounded above complex of free $B \otimes_A \kappa(\mathfrak{p})$-modules we see that

$$\begin{aligned} K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{q}) &= K^\bullet \otimes_B \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes^{\mathbf{L}}_{B \otimes_A \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \end{aligned}$$

Hence the arguments above imply that $H^i(K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{q})) = 0$ for $i \not\in [a-d, b]$. We conclude by Lemma 15.77.2. □

The following lemma is a local version of Lemma 15.77.5. It can be used to find perfect complexes over regular local rings.

09PC Lemma 15.77.6. Let $A \to B$ be a local ring homomorphism. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume

(1) the ring map $A \to B$ is flat,
(2) the ring $B/\mathfrak{m}_A B$ is regular of dimension d,
(3) $K^\bullet$ is pseudo-coherent as a complex of B-modules, and
(4) $K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules, in fact it suffices if $H^i(K^\bullet \otimes_A^{\mathbf{L}} \kappa(\mathfrak{m}_A))$ is nonzero only for $i \in [a, b]$.

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a - d, b]$.

Proof. By (3) we may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. We compute

$$\begin{aligned} K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{m}_B) &= K^\bullet \otimes_B \kappa(\mathfrak{m}_B) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{m}_A)) \otimes_{B/\mathfrak{m}_A B} \kappa(\mathfrak{m}_B) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{m}_A)) \otimes_{B/\mathfrak{m}_A B}^{\mathbf{L}} \kappa(\mathfrak{m}_B) \end{aligned}$$

The first equality because $K^\bullet$ is a bounded above complex of flat B-modules. The second equality follows from basic properties of the tensor product. The third equality holds because $K^\bullet \otimes_A \kappa(\mathfrak{m}_A) = K^\bullet/\mathfrak{m}_A K^\bullet$ is a bounded above complex of flat $B/\mathfrak{m}_A B$-modules. Since $K^\bullet$ is a bounded above complex of flat A-modules by (1), the cohomology modules H^i of the complex $K^\bullet \otimes_A \kappa(\mathfrak{m}_A)$ are nonzero only for $i \in [a, b]$ by assumption (4). Thus the spectral sequence of Example 15.62.1 and the fact that $B/\mathfrak{m}_A B$ has finite global dimension d (by (2) and Algebra, Proposition 10.110.1) shows that $H^j(K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{m}_B))$ is zero for $j \notin [a - d, b]$. This finishes the proof by Lemma 15.77.2. $\square$

15.78. Characterizing perfect complexes

07LQ In this section we prove that the perfect complexes are exactly the compact objects of the derived category of a ring. First we show the following.

0ATI Lemma 15.78.1. Let R be a ring. The full subcategory $D_{perf}(R) \subset D(R)$ of perfect objects is the smallest strictly full, saturated, triangulated subcategory containing $R = R[0]$. In other words $D_{perf}(R) = \langle R \rangle$. In particular, R is a classical generator for $D_{perf}(R)$.

Proof. To see what the statement means, please look at Derived Categories, Definitions 13.6.1 and 13.36.3. It was shown in Lemmas 15.74.4 and 15.74.5 that $D_{perf}(R) \subset D(R)$ is a strictly full, saturated, triangulated subcategory of $D(R)$. Of course $R \in D_{perf}(R)$.

Recall that $\langle R \rangle = \bigcup \langle R \rangle_n$. To finish the proof we will show that if $M \in D_{perf}(R)$ is represented by

$$\ldots \to 0 \to M^a \to M^{a+1} \to \ldots \to M^b \to 0 \to \ldots$$

with M^i finite projective, then $M \in \langle R \rangle_{b-a+1}$. The proof is by induction on $b - a$. By definition $\langle R \rangle_1$ contains any finite projective R-module placed in any degree; this deals with the base case $b - a = 0$ of the induction. In general, we consider the distinguished triangle

$$M_b[-b] \to M^\bullet \to \sigma_{\leq b-1} M^\bullet \to M_b[-b+1]$$

By induction the truncated complex $\sigma_{\leq b-1}M^\bullet$ is in $\langle R\rangle_{b-a}$ and $M_b[-b]$ is in $\langle R\rangle_1$. Hence $M^\bullet \in \langle R\rangle_{b-a+1}$ by definition. $\square$

Let R be a ring. Recall that $D(R)$ has direct sums which are given simply by taking direct sums of complexes, see Derived Categories, Lemma 13.33.5. We will use this in the lemmas of this section without further mention.

07LR Lemma 15.78.2. Let R be a ring. Let $K \in D(R)$ be an object such that for every countable set of objects $E_n \in D(R)$ the canonical map

$$\bigoplus \mathrm{Hom}_{D(R)}(K, E_n) \longrightarrow \mathrm{Hom}_{D(R)}(K, \bigoplus E_n)$$

is a bijection. Then, given any system $L_n^\bullet$ of complexes over $\mathbf{N}$ we have that

$$\mathrm{colim}\,\mathrm{Hom}_{D(R)}(K, L_n^\bullet) \longrightarrow \mathrm{Hom}_{D(R)}(K, L^\bullet)$$

is a bijection, where $L^\bullet$ is the termwise colimit, i.e., $L^m = \mathrm{colim}\, L_n^m$ for all $m \in \mathbf{Z}$.

Proof. Consider the short exact sequence of complexes

$$0 \to \bigoplus L_n^\bullet \to \bigoplus L_n^\bullet \to L^\bullet \to 0$$

where the first map is given by $1 - t_n$ in degree n where $t_n : L_n^\bullet \to L_{n+1}^\bullet$ is the transition map. By Derived Categories, Lemma 13.12.1 this is a distinguished triangle in $D(R)$. Apply the homological functor $\mathrm{Hom}_{D(R)}(K, -)$, see Derived Categories, Lemma 13.4.2. Thus a long exact cohomology sequence

$$\begin{tikzcd}
 & \ldots \arrow[r] & \mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet[-1]) \arrow[dll] \\
\mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet) \arrow[r] & \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet) \arrow[r] & \mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet) \arrow[dll] \\
\mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet[1]) \arrow[r] & \ldots &
\end{tikzcd}$$

Since we have assumed that $\mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet)$ is equal to $\bigoplus \mathrm{Hom}_{D(R)}(K, L_n^\bullet)$ we see that the first map on every row of the diagram is injective (by the explicit description of this map as the sum of the maps induced by $1 - t_n$). Hence we conclude that $\mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet)$ is the cokernel of the first map of the middle row in the diagram above which is what we had to show. $\square$

The following proposition, characterizing perfect complexes as the compact objects (Derived Categories, Definition 13.37.1) of the derived category, shows up in various places. See for example [Ric89b, proof of Proposition 6.3] (this treats the bounded case), [TT90, Theorem 2.4.3] (the statement doesn't match exactly), and [BN93, Proposition 6.4] (watch out for horrendous notational conventions).

07LT Proposition 15.78.3. Let R be a ring. For an object K of $D(R)$ the following are equivalent

(1) K is perfect, and
(2) K is a compact object of $D(R)$.

Proof. Assume K is perfect, i.e., K is quasi-isomorphic to a bounded complex $P^\bullet$ of finite projective modules, see Definition 15.74.1. If E_i is represented by the complex $E_i^\bullet$, then $\bigoplus E_i$ is represented by the complex whose degree n term is $\bigoplus E_i^n$. On the other hand, as P^n is projective for all n we have $\mathrm{Hom}_{D(R)}(P^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(P^\bullet, K^\bullet)$ for every complex of R-modules $K^\bullet$, see Derived Categories, Lemma 13.19.8. Thus $\mathrm{Hom}_{D(R)}(P^\bullet, E^\bullet)$ is the cohomology of the complex

$$\prod \mathrm{Hom}_R(P^n, E^{n-1}) \to \prod \mathrm{Hom}_R(P^n, E^n) \to \prod \mathrm{Hom}_R(P^n, E^{n+1}).$$

Since $P^\bullet$ is bounded we see that we may replace the $\prod$ signs by $\bigoplus$ signs in the complex above. Since each P^n is a finite R-module we see that $\mathrm{Hom}_R(P^n, \bigoplus_i E_i^m) = \bigoplus_i \mathrm{Hom}_R(P^n, E_i^m)$ for all n, m. Combining these remarks we see that the map of Derived Categories, Definition 13.37.1 is a bijection.

Conversely, assume K is compact. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 12.15. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

Since $K \in D^-(R)$ and since every R-module is a quotient of a free module, we may represent K by a bounded above complex $K^\bullet$ of free R-modules, see Derived Categories, Lemma 13.15.4. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 12.15. Hence by Lemma 15.78.2 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(R)$ for some complex $L^\bullet$ which is bounded and whose terms are free R-modules. Say $L^i = 0$ for $i \not\in [a, b]$. Fix a, b from now on. Let c be the largest integer $\leq b + 1$ such that we can find a factorization of $1_{K^\bullet}$ as above with L^i is finite free for $i < c$. We will show by induction that $c = b + 1$. Namely, write $L^c = \bigoplus_{\lambda \in \Lambda} R$. Since L^{c-1} is finite free we can find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda \in \Lambda'} R \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda \in \Lambda \setminus \Lambda'} R)[-c]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(R)$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(R)$ (see Derived Categories, Lemma 13.12.1). Since $\mathrm{Hom}_{D(R)}(K, -)$ is homological (see Derived Categories, Lemma 13.4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(R)$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda \in \Lambda'} R$ the induction step is proved.

The conclusion of the discussion of the preceding paragraph is that $1_K : K \to K$ factors as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(R)$ where L can be represented by a finite complex of free R-modules. In particular we see that L is perfect. Note that $e = \varphi \circ \psi \in \text{End}_{D(R)}(L)$ is an idempotent. By Derived Categories, Lemma 13.4.14 we see that $L = \text{Ker}(e) \oplus \text{Ker}(1-e)$. The map $\varphi : K \to L$ induces an isomorphism with $\text{Ker}(1-e)$ in $D(R)$. Hence we finally conclude that K is perfect by Lemma 15.74.5. □

07LU Lemma 15.78.4. Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$. Assume that

(1) $K \otimes_R^{\mathbf{L}} R/I$ is perfect in $D(R/I)$, and
(2) I is a nilpotent ideal.

Then K is perfect in $D(R)$.

Proof. Choose a finite complex $\overline{P}^\bullet$ of finite projective R/I-modules representing $K \otimes_R^{\mathbf{L}} R/I$, see Definition 15.74.1. By Lemma 15.75.3 there exists a complex $P^\bullet$ of projective R-modules representing K such that $\overline{P}^\bullet = P^\bullet/IP^\bullet$. It follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that $P^\bullet$ is a finite complex of finite projective R-modules. □

09AS Lemma 15.78.5. Let R be a ring. Let $I, J \subset R$ be ideals. Let K be an object of $D(R)$. Assume that

(1) $K \otimes_R^{\mathbf{L}} R/I$ is perfect in $D(R/I)$, and
(2) $K \otimes_R^{\mathbf{L}} R/J$ is perfect in $D(R/J)$.

Then $K \otimes_R^{\mathbf{L}} R/IJ$ is perfect in $D(R/IJ)$.

Proof. It is clear that we may assume replace R by R/IJ and K by $K \otimes_R^{\mathbf{L}} R/IJ$. Then $R \to R/(I \cap J)$ is a surjection whose kernel has square zero. Hence by Lemma 15.78.4 it suffices to prove that $K \otimes_R^{\mathbf{L}} R/(I \cap J)$ is perfect. Thus we may assume that $I \cap J = 0$.

We prove the lemma in case $I \cap J = 0$. First, we may represent K by a K-flat complex $K^\bullet$ with all K^n flat, see Lemma 15.59.10. Then we see that we have a short exact sequence of complexes

$$0 \to K^\bullet \to K^\bullet/IK^\bullet \oplus K^\bullet/JK^\bullet \to K^\bullet/(I+J)K^\bullet \to 0$$

Note that $K^\bullet/IK^\bullet$ represents $K \otimes_R^{\mathbf{L}} R/I$ by construction of the derived tensor product. Similarly for $K^\bullet/JK^\bullet$ and $K^\bullet/(I+J)K^\bullet$. Note that $K^\bullet/(I+J)K^\bullet$ is a perfect complex of $R/(I+J)$-modules, see Lemma 15.74.9. Hence the complexes $K^\bullet/IK^\bullet$, and $K^\bullet/JK^\bullet$ and $K^\bullet/(I+J)K^\bullet$ have finitely many nonzero cohomology groups (since a perfect complex has finite Tor-amplitude, see Lemma 15.74.2). We conclude that $K \in D^b(R)$ by the long exact cohomology sequence associated to short exact sequence of complexes displayed above. In particular we assume $K^\bullet$ is a bounded above complex of free R-modules (see Derived Categories, Lemma 13.15.4).

We will now show that K is perfect using the criterion of Proposition 15.78.3. Thus we let $E_j \in D(R)$ be a family of objects parametrized by a set J. We choose complexes $E_j^\bullet$ with flat terms representing E_j, see for example Lemma 15.59.10. It is clear that

$$0 \to E_j^\bullet \to E_j^\bullet/IE_j^\bullet \oplus E_j^\bullet/JE_j^\bullet \to E_j^\bullet/(I+J)E_j^\bullet \to 0$$

is a short exact sequence of complexes. Taking direct sums we obtain a similar short exact sequence

$$0 \to \bigoplus E_j^\bullet \to \bigoplus E_j^\bullet/IE_j^\bullet \oplus E_j^\bullet/JE_j^\bullet \to \bigoplus E_j^\bullet/(I+J)E_j^\bullet \to 0$$

(Note that $- \otimes_R R/I$ commutes with direct sums.) This short exact sequence determines a distinguished triangle in $D(R)$, see Derived Categories, Lemma 13.12.1. Apply the homological functor $\mathrm{Hom}_{D(R)}(K, -)$ (see Derived Categories, Lemma 13.4.2) to get a commutative diagram

$$\begin{tikzcd}
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/(I+J))[-1] \arrow[r] \arrow[d] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/(I+J))[-1] \arrow[d] \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/I \oplus E_j^\bullet/J)[-1] \arrow[r] \arrow[d] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/I \oplus E_j^\bullet/J)[-1] \arrow[d] \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet) \arrow[r] \arrow[d] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet) \arrow[d] \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/I \oplus E_j^\bullet/J) \arrow[r] \arrow[d] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/I \oplus E_j^\bullet/J) \arrow[d] \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/(I+J)) \arrow[r] & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/(I+J))
\end{tikzcd}$$

with exact columns. It is clear that, for any complex $E^\bullet$ of R-modules we have

$$\begin{aligned}
\mathrm{Hom}_{D(R)}(K^\bullet, E^\bullet/I) &= \mathrm{Hom}_{K(R)}(K^\bullet, E^\bullet/I) \\
&= \mathrm{Hom}_{K(R/I)}(K^\bullet/IK^\bullet, E^\bullet/I) \\
&= \mathrm{Hom}_{D(R/I)}(K^\bullet/IK^\bullet, E^\bullet/I)
\end{aligned}$$

and similarly for when dividing by J or $I + J$, see Derived Categories, Lemma 13.19.8. Derived Categories. Thus all the horizontal arrows, except for possibly the middle one, are isomorphisms as the complexes $K^\bullet/IK^\bullet$, $K^\bullet/JK^\bullet$, $K^\bullet/(I+J)K^\bullet$ are perfect complexes of R/I, R/J, $R/(I+J)$-modules, see Proposition 15.78.3. It follows from the 5-lemma (Homology, Lemma 12.5.20) that the middle map is an isomorphism and the lemma follows by Proposition 15.78.3. □

15.79. Strong generators and regular rings

0FXG Let R be a ring. Denote $D(R)_c$ the saturated full triangulated subcategory of $D(R)$. We already know that

$$\langle R \rangle = D_{perf}(R) = D(R)_c$$

See Lemma 15.78.1 and Proposition 15.78.3. It turns out that if R is regular, then R is a strong generator (Derived Categories, Definition 13.36.3).

0FXH **Lemma 15.79.1.** Let R be a ring. Let $n \geq 1$. Let $K \in \langle R \rangle_n$ with notation as in Derived Categories, Section 13.36. Consider maps [Kel65]

$$K \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \ldots \xrightarrow{f_n} K_n$$

in $D(R)$. If $H^i(f_j) = 0$ for all i, j, then $f_n \circ \ldots \circ f_1 = 0$.

Proof. If $n = 1$, then K is a direct summand in $D(R)$ of a bounded complex $P^\bullet$ whose terms are finite free R-modules and whose differentials are zero. Thus it suffices to show any morphism $f : P^\bullet \to K_1$ in $D(R)$ with $H^i(f) = 0$ for all i is zero. Since $P^\bullet$ is a finite direct sum $P^\bullet = \bigoplus R[m_j]$ it suffices to show any morphism $g : R[m] \to K_1$ with $H^{-m}(g) = 0$ in $D(R)$ is zero. This follows from the fact that $\mathrm{Hom}_{D(R)}(R[-m], K) = H^m(K)$.

For $n > 1$ we proceed by induction on n. Namely, we know that K is a summand in $D(R)$ of an object P which sits in a distinguished triangle

$$P' \xrightarrow{i} P \xrightarrow{p} P'' \to P'[1]$$

with $P' \in \langle R \rangle_1$ and $P'' \in \langle R \rangle_{n-1}$. As above we may replace K by P and assume that we have

$$P \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \ldots \xrightarrow{f_n} K_n$$

in $D(R)$ with f_j zero on cohomology. By the case $n = 1$ the composition $f_1 \circ i$ is zero. Hence by Derived Categories, Lemma 13.4.2 we can find a morphism $h : P'' \to K_1$ such that $f_1 = h \circ p$. Observe that $f_2 \circ h$ is zero on cohomology. Hence by induction we find that $f_n \circ \ldots \circ f_2 \circ h = 0$ which implies $f_n \circ \ldots \circ f_1 = f_n \circ \ldots \circ f_2 \circ h \circ p = 0$ as desired. $\square$

0FXI **Lemma 15.79.2.** Let R be a Noetherian ring. If R is a strong generator for $D_{perf}(R)$, then R is regular of finite dimension.

Proof. Assume $D_{perf}(R) = \langle R \rangle_n$ for some $n \geq 1$. For any finite R-module M we can choose a complex

$$P = (P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \xrightarrow{d^{1}} \ldots \xrightarrow{d^{-1}} P^0)$$

of finite free R-modules with $H^i(P) = 0$ for $i = -n, \ldots, -1$ and $M \cong \mathrm{Coker}(d^{-1})$. Note that P is in $D_{perf}(R)$. For any R-module N we can compute $\mathrm{Ext}^n_R(M, N)$ the finite free resolution P of M, see Algebra, Section 10.71 and compare with Derived Categories, Section 13.27. In particular, the sequence above defines an element

$$\xi \in \mathrm{Ext}^n_R(\mathrm{Coker}(d^{-1}), \mathrm{Coker}(d^{-n-1})) = \mathrm{Ext}^n_R(M, \mathrm{Coker}(d^{-n-1}))$$

and for any element $\overline{\xi}$ in $\mathrm{Ext}^n_R(M, N)$ there is a R-module map $\varphi : \mathrm{Coker}(d^{-n-1}) \to N$ such that φ maps ξ to $\overline{\xi}$. For $j = 1, \ldots, n-1$ consider the complexes

$$K_j = (\mathrm{Coker}(d^{-n-1}) \to P^{-n+1} \to \ldots \to P^{-j})$$

with $\mathrm{Coker}(d^{-n-1})$ in degree $-n$ and P^t in degree t. We also set $K_n = \mathrm{Coker}(d^{-n-1})[n]$. Then we have maps

$$P \to K_1 \to K_2 \to \ldots \to K_n$$

which induce vanishing maps on cohomology. By Lemma 15.79.1 since $P \in D_{perf}(R) = \langle R \rangle_n$ we find that the composition of this maps is zero in $D(R)$. Since $\mathrm{Hom}_{D(R)}(P, K_n) = \mathrm{Hom}_{K(R)}(P, K_n)$ by Derived Categories, Lemma 13.19.8 we conclude $\xi = 0$. Hence $\mathrm{Ext}^n_R(M, N) = 0$ for all R-modules N, see discussion above. It follows that M has projective dimension $\leq n - 1$ by Algebra, Lemma 10.109.8. Since this holds for all finite R-modules M we conclude that R has finite global dimension, see Algebra, Lemma 10.109.12. We finally conclude by Algebra, Lemma 10.110.8. $\square$

0FXJ Lemma 15.79.3. Let R be a Noetherian regular ring of dimension $d < \infty$. Let $K, L \in D^-(R)$. Assume there exists an k such that $H^i(K) = 0$ for $i \leq k$ and $H^i(L) = 0$ for $i \geq k - d + 1$. Then $\mathrm{Hom}_{D(R)}(K, L) = 0$.

Proof. Let $K^\bullet$ be a bounded above complex representing K, say $K^i = 0$ for $i \geq n + 1$. After replacing $K^\bullet$ by $\tau_{\geq k+1}K^\bullet$ we may assume $K^i = 0$ for $i \leq k$. Then we may use the distinguished triangle

$$K^n[-n] \to K^\bullet \to \sigma_{\leq n-1}K^\bullet$$

to see it suffices to prove the lemma for $K^n[-n]$ and $\sigma_{\leq n-1}K^\bullet$. By induction on n, we conclude that it suffices to prove the lemma in case K is represented by the complex $M[-m]$ for some R-module M and some $m \geq k + 1$. Since R has global dimension d by Algebra, Lemma 10.110.8 we see that M has a projective resolution $0 \to P_d \to \ldots \to P_0 \to M \to 0$. Then the complex $P^\bullet$ having P_i in degree $m - i$ is a bounded complex of projectives representing $M[-m]$. On the other hand, we can choose a complex $L^\bullet$ representing L with $L^i = 0$ for $i \geq k - d + 1$. Hence any map of complexes $P^\bullet \to L^\bullet$ is zero. This implies the lemma by Derived Categories, Lemma 13.19.8. $\square$

0FXK Lemma 15.79.4. Let R be a Noetherian regular ring of dimension $1 \leq d < \infty$. Let $K \in D(R)$ be perfect and let $k \in \mathbf{Z}$ such that $H^i(K) = 0$ for $i = k - d + 2, \ldots, k$ (empty condition if $d = 1$). Then $K = \tau_{\leq k-d+1}K \oplus \tau_{\geq k+1}K$.

Proof. The vanishing of cohomology shows that we have a distinguished triangle

$$\tau_{\leq k-d+1}K \to K \to \tau_{\geq k+1}K \to (\tau_{\leq k-d+1}K)[1]$$

By Derived Categories, Lemma 13.4.11 it suffices to show that the third arrow is zero. Thus it suffices to show that $\mathrm{Hom}_{D(R)}(\tau_{\geq k+1}K, (\tau_{\leq k-d+1}K)[1]) = 0$ which follows from Lemma 15.79.3. $\square$

0FXL Lemma 15.79.5. Let R be a Noetherian regular ring of finite dimension. Then R is a strong generator for the full subcategory $D_{perf}(R) \subset D(R)$ of perfect objects.

Proof. We will use that an object K of $D(R)$ is perfect if and only if K is bounded and has finite cohomology modules, see Lemma 15.74.14. Strong generators of triangulated categories are defined in Derived Categories, Definition 13.36.3. Let $d = \dim(R)$.

Let $K \in D_{perf}(R)$. We will show $K \in \langle R \rangle_{d+1}$. By Algebra, Lemma 10.110.8 every finite R-module has projective dimension $\leq d$. We will show by induction on $0 \leq i \leq d$ that if $H^n(K)$ has projective dimension $\leq i$ for all $n \in \mathbf{Z}$, then K is in $\langle R \rangle_{i+1}$.

Base case $i = 0$. In this case $H^n(K)$ is a finite R-module of projective dimension 0. In other words, each cohomology is a projective R-module. Thus $\mathrm{Ext}^i_R(H^n(K), H^m(K)) = 0$ for all $i > 0$ and $m, n \in \mathbf{Z}$. By Derived Categories, Lemma 13.27.9 we find that K is isomorphic to the direct sum of the shifts of its cohomology modules. Since each cohomology module is a finite projective R-module, it is a direct summand of a direct sum of copies of R. Hence by definition we see that K is contained in $\langle R \rangle_1$.

Induction step. Assume the claim holds for $i < d$ and let $K \in D_{perf}(R)$ have the property that $H^n(K)$ has projective dimension $\leq i + 1$ for all $n \in \mathbf{Z}$. Choose $a \leq b$ such that $H^n(K)$ is zero for $n \notin [a, b]$. For each $n \in [a, b]$ choose a surjection $F^n \to H^n(K)$ where F^n is a finite free R-module. Since F^n is projective, we can lift $F^n \to H^n(K)$ to a map $F^n[-n] \to K$ in $D(R)$ (small detail omitted). Thus we obtain a morphism $\bigoplus_{a \leq n \leq b} F^n[-n] \to K$ which is surjective on cohomology modules. Choose a distinguished triangle

$$K' \to \bigoplus_{a \leq n \leq b} F^n[-n] \to K \to K'[1]$$

in $D(R)$. Of course, the object K' is bounded and has finite cohomology modules. The long exact sequence of cohomology breaks into short exact sequences

$$0 \to H^n(K') \to F^n \to H^n(K) \to 0$$

by the choices we made. By Algebra, Lemma 10.109.9 we see that the projective dimension of $H^n(K')$ is $\leq \max(0, i)$. Thus $K' \in \langle R \rangle_{i+1}$. By definition this means that K is in $\langle R \rangle_{i+1+1}$ as desired. ☐

0FXM Proposition 15.79.6. Let R be a Noetherian ring. The following are equivalent

(1) R is regular of finite dimension,
(2) $D_{perf}(R)$ has a strong generator, and
(3) R is a strong generator for $D_{perf}(R)$.

Proof. This is a formal consequence of Lemmas 15.78.1, 15.79.2, and 15.79.5 as well as Derived Categories, Lemma 13.36.6. ☐

15.80. Relatively finitely presented modules

0659 Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a finite B-module. In this case it is not true that

$$M \text{ of finite presentation over } B \Leftrightarrow M \text{ of finite presentation over } A$$

A counter example is $R = k[x_1, x_2, x_3, \ldots]$, $A = R$, $B = R/(x_i)$, and $M = B$. To "fix" this we introduce a relative notion of finite presentation.

05GY Lemma 15.80.1. Let $R \to A$ be a ring map of finite type. Let M be an A-module. The following are equivalent

(1) for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module,
(2) for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module, and
(3) for any surjection $A' \to A$ where A' is a finitely presented R-algebra, the module M is finitely presented as A'-module.

In this case M is a finitely presented A-module.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\ R[y_1, \ldots, y_m] \arrow[r] & A \end{tikzcd}$$

Hence the equivalence of (1) and (2) follows by applying Algebra, Lemmas 10.6.4 and 10.36.23. The equivalence of (2) and (3) follows by choosing a presentation $A' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and using Algebra, Lemma 10.36.23 to show that M is finitely presented as A'-module if and only if M is finitely presented as a $R[x_1, \ldots, x_n]$-module. □

05GZ Definition 15.80.2. Let $R \to A$ be a finite type ring map. Let M be an A-module. We say M is an A-module finitely presented relative to R if the equivalent conditions of Lemma 15.80.1 hold.

Note that if $R \to A$ is of finite presentation, then M is an A-module finitely presented relative to R if and only if M is a finitely presented A-module. It is equally clear that A as an A-module is finitely presented relative to R if and only if A is of finite presentation over R. If R is Noetherian the notion is uninteresting. Now we can formulate the result we were looking for.

05H0 Lemma 15.80.3. Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a B-module. Then M is an A-module finitely presented relative to R if and only if M is a B-module finitely presented relative to R.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n] \arrow[r] \arrow[d] & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

Since the top arrow is a finite and finitely presented ring map we conclude by Algebra, Lemma 10.36.23 and the definition. □

With this result in hand we see that the relative notion makes sense and behaves well with regards to finite maps of rings of finite type over R. It is also stable under localization, stable under base change, and "glues" well.

065A Lemma 15.80.4. Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and M an A-module. If M of finite presentation relative to R_f, then M_g is an A_g-module of finite presentation relative to R.

Proof. Choose a presentation $R_f[x_1, \ldots, x_n] \to A$. We write $R_f = R[x_0]/(fx_0 - 1)$. Consider the presentation $R[x_0, x_1, \ldots, x_n, x_{n+1}] \to A_g$ which extends the given map, maps x_0 to the image of $1/f$, and maps x_{n+1} to $1/g$. Choose $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to g (this is possible). Suppose that

$$R_f[x_1, \ldots, x_n]^{\oplus s} \to R_f[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Pick $h'_{ij} \in R[x_0, x_1, \ldots, x_n]$ which map to h_{ij}. Then

$$R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus s+2t} \to R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus t} \to M_g \to 0$$

is a presentation of M_f. Here the $t \times (s + 2t)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h'_{ij}, a second $t \times t$ block which is $(x_0 f -)I_t$, and a third block which is $(x_{n+1}g' - 1)I_t$. □

065B Lemma 15.80.5. Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. For any ring map $R \to R'$ the $A \otimes_R R'$-module

$$M \otimes_A A' = M \otimes_R R'$$

is finitely presented relative to R'.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

Then

$$R'[x_1, \ldots, x_n]^{\oplus s} \to R'[x_1, \ldots, x_n]^{\oplus t} \to M \otimes_R R' \to 0$$

is a presentation of the base change and we win. $\square$

0670 Lemma 15.80.6. Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. Let $A \to A'$ be a ring map of finite presentation. The A'-module $M \otimes_A A'$ is finitely presented relative to R.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation $A' = A[y_1, \ldots, y_m]/(g_1, \ldots, g_l)$. Pick $g'_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to g_i. Say

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Then

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus s+tl} \to R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus t} \to M \otimes_A A' \to 0$$

is a presentation of $M \otimes_A A'$. Here the $t \times (s + lt)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h_{ij}, followed by l blocks of size $t \times t$ which are $g'_i I_t$. $\square$

065C Lemma 15.80.7. Let $R \to A \to B$ be finite type ring maps. Let M be a B-module. If M is finitely presented relative to A and A is of finite presentation over R, then M is finitely presented relative to R.

Proof. Choose a surjection $A[x_1, \ldots, x_n] \to B$. Choose a presentation

$$A[x_1, \ldots, x_n]^{\oplus s} \to A[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

given by a matrix (h_{ij}). Choose a presentation

$$A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_u).$$

Choose $h'_{ij} \in R[y_1, \ldots, y_m, x_1, \ldots, x_n]$ mapping to h_{ij}. Then we obtain the presentation

$$R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus s+tu} \to R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

where the $t \times (s+tu)$-matrix is given by a first $t \times s$ block consisting of h'_{ij} followed by u blocks of size $t \times t$ given by $g_i I_t$, $i = 1, \ldots, u$. $\square$

065D Lemma 15.80.8. Let $R \to A$ be a finite type ring map. Let M be an A-module. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent

(1) each M_{f_i} is finitely presented relative to R, and
(2) M is finitely presented relative to R.

Proof. The implication (2) $\Rightarrow$ (1) is in Lemma 15.80.4. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

By Lemma 15.80.1 we see that M_{f_i} is a finitely presented A_{f_i}-module, hence by Algebra, Lemma 10.23.2 we see that M is a finitely presented A-module. Hence M is a finite P-module (with P as above). Choose a surjection $P^{\oplus t} \to M$. We have to show that the kernel K of this map is a finite P-module. Since P_{y_i} surjects onto A_{f_i} we see by Lemma 15.80.1 and Algebra, Lemma 10.5.3 that the localization K_{y_i} is a finitely generated P_{y_i}-module. Choose elements $k_{i,j} \in K$, $i = 1, \ldots, r$, $j = 1, \ldots, s_i$ such that the images of $k_{i,j}$ in K_{y_i} generate. Set $K' \subset K$ equal to the P-module generated by the elements $k_{i,j}$. Then K/K' is a module whose localization at y_i is zero for all i. Since $(y_1, \ldots, y_r) = P$ we see that $K/K' = 0$ as desired. □

0671 Lemma 15.80.9. Let $R \to A$ be a finite type ring map. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules.

(1) If M', M'' are finitely presented relative to R, then so is M.
(2) If M' is a finite type A-module and M is finitely presented relative to R, then M'' is finitely presented relative to R.

Proof. Follows immediately from Algebra, Lemma 10.5.3. □

0672 Lemma 15.80.10. Let $R \to A$ be a finite type ring map. Let M, M' be A-modules. If $M \oplus M'$ is finitely presented relative to R, then so are M and M'.

Proof. Omitted. □

15.81. Relatively pseudo-coherent modules

065E This section is the analogue of Section 15.80 for pseudo-coherence.

065F Lemma 15.81.1. Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then

$$K^\bullet \otimes_{R[x]}^{\mathbf{L}} R \cong K^\bullet \oplus K^\bullet[1]$$

in $D(R)$.

Proof. Choose a K-flat resolution $P^\bullet \to K^\bullet$ over R such that P^n is a flat R-module for all n, see Lemma 15.59.10. Then $P^\bullet \otimes_R R[x]$ is a K-flat complex of $R[x]$-modules whose terms are flat $R[x]$-modules, see Lemma 15.59.3 and Algebra, Lemma 10.39.7. In particular $x : P^n \otimes_R R[x] \to P^n \otimes_R R[x]$ is injective with cokernel isomorphic to P^n. Thus

$$P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]$$

is a double complex of $R[x]$-modules whose associated total complex is quasi-isomorphic to $P^\bullet$ and hence $K^\bullet$. Moreover, this associated total complex is a K-flat complex of $R[x]$-modules for example by Lemma 15.59.4 or by Lemma 15.59.5. Hence

$$\begin{aligned} K^\bullet \otimes_{R[x]}^{\mathbf{L}} R &\cong \operatorname{Tot}(P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]) \otimes_{R[x]} R = \operatorname{Tot}(P^\bullet \xrightarrow{0} P^\bullet) \\ &= P^\bullet \oplus P^\bullet[1] \cong K^\bullet \oplus K^\bullet[1] \end{aligned}$$

as desired. □

065G Lemma 15.81.2. Let R be a ring and $K^\bullet$ a complex of R-modules. Let $m \in \mathbf{Z}$. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then $K^\bullet$ is m-pseudo-coherent as a complex of R-modules if and only if $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules.

Proof. This is a special case of Lemma 15.64.11. We also prove it in another way as follows.

Note that $0 \to R[x] \to R[x] \to R \to 0$ is exact. Hence R is pseudo-coherent as an $R[x]$-module. Thus one implication of the lemma follows from Lemma 15.64.11. To prove the other implication, assume that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules. By Lemma 15.64.12 we see that $K^\bullet \otimes^{\mathbf{L}}_{R[x]} R$ is m-pseudo-coherent as a complex of R-modules. By Lemma 15.81.1 we see that $K^\bullet \oplus K^\bullet[1]$ is m-pseudo-coherent as a complex of R-modules. Finally, we conclude that $K^\bullet$ is m-pseudo-coherent as a complex of R-modules from Lemma 15.64.8. $\square$

065H Lemma 15.81.3. Let $R \to A$ be a ring map of finite type. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. The following are equivalent

(1) for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules,
(2) for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules.

In particular the same equivalence holds for pseudo-coherence.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\ R[y_1, \ldots, y_m] \arrow[r] & A \end{tikzcd}$$

After a change of coordinates the ring homomorphism $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to R[x_1, \ldots, x_n]$ is isomorphic to the ring homomorphism which maps each y_i to zero. Similarly for the left vertical map in the diagram. Hence, by induction on the number of variables this lemma follows from Lemma 15.81.2. The pseudo-coherent case follows from this and Lemma 15.64.5. $\square$

065I Definition 15.81.4. Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let M be an A-module. Let $m \in \mathbf{Z}$.

(1) We say $K^\bullet$ is m-pseudo-coherent relative to R if the equivalent conditions of Lemma 15.81.3 hold.
(2) We say $K^\bullet$ is pseudo-coherent relative to R if $K^\bullet$ is m-pseudo-coherent relative to R for all $m \in \mathbf{Z}$.
(3) We say M is m-pseudo-coherent relative to R if $M[0]$ is m-pseudo-coherent relative to R.
(4) We say M is pseudo-coherent relative to R if $M[0]$ is pseudo-coherent relative to R.

Part (2) means that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n]$-modules for any surjection $R[y_1, \ldots, y_m] \to A$, see Lemma 15.64.5. This definition has the following pleasing property.

0673 Lemma 15.81.5. Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R if and only if $K^\bullet$ seen as a complex of A-modules is m-pseudo-coherent (pseudo-coherent) relative to R.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{tikzcd} & R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[d] \\ R[x_1, \ldots, x_n] \arrow[r] \arrow[d] & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

The top horizontal arrow and the top right vertical arrow satisfy the assumptions of Lemma 15.64.11. Hence $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n]$-modules if and only if $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-modules. □

0674 Lemma 15.81.6. Let R be a ring. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(A)$.

(1) If $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $L^\bullet$ is m-pseudo-coherent relative to R then $M^\bullet$ is m-pseudo-coherent relative to R.
(2) If $K^\bullet, M^\bullet$ are m-pseudo-coherent relative to R, then $L^\bullet$ is m-pseudo-coherent relative to R.
(3) If $L^\bullet$ is $(m + 1)$-pseudo-coherent relative to R and $M^\bullet$ is m-pseudo-coherent relative to R, then $K^\bullet$ is $(m + 1)$-pseudo-coherent relative to R.

Moreover, if two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent relative to R, the so is the third.

Proof. Follows immediately from Lemma 15.64.2 and the definitions. □

0675 Lemma 15.81.7. Let $R \to A$ be a finite type ring map. Let M be an A-module. Then

(1) M is 0-pseudo-coherent relative to R if and only if M is a finite type A-module,
(2) M is (-1)-pseudo-coherent relative to R if and only if M is a finitely presented relative to R,
(3) M is $(-d)$-pseudo-coherent relative to R if and only if for every surjection $R[x_1, \ldots, x_n] \to A$ there exists a resolution

$$R[x_1, \ldots, x_n]^{\oplus a_d} \to R[x_1, \ldots, x_n]^{\oplus a_{d-1}} \to \ldots \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$

of length d, and

(4) M is pseudo-coherent relative to R if and only if for every presentation $R[x_1, \ldots, x_n] \to A$ there exists an infinite resolution

$$\ldots \to R[x_1, \ldots, x_n]^{\oplus a_1} \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$

by finite free $R[x_1, \ldots, x_n]$-modules.

Proof. Follows immediately from Lemma 15.64.4 and the definitions. □

0676 Lemma 15.81.8. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet, L^\bullet \in D(A)$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R so are $K^\bullet$ and $L^\bullet$.

Proof. Immediate from Lemma 15.64.8 and the definitions. □

0677 Lemma 15.81.9. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of A-modules such that K^i is $(m - i)$-pseudo-coherent relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent relative to R. In particular, if $K^\bullet$ is a bounded above complex of A-modules pseudo-coherent relative to R, then $K^\bullet$ is pseudo-coherent relative to R.

Proof. Immediate from Lemma 15.64.9 and the definitions. □

0678 Lemma 15.81.10. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(A)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.

Proof. Immediate from Lemma 15.64.10 and the definitions. □

0679 Lemma 15.81.11. Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and $K^\bullet$ a complex of A-modules. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R_f, then $K^\bullet \otimes_A A_g$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.

Proof. First we show that $K^\bullet$ is m-pseudo-coherent relative to R. Namely, suppose $R_f[x_1, \ldots, x_n] \to A$ is surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Hence by Lemma 15.64.11 we see that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n]$-modules.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. By Lemma 15.64.12 we see that

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_{R[x_0,x_1,\ldots,x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] &= K^\bullet \otimes_{R[x_0,x_1,\ldots,x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] \\ &= K^\bullet \otimes_A A_f \end{aligned}$$

is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$-modules. write

$$R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] = R[x_0, \ldots, x_n, x_{n+1}]/(x_{n+1}g' - 1).$$

As $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$ is pseudo-coherent as a $R[x_0, \ldots, x_n, x_{n+1}]$-module we conclude (see Lemma 15.64.11) that $K^\bullet \otimes_A A_g$ is m-pseudo-coherent as a complex of $R[x_0, \ldots, x_n, x_{n+1}]$-modules as desired. □

067A Lemma 15.81.12. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules which is m-pseudo-coherent (resp. pseudo-coherent) relative to R. Let $R \to R'$ be a ring map such that A and R' are Tor independent over R. Set $A' = A \otimes_R R'$. Then $K^\bullet \otimes_A^{\mathbf{L}} A'$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R'.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Note that

$$K^\bullet \otimes_A^{\mathbf{L}} A' = K^\bullet \otimes_R^{\mathbf{L}} R' = K^\bullet \otimes_{R[x_1,\ldots,x_n]}^{\mathbf{L}} R'[x_1, \ldots, x_n]$$

by Lemma 15.61.2 applied twice. Hence we win by Lemma 15.64.12. $\square$

067B Lemma 15.81.13. Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules. Assume B as a B-module is pseudo-coherent relative to A. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $K^\bullet \otimes_A^{\mathbf{L}} B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.

Proof. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Combined we get a surjection $R[x_1, \ldots, x_n, y_1, \ldots y_m] \to B$. Choose a resolution $E^\bullet \to B$ of B by a complex of finite free $A[y_1, \ldots, y_n]$-modules (which is possible by our assumption on the ring map $A \to B$). We may assume that $K^\bullet$ is a bounded above complex of flat A-modules. Then

$$\begin{aligned} K^\bullet \otimes_A^{\mathbf{L}} B &= \mathrm{Tot}(K^\bullet \otimes_A B[0]) \\ &= \mathrm{Tot}(K^\bullet \otimes_A A[y_1, \ldots, y_m] \otimes_{A[y_1,\ldots,y_m]} B[0]) \\ &\cong \mathrm{Tot}\left((K^\bullet \otimes_A A[y_1, \ldots, y_m]) \otimes_{A[y_1,\ldots,y_m]} E^\bullet\right) \\ &= \mathrm{Tot}(K^\bullet \otimes_A E^\bullet) \end{aligned}$$

in $D(A[y_1, \ldots, y_m])$. The quasi-isomorphism $\cong$ comes from an application of Lemma 15.59.7. Thus we have to show that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ is m-pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots y_m]$-modules. Note that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $K^\bullet \otimes_A E^i[-i]$. Note that for $i \ll 0$ the complexes $K^\bullet \otimes_A E^i[-i]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent (over any ring). Hence, applying Lemma 15.81.6 and induction, it suffices to show that $K^\bullet \otimes_A E^i[-i]$ is pseudo-coherent relative to R for all i. Note that $E^i = 0$ for $i > 0$. Since also E^i is finite free this reduces to proving that $K^\bullet \otimes_A A[y_1, \ldots, y_m]$ is m-pseudo-coherent relative to R which follows from Lemma 15.81.12 for instance. $\square$

067C Lemma 15.81.14. Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let M be an A-module. Assume B is flat over A and B as a B-module is pseudo-coherent relative to A. If M is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $M \otimes_A B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.

Proof. Immediate from Lemma 15.81.13. $\square$

067D Lemma 15.81.15. Let R be a ring. Let $A \to B$ be a map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Assume A is pseudo-coherent relative to R. Then the following are equivalent

(1) $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to A, and
(2) $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Then we get a surjection

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to A[y_1, \ldots, y_m]$$

which is a flat base change of $R[x_1, \ldots, x_n] \to A$. By assumption A is a pseudo-coherent module over $R[x_1, \ldots, x_n]$ hence by Lemma 15.64.13 we see that $A[y_1, \ldots, y_m]$ is pseudo-coherent over $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Thus the lemma follows from Lemma 15.64.11 and the definitions. □

067E Lemma 15.81.16. Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent

(1) each $K^\bullet \otimes_A A_{f_i}$ is m-pseudo-coherent relative to R, and
(2) $K^\bullet$ is m-pseudo-coherent relative to R.

The same equivalence holds for pseudo-coherence relative to R.

Proof. The implication (2) $\Rightarrow$ (1) is in Lemma 15.81.11. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

Note that P is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-module and that $P[1/y_i]$ is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r, 1/y_i]$-module. Hence by Lemma 15.64.11 we see that $K^\bullet \otimes_A A_{f_i}$ is an m-pseudo-coherent complex of $P[1/y_i]$-modules for each i. Thus by Lemma 15.64.14 we see that $K^\bullet$ is pseudo-coherent as a complex of P-modules, and Lemma 15.64.11 shows that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-modules. □

067F Lemma 15.81.17. Let R be a Noetherian ring. Let $R \to A$ be a finite type ring map. Then

(1) A complex of A-modules $K^\bullet$ is m-pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for $i \geq m$.
(2) A complex of A-modules $K^\bullet$ is pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for all i.
(3) An A-module is pseudo-coherent relative to R if and only if it is finite.

Proof. Immediate consequence of Lemma 15.64.17 and the definitions. □

15.82. Pseudo-coherent and perfect ring maps

067G We can define these types of ring maps as follows.

067H Definition 15.82.1. Let $A \to B$ be a ring map.

(1) We say $A \to B$ is a pseudo-coherent ring map if it is of finite type and B, as a B-module, is pseudo-coherent relative to A.
(2) We say $A \to B$ is a perfect ring map if it is a pseudo-coherent ring map such that B as an A-module has finite tor dimension.

This terminology may be nonstandard. Using Lemma 15.81.7 we see that $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules. The motivation for the definition of a perfect ring map is Lemma 15.74.2. The following lemmas gives a more useful and intuitive characterization of a perfect ring map.

068Y Lemma 15.82.2. A ring map $A \to B$ is perfect if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules.

Proof. If $A \to B$ is perfect, then $B = A[x_1, \ldots, x_n]/I$ and B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module and has finite tor dimension as an A-module. Hence Lemma 15.77.5 implies that B is perfect as a $A[x_1, \ldots, x_n]$-module, i.e., it has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules (Lemma 15.74.3). Conversely, if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules then B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module, hence $A \to B$ is pseudo-coherent. Moreover, the given resolution over $A[x_1, \ldots, x_n]$ is a finite resolution by flat A-modules and hence B has finite tor dimension as an A-module. $\square$

Lots of the results of the preceding sections can be reformulated in terms of this terminology. We also refer to More on Morphisms, Sections 37.60 and 37.61 for the corresponding discussion concerning morphisms of schemes.

067I Lemma 15.82.3. A finite type ring map of Noetherian rings is pseudo-coherent.

Proof. See Lemma 15.81.17. $\square$

067J Lemma 15.82.4. A ring map which is flat and of finite presentation is perfect.

Proof. Let $A \to B$ be a ring map which is flat and of finite presentation. It is clear that B has finite tor dimension. By Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$-algebra $A_0 \subset A$ and a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{A_0} A$. By Lemma 15.81.17 we see that $A_0 \to B_0$ is pseudo-coherent. As $A_0 \to B_0$ is flat we see that B_0 and A are tor independent over A_0, hence we may use Lemma 15.81.12 to conclude that $A \to B$ is pseudo-coherent. $\square$

067K Lemma 15.82.5. Let $A \to B$ be a finite type ring map with A a regular ring of finite dimension. Then $A \to B$ is perfect.

Proof. By Algebra, Lemma 10.110.8 the assumption on A means that A has finite global dimension. Hence every module has finite tor dimension, see Lemma 15.66.19, in particular B does. By Lemma 15.82.3 the map is pseudo-coherent. $\square$

07EN Lemma 15.82.6. A local complete intersection homomorphism is perfect.

Proof. Let $A \to B$ be a local complete intersection homomorphism. By Definition 15.33.2 this means that $B = A[x_1, \ldots, x_n]/I$ where I is a Koszul ideal in $A[x_1, \ldots, x_n]$. By Lemmas 15.82.2 and 15.74.3 it suffices to show that I is a perfect module over $A[x_1, \ldots, x_n]$. By Lemma 15.74.12 this is a local question. Hence we may assume that I is generated by a Koszul-regular sequence (by Definition 15.32.1). Of course this means that I has a finite free resolution and we win. $\square$

0DHQ Lemma 15.82.7. Let $R \to A$ be a pseudo-coherent ring map. Let $K \in D(A)$. The following are equivalent

(1) K is m-pseudo-coherent (resp. pseudo-coherent) relative to R, and
(2) K is m-pseudo-coherent (resp. pseudo-coherent) in $D(A)$.

Proof. Reformulation of a special case of Lemma 15.81.15. $\square$

0E1T Lemma 15.82.8. Let $R \to B \to A$ be ring maps with $\varphi : B \to A$ surjective and $R \to B$ and $R \to A$ flat and of finite presentation. For $K \in D(A)$ denote $\varphi_*K \in D(B)$ the restriction. The following are equivalent

(1) K is pseudo-coherent,
(2) K is pseudo-coherent relative to R,
(3) K is pseudo-coherent relative to A,
(4) φ_*K is pseudo-coherent,
(5) φ_*K is pseudo-coherent relative to R.

Similar holds for m-pseudo-coherence.

Proof. Observe that $R \to A$ and $R \to B$ are perfect ring maps (Lemma 15.82.4) hence a fortiori pseudo-coherent ring maps. Thus (1) $\Leftrightarrow$ (2) and (4) $\Leftrightarrow$ (5) by Lemma 15.82.7.

Using that A is pseudo-coherent relative to R we use Lemma 15.81.15 to see that (2) $\Leftrightarrow$ (3). However, since $A \to B$ is surjective, we see directly from Definition 15.81.4 that (3) is equivalent with (4). $\square$

15.83. Relatively perfect modules

0DHR This section is the analogue of Section 15.81 for perfect objects of the derived category. we only define this notion in a limited generality as we are not sure what the correct definition is in general. See Derived Categories of Schemes, Remark 36.35.14 for a discussion.

0DHS Definition 15.83.1. Let $R \to A$ be a flat ring map of finite presentation. An object K of $D(A)$ is R-perfect or perfect relative to R if K is pseudo-coherent (Definition 15.64.1) and has finite tor dimension over R (Definition 15.66.1).

By Lemma 15.82.8 it would have been the same thing to ask K to be pseudo-coherent relative to R. Here are some obligatory lemmas.

0DHT Lemma 15.83.2. Let $R \to A$ be a flat ring map of finite presentation. The R-perfect objects of $D(A)$ form a saturated[9] triangulated strictly full subcategory.

Proof. This follows from Lemmas 15.64.2, 15.64.8, 15.66.5, and 15.66.7. $\square$

0DHU Lemma 15.83.3. Let $R \to A$ be a flat ring map of finite presentation. A perfect object of $D(A)$ is R-perfect. If $K, M \in D(A)$ then $K \otimes_A^{\mathbf{L}} M$ is R-perfect if K is perfect and M is R-perfect.

Proof. The first statement follows from the second by taking $M = A$. The second statement follows from Lemmas 15.74.2, 15.66.10, and 15.64.16. $\square$

0DHV Lemma 15.83.4. Let $R \to A$ be a flat ring map of finite presentation. Let $K \in D(A)$. The following are equivalent

(1) K is R-perfect, and
(2) K is isomorphic to a finite complex of R-flat, finitely presented A-modules.

[9]Derived Categories, Definition 13.6.1.

Proof. To prove (2) implies (1) it suffices by Lemma 15.83.2 to show that an R-flat, finitely presented A-module M defines an R-perfect object of $D(A)$. Since M has finite tor dimension over R, it suffices to show that M is pseudo-coherent. By Algebra, Lemma 10.168.1 there exists a finite type $\mathbf{Z}$-algebra $R_0 \subset R$ and a flat finite type ring map $R_0 \to A_0$ and a finite A_0-module M_0 flat over R_0 such that $A = A_0 \otimes_{R_0} R$ and $M = M_0 \otimes_{R_0} R$. By Lemma 15.64.17 we see that M_0 is pseudo-coherent A_0-module. Choose a resolution $P_0^\bullet \to M_0$ by finite free A_0-modules P_0^n. Since A_0 is flat over R_0, this is a flat resolution. Since M_0 is flat over R_0 we find that $P^\bullet = P_0^\bullet \otimes_{R_0} R$ still resolves $M = M_0 \otimes_{R_0} R$. (You can use Lemma 15.61.2 to see this.) Hence $P^\bullet$ is a finite free resolution of M over A and we conclude that M is pseudo-coherent.

Assume (1). We can represent K by a bounded above complex $P^\bullet$ of finite free A-modules. Assume that K viewed as an object of $D(R)$ has tor amplitude in $[a, b]$. By Lemma 15.66.2 we see that $\tau_{\geq a} P^\bullet$ is a complex of R-flat, finitely presented A-modules representing K. $\square$

0DHW **Lemma 15.83.5.** Let $R \to A$ be a flat ring map of finite presentation. Let $R \to R'$ be a ring map and set $A' = A \otimes_R R'$. If $K \in D(A)$ is R-perfect, then $K \otimes_A^{\mathbf{L}} A'$ is R'-perfect.

Proof. By Lemma 15.64.12 we see that $K \otimes_A^{\mathbf{L}} A'$ is pseudo-coherent. By Lemma 15.61.2 we see that $K \otimes_A^{\mathbf{L}} A'$ is equal to $K \otimes_R^{\mathbf{L}} R'$ in $D(R')$. Then we can apply Lemma 15.66.13 to see that $K \otimes_R^{\mathbf{L}} R'$ in $D(R')$ has finite tor dimension. $\square$

0E1U **Lemma 15.83.6.** Let $R \to A$ be a flat ring map. Let $K, L \in D(A)$ with K pseudo-coherent and L finite tor dimension over R. We may choose

(1) a bounded above complex $P^\bullet$ of finite free A-modules representing K, and
(2) a bounded complex of R-flat A-modules $F^\bullet$ representing L.

Given these choices we have

(a) $E^\bullet = \mathrm{Hom}^\bullet(P^\bullet, F^\bullet)$ is a bounded below complex of R-flat A-modules representing $R\,\mathrm{Hom}_A(K, L)$,
(b) for any ring map $R \to R'$ with $A' = A \otimes_R R'$ the complex $E^\bullet \otimes_R R'$ represents $R\,\mathrm{Hom}_{A'}(K \otimes_A^{\mathbf{L}} A', L \otimes_A^{\mathbf{L}} A')$.

If in addition $R \to A$ is of finite presentation and L is R-perfect, then we may choose F^p to be finitely presented A-modules and consequently E^n will be finitely presented A-modules as well.

Proof. The existence of $P^\bullet$ is the definition of a pseudo-coherent complex. We first represent L by a bounded above complex $F^\bullet$ of free A-modules (this is possible because bounded tor dimension in particular implies bounded). Next, say L viewed as an object of $D(R)$ has tor amplitude in $[a, b]$. Then, after replacing $F^\bullet$ by $\tau_{\geq a} F^\bullet$, we get a complex as in (2). This follows from Lemma 15.66.2.

Proof of (a). Since $F^\bullet$ is bounded an since $P^\bullet$ is bounded above, we see that $E^n = 0$ for $n \ll 0$ and that E^n is a finite (!) direct sum

$$E^n = \bigoplus\nolimits_{p+q=n} \mathrm{Hom}_A(P^{-q}, F^p)$$

and since P^{-q} is finite free, this is indeed an R-flat A-module. The fact that $E^\bullet$ represents $R\,\mathrm{Hom}_A(K, L)$ follows from Lemma 15.73.2.

Proof of (b). Let $R \to R'$ be a ring map and $A' = A \otimes_R R'$. By Lemma 15.61.2 the object $L \otimes_A^{\mathbf{L}} A'$ is represented by $F^\bullet \otimes_R R'$ viewed as a complex of A'-modules (by flatness of F^p over R). Similarly for $P^\bullet \otimes_R R'$. As above $R\operatorname{Hom}_{A'}(K \otimes_A^{\mathbf{L}} A', L \otimes_A^{\mathbf{L}} A')$ is represented by

$$\operatorname{Hom}^\bullet(P^\bullet \otimes_R R', F^\bullet \otimes_R R') = E^\bullet \otimes_R R'$$

The equality holds by looking at the terms of the complex individually and using that $\operatorname{Hom}_{A'}(P^{-q} \otimes_R R', F^p \otimes_R R') = \operatorname{Hom}_A(P^{-q}, F^p) \otimes_R R'$. □

0DHX Lemma 15.83.7. Let $R = \operatorname{colim}_{i \in I} R_i$ be a filtered colimit of rings. Let $0 \in I$ and $R_0 \to A_0$ be a flat ring map of finite presentation. For $i \geq 0$ set $A_i = R_i \otimes_{R_0} A_0$ and set $A = R \otimes_{R_0} A_0$.

(1) Given an R-perfect K in $D(A)$ there exists an $i \in I$ and an R_i-perfect K_i in $D(A_i)$ such that $K \cong K_i \otimes_{A_i}^{\mathbf{L}} A$ in $D(A)$.
(2) Given $K_0, L_0 \in D(A_0)$ with K_0 pseudo-coherent and L_0 finite tor dimension over R_0, then we have

$$\operatorname{Hom}_{D(A)}(K_0 \otimes_{A_0}^{\mathbf{L}} A, L_0 \otimes_{A_0}^{\mathbf{L}} A) = \operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(A_i)}(K_0 \otimes_{A_0}^{\mathbf{L}} A_i, L_0 \otimes_{A_0}^{\mathbf{L}} A_i)$$

In particular, the triangulated category of R-perfect complexes over A is the colimit of the triangulated categories of R_i-perfect complexes over A_i.

Proof. By Algebra, Lemma 10.127.6 the category of finitely presented A-modules is the colimit of the categories of finitely presented A_i-modules. Given this, Algebra, Lemma 10.168.1 tells us that category of R-flat, finitely presented A-modules is the colimit of the categories of R_i-flat, finitely presented A_i-modules. Thus the characterization in Lemma 15.83.4 proves that (1) is true.

To prove (2) we choose $P_0^\bullet$ representing K_0 and $F_0^\bullet$ representing L_0 as in Lemma 15.83.6. Then $E_0^\bullet = \operatorname{Hom}^\bullet(P_0^\bullet, F_0^\bullet)$ satisfies

$$H^0(E_0^\bullet \otimes_{R_0} R_i) = \operatorname{Hom}_{D(A_i)}(K_0 \otimes_{A_0}^{\mathbf{L}} A_i, L_0 \otimes_{A_0}^{\mathbf{L}} A_i)$$

and

$$H^0(E_0^\bullet \otimes_{R_0} R) = \operatorname{Hom}_{D(A)}(K_0 \otimes_{A_0}^{\mathbf{L}} A, L_0 \otimes_{A_0}^{\mathbf{L}} A)$$

by the lemma. Thus the result because tensor product commutes with colimits and filtered colimits are exact (Algebra, Lemma 10.8.8). □

0DJG Lemma 15.83.8. Let $R' \to A'$ be a flat ring map of finite presentation. Let $R' \to R$ be a surjective ring map whose kernel is a nilpotent ideal. Set $A = A' \otimes_{R'} R$. Let $K' \in D(A')$ and set $K = K' \otimes_{A'}^{\mathbf{L}} A$ in $D(A)$. If K is R-perfect, then K' is R'-perfect.

Proof. We can represent K by a bounded above complex of finite free A-modules $E^\bullet$, see Lemma 15.64.5. By Lemma 15.75.3 we conclude that K' is pseudo-coherent because it can be represented by a bounded above complex $P^\bullet$ of finite free A'-modules with $P^\bullet \otimes_{A'} A = E^\bullet$. Observe that this also means $P^\bullet \otimes_{R'} R = E^\bullet$ (since $A = A' \otimes_{R'} R$).

Let $I = \operatorname{Ker}(R' \to R)$. Then $I^n = 0$ for some n. Choose $[a, b]$ such that K has tor amplitude in $[a, b]$ as a complex of R-modules. We will show K' has tor amplitude in $[a, b]$. To do this, let M' be an R'-module. If $IM' = 0$, then

$$K' \otimes_{R'}^{\mathbf{L}} M' = P^\bullet \otimes_{R'} M' = E^\bullet \otimes_R M' = K \otimes_R^{\mathbf{L}} M'$$

(because A' is flat over R' and A is flat over R) which has nonzero cohomology only for degrees in $[a, b]$ by choice of a, b. If $I^{t+1}M' = 0$, then we consider the short exact sequence

$$0 \to IM' \to M' \to M'/IM' \to 0$$

with $M = M'/IM'$. By induction on t we have that both $K' \otimes_{R'}^{\mathbf{L}} IM'$ and $K' \otimes_{R'}^{\mathbf{L}} M'/IM'$ have nonzero cohomology only for degrees in $[a, b]$. Then the distinguished triangle

$$K' \otimes_{R'}^{\mathbf{L}} IM' \to K' \otimes_{R'}^{\mathbf{L}} M' \to K' \otimes_{R'}^{\mathbf{L}} M'/IM' \to (K' \otimes_{R'}^{\mathbf{L}} IM')[1]$$

proves the same is true for $K' \otimes_{R'}^{\mathbf{L}} M'$. This proves the desired bound for all M' and hence the desired bound on the tor amplitude of K'. □

0DJH Lemma 15.83.9. Let R be a ring. Let $A = R[x_1, \ldots, x_d]/I$ be flat and of finite presentation over R. Let $\mathfrak{q} \subset A$ be a prime ideal lying over $\mathfrak{p} \subset R$. Let $K \in D(A)$ be pseudo-coherent. Let $a, b \in \mathbf{Z}$. If $H^i(K_\mathfrak{q} \otimes_{R_\mathfrak{p}}^{\mathbf{L}} \kappa(\mathfrak{p}))$ is nonzero only for $i \in [a, b]$, then $K_\mathfrak{q}$ has tor amplitude in $[a - d, b]$ over R.

Proof. By Lemma 15.82.8 K is pseudo-coherent as a complex of $R[x_1, \ldots, x_d]$-modules. Therefore we may assume $A = R[x_1, \ldots, x_d]$. Applying Lemma 15.77.6 to $R_\mathfrak{p} \to A_\mathfrak{q}$ and the complex $K_\mathfrak{q}$ using our assumption, we find that $K_\mathfrak{q}$ is perfect in $D(A_\mathfrak{q})$ with tor amplitude in $[a - d, b]$. Since $R_\mathfrak{p} \to A_\mathfrak{q}$ is flat, we conclude by Lemma 15.66.11. □

0GHJ Lemma 15.83.10. Let $R \to A$ be a ring map which is flat and of finite presentation. Let $K \in D(A)$ be pseudo-coherent. The following are equivalent

(1) K is R-perfect, and
(2) K is bounded below and for every prime ideal $\mathfrak{p} \subset R$ the object $K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})$ is bounded below.

Proof. Observe that (1) implies (2) as an R-perfect complex has bounded tor dimension as a complex of R-modules by definition. Let us prove the other implication.

Write $A = R[x_1, \ldots, x_d]/I$. Denote L in $D(R[x_1, \ldots, x_d])$ the restriction of K. By Lemma 15.82.8 we see that L is pseudo-coherent. Since L and K have the same image in $D(R)$ we see that L is R-perfect if and only if K is R-perfect. Also $L \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})$ and $K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p})$ are the same objects of $D(\kappa(\mathfrak{p}))$. This reduces us to the case $A = R[x_1, \ldots, x_d]$.

Say $A = R[x_1, \ldots, x_d]$ and K satisfies (2). Let $\mathfrak{q} \subset A$ be a prime lying over a prime $\mathfrak{p} \subset R$. By Lemma 15.77.6 applied to $R_\mathfrak{p} \to A_\mathfrak{q}$ and the complex $K_\mathfrak{q}$ using our assumption, we find that $K_\mathfrak{q}$ is perfect in $D(A_\mathfrak{q})$. Since K is bounded below, we see that K is perfect in $D(A)$ by Lemma 15.77.3. This implies that K is R-perfect by Lemma 15.83.3 and the proof is complete. □

15.84. Two term complexes

0G9B In this section we prove some results on two term complexes of modules which will help us understand conditions on the naive cotangent complex.

0G9C Lemma 15.84.1. Let R be a ring. Let $K \in D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. The following are equivalent

(1) $H^{-1}(K) = 0$ and $H^0(K)$ is a projective module and
(2) $\mathrm{Ext}^1_R(K, M) = 0$ for every R-module M.

If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then these are also equivalent to

(3) $\text{Ext}^1_R(K, M) = 0$ for every finite R-module M.

Proof. The equivalence of (1) and (2) follows from Lemma 15.68.2. If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then K is pseudo-coherent, see Lemma 15.64.17. Thus the equivalence of (1) and (3) follows from Lemma 15.77.4. $\square$

0G9D Remark 15.84.2. The following two statements follow from Lemma 15.84.1, Algebra, Definition 10.137.1, and Algebra, Proposition 10.138.8.

(1) A ring map $A \to B$ is smooth if and only if $A \to B$ is of finite presentation and $\text{Ext}^1_B(NL_{B/A}, N) = 0$ for every B-module N.
(2) A ring map $A \to B$ is formally smooth if and only if $\text{Ext}^1_B(NL_{B/A}, N) = 0$ for every B-module N.

0G9E Lemma 15.84.3. Let R be a ring. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \not\in \{-1, 0\}$. Then

(1) K can be represented by a two term complex $K^{-1} \to K^0$ with K^0 a free module, and
(2) if R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then K can be represented by a two term complex $K^{-1} \to K^0$ with K^0 a finite free module and K^{-1} finite.

Proof. Proof of (1). Suppose K is given by the complex of modules $M^\bullet$. We may first replace $M^\bullet$ by $\tau_{\leq 0}M^\bullet$. Thus we may assume $M^i = 0$ for $i > 0$, Next, we may choose a free resolution $P^\bullet \to M^\bullet$ with $P^i = 0$ for $i > 0$, see Derived Categories, Lemma 13.15.4. Finally, we can set $K^\bullet = \tau_{\geq -1}P^\bullet$.

Proof of (2). Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. By Lemma 15.64.5 we can choose a quasi-isomorphism $F^\bullet \to M^\bullet$ with $F^i = 0$ for $i > 0$ and F^i finite free. Then we can set $K^\bullet = \tau_{\geq -1}F^\bullet$. $\square$

Maps in the derived category out of the naive cotangent complex $NL_{B/A}$ or $NL(\alpha)$ (see Algebra, Section 10.134) are easy to understand by the result of the following lemma.

0ALN Lemma 15.84.4. Let R be a ring. Let $M^\bullet$ be a complex of modules over R with $M^i = 0$ for $i > 0$ and M^0 a projective R-module. Let $K^\bullet$ be a second complex.

(1) Assume $K^i = 0$ for $i \leq -2$. Then $\text{Hom}_{D(R)}(M^\bullet, K^\bullet) = \text{Hom}_{K(R)}(M^\bullet, K^\bullet)$.
(2) Assume $K^i = 0$ for $i \not\in [-1, 0]$ and K^0 a projective R-module. Then for a map of complexes $a^\bullet : M^\bullet \to K^\bullet$, the following are equivalent
    (a) $a^\bullet$ induces the zero map $\text{Ext}^1_R(K^\bullet, N) \to \text{Ext}^1_R(M^\bullet, N)$ for all R-modules N, and
    (b) there is a map $h^0 : M^0 \to K^{-1}$ such that $a^{-1} + h^0 \circ d_K^{-1} = 0$.
(3) Assume $K^i = 0$ for $i \leq -3$. Let $\alpha \in \text{Hom}_{D(R)}(M^\bullet, K^\bullet)$. If the composition of α with $K^\bullet \to K^{-2}[2]$ comes from an R-module map $a : M^{-2} \to K^{-2}$ with $a \circ d_M^{-3} = 0$, then α can be represented by a map of complexes $a^\bullet : M^\bullet \to K^\bullet$ with $a^{-2} = a$.

(4) In (2) for any second map of complexes $(a')^\bullet : M^\bullet \to K^\bullet$ representing α with $a = (a')^{-2}$ there exist $h^i : M^i \to K^{i-1}$ for $i = 0, -1$ such that

$$h^{-1} \circ d_M^{-2} = 0, \quad (a')^{-1} = a^{-1} + d_K^{-2} \circ h^{-1} + h^0 \circ d_M^{-1}, \quad (a')^0 = a^0 + d_K^{-1} \circ h^0$$

Proof. Set $F^0 = M^0$. Choose a free R-module F^{-1} and a surjection $F^{-1} \to M^{-1}$. Choose a free R-module F^{-2} and a surjection $F^{-2} \to M^{-2} \times_{M^{-1}} F^{-1}$. Continuing in this way we obtain a quasi-isomorphism $p^\bullet : F^\bullet \to M^\bullet$ which is termwise surjective and with F^i projective for all i.

Proof of (1). By Derived Categories, Lemma 13.19.8 we have

$$\mathrm{Hom}_{D(R)}(M^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(F^\bullet, K^\bullet)$$

If $K^i = 0$ for $i \leq -2$, then any morphism of complexes $F^\bullet \to K^\bullet$ factors through $p^\bullet$. Similarly, any homotopy $\{h^i : F^i \to K^{i-1}\}$ factors through $p^\bullet$. Thus (1) holds.

Proof of (2). If (2)(b) holds, then $a^\bullet$ is homotopic to a map of complexes $(a')^\bullet : M^\bullet \to K^\bullet$ which is zero in degree -1. On the other hand, let $N \to I^\bullet$ be an injective resolution. We have

$$\mathrm{Ext}^1_R(K^\bullet, N) = \mathrm{Hom}_{D(R)}(K^\bullet, I^\bullet[1]) = \mathrm{Hom}_{K(R)}(K^\bullet, I^\bullet[1])$$

by Derived Categories, Lemma 13.18.8. Let $b^\bullet : K^\bullet \to I^\bullet[1]$ be a map of complexes. Since $K^1 = 0$ the map $b^0 : K^0 \to I^1$ maps into the kernel of $I^1 \to I^2$ which is the image of $I^0 \to I^1$. Since K^0 is projective we can lift b^0 to a map $h : K^0 \to I^0$. Thus we see that $b^\bullet$ is homotopic to a map of complexes $(b')^\bullet$ with $(b')^0 = 0$. Since $K^i = 0$ for $i \not\in [-1, 0]$ it follows that $(b')^\bullet \circ (a')^\bullet = 0$ as a map of complexes. Hence the map $\mathrm{Ext}^1_R(K^\bullet, N) \to \mathrm{Ext}^1_R(M^\bullet, N)$ is zero. In this way we see that (2)(b) implies (2)(a). Conversely, assume (2)(a). We see that the canonical element in $\mathrm{Ext}^1_R(K^\bullet, K^{-1})$ maps to zero in $\mathrm{Ext}^1_R(M^\bullet, K^{-1})$. Using (1) we see immediately that we get a map h^0 as in (2)(b).

Proof of (3). Choose $b^\bullet : F^\bullet \to K^\bullet$ representing α. The composition of α with $K^\bullet \to K^{-2}[2]$ is represented by $b^{-2} : F^{-2} \to K^{-2}$. As this is homotopic to $a \circ p^{-2} : F^{-2} \to M^{-2} \to K^{-2}$, there is a map $h : F^{-1} \to K^{-2}$ such that $b^{-2} = a \circ p^{-2} + h \circ d_F^{-2}$. Adjusting $b^\bullet$ by h viewed as a homotopy from $F^\bullet$ to $K^\bullet$, we find that $b^{-2} = a \circ p^{-2}$. Hence b^{-2} factors through p^{-2}. Since $F^0 = M^0$ the kernel of p^{-2} surjects onto the kernel of p^{-1} (for example because the kernel of $p^\bullet$ is an acyclic complex or by a diagram chase). Hence b^{-1} necessarily factors through p^{-1} as well and we see that (3) holds for these factorizations and $a^0 = b^0$.

Proof of (4) is omitted. Hint: There is a homotopy between $a^\bullet \circ p^\bullet$ and $(a')^\bullet \circ p^\bullet$ and we argue as before that this homotopy factors through $p^\bullet$. $\square$

Let $A \to B$ be a finitely presented ring map. Given an ideal $I \subset B$ we can consider the condition

(*) $\mathrm{Ext}^1_B(NL_{B/A}, N)$ is annihilated by I for all B-modules N.

This condition is one possible precise mathematical formulation of the notion "the singular locus of $A \to B$ is scheme theoretically contained in $V(I)$". Please compare with Remark 15.84.2 and the following lemmas.

0G9F Lemma 15.84.5. Let R be a ring and let $I \subset R$ be an ideal. Let $K \in D(R)$. Assume $H^i(K) = 0$ for $i \not\in \{-1, 0\}$. The following are equivalent

(1) $\mathrm{Ext}^1_R(K, N)$ is annihilated by I for all R-modules N,

(2) K can be represented by a complex $K^{-1} \to K^0$ with K^0 free such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$,
(3) whenever K is represented by a two term complex $K^{-1} \to K^0$ with K^0 projective, then for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$.

If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then these are also equivalent to

(4) $\mathrm{Ext}^1_R(K, N)$ is annihilated by I for every finite R-module N,
(5) K can be represented by a complex $K^{-1} \to K^0$ with K^0 finite free and K^{-1} finite such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ factors through $d_K^{-1} : K^{-1} \to K^0$.

Proof. Assume (1) and let $K^{-1} \to K^0$ be a two term complex representing K with K^0 projective. We will use the description of maps in $D(R)$ out of $K^\bullet$ given in Lemma 15.84.4 without further mention. Choosing $N = K^{-1}$ consider the element ξ of $\mathrm{Ext}^1_R(K, N)$ given by $\mathrm{id}_{K^{-1}} : K^{-1} \to K^{-1}$. Since is annihilated by $a \in I$ we see that we get the dotted arrow fitting into the following commutative diagram

$$\begin{tikzcd} K^{-1} \arrow[r, "d_K^{-1}"] \arrow[d, "a"] & K^0 \arrow[dl, dotted, "h"] \\ K^{-1} & \end{tikzcd}$$

This proves that (3) holds. Part (3) implies (2) in view of Lemma 15.84.3 part (1). Assume $K^\bullet$ is as in (2) and N is an arbitrary R-module. Any element ξ of $\mathrm{Ext}^1_R(K, N)$ is given as the class of a map $\varphi : K^{-1} \to N$. Then for $a \in I$ by assumption we may choose a map h as in the diagram above and we see that $a\varphi = \varphi \circ a = \varphi \circ h \circ \mathrm{d}_K^{-1}$ which proves that $a\xi$ is zero in $\mathrm{Ext}^1_R(K, N)$. Thus (1), (2), and (3) are equivalent.

Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. Part (3) implies (5) in view of Lemma 15.84.3 part (2). It is clear that (5) implies (2). Trivially (1) implies (4). Thus to finish the proof it suffices to show that (4) implies any of the other conditions. Let $K^{-1} \to K^0$ be a complex representing K with K^0 finite free and K^{-1} finite as in Lemma 15.84.3 part (2). The argument given in the proof of (2) $\Rightarrow$ (1) shows that if $\mathrm{Ext}^1_R(K, K^{-1})$ is annihilated by I, then (1) holds. In this way we see that (4) implies (1) and the proof is complete. ☐

0G9G Lemma 15.84.6. Let R be a ring. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. Let $K^{-1} \to K^0$ be a two term complex of R-modules representing K such that K^0 is a flat R-module (for example projective or free). Let $R \to R'$ be a ring map. Then the complex $K^\bullet \otimes_R R'$ represents $\tau_{\geq -1}(K \otimes_R^{\mathbf{L}} R')$.

Proof. We have a distinguished triangle

$$K^0 \to K^\bullet \to K^{-1}[1] \to K^0[1]$$

in $D(R)$. This determines a map of distinguished triangles

$$\begin{tikzcd} K^0 \otimes_R^{\mathbf{L}} R' \arrow[r] \arrow[d] & K^\bullet \otimes_R^{\mathbf{L}} R' \arrow[r] \arrow[d] & K^{-1} \otimes_R^{\mathbf{L}} R'[1] \arrow[r] \arrow[d] & K^0 \otimes_R^{\mathbf{L}} R'[1] \arrow[d] \\ K^0 \otimes_R R' \arrow[r] & K^\bullet \otimes_R R' \arrow[r] & K^{-1} \otimes_R R'[1] \arrow[r] & K^0 \otimes_R R'[1] \end{tikzcd}$$

The left and right vertical arrows are isomorphisms as K^0 is flat. Since $K^{-1} \otimes_R^{\mathbf{L}} R' \to K^{-1} \otimes_R R'$ is an isomorphism on cohomology in degree 0 we conclude. □

0G9H Lemma 15.84.7. Let I be an ideal of a ring R. Let K be an object of $D(R)$ with $H^i(K) = 0$ for $i \notin \{-1, 0\}$. Let $R \to R'$ be a ring map. If K satisfies the equivalent conditions (1), (2), and (3) of Lemma 15.84.5 with respect to (R, I), then $\tau_{\geq -1}(K \otimes_R^{\mathbf{L}} R')$ satisfies the equivalent conditions (1), (2), and (3) of Lemma 15.84.5 with respect to (R', IR')

Proof. We may assume K is represented by a two term complex $K^{-1} \to K^0$ with K^0 free such that for any $a \in I$ the map $a : K^{-1} \to K^{-1}$ is equal to $h_a \circ d_K^{-1}$ for some map $h_a : K^0 \to K^{-1}$. By Lemma 15.84.6 we see that $\tau_{\geq -1}(K \otimes_R^{\mathbf{L}} R')$ is represented by $K^\bullet \otimes_R R'$. Then of course for every $a \in I$ we see that $a \otimes 1 : K^{-1} \otimes_R R' \to K^{-1} \otimes_R R'$ is equal to $(h_a \otimes 1) \circ (d_K^{-1} \otimes 1)$. Since the collection of maps $K^{-1} \otimes_R R' \to K^{-1} \otimes_R R'$ which factor through $\mathrm{d}_K^{-1} \otimes 1$ forms an R'-module we conclude. □

0G9I Lemma 15.84.8. Let R be a ring. Let $\alpha : K \to K'$ be a morphism of $D(R)$. Assume

(1) $H^i(K) = H^i(K') = 0$ for $i \notin \{-1, 0\}$
(2) $H^0(\alpha)$ is an isomorphism and $H^{-1}(\alpha)$ is surjective.

For any $f \in R$ if $f : K \to K$ is 0, then $f : K' \to K'$ is 0.

Proof. Set $M = \mathrm{Ker}(H^{-1}(\alpha))$. Then α fits into a distinguished triangle

$$M[1] \to K \to K' \to M[2]$$

Since $K \to K' \xrightarrow{f} K'$ is zero by our assumption, we see that $f : K' \to K'$ factors over a map $M[2] \to K'$. However $\mathrm{Hom}(M[2], K') = 0$ for example by Derived Categories, Lemma 13.27.3. □

0G9J Lemma 15.84.9. Let I be an ideal of a ring R. Let $\alpha : K \to K'$ be a morphism of $D(R)$. Assume

(1) $H^i(K) = H^i(K') = 0$ for $i \notin \{-1, 0\}$
(2) $H^0(\alpha)$ is an isomorphism and $H^{-1}(\alpha)$ is surjective.

If K satisfies the equivalent conditions (1), (2), and (3) of Lemma 15.84.5, then K' does too.

Proof. Set $M = \mathrm{Ker}(H^{-1}(\alpha))$. Then α fits into a distinguished trangle

$$M[1] \to K \to K' \to M[2]$$

For any R-module N this determines an exact sequence

$$\mathrm{Ext}^0_R(M[1], N) \to \mathrm{Ext}^1_R(K', N) \to \mathrm{Ext}^1_R(K, N)$$

Since $\mathrm{Ext}^0_R(M[1], N) = \mathrm{Ext}^{-1}_R(M, N) = 0$ we see that $\mathrm{Ext}^1_R(K', N)$ is a submodule of $\mathrm{Ext}^1_R(K, N)$. Hence if $\mathrm{Ext}^1_R(K, N)$ is annihilated by I so is $\mathrm{Ext}^1_R(K', N)$. □

0G9K Lemma 15.84.10. Let R be ring and let $I \subset R$ be an ideal. Let $K \in D(R)$ with $H^i(K) = 0$ for $i \not\in \{-1, 0\}$. The following are equivalent

(1) there exists a $c \geq 0$ such that the equivalent conditions (1), (2), (3) of Lemma 15.84.5 hold for K and the ideal I^c,
(2) there exists a $c \geq 0$ such that (a) I^c annihilates $H^{-1}(K)$ and (b) $H^0(K)$ is an I^c-projective module (see Section 15.70).

If R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$, then these are also equivalent to

(3) there exists a $c \geq 0$ such that the equivalent conditions (4), (5) of Lemma 15.84.5 hold for K and the ideal I^c,
(4) $H^{-1}(K)$ is I-power torsion and there exist $f_1, \ldots, f_s \in R$ with $V(f_1, \ldots, f_s) \subset V(I)$ such that the localizations $H^0(K)_{f_i}$ are projective R_{f_i}-modules,
(5) $H^{-1}(K)$ is I-power torsion and there exist $f_1, \ldots, f_s \in I$ with $V(f_1, \ldots, f_s) = V(I)$ such that the localizations $H^0(K)_{f_i}$ are projective R_{f_i}-modules.

Proof. The distinguished triangle $H^{-1}(K)[1] \to K \to H^0(K)[0] \to H^{-1}(K)[2]$ determines an exact sequence

$$0 \to \text{Ext}^1_R(H^0(K), N) \to \text{Ext}^1_R(K, N) \to \text{Hom}_R(H^{-1}(K), N) \to \text{Ext}^2_R(H^0(K), N)$$

Thus (2) implies that I^{2c} annihilates $\text{Ext}^1_R(K, N)$ for every R-module N. Assuming (1) we immediately see that $H^0(K)$ is I^c-projective. On the other hand, we may choose an injective map $H^{-1}(K) \to N$ for some injective R-module N. Then this map is the image of an element of $\text{Ext}^1_R(K, N)$ by the vanishing of the Ext^2 in the sequence and we conclude $H^{-1}(K)$ is annihilated by I^c.

Assume R is Noetherian and $H^i(K)$ is a finite R-module for $i = -1, 0$. By Lemma 15.84.5 we see that (3) is equivalent to (1) and (2). Also, if (3) holds then for $f \in I$ the multiplication by f on $H^0(K)$ factors through a projective module, which implies that $H^0(K)_f$ is a summand of a projective R_f-module and hence itself a projective R_f-module. Choosing $f_1, \ldots, f_s$ to be generators of I we find the equivalent conditions (1), (2), and (3) imply (5). Of course (5) trivially implies (4).

Assume (4). Since $H^{-1}(K)$ is a finite R-module and I-power torsion we see that I^{c_1} annihilates $H^{-1}(K)$ for some $c_1 \geq 0$. Choose a short exact sequence

$$0 \to M \to R^{\oplus r} \to H^0(K) \to 0$$

which determines an element $\xi \in \text{Ext}^1_R(H^0(K), M)$. For any $f \in I$ we have $\text{Ext}^1_R(H^0(K), M)_f = \text{Ext}^1_{R_f}(H^0(K)_f, M_f)$ by Lemma 15.65.4. Hence if $H^0(K)_f$ is projective, then a power of f annihilates ξ. We conclude that ξ is annihilated by $(f_1, \ldots, f_s)^{c_2}$ for some $c_2 \geq 0$. Since $V(f_1, \ldots, f_s) \subset V(I)$ we have $\sqrt{I} \subset (f_1, \ldots, f_s)$ (Algebra, Lemma 10.17.2). Since R is Noetherian we find $I^{c_3} \subset (f_1, \ldots, f_s)$ for some $c_3 \geq 0$ (Algebra, Lemma 10.32.5). Hence I^{c2c3} annihilates ξ. This in turn says that $H^0(K)$ is $I^{c_2c_3}$-projective (as multiplication by $a \in I$ which annihilate ξ factor through $R^{\oplus r}$). Hence taking $c = \max(c_1, c_2c_3)$ we see that (2) holds. $\square$

0AJT Lemma 15.84.11. Let R be a ring. Let $K_j \in D(R)$, $j = 1, 2, 3$ with $H^i(K_j) = 0$ for $i \not\in \{-1, 0\}$. Let $\varphi : K_1 \to K_2$ and $\psi : K_2 \to K_3$ be maps in $D(R)$. If $H^0(\varphi) = 0$ and $H^{-1}(\psi) = 0$, then $\varphi \circ \psi = 0$.

Proof. Apply Derived Categories, Lemma 13.12.5 to see that $\varphi \circ \psi$ factors through $\tau_{\leq -2} K_2 = 0$. $\square$

0G9L Lemma 15.84.12. Let R be a ring. Let $K \in D(R)$ be given by a two term complex of the form $R^{\oplus n} \to R^{\oplus n}$. Denote $A \in \text{Mat}(n \times n, R)$ the matrix of the differential. Then $\det(a) : K \to K$ is zero in $D(R)$.

Proof. Omitted. Good exercise. $\square$

15.85. The naive cotangent complex

0FUX In this section we continue the discussion started in Algebra, Section 10.134. We begin with a discussion of base change. The first lemma shows that taking the naive tensor product of the naive cotangent complex with a ring extension isn't quite as naive as one might think.

0FUY Lemma 15.85.1. Let $R \to S$ and $S \to S'$ be ring maps. The canonical map $NL_{S/R} \otimes_S^{\mathbf{L}} S' \to NL_{S/R} \otimes_S S'$ induces an isomorphism $\tau_{\geq -1}(NL_{S/R} \otimes_S^{\mathbf{L}} S') \to NL_{S/R} \otimes_S S'$ in $D(S')$. Similarly, given a presentation α of S over R the canonical map $NL(\alpha) \otimes_S^{\mathbf{L}} S' \to NL(\alpha) \otimes_S S'$ induces an isomorphism $\tau_{\geq -1}(NL(\alpha) \otimes_S^{\mathbf{L}} S') \to NL(\alpha) \otimes_S S'$ in $D(S')$.

Proof. Special case of Lemma 15.84.6. $\square$

0FUZ Lemma 15.85.2. Let $R \to S$ and $R \to R'$ be ring maps. Let $\alpha : P \to S$ be a presentation of S over R. Then $\alpha' : P \otimes_R R' \to S \otimes_R R'$ is a presentation of $S' = S \otimes_R R'$ over R'. The canonical map

$$NL(\alpha) \otimes_S S' \to NL(\alpha')$$

is an isomorphism on H^0 and surjective on H^{-1}. In particular, the canonical map

$$NL_{S/R} \otimes_S S' \to NL_{S'/R'}$$

is an isomorphism on H^0 and surjective on H^{-1}.

Proof. Denote $I = \text{Ker}(P \to S)$. Denote $P' = P \otimes_R R'$ and $I' = \text{Ker}(P' \to S')$. Suppose P is a polynomial algebra on x_j for $j \in J$. The map displayed in the lemma becomes

$$\begin{tikzcd} \bigoplus_{j \in J} S' \text{d}x_j \arrow[r] & \bigoplus_{j \in J} S' \text{d}x_j \\ I/I^2 \otimes_S S' \arrow[r] \arrow[u] & I'/(I')^2 \arrow[u] \end{tikzcd}$$

where the left column is $NL(\alpha) \otimes_S S'$ and the right column is $NL(\alpha')$. By right exactness of tensor product we see that $I \otimes_R R' \to I'$ is surjective. Hence the bottom arrow is a surjection. This proves the first statement of the lemma. The statement for $NL_{S/R} \otimes_S S' \to NL_{S'/R'}$ follows as these complexes are homotopic to $NL(\alpha) \otimes_S S'$ and $NL(\alpha')$. $\square$

0FJU Lemma 15.85.3. Consider a cocartesian diagram of rings

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[r] \arrow[u] & A' \arrow[u] \end{tikzcd}$$

If B is flat over A, then the canonical map $NL_{B/A} \otimes_B B' \to NL_{B'/A'}$ is a quasi-isomorphism. If in addition $NL_{B/A}$ has tor-amplitude in $[-1, 0]$ then $NL_{B/A} \otimes_B^{\mathbf{L}} B' \to NL_{B'/A'}$ is a quasi-isomorphism too.

Proof. Choose a presentation $\alpha : P \to B$ as in Algebra, Section 10.134. Let $I = \mathrm{Ker}(\alpha)$. Set $P' = P \otimes_A A'$ and denote $\alpha' : P' \to B'$ the corresponding presentation of B' over A'. As B is flat over A we see that $I' = \mathrm{Ker}(\alpha')$ is equal to $I \otimes_A A'$. Hence

$$I'/(I')^2 = \mathrm{Coker}(I^2 \otimes_A A' \to I \otimes_A A') = I/I^2 \otimes_A A' = I/I^2 \otimes_B B'$$

We have $\Omega_{P'/A'} = \Omega_{P/A} \otimes_A A'$ because both sides have the same basis. It follows that $\Omega_{P'/A'} \otimes_{P'} B' = \Omega_{P/A} \otimes_P B \otimes_B B'$. This proves that $NL(\alpha) \otimes_B B' \to NL(\alpha')$ is an isomorphism of complexes and hence the first statement holds.

We have

$$NL(\alpha) = I/I^2 \longrightarrow \Omega_{P/A} \otimes_P B$$

as a complex of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see Lemma 15.66.2. If this holds, then $NL(\alpha) \otimes_B^{\mathbf{L}} B' = NL(\alpha) \otimes_B B'$ and we get the second statement. $\square$

0FV0 Lemma 15.85.4. Let $A \to B$ be a local complete intersection as in Definition 15.33.2. Then $NL_{B/A}$ is a perfect object of $D(B)$ with tor amplitude in $[-1, 0]$.

Proof. Write $B = A[x_1, \ldots, x_n]/I$. Then $NL_{B/A}$ is represented by the complex

$$I/I^2 \longrightarrow \bigoplus B \mathrm{d}x_i$$

of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is finite free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see Lemma 15.66.2. By definition I is a Koszul regular ideal and hence a quasi-regular ideal, see Section 15.32. Thus I/I^2 is a finite projective B-module (Lemma 15.32.3) and we conclude both that $NL_{B/A}$ is perfect and that it has tor amplitude in $[-1, 0]$. $\square$

0FV1 Lemma 15.85.5. Consider a cocartesian diagram of rings

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

If $A \to B$ and $A' \to B'$ are local complete intersections as in Definition 15.33.2, then the kernel of $H^{-1}(NL_{B/A} \otimes_B B') \to H^{-1}(NL_{B'/A'})$ is a finite projective B'-module.

Proof. By Lemma 15.85.4 the complexes $NL_{B/A}$ and $NL_{B'/A'}$ are perfect of tor-amplitude in $[-1, 0]$. Combining Lemmas 15.85.1, 15.74.9, and 15.66.13 we have $NL_{B/A} \otimes_B B' = NL_{B/A} \otimes_B^{\mathbf{L}} B'$ and this complex is also perfect of tor-amplitude in $[-1, 0]$. Choose a distinguished triangle

$$C \to NL_{B/A} \otimes_B B' \to NL_{B'/A'} \to C[1]$$

in $D(B')$. By Lemmas 15.74.4 and 15.66.5 we conclude that C is perfect with tor-amplitude in $[-1, 1]$. By Lemma 15.85.2 the complex C has only one nonzero cohomology module, namely the module of the lemma sitting in degree -1. This module is of finite presentation (Lemma 15.64.4) and flat (Lemma 15.66.6). Hence it is finite projective by Algebra, Lemma 10.78.2. □

15.86. Rlim of abelian groups

07KV We briefly discuss $R\lim$ on abelian groups. In this section we will denote $\text{Ab}(\mathbf{N})$ the abelian category of inverse systems of abelian groups. The notation is compatible with the notation for sheaves of abelian groups on a site, as an inverse system of abelian groups is the same thing as a sheaf of groups on the category $\mathbf{N}$ (with a unique morphism $i \to j$ if $i \leq j$), see Remark 15.86.6. Many of the arguments in this section duplicate the arguments used to construct the cohomological machinery for sheaves of abelian groups on sites.

07KW **Lemma 15.86.1.** The functor $\lim : \text{Ab}(\mathbf{N}) \to \text{Ab}$ has a right derived functor

08U4 (15.86.1.1)
$$R\lim : D(\text{Ab}(\mathbf{N})) \longrightarrow D(\text{Ab})$$

As usual we set $R^p\lim(K) = H^p(R\lim(K))$. Moreover, we have

(1) for any (A_n) in $\text{Ab}(\mathbf{N})$ we have $R^p \lim A_n = 0$ for $p > 1$,
(2) the object $R\lim A_n$ of $D(\text{Ab})$ is represented by the complex
$$\prod A_n \to \prod A_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$
sitting in degrees 0 and 1,
(3) if (A_n) is ML, then $R^1 \lim A_n = 0$, i.e., (A_n) is right acyclic for lim,
(4) every $K^\bullet \in D(\text{Ab}(\mathbf{N}))$ is quasi-isomorphic to a complex whose terms are right acyclic for lim, and
(5) if each $K^p = (K_n^p)$ is right acyclic for lim, i.e., of $R^1 \lim_n K_n^p = 0$, then $R\lim K$ is represented by the complex whose term in degree p is $\lim_n K_n^p$.

Proof. Let (A_n) be an arbitrary inverse system. Let (B_n) be the inverse system with
$$B_n = A_n \oplus A_{n-1} \oplus \ldots \oplus A_1$$
and transition maps given by projections. Let $A_n \to B_n$ be given by $(1, f_n, f_{n-1} \circ f_n, \ldots, f_2 \circ \ldots \circ f_n$ where $f_i : A_i \to A_{i-1}$ are the transition maps. In this way we see that every inverse system is a subobject of a ML system (Homology, Section 12.31). It follows from Derived Categories, Lemma 13.15.6 using Homology, Lemma 12.31.3 that every ML system is right acyclic for lim, i.e., (3) holds. This already implies that RF is defined on $D^+(\text{Ab}(\mathbf{N}))$, see Derived Categories, Proposition 13.16.8. Set $C_n = A_{n-1} \oplus \ldots \oplus A_1$ for $n > 1$ and $C_1 = 0$ with transition maps given by projections as well. Then there is a short exact sequence of inverse systems $0 \to (A_n) \to (B_n) \to (C_n) \to 0$ where $B_n \to C_n$ is given by $(x_i) \mapsto (x_i - f_{i+1}(x_{i+1}))$. Since (C_n) is ML as well, we conclude that (2) holds (by proposition reference above) which also implies (1). Finally, this implies by Derived Categories, Lemma 13.32.2 that $R\lim$ is in fact defined on all of $D(\text{Ab}(\mathbf{N}))$. In fact, the proof of Derived Categories, Lemma 13.32.2 proceeds by proving assertions (4) and (5). □

0H31 **Lemma 15.86.2.** Let
$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$
be a short exact sequence of inverse systems of abelian groups. Then there is an associated 6 term exact sequence $0 \to \lim A_i \to \lim B_i \to \lim C_i \to R^1 \lim A_i \to R^1 \lim B_i \to R^1 \lim C_i \to 0$.

Proof. Follows from the vanishing in Lemma 15.86.1. $\square$

Here is the "correct" formulation of Homology, Lemma 12.31.7.

0918 Lemma 15.86.3. Let
$$(A_n^{-2} \to A_n^{-1} \to A_n^0 \to A_n^1)$$
be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ its limit. Denote (H_n^{-1}), (H_n^0) the inverse systems of cohomologies, and denote H^{-1}, H^0 the cohomologies of $A^{-2} \to A^{-1} \to A^0 \to A^1$. If

(1) (A_n^{-2}) and (A_n^{-1}) have vanishing $R^1\lim$,
(2) (H_n^{-1}) has vanishing $R^1\lim$,

then $H^0 = \lim H_n^0$.

Proof. Let $K \in D(\text{Ab}(\mathbf{N}))$ be the object represented by the system of complexes whose nth constituent is the complex $A_n^{-2} \to A_n^{-1} \to A_n^0 \to A_n^1$. We will compute $H^0(R\lim K)$ using both spectral sequences[10] of Derived Categories, Lemma 13.21.3. The first has E_1-page
$$\begin{matrix} 0 & 0 & R^1\lim A_n^0 & R^1\lim A_n^1 \\ A^{-2} & A^{-1} & A^0 & A^1 \end{matrix}$$
with horizontal differentials and all higher differentials are zero. The second has E_2 page
$$\begin{matrix} R^1\lim H_n^{-2} & 0 & R^1\lim H_n^0 & R^1\lim H_n^1 \\ \lim H_n^{-2} & \lim H_n^{-1} & \lim H_n^0 & \lim H_n^1 \end{matrix}$$
and degenerates at this point. The result follows. $\square$

0919 Lemma 15.86.4. Let $\mathcal{D}$ be a triangulated category. Let (K_n) be an inverse system of objects of $\mathcal{D}$. Let K be a derived limit of the system (K_n). Then for every L in $\mathcal{D}$ we have a short exact sequence
$$0 \to R^1\lim \text{Hom}_\mathcal{D}(L, K_n[-1]) \to \text{Hom}_\mathcal{D}(L, K) \to \lim \text{Hom}_\mathcal{D}(L, K_n) \to 0$$

Proof. This follows from Derived Categories, Definition 13.34.1 and Lemma 13.4.2, and the description of lim and $R^1\lim$ in Lemma 15.86.1 above. $\square$

0CQX Lemma 15.86.5. Let $\mathcal{D}$ be a triangulated category. Let (K_n) be a system of objects of $\mathcal{D}$. Let K be a derived colimit of the system (K_n). Then for every L in $\mathcal{D}$ we have a short exact sequence
$$0 \to R^1\lim \text{Hom}_\mathcal{D}(K_n, L[-1]) \to \text{Hom}_\mathcal{D}(K, L) \to \lim \text{Hom}_\mathcal{D}(K_n, L) \to 0$$

Proof. This follows from Derived Categories, Definition 13.33.1 and Lemma 13.4.2, and the description of lim and $R^1\lim$ in Lemma 15.86.1 above. $\square$

091A Remark 15.86.6 (Rlim as cohomology). Consider the category $\mathbf{N}$ whose objects are natural numbers and whose morphisms are unique arrows $i \to j$ if $j \geq i$. Endow $\mathbf{N}$ with the chaotic topology (Sites, Example 7.6.6) so that a sheaf $\mathcal{F}$ is the same thing as an inverse system
$$\mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \mathcal{F}_3 \leftarrow \ldots$$

[10] To use these spectral sequences we have to show that $\text{Ab}(\mathbf{N})$ has enough injectives. A inverse system (I_n) of abelian groups is injective if and only if each I_n is an injective abelian group and the transition maps are split surjections. Every system embeds in one of these. Details omitted.

of sets over $\mathbf{N}$. Note that $\Gamma(\mathbf{N}, \mathcal{F}) = \lim \mathcal{F}_n$. For an inverse system of abelian groups $\mathcal{F}_n$ we have

$$R^p \lim \mathcal{F}_n = H^p(\mathbf{N}, \mathcal{F})$$

because both sides are the higher right derived functors of $\mathcal{F} \mapsto \lim \mathcal{F}_n = H^0(\mathbf{N}, \mathcal{F})$. Thus the existence of $R\lim$ also follows from the general material in Cohomology on Sites, Sections 21.2 and 21.19.

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category $D(\text{Ab})$, see Derived Categories, Lemma 13.34.2.

07KX Lemma 15.86.7. Let $K = (K_n^\bullet)$ be an object of $D(\text{Ab}(\mathbf{N}))$. There exists a canonical distinguished triangle

$$R\lim K \to \prod\nolimits_n K_n^\bullet \to \prod\nolimits_n K_n^\bullet \to R\lim K[1]$$

in $D(\text{Ab})$. In other words, $R\lim K$ is a derived limit of the inverse system $(K_n^\bullet)$ of $D(\text{Ab})$, see Derived Categories, Definition 13.34.1.

Proof. Suppose that for each p the inverse system (K_n^p) is right acyclic for lim. By Lemma 15.86.1 this gives a short exact sequence

$$0 \to \lim\nolimits_n K_n^p \to \prod\nolimits_n K_n^p \to \prod\nolimits_n K_n^p \to 0$$

for each p. Since the complex consisting of $\lim_n K_n^p$ computes $R\lim K$ by Lemma 15.86.1 we see that the lemma holds in this case.

Next, assume $K = (K_n^\bullet)$ is general. By Lemma 15.86.1 there is a quasi-isomorphism $K \to L$ in $D(\text{Ab}(\mathbf{N}))$ such that (L_n^p) is acyclic for each p. Then $\prod K_n^\bullet$ is quasi-isomorphic to $\prod L_n^\bullet$ as products are exact in Ab, whence the result for L (proved above) implies the result for K. $\square$

07KY Lemma 15.86.8. With notation as in Lemma 15.86.7 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences

$$0 \to R^1 \lim\nolimits_n H^{p-1}(K_n^\bullet) \to H^p(R\lim K) \to \lim\nolimits_n H^p(K_n^\bullet) \to 0$$

Proof. The long exact sequence of the distinguished triangle is

$$\ldots \to H^p(R\lim K) \to \prod\nolimits_n H^p(K_n^\bullet) \to \prod\nolimits_n H^p(K_n^\bullet) \to H^{p+1}(R\lim K) \to \ldots$$

The map in the middle has kernel $\lim_n H^p(K_n^\bullet)$ by its explicit description given in the lemma. The cokernel of this map is $R^1 \lim_n H^p(K_n^\bullet)$ by Lemma 15.86.1. $\square$

Warning. An object of $D(\text{Ab}(\mathbf{N}))$ is a complex of inverse systems of abelian groups. You can also think of this as an inverse system $(K_n^\bullet)$ of complexes. However, this is not the same thing as an inverse system of objects of $D(\text{Ab})$; the following lemma and remark explain the difference.

0CQ9 Lemma 15.86.9. Let (K_n) be an inverse system of objects of $D(\text{Ab})$. Then there exists an object $M = (M_n^\bullet)$ of $D(\text{Ab}(\mathbf{N}))$ and isomorphisms $M_n^\bullet \to K_n$ in $D(\text{Ab})$ such that the diagrams

$$\begin{tikzcd} M_{n+1}^\bullet \arrow[r] \arrow[d] & M_n^\bullet \arrow[d] \\ K_{n+1} \arrow[r] & K_n \end{tikzcd}$$

commute in $D(\text{Ab})$.

Proof. Namely, let $M_1^\bullet$ be a complex of abelian groups representing K_1. Suppose we have constructed $M_e^\bullet \to M_{e-1}^\bullet \to \ldots \to M_1^\bullet$ and maps $\psi_i : M_i^\bullet \to K_i$ such that the diagrams in the statement of the lemma commute for all $n < e$. Then we consider the diagram

$$\begin{tikzcd} & M_n^\bullet \arrow[d, "\psi_n"] \\ K_{n+1} \arrow[r] & K_n \end{tikzcd}$$

in $D(\text{Ab})$. By the definition of morphisms in $D(\text{Ab})$ we can find a complex $M_{n+1}^\bullet$ of abelian groups, an isomorphism $M_{n+1}^\bullet \to K_{n+1}$ in $D(\text{Ab})$, and a morphism of complexes $M_{n+1}^\bullet \to M_n^\bullet$ representing the composition

$$K_{n+1} \to K_n \xrightarrow{\psi_n^{-1}} M_n^\bullet$$

in $D(\text{Ab})$. Thus the lemma holds by induction. □

08U5 Remark 15.86.10. Let (K_n) be an inverse system of objects of $D(\text{Ab})$. Let $K = R\lim K_n$ be a derived limit of this system (see Derived Categories, Section 13.34). Such a derived limit exists because $D(\text{Ab})$ has countable products (Derived Categories, Lemma 13.34.2). By Lemma 15.86.9 we can also lift (K_n) to an object M of $D(\mathbf{N})$. Then $K \cong R\lim M$ where $R\lim$ is the functor (15.86.1.1) because $R\lim M$ is also a derived limit of the system (K_n) by Lemma 15.86.7. Thus, although there may be many isomorphism classes of lifts M of the system (K_n), the isomorphism type of $R\lim M$ is independent of the choice because it is isomorphic to the derived limit $K = R\lim K_n$ of the system. Thus we may apply results on $R\lim$ proved in this section to derived limits. For example, for every $p \in \mathbf{Z}$ there is a canonical short exact sequence

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(K) \to \lim H^p(K_n) \to 0$$

because we may apply Lemma 15.86.7 to M. This can also been seen directly, without invoking the existence of M, by applying the argument of the proof of Lemma 15.86.7 to the (defining) distinguished triangle $K \to \prod K_n \to \prod K_n \to K[1]$.

091B Lemma 15.86.11. Let $E \to D$ be a morphism of $D(\text{Ab}(\mathbf{N}))$. Let (E_n), resp. (D_n) be the system of objects of $D(\text{Ab})$ associated to E, resp. D. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then $R\lim E \to R\lim D$ is an isomorphism in $D(\text{Ab})$.

Proof. The assumption in particular implies that the pro-objects $H^p(E_n)$ and $H^p(D_n)$ are isomorphic. By the short exact sequences of Lemma 15.86.8 it suffices to show that given a map $(A_n) \to (B_n)$ of inverse systems of abelian groupsc which induces an isomorphism of pro-objects, then $\lim A_n \cong \lim B_n$ and $R^1 \lim A_n \cong R^1 \lim B_n$.

The assumption implies there are $1 \leq m_1 < m_2 < m_3 < \ldots$ and maps $\varphi_n : B_{m_n} \to A_n$ such that $(\varphi_n) : (B_{m_n}) \to (A_n)$ is a map of systems which is inverse to the given map $\psi = (\psi_n) : (A_n) \to (B_n)$ as a morphism of pro-objects. What this means is that (after possibly replacing m_n by larger integers) we may assume that the compositions $A_{m_n} \to B_{m_n} \to A_n$ and $B_{m_n} \to A_n \to B_n$ are equal to the transition maps of the inverse systems. Now, if $(b_n) \in \lim B_n$ we can set $a_n = \varphi_{m_n}(b_{m_n})$.

This defines an inverse $\lim B_n \to \lim A_n$ (computation omitted). Let us use the cokernel of the map

$$\prod B_n \longrightarrow \prod B_n$$

as an avatar of $R^1 \lim B_n$ (Lemma 15.86.1). Any element in this cokernel can be represented by an element (b_i) with $b_i = 0$ if $i \neq m_n$ for some n (computation omitted). We can define a map $R^1 \lim B_n \to R^1 \lim A_n$ by mapping the class of such a special element (b_n) to the class of $(\varphi_n(b_{m_n}))$. We omit the verification this map is inverse to the map $R^1 \lim A_n \to R^1 \lim B_n$. $\square$

0CQA **Lemma 15.86.12** (Emmanouil). Let (A_n) be an inverse system of abelian groups. The following are equivalent

Taken from [Emm96].

(1) (A_n) is Mittag-Leffler,
(2) $R^1 \lim A_n = 0$ and the same holds for $\bigoplus_{i \in \mathbf{N}} (A_n)$.

Proof. Set $B = \bigoplus_{i \in \mathbf{N}} (A_n)$ and hence $B = (B_n)$ with $B_n = \bigoplus_{i \in \mathbf{N}} A_n$. If (A_n) is ML, then B is ML and hence $R^1 \lim A_n = 0$ and $R^1 \lim B_n = 0$ by Lemma 15.86.1.

Conversely, assume (A_n) is not ML. Then we can pick an m and a sequence of integers $m < m_1 < m_2 < \ldots$ and elements $x_i \in A_{m_i}$ whose image $y_i \in A_m$ is not in the image of $A_{m_i+1} \to A_m$. We will use the elements x_i and y_i to show that $R^1 \lim B_n \neq 0$ in two ways. This will finish the proof of the lemma.

First proof. Set $C = (C_n)$ with $C_n = \prod_{i \in \mathbf{N}} A_n$. There is a canonical injective map $B_n \to C_n$ with cokernel Q_n. Set $Q = (Q_n)$. We may and do think of elements q_n of Q_n as sequences of elements $q_n = (q_{n,1}, q_{n,2}, \ldots)$ with $q_{n,i} \in A_n$ modulo sequences whose tail is zero (in other words, we identify sequences which differ in finitely many places). We have a short exact sequence of inverse systems

$$0 \to (B_n) \to (C_n) \to (Q_n) \to 0$$

Consider the element $q_n \in Q_n$ given by

$$q_{n,i} = \begin{cases} \text{image of } x_i & \text{if} \quad m_i \geq n \\ 0 & \text{else} \end{cases}$$

Then it is clear that q_{n+1} maps to q_n. Hence we obtain $q = (q_n) \in \lim Q_n$. On the other hand, we claim that q is not in the image of $\lim C_n \to \lim Q_n$. Namely, say that $c = (c_n)$ maps to q. Then we can write $c_n = (c_{n,i})$ and since $c_{n',i} \mapsto c_{n,i}$ for $n' \geq n$, we see that $c_{n,i} \in \mathrm{Im}(C_{n'} \to C_n)$ for all $n, i, n' \geq n$. In particular, the image of $c_{m,i}$ in A_m is in $\mathrm{Im}(A_{m_i+1} \to A_m)$ whence cannot be equal to y_i. Thus c_m and $q_m = (y_1, y_2, y_3, \ldots)$ differ in infinitely many spots, which is a contradiction. Considering the long exact cohomology sequence

$$0 \to \lim B_n \to \lim C_n \to \lim Q_n \to R^1 \lim B_n$$

we conclude that the last group is nonzero as desired.

Second proof. For $n' \geq n$ we denote $A_{n,n'} = \mathrm{Im}(A_{n'} \to A_n)$. Then we have $y_i \in A_m$, $y_i \notin A_{m,m_i+1}$. Let $\xi = (\xi_n) \in \prod B_n$ be the element with $\xi_n = 0$ unless $n = m_i$ and $\xi_{m_i} = (0, \ldots, 0, x_i, 0, \ldots)$ with x_i placed in the ith summand. We claim that ξ is not in the image of the map $\prod B_n \to \prod B_n$ of Lemma 15.86.1. This shows that $R^1 \lim B_n$ is nonzero and finishes the proof. Namely, suppose that ξ is the image of $\eta = (z_1, z_2, \ldots)$ with $z_n = \sum z_{n,i} \in \bigoplus_i A_n$. Observe that $x_i = z_{m_i,i} \bmod A_{m_i,m_i+1}$. Then $z_{m_i-1,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_{m_i-1}$, and so on, and we conclude that $z_{m,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_m$. We conclude that $z_{m,i}$ is congruent to y_i modulo A_{m,m_i+1}. In particular $z_{m,i} \neq 0$. This is impossible as $\sum z_{m,i} \in \bigoplus_i A_m$ hence only a finite number of $z_{m,i}$ can be nonzero. $\square$

0CQB Lemma 15.86.13. Let

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups. If (A_i) and (C_i) are ML, then so is (B_i).

Proof. This follows from Lemma 15.86.12, the fact that taking infinite direct sums is exact, and the long exact sequence of cohomology associated to $R\lim$. $\square$

091C Lemma 15.86.14. Let (A_n) be an inverse system of abelian groups. The following are equivalent

(1) (A_n) is zero as a pro-object,
(2) $\lim A_n = 0$ and $R^1 \lim A_n = 0$ and the same holds for $\bigoplus_{i \in \mathbf{N}}(A_n)$.

Proof. It follows from Lemma 15.86.11 that (1) implies (2). Assume (2). Then (A_n) is ML by Lemma 15.86.12. For $m \geq n$ let $A_{n,m} = \text{Im}(A_m \to A_n)$ so that $A_n = A_{n,n} \supset A_{n,n+1} \supset \ldots$. Note that (A_n) is zero as a pro-object if and only if for every n there is an $m \geq n$ such that $A_{n,m} = 0$. Note that (A_n) is ML if and only if for every n there is an $m_n \geq n$ such that $A_{n,m} = A_{n,m+1} = \ldots$. In the ML case it is clear that $\lim A_n = 0$ implies that $A_{n,m_n} = 0$ because the maps $A_{n+1,m_{n+1}} \to A_{n,m}$ are surjective. This finishes the proof. $\square$

15.87. Rlim of modules

0CQC We briefly discuss $R\lim$ on modules. Many of the arguments in this section duplicate the arguments used to construct the cohomological machinery for modules on ringed sites.

Let (A_n) be an inverse system of rings. We will denote $\text{Mod}(\mathbf{N}, (A_n))$ the category of inverse systems (M_n) of abelian groups such that each M_n is given the structure of a A_n-module and the transition maps $M_{n+1} \to M_n$ are A_{n+1}-module maps. This is an abelian category. Set $A = \lim A_n$. Given an object (M_n) of $\text{Mod}(\mathbf{N}, (A_n))$ the limit $\lim M_n$ is an A-module.

091D Lemma 15.87.1. In the situation above. The functor $\lim : \text{Mod}(\mathbf{N}, (A_n)) \to \text{Mod}_A$ has a right derived functor

$$R\lim : D(\text{Mod}(\mathbf{N}, (A_n))) \longrightarrow D(A)$$

As usual we set $R^p\lim(K) = H^p(R\lim(K))$. Moreover, we have

(1) for any (M_n) in $\text{Mod}(\mathbf{N}, (A_n))$ we have $R^p \lim M_n = 0$ for $p > 1$,
(2) the object $R\lim M_n$ of $D(\text{Mod}_A)$ is represented by the complex
$$\prod M_n \to \prod M_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$
sitting in degrees 0 and 1,
(3) if (M_n) is ML, then $R^1 \lim M_n = 0$, i.e., (M_n) is right acyclic for lim,
(4) every $K^\bullet \in D(\text{Mod}(\mathbf{N}, (A_n)))$ is quasi-isomorphic to a complex whose terms are right acyclic for lim, and
(5) if each $K^p = (K_n^p)$ is right acyclic for lim, i.e., of $R^1 \lim_n K_n^p = 0$, then $R\lim K$ is represented by the complex whose term in degree p is $\lim_n K_n^p$.

Proof. The proof of this is word for word the same as the proof of Lemma 15.86.1. ☐

091E Remark 15.87.2. This remark is a continuation of Remark 15.86.6. A sheaf of rings on $\mathbf{N}$ is just an inverse system of rings (A_n). A sheaf of modules over (A_n) is exactly the same thing as an object of the category $\text{Mod}(\mathbf{N}, (A_n))$ defined above. The derived functor $R\lim$ of Lemma 15.87.1 is simply $R\Gamma(\mathbf{N}, -)$ from the derived category of modules to the derived category of modules over the global sections of the structure sheaf. It is true in general that cohomology of groups and modules agree, see Cohomology on Sites, Lemma 21.12.4.

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category $D(A)$, see Derived Categories, Lemma 13.34.2.

0CQD Lemma 15.87.3. Let $K = (K_n^\bullet)$ be an object of $D(\text{Mod}(\mathbf{N}, (A_n)))$. There exists a canonical distinguished triangle

$$R\lim K \to \prod\nolimits_n K_n^\bullet \to \prod\nolimits_n K_n^\bullet \to R\lim K[1]$$

in $D(A)$. In other words, $R\lim K$ is a derived limit of the inverse system $(K_n^\bullet)$ of $D(A)$, see Derived Categories, Definition 13.34.1.

Proof. The proof is exactly the same as the proof of Lemma 15.86.7 using Lemma 15.87.1 in stead of Lemma 15.86.1. ☐

0CQE Lemma 15.87.4. With notation as in Lemma 15.87.3 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences

$$0 \to R^1\lim\nolimits_n H^{p-1}(K_n^\bullet) \to H^p(R\lim K) \to \lim\nolimits_n H^p(K_n^\bullet) \to 0$$

of A-modules.

Proof. The proof is exactly the same as the proof of Lemma 15.86.8 using Lemma 15.87.1 in stead of Lemma 15.86.1. ☐

Warning. As in the case of abelian groups an object $M = (M_n^\bullet)$ of $D(\text{Mod}(\mathbf{N}, (A_n)))$ is an inverse system of complexes of modules, which is not the same thing as an inverse system of objects in the derived categories. In the following lemma we show how an inverse system of objects in derived categories always lifts to an object of $D(\text{Mod}(\mathbf{N}, (A_n)))$.

091I Lemma 15.87.5. Let (A_n) be an inverse system of rings. Suppose that we are given

(1) for every n an object K_n of $D(A_n)$, and
(2) for every n a map $\varphi_n : K_{n+1} \to K_n$ of $D(A_{n+1})$ where we think of K_n as an object of $D(A_{n+1})$ by restriction via $A_{n+1} \to A_n$.

There exists an object $M = (M_n^\bullet) \in D(\text{Mod}(\mathbf{N}, (A_n)))$ and isomorphisms $\psi_n : M_n^\bullet \to K_n$ in $D(A_n)$ such that the diagrams

$$\begin{tikzcd} M_{n+1}^\bullet \arrow[r] \arrow[d, "\psi_{n+1}"'] & M_n^\bullet \arrow[d, "\psi_n"] \\ K_{n+1} \arrow[r, "\varphi_n"] & K_n \end{tikzcd}$$

commute in $D(A_{n+1})$.

Proof. We write out the proof in detail. For an A_n-module T we write $T_{A_{n+1}}$ for the same module viewd as an A_{n+1}-module. Suppose that $K_n^\bullet$ is a complex of A_n-modules representing K_n. Then $K_{n,A_{n+1}}^\bullet$ is the same complex, but viewed as a complex of A_{n+1}-modules. By the construction of the derived category, the map ψ_n can be given as

$$\psi_n = \tau_n \circ \sigma_n^{-1}$$

where $\sigma_n : L_{n+1}^\bullet \to K_{n+1}^\bullet$ is a quasi-isomorphism of complexes of A_{n+1}-modules and $\tau_n : L_{n+1}^\bullet \to K_{n,A_{n+1}}^\bullet$ is a map of complexes of A_{n+1}-modules.

Now we construct the complexes $M_n^\bullet$ by induction. As base case we let $M_1^\bullet = K_1^\bullet$. Suppose we have already constructed $M_e^\bullet \to M_{e-1}^\bullet \to \ldots \to M_1^\bullet$ and maps of complexes $\psi_i : M_i^\bullet \to K_i^\bullet$ such that the diagrams

$$\begin{tikzcd} M_{n+1}^\bullet \arrow[rr] \arrow[d, "\psi_{n+1}"'] & & M_{n,A_{n+1}}^\bullet \arrow[d, "\psi_{n,A_{n+1}}"] \\ K_{n+1}^\bullet & L_{n+1}^\bullet \arrow[l, "\sigma_n"'] \arrow[r, "\tau_n"] & K_{n,A_{n+1}}^\bullet \end{tikzcd}$$

above commute in $D(A_{n+1})$ for all $n < e$. Then we consider the diagram

$$\begin{tikzcd} & & M_{e,A_{e+1}}^\bullet \arrow[d, "\psi_{e,A_{e+1}}"] \\ K_{e+1}^\bullet & L_{e+1}^\bullet \arrow[l, "\sigma_e"'] \arrow[r, "\tau_e"] & K_{e,A_{e+1}}^\bullet \end{tikzcd}$$

in $D(A_{e+1})$. Because ψ_e is a quasi-isomorphism, we see that $\psi_{e,A_{e+1}}$ is a quasi-isomorphism too. By the definition of morphisms in $D(A_{e+1})$ we can find a quasi-isomorphism $\psi_{e+1} : M_{e+1}^\bullet \to K_{e+1}^\bullet$ of complexes of A_{e+1}-modules such that there exists a morphism of complexes $M_{e+1}^\bullet \to M_{e,A_{e+1}}^\bullet$ of A_{e+1}-modules representing the composition $\psi_{e,A_{e+1}}^{-1} \circ \tau_e \circ \sigma_e^{-1}$ in $D(A_{e+1})$. Thus the lemma holds by induction. □

07KZ Remark 15.87.6. With assumptions as in Lemma 15.87.5. A priori there are many isomorphism classes of objects M of $D(\text{Mod}(\mathbf{N}, (A_n)))$ which give rise to the system (K_n, φ_n) of the lemma. For each such M we can consider the complex $R\lim M \in D(A)$ where $A = \lim A_n$. By Lemma 15.87.3 we see that $R\lim M$ is a derived limit of the inverse system (K_n) of $D(A)$. Hence we see that the isomorphism class of $R\lim M$ in $D(A)$ is independent of the choices made in constructing M. In particular, we may apply results on $R\lim$ proved in this section to derived limits of inverse systems in $D(A)$. For example, for every $p \in \mathbf{Z}$ there is a canonical short exact sequence

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(R\lim K_n) \to \lim H^p(K_n) \to 0$$

because we may apply Lemma 15.87.3 to M. This can also been seen directly, without invoking the existence of M, by applying the argument of the proof of Lemma 15.87.3 to the (defining) distinguished triangle $R\lim K_n \to \prod K_n \to \prod K_n \to (R\lim K_n)[1]$ of the derived limit.

091F Lemma 15.87.7. Let (A_n) be an inverse system of rings. Every $K \in D(\text{Mod}(\mathbf{N}, (A_n)))$ can be represented by a system of complexes $(M_n^\bullet)$ such that all the transition maps $M_{n+1}^\bullet \to M_n^\bullet$ are surjective.

Proof. Let K be represented by the system $(K_n^\bullet)$. Set $M_1^\bullet = K_1^\bullet$. Suppose we have constructed surjective maps of complexes $M_n^\bullet \to M_{n-1}^\bullet \to \ldots \to M_1^\bullet$ and homotopy equivalences $\psi_e : K_e^\bullet \to M_e^\bullet$ such that the diagrams

$$\begin{tikzcd} K_{e+1}^\bullet \arrow[r] \arrow[d] & K_e^\bullet \arrow[d] \\ M_{e+1}^\bullet \arrow[r] & M_e^\bullet \end{tikzcd}$$

commute for all $e < n$. Then we consider the diagram

$$\begin{tikzcd} K_{n+1}^\bullet \arrow[r] & K_n^\bullet \arrow[d] \\ & M_n^\bullet \end{tikzcd}$$

By Derived Categories, Lemma 13.9.8 we can factor the composition $K_{n+1}^\bullet \to M_n^\bullet$ as $K_{n+1}^\bullet \to M_{n+1}^\bullet \to M_n^\bullet$ such that the first arrow is a homotopy equivalence and the second a termwise split surjection. The lemma follows from this and induction. $\square$

091G Lemma 15.87.8. Let (A_n) be an inverse system of rings. Every $K \in D(\text{Mod}(\mathbf{N}, (A_n)))$ can be represented by a system of complexes $(K_n^\bullet)$ such that each $K_n^\bullet$ is K-flat.

Proof. First use Lemma 15.87.7 to represent K by a system of complexes $(M_n^\bullet)$ such that all the transition maps $M_{n+1}^\bullet \to M_n^\bullet$ are surjective. Next, let $K_1^\bullet \to M_1^\bullet$ be a quasi-isomorphism with $K_1^\bullet$ a K-flat complex of A_1-modules (Lemma 15.59.10). Suppose we have constructed $K_n^\bullet \to K_{n-1}^\bullet \to \ldots \to K_1^\bullet$ and maps of complexes $\psi_e : K_e^\bullet \to M_e^\bullet$ such that

$$\begin{tikzcd} K_{e+1}^\bullet \arrow[r] \arrow[d] & K_e^\bullet \arrow[d] \\ M_{e+1}^\bullet \arrow[r] & M_e^\bullet \end{tikzcd}$$

commutes for all $e < n$. Then we consider the diagram

$$\begin{tikzcd} C^\bullet \arrow[r, dotted] \arrow[d, dotted] & K_n^\bullet \arrow[d, "\psi_n"] \\ M_{n+1}^\bullet \arrow[r, "\varphi_n"] & M_n^\bullet \end{tikzcd}$$

in $D(A_{n+1})$. As $M_{n+1}^\bullet \to M_n^\bullet$ is termwise surjective, the complex $C^\bullet$ fitting into the left upper corner with terms

$$C^p = M_{n+1}^p \times_{M_n^p} K_n^p$$

is quasi-isomorphic to $M_{n+1}^\bullet$ (details omitted). Choose a quasi-isomorphism $K_{n+1}^\bullet \to C^\bullet$ with $K_{n+1}^\bullet$ K-flat. Thus the lemma holds by induction. $\square$

091H Lemma 15.87.9. Let (A_n) be an inverse system of rings. Given $K, L \in D(\text{Mod}(\mathbf{N}, (A_n)))$ there is a canonical derived tensor product $K \otimes^{\mathbf{L}} L$ in $D(\mathbf{N}, (A_n))$ compatible with the maps to $D(A_n)$. The construction is symmetric in K and L and an exact functor of triangulated categories in each variable.

Proof. Choose a representative $(K_n^\bullet)$ for K such that each $K_n^\bullet$ is a K-flat complex (Lemma 15.87.8). Then you can define $K \otimes^{\mathbf{L}} L$ as the object represented by the system of complexes

$$(\text{Tot}(K_n^\bullet \otimes_{A_n} L_n^\bullet))$$

for any choice of representative $(L_n^\bullet)$ for L. This is well defined in both variables by Lemmas 15.59.2 and 15.59.12. Compatibility with the map to $D(A_n)$ is clear. Exactness follows exactly as in Lemma 15.58.4. $\square$

091J Remark 15.87.10. Let A be a ring. Let (E_n) be an inverse system of objects of $D(A)$. We've seen above that a derived limit $R\lim E_n$ exists. Thus for every object K of $D(A)$ also the derived limit $R\lim(K \otimes_A^{\mathbf{L}} E_n)$ exists. It turns out that we can construct these derived limits functorially in K and obtain an exact functor

$$R\lim(- \otimes_A^{\mathbf{L}} E_n) : D(A) \longrightarrow D(A)$$

of triangulated categories. Namely, we first lift (E_n) to an object E of $D(\mathbf{N}, A)$, see Lemma 15.87.5. (The functor will depend on the choice of this lift.) Next, observe that there is a "diagonal" or "constant" functor

$$\Delta : D(A) \longrightarrow D(\mathbf{N}, A)$$

mapping the complex $K^\bullet$ to the constant inverse system of complexes with value $K^\bullet$. Then we simply define

$$R\lim(K \otimes_A^{\mathbf{L}} E_n) = R\lim(\Delta(K) \otimes^{\mathbf{L}} E)$$

where on the right hand side we use the functor $R\lim$ of Lemma 15.87.1 and the functor $- \otimes^{\mathbf{L}} -$ of Lemma 15.87.9.

091K Lemma 15.87.11. Let A be a ring. Let $E \to D \to F \to E[1]$ be a distinguished triangle of $D(\mathbf{N}, A)$. Let (E_n), resp. (D_n), resp. (F_n) be the system of objects of $D(A)$ associated to E, resp. D, resp. F. Then for every $K \in D(A)$ there is a canonical distinguished triangle

$$R\lim(K \otimes_A^{\mathbf{L}} E_n) \to R\lim(K \otimes_A^{\mathbf{L}} D_n) \to R\lim(K \otimes_A^{\mathbf{L}} F_n) \to R\lim(K \otimes_A^{\mathbf{L}} E_n)[1]$$

in $D(A)$ with notation as in Remark 15.87.10.

Proof. This is clear from the construction in Remark 15.87.10 and the fact that $\Delta : D(A) \to D(\mathbf{N}, A)$, $- \otimes^{\mathbf{L}} -$, and $R\lim$ are exact functors of triangulated categories. $\square$

091L Lemma 15.87.12. Let A be a ring. Let $E \to D$ be a morphism of $D(\mathbf{N}, A)$. Let (E_n), resp. (D_n) be the system of objects of $D(A)$ associated to E, resp. D. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then for every $K \in D(A)$ the corresponding map

$$R\lim(K \otimes_A^{\mathbf{L}} E_n) \longrightarrow R\lim(K \otimes_A^{\mathbf{L}} D_n)$$

in $D(A)$ is an isomorphism (notation as in Remark 15.87.10).

Proof. Follows from the definitions and Lemma 15.86.11. $\square$

15.88. Torsion modules

0ALX In this section "torsion modules" will refer to modules supported on a given closed subset $V(I)$ of an affine scheme $\mathrm{Spec}(R)$. This is different, but analogous to, the notion of a torsion module over a domain (Definition 15.22.1).

05E6 Definition 15.88.1. Let R be a ring. Let M be an R-module.

(1) Let $I \subset R$ be an ideal. We say M is an I-power torsion module if for every $m \in M$ there exists an $n > 0$ such that $I^n m = 0$.
(2) Let $f \in R$. We say M is an f-power torsion module if for each $m \in M$, there exists an $n > 0$ such that $f^n m = 0$.

Thus an f-power torsion module is the same thing as an I-power torsion module for $I = (f)$. We will use the notation

$$M[I^n] = \{m \in M \mid I^n m = 0\}$$

and

$$M[I^\infty] = \bigcup M[I^n]$$

for an R-module M. Thus M is I-power torsion if and only if $M = M[I^\infty]$ if and only if $M = \bigcup M[I^n]$.

05E8 Lemma 15.88.2. Let R be a ring. Let I be an ideal of R. Let M be an I-power torsion module. Then M admits a resolution

$$\ldots \to K_2 \to K_1 \to K_0 \to M \to 0$$

with each K_i a direct sum of copies of R/I^n for n variable.

Proof. There is a canonical surjection

$$\oplus_{m \in M} R/I^{n_m} \to M \to 0$$

where n_m is the smallest positive integer such that $I^{n_m} \cdot m = 0$. The kernel of the preceding surjection is also an I-power torsion module. Proceeding inductively, we construct the desired resolution of M. □

05EA Lemma 15.88.3. Let R be a ring. Let I be an ideal of R. For any R-module M set $M[I^n] = \{m \in M \mid I^n m = 0\}$. If I is finitely generated then the following are equivalent

(1) $M[I] = 0$,
(2) $M[I^n] = 0$ for all $n \geq 1$, and
(3) if $I = (f_1, \ldots, f_t)$, then the map $M \to \bigoplus M_{f_i}$ is injective.

Proof. This follows from Algebra, Lemma 10.24.4. □

05EB Lemma 15.88.4. Let R be a ring. Let I be a finitely generated ideal of R.

(1) For any R-module M we have $(M/M[I^\infty])[I] = 0$.
(2) An extension of I-power torsion modules is I-power torsion.

Proof. Let $m \in M$. If m maps to an element of $(M/M[I^\infty])[I]$ then $Im \subset M[I^\infty]$. Write $I = (f_1, \ldots, f_t)$. Then we see that $f_i m \in M[I^\infty]$, i.e., $I^{n_i} f_i m = 0$ for some $n_i > 0$. Thus we see that $I^N m = 0$ with $N = \sum n_i + 2$. Hence m maps to zero in $(M/M[I^\infty])$ which proves the first statement of the lemma.

For the second, suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules with M' and M'' both I-power torsion modules. Then $M[I^\infty] \supset M'$

and hence $M/M[I^\infty]$ is a quotient of M'' and therefore I-power torsion. Combined with the first statement and Lemma 15.88.3 this implies that it is zero $\square$

0A6K Lemma 15.88.5. Let I be a finitely generated ideal of a ring R. The I-power torsion modules form a Serre subcategory of the abelian category Mod_R, see Homology, Definition 12.10.1.

Proof. It is clear that a submodule and a quotient module of an I-power torsion module is I-power torsion. Moreover, the extension of two I-power torsion modules is I-power torsion by Lemma 15.88.4. Hence the statement of the lemma by Homology, Lemma 12.10.2. $\square$

0953 Lemma 15.88.6. Let R be a ring and let $I \subset R$ be a finitely generated ideal. The subcategory I^∞-torsion $\subset \text{Mod}_R$ depends only on the closed subset $Z = V(I) \subset \text{Spec}(R)$. In fact, an R-module M is I-power torsion if and only if its support is contained in Z.

Proof. Let M be an R-module. Let $x \in M$. If $x \in M[I^\infty]$, then x maps to zero in M_f for all $f \in I$. Hence x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \not\supset I$. Conversely, if x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \not\supset I$, then x maps to zero in M_f for all $f \in I$. Hence if $I = (f_1, \ldots, f_r)$, then $f_i^{n_i} x = 0$ for some $n_i \geq 1$. It follows that $x \in M[I^{\sum n_i}]$. Thus $M[I^\infty]$ is the kernel of $M \to \prod_{\mathfrak{p} \not\in Z} M_\mathfrak{p}$. The second statement of the lemma follows and it implies the first. $\square$

The next lemma should probably go somewhere else.

0G1T Lemma 15.88.7. Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$ such hat $K \otimes_R^\mathbf{L} R/I = 0$ in $D(R)$. Then

(1) $K \otimes_R^\mathbf{L} R/I^n = 0$ for all $n \geq 1$,
(2) $K \otimes_R^\mathbf{L} N = 0$ for any I-power torsion R-module N,
(3) $K \otimes_R^\mathbf{L} M = 0$ for any $M \in D^b(R)$ whose cohomology modules are I-power torsion.

Proof. Proof of (2). We can write $N = \bigcup N[I^n]$. We have $K \otimes_R^\mathbf{L} N = \text{hocolim}_n K \otimes_R^\mathbf{L} N[I^n]$ as tensor products commute with colimits (details omitted; hint: represent K by a K-flat complex and compute directly). Hence we may assume N is annihilated by I^n. Consider the R-algebra $R' = R/I^n \oplus N$ where N is an ideal of square zero. It suffices to show that $K' = K \otimes_R^\mathbf{L} R'$ is 0 in $D(R')$. We have a surjection $R' \to R/I$ of R-algebras whose kernel J is nilpotent (any product of n elements in the kernel is zero). We have

$$0 = K \otimes_R^\mathbf{L} R/I = (K \otimes_R^\mathbf{L} R') \otimes_{R'}^\mathbf{L} R/I = K' \otimes_{R'}^\mathbf{L} R/I$$

by Lemma 15.60.5. Hence by Lemma 15.78.4 we find that K' is a perfect complex of R'-modules. In particular K' is bounded above and if $H^b(K')$ is the right-most nonvanishing cohomology module (if it exists), then $H^b(K')$ is a finite R'-module (use Lemmas 15.74.2 and 15.64.3) with $H^b(K') \otimes_{R'} R'/J = H^b(K')/JH^b(K') = 0$ (because $K' \otimes_{R'}^\mathbf{L} R'/J = 0$). By Nakayama's lemma (Algebra, Lemma 10.20.1) we find $H^b(K') = 0$, i.e., $K' = 0$ as desired.

Part (1) follows trivially from part (2). Part (3) follows from part (2), induction on the number of nonzero cohomology modules of M, and the distinguished triangles of truncation from Derived Categories, Remark 13.12.4. Details omitted. $\square$

15.89. Formal glueing of module categories

05E5 Fix a Noetherian scheme X, and a closed subscheme Z with complement U. Our goal is to explain how coherent sheaves on X can be constructed (uniquely) from coherent sheaves on the formal completion of X along Z, and those on U with a suitable compatibility on the overlap. We first do this using only commutative algebra (this section) and later we explain this in the setting of algebraic spaces (Pushouts of Spaces, Section 81.10).

Here are some references treating some of the material in this section: [Art70, Section 2], [FR70, Appendix], [BL95], [MB96], and [dJ95, Section 4.6].

05E7 **Lemma 15.89.1.** *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The following are equivalent*

1. *φ is flat and $R/I \to S/IS$ is faithfully flat,*
2. *φ is flat, and the map $\operatorname{Spec}(S/IS) \to \operatorname{Spec}(R/I)$ is surjective.*
3. *φ is flat, and the base change functor $M \mapsto M \otimes_R S$ is faithful on modules annihilated by I, and*
4. *φ is flat, and the base change functor $M \mapsto M \otimes_R S$ is faithful on I-power torsion modules.*

Proof. If $R \to S$ is flat, then $R/I^n \to S/I^nS$ is flat for every n, see Algebra, Lemma 10.39.7. Hence (1) and (2) are equivalent by Algebra, Lemma 10.39.16. The equivalence of (1) with (3) follows by identifying I-torsion R-modules with R/I-modules, using that

$$M \otimes_R S = M \otimes_{R/I} S/IS$$

for R-modules M annihilated by I, and Algebra, Lemma 10.39.14. The implication (4) $\Rightarrow$ (3) is immediate. Assume (3). We have seen above that $R/I^n \to S/I^nS$ is flat, and by assumption it induces a surjection on spectra, as $\operatorname{Spec}(R/I^n) = \operatorname{Spec}(R/I)$ and similarly for S. Hence the base change functor is faithful on modules annihilated by I^n. Since any I-power torsion module M is the union $M = \bigcup M_n$ where M_n is annihilated by I^n we see that the base change functor is faithful on the category of all I-power torsion modules (as tensor product commutes with colimits). $\square$

05E9 **Lemma 15.89.2.** *Assume $(\varphi : R \to S, I)$ satisfies the equivalent conditions of Lemma 15.89.1. The following are equivalent*

1. *for any I-power torsion module M, the natural map $M \to M \otimes_R S$ is an isomorphism, and*
2. *$R/I \to S/IS$ is an isomorphism.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). First assume that M is annihilated by I. In this case, M is an R/I-module. Hence, we have an isomorphism

$$M \otimes_R S = M \otimes_{R/I} S/IS = M \otimes_{R/I} R/I = M$$

proving the claim. Next we prove by induction that $M \to M \otimes_R S$ is an isomorphism for any module M is annihilated by I^n. Assume the induction hypothesis holds for n and assume M is annihilated by I^{n+1}. Then we have a short exact sequence

$$0 \to I^nM \to M \to M/I^nM \to 0$$

and as $R \to S$ is flat this gives rise to a short exact sequence

$$0 \to I^n M \otimes_R S \to M \otimes_R S \to M/I^n M \otimes_R S \to 0$$

Using that the canonical map is an isomorphism for $M' = I^n M$ and $M'' = M/I^n M$ (by induction hypothesis) we conclude the same thing is true for M. Finally, suppose that M is a general I-power torsion module. Then $M = \bigcup M_n$ where M_n is annihilated by I^n and we conclude using that tensor products commute with colimits. $\square$

05EC Lemma 15.89.3. Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Then

(1) for any R-module M the map $M \to M \otimes_R S$ induces an isomorphism $M[I^\infty] \to (M \otimes_R S)[(IS)^\infty]$ of I-power torsion submodules,
(2) the natural map
$$\mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$
is an isomorphism if either M or N is I-power torsion, and
(3) the base change functor $M \mapsto M \otimes_R S$ defines an equivalence of categories between I-power torsion modules and IS-power torsion modules.

Proof. Note that the equivalent conditions of both Lemma 15.89.1 and Lemma 15.89.2 are satisfied. We will use these without further mention. We first prove (1). Let M be any R-module. Set $M' = M/M[I^\infty]$ and consider the exact sequence

$$0 \to M[I^\infty] \to M \to M' \to 0$$

As $M[I^\infty] = M[I^\infty] \otimes_R S$ we see that it suffices to show that $(M' \otimes_R S)[(IS)^\infty] = 0$. Write $I = (f_1, \ldots, f_t)$. By Lemma 15.88.4 we see that $M'[I^\infty] = 0$. Hence for every $n > 0$ the map

$$M' \longrightarrow \bigoplus\nolimits_{i=1,\ldots t} M', \quad x \longmapsto (f_1^n x, \ldots, f_t^n x)$$

is injective. As S is flat over R also the corresponding map $M' \otimes_R S \to \bigoplus_{i=1,\ldots t} M' \otimes_R S$ is injective. This means that $(M' \otimes_R S)[I^n] = 0$ as desired.

Next we prove (2). If N is I-power torsion, then $N \otimes_R S = N$ and the displayed map of (2) is an isomorphism by Algebra, Lemma 10.14.3. If M is I-power torsion, then the image of any map $M \to N$ factors through $M[I^\infty]$ and the image of any map $M \otimes_R S \to N \otimes_R S$ factors through $(N \otimes_R S)[(IS)^\infty]$. Hence in this case part (1) guarantees that we may replace N by $N[I^\infty]$ and the result follows from the case where N is I-power torsion we just discussed.

Next we prove (3). The functor is fully faithful by (2). For essential surjectivity, we simply note that for any IS-power torsion S-module N, the natural map $N \otimes_R S \to N$ is an isomorphism. $\square$

091M Lemma 15.89.4. Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. For any $f_1, \ldots, f_r \in R$ such that $V(f_1, \ldots, f_r) = V(I)$

(1) the map of Koszul complexes $K(R, f_1, \ldots, f_r) \to K(S, f_1, \ldots, f_r)$ is a quasi-isomorphism, and

(2) The map of extended alternating Čech complexes

$$
\begin{array}{c}
R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0<i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1\ldots f_r} \\
\downarrow \\
S \to \prod_{i_0} S_{f_{i_0}} \to \prod_{i_0<i_1} S_{f_{i_0}f_{i_1}} \to \ldots \to S_{f_1\ldots f_r}
\end{array}
$$

is a quasi-isomorphism.

Proof. In both cases we have a complex $K_\bullet$ of R modules and we want to show that $K_\bullet \to K_\bullet \otimes_R S$ is a quasi-isomorphism. By Lemma 15.89.2 and the flatness of $R \to S$ this will hold as soon as all homology groups of K are I-power torsion. This is true for the Koszul complex by Lemma 15.28.6 and for the extended alternating Čech complex by Lemma 15.29.5. $\square$

05ED Lemma 15.89.5. Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. Let M be the R-module generated by elements $e_1, \ldots, e_n$ subject to the relations $f_ie_j - f_je_i = 0$. There exists a short exact sequence

$$0 \to K \to M \to I \to 0$$

such that K is annihilated by I.

Proof. This is just a truncation of the Koszul complex. The map $M \to I$ is determined by the rule $e_i \mapsto f_i$. If $m = \sum a_ie_i$ is in the kernel of $M \to I$, i.e., $\sum a_if_i = 0$, then $f_jm = \sum f_ja_ie_i = (\sum f_ia_i)e_j = 0$. $\square$

05EE Lemma 15.89.6. Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N set

$$H_1(N, f_\bullet) = \frac{\{(x_1, \ldots, x_n) \in N^{\oplus n} \mid f_ix_j = f_jx_i\}}{\{f_1x, \ldots, f_nx) \mid x \in N\}}$$

For any R-module N there exists a canonical short exact sequence

$$0 \to \mathrm{Ext}_R(R/I, N) \to H_1(N, f_\bullet) \to \mathrm{Hom}_R(K, N)$$

where K is as in Lemma 15.89.5.

Proof. The notation above indicates the Ext-groups in Mod_R as defined in Homology, Section 12.6. These are denoted $\mathrm{Ext}_R(M, N)$. Using the long exact sequence of Homology, Lemma 12.6.4 associated to the short exact sequence $0 \to I \to R \to R/I \to 0$ and the fact that $\mathrm{Ext}_R(R, N) = 0$ we see that

$$\mathrm{Ext}_R(R/I, N) = \mathrm{Coker}(N \longrightarrow \mathrm{Hom}(I, N))$$

Using the short exact sequence of Lemma 15.89.5 we see that we get a complex

$$N \to \mathrm{Hom}(M, N) \to \mathrm{Hom}_R(K, N)$$

whose homology in the middle is canonically isomorphic to $\mathrm{Ext}_R(R/I, N)$. The proof of the lemma is now complete as the cokernel of the first map is canonically isomorphic to $H_1(N, f_\bullet)$. $\square$

05EF Lemma 15.89.7. Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N the Koszul homology group $H_1(N, f_\bullet)$ defined in Lemma 15.89.6 is annihilated by I.

Proof. Let $(x_1, \ldots, x_n) \in N^{\oplus n}$ with $f_i x_j = f_j x_i$. Then we have $f_i(x_1, \ldots, x_n) = (f_i x_i, \ldots, f_i x_n)$. In other words f_i annihilates $H_1(N, f_\bullet)$. $\square$

We can improve on the full faithfulness of Lemma 15.89.3 by showing that Ext-groups whose source is I-power torsion are insensitive to passing to S as well. See Dualizing Complexes, Lemma 47.9.8 for a derived version of the following lemma.

05EG Lemma 15.89.8. Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Let M, N be R-modules. Assume M is I-power torsion. Given an short exact sequence

$$0 \to N \otimes_R S \to \tilde{E} \to M \otimes_R S \to 0$$

there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N \otimes_R S \arrow[r] & \tilde{E} \arrow[r] & M \otimes_R S \arrow[r] & 0 \end{tikzcd}$$

with exact rows.

Proof. As M is I-power torsion we see that $M \otimes_R S = M$, see Lemma 15.89.2. We will use this identification without further mention. As $R \to S$ is flat, the base change functor is exact and we obtain a functorial map of Ext-groups

$$\mathrm{Ext}_R(M, N) \longrightarrow \mathrm{Ext}_S(M \otimes_R S, N \otimes_R S),$$

see Homology, Lemma 12.7.3. The claim of the lemma is that this map is surjective when M is I-power torsion. In fact we will show that it is an isomorphism. By Lemma 15.88.2 we can find a surjection $M' \to M$ with M' a direct sum of modules of the form R/I^n. Using the long exact sequence of Homology, Lemma 12.6.4 we see that it suffices to prove the lemma for M'. Using compatibility of Ext with direct sums (details omitted) we reduce to the case where $M = R/I^n$ for some n.

Let $f_1, \ldots, f_t$ be generators for I^n. By Lemma 15.89.6 we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}_R(R/I^n, N) \arrow[r] \arrow[d] & H_1(N, f_\bullet) \arrow[r] \arrow[d] & \mathrm{Hom}_R(K, N) \arrow[d] \\ 0 \arrow[r] & \mathrm{Ext}_S(S/I^nS, N \otimes S) \arrow[r] & H_1(N \otimes S, f_\bullet) \arrow[r] & \mathrm{Hom}_S(K \otimes S, N \otimes S) \end{tikzcd}$$

with exact rows where K is as in Lemma 15.89.5. Hence it suffices to prove that the two right vertical arrows are isomorphisms. Since K is annihilated by I^n we see that $\mathrm{Hom}_R(K, N) = \mathrm{Hom}_S(K \otimes_R S, N \otimes_R S)$ by Lemma 15.89.3. As $R \to S$ is flat we have $H_1(N, f_\bullet) \otimes_R S = H_1(N \otimes_R S, f_\bullet)$. As $H_1(N, f_\bullet)$ is annihilated by I^n, see Lemma 15.89.7 we have $H_1(N, f_\bullet) \otimes_R S = H_1(N, f_\bullet)$ by Lemma 15.89.2. $\square$

Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Then for any R-module M we can define a complex

05EJ (15.89.8.1) $$0 \to M \xrightarrow{\alpha} M \otimes_R S \times \prod M_{f_i} \xrightarrow{\beta} \prod (M \otimes_R S)_{f_i} \times \prod M_{f_i f_j}$$

where $\alpha(m) = (m \otimes 1, m/1, \ldots, m/1)$ and

$\beta(m', m_1, \ldots, m_t) = ((m'/1 - m_1 \otimes 1, \ldots, m'/1 - m_t \otimes 1), (m_1 - m_2, \ldots, m_{t-1} - m_t)$.

We would like to know when this complex is exact.

05EK Lemma 15.89.9. Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Let M be an R-module. Then the complex (15.89.8.1) is exact.

Proof. First proof. Denote $\check{\mathcal{C}}_R \to \check{\mathcal{C}}_S$ the quasi-isomorphism of extended alternating Čech complexes of Lemma 15.89.4. Since these complexes are bounded with flat terms, we see that $M \otimes_R \check{\mathcal{C}}_R \to M \otimes_R \check{\mathcal{C}}_S$ is a quasi-isomorphism too (Lemmas 15.59.7 and 15.59.12). Now the complex (15.89.8.1) is a truncation of the cone of the map $M \otimes_R \check{\mathcal{C}}_R \to M \otimes_R \check{\mathcal{C}}_S$ and we win.

Second computational proof. Let $m \in M$. If $\alpha(m) = 0$, then $m \in M[I^\infty]$, see Lemma 15.88.3. Pick n such that $I^n m = 0$ and consider the map $\varphi : R/I^n \to M$. If $m \otimes 1 = 0$, then $\varphi \otimes 1_S = 0$, hence $\varphi = 0$ (see Lemma 15.89.3) hence $m = 0$. In this way we see that α is injective.

Let $(m', m'_1, \ldots, m'_t) \in \text{Ker}(\beta)$. Write $m'_i = m_i/f_i^n$ for some $n > 0$ and $m_i \in M$. We may, after possibly enlarging n assume that $f_i^n m' = m_i \otimes 1$ in $M \otimes_R S$ and $f_j^n m_i - f_i^n m_j = 0$ in M. In particular we see that $(m_1, \ldots, m_t)$ defines an element ξ of $H_1(M, (f_1^n, \ldots, f_t^n))$. Since $H_1(M, (f_1^n, \ldots, f_t^n))$ is annihilated by I^{tn+1} (see Lemma 15.89.7) and since $R \to S$ is flat we see that

$$H_1(M, (f_1^n, \ldots, f_t^n)) = H_1(M, (f_1^n, \ldots, f_t^n)) \otimes_R S = H_1(M \otimes_R S, (f_1^n, \ldots, f_t^n))$$

by Lemma 15.89.2 The existence of m' implies that ξ maps to zero in the last group, i.e., the element ξ is zero. Thus there exists an $m \in M$ such that $m_i = f_i^n m$. Then $(m', m'_1, \ldots, m'_t) - \alpha(m) = (m'', 0, \ldots, 0)$ for some $m'' \in (M \otimes_R S)[(IS)^\infty]$. By Lemma 15.89.3 we conclude that $m'' \in M[I^\infty]$ and we win. $\square$

05EL Remark 15.89.10. In this remark we define a category of glueing data. Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Consider the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ as the category whose

(1) objects are systems $(M', M_i, \alpha_i, \alpha_{ij})$, where M' is an S-module, M_i is an R_{f_i}-module, $\alpha_i : (M')_{f_i} \to M_i \otimes_R S$ is an isomorphism, and $\alpha_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ are isomorphisms such that
   (a) $\alpha_{ij} \circ \alpha_i = \alpha_j$ as maps $(M')_{f_i f_j} \to (M_j)_{f_i}$, and
   (b) $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ as maps $(M_i)_{f_j f_k} \to (M_k)_{f_i f_j}$ (cocycle condition).
(2) morphisms $(M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ are given by maps $\varphi' : M' \to N'$ and $\varphi_i : M_i \to N_i$ compatible with the given maps $\alpha_i, \beta_i, \alpha_{ij}, \beta_{ij}$.

There is a canonical functor

$$\text{Can} : \text{Mod}_R \longrightarrow \text{Glue}(R \to S, f_1, \ldots, f_t), \quad M \longmapsto (M \otimes_R S, M_{f_i}, \text{can}_i, \text{can}_{ij})$$

where $\text{can}_i : (M \otimes_R S)_{f_i} \to M_{f_i} \otimes_R S$ and $\text{can}_{ij} : (M_{f_i})_{f_j} \to (M_{f_j})_{f_i}$ are the canonical isomorphisms. For any object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ we define

$$H^0(\mathbf{M}) = \{(m', m_i) \mid \alpha_i(m') = m_i \otimes 1, \alpha_{ij}(m_i) = m_j\}$$

in other words defined by the exact sequence

$$0 \to H^0(\mathbf{M}) \to M' \times \prod M_i \to \prod M'_{f_i} \times \prod (M_i)_{f_j}$$

similar to (15.89.8.1). We think of $H^0(\mathbf{M})$ as an R-module. Thus we also get a functor

$$H^0 : \mathrm{Glue}(R \to S, f_1, \ldots, f_t) \longrightarrow \mathrm{Mod}_R$$

Our next goal is to show that the functors Can and H^0 are sometimes quasi-inverse to each other.

05EM Lemma 15.89.11. Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then the functor H^0 is a left quasi-inverse to the functor Can of Remark 15.89.10.

Proof. This is a reformulation of Lemma 15.89.9. $\square$

05EN Lemma 15.89.12. Assume $\varphi : R \to S$ is a flat ring map and let $I = (f_1, \ldots, f_t) \subset R$ be an ideal. Then $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ is an abelian category, and the functor Can is exact and commutes with arbitrary colimits.

Proof. Given a morphism $(\varphi', \varphi_i) : (M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ of the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ we see that its kernel exists and is equal to the object $(\mathrm{Ker}(\varphi'), \mathrm{Ker}(\varphi_i), \alpha_i, \alpha_{ij})$ and its cokernel exists and is equal to the object $(\mathrm{Coker}(\varphi'), \mathrm{Coker}(\varphi_i), \beta_i, \beta_{ij})$. This works because $R \to S$ is flat, hence taking kernels/cokernels commutes with $- \otimes_R S$. Details omitted. The exactness follows from the R-flatness of R_{f_i} and S, while commuting with colimits follows as tensor products commute with colimits. $\square$

05EP Lemma 15.89.13. Let $\varphi : R \to S$ be a flat ring map and $(f_1, \ldots, f_t) = R$. Then Can and H^0 are quasi-inverse equivalences of categories

$$\mathrm{Mod}_R = \mathrm{Glue}(R \to S, f_1, \ldots, f_t)$$

Proof. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$. By Algebra, Lemma 10.24.5 there exists a unique module M and isomorphisms $M_{f_i} \to M_i$ which recover the glueing data α_{ij}. Then both M' and $M \otimes_R S$ are S-modules which recover the modules $M_i \otimes_R S$ upon localizing at f_i. Whence there is a canonical isomorphism $M \otimes_R S \to M'$. This shows that $\mathbf{M}$ is in the essential image of Can. Combined with Lemma 15.89.11 the lemma follows. $\square$

05EQ Lemma 15.89.14. Let $\varphi : R \to S$ be a flat ring map and $I = (f_1, \ldots, f_t)$ and ideal. Let $R \to R'$ be a flat ring map, and set $S' = S \otimes_R R'$. Then we obtain a commutative diagram of categories and functors

$$\begin{tikzcd}
\mathrm{Mod}_R \arrow[r, "\mathrm{Can}"'] \arrow[d, "-\otimes_R R'"'] & \mathrm{Glue}(R \to S, f_1, \ldots, f_t) \arrow[r, "H^0"'] \arrow[d, "-\otimes_R R'"] & \mathrm{Mod}_R \arrow[d, "-\otimes_R R'"] \\
\mathrm{Mod}_{R'} \arrow[r, "\mathrm{Can}"] & \mathrm{Glue}(R' \to S', f_1, \ldots, f_t) \arrow[r, "H^0"] & \mathrm{Mod}_{R'}
\end{tikzcd}$$

Proof. Omitted. $\square$

05ER Proposition 15.89.15. Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then Can and H^0 are quasi-inverse equivalences of categories

$$\mathrm{Mod}_R = \mathrm{Glue}(R \to S, f_1, \ldots, f_t)$$

Proof. We have already seen that $H^0 \circ \text{Can}$ is isomorphic to the identity functor, see Lemma 15.89.11. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\text{Glue}(R \to S, f_1, \ldots, f_t)$. We get a natural morphism

$$\Psi : (H^0(\mathbf{M}) \otimes_R S, H^0(\mathbf{M})_{f_i}, \text{can}_i, \text{can}_{ij}) \longrightarrow (M', M_i, \alpha_i, \alpha_{ij}).$$

Namely, by definition $H^0(\mathbf{M})$ comes equipped with compatible R-module maps $H^0(\mathbf{M}) \to M'$ and $H^0(\mathbf{M}) \to M_i$. We have to show that this map is an isomorphism.

Pick an index i and set $R' = R_{f_i}$. Combining Lemmas 15.89.14 and 15.89.13 we see that $\Psi \otimes_R R'$ is an isomorphism. Hence the kernel, resp. cokernel of Ψ is a system of the form $(K, 0, 0, 0)$, resp. $(Q, 0, 0, 0)$. Note that $H^0((K, 0, 0, 0)) = K$, that H^0 is left exact, and that by construction $H^0(\Psi)$ is bijective. Hence we see $K = 0$, i.e., the kernel of Ψ is zero.

The conclusion of the above is that we obtain a short exact sequence

$$0 \to H^0(\mathbf{M}) \otimes_R S \to M' \to Q \to 0$$

and that $M_i = H^0(\mathbf{M})_{f_i}$. Note that we may think of Q as an R-module which is I-power torsion so that $Q = Q \otimes_R S$. By Lemma 15.89.8 we see that there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbf{M}) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(\mathbf{M}) \otimes_R S \arrow[r] & M' \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

with exact rows. This clearly determines an isomorphism $\text{Can}(E) \to (M', M_i, \alpha_i, \alpha_{ij})$ in the category $\text{Glue}(R \to S, f_1, \ldots, f_t)$ and we win. (Of course, a posteriori we have $Q = 0$.) $\square$

0ALK Lemma 15.89.16. Let $\varphi : R \to S$ be a flat ring map and let $I \subset R$ be a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism.

(1) Given an R-module N, an S-module M' and an S-module map $\varphi : M' \to N \otimes_R S$ whose kernel and cokernel are I-power torsion, there exists an R-module map $\psi : M \to N$ and an isomorphism $M \otimes_R S = M'$ compatible with φ and ψ.
(2) Given an R-module M, an S-module N' and an S-module map $\varphi : M \otimes_R S \to N'$ whose kernel and cokernel are I-power torsion, there exists an R-module map $\psi : M \to N$ and an isomorphism $N \otimes_R S = N'$ compatible with φ and ψ.

In both cases we have $\text{Ker}(\varphi) \cong \text{Ker}(\psi)$ and $\text{Coker}(\varphi) \cong \text{Coker}(\psi)$.

Proof. Proof of (1). Say $I = (f_1, \ldots, f_t)$. It is clear that the localization φ_{f_i} is an isomorphism. Thus we see that $(M', N_{f_i}, \varphi_{f_i}, can_{ij})$ is an object of $\text{Glue}(R \to S, f_1, \ldots, f_t)$, see Remark 15.89.10. By Proposition 15.89.15 we conclude that there exists an R-module M such that $M' = M \otimes_R S$ and $N_{f_i} = M_{f_i}$ compatibly with the isomorphisms φ_{f_i} and can_{ij}. There is a morphism

$$(M \otimes_R S, M_{f_i}, can_i, can_{ij}) = (M', N_{f_i}, \varphi_{f_i}, can_{ij}) \to (N \otimes_R S, N_{f_i}, can_i, can_{ij})$$

of $\text{Glue}(R \to S, f_1, \ldots, f_t)$ which uses φ in the first component. This corresponds to an R-module map $\psi : M \to N$ (by the equivalence of categories of Proposition

15.89.15). The composition of the base change of $M \to N$ with the isomorphism $M' \cong M \otimes_R S$ is φ, in other words $M \to N$ is compatible with φ.

Proof of (2). This is just the dual of the argument above. Namely, the localization φ_{f_i} is an isomorphism. Thus we see that $(N', M_{f_i}, \varphi_{f_i}^{-1}, can_{ij})$ is an object of $\text{Glue}(R \to S, f_1, \ldots, f_t)$, see Remark 15.89.10. By Proposition 15.89.15 we conclude that there exists an R-module N such that $N' = N \otimes_R S$ and $N_{f_i} = M_{f_i}$ compatibly with the isomorphisms $\varphi_{f_i}^{-1}$ and can_{ij}. There is a morphism

$$(M \otimes_R S, M_{f_i}, can_i, can_{ij}) \to (N', M_{f_i}, \varphi_{f_i}, can_{ij}) = (N \otimes_R S, N_{f_i}, can_i, can_{ij})$$

of $\text{Glue}(R \to S, f_1, \ldots, f_t)$ which uses φ in the first component. This corresponds to an R-module map $\psi : M \to N$ (by the equivalence of categories of Proposition 15.89.15). The composition of the base change of $M \to N$ with the isomorphism $N' \cong N \otimes_R S$ is φ, in other words $M \to N$ is compatible with φ.

The final statement follows for example from Lemma 15.89.3. $\square$

Next, we specialize Proposition 15.89.15 to get something more useable. Namely, if $I = (f)$ is a principal ideal then the objects of $\text{Glue}(R \to S, f)$ are simply triples (M', M_1, α_1) and there is no cocycle condition to check!

05ES Theorem 15.89.17. Let R be a ring, and let $f \in R$. Let $\varphi : R \to S$ be a flat ring map inducing an isomorphism $R/fR \to S/fS$. Then the functor

$$\text{Mod}_R \longrightarrow \text{Mod}_S \times_{\text{Mod}_{S_f}} \text{Mod}_{R_f}, \quad M \longmapsto (M \otimes_R S, M_f, \text{can})$$

is an equivalence.

Proof. The category appearing on the right side of the arrow is the category of triples (M', M_1, α_1) where M' is an S-module, M_1 is a R_f-module, and $\alpha_1 : M'_f \to M_1 \otimes_R S$ is a S_f-isomorphism, see Categories, Example 4.31.3. Hence this theorem is a special case of Proposition 15.89.15. $\square$

A useful special case of Theorem 15.89.17 is when R is Noetherian, and S is a completion of R at an element f. The completion $R \to S$ is flat, and the functor $M \mapsto M \otimes_R S$ can be identified with the f-adic completion functor when M is finitely generated. To state this more precisely, let Mod_R^{fg} denote the category of finitely generated R-modules.

05ET Proposition 15.89.18. Let R be a Noetherian ring. Let $f \in R$ be an element. Let $R^\wedge$ be the f-adic completion of R. Then the functor $M \mapsto (M^\wedge, M_f, \text{can})$ defines an equivalence

$$\text{Mod}_R^{fg} \longrightarrow \text{Mod}_{R^\wedge}^{fg} \times_{\text{Mod}_{(R^\wedge)_f}^{fg}} \text{Mod}_{R_f}^{fg}$$

Proof. The ring map $R \to R^\wedge$ is flat by Algebra, Lemma 10.97.2. It is clear that $R/fR = R^\wedge/fR^\wedge$. By Algebra, Lemma 10.97.1 the completion of a finite R-module M is equal to $M \otimes_R R^\wedge$. Hence the displayed functor of the proposition is equal to the functor occurring in Theorem 15.89.17. In particular it is fully faithful. Let (M_1, M_2, ψ) be an object of the right hand side. By Theorem 15.89.17 there exists an R-module M such that $M_1 = M \otimes_R R^\wedge$ and $M_2 = M_f$. As $R \to R^\wedge \times R_f$ is faithfully flat we conclude from Algebra, Lemma 10.23.2 that M is finitely generated, i.e., $M \in \text{Mod}_R^{fg}$. This proves the proposition. $\square$

05EU Remark 15.89.19. The equivalences of Proposition 15.89.15, Theorem 15.89.17, and Proposition 15.89.18 preserve properties of modules. For example if M corresponds to $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ then M is finite, or finitely presented, or flat, or projective over R if and only if M' and M_i have the corresponding property over S and R_{f_i}. This follows from the fact that $R \to S \times \prod R_{f_i}$ is faithfully flat and descend and ascent of these properties along faithfully flat maps, see Algebra, Lemma 10.83.2 and Theorem 10.95.6. These functors also preserve the $\otimes$-structures on either side. Thus, it defines equivalences of various categories built out of the pair $(\text{Mod}_R, \otimes)$, such as the category of algebras.

05EV Remark 15.89.20. Given a differential manifold X with a compact closed submanifold Z having complement U, specifying a sheaf on X is the same as specifying a sheaf on U, a sheaf on an unspecified tubular neighbourhood T of Z in X, and an isomorphism between the two resulting sheaves along $T \cap U$. Tubular neighbourhoods do not exist in algebraic geometry as such, but results such as Proposition 15.89.15, Theorem 15.89.17, and Proposition 15.89.18 allow us to work with formal neighbourhoods instead.

15.90. The Beauville-Laszlo theorem

0BNI Let R be a ring and let f be an element of R. Denote $R^\wedge = \lim R/f^nR$ the f-adic completion of R. In this section we discuss and slightly generalize a theorem of Beauville and Laszlo, see [BL95]. The theorem asserts that under suitable conditions, a module over R can be constructed by "glueing together" modules over $R^\wedge$ and R_f along an isomorphism between the base extensions to $(R^\wedge)_f$.

In [BL95] it is assumed that f is a nonzerodivisor on both R and M. In fact, one only needs to assume that

$$R[f^\infty] \longrightarrow R^\wedge[f^\infty]$$

is bijective and that

$$M[f^\infty] \longrightarrow M \otimes_R R^\wedge$$

is injective. This optimization was partly inspired by an alternate approach to glueing introduced in [KL15, §1.3] for use in the theory of nonarchimedean analytic spaces.

In fact, we will establish the Beauville-Laszlo theorem in the more general setting of a ring map

$$R \longrightarrow R'$$

which induces isomorphisms $R/f^nR \to R'/f^nR'$ for every $n > 0$ and an isomorphism $R[f^\infty] \to R'[f^\infty]$. This is better suited for globalizing and does not formally follow from the case when R' is the completion of R because, for instance, the condition that $R[f^\infty] \to R'[f^\infty]$ is a bijection does not imply that $R[f^\infty] \to R^\wedge[f^\infty]$ is a bijection.

The theorem of Beauville and Laszlo as proved in this section can be viewed as a non-flat version of Theorem 15.89.17 and in the case where $R' = R^\wedge$ can be viewed as a non-Noetherian version of Proposition 15.89.18. For a comparison with flat descent, please see Remark 15.90.6.

One can establish even stronger results (without imposing restrictions on M for example) but for this one must work at the level of derived categories. See [Bha16, §5] for more details.

0BNJ **Lemma 15.90.1.** Let R be a ring and let $f \in R$. For every positive integer n the map $R/f^nR \to R^\wedge/f^nR^\wedge$ is an isomorphism.

Proof. This is a special case of Algebra, Lemma 10.96.3. ☐

We will use the notation introduced in Section 15.88. Thus for an R-module M, we denote $M[f^n]$ the submodule of M annihilated by f^n and we put

$$M[f^\infty] = \bigcup\nolimits_{n=1}^{\infty} M[f^n] = \mathrm{Ker}(M \to M_f).$$

If $M = M[f^\infty]$, we say that M is an f-power torsion module.

0BNK **Lemma 15.90.2.** Let R be a ring, let $f \in R$ be an element, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. For any f-power torsion R-module M the map $M \to M \otimes_R R'$ is an isomorphism. For example, we have $M \cong M \otimes_R R^\wedge$. Slight generalization of [BL95, Lemme 1].

Proof. If M is annihilated by f^n, then

$$M \otimes_R R' \cong M \otimes_{R/f^nR} R'/f^nR' \cong M \otimes_{R/f^nR} R/f^nR \cong M.$$

Since $M = \bigcup M[f^n]$ and since tensor products commute with direct limits (Algebra, Lemma 10.12.9), we obtain the desired isomorphism. The last statement is a special case of the first statement by Lemma 15.90.1. ☐

0BNL **Lemma 15.90.3.** Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The R-module $R' \oplus R_f$ is faithful: for every nonzero R-module M, the module $M \otimes_R (R' \oplus R_f)$ is also nonzero. For example, if M is nonzero, then $M \otimes_R (R^\wedge \oplus R_f)$ is nonzero.

However, the map $M \to M \otimes_R (R' \oplus R_f)$ need not be injective; see Example 15.90.10.

Proof. If $M \neq 0$ but $M \otimes_R R_f = 0$, then M is f-power torsion. By Lemma 15.90.2 we find that $M \otimes_R R' \cong M \neq 0$. The last statement is a special case of the first statement by Lemma 15.90.1. ☐

0BNM **Lemma 15.90.4.** Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces an isomorphism $R/fR \to R'/fR'$. The map $\mathrm{Spec}(R') \amalg \mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ is surjective. For example, the map $\mathrm{Spec}(R^\wedge) \amalg \mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ is surjective.

Proof. Recall that $\mathrm{Spec}(R) = V(f) \amalg D(f)$ where $V(f) = \mathrm{Spec}(R/fR)$ and $D(f) = \mathrm{Spec}(R_f)$, see Algebra, Section 10.17 and especially Lemmas 10.17.7 and 10.17.6. Thus the lemma follows as the map $R \to R/fR$ factors through R'. The last statement is a special case of the first statement by Lemma 15.90.1. ☐

0BNN **Lemma 15.90.5.** Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. An R-module M is finitely generated if and only if the $(R' \oplus R_f)$-module $M \otimes_R (R' \oplus R_f)$ is finitely generated. For example, if $M \otimes_R (R^\wedge \oplus R_f)$ is finitely generated as a module over $R^\wedge \oplus R_f$, then M is a finitely generated R-module. Slight generalization of [BL95, Lemme 2(a)].

Proof. The 'only if' is clear, so we assume that $M \otimes_R (R' \oplus R_f)$ is finitely generated. In this case, by writing each generator as a sum of simple tensors, $M \otimes_R (R' \oplus R_f)$ admits a finite generating set consisting of elements of M. That is, there exists a morphism from a finite free R-module to M whose cokernel is killed by tensoring with $R' \oplus R_f$; we may thus deduce M is finite generated by applying Lemma 15.90.3 to this cokernel. The last statement is a special case of the first statement by Lemma 15.90.1. $\square$

0BNP Remark 15.90.6. While $R \to R_f$ is always flat, $R \to R^\wedge$ is typically not flat unless R is Noetherian (see Algebra, Lemma 10.97.2 and the discussion in Examples, Section 110.12). Consequently, we cannot in general apply faithfully flat descent as discussed in Descent, Section 35.3 to the morphism $R \to R^\wedge \oplus R_f$. Moreover, even in the Noetherian case, the usual definition of a descent datum for this morphism refers to the ring $R^\wedge \otimes_R R^\wedge$, which we will avoid considering in this section.

Glueing pairs. Let $R \to R'$ be a ring map that induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. Consider the sequence

0F1Q (15.90.6.1) $$0 \to R \to R' \oplus R_f \to R'_f \to 0,$$

in which the map on the right is the difference between the two canonical homomorphisms. If this sequence is exact, then we say that $(R \to R', f)$ is a glueing pair. We will say that (R, f) is a glueing pair if $(R \to R^\wedge, f)$ is a glueing pair; this makes sense by Lemma 15.90.1. Thus (R, f) is a glueing pair if and only if the sequence

0BNQ (15.90.6.2) $$0 \to R \to R^\wedge \oplus R_f \to (R^\wedge)_f \to 0,$$

is exact.

0BNR Lemma 15.90.7. Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The sequence (15.90.6.1) is

(1) exact on the right,
(2) exact on the left if and only if $R[f^\infty] \to R'[f^\infty]$ is injective, and
(3) exact in the middle if and only if $R[f^\infty] \to R'[f^\infty]$ is surjective.

In particular, $(R \to R', f)$ is a glueing pair if and only if $R[f^\infty] \to R'[f^\infty]$ is bijective. For example, (R, f) is a glueing pair if and only if $R[f^\infty] \to R^\wedge[f^\infty]$ is bijective.

Proof. Let $x \in R'_f$. Write $x = x'/f^n$ with $x' \in R'$. Write $x' = x'' + f^ny$ with $x'' \in R$ and $y \in R'$. Then we see that $(y, -x''/f^n)$ maps to x. Thus (1) holds.

Part (2) follows from the fact that $\text{Ker}(R \to R_f) = R[f^\infty]$.

If the sequence is exact in the middle, then elements of the form $(x, 0)$ with $x \in R'[f^\infty]$ are in the image of the first arrow. This implies that $R[f^\infty] \to R'[f^\infty]$ is surjective. Conversely, assume that $R[f^\infty] \to R'[f^\infty]$ is surjective. Let (x, y) be an element in the middle which maps to zero on the right. Write $y = y'/f^n$ for some $y' \in R$. Then we see that $f^nx - y'$ is annihilated by some power of f in R'. By assumption we can write $f^nx - y' = z$ for some $z \in R[f^\infty]$. Then $y = y''/f^n$ where $y'' = y' + z$ is in the kernel of $R \to R/f^nR$. Hence we see that y can be represented as $y'''/1$ for some $y''' \in R$. Then $x - y'''$ is in $R'[f^\infty]$. Thus $x - y''' = z' \in R[f^\infty]$. Then $(x, y'''/1) = (y''' + z', (y''' + z')/1)$ as desired.

The last statement of the lemma is a special case of the penultimate statement by Lemma 15.90.1. $\square$

0BNS Remark 15.90.8. Suppose that f is a nonzerodivisor. Then Algebra, Lemma 10.96.4 shows that f is a nonzerodivisor in $R^\wedge$. Hence (R, f) is a glueing pair.

0BNT Remark 15.90.9. If $R \to R^\wedge$ is flat, then for each positive integer n tensoring the sequence $0 \to R[f^n] \to R \to R$ with $R^\wedge$ gives the sequence $0 \to R[f^n] \otimes_R R^\wedge \to R^\wedge \to R^\wedge$. Combined with Lemma 15.90.2 we conclude that $R[f^n] \to R^\wedge[f^n]$ is an isomorphism. Thus (R, f) is a glueing pair. This holds in particular if R is Noetherian, see Algebra, Lemma 10.97.2.

0BNU Example 15.90.10. Let k be a field and put

$$R = k[f, T_1, T_2, \ldots]/(fT_1, fT_2 - T_1, fT_3 - T_2, \ldots).$$

Then (R, f) is not a glueing pair because the map $R[f^\infty] \to R^\wedge[f^\infty]$ is not injective as the image of T_1 is f-divisible in $R^\wedge$. For

$$R = k[f, T_1, T_2, \ldots]/(fT_1, f^2T_2, \ldots),$$

the map $R[f^\infty] \to R^\wedge[f^\infty]$ is not surjective as the element $T_1 + fT_2 + f^2T_3 + \ldots$ is not in the image. In particular, by Remark 15.90.9, these are both examples where $R \to R^\wedge$ is not flat.

Glueable modules. Let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. For any R-module M, we may tensor (15.90.6.1) with M to obtain a sequence

0F1R (15.90.10.1) $$0 \to M \to (M \otimes_R R') \oplus (M \otimes_R R_f) \to M \otimes_R R'_f \to 0$$

Observe that $M \otimes_R R_f = M_f$ and that $M \otimes_R R'_f = (M \otimes_R R')_f$. If this sequence is exact, we say that M is glueable for $(R \to R', f)$. If R is a ring and $f \in R$, then we say an R-module is glueable if M is glueable for $(R \to R^\wedge, f)$. Thus M is glueable if and only if the sequence

0BNV (15.90.10.2) $$0 \to M \to (M \otimes_R R^\wedge) \oplus (M \otimes_R R_f) \to M \otimes_R (R^\wedge)_f \to 0$$

is exact.

0BNW Lemma 15.90.11. Let R be a ring, let $f \in R$, and let $R \to R'$ be a ring map which induces isomorphisms $R/f^nR \to R'/f^nR'$ for $n > 0$. The sequence (15.90.10.1) is

(1) exact on the right,
(2) exact on the left if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is injective, and
(3) exact in the middle if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective.

Thus M is glueable for $(R \to R', f)$ if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is bijective. If $(R \to R', f)$ is a glueing pair, then M is glueable for $(R \to R', f)$ if and only if $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is injective. For example, if (R, f) is a glueing pair, then M is glueable if and only if $M[f^\infty] \to (M \otimes_R R^\wedge)[f^\infty]$ is injective.

Proof. We will use the results of Lemma 15.90.7 without further mention. The functor $M \otimes_R -$ is right exact (Algebra, Lemma 10.12.10) hence we get (1).

The kernel of $M \to M \otimes_R R_f = M_f$ is $M[f^\infty]$. Thus (2) follows.

If the sequence is exact in the middle, then elements of the form $(x, 0)$ with $x \in (M \otimes_R R')[f^\infty]$ are in the image of the first arrow. This implies that $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective. Conversely, assume that $M[f^\infty] \to (M \otimes_R R')[f^\infty]$ is surjective. Let (x, y) be an element in the middle which maps to zero on the right. Write $y = y'/f^n$ for some $y' \in M$. Then we see that $f^nx - y'$ is annihilated by some power of f in $M \otimes_R R'$. By assumption we can write $f^n x - y' = z$ for some $z \in M[f^\infty]$. Then $y = y''/f^n$ where $y'' = y' + z$ is in the kernel of $M \to M/f^nM$. Hence we see that y can be represented as $y'''/1$ for some $y''' \in M$. Then $x - y'''$ is in $(M \otimes_R R')[f^\infty]$. Thus $x - y''' = z' \in M[f^\infty]$. Then $(x, y'''/1) = (y''' + z', (y''' + z')/1)$ as desired.

If $(R \to R', f)$ is a glueing pair, then (15.90.10.1) is exact in the middle for any M by Algebra, Lemma 10.12.10. This gives the penultimate statement of the lemma. The final statement of the lemma follows from this and the fact that (R, f) is a glueing pair if and only if $(R \to R^\wedge, f)$ is a glueing pair. $\square$

0BNX Remark 15.90.12. Let $(R \to R', f)$ be a glueing pair and let M be an R-module. Here are some observations which can be used to determine whether M is glueable for $(R \to R', f)$.

(1) By Lemma 15.90.11 we see that M is glueable for $(R \to R^\wedge, f)$ if and only if $M[f^\infty] \to M \otimes_R R^\wedge$ is injective. This holds if $M[f] \to M^\wedge$ is injective, i.e., when $M[f] \cap \bigcap_{n=1}^{\infty} f^n M = 0$.
(2) If $\text{Tor}_1^R(M, R'_f) = 0$, then M is glueable for $(R \to R', f)$ (use Algebra, Lemma 10.75.2). This is equivalent to saying that $\text{Tor}_1^R(M, R')$ is f-power torsion. In particular, any flat R-module is glueable for $(R \to R', f)$.
(3) If $R \to R'$ is flat, then $\text{Tor}_1^R(M, R') = 0$ for every R-module so every R-module is glueable for $(R \to R', f)$. This holds in particular when R is Noetherian and $R' = R^\wedge$, see Algebra, Lemma 10.97.2

0BNY Example 15.90.13 (Non glueable module). Let R be the ring of germs at 0 of C^∞ functions on $\mathbf{R}$. Let $f \in R$ be the function $f(x) = x$. Then f is a nonzerodivisor in R, so (R, f) is a glueing pair and $R^\wedge \cong \mathbf{R}[[x]]$. Let $\varphi \in R$ be the function $\varphi(x) = \exp(-1/x^2)$. Then φ has zero Taylor series, so $\varphi \in \text{Ker}(R \to R^\wedge)$. Since $\varphi(x) \neq 0$ for $x \neq 0$, we see that φ is a nonzerodivisor in R. The function φ/f also has zero Taylor series, so its image in $M = R/\varphi R$ is a nonzero element of $M[f]$ which maps to zero in $M \otimes_R R^\wedge = R^\wedge/\varphi R^\wedge = R^\wedge$. Hence M is not glueable. [BL95, §4, Remarques]

We next make some calculations of Tor groups.

0BNZ Lemma 15.90.14. Let $(R \to R', f)$ be a glueing pair. Then $\text{Tor}_1^R(R', f^nR) = 0$ for each $n > 0$.

Proof. From the exact sequence $0 \to R[f^n] \to R \to f^nR \to 0$ we see that it suffices to check that $R[f^n] \otimes_R R' \to R'$ is injective. By Lemma 15.90.2 we have $R[f^n] \otimes_R R' = R[f^n]$ and by Lemma 15.90.7 we see that $R[f^n] \to R'$ is injective as $(R \to R', f)$ is a glueing pair. $\square$

0BP0 Lemma 15.90.15. Let $(R \to R', f)$ be a glueing pair. Then $\text{Tor}_1^R(R', R/R[f^\infty]) = 0$.

Proof. We have $R/R[f^\infty] = \text{colim}\, R/R[f^n] = \text{colim}\, f^nR$. As formation of Tor groups commutes with filtered colimits (Algebra, Lemma 10.76.2) we may apply Lemma 15.90.14. $\square$

0BP1 Lemma 15.90.16. Let $(R \to R', f)$ be a glueing pair. For every R-module M, we have $\text{Tor}_1^R(R', \text{Coker}(M \to M_f)) = 0$. Slight generalization of [BL95, Lemme 3(a)]

Proof. Set $\overline{M} = M/M[f^\infty]$. Then $\text{Coker}(M \to M_f) \cong \text{Coker}(\overline{M} \to \overline{M}_f)$ hence we may and do assume that f is a nonzerodivisor on M. In this case $M \subset M_f$

and $M_f/M = \operatorname{colim} M/f^nM$ where the transition maps are given by multiplication by f. Since formation of Tor groups commutes with colimits (Algebra, Lemma 10.76.2) it suffices to show that $\operatorname{Tor}_1^R(R', M/f^nM) = 0$.

We first treat the case $M = R/R[f^\infty]$. By Lemma 15.90.7 we have $M \otimes_R R' = R'/R'[f^\infty]$. From the short exact sequence $0 \to M \to M \to M/f^nM \to 0$ we obtain the exact sequence

$$\begin{tikzcd} \operatorname{Tor}_1^R(R', R/R[f^\infty]) \arrow[r] & \operatorname{Tor}_1^R(R', M/f^nM) \arrow[r] & R'/R'[f^\infty] \arrow[dll, "f^n"'] \\ R'/R'[f^\infty] \arrow[r] & (R'/R'[f^\infty])/(f^n(R'/R'[f^\infty])) \arrow[r] & 0 \end{tikzcd}$$

by Algebra, Lemma 10.75.2. Here the diagonal arrow is injective. Since the first group $\operatorname{Tor}_1^R(R', R/R[f^\infty])$ is zero by Lemma 15.90.15, we deduce that $\operatorname{Tor}_1^R(R', M/f^nM) = 0$ as desired.

To treat the general case, choose a surjection $F \to M$ with F a free $R/R[f^\infty]$-module, and form an exact sequence

$$0 \to N \to F/f^nF \to M/f^nM \to 0.$$

By Lemma 15.90.2 this sequence remains unchanged, and hence exact, upon tensoring with R'. Since $\operatorname{Tor}_1^R(R', F/f^nF) = 0$ by the previous paragraph, we deduce that $\operatorname{Tor}_1^R(R', M/f^nM) = 0$ as desired. $\square$

Let $(R \to R', f)$ be a glueing pair. This means that $R/f^nR \to R'/f^nR'$ is an isomorphism for $n > 0$ and the sequence

$$0 \to R \to R' \oplus R_f \to R'_f \to 0$$

is exact. Consider the category $\operatorname{Glue}(R \to R', f)$ introduced in Remark 15.89.10. We will call an object (M', M_1, α_1) of $\operatorname{Glue}(R \to R', f)$ a glueing datum. It consists of an R'-module M', an R_f-module M_1, and an isomorphism $\alpha_1 : (M')_f \to M_1 \otimes_R R'$. There is an obvious functor

$$\operatorname{Can} : \operatorname{Mod}_R \longrightarrow \operatorname{Glue}(R \to R', f), \quad M \longmapsto (M \otimes_R R', M_f, \operatorname{can}),$$

and there is a functor

$$H^0 : \operatorname{Glue}(R \to R', f) \longrightarrow \operatorname{Mod}_R, \quad (M', M_1, \alpha_1) \longmapsto \operatorname{Ker}(M' \oplus M_1 \to (M')_f)$$

in the reverse direction, see Remark 15.89.10 for the precise definition.

0BP2 Theorem 15.90.17. Let $(R \to R', f)$ be a glueing pair. The functor $\operatorname{Can} : \operatorname{Mod}_R \longrightarrow \operatorname{Glue}(R \to R', f)$ determines an equivalence of the category of R-modules glueable for $(R \to R', f)$ and the category $\operatorname{Glue}(R \to R', f)$ of glueing data.

Slight generalization of the main theorem of [BL95].

Proof. The functor is fully faithful due to the exactness of (15.90.10.1) for glueable modules, which tells us exactly that $H^0 \circ \operatorname{Can} = \operatorname{id}$ on the full subcategory of glueable modules. Hence it suffices to check essential surjectivity. That is, we must show that an arbitrary glueing datum (M', M_1, α_1) arises from some glueable R-module.

We first check that the map $\mathrm{d} : M' \oplus M_1 \to (M')_f$ used in the definition of the functor H^0 is surjective. Observe that $(x, y) \in M' \oplus M_1$ maps to $\mathrm{d}(x, y) = x/1 - \alpha_1^{-1}(y \otimes 1)$ in $(M')_f$. If $z \in (M')_f$, then we can write $\alpha_1(z) = \sum y_i \otimes g_i$ with

$g_i \in R'$ and $y_i \in M_1$. Write $\alpha_1^{-1}(y_i \otimes 1) = y_i'/f^n$ for some $y_i' \in M'$ and $n \geq 0$ (we can pick the same n for all i). Write $g_i = a_i + f^n b_i$ with $a_i \in R$ and $b_i \in R'$. Then with $y = \sum a_i y_i \in M_1$ and $x = \sum b_i y_i' \in M'$ we have $\mathrm{d}(x, -y) = z$ as desired.

Put $M = H^0((M', M_1, \alpha_1)) = \mathrm{Ker}(\mathrm{d})$. We obtain an exact sequence of R-modules

0BP3 (15.90.17.1)
$$0 \to M \to M' \oplus M_1 \to (M')_f \to 0.$$

We will prove that the maps $M \to M'$ and $M \to M_1$ induce isomorphisms $M \otimes_R R' \to M'$ and $M \otimes_R R_f \to M_1$. This will imply that M is glueable for $(R \to R', f)$ and gives rise to the original glueing datum.

Since f is a nonzerodivisor on M_1, we have $M[f^\infty] \cong M'[f^\infty]$. This yields an exact sequence

0BP4 (15.90.17.2)
$$0 \to M/M[f^\infty] \to M_1 \to (M')_f/M' \to 0.$$

Since $R \to R_f$ is flat, we may tensor this exact sequence with R_f to deduce that $M \otimes_R R_f = (M/M[f^\infty]) \otimes_R R_f \to M_1$ is an isomorphism.

By Lemma 15.90.16 we have $\mathrm{Tor}_1^R(R', \mathrm{Coker}(M' \to (M')_f)) = 0$. The sequence (15.90.17.2) thus remains exact upon tensoring over R with R'. Using α_1 and Lemma 15.90.2 the resulting exact sequence can be written as

0BP5 (15.90.17.3)
$$0 \to (M/M[f^\infty]) \otimes_R R' \to (M')_f \to (M')_f/M' \to 0$$

This yields an isomorphism $(M/M[f^\infty]) \otimes_R R' \cong M'/M'[f^\infty]$. This implies that in the diagram

$$\begin{tikzcd}
 & M[f^\infty] \otimes_R R' \arrow[r] \arrow[d] & M \otimes_R R' \arrow[r] \arrow[d] & (M/M[f^\infty]) \otimes_R R' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'[f^\infty] \arrow[r] & M' \arrow[r] & M'/M'[f^\infty] \arrow[r] & 0,
\end{tikzcd}$$

the third vertical arrow is an isomorphism. Since the rows are exact and the first vertical arrow is an isomorphism by Lemma 15.90.2 and $M[f^\infty] = M'[f^\infty]$, the five lemma implies that $M \otimes_R R' \to M'$ is an isomorphism. This completes the proof. □

0BP9 Remark 15.90.18. Let $(R \to R', f)$ be a glueing pair. Let M be an R-module that is not necessarily glueable for $(R \to R', f)$. Setting $M' = M \otimes_R R'$ and $M_1 = M_f$ we obtain the glueing datum $\mathrm{Can}(M) = (M', M_1, \mathrm{can})$. Then $\tilde{M} = H^0(M', M_1, \mathrm{can})$ is an R-module that is glueable for $(R \to R', f)$ and the canonical map $M \to \tilde{M}$ gives isomorphisms $M \otimes_R R' \to \tilde{M} \otimes_R R'$ and $M_f \to \tilde{M}_f$, see Theorem 15.90.17. From the exactness of the sequences

$$M \to (M \otimes_R R') \oplus M_f \to M \otimes_R (R')_f \to 0$$

and

$$0 \to \tilde{M} \to (\tilde{M} \otimes_R R') \oplus \tilde{M}_f \to \tilde{M} \otimes_R (R')_f \to 0$$

we conclude that the map $M \to \tilde{M}$ is surjective.

Recall that flat R-modules over a glueing pair $(R \to R', f)$ are glueable (Remark 15.90.12). Hence the following lemma shows that Theorem 15.90.17 determines an equivalence between the category of flat R-modules and the category of glueing data (M', M_1, α_1) where M' and M_1 are flat over R' and R_f.

0BP7 **Lemma 15.90.19.** Let $(R \to R', f)$ be a glueing pair. Let M be an R-module which is not necessarily glueable for $(R \to R', f)$. Then M is flat over R if and only if $M \otimes_R R'$ is flat over R' and M_f is flat over R_f.

Proof. One direction of the lemma follows from Algebra, Lemma 10.39.7. For the other direction, assume $M \otimes_R R'$ is flat over R' and M_f is flat over R_f. Let $\tilde{M}$ be as in Remark 15.90.18. If $\tilde{M}$ is flat over R, then applying Algebra, Lemma 10.39.12 to the short exact sequence $0 \to \mathrm{Ker}(M \to \tilde{M}) \to M \to \tilde{M} \to 0$ we find that $\mathrm{Ker}(M \to \tilde{M}) \otimes_R (R' \oplus R_f)$ is zero. Hence $M = \tilde{M}$ by Lemma 15.90.3 and we conclude. In other words, we may replace M by $\tilde{M}$ and assume M is glueable for $(R \to R', f)$. Let N be a second R-module. It suffices to prove that $\mathrm{Tor}_1^R(M, N) = 0$, see Algebra, Lemma 10.75.8.

The long the exact sequence of Tors associated to the short exact sequence $0 \to R \to R' \oplus R_f \to (R')_f \to 0$ and N gives an exact sequence

$$0 \to \mathrm{Tor}_1^R(R', N) \to \mathrm{Tor}_1^R((R')_f, N)$$

and isomorphisms $\mathrm{Tor}_i^R(R', N) = \mathrm{Tor}_i^R((R')_f, N)$ for $i \geq 2$. Since $\mathrm{Tor}_i^R((R')_f, N) = \mathrm{Tor}_i^R(R', N)_f$ we conclude that f is a nonzerodivisor on $\mathrm{Tor}_1^R(R', N)$ and invertible on $\mathrm{Tor}_i^R(R', N)$ for $i \geq 2$. Since $M \otimes_R R'$ is flat over R' we have

$$\mathrm{Tor}_i^R(M \otimes_R R', N) = (M \otimes_R R') \otimes_{R'} \mathrm{Tor}_i^R(R', N)$$

by the spectral sequence of Example 15.62.2. Writing $M \otimes_R R'$ as a filtered colimit of finite free R'-modules (Algebra, Theorem 10.81.4) we conclude that f is a nonzerodivisor on $\mathrm{Tor}_1^R(M \otimes_R R', N)$ and invertible on $\mathrm{Tor}_i^R(M \otimes_R R', N)$. Next, we consider the exact sequence $0 \to M \to M \otimes_R R' \oplus M_f \to M \otimes_R (R')_f \to 0$ coming from the fact that M is glueable and the associated long exact sequence of Tor. The relevant part is

$$\begin{tikzcd}
\mathrm{Tor}_1^R(M, N) \arrow[r] & \mathrm{Tor}_1^R(M \otimes_R R', N) \arrow[r] & \mathrm{Tor}_1^R(M \otimes_R (R')_f, N) \\
& \mathrm{Tor}_2^R(M \otimes_R R', N) \arrow[r] & \mathrm{Tor}_2^R(M \otimes_R (R')_f, N) \arrow[llu]
\end{tikzcd}$$

We conclude that $\mathrm{Tor}_1^R(M, N) = 0$ by our remarks above on the action on f on $\mathrm{Tor}_i^R(M \otimes_R R', N)$. $\square$

Observe that we have seen the result of the following lemma for "finitely generated" in Lemma 15.90.5.

0BP6 **Lemma 15.90.20.** Let $(R \to R', f)$ be a glueing pair. Let M be an R-module which is not necessarily glueable for $(R \to R', f)$. Then M is a finite projective R-module if and only if $M \otimes_R R'$ is finite projective over R' and M_f is finite projective over R_f.

Proof. Assume that $M \otimes_R R'$ is a finite projective module over R' and that M_f is a finite projective module over R_f. Our task is to prove that M is finite projective over R. We will use Algebra, Lemma 10.78.2 without further mention. By Lemma 15.90.19 we see that M is flat. By Lemma 15.90.5 we see that M is finite. Choose a short exact sequence $0 \to K \to R^{\oplus n} \to M \to 0$. Since a finite projective module is of finite presentation and since the sequence remains exact after tensoring with

R' (by Algebra, Lemma 10.39.12) and R_f, we conclude that $K \otimes_R R'$ and K_f are finite modules. Using the lemma above we conclude that K is finitely generated. Hence M is finitely presented and hence finite projective. □

0BP8 Remark 15.90.21. In [BL95] it is assumed that f is a nonzerodivisor in R and $R' = R^\wedge$, which gives a glueing pair by Lemma 15.90.7. Even in this setting Theorem 15.90.17 says something new: the results of [BL95] only apply to modules on which f is a nonzerodivisor (and hence glueable in our sense, see Lemma 15.90.11). Lemma 15.90.20 also provides a slight extension of the results of [BL95]: not only can we allow M to have nonzero f-power torsion, we do not even require it to be glueable.

15.91. Derived Completion

091N Some references for the material in this section are [DG02], [GM92], [PSY14a], [Lur11] (especially Chapter 4). Our exposition follows [BS13]. The analogue (or "dual") of this section for torsion modules is Dualizing Complexes, Section 47.9. The relationship between the derived category of complexes with torsion cohomology and derived complete complexes can be found in Dualizing Complexes, Section 47.12.

Let $K \in D(A)$. Let $f \in A$. We denote $T(K, f)$ a derived limit of the system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(A)$.

091P Lemma 15.91.1. Let A be a ring. Let $f \in A$. Let $K \in D(A)$. The following are equivalent

(1) $\text{Ext}^n_A(A_f, K) = 0$ for all n,
(2) $\text{Hom}_{D(A)}(E, K) = 0$ for all E in $D(A_f)$,
(3) $T(K, f) = 0$,
(4) for every $p \in \mathbf{Z}$ we have $T(H^p(K), f) = 0$,
(5) for every $p \in \mathbf{Z}$ we have $\text{Hom}_A(A_f, H^p(K)) = 0$ and $\text{Ext}^1_A(A_f, H^p(K)) = 0$,
(6) $R\,\text{Hom}_A(A_f, K) = 0$,
(7) the map $\prod_{n \geq 0} K \to \prod_{n \geq 0} K$, $(x_0, x_1, \ldots) \mapsto (x_0 - fx_1, x_1 - fx_2, \ldots)$ is an isomorphism in $D(A)$, and
(8) add more here.

Proof. It is clear that (2) implies (1) and that (1) is equivalent to (6). Assume (1). Let $I^\bullet$ be a K-injective complex of A-modules representing K. Condition (1) signifies that $\text{Hom}_A(A_f, I^\bullet)$ is acyclic. Let $M^\bullet$ be a complex of A_f-modules representing E. Then

$$\text{Hom}_{D(A)}(E, K) = \text{Hom}_{K(A)}(M^\bullet, I^\bullet) = \text{Hom}_{K(A_f)}(M^\bullet, \text{Hom}_A(A_f, I^\bullet))$$

by Algebra, Lemma 10.14.4. As $\text{Hom}_A(A_f, I^\bullet)$ is a K-injective complex of A_f-modules by Lemma 15.56.3 the fact that it is acyclic implies that it is homotopy equivalent to zero (Derived Categories, Lemma 13.31.2). Thus we get (2).

A free resolution of the A-module A_f is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} A \to \bigoplus\nolimits_{n \in \mathbf{N}} A \to A_f \to 0$$

where the first map sends the $(a_0, a_1, a_2, \ldots)$ to $(a_0, a_1 - fa_0, a_2 - fa_1, \ldots)$ and the second map sends $(a_0, a_1, a_2, \ldots)$ to $a_0 + a_1/f + a_2/f^2 + \ldots$. Applying $\mathrm{Hom}_A(-, I^\bullet)$ we get

$$0 \to \mathrm{Hom}_A(A_f, I^\bullet) \to \prod I^\bullet \to \prod I^\bullet \to 0$$

Since $\prod I^\bullet$ represents $\prod_{n \geq 0} K$ this proves the equivalence of (1) and (7). On the other hand, by construction of derived limits in Derived Categories, Section 13.34 the displayed exact sequence shows the object $T(K, f)$ is a representative of $R\mathrm{Hom}_A(A_f, K)$ in $D(A)$. Thus the equivalence of (1) and (3).

There is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_A(A_f, H^q(K)) \Rightarrow \mathrm{Ext}^{p+q}_A(A_f, K)$$

See Equation (15.67.0.1). This spectral sequence degenerates at E_2 because A_f has a length 1 resolution by projective A-modules (see above) hence the E_2-page has only 2 nonzero columns. Thus we obtain short exact sequences

$$0 \to \mathrm{Ext}^1_A(A_f, H^{p-1}(K)) \to \mathrm{Ext}^p_A(A_f, K) \to \mathrm{Hom}_A(A_f, H^p(K)) \to 0$$

This proves (4) and (5) are equivalent to (1). $\square$

091Q Lemma 15.91.2. Let A be a ring. Let $K \in D(A)$. The set I of $f \in A$ such that $T(K, f) = 0$ is a radical ideal of A.

Proof. We will use the results of Lemma 15.91.1 without further mention. If $f \in I$, and $g \in A$, then A_{gf} is an A_f-module hence $\mathrm{Ext}^n_A(A_{gf}, K) = 0$ for all n, hence $gf \in I$. Suppose $f, g \in I$. Then there is a short exact sequence

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{gf(f+g)} \to 0$$

because f, g generate the unit ideal in A_{f+g}. This follows from Algebra, Lemma 10.24.2 and the easy fact that the last arrow is surjective. From the long exact sequence of Ext and the vanishing of $\mathrm{Ext}^n_A(A_{f(f+g)}, K)$, $\mathrm{Ext}^n_A(A_{g(f+g)}, K)$, and $\mathrm{Ext}^n_A(A_{gf(f+g)}, K)$ for all n we deduce the vanishing of $\mathrm{Ext}^n_A(A_{f+g}, K)$ for all n. Finally, if $f^n \in I$ for some $n > 0$, then $f \in I$ because $T(K, f) = T(K, f^n)$ or because $A_f \cong A_{f^n}$. $\square$

091R Lemma 15.91.3. Let A be a ring. Let $I \subset A$ be an ideal. Let M be an A-module.

(1) If M is I-adically complete, then $T(M, f) = 0$ for all $f \in I$.
(2) Conversely, if $T(M, f) = 0$ for all $f \in I$ and I is finitely generated, then $M \to \lim M/I^nM$ is surjective.

Proof. Proof of (1). Assume M is I-adically complete. By Lemma 15.91.1 it suffices to prove $\mathrm{Ext}^1_A(A_f, M) = 0$ and $\mathrm{Hom}_A(A_f, M) = 0$. Since $M = \lim M/I^nM$ and since $\mathrm{Hom}_A(A_f, M/I^nM) = 0$ it follows that $\mathrm{Hom}_A(A_f, M) = 0$. Suppose we have an extension

$$0 \to M \to E \to A_f \to 0$$

For $n \geq 0$ pick $e_n \in E$ mapping to $1/f^n$. Set $\delta_n = fe_{n+1} - e_n \in M$ for $n \geq 0$. Replace e_n by

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \ldots$$

The infinite sum exists as M is complete with respect to I and $f \in I$. A simple calculation shows that $fe'_{n+1} = e'_n$. Thus we get a splitting of the extension by mapping $1/f^n$ to e'_n.

Proof of (2). Assume that $I = (f_1, \ldots, f_r)$ and that $T(M, f_i) = 0$ for $i = 1, \ldots, r$. By Algebra, Lemma 10.96.7 we may assume $I = (f)$ and $T(M, f) = 0$. Let $x_n \in M$ for $n \geq 0$. Consider the extension

$$0 \to M \to E \to A_f \to 0$$

given by

$$E = M \oplus \bigoplus Ae_n \Big/ \langle x_n - fe_{n+1} + e_n \rangle$$

mapping e_n to $1/f^n$ in A_f (see above). By assumption and Lemma 15.91.1 this extension is split, hence we obtain an element $x + e_0$ which generates a copy of A_f in E. Then

$$x + e_0 = x - x_0 + fe_1 = x - x_0 - fx_1 + f^2 e_2 = \ldots$$

Since $M/f^nM = E/f^nE$ by the snake lemma, we see that $x = x_0 + fx_1 + \ldots + f^{n-1}x_{n-1}$ modulo f^nM. In other words, the map $M \to \lim M/f^nM$ is surjective as desired. □

Motivated by the results above we make the following definition.

091S Definition 15.91.4. Let A be a ring. Let $K \in D(A)$. Let $I \subset A$ be an ideal. We say K is derived complete with respect to I if for every $f \in I$ we have $T(K, f) = 0$. If M is an A-module, then we say M is derived complete with respect to I if $M[0] \in D(A)$ is derived complete with respect to I.

The full subcategory $D_{comp}(A) = D_{comp}(A, I) \subset D(A)$ consisting of derived complete objects is a strictly full, saturated triangulated subcategory, see Derived Categories, Definitions 13.3.4 and 13.6.1. By Lemma 15.91.2 the subcategory $D_{comp}(A, I)$ depends only on the radical $\sqrt{I}$ of I, in other words it depends only on the closed subset $Z = V(I)$ of $\mathrm{Spec}(A)$. The subcategory $D_{comp}(A, I)$ is preserved under products and homotopy limits in $D(A)$. But it is not preserved under countable direct sums in general. We will often simply say M is a derived complete module if the choice of the ideal I is clear from the context.

091T Proposition 15.91.5. Let $I \subset A$ be a finitely generated ideal of a ring A. Let M be an A-module. The following are equivalent

(1) M is I-adically complete, and
(2) M is derived complete with respect to I and $\bigcap I^nM = 0$.

Proof. This is clear from the results of Lemma 15.91.3. □

The next lemma shows that the category $\mathcal{C}$ of derived complete modules is abelian. It turns out that $\mathcal{C}$ is not a Grothendieck abelian category, see Examples, Section 110.11.

091U Lemma 15.91.6. Let I be an ideal of a ring A.

(1) The derived complete A-modules form a weak Serre subcategory $\mathcal{C}$ of Mod_A.
(2) $D_{\mathcal{C}}(A) \subset D(A)$ is the full subcategory of derived complete objects.

Proof. Part (2) is immediate from Lemma 15.91.1 and the definitions. For part (1), suppose that $M \to N$ is a map of derived complete modules. Denote $K = (M \to N)$ the corresponding object of $D(A)$. Pick $f \in I$. Then $\mathrm{Ext}^n_A(A_f, K)$ is zero for all n because $\mathrm{Ext}^n_A(A_f, M)$ and $\mathrm{Ext}^n_A(A_f, N)$ are zero for all n. Hence K is derived complete. By (2) we see that $\mathrm{Ker}(M \to N)$ and $\mathrm{Coker}(M \to N)$ are objects of $\mathcal{C}$. Finally, suppose that $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of A-modules and M_1, M_3 are derived complete. Then it follows from the long exact sequence of Ext's that M_2 is derived complete. Thus $\mathcal{C}$ is a weak Serre subcategory by Homology, Lemma 12.10.3. □

We will generalize the following lemma in Lemma 15.91.19.

09B9 Lemma 15.91.7. Let I be a finitely generated ideal of a ring A. Let M be a derived complete A-module. If $M/IM = 0$, then $M = 0$.

Proof. Assume that M/IM is zero. Let $I = (f_1, \ldots, f_r)$. Let $i < r$ be the largest integer such that $N = M/(f_1, \ldots, f_i)M$ is nonzero. If i does not exist, then $M = 0$ which is what we want to show. Then N is derived complete as a cokernel of a map between derived complete modules, see Lemma 15.91.6. By our choice of i we have that $f_{i+1} : N \to N$ is surjective. Hence

$$\lim(\ldots \to N \xrightarrow{f_{i+1}} N \xrightarrow{f_{i+1}} N)$$

is nonzero, contradicting the derived completeness of N. □

If the ring is I-adically complete, then one obtains an ample supply of derived complete complexes.

0A05 Lemma 15.91.8. Let A be a ring and $I \subset A$ an ideal. If A is derived complete (eg. I-adically complete) then any pseudo-coherent object of $D(A)$ is derived complete.

Proof. (Lemma 15.91.3 explains the parenthetical statement of the lemma.) Let K be a pseudo-coherent object of $D(A)$. By definition this means K is represented by a bounded above complex $K^\bullet$ of finite free A-modules. Since A is derived complete it follows that $H^n(K)$ is derived complete for all n, by part (1) of Lemma 15.91.6. This in turn implies that K is derived complete by part (2) of the same lemma. □

0A6C Lemma 15.91.9. Let A be a ring. Let $f, g \in A$. Then for $K \in D(A)$ we have $R\,\mathrm{Hom}_A(A_f, R\,\mathrm{Hom}_A(A_g, K)) = R\,\mathrm{Hom}_A(A_{fg}, K)$.

Proof. This follows from Lemma 15.73.1. □

091V Lemma 15.91.10. Let I be a finitely generated ideal of a ring A. The inclusion functor $D_{comp}(A, I) \to D(A)$ has a left adjoint, i.e., given any object K of $D(A)$ there exists a map $K \to K^\wedge$ of K into a derived complete object of $D(A)$ such that the map

$$\mathrm{Hom}_{D(A)}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(A)}(K, E)$$

is bijective whenever E is a derived complete object of $D(A)$. In fact, if I is generated by $f_1, \ldots, f_r \in A$, then we have

$$K^\wedge = R\,\mathrm{Hom}\Big((A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}), K\Big)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \longrightarrow A$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.

Let $f \in A$. By Lemma 15.91.9 the object $R\operatorname{Hom}_A(A_f, K^\wedge)$ is equal to

$$R\operatorname{Hom}\left((A_f \to \prod\nolimits_{i_0} A_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{ff_{i_0}f_{i_1}} \to \ldots \to A_{ff_1\ldots f_r}), K\right)$$

If $f \in I$, then $f_1, \ldots, f_r$ generate the unit ideal in A_f, hence the extended alternating Čech complex

$$A_f \to \prod\nolimits_{i_0} A_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{ff_{i_0}f_{i_1}} \to \ldots \to A_{ff_1\ldots f_r}$$

is zero in $D(A)$ by Lemma 15.29.5. (In fact, if $f = f_i$ for some i, then this complex is homotopic to zero by Lemma 15.29.4; this is the only case we need.) Hence $R\operatorname{Hom}_A(A_f, K^\wedge) = 0$ and we conclude that $K^\wedge$ is derived complete by Lemma 15.91.1.

Conversely, if K is derived complete, then $R\operatorname{Hom}_A(A_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(A)$. □

0G3E Remark 15.91.11. Let A be a ring and let $I \subset A$ be a finitely generated ideal. The left adjoint to the inclusion functor $D_{comp}(A, I) \to D(A)$ which exists by Lemma 15.91.10 is called the derived completion. To indicate this we will say "let $K^\wedge$ be the derived completion of K". Please keep in mind that the unit of the adjunction is a functorial map $K \to K^\wedge$.

0A6D Lemma 15.91.12. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of A_f-modules. Then the derived completion of $K^\bullet$ is zero.

Proof. Indeed, in this case the $R\operatorname{Hom}_A(K, L)$ is zero for any derived complete complex L, see Lemma 15.91.1. Hence $K^\wedge$ is zero by the universal property in Lemma 15.91.10. □

0A6E Lemma 15.91.13. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K, L \in D(A)$. Then

$$R\operatorname{Hom}_A(K, L)^\wedge = R\operatorname{Hom}_A(K, L^\wedge) = R\operatorname{Hom}_A(K^\wedge, L^\wedge)$$

Proof. By Lemma 15.91.10 we know that derived completion is given by $R\operatorname{Hom}_A(C, -)$ for some $C \in D(A)$. Then

$$\begin{aligned} R\operatorname{Hom}_A(C, R\operatorname{Hom}_A(K, L)) &= R\operatorname{Hom}_A(C \otimes_A^{\mathbf{L}} K, L) \\ &= R\operatorname{Hom}_A(K, R\operatorname{Hom}_A(C, L)) \end{aligned}$$

by Lemma 15.73.1. This proves the first equation. The map $K \to K^\wedge$ induces a map

$$R\operatorname{Hom}_A(K^\wedge, L^\wedge) \to R\operatorname{Hom}_A(K, L^\wedge)$$

which is an isomorphism in $D(A)$ by definition of the derived completion as the left adjoint to the inclusion functor. □

091W Lemma 15.91.14. Let A be a ring and let $I \subset A$ be an ideal. Let (K_n) be an inverse system of objects of $D(A)$ such that for all $f \in I$ and n there exists an $e = e(n, f)$ such that f^e is zero on K_n. Then for $K \in D(A)$ the object $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n)$ is derived complete with respect to I.

Proof. Since the category of derived complete objects is preserved under $R\lim$ it suffices to show that each $K \otimes_A^{\mathbf{L}} K_n$ is derived complete. By assumption for all $f \in I$ there is an e such that f^e is zero on $K \otimes_A^{\mathbf{L}} K_n$. Of course this implies that $T(K \otimes_A^{\mathbf{L}} K_n, f) = 0$ and we win. $\square$

0BKC Situation 15.91.15. Let A be a ring. Let $I = (f_1, \ldots, f_r) \subset A$. Let $K_n^\bullet = K_\bullet(A, f_1^n, \ldots, f_r^n)$ be the Koszul complex on $f_1^n, \ldots, f_r^n$ viewed as a cochain complex in degrees $-r, -r+1, \ldots, 0$. Using the functoriality of Lemma 15.28.3 we obtain an inverse system

$$\ldots \to K_3^\bullet \to K_2^\bullet \to K_1^\bullet$$

compatible with the inverse system $H^0(K_n^\bullet) = A/(f_1^n, \ldots, f_r^n)$ and compatible with the maps $A \to K_n^\bullet$.

A key feature of the discussion below will use that for $m > n$ the map

$$K_m^{-p} = \wedge^p(A^{\oplus r}) \to \wedge^p(A^{\oplus r}) = K_n^{-p}$$

is given by multiplication by $f_{i_1}^{m-n} \ldots f_{i_p}^{m-n}$ on the basis element $e_{i_1} \wedge \ldots \wedge e_{i_p}$.

091Y Lemma 15.91.16. In Situation 15.91.15. For $K \in D(A)$ the object $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is derived complete with respect to I.

Proof. This is a special case of Lemma 15.91.14 because f_i^n acts by an endomorphism of $K_n^\bullet$ which is homotopic to zero by Lemma 15.28.6. $\square$

091Z Lemma 15.91.17. In Situation 15.91.15. Let $K \in D(A)$. The following are equivalent

(1) K is derived complete with respect to I, and
(2) the canonical map $K \to R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is an isomorphism of $D(A)$.

Proof. If (2) holds, then K is derived complete with respect to I by Lemma 15.91.16. Conversely, assume that K is derived complete with respect to I. Consider the filtrations

$$K_n^\bullet \supset \sigma_{\geq -r+1} K_n^\bullet \supset \sigma_{\geq -r+2} K_n^\bullet \supset \ldots \supset \sigma_{\geq -1} K_n^\bullet \supset \sigma_{\geq 0} K_n^\bullet = A$$

by stupid truncations (Homology, Section 12.15). Because the construction $R\lim(K \otimes E)$ is exact in the second variable (Lemma 15.87.11) we see that it suffices to show

$$R\lim\left(K \otimes_A^{\mathbf{L}} (\sigma_{\geq p} K_n^\bullet / \sigma_{\geq p+1} K_n^\bullet)\right) = 0$$

for $p < 0$. The explicit description of the Koszul complexes above shows that

$$R\lim\left(K \otimes_A^{\mathbf{L}} (\sigma_{\geq p} K_n^\bullet / \sigma_{\geq p+1} K_n^\bullet)\right) = \bigoplus\nolimits_{i_1, \ldots, i_{-p}} T(K, f_{i_1} \ldots f_{i_{-p}})$$

which is zero for $p < 0$ by assumption on K. $\square$

0920 Lemma 15.91.18. In Situation 15.91.15. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 15.91.10.

First proof. The assignment $K \rightsquigarrow K'$ is a functor and K' is derived complete with respect to I by Lemma 15.91.16. By a formal argument (omitted) we see that it suffices to show $K \to K'$ is an isomorphism if K is derived complete with respect to I. This is Lemma 15.91.17. $\square$

Second proof. Denote $K \mapsto K^\wedge$ the adjoint constructed in Lemma 15.91.10. By that lemma we have

$$K^\wedge = R\operatorname{Hom}\left((A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}), K\right)$$

In Lemma 15.29.6 we have seen that the extended alternating Čech complex

$$A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(A, f_1^n, \ldots, f_r^n)$ sitting in degrees $0, \ldots, r$. Note that K^n is a finite chain complex of finite free A-modules with dual (as in Lemma 15.74.15) $R\operatorname{Hom}_A(K^n, A) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$ (as usual). Thus it suffices to show that

$$R\operatorname{Hom}_A(\operatorname{hocolim} K^n, K) = R\lim(K \otimes_A^{\mathbf{L}} K_n)$$

This follows from Lemma 15.74.16. $\square$

0G1U Lemma 15.91.19. Let I be a finitely generated ideal of a ring A. Let K be a derived complete object of $D(A)$. If $K \otimes_A^{\mathbf{L}} A/I = 0$, then $K = 0$.

A related result is [DG02, Proposition 6.5]. The derived Nakayama lemma can for example be found in Bhatt's 3rd lecture on Prismatic cohomology at Columbia University in Fall 2018 as Section 2 property (2). Leonid Positselski proposed a proof in `https://mathoverflow.net/a/331501`. However, we follow the proof suggested by Anonymous in the comments.

Proof. Choose generators $f_1, \ldots, f_r$ of I. Denote K_n the Koszul complex on $f_1^n, \ldots, f_r^n$ over A. Recall that K_n is bounded and that the cohomology modules of K_n are annihilated by $f_1^n, \ldots, f_r^n$ and hence by I^{nr}. By Lemma 15.88.7 we see that $K \otimes_A^{\mathbf{L}} K_n = 0$. Since K is derived complete by Lemma 15.91.18 we have $K = R\lim K \otimes_A^{\mathbf{L}} K_n = 0$ as desired. $\square$

As an application of the relationship with the Koszul complex we obtain that derived completion has finite cohomological dimension.

0AAJ Lemma 15.91.20. Let A be a ring and let $I \subset A$ be an ideal which can be generated by r elements. Then derived completion has finite cohomological dimension:

1. Let $K \to L$ be a morphism in $D(A)$ such that $H^i(K) \to H^i(L)$ is an isomorphism for $i \geq 1$ and surjective for $i = 0$. Then $H^i(K^\wedge) \to H^i(L^\wedge)$ is an isomorphism for $i \geq 1$ and surjective for $i = 0$.
2. Let $K \to L$ be a morphism of $D(A)$ such that $H^i(K) \to H^i(L)$ is an isomorphism for $i \leq -1$ and injective for $i = 0$. Then $H^i(K^\wedge) \to H^i(L^\wedge)$ is an isomorphism for $i \leq -r-1$ and injective for $i = -r$.

Proof. Say I is generated by $f_1, \ldots, f_r$. For any $K \in D(A)$ by Lemma 15.91.18 we have $K^\wedge = R\lim K \otimes_A^{\mathbf{L}} K_n$ where K_n is the Koszul complex on $f_1^n, \ldots, f_r^n$ and hence we obtain a short exact sequence

$$0 \to R^1\lim H^{i-1}(K \otimes_A^{\mathbf{L}} K_n) \to H^i(K^\wedge) \to \lim H^i(K \otimes_A^{\mathbf{L}} K_n) \to 0$$

by Lemma 15.87.4.

Proof of (1). Pick a distinguished triangle $K \to L \to C \to K[1]$. Then $H^i(C) = 0$ for $i \geq 0$. Since K_n is sitting in degrees ≤ 0 we see that $H^i(C \otimes_A^{\mathbf{L}} K_n) = 0$ for $i \geq 0$ and that $H^{-1}(C \otimes_A^{\mathbf{L}} K_n) = H^{-1}(C) \otimes_A A/(f_1^n, \ldots, f_r^n)$ is a system with surjective transition maps. The displayed equation above shows that $H^i(C^\wedge) = 0$ for $i \geq 0$. Applying the distinguished triangle $K^\wedge \to L^\wedge \to C^\wedge \to K^\wedge[1]$ we get (1).

Proof of (2). Pick a distinguished triangle $K \to L \to C \to K[1]$. Then $H^i(C) = 0$ for $i < 0$. Since K_n is sitting in degrees $-r, \ldots, 0$ we see that $H^i(C \otimes_A^{\mathbf{L}} K_n) = 0$ for $i < -r$. The displayed equation above shows that $H^i(C^\wedge) = 0$ for $i < r$. Applying the distinguished triangle $K^\wedge \to L^\wedge \to C^\wedge \to K^\wedge[1]$ we get (2). $\square$

0BKD Lemma 15.91.21. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a filtered complex of A-modules. There exists a canonical spectral sequence $(E_r, \mathrm{d}_r)_{r \geq 1}$ of bigraded derived complete A-modules with d_r of bidegree $(r, -r+1)$ and with

$$E_1^{p,q} = H^{p+q}((\mathrm{gr}^p K^\bullet)^\wedge)$$

If the filtration on each K^n is finite, then the spectral sequence is bounded and converges to $H^*((K^\bullet)^\wedge)$.

Proof. By Lemma 15.91.10 we know that derived completion is given by $R\operatorname{Hom}_A(C, -)$ for some $C \in D^b(A)$. By Lemmas 15.91.20 and 15.68.2 we see that C has finite projective dimension. Thus we may choose a bounded complex of projective modules $P^\bullet$ representing C. Then

$$M^\bullet = \operatorname{Hom}^\bullet(P^\bullet, K^\bullet)$$

is a complex of A-modules representing $(K^\bullet)^\wedge$. It comes with a filtration given by $F^pM^\bullet = \operatorname{Hom}^\bullet(P^\bullet, F^pK^\bullet)$. We see that $F^pM^\bullet$ represents $(F^pK^\bullet)^\wedge$ and hence $\mathrm{gr}^pM^\bullet$ represents $(\mathrm{gr}K^\bullet)^\wedge$. Thus we find our spectral sequence by taking the spectral sequence of the filtered complex $M^\bullet$, see Homology, Section 12.24. If the filtration on each K^n is finite, then the filtration on each M^n is finite because $P^\bullet$ is a bounded complex. Hence the final statement follows from Homology, Lemma 12.24.11. □

0BKE Example 15.91.22. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $K^\bullet$ be a complex of A-modules. We can apply Lemma 15.91.21 with $F^pK^\bullet = \tau_{\leq -p}K^\bullet$. Then we get a bounded spectral sequence

$$E_1^{p,q} = H^{p+q}(H^{-p}(K^\bullet)^\wedge[p]) = H^{2p+q}(H^{-p}(K^\bullet)^\wedge)$$

converging to $H^{p+q}((K^\bullet)^\wedge)$. After renumbering $p = -j$ and $q = i + 2j$ we find that for any $K \in D(A)$ there is a bounded spectral sequence $(E'_r, d'_r)_{r \geq 2}$ of bigraded derived complete modules with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = H^i(H^j(K)^\wedge)$$

and converging to $H^{i+j}(K^\wedge)$.

0924 Lemma 15.91.23. Let $A \to B$ be a ring map. Let $I \subset A$ be an ideal. The inverse image of $D_{comp}(A, I)$ under the restriction functor $D(B) \to D(A)$ is $D_{comp}(B, IB)$.

Proof. Using Lemma 15.91.2 we see that $L \in D(B)$ is in $D_{comp}(B, IB)$ if and only if $T(L, f)$ is zero for every local section $f \in I$. Observe that the cohomology of $T(L, f)$ is computed in the category of abelian groups, so it doesn't matter whether we think of f as an element of A or take the image of f in B. The lemma follows immediately from this and the definition of derived complete objects. □

0925 Lemma 15.91.24. Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. If $A \to B$ is flat and $A/I \cong B/IB$, then the restriction functor $D(B) \to D(A)$ induces an equivalence $D_{comp}(B, IB) \to D_{comp}(A, I)$.

Proof. Choose generators $f_1, \ldots, f_r$ of I. Denote $\check{\mathcal{C}}_A^\bullet \to \check{\mathcal{C}}_B^\bullet$ the quasi-isomorphism of extended alternating Čech complexes of Lemma 15.89.4. Let $K \in D_{comp}(A, I)$. Let $I^\bullet$ be a K-injective complex of A-modules representing K. Since $\operatorname{Ext}^n_A(A_f, K)$

and $\text{Ext}^n_A(B_f, K)$ are zero for all $f \in I$ and $n \in \mathbf{Z}$ (Lemma 15.91.1) we conclude that $\check{\mathcal{C}}_A^\bullet \to A$ and $\check{\mathcal{C}}_B^\bullet \to B$ induce quasi-isomorphisms

$$I^\bullet = \text{Hom}_A(A, I^\bullet) \longrightarrow \text{Tot}(\text{Hom}_A(\check{\mathcal{C}}_A^\bullet, I^\bullet))$$

and

$$\text{Hom}_A(B, I^\bullet) \longrightarrow \text{Tot}(\text{Hom}_A(\check{\mathcal{C}}_B^\bullet, I^\bullet))$$

Some details omitted. Since $\check{\mathcal{C}}_A^\bullet \to \check{\mathcal{C}}_B^\bullet$ is a quasi-isomorphism and $I^\bullet$ is K-injective we conclude that $\text{Hom}_A(B, I^\bullet) \to I^\bullet$ is a quasi-isomorphism. As the complex $\text{Hom}_A(B, I^\bullet)$ is a complex of B-modules we conclude that K is in the image of the restriction map, i.e., the functor is essentially surjective

In fact, the argument shows that $F : D_{comp}(A, I) \to D_{comp}(B, IB)$, $K \mapsto \text{Hom}_A(B, I^\bullet)$ is a left inverse to restriction. Finally, suppose that $L \in D_{comp}(B, IB)$. Represent L by a K-injective complex $J^\bullet$ of B-modules. Then $J^\bullet$ is also K-injective as a complex of A-modules (Lemma 15.56.1) hence $F(\text{restriction of } L) = \text{Hom}_A(B, J^\bullet)$. There is a map $J^\bullet \to \text{Hom}_A(B, J^\bullet)$ of complexes of B-modules, whose composition with $\text{Hom}_A(B, J^\bullet) \to J^\bullet$ is the identity. We conclude that F is also a right inverse to restriction and the proof is finished. $\square$

15.92. The category of derived complete modules

0GLN Let A be a ring and let I be an ideal. Denote $\mathcal{C}$ the category of derived complete modules, see Definition 15.91.4. In this section we discuss some properties of this category. In Examples, Section 110.11 we show that $\mathcal{C}$ isn't a Grothendieck abelian category in general.

By Lemma 15.91.6 the category $\mathcal{C}$ is abelian and the inclusion functor $\mathcal{C} \to \text{Mod}_A$ is exact.

Since $D_{comp}(A) \subset D(A)$ is closed under products (see discussion following Definition 15.91.4) and since products in $D(A)$ are computed on the level of complexes, we see that $\mathcal{C}$ has products which agree with products in Mod_A. Thus $\mathcal{C}$ in fact has arbitrary limits and the inclusion functor $\mathcal{C} \to \text{Mod}_A$ commutes with them, see Categories, Lemma 4.14.11.

Assume I is finitely generated. Let $^\wedge : D(A) \to D(A)$ denote the derived completion functor of Lemma 15.91.10. Let us show the functor

$$\text{Mod}_A \longrightarrow \mathcal{C}, \quad M \longmapsto H^0(M^\wedge)$$

is a left adjoint to the inclusion functor $\mathcal{C} \to \text{Mod}_A$. Note that $H^i(M^\wedge) = 0$ for $i > 0$ for example by Lemma 15.91.20. Hence, if N is a derived complete A-module, then we have

$$\begin{aligned} \text{Hom}_\mathcal{C}(H^0(M^\wedge), N) &= \text{Hom}_{D_{comp}(A)}(M^\wedge, N) \\ &= \text{Hom}_{D(A)}(M, N) \\ &= \text{Hom}_A(M, N) \end{aligned}$$

as desired.

Let T be a preordered set and let $t \mapsto M_t$ be a system of derived complete A-modules, i.e., a system over T in $\mathcal{C}$, see Categories, Section 4.21. Denote $\text{colim}_{t \in T} M_t$ the colimit of the system in Mod_A. It follows formally from the above that

$$H^0((\text{colim}_{t \in T} M_t)^\wedge)$$

is the colimit of the system in $\mathcal{C}$. In this way we see that $\mathcal{C}$ has all colimits. In general the inclusion functor $\mathcal{C} \to \text{Mod}_A$ will not commute with colimits, see Examples, Section 110.11.

0GLP Lemma 15.92.1. Let A be a ring and let $I \subset A$ be an ideal. The category $\mathcal{C}$ of derived complete modules is abelian, has arbitrary limits, and the inclusion functor $F : \mathcal{C} \to \text{Mod}_A$ is exact and commutes with limits. If I is finitely generated, then $\mathcal{C}$ has arbitrary colimits and F has a left adjoint

Proof. This summarizes the discussion above. $\square$

15.93. Derived completion for a principal ideal

0BKF In this section we discuss what happens with derived completion when the ideal is generated by a single element.

091X Lemma 15.93.1. Let A be a ring. Let $f \in A$. If there exists an integer $c \geq 1$ such that $A[f^c] = A[f^{c+1}] = A[f^{c+2}] = \ldots$ (for example if A is Noetherian), then for all $n \geq 1$ there exist maps

$$(A \xrightarrow{f^n} A) \longrightarrow A/(f^n), \quad \text{and} \quad A/(f^{n+c}) \longrightarrow (A \xrightarrow{f^n} A)$$

in $D(A)$ inducing an isomorphism of the pro-objects $\{A/(f^n)\}$ and $\{(f^n : A \to A)\}$ in $D(A)$.

Proof. The first displayed arrow is obvious. We can define the second arrow of the lemma by the diagram

$$\begin{tikzcd} A/A[f^c] \arrow[r, "f^{n+c}"'] \arrow[d, "f^c"'] & A \arrow[d, "1"] \\ A \arrow[r, "f^n"] & A \end{tikzcd}$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $A/f^{n+c}A$. We omit the calculation of compositions needed to show the statement on pro objects. $\square$

0923 Lemma 15.93.2. Let A be a ring and $f \in A$. Set $I = (f)$. In this situation we have the naive derived completion $K \mapsto K' = R\lim(K \otimes_A^{\mathbf{L}} A/f^nA)$ and the derived completion

$$K \mapsto K^\wedge = R\lim(K \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A))$$

of Lemma 15.91.18. The natural transformation of functors $K^\wedge \to K'$ is an isomorphism if and only if the f-power torsion of A is bounded.

Proof. If the f-power torsion is bounded, then the pro-objects $\{(f^n : A \to A)\}$ and $\{A/f^nA\}$ are isomorphic by Lemma 15.93.1. Hence the functors are isomorphic by Lemma 15.86.11. Conversely, we see from Lemma 15.87.11 that the condition is exactly that

$$R\lim(K \otimes_A^{\mathbf{L}} A[f^n])$$

is zero for all $K \in D(A)$. Here the maps of the system $(A[f^n])$ are given by multiplication by f. Taking $K = A$ and $K = \bigoplus_{i \in \mathbf{N}} A$ we see from Lemma 15.86.14 this implies $(A[f^n])$ is zero as a pro-object, i.e., $f^{n-1}A[f^n] = 0$ for some n, i.e., $A[f^{n-1}] = A[f^n]$, i.e., the f-power torsion is bounded. $\square$

09AT Example 15.93.3. Let A be a ring. Let $f \in A$ be a nonzerodivisor. An example to keep in mind is $A = \mathbf{Z}_p$ and $f = p$. Let M be an A-module. Claim: M is derived complete with respect to f if and only if there exists a short exact sequence

$$0 \to K \to L \to M \to 0$$

where K, L are f-adically complete modules whose f-torsion is zero. Namely, if there is a such a short exact sequence, then

$$M \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A) = (K/f^nK \to L/f^nL)$$

because f is a nonzerodivisor on K and L and we conclude that $R\lim(M \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A))$ is quasi-isomorphic to $K \to L$, i.e., M. This shows that M is derived complete by Lemma 15.91.17. Conversely, suppose that M is derived complete. Choose a surjection $F \to M$ where F is a free A-module. Since f is a nonzerodivisor on F the derived completion of F is $L = \lim F/f^nF$. Note that L is f-torsion free: if (x_n) with $x_n \in F$ represents an element ξ of L and $f\xi = 0$, then $x_n = x_{n+1} + f^nz_n$ and $fx_n = f^ny_n$ for some $z_n, y_n \in F$. Then $f^ny_n = fx_n = fx_{n+1} + f^{n+1}z_n = f^{n+1}y_{n+1} + f^{n+1}z_n$ and since f is a nonzerodivisor on F we see that $y_n \in fF$ which implies that $x_n \in f^nF$, i.e., $\xi = 0$. Since L is the derived completion, the universal property gives a map $L \to M$ factoring $F \to M$. Let $K = \mathrm{Ker}(L \to M)$ be the kernel. Again K is f-torsion free, hence the derived completion of K is $\lim K/f^nK$. On the other hand, both M and L are derived complete, hence K is too by Lemma 15.91.6. It follows that $K = \lim K/f^nK$ and the claim is proved.

0G3F Example 15.93.4. Let p be a prime number. Consider the map $\mathbf{Z}_p[x] \to \mathbf{Z}_p[y]$ of polynomial algebras sending x to py. Consider the cokernel $M = \mathrm{Coker}(\mathbf{Z}_p[x]^\wedge \to \mathbf{Z}_p[y]^\wedge)$ of the induced map on (ordinary) p-adic completions. Then M is a derived complete $\mathbf{Z}_p$-module by Proposition 15.91.5 and Lemma 15.91.6; see also discussion in Example 15.93.3. However, M is not p-adically complete as $1 + py + p^2y^2 + \ldots$ maps to a nonzero element of M which is contained in $\bigcap p^nM$.

0BKG Example 15.93.5. Let A be a ring and let $f \in A$. Denote $K \mapsto K^\wedge$ the derived completion with respect to (f). Let M be an A-module. Using that

$$M^\wedge = R\lim(M \xrightarrow{f^n} M)$$

by Lemma 15.91.18 and using Lemma 15.87.4 we obtain

$$H^{-1}(M^\wedge) = \lim M[f^n] = T_f(M)$$

the f-adic Tate module of M. Here the maps $M[f^n] \to M[f^{n-1}]$ are given by multiplication by f. Then there is a short exact sequence

$$0 \to R^1\lim M[f^n] \to H^0(M^\wedge) \to \lim M/f^nM \to 0$$

describing $H^0(M^\wedge)$. We have $H^1(M^\wedge) = R^1\lim M/f^nM = 0$ as the transition maps are surjective (Lemma 15.87.1). All the other cohomologies of $M^\wedge$ are zero for trivial reasons. Finally, for $K \in D(A)$ and $p \in \mathbf{Z}$ there is a short exact sequence

$$0 \to H^0(H^p(K)^\wedge) \to H^p(K^\wedge) \to T_f(H^{p+1}(K)) \to 0$$

This follows from the spectral sequence of Example 15.91.22 because it degenerates at E_2 (as only $i = -1, 0$ give nonzero terms); the next lemma gives more information.

0H32 Lemma 15.93.6. Let A be a ring and let $f \in A$. Let K be an object of $D(A)$. Denote $K_n = K \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A)$. For all $p \in \mathbf{Z}$ there is a commutative diagram

$$\begin{tikzcd} & 0 & 0 & & \\ 0 \arrow[r] & \widehat{H^p(K)} \arrow[r] \arrow[u] & \lim H^p(K_n) \arrow[r] \arrow[u] & T_f(H^{p+1}(K)) \arrow[r] & 0 \\ 0 \arrow[r] & H^0(H^p(K)^\wedge) \arrow[r] \arrow[u] & H^p(K^\wedge) \arrow[r] \arrow[u] & T_f(H^{p+1}(K)) \arrow[r] \arrow[u, equal] & 0 \\ & R^1 \lim H^p(K)[f^n] \arrow[r, "\cong"] \arrow[u] & R^1 \lim H^{p-1}(K_n) \arrow[u] & & \\ & 0 \arrow[u] & 0 \arrow[u] & & \end{tikzcd}$$

with exact rows and columns where $\widehat{H^p(K)} = \lim H^p(K)/f^n H^p(K)$ is the usual f-adic completion. The left vertical short exact sequence and the middle horizontal short exact sequence are taken from Example 15.93.5 The middle vertical short exact sequence is the one from Lemma 15.87.4.

Proof. To construct the top horizontal short exact sequence, observe that we have the following inverse system short exact sequences

$$0 \to H^p(K)/f^n H^p(K) \to H^p(K_n) \to H^{p+1}(K)[f^n] \to 0$$

coming from the construction of K_n as a shift of the cone on $f^n : K \to K$. Taking the inverse limit of these we obtain the top horizontal short exact sequence, see Homology, Lemma 12.31.3.

Let us prove that we have a commutative diagram as in the lemma. We consider the map $L = \tau_{\leq p} K \to K$. Setting $L_n = L \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A)$ we obtain a map $(L_n) \to (K_n)$ of inverse systems which induces a map of short exact sequences

$$\begin{tikzcd} 0 & 0 \\ \lim H^p(L_n) \arrow[r] \arrow[u] & \lim H^p(K_n) \arrow[u] \\ H^p(L^\wedge) \arrow[r] \arrow[u] & H^p(K^\wedge) \arrow[u] \\ R^1 \lim H^{p-1}(L_n) \arrow[r] \arrow[u] & R^1 \lim H^{p-1}(K_n) \arrow[u] \\ 0 \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

Since $H^i(L) = 0$ for $i > p$ and $H^p(L) = H^p(K)$, a computation using the references in the statement of the lemma shows that $H^p(L^\wedge) = H^0(H^p(K)^\wedge)$ and that $H^p(L_n) = H^p(K)/f^nH^p(K)$. On the other hand, we have $H^{p-1}(L_n) = H^{p-1}(K_n)$ and hence we see that we get the isomorphism as indicated in the statement of the lemma since we already know the kernel of $H^0(H^p(K)^\wedge) \to \widehat{H^p(K)}$ is equal to $R^1 \lim H^p(K)[f^n]$. We omit the verification that the rightmost square in the diagram commutes if we define the top row by the construction in the first paragraph of the proof. □

0H33 Remark 15.93.7. With notation as in Lemma 15.93.6 we also see that the inverse system $H^p(K_n)$ has ML if and only if the inverse system $H^{p+1}(K)[f^n]$ has ML. This follows from the inverse system of short exact sequences $0 \to H^p(K)/f^nH^p(K) \to H^p(K_n) \to H^{p+1}(K)[f^n] \to 0$ (see proof of the lemma) combined with Homology, Lemma 12.31.3 and Lemma 15.86.13.

0CQY Lemma 15.93.8 (Bhatt). Let I be a finitely generated ideal in a ring A. Let M be a derived complete A-module. If M is an I-power torsion module, then $I^nM = 0$ for some n.

Proof. Say $I = (f_1, \ldots, f_r)$. It suffices to show that for each i there is an n_i such that $f_i^{n_i}M = 0$. Hence we may assume that $I = (f)$ is a principal ideal. Let $B = \mathbf{Z}[x] \to A$ be the ring map sending x to f. By Lemma 15.91.23 we see that M is derived complete as a B-module with respect to the ideal (x). After replacing A by B, we may assume that f is a nonzerodivisor in A.

Assume $I = (f)$ with $f \in A$ a nonzerodivisor. According to Example 15.93.3 there exists a short exact sequence

$$0 \to K \xrightarrow{u} L \to M \to 0$$

where K and L are I-adically complete A-modules whose f-torsion is zero[11]. Consider K and L as topological modules with the I-adic topology. Then u is continuous. Let

$$L_n = \{x \in L \mid f^nx \in u(K)\}$$

Since M is f-power torsion we see that $L = \bigcup L_n$. Let N_n be the closure of L_n in L. By Lemma 15.36.4 we see that N_n is open in L for some n. Fix such an n. Since $f^{n+m} : L \to L$ is a continuous open map, and since $f^{n+m}L_n \subset u(f^mK)$ we conclude that the closure of $u(f^mK)$ is open for all $m \geq 1$. Thus by Lemma 15.36.5 we conclude that u is open. Hence $f^tL \subset \mathrm{Im}(u)$ for some t and we conclude that f^t annihilates M as desired. □

0G3G Lemma 15.93.9. Let $f \in A$ be an element of a ring. Set $J = \bigcap f^nA$. Let M be an A-module derived complete with respect to f. Then $JM' = 0$ where $M' = \mathrm{Ker}(M \to \lim M/f^nM)$. In particular, if A is derived complete then J is an ideal of square zero.

[11] For the proof it is enough to show that there exists a sequence $K \xrightarrow{u} L \to M \to 0$ where K and L are I-adically complete A-modules. This can be shown by choosing a presentation $F_1 \to F_0 \to M \to 0$ with F_i free and then setting K and L equal to the f-adic completions of F_1 and F_0. Namely, as f is a nonzerodivisor these completions will be the derived completions and the sequence will remain exact.

Proof. Take $x \in M'$ and $g \in J$. For every $n \geq 1$ we may write $x = f^n x_n$. Since g is in $f^n A$ we see that the element $y_n = g x_n$ in M' is independent of the choice of x_n. In particular, we may take $x_n = f x_{n+1}$ and we find that $y_n = f y_{n+1}$. Thus we obtain a map $A_f \to M$ sending $1/f^n$ to y_n. This map has to be zero as M is derived complete (Lemma 15.91.1) and hence $y_n = 0$ for all n. Since $gx = gfx_1 = fy_1$ this completes the proof. □

0G3H Lemma 15.93.10. Let A be a ring derived complete with respect to an ideal I. Then (A, I) is a henselian pair.

Proof. Let $f \in I$. By Lemma 15.11.15 it suffices to show that (A, fA) is a henselian pair. Observe that A is derived complete with respect to fA (follows immediately from Definition 15.91.4). By Lemma 15.91.3 the map from A to the f-adic completion A' of A is surjective. By Lemma 15.11.4 the pair (A', fA') is henselian. Thus it suffices to show that $(A, \bigcap f^n A)$ is a henselian pair, see Lemma 15.11.9. This follows from Lemmas 15.93.9 and 15.11.2. □

0G3I Lemma 15.93.11. Let A be a ring derived complete with respect to an ideal I. Set $J = \bigcap I^n$. If I can be generated by r elements then $J^N = 0$ where $N = 2^r$.

Proof. When $r = 1$ this is Lemma 15.93.9. Say $I = (f_1, \ldots, f_r)$ with $r > 1$. By Lemma 15.91.6 the ring $A_t = A/f_r^t A$ is derived complete with respect to I and hence a fortiori derived complete with respect to $I_t = (f_1, \ldots, f_{r-1})A_t$. Observe that $A \to A_t$ sends J into $J_t = \bigcap I_t^n$. By induction $J_t^{N/2} = 0$ with $N = 2^r$. The ideal $\bigcap \text{Ker}(A \to A_t) = \bigcap f_r^t A$ has square zero by the case $r = 1$. This finishes the proof. □

0G5V Lemma 15.93.12. Let A be a reduced ring derived complete with respect to a finitely generated ideal I. Then A is I-adically complete.

Proof. Follows from Lemma 15.93.11 and Proposition 15.91.5. □

15.94. Derived completion for Noetherian rings

0BKH Let A be a ring and let $I \subset A$ be an ideal. For any $K \in D(A)$ we can consider the derived limit
$$K' = R\lim(K \otimes_A^{\mathbf{L}} A/I^n)$$
This is a functor in K, see Remark 15.87.10. The system of maps $A \to A/I^n$ induces a map $K \to K'$ and K' is derived complete with respect to I (Lemma 15.91.14). This "naive" derived completion construction does not agree with the adjoint of Lemma 15.91.10 in general. For example, if $A = \mathbf{Z}_p \oplus \mathbf{Q}_p/\mathbf{Z}_p$ with the second summand an ideal of square zero, $K = A[0]$, and $I = (p)$, then the naive derived completion gives $\mathbf{Z}_p[0]$, but the construction of Lemma 15.91.10 gives $K^\wedge \cong \mathbf{Z}_p[1] \oplus \mathbf{Z}_p[0]$ (computation omitted). Lemma 15.93.2 characterizes when the two functors agree in the case I is generated by a single element.

The main goal of this section is the show that the naive derived completion is equal to derived completion if A is Noetherian.

0921 Lemma 15.94.1. In Situation 15.91.15. If A is Noetherian, then the pro-objects $\{K_n^\bullet\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic[12].

[12]In particular, for every n there exists an $m \geq n$ such that $K_m^\bullet \to K_n^\bullet$ factors through the map $K_m^\bullet \to A/(f_1^m, \ldots, f_r^m)$.

Proof. We have an inverse system of distinguished triangles

$$\tau_{\leq -1}K_n^\bullet \to K_n^\bullet \to A/(f_1^m, \ldots, f_r^m) \to (\tau_{\leq -1}K_n^\bullet)[1]$$

See Derived Categories, Remark 13.12.4. By Derived Categories, Lemma 13.42.4 it suffices to show that the inverse system $\tau_{\leq -1}K_n^\bullet$ is pro-zero. Recall that $K_n^\bullet$ has nonzero terms only in degrees i with $-r \leq i \leq 0$. Thus by Derived Categories, Lemma 13.42.3 it suffices to show that $H^p(K_n^\bullet)$ is pro-zero for $p \leq -1$. In other words, for every $n \in \mathbf{N}$ we have to show there exists an $m \geq n$ such that $H^p(K_m^\bullet) \to H^p(K_n^\bullet)$ is zero. Since A is Noetherian, we see that

$$H^p(K_n^\bullet) = \frac{\mathrm{Ker}(K_n^p \to K_n^{p+1})}{\mathrm{Im}(K_n^{p-1} \to K_n^p)}$$

is a finite A-module. Moreover, the map $K_m^p \to K_n^p$ is given by a diagonal matrix whose entries are in the ideal $(f_1^{m-n}, \ldots, f_r^{m-n})$ as $p < 0$. Note that $H^p(K_n^\bullet)$ is annihilated by $J = (f_1^n, \ldots, f_r^n)$, see Lemma 15.28.6. Now $(f_1^{m-n}, \ldots, f_r^{m-n}) \subset J^t$ for $m - n \geq tn$. Thus by Algebra, Lemma 10.51.2 (Artin-Rees) applied to the ideal J and the module $M = K_n^p$ with submodule $N = \mathrm{Ker}(K_n^p \to K_n^{p+1})$ for m large enough the image of $K_m^p \to K_n^p$ intersected with $\mathrm{Ker}(K_n^p \to K_n^{p+1})$ is contained in $J\,\mathrm{Ker}(K_n^p \to K_n^{p+1})$. For such m we get the zero map. $\square$

0922 Proposition 15.94.2. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim(K \otimes_A^{\mathbf{L}} A/I^n)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 15.91.10.

Proof. Say $(f_1, \ldots, f_r) = I$ and let $K_n^\bullet$ be the Koszul complex with respect to $f_1^n, \ldots, f_r^n$. By Lemma 15.91.18 it suffices to prove that

$$R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet) = R\lim(K \otimes_A^{\mathbf{L}} A/(f_1^n, \ldots, f_r^n)) = R\lim(K \otimes_A^{\mathbf{L}} A/I^n).$$

By Lemma 15.94.1 the pro-objects $\{K_n^\bullet\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic. It is clear that the pro-objects $\{A/(f_1^n, \ldots, f_r^n)\}$ and $\{A/I^n\}$ are isomorphic. Thus the map from left to right is an isomorphism by Lemma 15.87.12. $\square$

0EET Lemma 15.94.3. Let I be an ideal of a Noetherian ring A. Let M be an A-module with derived completion $M^\wedge$. Then there are short exact sequences

$$0 \to R^1\lim \mathrm{Tor}_{i+1}^A(M, A/I^n) \to H^{-i}(M^\wedge) \to \lim \mathrm{Tor}_i^A(M, A/I^n) \to 0$$

A similar result holds for $M \in D^-(A)$.

Proof. Immediate consequence of Proposition 15.94.2 and Lemma 15.87.4. $\square$

As an application of the proposition above we identify the derived completion in the Noetherian case for pseudo-coherent complexes.

0A06 Lemma 15.94.4. Let A be a Noetherian ring and $I \subset A$ an ideal. Let K be an object of $D(A)$ such that $H^n(K)$ a finite A-module for all $n \in \mathbf{Z}$. Then the cohomology modules $H^n(K^\wedge)$ of the derived completion are the I-adic completions of the cohomology modules $H^n(K)$.

Proof. The complex $\tau_{\leq m}K$ is pseudo-coherent for all m by Lemma 15.64.17. Thus $\tau_{\leq m}K$ is represented by a bounded above complex $P^\bullet$ of finite free A-modules. Then $\tau_{\leq m}K \otimes_A^{\mathbf{L}} A/I^n = P^\bullet/I^nP^\bullet$. Hence $(\tau_{\leq m}K)^\wedge = R\lim P^\bullet/I^nP^\bullet$ (Proposition 15.94.2) and since the $R\lim$ is just given by termwise lim (Lemma 15.87.1) and since I-adic completion is an exact functor on finite A-modules (Algebra, Lemma 10.97.2) we conclude the result holds for $\tau_{\leq m}K$. Hence the result holds for K as derived completion has finite cohomological dimension, see Lemma 15.91.20. $\square$

09BA Lemma 15.94.5. Let I be an ideal of a Noetherian ring A. Let M be a derived complete A-module. If M/IM is a finite A/I-module, then $M = \lim M/I^nM$ and M is a finite $A^\wedge$-module.

Proof. Assume M/IM is finite. Pick $x_1, \ldots, x_t \in M$ which map to generators of M/IM. We obtain a map $A^{\oplus t} \to M$ mapping the ith basis vector to x_i. By Proposition 15.94.2 the derived completion of A is $A^\wedge = \lim A/I^n$. As M is derived complete, we see that our map factors through a map $q : (A^\wedge)^{\oplus t} \to M$. The module $\text{Coker}(q)$ is zero by Lemma 15.91.7. Thus M is a finite $A^\wedge$-module. Since $A^\wedge$ is Noetherian and complete with respect to $IA^\wedge$, it follows that M is I-adically complete (use Algebra, Lemmas 10.97.5, 10.96.11, and 10.51.2). $\square$

0EEU Lemma 15.94.6. Let I be an ideal in a Noetherian ring A.

(1) If M is a finite A-module and N is a flat A-module, then the derived I-adic completion of $M \otimes_A N$ is the usual I-adic completion of $M \otimes_A N$.
(2) If M is a finite A-module and $f \in A$, then the derived I-adic completion of M_f is the usual I-adic completion of M_f.

Proof. For an A-module M denote $M^\wedge$ the derived completion and $\lim M/I^nM$ the usual completion. Assume M is finite. The system $\text{Tor}_i^A(M, A/I^n)$ is pro-zero for $i > 0$, see Lemma 15.27.3. Since $\text{Tor}_i^A(M \otimes_A N, A/I^n) = \text{Tor}_i^A(M, A/I^n) \otimes_A N$ as N is flat, the same is true for the system $\text{Tor}_i^A(M \otimes_A N, A/I^n)$. By Lemma 15.94.3 we conclude $R\lim(M \otimes_A N) \otimes_A^{\mathbf{L}} A/I^n$ only has cohomology in degree 0 given by the usual completion $\lim M \otimes_A N/I^n(M \otimes_A N)$. This proves (1). Part (2) follows from (1) and the fact that $M_f = M \otimes_A A_f$. $\square$

0EEV Lemma 15.94.7. Let I be an ideal in a Noetherian ring A. Let $^\wedge$ denote derived completion with respect to I. Let $K \in D^-(A)$.

(1) If M is a finite A-module, then $(K \otimes_A^{\mathbf{L}} M)^\wedge = K^\wedge \otimes_A^{\mathbf{L}} M$.
(2) If $L \in D(A)$ is pseudo-coherent, then $(K \otimes_A^{\mathbf{L}} L)^\wedge = K^\wedge \otimes_A^{\mathbf{L}} L$.

Proof. Let L be as in (2). We may represent K by a bounded above complex $P^\bullet$ of free A-modules. We may represent L by a bounded above complex $F^\bullet$ of finite free A-modules. Since $\text{Tot}(P^\bullet \otimes_A F^\bullet)$ represents $K \otimes_A^{\mathbf{L}} L$ we see that $(K \otimes_A^{\mathbf{L}} L)^\wedge$ is represented by

$$\text{Tot}((P^\bullet)^\wedge \otimes_A F^\bullet)$$

where $(P^\bullet)^\wedge$ is the complex whose terms are the usual = derived completions $(P^n)^\wedge$, see for example Proposition 15.94.2 and Lemma 15.94.6. This proves (2). Part (1) is a special case of (2). $\square$

15.95. An operator introduced by Berthelot and Ogus

0F7N In this section we discuss a construction introduced in [BO78, Section 8] and generalized in [BMS18, Section 6]. We urge the reader to look at the original papers discussing this notion.

Let A be a ring and let $f \in A$ be a nonzerodivisor. If M is a A-module then by Lemma 15.88.3 following are equivalent

(1) f is a nonzerodivisor on M,
(2) $M[f] = 0$,
(3) $M[f^n] = 0$ for all $n \geq 1$, and
(4) the map $M \to M_f$ is injective.

If these equivalent conditions hold, then (in this section) we will say M is f-torsion free. If so, then we denote $f^i M \subset M_f$ the submodule consisting of elements of the form $f^i x$ with $x \in M$. Of course $f^i M$ is isomorphic to M as an A-module. Let $M^\bullet$ be a complex of f-torsion free A-modules with differentials $d^i : M^i \to M^{i+1}$. In this case we define $\eta_f M^\bullet$ to be the complex with terms

$$(\eta_f M)^i = \{x \in f^i M^i \mid d^i(x) \in f^{i+1} M^{i+1}\}$$

and differential induced by d^i. Observe that $\eta_f M^\bullet$ is another complex of f-torsion free A-modules. If $a^\bullet : M^\bullet \to N^\bullet$ is a map of complexes of f-torsion free A-modules, then we obtain a map of complexes

$$\eta_f a^\bullet : \eta_f M^\bullet \longrightarrow \eta_f N^\bullet$$

induced by the maps $f^i M^i \to f^i N^i$. The reader checks that we obtain an endofunctor on the category of complexes of f-torsion free A-modules. If $a^\bullet, b^\bullet : M^\bullet \to N^\bullet$ are two maps of complexes of f-torsion free A-modules and $h = \{h^i : M^i \to N^{i-1}\}$ is a homotopy between $a^\bullet$ and $b^\bullet$, then we define $\eta_f h$ to be the family of maps $(\eta_f h)^i : (\eta_f M)^i \to (\eta_f N)^{i-1}$ which sends x to $h^i(x)$; this makes sense as $x \in f^i M^i$ implies $h^i(x) \in f^i N^{i-1}$ which is certainly contained in $(\eta_f N)^{i-1}$. The reader checks that $\eta_f h$ is a homotopy between $\eta_f a^\bullet$ and $\eta_f b^\bullet$. All in all we see that we obtain a functor

$$\eta_f : K(f\text{-torsion free } A\text{-modules}) \longrightarrow K(f\text{-torsion free } A\text{-modules})$$

on the homotopy category (Derived Categories, Section 13.8) of the additive category of f-torsion free A-modules. There is no sense in which η_f is an exact functor of triangulated categories, see Example 15.95.1.

0GSN Example 15.95.1. Let A be a ring. Let $f \in A$ be a nonzerodivisor. Consider the functor $\eta_f : K(f\text{-torsion free } A\text{-modules}) \to K(f\text{-torsion free } A\text{-modules})$. Let $M^\bullet$ be a complex of f-torsion free A-modules. Multiplication by f defines an isomorphism $\eta_f(M^\bullet[1]) \to (\eta_f M^\bullet)[1]$, so in this sense η_f is compatible with shifts. However, consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"'] & A \arrow[r, "1"'] & A \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & A \arrow[r, "-1"] \arrow[u, "f"] & A \arrow[u]
\end{tikzcd}$$

Think of each column as a complex of f-torsion free A-modules with the module on top in degree 1 and the module under it in degree 0. Then this diagram provides us with a distinguished triangle in $K(f\text{-torsion free } A\text{-modules})$ with triangulated structure as given in Derived Categories, Section 13.10. Namely the third complex is the cone of the map between the first two complexes. However, applying η_f to each column we obtain

$$\begin{array}{ccccccc} fA & \xrightarrow{f} & fA & \xrightarrow{1} & fA & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle f} & & \uparrow \\ 0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{-1} & A \end{array}$$

However, the third complex is acyclic and even homotopic to zero. Hence if this were a distinguished triangle, then the first arrow would have to be an isomorphism in the homotopy category, which is not true unless f is a unit.

0F7P Lemma 15.95.2. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. There is a canonical isomorphism

$$f^i : H^i(M^\bullet)/H^i(M^\bullet)[f] \longrightarrow H^i(\eta_f M^\bullet)$$

given by multiplication by f^i.

Proof. Observe that $\text{Ker}(d^i : (\eta_f M)^i \to (\eta_f M)^{i+1})$ is equal to $\text{Ker}(d^i : f^i M^i \to f^i M^{i+1}) = f^i \text{Ker}(d^i : M^i \to M^{i+1})$. This we get a surjection $f^i : H^i(M^\bullet) \to H^i(\eta_f M^\bullet)$ by sending the class of $z \in \text{Ker}(d^i : M^i \to M^{i+1})$ to the class of $f^i z$. If we obtain the zero class in $H^i(\eta_f M^\bullet)$ then we see that $f^i z = d^{i-1}(f^{i-1} y)$ for some $y \in M^{i-1}$. Since f is a nonzerodivisor on all the modules involved, this means $fz = d^{i-1}(y)$ which exactly means that the class of z is f-torsion as desired. □

0F7Q Lemma 15.95.3. Let A be a ring and let $f \in A$ be a nonzerodivisor. If $M^\bullet \to N^\bullet$ is a quasi-isomorphism of complexes of f-torsion free A-modules, then the induced map $\eta_f M^\bullet \to \eta_f N^\bullet$ is a quasi-isomorphism too.

Proof. This is true because the isomorphisms of Lemma 15.95.2 are compatible with maps of complexes. □

0F7R Lemma 15.95.4. Let A be a ring and let $f \in A$ be a nonzerodivisor. There is an additive functor[13] $L\eta_f : D(A) \to D(A)$ such that if $M \in D(A)$ is represented by a complex $M^\bullet$ of f-torsion free A-modules, then $L\eta_f M = \eta_f M^\bullet$ and similarly for morphisms.

Proof. Denote $\mathcal{T} \subset \text{Mod}_A$ the full subcategory of f-torsion free A-modules. We have a corresponding inclusion

$$K(\mathcal{T}) \quad \subset \quad K(\text{Mod}_A) = K(A)$$

of $K(\mathcal{T})$ as a full triangulated subcategory of $K(A)$. Let $S \subset \text{Arrows}(K(\mathcal{T}))$ be the quasi-isomorphisms. We will apply Derived Categories, Lemma 13.5.8 to show that the map

$$S^{-1}K(\mathcal{T}) \longrightarrow D(A)$$

is an equivalence of triangulated categories. The lemma shows that it suffices to prove: given a complex $M^\bullet$ of A-modules, there exists a quasi-isomorphism $K^\bullet \to M^\bullet$ with $K^\bullet$ a complex of f-torsion free modules. By Lemma 15.59.10 we can find a quasi-isomorphism $K^\bullet \to M^\bullet$ such that the complex $K^\bullet$ is K-flat (we won't use this) and consists of flat A-modules K^i. In particular, f is a nonzerodivisor on K^i for all i as desired.

[13] Beware that this functor isn't exact, i.e., does not transform distinguished triangles into distinguished triangles. See Example 15.95.1.

With these preliminaries out of the way we can define $L\eta_f$. Namely, by the discussion at the start of this section we have already a well defined functor

$$K(\mathcal{T}) \xrightarrow{\eta_f} K(\mathcal{T}) \to K(A) \to D(A)$$

which according to Lemma 15.95.3 sends quasi-isomorphisms to quasi-isomorphisms. Hence this functor factors over $S^{-1}K(\mathcal{T}) = D(A)$ by Categories, Lemma 4.27.8. $\square$

0F7S Remark 15.95.5. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For every i set $\overline{M}^i = M^i/fM^i$. Denote $B^i \subset Z^i \subset \overline{M}^i$ the boundaries and cocycles for the differentials on the complex $\overline{M}^\bullet = M^\bullet \otimes_A A/fA$. We claim that there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B^{i+1} \arrow[r] \arrow[d, equal] & B^{i+1} \oplus B^i \arrow[r] \arrow[d, "{s,s'}"] & B^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^{i+1} \arrow[r, "s"] & (\eta_f M)^i/f(\eta_f M)^i \arrow[r, "t"] & Z^i \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Here are the constructions of the maps

1. If $x \in (\eta_f M)^i$ then $x = f^i x'$ with $d^i(x') = 0$ in $\overline{M}^{i+1}$. Hence we can define the map t by sending x to the class of x'.
2. If $y \in M^{i+1}$ has class $\overline{y}$ in $B^{i+1} \subset \overline{M}^{i+1}$ then we can write $y = fy' + d^i(x)$ for $y' \in M^{i+1}$ and $x \in M^i$. Hence we can define the map s sending $\overline{y}$ to the class of $f^{i+1}x$ in $(\eta_f M)^i/f(\eta_f M)^i$; we omit the verification that this is well defined.
3. If $x \in M^i$ has class $\overline{x}$ in $B^i \subset \overline{M}^i$ then we can write $x = fx' + d^{i-1}(z)$ for $x' \in M^i$ and $z \in M^{i-1}$. We define the map s' by sending $\overline{x}$ to the class of $f^i d^{i-1}(z)$ in $(\eta_f M)^i/f(\eta_f M)^i$. This is well defined because if $fx' + d^{i-1}(z) = 0$, then $f^i x'$ is in $(\eta_f M)^i$ and consequently $f^i d^{i-1}(z)$ is in $f(\eta_f M)^i$.

We omit the verification that the lower row in the displayed diagram is a short exact sequence of modules. It is immediately clear from these constructions that we have commutative diagrams

$$\begin{tikzcd}
B^{i+1} \oplus B^i \arrow[r] \arrow[d, "{s,s'}"] & B^{i+2} \oplus B^{i+1} \arrow[d, "{s,s'}"] \\
(\eta_f M)^i/f(\eta_f M)^i \arrow[r] & (\eta_f M)^{i+1}/f(\eta_f M)^{i+1}
\end{tikzcd}$$

where the upper horizontal arrow is given by the identification of the summands B^{i+1} in source and target. In other words, we have found an acyclic subcomplex of $\eta_f M^\bullet/f(\eta_f M^\bullet) = \eta_f M^\bullet \otimes_A A/fA$ and the quotient by this subcomplex is a complex whose terms Z^i/B^i are the cohomology modules of the complex $\overline{M}^\bullet = M^\bullet \otimes_A A/fA$.

To explain the phenomenon observed in Remark 15.95.5 in a more canonical manner, we are going to construct the Bockstein operators. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For every $i \in \mathbf{Z}$ there is a commutative diagram (with tensor products over A)

$$\begin{tikzcd} 0 \arrow[r] & M^\bullet \otimes f^{i+1}A \arrow[r] \arrow[d] & M^\bullet \otimes f^iA \arrow[r] \arrow[d] & M^\bullet \otimes f^iA/f^{i+1}A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M^\bullet \otimes f^{i+1}A/f^{i+2}A \arrow[r] & M^\bullet \otimes f^iA/f^{i+2}A \arrow[r] & M^\bullet \otimes f^iA/f^{i+1}A \arrow[r] & 0 \end{tikzcd}$$

whose rows are short exact sequences of complexes. Of course these short exact sequences for different i are all isomorphic to each other by suitably multiplying with powers of f. The long exact sequence of cohomology of the bottom sequence in particular determines the Bockstein operator

$$\beta = \beta^i : H^i(M^\bullet \otimes f^iA/f^{i+1}A) \to H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A)$$

for all $i \in \mathbf{Z}$. For later use we record here that by the commutative diagram above there is a factorization

0GSP (15.95.5.1)
$$\begin{tikzcd} H^i(M^\bullet \otimes f^iA/f^{i+1}A) \arrow[r, "\delta"'] \arrow[dr, "\beta"'] & H^{i+1}(M^\bullet \otimes f^{i+1}A) \arrow[d] \\ & H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \end{tikzcd}$$

of the Bockstein operator where δ is the boundary operator coming from the top row in the commutative diagram above. Let us show that we obtain a complex

0GSQ (15.95.5.2)
$$H^\bullet(M^\bullet/f) = \left[\begin{array}{c} \ldots \\ \downarrow \\ H^{i-1}(M^\bullet \otimes f^{i-1}A/f^iA) \\ \downarrow \beta \\ H^i(M^\bullet \otimes f^iA/f^{i+1}A) \\ \downarrow \beta \\ H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \\ \downarrow \\ \ldots \end{array}\right]$$

i.e., that $\beta \circ \beta = 0$[14]. Namely, using the factorization (15.95.5.1) we see that it suffices to show that

$$H^{i+1}(M^\bullet \otimes f^{i+1}A) \to H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+2}A) \xrightarrow{\beta^{i+1}} H^{i+2}(M^\bullet \otimes f^{i+2}A/f^{i+3}A)$$

is zero. This is true because the kernel of β^{i+1} consists of the cohomology classes which can be lifted to $H^{i+1}(M^\bullet \otimes f^{i+1}A/f^{i+3}A)$ and those in the image of the first map certainly can!

[14] An alternative is to argue that β occurs as the differential for the spectral sequence for the complex $(M^\bullet)_f$ filtered by the subcomplexes $f^iM^\bullet$. Yet another argument, which proves something stronger, is to first consider the case $M^\bullet = A$. Here the short exact sequences $0 \to f^{i+1}A/f^{i+2}A \to f^iA/f^{i+2}A \to f^iA/f^{i+1}A \to 0$ define maps $\beta^i : f^iA/f^{i+1}A \to f^{i+1}A/f^{i+2}A[1]$ in $D(A)$. Then one computes (arguing similarly to the text) that the composition $f^iA/f^{i+1}A \to f^{i+1}A/f^{i+2}A[1] \to f^{i+2}A/f^{i+3}A[2]$ is zero in $D(A)$. Since $M^\bullet \otimes f^iA/f^{i+1}A = M^\bullet \otimes^{\mathbf{L}} f^iA/f^{i+1}A$ by our assumption on $M^\bullet$ having f-torsion free terms, we conclude the compostion

$$(M^\bullet \otimes f^iA/f^{i+1}A) \to (M^\bullet \otimes f^{i+1}A/f^{i+2}A)[1] \to (M^\bullet \otimes f^{i+2}A/f^{i+3}A)[2]$$

in $D(A)$ is zero as well.

0F7T **Lemma 15.95.6.** Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. There is a canonical map of complexes

$$\eta_f M^\bullet \otimes_A A/fA \longrightarrow H^\bullet(M^\bullet/f)$$

which is a quasi-isomorphism where the right hand side is the complex (15.95.5.2).

Proof. Let $x \in (\eta_f M)^i$. Then $x = f^i x' \in f^i M$ and $d^i(x) = f^{i+1}y \in f^{i+1}M^{i+1}$. Thus d^i maps $x' \otimes f^i$ to zero in $M^{i+1} \otimes f^i A/f^{i+1}A$. All tensor products are over A in this proof. Hence we may map x to the class of $x' \otimes f^i$ in $H^i(M^\bullet \otimes f^i A/f^{i+1}A)$. It is clear that this rule defines a map

$$(\eta_f M)^i \otimes A/fA \longrightarrow H^i(M^\bullet \otimes f^i A/f^{i+1}A)$$

of A/fA-modules. Observe that in the situation above, we may view $x' \otimes f^i$ as an element of $M^i \otimes f^i A/f^{i+2}A$ with differential $d^i(x' \otimes f^i) = y \otimes f^{i+1}$. By the construction of β above we find that $\beta(x' \otimes f^i) = y \otimes f^{i+1}$ and we conclude that our maps are compatible with differentials, i.e., we have a map of complexes.

To finish the proof, we observe that the construction given in the previous paragraph agrees with the maps $(\eta_f M)^i \otimes A/fA \to Z^i/B^i$ discussed in Remark 15.95.5. Since we have seen that the kernel of these maps is an acyclic subcomplex of $\eta_f M^\bullet \otimes A/fA$, the lemma is proved. $\square$

0F7Y **Lemma 15.95.7.** Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. For $i \in \mathbf{Z}$ the following are equivalent

1. $\text{Ker}(d^i \bmod f^2)$ surjects onto $\text{Ker}(d^i \bmod f)$,
2. $\beta : H^i(M^\bullet \otimes_A f^i A/f^{i+1}A) \to H^{i+1}(M^\bullet \otimes_A f^{i+1}A/f^{i+2}A)$ is zero.

These equivalent conditions are implied by the condition $H^{i+1}(M^\bullet)[f] = 0$.

Proof. The equivalence of (1) and (2) follows from the definition of β as the boundary map on cohomology of a short exact sequence of complexes isomorphic to the short exact sequence of complexes $0 \to fM^\bullet/f^2M^\bullet \to M^\bullet/f^2M^\bullet \to M^\bullet/fM^\bullet \to 0$. If $\beta \neq 0$, then $H^{i+1}(M^\bullet)[f] \neq 0$ because of the factorization (15.95.5.1). $\square$

0F7Z **Lemma 15.95.8.** Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a complex of f-torsion free A-modules. If $\text{Ker}(d^i \bmod f^2)$ surjects onto $\text{Ker}(d^i \bmod f)$, then the canonical map

$$(1, d^i) : (\eta_f M)^i/f(\eta_f M)^i \longrightarrow f^i M^i/f^{i+1}M^i \oplus f^{i+1}M^{i+1}/f^{i+2}M^{i+1}$$

identifies the left hand side with a direct sum of submodules of the right hand side.

Proof. With notation as in Remark 15.95.5 we define a map $t^{-1} : Z^i \to (\eta_f M)^i/f(\eta_f M)^i$. Namely, for $x \in M^i$ with $d^i(x) = f^2 y$ we send the class of x in Z^i to the class of $f^i x$ in $(\eta_f M)^i/f(\eta_f M)^i$. We omit the verification that this is well defined; the assumption of the lemma exactly signifies that the domain of this operation is all of Z^i. Then $t \circ t^{-1} = \text{id}_{Z^i}$. Hence t^{-1} defines a splitting of the short exact sequence in Remark 15.95.5 and the resulting direct sum decomposition

$$(\eta_f M)^i/f(\eta_f M)^i = Z^i \oplus B^{i+1}$$

is compatible with the map displayed in the lemma. $\square$

0F7U **Lemma 15.95.9.** Let A be a ring and let $f, g \in A$ be nonzerodivisors. Let $M^\bullet$ be a complex of A-modules such that fg is a nonzerodivisor on all M^i. Then $\eta_f \eta_g M^\bullet = \eta_{fg} M^\bullet$.

Proof. The statement means that in degree i we obtain the same submodule of the localization $M^i_{fg} = (M^i_g)_f$. We omit the details. $\square$

0GSR Lemma 15.95.10. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $A \to B$ be a flat ring map and let $g \in B$ the image of f. Let $M^\bullet$ be a complex of f-torsion free A-modules. Then g is a nonzerodivisor, $M^\bullet \otimes_A B$ is a complex of g-torsion free modules, and $\eta_f M^\bullet \otimes_A B = \eta_g(M^\bullet \otimes_A B)$.

Proof. Omitted. $\square$

15.96. Perfect complexes and the eta operator

0F7V In this section we do some algebra to prepare for our version of Macpherson's graph construction, see More on Flatness, Section 38.44. We will use the η_f operator introduced in Section 15.95.

Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. For each i let r_i be the rank of M^i and set

$$I_i(M^\bullet, f) = \text{ideal generated by the } r_i \times r_i\text{-minors of } (f, d^i) : M^i \to M^i \oplus M^{i+1}$$

Observe that $f^{r_i} \in I_i(M^\bullet, f)$.

0GSS Lemma 15.96.1. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ and $N^\bullet$ be two bounded complexes of finite free A-modules representing the same object of $D(A)$. Then

$$f^m I_i(M^\bullet, f) = f^n I_i(N^\bullet, f)$$

as ideals of A for integers $n, m \geq 0$ such that

$$m + \sum\nolimits_{j \geq i} (-1)^{j-i} rk(M^j) = n + \sum\nolimits_{j \geq i} (-1)^{j-i} rk(N^j)$$

Proof. It suffices to prove the equality after localization at every prime ideal of A. Thus by Lemma 15.75.7 and an induction argument we omit we may assume $N^\bullet = M^\bullet \oplus Q^\bullet$ for some trivial complex $Q^\bullet$, i.e.,

$$Q^\bullet = \ldots \to 0 \to A \xrightarrow{1} A \to 0 \to \ldots$$

where A is placed in degree j and $j+1$. If $j \neq i-1, i, i+1$ then we clearly have equality $I_i(M^\bullet, f) = I_i(N^\bullet, f)$ and $m = n$ and we have the desired equality. If $j = i+1$ then the maps

$$(f, d^i) : M^i \to M^i \oplus M^{i+1} \quad \text{and} \quad (f, d^i, 0) : M^i \to M^i \oplus M^{i+1} \oplus A$$

have the same nonzero minors hence in this case we also have $I_i(M^\bullet, f) = I_i(N^\bullet, f)$ and $m = n$. If $j = i$, then $I_i(M^\bullet, f)$ is the ideal generated by the $r_i \times r_i$-minors of

$$(f, d^i) : M^i \to M^i \oplus M^{i+1}$$

and $I_i(N^\bullet, f)$ is the ideal generated by the $(r_i+1) \times (r_i+1)$-minors of

$$(f \oplus f, d^i \oplus 1) : (M^i \oplus A) \to (M^i \oplus A) \oplus (M^{i+1} \oplus A)$$

With suitable choice of coordinates we see that the matrix of the second map is in block form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_1 = \text{matrix of first map}, \quad T_2 = \begin{pmatrix} f \\ 1 \end{pmatrix}$$

With notation as in Lemma 15.8.1 we have $I_0(T_2) = A$, $I_1(T_2) = A$, $I_p(T_2) = 0$ for $p \geq 2$ and hence $I_{r_i+1}(T) = I_{r_i+1}(T_1) + I_{r_i}(T_1) = I_{r_i}(T_1)$ which means that

$I_i(M^\bullet, f) = I_i(N^\bullet, f)$. We also have $m = n$ so this finishes the case $j = i$. Finally, say $j = i-1$. Then we see that $m = n+1$, thus we have to show that $fI_i(M^\bullet, f) = I_i(N^\bullet, f)$. In this case $I_i(M^\bullet, f)$ is the ideal generated by the $r_i \times r_i$-minors of

$$(f, d^i) : M^i \to M^i \oplus M^{i+1}$$

and $I_i(N^\bullet, f)$ is the ideal generated by the $(r_i + 1) \times (r_i + 1)$-minors of

$$(f \oplus f, d^i) : (M^i \oplus A) \to (M^i \oplus A) \oplus M^{i+1}$$

With suitable choice of coordinates we see that the matrix of the second map is in block form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_1 = \text{matrix of first map}, \quad T_2 = \begin{pmatrix} f \end{pmatrix}$$

Arguing as above we find that indeed $fI_i(M^\bullet, f) = I_i(N^\bullet, f)$. □

0GST Lemma 15.96.2. Let $f \in A$ be a nonzerodivisor of a ring A. Let $u \in A$ be a unit. Let $M^\bullet$ be a bounded complex of finite free A-modules. Then $I_i(M^\bullet, f) = I_i(M^\bullet, uf)$.

Proof. Omitted. □

0GSU Lemma 15.96.3. Let $A \to B$ be a ring map. Let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume f maps to a nonzerodivisor g in B. Then $I_i(M^\bullet, f)B = I_i(M^\bullet \otimes_A B, g)$.

Proof. The minors of $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ map to the corresponding minors of $(g, d^i) : M^i \otimes_A B \to M^i \otimes_A B \oplus M^{i+1} \otimes_A B$. □

0GSV Lemma 15.96.4. Let A be a ring, let $\mathfrak p \subset A$ be a prime ideal, and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. If $H^i(M^\bullet)_{\mathfrak p}$ is free for all i, then $I_i(M^\bullet, f)_{\mathfrak p}$ is a principal ideal and in fact generated by a power of f for all i.

Proof. We may assume A is local with maximal ideal $\mathfrak p$ by Lemma 15.96.3. We may also replace $M^\bullet$ with a quasi-isomorphic complex by Lemma 15.96.1. By our assumption on the freeness of cohomology modules we see that $M^\bullet$ is quasi-isomorphic to the complex whose term in degree i is $H^i(M^\bullet)$ with vanishing differentials, see for example Derived Categories, Lemma 13.27.9. In other words, we may assume the differentials in the complex $M^\bullet$ are all zero. In this case it is clear that $I_i(M^\bullet, f) = (f^{r_i})$ is principal. □

0F7W Lemma 15.96.5. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume $I_i(M^\bullet, f)$ is a principal ideal. Then $(\eta_f M)^i$ is locally free of rank r_i and the map $(1, d^i) : (\eta_f M)^i \to f^i M^i \oplus f^{i+1} M^{i+1}$ is the inclusion of a direct summand.

Proof. Choose a generator g for $I_i(M^\bullet, f)$. Since $f^{r_i} \in I_i(M^\bullet, f)$ we see that g divides a power of f. In particular g is a nonzerodivisor in A. The $r_i \times r_i$-minors of the map $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ generate the ideal $I_i(M^\bullet, f)$ and the $(r_i + 1) \times (r_i + 1)$-minors of (f, d^i) are zero: we may check this after localizing at f where the rank of the map is equal to r_i. Consider the surjection

$$M^i \oplus M^{i+1} \longrightarrow Q = \text{Coker}(f, d^i)/g\text{-torsion}$$

By Lemma 15.8.9 the module Q is finite locally free of rank r_{i+1}. Hence Q is f-torsion free and we conclude the cokernel of (f, d^i) modulo f-power torsion is Q as well.

Consider the complex of finite free A-modules

$$0 \to f^{i+1}M^i \xrightarrow{1,d^i} f^iM^i \oplus f^{i+1}M^{i+1} \xrightarrow{d^i,-1} f^iM^{i+1} \to 0$$

which becomes split exact after localizing at f. The map $(1, d^i) : f^{i+1}M^i \to f^iM^i \oplus f^{i+1}M^{i+1}$ is isomorphic to the map $(f, d^i) : M^i \to M^i \oplus M^{i+1}$ we studied above. Hence the image

$$Q' = \mathrm{Im}(f^iM^i \oplus f^{i+1}M^{i+1} \xrightarrow{d^i,-1} f^iM^{i+1})$$

is isomorphic to Q in particular projective. On the other hand, by construction of η_f in Section 15.95 the image of the injective map $(1, d^i) : (\eta_f M)^i \to f^iM^i \oplus f^{i+1}M^{i+1}$ is the kernel of $(d^i, -1)$. We conclude that we obtain an isomorphism $(\eta_f M)^i \oplus Q' = f^iM^i \oplus f^{i+1}M^{i+1}$ and we see that indeed $\eta_f M^i$ is finite locally free of rank r_i and that $(1, d^i)$ is the inclusion of a direct summand. $\square$

0F7X Lemma 15.96.6. Let $A \to B$ be a ring map. Let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume f maps to a nonzerodivisor g in B and $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then there is a canonical isomorphism $\eta_f M^\bullet \otimes_A B = \eta_g(M^\bullet \otimes_A B)$.

Proof. Set $N^i = M^i \otimes_A B$. Observe that $f^iM^i \otimes_A B = g^iN^i$ as submodules of $(N^i)_g$. The maps

$$(\eta_f M)^i \otimes_A B \to g^iN^i \otimes g^{i+1}N^{i+1} \quad \text{and} \quad (\eta_g N)^i \to g^iN^i \otimes g^{i+1}N^{i+1}$$

are inclusions of direct summands by Lemma 15.96.5. Since their images agree after localizing at g we conclude. $\square$

0F80 Lemma 15.96.7. Let A be a ring. Let M, N_1, N_2 be finite projective A-modules. Let $s : M \to N_1 \oplus N_2$ be a split injection. There exists a finitely generated ideal $J \subset A$ with the following property: a ring map $A \to B$ factors through A/J if and only if $s \otimes \mathrm{id}_B$ identifies $M \otimes_A B$ with a direct sum of submodules of $N_1 \otimes_A B \oplus N_2 \otimes_A B$.

Proof. Choose a splitting $\pi : N_1 \oplus N_2 \to M$ of s. Denote $q_i : N_1 \oplus N_2 \to N_1 \oplus N_2$ the projector onto N_i. Set $p_i = \pi \circ q_i \circ s$. Observe that $p_1 + p_2 = \mathrm{id}_M$. We claim M is a direct sum of submodules of $N_1 \oplus N_2$ if and only if p_1 and p_2 are orthogonal projectors. Thus J is the smallest ideal of A such that $p_1 \circ p_1 - p_1$, $p_2 \circ p_2 - p_2$, $p_1 \circ p_2$, and $p_2 \circ p_1$ are contained in $J \otimes_A \mathrm{End}_A(M)$. Some details omitted. $\square$

Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume the ideals $I_i(M^\bullet, f)$ are principal for all $i \in \mathbf{Z}$. Then the maps

$$(1, d^i) : (\eta_f M)^i / f(\eta_f M)^i \longrightarrow f^iM^i/f^{i+1}M^i \oplus f^{i+1}M^{i+1}/f^{i+2}M^{i+1}$$

are split injections by Lemma 15.96.5. Denote $J_i(M^\bullet, f) \subset A/fA$ the finitely generated ideal of Lemma 15.96.7 corresponding to the split injection $(1, d^i)$ displayed above.

0GSW Lemma 15.96.8. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ and $N^\bullet$ be two bounded complexes of finite free A-modules representing the same object in $D(A)$. Assume $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Then $J_i(M^\bullet, f) = J_i(N^\bullet, f)$ as ideals in A/fA.

Proof. Observe that the fact that $I_i(M^\bullet, f)$ is a principal ideal implies that $I_i(M^\bullet, f)$ is a principal ideal by Lemma 15.96.1 and hence the statement makes sense. As in the proof of Lemma 15.96.1 we may assume $N^\bullet = M^\bullet \oplus Q^\bullet$ for some trivial complex $Q^\bullet$, i.e.,

$$Q^\bullet = \ldots \to 0 \to A \xrightarrow{1} A \to 0 \to \ldots$$

where A is placed in degree j and $j+1$. Since η_f is compatible with direct sums, we see that the map

$$(1, d^i) : (\eta_f N)^i / f(\eta_f N)^i \longrightarrow f^i N^i / f^{i+1} N^i \oplus f^{i+1} N^{i+1} / f^{i+2} N^{i+1}$$

is the direct sum of the corresponding map for $M^\bullet$ and for $Q^\bullet$. By the universal property defining the ideals in question, we conclude that $J_i(N^\bullet, f) = J_i(M^\bullet, f) + J_i(Q^\bullet, f)$. Hence it suffices to show that $J_i(Q^\bullet, f) = 0$ for all i. This is a computation that we omit. □

0F81 Lemma 15.96.9. Let A be a ring and let $f \in A$ be a nonzerodivisor. Let $M^\bullet$ be a bounded complex of finite free A-modules. Assume $I_i(M^\bullet, f)$ is a principal ideal for all $i \in \mathbf{Z}$. Consider the ideal $J(M^\bullet, f) = \sum_i J_i(M^\bullet, f)$ of A/fA. Consider the set of prime ideals

$$\begin{aligned} E &= \{f \in \mathfrak{p} \subset A \mid \text{Ker}(d^i \bmod f^2)_\mathfrak{p} \text{ surjects onto } \text{Ker}(d^i \bmod f)_\mathfrak{p} \text{ for all } i \in \mathbf{Z}\} \\ &= \{f \in \mathfrak{p} \subset A \mid \text{the localizations } \beta_\mathfrak{p} \text{ of the Bockstein operators are zero}\} \end{aligned}$$

Then we have

(1) $J(M^\bullet, f)$ is finitely generated,
(2) $A/fA \to C = (A/fA)/J(M^\bullet, f)$ is surjective of finite presentation,
(3) $J(M^\bullet, f)_\mathfrak{p} = 0$ for $\mathfrak{p} \in E$,
(4) if $f \in \mathfrak{p}$ and $H^i(M^\bullet)_\mathfrak{p}$ is free for all $i \in \mathbf{Z}$, then $\mathfrak{p} \in E$, and
(5) the cohomology modules of $\eta_f M^\bullet \otimes_A C$ are finite locally free C-modules.

Proof. The equality in the definition of E follows from Lemma 15.95.7 and in addition the final statement of that lemma implies part (4).

Part (1) is true because the ideals $J_i(M^\bullet, f)$ are finitely generated and because $M^\bullet$ is bounded and hence $J_i(M^\bullet, f)$ is zero for almost all i. Part (2) is just a reformulation of part (1).

Proof of (3). By Lemma 15.96.5 we find that $(\eta_f M)^i$ is finite locally free of rank r_i for all i. Consider the map

$$(1, d^i) : (\eta_f M)^i / f(\eta_f M)^i \longrightarrow f^i M^i / f^{i+1} M^i \oplus f^{i+1} M^{i+1} / f^{i+2} M^{i+1}$$

Pick $\mathfrak{p} \in E$. By Lemma 15.95.8 and the local freeness of the modules $(\eta_f M)^i$ we may write

$$\left((\eta_f M)^i / f(\eta_f M)^i\right)_\mathfrak{p} = (A/fA)_\mathfrak{p}^{\oplus m_i} \oplus (A/fA)_\mathfrak{p}^{\oplus n_i}$$

compatible with the arrow $(1, d^i)$ above. By the universal property of the ideal $J_i(M^\bullet, f)$ we conclude that $J_i(M^\bullet, f)_\mathfrak{p} = 0$. Hence $I_\mathfrak{p} = fA_\mathfrak{p}$ for $\mathfrak{p} \in E$.

Proof of (5). Observe that the differential on $\eta_f M^\bullet$ fits into a commutative diagram

$$\begin{tikzcd} (\eta_f M)^i \arrow[r] \arrow[d] & f^i M^i \oplus f^{i+1} M^{i+1} \arrow[d, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \\ (\eta_f M)^{i+1} \arrow[r] & f^{i+1} M^i \oplus f^{i+2} M^{i+2} \end{tikzcd}$$

By construction, after tensoring with C, the modules on the left are direct sums of direct summands of the summands on the right. Picture

$$\begin{tikzcd} (\eta_f M)^i \otimes_A C \arrow[r, equal] \arrow[d] & K^i \oplus L^i \arrow[r] \arrow[d] & f^i M^i \otimes_A C \oplus f^{i+1} M^{i+1} \otimes_A C \arrow[d, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \\ (\eta_f M)^{i+1} \otimes_A C \arrow[r, equal] & K^{i+1} \oplus L^{i+1} \arrow[r] & f^{i+1} M^i \otimes_A C \oplus f^{i+2} M^{i+2} \otimes_A C \end{tikzcd}$$

where the horizontal arrows are compatible with direct sum decompositions as well as inclusions of direct summands. It follows that the differential identifies L^i with a direct summand of K^{i+1} and we conclude that the cohomology of $\eta_f M^\bullet \otimes_A C$ in degree i is the module K^{i+1}/L^i which is finite projective as desired. □

15.97. Taking limits of complexes

09B6 In this section we discuss what happens when we have a "formal deformation" of a complex and we take its limit. We will consider two cases

(1) we have a limit $A = \lim A_n$ of an inverse system of rings whose transition maps are surjective with locally nilpotent kernels and objects $K_n \in D(A_n)$ which fit together in the sense that $K_n = K_{n+1} \otimes^{\mathbf{L}}_{A_{n+1}} A_n$, or
(2) we have a ring A, an ideal I, and objects $K_n \in D(A/I^n)$ which fit together in the sense that $K_n = K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n$.

Under additional hypotheses we can show that $K = R\lim K_n$ reproduces the system in the sense that $K_n = K \otimes^{\mathbf{L}}_A A_n$ or $K_n = K \otimes^{\mathbf{L}}_A A/I^n$.

0CQF Lemma 15.97.1. Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given $K_n \in D(A_n)$ and maps $K_{n+1} \to K_n$ in $D(A_{n+1})$. Assume

(1) the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,
(2) K_1 is pseudo-coherent, and
(3) the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A_{n+1}} A_n \to K_n$.

Then $K = R\lim K_n$ is a pseudo-coherent object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A_n \to K_n$ is an isomorphism for all n.

Proof. By assumption we can find a bounded above complex of finite free A_1-modules $P_1^\bullet$ representing K_1, see Definition 15.64.1. By Lemma 15.75.4 we can, by induction on $n > 1$, find complexes $P_n^\bullet$ of finite free A_n-modules representing K_n and maps $P_n^\bullet \to P_{n-1}^\bullet$ representing the maps $K_n \to K_{n-1}$ inducing isomorphisms (!) of complexes $P_n^\bullet \otimes_{A_n} A_{n-1} \to P_{n-1}^\bullet$. Thus $K = R\lim K_n$ is represented by $P^\bullet = \lim P_n^\bullet$, see Lemma 15.87.1 and Remark 15.87.6. Since P_n^i is a finite free A_n-module for each n and $A = \lim A_n$ we see that P^i is finite free of the same rank as P_1^i for each i. This means that K is pseudo-coherent. It also follows that $K \otimes^{\mathbf{L}}_A A_n$ is represented by $P^\bullet \otimes_A A_n = P_n^\bullet$ which proves the final assertion. □

09AV Lemma 15.97.2. Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. Assume

(1) A is I-adically complete,
(2) K_1 is pseudo-coherent, and
(3) the maps induce isomorphisms $K_{n+1} \otimes_{A/I^{n+1}}^{\mathbf{L}} A/I^n \to K_n$.

Then $K = R\lim K_n$ is a pseudo-coherent, derived complete object of $D(A)$ and $K \otimes_A^{\mathbf{L}} A/I^n \to K_n$ is an isomorphism for all n.

Proof. We already know that K is pseudo-coherent and that $K \otimes_A^{\mathbf{L}} A/I^n \to K_n$ is an isomorphism for all n, see Lemma 15.97.1. Finally, K is derived complete by Lemma 15.91.14. □

0CQG Lemma 15.97.3. Let $A = \lim A_n$ be a limit of an inverse system (A_n) of rings. Suppose given $K_n \in D(A_n)$ and maps $K_{n+1} \to K_n$ in $D(A_{n+1})$. Assume [Bha16, Lemma 4.2]

(1) the transition maps $A_{n+1} \to A_n$ are surjective with locally nilpotent kernels,
(2) K_1 is a perfect object, and
(3) the maps induce isomorphisms $K_{n+1} \otimes_{A_{n+1}}^{\mathbf{L}} A_n \to K_n$.

Then $K = R\lim K_n$ is a perfect object of $D(A)$ and $K \otimes_A^{\mathbf{L}} A_n \to K_n$ is an isomorphism for all n.

Proof. We already know that K is pseudo-coherent and that $K \otimes_A^{\mathbf{L}} A_n \to K_n$ is an isomorphism for all n by Lemma 15.97.1. Thus it suffices to show that $H^i(K \otimes_A^{\mathbf{L}} \kappa) = 0$ for $i \ll 0$ and every surjective map $A \to \kappa$ whose kernel is a maximal ideal $\mathfrak{m}$, see Lemma 15.77.3. Any element of A which maps to a unit in A_1 is a unit in A by Algebra, Lemma 10.32.4 and hence $\text{Ker}(A \to A_1)$ is contained in the Jacobson radical of A by Algebra, Lemma 10.19.1. Hence $A \to \kappa$ factors as $A \to A_1 \to \kappa$. Hence

$$K \otimes_A^{\mathbf{L}} \kappa = K \otimes_A^{\mathbf{L}} A_1 \otimes_{A_1}^{\mathbf{L}} \kappa = K_1 \otimes_{A_1}^{\mathbf{L}} \kappa$$

and we get what we want as K_1 has finite tor dimension by Lemma 15.74.2. □

09AW Lemma 15.97.4. Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. Assume

(1) A is I-adically complete,
(2) K_1 is a perfect object, and
(3) the maps induce isomorphisms $K_{n+1} \otimes_{A/I^{n+1}}^{\mathbf{L}} A/I^n \to K_n$.

Then $K = R\lim K_n$ is a perfect, derived complete object of $D(A)$ and $K \otimes_A^{\mathbf{L}} A/I^n \to K_n$ is an isomorphism for all n.

Proof. Combine Lemmas 15.97.3 and 15.97.2 (to get derived completeness). □

We do not know if the following lemma holds for unbounded complexes.

09AU Lemma 15.97.5. Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. If

(1) A is Noetherian,
(2) K_1 is bounded above, and
(3) the maps induce isomorphisms $K_{n+1} \otimes_{A/I^{n+1}}^{\mathbf{L}} A/I^n \to K_n$,

then $K = R\lim K_n$ is a derived complete object of $D^-(A)$ and $K \otimes_A^{\mathbf{L}} A/I^n \to K_n$ is an isomorphism for all n.

Proof. The object K of $D(A)$ is derived complete by Lemma 15.91.14.

Suppose that $H^i(K_1) = 0$ for $i > b$. Then we can find a complex of free A/I-modules $P_1^\bullet$ representing K_1 with $P_1^i = 0$ for $i > b$. By Lemma 15.75.3 we can, by induction on $n > 1$, find complexes $P_n^\bullet$ of free A/I^n-modules representing K_n and maps $P_n^\bullet \to P_{n-1}^\bullet$ representing the maps $K_n \to K_{n-1}$ inducing isomorphisms (!) of complexes $P_n^\bullet/I^{n-1}P_n^\bullet \to P_{n-1}^\bullet$.

Thus we have arrived at the situation where $R\lim K_n$ is represented by $P^\bullet = \lim P_n^\bullet$, see Lemma 15.87.1 and Remark 15.87.6. The complexes $P_n^\bullet$ are uniformly bounded above complexes of flat A/I^n-modules and the transition maps are termwise surjective. Then $P^\bullet$ is a bounded above complex of flat A-modules by Lemma 15.27.4. It follows that $K \otimes_A^{\mathbf{L}} A/I^t$ is represented by $P^\bullet \otimes_A A/I^t$. We have $P^\bullet \otimes_A A/I^t = \lim P_n^\bullet \otimes_A A/I^t$ termwise by Lemma 15.27.4. The transition maps $P_{n+1}^\bullet \otimes_A A/I^t \to P_n^\bullet \otimes_A A/I^t$ are isomorphisms for $n \geq t$ by our choice of $P_n^\bullet$, hence we have $\lim P_n^\bullet \otimes_A A/I^t = P_t^\bullet \otimes_A A/I^t = P_t^\bullet$. Since $P_t^\bullet$ represents K_t, we see that $K \otimes_A^{\mathbf{L}} A/I^t \to K_t$ is an isomorphism. □

Here is a different type of result.

0EGS Lemma 15.97.6 (Kollár-Kovács). Let I be an ideal of a Noetherian ring A. Let $K \in D(A)$. Set $K_n = K \otimes_A^{\mathbf{L}} A/I^n$. Assume for all $i \in \mathbf{Z}$ we have Email from Kovacs of 23/02/2018.

(1) $H^i(K)$ is a finite A-module, and
(2) the system $H^i(K_n)$ satisfies Mittag-Leffler.

Then $\lim H^i(K)/I^nH^i(K)$ is equal to $\lim H^i(K_n)$ for all $i \in \mathbf{Z}$.

Proof. Recall that $K^\wedge = R\lim K_n$ is the derived completion of K, see Proposition 15.94.2. By Lemma 15.94.4 we have $H^i(K^\wedge) = \lim H^i(K)/I^nH^i(K)$. By Lemma 15.87.4 we get short exact sequences

$$0 \to R^1\lim H^{i-1}(K_n) \to H^i(K^\wedge) \to \lim H^i(K_n) \to 0$$

The Mittag-Leffler condition guarantees that the left terms are zero (Lemma 15.87.1) and we conclude the lemma is true. □

15.98. Some evaluation maps

0ATJ In this section we prove that certain canonical maps of $R\operatorname{Hom}$'s are isomorphisms for suitable types of complexes.

0A68 Lemma 15.98.1. Let R be a ring. Let K, L, M be objects of $D(R)$. the map

$$R\operatorname{Hom}_R(L, M) \otimes_R^{\mathbf{L}} K \longrightarrow R\operatorname{Hom}_R(R\operatorname{Hom}_R(K, L), M)$$

of Lemma 15.73.3 is an isomorphism in the following two cases

(1) K perfect, or
(2) K is pseudo-coherent, $L \in D^+(R)$, and M finite injective dimension.

Proof. Choose a K-injective complex $I^\bullet$ representing M, a K-injective complex $J^\bullet$ representing L, and a bounded above complex of finite projective modules $K^\bullet$ representing K. Consider the map of complexes

$$\operatorname{Tot}(\operatorname{Hom}^\bullet(J^\bullet, I^\bullet) \otimes_R K^\bullet) \longrightarrow \operatorname{Hom}^\bullet(\operatorname{Hom}^\bullet(K^\bullet, J^\bullet), I^\bullet)$$

of Lemma 15.71.6. Note that

$$\left(\prod\nolimits_{p+r=t} \operatorname{Hom}_R(J^{-r}, I^p)\right) \otimes_R K^s = \prod\nolimits_{p+r=t} \operatorname{Hom}_R(J^{-r}, I^p) \otimes_R K^s$$

because K^s is finite projective. The map is given by the maps

$$c_{p,r,s} : \mathrm{Hom}_R(J^{-r}, I^p) \otimes_R K^s \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(K^s, J^{-r}), I^p)$$

which are isomorphisms as K^s is finite projective. For every element $\alpha = (\alpha^{p,r,s})$ of degree n of the left hand side, there are only finitely many values of s such that $\alpha^{p,r,s}$ is nonzero (for some p, r with $n = p+r+s$). Hence our map is an isomorphism if the same vanishing condition is forced on the elements $\beta = (\beta^{p,r,s})$ of the right hand side. If $K^\bullet$ is a bounded complex of finite projective modules, this is clear. On the other hand, if we can choose $I^\bullet$ bounded and $J^\bullet$ bounded below, then $\beta^{p,r,s}$ is zero for p outside a fixed range, for $s \gg 0$, and for $r \gg 0$. Hence among solutions of $n = p + r + s$ with $\beta^{p,r,s}$ nonzero only a finite number of s values occur. □

0A69 Lemma 15.98.2. Let R be a ring. Let K, L, M be objects of $D(R)$. the map

$$R\,\mathrm{Hom}_R(L, M) \otimes_R^{\mathbf{L}} K \longrightarrow R\,\mathrm{Hom}_R(R\,\mathrm{Hom}_R(K, L), M)$$

of Lemma 15.73.3 is an isomorphism if the following three conditions are satisfied

(1) L, M have finite injective dimension,
(2) $R\,\mathrm{Hom}_R(L, M)$ has finite tor dimension,
(3) for every $n \in \mathbf{Z}$ the truncation $\tau_{\leq n} K$ is pseudo-coherent

Proof. Pick an integer n and consider the distinguished triangle

$$\tau_{\leq n} K \to K \to \tau_{\geq n+1} K \to \tau_{\leq n} K[1]$$

see Derived Categories, Remark 13.12.4. By assumption (3) and Lemma 15.98.1 the map is an isomorphism for $\tau_{\leq n} K$. Hence it suffices to show that both

$$R\,\mathrm{Hom}_R(L, M) \otimes_R^{\mathbf{L}} \tau_{\geq n+1} K \quad \text{and} \quad R\,\mathrm{Hom}_R(R\,\mathrm{Hom}_R(\tau_{\geq n+1} K, L), M)$$

have vanishing cohomology in degrees $\leq n - c$ for some c. This follows immediately from assumptions (2) and (1). □

0ATK Lemma 15.98.3. Let R be a ring. Let K, L, M be objects of $D(R)$. The map

$$K \otimes_R^{\mathbf{L}} R\,\mathrm{Hom}_R(M, L) \longrightarrow R\,\mathrm{Hom}_R(M, K \otimes_R^{\mathbf{L}} L)$$

of Lemma 15.73.5 is an isomorphism in the following cases

(1) M perfect, or
(2) K is perfect, or
(3) M is pseudo-coherent, $L \in D^+(R)$, and K has tor amplitude in $[a, \infty]$.

Proof. Proof in case M is perfect. Note that both sides of the arrow transform distinguished triangles in M into distinguished triangles and commute with direct sums. Hence it suffices to check it holds when $M = R[n]$, see Derived Categories, Remark 13.36.7 and Lemma 15.78.1. In this case the result is obvious.

Proof in case K is perfect. Same argument as in the previous case.

Proof in case (3). We may represent K and L by bounded below complexes of R-modules $K^\bullet$ and $L^\bullet$. We may assume that $K^\bullet$ is a K-flat complex consisting of flat R-modules, see Lemma 15.66.4. We may represent M by a bounded above complex $M^\bullet$ of finite free R-modules, see Definition 15.64.1. Then the object on the LHS is represented by

$$\mathrm{Tot}(K^\bullet \otimes_R \mathrm{Hom}^\bullet(M^\bullet, L^\bullet))$$

and the object on the RHS by

$$\mathrm{Hom}^\bullet(M^\bullet, \mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$$

This uses Lemma 15.73.2. Both complexes have in degree n the module

$$\bigoplus\nolimits_{p+q+r=n} K^p \otimes \mathrm{Hom}_R(M^{-r}, L^q) = \bigoplus\nolimits_{p+q+r=n} \mathrm{Hom}_R(M^{-r}, K^p \otimes_R L^q)$$

because M^{-r} is finite free (as well these are finite direct sums). The map defined in Lemma 15.73.5 comes from the map of complexes defined in Lemma 15.71.4 which uses the canonical isomorphisms between these modules. □

0BYQ Lemma 15.98.4. Let R be a ring. Let $P^\bullet$ be a bounded above complex of projective R-modules. Let $K^\bullet$ be a K-flat complex of R-modules. If $P^\bullet$ is a perfect object of $D(R)$, then $\mathrm{Hom}^\bullet(P^\bullet, K^\bullet)$ is K-flat and represents $R\,\mathrm{Hom}_R(P^\bullet, K^\bullet)$.

Proof. The last statement is Lemma 15.73.2. Since $P^\bullet$ represents a perfect object, there exists a finite complex of finite projective R-modules $F^\bullet$ such that $P^\bullet$ and $F^\bullet$ are isomorphic in $D(R)$, see Definition 15.74.1. Then $P^\bullet$ and $F^\bullet$ are homotopy equivalent, see Derived Categories, Lemma 13.19.8. Then $\mathrm{Hom}^\bullet(P^\bullet, K^\bullet)$ and $\mathrm{Hom}^\bullet(F^\bullet, K^\bullet)$ are homotopy equivalent. Hence the first is K-flat if and only if the second is (follows from Definition 15.59.1 and Lemma 15.58.2). It is clear that

$$\mathrm{Hom}^\bullet(F^\bullet, K^\bullet) = \mathrm{Tot}(E^\bullet \otimes_R K^\bullet)$$

where $E^\bullet$ is the dual complex to $F^\bullet$ with terms $E^n = \mathrm{Hom}_R(F^{-n}, R)$, see Lemma 15.74.15 and its proof. Since $E^\bullet$ is a bounded complex of projectives we find that it is K-flat by Lemma 15.59.7. Then we conclude by Lemma 15.59.4. □

15.99. Base change for derived hom

0E1V We have already seen some material discussing this in Lemma 15.65.4 and in Algebra, Section 10.73.

0E1W Lemma 15.99.1. Let $R \to R'$ be a ring map. For $K \in D(R)$ and $M \in D(R')$ there is a canonical isomorphism

$$R\,\mathrm{Hom}_R(K, M) = R\,\mathrm{Hom}_{R'}(K \otimes_R^{\mathbf{L}} R', M)$$

Proof. Choose a K-injective complex of R'-modules $J^\bullet$ representing M. Choose a quasi-isomorphism $J^\bullet \to I^\bullet$ where $I^\bullet$ is a K-injective complex of R-modules. Choose a K-flat complex $K^\bullet$ of R-modules representing K. Consider the map

$$\mathrm{Hom}^\bullet(K^\bullet \otimes_R R', J^\bullet) \longrightarrow \mathrm{Hom}^\bullet(K^\bullet, I^\bullet)$$

The map on degree n terms is given by the map

$$\prod\nolimits_{n=p+q} \mathrm{Hom}_{R'}(K^{-q} \otimes_R R', J^p) \longrightarrow \prod\nolimits_{n=p+q} \mathrm{Hom}_R(K^{-q}, I^p)$$

coming from precomposing by $K^{-q} \to K^{-q} \otimes_R R'$ and postcomposing by $J^p \to I^p$. To finish the proof it suffices to show that we get isomorphisms on cohomology groups:

$$\mathrm{Hom}_{D(R)}(K, M) = \mathrm{Hom}_{D(R')}(K \otimes_R^{\mathbf{L}} R', M)$$

which is true because base change $- \otimes_R^{\mathbf{L}} R' : D(R) \to D(R')$ is left adjoint to the restriction functor $D(R') \to D(R)$ by Lemma 15.60.3. □

Let $R \to R'$ be a ring map. There is a base change map

0E1X (15.99.1.1) $$R\operatorname{Hom}_R(K, M) \otimes_R^{\mathbf{L}} R' \longrightarrow R\operatorname{Hom}_{R'}(K \otimes_R^{\mathbf{L}} R', M \otimes_R^{\mathbf{L}} R')$$

in $D(R')$ functorial in $K, M \in D(R)$. Namely, by adjointness of $- \otimes_R^{\mathbf{L}} R' : D(R) \to D(R')$ and the restriction functor $D(R') \to D(R)$, this is the same thing as a map

$$R\operatorname{Hom}_R(K, M) \longrightarrow R\operatorname{Hom}_{R'}(K \otimes_R^{\mathbf{L}} R', M \otimes_R^{\mathbf{L}} R') = R\operatorname{Hom}_R(K, M \otimes_R^{\mathbf{L}} R')$$

(equality by Lemma 15.99.1) for which we can use the canonical map $M \to M \otimes_R^{\mathbf{L}} R'$ (unit of the adjunction).

0A6A **Lemma 15.99.2.** Let $R \to R'$ be a ring map. Let $K, M \in D(R)$. The map (15.99.1.1)

$$R\operatorname{Hom}_R(K, M) \otimes_R^{\mathbf{L}} R' \longrightarrow R\operatorname{Hom}_{R'}(K \otimes_R^{\mathbf{L}} R', M \otimes_R^{\mathbf{L}} R')$$

is an isomorphism in $D(R')$ in the following cases

(1) K is perfect,
(2) R' is perfect as an R-module,
(3) $R \to R'$ is flat, K is pseudo-coherent, and $M \in D^+(R)$, or
(4) R' has finite tor dimension as an R-module, K is pseudo-coherent, and $M \in D^+(R)$

Proof. We may check the map is an isomorphism after applying the restriction functor $D(R') \to D(R)$. After applying this functor our map becomes the map

$$R\operatorname{Hom}_R(K, L) \otimes_R^{\mathbf{L}} R' \longrightarrow R\operatorname{Hom}_R(K, L \otimes_R^{\mathbf{L}} R')$$

of Lemma 15.73.5. See discussion above the lemma to match the left and right hand sides; in particular, this uses Lemma 15.99.1. Thus we conclude by Lemma 15.98.3. $\square$

15.100. Systems of modules

0EGT Let I be an ideal of a Noetherian ring A. In this section we add to our knowledge of the relationship between finite modules over A and systems of finite A/I^n-modules.

0EGU **Lemma 15.100.1.** Let I be an ideal of a Noetherian ring A. Let $K \xrightarrow{\alpha} L \xrightarrow{\beta} M$ be a complex of finite A-modules. Set $H = \operatorname{Ker}(\beta)/\operatorname{Im}(\alpha)$. For $n \geq 0$ let

$$K/I^nK \xrightarrow{\alpha_n} L/I^nL \xrightarrow{\beta_n} M/I^nM$$

be the induced complex. Set $H_n = \operatorname{Ker}(\beta_n)/\operatorname{Im}(\alpha_n)$. Then there are canonical A-module maps giving a commutative diagram

$$\begin{tikzcd} & & & H \arrow[dll] \arrow[dl] \arrow[d] \\ \ldots \arrow[r] & H_3 \arrow[r] & H_2 \arrow[r] & H_1 \end{tikzcd}$$

Moreover, there exists a $c > 0$ and canonical A-module maps $H_n \to H/I^{n-c}H$ for $n \geq c$ such that the compositions

$$H/I^nH \to H_n \to H/I^{n-c}H \quad \text{and} \quad H_n \to H/I^{n-c}H \to H_{n-c}$$

are the canonical ones. Moreover, we have

(1) (H_n) and (H/I^nH) are isomorphic as pro-objects of Mod_A,

(2) $\lim H_n = \lim H/I^nH$,
(3) the inverse system (H_n) is Mittag-Leffler,
(4) the image of $H_{n+c} \to H_n$ is equal to the image of $H \to H_n$,
(5) the composition $I^cH_n \to H_n \to H/I^{n-c}H \to H_n/I^{n-c}H_n$ is the inclusion $I^cH_n \to H_n$ followed by the quotient map $H_n \to H_n/I^{n-c}H_n$, and
(6) the kernel and cokernel of $H/I^nH \to H_n$ is annihilated by I^c.

Proof. Observe that $H_n = \beta^{-1}(I^nM)/\operatorname{Im}(\alpha)+I^nL$. For $n \geq 2$ we have $\beta^{-1}(I^nM) \subset \beta^{-1}(I^{n-1}M)$ and $\operatorname{Im}(\alpha)+I^nL \subset \operatorname{Im}(\alpha)+I^{n-1}L$. Thus we obtain our canonical map $H_n \to H_{n-1}$. Similarly, we have $\operatorname{Ker}(\beta) \subset \beta^{-1}(I^nM)$ and $\operatorname{Im}(\alpha) \subset \operatorname{Im}(\alpha) + I^nL$ which produces the canonical map $H \to H_n$. We omit the verification that the diagram commutes.

By Artin-Rees we may choose $c_1, c_2 \geq 0$ such that $\beta^{-1}(I^nM) \subset \operatorname{Ker}(\beta) + I^{n-c_1}L$ for $n \geq c_1$ and $\operatorname{Ker}(\beta) \cap I^nL \subset I^{n-c_2}\operatorname{Ker}(\beta)$ for $n \geq c_2$, see Algebra, Lemmas 10.51.3 and 10.51.2. Set $c = c_1 + c_2$.

Let $n \geq c$. We define $\psi_n : H_n \to H/I^{n-c}H$ as follows. Say $x \in H_n$. Choose $y \in \beta^{-1}(I^nM)$ representing x. Write $y = z + w$ with $z \in \operatorname{Ker}(\beta)$ and $w \in I^{n-c_1}L$ (this is possible by our choice of c_1). We set $\psi_n(x)$ equal to the class of z in $H/I^{n-c}H$. To see this is well defined, suppose we have a second set of choices y', z', w' as above for x with obvious notation. Then $y' - y \in \operatorname{Im}(\alpha) + I^nL$, say $y' - y = \alpha(v) + u$ with $v \in K$ and $u \in I^nL$. Thus

$$y' = z' + w' = \alpha(v) + u + z + w \Rightarrow z' = z + \alpha(v) + u + w - w'$$

Since $\beta(z' - z - \alpha(v)) = 0$ we find that $u + w - w' \in \operatorname{Ker}(\beta) \cap I^{n-c_1}L$ which is contained in $I^{n-c_1-c_2}\operatorname{Ker}(\beta) = I^{n-c}\operatorname{Ker}(\beta)$ by our choice of c_2. Thus z' and z have the same image in $H/I^{n-c}H$ as desired.

The composition $H/I^nH \to H_n \to H/I^{n-c}H$ is the canonical map because if $z \in \operatorname{Ker}(\beta)$ represents an element x in $H/I^nH = \operatorname{Ker}(\beta)/\operatorname{Im}(\alpha) + I^n\operatorname{Ker}(\beta)$ then it is clear from the above that x maps to the class of z in $H/I^{n-c}H$ under the maps constructed above.

Let us consider the composition $H_n \to H/I^{n-c}H \to H_{n-c}$. Given x, y, z, w as in the construction of ψ_n above, we see that x is mapped to the cass of z in H_{n-c}. On the other hand, the canonical map $H_n \to H_{n-c}$ from the first paragraph of the proof sends x to the class of y. Thus we have to show that $y - z \in \operatorname{Im}(\alpha) + I^{n-c}L$ which is the case because $y - z = w \in I^{n-c_1}L \subset I^{n-c}L$.

Statements (1) – (4) are formal consequences of what we just proved. Namely, (1) follows from the existence of the maps and the definition of morphisms of pro-objects in Categories, Remark 4.22.5. Part (2) holds because isomorphic pro-objects have isomorphic limits. Part (3) is immediate from part (4). Part (4) follows from the factorization $H_{n+c} \to H/I^nH \to H_n$ of the canonical map $H_{n+c} \to H_n$.

Proof of part (5). Let $x \in I^cH_n$. Write $x = \sum f_ix_i$ with $x_i \in H_n$ and $f_i \in I^c$. Choose y_i, z_i, w_i as in the construction of ψ_n for x_i. Then for the computation of ψ_n of x we may choose $y = \sum f_iy_i$, $z = \sum f_iz_i$ and $w = \sum f_iw_i$ and we see that $\psi_n(x)$ is given by the class of z. The image of this in $H_n/I^{n-c}H_n$ is equal to the class of y as $w = \sum f_iw_i$ is in I^nL. This proves (5).

Proof of part (6). Let $y \in \operatorname{Ker}(\beta)$ whose class is x in H. If x maps to zero in H_n, then $y \in I^nL + \operatorname{Im}(\alpha)$. Hence $y - \alpha(v) \in \operatorname{Ker}(\beta) \cap I^nL$ for some $v \in K$. Then

$y - \alpha(v) \in I^{n-c_2}\operatorname{Ker}(\beta)$ and hence the class of y in H/I^nH is annihilated by I^{c_2}. Finally, let $x \in H_n$ be the class of $y \in \beta^{-1}(I^nM)$. Then we write $y = z + w$ with $z \in \operatorname{Ker}(\beta)$ and $w \in I^{n-c_1}L$ as above. Clearly, if $f \in I^{c_1}$ then fx is the class of $fy + fw \equiv fy$ modulo $\operatorname{Im}(\alpha) + I^nL$ and hence fx is the image of the class of fy in H as desired. □

0EGV Lemma 15.100.2. Let I be an ideal of a Noetherian ring A. Let $K \in D(A)$ be pseudo-coherent. Set $K_n = K \otimes_A^{\mathbf{L}} A/I^n$. Then for all $i \in \mathbf{Z}$ the system $H^i(K_n)$ satisfies Mittag-Leffler and $\lim H^i(K)/I^nH^i(K)$ is equal to $\lim H^i(K_n)$. Email from Kovacs of 23/02/2018.

Proof. We may represent K by a bounded above complex $P^\bullet$ of finite free A-modules. Then K_n is represented by $P^\bullet/I^nP^\bullet$. Hence the Mittag-Leffler property by Lemma 15.100.1. The final statement follows then from Lemma 15.97.6. □

0G9M Lemma 15.100.3. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M^\bullet$ be a bounded complex of finite A-modules. The inverse system of maps

$$M^\bullet \otimes_A^{\mathbf{L}} A/I^n \longrightarrow M^\bullet/I^nM^\bullet$$

defines an isomorphism of pro-objects of $D(A)$.

Proof. Say $I = (f_1, \ldots, f_r)$. Let $K_n \in D(A)$ be the object represented by the Koszul complex on $f_1^n, \ldots, f_r^n$. Recall that we have maps $K_n \to A/I^n$ which induce a pro-isomorphism of inverse systems, see Lemma 15.94.1. Hence it suffices to show that

$$M^\bullet \otimes_A^{\mathbf{L}} K_n \longrightarrow M^\bullet/I^nM^\bullet$$

defines an isomorphism of pro-objects of $D(A)$. Since K_n is represented by a complex of finite free A-modules sitting in degrees $-r, \ldots, 0$ there exist $a, b \in \mathbf{Z}$ such that the source and target of the displayed arrow have vanishing cohomology in degrees outside $[a, b]$ for all n. Thus we may apply Derived Categories, Lemma 13.42.5 and we find that it suffices to show that the maps

$$H^i(M^\bullet \otimes_A^{\mathbf{L}} A/I^n) \to H^i(M^\bullet/I^nM^\bullet)$$

define isomorphisms of pro-systems of A-modules for any $i \in \mathbf{Z}$. To see this choose a quasi-isomorphism $P^\bullet \to M^\bullet$ where $P^\bullet$ is a bounded above complex of finite free A-modules. The arrows above are given by the maps

$$H^i(P^\bullet/I^nP^\bullet) \to H^i(M^\bullet/I^nM^\bullet)$$

These define an isomorphism of pro-systems by Lemma 15.100.1. Namely, the lemma shows both are isomorphic to the pro-system H^i/I^nH^i with $H^i = H^i(M^\bullet) = H^i(P^\bullet)$. □

09BB Lemma 15.100.4. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules. Set $M_n = M/I^nM$ and $N_n = N/I^nN$. Then

(1) the systems $(\operatorname{Hom}_A(M_n, N_n))$ and $(\operatorname{Isom}_A(M_n, N_n))$ are Mittag-Leffler,
(2) there exists a $c \geq 0$ such that the kernels and cokernels of

$$\operatorname{Hom}_A(M, N)/I^n \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M_n, N_n)$$

are killed by I^c for all n,
(3) we have $\lim \operatorname{Hom}_A(M_n, N_n) = \operatorname{Hom}_A(M, N)^\wedge = \operatorname{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$
(4) $\lim \operatorname{Isom}_A(M_n, N_n) = \operatorname{Isom}_{A^\wedge}(M^\wedge, N^\wedge)$.

Here $^\wedge$ denotes usual I-adic completion.

Proof. Note that $\mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N_n)$. Choose a presentation

$$A^{\oplus t} \to A^{\oplus s} \to M \to 0$$

Applying the right exact functor $\mathrm{Hom}_A(-, N)$ we obtain a complex

$$0 \xrightarrow{\alpha} N^{\oplus s} \xrightarrow{\beta} N^{\oplus t}$$

whose cohomology in the middle is $\mathrm{Hom}_A(M, N)$ and such that for $n \geq 0$ the cohomology of

$$0 \xrightarrow{\alpha_n} N_n^{\oplus s} \xrightarrow{\beta_n} N_n^{\oplus t}$$

is $\mathrm{Hom}_A(M_n, N_n)$. Let $c \geq 0$ be as in Lemma 15.100.1 for this A, I, α, and β. By part (3) of the lemma we deduce the Mittag-Leffler property for $(\mathrm{Hom}_A(M_n, N_n))$. The kernel and cokernel of the maps $\mathrm{Hom}_A(M, N)/I^n \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M_n, N_n)$ are killed by I^c by [art part (6) of the lemma. We find that $\lim \mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N)^\wedge$ by part (2) of the lemma. The equality

$$\mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge) = \lim \mathrm{Hom}_A(M_n, N_n)$$

follows formally from the fact that $M^\wedge = \lim M_n$ and $M_n = M^\wedge/I^n M^\wedge$ and the corresponding facts for N, see Algebra, Lemma 10.97.4.

The result for isomorphisms follows from the case of homomorphisms applied to both $(\mathrm{Hom}(M_n, N_n))$ and $(\mathrm{Hom}(N_n, M_n))$ and the following fact: for $n > m > 0$, if we have maps $\alpha : M_n \to N_n$ and $\beta : N_n \to M_n$ which induce an isomorphisms $M_m \to N_m$ and $N_m \to M_m$, then α and β are isomorphisms. Namely, then $\alpha \circ \beta$ is surjective by Nakayama's lemma (Algebra, Lemma 10.20.1) hence $\alpha \circ \beta$ is an isomorphism by Algebra, Lemma 10.16.4. □

09BC Lemma 15.100.5. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules. Set $M_n = M/I^n M$ and $N_n = N/I^n N$. If $M_n \cong N_n$ for all n, then $M^\wedge \cong N^\wedge$ as $A^\wedge$-modules.

Proof. By Lemma 15.100.4 the system $(\mathrm{Isom}_A(M_n, N_n))$ is Mittag-Leffler. By assumption each of the sets $\mathrm{Isom}_A(M_n, N_n)$ is nonempty. Hence $\lim \mathrm{Isom}_A(M_n, N_n)$ is nonempty. Since $\lim \mathrm{Isom}_A(M_n, N_n) = \mathrm{Isom}_{A^\wedge}(M^\wedge, N^\wedge)$ we obtain an isomorphism. □

0EGW Remark 15.100.6. Let I be an ideal of a Noetherian ring A. Set $A_n = A/I^n$ for $n \geq 1$. Consider the following category:

(1) An object is a sequence $\{E_n\}_{n \geq 1}$ where E_n is a finite A_n-module.
(2) A morphism $\{E_n\} \to \{E'_n\}$ is given by maps

$$\varphi_n : I^c E_n \longrightarrow E'_n/E'_n[I^c] \quad \text{for } n \geq c$$

where $E'_n[I^c]$ is the torsion submodule (Section 15.88) up to equivalence: we say (c, φ_n) is the same as $(c{+}1, \overline{\varphi}_n)$ where $\overline{\varphi}_n : I^{c+1} E_n \longrightarrow E'_n/E'_n[I^{c+1}]$ is the induced map.

Composition of $(c, \varphi_n) : \{E_n\} \to \{E'_n\}$ and $(c', \varphi'_n) : \{E'_n\} \to \{E''_n\}$ is defined by the obvious compositions

$$I^{c+c'} E_n \to I^{c'} E'_n/E'_n[I^c] \to E''_n/E''_n[I^{c+c'}]$$

for $n \geq c + c'$. We omit the verification that this is a category.

0EGX **Lemma 15.100.7.** A morphism (c, φ_n) of the category of Remark 15.100.6 is an isomorphism if and only if there exists a $c' \geq 0$ such that $\mathrm{Ker}(\varphi_n)$ and $\mathrm{Coker}(\varphi_n)$ are $I^{c'}$-torsion for all $n \gg 0$.

Proof. We may and do assume $c' \geq c$ and that the $\mathrm{Ker}(\varphi_n)$ and $\mathrm{Coker}(\varphi_n)$ are $I^{c'}$-torsion for all n. For $n \geq c'$ and $x \in I^{c'}E'_n$ we can choose $y \in I^cE_n$ with $x = \varphi_n(y) \bmod E'_n[I^c]$ as $\mathrm{Coker}(\varphi_n)$ is annihilated by $I^{c'}$. Set $\psi_n(x)$ equal to the class of y in $E_n/E_n[I^{c'}]$. For a different choice $y' \in I^cE_n$ with $x = \varphi_n(y') \bmod E'_n[I^c]$ the difference $y - y'$ maps to zero in $E'_n/E'_n[I^c]$ and hence is annihilated by $I^{c'}$ in I^cE_n. Thus the maps $\psi_n : I^{c'}E'_n \to E_n/E_n[I^{c'}]$ are well defined. We omit the verification that (c', ψ_n) is the inverse of (c, φ_n) in the category. $\square$

0EGY **Lemma 15.100.8.** Let I be an ideal of the Noetherian ring A. Let M and N be finite A-modules. Write $A_n = A/I^n$, $M_n = M/I^nM$, and $N_n = N/I^nN$. For every $i \geq 0$ the objects

Email correspondence between Janos Kollar, Sandor Kovacs, and Johan de Jong of 23/02/2018.

$$\{\mathrm{Ext}^i_A(M, N)/I^n \mathrm{Ext}^i_A(M, N)\}_{n \geq 1} \quad \text{and} \quad \{\mathrm{Ext}^i_{A_n}(M_n, N_n)\}_{n \geq 1}$$

are isomorphic in the category $\mathcal{C}$ of Remark 15.100.6.

Proof. Choose a short exact sequence

$$0 \to K \to A^{\oplus r} \to M \to 0$$

and set $K_n = K/I^nK$. For $n \geq 1$ define $K(n) = \mathrm{Ker}(A_n^{\oplus r} \to M_n)$ so that we have exact sequences

$$0 \to K(n) \to A_n^{\oplus r} \to M_n \to 0$$

and surjections $K_n \to K(n)$. In fact, by Lemma 15.100.1 there is a $c \geq 0$ and maps $K(n) \to K_n/I^{n-c}K_n$ which are "almost inverse". Since $I^{n-c}K_n \subset K_n[I^c]$ these maps which witness the fact that the systems $\{K(n)\}_{n \geq 1}$ and $\{K_n\}_{n \geq 1}$ are isomorphic in $\mathcal{C}$.

We claim the systems

$$\{\mathrm{Ext}^i_{A_n}(K(n), N_n)\}_{n \geq 1} \quad \text{and} \quad \{\mathrm{Ext}^i_{A_n}(K_n, N_n)\}_{n \geq 1}$$

are isomorphic in the category $\mathcal{C}$. Namely, the surjective maps $K_n \to K(n)$ have kernels annihilated by I^c and therefore determine maps

$$\mathrm{Ext}^i_{A_n}(K(n), N_n) \to \mathrm{Ext}^i_{A_n}(K_n, N_n)$$

whose kernel and cokernel are annihilated by I^c. Hence the claim by Lemma 15.100.7.

For $i \geq 2$ we have isomorphisms

$$\mathrm{Ext}^{i-1}_A(K, N) = \mathrm{Ext}^i_A(M, N) \quad \text{and} \quad \mathrm{Ext}^{i-1}_{A_n}(K(n), N_n) = \mathrm{Ext}^i_{A_n}(M_n, N_n)$$

In this way we see that it suffices to prove the lemma for $i = 0, 1$.

For $i = 0, 1$ we consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(M, N) \arrow[r] \arrow[dd] & N^{\oplus r} \arrow[r, "\varphi"'] \arrow[dd] & \mathrm{Hom}(K, N) \arrow[r] \arrow[d] & \mathrm{Ext}^1(M, N) \arrow[r] & 0 \\
 & & & \mathrm{Hom}(K_n, N_n) & & \\
0 \arrow[r] & \mathrm{Hom}(M_n, N_n) \arrow[r] & N_n^{\oplus r} \arrow[r] & \mathrm{Hom}(K(n), N_n) \arrow[r] \arrow[u] & \mathrm{Ext}^1(M_n, N_n) \arrow[r] & 0
\end{tikzcd}$$

By Lemma 15.100.4 we see that the kernel and cokernel of $\mathrm{Hom}(M, N)/I^n \mathrm{Hom}(M, N) \to \mathrm{Hom}(M_n, N_n)$ and $\mathrm{Hom}(K, N)/I^n \mathrm{Hom}(K, N) \to \mathrm{Hom}(K_n, N_n)$ and are I^c-torsion for some $c \geq 0$ independent of n. Above we have seen the cokernel of the injective maps $\mathrm{Hom}(K(n), N_n) \to \mathrm{Hom}(K_n, N_n)$ are annihilated by I^c after possibly increasing c. For such a c we obtain maps $\delta_n : I^c \mathrm{Hom}(K, N)/I^n \mathrm{Hom}(K, N) \to \mathrm{Hom}(K(n), N_n)$ fitting into the diagram (precise formulation omitted). The kernel and cokernel of δ_n are annihilated by I^c after possibly increasing c since we know that the same thing is true for $\mathrm{Hom}(K, N)/I^n \mathrm{Hom}(K, N) \to \mathrm{Hom}(K_n, N_n)$ and $\mathrm{Hom}(K(n), N_n) \to \mathrm{Hom}(K_n, N_n)$. Then we can use commutativity of the solid diagram

$$\begin{tikzcd}
\varphi^{-1}(I^c \mathrm{Hom}(K, N)) \arrow[r, "\varphi"'] \arrow[d] & I^c \mathrm{Hom}(K, N)/I^n \mathrm{Hom}(K, N) \arrow[r] \arrow[d, "\delta_n"] & I^c \mathrm{Ext}^1(M, N)/I^n \mathrm{Ext}^1(M, N) \arrow[r] \arrow[d, dotted] & 0 \\
N_n^{\oplus r} \arrow[r] & \mathrm{Hom}(K(n), N_n) \arrow[r] & \mathrm{Ext}^1(M_n, N_n) \arrow[r] & 0
\end{tikzcd}$$

to define the dotted arrow. A straightforward diagram chase (omitted) shows that the kernel and cokernel of the dotted arrow are annihilated buy I^c after possibly increasing c one final time. $\square$

0EGZ Remark 15.100.9. The awkwardness in the statement of Lemma 15.100.8 is partly due to the fact that there are no obvious maps between the modules $\mathrm{Ext}^i_{A_n}(M_n, N_n)$ for varying n. What we may conclude from the lemma is that there exists a $c \geq 0$ such that for $m \gg n \gg 0$ there are (canonical) maps

$$I^c \mathrm{Ext}^i_{A_n}(M_m, N_m)/I^n \mathrm{Ext}^i_{A_n}(M_m, N_m) \to \mathrm{Ext}^i_{A_n}(M_n, N_n)/\mathrm{Ext}^i_{A_n}(M_n, N_n)[I^c]$$

whose kernel and cokernel are annihilated by I^c. This is the (weak) sense in which we get a system of modules.

0EH2 Example 15.100.10. Let k be a field. Let $A = k[[x, y]]/(xy)$. By abuse of notation we denote x and y the images of x and y in A. Let $I = (x)$. Let $M = A/(y)$. There is a free resolution

$$\ldots \to A \xrightarrow{y} A \xrightarrow{x} A \xrightarrow{y} A \to M \to 0$$

We conclude that

$$\mathrm{Ext}^2_A(M, N) = N[y]/xN$$

where $N[y] = \mathrm{Ker}(y : N \to N)$. We denote $A_n = A/I^n$, $M_n = M/I^n M$, and $N_n = N/I^n N$. For each n we have a free resolution

$$\ldots \to A_n^{\oplus 2} \xrightarrow{y, x^{n-1}} A_n \xrightarrow{x} A_n \xrightarrow{y} A_n \to M_n \to 0$$

We conclude that

$$\mathrm{Ext}^2_{A_n}(M_n, N_n) = (N_n[y] \cap N_n[x^{n-1}])/xN_n$$

where $N_n[y] = \mathrm{Ker}(y : N_n \to N_n)$ and $N[x^{n-1}] = \mathrm{Ker}(x^{n-1} : N_n \to N_n)$. Take $N = A/(y)$. Then we see that

$$\mathrm{Ext}^2_A(M, N) = N[y]/xN = N/xN \cong k$$

but

$$\mathrm{Ext}^2_{A_n}(M_n, N_n) = (N_n[y] \cap N_n[x^{n-1}])/xN_n = N_n[x^{n-1}]/xN_n = 0$$

for all r because $N_n = k[x]/(x^n)$ and the sequence

$$N_n \xrightarrow{x} N_n \xrightarrow{x^{n-1}} N_n$$

is exact. Thus ignoring some kind of I-power torsion is necessary to get a result as in Lemma 15.100.8.

0EH0 Lemma 15.100.11. Let $A \to B$ be a flat homomorphism of Noetherian rings. Let $I \subset A$ be an ideal. Let M, N be A-modules. Set $B_n = B/I^nB$, $M_n = M/I^nM$, $N_n = N/I^nN$. If M is flat over A, then we have

$$\lim \mathrm{Ext}^i_B(M, N)/I^n \mathrm{Ext}^i_B(M, N) = \lim \mathrm{Ext}^i_{B_n}(M_n, N_n)$$

for all $i \in \mathbf{Z}$.

Email correspondence between Janos Kollar, Sandor Kovacs, and Johan de Jong of 23/02/2018.

Proof. Choose a resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

by finite free B-modues P_i. Set $P_{i,n} = P_i/I^nP_i$. Since M and B are flat over A, the sequence

$$\ldots \to P_{2,n} \to P_{1,n} \to P_{0,n} \to M_n \to 0$$

is exact. We see that on the one hand the complex

$$\mathrm{Hom}_B(P_0, N) \to \mathrm{Hom}_B(P_1, N) \to \mathrm{Hom}_B(P_2, N) \to \ldots$$

computes the modules $\mathrm{Ext}^i_B(M, N)$ and on the other hand the complex

$$\mathrm{Hom}_{B_n}(P_{0,n}, N_n) \to \mathrm{Hom}_{B_n}(P_{1,n}, N_n) \to \mathrm{Hom}_{B_n}(P_{2,n}, N_n) \to \ldots$$

computes the modules $\mathrm{Ext}^i_{B_n}(M_n, N_n)$. Since

$$\mathrm{Hom}_{B_n}(P_{i,n}, N_n) = \mathrm{Hom}_B(P_i, N)/I^n \mathrm{Hom}_B(P_i, N)$$

we obtain the result from Lemma 15.100.1 part (2). □

15.101. Systems of modules, bis

0G3J Let I be an ideal of a Noetherian ring A. In Section 15.100 we considered what happens when considering systems of the form M/I^nM for finite A-modules M. In this section we consider the systems I^nM instead.

0G3K Lemma 15.101.1. Let I be an ideal of a Noetherian ring A. Let $K \xrightarrow{\alpha} L \xrightarrow{\beta} M$ be a complex of finite A-modules. Set $H = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$. For $n \geq 0$ let

$$I^nK \xrightarrow{\alpha_n} I^nL \xrightarrow{\beta_n} I^nM$$

be the induced complex. Set $H_n = \mathrm{Ker}(\beta_n)/\mathrm{Im}(\alpha_n)$. Then there are canonical A-module maps

$$\ldots \to H_3 \to H_2 \to H_1 \to H$$

There exists a $c > 0$ such that for $n \geq c$ the image of $H_n \to H$ is contained in $I^{n-c}H$ and there is a canonical A-module map $I^n H \to H_{n-c}$ such that the compositions

$$I^n H \to H_{n-c} \to I^{n-2c}H \quad \text{and} \quad H_n \to I^{n-c}H \to H_{n-2c}$$

are the canonical ones. In particular, the inverse systems (H_n) and $(I^n H)$ are isomorphic as pro-objects of Mod_A.

Proof. We have $H_n = \text{Ker}(\beta) \cap I^n L/\alpha(I^n K)$. Since $\text{Ker}(\beta) \cap I^n L \subset \text{Ker}(\beta) \cap I^{n-1}L$ and $\alpha(I^n K) \subset \alpha(I^{n-1}K)$ we get the maps $H_n \to H_{n-1}$. Similarly for the map $H_1 \to H$.

By Artin-Rees we may choose $c_1, c_2 \geq 0$ such that $\text{Im}(\alpha) \cap I^n L \subset \alpha(I^{n-c_1}K)$ for $n \geq c_1$ and $\text{Ker}(\beta) \cap I^n L \subset I^{n-c_2}\text{Ker}(\beta)$ for $n \geq c_2$, see Algebra, Lemmas 10.51.3 and 10.51.2. Set $c = c_1 + c_2$.

It follows immediately from our choice of $c \geq c_2$ that for $n \geq c$ the image of $H_n \to H$ is contained in $I^{n-c}H$.

Let $n \geq c$. We define $\psi_n : I^n H \to H_{n-c}$ as follows. Say $x \in I^n H$. Choose $y \in I^n \text{Ker}(\beta)$ representing x. We set $\psi_n(x)$ equal to the class of y in H_{n-c}. To see this is well defined, suppose we have a second choice y' as above for x. Then $y' - y \in \text{Im}(\alpha)$. By our choice of $c \geq c_1$ we conclude that $y' - y \in \alpha(I^{n-c}K)$ which implies that y and y' represent the same element of H_{n-c}. Thus ψ_n is well defined.

The statements on the compositions $I^n H \to H_{n-c} \to I^{n-2c}H$ and $H_n \to I^{n-c}H \to H_{n-2c}$ follow immediately from our definitions. $\square$

0G3L Lemma 15.101.2. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be A-modules with M finite. For each $p > 0$ there exists a $c \geq 0$ such that for $n \geq c$ the map $\text{Ext}^p_A(M, N) \to \text{Ext}^p_A(I^n M, N)$ factors through $\text{Ext}^p_A(I^n M, I^{n-c}N) \to \text{Ext}^p_A(I^n M, N)$.

Proof. For $p = 0$, if $\varphi : M \to N$ is an A-linear map, then $\varphi(\sum f_i m_i) = \sum f_i \varphi(m_i)$ for $f_i \in A$ and $m_i \in M$. Hence φ induces a map $I^n M \to I^n N$ for all n and the result is true with $c = 0$.

Choose a short exact sequence $0 \to K \to A^{\oplus t} \to M \to 0$. For each n we pick a short exact sequence $0 \to L_n \to A^{\oplus s_n} \to I^n M \to 0$. It is clear that we can construct a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & L_n \arrow[r] \arrow[d] & A^{\oplus s_n} \arrow[r] \arrow[d] & I^n M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & A^{\oplus t} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

such that $A^{\oplus s_n} \to A^{\oplus t}$ has image in $(I^n)^{\oplus t}$. By Artin-Rees (Algebra, Lemma 10.51.2) there exists a $c \geq 0$ such that $L_n \to K$ factors through $I^{n-c}K$ if $n \geq c$.

For $p = 1$ our choices above induce a solid commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(A^{\oplus s_n}, N) \arrow[r] & \mathrm{Hom}_A(L_n, N) \arrow[r] & \mathrm{Ext}^1_A(I^nM, N) \arrow[r] & 0 \\
\mathrm{Hom}_A((I^n)^{\oplus t}, I^{n-c}N) \arrow[r] \arrow[u] & \mathrm{Hom}_A(K \cap (I^n)^{\oplus t}, I^{n-c}N) \arrow[r] \arrow[u] & \mathrm{Ext}^1_A(I^nM, I^{n-c}N) \arrow[u] \\
\mathrm{Hom}_A(A^{\oplus t}, N) \arrow[r] \arrow[u] & \mathrm{Hom}_A(K, N) \arrow[r] \arrow[u] & \mathrm{Ext}^1_A(M, N) \arrow[r] \arrow[u, dotted] & 0
\end{tikzcd}$$

whose horizontal arrows are exact. The lower middle vertical arrow arises because $K \cap (I^n)^{\oplus t} \subset I^{n-c}K$ and hence any A-linear map $K \to N$ induces an A-linear map $(I^n)^{\oplus t} \to I^{n-c}N$ by the argument of the first paragraph. Thus we obtain the dotted arrow as desired.

For $p > 1$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^{p-1}_A(I^{n-c}K, N) \arrow[r] & \mathrm{Ext}^{p-1}_A(L_n, N) \arrow[r] & \mathrm{Ext}^p_A(I^nM, N) \\
\mathrm{Ext}^{p-1}_A(K, N) \arrow[rr] \arrow[u] & & \mathrm{Ext}^p_A(M, N) \arrow[u]
\end{tikzcd}$$

whose bottom horizontal arrow is an isomorphism. By induction on p the left vertical map factors through $\mathrm{Ext}^{p-1}_A(I^{n-c}K, I^{n-c-c'}N)$ for some $c' \geq 0$ and all $n \geq c + c'$. Using the composition $\mathrm{Ext}^{p-1}_A(I^{n-c}K, I^{n-c-c'}N) \to \mathrm{Ext}^{p-1}_A(L_n, I^{n-c-c'}N) \to \mathrm{Ext}^p_A(I^nM, I^{n-c-c'}N)$ we obtain the desired factorization (for $n \geq c + c'$ and with c replaced by $c + c'$). $\square$

0927 Lemma 15.101.3. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be A-modules with M finite and N annihilated by a power of I. For each $p > 0$ there exists an n such that the map $\mathrm{Ext}^p_A(M, N) \to \mathrm{Ext}^p_A(I^nM, N)$ is zero.

Proof. Immediate consequence of Lemma 15.101.2 and the fact that $I^mN = 0$ for some $m > 0$. $\square$

0DYI Lemma 15.101.4. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $K \in D(A)$ be pseudo-coherent and let M be a finite A-module. For each $p \in \mathbf{Z}$ there exists an c such that the image of $\mathrm{Ext}^p_A(K, I^nM) \to \mathrm{Ext}^p_A(K, M)$ is contained in $I^{n-c}\mathrm{Ext}^p_A(K, M)$ for $n \geq c$.

Proof. Choose a bounded above complex $P^\bullet$ of finite free A-modules representing K. Then $\mathrm{Ext}^p_A(K, M)$ is the cohomology of

$$\mathrm{Hom}_A(F^{-p+1}, M) \xrightarrow{a} \mathrm{Hom}_A(F^{-p}, M) \xrightarrow{b} \mathrm{Hom}_A(F^{-p-1}, M)$$

and $\mathrm{Ext}^p_A(K, I^nM)$ is computed by replacing these finite A-modules by I^n times themselves. Thus the result by Lemma 15.101.1 (and much more is true). $\square$

In Situation 15.91.15 we define complexes $I_n^\bullet$ such that we have distinguished triangles

$$I_n^\bullet \to A \to K_n^\bullet \to I_n^\bullet[1]$$

in the triangulated category $K(A)$ of complexes of A-modules up to homotopy. Namely, we set $I_n^\bullet = \sigma_{\leq -1} K_n^\bullet[-1]$. We have termwise split short exact sequences of complexes

$$0 \to A \to K_n^\bullet \to I_n^\bullet[1] \to 0$$

defining distinguished triangles by definition of the triangulated structure on $K(A)$. Their rotations determine the desired distinguished triangles above. Note that $I_n^0 = A^{\oplus r} \to A$ is given by multiplication by f_i^n on the ith factor. Hence $I_n^\bullet \to A$ factors as

$$I_n^\bullet \to (f_1^n, \ldots, f_r^n) \to A$$

In fact, there is a short exact sequence

$$0 \to H^{-1}(K_n^\bullet) \to H^0(I_n^\bullet) \to (f_1^n, \ldots, f_r^n) \to 0$$

and for every $i < 0$ we have $H^i(I_n^\bullet) = H^{i-1}(K_n^\bullet$. The maps $K_{n+1}^\bullet \to K_n^\bullet$ induce maps $I_{n+1}^\bullet \to I_n^\bullet$ and we obtain a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & I_3^\bullet \arrow[r] \arrow[d] & I_2^\bullet \arrow[r] \arrow[d] & I_1^\bullet \arrow[d] \\
\ldots \arrow[r] & (f_1^3, \ldots, f_r^3) \arrow[r] & (f_1^2, \ldots, f_r^2) \arrow[r] & (f_1, \ldots, f_r)
\end{tikzcd}$$

in $K(A)$.

0G3M Lemma 15.101.5. In Situation 15.91.15 assume A is Noetherian. With notation as above, the inverse system (I^n) is pro-isomorphic in $D(A)$ to the inverse system $(I_n^\bullet)$.

Proof. It is elementary to show that the inverse system I^n is pro-isomorphic to the inverse system $(f_1^n, \ldots, f_r^n)$ in the category of A-modules. Consider the inverse system of distinguished triangles

$$I_n^\bullet \to (f_1^n, \ldots, f_r^n) \to C_n^\bullet \to I_n^\bullet[1]$$

where $C_n^\bullet$ is the cone of the first arrow. By Derived Categories, Lemma 13.42.4 it suffices to show that the inverse system $C_n^\bullet$ is pro-zero. The complex $I_n^\bullet$ has nonzero terms only in degrees i with $-r+1 \leq i \leq 0$ hence $C_n^\bullet$ is bounded similarly. Thus by Derived Categories, Lemma 13.42.3 it suffices to show that $H^p(C_n^\bullet)$ is pro-zero. By the discussion above we have $H^p(C_n^\bullet) = H^p(K_n^\bullet)$ for $p \leq -1$ and $H^p(C_n^\bullet) = 0$ for $p \geq 0$. The fact that the inverse systems $H^p(K_n^\bullet)$ are pro-zero was shown in the proof of Lemma 15.94.1 (and this is where the assumption that A is Noetherian is used). ☐

0G3N Lemma 15.101.6. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $M^\bullet$ be a bounded complex of finite A-modules. The inverse system of maps

$$I^n \otimes_A^{\mathbf{L}} M^\bullet \longrightarrow I^n M^\bullet$$

defines an isomorphism of pro-objects of $D(A)$.

Proof. Choose generators $f_1, \ldots, f_r \in I$ of I. The inverse system I^n is pro-isomorphic to the inverse system $(f_1^n, \ldots, f_r^n)$ in the category of A-modules. With notation as in Lemma 15.101.5 we find that it suffices to prove the inverse system of maps

$$I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet \longrightarrow (f_1^n, \ldots, f_r^n) M^\bullet$$

defines an isomorphism of pro-objects of $D(A)$. Say we have $a \leq b$ such that $M^i = 0$ if $i \notin [a, b]$. Then source and target of the arrows above have cohomology only in degrees $[-r + a, b]$. Thus it suffices to show that for any $p \in \mathbf{Z}$ the inverse system of maps

$$H^p(I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet) \longrightarrow H^p((f_1^n, \ldots, f_r^n)M^\bullet)$$

defines an isomorphism of pro-objects of A-modules, see Derived Categories, Lemma 13.42.5. Using the pro-isomorphism between $I_n^\bullet \otimes_A^{\mathbf{L}} M^\bullet$ and $I^n \otimes_A^{\mathbf{L}} M^\bullet$ and the pro-isomorphism between $(f_1^n, \ldots, f_r^n)M^\bullet$ and $I^n M^\bullet$ this is equivalent to showing that the inverse system of maps

$$H^p(I^n \otimes_A^{\mathbf{L}} M^\bullet) \longrightarrow H^p(I^n M^\bullet)$$

defines an isomorphism of pro-objects of A-modules Choose a bounded above complex of finite free A-modules $P^\bullet$ and a quasi-isomorphism $P^\bullet \to M^\bullet$. Then it suffices to show that the inverse system of maps

$$H^p(I^n P^\bullet) \longrightarrow H^p(I^n M^\bullet)$$

is a pro-isomorphism. This follows from Lemma 15.101.1 as $H^p(P^\bullet) = H^p(M^\bullet)$. □

0928 Lemma 15.101.7. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. There exists an integer $n > 0$ such that $I^n M \to M$ factors through the map $I \otimes_A^{\mathbf{L}} M \to M$ in $D(A)$.

Proof. This follows from Lemma 15.101.6. It can also been seen directly as follows. Consider the distinguished triangle

$$I \otimes_A^{\mathbf{L}} M \to M \to A/I \otimes_A^{\mathbf{L}} M \to I \otimes_A^{\mathbf{L}} M[1]$$

By the axioms of a triangulated category it suffices to prove that $I^n M \to A/I \otimes_A^{\mathbf{L}} M$ is zero in $D(A)$ for some n. Choose generators $f_1, \ldots, f_r$ of I and let $K = K_\bullet(A, f_1, \ldots, f_r)$ be the Koszul complex and consider the factorization $A \to K \to A/I$ of the quotient map. Then we see that it suffices to show that $I^n M \to K \otimes_A M$ is zero in $D(A)$ for some $n > 0$. Suppose that we have found an $n > 0$ such that $I^n M \to K \otimes_A M$ factors through $\tau_{\geq t}(K \otimes_A M)$ in $D(A)$. Then the obstruction to factoring through $\tau_{\geq t+1}(K \otimes_A M)$ is an element in $\text{Ext}^t(I^n M, H_t(K \otimes_A M))$. The finite A-module $H_t(K \otimes_A M)$ is annihilated by I. Then by Lemma 15.101.3 we can after increasing n assume this obstruction element is zero. Repeating this a finite number of times we find n such that $I^n M \to K \otimes_A M$ factors through $0 = \tau_{\geq r+1}(K \otimes_A M)$ in $D(A)$ and we win. □

15.102. Miscellany

0926 Some results which do not fit anywhere else.

0DYJ Lemma 15.102.1. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $K \in D(A)$ be pseudo-coherent. Let $a \in \mathbf{Z}$. Assume that for every finite A-module M the modules $\text{Ext}^i_A(K, M)$ are I-power torsion for $i \geq a$. Then for $i \geq a$ and M finite the system $\text{Ext}^i_A(K, M/I^n M)$ is essentially constant with value

$$\text{Ext}^i_A(K, M) = \lim \text{Ext}^i_A(K, M/I^n M)$$

Proof. Let M be a finite A-module. Since K is pseudo-coherent we see that $\text{Ext}^i_A(K, M)$ is a finite A-module. Thus for $i \geq a$ it is annihilated by I^t for some $t \geq 0$. By Lemma 15.101.4 we see that the image of $\text{Ext}^i_A(K, I^n M) \to \text{Ext}^i_A(K, M)$ is zero for some $n > 0$. The short exact sequence $0 \to I^n M \to M \to M/I^n M \to 0$ gives a long exact sequence

$$\text{Ext}^i_A(K, I^n M) \to \text{Ext}^i_A(K, M) \to \text{Ext}^i_A(K, M/I^n M) \to \text{Ext}^{i+1}_A(K, I^n M)$$

The systems $\text{Ext}^i_A(K, I^n M)$ and $\text{Ext}^{i+1}_A(K, I^n M)$ are essentially constant with value 0 by what we just said (applied to the finite A-modules $I^m M$). A diagram chase shows $\text{Ext}^i_A(K, M/I^n M)$ is essentially constant with value $\text{Ext}^i_A(K, M)$. $\square$

0FXN Lemma 15.102.2. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. Let N be an A-module annihilated by I. There exists an integer $n > 0$ such that $\text{Tor}^A_p(I^n M, N) \to \text{Tor}^A_p(M, N)$ is zero for all $p \geq 0$.

Proof. By Lemma 15.101.7 we can factor $I^n M \to M$ as $I^n M \to M \otimes^{\mathbf{L}}_A I \to M$. We claim the composition

$$I^n M \otimes^{\mathbf{L}}_A N \to (M \otimes^{\mathbf{L}}_A I) \otimes^{\mathbf{L}}_A N \to M \otimes^{\mathbf{L}}_A N$$

is zero. Namely, the diagram

$$\begin{tikzcd}
(M \otimes^{\mathbf{L}}_A I) \otimes^{\mathbf{L}}_A N \arrow[rr] \arrow[dr] & & M \otimes^{\mathbf{L}}_A (I \otimes^{\mathbf{L}}_A N) \arrow[dl] \\
& M \otimes^{\mathbf{L}}_A N &
\end{tikzcd}$$

commutes (details omitted) and the map $I \otimes^{\mathbf{L}}_A N \to N$ is zero as N is annihilated by I. $\square$

0D2L Lemma 15.102.3. Let R be a ring. Let $K \in D(R)$ be pseudo-coherent. Let (M_n) be an inverse system of R-modules. Then $R\lim K \otimes^{\mathbf{L}}_R M_n = K \otimes^{\mathbf{L}}_R R\lim M_n$.

Proof. Consider the defining distinguished triangle

$$R\lim M_n \to \prod M_n \to \prod M_n \to R\lim M_n[1]$$

and apply Lemma 15.65.5. $\square$

0929 Lemma 15.102.4. Let R be a Noetherian local ring. Let $I \subset R$ be an ideal and let E be a nonzero module over R/I. If R/I has finite projective dimension and E has finite projective dimension over R/I, then E has finite projective dimension over R and

$$\text{pd}_R(E) = \text{pd}_R(R/I) + \text{pd}_{R/I}(E)$$

Proof. We will use that, for a finite module, having finite projective dimension over R, resp. R/I is the same as being a perfect module, see discussion following Definition 15.74.1. We see that E has finite projective dimension over R by Lemma 15.74.7. Thus we can apply Auslander-Buchsbaum (Algebra, Proposition 10.111.1) to see that

$$\text{pd}_R(E) + \text{depth}(E) = \text{depth}(R), \quad \text{pd}_{R/I}(E) + \text{depth}(E) = \text{depth}(R/I),$$

and

$$\text{pd}_R(R/I) + \text{depth}(R/I) = \text{depth}(R)$$

Note that in the first equation we take the depth of E as an R-module and in the second as an R/I-module. However these depths are the same (this is trivial but also follows from Algebra, Lemma 10.72.11). This concludes the proof. $\square$

0GYI Lemma 15.102.5. Let $A \to B$ be a ring map. There exists a cardinal $\kappa = \kappa(A \to B)$ with the following property: Let $M^\bullet$, resp. $N^\bullet$ be a complex of A-modules, resp. B-modules. Let $a : M^\bullet \to N^\bullet$ be a map of complexes of A-modules which induces an isomorphism $M^\bullet \otimes_A^{\mathbf{L}} B \to N^\bullet$ in $D(B)$. Let $M_1^\bullet \subset M^\bullet$, resp. $N_1^\bullet \subset N^\bullet$ be a subcomplex of A-modules, resp. B-modules such that $a(M_1^\bullet) \subset N_1^\bullet$. Then there exist subcomplexes

$$M_1^\bullet \subset M_2^\bullet \subset M^\bullet \quad \text{and} \quad N_1^\bullet \subset N_2^\bullet \subset N^\bullet$$

such that $a(M_2^\bullet) \subset N_2^\bullet$ with the following properties:

(1) $\mathrm{Ker}(H^i(M_1^\bullet \otimes_A^{\mathbf{L}} B) \to H^i(N_1^\bullet))$ maps to zero in $H^i(M_2^\bullet \otimes_A^{\mathbf{L}} B)$,
(2) $\mathrm{Im}(H^i(N_1^\bullet) \to H^i(N_2^\bullet))$ is contained in $\mathrm{Im}(H^i(M_2^\bullet \otimes_A^{\mathbf{L}} B) \to H^2(N_2^\bullet))$,
(3) $|\bigcup M_2^i \cup \bigcup N_2^i| \leq \max(\kappa, |\bigcup M_1^i \cup \bigcup N_1^i|)$.

Proof. Let $\kappa = \max(|A|, |B|, \aleph_0)$. Set $|M^\bullet| = |\bigcup M^i|$ and similarly for other complexes. With this notation we have

$$\max(\kappa, |\bigcup M_1^i \cup \bigcup N_1^i|) = \max(\kappa, |M_1^\bullet|, |M_2^\bullet|)$$

for the quantity used in the statement of the lemma. We are going to use this and other observations coming from arithmetic of cardinals without further mention.

First, let us show that there are plenty of "small" subcomplexes. For every pair of collections $E = \{E^i\}$ and $F = \{F^i\}$ of finite subsets $E^i \subset M^i$, $i \in \mathbf{Z}$ and $F^i \subset N^i$, $i \in \mathbf{Z}$ we can let

$$M_1^\bullet \subset M_1(E, F)^\bullet \subset M^\bullet \quad \text{and} \quad N_1^\bullet \subset N_1(E, F)^\bullet \subset N^\bullet$$

be the smallest subcomplexes of A and B-modules such that $a(M_1(E, F)^\bullet) \subset N_1(E, F)^\bullet$ and such that $E^i \subset M_1(E, F)^i$ and $F^i \subset M_2(E, F)^i$. Then it is easy to see that

$$|M_1(E, F)^\bullet| \leq \max(\kappa, |M_1^\bullet|) \quad \text{and} \quad |M_2(E, F)^\bullet| \leq \max(\kappa, |M_2^\bullet|)$$

Details omitted. It is clear that we have

$$M^\bullet = \mathrm{colim}_{(E,F)} M_1(E, F)^\bullet \quad \text{and} \quad N^\bullet = \mathrm{colim}_{(E,F)} N_1(E, F)^\bullet$$

and the colimits are (termwise) filtered colimits.

There exists a resolution $\ldots \to F^{-1} \to F^0 \to B$ by free A-modules F_i with $|F_i| \leq \kappa$ (details omitted). The cohomology modules of $M_1^\bullet \otimes_A^{\mathbf{L}} B$ are computed by $\mathrm{Tot}(M_1^\bullet \otimes_A F^\bullet)$. It follows that $|H^i(M_1^\bullet \otimes_A^{\mathbf{L}} B)| \leq \max(\kappa, |M_1^\bullet|)$.

Let $i \in \mathbf{Z}$ and let $\xi \in H^i(M_1^\bullet \otimes_A^{\mathbf{L}} B)$ be an element which maps to zero in $H^i(N_1^\bullet)$. Then ξ maps to zero in $H^i(N^\bullet)$ and hence ξ maps to zero in $H^i(M^\bullet \otimes_A^{\mathbf{L}} B)$. Since derived tensor product commutes with filtered colimits, we can find finite collections E_ξ and F_ξ as above such that ξ maps to zero in $H^i(M_1(E_\xi, F_\xi)^\bullet \otimes_A^{\mathbf{L}} B)$.

Let $i \in \mathbf{Z}$ and let $\eta \in H^i(N_1^\bullet)$. Then the image of η in $H^i(N^\bullet)$ is in the image of $H^i(M^\bullet \otimes_A^{\mathbf{L}} B) \to H^i(N^\bullet)$. Hence as before, we can find finite collections E_η and F_η as above such that η maps to an element of $H^i(N_1(E_\eta, F_\eta)$ which is in the image of the map $H^i(M_1(E_\eta, F_\eta)^\bullet \otimes_A^{\mathbf{L}} B) \to H^i(N_1(E_\eta, F_\eta)$.

Now we simply define

$$M_2^\bullet = \sum\nolimits_\xi M_1(E_\xi, F_\xi)^\bullet + \sum\nolimits_\eta M_1(E_\eta, F_\eta)^\bullet$$

where the sum is over ξ and η as in the previous two paragraphs and the sum is taken inside $M^\bullet$. Similiarly we set

$$N_2^\bullet = \sum\nolimits_\xi N_1(E_\xi, F_\xi)^\bullet + \sum\nolimits_\eta N_1(E_\eta, F_\eta)^\bullet$$

where the sum is taken inside $N^\bullet$. By construction we will have properties (1) and (2) with these choices. The bound (3) also follows as the set of ξ and η has cardinality at most $\max(\kappa, |M_1^\bullet|, |N_1^\bullet|)$. □

15.103. Tricks with double complexes

0EYW This section continues the discussion in Homology, Section 12.26.

0H0Q **Lemma 15.103.1.** Let $A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to \ldots$ be a complex of complexes of abelian groups. Assume $H^{-p}(A_p^\bullet) = 0$ for all $p \geq 0$. Set $A^{p,q} = A_p^q$ and view $A^{\bullet,\bullet}$ as a double complex. Then $H^0(\text{Tot}_\pi(A^{\bullet,\bullet})) = 0$.

Proof. Denote $f_p : A_p^\bullet \to A_{p+1}^\bullet$ the given maps of complexes. Recall that the differential on $\text{Tot}_\pi(A^{\bullet,\bullet})$ is given by

$$\prod\nolimits_{p+q=n} (f_p^q + (-1)^p \text{d}^q_{A_p^\bullet})$$

on elements in degree n. Let $\xi \in H^0(\text{Tot}_\pi(A^{\bullet,\bullet}))$ be a cohomology class. We will show ξ is zero. Represent ξ as the class of an cocycle $x = (x_p) \in \prod A^{p,-p}$. Since $\text{d}(x) = 0$ we find that $\text{d}_{A_0^\bullet}(x_0) = 0$. Since $H^0(A_0^\bullet) = 0$ there exists a $y_{-1} \in A^{0,-1}$ with $\text{d}_{A_0^\bullet}(y_{-1}) = x_0$. Then we see that $\text{d}_{A_1^\bullet}(x_1 + f_0(y_{-1})) = 0$. Since $H^{-1}(A_1^\bullet) = 0$ we can find a $y_{-2} \in A^{1,-2}$ such that $-\text{d}_{A_1^\bullet}(y_{-2}) = x_1 + f_0(y_{-1})$. By induction we can find $y_{-p-1} \in A^{p,-p-1}$ such that

$$(-1)^p \text{d}_{A_p^\bullet}(y_{-p-1}) = x_p + f_{p-1}(y_{-p})$$

This implies that $\text{d}(y) = x$ where $y = (y_{-p-1})$. □

0EYX **Lemma 15.103.2.** Let

$$(A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to \ldots) \longrightarrow (B_0^\bullet \to B_1^\bullet \to B_2^\bullet \to \ldots)$$

be a map between two complexes of complexes of abelian groups. Set $A^{p,q} = A_p^q$, $B^{p,q} = B_p^q$ to obtain double complexes. Let $\text{Tot}_\pi(A^{\bullet,\bullet})$ and $\text{Tot}_\pi(B^{\bullet,\bullet})$ be the product total complexes associated to the double complexes. If each $A_p^\bullet \to B_p^\bullet$ is a quasi-isomorphism, then $\text{Tot}_\pi(A^{\bullet,\bullet}) \to \text{Tot}_\pi(B^{\bullet,\bullet})$ is a quasi-isomorphism.

Proof. Recall that $\text{Tot}_\pi(A^{\bullet,\bullet})$ in degree n is given by $\prod_{p+q=n} A^{p,q} = \prod_{p+1=n} A_p^q$. Let $C_p^\bullet$ be the cone on the map $A_p^\bullet \to B_p^\bullet$, see Derived Categories, Section 13.9. By the functoriality of the cone construction we obtain a complex of complexes

$$C_0^\bullet \to C_1^\bullet \to C_2^\bullet \to \ldots$$

Then we see $\text{Tot}_\pi(C^{\bullet,\bullet})$ in degree n is given by

$$\prod_{p+q=n} C^{p,q} = \prod_{p+q=n} C_p^q = \prod_{p+q=n} (B_p^q \oplus A_p^{q+1}) = \prod_{p+q=n} B_p^q \oplus \prod_{p+q=n} A_p^{q+1}$$

We conclude that $\text{Tot}_\pi(C^{\bullet,\bullet})$ is the cone of the map $\text{Tot}_\pi(A^{\bullet,\bullet}) \to \text{Tot}_\pi(B^{\bullet,\bullet})$ (We omit the verification that the differentials agree.) Thus it suffices to show $\text{Tot}_\pi(A^{\bullet,\bullet})$ is acyclic if each $A_p^\bullet$ is acyclic. This follows from Lemma 15.103.1. □

15.104. Weakly étale ring maps

092A Most of the results in this section are from the paper [Oli83] by Olivier. See also the related paper [Fer67a].

092B Definition 15.104.1. A ring A is called absolutely flat if every A-module is flat over A. A ring map $A \to B$ is weakly étale or absolutely flat if both $A \to B$ and $B \otimes_A B \to B$ are flat.

Absolutely flat rings are sometimes called von Neumann regular rings (often in the setting of noncommutative rings). A localization is a weakly étale ring map. An étale ring map is weakly étale. Here is a simple, yet key property.

092C Lemma 15.104.2. Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Let N be a B-module. If N is flat as an A-module, then N is flat as a B-module.

Proof. Assume N is a flat as an A-module. Then the functor

$$\text{Mod}_B \longrightarrow \text{Mod}_{B \otimes_A B}, \quad N' \mapsto N \otimes_A N'$$

is exact. As $B \otimes_A B \to B$ is flat we conclude that the functor

$$\text{Mod}_B \longrightarrow \text{Mod}_B, \quad N' \mapsto (N \otimes_A N') \otimes_{B \otimes_A B} B = N \otimes_B N'$$

is exact, hence N is flat over B. □

092D Definition 15.104.3. Let A be a ring. Let $d \geq 0$ be an integer. We say that A has weak dimension $\leq d$ if every A-module has tor dimension $\leq d$.

092E Lemma 15.104.4. Let $A \to B$ be a weakly étale ring map. If A has weak dimension at most d, then so does B.

Proof. Let N be a B-module. If $d = 0$, then N is flat as an A-module, hence flat as a B-module by Lemma 15.104.2. Assume $d > 0$. Choose a resolution $F_\bullet \to N$ by free B-modules. Our assumption implies that $K = \text{Im}(F_d \to F_{d-1})$ is A-flat, see Lemma 15.66.2. Hence it is B-flat by Lemma 15.104.2. Thus $0 \to K \to F_{d-1} \to \ldots \to F_0 \to N \to 0$ is a flat resolution of length d and we see that N has tor dimension at most d. □

092F Lemma 15.104.5. Let A be a ring. The following are equivalent

1. A has weak dimension ≤ 0,
2. A is absolutely flat, and
3. A is reduced and every prime is maximal.

In this case every local ring of A is a field.

Proof. The equivalence of (1) and (2) is immediate. Assume A is absolutely flat. This implies every ideal of A is pure, see Algebra, Definition 10.108.1. Hence every finitely generated ideal is generated by an idempotent by Algebra, Lemma 10.108.5. If $f \in A$, then $(f) = (e)$ for some idempotent $e \in A$ and $D(f) = D(e)$ is open and closed (Algebra, Lemma 10.21.1). This already implies every ideal of A is maximal for example by Algebra, Lemma 10.26.5. Moreover, if f is nilpotent, then $e = 0$ hence $f = 0$. Thus A is reduced.

Assume A is reduced and every prime of A is maximal. Let M be an A-module. Our goal is to show that M is flat. We may write M as a filtered colimit of finite A-modules, hence we may assume M is finite (Algebra, Lemma 10.39.3). There is a finite filtration of M by modules of the form A/I (Algebra, Lemma 10.5.4), hence we may assume that $M = A/I$ (Algebra, Lemma 10.39.13). Thus it suffices to show every ideal of A is pure. Since every local ring of A is a field (by Algebra, Lemma 10.25.1 and the fact that every prime of A is minimal), we see that every ideal $I \subset A$ is radical. Note that every closed subset of $\mathrm{Spec}(A)$ is closed under generalization. Thus every (radical) ideal of A is pure by Algebra, Lemma 10.108.4. □

092G Lemma 15.104.6. A product of fields is an absolutely flat ring.

Proof. Let K_i be a family of fields. If $f = (f_i) \in \prod K_i$, then the ideal generated by f is the same as the ideal generated by the idempotent $e = (e_i)$ with $e_i = 0, 1$ according to whether f_i is 0 or not. Thus $D(f) = D(e)$ is open and closed and we conclude by Lemma 15.104.5 and Algebra, Lemma 10.26.5. □

092H Lemma 15.104.7. Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B.

(1) If $B \otimes_A B \to B$ is flat, then $B' \otimes_{A'} B' \to B'$ is flat.
(2) If $A \to B$ is weakly étale, then $A' \to B'$ is weakly étale.

Proof. Assume $B \otimes_A B \to B$ is flat. The ring map $B' \otimes_{A'} B' \to B'$ is the base change of $B \otimes_A B \to B$ by $A \to A'$. Hence it is flat by Algebra, Lemma 10.39.7. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat ring map is flat. □

092I Lemma 15.104.8. Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat.

(1) If A is an absolutely flat ring, then so is B.
(2) If A is reduced and $A \to B$ is weakly étale, then B is reduced.

Proof. Part (1) follows immediately from Lemma 15.104.2 and the definitions. If A is reduced, then there exists an injection $A \to A' = \prod_{\mathfrak{p} \subset A \text{ minimal}} A_{\mathfrak{p}}$ of A into an absolutely flat ring (Algebra, Lemma 10.25.2 and Lemma 15.104.6). If $A \to B$ is flat, then the induced map $B \to B' = B \otimes_A A'$ is injective too. By Lemma 15.104.7 the ring map $A' \to B'$ is weakly étale. By part (1) we see that B' is absolutely flat. By Lemma 15.104.5 the ring B' is reduced. Hence B is reduced. □

092J Lemma 15.104.9. Let $A \to B$ and $B \to C$ be ring maps.

(1) If $B \otimes_A B \to B$ and $C \otimes_B C \to C$ are flat, then $C \otimes_A C \to C$ is flat.
(2) If $A \to B$ and $B \to C$ are weakly étale, then $A \to C$ is weakly étale.

Proof. Part (1) follows from the factorization

$$C \otimes_A C \longrightarrow C \otimes_B C \longrightarrow C$$

of the multiplication map, the fact that

$$C \otimes_B C = (C \otimes_A C) \otimes_{B \otimes_A B} B,$$

the fact that a base change of a flat map is flat, and the fact that the composition of flat ring maps is flat. See Algebra, Lemmas 10.39.7 and 10.39.4. Part (2) follows from (1) and the fact (just used) that the composition of flat ring maps is flat. □

092K Lemma 15.104.10. Let $A \to B \to C$ be ring maps.

(1) If $B \to C$ is faithfully flat and $C \otimes_A C \to C$ is flat, then $B \otimes_A B \to B$ is flat.
(2) If $B \to C$ is faithfully flat and $A \to C$ is weakly étale, then $A \to B$ is weakly étale.

Proof. Assume $B \to C$ is faithfully flat and $C \otimes_A C \to C$ is flat. Consider the commutative diagram

$$\begin{tikzcd} C \otimes_A C \arrow[r] & C \\ B \otimes_A B \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

The vertical arrows are flat, the top horizontal arrow is flat. Hence C is flat as a $B \otimes_A B$-module. The map $B \to C$ is faithfully flat and $C = B \otimes_B C$. Hence B is flat as a $B \otimes_A B$-module by Algebra, Lemma 10.39.9. This proves (1). Part (2) follows from (1) and the fact that $A \to B$ is flat if $A \to C$ is flat and $B \to C$ is faithfully flat (Algebra, Lemma 10.39.9). ☐

092L Lemma 15.104.11. Let A be a ring. Let $B \to C$ be an A-algebra map of weakly étale A-algebras. Then $B \to C$ is weakly étale.

Proof. The ring map $B \to C$ is flat by Lemma 15.104.2. The ring map $C \otimes_A C \to C \otimes_B C$ is surjective, hence an epimorphism. Thus Lemma 15.104.2 implies, that since C is flat over $C \otimes_A C$ also C is flat over $C \otimes_B C$. ☐

092M Lemma 15.104.12. Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Then $\Omega_{B/A} = 0$, i.e., B is formally unramified over A.

Proof. Let $I \subset B \otimes_A B$ be the kernel of the flat surjective map $B \otimes_A B \to B$. Then I is a pure ideal (Algebra, Definition 10.108.1), so $I^2 = I$ (Algebra, Lemma 10.108.2). Since $\Omega_{B/A} = I/I^2$ (Algebra, Lemma 10.131.13) we obtain the vanishing. This means B is formally unramified over A by Algebra, Lemma 10.148.2. ☐

0CKP Lemma 15.104.13. Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat.

(1) If $A \to B$ is of finite type, then $A \to B$ is unramified.
(2) If $A \to B$ is of finite presentation and flat, then $A \to B$ is étale.

In particular a weakly étale ring map of finite presentation is étale.

Proof. Part (1) follows from Lemma 15.104.12 and Algebra, Definition 10.151.1. Part (2) follows from part (1) and Algebra, Lemma 10.151.8. ☐

092N Lemma 15.104.14. Let $A \to B$ be a ring map. Then $A \to B$ is weakly étale in each of the following cases

(1) $B = S^{-1}A$ is a localization of A,
(2) $A \to B$ is étale,
(3) B is a filtered colimit of weakly étale A-algebras.

Proof. An étale ring map is flat and the map $B \otimes_A B \to B$ is also étale as a map between étale A-algebras (Algebra, Lemma 10.143.8). This proves (2).

Let B_i be a directed system of weakly étale A-algebras. Then $B = \operatorname{colim} B_i$ is flat over A by Algebra, Lemma 10.39.3. Note that the transition maps $B_i \to B_{i'}$ are flat by Lemma 15.104.11. Hence B is flat over B_i for each i, and we see that B is flat over

$B_i \otimes_A B_i$ by Algebra, Lemma 10.39.4. Thus B is flat over $B \otimes_A B = \operatorname{colim} B_i \otimes_A B_i$ by Algebra, Lemma 10.39.6.

Part (1) can be proved directly, but also follows by combining (2) and (3). $\square$

092P Lemma 15.104.15. Let L/K be an extension of fields. If $L \otimes_K L \to L$ is flat, then L is an algebraic separable extension of K.

Proof. By Lemma 15.104.10 we see that any subfield $K \subset L' \subset L$ the map $L' \otimes_K L' \to L'$ is flat. Thus we may assume L is a finitely generated field extension of K. In this case the fact that L/K is formally unramified (Lemma 15.104.12) implies that L/K is finite separable, see Algebra, Lemma 10.158.1. $\square$

092Q Lemma 15.104.16. Let B be an algebra over a field K. The following are equivalent

(1) $B \otimes_K B \to B$ is flat,
(2) $K \to B$ is weakly étale, and
(3) B is a filtered colimit of étale K-algebras.

Moreover, every finitely generated K-subalgebra of B is étale over K.

Proof. Parts (1) and (2) are equivalent because every K-algebra is flat over K. Part (3) implies (1) and (2) by Lemma 15.104.14

Assume (1) and (2) hold. We will prove (3) and the finite statement of the lemma. A field is absolutely flat ring, hence B is a absolutely flat ring by Lemma 15.104.8. Hence B is reduced and every local ring is a field, see Lemma 15.104.5.

Let $\mathfrak{q} \subset B$ be a prime. The ring map $B \to B_\mathfrak{q}$ is weakly étale, hence $B_\mathfrak{q}$ is weakly étale over K (Lemma 15.104.9). Thus $B_\mathfrak{q}$ is a separable algebraic extension of K by Lemma 15.104.15.

Let $K \subset A \subset B$ be a finitely generated K-sub algebra. We will show that A is étale over K which will finish the proof of the lemma. Then every minimal prime $\mathfrak{p} \subset A$ is the image of a prime $\mathfrak{q}$ of B, see Algebra, Lemma 10.30.5. Thus $\kappa(\mathfrak{p})$ as a subfield of $B_\mathfrak{q} = \kappa(\mathfrak{q})$ is separable algebraic over K. Hence every generic point of $\text{Spec}(A)$ is closed (Algebra, Lemma 10.35.9). Thus $\dim(A) = 0$. Then A is the product of its local rings, e.g., by Algebra, Proposition 10.60.7. Moreover, since A is reduced, all local rings are equal to their residue fields wich are finite separable over K. This means that A is étale over K by Algebra, Lemma 10.143.4 and finishes the proof. $\square$

092R Lemma 15.104.17. Let $A \to B$ be a ring map. If $A \to B$ is weakly étale, then $A \to B$ induces separable algebraic residue field extensions.

Proof. Let $\mathfrak{p}$ be a prime of A. Then $\kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$ is weakly étale by Lemma 15.104.7. Hence $B \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of étale $\kappa(\mathfrak{p})$-algebras by Lemma 15.104.16. Hence for $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ the extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is a filtered colimit of finite separable extensions by Algebra, Lemma 10.143.4. $\square$

092S Lemma 15.104.18. Let A be a ring. The following are equivalent

(1) A has weak dimension ≤ 1,
(2) every ideal of A is flat,
(3) every finitely generated ideal of A is flat,
(4) every submodule of a flat A-module is flat, and
(5) every local ring of A is a valuation ring.

Proof. If A has weak dimension ≤ 1, then the resolution $0 \to I \to A \to A/I \to 0$ shows that every ideal I is flat by Lemma 15.66.2. Hence (1) $\Rightarrow$ (2).

Assume (4). Let M be an A-module. Choose a surjection $F \to M$ where F is a free A-module. Then $\text{Ker}(F \to M)$ is flat by assumption, and we see that M has tor dimension ≤ 1 by Lemma 15.66.6. Hence (4) $\Rightarrow$ (1).

Every ideal is the union of the finitely generated ideals contained in it. Hence (3) implies (2) by Algebra, Lemma 10.39.3. Thus (3) $\Leftrightarrow$ (2).

Assume (2). Suppose that $N \subset M$ with M a flat A-module. We will prove that N is flat. We can write $M = \text{colim}\, M_i$ with each M_i finite free, see Algebra, Theorem 10.81.4. Setting $N_i \subset M_i$ the inverse image of N we see that $N = \text{colim}\, N_i$. By Algebra, Lemma 10.39.3. it suffices to prove N_i is flat and we reduce to the case $M = R^{\oplus n}$. In this case the module N has a finite filtration by the submodules $R^{\oplus j} \cap N$ whose subquotients are ideals. By (2) these ideals are flat and hence N is flat by Algebra, Lemma 10.39.13. Thus (2) $\Rightarrow$ (4).

Assume A satisfies (1) and let $\mathfrak{p} \subset A$ be a prime ideal. By Lemmas 15.104.14 and 15.104.4 we see that $A_\mathfrak{p}$ satisfies (1). We will show A is a valuation ring if A is a local ring satisfying (3). Let $f \in \mathfrak{m}$ be a nonzero element. Then (f) is a flat nonzero module generated by one element. Hence it is a free A-module by Algebra, Lemma 10.78.5. It follows that f is a nonzerodivisor and A is a domain. If $I \subset A$ is a finitely generated ideal, then we similarly see that I is a finite free A-module, hence (by considering the rank) free of rank 1 and I is a principal ideal. Thus A is a valuation ring by Algebra, Lemma 10.50.15. Thus (1) $\Rightarrow$ (5).

Assume (5). Let $I \subset A$ be a finitely generated ideal. Then $I_\mathfrak{p} \subset A_\mathfrak{p}$ is a finitely generated ideal in a valuation ring, hence principal (Algebra, Lemma 10.50.15), hence flat. Thus I is flat by Algebra, Lemma 10.39.18. Thus (5) $\Rightarrow$ (3). This finishes the proof of the lemma. $\square$

092T **Lemma 15.104.19.** Let J be a set. For each $j \in J$ let A_j be a valuation ring with fraction field K_j. Set $A = \prod A_j$ and $K = \prod K_j$. Then A has weak dimension at most 1 and $A \to K$ is a localization.

Proof. Let $I \subset A$ be a finitely generated ideal. By Lemma 15.104.18 it suffices to show that I is a flat A-module. Let $I_j \subset A_j$ be the image of I. Observe that $I_j = I \otimes_A A_j$, hence $I \to \prod I_j$ is surjective by Algebra, Proposition 10.89.2. Thus $I = \prod I_j$. Since A_j is a valuation ring, the ideal I_j is generated by a single element (Algebra, Lemma 10.50.15). Say $I_j = (f_j)$. Then I is generated by the element $f = (f_j)$. Let $e \in A$ be the idempotent which has a 0 or 1 in A_j depending on whether f_j is 0 or not. Then $f = ge$ for some nonzerodivisor $g \in A$: take $g = (g_j)$ with $g_j = 1$ if $f_j = 0$ and $g_j = f_j$ else. Thus $I \cong (e)$ as a module. We conclude I is flat as (e) is a direct summand of A. The final statement is true because $K = S^{-1}A$ where $S = \prod (A_j \setminus \{0\})$. $\square$

092U **Lemma 15.104.20.** Let A be a normal domain with fraction field K. There exists a cartesian diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & K \arrow[d] \\ V \arrow[r] & L \end{tikzcd}$$

of rings where V has weak dimension at most 1 and $V \to L$ is a flat, injective, epimorphism of rings.

Proof. For every $x \in K$, $x \notin A$ pick $V_x \subset K$ as in Algebra, Lemma 10.50.11. Set $V = \prod_{x \in K \setminus A} V_x$ and $L = \prod_{x \in K \setminus A} K$. The ring V has weak dimension at most 1 by Lemma 15.104.19 which also shows that $V \to L$ is a localization. A localization is flat and an epimorphism, see Algebra, Lemmas 10.39.18 and 10.107.5. $\square$

092V Lemma 15.104.21. Let A be a ring of weak dimension at most 1. If $A \to B$ is a flat, injective, epimorphism of rings, then A is integrally closed in B.

Proof. Let $x \in B$ be integral over A. Let $A' = A[x] \subset B$. Then A' is a finite ring extension of A by Algebra, Lemma 10.36.5. To show $A = A'$ it suffices to show $A \to A'$ is an epimorphism by Algebra, Lemma 10.107.6. Note that A' is flat over A by assumption on A and the fact that B is flat over A (Lemma 15.104.18). Hence the composition

$$A' \otimes_A A' \to B \otimes_A A' \to B \otimes_A B \to B$$

is injective, i.e., $A' \otimes_A A' \cong A'$ and the lemma is proved. $\square$

092W Lemma 15.104.22. Let A be a normal domain with fraction field K. Let $A \to B$ be weakly étale. Then B is integrally closed in $B \otimes_A K$.

Proof. Choose a diagram as in Lemma 15.104.20. As $A \to B$ is flat, the base change gives a cartesian diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B \otimes_A K \arrow[d] \\ B \otimes_A V \arrow[r] & B \otimes_A L \end{tikzcd}$$

of rings. Note that $V \to B \otimes_A V$ is weakly étale (Lemma 15.104.7), hence $B \otimes_A V$ has weak dimension at most 1 by Lemma 15.104.4. Note that $B \otimes_A V \to B \otimes_A L$ is a flat, injective, epimorphism of rings as a flat base change of such (Algebra, Lemmas 10.39.7 and 10.107.3). By Lemma 15.104.21 we see that $B \otimes_A V$ is integrally closed in $B \otimes_A L$. It follows from the cartesian property of the diagram that B is integrally closed in $B \otimes_A K$. $\square$

092X Lemma 15.104.23. Let $A \to B$ be a ring homomorphism. Assume

(1) A is a henselian local ring,
(2) $A \to B$ is integral,
(3) B is a domain.

Then B is a henselian local ring and $A \to B$ is a local homomorphism. If A is strictly henselian, then B is a strictly henselian local ring and the extension $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ of residue fields is purely inseparable.

Proof. Write B as a filtered colimit $B = \operatorname{colim} B_i$ of finite A-sub algebras. If we prove the results for each B_i, then the result follows for B. See Algebra, Lemma 10.154.8. If $A \to B$ is finite, then B is a product of local henselian rings by Algebra, Lemma 10.153.4. Since B is a domain we see that B is a local ring. The maximal ideal of B lies over the maximal ideal of A by going up for $A \to B$ (Algebra, Lemma 10.36.22). If A is strictly henselian, then the field extension $\kappa(\mathfrak{m}_B)/\kappa(\mathfrak{m}_A)$ being algebraic, has to be purely inseparable. Of course, then $\kappa(\mathfrak{m}_B)$ is separably algebraically closed and B is strictly henselian. $\square$

092Z Theorem 15.104.24 (Olivier). Let $A \to B$ be a local homomorphism of local rings. If A is strictly henselian and $A \to B$ is weakly étale, then $A = B$.

Proof. We will show that for all $\mathfrak{p} \subset A$ there is a unique prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. This implies that $B \otimes_A B \to B$ is bijective on spectra as well as surjective and flat. Hence it is an isomorphism for example by the description of pure ideals in Algebra, Lemma 10.108.4. Hence $A \to B$ is a faithfully flat epimorphism of rings. We get $A = B$ by Algebra, Lemma 10.107.7.

Note that the fibre ring $B \otimes_A \kappa(\mathfrak{p})$ is a colimit of étale extensions of $\kappa(\mathfrak{p})$ by Lemmas 15.104.7 and 15.104.16. Hence, if there exists more than one prime lying over $\mathfrak{p}$ or if $\kappa(\mathfrak{p}) \neq \kappa(\mathfrak{q})$ for some $\mathfrak{q}$, then $B \otimes_A L$ has a nontrivial idempotent for some (separable) algebraic field extension $L/\kappa(\mathfrak{p})$.

Let $L/\kappa(\mathfrak{p})$ be an algebraic field extension. Let $A' \subset L$ be the integral closure of $A/\mathfrak{p}$ in L. By Lemma 15.104.23 we see that A' is a strictly henselian local ring whose residue field is a purely inseparable extension of the residue field of A. Thus $B \otimes_A A'$ is a local ring by Algebra, Lemma 10.156.5. On the other hand, $B \otimes_A A'$ is integrally closed in $B \otimes_A L$ by Lemma 15.104.22. Since $B \otimes_A A'$ is local, it follows that the ring $B \otimes_A L$ does not have nontrivial idempotents which is what we wanted to prove. $\square$

15.105. Weakly étale algebras over fields

0CKQ If K is a field, then an algebra B is weakly étale over K if and only if it is a filtered colimit of étale K-algebras. This is Lemma 15.104.16.

0CKR Lemma 15.105.1. Let K be a field. If B is weakly étale over K, then

(1) B is reduced,
(2) B is integral over K,
(3) any finitely generated K-subalgebra of B is a finite product of finite separable extensions of K,
(4) B is a field if and only if B does not have nontrivial idempotents and in this case it is a separable algebraic extension of K,
(5) any sub or quotient K-algebra of B is weakly étale over K,
(6) if B' is weakly étale over K, then $B \otimes_K B'$ is weakly étale over K.

Proof. Part (1) follows from Lemma 15.104.8 but of course it follows from part (3) as well. Part (3) follows from Lemma 15.104.16 and the fact that étale K-algebras are finite products of finite separable extensions of K, see Algebra, Lemma 10.143.4. Part (3) implies (2). Part (4) follows from (3) as a product of fields is a field if and only if it has no nontrivial idempotents.

If $S \subset B$ is a subalgebra, then it is the filtered colimit of its finitely generated subalgebras which are all étale over K by the above and hence S is weakly étale over K by Lemma 15.104.16. If $B \to Q$ is a quotient algebra, then Q is the filtered colimit of K-algebra quotients of finite products $\prod_{i \in I} L_i$ of finite separable extensions L_i/K. Such a quotient is of the form $\prod_{i \in J} L_i$ for some subset $J \subset I$ and hence the result holds for quotients by the same reasoning.

The statement on tensor products follows in a similar manner or by combining Lemmas 15.104.7 and 15.104.9. $\square$

0CKS Lemma 15.105.2. Let K be a field. Let A be a K-algebra. There exists a maximal weakly étale K-subalgebra $B_{max} \subset A$.

Proof. Let $B_1, B_2 \subset A$ be weakly étale K-subalgebras. Then $B_1 \otimes_K B_2$ is weakly étale over K and so is the image of $B_1 \otimes_K B_2 \to A$ (Lemma 15.105.1). Thus the collection $\mathcal{B}$ of weakly étale K-subalgebras $B \subset A$ is directed and the colimit $B_{max} = \text{colim}_{B \in \mathcal{B}} B$ is a weakly étale K-algebra by Lemma 15.104.14. Hence the image of $B_{max} \to A$ is weakly étale over K (previous lemma cited). It follows that this image is in $\mathcal{B}$ and hence $\mathcal{B}$ has a maximal element (and the image is the same as B_{max}). $\square$

0CKT Lemma 15.105.3. Let K be a field. For a K-algebra A denote $B_{max}(A)$ the maximal weakly étale K-subalgebra of A as in Lemma 15.105.2. Then

(1) any K-algebra map $A' \to A$ induces a K-algebra map $B_{max}(A') \to B_{max}(A)$,
(2) if $A' \subset A$, then $B_{max}(A') = B_{max}(A) \cap A'$,
(3) if $A = \text{colim}\, A_i$ is a filtered colimit, then $B_{max}(A) = \text{colim}\, B_{max}(A_i)$,
(4) the map $B_{max}(A) \to B_{max}(A_{red})$ is an isomorphism,
(5) $B_{max}(A_1 \times \ldots \times A_n) = B_{max}(A_1) \times \ldots \times B_{max}(A_n)$,
(6) if A has no nontrivial idempotents, then $B_{max}(A)$ is a field and a separable algebraic extension of K,
(7) add more here.

Proof. Proof of (1). This is true because the image of $B_{max}(A') \to A$ is weakly étale over K by Lemma 15.105.1.

Proof of (2). By (1) we have $B_{max}(A') \subset B_{max}(A)$. Conversely, $B_{max}(A) \cap A'$ is a weakly étale K-algebra by Lemma 15.105.1 and hence contained in $B_{max}(A')$.

Proof of (3). By (1) there is a map $\text{colim}\, B_{max}(A_i) \to A$ which is injective because the system is filtered and $B_{max}(A_i) \subset A_i$. The colimit $\text{colim}\, B_{max}(A_i)$ is weakly étale over K by Lemma 15.104.14. Hence we get an injective map $\text{colim}\, B_{max}(A_i) \to B_{max}(A)$. Suppose that $a \in B_{max}(A)$. Then a generates a finitely presented K-subalgebra $B \subset B_{max}(A)$. By Algebra, Lemma 10.127.3 there is an i and a K-algebra map $f : B \to A_i$ lifting the given map $B \to A$. Since B is weakly étale by Lemma 15.105.1, we see that $f(B) \subset B_{max}(A_i)$ and we conclude that a is in the image of $\text{colim}\, B_{max}(A_i) \to B_{max}(A)$.

Proof of (4). Write $B_{max}(A_{red}) = \text{colim}\, B_i$ as a filtered colimit of étale K-algebras (Lemma 15.104.16). By Algebra, Lemma 10.138.17 for each i there is a K-algebra map $f_i : B_i \to A$ lifting the given map $B_i \to A_{red}$. It follows that the canonical map $B_{max}(A_{red}) \to B_{max}(A)$ is surjective. The kernel consists of nilpotent elements and hence is zero as $B_{max}(A_{red})$ is reduced (Lemma 15.105.1).

Proof of (5). Omitted.

Proof of (6). Follows from Lemma 15.105.1 part (4). $\square$

0CKU Lemma 15.105.4. Let L/K be an extension of fields. Let A be a K-algebra. Let $B \subset A$ be the maximal weakly étale K-subalgebra of A as in Lemma 15.105.2. Then $B \otimes_K L$ is the maximal weakly étale L-subalgebra of $A \otimes_K L$.

Proof. For an algebra A over K we write $B_{max}(A/K)$ for the maximal weakly étale K-subalgebra of A. Similarly we write $B_{max}(A'/L)$ for the maximal weakly étale L-subalgebra of A' if A' is an L-algebra. Since $B_{max}(A/K) \otimes_K L$ is weakly étale over L (Lemma 15.104.7) and since $B_{max}(A/K) \otimes_K L \subset A \otimes_K L$ we obtain a canonical injective map

$$B_{max}(A/K) \otimes_K L \to B_{max}((A \otimes_K L)/L)$$

The lemma states that this map is an isomorphism.

To prove the lemma for L and our K-algebra A, it suffices to prove the lemma for any field extension L' of L. Namely, we have the factorization

$$B_{max}(A/K) \otimes_K L' \to B_{max}((A \otimes_K L)/L) \otimes_L L' \to B_{max}((A \otimes_K L')/L')$$

hence the composition cannot be surjective without $B_{max}(A/K) \otimes_K L \to B_{max}((A \otimes_K L)/L)$ being surjective. Thus we may assume L is algebraically closed.

Reduction to finite type K-algebra. We may write A is the filtered colimit of its finite type K-subalgebras. Using Lemma 15.105.3 we see that it suffices to prove the lemma for finite type K-algebras.

Assume A is a finite type K-algebra. Since the kernel of $A \to A_{red}$ is nilpotent, the same is true for $A \otimes_K L \to A_{red} \otimes_K L$. Then

$$B_{max}((A \otimes_K L)/L) \to B_{max}((A_{red} \otimes_K L)/L)$$

is injective because the kernel is nilpotent and the weakly étale L-algebra $B_{max}((A \otimes_K L)/L)$ is reduced (Lemma 15.105.1). Since $B_{max}(A/K) = B_{max}(A_{red}/K)$ by Lemma 15.105.3 we conclude that it suffices to prove the lemma for A_{red}.

Assume A is a reduced finite type K-algebra. Let $Q = Q(A)$ be the total quotient ring of A. Then $A \subset Q$ and $A \otimes_K L \subset Q \otimes_A L$ and hence

$$B_{max}(A/K) = A \cap B_{max}(Q/K)$$

and

$$B_{max}((A \otimes_K L)/L) = (A \otimes_K L) \cap B_{max}((Q \otimes_K L)/L)$$

by Lemma 15.105.3. Since $- \otimes_K L$ is an exact functor, it follows that if we prove the result for Q, then the result follows for A. Since Q is a finite product of fields (Algebra, Lemmas 10.25.4, 10.25.1, 10.31.6, and 10.31.1) and since B_{max} commutes with products (Lemma 15.105.3) it suffices to prove the lemma when A is a field.

Assume A is a field. We reduce to A being finitely generated over K by the argument in the third paragraph of the proof. (In fact the way we reduced to the case of a field produces a finitely generated field extension of K.)

Assume A is a finitely generated field extension of K. Then $K' = B_{max}(A/K)$ is a field separable algebraic over K by Lemma 15.105.3 part (6). Hence K' is a finite separable field extension of K and A is geometrically irreducible over K' by Algebra, Lemma 10.47.13. Since L is algebraically closed and K'/K finite separable we see that

$$K' \otimes_K L \to \prod\nolimits_{\sigma \in \mathrm{Hom}_K(K', L)} L, \quad \alpha \otimes \beta \mapsto (\sigma(\alpha)\beta)_\sigma$$

is an isomorphism (Fields, Lemma 9.13.4). We conclude

$$A \otimes_K L = A \otimes_{K'} (K' \otimes_K L) = \prod\nolimits_{\sigma \in \mathrm{Hom}_K(K', L)} A \otimes_{K', \sigma} L$$

Since A is geometrically irreducible over K' we see that $A \otimes_{K',\sigma} L$ has a unique minimal prime. Since L is algebraically closed it follows that $B_{max}((A \otimes_{K',\sigma} L)/L) = L$

because this L-algebra is a field algebraic over L by Lemma 15.105.3 part (6). It follows that the maximal weakly étale $K' \otimes_K L$-subalgebra of $A \otimes_K L$ is $K' \otimes_K L$ because we can decompose these subalgebras into products as above. Hence the inclusion $K' \otimes_K L \subset B_{max}((A \otimes_K L)/L)$ is an equality: the ring map $K' \otimes_K L \to B_{max}((A \otimes_K L)/L)$ is weakly étale by Lemma 15.104.11. □

15.106. Local irreducibility

06DT The following definition seems to be the generally accepted one. To parse it, observe that if $A \subset B$ is an integral extension of local domains, then $A \to B$ is a local ring homomorphism by going up (Algebra, Lemma 10.36.22).

0BPZ **Definition 15.106.1.** Let A be a local ring. We say A is *unibranch* if the reduction A_{red} is a domain and if the integral closure A' of A_{red} in its field of fractions is local. We say A is *geometrically unibranch* if A is unibranch and moreover the residue field of A' is purely inseparable over the residue field of A. [GD67, Chapter 0 (23.2.1)]

Let A be a local ring. Here is an equivalent formulation

1. A is unibranch if A has a unique minimal prime $\mathfrak{p}$ and the integral closure of $A/\mathfrak{p}$ in its fraction field is a local ring, and
2. A is geometrically unibranch if A has a unique minimal prime $\mathfrak{p}$ and the integral closure of $A/\mathfrak{p}$ in its fraction field is a local ring whose residue field is purely inseparable over the residue field of A.

A local ring which is normal is geometrically unibranch (follows from Definition 15.106.1 and Algebra, Definition 10.37.11). Lemmas 15.106.3 and 15.106.5 suggest that being (geometrically) unibranch is a reasonable property to look at.

0C24 **Lemma 15.106.2.** Let A be a local ring. Assume A has finitely many minimal prime ideals. Let A' be the integral closure of A in the total ring of fractions of A_{red}. Let A^h be the henselization of A. Consider the maps

$$\mathrm{Spec}(A') \leftarrow \mathrm{Spec}((A')^h) \to \mathrm{Spec}(A^h)$$

where $(A')^h = A' \otimes_A A^h$. Then

1. the left arrow is bijective on maximal ideals,
2. the right arrow is bijective on minimal primes,
3. every minimal prime of $(A')^h$ is contained in a unique maximal ideal and every maximal ideal contains exactly one minimal prime.

Proof. Let $I \subset A$ be the ideal of nilpotents. We have $(A/I)^h = A^h/IA^h$ by (Algebra, Lemma 10.156.2). The spectra of A, A^h, A', and $(A')^h$ are the same as the spectra of A/I, A^h/IA^h, A', and $(A')^h = A' \otimes_{A/I} A^h/IA^h$. Thus we may replace A by $A_{red} = A/I$ and assume A is reduced. Then $A \subset A'$ which we will use below without further mention.

Proof of (1). As A' is integral over A we see that $(A')^h$ is integral over A^h. By going up (Algebra, Lemma 10.36.22) every maximal ideal of A', resp. $(A')^h$ lies over the maximal ideal $\mathfrak{m}$, resp. $\mathfrak{m}^h$ of A, resp. A^h. Thus (1) follows from the isomorphism

$$(A')^h \otimes_{A^h} \kappa^h = A' \otimes_A A^h \otimes_{A^h} \kappa^h = A' \otimes_A \kappa$$

because the residue field extension κ^h/κ induced by $A \to A^h$ is trivial. We will use below that the displayed ring is integral over a field hence spectrum of this ring is a profinite space, see Algebra, Lemmas 10.36.19 and 10.26.5.

Proof of (3). The ring A' is a normal ring and in fact a finite product of normal domains, see Algebra, Lemma 10.37.16. Since A^h is a filtered colimit of étale A-algebras, $(A')^h$ is filtered colimit of étale A'-algebras hence $(A')^h$ is a normal ring by Algebra, Lemmas 10.163.9 and 10.37.17. Thus every local ring of $(A')^h$ is a normal domain and we see that every maximal ideal contains a unique minimal prime. By Lemma 15.11.8 applied to $A^h \to (A')^h$ we see that $((A')^h, \mathfrak{m}(A')^h)$ is a henselian pair. If $\mathfrak{q} \subset (A')^h$ is a minimal prime (or any prime), then the intersection of $V(\mathfrak{q})$ with $V(\mathfrak{m}(A')^h)$ is connected by Lemma 15.11.16 Since $V(\mathfrak{m}(A')^h) = \mathrm{Spec}((A')^h \otimes \kappa^h)$ is a profinite space by we see there is a unique maximal ideal containing $\mathfrak{q}$.

Proof of (2). The minimal primes of A' are exactly the primes lying over a minimal prime of A (by construction). Since $A' \to (A')^h$ is flat by going down (Algebra, Lemma 10.39.19) every minimal prime of $(A')^h$ lies over a minimal prime of A'. Conversely, any prime of $(A')^h$ lying over a minimal prime of A' is minimal because $(A')^h$ is a filtered colimit of étale hence quasi-finite algebras over A' (small detail omitted). We conclude that the minimal primes of $(A')^h$ are exactly the primes which lie over a minimal prime of A. Similarly, the minimal primes of A^h are exactly the primes lying over minimal primes of A. By construction we have $A' \otimes_A Q(A) = Q(A)$ where $Q(A)$ is the total fraction ring of our reduced local ring A. Of course $Q(A)$ is the finite product of residue fields of the minimal primes of A. It follows that

$$(A')^h \otimes_A Q(A) = A^h \otimes_A A' \otimes_A Q(A) = A^h \otimes_A Q(A)$$

Our discussion above shows the spectrum of the ring on the left is the set of minimal primes of $(A')^h$ and the spectrum of the ring on the right is the is the set of minimal primes of A^h. This finishes the proof. □

0BQ0 Lemma 15.106.3. Let A be a local ring. Let A^h be the henselization of A. The following are equivalent [GD67, Chapter IV Proposition 18.6.12]

(1) A is unibranch, and
(2) A^h has a unique minimal prime.

Proof. This follows from Lemma 15.106.2 but we will also give a direct proof. Denote $\mathfrak{m}$ the maximal ideal of the ring A. Recall that the residue field $\kappa = A/\mathfrak{m}$ is the same as the residue field of A^h.

Assume (2). Let $\mathfrak{p}^h$ be the unique minimal prime of A^h. The flatness of $A \to A^h$ implies that $\mathfrak{p} = A \cap \mathfrak{p}^h$ is the unique minimal prime of A (by going down, see Algebra, Lemma 10.39.19). Also, since $A^h/\mathfrak{p}A^h = (A/\mathfrak{p})^h$ (see Algebra, Lemma 10.156.2) is reduced by Lemma 15.45.4 we see that $\mathfrak{p}^h = \mathfrak{p}A^h$. Let A' be the integral closure of $A/\mathfrak{p}$ in its fraction field. We have to show that A' is local. Since $A \to A'$ is integral, every maximal ideal of A' lies over $\mathfrak{m}$ (by going up for integral ring maps, see Algebra, Lemma 10.36.22). If A' is not local, then we can find distinct maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$. Choose elements $f_1, f_2 \in A'$ with $f_i \in \mathfrak{m}_i$ and $f_i \notin \mathfrak{m}_{3-i}$. We find a finite subalgebra $B = A[f_1, f_2] \subset A'$ with distinct maximal ideals $B \cap \mathfrak{m}_i$, $i = 1, 2$. Note that the inclusions

$$A/\mathfrak{p} \subset B \subset \kappa(\mathfrak{p})$$

give, on tensoring with the flat ring map $A \to A^h$ the inclusions

$$A^h/\mathfrak{p}^h \subset B \otimes_A A^h \subset \kappa(\mathfrak{p}) \otimes_A A^h \subset \kappa(\mathfrak{p}^h)$$

the last inclusion because $\kappa(\mathfrak p) \otimes_A A^h = \kappa(\mathfrak p) \otimes_{A/\mathfrak p} A^h/\mathfrak p^h$ is a localization of the domain $A^h/\mathfrak p^h$. Note that $B \otimes_A \kappa$ has at least two maximal ideals because $B/\mathfrak mB$ has two maximal ideals. Hence, as A^h is henselian we see that $B \otimes_A A^h$ is a product of ≥ 2 local rings, see Algebra, Lemma 10.153.5. But we've just seen that $B \otimes_A A^h$ is a subring of a domain and we get a contradiction.

Assume (1). Let $\mathfrak p \subset A$ be the unique minimal prime and let A' be the integral closure of $A/\mathfrak p$ in its fraction field. Let $A \to B$ be a local map of local rings inducing an isomorphism of residue fields which is a localization of an étale A-algebra. In particular $\mathfrak m_B$ is the unique prime containing $\mathfrak mB$. Then $B' = A' \otimes_A B$ is integral over B and the assumption that $A \to A'$ is local implies that B' is local (Algebra, Lemma 10.156.5). On the other hand, $A' \to B'$ is the localization of an étale ring map, hence B' is normal, see Algebra, Lemma 10.163.9. Thus B' is a (local) normal domain. Finally, we have

$$B/\mathfrak pB \subset B \otimes_A \kappa(\mathfrak p) = B' \otimes_{A'} (\text{fraction field of } A') \subset \text{fraction field of } B'$$

Hence $B/\mathfrak pB$ is a domain, which implies that B has a unique minimal prime (since by flatness of $A \to B$ these all have to lie over $\mathfrak p$). Since A^h is a filtered colimit of the local rings B it follows that A^h has a unique minimal prime. Namely, if $fg = 0$ in A^h for some non-nilpotent elements f, g, then we can find a B as above containing both f and g which leads to a contradiction. □

0C25 Lemma 15.106.4. Let $(A, \mathfrak m, \kappa)$ be a local ring. Assume A has finitely many minimal prime ideals. Let A' be the integral closure of A in the total ring of fractions of A_{red}. Choose an algebraic closure $\overline{\kappa}$ of κ and denote $\kappa^{sep} \subset \overline{\kappa}$ the separable algebraic closure of κ. Let A^{sh} be the strict henselization of A with respect to κ^{sep}. Consider the maps

$$\text{Spec}(A') \xleftarrow{c} \text{Spec}((A')^{sh}) \xrightarrow{e} \text{Spec}(A^{sh})$$

where $(A')^{sh} = A' \otimes_A A^{sh}$. Then

1. (1) for $\mathfrak m' \subset A'$ maximal the residue field κ' is algebraic over κ and the fibre of c over $\mathfrak m'$ can be canonically identified with $\text{Hom}_\kappa(\kappa', \overline{\kappa})$,
2. (2) the right arrow is bijective on minimal primes,
3. (3) every minimal prime of $(A')^{sh}$ is contained in a unique maximal ideal and every maximal ideal contains a unique minimal prime.

Proof. The proof is almost exactly the same as for Lemma 15.106.2. Let $I \subset A$ be the ideal of nilpotents. We have $(A/I)^{sh} = A^{sh}/IA^{sh}$ by (Algebra, Lemma 10.156.2). The spectra of A, A^{sh}, A', and $(A')^h$ are the same as the spectra of A/I, A^{sh}/IA^{sh}, A', and $(A')^{sh} = A' \otimes_{A/I} A^{sh}/IA^{sh}$. Thus we may replace A by $A_{red} = A/I$ and assume A is reduced. Then $A \subset A'$ which we will use below without further mention.

Proof of (1). The field extension κ'/κ is algebraic because A' is integral over A. Since A' is integral over A, we see that $(A')^{sh}$ is integral over A^{sh}. By going up (Algebra, Lemma 10.36.22) every maximal ideal of A', resp. $(A')^{sh}$ lies over the maximal ideal $\mathfrak m$, resp. $\mathfrak m^{sh}$ of A, resp. A^h. We have

$$(A')^{sh} \otimes_{A^{sh}} \kappa^{sep} = A' \otimes_A A^h \otimes_{A^h} \kappa^{sep} = (A' \otimes_A \kappa) \otimes_\kappa \kappa^{sep}$$

because the residue field of A^{sh} is κ^{sep}. Thus the fibre of c over $\mathfrak m'$ is the spectrum of $\kappa' \otimes_\kappa \kappa^{sep}$. We conclude (1) is true because there is a bijection

$$\text{Hom}_\kappa(\kappa', \overline{\kappa}) \to \text{Spec}(\kappa' \otimes_\kappa \kappa^{sep}), \quad \sigma \mapsto \text{Ker}(\sigma \otimes 1 : \kappa' \otimes_\kappa \kappa^{sep} \to \overline{\kappa})$$

We will use below that the displayed ring is integral over a field hence spectrum of this ring is a profinite space, see Algebra, Lemmas 10.36.19 and 10.26.5.

Proof of (3). The ring A' is a normal ring and in fact a finite product of normal domains, see Algebra, Lemma 10.37.16. Since A^{sh} is a filtered colimit of étale A-algebras, $(A')^{sh}$ is filtered colimit of étale A'-algebras hence $(A')^{sh}$ is a normal ring by Algebra, Lemmas 10.163.9 and 10.37.17. Thus every local ring of $(A')^{sh}$ is a normal domain and we see that every maximal ideal contains a unique minimal prime. By Lemma 15.11.8 applied to $A^{sh} \to (A')^{sh}$ to see that $((A')^{sh}, \mathfrak m(A')^{sh})$ is a henselian pair. If $\mathfrak q \subset (A')^{sh}$ is a minimal prime (or any prime), then the intersection of $V(\mathfrak q)$ with $V(\mathfrak m(A')^{sh})$ is connected by Lemma 15.11.16 Since $V(\mathfrak m(A')^{sh}) = \text{Spec}((A')^{sh} \otimes \kappa^{sh})$ is a profinite space by we see there is a unique maximal ideal containing $\mathfrak q$.

Proof of (2). The minimal primes of A' are exactly the primes lying over a minimal prime of A (by construction). Since $A' \to (A')^{sh}$ is flat by going down (Algebra, Lemma 10.39.19) every minimal prime of $(A')^{sh}$ lies over a minimal prime of A'. Conversely, any prime of $(A')^{sh}$ lying over a minimal prime of A' is minimal because $(A')^{sh}$ is a filtered colimit of étale hence quasi-finite algebras over A' (small detail omitted). We conclude that the minimal primes of $(A')^{sh}$ are exactly the primes which lie over a minimal prime of A. Similarly, the minimal primes of A^{sh} are exactly the primes lying over minimal primes of A. By construction we have $A' \otimes_A Q(A) = Q(A)$ where $Q(A)$ is the total fraction ring of our reduced local ring A. Of course $Q(A)$ is the finite product of residue fields of the minimal primes of A. It follows that

$$(A')^{sh} \otimes_A Q(A) = A^{sh} \otimes_A A' \otimes_A Q(A) = A^{sh} \otimes_A Q(A)$$

Our discussion above shows the spectrum of the ring on the left is the set of minimal primes of $(A')^{sh}$ and the spectrum of the ring on the right is the is the set of minimal primes of A^{sh}. This finishes the proof. □

06DM Lemma 15.106.5. Let A be a local ring. Let A^{sh} be a strict henselization of A. The following are equivalent [Art66, Lemma 2.2] and [GD67, Chapter IV Proposition 18.8.15]

(1) A is geometrically unibranch, and
(2) A^{sh} has a unique minimal prime.

Proof. This follows from Lemma 15.106.4 but we will also give a direct proof; this direct proof is almost exactly the same as the direct proof of Lemma 15.106.3. Denote $\mathfrak m$ the maximal ideal of the ring A. Denote κ, κ^{sh} the residue field of A, A^{sh}.

Assume (2). Let $\mathfrak p^{sh}$ be the unique minimal prime of A^{sh}. The flatness of $A \to A^{sh}$ implies that $\mathfrak p = A \cap \mathfrak p^{sh}$ is the unique minimal prime of A (by going down, see Algebra, Lemma 10.39.19). Also, since $A^{sh}/\mathfrak p A^{sh} = (A/\mathfrak p)^{sh}$ (see Algebra, Lemma 10.156.4) is reduced by Lemma 15.45.4 we see that $\mathfrak p^{sh} = \mathfrak p A^{sh}$. Let A' be the integral closure of $A/\mathfrak p$ in its fraction field. We have to show that A' is local and that its residue field is purely inseparable over κ. Since $A \to A'$ is integral, every maximal ideal of A' lies over $\mathfrak m$ (by going up for integral ring maps, see Algebra, Lemma 10.36.22). If A' is not local, then we can find distinct maximal ideals $\mathfrak m_1$, $\mathfrak m_2$. Choosing elements $f_1, f_2 \in A'$ with $f_i \in \mathfrak m_i$, $f_i \not\in \mathfrak m_{3-i}$ we find a finite subalgebra $B = A[f_1, f_2] \subset A'$ with distinct maximal ideals $B \cap \mathfrak m_i$, $i = 1, 2$. If A' is local with maximal ideal $\mathfrak{m}'$, but $A/\mathfrak{m} \subset A'/\mathfrak{m}'$ is not purely inseparable, then we can find $f \in A'$ whose image in $A'/\mathfrak{m}'$ generates a finite, not purely inseparable extension of $A/\mathfrak{m}$ and we find a finite local subalgebra $B = A[f] \subset A'$ whose residue field is not a purely inseparable extension of $A/\mathfrak{m}$. Note that the inclusions

$$A/\mathfrak{p} \subset B \subset \kappa(\mathfrak{p})$$

give, on tensoring with the flat ring map $A \to A^{sh}$ the inclusions

$$A^{sh}/\mathfrak{p}^{sh} \subset B \otimes_A A^{sh} \subset \kappa(\mathfrak{p}) \otimes_A A^{sh} \subset \kappa(\mathfrak{p}^{sh})$$

the last inclusion because $\kappa(\mathfrak{p}) \otimes_A A^{sh} = \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{p}} A^{sh}/\mathfrak{p}^{sh}$ is a localization of the domain $A^{sh}/\mathfrak{p}^{sh}$. Note that $B \otimes_A \kappa^{sh}$ has at least two maximal ideals because $B/\mathfrak{m}B$ either has two maximal ideals or one whose residue field is not purely inseparable over κ, and because κ^{sh} is separably algebraically closed. Hence, as A^{sh} is strictly henselian we see that $B \otimes_A A^{sh}$ is a product of ≥ 2 local rings, see Algebra, Lemma 10.153.6. But we've just seen that $B \otimes_A A^{sh}$ is a subring of a domain and we get a contradiction.

Assume (1). Let $\mathfrak{p} \subset A$ be the unique minimal prime and let A' be the integral closure of $A/\mathfrak{p}$ in its fraction field. Let $A \to B$ be a local map of local rings which is a localization of an étale A-algebra. In particular $\mathfrak{m}_B$ is the unique prime containing $\mathfrak{m}_A B$. Then $B' = A' \otimes_A B$ is integral over B and the assumption that $A \to A'$ is local with purely inseparable residue field extension implies that B' is local (Algebra, Lemma 10.156.5). On the other hand, $A' \to B'$ is the localization of an étale ring map, hence B' is normal, see Algebra, Lemma 10.163.9. Thus B' is a (local) normal domain. Finally, we have

$$B/\mathfrak{p}B \subset B \otimes_A \kappa(\mathfrak{p}) = B' \otimes_{A'} (\text{fraction field of } A') \subset \text{fraction field of } B'$$

Hence $B/\mathfrak{p}B$ is a domain, which implies that B has a unique minimal prime (since by flatness of $A \to B$ these all have to lie over $\mathfrak{p}$). Since A^{sh} is a filtered colimit of the local rings B it follows that A^{sh} has a unique minimal prime. Namely, if $fg = 0$ in A^{sh} for some non-nilpotent elements f, g, then we can find a B as above containing both f and g which leads to a contradiction. $\square$

0C26 Definition 15.106.6. Let A be a local ring with henselization A^h and strict henselization A^{sh}. The number of branches of A is the number of minimal primes of A^h if finite and ∞ otherwise. The number of geometric branches of A is the number of minimal primes of A^{sh} if finite and ∞ otherwise.

We spell out the relationship with Definition 15.106.1.

0C37 Lemma 15.106.7. Let $(A, \mathfrak{m}, \kappa)$ be a local ring.

(1) If A has infinitely many minimal prime ideals, then the number of (geometric) branches of A is ∞.
(2) The number of branches of A is 1 if and only if A is unibranch.
(3) The number of geometric branches of A is 1 if and only if A is geometrically unibranch.

Assume A has finitely many minimal primes and let A' be the integral closure of A in the total ring of fractions of A_{red}. Then

(4) the number of branches of A is the number of maximal ideals $\mathfrak{m}'$ of A',

(5) to get the number of geometric branches of A we have to count each maximal ideal $\mathfrak{m}'$ of A' with multiplicity given by the separable degree of $\kappa(\mathfrak{m}')/\kappa$.

Proof. This lemma follows immediately from the definitions, Lemma 15.106.2, Lemma 15.106.4, and Fields, Lemma 9.14.8. □

0DQ1 Lemma 15.106.8. Let $A \to B$ be a local homomorphism of local rings which is the localization of a smooth ring map.

(1) The number of geometric branches of A is equal to the number of geometric branches of B.
(2) If $A \to B$ induces a purely inseparable extension of residue fields, then the number of branches of A is the number of branches of B.

Proof. We will use that smooth ring maps are flat (Algebra, Lemma 10.137.10), that localizations are flat (Algebra, Lemma 10.39.18), that compositions of flat ring maps are flat (Algebra, Lemma 10.39.4), that base change of a flat ring map is flat (Algebra, Lemma 10.39.7), that flat local homomorphisms are faithfully flat (Algebra, Lemma 10.39.17), that (strict) henselization is flat (Lemma 15.45.1), and Going down for flat ring maps (Algebra, Lemma 10.39.19).

Proof of (2). Let A^h, B^h be the henselizations of A, B. Then B^h is the henselization of $A^h \otimes_A B$ at the unique maximal ideal lying over $\mathfrak{m}_B$, see Algebra, Lemma 10.155.8. Thus we may and do assume A is henselian. Since $A \to B \to B^h$ is flat, every minimal prime of B^h lies over a minimal prime of A and since $A \to B^h$ is faithfully flat, every minimal prime of A does lie under a minimal prime of B^h; in both cases use going down for flat ring maps. Therefore it suffices to show that given a minimal prime $\mathfrak{p} \subset A$, there is at most one minimal prime of B^h lying over $\mathfrak{p}$. After replacing A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ we may assume that A is a domain; the A is still henselian by Algebra, Lemma 10.156.2. By Lemma 15.106.3 we see that the integral closure A' of A in its field of fractions is a local domain. Of course A' is a normal domain. By Algebra, Lemma 10.163.9 we see that $A' \otimes_A B^h$ is a normal ring (the lemma just gives it for $A' \otimes_A B$, to go up to $A' \otimes_A B^h$ use that B^h is a colimit of étale B-algebras and use Algebra, Lemma 10.37.17). By Algebra, Lemma 10.156.5 we see that $A' \otimes_A B^h$ is local (this is where we use the assumption on the residue fields of A and B). Hence $A' \otimes_A B^h$ is a local normal ring, hence a local domain. Since $B^h \subset A' \otimes_A B^h$ by flatness of $A \to B^h$ we conclude that B^h is a domain as desired.

Proof of (1). Let A^{sh}, B^{sh} be strict henselizations of A, B. Then B^{sh} is a strict henselization of $A^h \otimes_A B$ at a maximal ideal lying over $\mathfrak{m}_B$ and $\mathfrak{m}_{A^h}$, see Algebra, Lemma 10.155.12. Thus we may and do assume A is strictly henselian. Since $A \to B \to B^{sh}$ is flat, every minimal prime of B^{sh} lies over a minimal prime of A and since $A \to B^{sh}$ is faithfully flat, every minimal prime of A does lie under a minimal prime of B^{sh}; in both cases use going down for flat ring maps. Therefore it suffices to show that given a minimal prime $\mathfrak{p} \subset A$, there is at most one minimal prime of B^{sh} lying over $\mathfrak{p}$. After replacing A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ we may assume that A is a domain; then A is still strictly henselian by Algebra, Lemma 10.156.4. By Lemma 15.106.5 we see that the integral closure A' of A in its field of fractions is a local domain whose residue field is a purely inseparable extension of the residue field of A. Of course A' is a normal domain. By Algebra, Lemma 10.163.9 we see that $A' \otimes_A B^{sh}$ is a normal ring (the lemma just gives it for $A' \otimes_A B$, to go up to $A' \otimes_A B^{sh}$ use that B^{sh} is a colimit of étale B-algebras and use Algebra, Lemma 10.37.17). By Algebra, Lemma 10.156.5 we see that $A' \otimes_A B^{sh}$ is local (since $A \subset A'$ induces a purely inseparable residue field extension). Hence $A' \otimes_A B^{sh}$ is a local normal ring, hence a local domain. Since $B^{sh} \subset A' \otimes_A B^{sh}$ by flatness of $A \to B^{sh}$ we conclude that B^{sh} is a domain as desired. $\square$

15.107. Miscellaneous on branches

0GS4 Some results related to branches of local rings as defined in Section 15.106.

0GS5 Lemma 15.107.1. Let A and B be domains and let $A \to B$ be a ring map. Assume $A \to B$ has additionally at least one of the following properties

(1) it is the localization of an étale ring map,
(2) it is flat and the localization of an unramified ring map,
(3) it is flat and the localization of a quasi-finite ring map,
(4) it is flat and the localization of an integral ring map,
(5) it is flat and there are no nontrivial specializations between points of fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$,
(6) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ maps the generic point to the generic point and there are no nontrivial specializations between points of fibres, or
(7) exactly one point of $\mathrm{Spec}(B)$ is mapped to the generic point of $\mathrm{Spec}(A)$.

Then $A \cap J$ is nonzero for every nonzero ideal J of B.

Proof. Proof in case (7). Let K, resp. L be the fraction field of A, resp. B. By Algebra, Lemma 10.30.7 we see that the unique point of $\mathrm{Spec}(B)$ which maps to the generic point $(0) \in \mathrm{Spec}(A)$ is $(0) \in \mathrm{Spec}(B)$. We conclude that $B \otimes_A K$ is a ring with a unique prime ideal whose residue field is L (in fact it is equal to L but we do not need this). Choose $b \in J$ nonzero. Then b maps to a unit of L. Hence b maps to a unit of $B \otimes_A K$ (Algebra, Lemma 10.19.2). Since $B \otimes_A K = \mathrm{colim}_{f \in A \setminus \{0\}} B_f$ we see that b maps to a unit of B_f for some $f \in A$ nonzero. This means that $bb' = f^n$ for some $b' \in B$ and $n \geq 1$. Thus $f^n \in A \cap J$ as desired.

In the rest of the proof, we show that each of the other assumptions imply (7). Under assumptions (1) − (5), the ring map $A \to B$ is flat and hence $A \to B$ is injective (since flat local homomorphisms are faithfully flat by Algebra, Lemma 10.39.17). Hence the generic point of $\mathrm{Spec}(B)$ maps to the generic point of $\mathrm{Spec}(A)$. Now, if there are no nontrivial specializations between points of fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, then of course this generic point of $\mathrm{Spec}(B)$ has to be the unique point mapping to the generic point of $\mathrm{Spec}(A)$. So (6) implies (7). Finally, to finish we show that in cases (1) − (5) there are no nontrivial specializations between the points of fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Namely, see Algebra, Lemma 10.36.20 for the integral case, Algebra, Definition 10.122.3 for the quasi-finite case, and use that unramified and étale ring maps are quasi-finite (Algebra, Lemmas 10.151.6 and 10.143.6). $\square$

0GSC Lemma 15.107.2. Let $A \to B$ be a ring map. Let $\mathfrak{q} \subset B$ be a prime ideal lying over the prime $\mathfrak{p} \subset A$. Assume

(1) A is a domain,
(2) $A_\mathfrak{p}$ is geometrically unibranch,
(3) $A \to B$ is unramified at $\mathfrak{q}$, and

(4) $A_{\mathfrak p} \to B_{\mathfrak q}$ is injective.

Then there exists a $g \in B$, $g \notin \mathfrak q$ such that B_g is étale over A.

Proof. By Algebra, Proposition 10.152.1 after replacing B by a principal localization, we can find a standard étale ring map $A \to B'$ and a surjection $B' \to B$. Denote $\mathfrak q' \subset B'$ the inverse image of $\mathfrak q$. We will show that $B' \to B$ is injective after possibly replacing B' by a principal localization.

In this paragraph we reduce to the case that B' is a domain. Since A is a domain, the ring B' is reduced, see Algebra, Lemma 15.42.1. Let K be the fraction field of A. Then $B' \otimes_A K$ is étale over a field, hence is a finite product of fields, see Algebra, Lemma 10.143.4. Since $A \to B'$ is étale (hence flat) the minimal primes of B' are lie over $(0) \subset A$ (by going down for flat ring maps). We conclude that B' has finitely many minimal primes, say $\mathfrak r_1, \ldots, \mathfrak r_r \subset B'$. Since $A_{\mathfrak p}$ is geometrically unibranch and $A \to B'$ étale, the ring $B'_{\mathfrak q'}$ is a domain, see Lemmas 15.106.8 and 15.106.7. Hence $\mathfrak q' \supset \mathfrak r_i$ for exactly one $i = i_0$. Choose $g' \in B'$, $g' \notin \mathfrak r_{i_0}$ but $g' \in \mathfrak r_i$ for $i \neq i_0$, see Algebra, Lemma 10.15.2. After replacing B' and B by $B'_{g'}$ and $B_{g'}$ we obtain that B' is a domain.

Assume B' is a domain, in particular $B' \subset B'_{\mathfrak q'}$. If $B' \to B$ is not injective, then $J = \mathrm{Ker}(B'_{\mathfrak q'} \to B_{\mathfrak q})$ is nonzero. By Lemma 15.107.1 applied to $A_{\mathfrak p} \to B'_{\mathfrak q'}$ we find a nonzero element $a \in A_{\mathfrak p}$ mapping to zero in $B_{\mathfrak q}$ contradicting assumption (4). This finishes the proof. □

0GS6 Lemma 15.107.3. Let $(A, \mathfrak m)$ be a geometrically unibranch local domain. Let $A \to B$ be an injective local homomorphism of local rings, which is essentially of finite type. If $\mathfrak m B$ is the maximal ideal of B and the induced extension of residue fields is separable, then $A \to B$ is the localization of an étale ring map. Generalization of [Gro71, Expose I, Theorem 9.5 part (ii)]

Proof. We may write $B = C_{\mathfrak q}$ where $A \to C$ is a finite type ring map and $\mathfrak q \subset C$ is a prime ideal lying over $\mathfrak m$. By Algebra, Lemma 10.151.7 the ring map $A \to C$ is unramified at $\mathfrak q$. By Algebra, Proposition 10.152.1 after replacing C by a principal localization, we can find a standard étale ring map $A \to C'$ and a surjection $C' \to C$. Denote $\mathfrak q' \subset C'$ the inverse image of $\mathfrak q$ and set $B' = C'_{\mathfrak q'}$. Then $B' \to B$ is surjective. It suffices to show that $B' \to B$ is also injective.

Since A is a domain, the rings C' and B' are reduced, see Algebra, Lemma 15.42.1. Since A is geometrically unibranch, the ring B' is a domain, see by Lemmas 15.106.8 and 15.106.7. If $B' \to B$ is not injective, then $A \cap \mathrm{Ker}(B' \to B)$ is nonzero by Lemma 15.107.1 which contradicts the assumption that $A \to B$ is injective. □

06DU Lemma 15.107.4. Let k be an algebraically closed field. Let A, B be strictly henselian local k-algebras with residue field equal to k. Let C be the strict henselization of $A \otimes_k B$ at the maximal ideal $\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B$. Then the minimal primes of C correspond 1-to-1 to pairs of minimal primes of A and B.

Proof. First note that a minimal prime $\mathfrak r$ of C maps to a minimal prime $\mathfrak p$ in A and to a minimal prime $\mathfrak q$ of B because the ring maps $A \to C$ and $B \to C$ are flat (by going down for flat ring map Algebra, Lemma 10.39.19). Hence it suffices to show that the strict henselization of $(A/\mathfrak p \otimes_k B/\mathfrak q)_{\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B}$ has a unique minimal prime ideal. By Algebra, Lemma 10.156.4 the rings $A/\mathfrak p$, $B/\mathfrak q$ are strictly henselian. Hence we may assume that A and B are strictly henselian local domains and our goal is to show that C has a unique minimal prime. By Lemma 15.106.5

the integral closure A' of A in its fraction field is a normal local domain with residue field k. Similarly for the integral closure B' of B into its fraction field. By Algebra, Lemma 10.165.5 we see that $A' \otimes_k B'$ is a normal ring. Hence its localization

$$R = (A' \otimes_k B')_{\mathfrak m_{A'} \otimes_k B' + A' \otimes_k \mathfrak m_{B'}}$$

is a normal local domain. Note that $A \otimes_k B \to A' \otimes_k B'$ is integral (hence gong up holds – Algebra, Lemma 10.36.22) and that $\mathfrak m_{A'} \otimes_k B' + A' \otimes_k \mathfrak m_{B'}$ is the unique maximal ideal of $A' \otimes_k B'$ lying over $\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B$. Hence we see that

$$R = (A' \otimes_k B')_{\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B}$$

by Algebra, Lemma 10.41.11. It follows that

$$(A \otimes_k B)_{\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B} \longrightarrow R$$

is integral. We conclude that R is the integral closure of $(A \otimes_k B)_{\mathfrak m_A \otimes_k B + A \otimes_k \mathfrak m_B}$ in its fraction field, and by Lemma 15.106.5 once again we conclude that C has a unique prime ideal. $\square$

15.108. Branches of the completion

0C27 Let $(A, \mathfrak m)$ be a Noetherian local ring. Consider the maps $A \to A^h \to A^\wedge$. In general the map $A^h \to A^\wedge$ need not induce a bijection on minimal primes, see Examples, Section 110.19. In other words, the number of branches of A (as defined in Definition 15.106.6) may be different from the number of branches of $A^\wedge$. However, under some conditions the number of branches is the same, for example if the dimension of A is 1.

0C28 **Lemma 15.108.1.** Let $(A, \mathfrak m)$ be a Noetherian local ring.

(1) The map $A^h \to A^\wedge$ defines a surjective map from minimal primes of $A^\wedge$ to minimal primes of A^h.
(2) The number of branches of A is at most the number of branches of $A^\wedge$.
(3) The number of geometric branches of A is at most the number of geometric branches of $A^\wedge$.

Proof. By Lemma 15.45.3 the map $A^h \to A^\wedge$ is flat and injective. Combining going down (Algebra, Lemma 10.39.19) and Algebra, Lemma 10.30.5 we see that part (1) holds. Part (2) follows from this, Definition 15.106.6, and the fact that $A^\wedge$ is henselian (Algebra, Lemma 10.153.9). By Lemma 15.45.3 we have $(A^\wedge)^{sh} = A^{sh} \otimes_{A^h} A^\wedge$. Thus we can repeat the arguments above using the flat injective map $A^{sh} \to (A^\wedge)^{sh}$ to prove (3). $\square$

0C29 **Lemma 15.108.2.** Let $(A, \mathfrak m)$ be a Noetherian local ring. The number of branches of A is the same as the number of branches of $A^\wedge$ if and only if $\sqrt{\mathfrak q A^\wedge}$ is prime for every minimal prime $\mathfrak q \subset A^h$ of the henselization.

Proof. Follows from Lemma 15.108.1 and the fact that there are only a finite number of branches for both A and $A^\wedge$ by Algebra, Lemma 10.31.6 and the fact that A^h and $A^\wedge$ are Noetherian (Lemma 15.45.3). $\square$

A simple glueing lemma.

0C2A Lemma 15.108.3. Let A be a ring and let I be a finitely generated ideal. Let $A \to C$ be a ring map such that for all $f \in I$ the ring map $A_f \to C_f$ is localization at an idempotent. Then there exists a surjection $A \to C'$ such that $A_f \to (C \times C')_f$ is an isomorphism for all $f \in I$.

Proof. Choose generators $f_1, \ldots, f_r$ of I. Write

$$C_{f_i} = (A_{f_i})_{e_i}$$

for some idempotent $e_i \in A_{f_i}$. Write $e_i = a_i/f_i^n$ for some $a_i \in A$ and $n \geq 0$; we may use the same n for all $i = 1, \ldots, r$. After replacing a_i by $f_i^m a_i$ and n by $n+m$ for a suitable $m \gg 0$, we may assume $a_i^2 = f_i^n a_i$ for all i. Since e_i maps to 1 in $C_{f_i f_j} = (A_{f_i f_j})_{e_j} = A_{f_i f_j a_j}$ we see that

$$(f_i f_j a_j)^N (f_j^n a_i - f_i^n a_j) = 0$$

for some N (we can pick the same N for all pairs i, j). Using $a_j^2 = f_j^n a_j$ this gives

$$f_i^{N+n} f_j^{N+nN} a_j = f_i^N f_j^{N+n} a_i a_j^N$$

After increasing n to $n + N + nN$ and replacing a_i by $f_i^{N+nN} a_i$ we see that $f_i^n a_j$ is in the ideal of a_i for all pairs i, j. Let $C' = A/(a_1, \ldots, a_r)$. Then

$$C'_{f_i} = A_{f_i}/(a_i) = A_{f_i}/(e_i)$$

because a_j is in the ideal generated by a_i after inverting f_i. Since for an idempotent e of a ring B we have $B = B_e \times B/(e)$ we see that the conclusion of the lemma holds for f equal to one of $f_1, \ldots, f_r$. Using glueing of functions, in the form of Algebra, Lemma 10.23.2, we conclude that the result holds for all $f \in I$. Namely, for $f \in I$ the elements $f_1, \ldots, f_r$ generate the unit ideal in A_f so $A_f \to (C \times C')_f$ is an isomorphism if and only if this is the case after localizing at $f_1, \ldots, f_r$. □

Lemma 15.108.4 can be used to construct finite type extensions from given finite type extensions of the formal completion. We will generalize this lemma in Algebraization of Formal Spaces, Lemma 88.10.3.

0ALR Lemma 15.108.4. Let A be a Noetherian ring and I an ideal. Let B be a finite type A-algebra. Let $B^\wedge \to C$ be a surjective ring map with kernel J where $B^\wedge$ is the I-adic completion. If J/J^2 is annihilated by I^c for some $c \geq 0$, then C is isomorphic to the completion of a finite type A-algebra.

Proof. Let $f \in I$. Since $B^\wedge$ is Noetherian (Algebra, Lemma 10.97.6), we see that J is a finitely generated ideal. Hence we conclude from Algebra, Lemma 10.21.5 that

$$C_f = ((B^\wedge)_f)_e$$

for some idempotent $e \in (B^\wedge)_f$. By Lemma 15.108.3 we can find a surjection $B^\wedge \to C'$ such that $B^\wedge \to C \times C'$ becomes an isomorphism after inverting any $f \in I$. Observe that $C \times C'$ is a finite $B^\wedge$-algebra.

Choose generators $f_1, \ldots, f_r \in I$. Denote $\alpha_i : (C \times C')_{f_i} \to B_{f_i} \otimes_B B^\wedge$ the inverse of the isomorphism of $(B^\wedge)_{f_i}$-algebras we obtained above. Denote $\alpha_{ij} : (B_{f_i})_{f_j} \to (B_{f_j})_{f_i}$ the obvious B-algebra isomorphism. Consider the object

$$(C \times C', B_{f_i}, \alpha_i, \alpha_{ij})$$

of the category $\text{Glue}(B \to B^\wedge, f_1, \ldots, f_r)$ introduced in Remark 15.89.10. We omit the verification of conditions (1)(a) and (1)(b). Since $B \to B^\wedge$ is a flat map (Algebra, Lemma 10.97.2) inducing an isomorphism $B/IB \to B^\wedge/IB^\wedge$ we may apply Proposition 15.89.15 and Remark 15.89.19. We conclude that $C \times C'$ is isomorphic to $D \otimes_B B^\wedge$ for some finite B-algebra D. Then $D/ID \cong C/IC \times C'/IC'$. Let $\overline{e} \in D/ID$ be the idempotent corresponding to the factor C/IC. By Lemma 15.9.10 there exists an étale ring map $B \to B'$ which induces an isomorphism $B/IB \to B'/IB'$ such that $D' = D \otimes_B B'$ contains an idempotent e lifting $\overline{e}$. Since $C \times C'$ is I-adically complete the pair $(C \times C', IC \times IC')$ is henselian (Lemma 15.11.4). Thus we can factor the map $B \to C \times C'$ through B'. Doing so we may replace B by B' and D by D'. Then we find that $D = D_e \times D_{1-e} = D/(1-e) \times D/(e)$ is a product of finite type A-algebras and the completion of the first part is C and the completion of the second part is C'. □

0C2B Lemma 15.108.5. Let $(A, \mathfrak m)$ be a Noetherian local ring with henselization A^h. Let $\mathfrak q \subset A^\wedge$ be a minimal prime with $\dim(A^\wedge/\mathfrak q) = 1$. Then there exists a minimal prime $\mathfrak q^h$ of A^h such that $\mathfrak q = \sqrt{\mathfrak q^h A^\wedge}$.

Proof. Since the completion of A and A^h are the same, we may assume that A is henselian (Lemma 15.45.3). We will apply Lemma 15.108.4 to $A^\wedge \to A^\wedge/J$ where $J = \text{Ker}(A^\wedge \to (A^\wedge)_\mathfrak q)$. Since $\dim((A^\wedge)_\mathfrak q) = 0$ we see that $\mathfrak q^n \subset J$ for some n. Hence J/J^2 is annihilated by $\mathfrak q^n$. On the other hand $(J/J^2)_\mathfrak q = 0$ because $J_\mathfrak q = 0$. Hence $\mathfrak m$ is the only associated prime of J/J^2 and we find that a power of $\mathfrak m$ annihilates J/J^2. Thus the lemma applies and we find that $A^\wedge/J = C^\wedge$ for some finite type A-algebra C.

Then $C/\mathfrak m C = A/\mathfrak m$ because $A^\wedge/J$ has the same property. Hence $\mathfrak m_C = \mathfrak m C$ is a maximal ideal and $A \to C$ is unramified at $\mathfrak m_C$ (Algebra, Lemma 10.151.7). After replacing C by a principal localization we may assume that C is a quotient of an étale A-algebra B, see Algebra, Proposition 10.152.1. However, since the residue field extension of $A \to C_{\mathfrak m_C}$ is trivial and A is henselian, we conclude that $B = A$ again after a localization. Thus $C = A/I$ for some ideal $I \subset A$ and it follows that $J = IA^\wedge$ (because completion is exact in our situation by Algebra, Lemma 10.97.2) and $I = J \cap A$ (by flatness of $A \to A^\wedge$). Since $\mathfrak q^n \subset J \subset \mathfrak q$ we see that $\mathfrak p = \mathfrak q \cap A$ satisfies $\mathfrak p^n \subset I \subset \mathfrak p$. Then $\sqrt{\mathfrak p A^\wedge} = \mathfrak q$ and the proof is complete. □

0C2C Lemma 15.108.6. Let $(A, \mathfrak m)$ be a Noetherian local ring. The punctured spectrum of $A^\wedge$ is disconnected if and only if the punctured spectrum of A^h is disconnected.

Proof. Since the completion of A and A^h are the same, we may assume that A is henselian (Lemma 15.45.3).

Since $A \to A^\wedge$ is faithfully flat (see reference just given) the map from the punctured spectrum of $A^\wedge$ to the punctured spectrum of A is surjective (see Algebra, Lemma 10.39.16). Hence if the punctured spectrum of A is disconnected, then the same is true for $A^\wedge$.

Assume the punctured spectrum of $A^\wedge$ is disconnected. This means that

$$\text{Spec}(A^\wedge) \setminus \{\mathfrak m^\wedge\} = Z \amalg Z'$$

with Z and Z' closed. Let $\overline{Z}, \overline{Z}' \subset \text{Spec}(A^\wedge)$ be the closures. Say $\overline{Z} = V(J)$, $\overline{Z}' = V(J')$ for some ideals $J, J' \subset A^\wedge$. Then $V(J + J') = \{\mathfrak m^\wedge\}$ and $V(JJ') = \text{Spec}(A^\wedge)$. The first equality means that $\mathfrak m^\wedge = \sqrt{J + J'}$ which implies $(\mathfrak m^\wedge)^e \subset J + J'$ for some $e \geq 1$. The second equality implies every element of JJ' is nilpotent hence

$(JJ')^n = 0$ for some $n \geq 1$. Combined this means that J^n/J^{2n} is annihilated by J^n and $(J')^n$ and hence by $(\mathfrak{m}^\wedge)^{2en}$. Thus we may apply Lemma 15.108.4 to see that there is a finite type A-algebra C and an isomorphism $A^\wedge/J^n = C^\wedge$.

The rest of the proof is exactly the same as the second part of the proof of Lemma 15.108.5; of course that lemma is a special case of this one! We have $C/\mathfrak{m}C = A/\mathfrak{m}$ because $A^\wedge/J^n$ has the same property. Hence $\mathfrak{m}_C = \mathfrak{m}C$ is a maximal ideal and $A \to C$ is unramified at $\mathfrak{m}_C$ (Algebra, Lemma 10.151.7). After replacing C by a principal localization we may assume that C is a quotient of an étale A-algebra B, see Algebra, Proposition 10.152.1. However, since the residue field extension of $A \to C_{\mathfrak{m}_C}$ is trivial and A is henselian, we conclude that $B = A$ again after a localization. Thus $C = A/I$ for some ideal $I \subset A$ and it follows that $J^n = IA^\wedge$ (because completion is exact in our situation by Algebra, Lemma 10.97.2) and $I = J^n \cap A$ (by flatness of $A \to A^\wedge$). By symmetry $I' = (J')^n \cap A$ satisfies $(J')^n = I'A^\wedge$. Then $\mathfrak{m}^e \subset I + I'$ and $II' = 0$ and we conclude that $V(I)$ and $V(I')$ are closed subschemes which give the desired disjoint union decomposition of the punctured spectrum of A. □

0C2D Lemma 15.108.7. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Then the number of (geometric) branches of A and $A^\wedge$ is the same.

Proof. To see this for the number of branches, combine Lemmas 15.108.1, 15.108.2, and 15.108.5 and use that the dimension of $A^\wedge$ is one, see Lemma 15.43.1. To see this is true for the number of geometric branches we use the result for branches, the fact that the dimension does not change under strict henselization (Lemma 15.45.7), and the fact that $(A^{sh})^\wedge = ((A^\wedge)^{sh})^\wedge$ by Lemma 15.45.3. □

0C2E Lemma 15.108.8. Let $(A, \mathfrak{m})$ be a Noetherian local ring. If the formal fibres of A are geometrically normal (for example if A is excellent or quasi-excellent), then A is Nagata and the number of (geometric) branches of A and $A^\wedge$ is the same. [Bed13, Theorem 2.3]

Proof. Since a normal ring is reduced, we see that A is Nagata by Lemma 15.52.4. In the rest of the proof we will use Lemma 15.51.10, Proposition 15.51.5, and Lemma 15.51.4. This tells us that A is a P-ring where $P(k \to R) =$"R is geometrically normal over k" and the same is true for any (essentially of) finite type A-algebra.

Let $\mathfrak{q} \subset A$ be a minimal prime. Then $A^\wedge/\mathfrak{q}A^\wedge = (A/\mathfrak{q})^\wedge$ and $A^h/\mathfrak{q}A^h = (A/\mathfrak{q})^h$ (Algebra, Lemma 10.156.2). Hence the number of branches of A is the sum of the number of branches of the rings $A/\mathfrak{q}$ and similarly for $A^\wedge$. In this way we reduce to the case that A is a domain.

Assume A is a domain. Let A' be the integral closure of A in the fraction field K of A. Since A is Nagata, we see that $A \to A'$ is finite. Recall that the number of branches of A is the number of maximal ideals $\mathfrak{m}'$ of A' (Lemma 15.106.2). Also, recall that

$$(A')^\wedge = A' \otimes_A A^\wedge = \prod\nolimits_{\mathfrak{m}' \subset A'} (A'_{\mathfrak{m}'})^\wedge$$

by Algebra, Lemma 10.97.8. Because $A'_{\mathfrak{m}'}$ is a local ring whose formal fibres are geometrically normal, we see that $(A'_{\mathfrak{m}'})^\wedge$ is normal (Lemma 15.52.6). Hence the minimal primes of $A' \otimes_A A^\wedge$ are in 1-to-1 correspondence with the factors in the decomposition above. By flatness of $A \to A^\wedge$ we have

$$A^\wedge \subset A' \otimes_A A^\wedge \subset K \otimes_A A^\wedge$$

Since the left and the right ring have the same set of minimal primes, the same is true for the ring in the middle (small detail omitted) and this finishes the proof.

To see this is true for the number of geometric branches we use the result for branches, the fact that the formal fibres of A^{sh} are geometrically normal (Lemmas 15.51.10 and 15.51.8) and the fact that $(A^{sh})^\wedge = ((A^\wedge)^{sh})^\wedge$ by Lemma 15.45.3. □

15.109. Formally catenary rings

0AW1 In this section we prove a theorem of Ratliff [Rat71] that a Noetherian local ring is universally catenary if and only if it is formally catenary.

0AW2 Definition 15.109.1. A Noetherian local ring A is formally catenary if for every minimal prime $\mathfrak{p} \subset A$ the spectrum of $A^\wedge/\mathfrak{p}A^\wedge$ is equidimensional.

Let A be a Noetherian local ring which is formally catenary. By Ratliff's result (Proposition 15.109.5) we see that any quotient of A is also formally catenary (because the class of universally catenary rings is stable under quotients). We conclude that the spectrum of $A^\wedge/\mathfrak{p}A^\wedge$ is equidimensional for every prime ideal $\mathfrak{p}$ of A.

0AW3 Lemma 15.109.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring which is not formally catenary. Then A is not universally catenary.

Proof. By assumption there exists a minimal prime $\mathfrak{p} \subset A$ such that the spectrum of $A^\wedge/\mathfrak{p}A^\wedge$ is not equidimensional. After replacing A by $A/\mathfrak{p}$ we may assume that A is a domain and that the spectrum of $A^\wedge$ is not equidimensional. Let $\mathfrak{q}$ be a minimal prime of $A^\wedge$ such that $d = \dim(A^\wedge/\mathfrak{q})$ is minimal and hence $0 < d < \dim(A)$. We prove the lemma by induction on d.

The case $d = 1$. In this case $\dim(A^\wedge_\mathfrak{q}) = 0$. Hence $A^\wedge_\mathfrak{q}$ is Artinian local and we see that for some $n > 0$ the ideal $J = \mathfrak{q}^n$ maps to zero in $A^\wedge_\mathfrak{q}$. It follows that $\mathfrak{m}$ is the only associated prime of J/J^2, whence $\mathfrak{m}^m$ annihilates J/J^2 for some $m > 0$. Thus we can use Lemma 15.108.4 to find $A \to B$ of finite type such that $B^\wedge \cong A^\wedge/J$. It follows that $\mathfrak{m}_B = \sqrt{\mathfrak{m}B}$ is a maximal ideal with the same residue field as $\mathfrak{m}$ and $B^\wedge$ is the $\mathfrak{m}_B$-adic completion (Algebra, Lemma 10.97.7). Then

$$\dim(B_{\mathfrak{m}_B}) = \dim(B^\wedge) = 1 = d.$$

Since we have the factorization $A \to B \to A^\wedge/J$ the inverse image of $\mathfrak{q}/J$ is a prime $\mathfrak{q}' \subset \mathfrak{m}_B$ lying over (0) in A. Thus, if A were universally catenary, the dimension formula (Algebra, Lemma 10.113.1) would give

$$\begin{aligned}
\dim(B_{\mathfrak{m}_B}) &\geq \dim((B/\mathfrak{q}')_{\mathfrak{m}_B}) \\
&= \dim(A) + \mathrm{trdeg}_A(B/\mathfrak{q}') - \mathrm{trdeg}_{\kappa(\mathfrak{m})}(\kappa(\mathfrak{m}_B)) \\
&= \dim(A) + \mathrm{trdeg}_A(B/\mathfrak{q}')
\end{aligned}$$

This contradiction finishes the argument in case $d = 1$.

Assume $d > 1$. Let $Z \subset \mathrm{Spec}(A^\wedge)$ be the union of the irreducible components distinct from $V(\mathfrak{q})$. Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_m \subset A^\wedge$ be the prime ideals corresponding to irreducible components of $V(\mathfrak{q}) \cap Z$ of dimension > 0. Choose $f \in \mathfrak{m}$, $f \notin A \cap \mathfrak{r}_j$ using prime avoidance (Algebra, Lemma 10.15.2). Then $\dim(A/fA) = \dim(A) - 1$ and there is some irreducible component of $V(\mathfrak{q}, f)$ of dimension $d - 1$. Thus A/fA

is not formally catenary and the invariant d has decreased. By induction A/fA is not universally catenary, hence A is not universally catenary. $\square$

0AW4 Lemma 15.109.3. Let $A \to B$ be a flat local ring map of local Noetherian rings. Assume B is catenary and is $\mathrm{Spec}(B)$ equidimensional. Then

(1) $\mathrm{Spec}(B/\mathfrak{p}B)$ is equidimensional for all $\mathfrak{p} \subset A$ and
(2) A is catenary and $\mathrm{Spec}(A)$ is equidimensional.

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal. Let $\mathfrak{q} \subset B$ be a prime minimal over $\mathfrak{p}B$. Then $\mathfrak{q} \cap A = \mathfrak{p}$ by going down for $A \to B$ (Algebra, Lemma 10.39.19). Hence $A_\mathfrak{p} \to B_\mathfrak{q}$ is a flat local ring map with special fibre of dimension 0 and hence

$$\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q}) = \dim(B) - \dim(B/\mathfrak{q})$$

(Algebra, Lemma 10.112.7). The second equality because $\mathrm{Spec}(B)$ is equidimensional and B is catenary. Thus $\dim(B/\mathfrak{q})$ is independent of the choice of $\mathfrak{q}$ and we conclude that $\mathrm{Spec}(B/\mathfrak{p}B)$ is equidimensional of dimension $\dim(B) - \dim(A_\mathfrak{p})$. On the other hand, we have $\dim(B/\mathfrak{p}B) = \dim(A/\mathfrak{p}) + \dim(B/\mathfrak{m}_AB)$ and $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_AB)$ by flatness (see lemma cited above) and we get

$$\dim(A_\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p})$$

for all $\mathfrak{p}$ in A. Applying this to all minimal primes in A we see that A is equidimensional. If $\mathfrak{p} \subset \mathfrak{p}'$ is a strict inclusion with no primes in between, then we may apply the above to the prime $\mathfrak{p}'/\mathfrak{p}$ in $A/\mathfrak{p}$ because $A/\mathfrak{p} \to B/\mathfrak{p}B$ is flat and $\mathrm{Spec}(B/\mathfrak{p}B)$ is equidimensional, to get

$$1 = \dim((A/\mathfrak{p})_{\mathfrak{p}'}) = \dim(A/\mathfrak{p}) - \dim(A/\mathfrak{p}')$$

Thus $\mathfrak{p} \mapsto \dim(A/\mathfrak{p})$ is a dimension function and we conclude that A is catenary. $\square$

0AW5 Lemma 15.109.4. Let A be a formally catenary Noetherian local ring. Then A is universally catenary.

Proof. We may replace A by $A/\mathfrak{p}$ where $\mathfrak{p}$ is a minimal prime of A, see Algebra, Lemma 10.105.8. Thus we may assume that the spectrum of $A^\wedge$ is equidimensional. It suffices to show that every local ring essentially of finite type over A is catenary (see for example Algebra, Lemma 10.105.6). Hence it suffices to show that $A[x_1, \ldots, x_n]_\mathfrak{m}$ is catenary where $\mathfrak{m} \subset A[x_1, \ldots, x_n]$ is a maximal ideal lying over $\mathfrak{m}_A$, see Algebra, Lemma 10.54.5 (and Algebra, Lemmas 10.105.7 and 10.105.4). Let $\mathfrak{m}' \subset A^\wedge[x_1, \ldots, x_n]$ be the unique maximal ideal lying over $\mathfrak{m}$. Then

$$A[x_1, \ldots, x_n]_\mathfrak{m} \to A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$$

is local and flat (Algebra, Lemma 10.97.2). Hence it suffices to show that the ring on the right hand side catenary with equidimensional spectrum, see Lemma 15.109.3. It is catenary because complete local rings are universally catenary (Algebra, Remark 10.160.9). Pick any minimal prime $\mathfrak{q}$ of $A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$. Then $\mathfrak{q} = \mathfrak{p}A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}$ for some minimal prime $\mathfrak{p}$ of $A^\wedge$ (small detail omitted). Hence

$$\dim(A^\wedge[x_1, \ldots, x_n]_{\mathfrak{m}'}/\mathfrak{q}) = \dim(A^\wedge/\mathfrak{p}) + n = \dim(A^\wedge) + n$$

the first equality by Algebra, Lemma 10.112.7 and the second because the spectrum of $A^\wedge$ is equidimensional. This finishes the proof. $\square$

0AW6 Proposition 15.109.5 (Ratliff). A Noetherian local ring is universally catenary if and only if it is formally catenary. [Rat71]

Proof. Combine Lemmas 15.109.2 and 15.109.4. ☐

0C2F Lemma 15.109.6. Let $(A, \mathfrak{m})$ be a Noetherian local ring with geometrically normal formal fibres. Then [HRW04, Corollary 2.3]

(1) A^h is universally catenary, and
(2) if A is unibranch (for example normal), then A is universally catenary.

Proof. By Lemma 15.108.8 the number of branches of A and $A^\wedge$ are the same, hence Lemma 15.108.2 applies. Then for any minimal prime $\mathfrak{q} \subset A^h$ we see that $A^\wedge/\mathfrak{q}A^\wedge$ has a unique minimal prime. Thus A^h is formally catenary (by definition) and hence universally catenary by Proposition 15.109.5. If A is unibranch, then A^h has a unique minimal prime, hence $A^\wedge$ has a unique minimal prime, hence A is formally catenary and we conclude in the same way. ☐

15.110. Group actions and integral closure

0BRE This section is in some sense a continuation of Algebra, Section 10.38. More material of a similar kind can be found in Fundamental Groups, Section 58.12

0BRF Lemma 15.110.1. Let $\varphi : A \to B$ be a surjection of rings. Let G be a finite group of order n acting on $\varphi : A \to B$. If $b \in B^G$, then there exists a monic polynomial $P \in A^G[T]$ which maps to $(T - b)^n$ in $B^G[T]$.

Proof. Choose $a \in A$ lifting b and set $P = \prod_{\sigma \in G}(T - \sigma(a))$. ☐

09EG Lemma 15.110.2. Let R be a ring. Let G be a finite group acting on R. Let $I \subset R$ be an ideal such that $\sigma(I) \subset I$ for all $\sigma \in G$. Then $R^G/I^G \subset (R/I)^G$ is an integral extension of rings which induces a homeomorphism on spectra and purely inseparable extensions of residue fields.

Proof. Since $I^G = R^G \cap I$ it is clear that the map is injective. Lemma 15.110.1 shows that Algebra, Lemma 10.46.11 applies. ☐

0H34 Lemma 15.110.3. Let G be a finite group of order n acting on a ring R. Let $J \subset R^G$ be an ideal. For $x \in JR$ we have $\prod_{\sigma \in G}(T - \sigma(x)) = T^n + a_1 T^{n-1} + \ldots + a_n$ with $a_i \in J$.

Proof. Observe that the polynomial is indeed monic and has coefficients in R^G. We can write $x = f_1 b_1 + \ldots + f_m b_m$ with $f_j \in J$ and $b_j \in R$. Thus, arguing by induction on m, we may assume that $x = y - fb$ with $f \in J$, $b \in R$, and $y \in JR$ such that the result holds for y. Then we see that

$$\prod_{\sigma \in G}(T - \sigma(x)) = \prod_{\sigma \in G}(T - \sigma(y) + f\sigma(b)) = \prod_{\sigma \in G}(T - \sigma(y)) + \sum_{i=1,\ldots,n} f^i a_i$$

where we have

$$a_i = \sum_{S \subset G,\ |S| = i} \prod_{\sigma \in S} \sigma(b) \prod_{\sigma \not\in S}(T - \sigma(y))$$

A computation we omit shows that $a_i \in R^G$ (hint: the given expression is symmetric). Thus the polynomial of the statement of the lemma for x is congruent modulo J to the polynomial for y and this proves the induction step. ☐

0H35 Lemma 15.110.4. Let R be a ring. Let G be a finite group of order n acting on R. Let $J \subset R^G$ be an ideal. Then $R^G/J \to (R/JR)^G$ is ring map such that

(1) for $b \in (R/JR)^G$ there is a monic polynomial $P \in R^G/J[T]$ whose image in $(R/JR)^G[T]$ is $(T-b)^n$,
(2) for $a \in \mathrm{Ker}(R^G/J \to (R/JR)^G)$ we have $(T-a)^n = T^n$ in $R^G/J[T]$.

In particular, $R^G/J \to (R/JR)^G$ is an integral ring map which induces homeomorphisms on spectra and purely inseparable extensions of residue fields.

Proof. Part (1) follow from Lemma 15.110.1 with $I = JR$. If a is as in part (2), then a is the image of $x \in R^G \cap JR$. Hence $(T-x)^n = \prod_{\sigma \in G}(T - \sigma(x))$ is congruent to T^n modulo J by Lemma 15.110.3. This proves part (2). To see the final statement we may apply Algebra, Lemma 10.46.11. $\square$

0H36 Remark 15.110.5. In Lemma 15.110.4 we see that the map $R^G/J \to (R/JR)^G$ is an isomorphism if n is invertible in R.

0BRG Lemma 15.110.6. Let R be a ring. Let G be a finite group of order n acting on R. Let A be an R^G-algebra.

(1) for $b \in (A \otimes_{R^G} R)^G$ there exists a monic polynomial $P \in A[T]$ whose image in $(A \otimes_{R^G} R)^G[T]$ is $(T-b)^n$,
(2) for $a \in \mathrm{Ker}(A \to (A \otimes_{R^G} R)^G)$ we have $(T-a)^n = T^n$ in $A[T]$.

Proof. Choose a surjection $E \to A$ where E is a polynomial algebra over R^G. Then $(E \otimes_{R^G} R)^G = E$ because E is free as an R^G-module. Denote $J = \mathrm{Ker}(E \to A)$. Since tensor product is right exact we see that $A \otimes_{R^G} R$ is the quotient of $E \otimes_{R^G} R$ by the ideal generated by J. In this way we see that our lemma is a special case of Lemma 15.110.4. $\square$

0BRH Lemma 15.110.7. Let R be a ring. Let G be a finite group acting on R. Let $R^G \to A$ be a ring map. The map

$$A \to (A \otimes_{R^G} R)^G$$

is an isomorphism if $R^G \to A$ is flat. In general the map is integral, induces a homeomorphism on spectra, and induces purely inseparable residue field extensions.

Proof. To see the first statement consider the exact sequence $0 \to R^G \to R \to \bigoplus_{\sigma \in G} R$ where the second map sends x to $(\sigma(x) - x)_{\sigma \in G}$. Tensoring with A the sequence remains exact if $R^G \to A$ is flat. Thus A is the G-invariants in $(A \otimes_{R^G} R)^G$.

The second statement follows from Lemma 15.110.6 and Algebra, Lemma 10.46.11. $\square$

0BRI Lemma 15.110.8. Let G be a finite group acting on a ring R. For any two primes $\mathfrak{q}, \mathfrak{q}' \subset R$ lying over the same prime in R^G there exists a $\sigma \in G$ with $\sigma(\mathfrak{q}) = \mathfrak{q}'$.

Proof. The extension $R^G \subset R$ is integral because every $x \in R$ is a root of the monic polynomial $\prod_{\sigma \in G}(T - \sigma(x))$ in $R^G[T]$. Thus there are no inclusion relations among the primes lying over a given prime $\mathfrak{p}$ (Algebra, Lemma 10.36.20). If the lemma is wrong, then we can choose $x \in \mathfrak{q}'$, $x \notin \sigma(\mathfrak{q})$ for all $\sigma \in G$. See Algebra, Lemma 10.15.2. Then $y = \prod_{\sigma \in G} \sigma(x)$ is in R^G and in $\mathfrak{p} = R^G \cap \mathfrak{q}'$. On the other hand, $x \notin \sigma(\mathfrak{q})$ for all σ means $\sigma(x) \notin \mathfrak{q}$ for all σ. Hence $y \notin \mathfrak{q}$ as $\mathfrak{q}$ is a prime ideal. This is impossible as $y \in \mathfrak{p} \subset \mathfrak{q}$. $\square$

0BRJ Lemma 15.110.9. Let G be a finite group acting on a ring R. Let $\mathfrak q \subset R$ be a prime lying over $\mathfrak p \subset R^G$. Then $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is an algebraic normal extension and the map

$$D = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q\} \longrightarrow \text{Aut}(\kappa(\mathfrak q)/\kappa(\mathfrak p))$$

is surjective[15].

Proof. With $A = (R^G)_{\mathfrak p}$ and $B = A \otimes_{R^G} R$ we see that $A = B^G$ as localization is flat, see Lemma 15.110.7. Observe that $\mathfrak pA$ and $\mathfrak qB$ are prime ideals, D is the stabilizer of $\mathfrak qB$, and $\kappa(\mathfrak p) = \kappa(\mathfrak pA)$ and $\kappa(\mathfrak q) = \kappa(\mathfrak qB)$. Thus we may replace R by B and assume that $\mathfrak p$ is a maximal ideal. Since $R^G \subset R$ is an integral ring extension, we find that the maximal ideals of R are exactly the primes lying over $\mathfrak p$ (follows from Algebra, Lemmas 10.36.20 and 10.36.22). By Lemma 15.110.8 there are finitely many of them $\mathfrak q = \mathfrak q_1, \mathfrak q_2, \ldots, \mathfrak q_m$ and they form a single orbit for G. By the Chinese remainder theorem (Algebra, Lemma 10.15.4) the map $R \to \prod_{j=1,\ldots,m} R/\mathfrak q_j$ is surjective.

First we prove that the extension is normal. Pick an element $\alpha \in \kappa(\mathfrak q)$. We have to show that the minimal polynomial P of α over $\kappa(\mathfrak p)$ splits completely. By the above we can choose $a \in \mathfrak q_2 \cap \ldots \cap \mathfrak q_m$ mapping to α in $\kappa(\mathfrak q)$. Consider the polynomial $Q = \prod_{\sigma \in G}(T - \sigma(a))$ in $R^G[T]$. The image of Q in $R[T]$ splits completely into linear factors, hence the same is true for its image in $\kappa(\mathfrak q)[T]$. Since P divides the image of Q in $\kappa(\mathfrak p)[T]$ we conclude that P splits completely into linear factors over $\kappa(\mathfrak q)$ as desired.

Since $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is normal we may assume $\kappa(\mathfrak q) = \kappa_1 \otimes_{\kappa(\mathfrak p)} \kappa_2$ with $\kappa_1/\kappa(\mathfrak p)$ purely inseparable and $\kappa_2/\kappa(\mathfrak p)$ Galois, see Fields, Lemma 9.27.3. Pick $\alpha \in \kappa_2$ which generates κ_2 over $\kappa(\mathfrak p)$ if it is finite and a subfield of degree $> |G|$ if it is infinite (to get a contradiction). This is possible by Fields, Lemma 9.19.1. Pick a, P, and Q as in the previous paragraph. If $\alpha' \in \kappa_2$ is a Galois conjugate of α over $\kappa(\mathfrak p)$, then the fact that P divides the image of P in $\kappa(\mathfrak p)[T]$ shows there exists a $\sigma \in G$ such that $\sigma(a)$ maps to α'. By our choice of a (vanishing at other maximal ideals) this implies $\sigma \in D$ and that the image of σ in $\text{Aut}(\kappa(\mathfrak q)/\kappa(\mathfrak p))$ maps α to α'. Hence the surjectivity or the desired absurdity in case α has degree $> |G|$ over $\kappa(\mathfrak p)$. □

0BRK Lemma 15.110.10. Let A be a normal domain with fraction field K. Let L/K be a (possibly infinite) Galois extension. Let $G = \text{Gal}(L/K)$ and let B be the integral closure of A in L.

(1) For any two primes $\mathfrak q, \mathfrak q' \subset B$ lying over the same prime in A there exists a $\sigma \in G$ with $\sigma(\mathfrak q) = \mathfrak q'$.
(2) Let $\mathfrak q \subset B$ be a prime lying over $\mathfrak p \subset A$. Then $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is an algebraic normal extension and the map

$$D = \{\sigma \in G \mid \sigma(\mathfrak q) = \mathfrak q\} \longrightarrow \text{Aut}(\kappa(\mathfrak q)/\kappa(\mathfrak p))$$

is surjective.

Proof. Proof of (1). Consider pairs (M, σ) where $K \subset M \subset L$ is a subfield such that M/K is Galois, $\sigma \in \text{Gal}(M/K)$ with $\sigma(\mathfrak q \cap M) = \mathfrak q' \cap M$. We say $(M', \sigma') \geq (M, \sigma)$ if and only if $M \subset M'$ and $\sigma'|_M = \sigma$. Observe that (K, id_K) is such a pair as $A = K \cap B$ since A is a normal domain. The collection of these pairs satisfies the hypotheses of Zorn's lemma, hence there exists a maximal pair (M, σ). If $M \neq L$,

[15]Recall that we use the notation Gal only in the case of Galois extensions.

then we can find $M \subset M' \subset L$ with M'/M nontrivial and finite and M'/K Galois (Fields, Lemma 9.16.5). Choose $\sigma' \in \text{Gal}(M'/K)$ whose restriction to M is σ (Fields, Lemma 9.22.2). Then the primes $\sigma'(\mathfrak q \cap M')$ and $\mathfrak q' \cap M'$ restrict to the same prime of $B \cap M$. Since $B \cap M = (B \cap M')^{\text{Gal}(M'/M)}$ we can use Lemma 15.110.8 to find $\tau \in \text{Gal}(M'/M)$ with $\tau(\sigma'(\mathfrak q \cap M')) = \mathfrak q' \cap M'$. Hence $(M', \tau \circ \sigma') > (M, \sigma)$ contradicting the maximality of (M, σ).

Part (2) is proved in exactly the same manner as part (1). We write out the details. Pick $\overline{\sigma} \in \text{Aut}(\kappa(\mathfrak q)/\kappa(\mathfrak p))$. Consider pairs (M, σ) where $K \subset M \subset L$ is a subfield such that M/K is Galois, $\sigma \in \text{Gal}(M/K)$ with $\sigma(\mathfrak q \cap M) = \mathfrak q \cap M$ and

$$\begin{tikzcd} \kappa(\mathfrak q \cap M) \arrow[r] \arrow[d, "\sigma"'] & \kappa(\mathfrak q) \arrow[d, "\overline{\sigma}"'] \\ \kappa(\mathfrak q \cap M) \arrow[r] & \kappa(\mathfrak q) \end{tikzcd}$$

commutes. We say $(M', \sigma') \geq (M, \sigma)$ if and only if $M \subset M'$ and $\sigma'|_M = \sigma$. As above (K, id_K) is such a pair. The collection of these pairs satisfies the hypotheses of Zorn's lemma, hence there exists a maximal pair (M, σ). If $M \neq L$, then we can find $M \subset M' \subset L$ with M'/M finite and M'/K Galois (Fields, Lemma 9.16.5). Choose $\sigma' \in \text{Gal}(M'/K)$ whose restriction to M is σ (Fields, Lemma 9.22.2). Then the primes $\sigma'(\mathfrak q \cap M')$ and $\mathfrak q \cap M'$ restrict to the same prime of $B \cap M$. Adjusting the choice of σ' as in the first paragraph, we may assume that $\sigma'(\mathfrak q \cap M') = \mathfrak q \cap M'$. Then σ' and $\overline{\sigma}$ define maps $\kappa(\mathfrak q \cap M') \to \kappa(\mathfrak q)$ which agree on $\kappa(\mathfrak q \cap M)$. Since $B \cap M = (B \cap M')^{\text{Gal}(M'/M)}$ we can use Lemma 15.110.9 to find $\tau \in \text{Gal}(M'/M)$ with $\tau(\mathfrak q \cap M') = \mathfrak q \cap M'$ such that $\tau \circ \sigma$ and $\overline{\sigma}$ induce the same map on $\kappa(\mathfrak q \cap M')$. There is a small detail here in that the lemma first guarantees that $\kappa(\mathfrak q \cap M')/\kappa(\mathfrak q \cap M)$ is normal, which then tells us that the difference between the maps is an automorphism of this extension (Fields, Lemma 9.15.10), to which we can apply the lemma to get τ. Hence $(M', \tau \circ \sigma') > (M, \sigma)$ contradicting the maximality of (M, σ). $\square$

0BSX Lemma 15.110.11. Let A be a normal domain with fraction field K. Let $M/L/K$ be a tower of (possibly infinite) Galois extensions of K. Let $H = \text{Gal}(M/K)$ and $G = \text{Gal}(L/K)$ and let C and B be the integral closure of A in M and L. Let $\mathfrak r \subset C$ and $\mathfrak q = B \cap \mathfrak r$. Set $D_{\mathfrak r} = \{\tau \in H \mid \tau(\mathfrak r) = \mathfrak r\}$ and $I_{\mathfrak r} = \{\tau \in D_{\mathfrak r} \mid \tau \bmod \mathfrak r = \text{id}_{\kappa(\mathfrak r)}\}$ and similarly for $D_{\mathfrak q}$ and $I_{\mathfrak q}$. Under the map $H \to G$ the induced maps $D_{\mathfrak r} \to D_{\mathfrak q}$ and $I_{\mathfrak r} \to I_{\mathfrak q}$ are surjective.

Proof. Let $\sigma \in D_{\mathfrak q}$. Pick $\tau \in H$ mapping to σ. This is possible by Fields, Lemma 9.22.2. Then $\tau(\mathfrak r)$ and $\mathfrak r$ both lie over $\mathfrak q$. Hence by Lemma 15.110.10 there exists a $\sigma' \in \text{Gal}(M/L)$ with $\sigma'(\tau(\mathfrak r)) = \mathfrak r$. Hence $\sigma'\tau \in D_{\mathfrak r}$ maps to σ. The case of inertia groups is proved in exactly the same way using surjectivity onto automorphism groups. $\square$

15.111. Extensions of discrete valuation rings

0EXQ In this section and the next few we use the following definitions.

09E4 Definition 15.111.1. We say that $A \to B$ or $A \subset B$ is an extension of discrete valuation rings if A and B are discrete valuation rings and $A \to B$ is injective and local. In particular, if π_A and π_B are uniformizers of A and B, then $\pi_A = u\pi_B^e$ for some $e \geq 1$ and unit u of B. The integer e does not depend on the choice of the uniformizers as it is also the unique integer ≥ 1 such that

$$\mathfrak{m}_A B = \mathfrak{m}_B^e$$

The integer e is called the ramification index of B over A. We say that B is weakly unramified over A if $e = 1$. If the extension of residue fields $\kappa_A = A/\mathfrak{m}_A \subset \kappa_B = B/\mathfrak{m}_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the residual degree or residue degree of the extension $A \subset B$.

Note that we do not require the extension of fraction fields to be finite.

09E5 **Lemma 15.111.2.** Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the extension L/K is finite, then the residue field extension is finite and we have $ef \leq [L : K]$.

Proof. Finiteness of the residue field extension is Algebra, Lemma 10.119.10. The inequality follows from Algebra, Lemmas 10.119.9 and 10.52.12. $\square$

0BRL **Lemma 15.111.3.** Let $A \subset B \subset C$ be extensions of discrete valuation rings. Then the ramification indices of B/A and C/B multiply to give the ramification index of C/A. In a formula $e_{C/A} = e_{B/A} e_{C/B}$. Similarly for the residual degrees in case they are finite.

Proof. This is immediate from the definitions and Fields, Lemma 9.7.7. $\square$

09E6 **Lemma 15.111.4.** Let $A \subset B$ be an extension of discrete valuation rings inducing the field extension $K \subset L$. If the characteristic of K is $p > 0$ and L is purely inseparable over K, then the ramification index e is a power of p.

Proof. Write $\pi_A = u\pi_B^e$ for some $u \in B^*$. On the other hand, we have $\pi_B^q \in K$ for some p-power q. Write $\pi_B^q = v\pi_A^k$ for some $v \in A^*$ and $k \in \mathbf{Z}$. Then $\pi_A^q = u^q \pi_B^{qe} = u^q v^e \pi_A^{ke}$. Taking valuations in B we conclude that $ke = q$. $\square$

In the following lemma we discuss what it means for an extension $A \subset B$ of discrete valuation rings to be "unramified", i.e., have ramification index 1 and separable (possibly nonalgebraic) extension of residue fields. However, we cannot use the term "unramified" itself because there already exists a notion of an unramified ring map, see Algebra, Section 10.151.

09E7 **Lemma 15.111.5.** Let $A \subset B$ be an extension of discrete valuation rings. The following are equivalent

(1) $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology, and
(2) $A \to B$ is weakly unramified and κ_B/κ_A is a separable field extension.

Proof. This follows from Proposition 15.40.5 and Algebra, Proposition 10.158.9. $\square$

09E8 **Remark 15.111.6.** Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable field extension. Let $B \subset L$ be the integral closure of A in L. Picture:

$$\begin{tikzcd} B \arrow[r] & L \\ A \arrow[u] \arrow[r] & K \arrow[u] \end{tikzcd}$$

By Algebra, Lemma 10.161.8 the ring extension $A \subset B$ is finite, hence B is Noetherian. By Algebra, Lemma 10.112.4 the dimension of B is 1, hence B is a Dedekind domain, see Algebra, Lemma 10.120.17. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of B (i.e., the primes lying over $\mathfrak{m}_A$). We obtain extensions of discrete valuation rings

$$A \subset B_{\mathfrak{m}_i}$$

and hence ramification indices e_i and residue degrees f_i. We have

$$[L : K] = \sum\nolimits_{i=1,\ldots,n} e_i f_i$$

by Algebra, Lemma 10.121.8 applied to a uniformizer in A. We observe that $n = 1$ if A is henselian (by Algebra, Lemma 10.153.4), e.g. if A is complete.

09E9 Definition 15.111.7. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite separable extension. With B and $\mathfrak{m}_i$, $i = 1, \ldots, n$ as in Remark 15.111.6 we say the extension L/K is

(1) unramified with respect to A if $e_i = 1$ and the extension $\kappa(\mathfrak{m}_i)/\kappa_A$ is separable for all i,
(2) tamely ramified with respect to A if either the characteristic of κ_A is 0 or the characteristic of κ_A is $p > 0$, the field extensions $\kappa(\mathfrak{m}_i)/\kappa_A$ are separable, and the ramification indices e_i are prime to p, and
(3) totally ramified with respect to A if $n = 1$ and the residue field extension $\kappa(\mathfrak{m}_1)/\kappa_A$ is trivial.

If the discrete valuation ring A is clear from context, then we sometimes say L/K is unramified, totally ramified, or tamely ramified for short.

For unramified extensions we have the following basic lemma.

0EXR Lemma 15.111.8. Let A be a discrete valuation ring with fraction field K.

(1) If $M/L/K$ are finite separable extensions and M is unramified with respect to A, then L is unramified with respect to A.
(2) If L/K is a finite separable extension which is unramified with respect to A, then there exists a Galois extension M/K containing L which is unramified with respect to A.
(3) If L_1/K, L_2/K are finite separable extensions which are unramified with respect to A, then there exists a a finite separable extension L/K which is unramified with respect to A containing L_1 and L_2.

Proof. We will use the results of the discussion in Remark 15.111.6 without further mention.

Proof of (1). Let $C/B/A$ be the integral closures of A in $M/L/K$. Since C is a finite ring extension of B, we see that $\text{Spec}(C) \to \text{Spec}(B)$ is surjective. Hence for ever maximal ideal $\mathfrak{m} \subset B$ there is a maximal ideal $\mathfrak{m}' \subset C$ lying over $\mathfrak{m}$. By the multiplicativity of ramification indices (Lemma 15.111.3) and the assumption, we conclude that the ramification index of $B_\mathfrak{m}$ over A is 1. Since $\kappa(\mathfrak{m}')/\kappa_A$ is finite separable, the same is true for $\kappa(\mathfrak{m})/\kappa_A$.

Proof of (2). Let M be the normal closure of L over K, see Fields, Definition 9.16.4. Then M/K is Galois by Fields, Lemma 9.21.5. On the other hand, there is a surjection

$$L \otimes_K \ldots \otimes_K L \longrightarrow M$$

of K-algebras, see Fields, Lemma 9.16.6. Let B be the integral closure of A in L as in Remark 15.111.6. The condition that L is unramified with respect to A exactly means that $A \to B$ is an étale ring map, see Algebra, Lemma 10.143.7. By permanence properties of étale ring maps we see that

$$B \otimes_A \ldots \otimes_A B$$

is étale over A, see Algebra, Lemma 10.143.3. Hence the displayed ring is a product of Dedekind domains, see Lemma 15.44.4. We conclude that M is the fraction field of a Dedekind domain finite étale over A. This means that M is unramified with respect to A as desired.

Proof of (3). Let $B_i \subset L_i$ be the integral closure of A. Argue in the same manner as above to show that $B_1 \otimes_A B_2$ is finite étale over A. Details omitted. □

0EXS Lemma 15.111.9. Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be finite separable extensions. Let B be the integral closure of A in L. If L/K is unramified with respect to A and M/L is unramified with respect to $B_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of B, then M/K is unramified with respect to A.

Proof. Let C be the integral closure of A in M. Every maximal ideal $\mathfrak{m}'$ of C lies over a maximal ideal $\mathfrak{m}$ of B. Then the lemma follows from the multiplicativity of ramification indices (Lemma 15.111.3) and the fact that we have the tower $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})/\kappa_A$ of finite extensions of fields. □

15.112. Galois extensions and ramification

09E3 In the case of Galois extensions, we can elaborate on the discussion in Section 15.111.

09EA Lemma 15.112.1. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension with Galois group G. Then G acts on the ring B of Remark 15.111.6 and acts transitively on the set of maximal ideals of B.

Proof. Observe that $A = B^G$ as A is integrally closed in K and $K = L^G$. Hence this lemma is a special case of Lemma 15.110.8. □

09EB Lemma 15.112.2. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Then there are $e \geq 1$ and $f \geq 1$ such that $e_i = e$ and $f_i = f$ for all i (notation as in Remark 15.111.6). In particular $[L : K] = nef$.

Proof. Immediate consequence of Lemma 15.112.1 and the definitions. □

09EC Definition 15.112.3. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{m} \subset B$ be a maximal ideal.

(1) The decomposition group of $\mathfrak{m}$ is the subgroup $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$.
(2) The inertia group of $\mathfrak{m}$ is the kernel I of the map $D \to \text{Aut}(\kappa(\mathfrak{m})/\kappa_A)$.

Note that the field $\kappa(\mathfrak{m})$ may be inseparable over κ_A. In particular the field extension $\kappa(\mathfrak{m})/\kappa_A$ need not be Galois. If κ_A is perfect, then it is.

09ED Lemma 15.112.4. Let A be a discrete valuation ring with fraction field K and residue field κ. Let L/K be a finite Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{m}$ be a maximal ideal of B. Then

(1) the field extension $\kappa(\mathfrak m)/\kappa$ is normal, and
(2) $D \to \text{Aut}(\kappa(\mathfrak m)/\kappa)$ is surjective.

If for some (equivalently all) maximal ideal(s) $\mathfrak m \subset B$ the field extension $\kappa(\mathfrak m)/\kappa$ is separable, then

(3) $\kappa(\mathfrak m)/\kappa$ is Galois, and
(4) $D \to \text{Gal}(\kappa(\mathfrak m)/\kappa)$ is surjective.

Here $D \subset G$ is the decomposition group of $\mathfrak m$.

Proof. Observe that $A = B^G$ as A is integrally closed in K and $K = L^G$. Thus parts (1) and (2) follow from Lemma 15.110.9. The "equivalently all" part of the lemma follows from Lemma 15.112.1. Assume $\kappa(\mathfrak m)/\kappa$ is separable. Then parts (3) and (4) follow immediately from (1) and (2). $\square$

09EE Lemma 15.112.5. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak m \subset B$ be a maximal ideal. The inertia group I of $\mathfrak m$ sits in a canonical exact sequence

$$1 \to P \to I \to I_t \to 1$$

such that

(1) $P = \{\sigma \in D \mid \sigma|_{B/\mathfrak m^2} = \text{id}_{B/\mathfrak m^2}\}$ where D is the decomposition group,
(2) P is a normal subgroup of D,
(3) P is a p-group if the characteristic of κ_A is $p > 0$ and $P = \{1\}$ if the characteristic of κ_A is zero,
(4) I_t is cyclic of order the prime to p part of the integer e, and
(5) there is a canonical isomorphism $\theta : I_t \to \mu_e(\kappa(\mathfrak m))$.

Here e is the integer of Lemma 15.112.2.

Proof. Recall that $|G| = [L : K] = nef$, see Lemma 15.112.2. Since G acts transitively on the set $\{\mathfrak m_1, \ldots, \mathfrak m_n\}$ of maximal ideals of B (Lemma 15.112.1) and since D is the stabilizer of an element we see that $|D| = ef$. By Lemma 15.112.4 we have

$$ef = |D| = |I| \cdot |\text{Aut}(\kappa(\mathfrak m)/\kappa)|$$

where κ is the residue field of A. As $\kappa(\mathfrak m)$ is normal over κ the order of $\text{Aut}(\kappa(\mathfrak m)/\kappa)$ differs from f by a power of p (see Fields, Lemma 9.15.9 and discussion following Fields, Definition 9.14.7). Hence the prime to p part of $|I|$ is equal to the prime to p part of e.

Set $C = B_{\mathfrak m}$. Then I acts on C over A and trivially on the residue field of C. Let $\pi_A \in A$ and $\pi_C \in C$ be uniformizers. Write $\pi_A = u\pi_C^e$ for some unit u in C. For $\sigma \in I$ write $\sigma(\pi_C) = \theta_\sigma \pi_C$ for some unit θ_σ in C. Then we have

$$\pi_A = \sigma(\pi_A) = \sigma(u)(\theta_\sigma\pi_C)^e = \sigma(u)\theta_\sigma^e\pi_C^e = \frac{\sigma(u)}{u}\theta_\sigma^e\pi_A$$

Since $\sigma(u) \equiv u \bmod \mathfrak m_C$ as $\sigma \in I$ we see that the image $\overline{\theta}_\sigma$ of θ_σ in $\kappa_C = \kappa(\mathfrak m)$ is an eth root of unity. We obtain a map

0BU3 (15.112.5.1) $$\theta : I \longrightarrow \mu_e(\kappa(\mathfrak m)), \quad \sigma \mapsto \overline{\theta}_\sigma$$

We claim that θ is a homomorphism of groups and independent of the choice of uniformizer π_C. Namely, if τ is a second element of I, then $\tau(\sigma(\pi_C)) = \tau(\theta_\sigma\pi_C) = \tau(\theta_\sigma)\theta_\tau\pi_C$, hence $\theta_{\tau\sigma} = \tau(\theta_\sigma)\theta_\tau$ and since $\tau \in I$ we conclude that $\overline{\theta}_{\tau\sigma} = \overline{\theta}_\sigma\overline{\theta}_\tau$. If π'_C is a second uniformizer, then we see that $\pi'_C = w\pi_C$ for some unit w of C and

$\sigma(\pi'_C) = w^{-1}\sigma(w)\theta_\sigma\pi'_C$, hence $\theta'_\sigma = w^{-1}\sigma(w)\theta_\sigma$, hence θ'_σ and θ_σ map to the same element of the residue field as before.

Since $\kappa(\mathfrak m)$ has characteristic p, the group $\mu_e(\kappa(\mathfrak m))$ is cyclic of order at most the prime to p part of e (see Fields, Section 9.17).

Let $P = \text{Ker}(\theta)$. The elements of P are exactly the elements of D acting trivially on $C/\pi_C^2 C \cong B/\mathfrak m^2$. Thus (a) is true. This implies (b) as P is the kernel of the map $D \to \text{Aut}(B/\mathfrak m^2)$. If we can prove (c), then parts (d) and (e) will follow as I_t will be isomorphic to $\mu_e(\kappa(\mathfrak m))$ as the arguments above show that $|I_t| \geq |\mu_e(\kappa(\mathfrak m))|$.

Thus it suffices to prove that the kernel P of θ is a p-group. Let σ be a nontrivial element of the kernel. Then $\sigma - \text{id}$ sends $\mathfrak m_C^i$ into $\mathfrak m_C^{i+1}$ for all i. Let m be the order of σ. Pick $c \in C$ such that $\sigma(c) \neq c$. Then $\sigma(c) - c \in \mathfrak m_C^i$, $\sigma(c) - c \notin \mathfrak m_C^{i+1}$ for some i and we have

$$\begin{aligned} 0 &= \sigma^m(c) - c \\ &= \sigma^m(c) - \sigma^{m-1}(c) + \ldots + \sigma(c) - c \\ &= \sum\nolimits_{j=0,\ldots,m-1} \sigma^j(\sigma(c) - c) \\ &\equiv m(\sigma(c) - c) \bmod \mathfrak m_C^{i+1} \end{aligned}$$

It follows that $p|m$ (or $m = 0$ if $p = 1$). Thus every element of the kernel of θ has order divisible by p, i.e., $\text{Ker}(\theta)$ is a p-group. □

0BU4 Definition 15.112.6. With assumptions and notation as in Lemma 15.112.5.

(1) The wild inertia group of $\mathfrak m$ is the subgroup P.
(2) The tame inertia group of $\mathfrak m$ is the quotient $I \to I_t$.

We denote $\theta : I \to \mu_e(\kappa(\mathfrak m))$ the surjective map (15.112.5.1) whose kernel is P and which induces the isomorphism $I_t \to \mu_e(\kappa(\mathfrak m))$.

0BU5 Lemma 15.112.7. With assumptions and notation as in Lemma 15.112.5. The inertia character $\theta : I \to \mu_e(\kappa(\mathfrak m))$ satisfies the following property

$$\theta(\tau\sigma\tau^{-1}) = \tau(\theta(\sigma))$$

for $\tau \in D$ and $\sigma \in I$.

Proof. The formula makes sense as I is a normal subgroup of D and as τ acts on $\kappa(\mathfrak m)$ via the map $D \to \text{Aut}(\kappa(\mathfrak m))$ discussed in Lemma 15.112.4 for example. Recall the construction of θ. Choose a uniformizer π of $B_{\mathfrak m}$ and for $\sigma \in I$ write $\sigma(\pi) = \theta_\sigma\pi$. Then $\theta(\sigma)$ is the image $\overline{\theta}_\sigma$ of θ_σ in the residue field. For any $\tau \in D$ we can write $\tau(\pi) = \theta_\tau\pi$ for some unit θ_τ. Then $\theta_{\tau^{-1}} = \tau^{-1}(\theta_\tau^{-1})$. We compute

$$\begin{aligned} \theta_{\tau\sigma\tau^{-1}} &= \tau(\sigma(\tau^{-1}(\pi)))/\pi \\ &= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1})\pi))/\pi \\ &= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1}))\theta_\sigma\pi)/\pi \\ &= \tau(\sigma(\tau^{-1}(\theta_\tau^{-1})))\tau(\theta_\sigma)\theta_\tau \end{aligned}$$

However, since σ acts trivially modulo π we see that the product $\tau(\sigma(\tau^{-1}(\theta_\tau^{-1})))\theta_\tau$ maps to 1 in the residue field. This proves the lemma. □

We will generalize the following lemma in Fundamental Groups, Lemma 58.12.5.

09EH Lemma 15.112.8. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Let $\mathfrak{m} \subset B$ be a maximal ideal of the integral closure of A in L. Let $I \subset G$ be the inertia group of $\mathfrak{m}$. Then B^I is the integral closure of A in L^I and $A \to (B^I)_{B^I \cap \mathfrak{m}}$ is étale.

Proof. Write $B' = B^I$. It follows from the definitions that $B' = B^I$ is the integral closure of A in L^I. Write $\mathfrak{m}' = B^I \cap \mathfrak{m} = B' \cap \mathfrak{m} \subset B'$. By Lemma 15.110.8 the maximal ideal $\mathfrak{m}$ is the unique prime ideal of B lying over $\mathfrak{m}'$. As I acts trivially on $\kappa(\mathfrak{m})$ we see from Lemma 15.110.2 that the extension $\kappa(\mathfrak{m})/\kappa(\mathfrak{m}')$ is purely inseparable (perhaps an easier alternative is to apply the result of Lemma 15.110.9). Since D/I acts faithfully on $\kappa(\mathfrak{m}')$, we conclude that D/I acts faithfully on $\kappa(\mathfrak{m})$. Of course the elements of the residue field κ of A are fixed by this action. By Galois theory we see that $[\kappa(\mathfrak{m}') : \kappa] \geq |D/I|$, see Fields, Lemma 9.21.6.

Let π be the uniformizer of A. Since $\text{Norm}_{L/K}(\pi) = \pi^{[L:K]}$ we see from Algebra, Lemma 10.121.8 that

$$|G| = [L : K] = [L : K] \text{ ord}_A(\pi) = |G/D| \; [\kappa(\mathfrak{m}) : \kappa] \text{ ord}_{B_\mathfrak{m}}(\pi)$$

as there are $n = |G/D|$ maximal ideals of B which are all conjugate under G, see Remark 15.111.6 and Lemma 15.112.1. Applying the same reasoning to the finite extension the finite extension L/L^I of degree $|I|$ we find

$$|I| \text{ ord}_{B'_{\mathfrak{m}'}}(\pi) = [\kappa(\mathfrak{m}) : \kappa(\mathfrak{m}')] \text{ ord}_{B_\mathfrak{m}}(\pi)$$

We conclude that

$$\text{ord}_{B'_{\mathfrak{m}'}}(\pi) = \frac{|D/I|}{[\kappa(\mathfrak{m}') : \kappa]}$$

Since the left hand side is a positive integer and since the right hand side is ≤ 1 by the above, we conclude that we have equality, $\text{ord}_{B'_{\mathfrak{m}'}}(\pi) = 1$ and $\kappa(\mathfrak{m}')/\kappa$ has degree $|D/I|$. Thus $\pi B'_{\mathfrak{m}'} = \mathfrak{m}' B'_\mathfrak{m}$ and $\kappa(\mathfrak{m}')$ is Galois over κ with Galois group D/I, in particular separable, see Fields, Lemma 9.21.2. By Algebra, Lemma 10.143.7 we find that $A \to B'_{\mathfrak{m}'}$ is étale as desired. ☐

0BU6 Remark 15.112.9. Let A be a discrete valuation ring with fraction field K. Let L/K be a finite Galois extension. Let $\mathfrak{m} \subset B$ be a maximal ideal of the integral closure of A in L. Let

$$P \subset I \subset D \subset G$$

be the wild inertia, inertia, decomposition group of $\mathfrak{m}$. Consider the diagram

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, dash] \arrow[d, dash] & \mathfrak{m}^P \arrow[r, dash] \arrow[d, dash] & \mathfrak{m}^I \arrow[r, dash] \arrow[d, dash] & \mathfrak{m}^D \arrow[r, dash] \arrow[d, dash] & A \cap \mathfrak{m} \arrow[d, dash] \\
B & B^P \arrow[l] & B^I \arrow[l] & B^D \arrow[l] & A \arrow[l]
\end{tikzcd}$$

Observe that B^P, B^I, B^D are the integral closures of A in the fields L^P, L^I, L^D. Thus we also see that B^P is the integral closure of B^I in L^P and so on. Observe that $\mathfrak{m}^P = \mathfrak{m} \cap B^P$, $\mathfrak{m}^I = \mathfrak{m} \cap B^I$, and $\mathfrak{m}^D = \mathfrak{m} \cap B^D$. Hence the top line of the diagram corresponds to the images of $\mathfrak{m} \in \text{Spec}(B)$ under the induced maps of spectra. Having said all of this we have the following

(1) the extension L^I/L^D is Galois with group D/I,
(2) the extension L^P/L^I is Galois with group $I_t = I/P$,
(3) the extension L^P/L^D is Galois with group D/P,

(4) $\mathfrak{m}^I$ is the unique prime of B^I lying over $\mathfrak{m}^D$,
(5) $\mathfrak{m}^P$ is the unique prime of B^P lying over $\mathfrak{m}^I$,
(6) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^P$,
(7) $\mathfrak{m}^P$ is the unique prime of B^P lying over $\mathfrak{m}^D$,
(8) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^I$,
(9) $\mathfrak{m}$ is the unique prime of B lying over $\mathfrak{m}^D$,
(10) $A \to B^D_{\mathfrak{m}^D}$ is étale and induces a trivial residue field extension,
(11) $B^D_{\mathfrak{m}^D} \to B^I_{\mathfrak{m}^I}$ is étale and induces a Galois extension of residue fields with Galois group D/I,
(12) $A \to B^I_{\mathfrak{m}^I}$ is étale,
(13) $B^I_{\mathfrak{m}^I} \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a trivial residue field extension,
(14) $B^D_{\mathfrak{m}^D} \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a separable residue field extension,
(15) $A \to B^P_{\mathfrak{m}^P}$ has ramification index $|I/P|$ prime to p and induces a separable residue field extension.

Statements (1), (2), and (3) are immediate from Galois theory (Fields, Section 9.21) and Lemma 15.112.5. Statements (4) – (9) are clear from Lemma 15.112.1. Part (12) is Lemma 15.112.8. Since we have the factorization $A \to B^D_{\mathfrak{m}^D} \to B^I_{\mathfrak{m}^I}$ we obtain the étaleness in (10) and (11) as a consequence. The residue field extension in (10) must be trivial because it is separable and D/I maps onto $\text{Aut}(\kappa(\mathfrak{m})/\kappa_A)$ as shown in Lemma 15.112.4. The same argument provides the proof of the statement on residue fields in (11). To see (13), (14), and (15) it suffices to prove (13). By the above, the extension L^P/L^I is Galois with a cyclic Galois group of order prime to p, the prime $\mathfrak{m}^P$ is the unique prime lying over $\mathfrak{m}^I$ and the action of I/P on the residue field is trivial. Thus we can apply Lemma 15.112.5 to this extension and the discrete valuation ring $B^I_{\mathfrak{m}^I}$ to see that (13) holds.

0BU7 Lemma 15.112.10. Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be a tower with M/K and L/K finite Galois. Let C, B be the integral closure of A in M, L. Let $\mathfrak{m}' \subset C$ be a maximal ideal and set $\mathfrak{m} = \mathfrak{m}' \cap B$. Let

$$P \subset I \subset D \subset \text{Gal}(L/K) \quad \text{and} \quad P' \subset I' \subset D' \subset \text{Gal}(M/K)$$

be the wild inertia, inertia, decomposition group of $\mathfrak{m}$ and $\mathfrak{m}'$. Then the canonical surjection $\text{Gal}(M/K) \to \text{Gal}(L/K)$ induces surjections $P' \to P$, $I' \to I$, and $D' \to D$. Moreover these fit into commutative diagrams

$$\begin{tikzcd} D' \arrow[r] \arrow[d] & \text{Aut}(\kappa(\mathfrak{m}')/\kappa_A) \arrow[d] \\ D \arrow[r] & \text{Aut}(\kappa(\mathfrak{m})/\kappa_A) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I' \arrow[r, "\theta'"'] \arrow[d] & \mu_{e'}(\kappa(\mathfrak{m}')) \arrow[d, "(-)^{e'/e}"] \\ I \arrow[r, "\theta"] & \mu_e(\kappa(\mathfrak{m})) \end{tikzcd}$$

where e' and e are the ramification indices of $A \to C_{\mathfrak{m}'}$ and $A \to B_{\mathfrak{m}}$.

Proof. The fact that under the map $\text{Gal}(M/K) \to \text{Gal}(L/K)$ the groups P', I', D' map into P, I, D is immediate from the definitions of these groups. The commutativity of the first diagram is clear (observe that since $\kappa(\mathfrak{m})/\kappa_A$ is normal every automorphism of $\kappa(\mathfrak{m}')$ over κ_A indeed induces an automorphism of $\kappa(\mathfrak{m})$ over κ_A and hence we obtain the right vertical arrow in the first diagram, see Lemma 15.112.4 and Fields, Lemma 9.15.7).

The maps $I' \to I$ and $D' \to D$ are surjective by Lemma 15.110.11. The surjectivity of $P' \to P$ follows as P' and P are p-Sylow subgroups of I' and I.

To see the commutativity of the second diagram we choose a uniformizer π' of $C_{\mathfrak m'}$ and a uniformizer π of $B_{\mathfrak m}$. Then $\pi = c'(\pi')^{e'/e}$ for some unit c' of $C_{\mathfrak m'}$. For $\sigma' \in I'$ the image $\sigma \in I$ is simply the restriction of σ' to L. Write $\sigma'(\pi') = c\pi'$ for a unit $c \in C_{\mathfrak m'}$ and write $\sigma(\pi) = b\pi$ for a unit b of $B_{\mathfrak m}$. Then $\sigma'(\pi) = b\pi$ and we obtain

$$b\pi = \sigma'(\pi) = \sigma'(c'(\pi')^{e'/e}) = \sigma'(c')c^{e'/e}(\pi')^{e'/e} = \frac{\sigma'(c')}{c'}c^{e'/e}\pi$$

As $\sigma' \in I'$ we see that b and $c^{e'/e}$ have the same image in the residue field which proves what we want. □

0BU8 Remark 15.112.11. In order to use the inertia character $\theta : I \to \mu_e(\kappa(\mathfrak m))$ for infinite Galois extensions, it is convenient to scale it. Let $A, K, L, B, \mathfrak m, G, P, I, D, e, \theta$ be as in Lemma 15.112.5 and Definition 15.112.6. Then $e = q|I_t|$ with q is a power of the characteristic p of $\kappa(\mathfrak m)$ if positive or 1 if zero. Note that $\mu_e(\kappa(\mathfrak m)) = \mu_{|I_t|}(\kappa(\mathfrak m))$ because the characteristic of $\kappa(\mathfrak m)$ is p. Consider the map

$$\theta_{can} = q\theta : I \longrightarrow \mu_{|I_t|}(\kappa(\mathfrak m))$$

This map induces an isomorphism $\theta_{can} : I_t \to \mu_{|I_t|}(\kappa(\mathfrak m))$. We have $\theta_{can}(\tau\sigma\tau^{-1}) = \tau(\theta_{can}(\sigma))$ for $\tau \in D$ and $\sigma \in I$ by Lemma 15.112.7. Finally, if M/L is an extension such that M/K is Galois and $\mathfrak m'$ is a prime of the integral closure of A in M lying over $\mathfrak m$, then we get the commutative diagram

$$\begin{tikzcd} I' \arrow[r, "\theta'_{can}"'] \arrow[d] & \mu_{|I'_t|}(\kappa(\mathfrak m')) \arrow[d, "(-)^{|I'_t|/|I_t|}"] \\ I \arrow[r, "\theta_{can}"] & \mu_{|I_t|}(\kappa(\mathfrak m)) \end{tikzcd}$$

by Lemma 15.112.10.

15.113. Krasner's lemma

0BU9 Here is Krasner's lemma in the case of discretely valued fields.

09EI Lemma 15.113.1 (Krasner's lemma). Let A be a complete local domain of dimension 1. Let $P(t) \in A[t]$ be a polynomial with coefficients in A. Let $\alpha \in A$ be a root of P but not a root of the derivative $P' = \mathrm{d}P/\mathrm{d}t$. For every $c \geq 0$ there exists an integer n such that for any $Q \in A[t]$ whose coefficients are in $\mathfrak m_A^n$ the polynomial $P + Q$ has a root $\beta \in A$ with $\beta - \alpha \in \mathfrak m_A^c$.

Proof. Choose a nonzero $\pi \in \mathfrak m$. Since the dimension of A is 1 we have $\mathfrak m = \sqrt{(\pi)}$. By assumption we may write $P'(\alpha)^{-1} = \pi^{-m}a$ for some $m \geq 0$ and $a \in A$. We may and do assume that $c \geq m + 1$. Pick n such that $\mathfrak m_A^n \subset (\pi^{c+m})$. Pick any Q as in the statement. For later use we observe that we can write

$$P(x + y) = P(x) + P'(x)y + R(x, y)y^2$$

for some $R(x, y) \in A[x, y]$. We will show by induction that we can find a sequence $\alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots$ such that

(1) $\alpha_k \equiv \alpha \bmod \pi^c$,
(2) $\alpha_{k+1} - \alpha_k \in (\pi^k)$, and
(3) $(P + Q)(\alpha_k) \in (\pi^{m+k})$.

Setting $\beta = \lim \alpha_k$ will finish the proof.

Base case. Since the coefficients of Q are in (π^{c+m}) we have $(P+Q)(\alpha) \in (\pi^{c+m})$. Hence $\alpha_m = \alpha$ works. This choice guarantees that $\alpha_k \equiv \alpha \bmod \pi^c$ for all $k \geq m$.

Induction step. Given α_k we write $\alpha_{k+1} = \alpha_k + \delta$ for some $\delta \in (\pi^k)$. Then we have

$$(P+Q)(\alpha_{k+1}) = P(\alpha_k + \delta) + Q(\alpha_k + \delta)$$

Because the coefficients of Q are in (π^{c+m}) we see that $Q(\alpha_k + \delta) \equiv Q(\alpha_k) \bmod \pi^{c+m+k}$. On the other hand we have

$$P(\alpha_k + \delta) = P(\alpha_k) + P'(\alpha_k)\delta + R(\alpha_k, \delta)\delta^2$$

Note that $P'(\alpha_k) \equiv P'(\alpha) \bmod (\pi^{m+1})$ as $\alpha_k \equiv \alpha \bmod \pi^{m+1}$. Hence we obtain

$$P(\alpha_k + \delta) \equiv P(\alpha_k) + P'(\alpha)\delta \bmod \pi^{k+m+1}$$

Recombining the two terms we see that

$$(P+Q)(\alpha_{k+1}) \equiv (P+Q)(\alpha_k) + P'(\alpha)\delta \bmod \pi^{k+m+1}$$

Thus a solution is to take $\delta = -P'(\alpha)^{-1}(P+Q)(\alpha_k) = -\pi^{-m}a(P+Q)(\alpha_k)$ which is contained in (π^k) by induction assumption. □

09EJ Lemma 15.113.2. Let A be a discrete valuation ring with field of fractions K. Let $A^\wedge$ be the completion of A with fraction field $K^\wedge$. If $M/K^\wedge$ is a finite separable extension, then there exists a finite separable extension L/K such that $M = K^\wedge \otimes_K L$.

Proof. Note that $A^\wedge$ is a discrete valuation ring too (by Lemmas 15.43.4 and 15.43.1). In particular $A^\wedge$ is a domain. The proof will work more generally for Noetherian local rings A such that $A^\wedge$ is a local domain of dimension 1.

Let $\theta \in M$ be an element that generates M over $K^\wedge$. (Theorem of the primitive element.) Let $P(t) \in K^\wedge[t]$ be the minimal polynomial of θ over $K^\wedge$. Let $\pi \in \mathfrak{m}_A$ be a nonzero element. After replacing θ by $\pi^n\theta$ we may assume that the coefficients of $P(t)$ are in $A^\wedge$. Let $B = A^\wedge[\theta] = A^\wedge[t]/(P(t))$. Note that B is a complete local domain of dimension 1 because it is finite over A and contained in M. Since M is separable over K the element θ is not a root of the derivative of P. For any integer n we can find a monic polynomial $P_1 \in A[t]$ such that $P - P_1$ has coefficients in $\pi^n A^\wedge[t]$. By Krasner's lemma (Lemma 15.113.1) we see that P_1 has a root β in B for n sufficiently large. Moreover, we may assume (if n is chosen large enough) that $\theta - \beta \in \pi B$. Consider the map $\Phi : A^\wedge[t]/(P_1) \to B$ of $A^\wedge$-algebras which maps t to β. Since $B = \pi B + \sum_{i<\deg(P)} A^\wedge\theta^i$, the map Φ is surjective by Nakayama's lemma. As $\deg(P_1) = \deg(P)$ it follows that Φ is an isomorphism. We conclude that the ring extension $L = K[t]/(P_1(t))$ satisfies $K^\wedge \otimes_K L \cong M$. This implies that L is a field and the proof is complete. □

09EK Definition 15.113.3. Let A be a discrete valuation ring. We say A has mixed characteristic if the characteristic of the residue field of A is $p > 0$ and the characteristic of the fraction field of A is 0. In this case we obtain an extension of discrete valuation rings $\mathbf{Z}_{(p)} \subset A$ and the absolute ramification index of A is the ramification index of this extension.

15.114. Abhyankar's lemma and tame ramification

0EXT In this section we prove what we think is the most general version of Abhyankar's lemma for discrete valuation rings. After doing so, we apply this to prove some results about tamely ramified extensions of the fraction field of a discrete valuation ring.

09EM **Remark 15.114.1.** Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let K_1/K be a finite extension of fields. Let $A_1 \subset K_1$ be the integral closure of A in K_1. On the other hand, let $L_1 = (L \otimes_K K_1)_{red}$. Then L_1 is a nonempty finite product of finite field extensions of L. Let B_1 be the integral closure of B in L_1. We obtain compatible commutative diagrams

$$\begin{tikzcd} L \arrow[r] & L_1 \\ K \arrow[u] \arrow[r] & K_1 \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r] & B_1 \\ A \arrow[u] \arrow[r] & A_1 \arrow[u] \end{tikzcd}$$

In this situation we have the following

(1) By Algebra, Lemma 10.120.18 the ring A_1 is a Dedekind domain and B_1 is a finite product of Dedekind domains.
(2) Note that $L \otimes_K K_1 = (B \otimes_A A_1)_\pi$ where $\pi \in A$ is a uniformizer and that π is a nonzerodivisor on $B \otimes_A A_1$. Thus the ring map $B \otimes_A A_1 \to B_1$ is integral with kernel consisting of nilpotent elements. Hence $\text{Spec}(B_1) \to \text{Spec}(B \otimes_A A_1)$ is surjective on spectra (Algebra, Lemma 10.36.17). The map $\text{Spec}(B \otimes_A A_1) \to \text{Spec}(A_1)$ is surjective as $A_1/\mathfrak m_A A_1 \to B/\mathfrak m_A B \otimes_{\kappa_A} A_1/\mathfrak m_A A_1$ is an injective ring map with $A_1/\mathfrak m_A A_1$ Artinian. We conclude that $\text{Spec}(B_1) \to \text{Spec}(A_1)$ is surjective.
(3) Let $\mathfrak m_i$, $i = 1, \ldots n$ with $n \geq 1$ be the maximal ideals of A_1. For each $i = 1, \ldots, n$ let $\mathfrak m_{ij}$, $j = 1, \ldots, m_i$ with $m_i \geq 1$ be the maximal ideals of B_1 lying over $\mathfrak m_i$. We obtain diagrams

$$\begin{tikzcd} B \arrow[r] & (B_1)_{\mathfrak m_{ij}} \\ A \arrow[u] \arrow[r] & (A_1)_{\mathfrak m_i} \arrow[u] \end{tikzcd}$$

of extensions of discrete valuation rings.
(4) If A is henselian (for example complete), then A_1 is a discrete valuation ring, i.e., $n = 1$. Namely, A_1 is a union of finite extensions of A which are domains, hence local by Algebra, Lemma 10.153.4.
(5) If B is henselian (for example complete), then B_1 is a product of discrete valuation rings, i.e., $m_i = 1$ for $i = 1, \ldots, n$.
(6) If $K \subset K_1$ is purely inseparable, then A_1 and B_1 are both discrete valuation rings, i.e., $n = 1$ and $m_1 = 1$. This is true because for every $b \in B_1$ a p-power power of b is in B, hence B_1 can only have one maximal ideal.
(7) If $K \subset K_1$ is finite separable, then $L_1 = L \otimes_K K_1$ and is a finite product of finite separable extensions too. Hence $A \subset A_1$ and $B \subset B_1$ are finite by Algebra, Lemma 10.161.8.
(8) If A is Nagata, then $A \subset A_1$ is finite.
(9) If B is Nagata, then $B \subset B_1$ is finite.

09EV Lemma 15.114.2. Let A be a discrete valuation ring with uniformizer π. Let $n \geq 2$. Then $K_1 = K[\pi^{1/n}]$ is a degree n extension of K and the integral closure A_1 of A in K_1 is the ring $A[\pi^{1/n}]$ which is a discrete valuation ring with ramification index n over A.

Proof. This lemma proves itself. $\square$

09EQ Lemma 15.114.3. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume that $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then for any finite extension K_1/K we have $L_1 = L \otimes_K K_1$, $B_1 = B \otimes_A A_1$, and each extension $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ (see Remark 15.114.1) is formally smooth in the $\mathfrak{m}_{ij}$-adic topology.

Proof. We will use the equivalence of Lemma 15.111.5 without further mention. Let $\pi \in A$ and $\pi_i \in (A_1)_{\mathfrak{m}_i}$ be uniformizers. As $\kappa_A \subset \kappa_B$ is separable, the ring

$$(B \otimes_A (A_1)_{\mathfrak{m}_i})/\pi_i(B \otimes_A (A_1)_{\mathfrak{m}_i}) = B/\pi B \otimes_{A/\pi A} (A_1)_{\mathfrak{m}_i}/\pi_i(A_1)_{\mathfrak{m}_i}$$

is a product of fields each separable over $\kappa_{\mathfrak{m}_i}$. Hence the element π_i in $B \otimes_A (A_1)_{\mathfrak{m}_i}$ is a nonzerodivisor and the quotient by this element is a product of fields. It follows that $B \otimes_A A_1$ is a Dedekind domain in particular reduced. Thus $B \otimes_A A_1 \subset B_1$ is an equality. $\square$

The following lemma is our version of Abhyankar's lemma for discrete valuation rings. Observe that κ_B/κ_A is not assumed to be an algebraic extension of fields.

0BRM Lemma 15.114.4 (Abhyankar's lemma). Let $A \subset B$ be an extension of discrete valuation rings. Assume that either the residue characteristic of A is 0 or it is p, the ramification index e is prime to p, and κ_B/κ_A is a separable field extension. Let K_1/K be a finite extension. Using the notation of Remark 15.114.1 assume e divides the ramification index of $A \subset (A_1)_{\mathfrak{m}_i}$ for some i. Then $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ is formally smooth in the $\mathfrak{m}_{ij}$-adic topology for all $j = 1, \ldots, m_i$.

Proof. Let $\pi \in A$ be a uniformizer. Let π_1 be a uniformizer of $(A_1)_{\mathfrak{m}_i}$. Write $\pi = u\pi_1^{e_1}$ with u a unit of $(A_1)_{\mathfrak{m}_i}$ and e_1 the ramification index of $A \subset (A_1)_{\mathfrak{m}_i}$.

Claim: we may assume that u is an eth power in K_1. Namely, let K_2 be an extension of K_1 obtained by adjoining a root of $x^e = u$; thus K_2 is a factor of $K_1[x]/(x^e - u)$. Then K_2/K_1 is a finite separable extension (by our assumption on e) and hence $A_1 \subset A_2$ is finite. Since $(A_1)_{\mathfrak{m}_i} \to (A_1)_{\mathfrak{m}_i}[x]/(x^e - u)$ is finite étale (as e is prime to the residue characteristic and u a unit) we conclude that $(A_2)_{\mathfrak{m}_i}$ is a factor of a finite étale extension of $(A_1)_{\mathfrak{m}_i}$ hence finite étale over $(A_1)_{\mathfrak{m}_i}$ itself. The same reasoning shows that $B_1 \subset B_2$ induces finite étale extensions $(B_1)_{\mathfrak{m}_{ij}} \subset (B_2)_{\mathfrak{m}_{ij}}$. Pick a maximal ideal $\mathfrak{m}'_{ij} \subset B_2$ lying over $\mathfrak{m}_{ij} \subset B_1$ (of course there may be more than one) and consider

$$\begin{tikzcd}
(B_1)_{\mathfrak{m}_{ij}} \arrow[r] & (B_2)_{\mathfrak{m}'_{ij}} \\
(A_1)_{\mathfrak{m}_i} \arrow[r] \arrow[u] & (A_2)_{\mathfrak{m}'_i} \arrow[u]
\end{tikzcd}$$

where $\mathfrak{m}'_i \subset A_2$ is the image. Now the horizontal arrows have ramification index 1 and induce finite separable residue field extensions. Thus, using the equivalence of Lemma 15.111.5, we see that it suffices to show that the right vertical arrow is formally smooth in the $\mathfrak m'_{ij}$-adic topology. Since u has a eth root in K_2 we obtain the claim.

Assume u has an eth root in K_1. Since $e|e_1$ and since u has a eth root in K_1 we see that $\pi = \theta^e$ for some $\theta \in K_1$. Let $K_1' = K[\theta] \subset K_1$ be the subfield generated by θ. By Lemma 15.114.2 the integral closure A_1' of A in $K[\theta]$ is the discrete valuation ring $A_1' = A[\theta]$ which has ramification index e over A. If we can prove the lemma for the extension K_1'/K, then we conclude by Lemma 15.114.3 applied to the diagram

$$\begin{tikzcd} (B_1')_{B_1' \cap \mathfrak m_{ij}} \arrow[r] & (B_1)_{\mathfrak m_{ij}} \\ A_1' \arrow[u] \arrow[r] & (A_1)_{\mathfrak m_i} \arrow[u] \end{tikzcd}$$

for all $j = 1, \ldots, m_i$. This reduces us to the case discussed in the next paragraph.

Assume $K_1 = K[\pi^{1/e}]$ and set $\theta = \pi^{1/e}$. Let π_B be a uniformizer for B and write $\pi = w\pi_B^e$ for some unit w of B. Then we see that $L_1 = L \otimes_K K_1$ is obtained by adjoining π_B/θ which is an eth root of the unit w. Thus $B \subset B_1$ is finite étale. Thus for any maximal ideal $\mathfrak m \subset B_1$ consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "1"'] & (B_1)_{\mathfrak m} \\ A \arrow[u, "e"] \arrow[r, "e"] & A_1 \arrow[u, "e_{\mathfrak m}"'] \end{tikzcd}$$

Here the numbers along the arrows are the ramification indices. By multiplicativity of ramification indices (Lemma 15.111.3) we conclude $e_{\mathfrak m} = 1$. Looking at the residue field extensions we find that $\kappa(\mathfrak m)$ is a finite separable extension of κ_B which is separable over κ_A. Therefore $\kappa(\mathfrak m)$ is separable over κ_A which is equal to the residue field of A_1 and we win by Lemma 15.111.5. □

0EXU Lemma 15.114.5. Let A be a discrete valuation ring with fraction field K. Let $M/L/K$ be finite separable extensions. Let B be the integral closure of A in L. If L/K is tamely ramified with respect to A and M/L is tamely ramified with respect to $B_{\mathfrak m}$ for every maximal ideal $\mathfrak m$ of B, then M/K is tamely ramified with respect to A.

Proof. Let C be the integral closure of A in M. Every maximal ideal $\mathfrak m'$ of C lies over a maximal ideal $\mathfrak m$ of B. Then the lemma follows from the multiplicativity of ramification indices (Lemma 15.111.3) and the fact that we have the tower $\kappa(\mathfrak m')/\kappa(\mathfrak m)/\kappa_A$ of finite extensions of fields. □

0EXV Lemma 15.114.6. Let A be a discrete valuation ring with fraction field K. If $M/L/K$ are finite separable extensions and M is tamely ramified with respect to A, then L is tamely ramified with respect to A.

Proof. We will use the results of the discussion in Remark 15.111.6 without further mention. Let $C/B/A$ be the integral closures of A in $M/L/K$. Since C is a finite ring extension of B, we see that $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ is surjective. Hence for ever maximal ideal $\mathfrak m \subset B$ there is a maximal ideal $\mathfrak m' \subset C$ lying over $\mathfrak m$. By the multiplicativity of ramification indices (Lemma 15.111.3) and the assumption, we conclude that the ramification index of $B_\mathfrak{m}$ over A is prime to the residue characteristic. Since $\kappa(\mathfrak{m}')/\kappa_A$ is finite separable, the same is true for $\kappa(\mathfrak{m})/\kappa_A$. $\square$

0EXW Lemma 15.114.7. Let A be a discrete valuation ring with fraction field K. Let $\pi \in A$ be a uniformizer. Let L/K be a finite separable extension. The following are equivalent

(1) L is tamely ramified with respect to A,
(2) there exists an $e \geq 1$ invertible in κ_A and an extension $L'/K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ such that L is contained in L', and
(3) there exists an $e_0 \geq 1$ invertible in κ_A such that for every $d \geq 1$ invertible in κ_A (2) holds with $e = de_0$.

Proof. Observe that A' is a discrete valuation ring with fraction field K', see Lemma 15.114.2. Of course the ramification index of A' over A is e. Thus if (2) holds, then L' is tamely ramified with respect to A by Lemma 15.114.5. Hence L is tamely ramified with respect to A by Lemma 15.114.6.

The implication (3) $\Rightarrow$ (2) is immediate.

Assume that (1) holds. Let B be the integral closure of A in L and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be its maximal ideals. Denote e_i the ramification index of $A \to B_{\mathfrak{m}_i}$. Let e_0 be the least common multiple of $e_1, \ldots, e_r$. This is invertible in κ_A by our assumption (1). Let $e = de_0$ as in (3). Set $A' = A[\pi^{1/e}]$. Then $A \to A'$ is an extension of discrete valuation rings with fraction field $K' = K[\pi^{1/e}]$, see Lemma 15.114.2. Choose a product decomposition

$$L \otimes_K K' = \prod L'_j$$

where L'_j are fields. Let B'_j be the integral closure of A in L'_j. Let $\mathfrak{m}_{ijk}$ be the maximal ideals of B'_j lying over $\mathfrak{m}_i$. Observe that $(B'_j)_{\mathfrak{m}_i}$ is the integral closure of $B_{\mathfrak{m}_i}$ in L'_j. By Abhyankar's lemma (Lemma 15.114.4) applied to $A \subset B_{\mathfrak{m}_i}$ and the extension K'/K we see that $A' \to (B'_j)_{\mathfrak{m}_{ijk}}$ is formally smooth in the $\mathfrak{m}_{ijk}$-adic topology. This implies that the ramification index is 1 and that the residue field extension is separable (Lemma 15.111.5). In this way we see that L'_j is unramified with respect to A'. This finishes the proof: we take $L' = L'_j$ for some j. $\square$

0EXX Lemma 15.114.8. Let A be a discrete valuation ring with fraction field K.

(1) If L/K is a finite separable extension which is tamely ramified with respect to A, then there exists a Galois extension M/K containing L which is tamely ramified with respect to A.
(2) If L_1/K, L_2/K are finite separable extensions which are tamely ramified with respect to A, then there exists a a finite separable extension L/K which is tamely ramified with respect to A containing L_1 and L_2.

Proof. Proof of (2). Choose a uniformizer $\pi \in A$. We can choose an integer e invertible in κ_A and extensions $L'_i/K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ with L'_i/L_i as extensions of K, see Lemma 15.114.7. By Lemma 15.111.8 we can find an extension L'/K' which is unramified with respect to A' such that L'_i/K is isomorphic to a subextension of L'/K' for $i = 1, 2$. This finishes the proof of (3) as L'/K is tamely ramified (use same lemma as above).

Proof of (1). We may first replace L by a larger extension and assume that L is an extension of $K' = K[\pi^{1/e}]$ unramified with respect to $A' = A[\pi^{1/e}]$ where e is invertible in κ_A, see Lemma 15.114.7. Let M be the normal closure of L over K, see Fields, Definition 9.16.4. Then M/K is Galois by Fields, Lemma 9.21.5. On the other hand, there is a surjection

$$L \otimes_K \ldots \otimes_K L \longrightarrow M$$

of K-algebras, see Fields, Lemma 9.16.6. Let B be the integral closure of A in L as in Remark 15.111.6. The condition that L is unramified with respect to $A' = A[\pi^{1/e}]$ exactly means that $A' \to B$ is an étale ring map, see Algebra, Lemma 10.143.7. Claim:

$$K' \otimes_K \ldots \otimes_K K' = \prod K'_i$$

is a product of field extensions K'_i/K tamely ramified with respect to A. Then if A'_i is the integral closure of A in K'_i we see that

$$\prod A'_i \otimes_{(A' \otimes_A \ldots \otimes_A A')} (B \otimes_A \ldots \otimes_A B)$$

is finite étale over $\prod A'_i$ and hence a product of Dedekind domains (Lemma 15.44.4). We conclude that M is the fraction field of one of these Dedekind domains which is finite étale over A'_i for some i. It follows that M/K'_i is unramified with respect to every maximal ideal of A'_i and hence M/K is tamely ramified by Lemma 15.114.5.

It remains the prove the claim. For this we write $A' = A[x]/(x^e - \pi)$ and we see that

$$A' \otimes_A \ldots \otimes_A A' = A'[x_1, \ldots, x_r]/(x_1^e - \pi, \ldots, x_r^e - \pi)$$

The normalization of this ring certainly contains the elements $y_i = x_i/x_1$ for $i = 2, \ldots, r$ subject to the relations $y_i^e - 1 = 0$ and we obtain

$$A[x_1, y_2, \ldots, y_r]/(x_1^e - \pi, y_2^e - 1, \ldots, y_r - 1) = A'[y_2, \ldots, y_r]/(y_2^e - 1, \ldots, y_r^e - 1)$$

This ring is finite étale over A' because e is invertible in A'. Hence it is a product of Dedekind domains each unramified over A' as desired (see references given above in case of confusion). ☐

0EXY Lemma 15.114.9. Let $A \subset B$ be an extension of discrete valuation rings. Denote L/K the corresponding extension of fraction fields. Let K'/K be a finite separable extension. Then

$$K' \otimes_K L = \prod L'_i$$

is a finite product of fields and the following is true

(1) If K' is unramified with respect to A, then each L'_i is unramified with respect to B.
(2) If K' is tamely ramified with respect to A, then each L'_i is tamely ramified with respect to B.

Proof. The algebra $K' \otimes_K L$ is a finite product of fields as it is a finite étale algebra over L. Let A' be the integral closure of A in K'.

In case (1) the ring map $A \to A'$ is finite étale. Hence $B' = B \otimes_A A'$ is finite étale over B and is a finite product of Dedekind domains (Lemma 15.44.4). Hence B' is the integral closure of B in $K' \otimes_K L$. It follows immediately that each L'_i is unramified with respect to B.

Choose a uniformizer $\pi \in A$. To prove (2) we may replace K' by a larger extension tame ramified with respect to A (details omitted; hint: use Lemma 15.114.6). Thus by Lemma 15.114.7 we may assume there exists some $e \geq 1$ invertible in κ_A such that K' contains $K[\pi^{1/e}]$ and such that K' is unramified with respect to $A[\pi^{1/e}]$. Choose a product decomposition

$$K[\pi^{1/e}] \otimes_K L = \prod L_{e,j}$$

For every i there exists a j_i such that $L'_i/L_{e,j_i}$ is a finite separable extension. Let $B_{e,j}$ be the integral closure of B in $L_{e,j}$. By (1) applied to $K'/K[\pi^{1/e}]$ and $A[\pi^{1/e}] \subset (B_{e,j_i})_\mathfrak{m}$ we see that L'_i is unramified with respect to $(B_{e,j_i})_\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subset B_{e,j_i}$. Hence the proof will be complete if we can show that $L_{e,j}$ is tamely ramified with respect to B, see Lemma 15.114.5.

Choose a uniformizer θ in B. Write $\pi = u\theta^t$ where u is a unit of B and $t \geq 1$. Then we have

$$A[\pi^{1/e}] \otimes_A B = B[x]/(x^e - u\theta^t) \subset B[y,z]/(y^{e'} - \theta, z^e - u)$$

where $e' = e/\gcd(e,t)$. The map sends x to $zy^{t/\gcd(e,t)}$. Since the right hand side is a product of Dedekind domains each tamely ramified over B the proof is complete (details omitted). $\square$

15.115. Eliminating ramification

09EL In this section we discuss a result of Helmut Epp, see [Epp73]. We strongly encourage the reader to read the original. Our approach is slightly different as we try to handle the mixed and equicharacteristic cases by the same method. For related results, see also [Pon98], [Pon99], [Kuh03], and [ZK99].

Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. The goal in this section is to find a finite extension K_1/K such that with

$$\begin{tikzcd} L \arrow[r] & L_1 \\ K \arrow[u] \arrow[r] & K_1 \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r] & B_1 \arrow[r] & (B_1)_{\mathfrak{m}_{ij}} \\ A \arrow[u] \arrow[r] & A_1 \arrow[u] \arrow[r] & (A_1)_{\mathfrak{m}_i} \arrow[u] \end{tikzcd}$$

as in Remark 15.114.1 the extensions $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ are all weakly unramified or even formally smooth in the relevant adic topologies. The simplest (but nontrivial) example of this is Abhyankar's lemma, see Lemma 15.114.4.

09EN Definition 15.115.1. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$.

(1) We say a finite field extension K_1/K is a weak solution for $A \subset B$ if all the extensions $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ of Remark 15.114.1 are weakly unramified.
(2) We say a finite field extension K_1/K is a solution for $A \subset B$ if each extension $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ of Remark 15.114.1 is formally smooth in the $\mathfrak{m}_{ij}$-adic topology.

We say a solution K_1/K is a separable solution if K_1/K is separable.

In general (weak) solutions do not exist; there is an example in [Epp73]. Under a mild hypothesis on the residue field extension, we will prove the existence of weak solutions in Theorem 15.115.18 following [Epp73]. In the next section, we will deduce the existence of solutions and sometimes separable solutions in geometrically meaningful cases, see Proposition 15.116.8 and Lemma 15.116.9. However, the following example shows that in general one needs inseparable extensions to get even a weak solution.

09EP Example 15.115.2. Let k be a perfect field of characteristic $p > 0$. Let $A = k[[x]]$ and $K = k((x))$. Let $B = A[x^{1/p}]$. Any weak solution K_1/K for $A \to B$ is inseparable (and any finite inseparable extension of K is a solution). We omit the proof.

Solutions are stable under further extensions, see Lemma 15.116.1. This may not be true for weak solutions. Weak solutions are in some sense stable under totally ramified extensions, see Lemma 15.115.3.

09ER Lemma 15.115.3. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume that $A \to B$ is weakly unramified. Then for any finite separable extension K_1/K totally ramified with respect to A we have that $L_1 = L \otimes_K K_1$ is a field, A_1 and $B_1 = B \otimes_A A_1$ are discrete valuation rings, and the extension $A_1 \subset B_1$ (see Remark 15.114.1) is weakly unramified.

Proof. Let $\pi \in A$ and $\pi_1 \in A_1$ be uniformizers. As K_1/K is totally ramified with respect to A we have $\pi_1^e = u_1\pi$ for some unit u_1 in A_1. Hence A_1 is generated by π_1 over A and the minimal polynomial $P(t)$ of π_1 over K has the form

$$P(t) = t^e + a_{e-1}t^{e-1} + \ldots + a_0$$

with $a_i \in (\pi)$ and $a_0 = u\pi$ for some unit u of A. Note that $e = [K_1 : K]$ as well. Since $A \to B$ is weakly unramified we see that π is a uniformizer of B and hence $B_1 = B[t]/(P(t))$ is a discrete valuation ring with uniformizer the class of t. Thus the lemma is clear. $\square$

09ES Lemma 15.115.4. Let $A \to B \to C$ be extensions of discrete valuation rings with fraction fields $K \subset L \subset M$. Let K_1/K be a finite extension.

(1) If K_1 is a (weak) solution for $A \to C$, then K_1 is a (weak) solution for $A \to B$.
(2) If K_1 is a (weak) solution for $A \to B$ and $L_1 = (L \otimes_K K_1)_{red}$ is a product of fields which are (weak) solutions for $B \to C$, then K_1 is a (weak) solution for $A \to C$.

Proof. Let $L_1 = (L \otimes_K K_1)_{red}$ and $M_1 = (M \otimes_K K_1)_{red}$ and let $B_1 \subset L_1$ and $C_1 \subset M_1$ be the integral closure of B and C. Note that $M_1 = (M \otimes_L L_1)_{red}$ and that L_1 is a (nonempty) finite product of finite extensions of L. Hence the ring map $B_1 \to C_1$ is a finite product of ring maps of the form discussed in Remark 15.114.1. In particular, the map $\text{Spec}(C_1) \to \text{Spec}(B_1)$ is surjective. Choose a maximal ideal $\mathfrak{m} \subset C_1$ and consider the extensions of discrete valuation rings

$$(A_1)_{A_1 \cap \mathfrak{m}} \to (B_1)_{B_1 \cap \mathfrak{m}} \to (C_1)_{\mathfrak{m}}$$

If the composition is weakly unramified, so is the map $(A_1)_{A_1 \cap \mathfrak{m}} \to (B_1)_{B_1 \cap \mathfrak{m}}$. If the residue field extension $\kappa_{A_1 \cap \mathfrak{m}} \to \kappa_{\mathfrak{m}}$ is separable, so is the subextension $\kappa_{A_1 \cap \mathfrak{m}} \to \kappa_{B_1 \cap \mathfrak{m}}$. Taking into account Lemma 15.111.5 this proves (1). A similar argument works for (2). $\square$

09ET **Lemma 15.115.5.** Let $A \to B$ be an extension of discrete valuation rings. There exists a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

of extensions of discrete valuation rings such that

(1) the extensions K'/K and L'/L of fraction fields are separable algebraic,
(2) the residue fields of A' and B' are separable algebraic closures of the residue fields of A and B, and
(3) if a solution, weak solution, or separable solution exists for $A' \to B'$, then a solution, weak solution, or separable solution exists for $A \to B$.

Proof. By Algebra, Lemma 10.159.2 there exists an extension $A \subset A'$ which is a filtered colimit of finite étale extensions such that the residue field of A' is a separable algebraic closure of the residue field of A. Then $A \subset A'$ is an extension of discrete valuation rings such that the induced extension K'/K of fraction fields is separable algebraic.

Let $B \subset B'$ be a strict henselization of B. Then $B \subset B'$ is an extension of discrete valuation rings whose fraction field extension is separable algebraic. By Algebra, Lemma 10.155.9 there exists a commutative diagram as in the statement of the lemma. Parts (1) and (2) of the lemma are clear.

Let K'_1/K' be a (weak) solution for $A' \to B'$. Since A' is a colimit, we can find a finite étale extension $A \subset A'_1$ and a finite extension K_1 of the fraction field F of A'_1 such that $K'_1 = K' \otimes_F K_1$. As $A \subset A'_1$ is finite étale and B' strictly henselian, it follows that $B' \otimes_A A'_1$ is a finite product of rings isomorphic to B'. Hence

$$L' \otimes_K K_1 = L' \otimes_K F \otimes_F K_1$$

is a finite product of rings isomorphic to $L' \otimes_{K'} K'_1$. Thus we see that K_1/K is a (weak) solution for $A \to B'$. Hence it is also a (weak) solution for $A \to B$ by Lemma 15.115.4. $\square$

09EU **Lemma 15.115.6.** Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let K_1/K be a normal extension. Say $G = \text{Aut}(K_1/K)$. Then G acts on the rings K_1, L_1, A_1 and B_1 of Remark 15.114.1 and acts transitively on the set of maximal ideals of B_1.

Proof. Everything is clear apart from the last assertion. If there are two or more orbits of the action, then we can find an element $b \in B_1$ which vanishes at all the maximal ideals of one orbit and has residue 1 at all the maximal ideals in another orbit. Then $b' = \prod_{\sigma \in G} \sigma(b)$ is a G-invariant element of $B_1 \subset L_1 = (L \otimes_K K_1)_{red}$ which is in some maximal ideals of B_1 but not in all maximal ideals of B_1. Lifting it to an element of $L \otimes_K K_1$ and raising to a high power we obtain a G-invariant element b'' of $L \otimes_K K_1$ mapping to $(b')^N$ for some $N > 0$; in fact, we only need to do this in case the characteristic is $p > 0$ and in this case raising to a suitably large p-power q defines a canonical map $(L \otimes_K K_1)_{red} \to L \otimes_K K_1$. Since $K = (K_1)^G$ we conclude that $b'' \in L$. Since b'' maps to an element of B_1 we see that $b'' \in B$ (as B is normal). Then on the one hand it must be true that $b'' \in \mathfrak{m}_B$ as b' is in some maximal ideal of B_1 and on the other hand it must be true that $b'' \notin \mathfrak{m}_B$ as

b' is not in all maximal ideals of B_1. This contradiction finishes the proof of the lemma. □

09EW Lemma 15.115.7. Let A be a discrete valuation ring with uniformizer π. If the residue characteristic of A is $p > 0$, then for every $n > 1$ and p-power q there exists a degree q separable extension L/K totally ramified with respect to A such that the integral closure B of A in L has ramification index q and a uniformizer π_B such that $\pi_B^q = \pi + \pi^n b$ and $\pi_B^q = \pi + (\pi_B)^{nq} b'$ for some $b, b' \in B$.

Proof. If the characteristic of K is zero, then we can take the extension given by $\pi_B^q = \pi$, see Lemma 15.114.2. If the characteristic of K is $p > 0$, then we can take the extension of K given by $z^q - \pi^n z = \pi^{1-q}$. Namely, then we see that $y^q - \pi^{n+q-1} y = \pi$ where $y = \pi z$. Taking $\pi_B = y$ we obtain the desired result. □

09EX Lemma 15.115.8. Let A be a discrete valuation ring. Assume the reside field κ_A has characteristic $p > 0$ and that $a \in A$ is an element whose residue class in κ_A is not a pth power. Then a is not a pth power in K and the integral closure of A in $K[a^{1/p}]$ is the ring $A[a^{1/p}]$ which is a discrete valuation ring weakly unramified over A.

Proof. This lemma proves itself. □

09EY Lemma 15.115.9. Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Let $\pi \in A$ be a uniformizer. Assume

(1) B is a Nagata ring,
(2) $A \subset B$ is weakly unramified,
(3) M is a degree p purely inseparable extension of L.

Then either

(1) $A \to C$ is weakly unramified, or
(2) $C = B[\pi^{1/p}]$, or
(3) there exists a degree p separable extension K_1/K totally ramified with respect to A such that $L_1 = L \otimes_K K_1$ and $M_1 = M \otimes_K K_1$ are fields and the maps of integral closures $A_1 \to B_1 \to C_1$ are weakly unramified extensions of discrete valuation rings.

Proof. Let e be the ramification index of C over B. If $e = 1$, then we are done. If not, then $e = p$ by Lemmas 15.111.2 and 15.111.4. This in turn implies that the residue fields of B and C agree. Choose a uniformizer π_C of C. Write $\pi_C^p = u\pi$ for some unit u of C. Since $\pi_C^p \in L$, we see that $u \in B^*$. Also $M = L[\pi_C]$.

Suppose there exists an integer $m \geq 0$ such that

$$u = \sum\nolimits_{0 \leq i < m} b_i^p \pi^i + b\pi^m$$

with $b_i \in B$ and with $b \in B$ an element whose image in κ_B is not a pth power. Choose an extension K_1/K as in Lemma 15.115.7 with $n = m + 2$ and denote π' the uniformizer of the integral closure A_1 of A in K_1 such that $\pi = (\pi')^p + (\pi')^{np} a$ for some $a \in A_1$. Let B_1 be the integral closure of B in $L \otimes_K K_1$. Observe that $A_1 \to B_1$ is weakly unramified by Lemma 15.115.3. In B_1 we have

$$u\pi = \left(\sum\nolimits_{0 \leq i < m} b_i (\pi')^{i+1}\right)^p + b(\pi')^{(m+1)p} + (\pi')^{np} b_1$$

for some $b_1 \in B_1$ (computation omitted). We conclude that M_1 is obtained from L_1 by adjoining a pth root of

$$b + (\pi')^{n-m-1}b_1$$

Since the residue field of B_1 equals the residue field of B we see from Lemma 15.115.8 that M_1/L_1 has degree p and the integral closure C_1 of B_1 is weakly unramified over B_1. Thus we conclude in this case.

If there does not exist an integer m as in the preceding paragraph, then u is a pth power in the π-adic completion of B_1. Since B is Nagata, this means that u is a pth power in B_1 by Algebra, Lemma 10.162.18. Whence the second case of the statement of the lemma holds. □

09EZ Lemma 15.115.10. Let A be a local ring annihilated by a prime p whose maximal ideal is nilpotent. There exists a ring map $\sigma : \kappa_A \to A$ which is a section to the residue map $A \to \kappa_A$. If $A \to A'$ is a local homomorphism of local rings, then we can choose a similar ring map $\sigma' : \kappa_{A'} \to A'$ compatible with σ provided that the extension $\kappa_{A'}/\kappa_A$ is separable.

Proof. Separable extensions are formally smooth by Algebra, Proposition 10.158.9. Thus the existence of σ follows from the fact that $\mathbf{F}_p \to \kappa_A$ is separable. Similarly for the existence of σ' compatible with σ. □

09F0 Lemma 15.115.11. Let A be a discrete valuation ring with fraction field K of characteristic $p > 0$. Let $\xi \in K$. Let L be an extension of K obtained by adjoining a root of $z^p - z = \xi$. Then L/K is Galois and one of the following happens

(1) $L = K$,
(2) L/K is unramified with respect to A of degree p,
(3) L/K is totally ramified with respect to A with ramification index p, and
(4) the integral closure B of A in L is a discrete valuation ring, $A \subset B$ is weakly unramified, and $A \to B$ induces a purely inseparable residue field extension of degree p.

Let π be a uniformizer of A. We have the following implications:

(A) If $\xi \in A$, then we are in case (1) or (2).
(B) If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by p and a is a unit in A, then we are in case (3)
(C) If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by p and the image of a in κ_A is not a pth power, then we are in case (4).

Proof. The extension is Galois of order dividing p by the discussion in Fields, Section 9.25. It immediately follows from the discussion in Section 15.112 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \to A[x]/(x^p - x - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where p does not divide n. Let $B \subset L$ be the integral closure of A in L. If $C = B_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then it is clear that $p\text{ord}_C(z) = -n\text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by p. It follows that it is p and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) = a$$

Since $A[y]/(y^p - \pi^{n-k}y - a)$ is a discrete valuation ring weakly unramified over A, the lemma follows. $\square$

09F1 Lemma 15.115.12. Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume

(1) $A \subset B$ weakly unramified,
(2) the characteristic of K is p,
(3) M is a degree p Galois extension of L, and
(4) $\kappa_A = \bigcap_{n \geq 1} \kappa_B^{p^n}$.

Then there exists a finite Galois extension K_1/K totally ramified with respect to A which is a weak solution for $A \to C$.

Proof. Since the characteristic of L is p we know that M is an Artin-Schreier extension of L (Fields, Lemma 9.25.1). Thus we may pick $z \in M$, $z \notin L$ such that $\xi = z^p - z \in L$. Choose $n \geq 0$ such that $\pi^n \xi \in B$. We pick z such that n is minimal. If $n = 0$, then M/L is unramified with respect to B (Lemma 15.115.11) and we are done. Thus we have $n > 0$.

Assumption (4) implies that κ_A is perfect. Thus we may choose compatible ring maps $\overline{\sigma} : \kappa_A \to A/\pi^n A$ and $\overline{\sigma} : \kappa_B \to B/\pi^n B$ as in Lemma 15.115.10. We lift the second of these to a map of sets $\sigma : \kappa_B \to B$[16]. Then we can write

$$\xi = \sum\nolimits_{i=n,\ldots,1} \sigma(\lambda_i)\pi^{-i} + b$$

for some $\lambda_i \in \kappa_B$ and $b \in B$. Let

$$I = \{i \in \{n, \ldots, 1\} \mid \lambda_i \in \kappa_A\}$$

and

$$J = \{j \in \{n, \ldots, 1\} \mid \lambda_i \notin \kappa_A\}$$

We will argue by induction on the size of the finite set J.

The case $J = \emptyset$. Here for all $i \in \{n, \ldots, 1\}$ we have $\sigma(\lambda_i) = a_i + \pi^n b_i$ for some $a_i \in A$ and $b_i \in B$ by our choice of σ. Thus $\xi = \pi^{-n}a + b$ for some $a \in A$ and $b \in B$. If $p|n$, then we write $a = a_0^p + \pi a_1$ for some $a_0, a_1 \in A$ (as the residue field of A is perfect). We compute

$$(z - \pi^{-n/p}a_0)^p - (z - \pi^{-n/p}a_0) = \pi^{-(n-1)}(a_1 + \pi^{n-1-n/p}a_0) + b'$$

for some $b' \in B$. This would contradict the minimality of n. Thus p does not divide n. Consider the degree p extension K_1 of K given by $w^p - w = \pi^{-n}a$. By Lemma 15.115.11 this extension is Galois and totally ramified with respect to A. Thus $L_1 = L \otimes_K K_1$ is a field and $A_1 \subset B_1$ is weakly unramified (Lemma 15.115.3). By Lemma 15.115.11 the ring $M_1 = M \otimes_K K_1$ is either a product of p copies of L_1 (in which case we are done) or a field extension of L_1 of degree p. Moreover, in the second case, either C_1 is weakly unramified over B_1 (in which case we are done) or M_1/L_1 is degree p, Galois, and totally ramified with respect to B_1. In this last case the extension M_1/L_1 is generated by the element $z - w$ and

$$(z - w)^p - (z - w) = z^p - z - (w^p - w) = b$$

[16]If B is complete, then we can choose σ to be a ring map. If A is also complete and σ is a ring map, then σ maps κ_A into A.

with $b \in B$ (see above). Thus by Lemma 15.115.11 once more the extension M_1/L_1 is unramified with respect to B_1 and we conclude that K_1 is a weak solution for $A \to C$. From now on we assume $J \neq \emptyset$.

Suppose that $j', j \in J$ such that $j' = p^r j$ for some $r > 0$. Then we change our choice of z into

$$z' = z - (\sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^p)\pi^{-pj} + \ldots + \sigma(\lambda_j^{p^{r-1}})\pi^{-p^{r-1}j})$$

Then ξ changes into $\xi' = (z')^p - (z')$ as follows

$$\xi' = \xi - \sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^{p^r})\pi^{-j'} + \text{something in } B$$

Writing $\xi' = \sum_{i=n,\ldots,1} \sigma(\lambda'_i)\pi^{-i} + b'$ as before we find that $\lambda'_i = \lambda_i$ for $i \neq j, j'$ and $\lambda'_j = 0$. Thus the set J has gotten smaller. By induction on the size of J we may assume no such pair j, j' exists. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \kappa_B$ which is not a pth power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that jp^{-r_j} is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r-r_j} > n$ for $j \in J$. Choose a separable extension K_1/K totally ramified with respect to A of degree p^r such that the corresponding discrete valuation ring $A_1 \subset K_1$ has uniformizer π' with $(\pi')^{p^r} = \pi + \pi^{n+1}a$ for some $a \in A_1$ (Lemma 15.115.7). Observe that $L_1 = L \otimes_K K_1$ is a field and that L_1/L is totally ramified with respect to B (Lemma 15.115.3). Computing in the integral closure B_1 we get

$$\xi = \sum\nolimits_{i \in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a qth power modulo π^n, i.e., modulo $(\pi')^{np^r}$. Hence we can rewrite the above as

$$\xi = \sum\nolimits_{i \in I} x_i^{p^r}(\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + b_1$$

As in the previous paragraph we change our choice of z into

$$\begin{aligned} z' = z & \\ & - \sum\nolimits_{i \in I} \left(x_i(\pi')^{-i} + \ldots + x_i^{p^{r-1}}(\pi')^{-ip^{r-1}}\right) \\ & - \sum\nolimits_{j \in J} \left(\sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + \ldots + \sigma(\mu_j)^{p^{r_j-1}}(\pi')^{-jp^{r-1}}\right) \end{aligned}$$

to obtain

$$(z')^p - z' = \sum\nolimits_{i \in I} x_i(\pi')^{-i} + \sum\nolimits_{j \in J} \sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + b'_1$$

for some $b'_1 \in B_1$. Since there is a unique j such that jp^{r-r_j} is maximal and since jp^{r-r_j} is bigger than $i \in I$ and divisible by p, we see that M_1/L_1 falls into case (C) of Lemma 15.115.11. This finishes the proof. □

09F2 **Lemma 15.115.13.** Let A be a ring which contains a primitive pth root of unity ζ. Set $w = 1 - \zeta$. Then

$$P(z) = \frac{(1 + wz)^p - 1}{w^p} = z^p - z + \sum\nolimits_{0 < i < p} a_i z^i$$

is an element of $A[z]$ and in fact $a_i \in (w)$. Moreover, we have

$$P(z_1 + z_2 + wz_1z_2) = P(z_1) + P(z_2) + w^p P(z_1)P(z_2)$$

in the polynomial ring $A[z_1, z_2]$.

Proof. It suffices to prove this when

$$A = \mathbf{Z}[\zeta] = \mathbf{Z}[x]/(x^{p-1} + \ldots + x + 1)$$

is the ring of integers of the cyclotomic field. The polynomial identity $t^p - 1 = (t-1)(t-\zeta)\ldots(t-\zeta^{p-1})$ (which is proved by looking at the roots on both sides) shows that $t^{p-1} + \ldots + t + 1 = (t-\zeta)\ldots(t-\zeta^{p-1})$. Substituting $t = 1$ we obtain $p = (1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1})$. The maximal ideal $(p, w) = (w)$ is the unique prime ideal of A lying over p (as fields of characteristic p do not have nontrivial pth roots of 1). It follows that $p = uw^{p-1}$ for some unit u. This implies that

$$a_i = \frac{1}{p}\binom{p}{i} uw^{i-1}$$

for $p > i > 1$ and $-1 + a_1 = pw/w^p = u$. Since $P(-1) = 0$ we see that $0 = (-1)^p - u$ modulo (w). Hence $a_1 \in (w)$ and the proof if the first part is done. The second part follows from a direct computation we omit. □

09F3 Lemma 15.115.14. Let A be a discrete valuation ring of mixed characteristic $(0, p)$ which contains a primitive pth root of 1. Let $P(t) \in A[t]$ be the polynomial of Lemma 15.115.13. Let $\xi \in K$. Let L be an extension of K obtained by adjoining a root of $P(z) = \xi$. Then L/K is Galois and one of the following happens

(1) $L = K$,
(2) L/K is unramified with respect to A of degree p,
(3) L/K is totally ramified with respect to A with ramification index p, and
(4) the integral closure B of A in L is a discrete valuation ring, $A \subset B$ is weakly unramified, and $A \to B$ induces a purely inseparable residue field extension of degree p.

Let π be a uniformizer of A. We have the following implications:

(A) If $\xi \in A$, then we are in case (1) or (2).
(B) If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by p and a is a unit in A, then we are in case (3)
(C) If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by p and the image of a in κ_A is not a pth power, then we are in case (4).

Proof. Adjoining a root of $P(z) = \xi$ is the same thing as adjoining a root of $y^p = w^p(1 + \xi)$. Since K contains a primitive pth root of 1 the extension is Galois of order dividing p by the discussion in Fields, Section 9.24. It immediately follows from the discussion in Section 15.112 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \to A[x]/(P(x) - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where p does not divide n. Let $B \subset L$ be the integral closure of A in L. If $C = B_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then it is clear that $p\text{ord}_C(z) = -n\text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by p. It follows that it is p and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) + \sum a_i \pi^{n-ik}(\pi^k z)^i = a$$

Since $A[y]/(y^p - \pi^{n-k}y - \sum a_i \pi^{n-ik} y^i - a)$ is a discrete valuation ring weakly unramified over A, the lemma follows. $\square$

Let A be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive pth root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 15.115.13. Let L be a finite extension of K. We say L/K is a degree p extension of finite level if L is a degree p extension of K obtained by adjoining a root of the equation $P(z) = \xi$ where $\xi \in K$ is an element with $w^p\xi \in \mathfrak{m}_A$.

This definition is relevant to the discussion in this section due to the following straightforward lemma.

09F4 Lemma 15.115.15. Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume that

(1) A has mixed characteristic $(0, p)$,
(2) $A \subset B$ is weakly unramified,
(3) B contains a primitive pth root of 1, and
(4) M/L is Galois of degree p.

Then there exists a finite Galois extension K_1/K totally ramified with respect to A which is either a weak solution for $A \to C$ or is such that M_1/L_1 is a degree p extension of finite level.

Proof. Let $\pi \in A$ be a uniformizer. By Kummer theory (Fields, Lemma 9.24.1) M is obtained from L by adjoining the root of $y^p = b$ for some $b \in L$.

If $\text{ord}_B(b)$ is prime to p, then we choose a degree p separable extension K_1/K totally ramified with respect to A (for example using Lemma 15.115.7). Let A_1 be the integral closure of A in K_1. By Lemma 15.115.3 the integral closure B_1 of B in $L_1 = L \otimes_K K_1$ is a discrete valuation ring weakly unramified over A_1. If K_1/K is not a weak solution for $A \to C$, then the integral closure C_1 of C in $M_1 = M \otimes_K K_1$ is a discrete valuation ring and $B_1 \to C_1$ has ramification index p. In this case, the field M_1 is obtained from L_1 by adjoining the pth root of b with $\text{ord}_{B_1}(b)$ divisible by p. Replacing A by A_1, etc we may assume that $b = \pi^n u$ where $u \in B$ is a unit and n is divisible by p. Of course, in this case the extension M is obtained from L by adjoining the pth root of a unit.

Suppose M is obtained from L by adjoining the root of $y^p = u$ for some unit u of B. If the residue class of u in κ_B is not a pth power, then $B \subset C$ is weakly unramified (Lemma 15.115.8) and we are done. Otherwise, we can replace our choice of y by y/v where v^p and u have the same image in κ_B. After such a replacement we have

$$y^p = 1 + \pi b$$

for some $b \in B$. Then we see that $P(z) = \pi b/w^p$ where $z = (y - 1)/w$. Thus we see that the extension is a degree p extension of finite level with $\xi = \pi b/w^p$. $\square$

Let A be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive pth root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 15.115.13. Let L be a degree p extension of K of finite level. Choose $z \in L$ generating L over K with

$\xi = P(z) \in K$. Choose a uniformizer π for A and write $w = u\pi^{e_1}$ for some integer $e_1 = \text{ord}_A(w)$ and unit $u \in A$. Finally, pick $n \geq 0$ such that

$$\pi^n \xi \in A$$

The level of L/K is the smallest value of the quantity n/e_1 taking over all z generating L/K with $\xi = P(z) \in K$.

We make a couple of remarks. Since the extension is of finite level we know that we can choose z such that $n < pe_1$. Thus the level is a rational number contained in $[0, p)$. If the level is zero then L/K is unramified with respect to A by Lemma 15.115.14. Our next goal is to lower the level.

09F5 Lemma 15.115.16. Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume

1. (1) A has mixed characteristic $(0, p)$,
2. (2) $A \subset B$ weakly unramified,
3. (3) B contains a primitive pth root of 1,
4. (4) M/L is a degree p extension of finite level $l > 0$,
5. (5) $\kappa_A = \bigcap_{n \geq 1} \kappa_B^{p^n}$.

Then there exists a finite separable extension K_1 of K totally ramified with respect to A such that either K_1 is a weak solution for $A \to C$, or the extension M_1/L_1 is a degree p extension of finite level $\leq \max(0, l-1, 2l-p)$.

Proof. Let $\pi \in A$ be a uniformizer. Let $w \in B$ and $P \in B[t]$ be as in Lemma 15.115.13 (for B). Set $e_1 = \text{ord}_B(w)$, so that w and π^{e_1} are associates in B. Pick $z \in M$ generating M over L with $\xi = P(z) \in K$ and n such that $\pi^n \xi \in B$ as in the definition of the level of M over L, i.e., $l = n/e_1$.

The proof of this lemma is completely similar to the proof of Lemma 15.115.12. To explain what is going on, observe that

09F6 (15.115.16.1) $$P(z) \equiv z^p - z \bmod \pi^{-n+e_1}B$$

for any $z \in L$ such that $\pi^{-n}P(z) \in B$ (use that z has valuation at worst $-n/p$ and the shape of the polynomial P). Moreover, we have

09F7 (15.115.16.2) $$\xi_1 + \xi_2 + w^p\xi_1\xi_2 \equiv \xi_1 + \xi_2 \bmod \pi^{-2n+pe_1}B$$

for $\xi_1, \xi_2 \in \pi^{-n}B$. Finally, observe that $n - e_1 = (l-1)/e_1$ and $-2n + pe_1 = -(2l-p)e_1$. Write $m = n - e_1 \max(0, l-1, 2l-p)$. The above shows that doing calculations in $\pi^{-n}B/\pi^{-n+m}B$ the polynomial P behaves exactly as the polynomial $z^p - z$. This explains why the lemma is true but we also give the details below.

Assumption (4) implies that κ_A is perfect. Observe that $m \leq e_1$ and hence A/π^m is annihilated by w and hence p. Thus we may choose compatible ring maps $\overline{\sigma} : \kappa_A \to A/\pi^m A$ and $\overline{\sigma} : \kappa_B \to B/\pi^m B$ as in Lemma 15.115.10. We lift the second of these to a map of sets $\sigma : \kappa_B \to B$. Then we can write

$$\xi = \sum\nolimits_{i=n,\ldots,n-m+1} \sigma(\lambda_i)\pi^{-i} + \pi^{-n+m)}b$$

for some $\lambda_i \in \kappa_B$ and $b \in B$. Let

$$I = \{i \in \{n, \ldots, n-m+1\} \mid \lambda_i \in \kappa_A\}$$

and

$$J = \{j \in \{n, \ldots, n-m+1\} \mid \lambda_i \notin \kappa_A\}$$

We will argue by induction on the size of the finite set J.

The case $J = \emptyset$. Here for all $i \in \{n, \ldots, n-m+1\}$ we have $\sigma(\lambda_i) = a_i + \pi^{n-m}b_i$ for some $a_i \in A$ and $b_i \in B$ by our choice of $\overline{\sigma}$. Thus $\xi = \pi^{-n}a + \pi^{-n+m}b$ for some $a \in A$ and $b \in B$. If $p|n$, then we write $a = a_0^p + \pi a_1$ for some $a_0, a_1 \in A$ (as the residue field of A is perfect). Set $z_1 = -\pi^{-n/p}a_0$. Note that $P(z_1) \in \pi^{-n}B$ and that $z + z_1 + wzz_1$ is an element generating M over L (note that $wz_1 \neq -1$ as $n < pe_1$). Moreover, by Lemma 15.115.13 we have

$$P(z + z_1 + wzz_1) = P(z) + P(z_1) + w^pP(z)P(z_1) \in K$$

and by equations (15.115.16.1) and (15.115.16.2) we have

$$P(z) + P(z_1) + w^pP(z)P(z_1) \equiv \xi + z_1^p - z_1 \bmod \pi^{-n+m}B$$

for some $b' \in B$. This contradict the minimality of n! Thus p does not divide n. Consider the degree p extension K_1 of K given by $P(y) = -\pi^{-n}a$. By Lemma 15.115.14 this extension is separable and totally ramified with respect to A. Thus $L_1 = L \otimes_K K_1$ is a field and $A_1 \subset B_1$ is weakly unramified (Lemma 15.115.3). By Lemma 15.115.14 the ring $M_1 = M \otimes_K K_1$ is either a product of p copies of L_1 (in which case we are done) or a field extension of L_1 of degree p. Moreover, in the second case, either C_1 is weakly unramified over B_1 (in which case we are done) or M_1/L_1 is degree p, Galois, totally ramified with respect to B_1. In this last case the extension M_1/L_1 is generated by the element $z + y + wzy$ and we see that $P(z + y + wzy) \in L_1$ and

$$\begin{aligned} P(z + y + wzy) &= P(z) + P(y) + w^pP(z)P(y) \\ &\equiv \xi - \pi^{-n}a \bmod \pi^{-n+m}B_1 \\ &\equiv 0 \bmod \pi^{-n+m}B_1 \end{aligned}$$

in exactly the same manner as above. By our choice of m this means exactly that M_1/L_1 has level at most $\max(0, l-1, 2l-p)$. From now on we assume that $J \neq \emptyset$.

Suppose that $j', j \in J$ such that $j' = p^rj$ for some $r > 0$. Then we set

$$z_1 = -\sigma(\lambda_j)\pi^{-j} - \sigma(\lambda_j^p)\pi^{-pj} - \ldots - \sigma(\lambda_j^{p^{r-1}})\pi^{-p^{r-1}j}$$

and we change z into $z' = z + z_1 + wzz_1$. Observe that $z' \in M$ generates M over L and that we have $\xi' = P(z') = P(z) + P(z_1) + wP(z)P(z_1) \in L$ with

$$\xi' \equiv \xi - \sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^{p^r})\pi^{-j'} \bmod \pi^{-n+m}B$$

by using equations (15.115.16.1) and (15.115.16.2) as above. Writing

$$\xi' = \sum\nolimits_{i=n,\ldots,n-m+1} \sigma(\lambda'_i)\pi^{-i} + \pi^{-n+m}b'$$

as before we find that $\lambda'_i = \lambda_i$ for $i \neq j, j'$ and $\lambda'_j = 0$. Thus the set J has gotten smaller. By induction on the size of J we may assume there is no pair j, j' of J such that j'/j is a power of p. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \kappa_B$ which is not a pth power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that jp^{-r_j} is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r-r_j} > n$ for $j \in J$. Let K_1/K be the extension of degree p^r, totally ramified with respect to A, defined by $(\pi')^{p^r} = \pi$. Observe that π' is the uniformizer of the corresponding discrete valuation ring $A_1 \subset K_1$. Observe that $L_1 = L \otimes_K K_1$ is a field and L_1/L is totally ramified with respect to B (Lemma 15.115.3). Computing in the integral closure B_1 we get

$$\xi = \sum\nolimits_{i \in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + \pi^{-n+m}b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a qth power modulo π^m, i.e., modulo $(\pi')^{mp^r}$. Hence we can rewrite the above as

$$\xi = \sum\nolimits_{i \in I} x_i^{p^r}(\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}}(\pi')^{-jp^r} + \pi^{-n+m}b_1$$

Similar to our choice in the previous paragraph we set

$$\begin{aligned} z_1 - &\sum\nolimits_{i \in I} \left(x_i(\pi')^{-i} + \ldots + x_i^{p^{r-1}}(\pi')^{-ip^{r-1}}\right) \\ &- \sum\nolimits_{j \in J} \left(\sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + \ldots + \sigma(\mu_j)^{p^{r_j-1}}(\pi')^{-jp^{r-1}}\right) \end{aligned}$$

and we change our choice of z into $z' = z + z_1 + wzz_1$. Then z' generates M_1 over L_1 and $\xi' = P(z') = P(z) + P(z_1) + w^pP(z)P(z_1) \in L_1$ and a calculation shows that

$$\xi' \equiv \sum\nolimits_{i \in I} x_i(\pi')^{-i} + \sum\nolimits_{j \in J} \sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + (\pi')^{(-n+m)p^r}b'_1$$

for some $b'_1 \in B_1$. There is a unique j such that jp^{r-r_j} is maximal and jp^{r-r_j} is bigger than $i \in I$. If $jp^{r-r_j} \leq (n-m)p^r$ then the level of the extension M_1/L_1 is less than $\max(0, l-1, 2l-p)$. If not, then, as p divides jp^{r-r_j}, we see that M_1/L_1 falls into case (C) of Lemma 15.115.14. This finishes the proof. □

09F8 Lemma 15.115.17. Let $A \subset B \subset C$ be extensions of discrete valuation rings with fraction fields $K \subset L \subset M$. Assume

(1) the residue field k of A is algebraically closed of characteristic $p > 0$,
(2) A and B are complete,
(3) $A \to B$ is weakly unramified,
(4) M is a finite extension of L,
(5) $k = \bigcap_{n \geq 1} \kappa_B^{p^n}$

Then there exists a finite extension K_1/K which is a weak solution for $A \to C$.

Proof. Let M' be any finite extension of L and consider the integral closure C' of B in M'. Then C' is finite over B as B is Nagata by Algebra, Lemma 10.162.8. Moreover, C' is a discrete valuation ring, see discussion in Remark 15.114.1. Moreover C' is complete as a B-module, hence complete as a discrete valuation ring, see Algebra, Section 10.96. It follows in particular that C is the integral closure of B in M (by definition of valuation rings as maximal for the relation of domination).

Let $M \subset M'$ be a finite extension and let $C' \subset M'$ be the integral closure of B as above. By Lemma 15.115.4 it suffices to prove the result for $A \to B \to C'$. Hence we may assume that M/L is normal, see Fields, Lemma 9.16.3.

If M/L is normal, we can find a chain of finite extensions

$$L = L^0 \subset L^1 \subset L^2 \subset \ldots \subset L^r = M$$

such that each extension L^{j+1}/L^j is either:

(a) purely inseparable of degree p,

(b) totally ramified with respect to B^j and Galois of degree p,
(c) totally ramified with respect to B^j and Galois cyclic of order prime to p,
(d) Galois and unramified with respect to B^j.

Here B^j is the integral closure of B in L^j. Namely, since M/L is normal we can write it as a compositum of a Galois extension and a purely inseparable extension (Fields, Lemma 9.27.3). For the purely inseparable extension the existence of the filtration is clear. In the Galois case, note that G is "the" decomposition group and let $I \subset G$ be the inertia group. Then on the one hand I is solvable by Lemma 15.112.5 and on the other hand the extension M^I/L is unramified with respect to B by Lemma 15.112.8. This proves we have a filtration as stated.

We are going to argue by induction on the integer r. Suppose that we can find a finite extension K_1/K which is a weak solution for $A \to B^1$ where B^1 is the integral closure of B in L^1. Let K_1' be the normal closure of K_1/K (Fields, Lemma 9.16.3). Since A is complete and the residue field of A is algebraically closed we see that K_1'/K_1 is separable and totally ramified with respect to A_1 (some details omitted). Hence K_1'/K is a weak solution for $A \to B^1$ as well by Lemma 15.115.3. In other words, we may and do assume that K_1 is a normal extension of K. Having done so we consider the sequence

$$L_1^0 = (L^0 \otimes_K K_1)_{red} \subset L_1^1 = (L^1 \otimes_K K_1)_{red} \subset \ldots \subset L_1^r = (L^r \otimes_K K_1)_{red}$$

and the corresponding integral closures B_1^i. Note that $C_1 = B_1^r$ is a product of discrete valuation rings which are transitively permuted by $G = \text{Aut}(K_1/K)$ by Lemma 15.115.6. In particular all the extensions of discrete valuation rings $A_1 \to (C_1)_{\mathfrak m}$ are isomorphic and a weak solution for one will be a weak solution for all of them. We can apply the induction hypothesis to the sequence

$$A_1 \to (B_1^1)_{B_1^1 \cap \mathfrak m} \to (B_1^2)_{B_1^2 \cap \mathfrak m} \to \ldots \to (B_1^r)_{B_1^r \cap \mathfrak m} = (C_1)_{\mathfrak m}$$

to get a weak solution K_2/K_1 for $A_1 \to (C_1)_{\mathfrak m}$. The extension K_2/K will then be a weak solution for $A \to C$ by what we said before. Note that the induction hypothesis applies: the ring map $A_1 \to (B_1^1)_{B_1^1 \cap \mathfrak m}$ is weakly unramified by our choice of K_1 and the sequence of fraction field extensions each still have one of the properties (a), (b), (c), or (d) listed above. Moreover, observe that for any finite extension $\kappa_B \subset \kappa$ we still have $k = \bigcap \kappa^{p^n}$.

Thus everything boils down to finding a weak solution for $A \subset C$ when the field extension M/L satisfies one of the properties (a), (b), (c), or (d).

Case (d). This case is trivial as here $B \to C$ is unramified already.

Case (c). Say M/L is cyclic of order n prime to p. Because M/L is totally ramified with respect to B we see that the ramification index of $B \subset C$ is n and hence the ramification index of $A \subset C$ is n as well. Choose a uniformizer $\pi \in A$ and set $K_1 = K[\pi^{1/n}]$. Then K_1/K is a solution for $A \subset C$ by Abhyankar's lemma (Lemma 15.114.4).

Case (b). We divide this case into the mixed characteristic case and the equicharacteristic case. In the equicharacteristic case this is Lemma 15.115.12. In the mixed characteristic case, we first replace K by a finite extension to get to the situation where M/L is a degree p extension of finite level using Lemma 15.115.15. Then the level is a rational number $l \in [0, p)$, see discussion preceding Lemma 15.115.16. If the level is 0, then $B \to C$ is weakly unramified and we're done. If not, then we can replacing the field K by a finite extension to obtain a new situation with level $l' \leq \max(0, l-1, 2l-p)$ by Lemma 15.115.16. If $l = p - \epsilon$ for $\epsilon < 1$ then we see that $l' \leq p - 2\epsilon$. Hence after a finite number of replacements we obtain a case with level $\leq p-1$. Then after at most $p-1$ more such replacements we reach the situation where the level is zero.

Case (a) is Lemma 15.115.9. This is the only case where we possibly need a purely inseparable extension of K, namely, in case (2) of the statement of the lemma we win by adjoining a pth power of the element π. This finishes the proof of the lemma. □

At this point we have collected all the lemmas we need to prove the main result of this section.

09F9 Theorem 15.115.18 (Epp). Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the characteristic of κ_A is $p > 0$, assume that every element of

$$\bigcap\nolimits_{n \geq 1} \kappa_B^{p^n}$$

is separable algebraic over κ_A. Then there exists a finite extension K_1/K which is a weak solution for $A \to B$ as defined in Definition 15.115.1.

Proof. If the characteristic of κ_A is zero or if the residue characteristic is p, the ramification index is prime to p, and the residue field extension is separable, then this follows from Abhyankar's lemma (Lemma 15.114.4). Namely, suppose the ramification index is e. Choose a uniformizer $\pi \in A$. Let K_1/K be the extension obtained by adjoining an eth root of π. By Lemma 15.114.2 we see that the integral closure A_1 of A in K_1 is a discrete valuation ring with ramification index over A. Thus $A_1 \to (B_1)_\mathfrak{m}$ is formally smooth in the $\mathfrak{m}$-adic topology for all maximal ideals $\mathfrak{m}$ of B_1 by Lemma 15.114.4 and a fortiori these are weakly unramified extensions of discrete valuation rings.

From now on we let p be a prime number and we assume that κ_A has characteristic p. We first apply Lemma 15.115.5 to reduce to the case that A and B have separably closed residue fields. Since κ_A and κ_B are replaced by their separable algebraic closures by this procedure we see that we obtain

$$\kappa_A \supset \bigcap\nolimits_{n \geq 1} \kappa_B^{p^n}$$

from the condition of the theorem.

Let $\pi \in A$ be a uniformizer. Let $A^\wedge$ and $B^\wedge$ be the completions of A and B. We have a commutative diagram

$$\begin{tikzcd} B \arrow[r] & B^\wedge \\ A \arrow[u] \arrow[r] & A^\wedge \arrow[u] \end{tikzcd}$$

of extensions of discrete valuation rings. Let $K^\wedge$ be the fraction field of $A^\wedge$. Suppose that we can find a finite extension $M/K^\wedge$ which is (a) a weak solution for $A^\wedge \to B^\wedge$ and (b) a compositum of a separable extension and an extension obtained by adjoining a p-power root of π. Then by Lemma 15.113.2 we can find a finite extension K_1/K such that $K^\wedge \otimes_K K_1 = M$. Let A_1, resp. $A_1^\wedge$ be the integral closure of A, resp. $A^\wedge$ in K_1, resp. M. Since $A \to A^\wedge$ is formally smooth in the

$\mathfrak{m}^\wedge$-adic topology (Lemma 15.111.5) we see that $A_1 \to A_1^\wedge$ is formally smooth in the $\mathfrak{m}_1^\wedge$-adic topology (Lemma 15.114.3 and A_1 and $A_1^\wedge$ are discrete valuation rings by discussion in Remark 15.114.1). We conclude from Lemma 15.115.4 part (2) that K_1/K is a weak solution for $A \to B^\wedge$. Applying Lemma 15.115.4 part (1) we see that K_1/K is a weak solution for $A \to B$.

Thus we may assume A and B are complete discrete valuation rings with separably closed residue fields of characteristic p and with $\kappa_A \supset \bigcap_{n \geq 1} \kappa_B^{p^n}$. We are also given a uniformizer $\pi \in A$ and we have to find a weak solution for $A \to B$ which is a compositum of a separable extension and a field obtained by taking p-power roots of π. Note that the second condition is automatic if A has mixed characteristic.

Set $k = \bigcap_{n \geq 1} \kappa_B^{p^n}$. Observe that k is an algebraically closed field of characteristic p. If A has mixed characteristic let Λ be a Cohen ring for k and in the equicharacteristic case set $\Lambda = k[[t]]$. We can choose a ring map $\Lambda \to A$ which maps t to π in the equicharacteristic case. In the equicharacteristic case this follows from the Cohen structure theorem (Algebra, Theorem 10.160.8) and in the mixed characteristic case this follows as $\mathbf{Z}_p \to \Lambda$ is formally smooth in the adic topology (Lemmas 15.111.5 and 15.37.5). Applying Lemma 15.115.4 we see that it suffices to prove the existence of a weak solution for $\Lambda \to B$ which in the equicharacteristic p case is a compositum of a separable extension and a field obtained by taking p-power roots of t. However, since $\Lambda = k[[t]]$ in the equicharacteristic case and any extension of $k((t))$ is such a compositum, we can now drop this requirement!

Thus we arrive at the situation where A and B are complete, the residue field k of A is algebraically closed of characteristic $p > 0$, we have $k = \bigcap \kappa_B^{p^n}$, and in the mixed characteristic case p is a uniformizer of A (i.e., A is a Cohen ring for k). If A has mixed characteristic choose a Cohen ring Λ for κ_B and in the equicharacteristic case set $\Lambda = \kappa_B[[t]]$. Arguing as above we may choose a ring map $A \to \Lambda$ lifting $k \to \kappa_B$ and mapping a uniformizer to a uniformizer. Since $k \subset \kappa_B$ is separable the ring map $A \to \Lambda$ is formally smooth in the adic topology (Lemma 15.111.5). Hence we can find a ring map $\Lambda \to B$ such that the composition $A \to \Lambda \to B$ is the given ring map $A \to B$ (see Lemma 15.37.5). Since Λ and B are complete discrete valuation rings with the same residue field, B is finite over Λ (Algebra, Lemma 10.96.12). This reduces us to the special case discussed in Lemma 15.115.17. □

15.116. Eliminating ramification, II

0GLQ In this section we use the results of Section 15.115 to obtain (separable) solutions in some cases.

0GLR Lemma 15.116.1. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If K_1/K is a solution for $A \subset B$, then for any finite extension K_2/K_1 the extension K_2/K is a solution for $A \subset B$.

Proof. This follows from Lemma 15.114.3. Details omitted. □

0GLS Lemma 15.116.2. Let $A \subset B$ be an extension of discrete valuation rings. If B is Nagata and the extension L/K of fraction fields is separable, then A is Nagata.

Proof. A discrete valuation ring is Nagata if and only if it is N-2. Let K_1/K be a finite purely inseparable field extension. We have to show that the integral closure A_1 of A in K_1 is finite over A, see Algebra, Lemma 10.161.12. Since L/K

is separable and K_1/K is purely inseparable, the algebra $L \otimes_K K_1$ is a field (by Algebra, Lemmas 10.43.6 and 10.46.10). Let B_1 be the integral closure of B in $L \otimes_K K_1$. Since B is Nagata, B_1 is finite over B. Since $B \otimes_A A_1 \subset B_1$ and B is Noetherian, we see that $B \otimes_A A_1$ is finite over B. As $A \to B$ is faithfully flat, this implies A_1 is finite over A, see Algebra, Lemma 10.83.2. □

0GLT Lemma 15.116.3. Let $A' \subset A$ be an extension of rings. Let $f \in A'$. Assume that (a) A is finite over A', (b) f is a nonzerodivisor on A, and (c) $A'_f = A_f$. Then there exists an integer $n_0 > 0$ such that for all $n \geq n_0$ the following is true: given a ring B', a nonzerodivisor $g \in B'$, and an isomorphism $\varphi' : A'/f^nA' \to B'/g^nB'$ with $\varphi'(f) \equiv g$, there is a finite extension $B' \subset B$ and an isomorphism $\varphi : A/fA \to B/gB$ compatible with φ'.

Proof. Since A is finite over A' and since $A'_f = A_f$ we can Cchoose $t > 0$ such that $f^tA \subset A'$. Set $n_0 = 2t$. Given n, B', g, φ' as in the statement of the lemma, denote $N \subset B'$ the set of elements $b \in B'$ such that $b \bmod g^nB' \in \varphi'(f^tA)$. Set $B = g^{-t}N$. As $f^tA' \subset f^tA$ and φ' sends f to g we have $g^tB' \subset N$, hence $B' \subset B$. Since $f^tA \cdot f^tA \subset f^t \cdot f^tA$ and φ' sends f to g, we see that $N \cdot N \subset g^tN$. Hence we obtain a multiplication on B extending the multiplication of B'. We have an isomorphism of A'/f^nA'-modules

$$A/f^tA' \xrightarrow{f^t} f^tA/f^nA' \xrightarrow{\varphi'} g^tB/g^nB' \xrightarrow{g^{-t}} B/g^tB'$$

where the module structures on the right are defined using φ'. Since A/f^tA' is a finite A'-module, we conclude that B/g^tB' is a finite B'-module and hence we see that $B' \to B$ is finite. Finally, we leave it to the reader to see that the displayed isomorphism of modules sends fA into gB and induces an isomorphism of rings $\varphi : A/fA \to B/gB$ compatible with φ' (it even induces an isomorphism $A/f^tA \to B/g^tB$ but we don't need this). □

0GLU Remark 15.116.4. The construction in Lemma 15.116.3 satisfies the following "functoriality". Suppose we have a commutative diagram

$$\begin{tikzcd} A'_2 \arrow[r] & A_2 \\ A'_1 \arrow[u] \arrow[r] & A_1 \arrow[u] \end{tikzcd}$$

with injective horizontal arrows. Suppose given an element $f \in A'_1$ such that $(A'_1 \subset A_1, f)$ and $(A'_2 \subset A_2, f)$ satisfy properties (a), (b), (c) of Lemma 15.116.3. Let $n_{0,1}$ and $n_{0,2}$ be the integers found in the lemma for these two situations. Finally, let $B'_1 \to B'_2$ be a ring map, let $g \in B'_1$ be a nonzerodivisor on B_1 and B_2, let $n \geq \max(n_{0,1}, n_{0,2})$, and let a commutative diagram

$$\begin{tikzcd} A'_2/f^nA'_2 \arrow[r, "\varphi'_2"'] & B'_2/g^nB'_2 \\ A'_1/f^nA'_1 \arrow[u] \arrow[r, "\varphi'_1"] & B'_2/g^nB'_2 \arrow[u] \end{tikzcd}$$

be given whose horizontal arrows are isomorphisms and where $\varphi_1'(f) \equiv g$. Then we obtain commutative diagrams

$$\begin{tikzcd} B_2' \arrow[r] & B_2 \\ B_1' \arrow[u] \arrow[r] & B_1 \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_2/fA_2 \arrow[r, "\varphi_2"'] & B_2/gB_2 \\ A_1/fA_1 \arrow[u] \arrow[r, "\varphi_1"] & B_2/gB_2 \arrow[u] \end{tikzcd}$$

where $(B_1' \subset B_1, \varphi_1)$ and $(B_2' \subset B_2, \varphi_2)$ are constructed as in the proof of Lemma 15.116.3. We omit the detailed verification.

0GLV Lemma 15.116.5. Let p be a prime number. Let $A \subset B$ be an extension of discrete valuation rings with fraction field extension L/K. Let $K_2/K_1/K$ be a tower of finite field extensions. Assume

(1) K has characteristic p,
(2) L/K is separable,
(3) B is Nagata,
(4) K_2 is a solution for $A \subset B$,
(5) K_2/K_1 is purely inseparable of degree p.

Then there exists a separable extension K_3/K_1 which is a solution for $A \subset B$.

Proof. Let us use notation as in Remark 15.114.1; we will use all the observations made there. Since L/K is separable, the algebra $L_1 = L \otimes_K K_1$ is reduced (Algebra, Lemma 10.43.6). Since B is Nagata, the ring extension $B \subset B_1$ is finite where B_1 is the integral closure of B in L_1 and B_1 is a Nagata ring. Similarly, the ring A is Nagata by Lemma 15.116.2 hence $A \subset A_1$ is finite and A_1 is a Nagata ring too. Moreover, the same assertions are true for K_2, i.e., $L_2 = L \otimes_K K_2$ is reduced, the ring extensions $A_1 \subset A_2$ and $B_1 \subset B_2$ are finite where A_2, resp. B_2 is the integral closure of A, resp. B in K_2, resp. L_2.

Let $\pi \in A$ be a uniformizer. Observe that π is a nonzerodivisor on K_1, K_2, A_1, A_2, L_1, L_2, B_1, and B_2 and we have $K_1 = (A_1)_\pi$, $K_2 = (A_2)_\pi$, $L_1 = (B_1)_\pi$, and $L_2 = (B_2)_\pi$. We may write $K_2 = K_1(\alpha)$ where $\alpha^p = a_1 \in K_1$, see Fields, Lemma 9.14.5. After multiplying α by a power of π we may and do assume $a_1 \in A_1$. For the rest of the proof it is convenient to write $K_2 = K_1[x]/(x^p - a_1)$ and $L_2 = L_1[x]/(x^p - a_1)$. Consider the extensions of rings

$$A_2' = A_1[x]/(x^p - a_1) \subset A_2 \quad \text{and} \quad B_2' = B_1[x]/(x^p - a_1) \subset B_2$$

We may apply Lemma 15.116.3 to $A_2' \subset A_2$ and $f = \pi^2$ and to $B_2' \subset B_2$ and $f = \pi^2$. Choose an integer n large enough which works for both of these.

Consider the algebras

$$K_3 = K_1[x]/(x^p - \pi^{2n}x - a_1) \quad \text{and} \quad L_3 = L_1[x]/(x^p - \pi^{2n}x - a_1)$$

Observe that K_3/K_1 and L_3/L_1 are finite étale algebra extensions of degree p. Consider the subrings

$$A_3' = A_1[x]/(x^p - \pi^n x - a_1) \quad \text{and} \quad B_3' = B_1[x]/(x^p - \pi^n x - a_1)$$

of $K_3 = (A'_2)_\pi$ and $L_3 = (B'_3)_\pi$. We are going to construct a commutative diagram

$$\begin{array}{ccc} B'_2/\pi^{2n}B'_2 & \xrightarrow[\psi']{} & B'_3/\pi^{2n}B'_3 \\ \uparrow & & \uparrow \\ A'_2/\pi^{2n}A'_2 & \xrightarrow{\varphi'} & A'_3/\pi^{2n}A'_3 \end{array}$$

Namely, φ' is the unique A_1-algebra isomorphism sending the class of x to the class of x. Simiarly, ψ' is the unique B_1-algebra isomorphism sending the class of x to the class of x. By our choice of n we obtain, via Lemma 15.116.3 and Remark 15.116.4 finite ring extensions $A'_3 \subset A_3$ and $B'_3 \subset B_3$ such that $A'_3 \to B'_3$ extends to a ring map $A_3 \to B_3$ and a commutative diagram

$$\begin{array}{ccc} B_2/\pi^2B_2 & \xrightarrow[\psi]{} & B_3/\pi^2B_3 \\ \uparrow & & \uparrow \\ A_2/\pi^2A_2 & \xrightarrow{\varphi} & A_3/\pi^2A_3 \end{array}$$

with all the properties asserted in the references mentioned above (in particular φ and ψ are isomorphisms).

With all of this data in hand, we can finish the proof. Namely, we first observe that A_3 and B_3 are finite products of Dedekind domains with π contained in all of the maximal ideals. Namely, if $\mathfrak p \subset A_3$ is a maximal ideal, then $\pi \in \mathfrak p$ as $A \to A_3$ is finite. Then $\mathfrak p/\pi^2A_3$ corresponds via φ to a maximal ideal in A_2/π^2A_2 which is principal as A_2 is a finite product of Dedekind domains. We conclude that $\mathfrak p/\pi^2A_3$ is principal and hence by Nakayama we see that $\mathfrak p(A_3)_\mathfrak p$ is principal. The same argument works for B_3. We conclude that A_3 is the integral closure of A in K_3 and that B_3 is the integral closure of B in L_3. Let $\mathfrak q \subset B_3$ be a maximal ideal lying over $\mathfrak p \subset A_3$. To finish the proof we have to show that $(A_3)_\mathfrak p \to (B_3)_\mathfrak q$ is formally smooth in the $\mathfrak q$-adic topology. By the criterion of Lemma 15.111.5 it suffices to show that $\mathfrak p(B_3)_\mathfrak q = \mathfrak q(B_3)_\mathfrak q$ and that the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is separable. This is true because we may check both assertions by looking at the ring map $A_3/\pi^2A_3 \to B_3/\pi^2B_3$ and this is isomorphic to the ring map $A_2/\pi^2A_2 \to B_2/\pi^2B_2$ where the corresponding statement holds by our assumption that K_2 is a solution for $A \subset B$. Some details omitted. ☐

0BRN Lemma 15.116.6. Let $A \subset B$ be an extension of discrete valuation rings. Assume

(1) the extension L/K of fraction fields is separable,
(2) B is Nagata, and
(3) there exists a solution for $A \subset B$.

Then there exists a separable solution for $A \subset B$.

Proof. The lemma is trivial if the characteristic of K is zero; thus we may and do assume that the characteristic of K is $p > 0$.

Let K_2/K be a solution for $A \to B$. We will use induction on the inseparable degree $[K_2 : K]_i$ (Fields, Definition 9.14.7) of K_2/K. If $[K_2 : K]_i = 1$, then K_2 is separable over K and we are done. If not, then there exists a subfield $K_2/K_1/K$ such that K_2/K_1 is purely inseparable of degree p (Fields, Lemmas 9.14.6 and 9.14.5). By Lemma 15.116.5 there exists a separable extension K_3/K_1 which is a solution for

$A \subset B$. Then $[K_3 : K]_i = [K_1 : K]_i = [K_2 : K]_i/p$ (Fields, Lemma 9.14.9) is smaller and we conclude by induction. $\square$

09IH Lemma 15.116.7. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume B is essentially of finite type over A. Let K'/K be an algebraic extension of fields such that the integral closure A' of A in K' is Noetherian. Then the integral closure B' of B in $L' = (L \otimes_K K')_{red}$ is Noetherian as well. Moreover, the map $\mathrm{Spec}(B') \to \mathrm{Spec}(A')$ is surjective and the corresponding residue field extensions are finitely generated field extensions.

Proof. Let $A \to C$ be a finite type ring map such that B is a localization of C at a prime $\mathfrak p$. Then $C' = C \otimes_A A'$ is a finite type A'-algebra, in particular Noetherian. Since $A \to A'$ is integral, so is $C \to C'$. Thus $B = C_{\mathfrak p} \subset C'_{\mathfrak p}$ is integral too. It follows that the dimension of $C'_{\mathfrak p}$ is 1 (Algebra, Lemma 10.112.4). Of course $C'_{\mathfrak p}$ is Noetherian. Let $\mathfrak q_1, \ldots, \mathfrak q_n$ be the minimal primes of $C'_{\mathfrak p}$. Let B'_i be the integral closure of $B = C_{\mathfrak p}$, or equivalently by the above of $C'_{\mathfrak p}$ in the field of fractions of $C'_{\mathfrak p'}/\mathfrak q_i$. It follows from Krull-Akizuki (Algebra, Lemma 10.119.12 applied to the finitely many localizations of $C'_{\mathfrak p}$ at its maximal ideals) that each B'_i is Noetherian. Moreover the residue field extensions in $C'_{\mathfrak p} \to B'_i$ are finite by Algebra, Lemma 10.119.10. Finally, we observe that $B' = \prod B'_i$ is the integral closure of B in $L' = (L \otimes_K K')_{red}$. $\square$

09II Proposition 15.116.8. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If B is essentially of finite type over A, then there exists a finite extension K_1/K which is a solution for $A \to B$ as defined in Definition 15.115.1.

See [dJ96, Lemma 2.13] for a special case.

Proof. Observe that a weak solution is a solution if the residue field of A is perfect, see Lemma 15.111.5. Thus the proposition follows immediately from Theorem 15.115.18 if the residue characteristic of A is 0 (and in fact we do not need the assumption that $A \to B$ is essentially of finite type). If the residue characteristic of A is $p > 0$ we will also deduce it from Epp's theorem.

Let $x_i \in A$, $i \in I$ be a set of elements mapping to a p-base of the residue field κ of A. Set

$$A' = \bigcup\nolimits_{n \geq 1} A[t_{i,n}]/(t_{i,n}^{p^n} - x_i)$$

where the transition maps send $t_{i,n+1}$ to $t_{i,n}^p$. Observe that A' is a filtered colimit of weakly unramified finite extensions of discrete valuation rings over A. Thus A' is a discrete valuation ring and $A \to A'$ is weakly unramified. By construction the residue field $\kappa' = A'/\mathfrak m_A A'$ is the perfection of κ.

Let K' be the fraction field of A'. We may apply Lemma 15.116.7 to the extension K'/K. Thus B' is a finite product of Dedekind domains. Let $\mathfrak m_1, \ldots, \mathfrak m_n$ be the maximal ideals of B'. Using Epp's theorem (Theorem 15.115.18) we find a weak solution K'_i/K' for each of the extensions $A' \subset B'_{\mathfrak m_i}$. Since the residue field of A' is perfect, these are actually solutions. Let K'_1/K' be a finite extension which contains each K'_i. Then K'_1/K' is still a solution for each $A' \subset B'_{\mathfrak m_i}$ by Lemma 15.116.1.

Let A'_1 be the integral closure of A in K'_1. Note that A'_1 is a Dedekind domain by the discussion in Remark 15.114.1 applied to $K' \subset K'_1$. Thus Lemma 15.116.7 applies to K'_1/K. Therefore the integral closure B'_1 of B in $L'_1 = (L \otimes_K K'_1)_{red}$ is a

Dedekind domain and because K'_1/K' is a solution for each $A' \subset B'_{\mathfrak{m}_i}$ we see that $(A'_1)_{A'_1 \cap \mathfrak{m}} \to (B'_1)_{\mathfrak{m}}$ is formally smooth in the $\mathfrak{m}$-adic topology for each maximal ideal $\mathfrak{m} \subset B'_1$.

By construction, the field K'_1 is a filtered colimit of finite extensions of K. Say $K'_1 = \mathrm{colim}_{i \in I} K_i$. For each i let A_i, resp. B_i be the integral closure of A, resp. B in K_i, resp. $L_i = (L \otimes_K K_i)_{red}$. Then it is clear that

$$A'_1 = \mathrm{colim}\, A_i \quad \text{and} \quad B'_1 = \mathrm{colim}\, B_i$$

Since the ring maps $A_i \to A'_1$ and $B_i \to B'_1$ are injective integral ring maps and since A'_1 and B'_1 have finite spectra, we see that for all i large enough the ring maps $A_i \to A'_1$ and $B_i \to B'_1$ are bijective on spectra. Once this is true, for all i large enough the maps $A_i \to A'_1$ and $B_i \to B'_1$ will be weakly unramified (once the uniformizer is in the image). It follows from multiplicativity of ramification indices that $A_i \to B_i$ induces weakly unramified maps on all localizations at maximal ideals of B_i for such i. Increasing i a bit more we see that

$$B_i \otimes_{A_i} A'_1 \longrightarrow B'_1$$

induces surjective maps on residue fields (because the residue fields of B'_1 are finitely generated over those of A'_1 by Lemma 15.116.7). Picture of residue fields at maximal ideals lying under a chosen maximal ideal of B'_1:

$$\begin{tikzcd}
\kappa_{B_i} \arrow[r] & \kappa_{B_{i'}} \arrow[r] & \ldots & \kappa_{B'_1} \\
\kappa_{A_i} \arrow[r] \arrow[u] & \kappa_{A_{i'}} \arrow[r] \arrow[u] & \ldots & \kappa_{A'_1} \arrow[u]
\end{tikzcd}$$

Thus κ_{B_i} is a finitely generated extension of κ_{A_i} such that the compositum of κ_{B_i} and $\kappa_{A'_1}$ in $\kappa_{B'_1}$ is separable over $\kappa_{A'_1}$. Then that happens already at a finite stage: for example, say $\kappa_{B'_1}$ is finite separable over $\kappa_{A'_1}(x_1, \ldots, x_n)$, then just increase i such that $x_1, \ldots, x_n$ are in κ_{B_i} and such that all generators satisfy separable polynomial equations over $\kappa_{A_i}(x_1, \ldots, x_n)$. This means that $A_i \to (B_i)_{\mathfrak{m}}$ is formally smooth in the $\mathfrak{m}$-adic topology for all maximal ideals $\mathfrak{m}$ of B_i and the proof is complete. □

0BRP Lemma 15.116.9. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume

(1) B is essentially of finite type over A,
(2) either A or B is a Nagata ring, and
(3) L/K is separable.

Then there exists a separable solution for $A \to B$ (Definition 15.115.1).

Proof. Observe that if A is Nagata, then so is B (Algebra, Lemma 10.162.6 and Proposition 10.162.15). Thus the lemma follows on combining Proposition 15.116.8 and Lemma 15.116.6. □

15.117. Picard groups of rings

0AFW We first define invertible modules as follows.

0B8H **Definition 15.117.1.** Let R be a ring. An R-module M is *invertible* if the functor

$$\text{Mod}_R \longrightarrow \text{Mod}_R, \quad N \longmapsto M \otimes_R N$$

is an equivalence of categories. An invertible R-module is said to be *trivial* if it is isomorphic to R as an R-module.

0B8I **Lemma 15.117.2.** Let R be a ring. Let M be an R-module. Equivalent are

(1) M is finite locally free module of rank 1,
(2) M is invertible, and
(3) there exists an R-module N such that $M \otimes_R N \cong R$.

Moreover, in this case the module N in (3) is isomorphic to $\text{Hom}_R(M, R)$.

Proof. Assume (1). Consider the module $N = \text{Hom}_R(M, R)$ and the evaluation map $M \otimes_R N = M \otimes_R \text{Hom}_R(M, R) \to R$. If $f \in R$ such that $M_f \cong R_f$, then the evaluation map becomes an isomorphism after localization at f (details omitted). Thus we see the evaluation map is an isomorphism by Algebra, Lemma 10.23.2. Thus (1) $\Rightarrow$ (3).

Assume (3). Then the functor $K \mapsto K \otimes_R N$ is a quasi-inverse to the functor $K \mapsto K \otimes_R M$. Thus (3) $\Rightarrow$ (2). Conversely, if (2) holds, then $K \mapsto K \otimes_R M$ is essentially surjective and we see that (3) holds.

Assume the equivalent conditions (2) and (3) hold. Denote $\psi : M \otimes_R N \to R$ the isomorphism from (3). Choose an element $\xi = \sum_{i=1,\ldots,n} x_i \otimes y_i$ such that $\psi(\xi) = 1$. Consider the isomorphisms

$$M \to M \otimes_R M \otimes_R N \to M$$

where the first arrow sends x to $\sum x_i \otimes x \otimes y_i$ and the second arrow sends $x \otimes x' \otimes y$ to $\psi(x' \otimes y)x$. We conclude that $x \mapsto \sum \psi(x \otimes y_i)x_i$ is an automorphism of M. This automorphism factors as

$$M \to R^{\oplus n} \to M$$

where the first arrow is given by $x \mapsto (\psi(x \otimes y_1), \ldots, \psi(x \otimes y_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. In this way we conclude that M is a direct summand of a finite free R-module. This means that M is finite locally free (Algebra, Lemma 10.78.2). Since the same is true for N by symmetry and since $M \otimes_R N \cong R$, we see that M and N both have to have rank 1. $\square$

The set of isomorphism classes of these modules is often called the class group or Picard group of R. The group structure is determined by assigning to the isomorphism classes of the invertible modules L and L' the isomorphism class of $L \otimes_R L'$. The inverse of an invertible module L is the module

$$L^{\otimes -1} = \text{Hom}_R(L, R),$$

because as seen in the proof of Lemma 15.117.2 the evaluation map $L \otimes_R L^{\otimes -1} \to R$ is an isomorphism. Let us denote the Picard group of R by $\text{Pic}(R)$.

0BCH **Lemma 15.117.3.** Let R be a UFD. Then $\text{Pic}(R)$ is trivial.

Proof. Let L be an invertible R-module. By Lemma 15.117.2 we see that L is a finite locally free R-module. In particular L is torsion free and finite over R. Pick a nonzero element $\varphi \in \text{Hom}_R(L, R)$ of the dual invertible module. Then

$I = \varphi(L) \subset R$ is an ideal which is an invertible module. Pick a nonzero $f \in I$ and let

$$f = up_1^{e_1} \ldots p_r^{e_r}$$

be the factorization into prime elements with p_i pairwise distinct. Since L is finite locally free there exist $a_i \in R$, $a_i \not\in (p_i)$ such that $I_{a_i} = (g_i)$ for some $g_i \in R_{a_i}$. Then p_i is still a prime element of the UFD R_{a_i} and we can write $g_i = p_i^{c_i} g'_i$ for some $g'_i \in R_{a_i}$ not divisible by p_i. Since $f \in I_{a_i}$ we see that $e_i \geq c_i$. We claim that I is generated by $h = p_1^{c_1} \ldots p_r^{c_r}$ which finishes the proof.

To prove the claim it suffices to show that I_a is generated by h for any $a \in R$ such that I_a is a principal ideal (Algebra, Lemma 10.23.2). Say $I_a = (g)$. Let $J \subset \{1, \ldots, r\}$ be the set of i such that p_i is a nonunit (and hence a prime element) in R_a. Because $f \in I_a = (g)$ we find the prime factorization $g = v \prod_{i \in J} p_j^{b_j}$ with v a unit and $b_j \leq e_j$. For each $j \in J$ we have $I_{aa_j} = gR_{aa_j} = g_j R_{aa_j}$, in other words g and g_j map to associates in R_{aa_j}. By uniqueness of factorization this implies that $b_j = c_j$ and the proof is complete. □

15.118. Determinants

0FJ9 Let R be a ring. Let M be a finite projective R-module. There exists a product decomposition $R = R_0 \times \ldots \times R_t$ such that in the corresponding decomposition $M = M_0 \times \ldots \times M_t$ of M we have that M_i is finite locally free of rank i over R_i. This follows from Algebra, Lemma 10.78.2 (to see that the rank is locally constant) and Algebra, Lemmas 10.21.3 and 10.24.3 (to decompose R into a product). In this situation we define

$$\det(M) = \wedge^0_{R_0}(M_0) \times \ldots \times \wedge^t_{R_t}(M_t)$$

as an R-module. This is a finite locally free module of rank 1 as each term is finite locally free of rank 1. If $\varphi : M \to N$ is an isomorphism of finite projective R-modules, then we obtain a canonical isomorphism

$$\det(\varphi) : \det(M) \longrightarrow \det(N)$$

of locally free modules of rank 1. More generally, if for all primes $\mathfrak{p}$ of R the ranks of the free modules $M_\mathfrak{p}$ and $N_\mathfrak{p}$ are the same, then any R-module homomorphism $\varphi : M \to N$ induces an R-module map $\det(\varphi) : \det(M) \to \det(N)$. Finally, if $M = N$ then $\det(\varphi) : \det(M) \to \det(M)$ is an endomorphism of an invertible R-module. Since $R = \mathrm{Hom}_R(L, L)$ for an invertible R-module we may and do view $\det(\varphi)$ as an element of R. In this way we obtain the determinant

$$\det : \mathrm{Hom}_R(M, M) \longrightarrow R$$

which is a multiplicative map.

0FJA Remark 15.118.1. Let R be a ring. Let M be a finite projective R-module. Then we can consider the graded commutative R-algebra exterior algebra $\wedge^*_R(M)$ on M over R. A formula for $\det(M)$ is that $\det(M) \subset \wedge^*_R(M)$ is the annihilator of $M \subset \wedge^*_R(M)$. This is sometimes useful as it does not refer to the decomposition of R into a product. Of course, to prove this satisfies the desired properties one has to either decompose R into a product (as above), or one has to look at the localizations at primes of R.

Next, we consider what happens to the determinant give a short exact sequence of finite projective modules.

0FJB Lemma 15.118.2. Let R be a ring. Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of finite projective R-modules. Then there is a canonical isomorphism

$$\gamma : \det(M') \otimes \det(M'') \longrightarrow \det(M)$$

First proof. First proof. Decompose R into a product of rings R_{ij} such that $M' = \prod M'_{ij}$ and $M'' = \prod M''_{ij}$ where M'_{ij} has rank i and M''_{ij} has rank j. Of course then $M = \prod M_{ij}$ and M_{ij} has rank $i + j$. This reduces us to the case where M' and M'' have constant rank say i and j. In this case we have to construct a canonical map

$$\wedge^i(M') \otimes \wedge^j(M'') \longrightarrow \wedge^{i+j}(M)$$

To do this choose $m'_1, \ldots, m'_i$ in M' and $m''_1, \ldots, m''_j$ in M''. Denote $m_1, \ldots, m_i \in M$ the images of $m'_1, \ldots, m'_i$ and denote $m_{i+1}, \ldots, m_{i+j} \in M$ elements mapping to $m''_1, \ldots, m''_j$ in M''. Our rule will be that

$$m'_1 \wedge \ldots \wedge m'_i \otimes m''_1 \wedge \ldots \wedge m''_j \longmapsto m_1 \wedge \ldots \wedge m_{i+j}$$

We omit the detailed proof that this is well defined and an isomorphism. $\square$

Second proof. We will use the description of $\det(M)$, $\det(M')$, and $\det(M'')$ given in Remark 15.118.1. Consider the R-algebra maps $\wedge^*_R(M') \to \wedge^*_R(M)$ and $\wedge^*_R(M) \to \wedge^*_R(M'')$. The first is injective and the second is surjective. Take an element $x' \in \det(M') \subset \wedge^*_R(M')$ and an element $x'' \in \det(M'') \subset \wedge^*_R(M'')$. Choose an element $y'' \in \wedge^*(M)$ mapping to x'' and set

$$\gamma(x' \otimes x'') = x' \wedge y'' \in \det(M) \subset \wedge^*_R(M)$$

The reader verifies easily by looking at localizations at primes that this well defined and an isomorphism. Moreover, this construction gives the same map as the construction given in the first proof. $\square$

0FJC Lemma 15.118.3. Let R be a ring. Let

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "u"] & M \arrow[r] \arrow[d, "v"] & M'' \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & K' \arrow[r] & K \arrow[r] & K'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of finite projective R-modules whose vertical arrows are isomorphisms. Then we get a commutative diagram of isomorphisms

$$\begin{tikzcd} \det(M') \otimes \det(M'') \arrow[r, "\gamma"'] \arrow[d, "\det(u)\otimes\det(w)"'] & \det(M) \arrow[d, "\det(v)"] \\ \det(K') \otimes \det(K'') \arrow[r, "\gamma"] & \det(K) \end{tikzcd}$$

where the horizontal arrows are the ones constructed in Lemma 15.118.2.

Proof. Omitted. Hint: use the second construction of the maps γ in Lemma 15.118.2. $\square$

0FJD Lemma 15.118.4. Let R be a ring. Let

$$K \subset L \subset M$$

be R-modules such that K, L/K, and M/L are finite projective R-modules. Then the diagram

$$\begin{tikzcd} \det(K) \otimes \det(L/K) \otimes \det(M/L) \arrow[r] \arrow[d] & \det(L) \otimes \det(M/L) \arrow[d] \\ \det(K) \otimes \det(M/K) \arrow[r] & \det(M) \end{tikzcd}$$

commutes where the maps are those of Lemma 15.118.2.

Proof. Omitted. Hint: after localizing at a prime of R we can assume $K \subset L \subset M$ is isomorphic to $R^{\oplus a} \subset R^{\oplus a+b} \subset R^{\oplus a+b+c}$ and in this case the result is an evident computation. ☐

0FJE Lemma 15.118.5. Let R be a ring. Let M' and M'' be two finite projective R-modules. Then the diagram

$$\begin{tikzcd} \det(M') \otimes \det(M'') \arrow[r] \arrow[d, "\epsilon\cdot(\text{switch tensors})"'] & \det(M' \oplus M'') \arrow[d, "\det(\text{swith summands})"] \\ \det(M'') \otimes \det(M') \arrow[r] & \det(M'' \oplus M') \end{tikzcd}$$

commutes where $\epsilon = \det(-\mathrm{id}_{M' \otimes M''}) \in R^*$ and the horizontal arrows are those of Lemma 15.118.2.

Proof. Omitted. ☐

0FJF Lemma 15.118.6. Let R be a ring. Let M, N be finite projective R-modules. Let $a : M \to N$ and $b : N \to M$ be R-linear maps. Then

$$\det(\mathrm{id} + a \circ b) = \det(\mathrm{id} + b \circ a)$$

as elements of R.

Proof. It suffices to prove the assertion after replacing R by a localization at a prime ideal. Thus we may assume R is local and M and N are finite free. In this case we have to prove the equality

$$\det(I_n + AB) = \det(I_m + BA)$$

of usual determinants of matrices where A has size $n \times m$ and B has size $m \times n$. This reduces to the case of the ring $R = \mathbf{Z}[a_{ij}, b_{ji}; 1 \leq i \leq n, 1 \leq j \leq m]$ where a_{ij} and b_{ij} are variables and the entries of the matrices A and B. Taking the fraction field, this reduces to the case of a field of characteristic zero. In characteristic zero there is a universal polynomial expressing the determinant of a matrix of size $\leq N$ in the traces of the powers of said matrix. Hence it suffices to prove

$$\mathrm{Trace}((I_n + AB)^k) = \mathrm{Trace}((I_m + BA)^k)$$

for all $k \geq 1$. Expanding we see that it suffices to prove $\mathrm{Trace}((AB)^k) = \mathrm{Trace}((BA)^k)$ for all $k \geq 0$. For $k = 1$ this is the well known fact that $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$. For $k > 1$ it follows from this by writing $(AB)^k = A(BA)^{k-1}B$ and $(BA)^k = (BA)^{k-1}AB$. ☐

Recall that we have defined in Algebra, Section 10.55 a group $K_0(R)$ as the free group on isomorphism classes of finite projective R-modules modulo the relations $[M'] + [M''] = [M' \oplus M'']$.

0AFX Lemma 15.118.7. Let R be a ring. There is a map

$$\det : K_0(R) \longrightarrow \text{Pic}(R)$$

which maps $[M]$ to the class of the invertible module $\wedge^n(M)$ if M is a finite locally free module of rank n.

Proof. This follows immediately from the constructions above and in particular Lemma 15.118.2 to see that the relations are mapped to 0. □

15.119. Perfect complexes and K-groups

0FJG We quickly show that the zeroth K-group of the derived category of perfect complexes of a ring R is the same as $K_0(R)$ defined in Algebra, Section 10.55.

0AFY Lemma 15.119.1. Let R be a ring. There is a map

$$c : \text{perfect complexes over } R \longrightarrow K_0(R)$$

with the following properties

1. $c(K[n]) = (-1)^n c(K)$ for a perfect complex K,
2. if $K \to L \to M \to K[1]$ is a distinguished triangle of perfect complexes, then $c(L) = c(K) + c(M)$,
3. if K is represented by a finite complex $M^\bullet$ consisting of finite projective modules, then $c(K) = \sum (-1)^i [M_i]$.

Proof. Let K be a perfect object of $D(R)$. By definition we can represent K by a finite complex $M^\bullet$ of finite projective R-modules. We define c by setting

$$c(K) = \sum (-1)^n [M^n]$$

in $K_0(R)$. Of course we have to show that this is well defined, but once it is well defined, then (1) and (3) are immediate. For the moment we view the map c as defined on complexes of finite projective R-modules.

Suppose that $L^\bullet \to M^\bullet$ is a surjective map of finite complexes of finite projective R-modules. Let $K^\bullet$ be the kernel. Then we obtain short exact sequences of R-modules

$$0 \to K^n \to L^n \to M^n \to 0$$

which are split because M^n is projective. Hence $K^\bullet$ is also a finite complex of finite projective R-modules and $c(L^\bullet) = c(K^\bullet) + c(M^\bullet)$ in $K_0(R)$.

Suppose given finite complex $M^\bullet$ of finite projective R-modules which is acyclic. Say $M^n = 0$ for $n \notin [a, b]$. Then we can break $M^\bullet$ into short exact sequences

$$\begin{gathered} 0 \to M^a \to M^{a+1} \to N^{a+1} \to 0, \\ 0 \to N^{a+1} \to M^{a+2} \to N^{a+3} \to 0, \\ \ldots \\ 0 \to N^{b-3} \to M^{b-2} \to N^{b-2} \to 0, \\ 0 \to N^{b-2} \to M^{b-1} \to M^b \to 0 \end{gathered}$$

Arguing by descending induction we see that $N^{b-2}, \ldots, N^{a+1}$ are finite projective R-modules, the sequences are split exact, and

$$c(M^\bullet) = \sum(-1)[M^n] = \sum(-1)^n([N^{n-1}] + [N^n]) = 0$$

Thus our construction gives zero on acyclic complexes.

It follows formally from the results of the preceding two paragraphs that c is well defined and satisfies (2). Namely, suppose the finite complexes $M^\bullet$ and $L^\bullet$ of finite projective R-modules represent the same object of $D(R)$. Then we can represent the isomorphism by a map $f : M^\bullet \to L^\bullet$ of complexes, see Derived Categories, Lemma 13.19.8. We obtain a short exact sequence of complexes

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

see Derived Categories, Definition 13.9.1. Since f is a quasi-isomorphism, the cone $C(f)^\bullet$ is acyclic (this follows for example from the discussion in Derived Categories, Section 13.12). Hence

$$0 = c(C(f)^\bullet) = c(L^\bullet) + c(K^\bullet[1]) = c(L^\bullet) - c(K^\bullet)$$

as desired. We omit the proof of (2) which is similar. $\square$

The following lemma shows that $K_0(R)$ is equal to $K_0(D_{perf}(R))$.

0FCU Lemma 15.119.2. Let R be a ring. Let $D_{perf}(R)$ be the derived category of perfect objects, see Lemma 15.78.1. The map c of Lemma 15.119.1 gives an isomorphism $K_0(D_{perf}(R)) = K_0(R)$.

Proof. It follows from the definition of $K_0(D_{perf}(R))$ (Derived Categories, Definition 13.28.1) that c induces a homomorphism $K_0(D_{perf}(R)) \to K_0(R)$.

Given a finite projective module M over R let us denote $M[0]$ the perfect complex over R which has M sitting in degree 0 and zero in other degrees. Given a short exact sequence $0 \to M \to M' \to M'' \to 0$ of finite projective modules we obtain a distinguished triangle $M[0] \to M'[0] \to M''[0] \to M[1]$, see Derived Categories, Section 13.12. This shows that we obtain a map $K_0(R) \to K_0(D_{perf}(R))$ by sending $[M]$ to $[M[0]]$ with apologies for the horrendous notation.

It is clear that $K_0(R) \to K_0(D_{perf}(R)) \to K_0(R)$ is the identity. On the other hand, if $M^\bullet$ is a bounded complex of finite projective R-modules, then the the existence of the distinguished triangles of "stupid truncations" (see Homology, Section 12.15)

$$\sigma_{\geq n}M^\bullet \to \sigma_{\geq n-1}M^\bullet \to M^{n-1}[-n+1] \to (\sigma_{\geq n}M^\bullet)[1]$$

and induction show that

$$[M^\bullet] = \sum(-1)^i[M^i[0]]$$

in $K_0(D_{perf}(R))$ (with again apologies for the notation). Hence the map $K_0(R) \to K_0(D_{perf}(R))$ is surjective which finishes the proof. $\square$

15.120. Determinants of endomorphisms of finite length modules

0GSX Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Consider the category of pairs (M, φ) consisting of a finite length R-module and an endomorphism $\varphi : M \to M$. This category is abelian and every object is Artinian as well as Noetherian. See Homology, Section 12.9 for definitions.

If (M, φ) is a simple object of this category, then M is annihilated by $\mathfrak{m}$ since otherwise $(\mathfrak{m}M, \varphi|_{\mathfrak{m}M})$ would be a nontrivial suboject. Also $\dim_\kappa(M) = \text{length}_R(M)$ is finite. Thus we may define the determinant and the trace

$$\det_\kappa(\varphi), \quad \text{Trace}_\kappa(\varphi)$$

as elements of κ using linear algebra. Simlarly for the characteristic polynomial of φ in this case.

By Homology, Lemma 12.9.6 for an arbitrary object (M, φ) of our category we have a finite filtration

$$0 \subset M_1 \subset \ldots \subset M_n = M$$

by submodules stable under φ such that $(M_i/M_{i-1}, \varphi_i)$ is a simple object of the category where $\varphi_i : M_i/M_{i-1} \to M_i/M_{i-1}$ is the induced map. We define the determinant of (M, φ) over κ as

$$\det_\kappa(\varphi) = \prod \det_\kappa(\varphi_i)$$

with $\det_\kappa(\varphi_i)$ as defined in the previous paragraph. We define the trace of (M, φ) over κ as

$$\text{Trace}_\kappa(\varphi) = \sum \text{Trace}_\kappa(\varphi_i)$$

with $\text{Trace}_\kappa(\varphi_i)$ as defined in the previous paragraph. We can similarly define the characteristic polynomial of φ over κ as the product of the characteristic polynomials of φ_i as defined in the previous paragraph. By Jordan-Hölder (Homology, Lemma 12.9.7) this is well defined.

0GSY **Lemma 15.120.1.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $0 \to (M, \varphi) \to (M', \varphi') \to (M'', \varphi'') \to 0$ be a short exact sequence in the category discussed above. Then

$$\det_\kappa(\varphi') = \det_\kappa(\varphi)\det_\kappa(\varphi''), \quad \text{Trace}_\kappa(\varphi') = \text{Trace}_\kappa(\varphi) + \text{Trace}_\kappa(\varphi'')$$

Also, the characteristic polynomial of φ' over κ is the product of the characteristic polynomials of φ and φ''.

Proof. Left as an exercise. $\square$

0GSZ **Lemma 15.120.2.** Let $(R, \mathfrak{m}, \kappa) \to (R', \mathfrak{m}', \kappa')$ be a local homomorphism of local rings. Assume that κ'/κ is a finite extension. Let $u \in R'$. Then for any finite length R'-module M' we have

$$\det_\kappa(u : M' \to M') = \text{Norm}_{\kappa'/\kappa}(u \bmod \mathfrak{m}')^m$$

where $m = \text{length}_{R'}(M')$.

Proof. Observe that the statement makes sense as $\text{length}_R(M') = \text{length}_{R'}(M')[\kappa' : \kappa]$. If $M' = \kappa'$, then the equality holds by definition of the norm as the determinant of the linear operator given by multiplication by u. In general one reduces to this case by chosing a suitable filtration and using the multiplicativity of Lemma 15.120.1. Some details omitted. $\square$

0GT0 Lemma 15.120.3. Let $(R, \mathfrak{m}, \kappa) \to (R', \mathfrak{m}', \kappa')$ be a flat local homomorphism of local rings such that $m = \text{length}_{R'}(R'/\mathfrak{m}R') < \infty$. For any (M, φ) as above, the element $\det_\kappa(\varphi)^m$ maps to $\det_{\kappa'}(\varphi \otimes 1 : M \otimes_R R' \to M \otimes_R R')$ in κ'.

Proof. The flatness of $R \to R'$ assures us that short exact sequences as in Lemma 15.120.1 base change to short exact sequences over R'. Hence by the multiplicativity of Lemma 15.120.1 we may assume that (M, φ) is a simple object of our category (see introduction to this section). In the simple case M is annihilated by $\mathfrak{m}$. Choose a filtration

$$0 \subset I_1 \subset I_2 \subset \ldots \subset I_{m-1} \subset R'/\mathfrak{m}R'$$

whose successive quotients are isomorphic to κ' as R'-modules. Then we obtain the filtration

$$0 \subset M \otimes_\kappa I_1 \subset M \otimes_\kappa I_2 \subset \ldots \subset M \otimes_\kappa I_{m-1} \subset M \otimes_\kappa R'/\mathfrak{m}R' = M \otimes_R R'$$

whose successive quotients are isomorphic to $M \otimes_\kappa \kappa'$. Also, these submodules are invariant under $\varphi \otimes 1$. By Lemma 15.120.1 we find

$$\det_{\kappa'}(\varphi \otimes 1 : M \otimes_R R' \to M \otimes_R R') = \det_{\kappa'}(\varphi \otimes 1 : M \otimes_\kappa \kappa' \to M \otimes_\kappa \kappa')^m = \det_\kappa(\varphi)^m$$

The last equality holds by the compatibility of determinants of linear maps with field extensions. This proves the lemma. □

15.121. A regular local ring is a UFD

0FJH We prove the result mentioned in the section title.

0AFZ Lemma 15.121.1. Let R be a regular local ring. Let $f \in R$. Then $\text{Pic}(R_f) = 0$.

Proof. Let L be an invertible R_f-module. In particular L is a finite R_f-module. There exists a finite R-module M such that $M_f \cong L$, see Algebra, Lemma 10.126.3. By Algebra, Proposition 10.110.1 we see that M has a finite free resolution $F_\bullet$ over R. It follows that L is quasi-isomorphic to a finite complex of free R_f-modules. Hence by Lemma 15.119.1 we see that $[L] = n[R_f]$ in $K_0(R)$ for some $n \in \mathbf{Z}$. Applying the map of Lemma 15.118.7 we see that L is trivial. □

0AG0 Lemma 15.121.2. A regular local ring is a UFD.

Proof. Recall that a regular local ring is a domain, see Algebra, Lemma 10.106.2. We will prove the unique factorization property by induction on the dimension of the regular local ring R. If $\dim(R) = 0$, then R is a field and in particular a UFD. Assume $\dim(R) > 0$. Let $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$. Then $R/(x)$ is regular by Algebra, Lemma 10.106.3, hence a domain by Algebra, Lemma 10.106.2, hence x is a prime element. Let $\mathfrak{p} \subset R$ be a height 1 prime. We have to show that $\mathfrak{p}$ is principal, see Algebra, Lemma 10.120.6. We may assume $x \notin \mathfrak{p}$, since if $x \in \mathfrak{p}$, then $\mathfrak{p} = (x)$ and we are done. For every nonmaximal prime $\mathfrak{q} \subset R$ the local ring $R_\mathfrak{q}$ is a regular local ring, see Algebra, Lemma 10.110.6. By induction we see that $\mathfrak{p}R_\mathfrak{q}$ is principal. In particular, the R_x-module $\mathfrak{p}_x = \mathfrak{p}R_x \subset R_x$ is a finitely presented R_x-module whose localization at any prime is free of rank 1. By Algebra, Lemma 10.78.2 we see that $\mathfrak{p}_x$ is an invertible R_x-module. By Lemma 15.121.1 we see that $\mathfrak{p}_x = (y)$ for some $y \in R_x$. We can write $y = x^e f$ for some $f \in \mathfrak{p}$ and $e \in \mathbf{Z}$. Factor $f = a_1 \ldots a_r$ into irreducible elements of R (Algebra, Lemma 10.120.3). Since $\mathfrak{p}$ is prime, we see that $a_i \in \mathfrak{p}$ for some i. Since $\mathfrak{p}_x = (y)$ is prime and $a_i | y$ in R_x, it follows that $\mathfrak{p}_x$ is generated by a_i in R_x, i.e., the image of a_i in R_x is prime. As x is a prime element, we find that a_i is prime in R by Algebra, Lemma 10.120.7. Since $(a_i) \subset \mathfrak{p}$ and $\mathfrak{p}$ has height 1 we conclude that $(a_i) = \mathfrak{p}$ as desired. □

0DLQ Lemma 15.121.3. Let R be a valuation ring with fraction field K and residue field κ. Let $R \to A$ be a homomorphism of rings such that

(1) A is local and $R \to A$ is local,
(2) A is flat and essentially of finite type over R,
(3) $A \otimes_R \kappa$ regular.

Then $\mathrm{Pic}(A \otimes_R K) = 0$.

Proof. Let L be an invertible $A \otimes_R K$-module. In particular L is a finite module. There exists a finite A-module M such that $M \otimes_R K \cong L$, see Algebra, Lemma 10.126.3. We may assume M is torsion free as an R-module. Thus M is flat as an R-module (Lemma 15.22.10). From Lemma 15.25.6 we deduce that M is of finite presentation as an A-module and A is essentially of finite presentation as an R-algebra. By Lemma 15.83.4 we see that M is perfect relative to R, in particular M is pseudo-coherent as an A-module. By Lemma 15.77.6 we see that M is perfect, hence M has a finite free resolution $F_\bullet$ over A. It follows that L is quasi-isomorphic to a finite complex of free $A \otimes_R K$-modules. Hence by Lemma 15.119.1 we see that $[L] = n[A \otimes_R K]$ in $K_0(A \otimes_R K)$ for some $n \in \mathbf{Z}$. Applying the map of Lemma 15.118.7 we see that L is trivial. □

15.122. Determinants of complexes

0FJI In Section 15.119 we have seen how to a perfect complex K over a ring R there is associated an isomorphism class of invertible R-modules, i.e., an element of $\mathrm{Pic}(R)$. In fact, analogously to Section 15.118 it turns out there is a functor

$$\det : \left\{\begin{matrix}\text{category of perfect complexes}\\ \text{morphisms are isomorphisms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{category of invertible modules}\\ \text{morphisms are isomorphisms}\end{matrix}\right\}$$

Moreover, given an object (L, F) of the filtered derived category $DF(R)$ of R whose filtration is finite and whose graded parts are perfect complexes, there is a canonical isomorphism $\det(\mathrm{gr}L) \to \det(L)$. See [KM76] for the original exposition. We will add this material later (insert future reference).

For the moment we will present an ad hoc construction in the case of perfect objects L in $D(R)$ of tor-amplitude in $[-1, 0]$. Such an object may be represented by a complex

$$L^\bullet = \ldots \to 0 \to L^{-1} \to L^0 \to 0 \to \ldots$$

with L^{-1} and L^0 finite projective R-modules, see Lemma 15.74.2. In this case we set

$$\det(L^\bullet) = \det(L^0) \otimes_R \det(L^{-1})^{\otimes -1} = \mathrm{Hom}_R(\det(L^{-1}), \det(L^0))$$

Let us say a complex of this form has rank 0 if $L^{-1}_\mathfrak{p}$ and $L^0_\mathfrak{p}$ have the same rank for all primes of R. If $L^\bullet$ has rank 0, then we have seen in Section 15.118 that there is a canonical element

$$\delta(L^\bullet) \in \det(L^\bullet)$$

which is simply the determininant of $d : L^{-1} \to L^0$. Note that $\delta(L^\bullet)$ is a trivialization of $\det(L^\bullet)$ if and only if $L^\bullet$ is acyclic.

Consider a map of complexes $a^\bullet : K^\bullet \to L^\bullet$ such that

(1) $a^\bullet$ is a quasi-isomorphism,

(2) $a^n : K^n \to L^n$ is surjective for all n,
(3) K^n, L^n are finite projective R-modules, nonzero only for $n \in \{-1, 0\}$.

In this situation we will construct an isomorphism

$$\det(a^\bullet) : \det(K^\bullet) \longrightarrow \det(L^\bullet)$$

Using the exact sequences $0 \to \mathrm{Ker}(a^i) \to K^i \to L^i \to 0$ we obtain isomorphisms

$$\gamma^i : \det(\mathrm{Ker}(a^i)) \otimes \det(L^i) \to \det(K^i)$$

for $i = -1, 0$ by Lemma 15.118.2. Since $a^\bullet$ is a quasi-isomorphism the complex $\mathrm{Ker}(a^\bullet)$ is acyclic and has rank 0. Hence the canonical element $\delta(\mathrm{Ker}(a^\bullet))$ is a trivialization of the invertible R-module $\det(\mathrm{Ker}(a^\bullet))$, see above. We define $\det(a^\bullet) : \det(K^\bullet) \to \det(L^\bullet)$ as the unique isomorphism such that the diagram

$$\begin{tikzcd}
\det(K^\bullet) \arrow[rr, "\det(a^\bullet)"'] \arrow[dr, "\delta(\mathrm{Ker}(a^\bullet))"'] & & \det(L^\bullet) \\
& \det(K^\bullet) \otimes \det(\mathrm{Ker}(a^\bullet)) \arrow[ur, "\gamma^0 \otimes (\gamma^{-1})^{\otimes -1}"'] &
\end{tikzcd}$$

commutes.

0FJJ **Lemma 15.122.1.** Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ be a map of complexes of R-modules satisfying (1), (2), (3) above. If $L^\bullet$ has rank 0, then $\det(a^\bullet)$ maps the canonical element $\delta(K^\bullet)$ to $\delta(L^\bullet)$.

Proof. Write $M^i = \mathrm{Ker}(a^i)$. Thus we have a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M^{-1} \arrow[r] \arrow[d, "d_M"'] & K^{-1} \arrow[r] \arrow[d, "d_K"'] & L^{-1} \arrow[r] \arrow[d, "d_L"'] & 0 \\
0 \arrow[r] & M^0 \arrow[r] & K^0 \arrow[r] & L^0 \arrow[r] & 0
\end{tikzcd}$$

By Lemma 15.118.3 we know that $\det(d_K)$ corresponds to $\det(d_M) \otimes \det(d_L)$ as maps. Unwinding the definitions this gives the required equality. $\square$

0FJK **Lemma 15.122.2.** Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ be a map of complexes of R-modules satisfying (1), (2), (3) above. Let $h : K^0 \to L^{-1}$ be a map such that $b^0 = a^0 + d \circ h$ and $b^{-1} = a^{-1} + h \circ d$ are surjective. Then $\det(a^\bullet) = \det(b^\bullet)$ as maps $\det(K^\bullet) \to \det(L^\bullet)$.

Proof. Suppose there exists a map $\tilde{h} : K^0 \to K^{-1}$ such that $h = a^{-1} \circ \tilde{h}$ and such that $k^0 = \mathrm{id} + d \circ \tilde{h} : K^0 \to K^0$ and $k^1 = \mathrm{id} + \tilde{h} \circ d : K^{-1} \to K^{-1}$ are isomorphisms. Then we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(b^\bullet) \arrow[r] \arrow[d, "c^\bullet"'] & K^\bullet \arrow[r, "b^\bullet"'] \arrow[d, "k^\bullet"'] & L^\bullet \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(a^\bullet) \arrow[r] & K^\bullet \arrow[r, "a^\bullet"] & L^\bullet \arrow[r] & 0
\end{tikzcd}$$

of complexes, where $c^\bullet$ is the induced isomorphism of kernels. Using Lemma 15.118.3 we see that

$$\begin{tikzcd} \det(\mathrm{Ker}(b^i)) \otimes \det(L^i) \arrow[r] \arrow[d, "\det(c^i)\otimes 1"'] & \det(K^i) \arrow[d, "\det(k^i)"] \\ \det(\mathrm{Ker}(a^i)) \otimes \det(L^i) \arrow[r] & \det(K^i) \end{tikzcd}$$

commutes. Since $\det(c^\bullet)$ maps the canonical trivialization of $\det(\mathrm{Ker}(a^\bullet))$ to the canonical trivializatio of $\mathrm{Ker}(b^\bullet)$ (Lemma 15.122.1) we see that we conclude if (and only if)

$$\det(k^0) = \det(k^{-1})$$

as elements of R which follows from Lemma 15.118.6.

Suppose there exists a direct summand $U \subset K^{-1}$ such that both $a^{-1}|_U : U \to L^{-1}$ and $b^{-1}|_U : U \to L^{-1}$ are isomorphisms. Define $\tilde{h}$ as the composition of h with the inverse of $a^{-1}|_U$. We claim that $\tilde{h}$ is a map as in the first paragraph of the proof. Namely, we have $h = a^{-1} \circ \tilde{h}$ by construction. To show that $k^{-1} : K^{-1} \to K^{-1}$ is an isomorphism it suffices to show that it is surjective (Algebra, Lemma 10.16.4). Let $u \in U$. We may choose $u' \in U$ such that $b^{-1}(u') = a^{-1}(u)$. Then $u = k^{-1}(u')$. Namely, both u and $k^{-1}(u')$ are in U and $a^{-1}(u) = a^{-1}(k^{-1}(u'))$ by a calculation[17] Since $a^{-1}|_U$ is an isomorphism we get the equality. Thus $U \subset \mathrm{Im}(k^{-1})$. On the other hand, if $x \in \mathrm{Ker}(a^{-1})$ then $x = k^{-1}(x) \bmod U$. Since $K^{-1} = \mathrm{Ker}(a^{-1}) + U$ we conclude k^{-1} is surjective. Finally, we show that $k^0 : K^0 \to K^0$ is surjective. First, since $a^0 \circ k^0 = b^0$ we see that $a^0 \circ k^0$ is surjective. If $x \in \mathrm{Ker}(a^0)$, then $x = d(y)$ for some $y \in \mathrm{Ker}(a^{-1})$. We may write $y = k^{-1}(z)$ for some $z \in K^{-1}$ by the above. Then $x = k^0(d(z))$ and we conclude.

Final step of the proof. It suffices to find U as in the preceding paragraph, but this may not always be possible. However, in order to show equality of two maps of R-modules, it suffices to do so after localization at primes of R. Hence we may assume R is local. Then we get the following problem: suppose

$$\alpha, \beta : R^{\oplus n} \longrightarrow R^{\oplus m}$$

are two surjective R-linear maps. Find a direct summand $U \subset R^{\oplus n}$ such that both $\alpha|_U$ and $\beta|_U$ are isomorphisms. If R is a field, this is possible by linear algebra. In general, one takes a solution over the residue field and lifts this to a solution over the local ring R. Some details omitted. $\square$

0FJL Lemma 15.122.3. Let R be a ring. Let $a^\bullet : K^\bullet \to L^\bullet$ and $b^\bullet : L^\bullet \to M^\bullet$ be maps of complexes of R-modules satisfying (1), (2), (3) above. Then we have $\det(b^\bullet) \circ \det(a^\bullet) = \det(b^\bullet \circ a^\bullet)$ as maps $\det(M^\bullet) \to \det(K^\bullet)$.

Proof. Omitted. Hints: Straightforward from Lemmas 15.118.2, 15.118.3, and 15.118.4. $\square$

0FJM Lemma 15.122.4. Let R be a ring. The constructions above determine a functor

$$\det : \left\{ \begin{matrix} \text{category of perfect complexes} \\ \text{with tor amplitude in } [-1, 0] \\ \text{morphisms are isomorphisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{category of invertible modules} \\ \text{morphisms are isomorphisms} \end{matrix} \right\}$$

[17] $a^{-1}(k^{-1}(u')) = a^{-1}(u') + a^{-1}(\tilde{h}(d(u'))) = a^{-1}(u') + h(d(u')) = b^{-1}(u') = a^{-1}(u)$

Moreover, given a rank 0 perfect object L of $D(R)$ with tor-amplitude in $[-1, 0]$ there is a canonical element $\delta(L) \in \det(L)$ such that for any isomorphism $a : L \to K$ in $D(R)$ we have $\det(a)(\delta(L)) = \delta(K)$.

Proof. By Lemma 15.74.2 every object of the source category may be represented by a complex

$$L^\bullet = \ldots \to 0 \to L^{-1} \to L^0 \to 0 \to \ldots$$

with L^{-1} and L^0 finite projective R-modules. Let us temporarily call a complex of this type good. By Derived Categories, Lemma 13.19.8 morphisms between good complexes in the derived category are homotopy classes of maps of complexes. Thus we may work with good complexes and we can use the determinant $\det(L^\bullet) = \det(L^0) \otimes \det(L^{-1})^{\otimes -1}$ we investigated above.

Let $a^\bullet : L^\bullet \to K^\bullet$ be a morphism of good complexes which is an isomorphism in $D(R)$, i.e., a quasi-isomorphism. We say that

$$\begin{tikzcd} L^\bullet \arrow[rr, "a^\bullet"'] & & K^\bullet \\ & M^\bullet \arrow[ul, "b^\bullet"] \arrow[ur, "c^\bullet"'] & \end{tikzcd}$$

is a good diagram if it commutes up to homotopy and $b^\bullet$ and $c^\bullet$ satisfy conditions (1), (2), (3) above. Whenever we have such a diagram it makes sense to define

$$\det(a^\bullet) = \det(c^\bullet) \circ \det(b^\bullet)^{-1}$$

where $\det(c^\bullet)$ and $\det(b^\bullet)$ are the isomorphisms constructed in the text above. We will show that good diagrams always exist and that the resulting map $\det(a^\bullet)$ is independent of the choice of good diagram.

Existence of good diagrams for a quasi-isomorphism $a^\bullet : L^\bullet \to K^\bullet$ of good complexes. Choose a surjection $p : R^{\oplus n} \to K^{-1}$. Then we can consider the new good complex

$$M^\bullet = \ldots \to 0 \to L^{-1} \oplus R^{\oplus n} \xrightarrow{d \oplus 1} L^0 \oplus R^{\oplus n} \to 0 \to \ldots$$

with the projection map $b^\bullet : M^\bullet \to L^\bullet$ and the map $c^\bullet : M^\bullet \to K^\bullet$ using $a^{-1} \oplus p$ in degree -1 and using $a^0 \oplus d \circ p$ in degree 0. The maps $b^\bullet : M^\bullet \to L^\bullet$ and $c^\bullet : M^\bullet \to K^\bullet$ satisfy conditions (1), (2), (3) above and we get a good diagram.

Suppose that we have a good diagram

$$\begin{tikzcd} L^\bullet \arrow[rr, "\mathrm{id}^\bullet"'] & & L^\bullet \\ & M^\bullet \arrow[ul, "b^\bullet"] \arrow[ur, "c^\bullet"'] & \end{tikzcd}$$

Then by Lemma 15.122.2 we see that $\det(c^\bullet) = \det(b^\bullet)$. Thus we see that $\det(\mathrm{id}^\bullet) = \mathrm{id}$ is independent of the choice of good diagram.

Before we prove independence in general, we think about composition. Suppose we have quasi-isomorphisms $L_1^\bullet \to L_2^\bullet$ and $L_2^\bullet \to L_3^\bullet$ of good complexes and good diagrams

$$\begin{tikzcd} L_1^\bullet \arrow[rr] & & L_2^\bullet \\ & M_{12}^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L_2^\bullet \arrow[rr] & & L_3^\bullet \\ & M_{23}^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We can extend this to a diagram

$$\begin{tikzcd} L_1^\bullet \arrow[rr] & & L_2^\bullet \arrow[rr] & & L_3^\bullet \\ & M_{12}^\bullet \arrow[ul] \arrow[ur] & & M_{23}^\bullet \arrow[ul] \arrow[ur] & \\ & & M_{123}^\bullet \arrow[ul] \arrow[ur] & & \end{tikzcd}$$

where $M_{123}^\bullet \to M_{12}^\bullet$ and $M_{123}^\bullet \to M_{23}^\bullet$ have properties (1), (2), (3) and the square in the diagram commutes: we can just take $M_{123}^n = M_{12}^n \times_{L_2^n} M_{23}^n$. Then Lemma 15.122.3 shows that

$$\begin{tikzcd} \det(L_2^\bullet) & \det(M_{23}^\bullet) \arrow[l] \\ \det(M_{12}^\bullet) \arrow[u] & \det(M_{123}^\bullet) \arrow[l] \arrow[u] \end{tikzcd}$$

commutes. A diagram chase shows that the composition $\det(L_1^\bullet) \to \det(L_2^\bullet) \to \det(L_3^\bullet)$ of the maps associated to the two good diagrams using $M_{12}^\bullet$ and $M_{23}^\bullet$ is equal to the map associated to the good diagram

$$\begin{tikzcd} L_1^\bullet \arrow[rr] & & L_3^\bullet \\ & M_{123}^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Thus if we can show that these maps are independent of choices, then the composition law is satisfied too and we obtain our functor.

Independence. Let a quasi-isomorphism $a^\bullet : L^\bullet \to K^\bullet$ of good complexes be given. Choose an inverse quasi-isomorphism $b^\bullet : K^\bullet \to L^\bullet$. Setting $L_1^\bullet = L$, $L_2^\bullet = K^\bullet$ and $L_3^\bullet = L^\bullet$ may fix our choice of good diagram for $b^\bullet$ and consider varying good diagrams for $a^\bullet$. Then the result of the previous paragraphs is that no matter what choices, the composition always equals the identity map on $\det(L^\bullet)$. This clearly proves indepence of those choices.

The statement on canonical elements follows immediately from Lemma 15.122.1 and our construction. □

15.123. Extensions of valuation rings

0ASF This section is the analogue of Section 15.111 for general valuation rings.

0ASG Definition 15.123.1. We say that $A \to B$ or $A \subset B$ is an extension of valuation rings if A and B are valuation rings and $A \to B$ is injective and local. Such an extension induces a commutative diagram

$$\begin{tikzcd} A \setminus \{0\} \arrow[r] \arrow[d, "v"] & B \setminus \{0\} \arrow[d, "v"] \\ \Gamma_A \arrow[r] & \Gamma_B \end{tikzcd}$$

where Γ_A and Γ_B are the value groups. We say that B is weakly unramified over A if the lower horizontal arrow is a bijection. If the extension of residue fields $\kappa_A = A/\mathfrak m_A \subset \kappa_B = B/\mathfrak m_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the residual degree or residue degree of the extension $A \subset B$.

Note that $\Gamma_A \to \Gamma_B$ is injective, because the units of A are the inverse of the units of B under the map $A \to B$. Note also, that we do not require the extension of fraction fields to be finite.

0ASH Lemma 15.123.2. Let $A \subset B$ be an extension of valuation rings with fraction fields $K \subset L$. If the extension L/K is finite, then the residue field extension is finite, the index of Γ_A in Γ_B is finite, and

$$[\Gamma_B : \Gamma_A][\kappa_B : \kappa_A] \leq [L : K].$$

Proof. Let $b_1, \ldots, b_n \in B$ be units whose images in κ_B are linearly independent over κ_A. Let $c_1, \ldots, c_m \in B$ be nonzero elements whose images in Γ_B/Γ_A are pairwise distinct. We claim that $b_i c_j$ are K-linearly independent in L. Namely, we claim a sum

$$\sum a_{ij} b_i c_j$$

with $a_{ij} \in K$ not all zero cannot be zero. Choose (i_0, j_0) with $v(a_{i_0j_0} b_{i_0} c_{j_0})$ minimal. Replace a_{ij} by $a_{ij}/a_{i_0j_0}$, so that $a_{i_0j_0} = 1$. Let

$$P = \{(i, j) \mid v(a_{ij} b_i c_j) = v(a_{i_0j_0} b_{i_0} c_{j_0})\}$$

By our choice of $c_1, \ldots, c_m$ we see that $(i, j) \in P$ implies $j = j_0$. Hence if $(i, j) \in P$, then $v(a_{ij}) = v(a_{i_0j_0}) = 0$, i.e., a_{ij} is a unit. By our choice of $b_1, \ldots, b_n$ we see that

$$\sum\nolimits_{(i,j) \in P} a_{ij} b_i$$

is a unit in B. Thus the valuation of $\sum_{(i,j) \in P} a_{ij} b_i c_j$ is $v(c_{j_0}) = v(a_{i_0j_0} b_{i_0} c_{j_0})$. Since the terms with $(i, j) \notin P$ in the first displayed sum have strictly bigger valuation, we conclude that this sum cannot be zero, thereby proving the lemma. □

0H37 Lemma 15.123.3. Let A be a valuation ring with fraction field K of characteristic $p > 0$. Let L/K be a purely inseparable extension. Then the integral closure B of A in L is a valuation ring with fraction field L and $A \subset B$ is an extension of valuation rings.

Proof. Omitted. Hints: use Algebra, Lemmas 10.50.5 and 10.36.17 for example. □

0ASI Lemma 15.123.4. Let $A \to B$ be a flat local homomorphism of Noetherian local normal domains. Let $f \in A$ and $h \in B$ such that $f = wh^n$ for some $n > 1$ and some unit w of B. Assume that for every height 1 prime $\mathfrak p \subset A$ there is a height 1 prime $\mathfrak q \subset B$ lying over $\mathfrak p$ such that the extension $A_{\mathfrak p} \subset B_{\mathfrak q}$ is weakly unramified. Then $f = ug^n$ for some $g \in A$ and unit u of A.

Proof. The local rings of A and B at height 1 primes are discrete valuation rings (Algebra, Lemma 10.119.7). Thus the assumption makes sense (via Definition 15.111.1). Let $\mathfrak p_1, \ldots, \mathfrak p_r$ be the primes of A minimal over f. These have height 1 by Algebra, Lemma 10.60.11. For each i let $\mathfrak q_{i,j} \subset B$, $j = 1, \ldots, r_i$ be the height 1 primes of B lying over $\mathfrak p_i$. Say we number them so that $A_{\mathfrak p_i} \to B_{\mathfrak q_{i,1}}$ is weakly unramified. Since f maps to an nth power times a unit in $B_{\mathfrak q_{i,1}}$ we see that the valuation v_i of f in $A_{\mathfrak p_i}$ is divisible by n. Say $v_i = nw_i$ for some $w_i \geq 0$. Consider the exact sequence

$$0 \to I \to A \to \prod\nolimits_{i=1,\ldots,r} A_{\mathfrak p_i}/\mathfrak p_i^{w_i}A_{\mathfrak p_i}$$

defining the ideal I. Applying the exact functor $- \otimes_A B$ we obtain an exact sequence

$$0 \to I \otimes_A B \to B \to \prod\nolimits_{i=1,\ldots,r} (A_{\mathfrak p_i}/\mathfrak p_i^{w_i}A_{\mathfrak p_i}) \otimes_A B$$

Fix i. We claim that the canonical map

$$(A_{\mathfrak p_i}/\mathfrak p_i^{w_i}A_{\mathfrak p_i}) \otimes_A B \to \prod\nolimits_{j=1,\ldots,r_i} B_{\mathfrak q_{i,j}}/\mathfrak q_{i,j}^{e_{i,j}w_i}B_{\mathfrak q_{i,j}}$$

is injective. Here $e_{i,j}$ is the ramification index of $A_{\mathfrak p_i} \to B_{\mathfrak q_{i,j}}$. The claim asserts that $\mathfrak p_i^{w_i}B_{\mathfrak p_i}$ is equal to the set of elements b of $B_{\mathfrak p_i}$ whose valuation at $\mathfrak q_{i,j}$ is $\geq e_{i,j}w_i$. Choose a generator $a \in A_{\mathfrak p_i}$ of the principal ideal $\mathfrak p_i^{w_i}$. Then the valuation of a at $\mathfrak q_{i,j}$ is equal to $e_{i,j}w_i$. Hence, as $B_{\mathfrak p_i}$ is a normal domain whose height one primes are the primes $\mathfrak q_{i,j}$, $j = 1, \ldots, r_i$, we see that, for b as above, we have $b/a \in B_{\mathfrak p_i}$ by Algebra, Lemma 10.157.6. Thus the claim.

The claim combined with the second exact sequence above determines an exact sequence

$$0 \to I \otimes_A B \to B \to \prod\nolimits_{i=1,\ldots,r} \prod\nolimits_{j=1,\ldots,r_i} B_{\mathfrak q_{i,j}}/\mathfrak q_{i,j}^{e_{i,j}w_i}B_{\mathfrak q_{i,j}}$$

It follows that $I \otimes_A B$ is the set of elements h' of B which have valuation $\geq e_{i,j}w_i$ at $\mathfrak q_{i,j}$. Since $f = wh^n$ in B we see that h has valuation $e_{i,j}w_i$ at $\mathfrak q_{i,j}$. Thus $h'/h \in B$ by Algebra, Lemma 10.157.6. It follows that $I \otimes_A B$ is a free B-module of rank 1 (generated by h). Therefore I is a free A-module of rank 1, see Algebra, Lemma 10.78.6. Let $g \in I$ be a generator. Then we see that g and h differ by a unit in B. Working backwards we conclude that the valuation of g in $A_{\mathfrak p_i}$ is $w_i = v_i/n$. Hence g^n and f differ by a unit in A (by Algebra, Lemma 10.157.6) as desired. □

0ASJ Lemma 15.123.5. Let A be a valuation ring. Let $A \to B$ be an étale ring map and let $\mathfrak m \subset B$ be a prime lying over the maximal ideal of A. Then $A \subset B_{\mathfrak m}$ is an extension of valuation rings which is weakly unramified.

Proof. The ring A has weak dimension ≤ 1 by Lemma 15.104.18. Then B has weak dimension ≤ 1 by Lemmas 15.104.4 and 15.104.14. hence the local ring $B_{\mathfrak m}$ is a valuation ring by Lemma 15.104.18. Since the extension $A \subset B_{\mathfrak m}$ induces a finite extension of fraction fields, we see that the Γ_A has finite index in the value group of $B_{\mathfrak m}$. Thus for every $h \in B_{\mathfrak m}$ there exists an $n > 0$, an element $f \in A$, and a unit $w \in B_{\mathfrak m}$ such that $f = wh^n$ in $B_{\mathfrak m}$. We will show that this implies $f = ug^n$ for some $g \in A$ and unit $u \in A$; this will show that the value groups of A and $B_{\mathfrak m}$ agree, as claimed in the lemma.

Write $A = \operatorname{colim} A_i$ as the colimit of its local subrings which are essentially of finite type over $\mathbf{Z}$. Since A is a normal domain (Algebra, Lemma 10.50.3), we may assume that each A_i is normal (here we use that taking normalizations the local rings remain essentially of finite type over $\mathbf{Z}$ by Algebra, Proposition 10.162.16). For some i we can find an étale extension $A_i \to B_i$ such that $B = A \otimes_{A_i} B_i$, see Algebra, Lemma 10.143.3. Let $\mathfrak{m}_i$ be the intersection of B_i with $\mathfrak{m}$. Then we may apply Lemma 15.123.4 to the ring map $A_i \to (B_i)_{\mathfrak{m}_i}$ to conclude. The hypotheses of the lemma are satisfied because:

(1) A_i and $(B_i)_{\mathfrak{m}_i}$ are Noetherian as they are essentially of finite type over $\mathbf{Z}$,
(2) $A_i \to (B_i)_{\mathfrak{m}_i}$ is flat as $A_i \to B_i$ is étale,
(3) B_i is normal as $A_i \to B_i$ is étale, see Algebra, Lemma 10.163.9,
(4) for every height 1 prime of A_i there exists a height 1 prime of $(B_i)_{\mathfrak{m}_i}$ lying over it by Algebra, Lemma 10.113.2 and the fact that $\mathrm{Spec}((B_i)_{\mathfrak{m}_i}) \to \mathrm{Spec}(A_i)$ is surjective,
(5) the induced extensions $(A_i)_{\mathfrak{p}} \to (B_i)_{\mathfrak{q}}$ are unramified for every prime $\mathfrak{q}$ lying over a prime $\mathfrak{p}$ as $A_i \to B_i$ is étale.

This concludes the proof of the lemma. □

0ASK Lemma 15.123.6. Let A be a valuation ring. Let A^h, resp. A^{sh} be its henselization, resp. strict henselization. Then

$$A \subset A^h \subset A^{sh}$$

are extensions of valuation rings which induce bijections on value groups, i.e., which are weakly unramified.

Proof. Write $A^h = \mathrm{colim}(B_i)_{\mathfrak{q}_i}$ where $A \to B_i$ is étale and $\mathfrak{q}_i \subset B_i$ is a prime ideal lying over $\mathfrak{m}_A$, see Algebra, Lemma 10.155.7. Then Lemma 15.123.5 tells us that $(B_i)_{\mathfrak{q}_i}$ is a valuation ring and that the induced map

$$(A \setminus \{0\})/A^* \longrightarrow ((B_i)_{\mathfrak{q}_i} \setminus \{0\})/(B_i)^*_{\mathfrak{q}_i}$$

is bijective. By Algebra, Lemma 10.50.6 we conclude that A^h is a valuation ring. It also follows that $(A \setminus \{0\})/A^* \to (A^h \setminus \{0\})/(A^h)^*$ is bijective. This proves the lemma for the inclusion $A \subset A^h$. To prove it for $A \subset A^{sh}$ we can use exactly the same argument except we replace Algebra, Lemma 10.155.7 by Algebra, Lemma 10.155.11. Since $A^{sh} = (A^h)^{sh}$ we see that this also proves the assertions of the lemma for the inclusion $A^h \subset A^{sh}$. □

15.124. Structure of modules over a PID

0ASL We work a little bit more generally (following the papers [War69] and [War70] by Warfield) so that the proofs work over valuation rings.

0ASM Lemma 15.124.1. Let P be a module over a ring R. The following are equivalent [War69, Corollary 1]

(1) P is a direct summand of a direct sum of modules of the form R/fR, for $f \in R$ varying.
(2) for every short exact sequence $0 \to A \to B \to C \to 0$ of R-modules such that $fA = A \cap fB$ for all $f \in R$ the map $\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ is surjective.

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence as in (2). To prove that (1) implies (2) it suffices to prove that $\mathrm{Hom}_R(R/fR, B) \to \mathrm{Hom}_R(R/fR, C)$ is surjective for every $f \in R$. Let $\psi : R/fR \to C$ be a map. Say $\psi(1)$ is the image of $b \in B$. Then $fb \in A$. Hence there exists an $a \in A$ such that $fa = fb$. Then

$f(b-a) = 0$ hence we get a morphism $\varphi : R/fR \to B$ mapping 1 to $b - a$ which lifts ψ.

Conversely, assume that (2) holds. Let I be the set of pairs (f, φ) where $f \in R$ and $\varphi : R/fR \to P$. For $i \in I$ denote (f_i, φ_i) the corresponding pair. Consider the map

$$B = \bigoplus\nolimits_{i \in I} R/f_iR \longrightarrow P$$

which sends the element r in the summand R/f_iR to $\varphi_i(r)$ in P. Let $A = \text{Ker}(B \to P)$. Then we see that (1) is true if the sequence

$$0 \to A \to B \to P \to 0$$

is an exact sequence as in (2). To see this suppose $f \in R$ and $a \in A$ maps to fb in B. Write $b = (r_i)_{i \in I}$ with almost all $r_i = 0$. Then we see that

$$f \sum \varphi_i(r_i) = 0$$

in P. Hence there is an $i_0 \in I$ such that $f_{i_0} = f$ and $\varphi_{i_0}(1) = \sum \varphi_i(r_i)$. Let $x_{i_0} \in R/f_{i_0}R$ be the class of 1. Then we see that

$$a' = (r_i)_{i \in I} - (0, \ldots, 0, x_{i_0}, 0, \ldots)$$

is an element of A and $fa' = a$ as desired. □

0ASN Lemma 15.124.2 (Generalized valuation rings). Let R be a nonzero ring. The following are equivalent [War70]

(1) For $a, b \in R$ either a divides b or b divides a.
(2) Every finitely generated ideal is principal and R is local.
(3) The set of ideals of R is linearly ordered by inclusion.

This holds in particular if R is a valuation ring.

Proof. Assume (2) and let $a, b \in R$. Then $(a, b) = (c)$. If $c = 0$, then $a = b = 0$ and a divides b. Assume $c \neq 0$. Write $c = ua + vb$ and $a = wc$ and $b = zc$. Then $c(1 - uw - vz) = 0$. Since R is local, this implies that $1 - uw - vz \in \mathfrak{m}$. Hence either w or z is a unit, so either a divides b or b divides a. Thus (2) implies (1).

Assume (1). If R has two maximal ideals $\mathfrak{m}_i$ we can choose $a \in \mathfrak{m}_1$ with $a \notin \mathfrak{m}_2$ and $b \in \mathfrak{m}_2$ with $b \notin \mathfrak{m}_1$. Then a does not divide b and b does not divide a. Hence R has a unique maximal ideal and is local. It follows easily from condition (1) and induction that every finitely generated ideal is principal. Thus (1) implies (2).

It is straightforward to prove that (1) and (3) are equivalent. The final statement is Algebra, Lemma 10.50.4. □

0ASP Lemma 15.124.3. Let R be a ring satisfying the equivalent conditions of Lemma 15.124.2. Then every finitely presented R-module is isomorphic to a finite direct sum of modules of the form R/fR. [War70, Theorem 1]

Proof. Let M be a finitely presented R-module. We will use all the equivalent properties of R from Lemma 15.124.2 without further mention. Denote $\mathfrak{m} \subset R$ the maximal ideal and $\kappa = R/\mathfrak{m}$ the residue field. Let $I \subset R$ be the annihilator of M. Choose a basis $y_1, \ldots, y_n$ of the finite dimensional κ-vector space $M/\mathfrak{m}M$. We will argue by induction on n.

By Nakayama's lemma any collection of elements $x_1, \ldots, x_n \in M$ lifting the elements $y_1, \ldots, y_n$ in $M/\mathfrak{m}M$ generate M, see Algebra, Lemma 10.20.1. This immediately proves the base case $n = 0$ of the induction.

We claim there exists an index i such that for any choice of $x_i \in M$ mapping to y_i the annihilator of x_i is I. Namely, if not, then we can choose $x_1, \ldots, x_n$ such that $I_i = \text{Ann}(x_i) \neq I$ for all i. But as $I \subset I_i$ for all i, ideals being totally ordered implies I_i is strictly bigger than I for $i = 1, \ldots, n$, and by total ordering once more we would see that $\text{Ann}(M) = I_1 \cap \ldots \cap I_n$ is bigger than I which is a contradiction. After renumbering we may assume that y_1 has the property: for any $x_1 \in M$ lifting y_1 the annihilator of x_1 is I.

We set $A = Rx_1 \subset M$. Consider the exact sequence $0 \to A \to M \to M/A \to 0$. Since A is finite, we see that M/A is a finitely presented R-module (Algebra, Lemma 10.5.3) with fewer generators. Hence $M/A \cong \bigoplus_{j=1,\ldots,m} R/f_jR$ by induction. On the other hand, we claim that $A \to M$ satisfies the property: if $f \in R$, then $fA = A \cap fM$. The inclusion $fA \subset A \cap fM$ is trivial. Conversely, if $x \in A \cap fM$, then $x = gx_1 = fy$ for some $g \in R$ and $y \in M$. If f divides g, then $x \in fA$ as desired. If not, then we can write $f = hg$ for some $h \in \mathfrak{m}$. The element $x_1' = x_1 - hy$ has annihilator I by the previous paragraph. Thus $g \in I$ and we see that $x = 0$ as desired. The claim and Lemma 15.124.1 imply the sequence $0 \to A \to M \to M/A \to 0$ is split and we find $M \cong A \oplus \bigoplus_{j=1,\ldots,m} R/f_jR$. Then $A = R/I$ is finitely presented (as a summand of M) and hence I is finitely generated, hence principal. This finishes the proof. □

0ASQ Lemma 15.124.4. Let R be a ring such that every local ring of R at a maximal ideal satisfies the equivalent conditions of Lemma 15.124.2. Then every finitely presented R-module is a summand of a finite direct sum of modules of the form R/fR for f in R varying. [War70, Theorem 3]

Proof. Let M be a finitely presented R-module. We first show that M is a summand of a direct sum of modules of the form R/fR and at the end we argue the direct sum can be taken to be finite. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules such that $fA = A \cap fB$ for all $f \in R$. By Lemma 15.124.1 we have to show that $\text{Hom}_R(M, B) \to \text{Hom}_R(M, C)$ is surjective. It suffices to prove this after localization at maximal ideals $\mathfrak{m}$, see Algebra, Lemma 10.23.1. Note that the localized sequences $0 \to A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m} \to 0$ satisfy the condition that $fA_\mathfrak{m} = A_\mathfrak{m} \cap fB_\mathfrak{m}$ for all $f \in R_\mathfrak{m}$ (because we can write $f = uf'$ with $u \in R_\mathfrak{m}$ a unit and $f' \in R$ and because localization is exact). Since M is finitely presented, we see that

$$\text{Hom}_R(M, B)_\mathfrak{m} = \text{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m}) \quad \text{and} \quad \text{Hom}_R(M, C)_\mathfrak{m} = \text{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, C_\mathfrak{m})$$

by Algebra, Lemma 10.10.2. The module $M_\mathfrak{m}$ is a finitely presented $R_\mathfrak{m}$-module. By Lemma 15.124.3 we see that $M_\mathfrak{m}$ is a direct sum of modules of the form $R_\mathfrak{m}/fR_\mathfrak{m}$. Thus we conclude by Lemma 15.124.1 that the map on localizations is surjective.

At this point we know that M is a summand of $\bigoplus_{i \in I} R/f_iR$. Consider the map $M \to \bigoplus_{i \in I} R/f_iR$. Since M is a finite R-module, the image is contained in $\bigoplus_{i \in I'} R/f_iR$ for some finite subset $I' \subset I$. This finishes the proof. □

0ASR Definition 15.124.5. Let R be a domain.

(1) We say R is a Bézout domain if every finitely generated ideal of R is principal.
(2) We say R is an elementary divisor domain if for all $n, m \geq 1$ and every $n \times m$ matrix A, there exist invertible matrices U, V of size $n \times n, m \times m$ such that
$$UAV = \begin{pmatrix} f_1 & 0 & 0 & \ldots \\ 0 & f_2 & 0 & \ldots \\ 0 & 0 & f_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$
with $f_1, \ldots, f_{\min(n,m)} \in R$ and $f_1 | f_2 | \ldots$.

It is apparently still an open question as to whether every Bézout domain R is an elementary divisor domain (or not). This is equivalent to the question of whether every finitely presented module over R is a direct sum of cyclic modules. The converse implication is true.

0ASS Lemma 15.124.6. An elementary divisor domain is Bézout.

Proof. Let $a, b \in R$ be nonzero. Consider the 1×2 matrix $A = (a\ b)$. Then we see that $u(a\ b)V = (f\ 0)$ with $u \in R$ invertible and $V = (g_{ij})$ an invertible 2×2 matrix. Then $f = uag_{11} + ubg_{21}$ and $(g_{11}, g_{21}) = R$. It follows that $(a, b) = (f)$. An induction argument (omitted) then shows any finitely generated ideal in R is generated by one element. $\square$

0AST Lemma 15.124.7. The localization of a Bézout domain is Bézout. Every local ring of a Bézout domain is a valuation ring. A local domain is Bézout if and only if it is a valuation ring.

Proof. We omit the proof of the statement on localizations. The final statement is Algebra, Lemma 10.50.15. The second statement follows from the other two. $\square$

0ASU Lemma 15.124.8. Let R be a Bézout domain.

(1) Every finite submodule of a free module is finite free.
(2) Every finitely presented R-module M is a direct sum of a finite free module and a torsion module M_{tors} which is a summand of a module of the form $\bigoplus_{i=1,\ldots,n} R/f_iR$ with $f_1, \ldots, f_n \in R$ nonzero.

Proof. Proof of (1). Let $M \subset F$ be a finite submodule of a free module F. Since M is finite, we may assume F is a finite free module (details omitted). Say $F = R^{\oplus n}$. We argue by induction on n. If $n = 1$, then M is a finitely generated ideal, hence principal by our assumption that R is Bézout. If $n > 1$, then we consider the image I of M under the projection $R^{\oplus n} \to R$ onto the last summand. If $I = (0)$, then $M \subset R^{\oplus n-1}$ and we are done by induction. If $I \neq 0$, then $I = (f) \cong R$. Hence $M \cong R \oplus \text{Ker}(M \to I)$ and we are done by induction as well.

Let M be a finitely presented R-module. Since the localizations of R are maximal ideals are valuation rings (Lemma 15.124.7) we may apply Lemma 15.124.4. Thus M is a summand of a module of the form $R^{\oplus r} \oplus \bigoplus_{i=1,\ldots,n} R/f_iR$ with $f_i \neq 0$. Since taking the torsion submodule is a functor we see that M_{tors} is a summand of the module $\bigoplus_{i=1,\ldots,n} R/f_iR$ and M/M_{tors} is a summand of $R^{\oplus r}$. By the first part of the proof we see that M/M_{tors} is finite free. Hence $M \cong M_{tors} \oplus M/M_{tors}$ as desired. $\square$

0ASV Lemma 15.124.9. Let R be a PID. Every finite R-module M is of isomorphic to a module of the form

$$R^{\oplus r} \oplus \bigoplus\nolimits_{i=1,\ldots,n} R/f_iR$$

for some $r, n \geq 0$ and $f_1, \ldots, f_n \in R$ nonzero.

Proof. A PID is a Noetherian Bézout ring. By Lemma 15.124.8 it suffices to prove the result if M is torsion. Since M is finite, this means that the annihilator of M is nonzero. Say $fM = 0$ for some $f \in R$ nonzero. Then we can think of M as a module over R/fR. Since R/fR is Noetherian of dimension 0 (small detail omitted) we see that $R/fR = \prod R_j$ is a finite product of Artinian local rings R_i (Algebra, Proposition 10.60.7). Each R_i, being a local ring and a quotient of a PID, is a generalized valuation ring in the sense of Lemma 15.124.2 (small detail omitted). Write $M = \prod M_j$ with $M_j = e_jM$ where $e_j \in R/fR$ is the idempotent corresponding to the factor R_j. By Lemma 15.124.3 we see that $M_j = \bigoplus_{i=1,\ldots,n_j} R_j/\overline{f}_{ji}R_j$ for some $\overline{f}_{ji} \in R_j$. Choose lifts $f_{ji} \in R$ and choose $g_{ji} \in R$ with $(g_{ji}) = (f_j, f_{ji})$. Then we conclude that

$$M \cong \bigoplus R/g_{ji}R$$

as an R-module which finishes the proof. □

One can also prove that a PID is a elementary divisor domain (insert future reference here), by proving lemmas similar to the following.

0ASW Lemma 15.124.10. Let R be a Bézout domain. Let $n \geq 1$ and $f_1, \ldots, f_n \in R$ generate the unit ideal. There exists an invertible $n \times n$ matrix in R whose first row is $f_1 \ldots f_n$.

Proof. This follows from Lemma 15.124.8 but we can also prove it directly as follows. By induction on n. The result holds for $n = 1$. Assume $n > 1$. We may assume $f_1 \neq 0$ after renumbering. Choose $f \in R$ such that $(f) = (f_1, \ldots, f_{n-1})$. Let A be an $(n-1) \times (n-1)$ matrix whose first row is $f_1/f, \ldots, f_{n-1}/f$. Choose $a, b \in R$ such that $af - bf_n = 1$ which is possible because $1 \in (f_1, \ldots, f_n) = (f, f_n)$. Then a solution is the matrix

$$\begin{pmatrix} f & 0 & \ldots & 0 & f_n \\ 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & 0 \\ b & 0 & \ldots & 0 & a \end{pmatrix} \begin{pmatrix} & & & 0 \\ & A & & \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Observe that the left matrix is invertible because it has determinant 1. □

15.125. Principal radical ideals

0BWR In this section we prove that a catenary Noetherian normal local domain there exists a nontrivial principal radical ideal. This result can be found in [Art86].

0BWS Lemma 15.125.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension one, and let $x \in \mathfrak{m}$ be an element not contained in any minimal prime of R. Then

(1) the function $P : n \mapsto \text{length}_R(R/x^nR)$ satisfies $P(n) \leq nP(1)$ for $n \geq 0$,
(2) if x is a nonzerodivisor, then $P(n) = nP(1)$ for $n \geq 0$.

Proof. Since $\dim(R) = 1$, we have $\dim(R/x^nR) = 0$ and so $\text{length}_R(R/x^nR)$ is finite for each n (Algebra, Lemma 10.62.3). To show the lemma we will induct on n. Since $x^0R = R$, we have that $P(0) = \text{length}_R(R/x^0R) = \text{length}_R 0 = 0$. The statement also holds for $n = 1$. Now let $n \geq 2$ and suppose the statement holds for $n - 1$. The following sequence is exact

$$R/x^{n-1}R \xrightarrow{x} R/x^nR \to R/xR \to 0$$

where x denotes the multiplication by x map. Since length is additive (Algebra, Lemma 10.52.3), we have that $P(n) \leq P(n-1) + P(1)$. By induction $P(n-1) \leq (n-1)P(1)$, whence $P(n) \leq nP(1)$. This proves the induction step.

If x is a nonzerodivisor, then the displayed exact sequence above is exact on the left also. Hence we get $P(n) = P(n-1) + P(1)$ for all $n \geq 1$. $\square$

0BWT **Lemma 15.125.2.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $x \in \mathfrak{m}$ be an element not contained in any minimal prime of R. Let t be the number of minimal prime ideals of R. Then $t \leq \text{length}_R(R/xR)$.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal prime ideals of R. Set $R' = R/\sqrt{0} = R/(\bigcap_{i=1}^t \mathfrak{p}_i)$. We claim it suffices to prove the lemma for R'. Namely, it is clear that R' has t minimal primes too and $\text{length}_{R'}(R'/xR') = \text{length}_R(R'/xR')$ is less than $\text{length}_R(R/xR)$ as there is a surjection $R/xR \to R'/xR'$. Thus we may assume R is reduced.

Assume R is reduced with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. This means there is an exact sequence

$$0 \to R \to \prod\nolimits_{i=1}^t R/\mathfrak{p}_i \to Q \to 0$$

Here Q is the cokernel of the first map. Write $M = \prod_{i=1}^t R/\mathfrak{p}_i$. Localizing at $\mathfrak{p}_j$ we see that

$$R_{\mathfrak{p}_j} \to M_{\mathfrak{p}_j} = \left(\prod\nolimits_{i=1}^t R/\mathfrak{p}_i\right)_{\mathfrak{p}_j} = (R/\mathfrak{p}_j)_{\mathfrak{p}_j}$$

is surjective. Thus $Q_{\mathfrak{p}_j} = 0$ for all j. We conclude that $\text{Supp}(Q) = \{\mathfrak{m}\}$ as $\mathfrak{m}$ is the only prime of R different from the $\mathfrak{p}_i$. It follows that Q has finite length (Algebra, Lemma 10.62.3). Since $\text{Supp}(Q) = \{\mathfrak{m}\}$ we can pick an $n \gg 0$ such that x^n acts as 0 on Q (Algebra, Lemma 10.62.4). Now consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, "x^n"] & M \arrow[r] \arrow[d, "x^n"] & Q \arrow[r] \arrow[d, "x^n"] & 0 \\ 0 \arrow[r] & R \arrow[r] & M \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are multiplication by x^n. This is injective on R and on M since x is not contained in any of the $\mathfrak{p}_i$. By the snake lemma (Algebra, Lemma 10.4.1), the following sequence is exact:

$$0 \to Q \to R/x^nR \to M/x^nM \to Q \to 0$$

Hence we find that $\text{length}_R(R/x^nR) = \text{length}_R(M/x^nM)$ for large enough n. Writing $R_i = R/\mathfrak{p}_i$ we see that $\text{length}(M/x^nM) = \sum_{i=1}^t \text{length}_R(R_i/x^nR_i)$. Applying Lemma 15.125.1 and the fact that x is a nonzerodivisor on R and R_i, we conclude that

$$n\text{length}_R(R/xR) = \sum\nolimits_{i=1}^t n\text{length}_{R_i}(R_i/xR_i)$$

Since $\text{length}_{R_i}(R_i/xR_i) \geq 1$ the lemma is proved. $\square$

0BWU Lemma 15.125.3. Let $(R, \mathfrak m)$ be a Noetherian local ring of dimension $d > 1$, let $f \in \mathfrak m$ be an element not contained in any minimal prime ideal of R, and let $k \in \mathbf{N}$. Then there exist elements $g_1, \ldots, g_{d-1} \in \mathfrak m^k$ such that $f, g_1, \ldots, g_{d-1}$ is a system of parameters.

Proof. We have $\dim(R/fR) = d-1$ by Algebra, Lemma 10.60.13. Choose a system of parameters $\overline{g}_1, \ldots, \overline{g}_{d-1}$ in R/fR (Algebra, Proposition 10.60.9) and take lifts $g_1, \ldots, g_{d-1}$ in R. It is straightforward to see that $f, g_1, \ldots, g_{d-1}$ is a system of parameters in R. Then $f, g_1^k, \ldots, g_{d-1}^k$ is also a system of parameters and the proof is complete. $\square$

0BWV Lemma 15.125.4. Let $(R, \mathfrak m)$ be a Noetherian local ring of dimension two, and let $f \in \mathfrak m$ be an element not contained in any minimal prime ideal of R. Then there exist $g \in \mathfrak m$ and $N \in \mathbf{N}$ such that

(a) f, g form a system of parameters for R.
(b) If $h \in \mathfrak m^N$, then $f+h, g$ is a system of parameters and $\text{length}_R(R/(f, g)) = \text{length}_R(R/(f + h, g))$.

Proof. By Lemma 15.125.3 there exists a $g \in \mathfrak m$ such that f, g is a system of parameters for R. Then $\mathfrak m = \sqrt{(f, g)}$. Thus there exists an n such that $\mathfrak m^n \subset (f, g)$, see Algebra, Lemma 10.32.5. We claim that $N = n+1$ works. Namely, let $h \in \mathfrak m^N$. By our choice of N we can write $h = af + bg$ with $a, b \in \mathfrak m$. Thus

$$(f + h, g) = (f + af + bg, g) = ((1 + a)f, g) = (f, g)$$

because $1 + a$ is a unit in R. This proves the equality of lengths and the fact that $f + h, g$ is a system of parameters. $\square$

0AXH Lemma 15.125.5. Let R be a Noetherian local normal domain of dimension 2. Let $\mathfrak p_1, \ldots, \mathfrak p_r$ be pairwise distinct primes of height 1. There exists a nonzero element $f \in \mathfrak p_1 \cap \ldots \cap \mathfrak p_r$ such that R/fR is reduced.

Proof. Let $f \in \mathfrak p_1 \cap \ldots \cap \mathfrak p_r$ be a nonzero element. We will modify f slightly to obtain an element that generates a radical ideal. The localization $R_\mathfrak p$ of R at each height one prime ideal $\mathfrak p$ is a discrete valuation ring, see Algebra, Lemma 10.119.7 or Algebra, Lemma 10.157.4. We denote by $\text{ord}_\mathfrak p(f)$ the corresponding valuation of f in $R_\mathfrak p$. Let $\mathfrak q_1, \ldots, \mathfrak q_s$ be the distinct height one prime ideals containing f. Write $\text{ord}_{\mathfrak q_j}(f) = m_j \geq 1$ for each j. Then we define $\text{div}(f) = \sum_{j=1}^s m_j \mathfrak q_j$ as a formal linear combination of height one primes with integer coefficients. Note for later use that each of the primes $\mathfrak p_i$ occurs among the primes $\mathfrak q_j$. The ring R/fR is reduced if and only if $m_j = 1$ for $j = 1, \ldots, s$. Namely, if m_j is 1 then $(R/fR)\mathfrak q_j$ is reduced and $R/fR \subset \prod (R/fR)_{\mathfrak q_j}$ as $\mathfrak q_1, \ldots, \mathfrak q_j$ are the associated primes of R/fR, see Algebra, Lemmas 10.63.19 and 10.157.6.

Choose and fix g and N as in Lemma 15.125.4. For a nonzero $y \in R$ denote $t(y)$ the number of primes minimal over y. Since R is a normal domain, these primes are height one and correspond 1-to-1 to the minimal primes of R/yR (Algebra, Lemmas 10.60.11 and 10.157.6). For example $t(f) = s$ is the number of primes $\mathfrak q_j$

occurring in $\text{div}(f)$. Let $h \in \mathfrak{m}^N$. By Lemma 15.125.2 we have

$$\begin{aligned} t(f+h) &\leq \text{length}_{R/(f+h)}(R/(f+h,g)) \\ &= \text{length}_R(R/(f+h,g)) \\ &= \text{length}_R(R/(f,g)) \end{aligned}$$

see Algebra, Lemma 10.52.5 for the first equality. Therefore we see that $t(f+h)$ is bounded independent of $h \in \mathfrak{m}^N$.

By the boundedness proved above we may pick $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ such that $t(f+h)$ is maximal among such h. Set $f' = f + h$. Given $h' \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ we see that the number $t(f'+h') \leq t(f+h)$. Thus after replacing f by f' we may assume that for every $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ we have $t(f+h) \leq s$.

Next, assume that we can find an element $h \in \mathfrak{m}^N$ such that for each j we have $\text{ord}_{\mathfrak{q}_j}(h) \geq 1$ and $\text{ord}_{\mathfrak{q}_j}(h) = 1 \Leftrightarrow m_j > 1$. Observe that $h \in \mathfrak{m}^N \cap \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$. Then $\text{ord}_{\mathfrak{q}_j}(f+h) = 1$ for every j by elementary properties of valuations. Thus

$$\text{div}(f+h) = \sum\nolimits_{j=1}^{s} \mathfrak{q}_j + \sum\nolimits_{k=1}^{v} e_k \mathfrak{r}_k$$

for some pairwise distinct height one prime ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_v$ and $e_k \geq 1$. However, since $s = t(f) \geq t(f+h)$ we see that $v = 0$ and we have found the desired element.

Now we will pick h that satisfies the above criteria. By prime avoidance (Algebra, Lemma 10.15.2) for each $1 \leq j \leq s$ we can find an element $a_j \in \mathfrak{q}_j$ such that $a_j \notin \mathfrak{q}_{j'}$ for $j' \neq j$ and $a_j \notin \mathfrak{q}_j^{(2)}$. Here $\mathfrak{q}_j^{(2)} = \{x \in R \mid \text{ord}_{\mathfrak{q}_j}(x) \geq 2\}$ is the second symbolic power of $\mathfrak{q}_j$. Then we take

$$h = \prod\nolimits_{m_j=1} a_j^2 \times \prod\nolimits_{m_j>1} a_j$$

Then h clearly satisfies the conditions on valuations imposed above. If $h \notin \mathfrak{m}^N$, then we multiply by an element of $\mathfrak{m}^N$ which is not contained in $\mathfrak{q}_j$ for all j. $\square$

0AXI Lemma 15.125.6. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian normal local domain of dimension 2. If $a \in \mathfrak{m}$ is nonzero, then there exists an element $c \in A$ such that A/cA is reduced and such that a divides c^n for some n.

Proof. Let $\text{div}(a) = \sum_{i=1,\ldots,r} n_i \mathfrak{p}_i$ with notation as in the proof of Lemma 15.125.5. Choose $c \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ with A/cA reduced, see Lemma 15.125.5. For $n \geq \max(n_i)$ we see that $-\text{div}(a) + \text{div}(c^n)$ is an effective divisor (all coefficients nonnegative). Thus $c^n/a \in A$ by Algebra, Lemma 10.157.6. $\square$

In the rest of this section we prove the result in dimension > 2.

0BWW Lemma 15.125.7. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, let $g_1, \ldots, g_d$ be a system of parameters, and let $I = (g_1, \ldots, g_d)$. If $e_I/d!$ is the leading coefficient of the numerical polynomial $n \mapsto \text{length}_R(R/I^{n+1})$, then $e_I \leq \text{length}_R(R/I)$.

Proof. The function is a numerical polynomial by Algebra, Proposition 10.59.5. It has degree d by Algebra, Proposition 10.60.9. If $d = 0$, then the result is trivial. If $d = 1$, then the result is Lemma 15.125.1. To prove it in general, observe that there is a surjection

$$\bigoplus\nolimits_{i_1,\ldots,i_d \geq 0,\ \sum i_j = n} R/I \longrightarrow I^n/I^{n+1}$$

sending the basis element corresponding to $i_1, \ldots, i_d$ to the class of $g_1^{i_1} \ldots g_d^{i_d}$ in I^n/I^{n+1}. Thus we see that

$$\text{length}_R(R/I^{n+1}) - \text{length}_R(R/I^n) \leq \text{length}_R(R/I)\binom{n+d-1}{d-1}$$

Since $d \geq 2$ the numerical polynomial on the left has degree $d-1$ with leading coefficient $e_I/(d-1)!$. The polynomial on the right has degree $d-1$ and its leading coefficient is $\text{length}_R(R/I)/(d-1)!$. This proves the lemma. □

0BWX Lemma 15.125.8. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, let t be the number of minimal prime ideals of R of dimension d, and let $(g_1, \ldots, g_d)$ be a system of parameters. Then $t \leq \text{length}_R(R/(g_1, \ldots, g_n))$.

Proof. If $d = 0$ the lemma is trivial. If $d = 1$ the lemma is Lemma 15.125.2. Thus we may assume $d > 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of R where the first t have dimension d, and denote $I = (g_1, \ldots, g_n)$. Arguing in exactly the same way as in the proof of Lemma 15.125.2 we can assume R is reduced.

Assume R is reduced with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. This means there is an exact sequence

$$0 \to R \to \prod\nolimits_{i=1}^t R/\mathfrak{p}_i \to Q \to 0$$

Here Q is the cokernel of the first map. Write $M = \prod_{i=1}^t R/\mathfrak{p}_i$. Localizing at $\mathfrak{p}_j$ we see that

$$R_{\mathfrak{p}_j} \to M_{\mathfrak{p}_j} = \left(\prod\nolimits_{i=1}^t R/\mathfrak{p}_i\right)_{\mathfrak{p}_j} = (R/\mathfrak{p}_j)_{\mathfrak{p}_j}$$

is surjective. Thus $Q_{\mathfrak{p}_j} = 0$ for all j. Therefore no height 0 prime of R is in the support of Q. It follows that the degree of the numerical polynomial $n \mapsto \text{length}_R(Q/I^nQ)$ equals $\dim(\text{Supp}(Q)) < d$, see Algebra, Lemma 10.62.6. By Algebra, Lemma 10.59.10 (which applies as R does not have finite length) the polynomial

$$n \longmapsto \text{length}_R(M/I^nM) - \text{length}_R(R/I^n) - \text{length}_R(Q/I^nQ)$$

has degree $< d$. Since $M = \prod R/\mathfrak{p}_i$ and since $n \to \text{length}_R(R/\mathfrak{p}_i + I^n)$ is a numerical polynomial of degree exactly(!) d for $i = 1, \ldots, t$ (by Algebra, Lemma 10.62.6) we see that the leading coefficient of $n \mapsto \text{length}_R(M/I^nM)$ is at least $t/d!$. Thus we conclude by Lemma 15.125.7. □

0BWY Lemma 15.125.9. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let $f \in \mathfrak{m}$ be an element not contained in any minimal prime ideal of R. Then there exist elements $g_1, \ldots, g_{d-1} \in \mathfrak{m}$ and $N \in \mathbf{N}$ such that

(1) $f, g_1, \ldots, g_{d-1}$ form a system of parameters for R
(2) If $h \in \mathfrak{m}^N$, then $f + h, g_1, \ldots, g_{d-1}$ is a system of parameters and we have $\text{length}_R R/(f, g_1, \ldots, g_{d-1}) = \text{length}_R R/(f + h, g_1, \ldots, g_{d-1})$.

Proof. By Lemma 15.125.3 there exist $g_1, \ldots, g_{d-1} \in \mathfrak{m}$ such that $f, g_1, \ldots, g_{d-1}$ is a system of parameters for R. Then $\mathfrak{m} = \sqrt{(f, g_1, \ldots, g_{d-1})}$. Thus there exists an n such that $\mathfrak{m}^n \subset (f, g)$, see Algebra, Lemma 10.32.5. We claim that $N = n + 1$

works. Namely, let $h \in \mathfrak{m}^N$. By our choice of N we can write $h = af + \sum b_i g_i$ with $a, b_i \in \mathfrak{m}$. Thus

$$\begin{aligned}(f + h, g_1, \ldots, g_{d-1}) &= (f + af + \sum b_i g_i, g_1, \ldots, g_{d-1}) \\ &= ((1 + a)f, g_1, \ldots, g_{d-1}) \\ &= (f, g_1, \ldots, g_{d-1})\end{aligned}$$

because $1 + a$ is a unit in R. This proves the equality of lengths and the fact that $f + h, g_1, \ldots, g_{d-1}$ is a system of parameters. □

0BWZ Proposition 15.125.10. Let R be a catenary Noetherian local normal domain. Let $J \subset R$ be a radical ideal. Then there exists a nonzero element $f \in J$ such that R/fR is reduced.

[Art86, Lemma 3.14] has this result without the assumption that the ring is catenary

Proof. The proof is the same as that of Lemma 15.125.5, using Lemma 15.125.8 instead of Lemma 15.125.2 and Lemma 15.125.9 instead of Lemma 15.125.4. We can use Lemma 15.125.8 because R is a catenary domain, so every height one prime ideal of R has dimension $d - 1$, and hence the spectrum of $R/(f + h)$ is equidimensional. For the convenience of the reader we write out the details.

Let $f \in J$ be a nonzero element. We will modify f slightly to obtain an element that generates a radical ideal. The localization $R_\mathfrak{p}$ of R at each height one prime ideal $\mathfrak{p}$ is a discrete valuation ring, see Algebra, Lemma 10.119.7 or Algebra, Lemma 10.157.4. We denote by $\text{ord}_\mathfrak{p}(f)$ the corresponding valuation of f in $R_\mathfrak{p}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the distinct height one prime ideals containing f. Write $\text{ord}_{\mathfrak{q}_j}(f) = m_j \geq 1$ for each j. Then we define $\text{div}(f) = \sum_{j=1}^s m_j \mathfrak{q}_j$ as a formal linear combination of height one primes with integer coefficients. The ring R/fR is reduced if and only if $m_j = 1$ for $j = 1, \ldots, s$. Namely, if m_j is 1 then $(R/fR)\mathfrak{q}_j$ is reduced and $R/fR \subset \prod (R/fR)_{\mathfrak{q}_j}$ as $\mathfrak{q}_1, \ldots, \mathfrak{q}_j$ are the associated primes of R/fR, see Algebra, Lemmas 10.63.19 and 10.157.6.

Choose and fix $g_2, \ldots, g_{d-1}$ and N as in Lemma 15.125.9. For a nonzero $y \in R$ denote $t(y)$ the number of primes minimal over y. Since R is a normal domain, these primes are height one and correspond 1-to-1 to the minimal primes of R/yR (Algebra, Lemmas 10.60.11 and 10.157.6). For example $t(f) = s$ is the number of primes $\mathfrak{q}_j$ occurring in $\text{div}(f)$. Let $h \in \mathfrak{m}^N$. Because R is catenary, for each height one prime $\mathfrak{p}$ of R we have $\dim(R/\mathfrak{p}) = d$. Hence by Lemma 15.125.8 we have

$$\begin{aligned}t(f + h) &\leq \text{length}_{R/(f+h)}(R/(f + h, g_1, \ldots, g_{d-1})) \\ &= \text{length}_R(R/(f + h, g_1, \ldots, g_{d-1})) \\ &= \text{length}_R(R/(f, g_1, \ldots, g_{d-1}))\end{aligned}$$

see Algebra, Lemma 10.52.5 for the first equality. Therefore we see that $t(f + h)$ is bounded independent of $h \in \mathfrak{m}^N$.

By the boundedness proved above we may pick $h \in \mathfrak{m}^N \cap J$ such that $t(f + h)$ is maximal among such h. Set $f' = f + h$. Given $h' \in \mathfrak{m}^N \cap J$ we see that the number $t(f' + h') \leq t(f + h)$. Thus after replacing f by f' we may assume that for every $h \in \mathfrak{m}^N \cap J$ we have $t(f + h) \leq s$.

Next, assume that we can find an element $h \in \mathfrak{m}^N \cap J$ such that for each j we have $\text{ord}_{\mathfrak{q}_j}(h) \geq 1$ and $\text{ord}_{\mathfrak{q}_j}(h) = 1 \Leftrightarrow m_j > 1$. Then $\text{ord}_{\mathfrak{q}_j}(f + h) = 1$ for every j by elementary properties of valuations. Thus

$$\operatorname{div}(f+h) = \sum\nolimits_{j=1}^{s} \mathfrak{q}_j + \sum\nolimits_{k=1}^{v} e_k \mathfrak{r}_k$$

for some pairwise distinct height one prime ideals $\mathfrak{r}_1, \ldots, \mathfrak{r}_v$ and $e_k \geq 1$. However, since $s = t(f) \geq t(f+h)$ we see that $v = 0$ and we have found the desired element.

Now we will pick h that satisfies the above criteria. By prime avoidance (Algebra, Lemma 10.15.2) for each $1 \leq j \leq s$ we can find an element $a_j \in \mathfrak{q}_j \cap J$ such that $a_j \notin \mathfrak{q}_{j'}$ for $j' \neq j$. Next, we can pick $b_j \in J \cap \mathfrak{q}_1 \cap \ldots \cap q_s$ with $b_j \notin \mathfrak{q}_j^{(2)}$. Here $\mathfrak{q}_j^{(2)} = \{x \in R \mid \operatorname{ord}_{\mathfrak{q}_j}(x) \geq 2\}$ is the second symbolic power of $\mathfrak{q}_j$. Prime avoidance applies because the ideal $J' = J \cap \mathfrak{q}_1 \cap \ldots \cap q_s$ is radical, hence R/J' is reduced, hence $(R/J')_{\mathfrak{q}_j}$ is reduced, hence J' contains an element x with $\operatorname{ord}_{\mathfrak{q}_j}(x) = 1$, hence $J' \not\subset \mathfrak{q}_j^{(2)}$. Then the element

$$c = \sum\nolimits_{j=1,\ldots,s} b_j \times \prod\nolimits_{j' \neq j} a_{j'}$$

is an element of J with $\operatorname{ord}_{\mathfrak{q}_j}(c) = 1$ for all $j = 1, \ldots, s$ by elementary properties of valuations. Finally, we let

$$h = c \times \prod\nolimits_{m_j = 1} a_j \times y$$

where $y \in \mathfrak{m}^N$ is an element which is not contained in $\mathfrak{q}_j$ for all j. □

15.126. Invertible objects in the derived category

0FNP We characterize invertible objects in the derived category of a ring.

0FNQ **Lemma 15.126.1.** Let R be a ring. The derived category $D(R)$ of R is a symmetric monoidal category with tensor product given by derived tensor product and associativity and commutativity constraints as in Section 15.72.

Proof. Omitted. Hints: The associativity constraint is the isomorphism of Lemma 15.59.15 and the commutativity constraint is the isomorphism of Lemma 15.59.14. Having said this the commutativity of various diagrams follows from the corresponding result for the category of complexes of R-modules, see Section 15.58. □

Thus we know what it means for an object of $D(R)$ to have a (left) dual or to be invertible. Before we can work out what this amounts to we need a simple lemma.

0FNR **Lemma 15.126.2.** Let R be a ring. Let $F^\bullet$ be a bounded above complex of free R-modules. Given pairs (n_i, f_i), $i = 1, \ldots, N$ with $n_i \in \mathbf{Z}$ and $f_i \in F^{n_i}$ there exists a subcomplex $G^\bullet \subset F^\bullet$ containing all f_i which is bounded and consists of finite free R-modules.

Proof. By descending induction on $a = \min(n_i; i = 1, \ldots, N)$. If $F^n = 0$ for $n \geq a$, then the result is true with $G^\bullet$ equal to the zero complex. In general, after renumbering we may assume there exists an $1 \leq r \leq N$ such that $n_1 = \ldots = n_r = a$ and $n_i > a$ for $i > r$. Choose a basis $b_j, j \in J$ for F^a. We can choose a finite subset $J' \subset J$ such that $f_i \in \bigoplus_{j \in J'} Rb_j$ for $i = 1, \ldots, r$. Choose a basis $c_k, k \in K$ for F^{a+1}. We can choose a finite subset $K' \subset K$ such that $\mathrm{d}_F^a(b_j) \in \bigoplus_{k \in K'} Rc_k$ for $j \in J'$. Then we can apply the induction hypothesis to find a subcomplex $H^\bullet \subset F^\bullet$ containing $c_k \in F^{a+1}$ for $k \in K'$ and $f_i \in F^{n_i}$ for $i > r$. Take $G^\bullet$ equal to $H^\bullet$ in degrees $> a$ and equal to $\bigoplus_{j \in J'} Rb_j$ in degree a. □

0FNS **Lemma 15.126.3.** Let R be a ring. Let M be an object of $D(R)$. The following are equivalent

(1) M has a left dual in $D(R)$ as in Categories, Definition 4.43.5,
(2) M is a perfect object of $D(R)$.

Moreover, in this case the left dual of M is the object $M^\vee$ of Lemma 15.74.15.

Proof. If M is perfect, then we can represent M by a bounded complex $M^\bullet$ of finite projective R-modules. In this case $M^\bullet$ has a left dual in the category of complexes by Lemma 15.72.2 which is a fortiori a left dual in $D(R)$.

Assume (1). Say N, $\eta : R \to M \otimes_R^{\mathbf{L}} N$, and $\epsilon : M \otimes_R^{\mathbf{L}} N \to R$ is a left dual as in Categories, Definition 4.43.5. Choose a complex $M^\bullet$ representing M. Choose a K-flat complexes $N^\bullet$ with flat terms representing N, see Lemma 15.59.10. Then η is given by a map of complexes

$$\eta : R \longrightarrow \text{Tot}(M^\bullet \otimes_R N^\bullet)$$

We can write the image of 1 as a finite sum

$$\eta(1) = \sum\nolimits_n \sum\nolimits_i m_{n,i} \otimes n_{-n,i}$$

with $m_{n,i} \in M^n$ and $n_{-n,i} \in N^{-n}$. Let $K^\bullet \subset M^\bullet$ be the subcomplex generated by all the elements $m_{n,i}$ and $\text{d}(m_{n,i})$. By our choice of $N^\bullet$ we find that $\text{Tot}(K^\bullet \otimes_R N^\bullet) \subset \text{Tot}(M^\bullet \otimes_R N^\bullet)$ and $\eta(1)$ is in the subcomplex by our choice above. Denote K the object of $D(R)$ represented by $K^\bullet$. Then we see that η factors over a map $\tilde{\eta} : R \longrightarrow K \otimes_R^{\mathbf{L}} N$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \text{id}_M$ we conclude that the identity on M factors through K by the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\eta\otimes 1"'] \arrow[dr, "\tilde{\eta}\otimes 1"'] & M \otimes_R^{\mathbf{L}} N \otimes_R^{\mathbf{L}} M \arrow[r, "1\otimes\epsilon"'] & M \\
 & K \otimes_R^{\mathbf{L}} N \otimes_R^{\mathbf{L}} M \arrow[u] \arrow[r, "1\otimes\epsilon"] & K \arrow[u]
\end{tikzcd}$$

Since K is bounded above it follows that $M \in D^-(R)$. Thus we can represent M by a bounded above complex $M^\bullet$ of free R-modules, see for example Derived Categories, Lemma 13.15.4. Write $\eta(1) = \sum_n \sum_i m_{n,i} \otimes n_{-n,i}$ as before. By Lemma 15.126.2 we can find a subcomplex $K^\bullet \subset M^\bullet$ containing all the elements $m_{n,i}$ which is bounded and consists of finite free R-modules. As above we find that the identity on M factors through K. Since K is perfect we conclude M is perfect too, see Lemma 15.74.5. $\square$

0FNT **Lemma 15.126.4.** Let R be a ring. Let M be an object of $D(R)$. The following are equivalent

(1) M is invertible in $D(R)$, see Categories, Definition 4.43.4, and
(2) for every prime ideal $\mathfrak{p} \subset R$ there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $M_f \cong R_f[-n]$ for some $n \in \mathbf{Z}$.

Moreover, in this case

(a) M is a perfect object of $D(R)$,
(b) $M = \bigoplus H^n(M)[-n]$ in $D(R)$,
(c) each $H^n(M)$ is a finite projective R-module,
(d) we can write $R = \prod_{a \leq n \leq b} R_n$ such that $H^n(M)$ corresponds to an invertible R_n-module.

Proof. Assume (2). Consider the object $R\operatorname{Hom}_R(M, R)$ and the composition map

$$R\operatorname{Hom}(M, R) \otimes_R^{\mathbf{L}} M \to R$$

Checking locally we see that this is an isomorphism; we omit the details. Because $D(R)$ is symmetric monoidal we see that M is invertible.

Assume (1). Observe that an invertible object of a monoidal category has a left dual, namely, its inverse. Thus M is perfect by Lemma 15.126.3. Consider a prime ideal $\mathfrak{p} \subset R$ with residue field κ. Then we see that $M \otimes_R^{\mathbf{L}} \kappa$ is an invertible object of $D(\kappa)$. Clearly this implies that $\dim H^i(M \otimes_R^{\mathbf{L}} \kappa)$ is nonzero exactly for one i and equal to 1 in that case. By Lemma 15.75.6 this gives (2).

In the proof above we have seen that (a) holds. Let $U_n \subset \operatorname{Spec}(R)$ be the union of the opens of the form $D(f)$ such that $M_f \cong R_f[-n]$. Clearly, $U_n \cap U_{n'} = \emptyset$ if $n \neq n'$. If M has tor amplitude in $[a, b]$, then $U_n = \emptyset$ if $n \notin [a, b]$. Hence we see that we have a product decomposition $R = \prod_{a \leq n \leq b} R_n$ as in (d) such that U_n corresponds to $\operatorname{Spec}(R_n)$, see Algebra, Lemma 10.24.3. Since $D(R) = \prod_{a \leq n \leq b} D(R_n)$ and similary for the category of modules parts (b), (c), and (d) follow immediately. □

15.127. Splitting off a free module

0GV7 The arguments in this section are due to Serre, see [Ser58].

0GV8 Situation 15.127.1. Here R is a ring and M is a finitely presented R-module. Denote $\Omega \subset \operatorname{Spec}(R)$ the set of closed points with the induced topology. For $x \in \Omega$ denote $M(x) = M/xM$ the fibre of M at x. This is a finite dimensional vector space over the residue field $\kappa(x)$ at x. Given $s \in M$ we denote $s(x)$ the image of s in $M(x)$.

0GV9 Lemma 15.127.2. In Situation 15.127.1 let $x \in \Omega$. There exists a canonical short exact sequence

$$0 \to B(x) \to M(x) \to V(x) \to 0$$

of $\kappa(x)$-vector spaces which the following property: for $s_1, \ldots, s_r \in M$ the following are equivalent

(1) there exists an $f \in R$, $f \notin x$ such that the map $s_1, \ldots, s_r : R^{\oplus r} \to M$ becomes the inclusion of a direct summand after inverting f, and
(2) $s_1(x), \ldots, s_r(x)$ map to linearly independent elements of $V(x)$.

Proof. Define $B(x) \subset M(x)$ as the perpendicular of the image of the map

$$\operatorname{Hom}_R(M, R) \to \operatorname{Hom}_{\kappa(x)}(M(x), \kappa(x))$$

and set $V(x) = M(x)/B(x)$. Then any R-linear map $\varphi : M \to R$ induces a map $\overline{\varphi} : V(x) \to \kappa(x)$ and conversely any $\kappa(x)$-linear map $\lambda : V(x) \to \kappa(x)$ is equal to $\overline{\varphi}$ for some φ. Let $s_1, \ldots, s_r \in M$.

Suppose $s_1, \ldots, s_r$ map to linearly independent elements of $V(x)$. Then we can find $\varphi_1, \ldots, \varphi_r \in \operatorname{Hom}_R(M, R)$ such that $\varphi_i(s_j)$ maps to δ_{ij}[18] in $\kappa(x)$. Hence the matrix of the composition

$$R^{\oplus r} \xrightarrow{s_1, \ldots, s_r} M \xrightarrow{\varphi_1, \ldots, \varphi_r} R^{\oplus r}$$

has a determinant $f \in R$ which maps to 1 in $\kappa(x)$ Clearly, this implies that $s_1, \ldots, s_r : R^{\oplus r} \to M$ is the inclusion of a direct summand after inverting f.

[18] Kronecker delta.

Conversely, suppose that we have an $f \in R$, $f \not\in x$ such that $s_1, \ldots, s_r : R^{\oplus r} \to M$ is the inclusion of a direct summand after inverting f. Hence we can find R_f-linear maps $\varphi_i : M_f \to R_f$ such that $\varphi_i(s_j) = \delta_{ij} \in R_f$. Since $\mathrm{Hom}_R(M, R)_f = \mathrm{Hom}_{R_f}(M_f, R_f)$ by Algebra, Lemma 10.10.2 we conclude that we can find $n \geq 0$ and $\varphi'_i \in \mathrm{Hom}_R(M, R)$ such that $\varphi'_i(s_j) = f^n\delta_{ij} \in R$. It follows that $s_1, \ldots, s_r$ map to linearly independent elements of $V(x)$ as $\overline{\varphi}'_i(s_j) = f^n\delta_{ij}$. □

In Situation 15.127.1 given $s_1, \ldots, s_r \in M$ we denote $Z(s_1, \ldots, s_r) \subset \Omega$ the set of $x \in \Omega$ such that $s_1(x), \ldots, s_r(x)$ map to linearly dependent elements of $V(x)$. By the lemma this is a closed subset of Ω.

0GVA Lemma 15.127.3. In Situation 15.127.1 let $x_1, \ldots, x_n \in \Omega$ be pairwise distinct. Let $v_i \in V(x_i)$. Then there exists an $s \in M$ such that $s(x_i)$ maps to v_i for $i = 1, \ldots, n$.

Proof. Since x_i is a maximal ideal of R we may use Algebra, Lemma 10.15.4 to see that $M(x_1) \oplus \ldots \oplus M(x_n)$ is a quotient of M. □

0GVB Proposition 15.127.4. In Situation 15.127.1 assume Ω is a Noetherian topological space. Let $s_1, \ldots, s_h \in M$. Let $Z(s_1, \ldots, s_h) \subset F \subset \Omega$ be closed. Let $x_1, \ldots, x_n \in F$ be pairwise distinct. Let $v_i \in V(x_i)$. Let $k \geq 0$ be an integer such that [Ser58, Theorem 2]

$$(*) \quad h + k \leq \dim_{\kappa(x)} V(x) \text{ for all } x \in \Omega$$

Then there exist $s \in M$ and $F' \subset \Omega$ closed such that

(a) $s(x_i)$ maps to v_i,
(b) $Z(s_1, \ldots, s_h, s) \subset F \cup F'$, and
(c) every irreducible component of F' has codimension $\geq k$ in Ω.

Proof. We note that codimension was defined in Topology, Section 5.11 and that we will use some results on Noetherian topological spaces contained in Topology, Section 5.9.

The proof is by induction on k. If $k = 0$, then we choose $s \in M$ as in Lemma 15.127.3 and we choose $F' = \Omega$.

Assume $k > 0$. By our induction hypothesis we may choose $u \in M$ and $G \subset \Omega$ closed satisfying (a), (b), (c) for $s_1, \ldots, s_h$, F, $x_1, \ldots, x_n$, $v_1, \ldots, v_n$, and $k - 1$.

Let $G = G_1 \cup \ldots \cup G_m$ be the decomposition of G into its irreducible components. If $G_j \subset F$, then we can remove it from the list. Thus we may assume G_j is not contained in F for $j = 1, \ldots, m$. For $j = 1, \ldots, m$ choose $y_j \in G_j$ with $y_j \not\in F$ and $y_j \not\in G_{j'}$ for $j' \neq j$. This is possible as there are no inclusions among the irreducible components of G. Choose $w_j \in V(y_j)$ not contained in the span of the images of $s_1(y_j), \ldots, s_h(y_j)$; this is possible because $h + k \leq \dim V(y_j)$ and $k > 0$.

Apply the induction hypothesis to the $h + 1$ sections $s_1, \ldots, s_h, u$, the closed set $F \cup G$, the points $x_1, \ldots, x_n, y_1, \ldots, y_m \in F \cup G$, the elements $0 \in V(x_i)$ and $w_j \in V(y_j)$, and the integer $k - 1$. Note that we have increased h by 1 and decreased k by 1 hence the assumption $(*)$ of the proposition remains valid. This produces $t \in M$ and $H \subset \Omega$ closed satisfying (a), (b), (c) for $s_1, \ldots, s_h, u$, $F \cup G$, $x_1, \ldots, x_n, y_1, \ldots, y_m$, $0, \ldots, 0, w_1, \ldots, w_m$, and $k - 1$.

Let $H_1, \ldots, H_p \subset H$ be the irreducible components of H which are not contained in $F \cup G$. As before pick $z_l \in H_l$, $z_l \not\in F \cup G$ and $z_l \not\in H_{l'}$ for $l' \neq l$. Using

Algebra, Lemma 10.15.4 we may choose $f \in R$ such that $f(y_j) = 1$, $j = 1, \ldots, m$ and $f(z_l) = 0$, $l = 1, \ldots, p$. Claim: the element $s = u + ft$ works.

First, the value $s(x_i)$ agrees with $u(x_i)$ because $t(x_i) = 0$ and hence we see that $s(x_i)$ maps to v_i. This proves (a). To finish the proof it suffices to show that every irreducible component Z of $Z(s_1, \ldots, s_h, s)$ not contained in F has codimension $\geq k$ in Ω. Namely, then we can set F' equal to the union of these and we get (b) and (c). We can see that irreducible components Z of $Z(s_1, \ldots, s_h, s)$ of codimension $\leq k - 1$ do not exist as follows:

(1) Observe that $Z(s_1, \ldots, s_h, s) \subset Z(s_1, \ldots, s_h, u, t) = F \cup H$ as $s = u + ft$. Hence $Z \subset H$.
(2) The irreducible components of H have codimension $\geq k - 1$. Hence Z is equal to an irreducible component of H as Z has codimension $\leq k - 1$. Hence $Z = H_l$ for some $l \in \{1, \ldots, p\}$ or $Z = G_j$ for some $j \in \{1, \ldots m\}$.
(3) But $Z = G_j$ is impossible as $s_1(y_j), \ldots, s_h(y_j)$ map to linearly independent elements of $V(y_j)$ and $s(y_j) = u(y_j) + f(y_j)t(y_j) = u(y_j) + t(y_j)$ maps to an element of the form
$$\text{linear combination images of } s_i(y_j) + w_j$$
which is linearly independent of the images of $s_1(y_j), \ldots, s_h(y_j)$ in $V(y_j)$ by our choice of w_j.
(4) Also $Z = Z_l$ is impossible. Namely, again $s_1(z_l), \ldots, s_h(z_l)$ map to linearly independent elements of $V(z_l)$ and $s(z_l) = u(z_l) + f(z_l)t(z_l) = u(z_l)$ maps to an element of $V(z_l)$ linearly independent of those as $z_l \notin F \cup G$.

This finishes the proof. □

0GVC Theorem 15.127.5. Let R be a ring whose max spectrum $\Omega \subset \mathrm{Spec}(R)$ is a Noetherian topological space of dimension $d < \infty$. Let M be a finitely presented R-module such that for all $\mathfrak{m} \in \Omega$ the $R_\mathfrak{m}$-module $M_\mathfrak{m}$ has a free direct summand of rank $> d$. Then $M \cong R \oplus M'$. [Ser58, Theorem 1]

Proof. For $\mathfrak{m} \in \Omega$ suppose that $R_\mathfrak{m}^{\oplus r}$ is a direct summand of $M_\mathfrak{m}$. Then by Algebra, Lemmas 10.9.9 and 10.127.6 we see that $R_f^{\oplus r}$ is a direct summand of M_f for some $f \in R$, $f \notin \mathfrak{m}$. Hence the assumption means that $\dim V(x) > d$ for all $x \in \Omega$ where $V(x)$ is as in Lemma 15.127.2. By Proposition 15.127.4 applied with $F = \emptyset$, $h = 0$ and no s_i, $n = 0$ and no x_i, v_i, and $k = d + 1$ we find an $s \in M$ and $F' \subset \Omega$ such that every irreducible component of F' has codimension $\geq d + 1$ and $Z(s) \subset F'$. Since $d = \dim(\Omega)$ this forces $F' = \emptyset$. Hence $s : R \to M$ is the inclusion of a direct summand at all maximal ideals. It follows that s is universally injective, see Algebra, Lemma 10.82.12. Then s is split injective by Algebra, Lemma 10.82.4. □

15.128. Big projective modules are free

0GVE In this section we discuss one of the results of [Bas63]; we suggest the reader look at the original paper. Our argument will use the slightly simplified proof given in the papers [Aka70] and [Hin63].

0GVF Lemma 15.128.1 (Eilenberg's lemma). If $P \oplus Q \cong F$ with F a nonfinitely generated free module, then $P \oplus F \cong F$. [Bas63, Eilenberg's lemma]

Proof.

$$F \cong F \oplus F \oplus \ldots \cong P \oplus Q \oplus P \oplus Q \oplus \ldots \cong P \oplus F \oplus F \oplus \ldots \cong P \oplus F$$

□

0GVG Lemma 15.128.2. Let R be a ring. Let P be a projective module. There exists a free module F such that $P \oplus F$ is free.

Proof. Since P is projective we see that $F_0 = P \oplus Q$ is a free module for some module Q. Set $F = \bigoplus_{n \geq 1} F_0$. Then $P \oplus F \cong F$ by Lemma 15.128.1. □

0GVH Lemma 15.128.3. Let R be a ring. Let P be a projective module. Let $s \in P$. There exists a finite free module F and a finite free direct summand $K \subset F \oplus P$ with $(0, s) \in K$.

Proof. By Lemma 15.128.2 we can find a (possibly infinite) free module F such that $F \oplus P$ is free. Then of course $(0, s)$ is contained in a finite free direct summand $K \subset F \oplus P$. In turn K is contained in $F' \oplus P$ where $F' \subset F$ is a finite free direct summand. □

0GVI Lemma 15.128.4. Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a projective R-module such that $P_\mathfrak{m}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Let $s \in P$ and $M \subset P$ such that $Rs + M = P$. Then we can find $m \in M$ such that $R(s + m)$ is a free direct summand of P.

Proof. The statement makes sense as $P_\mathfrak{m}$ is free by Algebra, Theorem 10.85.4.

Denote $M' \subset P/JP$ the image of M and $s' \in P/JP$ the image of s. Observe that $R/Js' + M' = P/JP$. Suppose we can find $m' \in M'$ such that $R/J(s' + m')$ is a free direct summand of M'. Choose $\varphi' : P/JP \to R/J$ which gives a splitting, i.e., we have $\varphi'(s' + m') = 1$ in R/J. Then since P is a projective R-module we can find a lift $\varphi : P \to R$ of φ'. Choose $m \in M$ mapping to m'. Then $\varphi(s + m) \in R$ is congruent to 1 modulo J and hence a unit in R (Algebra, Lemma 10.19.1). Whence $R(s + m)$ is a free direct summand of P. This reduces us to the case discussed in the next paragraph.

Assume R is Noetherian. Let $m \in M$ be an element and let $\varphi_1, \ldots, \varphi_n : P \to R$ be R-linear maps. Denote

$$Z(s + m, \varphi_1, \ldots, \varphi_n) \subset \text{Spec}(R)$$

the vanishing locus of $\varphi_1(s + m), \ldots, \varphi_n(s + m) \in R$.

Suppose $\mathfrak{m}$ is a maximal ideal of R and $\mathfrak{m} \in Z(s, \varphi_1, \ldots, \varphi_n)$. Set $K = M \cap \bigcap \text{Ker}(\varphi_i)$. We claim the image of

$$K/\mathfrak{m}K \to P/\mathfrak{m}P$$

has infinite dimension. Namely, the quotient P/K is a finite R-module as it is isomorphic to a submodule of $P/M \oplus R^{\oplus n}$. Thus we see that the kernel of the displayed arrow is a quotient of $\text{Tor}_1^R(P/K, \kappa(\mathfrak{m}))$ which is finite by Algebra, Lemma 10.75.7. Combined with the fact that $P/\mathfrak{m}P$ has infinite dimension we obtain our claim. Thus we can find a $t \in K$ which maps to a nonzero element $\overline{t}$ of the vector space $P/\mathfrak{m}P$. By linear algebra, we find an R-linear map $\overline{\varphi} : P \to \kappa(\mathfrak{m})$ such that $\overline{\varphi}(\overline{t}) = 1$. Since P is projective, we can find an R-linear map $\varphi : P \to R$ lifting $\overline{\varphi}$. Then we see that the vanishing locus $Z(s + m + t, \varphi_1, \ldots, \varphi_n, \varphi)$ is contained in $Z(s+m, \varphi_1, \ldots, \varphi_n)$ but does not contain $\mathfrak{m}$, i.e., it is strictly smaller than $Z(s+m, \varphi_1, \ldots, \varphi_n)$.

Since $\text{Spec}(R)$ is a Noetherian topological space, we see from the arguments above that we may find $m \in M$ and $\varphi_1, \ldots, \varphi_n : P \to R$ such that the closed subset $Z(s+m, \varphi_1, \ldots, \varphi_n)$ does not contain any closed points of $\text{Spec}(R)$. Hence $Z(s+m, \varphi_1, \ldots, \varphi_n) = \emptyset$. Hence we can find $r_1, \ldots, r_n \in R$ such that $\sum r_i\varphi_i(s+m) = 1$. Hence

$$R \xrightarrow{s+m} P \xrightarrow{\sum r_i\varphi_i} R$$

is the desired splitting. □

0GVJ Lemma 15.128.5. Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a projective R-module such that $P_\mathfrak{m}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Let $s \in P$. Then we can find a finite stably free direct summand $M \subset P$ such that $s \in M$.

Proof. By Lemma 15.128.3 we can find a finite free module F and a finite free direct summand $K \subset F \oplus P$ such that $(0, s) \in K$. By induction on the rank of F we reduce to the case discussed in the next paragraph.

Assume there exists a finite stably free direct summand $K \subset R \oplus P$ such that $(0, s) \in K$. Choose a complement K' of K, i.e., such that $R \oplus P = K \oplus K'$. The projection $\pi : R \oplus P \to K'$ is surjectve, hence by Lemma 15.128.4 we find a $p \in P$ such that $\pi(1, p) \in K'$ generates a free direct summand. Accordingly we write $K' = R\pi(1, p) \oplus K''$. We see that

$$R \oplus P = K \oplus K' = K \oplus R\pi(1, p) \oplus K''$$

The projection $\pi' : P \to K''$ is surjective[19] and hence split (as K'' is projective). Thus $\text{Ker}(\pi') \subset P$ is a direct summand containing s. Finally, by construction we have an isomorphism

$$R \oplus \text{Ker}(\pi') \cong K \oplus R\pi(1, p)$$

and hence since K is finite and stably free, so is $\text{Ker}(\pi')$. □

0GVK Theorem 15.128.6. Let R be a ring with Jacobson radical J such that R/J is Noetherian. Let P be a countably generated projective R-module such that $P_\mathfrak{m}$ has infinite rank for all maximal ideals $\mathfrak{m}$ of R. Then P is free. Commutative case of [Bas63, Theorem 3.1]

Proof. We first prove that P is a countable direct sum of finite stably free modules. Let $x_1, x_2, \ldots$ be a countable set of generators for P. We inductively construct finite stably free direct summands $F_1, F_2, \ldots$ of P such that for all n we have that $F_1 \oplus \ldots \oplus F_n$ is a direct summand of P which contains $x_1, \ldots, x_n$. Namely, given $F_1, \ldots, F_n$ with the desired properties, write

$$P = F_1 \oplus \ldots \oplus F_n \oplus P'$$

and let $s \in P'$ be the image of x_{n+1}. By Lemma 15.128.5 we can find a finite stably free direct summand $F_{n+1} \subset P'$ containing s. Then $P = \bigoplus_{i=1}^{\infty} F_i$.

[19]Namely, if $k'' \in K''$ then k'' viewed as an element of K' can be written as $k'' = \lambda\pi(1, 0) + \pi(0, q)$ for some $\lambda \in R$ and $q \in P$. This means $k'' = \lambda\pi(1, p) + \pi(0, q - \lambda p)$. This in turn means that $q - \lambda p$ maps to k'' by the composition $P \to R \oplus P \xrightarrow{\pi} K' \to K''$ since $K' \to K''$ annihilates $\pi(1, p)$.

Assume that P is an infinite direct sum $P = \bigoplus_{i=1}^{\infty} F_i$ of nonzero finite stably free modules. The stable freeness of the modules F_i will be used in the following manner: the rank of each F_i is constant (and positive). Hence we see that $P_{\mathfrak m}$ is free of countably infinite rank for each maximal ideal $\mathfrak m$ of R. By Lemma 15.128.4 applied with $s = 0$ and $M = P$, we can find a $t_1 \in P$ such that Rt_1 is a free direct summand of P. Then t_1 is contained in $F_1 \oplus \ldots \oplus F_{n_1}$ for some $n_1 > n_0 = 0$. The same reasoning applied to $\bigoplus_{n > n_1} F_n$ produces an $n_1 < n_2$ and $t_2 \in F_{n_1+1} \oplus \ldots \oplus F_{n_2}$ which generates a free direct summand. Continuing in this fashion we obtain a free direct summand

$$\bigoplus\nolimits_{i \geq 1} t_i : \bigoplus\nolimits_{i \geq 1} R \longrightarrow \bigoplus\nolimits_{i \geq 1} \bigoplus\nolimits_{n_i \geq n > n_{i-1}} F_n = P$$

of infinite rank. Thus we see that $P \cong Q \oplus F$ for some free R-module F of countable rank. Since Q is countably generated it follows that $Q \oplus Q' \cong F$ for some module Q'. Then the Eilenberg swindle (Lemma 15.128.1) implies that $Q \oplus F \cong F$ and P is free. $\square$

15.129. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

Topics in Scheme Theory

(42) Chow Homology
(43) Intersection Theory
(44) Picard Schemes of Curves
(45) Weil Cohomology Theories
(46) Adequate Modules
(47) Dualizing Complexes
(48) Duality for Schemes
(49) Discriminants and Differents
(50) de Rham Cohomology
(51) Local Cohomology
(52) Algebraic and Formal Geometry
(53) Algebraic Curves
(54) Resolution of Surfaces
(55) Semistable Reduction
(56) Functors and Morphisms

CHAPTER 16

Smoothing Ring Maps

07BW
16.1. Introduction

07BX The main result of this chapter is the following:

$$\boxed{\text{A regular map of Noetherian rings is a filtered colimit of smooth ones.}}$$

This theorem is due to Popescu, see [Pop90]. A readable exposition of Popescu's proof was given by Richard Swan, see [Swa98] who used notes by André and a paper of Ogoma, see [Ogo94].

Our exposition follows Swan's, but we first prove an intermediate result which lets us work in a slightly simpler situation. Here is an overview. We first solve the following "lifting problem": A flat infinitesimal deformation of a filtered colimit of smooth algebras is a filtered colimit of smooth algebras. This result essentially says that it suffices to prove the main theorem for maps between reduced Noetherian rings. Next we prove two very clever lemmas called the "lifting lemma" and the "desingularization lemma". We show that these lemmas combined reduce the main theorem to proving a Noetherian, geometrically regular algebra Λ over a field k is a filtered colimit of smooth k-algebras. Next, we discuss the necessary local tricks that go into the Popescu-Ogoma-Swan-André proof. Finally, in the last three sections we give the proof.

We end this introduction with some pointers to references. Let A be a henselian Noetherian local ring. We say A has the approximation property if for any $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ the system of equations $f_1 = 0, \ldots, f_m = 0$ has a solution in the completion of A if and only if it has a solution in A. This definition is due to Artin. Artin first proved the approximation property for analytic systems of equations, see [Art68]. In [Art69a] Artin proved the approximation property for local rings essentially of finite type over an excellent discrete valuation ring. Artin conjectured (page 26 of [Art69a]) that every excellent henselian local ring should have the approximation property.

At some point in time it became a conjecture that every regular homomorphism of Noetherian rings is a filtered colimit of smooth algebras (see for example [Ray72], [Pop81], [Art82], [AD83]). We're not sure who this conjecture[1] is due to. The relationship with the approximation property is that if $A \to A^\wedge$ is a colimit of smooth algebras with A as above, then the approximation property holds (insert future reference here). Moreover, the main theorem applies to the map $A \to A^\wedge$ if A is an excellent local ring, as one of the conditions of an excellent local ring

[1]The question/conjecture as formulated in [Art82], [AD83], and [Pop81] is stronger and was shown to be equivalent to the original version in [CP84].

is that the formal fibres are geometrically regular. Note that excellent local rings were defined by Grothendieck and their definition appeared in print in 1965.

In [Art82] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over a field. In [AR88] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over an excellent discrete valuation ring. Finally, the main theorem was shown in [Pop85], [Pop86], [Pop90], [Ogo94], and [Swa98] as discussed above.

Conversely, using some of the results above, in [Rot90] it was shown that any Noetherian local ring with the approximation property is excellent.

The paper [Spi99] provides an alternative approach to the main theorem, but it seems hard to read (for example [Spi99, Lemma 5.2] appears to be an incorrectly reformulated version of [Elk73, Lemma 3]). There is also a Bourbaki lecture about this material, see [Tei95].

16.2. Singular ideals

07C4 Let $R \to A$ be a ring map. The singular ideal of A over R is the radical ideal in A cutting out the singular locus of the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$. Here is a formal definition.

07C5 Definition 16.2.1. Let $R \to A$ be a ring map. The singular ideal of A over R, denoted $H_{A/R}$ is the unique radical ideal $H_{A/R} \subset A$ with

$$V(H_{A/R}) = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid R \to A \text{ not smooth at } \mathfrak{q}\}$$

This makes sense because the set of primes where $R \to A$ is smooth is open, see Algebra, Definition 10.137.11. In order to find an explicit set of generators for the singular ideal we first prove the following lemma.

07C6 Lemma 16.2.2. Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\mathfrak{q} \subset A$ be a prime ideal. Assume $R \to A$ is smooth at $\mathfrak{q}$. Then there exists an $a \in A$, $a \notin \mathfrak{q}$, an integer c, $0 \leq c \leq \min(n, m)$, subsets $U \subset \{1, \ldots, n\}$, $V \subset \{1, \ldots, m\}$ of cardinality c such that

$$a = a' \det(\partial f_j/\partial x_i)_{j \in V, i \in U}$$

for some $a' \in A$ and

$$af_\ell \in (f_j, j \in V) + (f_1, \ldots, f_m)^2$$

for all $\ell \in \{1, \ldots, m\}$.

Proof. Set $I = (f_1, \ldots, f_m)$ so that the naive cotangent complex of A over R is homotopy equivalent to $I/I^2 \to \bigoplus A\mathrm{d}x_i$, see Algebra, Lemma 10.134.2. We will use the formation of the naive cotangent complex commutes with localization, see Algebra, Section 10.134, especially Algebra, Lemma 10.134.13. By Algebra, Definitions 10.137.1 and 10.137.11 we see that $(I/I^2)_a \to \bigoplus A_a\mathrm{d}x_i$ is a split injection for some $a \in A$, $a \notin \mathfrak{q}$. After renumbering $x_1, \ldots, x_n$ and $f_1, \ldots, f_m$ we may assume that $f_1, \ldots, f_c$ form a basis for the vector space $I/I^2 \otimes_A \kappa(\mathfrak{q})$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $\Omega_{A/R} \otimes_A \kappa(\mathfrak{q})$. Hence after replacing a by aa' for some $a' \in A$, $a' \notin \mathfrak{q}$ we may assume $f_1, \ldots, f_c$ form a basis for $(I/I^2)_a$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $(\Omega_{A/R})_a$. In this situation a^N for some large integer N satisfies the conditions of the lemma (with $U = V = \{1, \ldots, c\}$). $\square$

We will use the notion of a strictly standard element in A over R. Our notion is slightly weaker than the one in Swan's paper [Swa98]. We also define an elementary standard element to be one of the type we found in the lemma above. We compare the different types of elements in Lemma 16.3.7.

07C7 Definition 16.2.3. Let $R \to A$ be a ring map of finite presentation. We say an element $a \in A$ is elementary standard in A over R if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

07C8 (16.2.3.1)
$$a = a' \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$$

for some $a' \in A$ and

07C9 (16.2.3.2)
$$af_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2$$

for $j = 1, \ldots, m-c$. We say $a \in A$ is strictly standard in A over R if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

07ER (16.2.3.3)
$$a = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} a_I \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i \in I}$$

for some $a_I \in A$ and

07ES (16.2.3.4)
$$af_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2$$

for $j = 1, \ldots, m-c$.

The following lemma is useful to find implications of (16.2.3.3).

07ET Lemma 16.2.4. Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and write $I = (f_1, \ldots, f_m)$. Let $a \in A$. Then (16.2.3.3) implies there exists an A-linear map $\psi : \bigoplus_{i=1,\ldots,n} A\mathrm{d}x_i \to A^{\oplus c}$ such that the composition

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} I/I^2 \xrightarrow{f \mapsto \mathrm{d}f} \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i \xrightarrow{\psi} A^{\oplus c}$$

is multiplication by a. Conversely, if such a ψ exists, then a^c satisfies (16.2.3.3).

Proof. This is a special case of Algebra, Lemma 10.15.5. □

07CA Lemma 16.2.5 (Elkik). Let $R \to A$ be a ring map of finite presentation. The singular ideal $H_{A/R}$ is the radical of the ideal generated by strictly standard elements in A over R and also the radical of the ideal generated by elementary standard elements in A over R.

Proof. Assume a is strictly standard in A over R. We claim that A_a is smooth over R, which proves that $a \in H_{A/R}$. Namely, let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, c, and $a' \in A$ be as in Definition 16.2.3. Write $I = (f_1, \ldots, f_m)$ so that the naive cotangent complex of A over R is given by $I/I^2 \to \bigoplus A\mathrm{d}x_i$. Assumption (16.2.3.4) implies that $(I/I^2)_a$ is generated by the classes of $f_1, \ldots, f_c$. Assumption (16.2.3.3) implies that the differential $(I/I^2)_a \to \bigoplus A_a\mathrm{d}x_i$ has a left inverse, see Lemma 16.2.4. Hence $R \to A_a$ is smooth by definition and Algebra, Lemma 10.134.13.

Let $H_e, H_s \subset A$ be the radical of the ideal generated by elementary, resp. strictly standard elements of A over R. By definition and what we just proved we have $H_e \subset H_s \subset H_{A/R}$. The inclusion $H_{A/R} \subset H_e$ follows from Lemma 16.2.2. □

07CB Example 16.2.6. The set of points where a finitely presented ring map is smooth needn't be a quasi-compact open. For example, let $R = k[x, y_1, y_2, y_3, \ldots]/(xy_i)$ and $A = R/(x)$. Then the smooth locus of $R \to A$ is $\bigcup D(y_i)$ which is not quasi-compact.

07CC Lemma 16.2.7. Let $R \to A$ be a ring map of finite presentation. Let $R \to R'$ be a ring map. If $a \in A$ is elementary, resp. strictly standard in A over R, then $a \otimes 1$ is elementary, resp. strictly standard in $A \otimes_R R'$ over R'.

Proof. If $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ is a presentation of A over R, then $A \otimes_R R' = R'[x_1, \ldots, x_n]/(f_1', \ldots, f_m')$ is a presentation of $A \otimes_R R'$ over R'. Here f_j' is the image of f_j in $R'[x_1, \ldots, x_n]$. Hence the result follows from the definitions. $\square$

07EU Lemma 16.2.8. Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Assume that $H_{A/R}\Lambda = \Lambda$. Then there exists a factorization $A \to B \to \Lambda$ with B smooth over R.

Proof. Choose $f_1, \ldots, f_r \in H_{A/R}$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $\sum f_i \lambda_i = 1$ in Λ. Set $B = A[x_1, \ldots, x_r]/(f_1 x_1 + \ldots + f_r x_r - 1)$ and define $B \to \Lambda$ by mapping x_i to λ_i. To check that B is smooth over R use that A_{f_i} is smooth over R by definition of $H_{A/R}$ and that B_{f_i} is smooth over A_{f_i}. Details omitted. $\square$

16.3. Presentations of algebras

07CD Some of the results in this section are due to Elkik. Note that the algebra C in the following lemma is a symmetric algebra over A. Moreover, if R is Noetherian, then C is of finite presentation over R.

07CE Lemma 16.3.1. Let R be a ring and let A be a finitely presented R-algebra. There exists finite type R-algebra map $A \to C$ which has a retraction with the following two properties

(1) for each $a \in A$ such that $R \to A_a$ is a local complete intersection (More on Algebra, Definition 15.33.2) the ring C_a is smooth over A_a and has a presentation $C_a = R[y_1, \ldots, y_m]/J$ such that J/J^2 is free over C_a, and
(2) for each $a \in A$ such that A_a is smooth over R the module $\Omega_{C_a/R}$ is free over C_a.

Proof. Choose a presentation $A = R[x_1, \ldots, x_n]/I$ and write $I = (f_1, \ldots, f_m)$. Define the A-module K by the short exact sequence

$$0 \to K \to A^{\oplus m} \to I/I^2 \to 0$$

where the jth basis vector e_j in the middle is mapped to the class of f_j on the right. Set

$$C = \text{Sym}^*_A(I/I^2).$$

The retraction is just the projection onto the degree 0 part of C. We have a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to C$ which maps y_j to the class of f_j in I/I^2. The kernel J of this map is generated by the elements $f_1, \ldots, f_m$ and by elements $\sum h_j y_j$ with $h_j \in R[x_1, \ldots, x_n]$ such that $\sum h_j e_j$ defines an element of K. By Algebra, Lemma 10.134.4 applied to $R \to A \to C$ and the presentations above and More on Algebra, Lemma 15.9.12 there is a short exact sequence

07EW (16.3.1.1) $$I/I^2 \otimes_A C \to J/J^2 \to K \otimes_A C \to 0$$

of C-modules. Let $h \in R[x_1, \ldots, x_n]$ be an element with image $a \in A$. We will use as presentations for the localized rings

$$A_a = R[x_0, x_1, \ldots, x_n]/I' \quad \text{and} \quad C_a = R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]/J'$$

where $I' = (hx_0 - 1, I)$ and $J' = (hx_0 - 1, J)$. Hence $I'/(I')^2 = A_a \oplus (I/I^2)_a$ as A_a-modules and $J'/(J')^2 = C_a \oplus (J/J^2)_a$ as C_a-modules. Thus we obtain

07EX (16.3.1.2)
$$C_a \oplus I/I^2 \otimes_A C_a \to C_a \oplus (J/J^2)_a \to K \otimes_A C_a \to 0$$

as the sequence of Algebra, Lemma 10.134.4 corresponding to $R \to A_a \to C_a$ and the presentations above.

Next, assume that $a \in A$ is such that A_a is a local complete intersection over R. Then $(I/I^2)_a$ is finite projective over A_a, see More on Algebra, Lemma 15.32.3. Hence we see $K_a \oplus (I/I^2)_a \cong A_a^{\oplus m}$ is free. In particular K_a is finite projective too. By More on Algebra, Lemma 15.33.6 the sequence (16.3.1.2) is exact on the left. Hence

$$J'/(J')^2 \cong C_a \oplus I/I^2 \otimes_A C_a \oplus K \otimes_A C_a \cong C_a^{\oplus m+1}$$

This proves (1). Finally, suppose that in addition A_a is smooth over R. Then the same presentation shows that $\Omega_{C_a/R}$ is the cokernel of the map

$$J'/(J')^2 \longrightarrow \bigoplus\nolimits_i C_a \text{d}x_i \oplus \bigoplus\nolimits_j C_a \text{d}y_j$$

The summand C_a of $J'/(J')^2$ in the decomposition above corresponds to $hx_0 - 1$ and hence maps isomorphically to the summand $C_a \text{d}x_0$. The summand $I/I^2 \otimes_A C_a$ of $J'/(J')^2$ maps injectively to $\bigoplus_{i=1,\ldots,n} C_a \text{d}x_i$ with quotient $\Omega_{A_a/R} \otimes_{A_a} C_a$. The summand $K \otimes_A C_a$ maps injectively to $\bigoplus_{j \geq 1} C_a \text{d}y_j$ with quotient isomorphic to $I/I^2 \otimes_A C_a$. Thus the cokernel of the last displayed map is the module $I/I^2 \otimes_A C_a \oplus \Omega_{A_a/R} \otimes_{A_a} C_a$. Since $(I/I^2)_a \oplus \Omega_{A_a/R}$ is free (from the definition of smooth ring maps) we see that (2) holds. $\square$

The following proposition was proved for smooth ring maps over henselian pairs by Elkik in [Elk73]. For smooth ring maps it can be found in [Ara01], where it is also proven that ring maps between smooth algebras can be lifted.

07M8 **Proposition 16.3.2.** Let $R \to R_0$ be a surjective ring map with kernel I.

(1) If $R_0 \to A_0$ is a syntomic ring map, then there exists a syntomic ring map $R \to A$ such that $A/IA \cong A_0$.
(2) If $R_0 \to A_0$ is a smooth ring map, then there exists a smooth ring map $R \to A$ such that $A/IA \cong A_0$.

Proof. Assume $R_0 \to A_0$ syntomic, in particular a local complete intersection (More on Algebra, Lemma 15.33.5). Choose a presentation $A_0 = R_0[x_1, \ldots, x_n]/J_0$. Set $C_0 = \text{Sym}^*_{A_0}(J_0/J_0^2)$. Note that J_0/J_0^2 is a finite projective A_0-module (Algebra, Lemma 10.136.16). By Lemma 16.3.1 the ring map $A_0 \to C_0$ is smooth and we can find a presentation $C_0 = R_0[y_1, \ldots, y_m]/K_0$ with K_0/K_0^2 free over C_0. By Algebra, Lemma 10.136.6 we can assume $C_0 = R_0[y_1, \ldots, y_m]/(\overline{f}_1, \ldots, \overline{f}_c)$ where $\overline{f}_1, \ldots, \overline{f}_c$ maps to a basis of K_0/K_0^2 over C_0. Choose $f_1, \ldots, f_c \in R[y_1, \ldots, y_c]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$ and set

$$C = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$$

By construction $C_0 = C/IC$. By Algebra, Lemma 10.136.10 we can after replacing C by C_g assume that C is a relative global complete intersection over R. We conclude that there exists a finite projective A_0-module P_0 such that $C_0 = \text{Sym}^*_{A_0}(P_0)$ is isomorphic to C/IC for some syntomic R-algebra C.

Choose an integer n and a direct sum decomposition $A_0^{\oplus n} = P_0 \oplus Q_0$. By More on Algebra, Lemma 15.9.11 we can find an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$ and a finite projective C'-module Q such that Q/IQ is isomorphic to $Q_0 \otimes_{A_0} C/IC$. Then $D = \text{Sym}^*_{C'}(Q)$ is a smooth C'-algebra (see More on Algebra, Lemma 15.9.13). Picture

$$\begin{tikzcd} R \arrow[rr] \arrow[d] & & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & D \arrow[d] \\ R/I \arrow[r] & A_0 \arrow[r] & C/IC \arrow[r, "\cong"] & C'/IC' \arrow[r] & D/ID \end{tikzcd}$$

Observe that our choice of Q gives

$$\begin{aligned} D/ID &= \text{Sym}^*_{C/IC}(Q_0 \otimes_{A_0} C/IC) \\ &= \text{Sym}^*_{A_0}(Q_0) \otimes_{A_0} C/IC \\ &= \text{Sym}^*_{A_0}(Q_0) \otimes_{A_0} \text{Sym}^*_{A_0}(P_0) \\ &= \text{Sym}^*_{A_0}(Q_0 \oplus P_0) \\ &= \text{Sym}^*_{A_0}(A_0^{\oplus n}) \\ &= A_0[x_1, \ldots, x_n] \end{aligned}$$

Choose $f_1, \ldots, f_n \in D$ which map to $x_1, \ldots, x_n$ in $D/ID = A_0[x_1, \ldots, x_n]$. Set $A = D/(f_1, \ldots, f_n)$. Note that $A_0 = A/IA$. We claim that $R \to A$ is syntomic in a neighbourhood of $V(IA)$. If the claim is true, then we can find a $f \in A$ mapping to $1 \in A_0$ such that A_f is syntomic over R and the proof of (1) is finished.

Proof of the claim. Observe that $R \to D$ is syntomic as a composition of the syntomic ring map $R \to C$, the étale ring map $C \to C'$ and the smooth ring map $C' \to D$ (Algebra, Lemmas 10.136.17 and 10.137.10). The question is local on $\text{Spec}(D)$, hence we may assume that D is a relative global complete intersection (Algebra, Lemma 10.136.15). Say $D = R[y_1, \ldots, y_m]/(g_1, \ldots, g_s)$. Let $f'_1, \ldots, f'_n \in R[y_1, \ldots, y_m]$ be lifts of $f_1, \ldots, f_n$. Then we can apply Algebra, Lemma 10.136.10 to get the claim.

Proof of (2). Since a smooth ring map is syntomic, we can find a syntomic ring map $R \to A$ such that $A_0 = A/IA$. By assumption the fibres of $R \to A$ are smooth over primes in $V(I)$ hence $R \to A$ is smooth in an open neighbourhood of $V(IA)$ (Algebra, Lemma 10.137.17). Thus we can replace A by a localization to obtain the result we want. $\square$

We know that any syntomic ring map $R \to A$ is locally a relative global complete intersection, see Algebra, Lemma 10.136.15. The next lemma says that a vector bundle over $\text{Spec}(A)$ is a relative global complete intersection.

07CG Lemma 16.3.3. Let $R \to A$ be a syntomic ring map. Then there exists a smooth R-algebra map $A \to C$ with a retraction such that C is a global relative complete intersection over R, i.e.,

$$C \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

flat over R and all fibres of dimension $n - c$.

Proof. Apply Lemma 16.3.1 to get $A \to C$. By Algebra, Lemma 10.136.6 we can write $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with f_i mapping to a basis of J/J^2. The ring map $R \to C$ is syntomic (hence flat) as it is a composition of a syntomic and a smooth ring map. The dimension of the fibres is $n-c$ by Algebra, Lemma 10.135.4 (the fibres are local complete intersections, so the lemma applies). $\square$

07CH Lemma 16.3.4. Let $R \to A$ be a smooth ring map. Then there exists a smooth R-algebra map $A \to B$ with a retraction such that B is standard smooth over R, i.e.,

$$B \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

and $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B.

Proof. Apply Lemma 16.3.3 to get a smooth R-algebra map $A \to C$ with a retraction such that $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. As C is smooth over R we have a short exact sequence

$$0 \to \bigoplus\nolimits_{j=1,\ldots,c} Cf_j \to \bigoplus\nolimits_{i=1,\ldots,n} C\mathrm{d}x_i \to \Omega_{C/R} \to 0$$

Since $\Omega_{C/R}$ is a projective C-module this sequence is split. Choose a left inverse t to the first map. Say $t(\mathrm{d}x_i) = \sum c_{ij} f_j$ so that $\sum_i \frac{\partial f_j}{\partial x_i} c_{i\ell} = \delta_{j\ell}$ (Kronecker delta). Let

$$B' = C[y_1, \ldots, y_c] = R[x_1, \ldots, x_n, y_1, \ldots, y_c]/(f_1, \ldots, f_c)$$

The R-algebra map $C \to B'$ has a retraction given by mapping y_j to zero. We claim that the map

$$R[z_1, \ldots, z_n] \longrightarrow B', \quad z_i \longmapsto x_i - \sum\nolimits_j c_{ij} y_j$$

is étale at every point in the image of $\mathrm{Spec}(C) \to \mathrm{Spec}(B')$. In $\Omega_{B'/R[z_1,\ldots,z_n]}$ we have

$$0 = \mathrm{d}f_j - \sum\nolimits_i \frac{\partial f_j}{\partial x_i}\mathrm{d}z_i \equiv \sum\nolimits_{i,\ell} \frac{\partial f_j}{\partial x_i} c_{i\ell}\mathrm{d}y_\ell \equiv \mathrm{d}y_j \bmod (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$$

Since $0 = \mathrm{d}z_i = \mathrm{d}x_i$ modulo $\sum B'\mathrm{d}y_j + (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$ we conclude that

$$\Omega_{B'/R[z_1,\ldots,z_n]}/(y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]} = 0.$$

As $\Omega_{B'/R[z_1,\ldots,z_n]}$ is a finite B'-module by Nakayama's lemma there exists a $g \in 1 + (y_1, \ldots, y_c)$ that $(\Omega_{B'/R[z_1,\ldots,z_n]})_g = 0$. This proves that $R[z_1, \ldots, z_n] \to B'_g$ is unramified, see Algebra, Definition 10.151.1. For any ring map $R \to k$ where k is a field we obtain an unramified ring map $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ between smooth k-algebras of dimension n. It follows that $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ is flat by Algebra, Lemmas 10.128.1 and 10.140.2. By the critère de platitude par fibre (Algebra, Lemma 10.128.8) we conclude that $R[z_1, \ldots, z_n] \to B'_g$ is flat. Finally, Algebra, Lemma 10.143.7 implies that $R[z_1, \ldots, z_n] \to B'_g$ is étale. Set $B = B'_g$. Note that $C \to B$ is smooth and has a retraction, so also $A \to B$ is smooth and has a retraction. Moreover, $R[z_1, \ldots, z_n] \to B$ is étale. By Algebra, Lemma 10.143.2 we can write

$$B = R[z_1, \ldots, z_n, w_1, \ldots, w_c]/(g_1, \ldots, g_c)$$

with $\det(\partial g_j/\partial w_i)$ invertible in B. This proves the lemma. $\square$

07CI **Lemma 16.3.5.** Let $R \to \Lambda$ be a ring map. If Λ is a filtered colimit of smooth R-algebras, then Λ is a filtered colimit of standard smooth R-algebras.

Proof. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation over R. According to Algebra, Lemma 10.127.4 we have to factor this map through a standard smooth algebra, and we know we can factor it as $A \to B \to \Lambda$ with B smooth over R. Choose an R-algebra map $B \to C$ with a retraction $C \to B$ such that C is standard smooth over R, see Lemma 16.3.4. Then the desired factorization is $A \to B \to C \to B \to \Lambda$. □

07EY **Lemma 16.3.6.** Let $R \to A$ be a standard smooth ring map. Let $E \subset A$ be a finite subset of order $|E| = n$. Then there exists a presentation $A = R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ with $c \geq n$, with $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A, and such that E is the set of congruence classes of $x_1, \ldots, x_n$.

Proof. Choose a presentation $A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$ such that the image of $\det(\partial g_j/\partial y_i)_{i,j=1,\ldots,d}$ is invertible in A. Choose an enumerations $E = \{a_1, \ldots, a_n\}$ and choose $h_i \in R[y_1, \ldots, y_m]$ whose image in A is a_i. Consider the presentation

$$A = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(x_1 - h_1, \ldots, x_n - h_n, g_1, \ldots, g_d)$$

and set $c = n + d$. □

07EZ **Lemma 16.3.7.** Let $R \to A$ be a ring map of finite presentation. Let $a \in A$. Consider the following conditions on a:

(1) A_a is smooth over R,
(2) A_a is smooth over R and $\Omega_{A_a/R}$ is stably free,
(3) A_a is smooth over R and $\Omega_{A_a/R}$ is free,
(4) A_a is standard smooth over R,
(5) a is strictly standard in A over R,
(6) a is elementary standard in A over R.

Then we have

(a) (4) ⇒ (3) ⇒ (2) ⇒ (1),
(b) (6) ⇒ (5),
(c) (6) ⇒ (4),
(d) (5) ⇒ (2),
(e) (2) ⇒ the elements a^e, $e \geq e_0$ are strictly standard in A over R,
(f) (4) ⇒ the elements a^e, $e \geq e_0$ are elementary standard in A over R.

Proof. Part (a) is clear from the definitions and Algebra, Lemma 10.137.7. Part (b) is clear from Definition 16.2.3.

Proof of (c). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (16.2.3.1) and (16.2.3.2) hold. Choose $h \in R[x_1, \ldots, x_n]$ mapping to a. Then

$$A_a = R[x_0, x_1, \ldots, x_n]/(x_0 h - 1, f_1, \ldots, f_m).$$

Write $J = (x_0 h - 1, f_1, \ldots, f_m)$. By (16.2.3.2) we see that the A_a-module J/J^2 is generated by $x_0 h - 1, f_1, \ldots, f_c$ over A_a. Hence, as in the proof of Algebra, Lemma 10.136.6, we can choose a $g \in 1 + J$ such that

$$A_a = R[x_0, \ldots, x_n, x_{n+1}]/(x_0 h - 1, f_1, \ldots, f_m, g x_{n+1} - 1).$$

At this point (16.2.3.1) implies that $R \to A_a$ is standard smooth (use the coordinates $x_0, x_1, \ldots, x_c, x_{n+1}$ to take derivatives).

Proof of (d). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (16.2.3.3) and (16.2.3.4) hold. Write $I = (f_1, \ldots, f_m)$. We already know that A_a is smooth over R, see Lemma 16.2.5. By Lemma 16.2.4 we see that $(I/I^2)_a$ is free on $f_1, \ldots, f_c$ and maps isomorphically to a direct summand of $\bigoplus A_a \mathrm{d}x_i$. Since $\Omega_{A_a/R} = (\Omega_{A/R})_a$ is the cokernel of the map $(I/I^2)_a \to \bigoplus A_a \mathrm{d}x_i$ we conclude that it is stably free.

Proof of (e). Choose a presentation $A = R[x_1, \ldots, x_n]/I$ with I finitely generated. By assumption we have a short exact sequence

$$0 \to (I/I^2)_a \to \bigoplus\nolimits_{i=1,\ldots,n} A_a \mathrm{d}x_i \to \Omega_{A_a/R} \to 0$$

which is split exact. Hence we see that $(I/I^2)_a \oplus \Omega_{A_a/R}$ is a free A_a-module. Since $\Omega_{A_a/R}$ is stably free we see that $(I/I^2)_a$ is stably free as well. Thus replacing the presentation chosen above by $A = R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r}]/J$ with $J = (I, x_{n+1}, \ldots, x_{n+r})$ for some r we get that $(J/J^2)_a$ is (finite) free. Choose $f_1, \ldots, f_c \in J$ which map to a basis of $(J/J^2)_a$. Extend this to a list of generators $f_1, \ldots, f_m \in J$. Consider the presentation $A = R[x_1, \ldots, x_{n+r}]/(f_1, \ldots, f_m)$. Then (16.2.3.4) holds for a^e for all sufficiently large e by construction. Moreover, since $(J/J^2)_a \to \bigoplus_{i=1,\ldots,n+r} A_a \mathrm{d}x_i$ is a split injection we can find an A_a-linear left inverse. Writing this left inverse in terms of the basis $f_1, \ldots, f_c$ and clearing denominators we find a linear map $\psi_0 : A^{\oplus n+r} \to A^{\oplus c}$ such that

$$A^{\oplus c} \xrightarrow{(f_1, \ldots, f_c)} J/J^2 \xrightarrow{f \mapsto \mathrm{d}f} \bigoplus\nolimits_{i=1,\ldots,n+r} A \mathrm{d}x_i \xrightarrow{\psi_0} A^{\oplus c}$$

is multiplication by a^{e_0} for some $e_0 \geq 1$. By Lemma 16.2.4 we see (16.2.3.3) holds for all a^{ce_0} and hence for a^e for all e with $e \geq ce_0$.

Proof of (f). Choose a presentation $A_a = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in A_a. We may assume that for some $m < n$ the classes of the elements $x_1, \ldots, x_m$ correspond $a_i/1$ where $a_1, \ldots, a_m \in A$ are generators of A over R, see Lemma 16.3.6. After replacing x_i by $a^N x_i$ for $m < i \leq n$ we may assume the class of x_i is $a_i/1 \in A_a$ for some $a_i \in A$. Consider the ring map

$$\Psi : R[x_1, \ldots, x_n] \longrightarrow A, \quad x_i \longmapsto a_i.$$

This is a surjective ring map. By replacing f_j by $a^N f_j$ we may assume that $f_j \in R[x_1, \ldots, x_n]$ and that $\Psi(f_j) = 0$ (since after all $f_j(a_1/1, \ldots, a_n/1) = 0$ in A_a). Let $J = \mathrm{Ker}(\Psi)$. Then $A = R[x_1, \ldots, x_n]/J$ is a presentation and $f_1, \ldots, f_c \in J$ are elements such that $(J/J^2)_a$ is freely generated by $f_1, \ldots, f_c$ and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ maps to an invertible element of A_a. It follows that (16.2.3.1) and (16.2.3.2) hold for a^e and all large enough e as desired. □

16.4. Intermezzo: Néron desingularization

0BJ1 We interrupt the attack on the general case of Popescu's theorem to an easier but already very interesting case, namely, when $R \to \Lambda$ is a homomorphism of discrete valuation rings. This is discussed in [Art69a, Section 4].

0BJ2 Situation 16.4.1. Here $R \subset \Lambda$ is an extension of discrete valuation rings with ramification index 1 (More on Algebra, Definition 15.111.1). We assume given a factorization

$$R \to A \xrightarrow{\varphi} \Lambda$$

with $R \to A$ flat and of finite type. Let $\mathfrak{q} = \mathrm{Ker}(\varphi)$ and $\mathfrak{p} = \varphi^{-1}(\mathfrak{m}_\Lambda)$.

In Situation 16.4.1 let $\pi \in R$ be a uniformizer. Recall that flatness of A over R signifies that π is a nonzerodivisor on A (More on Algebra, Lemma 15.22.10). By our assumption on $R \subset \Lambda$ we see that π maps to a uniformizer of Λ. Since $\pi \in \mathfrak p$ we can consider Néron's affine blowup algebra (see Algebra, Section 10.70)

$$\varphi' : A' = A[\tfrac{\mathfrak p}{\pi}] \longrightarrow \Lambda$$

which comes endowed with an induced map to Λ sending a/π^n, $a \in \mathfrak p^n$ to $\pi^{-n}\varphi(a)$ in Λ. We will denote $\mathfrak q' \subset \mathfrak p' \subset A'$ the corresponding prime ideals of A'. Observe that the isomorphism class of A' does not depend on our choice of uniformizer. Repeating the construction we obtain a sequence

$$A \to A' \to A'' \to \ldots \to \Lambda$$

0BJ3 Lemma 16.4.2. In Situation 16.4.1 Néron's blowup is functorial in the following sense

(1) if $a \in A$, $a \not\in \mathfrak p$, then Néron's blowup of A_a is A'_a, and
(2) if $B \to A$ is a surjection of flat finite type R-algebras with kernel I, then A' is the quotient of B'/IB' by its π-power torsion.

Proof. Both (1) and (2) are special cases of Algebra, Lemma 10.70.3. In fact, whenever we have $A_1 \to A_2 \to \Lambda$ such that $\mathfrak p_1 A_2 = \mathfrak p_2$, we have that A'_2 is the quotient of $A'_1 \otimes_{A_1} A_2$ by its π-power torsion. $\square$

0BJ4 Lemma 16.4.3. In Situation 16.4.1 assume that $R \to A$ is smooth at $\mathfrak p$ and that $R/\pi R \subset \Lambda/\pi\Lambda$ is a separable field extension. Then $R \to A'$ is smooth at $\mathfrak p'$ and there is a short exact sequence

$$0 \to \Omega_{A/R} \otimes_A A'_{\mathfrak p'} \to \Omega_{A'/R, \mathfrak p'} \to (A'/\pi A')^{\oplus c}_{\mathfrak p'} \to 0$$

where $c = \dim((A/\pi A)_{\mathfrak p})$.

Proof. By Lemma 16.4.2 we may replace A by a localization at an element not in $\mathfrak p$; we will use this without further mention. Write $\kappa = R/\pi R$. Since smoothness is stable under base change (Algebra, Lemma 10.137.4) we see that $A/\pi A$ is smooth over κ at $\mathfrak p$. Hence $(A/\pi A)_{\mathfrak p}$ is a regular local ring (Algebra, Lemma 10.140.3). Choose $g_1, \ldots, g_c \in \mathfrak p$ which map to a regular system of parameters in $(A/\pi A)_{\mathfrak p}$. Then we see that $\mathfrak p = (\pi, g_1, \ldots, g_c)$ after possibly replacing A by a localization. Note that $\pi, g_1, \ldots, g_c$ is a regular sequence in $A_{\mathfrak p}$ (first π is a nonzerodivisor and then Algebra, Lemma 10.106.3 for the rest of the sequence). After replacing A by a localization we may assume that $\pi, g_1, \ldots, g_c$ is a regular sequence in A (Algebra, Lemma 10.68.6). It follows that

$$A' = A[y_1, \ldots, y_c]/(\pi y_1 - g_1, \ldots, \pi y_c - g_c) = A[y_1, \ldots, y_c]/I$$

by More on Algebra, Lemma 15.31.2. In the following we will use the definition of smoothness using the naive cotangent complex (Algebra, Definition 10.137.1) and the criterion of Algebra, Lemma 10.137.12 without further mention. The exact sequence of Algebra, Lemma 10.134.4 for $R \to A[y_1, \ldots, y_c] \to A'$ looks like this

$$0 \to H_1(NL_{A'/R}) \to I/I^2 \to \Omega_{A/R} \otimes_A A' \oplus \bigoplus\nolimits_{i=1,\ldots,c} A'\text{d}y_i \to \Omega_{A'/R} \to 0$$

where the class of $\pi y_i - g_i$ in I/I^2 is mapped to $-\text{d}g_i + \pi \text{d}y_i$ in the next term. Here we have used Algebra, Lemma 10.134.6 to compute $NL_{A'/A[y_1,\ldots,y_c]}$ and we have used that $R \to A[y_1, \ldots, y_c]$ is smooth, so $H_1(NL_{A[y_1,\ldots,y_c]/R}) = 0$ and $\Omega_{A[y_1,\ldots,y_c]/R}$

is a finite projective (a fortiori flat) $A[y_1, \ldots, y_c]$-module which is in fact the direct sum of $\Omega_{A/R} \otimes_A A[y_1, \ldots, y_c]$ and a free module with basis $\text{d}y_i$. To finish the proof it suffices to show that $\text{d}g_1, \ldots, \text{d}g_c$ forms part of a basis for the finite free module $\Omega_{A/R,\mathfrak p}$. Namely, this will show $(I/I^2)_\mathfrak p$ is free on $\pi y_i - g_i$, the localization at $\mathfrak p$ of the middle map in the sequence is injective, so $H_1(NL_{A'/R})_\mathfrak p = 0$, and that the cokernel $\Omega_{A'/R,\mathfrak p}$ is finite free. To do this it suffices to show that the images of $\text{d}g_i$ are $\kappa(\mathfrak p)$-linearly independent in $\Omega_{A/R,\mathfrak p}/\pi = \Omega_{(A/\pi A)/\kappa,\mathfrak p}$ (equality by Algebra, Lemma 10.131.12). Since $\kappa \subset \kappa(\mathfrak p) \subset \Lambda/\pi\Lambda$ we see that $\kappa(\mathfrak p)$ is separable over κ (Algebra, Definition 10.42.1). The desired linear independence now follows from Algebra, Lemma 10.140.4. □

0BJ5 Lemma 16.4.4. In Situation 16.4.1 assume that $R \to A$ is smooth at $\mathfrak q$ and that we have a surjection of R-algebras $B \to A$ with kernel I. Assume $R \to B$ smooth at $\mathfrak p_B = (B \to A)^{-1}\mathfrak p$. If the cokernel of

$$I/I^2 \otimes_A \Lambda \to \Omega_{B/R} \otimes_B \Lambda$$

is a free Λ-module, then $R \to A$ is smooth at $\mathfrak p$.

Proof. The cokernel of the map $I/I^2 \to \Omega_{B/R} \otimes_B A$ is $\Omega_{A/R}$, see Algebra, Lemma 10.131.9. Let $d = \dim_\mathfrak q(A/R)$ be the relative dimension of $R \to A$ at $\mathfrak q$, i.e., the dimension of $\text{Spec}(A[1/\pi])$ at $\mathfrak q$. See Algebra, Definition 10.125.1. Then $\Omega_{A/R,\mathfrak q}$ is free over $A_\mathfrak q$ of rank d (Algebra, Lemma 10.140.3). Thus if the hypothesis of the lemma holds, then $\Omega_{A/R} \otimes_A \Lambda$ is free of rank d. It follows that $\Omega_{A/R} \otimes_A \kappa(\mathfrak p)$ has dimension d (as it is true upon tensoring with $\Lambda/\pi\Lambda$). Since $R \to A$ is flat and since $\mathfrak p$ is a specialization of $\mathfrak q$, we see that $\dim_\mathfrak p(A/R) \geq d$ by Algebra, Lemma 10.125.6. Then it follows that $R \to A$ is smooth at $\mathfrak p$ by Algebra, Lemmas 10.137.17 and 10.140.3. □

0BJ6 Lemma 16.4.5. In Situation 16.4.1 assume that $R \to A$ is smooth at $\mathfrak q$ and that $R/\pi R \subset \Lambda/\pi\Lambda$ is a separable extension of fields. Then after a finite number of affine Néron blowups the algebra A becomes smooth over R at $\mathfrak p$.

Proof. We choose an R-algebra B and a surjection $B \to A$. Set $\mathfrak p_B = (B \to A)^{-1}(\mathfrak p)$ and denote r the relative dimension of $R \to B$ at $\mathfrak p_B$. We choose B such that $R \to B$ is smooth at $\mathfrak p_B$. For example we can take B to be a polynomial algebra in r variables over R. Consider the complex

$$I/I^2 \otimes_A \Lambda \longrightarrow \Omega_{B/R} \otimes_B \Lambda$$

of Lemma 16.4.4. By the structure of finite modules over Λ (More on Algebra, Lemma 15.124.9) we see that the cokernel looks like

$$\Lambda^{\oplus d} \oplus \bigoplus\nolimits_{i=1,\ldots,n} \Lambda/\pi^{e_i}\Lambda$$

for some $d \geq 0$, $n \geq 0$, and $e_i \geq 1$. Observe that d is the relative dimension of A/R at $\mathfrak q$ (Algebra, Lemma 10.140.3). If the defect $e = \sum_{i=1,\ldots,n} e_i$ is zero, then we are done by Lemma 16.4.4.

Next, we consider what happens when we perform the Néron blowup. Recall that A' is the quotient of B'/IB' by its π-power torsion (Lemma 16.4.2) and that $R \to B'$ is smooth at $\mathfrak p_{B'}$ (Lemma 16.4.3). Thus after blowup we have exactly the same setup. Picture

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Since $I \subset \mathfrak{p}_B$, we see that $I \to I'$ factors through $\pi I'$. Looking at the induced map of complexes we get

$$\begin{array}{ccccc} I'/(I')^2 \otimes_{A'} \Lambda & \longrightarrow & \Omega_{B'/R} \otimes_{B'} \Lambda & = & M' \\ \uparrow & & \uparrow & & \\ I/I^2 \otimes_A \Lambda & \longrightarrow & \Omega_{B/R} \otimes_B \Lambda & = & M \end{array}$$

Then $M \subset M'$ are finite free Λ-modules with quotient M'/M annihilated by π, see Lemma 16.4.3. Let $N \subset M$ and $N' \subset M'$ be the images of the horizontal maps and denote $Q = M/N$ and $Q' = M'/N'$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & Q' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Then $N \subset N'$ are free Λ-modules of rank $r - d$. Since I maps into $\pi I'$ we see that $N \subset \pi N'$.

Let $K = \Lambda_\pi$ be the fraction field of Λ. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N'_K \cap M' & \longrightarrow & Q'_{tor} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N & \longrightarrow & N_K \cap M & \longrightarrow & Q_{tor} & \longrightarrow & 0 \end{array}$$

whose rows are short exact sequences. This shows that the change in defect is given by

$$e - e' = \text{length}(Q_{tor}) - \text{length}(Q'_{tor}) = \text{length}(N'/N) - \text{length}(N'_K \cap M'/N_K \cap M)$$

Since M'/M is annihilated by π, so is $N'_K \cap M'/N_K \cap M$, and its length is at most $\dim_K(N_K)$. Since $N \subset \pi N'$ we get $\text{length}(N'/N) \geq \dim_K(N_K)$, with equality if and only if $N = \pi N'$.

To finish the proof we have to show that N is strictly smaller than $\pi N'$ when A is not smooth at $\mathfrak{p}$; this is the key computation one has to do in Néron's argument. To do this, we consider the exact sequence

$$I/I^2 \otimes_B \kappa(\mathfrak{p}_B) \to \Omega_{B/R} \otimes_B \kappa(\mathfrak{p}_B) \to \Omega_{A/R} \otimes_A \kappa(\mathfrak{p}) \to 0$$

(follows from Algebra, Lemma 10.131.9). Since $R \to A$ is not smooth at $\mathfrak{p}$ we see that the dimension s of $\Omega_{A/R} \otimes_A \kappa(\mathfrak{p})$ is bigger than d. On the other hand the first arrow factors through the injective map

$$\mathfrak{p}B_\mathfrak{p}/\mathfrak{p}^2 B_\mathfrak{p} \to \Omega_{B/R} \otimes_B \kappa(\mathfrak{p}_B)$$

of Algebra, Lemma 10.140.4; note that $\kappa(\mathfrak{p})$ is separable over k by our assumption on $R/\pi R \subset \Lambda/\pi\Lambda$. Hence we conclude that we can find generators $g_1, \ldots, g_t \in I$

such that $g_j \in \mathfrak{p}^2$ for $j > r-s$. Then the images of g_j in A' are in $\pi^2 I'$ for $j > r-s$. Since $r-s < r-d$ we find that at least one of the minimal generators of N becomes divisible by π^2 in N'. Thus we see that e decreases by at least 1 and we win. $\square$

If $R \to \Lambda$ is an extension of discrete valuation rings, then $R \to \Lambda$ is regular if and only if (a) the ramification index is 1, (b) the extension of fraction fields is separable, and (c) $R/\mathfrak{m}_R \subset \Lambda/\mathfrak{m}_\Lambda$ is separable. Thus the following result is a special case of general Néron desingularization in Theorem 16.12.1.

0BJ7 Lemma 16.4.6. Let $R \subset \Lambda$ be an extension of discrete valuation rings which has ramification index 1 and induces a separable extension of residue fields and of fraction fields. Then Λ is a filtered colimit of smooth R-algebras.

Proof. By Algebra, Lemma 10.127.4 it suffices to show that any $R \to A \to \Lambda$ as in Situation 16.4.1 can be factored as $A \to B \to \Lambda$ with B a smooth R-algebra. After replacing A by its image in Λ we may assume that A is a domain whose fraction field K is a subfield of the fraction field of Λ. In particular, A is separable over the fraction field of R by our assumptions. Then $R \to A$ is smooth at $\mathfrak{q} = (0)$ by Algebra, Lemma 10.140.9. After a finite number of Néron blowups, we may assume $R \to A$ is smooth at $\mathfrak{p}$, see Lemma 16.4.5. Then, after replacing A by a localization at an element $a \in A$, $a \notin \mathfrak{p}$ it becomes smooth over R and the lemma is proved. $\square$

16.5. The lifting problem

07CJ The goal in this section is to prove (Proposition 16.5.3) that the collection of algebras which are filtered colimits of smooth algebras is closed under infinitesimal flat deformations. The proof is elementary and only uses the results on presentations of smooth algebras from Section 16.3.

07CK Lemma 16.5.1. Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that

(1) $I^2 = 0$, and
(2) $\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras.

Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. Then there exists a factorization
$$A \to B/J \to \Lambda$$
where B is a smooth R-algebra and $J \subset IB$ is a finitely generated ideal.

Proof. Choose a factorization
$$A/IA \to \bar{B} \to \Lambda/I\Lambda$$
with $\bar{B}$ standard smooth over R/I; this is possible by assumption and Lemma 16.3.5. Write
$$\bar{B} = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)$$
and say $\bar{B} \to \Lambda/I\Lambda$ maps t_i to the class of λ_i modulo $I\Lambda$. Choose $g_1, \ldots, g_s \in A[t_1, \ldots, t_r]$ lifting $\bar{g}_1, \ldots, \bar{g}_s$. Write $\varphi(g_i)(\lambda_1, \ldots, \lambda_r) = \sum \epsilon_{ij}\mu_{ij}$ for some $\epsilon_{ij} \in I$ and $\mu_{ij} \in \Lambda$. Define
$$A' = A[t_1, \ldots, t_r, \delta_{i,j}]/(g_i - \sum \epsilon_{ij}\delta_{ij})$$
and consider the map
$$A' \longrightarrow \Lambda, \quad a \longmapsto \varphi(a), \quad t_i \longmapsto \lambda_i, \quad \delta_{ij} \longmapsto \mu_{ij}$$

We have

$$A'/IA' = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)[\delta_{ij}] \cong \bar{B}[\delta_{ij}]$$

This is a standard smooth algebra over R/I as $\bar{B}$ is standard smooth. Choose a presentation $A'/IA' = R/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ with $\det(\partial \bar{f}_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A'/IA'. Choose lifts $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ of $\bar{f}_1, \ldots, \bar{f}_c$. Then

$$B = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1} \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} - 1)$$

is smooth over R. Since smooth ring maps are formally smooth (Algebra, Proposition 10.138.13) there exists an R-algebra map $B \to A'$ which is an isomorphism modulo I. Then $B \to A'$ is surjective by Nakayama's lemma (Algebra, Lemma 10.20.1). Thus $A' = B/J$ with $J \subset IB$ finitely generated (see Algebra, Lemma 10.6.3). □

07CL Lemma 16.5.2. Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that

(1) $I^2 = 0$,
(2) $\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras, and
(3) $R \to \Lambda$ is flat.

Let $\varphi : B \to \Lambda$ be an R-algebra map with B smooth over R. Let $J \subset IB$ be a finitely generated ideal such that $\varphi(J) = 0$. Then there exists R-algebra maps

$$B \xrightarrow{\alpha} B' \xrightarrow{\beta} \Lambda$$

such that B' is smooth over R, such that $\alpha(J) = 0$ and such that $\beta \circ \alpha = \varphi \bmod I\Lambda$.

Proof. If we can prove the lemma in case $J = (h)$, then we can prove the lemma by induction on the number of generators of J. Namely, suppose that J can be generated by n elements $h_1, \ldots, h_n$ and the lemma holds for all cases where J is generated by $n-1$ elements. Then we apply the case $n = 1$ to produce $B \to B' \to \Lambda$ where the first map kills of h_n. Then we let J' be the ideal of B' generated by the images of $h_1, \ldots, h_{n-1}$ and we apply the case for $n-1$ to produce $B' \to B'' \to \Lambda$. It is easy to verify that $B \to B'' \to \Lambda$ does the job.

Assume $J = (h)$ and write $h = \sum \epsilon_i b_i$ for some $\epsilon_i \in I$ and $b_i \in B$. Note that $0 = \varphi(h) = \sum \epsilon_i \varphi(b_i)$. As Λ is flat over R, the equational criterion for flatness (Algebra, Lemma 10.39.11) implies that we can find $\lambda_j \in \Lambda$, $j = 1, \ldots, m$ and $a_{ij} \in R$ such that $\varphi(b_i) = \sum_j a_{ij}\lambda_j$ and $\sum_i \epsilon_i a_{ij} = 0$. Set

$$C = B[x_1, \ldots, x_m]/(b_i - \sum a_{ij}x_j)$$

with $C \to \Lambda$ given by φ and $x_j \mapsto \lambda_j$. Choose a factorization

$$C \to B'/J' \to \Lambda$$

as in Lemma 16.5.1. Since B is smooth over R we can lift the map $B \to C \to B'/J'$ to a map $\psi : B \to B'$. We claim that $\psi(h) = 0$. Namely, the fact that ψ agrees with $B \to C \to B'/J'$ mod I implies that

$$\psi(b_i) = \sum a_{ij}\xi_j + \theta_i$$

for some $\xi_i \in B'$ and $\theta_i \in IB'$. Hence we see that

$$\psi(h) = \psi(\sum \epsilon_i b_i) = \sum \epsilon_i a_{ij}\xi_j + \sum \epsilon_i \theta_i = 0$$

because of the relations above and the fact that $I^2 = 0$. □

07CM **Proposition 16.5.3.** Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that

(1) I is nilpotent,
(2) $\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras, and
(3) $R \to \Lambda$ is flat.

Then Λ is a filtered colimit of smooth R-algebras.

Proof. Since $I^n = 0$ for some n, it follows by induction on n that it suffices to consider the case where $I^2 = 0$. Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Algebra, Lemma 10.127.4. By Lemma 16.5.1 we may assume that $A = B/J$ with B smooth over R and $J \subset IB$ a finitely generated ideal. By Lemma 16.5.2 we can find a (possibly noncommutative) diagram

$$\begin{tikzcd} B \arrow[rr, "\alpha"] \arrow[dr, "\varphi"'] & & B' \arrow[dl, "\beta"] \\ & \Lambda & \end{tikzcd}$$

of R-algebras which commutes modulo I and such that $\alpha(J) = 0$. The map

$$D : B \longrightarrow I\Lambda, \quad b \longmapsto \varphi(b) - \beta(\alpha(b))$$

is a derivation over R hence we can write it as $D = \xi \circ \mathrm{d}_{B/R}$ for some B-linear map $\xi : \Omega_{B/R} \to I\Lambda$. Since $\Omega_{B/R}$ is a finite projective B-module we can write $\xi = \sum_{i=1,\ldots,n} \epsilon_i \Xi_i$ for some $\epsilon_i \in I$ and B-linear maps $\Xi_i : \Omega_{B/R} \to \Lambda$. (Details omitted. Hint: write $\Omega_{B/R}$ as a direct sum of a finite free module to reduce to the finite free case.) We define

$$B'' = \mathrm{Sym}^*_{B'}\left(\bigoplus\nolimits_{i=1,\ldots,n} \Omega_{B/R} \otimes_{B,\alpha} B'\right)$$

and we define $\beta' : B'' \to \Lambda$ by β on B' and by

$$\beta'|_{i\text{th summand } \Omega_{B/R} \otimes_{B,\alpha} B'} = \Xi_i \otimes \beta$$

and $\alpha' : B \to B''$ by

$$\alpha'(b) = \alpha(b) \oplus \sum \epsilon_i \mathrm{d}_{B/R}(b) \otimes 1 \oplus 0 \oplus \ldots$$

At this point the diagram

$$\begin{tikzcd} B \arrow[rr, "\alpha'"] \arrow[dr, "\varphi"'] & & B'' \arrow[dl, "\beta'"] \\ & \Lambda & \end{tikzcd}$$

does commute. Moreover, it is direct from the definitions that $\alpha'(J) = 0$ as $I^2 = 0$. Hence the desired factorization. $\square$

16.6. The lifting lemma

07CN Here is a fiendishly clever lemma.

07CP **Lemma 16.6.1.** Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $\text{Ann}_R(\pi) = \text{Ann}_R(\pi^2)$ and $\text{Ann}_\Lambda(\pi) = \text{Ann}_\Lambda(\pi^2)$. Suppose we have R-algebra maps $R/\pi^2 R \to \bar{C} \to \Lambda/\pi^2\Lambda$ with $\bar{C}$ of finite presentation. Then there exists an R-algebra homomorphism $D \to \Lambda$ and a commutative diagram

$$\begin{tikzcd} R/\pi^2 R \arrow[r] \arrow[d] & \bar{C} \arrow[r] \arrow[d] & \Lambda/\pi^2\Lambda \arrow[d] \\ R/\pi R \arrow[r] & D/\pi D \arrow[r] & \Lambda/\pi\Lambda \end{tikzcd}$$

with the following properties

(a) D is of finite presentation,
(b) $R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \not\in \mathfrak{q}$,
(c) $R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \in \mathfrak{q}$ lying over a prime of $\bar{C}$ where $R/\pi^2 R \to \bar{C}$ is smooth, and
(d) $\bar{C}/\pi\bar{C} \to D/\pi D$ is smooth at any prime lying over a prime of $\bar{C}$ where $R/\pi^2 R \to \bar{C}$ is smooth.

Proof. We choose a presentation

$$\bar{C} = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

We also denote $I = (f_1, \ldots, f_m)$ and $\bar{I}$ the image of I in $R/\pi^2 R[x_1, \ldots, x_n]$. Since R is Noetherian, so is $\bar{C}$. Hence the smooth locus of $R/\pi^2 R \to \bar{C}$ is quasi-compact, see Topology, Lemma 5.9.2. Applying Lemma 16.2.2 we may choose a finite list of elements $a_1, \ldots, a_r \in R[x_1, \ldots, x_n]$ such that

(1) the union of the open subspaces $\text{Spec}(\bar{C}_{a_k}) \subset \text{Spec}(\bar{C})$ cover the smooth locus of $R/\pi^2 R \to \bar{C}$, and
(2) for each $k = 1, \ldots, r$ there exists a finite subset $E_k \subset \{1, \ldots, m\}$ such that $(\bar{I}/\bar{I}^2)_{a_k}$ is freely generated by the classes of f_j, $j \in E_k$.

Set $I_k = (f_j, j \in E_k) \subset I$ and denote $\bar{I}_k$ the image of I_k in $R/\pi^2 R[x_1, \ldots, x_n]$. By (2) and Nakayama's lemma we see that $(\bar{I}/\bar{I}_k)_{a_k}$ is annihilated by $1 + b'_k$ for some $b'_k \in \bar{I}_{a_k}$. Suppose b'_k is the image of $b_k/(a_k)^N$ for some $b_k \in I$ and some integer N. After replacing a_k by $a_k b_k$ we get

(3) $(\bar{I}_k)_{a_k} = (\bar{I})_{a_k}$.

Thus, after possibly replacing a_k by a high power, we may write

(4) $a_k f_\ell = \sum_{j \in E_k} h^j_{k,\ell} f_j + \pi^2 g_{k,\ell}$

for any $\ell \in \{1, \ldots, m\}$ and some $h^j_{i,\ell}, g_{i,\ell} \in R[x_1, \ldots, x_n]$. If $\ell \in E_k$ we choose $h^j_{k,\ell} = a_k \delta_{\ell,j}$ (Kronecker delta) and $g_{k,\ell} = 0$. Set

$$D = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, p_{k,\ell}).$$

Here $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, r\}$, $\ell \in \{1, \ldots, m\}$, and

$$p_{k,\ell} = a_k z_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j - \pi g_{k,\ell}.$$

Note that for $\ell \in E_k$ we have $p_{k,\ell} = 0$ by our choices above.

The map $R \to D$ is the given one. Say $\bar{C} \to \Lambda/\pi^2\Lambda$ maps x_i to the class of λ_i modulo π^2. For an element $f \in R[x_1, \ldots, x_n]$ we denote $f(\lambda) \in \Lambda$ the result of substituting λ_i for x_i. Then we know that $f_j(\lambda) = \pi^2\mu_j$ for some $\mu_j \in \Lambda$. Define $D \to \Lambda$ by the rules $x_i \mapsto \lambda_i$ and $z_j \mapsto \pi\mu_j$. This is well defined because

$$\begin{aligned} p_{k,\ell} &\mapsto a_k(\lambda)\pi\mu_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell}(\lambda)\pi\mu_j - \pi g_{k,\ell}(\lambda) \\ &= \pi\left(a_k(\lambda)\mu_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell}(\lambda)\mu_j - g_{k,\ell}(\lambda)\right) \end{aligned}$$

Substituting $x_i = \lambda_i$ in (4) above we see that the expression inside the brackets is annihilated by π^2, hence it is annihilated by π as we have assumed $\text{Ann}_\Lambda(\pi) = \text{Ann}_\Lambda(\pi^2)$. The map $\bar{C} \to D/\pi D$ is determined by $x_i \mapsto x_i$ (clearly well defined). Thus we are done if we can prove (b), (c), and (d).

Using (4) we obtain the following key equality

$$\begin{aligned} \pi p_{k,\ell} &= \pi a_k z_\ell - \sum\nolimits_{j \in E_k} \pi h^j_{k,\ell} z_j - \pi^2 g_{k,\ell} \\ &= -a_k(f_\ell - \pi z_\ell) + a_k f_\ell + \sum\nolimits_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j) - \sum\nolimits_{j \in E_k} h^j_{k,\ell} f_j - \pi^2 g_{k,\ell} \\ &= -a_k(f_\ell - \pi z_\ell) + \sum\nolimits_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j) \end{aligned}$$

The end result is an element of the ideal generated by $f_j - \pi z_j$. In particular, we see that $D[1/\pi]$ is isomorphic to $R[1/\pi][x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j)$ which is isomorphic to $R[1/\pi][x_1, \ldots, x_n]$ hence smooth over R. This proves (b).

For fixed $k \in \{1, \ldots, r\}$ consider the ring

$$D_k = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, j \in E_k, p_{k,\ell})$$

The number of equations is $m = |E_k| + (m - |E_k|)$ as $p_{k,\ell}$ is zero if $\ell \in E_k$. Also, note that

$$\begin{aligned} (D_k/\pi D_k)_{a_k} &= R/\pi R[x_1, \ldots, x_n, 1/a_k, z_1, \ldots, z_m]/(f_j, j \in E_k, p_{k,\ell}) \\ &= (\bar{C}/\pi\bar{C})_{a_k}[z_1, \ldots, z_m]/(a_k z_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j) \\ &\cong (\bar{C}/\pi\bar{C})_{a_k}[z_j, j \in E_k] \end{aligned}$$

In particular $(D_k/\pi D_k)_{a_k}$ is smooth over $(\bar{C}/\pi\bar{C})_{a_k}$. By our choice of a_k we have that $(\bar{C}/\pi\bar{C})_{a_k}$ is smooth over $R/\pi R$ of relative dimension $n - |E_k|$, see (2). Hence for a prime $\mathfrak{q}_k \subset D_k$ containing π and lying over $\text{Spec}(\bar{C}_{a_k})$ the fibre ring of $R \to D_k$ is smooth at $\mathfrak{q}_k$ of dimension n. Thus $R \to D_k$ is syntomic at $\mathfrak{q}_k$ by our count of the number of equations above, see Algebra, Lemma 10.136.10. Hence $R \to D_k$ is smooth at $\mathfrak{q}_k$, see Algebra, Lemma 10.137.17.

To finish the proof, let $\mathfrak{q} \subset D$ be a prime containing π lying over a prime where $R/\pi^2 R \to \bar{C}$ is smooth. Then $a_k \notin \mathfrak{q}$ for some k by (1). We will show that the surjection $D_k \to D$ induces an isomorphism on local rings at $\mathfrak{q}$. Since we know that the ring maps $\bar{C}/\pi\bar{C} \to D_k/\pi D_k$ and $R \to D_k$ are smooth at the corresponding prime $\mathfrak{q}_k$ by the preceding paragraph this will prove (c) and (d) and thus finish the proof.

First, note that for any ℓ the equation $\pi p_{k,\ell} = -a_k(f_\ell - \pi z_\ell) + \sum_{j \in E_k} h^j_{k,\ell}(f_j - \pi z_j)$ proved above shows that $f_\ell - \pi z_\ell$ maps to zero in $(D_k)_{a_k}$ and in particular in $(D_k)_{\mathfrak{q}_k}$.

The relations (4) imply that $a_k f_\ell = \sum_{j \in E_k} h^j_{k,\ell} f_j$ in I/I^2. Since $(\bar{I}_k/\bar{I}_k^2)_{a_k}$ is free on f_j, $j \in E_k$ we see that

$$a_{k'} h^j_{k,\ell} - \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell} h^j_{k,j'}$$

is zero in $\bar{C}_{a_k}$ for every k, k', ℓ and $j \in E_k$. Hence we can find a large integer N such that

$$a_k^N \left(a_{k'} h^j_{k,\ell} - \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell} h^j_{k,j'} \right)$$

is in $I_k + \pi^2 R[x_1, \ldots, x_n]$. Computing modulo π we have

$$\begin{aligned} & a_k p_{k',\ell} - a_{k'} p_{k,\ell} + \sum h^{j'}_{k',\ell} p_{k,j'} \\ & = -a_k \sum h^{j'}_{k',\ell} z_{j'} + a_{k'} \sum h^j_{k,\ell} z_j + \sum h^{j'}_{k',\ell} a_k z_{j'} - \sum \sum h^{j'}_{k',\ell} h^j_{k,j'} z_j \\ & = \sum \left(a_{k'} h^j_{k,\ell} - \sum h^{j'}_{k',\ell} h^j_{k,j'} \right) z_j \end{aligned}$$

with Einstein summation convention. Combining with the above we see $a_k^{N+1} p_{k',\ell}$ is contained in the ideal generated by I_k and π in $R[x_1, \ldots, x_n, z_1, \ldots, z_m]$. Thus $p_{k',\ell}$ maps into $\pi(D_k)_{a_k}$. On the other hand, the equation

$$\pi p_{k',\ell} = -a_{k'}(f_\ell - \pi z_\ell) + \sum\nolimits_{j' \in E_{k'}} h^{j'}_{k',\ell}(f_{j'} - \pi z_{j'})$$

shows that $\pi p_{k',\ell}$ is zero in $(D_k)_{a_k}$. Since we have assumed that $\text{Ann}_R(\pi) = \text{Ann}_R(\pi^2)$ and since $(D_k)_{\mathfrak{q}_k}$ is smooth hence flat over R we see that $\text{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi) = \text{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi^2)$. We conclude that $p_{k',\ell}$ maps to zero as well, hence $D_\mathfrak{q} = (D_k)_{\mathfrak{q}_k}$ and we win. $\square$

16.7. The desingularization lemma

07CQ Here is another fiendishly clever lemma.

07CR **Lemma 16.7.1.** Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $\text{Ann}_\Lambda(\pi) = \text{Ann}_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation. Assume

(1) the image of π is strictly standard in A over R, and
(2) there exists a section $\rho : A/\pi^4 A \to R/\pi^4 R$ which is compatible with the map to $\Lambda/\pi^4\Lambda$.

Then we can find R-algebra maps $A \to B \to \Lambda$ with B of finite presentation such that $\mathfrak{a}B \subset H_{B/R}$ where $\mathfrak{a} = \text{Ann}_R(\text{Ann}_R(\pi^2)/\text{Ann}_R(\pi))$.

Proof. Choose a presentation

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

and $0 \leq c \leq \min(n, m)$ such that (16.2.3.3) holds for π and such that

07CS (16.7.1.1) $$\pi f_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2$$

for $j = 1, \ldots, m-c$. Say ρ maps x_i to the class of $r_i \in R$. Then we can replace x_i by $x_i - r_i$. Hence we may assume $\rho(x_i) = 0$ in $R/\pi^4 R$. This implies that $f_j(0) \in \pi^4 R$ and that $A \to \Lambda$ maps x_i to $\pi^4 \lambda_i$ for some $\lambda_i \in \Lambda$. Write

$$f_j = f_j(0) + \sum\nolimits_{i=1,\ldots,n} r_{ji} x_i + \text{h.o.t.}$$

This implies that the constant term of $\partial f_j/\partial x_i$ is r_{ji}. Apply ρ to (16.2.3.3) for π and we see that

$$\pi = \sum\nolimits_{I\subset\{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i\in I} \bmod \pi^4 R$$

for some $r_I \in R$. Thus we have

$$u\pi = \sum\nolimits_{I\subset\{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i\in I}$$

for some $u \in 1 + \pi^3 R$. By Algebra, Lemma 10.15.5 this implies there exists a $n\times c$ matrix (s_{ik}) such that

$$u\pi\delta_{jk} = \sum\nolimits_{i=1,\ldots,n} r_{ji}s_{ik} \quad \text{for all } j,k = 1,\ldots,c$$

(Kronecker delta). We introduce auxiliary variables $v_1,\ldots,v_c,w_1,\ldots,w_n$ and we set

$$h_i = x_i - \pi^2\sum\nolimits_{j=1,\ldots c} s_{ij}v_j - \pi^3 w_i$$

In the following we will use that

$$R[x_1,\ldots,x_n,v_1,\ldots,v_c,w_1,\ldots,w_n]/(h_1,\ldots,h_n) = R[v_1,\ldots,v_c,w_1,\ldots,w_n]$$

without further mention. In $R[x_1,\ldots,x_n,v_1,\ldots,v_c,w_1,\ldots,w_n]/(h_1,\ldots,h_n)$ we have

$$\begin{aligned} f_j &= f_j(x_1-h_1,\ldots,x_n-h_n) \\ &= \pi^2\sum\nolimits_{k=1}^{c}\left(\sum\nolimits_{i=1}^{n} r_{ji}s_{ik}\right)v_k + \pi^3\sum\nolimits_{i=1}^{n} r_{ji}w_i \bmod \pi^4 \\ &= \pi^3 v_j + \pi^3\sum\nolimits_{i=1}^{n} r_{ji}w_i \bmod \pi^4 \end{aligned}$$

for $1 \leq j \leq c$. Hence we can choose elements $g_j \in R[v_1,\ldots,v_c,w_1,\ldots,w_n]$ such that $g_j = v_j + \sum r_{ji}w_i \bmod \pi$ and such that $f_j = \pi^3 g_j$ in the R-algebra $R[x_1,\ldots,x_n,v_1,\ldots,v_c,w_1,\ldots,w_n]/(h_1,\ldots,h_n)$. We set

$$B = R[x_1,\ldots,x_n,v_1,\ldots,v_c,w_1,\ldots,w_n]/(f_1,\ldots,f_m,h_1,\ldots,h_n,g_1,\ldots,g_c).$$

The map $A \to B$ is clear. We define $B \to \Lambda$ by mapping $x_i \to \pi^4\lambda_i$, $v_i \mapsto 0$, and $w_i \mapsto \pi\lambda_i$. Then it is clear that the elements f_j and h_i are mapped to zero in Λ. Moreover, it is clear that g_i is mapped to an element t of $\pi\Lambda$ such that $\pi^3 t = 0$ (as $f_i = \pi^3 g_i$ modulo the ideal generated by the h's). Hence our assumption that $\text{Ann}_\Lambda(\pi) = \text{Ann}_\Lambda(\pi^2)$ implies that $t = 0$. Thus we are done if we can prove the statement about smoothness.

Note that $B_\pi \cong A_\pi[v_1,\ldots,v_c]$ because the equations $g_i = 0$ are implied by $f_i = 0$. Hence B_π is smooth over R as A_π is smooth over R by the assumption that π is strictly standard in A over R, see Lemma 16.2.5.

Set $B' = R[v_1,\ldots,v_c,w_1,\ldots,w_n]/(g_1,\ldots,g_c)$. As $g_i = v_i + \sum r_{ji}w_i \bmod \pi$ we see that $B'/\pi B' = R/\pi R[w_1,\ldots,w_n]$. Hence $R \to B'$ is smooth of relative dimension n at every point of $V(\pi)$ by Algebra, Lemmas 10.136.10 and 10.137.17 (the first lemma shows it is syntomic at those primes, in particular flat, whereupon the second lemma shows it is smooth).

Let $\mathfrak q \subset B$ be a prime with $\pi \in \mathfrak q$ and for some $r \in \mathfrak a$, $r \notin \mathfrak q$. Denote $\mathfrak q' = B' \cap \mathfrak q$. We claim the surjection $B' \to B$ induces an isomorphism of local rings $(B')_{\mathfrak q'} \to B_{\mathfrak q}$. This will conclude the proof of the lemma. Note that $B_{\mathfrak q}$ is the quotient of $(B')_{\mathfrak q'}$ by the ideal generated by f_{c+j}, $j = 1,\ldots,m-c$. We observe two things: first the image of f_{c+j} in $(B')_{\mathfrak q'}$ is divisible by π^2 and second the image of πf_{c+j} in $(B')_{\mathfrak q'}$ can be written as $\sum b_{j_1j_2}f_{c+j_1}f_{c+j_2}$ by (16.7.1.1). Thus we see that the image of each πf_{c+j} is contained in the ideal generated by the elements $\pi^2 f_{c+j'}$. Hence $\pi f_{c+j} = 0$ in $(B')_{\mathfrak q'}$ as this is a Noetherian local ring, see Algebra, Lemma 10.51.4. As $R \to (B')_{\mathfrak q'}$ is flat we see that

$$\left(\mathrm{Ann}_R(\pi^2)/\mathrm{Ann}_R(\pi)\right) \otimes_R (B')_{\mathfrak q'} = \mathrm{Ann}_{(B')_{\mathfrak q'}}(\pi^2)/\mathrm{Ann}_{(B')_{\mathfrak q'}}(\pi)$$

Because $r \in \mathfrak a$ is invertible in $(B')_{\mathfrak q'}$ we see that this module is zero. Hence we see that the image of f_{c+j} is zero in $(B')_{\mathfrak q'}$ as desired. $\square$

07CT Lemma 16.7.2. Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$ and $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$. Let $A \to \Lambda$ and $D \to \Lambda$ be R-algebra maps with A and D of finite presentation. Assume

(1) π is strictly standard in A over R, and
(2) there exists an R-algebra map $A/\pi^4A \to D/\pi^4D$ compatible with the maps to $\Lambda/\pi^4\Lambda$.

Then we can find an R-algebra map $B \to \Lambda$ with B of finite presentation and R-algebra maps $A \to B$ and $D \to B$ compatible with the maps to Λ such that $H_{D/R}B \subset H_{B/D}$ and $H_{D/R}B \subset H_{B/R}$.

Proof. We apply Lemma 16.7.1 to

$$D \longrightarrow A \otimes_R D \longrightarrow \Lambda$$

and the image of π in D. By Lemma 16.2.7 we see that π is strictly standard in $A \otimes_R D$ over D. As our section $\rho : (A \otimes_R D)/\pi^4(A \otimes_R D) \to D/\pi^4D$ we take the map induced by the map in (2). Thus Lemma 16.7.1 applies and we obtain a factorization $A \otimes_R D \to B \to \Lambda$ with B of finite presentation and $\mathfrak aB \subset H_{B/D}$ where

$$\mathfrak a = \mathrm{Ann}_D(\mathrm{Ann}_D(\pi^2)/\mathrm{Ann}_D(\pi)).$$

For any prime $\mathfrak q$ of D such that $D_{\mathfrak q}$ is flat over R we have $\mathrm{Ann}_{D_{\mathfrak q}}(\pi^2)/\mathrm{Ann}_{D_{\mathfrak q}}(\pi) = 0$ because annihilators of elements commutes with flat base change and we assumed $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$. Because D is Noetherian we see that $\mathrm{Ann}_D(\pi^2)/\mathrm{Ann}_D(\pi)$ is a finite D-module, hence formation of its annihilator commutes with localization. Thus we see that $\mathfrak a \not\subset \mathfrak q$. Hence we see that $D \to B$ is smooth at any prime of B lying over $\mathfrak q$. Since any prime of D where $R \to D$ is smooth is one where $D_{\mathfrak q}$ is flat over R we conclude that $H_{D/R}B \subset H_{B/D}$. The final inclusion $H_{D/R}B \subset H_{B/R}$ follows because compositions of smooth ring maps are smooth (Algebra, Lemma 10.137.14). $\square$

07F0 Lemma 16.7.3. Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$ and $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation and assume π is strictly standard in A over R. Let

$$A/\pi^8A \to \bar C \to \Lambda/\pi^8\Lambda$$

be a factorization with $\bar C$ of finite presentation. Then we can find a factorization $A \to B \to \Lambda$ with B of finite presentation such that $R_\pi \to B_\pi$ is smooth and such that

$$H_{\bar C/(R/\pi^8R)} \cdot \Lambda/\pi^8\Lambda \subset \sqrt{H_{B/R}\Lambda} \bmod \pi^8\Lambda.$$

Proof. Apply Lemma 16.6.1 to get $R \to D \to \Lambda$ with a factorization $\bar{C}/\pi^4\bar{C} \to D/\pi^4 D \to \Lambda/\pi^4\Lambda$ such that $R \to D$ is smooth at any prime not containing π and at any prime lying over a prime of $\bar{C}/\pi^4\bar{C}$ where $R/\pi^8 R \to \bar{C}$ is smooth. By Lemma 16.7.2 we can find a finitely presented R-algebra B and factorizations $A \to B \to \Lambda$ and $D \to B \to \Lambda$ such that $H_{D/R}B \subset H_{B/R}$. We omit the verification that this is a solution to the problem posed by the lemma. □

16.8. Warmup: reduction to a base field

07F1 In this section we apply the lemmas in the previous sections to prove that it suffices to prove the main result when the base ring is a field, see Lemma 16.8.4.

07F2 Situation 16.8.1. Here $R \to \Lambda$ is a regular ring map of Noetherian rings.

Let $R \to \Lambda$ be as in Situation 16.8.1. We say PT holds for $R \to \Lambda$ if Λ is a filtered colimit of smooth R-algebras.

07F3 Lemma 16.8.2. Let $R_i \to \Lambda_i$, $i = 1, 2$ be as in Situation 16.8.1. If PT holds for $R_i \to \Lambda_i$, $i = 1, 2$, then PT holds for $R_1 \times R_2 \to \Lambda_1 \times \Lambda_2$.

Proof. Omitted. Hint: A product of filtered colimits is a filtered colimit. □

07F4 Lemma 16.8.3. Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Let $S \subset R$ be a multiplicative set. Let $S^{-1}A \to B' \to S^{-1}\Lambda$ be a factorization with B' smooth over $S^{-1}R$. Then we can find a factorization $A \to B \to \Lambda$ such that some $s \in S$ maps to an elementary standard element (Definition 16.2.3) in B over R.

Proof. We first apply Lemma 16.3.4 to $S^{-1}R \to B'$. Thus we may assume B' is standard smooth over $S^{-1}R$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $x_i \mapsto \lambda_i$ in Λ. We may write $B' = S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ where $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B' and such that $A \to B'$ is given by $x_i \mapsto x_i$, see Lemma 16.3.6. After multiplying x_i, $i > n$ by an element of S and correspondingly modifying the equations f_j we may assume $B' \to S^{-1}\Lambda$ maps x_i to $\lambda_i/1$ for some $\lambda_i \in \Lambda$ for $i > n$. Choose a relation

$$1 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $a_j \in S^{-1}R[x_1, \ldots, x_{n+m}]$. Since each element of S is invertible in B' we may (by clearing denominators) assume that $f_j, a_j \in R[x_1, \ldots, x_{n+m}]$ and that

$$s_0 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $s_0 \in S$. Since g_j maps to zero in $S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, x_c)$ we can find elements $s_j \in S$ such that $s_j g_j = 0$ in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Since f_j maps to zero in $S^{-1}\Lambda$ we can find $s'_j \in S$ such that $s'_j f_j(\lambda_1, \ldots, \lambda_{n+m}) = 0$ in Λ. Consider the ring

$$B = R[x_1, \ldots, x_{n+m}]/(s'_1 f_1, \ldots, s'_c f_c, g_1, \ldots, g_t)$$

and the factorization $A \to B \to \Lambda$ with $B \to \Lambda$ given by $x_i \mapsto \lambda_i$. We claim that $s = s_0 s_1 \ldots s_t s'_1 \ldots s'_c$ is elementary standard in B over R which finishes the proof. Namely, $s_j g_j \in (f_1, \ldots, f_c)$ and hence $s g_j \in (s'_1 f_1, \ldots, s'_c f_c)$. Finally, we have

$$a_0 \det(\partial s'_j f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} (s'_1 \ldots \hat{s'_j} \ldots s'_c) a_j s'_j f_j = s_0 s'_1 \ldots s'_c$$

which divides s as desired. □

07F5 Lemma 16.8.4. If for every Situation 16.8.1 where R is a field PT holds, then PT holds in general.

Proof. Assume PT holds for any Situation 16.8.1 where R is a field. Let $R \to \Lambda$ be as in Situation 16.8.1 arbitrary. Note that $R/I \to \Lambda/I\Lambda$ is another regular ring map of Noetherian rings, see More on Algebra, Lemma 15.41.3. Consider the set of ideals

$$\mathcal{I} = \{I \subset R \mid R/I \to \Lambda/I\Lambda \text{ does not have PT}\}$$

We have to show that $\mathcal{I}$ is empty. If this set is nonempty, then it contains a maximal element because R is Noetherian. Replacing R by R/I and Λ by Λ/I we obtain a situation where PT holds for $R/I \to \Lambda/I\Lambda$ for any nonzero ideal of R. In particular, we see by applying Proposition 16.5.3 that R is a reduced ring.

Let $A \to \Lambda$ be an R-algebra homomorphism with A of finite presentation. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Algebra, Lemma 10.127.4.

Let $S \subset R$ be the set of nonzerodivisors and consider the total ring of fractions $Q = S^{-1}R$ of R. We know that $Q = K_1 \times \ldots \times K_n$ is a product of fields, see Algebra, Lemmas 10.25.4 and 10.31.6. By Lemma 16.8.2 and our assumption PT holds for the ring map $S^{-1}R \to S^{-1}\Lambda$. Hence we can find a factorization $S^{-1}A \to B' \to S^{-1}\Lambda$ with B' smooth over $S^{-1}R$.

We apply Lemma 16.8.3 and find a factorization $A \to B \to \Lambda$ such that some $\pi \in S$ is elementary standard in B over R. After replacing A by B we may assume that π is elementary standard, hence strictly standard in A. We know that $R/\pi^8R \to \Lambda/\pi^8\Lambda$ satisfies PT. Hence we can find a factorization $R/\pi^8R \to A/\pi^8A \to \bar{C} \to \Lambda/\pi^8\Lambda$ with $R/\pi^8R \to \bar{C}$ smooth. By Lemma 16.6.1 we can find an R-algebra map $D \to \Lambda$ with D smooth over R and a factorization $R/\pi^4R \to A/\pi^4A \to D/\pi^4D \to \Lambda/\pi^4\Lambda$. By Lemma 16.7.2 we can find $A \to B \to \Lambda$ with B smooth over R which finishes the proof. □

16.9. Local tricks

07F6

07F7 Situation 16.9.1. We are given a Noetherian ring R and an R-algebra map $A \to \Lambda$ and a prime $\mathfrak{q} \subset \Lambda$. We assume A is of finite presentation over R. In this situation we denote $\mathfrak{h}_A = \sqrt{H_{A/R}\Lambda}$.

Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. We say $R \to A \to \Lambda \supset \mathfrak{q}$ can be resolved if there exists a factorization $A \to B \to \Lambda$ with B of finite presentation and $\mathfrak{h}_A \subset \mathfrak{h}_B \not\subset \mathfrak{q}$. In this case we will call the factorization $A \to B \to \Lambda$ a resolution of $R \to A \to \Lambda \supset \mathfrak{q}$.

07F8 Lemma 16.9.2. Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. Let $r \geq 1$ and $\pi_1, \ldots, \pi_r \in R$ map to elements of $\mathfrak{q}$. Assume

(1) for $i = 1, \ldots, r$ we have

$$\mathrm{Ann}_{R/(\pi_1^8, \ldots, \pi_{i-1}^8)R}(\pi_i) = \mathrm{Ann}_{R/(\pi_1^8, \ldots, \pi_{i-1}^8)R}(\pi_i^2)$$

and

$$\mathrm{Ann}_{\Lambda/(\pi_1^8, \ldots, \pi_{i-1}^8)\Lambda}(\pi_i) = \mathrm{Ann}_{\Lambda/(\pi_1^8, \ldots, \pi_{i-1}^8)\Lambda}(\pi_i^2)$$

(2) for $i = 1, \ldots, r$ the element π_i maps to a strictly standard element in A over R.

Then, if

$$R/(\pi_1^8, \ldots, \pi_r^8)R \to A/(\pi_1^8, \ldots, \pi_r^8)A \to \Lambda/(\pi_1^8, \ldots, \pi_r^8)\Lambda \supset \mathfrak{q}/(\pi_1^8, \ldots, \pi_r^8)\Lambda$$

can be resolved, so can $R \to A \to \Lambda \supset \mathfrak{q}$.

Proof. We are going to prove this by induction on r.

The case $r = 1$. Here the assumption is that there exists a factorization $A/\pi_1^8 \to \bar{C} \to \Lambda/\pi_1^8$ which resolves the situation modulo π_1^8. Conditions (1) and (2) are the assumptions needed to apply Lemma 16.7.3. Thus we can "lift" the resolution $\bar{C}$ to a resolution of $R \to A \to \Lambda \supset \mathfrak{q}$.

The case $r > 1$. In this case we apply the induction hypothesis for $r - 1$ to the situation $R/\pi_1^8 \to A/\pi_1^8 \to \Lambda/\pi_1^8 \supset \mathfrak{q}/\pi_1^8\Lambda$. Note that property (2) is preserved by Lemma 16.2.7. □

07F9 Lemma 16.9.3. Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Assume that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$ and that $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$ can be resolved. Then there exists a factorization $A \to C \to \Lambda$ with C of finite presentation such that $H_{C/R}\Lambda \not\subset \mathfrak{q}$. [Swa98, Lemma 12.2] or [Pop85, Lemma 2]

Proof. Let $A_\mathfrak{p} \to C \to \Lambda_\mathfrak{q}$ be a resolution of $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$. By our assumption that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$ this means that $H_{C/R_\mathfrak{p}}\Lambda_\mathfrak{q} = \Lambda_\mathfrak{q}$. By Lemma 16.2.8 we may assume that C is smooth over $R_\mathfrak{p}$. By Lemma 16.3.4 we may assume that C is standard smooth over $R_\mathfrak{p}$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $A \to \Lambda$ is given by $x_i \mapsto \lambda_i$. Write $C = R_\mathfrak{p}[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ such that $A \to C$ maps x_i to x_i and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in C, see Lemma 16.3.6. After clearing denominators we may assume $f_1, \ldots, f_c$ are elements of $R[x_1, \ldots, x_{n+m}]$. Of course $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is not invertible in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ but it becomes invertible after inverting some element $s_0 \in R$, $s_0 \notin \mathfrak{p}$. As g_j maps to zero under $R[x_1, \ldots, x_n] \to A \to C$ we can find $s_j \in R$, $s_j \notin \mathfrak{p}$ such that $s_j g_j$ is zero in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Write $f_j = F_j(x_1, \ldots, x_{n+m}, 1)$ for some polynomial $F_j \in R[x_1, \ldots, x_n, X_{n+1}, \ldots, X_{n+m+1}]$ homogeneous in $X_{n+1}, \ldots, X_{n+m+1}$. Pick $\lambda_{n+i} \in \Lambda$, $i = 1, \ldots, m+1$ with $\lambda_{n+m+1} \notin \mathfrak{q}$ such that x_{n+i} maps to $\lambda_{n+i}/\lambda_{n+m+1}$ in $\Lambda_\mathfrak{q}$. Then

$$\begin{aligned} F_j(\lambda_1, \ldots, \lambda_{n+m+1}) &= (\lambda_{n+m+1})^{\deg(F_j)} F_j(\lambda_1, \ldots, \lambda_n, \frac{\lambda_{n+1}}{\lambda_{n+m+1}}, \ldots, \frac{\lambda_{n+m}}{\lambda_{n+m+1}}, 1) \\ &= (\lambda_{n+m+1})^{\deg(F_j)} f_j(\lambda_1, \ldots, \lambda_n, \frac{\lambda_{n+1}}{\lambda_{n+m+1}}, \ldots, \frac{\lambda_{n+m}}{\lambda_{n+m+1}}) \\ &= 0 \end{aligned}$$

in $\Lambda_\mathfrak{q}$. Thus we can find $\lambda_0 \in \Lambda$, $\lambda_0 \notin \mathfrak{q}$ such that $\lambda_0 F_j(\lambda_1, \ldots, \lambda_{n+m+1}) = 0$ in Λ. Now we set B equal to

$$R[x_0, \ldots, x_{n+m+1}]/(g_1, \ldots, g_t, x_0F_1(x_1, \ldots, x_{n+m+1}), \ldots, x_0F_c(x_1, \ldots, x_{n+m+1}))$$

which we map to Λ by mapping x_i to λ_i. Let b be the image of $x_0 x_{n+m+1} s_0 s_1 \ldots s_t$ in B. Then B_b is isomorphic to

$$R_{s_0 s_1 \ldots s_t}[x_0, x_1, \ldots, x_{n+m+1}, 1/x_0 x_{n+m+1}]/(f_1, \ldots, f_c)$$

which is smooth over R by construction. Since b does not map to an element of $\mathfrak{q}$, we win. $\square$

07FA Lemma 16.9.4. Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Assume

(1) $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$,
(2) $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$ can be resolved, and
(3) $\dim(\Lambda_\mathfrak{q}) = 0$.

Then $R \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. By (3) the ring $\Lambda_\mathfrak{q}$ is Artinian local hence $\mathfrak{q}\Lambda_\mathfrak{q}$ is nilpotent. Thus $(\mathfrak{h}_A)^N\Lambda_\mathfrak{q} = 0$ for some $N > 0$. Thus there exists a $\lambda \in \Lambda$, $\lambda \notin \mathfrak{q}$ such that $\lambda(\mathfrak{h}_A)^N = 0$ in Λ. Say $H_{A/R} = (a_1, \ldots, a_r)$ so that $\lambda a_i^N = 0$ in Λ. By Lemma 16.9.3 we can find a factorization $A \to C \to \Lambda$ with C of finite presentation such that $\mathfrak{h}_C \not\subset \mathfrak{q}$. Write $C = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Set

$$B = A[x_1, \ldots, x_n, y_1, \ldots, y_r, z, t_{ij}]/(f_j - \sum y_i t_{ij}, zy_i)$$

where t_{ij} is a set of rm variables. Note that there is a map $B \to C[y_i, z]/(y_i z)$ given by setting t_{ij} equal to zero. The map $B \to \Lambda$ is the composition $B \to C[y_i, z]/(y_i z) \to \Lambda$ where $C[y_i, z]/(y_i z) \to \Lambda$ is the given map $C \to \Lambda$, maps z to λ, and maps y_i to the image of a_i^N in Λ.

We claim that B is a solution for $R \to A \to \Lambda \supset \mathfrak{q}$. First note that B_z is isomorphic to $C[y_1, \ldots, y_r, z, z^{-1}]$ and hence is smooth. On the other hand, $B_{y_\ell} \cong A[x_i, y_i, y_\ell^{-1}, t_{ij}, i \neq \ell]$ which is smooth over A. Thus we see that z and $a_\ell y_\ell$ (compositions of smooth maps are smooth) are all elements of $H_{B/R}$. This proves the lemma. $\square$

16.10. Separable residue fields

07FB In this section we explain how to solve a local problem in the case of a separable residue field extension.

07FC Lemma 16.10.1 (Ogoma). Let A be a Noetherian ring and let M be a finite A-module. Let $S \subset A$ be a multiplicative set. If $\pi \in A$ and $\mathrm{Ker}(\pi : S^{-1}M \to S^{-1}M) = \mathrm{Ker}(\pi^2 : S^{-1}M \to S^{-1}M)$ then there exists an $s \in S$ such that for any $n > 0$ we have $\mathrm{Ker}(s^n\pi : M \to M) = \mathrm{Ker}((s^n\pi)^2 : M \to M)$.

Proof. Let $K = \mathrm{Ker}(\pi : M \to M)$ and $K' = \{m \in M \mid \pi^2 m = 0 \text{ in } S^{-1}M\}$ and $Q = K'/K$. Note that $S^{-1}Q = 0$ by assumption. Since A is Noetherian we see that Q is a finite A-module. Hence we can find an $s \in S$ such that s annihilates Q. Then s works. $\square$

07FD Lemma 16.10.2. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $I \subset \mathfrak{q}$ be a prime. Let n, e be positive integers Assume that $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset I\Lambda_\mathfrak{q}$ and that $\Lambda_\mathfrak{q}$ is a regular local ring of dimension d. Then there exists an $n > 0$ and $\pi_1, \ldots, \pi_d \in \Lambda$ such that

(1) $(\pi_1, \ldots, \pi_d)\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$,
(2) $\pi_1^n, \ldots, \pi_d^n \in I$, and
(3) for $i = 1, \ldots, d$ we have

$$\mathrm{Ann}_{\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)\Lambda}(\pi_i) = \mathrm{Ann}_{\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)\Lambda}(\pi_i^2).$$

Proof. Set $S = \Lambda \setminus \mathfrak{q}$ so that $\Lambda_\mathfrak{q} = S^{-1}\Lambda$. First pick $\pi_1, \ldots, \pi_d$ with (1) which is possible as $\Lambda_\mathfrak{q}$ is regular. By assumption $\pi_i^n \in I\Lambda_\mathfrak{q}$. Thus we can find $s_1, \ldots, s_d \in S$ such that $s_i\pi_i^n \in I$. Replacing π_i by $s_i\pi_i$ we get (2). Note that (1) and (2) are preserved by further multiplying by elements of S. Suppose that (3) holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\pi_1, \ldots, \pi_d$ is a regular sequence in $S^{-1}\Lambda$, see Algebra, Lemma 10.106.3. In particular $\pi_1^e, \ldots, \pi_t^e, \pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda = \Lambda_\mathfrak{q}$ by Algebra, Lemma 10.68.9. Hence we see that

$$\mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)}(\pi_i) = \mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)}(\pi_i^2).$$

Thus we get (3) for $i = t+1$ after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 16.10.1. By induction on t this produces a sequence satisfying (1), (2), and (3). $\square$

07FE Lemma 16.10.3. Let $k \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1 where

(1) k is a field,
(2) Λ is Noetherian,
(3) $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$,
(4) $\Lambda_\mathfrak{q}$ is a regular local ring, and
(5) the field extension $\kappa(\mathfrak{q})/k$ is separable.

Then $k \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. Set $d = \dim \Lambda_\mathfrak{q}$. Set $R = k[x_1, \ldots, x_d]$. Choose $n > 0$ such that $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset \mathfrak{h}_A\Lambda_\mathfrak{q}$ which is possible as $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$. Choose generators $a_1, \ldots, a_r$ of $H_{A/R}$. Set

$$B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^n - \sum z_{ij}a_j)$$

Each B_{a_j} is smooth over R it is a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$ and A_{a_j} is smooth over k. Hence B_{x_i} is smooth over R. Let $B \to C$ be the R-algebra map constructed in Lemma 16.3.1 which comes with a R-algebra retraction $C \to B$. In particular a map $C \to \Lambda$ fitting into the diagram above. By construction C_{x_i} is a smooth R-algebra with $\Omega_{C_{x_i}/R}$ free. Hence we can find $c > 0$ such that x_i^c is strictly standard in C/R, see Lemma 16.3.7. Now choose $\pi_1, \ldots, \pi_d \in \Lambda$ as in Lemma 16.10.2 where $n = n$, $e = 8c$, $\mathfrak{q} = \mathfrak{q}$ and $I = \mathfrak{h}_A$. Write $\pi_i^n = \sum \lambda_{ij}a_j$ for some $\pi_{ij} \in \Lambda$. There is a map $B \to \Lambda$ given by $x_i \mapsto \pi_i$ and $z_{ij} \mapsto \lambda_{ij}$. Set $R = k[x_1, \ldots, x_d]$. Diagram

$$\begin{tikzcd} R \arrow[r] & B \arrow[dr] & \\ k \arrow[u] \arrow[r] & A \arrow[u] \arrow[r] & \Lambda \end{tikzcd}$$

Now we apply Lemma 16.9.2 to $R \to C \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $x_1^c, \ldots, x_d^c$ of R. Assumption (2) is clear. Assumption (1) holds for R by inspection and for Λ by our choice of $\pi_1, \ldots, \pi_d$. (Note that if $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$, then we have $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^c)$ for all $c > 0$.) Thus it suffices to resolve

$$R/(x_1^e, \ldots, x_d^e) \to C/(x_1^e, \ldots, x_d^e) \to \Lambda/(\pi_1^e, \ldots, \pi_d^e) \supset \mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

for $e = 8c$. By Lemma 16.9.4 it suffices to resolve this after localizing at $\mathfrak{q}$. But since $x_1, \ldots, x_d$ map to a regular sequence in $\Lambda_\mathfrak{q}$ we see that $R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat, see Algebra, Lemma 10.128.2. Hence

$$R_\mathfrak{p}/(x_1^e, \ldots, x_d^e) \to \Lambda_\mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

is a flat ring map of Artinian local rings. Moreover, this map induces a separable field extension on residue fields by assumption. Thus this map is a filtered colimit of smooth algebras by Algebra, Lemma 10.158.11 and Proposition 16.5.3. Existence of the desired solution follows from Algebra, Lemma 10.127.4. □

16.11. Inseparable residue fields

07FF In this section we explain how to solve a local problem in the case of an inseparable residue field extension.

07FG Lemma 16.11.1. Let k be a field of characteristic $p > 0$. Let $(\Lambda, \mathfrak{m}, K)$ be an Artinian local k-algebra. Assume that $\dim H_1(L_{K/k}) < \infty$. Then Λ is a filtered colimit of Artinian local k-algebras A with each map $A \to \Lambda$ flat, with $\mathfrak{m}_A\Lambda = \mathfrak{m}$, and with A essentially of finite type over k.

Proof. Note that the flatness of $A \to \Lambda$ implies that $A \to \Lambda$ is injective, so the lemma really tells us that Λ is a directed union of these types of subrings $A \subset \Lambda$. Let n be the minimal integer such that $\mathfrak{m}^n = 0$. We will prove this lemma by induction on n. The case $n = 1$ is clear as a field extension is a union of finitely generated field extensions.

Pick $\lambda_1, \ldots, \lambda_d \in \mathfrak{m}$ which generate $\mathfrak{m}$. As K is formally smooth over $\mathbf{F}_p$ (see Algebra, Lemma 10.158.7) we can find a ring map $\sigma : K \to \Lambda$ which is a section of the quotient map $\Lambda \to K$. In general σ is not a k-algebra map. Given σ we define

$$\Psi_\sigma : K[x_1, \ldots, x_d] \longrightarrow \Lambda$$

using σ on elements of K and mapping x_i to λ_i. Claim: there exists a $\sigma : K \to \Lambda$ and a subfield $k \subset F \subset K$ finitely generated over k such that the image of k in Λ is contained in $\Psi_\sigma(F[x_1, \ldots, x_d])$.

We will prove the claim by induction on the least integer n such that $\mathfrak{m}^n = 0$. It is clear for $n = 1$. If $n > 1$ set $I = \mathfrak{m}^{n-1}$ and $\Lambda' = \Lambda/I$. By induction we may assume given $\sigma' : K \to \Lambda'$ and $k \subset F' \subset K$ finitely generated such that the image of $k \to \Lambda \to \Lambda'$ is contained in $A' = \Psi_{\sigma'}(F'[x_1, \ldots, x_d])$. Denote $\tau' : k \to A'$ the induced map. Choose a lift $\sigma : K \to \Lambda$ of σ' (this is possible by the formal smoothness of $K/\mathbf{F}_p$ we mentioned above). For later reference we note that we can change σ to $\sigma + D$ for some derivation $D : K \to I$. Set $A = F[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n$. Then Ψ_σ induces a ring map $\Psi_\sigma : A \to \Lambda$. The composition with the quotient map $\Lambda \to \Lambda'$ induces a surjective map $A \to A'$ with nilpotent kernel. Choose a lift $\tau : k \to A$ of τ' (possible as $k/\mathbf{F}_p$ is formally smooth). Thus we obtain two maps $k \to \Lambda$, namely $\Psi_\sigma \circ \tau : k \to \Lambda$ and the given map $i : k \to \Lambda$. These maps agree modulo I, whence the difference is a derivation $\theta = i - \Psi_\sigma \circ \tau : k \to I$. Note that if we change σ into $\sigma + D$ then we change θ into $\theta - D|_k$.

Choose a set of elements $\{y_j\}_{j \in J}$ of k whose differentials $\text{d}y_j$ form a basis of $\Omega_{k/\mathbf{F}_p}$. The Jacobi-Zariski sequence for $\mathbf{F}_p \subset k \subset K$ is

$$0 \to H_1(L_{K/k}) \to \Omega_{k/\mathbf{F}_p} \otimes K \to \Omega_{K/\mathbf{F}_p} \to \Omega_{K/k} \to 0$$

As $\dim H_1(L_{K/k}) < \infty$ we can find a finite subset $J_0 \subset J$ such that the image of the first map is contained in $\bigoplus_{j \in J_0} K\text{d}y_j$. Hence the elements $\text{d}y_j$, $j \in J \setminus J_0$ map to

K-linearly independent elements of $\Omega_{K/\mathbf{F}_p}$. Therefore we can choose a $D : K \to I$ such that $\theta - D|_k = \xi \circ \mathrm{d}$ where ξ is a composition

$$\Omega_{k/\mathbf{F}_p} = \bigoplus\nolimits_{j \in J} k\mathrm{d}y_j \longrightarrow \bigoplus\nolimits_{j \in J_0} k\mathrm{d}y_j \longrightarrow I$$

Let $f_j = \xi(\mathrm{d}y_j) \in I$ for $j \in J_0$. Change σ into $\sigma + D$ as above. Then we see that $\theta(a) = \sum_{j \in J_0} a_j f_j$ for $a \in k$ where $\mathrm{d}a = \sum a_j \mathrm{d}y_j$ in $\Omega_{k/\mathbf{F}_p}$. Note that I is generated by the monomials $\lambda^E = \lambda_1^{e_1} \ldots \lambda_d^{e_d}$ of total degree $|E| = \sum e_i = n - 1$ in $\lambda_1, \ldots, \lambda_d$. Write $f_j = \sum_E c_{j,E} \lambda^E$ with $c_{j,E} \in K$. Replace F' by $F = F'(c_{j,E})$. Then the claim holds.

Choose σ and F as in the claim. The kernel of Ψ_σ is generated by finitely many polynomials $g_1, \ldots, g_t \in K[x_1, \ldots, x_d]$ and we may assume their coefficients are in F after enlarging F by adjoining finitely many elements. In this case it is clear that the map $A = F[x_1, \ldots, x_d]/(g_1, \ldots, g_t) \to K[x_1, \ldots, x_d]/(g_1, \ldots, g_t) = \Lambda$ is flat. By the claim A is a k-subalgebra of Λ. It is clear that Λ is the filtered colimit of these algebras, as K is the filtered union of the subfields F. Finally, these algebras are essentially of finite type over k by Algebra, Lemma 10.54.4. $\square$

07FH Lemma 16.11.2. Let k be a field of characteristic $p > 0$. Let Λ be a Noetherian geometrically regular k-algebra. Let $\mathfrak{q} \subset \Lambda$ be a prime ideal. Let $n \geq 1$ be an integer and let $E \subset \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ be a finite subset. Then we can find $m \geq 0$ and $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ with the following properties

(1) setting $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ we have $\mathfrak{q}\Lambda_\mathfrak{q} = \mathfrak{p}\Lambda_\mathfrak{q}$ and $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat,
(2) there is a factorization by homomorphisms of local Artinian rings

$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

where the first arrow is essentially smooth and the second is flat,
(3) E is contained in D modulo $\mathfrak{q}^n\Lambda_\mathfrak{q}$.

Proof. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Note that $\dim H_1(L_{\kappa(\mathfrak{q})/k}) < \infty$ by More on Algebra, Proposition 15.35.1. Pick $A \subset \bar{\Lambda}$ containing E such that A is local Artinian, essentially of finite type over k, the map $A \to \bar{\Lambda}$ is flat, and $\mathfrak{m}_A$ generates the maximal ideal of $\bar{\Lambda}$, see Lemma 16.11.1. Denote $F = A/\mathfrak{m}_A$ the residue field so that $k \subset F \subset K$. Pick $\lambda_1, \ldots, \lambda_t \in \Lambda$ which map to elements of A in $\bar{\Lambda}$ such that moreover the images of $\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_t$ form a basis of $\Omega_{F/k}$. Consider the map $\varphi' : k[y_1, \ldots, y_t] \to \Lambda$ sending y_j to λ_j. Set $\mathfrak{p}' = (\varphi')^{-1}(\mathfrak{q})$. By More on Algebra, Lemma 15.35.2 the ring map $k[y_1, \ldots, y_t]_{\mathfrak{p}'} \to \Lambda_\mathfrak{q}$ is flat and $\Lambda_\mathfrak{q}/\mathfrak{p}'\Lambda_\mathfrak{q}$ is regular. Thus we can choose further elements $\lambda_{t+1}, \ldots, \lambda_m \in \Lambda$ which map into $A \subset \bar{\Lambda}$ and which map to a regular system of parameters of $\Lambda_\mathfrak{q}/\mathfrak{p}'\Lambda_\mathfrak{q}$. We obtain $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ having property (1) such that $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to \bar{\Lambda}$ factors through A. Thus $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to A$ is flat by Algebra, Lemma 10.39.9. By construction the residue field extension $F/\kappa(\mathfrak{p})$ is finitely generated and $\Omega_{F/\kappa(\mathfrak{p})} = 0$. Hence it is finite separable by More on Algebra, Lemma 15.34.1. Thus $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to A$ is finite by Algebra, Lemma 10.54.4. Finally, we conclude that it is étale by Algebra, Lemma 10.143.7. Since an étale ring map is certainly essentially smooth we win. $\square$

07FI Lemma 16.11.3. Let $\varphi : k[y_1, \ldots, y_m] \to \Lambda$, n, $\mathfrak{q}$, $\mathfrak{p}$ and

$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

be as in Lemma 16.11.2. Then for any $\lambda \in \Lambda \setminus \mathfrak q$ there exists an integer $q > 0$ and a factorization

$$k[y_1, \ldots, y_m]_\mathfrak p/\mathfrak p^n \to D \to D' \to \Lambda_\mathfrak q/\mathfrak q^n\Lambda_\mathfrak q$$

such that $D \to D'$ is an essentially smooth map of local Artinian rings, the last arrow is flat, and λ^q is in D'.

Proof. Set $\bar\Lambda = \Lambda_\mathfrak q/\mathfrak q^n\Lambda_\mathfrak q$. Let $\bar\lambda$ be the image of λ in $\bar\Lambda$. Let $\alpha \in \kappa(\mathfrak q)$ be the image of λ in the residue field. Let $k \subset F \subset \kappa(\mathfrak q)$ be the residue field of D. If α is in F then we can find an $x \in D$ such that $x\bar\lambda = 1 \bmod \mathfrak q$. Hence $(x\bar\lambda)^q = 1 \bmod (\mathfrak q)^q$ if q is divisible by p. Hence $\bar\lambda^q$ is in D. If α is transcendental over F, then we can take $D' = (D[\bar\lambda])_\mathfrak m$ equal to the subring generated by D and $\bar\lambda$ localized at $\mathfrak m = D[\bar\lambda] \cap \mathfrak q\bar\Lambda$. This works because $D[\bar\lambda]$ is in fact a polynomial algebra over D in this case. Finally, if $\lambda \bmod \mathfrak q$ is algebraic over F, then we can find a p-power q such that α^q is separable algebraic over F, see Fields, Section 9.28. Note that D and $\bar\Lambda$ are henselian local rings, see Algebra, Lemma 10.153.10. Let $D \to D'$ be a finite étale extension whose residue field extension is $F(\alpha^q)/F$, see Algebra, Lemma 10.153.7. Since $\bar\Lambda$ is henselian and $F(\alpha^q)$ is contained in its residue field we can find a factorization $D' \to \bar\Lambda$. By the first part of the argument we see that $\bar\lambda^{qq'} \in D'$ for some $q' > 0$. □

07FJ Lemma 16.11.4. Let $k \to A \to \Lambda \supset \mathfrak q$ be as in Situation 16.9.1 where

(1) k is a field of characteristic $p > 0$,
(2) Λ is Noetherian and geometrically regular over k,
(3) $\mathfrak q$ is minimal over $\mathfrak h_A$.

Then $k \to A \to \Lambda \supset \mathfrak q$ can be resolved.

Proof. The lemma is proven by the following steps in the given order. We will justify each of these steps below.

07FK (1) Pick an integer $N > 0$ such that $\mathfrak q^N\Lambda_\mathfrak q \subset H_{A/k}\Lambda_\mathfrak q$.
07FL (2) Pick generators $a_1, \ldots, a_t \in A$ of the ideal $H_{A/R}$.
07FM (3) Set $d = \dim(\Lambda_\mathfrak q)$.
07FN (4) Set $B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^{2N} - \sum z_{ij}a_j)$.
07FP (5) Consider B as a $k[x_1, \ldots, x_d]$-algebra and let $B \to C$ be as in Lemma 16.3.1. We also obtain a section $C \to B$.
07FQ (6) Choose $c > 0$ such that each x_i^c is strictly standard in C over $k[x_1, \ldots, x_d]$.
07FR (7) Set $n = N + dc$ and $e = 8c$.
07FS (8) Let $E \subset \Lambda_\mathfrak q/\mathfrak q^n\Lambda_\mathfrak q$ be the images of generators of A as a k-algebra.
07FT (9) Choose an integer m and a k-algebra map $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ and a factorization by local Artinian rings

$$k[y_1, \ldots, y_m]_\mathfrak p/\mathfrak p^n k[y_1, \ldots, y_m]_\mathfrak p \to D \to \Lambda_\mathfrak q/\mathfrak q^n\Lambda_\mathfrak q$$

such that the first arrow is essentially smooth, the second is flat, E is contained in D, with $\mathfrak p = \varphi^{-1}(\mathfrak q)$ the map $k[y_1, \ldots, y_m]_\mathfrak p \to \Lambda_\mathfrak q$ is flat, and $\mathfrak p\Lambda_\mathfrak q = \mathfrak q\Lambda_\mathfrak q$.
07FU (10) Choose $\pi_1, \ldots, \pi_d \in \mathfrak p$ which map to a regular system of parameters of $k[y_1, \ldots, y_m]_\mathfrak p$.
07FV (11) Let $R = k[y_1, \ldots, y_m, t_1, \ldots, t_m]$ and $\gamma_i = \pi_i t_i$.

07FW (12) If necessary modify the choice of π_i such that for $i = 1, \ldots, d$ we have

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i^2)$$

07FX (13) There exist $\delta_1, \ldots, \delta_d \in \Lambda$, $\delta_i \not\in \mathfrak q$ and a factorization $D \to D' \to \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$ with D' local Artinian, $D \to D'$ essentially smooth, the map $D' \to \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$ flat such that, with $\pi'_i = \delta_i\pi_i$, we have for $i = 1, \ldots, d$
(a) $(\pi'_i)^{2N} = \sum a_j\lambda_{ij}$ in Λ where λ_{ij} mod $\mathfrak q^n\Lambda_{\mathfrak q}$ is an element of D',
(b) $\mathrm{Ann}_{\Lambda/(\pi'^e_1, \ldots, \pi'^e_{i-1})}(\pi'_i) = \mathrm{Ann}_{\Lambda/(\pi'^e_1, \ldots, \pi'^e_{i-1})}(\pi'^2_i)$,
(c) δ_i mod $\mathfrak q^n\Lambda_{\mathfrak q}$ is an element of D'.

07FY (14) Define $B \to \Lambda$ by sending x_i to π'_i and z_{ij} to λ_{ij} found above. Define $C \to \Lambda$ by composing the map $B \to \Lambda$ with the retraction $C \to B$.

07FZ (15) Map $R \to \Lambda$ by φ on $k[y_1, \ldots, y_m]$ and by sending t_i to δ_i. Further introduce a map

$$k[x_1, \ldots, x_d] \longrightarrow R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$$

by sending x_i to $\gamma_i = \pi_i t_i$.

07G0 (16) It suffices to resolve

$$R \to C \otimes_{k[x_1, \ldots, x_d]} R \to \Lambda \supset \mathfrak q$$

07G1 (17) Set $I = (\gamma_1^e, \ldots, \gamma_d^e) \subset R$.

07G2 (18) It suffices to resolve

$$R/I \to C \otimes_{k[x_1, \ldots, x_d]} R/I \to \Lambda/I\Lambda \supset \mathfrak q/I\Lambda$$

07G3 (19) We denote $\mathfrak r \subset R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$ the inverse image of $\mathfrak q$.

07G4 (20) It suffices to resolve

$$(R/I)_{\mathfrak r} \to C \otimes_{k[x_1, \ldots, x_d]} (R/I)_{\mathfrak r} \to \Lambda_{\mathfrak q}/I\Lambda_{\mathfrak q} \supset \mathfrak q\Lambda_{\mathfrak q}/I\Lambda_{\mathfrak q}$$

07G5 (21) Set $J = (\pi_1^e, \ldots, \pi_d^e)$ in $k[y_1, \ldots, y_m]$.

07G6 (22) It suffices to resolve

$$(R/JR)_{\mathfrak p} \to C \otimes_{k[x_1, \ldots, x_d]} (R/JR)_{\mathfrak p} \to \Lambda_{\mathfrak q}/J\Lambda_{\mathfrak q} \supset \mathfrak q\Lambda_{\mathfrak q}/J\Lambda_{\mathfrak q}$$

07G7 (23) It suffices to resolve

$$(R/\mathfrak p^nR)_{\mathfrak p} \to C \otimes_{k[x_1, \ldots, x_d]} (R/\mathfrak p^nR)_{\mathfrak p} \to \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q} \supset \mathfrak q\Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$$

07G8 (24) It suffices to resolve

$$(R/\mathfrak p^nR)_{\mathfrak p} \to B \otimes_{k[x_1, \ldots, x_d]} (R/\mathfrak p^nR)_{\mathfrak p} \to \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q} \supset \mathfrak q\Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$$

07G9 (25) The ring $D'[t_1, \ldots, t_d]$ is given the structure of an $R_{\mathfrak p}/\mathfrak p^nR_{\mathfrak p}$-algebra by the given map $k[y_1, \ldots, y_m]_{\mathfrak p}/\mathfrak p^nk[y_1, \ldots, y_m]_{\mathfrak p} \to D'$ and by sending t_i to t_i. It suffices to find a factorization

$$B \otimes_{k[x_1, \ldots, x_d]} (R/\mathfrak p^nR)_{\mathfrak p} \to D'[t_1, \ldots, t_d] \to \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$$

where the second arrow sends t_i to δ_i and induces the given homomorphism $D' \to \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$.

07GA (26) Such a factorization exists by our choice of D' above.

We now give the justification for each of the steps, except that we skip justifying the steps which just introduce notation.

Ad (1). This is possible as $\mathfrak q$ is minimal over $\mathfrak h_A = \sqrt{H_{A/k}\Lambda}$.

Ad (6). Note that A_{a_i} is smooth over k. Hence B_{a_j}, which is isomorphic to a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$, is smooth over $k[x_1, \ldots, x_d]$. Thus B_{x_i} is smooth over $k[x_1, \ldots, x_d]$. By Lemma 16.3.1 we see that C_{x_i} is smooth over $k[x_1, \ldots, x_d]$ with finite free module of differentials. Hence some power of x_i is strictly standard in C over $k[x_1, \ldots, x_n]$ by Lemma 16.3.7.

Ad (9). This follows by applying Lemma 16.11.2.

Ad (10). Since $k[y_1, \ldots, y_m]_{\mathfrak p} \to \Lambda_{\mathfrak q}$ is flat and $\mathfrak p\Lambda_{\mathfrak q} = \mathfrak q\Lambda_{\mathfrak q}$ by construction we see that $\dim(k[y_1, \ldots, y_m]_{\mathfrak p}) = d$ by Algebra, Lemma 10.112.7. Thus we can find $\pi_1, \ldots, \pi_d \in \Lambda$ which map to a regular system of parameters in $\Lambda_{\mathfrak q}$.

Ad (12). By Algebra, Lemma 10.106.3 any permutation of the sequence $\pi_1, \ldots, \pi_d$ is a regular sequence in $k[y_1, \ldots, y_m]_{\mathfrak p}$. Hence $\gamma_1 = \pi_1 t_1, \ldots, \gamma_d = \pi_d t_d$ is a regular sequence in $R_{\mathfrak p} = k[y_1, \ldots, y_m]_{\mathfrak p}[t_1, \ldots, t_d]$, see Algebra, Lemma 10.68.10. Let $S = k[y_1, \ldots, y_m] \setminus \mathfrak p$ so that $R_{\mathfrak p} = S^{-1}R$. Note that $\pi_1, \ldots, \pi_d$ and $\gamma_1, \ldots, \gamma_d$ remain regular sequences if we multiply our π_i by elements of S. Suppose that

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i^2)$$

holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\gamma_1^e, \ldots, \gamma_t^e, \gamma_{t+1}$ is a regular sequence in $S^{-1}R$ by Algebra, Lemma 10.68.9. Hence we see that

$$\mathrm{Ann}_{S^{-1}R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)}(\gamma_i) = \mathrm{Ann}_{S^{-1}R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)}(\gamma_i^2).$$

Thus we get

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_t^e)R}(\gamma_{t+1}) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_t^e)R}(\gamma_{t+1}^2)$$

after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 16.10.1. By induction on t this produces the desired sequence.

Ad (13). Let $S = \Lambda \setminus \mathfrak q$ so that $\Lambda_{\mathfrak q} = S^{-1}\Lambda$. Set $\bar\Lambda = \Lambda_{\mathfrak q}/\mathfrak q^n\Lambda_{\mathfrak q}$. Suppose that we have a $t \in \{0, \ldots, d\}$ and $\delta_1, \ldots, \delta_t \in S$ and a factorization $D \to D' \to \bar\Lambda$ as in (13) such that (a), (b), (c) hold for $i = 1, \ldots, t$. We have $\pi_{t+1}^N \in H_{A/k}\Lambda_{\mathfrak q}$ as $\mathfrak q^N\Lambda_{\mathfrak q} \subset H_{A/k}\Lambda_{\mathfrak q}$ by (1). Hence $\pi_{t+1}^N \in H_{A/k}\bar\Lambda$. Hence $\pi_{t+1}^N \in H_{A/k}D'$ as $D' \to \bar\Lambda$ is faithfully flat, see Algebra, Lemma 10.82.11. Recall that $H_{A/k} = (a_1, \ldots, a_t)$. Say $\pi_{t+1}^N = \sum a_j d_j$ in D' and choose $c_j \in \Lambda_{\mathfrak q}$ lifting $d_j \in D'$. Then $\pi_{t+1}^N = \sum c_j a_j + \epsilon$ with $\epsilon \in \mathfrak q^n\Lambda_{\mathfrak q} \subset \mathfrak q^{n-N}H_{A/k}\Lambda_{\mathfrak q}$. Write $\epsilon = \sum a_j c'_j$ for some $c'_j \in \mathfrak q^{n-N}\Lambda_{\mathfrak q}$. Hence $\pi_{t+1}^{2N} = \sum(\pi_{t+1}^N c_j + \pi_{t+1}^N c'_j)a_j$. Note that $\pi_{t+1}^N c'_j$ maps to zero in $\bar\Lambda$; this trivial but key observation will ensure later that (a) holds. Now we choose $s \in S$ such that there exist $\mu_{t+1j} \in \Lambda$ such that on the one hand $\pi_{t+1}^N c_j + \pi_{t+1}^N c'_j = \mu_{t+1j}/s^{2N}$ in $S^{-1}\Lambda$ and on the other $(s\pi_{t+1})^{2N} = \sum \mu_{t+1j} a_j$ in Λ (minor detail omitted). We may further replace s by a power and enlarge D' such that s maps to an element of D'. With these choices μ_{t+1j} maps to $s^{2N}d_j$ which is an element of D'. Note that $\pi_1, \ldots, \pi_d$ are a regular sequence of parameters in $S^{-1}\Lambda$ by our choice of φ. Hence $\pi_1, \ldots, \pi_d$ forms a regular sequence in $\Lambda_{\mathfrak q}$ by Algebra, Lemma 10.106.3. It follows that ${\pi'}_1^e, \ldots, {\pi'}_t^e, s\pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda$ by Algebra, Lemma 10.68.9. Thus we get

$$\mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(s\pi_{t+1}) = \mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s\pi_{t+1})^2).$$

Hence we may apply Lemma 16.10.1 to find an $s' \in S$ such that

$$\mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s')^q s\pi_{t+1}) = \mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(((s')^q s\pi_{t+1})^2).$$

for any $q > 0$. By Lemma 16.11.3 we can choose q and enlarge D' such that $(s')^q$ maps to an element of D'. Setting $\delta_{t+1} = (s')^q s$ and we conclude that (a), (b), (c) hold for $i = 1, \ldots, t+1$. For (a) note that $\lambda_{t+1j} = (s')^{2Nq}\mu_{t+1j}$ works. By induction on t we win.

Ad (16). By construction the radical of $H_{(C \otimes_{k[x_1,\ldots,x_d]} R)/R}\Lambda$ contains $\mathfrak{h}_A$. Namely, the elements $a_j \in H_{A/k}$ map to elements of $H_{B/k[x_1,\ldots,x_n]}$, hence map to elements of $H_{C/k[x_1,\ldots,x_n]}$, hence $a_j \otimes 1$ map to elements of $H_{C \otimes_{k[x_1,\ldots,x_d]} R/R}$. Moreover, if we have a solution $C \otimes_{k[x_1,\ldots,x_n]} R \to T \to \Lambda$ of

$$R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$$

then $H_{T/R} \subset H_{T/k}$ as R is smooth over k. Hence T will also be a solution for the original situation $k \to A \to \Lambda \supset \mathfrak{q}$.

Ad (18). Follows on applying Lemma 16.9.2 to $R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $\gamma_1^c, \ldots, \gamma_d^c$. We note that since x_i^c are strictly standard in C over $k[x_1, \ldots, x_d]$ the elements γ_i^c are strictly standard in $C \otimes_{k[x_1,\ldots,x_d]} R$ over R by Lemma 16.2.7. The other assumption of Lemma 16.9.2 holds by steps (12) and (13).

Ad (20). Apply Lemma 16.9.4 to the situation in (18). In the rest of the arguments the target ring is local Artinian, hence we are looking for a factorization by a smooth algebra T over the source ring.

Ad (22). Suppose that $C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$ is a solution to

$$(R/JR)_\mathfrak{p} \to C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$$

Then $C \otimes_{k[x_1,\ldots,x_d]} (R/I)_\mathfrak{r} \to T_\mathfrak{r} \to \Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q}$ is a solution to the situation in (20).

Ad (23). Our $n = N + dc$ is large enough so that $\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \subset J_\mathfrak{p}$ and $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset J\Lambda_\mathfrak{q}$. Hence if we have a solution $C \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ of (22 then we can take T/JT as the solution for (23).

Ad (24). This is true because we have a section $C \to B$ in the category of R-algebras.

Ad (25). This is true because D' is essentially smooth over the local Artinian ring $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}$ and

$$R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p} = k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}[t_1, \ldots, t_d].$$

Hence $D'[t_1, \ldots, t_d]$ is a filtered colimit of smooth $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-algebras and $B \otimes_{k[x_1,\ldots,x_d]} (R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p})$ factors through one of these.

Ad (26). The final twist of the proof is that we cannot just use the map $B \to D'$ which maps x_i to the image of π_i' in D' and z_{ij} to the image of λ_{ij} in D' because we need the diagram

$$\begin{tikzcd}
B \arrow[r] & D'[t_1, \ldots, t_d] \\
k[x_1, \ldots, x_d] \arrow[u] \arrow[r] & R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p} \arrow[u]
\end{tikzcd}$$

to commute and we need the composition $B \to D'[t_1, \ldots, t_d] \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ to be the map of (14). This requires us to map x_i to the image of $\pi_i t_i$ in $D'[t_1, \ldots, t_d]$. Hence we map z_{ij} to the image of $\lambda_{ij} t_i^{2N}/\delta_i^{2N}$ in $D'[t_1, \ldots, t_d]$ and everything is clear. $\square$

16.12. The main theorem

07GB In this section we wrap up the discussion.

07GC Theorem 16.12.1 (Popescu). Any regular homomorphism of Noetherian rings is a filtered colimit of smooth ring maps.

Proof. By Lemma 16.8.4 it suffices to prove this for $k \to \Lambda$ where Λ is Noetherian and geometrically regular over k. Let $k \to A \to \Lambda$ be a factorization with A a finite type k-algebra. It suffices to construct a factorization $A \to B \to \Lambda$ with B of finite type such that $\mathfrak{h}_B = \Lambda$, see Lemma 16.2.8. Hence we may perform Noetherian induction on the ideal $\mathfrak{h}_A$. Pick a prime $\mathfrak{q} \supset \mathfrak{h}_A$ such that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$. It now suffices to resolve $k \to A \to \Lambda \supset \mathfrak{q}$ (as defined in the text following Situation 16.9.1). If the characteristic of k is zero, this follows from Lemma 16.10.3. If the characteristic of k is $p > 0$, this follows from Lemma 16.11.4. $\square$

16.13. The approximation property for G-rings

07QX Let R be a Noetherian local ring. In this case R is a G-ring if and only if the ring map $R \to R^\wedge$ is regular, see More on Algebra, Lemma 15.50.7. In this case it is true that the henselization R^h and the strict henselization R^{sh} of R are G-rings, see More on Algebra, Lemma 15.50.8. Moreover, any algebra essentially of finite type over a field, over a complete local ring, over $\mathbf{Z}$, or over a characteristic zero Dedekind ring is a G-ring, see More on Algebra, Proposition 15.50.12. This gives an ample supply of rings to which the result below applies.

Let R be a ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Let S be an R-algebra. In this situation we say a vector $(a_1, \ldots, a_n) \in S^n$ is a solution in S if and only if

$$f_j(a_1, \ldots, a_n) = 0 \text{ in } S, \text{ for } j = 1, \ldots, m$$

Of course an important question in algebraic geometry is to see when systems of polynomial equations have solutions. The following theorem tells us that having solutions in the completion of a local Noetherian ring is often enough to show there exist solutions in the henselization of the ring.

07QY Theorem 16.13.1. Let R be a Noetherian local ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in (R^\wedge)^n$ is a solution in $R^\wedge$. If R is a henselian G-ring, then for every integer N there exists a solution $(b_1, \ldots, b_n) \in R^n$ in R such that $a_i - b_i \in \mathfrak{m}^N R^\wedge$.

Proof. Let $c_i \in R$ be an element such that $a_i - c_i \in \mathfrak{m}^N$. Choose generators $\mathfrak{m}^N = (d_1, \ldots, d_M)$. Write $a_i = c_i + \sum a_{i,l} d_l$. Consider the polynomial ring $R[x_{i,l}]$ and the elements

$$g_j = f_j(c_1 + \sum x_{1,l} d_l, \ldots, c_n + \sum x_{n,l} d_{n,l}) \in R[x_{i,l}]$$

The system of equations $g_j = 0$ has the solution $(a_{i,l})$. Suppose that we can show that g_j as a solution $(b_{i,l})$ in R. Then it follows that $b_i = c_i + \sum b_{i,l} d_l$ is a solution of $f_j = 0$ which is congruent to a_i modulo $\mathfrak{m}^N$. Thus it suffices to show that solvability over $R^\wedge$ implies solvability over R.

Let $A \subset R^\wedge$ be the R-subalgebra generated by $a_1, \ldots, a_n$. Since we've assumed R is a G-ring, i.e., that $R \to R^\wedge$ is regular, we see that there exists a factorization

$$A \to B \to R^\wedge$$

with B smooth over R, see Theorem 16.12.1. Denote $\kappa = R/\mathfrak m$ the residue field. It is also the residue field of $R^\wedge$, so we get a commutative diagram

$$\begin{tikzcd} B \arrow[r, dotted] \arrow[rd] & R' \arrow[d, dotted] \\ R \arrow[u] \arrow[r] & \kappa \end{tikzcd}$$

Since the vertical arrow is smooth, More on Algebra, Lemma 15.9.14 implies that there exists an étale ring map $R \to R'$ which induces an isomorphism $R/\mathfrak m \to R'/\mathfrak m R'$ and an R-algebra map $B \to R'$ making the diagram above commute. Since R is henselian we see that $R \to R'$ has a section, see Algebra, Lemma 10.153.3. Let $b_i \in R$ be the image of a_i under the ring maps $A \to B \to R' \to R$. Since all of these maps are R-algebra maps, we see that $(b_1, \ldots, b_n)$ is a solution in R. $\square$

Given a Noetherian local ring $(R, \mathfrak m)$, an étale ring map $R \to R'$, and a maximal ideal $\mathfrak m' \subset R'$ lying over $\mathfrak m$ with $\kappa(\mathfrak m) = \kappa(\mathfrak m')$, then we have inclusions

$$R \subset R_{\mathfrak m'} \subset R^h \subset R^\wedge,$$

by Algebra, Lemma 10.155.5 and More on Algebra, Lemma 15.45.3.

07QZ Theorem 16.13.2. Let R be a Noetherian local ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in (R^\wedge)^n$ is a solution. If R is a G-ring, then for every integer N there exist

(1) an étale ring map $R \to R'$,
(2) a maximal ideal $\mathfrak m' \subset R'$ lying over $\mathfrak m$
(3) a solution $(b_1, \ldots, b_n) \in (R')^n$ in R'

such that $\kappa(\mathfrak m) = \kappa(\mathfrak m')$ and $a_i - b_i \in (\mathfrak m')^N R^\wedge$.

Proof. We could deduce this theorem from Theorem 16.13.1 using that the henselization R^h is a G-ring by More on Algebra, Lemma 15.50.8 and writing R^h as a directed colimit of étale extension R'. Instead we prove this by redoing the proof of the previous theorem in this case.

Let $c_i \in R$ be an element such that $a_i - c_i \in \mathfrak m^N$. Choose generators $\mathfrak m^N = (d_1, \ldots, d_M)$. Write $a_i = c_i + \sum a_{i,l} d_l$. Consider the polynomial ring $R[x_{i,l}]$ and the elements

$$g_j = f_j(c_1 + \sum x_{1,l} d_l, \ldots, c_n + \sum x_{n,l} d_{n,l}) \in R[x_{i,l}]$$

The system of equations $g_j = 0$ has the solution $(a_{i,l})$. Suppose that we can show that g_j as a solution $(b_{i,l})$ in R' for some étale ring map $R \to R'$ endowed with a maximal ideal $\mathfrak m'$ such that $\kappa(\mathfrak m) = \kappa(\mathfrak m')$. Then it follows that $b_i = c_i + \sum b_{i,l} d_l$ is a solution of $f_j = 0$ which is congruent to a_i modulo $(\mathfrak m')^N$. Thus it suffices to show that solvability over $R^\wedge$ implies solvability over some étale ring extension which induces a trivial residue field extension at some prime over $\mathfrak m$.

Let $A \subset R^\wedge$ be the R-subalgebra generated by $a_1, \ldots, a_n$. Since we've assumed R is a G-ring, i.e., that $R \to R^\wedge$ is regular, we see that there exists a factorization

$$A \to B \to R^\wedge$$

with B smooth over R, see Theorem 16.12.1. Denote $\kappa = R/\mathfrak{m}$ the residue field. It is also the residue field of $R^\wedge$, so we get a commutative diagram

$$\begin{tikzcd} B \arrow[r, dotted] \arrow[rd] & R' \arrow[d, dotted] \\ R \arrow[u] \arrow[r] & \kappa \end{tikzcd}$$

Since the vertical arrow is smooth, More on Algebra, Lemma 15.9.14 implies that there exists an étale ring map $R \to R'$ which induces an isomorphism $R/\mathfrak{m} \to R'/\mathfrak{m}R'$ and an R-algebra map $B \to R'$ making the diagram above commute. Let $b_i \in R'$ be the image of a_i under the ring maps $A \to B \to R'$. Since all of these maps are R-algebra maps, we see that $(b_1, \ldots, b_n)$ is a solution in R'. □

0A1W Example 16.13.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring with henselization R^h. The map on completions $R^\wedge \to (R^h)^\wedge$ is an isomorphism, see More on Algebra, Lemma 15.45.3. Since also R^h is Noetherian (ibid.) we may think of R^h as a subring of its completion (because the completion is faithfully flat). In this way we see that we may identify R^h with a subring of $R^\wedge$.

Let us try to understand which elements of $R^\wedge$ are in R^h. For simplicity we assume R is a domain with fraction field K. Clearly, every element f of R^h is algebraic over R, in the sense that there exists an equation of the form $a_n f^n + \ldots + a_1 f + a_0 = 0$ for some $a_i \in R$ with $n > 0$ and $a_n \neq 0$.

Conversely, assume that $f \in R^\wedge$, $n \in \mathbf{N}$, and $a_0, \ldots, a_n \in R$ with $a_n \neq 0$ such that $a_n f^n + \ldots + a_1 f + a_0 = 0$. If R is a G-ring, then, for every $N > 0$ there exists an element $g \in R^h$ with $a_n g^n + \ldots + a_1 g + a_0 = 0$ and $f - g \in \mathfrak{m}^N R^\wedge$, see Theorem 16.13.2. We'd like to conclude that $f = g$ when $N \gg 0$. If this is not true, then we find infinitely many roots g of $P(T)$ in R^h. This is impossible because (1) $R^h \subset R^h \otimes_R K$ and (2) $R^h \otimes_R K$ is a finite product of field extensions of K. Namely, $R \to K$ is injective and $R \to R^h$ is flat, hence $R^h \to R^h \otimes_R K$ is injective and (2) follows from More on Algebra, Lemma 15.45.13.

Conclusion: If R is a Noetherian local domain with fraction field K and a G-ring, then $R^h \subset R^\wedge$ is the set of all elements which are algebraic over K.

Here is another variant of the main theorem of this section.

0CAR Lemma 16.13.4. Let R be a Noetherian ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in ((R_\mathfrak{p})^\wedge)^n$ is a solution. If $R_\mathfrak{p}$ is a G-ring, then for every integer N there exist

1. an étale ring map $R \to R'$,
2. a prime ideal $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$
3. a solution $(b_1, \ldots, b_n) \in (R')^n$ in R'

such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$ and $a_i - b_i \in (\mathfrak{p}')^N (R'_{\mathfrak{p}'})^\wedge$.

Proof. By Theorem 16.13.2 we can find a solution $(b'_1, \ldots, b'_n)$ in some ring R'' étale over $R_\mathfrak{p}$ which comes with a prime ideal $\mathfrak{p}''$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}'')$ and $a_i - b'_i \in (\mathfrak{p}'')^N (R''_{\mathfrak{p}''})^\wedge$. We can write $R'' = R' \otimes_R R_\mathfrak{p}$ for some étale R-algebra R' (see Algebra, Lemma 10.143.3). After replacing R' by a principal localization if necessary we may assume $(b'_1, \ldots, b'_n)$ come from a solution $(b_1, \ldots, b_n)$ in R'. Setting $\mathfrak{p}' = R' \cap \mathfrak{p}''$ we see that $R''_{\mathfrak{p}''} = R'_{\mathfrak{p}'}$ which finishes the proof. □

16.14. Approximation for henselian pairs

0AH4 We can generalize the discussion of Section 16.13 to the case of henselian pairs. Henselian pairs where defined in More on Algebra, Section 15.11.

0AH5 Lemma 16.14.1. Let (A, I) be a henselian pair with A Noetherian. Let $A^\wedge$ be the I-adic completion of A. Assume at least one of the following conditions holds

(1) $A \to A^\wedge$ is a regular ring map,
(2) A is a Noetherian G-ring, or
(3) (A, I) is the henselization (More on Algebra, Lemma 15.12.1) of a pair (B, J) where B is a Noetherian G-ring.

Given $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ and $\hat{a}_1, \ldots, \hat{a}_n \in A^\wedge$ such that $f_j(\hat{a}_1, \ldots, \hat{a}_n) = 0$ for $j = 1, \ldots, m$, for every $N \geq 1$ there exist $a_1, \ldots, a_n \in A$ such that $\hat{a}_i - a_i \in I^N$ and such that $f_j(a_1, \ldots, a_n) = 0$ for $j = 1, \ldots, m$.

Proof. By More on Algebra, Lemma 15.50.15 we see that (3) implies (2). By More on Algebra, Lemma 15.50.14 we see that (2) implies (1). Thus it suffices to prove the lemma in case $A \to A^\wedge$ is a regular ring map.

Let $\hat{a}_1, \ldots, \hat{a}_n$ be as in the statement of the lemma. By Theorem 16.12.1 we can find a factorization $A \to B \to A^\wedge$ with $A \to P$ smooth and $b_1, \ldots, b_n \in B$ with $f_j(b_1, \ldots, b_n) = 0$ in B. Denote $\sigma : B \to A^\wedge \to A/I^N$ the composition. By More on Algebra, Lemma 15.9.14 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I^N \to A'/I^N A'$ and an A-algebra map $\tilde{\sigma} : B \to A'$ lifting σ. Since (A, I) is henselian, there is an A-algebra map $\chi : A' \to A$, see More on Algebra, Lemma 15.11.6. Then setting $a_i = \chi(\tilde{\sigma}(b_i))$ gives a solution. $\square$

16.15. Other chapters

CHAPTER 17

Sheaves of Modules

01AC 17.1. Introduction

01AD In this chapter we work out basic notions of sheaves of modules. This in particular includes the case of abelian sheaves, since these may be viewed as sheaves of $\underline{\mathbf{Z}}$-modules. Basic references are [Ser55b], [DG67] and [AGV71].

We work out what happens for sheaves of modules on ringed topoi in another chapter (see Modules on Sites, Section 18.1), although there we will mostly just duplicate the discussion from this chapter.

17.2. Pathology

01AE A ringed space is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}$. We allow $\mathcal{O} = 0$ in the definition. In this case the category of modules has a single object (namely 0). It is still an abelian category etc, but it is a little degenerate. Similarly the sheaf $\mathcal{O}$ may be zero over open subsets of X, etc.

This doesn't happen when considering locally ringed spaces (as we will do later).

17.3. The abelian category of sheaves of modules

01AF Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Definition 6.25.1. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, see Sheaves, Definition 6.10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subset X$ is the sum of the maps induced by φ, ψ. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus $\text{Mod}(\mathcal{O}_X)$ is a pre-additive category, see Homology, Definition 12.3.1.

We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subset X$. Clearly this is both a final and an initial object of $\text{Mod}(\mathcal{O}_X)$. Given a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each open U, and (d) $\varphi_x = 0$ for all $x \in X$. See Sheaves, Lemma 6.16.1.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 12.3.5. Thus $\text{Mod}(\mathcal{O}_X)$ is an additive category, see Homology, Definition 12.3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define $\text{Ker}(\varphi)$ to be the subsheaf of $\mathcal{F}$ with sections

$$\text{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all open $U \subset X$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}_X$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\text{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$. Moreover, on the level of stalks we have $\text{Ker}(\varphi)_x = \text{Ker}(\varphi_x)$.

On the other hand, we define $\text{Coker}(\varphi)$ as the sheaf of $\mathcal{O}_X$-modules associated to the presheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \text{Coker}(\mathcal{G}(U) \to \mathcal{F}(U)) = \mathcal{F}(U)/\varphi(\mathcal{G}(U)).$$

Since taking stalks commutes with taking sheafification, see Sheaves, Lemma 6.17.2 we see that $\text{Coker}(\varphi)_x = \text{Coker}(\varphi_x)$. Thus the map $\mathcal{G} \to \text{Coker}(\varphi)$ is surjective (as a map of sheaves of sets), see Sheaves, Section 6.16. To show that this is a cokernel, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}_X$-modules such that $\beta \circ \varphi$ is zero, then you get for every open $U \subset X$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 6.20.1) we obtain a canonical map $\text{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \text{Coker}(\varphi) \to \mathcal{H}$. The morphism $\text{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

01AG Lemma 17.3.1. Let $(X, \mathcal{O}_X)$ be a ringed space. The category $\text{Mod}(\mathcal{O}_X)$ is an abelian category. Moreover a complex

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact at $\mathcal{G}_x$.

Proof. By Homology, Definition 12.5.1 we have to show that image and coimage agree. By Sheaves, Lemma 6.16.1 it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathcal{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. □

Actually the category $\text{Mod}(\mathcal{O}_X)$ has many more properties. Here are two constructions we can do.

(1) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the product
$$\prod\nolimits_{i \in I} \mathcal{F}_i$$
which is the sheaf that associates to each open U the product of the modules $\mathcal{F}_i(U)$. This is also the categorical product, as in Categories, Definition 4.14.6.
(2) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the direct sum
$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i$$
which is the sheafification of the presheaf that associates to each open U the direct sum of the modules $\mathcal{F}_i(U)$. This is also the categorical coproduct, as in Categories, Definition 4.14.7. To see this you use the universal property of sheafification.

Using these we conclude that all limits and colimits exist in $\text{Mod}(\mathcal{O}_X)$.

01AH Lemma 17.3.2. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) All limits exist in $\text{Mod}(\mathcal{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathcal{O}_X$-modules (i.e., commute with taking sections over opens).
(2) All colimits exist in $\text{Mod}(\mathcal{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.
(3) Filtered colimits are exact.
(4) Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathcal{O}_X$-modules.

Proof. As $\text{Mod}(\mathcal{O}_X)$ is abelian (Lemma 17.3.1) it has all finite limits and colimits (Homology, Lemma 12.5.5). Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description (see discussion above) and Categories, Lemmas 4.14.11 and 4.14.12. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks. Part (3) signifies that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of $\mathcal{O}_X$-modules over a directed set I the sequence $0 \to \operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact as well. Since we can check exactness on stalks (Lemma 17.3.1) this follows from the case of modules which is Algebra, Lemma 10.8.8. We omit the proof of (4). $\square$

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 4.23. See Homology, Lemma 12.7.2 for a description of exactness properties in terms of short exact sequences.

01AJ Lemma 17.3.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) The functor $f_* : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_Y)$ is left exact. In fact it commutes with all limits.
(2) The functor $f^* : \text{Mod}(\mathcal{O}_Y) \to \text{Mod}(\mathcal{O}_X)$ is right exact. In fact it commutes with all colimits.
(3) Pullback $f^{-1} : \text{Ab}(Y) \to \text{Ab}(X)$ on abelian sheaves is exact.

Proof. Parts (1) and (2) hold because (f^*, f_*) is an adjoint pair of functors, see Sheaves, Lemma 6.26.2 and Categories, Section 4.24. Part (3) holds because exactness can be checked on stalks (Lemma 17.3.1) and the description of stalks of the pullback, see Sheaves, Lemma 6.22.1. $\square$

01AK Lemma 17.3.4. Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_! : \text{Ab}(U) \to \text{Ab}(X)$ is exact.

Proof. Follows from the description of stalks given in Sheaves, Lemma 6.31.6. $\square$

01AI Lemma 17.3.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathcal{F}_i$ be a sheaf of $\mathcal{O}_X$-modules. For $U \subset X$ quasi-compact open the map

$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i(U) \longrightarrow \left(\bigoplus\nolimits_{i \in I} \mathcal{F}_i\right)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j \in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i \in I_j} s_{ji}$ with $s_{ji} \in \mathcal{F}_i(U_j)$. Because

U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j \in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i \in I'} \mathcal{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed, see Lemma 17.3.2 above. □

17.4. Sections of sheaves of modules

01AL Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $s \in \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ be a global section. There is a unique map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow \mathcal{F},\ f \longmapsto fs$$

associated to s. The notation above signifies that a local section f of $\mathcal{O}_X$, i.e., a section f over some open U, is mapped to the multiplication of f with the restriction of s to U. Conversely, any map $\varphi : \mathcal{O}_X \to \mathcal{F}$ gives rise to a section $s = \varphi(1)$ such that φ is the morphism associated to s.

01AM Definition 17.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is generated by global sections if there exist a set I, and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$$

which is the map associated to s_i on the summand corresponding to i, is surjective. In this case we say that the sections s_i generate $\mathcal{F}$.

We often use the abuse of notation introduced in Sheaves, Section 6.11 where, given a local section s of $\mathcal{F}$ defined in an open neighbourhood of a point $x \in X$, we denote s_x, or even s the image of s in the stalk $\mathcal{F}_x$.

01AN Lemma 17.4.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ be global sections. The sections s_i generate $\mathcal{F}$ if and only if for all $x \in X$ the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.

Proof. Omitted. □

01AO Lemma 17.4.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Proof. Omitted. □

01AP Lemma 17.4.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let s_i, $i \in I$ be a collection of local sections of $\mathcal{F}$, i.e., $s_i \in \mathcal{F}(U_i)$ for some opens $U_i \subset X$. There exists a unique smallest subsheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ such that each s_i corresponds to a local section of $\mathcal{G}$.

Proof. Consider the subpresheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \{\text{sums } \sum\nolimits_{i \in J} f_i(s_i|_U) \text{ where } J \text{ is finite, } U \subset U_i \text{ for } i \in J, \text{ and } f_i \in \mathcal{O}_X(U)\}$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf. This is a subsheaf of $\mathcal{F}$ by Sheaves, Lemma 6.16.3. Since all the finite sums clearly have to be in $\mathcal{G}$ this is the smallest subsheaf as desired. □

01AQ Definition 17.4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$ we say that the subsheaf $\mathcal{G}$ of Lemma 17.4.4 above is the subsheaf generated by the s_i.

01AR Lemma 17.4.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$. Let $\mathcal{G}$ be the subsheaf generated by the s_i and let $x \in X$. Then $\mathcal{G}_x$ is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $s_{i,x}$ for those i such that s_i is defined at x.

Proof. This is clear from the construction of $\mathcal{G}$ in the proof of Lemma 17.4.4. □

17.5. Supports of modules and sections

01AS

01AT Definition 17.5.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) The support of $\mathcal{F}$ is the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$.
(2) We denote $\text{Supp}(\mathcal{F})$ the support of $\mathcal{F}$.
(3) Let $s \in \Gamma(X, \mathcal{F})$ be a global section. The support of s is the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of s is not zero.

Of course the support of a local section is then defined also since a local section is a global section of the restriction of $\mathcal{F}$.

01AU Lemma 17.5.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ open.

(1) The support of $s \in \mathcal{F}(U)$ is closed in U.
(2) The support of fs is contained in the intersections of the supports of $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$.
(3) The support of $s + s'$ is contained in the union of the supports of $s, s' \in \mathcal{F}(U)$.
(4) The support of $\mathcal{F}$ is the union of the supports of all local sections of $\mathcal{F}$.
(5) If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then the support of $\varphi(s)$ is contained in the support of $s \in \mathcal{F}(U)$.

Proof. This is true because if $s_x = 0$, then s is zero in an open neighbourhood of x by definition of stalks. Similarly for f. Details omitted. □

In general the support of a sheaf of modules is not closed. Namely, the sheaf could be an abelian sheaf on $\mathbf{R}$ (with the usual archimedean topology) which is the direct sum of infinitely many nonzero skyscraper sheaves each supported at a single point p_i of $\mathbf{R}$. Then the support would be the set of points p_i which may not be closed.

Another example is to consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 6.31 the support of this sheaf is exactly U.

01AV Lemma 17.5.3. Let X be a topological space. The support of a sheaf of rings is closed.

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

17.6. Closed immersions and abelian sheaves

01AW Recall that we think of an abelian sheaf on a topological space X as a sheaf of $\underline{\mathbf{Z}}_X$-modules. Thus we may apply any results, definitions for sheaves of modules to abelian sheaves.

01AX Lemma 17.6.1. Let X be a topological space. Let $Z \subset X$ be a closed subset. Denote $i : Z \to X$ the inclusion map. The functor

$$i_* : \text{Ab}(Z) \longrightarrow \text{Ab}(X)$$

is exact, fully faithful, with essential image exactly those abelian sheaves whose support is contained in Z. The functor i^{-1} is a left inverse to i_*.

Proof. Exactness follows from the description of stalks in Sheaves, Lemma 6.32.1 and Lemma 17.3.1. The rest was shown in Sheaves, Lemma 6.32.3. □

Let $\mathcal{F}$ be an abelian sheaf on the topological space X. Given a closed subset Z, there is a canonical abelian subsheaf of $\mathcal{F}$ which consists of exactly those sections whose support is contained in Z. Here is the exact statement.

01AY Remark 17.6.2. Let X be a topological space. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be an abelian sheaf on X. For $U \subset X$ open set

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{ the support of } s \text{ is contained in } Z \cap U\}$$

Then $\mathcal{H}_Z(\mathcal{F})$ is an abelian subsheaf of $\mathcal{F}$. It is the largest abelian subsheaf of $\mathcal{F}$ whose support is contained in Z. By Lemma 17.6.1 we may (and we do) view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on Z. In this way we obtain a left exact functor

$$\text{Ab}(X) \longrightarrow \text{Ab}(Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as abelian sheaf on } Z$$

All of the statements made above follow directly from Lemma 17.5.2.

This seems like a good opportunity to show that the functor i_* has a right adjoint on abelian sheaves.

01AZ Lemma 17.6.3. Let $i : Z \to X$ be the inclusion of a closed subset into the topological space X. The functor $\text{Ab}(X) \to \text{Ab}(Z)$, $\mathcal{F} \mapsto \mathcal{H}_Z(\mathcal{F})$ of Remark 17.6.2 is a right adjoint to $i_* : \text{Ab}(Z) \to \text{Ab}(X)$. In particular i_* commutes with arbitrary colimits.

Proof. We have to show that for any abelian sheaf $\mathcal{F}$ on X and any abelian sheaf $\mathcal{G}$ on Z we have

$$\text{Hom}_{\text{Ab}(X)}(i_*\mathcal{G}, \mathcal{F}) = \text{Hom}_{\text{Ab}(Z)}(\mathcal{G}, \mathcal{H}_Z(\mathcal{F}))$$

This is clear because after all any section of $i_*\mathcal{G}$ has support in Z. Details omitted. □

01B0 Remark 17.6.4. In Sheaves, Remark 6.32.5 we showed that i_* as a functor on the categories of sheaves of sets does not have a right adjoint simply because it is not exact. However, it is very close to being true, in fact, the functor i_* is exact on sheaves of pointed sets, sections with support in Z can be defined for sheaves of pointed sets, and $\mathcal{H}_Z$ makes sense and is a right adjoint to i_*.

17.7. A canonical exact sequence

02US We give this exact sequence its own section.

02UT Lemma 17.7.1. Let X be a topological space. Let $U \subset X$ be an open subset with complement $Z \subset X$. Denote $j : U \to X$ the open immersion and $i : Z \to X$ the closed immersion. For any sheaf of abelian groups $\mathcal{F}$ on X the adjunction mappings $j_!j^{-1}\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^{-1}\mathcal{F}$ give a short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

of sheaves of abelian groups. For any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of abelian sheaves on X we obtain a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & j_!j^{-1}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & i_*i^{-1}\mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & j_!j^{-1}\mathcal{G} \arrow[r] & \mathcal{G} \arrow[r] & i_*i^{-1}\mathcal{G} \arrow[r] & 0 \end{tikzcd}$$

Proof. The functoriality of the short exact sequence is immediate from the naturality of the adjunction mappings. We may check exactness on stalks (Lemma 17.3.1). For a description of the stalks in question see Sheaves, Lemmas 6.31.6 and 6.32.1. □

17.8. Modules locally generated by sections

01B1 Let $(X, \mathcal{O}_X)$ be a ringed space. In this and the following section we will often restrict sheaves to open subspaces $U \subset X$, see Sheaves, Section 6.31. In particular, we will often denote the open subspace by $(U, \mathcal{O}_U)$ instead of the more correct notation $(U, \mathcal{O}_X|_U)$, see Sheaves, Definition 6.31.2.

Consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 6.31 the stalk of $j_!\underline{\mathbf{Z}}_U$ at $x = 0$ is 0. In fact the sections of this sheaf over any open interval containing 0 are 0. Thus there is no open neighbourhood of the point 0 over which the sheaf can be generated by sections.

01B2 Definition 17.8.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is locally generated by sections if for every $x \in X$ there exists an open neighbourhood U of x such that $\mathcal{F}|_U$ is globally generated as a sheaf of $\mathcal{O}_U$-modules.

In other words there exists a set I and for each i a section $s_i \in \mathcal{F}(U)$ such that the associated map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U$$

is surjective.

01B3 Lemma 17.8.2. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ is locally generated by sections if $\mathcal{G}$ is locally generated by sections.

Proof. Given an open subspace V of Y we may consider the commutative diagram of ringed spaces

$$\begin{tikzcd} (f^{-1}V, \mathcal{O}_{f^{-1}V}) \arrow[r, "j'"'] \arrow[d, "f'"'] & (X, \mathcal{O}_X) \arrow[d, "f"] \\ (V, \mathcal{O}_V) \arrow[r, "j"] & (Y, \mathcal{O}_Y) \end{tikzcd}$$

We know that $f^*\mathcal{G}|_{f^{-1}V} \cong (f')^*(\mathcal{G}|_V)$, see Sheaves, Lemma 6.26.3. Thus we may assume that $\mathcal{G}$ is globally generated.

We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. $\square$

17.9. Modules of finite type

01B4

01B5 Definition 17.9.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of finite type if for every $x \in X$ there exists an open neighbourhood U such that $\mathcal{F}|_U$ is generated by finitely many sections.

01B6 Lemma 17.9.2. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a finite type $\mathcal{O}_Y$-module is a finite type $\mathcal{O}_X$-module.

Proof. Arguing as in the proof of Lemma 17.8.2 we may assume $\mathcal{G}$ is globally generated by finitely many sections. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. $\square$

01B7 Lemma 17.9.3. Let X be a ringed space. The image of a morphism of $\mathcal{O}_X$-modules of finite type is of finite type. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite type, so is $\mathcal{F}_2$.

Proof. The statement on images is trivial. The statement on short exact sequences comes from the fact that sections of $\mathcal{F}_3$ locally lift to sections of $\mathcal{F}_2$ and the corresponding result in the category of modules over a ring (applied to the stalks for example). $\square$

01B8 Lemma 17.9.4. Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{F}$ of finite type and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ surjective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is surjective.

Proof. Choose an open neighbourhood $U \subset X$ of x such that $\mathcal{F}$ is generated by $s_1, \ldots, s_n \in \mathcal{F}(U)$ over U. By assumption of surjectivity of φ_x, after shrinking U we may assume that $s_i = \varphi(t_i)$ for some $t_i \in \mathcal{G}(U)$. Then U works. $\square$

01B9 Lemma 17.9.5. Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$. Assume $\mathcal{F}$ of finite type and $\mathcal{F}_x = 0$. Then there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is zero.

Proof. This is a special case of Lemma 17.9.4 applied to the morphism $0 \to \mathcal{F}$. $\square$

01BA Lemma 17.9.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then support of $\mathcal{F}$ is closed.

Proof. This is a reformulation of Lemma 17.9.5. $\square$

01BB Lemma 17.9.7. Let X be a ringed space. Let I be a preordered set and let $(\mathcal{F}_i, f_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 4.21). Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be the colimit. Assume (a) I is directed, (b) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and (c) X is quasi-compact. Then there exists an i such that $\mathcal{F}_i \to \mathcal{F}$ is surjective. If the transition maps $f_{ii'}$ are injective then we conclude that $\mathcal{F} = \mathcal{F}_i$ for some $i \in I$.

Proof. Let $x \in X$. There exists an open neighbourhood $U \subset X$ of x and finitely many sections $s_j \in \mathcal{F}(U)$, $j = 1, \ldots, m$ such that $s_1, \ldots, s_m$ generate $\mathcal{F}$ as $\mathcal{O}_U$-module. After possibly shrinking U to a smaller open neighbourhood of x we may assume that each s_j comes from a section of $\mathcal{F}_i$ for some $i \in I$. Hence, since X is quasi-compact we can find a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$, and for each j an index i_j and finitely many sections $s_{jl} \in \mathcal{F}_{i_j}(U_j)$ whose images generate the restriction of $\mathcal{F}$ to U_j. Clearly, the lemma holds for any index $i \in I$ which is $\geq$ all i_j. $\square$

01BC Lemma 17.9.8. Let X be a ringed space. There exists a set of $\mathcal{O}_X$-modules $\{\mathcal{F}_i\}_{i \in I}$ of finite type such that each finite type $\mathcal{O}_X$-module on X is isomorphic to exactly one of the $\mathcal{F}_i$.

Proof. For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that each restriction $\mathcal{F}|_{U_j}$ is a quotient of $\mathcal{O}_{U_j}^{\oplus r_j}$ for some $r_j \geq 0$. These are parametrized by subsheaves $\mathcal{K}_j \subset \mathcal{O}_{U_j}^{\oplus r_j}$ and glueing data

$$\varphi_{jj'} : \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_j} / (\mathcal{K}_j|_{U_j \cap U_{j'}}) \longrightarrow \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_{j'}} / (\mathcal{K}_{j'}|_{U_j \cap U_{j'}})$$

see Sheaves, Section 6.33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of $\mathcal{O}_X$-modules gotten from glueing quotients as above forms a set $\mathcal{I}$. By definition every finite type $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of sheaves (uses axiom of choice). $\square$

17.10. Quasi-coherent modules

01BD In this section we introduce an abstract notion of quasi-coherent $\mathcal{O}_X$-module. This notion is very useful in algebraic geometry, since quasi-coherent modules on a scheme have a good description on any affine open. However, we warn the reader that in the general setting of (locally) ringed spaces this notion is not well behaved at all. The category of quasi-coherent sheaves is not abelian in general, infinite direct sums of quasi-coherent sheaves aren't quasi-coherent, etc, etc.

01BE Definition 17.10.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_U$$

The category of quasi-coherent $\mathcal{O}_X$-modules is denoted $QCoh(\mathcal{O}_X)$.

The definition means that X is covered by open sets U such that $\mathcal{F}|_U$ has a presentation of the form

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by global sections.

01BF Lemma 17.10.2. Let $(X, \mathcal{O}_X)$ be a ringed space. The direct sum of two quasi-coherent $\mathcal{O}_X$-modules is a quasi-coherent $\mathcal{O}_X$-module.

Proof. Omitted. □

02CF Remark 17.10.3. Warning: It is not true in general that an infinite direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. For more esoteric behaviour of quasi-coherent modules see Example 17.10.9.

01BG Lemma 17.10.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a quasi-coherent $\mathcal{O}_Y$-module is quasi-coherent.

Proof. Arguing as in the proof of Lemma 17.8.2 we may assume $\mathcal{G}$ has a global presentation by direct sums of copies of $\mathcal{O}_Y$. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have an exact sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_Y \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_Y \longrightarrow \mathcal{G} \longrightarrow 0$$

then upon applying f^* we obtain the exact sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_X \longrightarrow f^*\mathcal{G} \longrightarrow 0.$$

This implies the lemma. □

This gives plenty of examples of quasi-coherent sheaves.

01BH Lemma 17.10.5. Let $(X, \mathcal{O}_X)$ be ringed space. Let $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism from a ring R into the ring of global sections on X. Let M be an R-module. The following three constructions give canonically isomorphic sheaves of $\mathcal{O}_X$-modules:

(1) Let $\pi : (X, \mathcal{O}_X) \longrightarrow (\{*\}, R)$ be the morphism of ringed spaces with $\pi : X \to \{*\}$ the unique map and with π-map $\pi^\sharp$ the given map $\alpha : R \to \Gamma(X, \mathcal{O}_X)$. Set $\mathcal{F}_1 = \pi^*M$.
(2) Choose a presentation $\bigoplus_{j\in J} R \to \bigoplus_{i\in I} R \to M \to 0$. Set
$$\mathcal{F}_2 = \text{Coker}\left(\bigoplus\nolimits_{j\in J} \mathcal{O}_X \to \bigoplus\nolimits_{i\in I} \mathcal{O}_X\right).$$
Here the map on the component $\mathcal{O}_X$ corresponding to $j \in J$ given by the section $\sum_i \alpha(r_{ij})$ where the r_{ij} are the matrix coefficients of the map in the presentation of M.
(3) Set $\mathcal{F}_3$ equal to the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_R M$, where the map $R \to \mathcal{O}_X(U)$ is the composition of α and the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.

This construction has the following properties:

(1) The resulting sheaf of $\mathcal{O}_X$-modules $\mathcal{F}_M = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$ is quasi-coherent.
(2) The construction gives a functor from the category of R-modules to the category of quasi-coherent sheaves on X which commutes with arbitrary colimits.
(3) For any $x \in X$ we have $\mathcal{F}_{M,x} = \mathcal{O}_{X,x} \otimes_R M$ functorial in M.
(4) Given any $\mathcal{O}_X$-module $\mathcal{G}$ we have

$$\mathrm{Mor}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{G}))$$

where the R-module structure on $\Gamma(X, \mathcal{G})$ comes from the $\Gamma(X, \mathcal{O}_X)$-module structure via α.

Proof. The isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_3$ comes from the fact that π^* is defined as the sheafification of the presheaf in (3), see Sheaves, Section 6.26. The isomorphism between the constructions in (2) and (1) comes from the fact that the functor π^* is right exact, so $\pi^*(\bigoplus_{j \in J} R) \to \pi^*(\bigoplus_{i \in I} R) \to \pi^* M \to 0$ is exact, π^* commutes with arbitrary direct sums, see Lemma 17.3.3, and finally the fact that $\pi^*(R) = \mathcal{O}_X$.

Assertion (1) is clear from construction (2). Assertion (2) is clear since π^* has these properties. Assertion (3) follows from the description of stalks of pullback sheaves, see Sheaves, Lemma 6.26.4. Assertion (4) follows from adjointness of π_* and π^*. $\square$

01BI Definition 17.10.6. In the situation of Lemma 17.10.5 we say $\mathcal{F}_M$ is the sheaf associated to the module M and the ring map α. If $R = \Gamma(X, \mathcal{O}_X)$ and $\alpha = \mathrm{id}_R$ we simply say $\mathcal{F}_M$ is the sheaf associated to the module M.

01BJ Lemma 17.10.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. Let $\mathcal{F}_M$ be the quasi-coherent sheaf of $\mathcal{O}_X$-modules associated to M. If $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces, then $g^*\mathcal{F}_M$ is the sheaf associated to the $\Gamma(Y, \mathcal{O}_Y)$-module $\Gamma(Y, \mathcal{O}_Y) \otimes_R M$.

Proof. The assertion follows from the first description of $\mathcal{F}_M$ in Lemma 17.10.5 as $\pi^* M$, and the following commutative diagram of ringed spaces

$$\begin{tikzcd}
(Y, \mathcal{O}_Y) \arrow[r, "\pi"'] \arrow[d, "g"'] & (\{*\}, \Gamma(Y, \mathcal{O}_Y)) \arrow[d, "\text{induced by } g^\sharp"] \\
(X, \mathcal{O}_X) \arrow[r, "\pi"] & (\{*\}, \Gamma(X, \mathcal{O}_X))
\end{tikzcd}$$

(Also use Sheaves, Lemma 6.26.3.) $\square$

01BK Lemma 17.10.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$ be a point. Assume that x has a fundamental system of quasi-compact neighbourhoods. Consider any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Then there exists an open neighbourhood U of x such that $\mathcal{F}|_U$ is isomorphic to the sheaf of modules $\mathcal{F}_M$ on $(U, \mathcal{O}_U)$ associated to some $\Gamma(U, \mathcal{O}_U)$-module M.

Proof. First we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is isomorphic to the cokernel of a map

$$\Psi : \bigoplus\nolimits_{j \in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_X.$$

The problem is that this map may not be given by a "matrix", because the module of global sections of a direct sum is in general different from the direct sum of the modules of global sections.

Let $x \in E \subset X$ be a quasi-compact neighbourhood of x (note: E may not be open). Let $x \in U \subset E$ be an open neighbourhood of x contained in E. Next, we proceed as in the proof of Lemma 17.3.5. For each $j \in J$ denote $s_j \in \Gamma(X, \bigoplus_{i \in I} \mathcal{O}_X)$ the image of the section 1 in the summand $\mathcal{O}_X$ corresponding to j. There exists a finite collection of opens U_{jk}, $k \in K_j$ such that $E \subset \bigcup_{k \in K_j} U_{jk}$ and such that each restriction $s_j|_{U_{jk}}$ is a finite sum $\sum_{i \in I_{jk}} f_{jki}$ with $I_{jk} \subset I$, and f_{jki} in the summand $\mathcal{O}_X$ corresponding to $i \in I$. Set $I_j = \bigcup_{k \in K_j} I_{jk}$. This is a finite set. Since $U \subset E \subset \bigcup_{k \in K_j} U_{jk}$ the section $s_j|_U$ is a section of the finite direct sum $\bigoplus_{i \in I_j} \mathcal{O}_X$. By Lemma 17.3.2 we see that actually $s_j|_U$ is a sum $\sum_{i \in I_j} f_{ij}$ and $f_{ij} \in \mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U)$.

At this point we can define a module M as the cokernel of the map

$$\bigoplus\nolimits_{j \in J} \Gamma(U, \mathcal{O}_U) \longrightarrow \bigoplus\nolimits_{i \in I} \Gamma(U, \mathcal{O}_U)$$

with matrix given by the (f_{ij}). By construction (2) of Lemma 17.10.5 we see that $\mathcal{F}_M$ has the same presentation as $\mathcal{F}|_U$ and therefore $\mathcal{F}_M \cong \mathcal{F}|_U$. $\square$

01BL Example 17.10.9. Let X be countably many copies $L_1, L_2, L_3, \ldots$ of the real line all glued together at 0; a fundamental system of neighbourhoods of 0 being the collection $\{U_n\}_{n \in \mathbf{N}}$, with $U_n \cap L_i = (-1/n, 1/n)$. Let $\mathcal{O}_X$ be the sheaf of continuous real valued functions. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function which is identically zero on $(-1, 1)$ and identically 1 on $(-\infty, -2) \cup (2, \infty)$. Denote f_n the continuous function on X which is equal to $x \mapsto f(nx)$ on each $L_j = \mathbf{R}$. Let 1_{L_j} be the characteristic function of L_j. We consider the map

$$\bigoplus\nolimits_{j \in \mathbf{N}} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{j,i \in \mathbf{N}} \mathcal{O}_X, \quad e_j \longmapsto \sum\nolimits_{i \in \mathbf{N}} f_j 1_{L_i} e_{ij}$$

with obvious notation. This makes sense because this sum is locally finite as f_j is zero in a neighbourhood of 0. Over U_n the image of e_j, for $j > 2n$ is not a finite linear combination $\sum g_{ij} e_{ij}$ with g_{ij} continuous. Thus there is no neighbourhood of $0 \in X$ such that the displayed map is given by a "matrix" as in the proof of Lemma 17.10.8 above.

Note that $\bigoplus_{j \in \mathbf{N}} \mathcal{O}_X$ is the sheaf associated to the free module with basis e_j and similarly for the other direct sum. Thus we see that a morphism of sheaves associated to modules in general even locally on X does not come from a morphism of modules. Similarly there should be an example of a ringed space X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}$ is not locally of the form $\mathcal{F}_M$. (Please email if you find one.) Moreover, there should be examples of locally compact spaces X and maps $\mathcal{F}_M \to \mathcal{F}_N$ which also do not locally come from maps of modules (the proof of Lemma 17.10.8 shows this cannot happen if N is free).

17.11. Modules of finite presentation

01BM Here is the definition.

01BN Definition 17.11.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of finite presentation if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$, and $n, m \in \mathbf{N}$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U$$

This means that X is covered by open sets U such that $\mathcal{F}|_U$ has a presentation of the form

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by finitely many global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by finitely many global sections.

01BO Lemma 17.11.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Any $\mathcal{O}_X$-module of finite presentation is quasi-coherent.

Proof. Immediate from definitions. □

01BP Lemma 17.11.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation.

(1) If $\psi : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ is a surjection, then $\mathrm{Ker}(\psi)$ is of finite type.
(2) If $\theta : \mathcal{G} \to \mathcal{F}$ is surjective with $\mathcal{G}$ of finite type, then $\mathrm{Ker}(\theta)$ is of finite type.

Proof. Proof of (1). Let $x \in X$. Choose an open neighbourhood $U \subset X$ of x such that there exists a presentation

$$\mathcal{O}_U^{\oplus m} \xrightarrow{\chi} \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F}|_U \to 0.$$

Let e_k be the section generating the kth factor of $\mathcal{O}_X^{\oplus r}$. For every $k = 1, \ldots, r$ we can, after shrinking U to a small neighbourhood of x, lift $\psi(e_k)$ to a section $\tilde{e}_k$ of $\mathcal{O}_U^{\oplus n}$ over U. This gives a morphism of sheaves $\alpha : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n}$ such that $\varphi \circ \alpha = \psi$. Similarly, after shrinking U, we can find a morphism $\beta : \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus r}$ such that $\psi \circ \beta = \varphi$. Then the map

$$\mathcal{O}_U^{\oplus m} \oplus \mathcal{O}_U^{\oplus r} \xrightarrow{\beta \circ \chi, 1 - \beta \circ \alpha} \mathcal{O}_U^{\oplus r}$$

is a surjection onto the kernel of ψ.

To prove (2) we may locally choose a surjection $\eta : \mathcal{O}_X^{\oplus r} \to \mathcal{G}$. By part (1) we see $\mathrm{Ker}(\theta \circ \eta)$ is of finite type. Since $\mathrm{Ker}(\theta) = \eta(\mathrm{Ker}(\theta \circ \eta))$ we win. □

01BQ Lemma 17.11.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a module of finite presentation is of finite presentation.

Proof. Exactly the same as the proof of Lemma 17.10.4 but with finite index sets. □

01BR Lemma 17.11.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. The $\mathcal{O}_X$-module $\mathcal{F}_M$ associated to M is a directed colimit of finitely presented $\mathcal{O}_X$-modules.

Proof. This follows immediately from Lemma 17.10.5 and the fact that any module is a directed colimit of finitely presented modules, see Algebra, Lemma 10.11.3. ☐

0B8J Lemma 17.11.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in X$ such that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$. Then there exists an open neighbourhood U of x such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$.

Proof. Choose $s_1, \ldots, s_r \in \mathcal{F}_x$ mapping to a basis of $\mathcal{O}_{X,x}^{\oplus r}$ by the isomorphism. Choose an open neighbourhood U of x such that s_i lifts to $s_i \in \mathcal{F}(U)$. After shrinking U we see that the induced map $\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$ is surjective (Lemma 17.9.4). By Lemma 17.11.3 we see that $\mathrm{Ker}(\psi)$ is of finite type. Then $\mathrm{Ker}(\psi)_x = 0$ implies that $\mathrm{Ker}(\psi)$ becomes zero after shrinking U once more (Lemma 17.9.5). ☐

17.12. Coherent modules

01BU A reference for this section is [Ser55b].

The category of coherent sheaves on a ringed space X is a more reasonable object than the category of quasi-coherent sheaves, in the sense that it is at least an abelian subcategory of $\mathrm{Mod}(\mathcal{O}_X)$ no matter what X is. On the other hand, the pullback of a coherent module is "almost never" coherent in the general setting of ringed spaces.

01BV Definition 17.12.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module if the following two conditions hold:

(1) $\mathcal{F}$ is of finite type, and
(2) for every open $U \subset X$ and every finite collection $s_i \in \mathcal{F}(U)$, $i = 1, \ldots, n$ the kernel of the associated map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type.

The category of coherent $\mathcal{O}_X$-modules is denoted $\mathrm{Coh}(\mathcal{O}_X)$.

01BW Lemma 17.12.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Any coherent $\mathcal{O}_X$-module is of finite presentation and hence quasi-coherent.

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Pick a point $x \in X$. By (1) of the definition of coherent, we may find an open neighbourhood U and sections s_i, $i = 1, \ldots, n$ of $\mathcal{F}$ over U such that $\Psi : \bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}$ is surjective. By (2) of the definition of coherent, we may find an open neighbourhood V, $x \in V \subset U$ and sections $t_1, \ldots, t_m$ of $\bigoplus_{i=1,\ldots,n} \mathcal{O}_V$ which generate the kernel of $\Psi|_V$. Then over V we get the presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}|_V \to 0$$

as desired. ☐

01BX Example 17.12.3. Suppose that X is a point. In this case the definition above gives a notion for modules over rings. What does the definition of coherent mean? It is closely related to the notion of Noetherian, but it is not the same: Namely, the ring $R = \mathbf{C}[x_1, x_2, x_3, \ldots]$ is coherent as a module over itself but not Noetherian as a module over itself. See Algebra, Section 10.90 for more discussion.

01BY Lemma 17.12.4. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) Any finite type subsheaf of a coherent sheaf is coherent.
(2) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism from a finite type sheaf $\mathcal{F}$ to a coherent sheaf $\mathcal{G}$. Then $\mathrm{Ker}(\varphi)$ is of finite type.

(3) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules. Then $\text{Ker}(\varphi)$ and $\text{Coker}(\varphi)$ are coherent.
(4) Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are coherent so is the third.
(5) The category $\text{Coh}(\mathcal{O}_X)$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_X)$. In particular, the category of coherent modules is abelian and the inclusion functor $\text{Coh}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ is exact.

Proof. Condition (2) of Definition 17.12.1 holds for any subsheaf of a coherent sheaf. Thus we get (1).

Assume the hypotheses of (2). Let us show that $\text{Ker}(\varphi)$ is of finite type. Pick $x \in X$. Choose an open neighbourhood U of x in X such that $\mathcal{F}|_U$ is generated by $s_1, \ldots, s_n$. By Definition 17.12.1 the kernel $\mathcal{K}$ of the induced map $\bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$, $e_i \mapsto \varphi(s_i)$ is of finite type. Hence $\text{Ker}(\varphi)$ which is the image of the composition $\mathcal{K} \to \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{F}$ is of finite type.

Assume the hypotheses of (3). By (2) the kernel of φ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that $\text{Coker}(\varphi)$ is coherent. Since $\mathcal{G}$ is of finite type so is $\text{Coker}(\varphi)$. Let $U \subset X$ be open and let $\overline{s}_i \in \text{Coker}(\varphi)(U)$, $i = 1, \ldots, n$ be sections. We have to show that the kernel of the associated morphism $\overline{\Psi} : \bigoplus_{i=1}^n \mathcal{O}_U \to \text{Coker}(\varphi)$ is of finite type. There exists an open covering of U such that on each open all the sections $\overline{s}_i$ lift to sections s_i of $\mathcal{G}$. Hence we may assume this is the case over U. We may in addition assume there are sections t_j, $j = 1, \ldots, m$ of $\text{Im}(\varphi)$ over U which generate $\text{Im}(\varphi)$ over U. Let $\Phi : \bigoplus_{j=1}^m \mathcal{O}_U \to \text{Im}(\varphi)$ be defined using t_j and $\Psi : \bigoplus_{j=1}^m \mathcal{O}_U \oplus \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$ using t_j and s_i. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{j=1}^m \mathcal{O}_U \arrow[r] \arrow[d, "\Phi"'] & \bigoplus_{j=1}^m \mathcal{O}_U \oplus \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d, "\Psi"'] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d, "\overline{\Psi}"'] & 0 \\
0 \arrow[r] & \text{Im}(\varphi) \arrow[r] & \mathcal{G} \arrow[r] & \text{Coker}(\varphi) \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we get an exact sequence $\text{Ker}(\Psi) \to \text{Ker}(\overline{\Psi}) \to 0$. Since $\text{Ker}(\Psi)$ is a finite type module, we see that $\text{Ker}(\overline{\Psi})$ has finite type.

Proof of part (4). Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. By part (3) it suffices to prove that if $\mathcal{F}_1$ and $\mathcal{F}_3$ are coherent so is $\mathcal{F}_2$. By Lemma 17.9.3 we see that $\mathcal{F}_2$ has finite type. Let $s_1, \ldots, s_n$ be finitely many local sections of $\mathcal{F}_2$ defined over a common open U of X. We have to show that the module of relations $\mathcal{K}$ between them is of finite type. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0
\end{tikzcd}$$

with obvious notation. By the snake lemma we get a short exact sequence $0 \to \mathcal{K} \to \mathcal{K}_3 \to \mathcal{F}_1$ where $\mathcal{K}_3$ is the module of relations among the images of the sections s_i

in $\mathcal{F}_3$. Since $\mathcal{F}_1$ is coherent we see that $\mathcal{K}$ is the kernel of a map from a finite type module to a coherent module and hence finite type by (2).

Proof of (5). This follows because (3) and (4) show that Homology, Lemma 12.10.3 applies. □

01BZ Lemma 17.12.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if it is of finite presentation.

Proof. Omitted. □

01C0 Lemma 17.12.6. Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{G}$ of finite type, $\mathcal{F}$ coherent and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ injective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is injective.

Proof. Denote $\mathcal{K} \subset \mathcal{G}$ the kernel of φ. By Lemma 17.12.4 we see that $\mathcal{K}$ is a finite type $\mathcal{O}_X$-module. Our assumption is that $\mathcal{K}_x = 0$. By Lemma 17.9.5 there exists an open neighbourhood U of x such that $\mathcal{K}|_U = 0$. Then U works. □

17.13. Closed immersions of ringed spaces

01C1 When do we declare a morphism of ringed spaces $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ to be a closed immersion?

Motivated by the example of a closed immersion of normal topological spaces (ringed with the sheaf of continuous functors), or differential manifolds (ringed with the sheaf of differentiable functions), it seems natural to assume at least:

(1) The map i is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.

Already these conditions imply a number of pleasing results: For example we prove that the category of $\mathcal{O}_Z$-modules is equivalent to the category of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ generalizing the result on abelian sheaves of Section 17.6

However, in the Stacks project we choose the definition that guarantees that if i is a closed immersion and $(X, \mathcal{O}_X)$ is a scheme, then also $(Z, \mathcal{O}_Z)$ is a scheme. Moreover, in this situation we want i_* and i^* to provide an equivalence between the category of quasi-coherent $\mathcal{O}_Z$-modules and the category of quasi-coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. A minimal condition is that $i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. A good way to guarantee that $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module is to assume that $\mathcal{I}$ is locally generated by sections. We can interpret this condition as saying "$(Z, \mathcal{O}_Z)$ is locally on $(X, \mathcal{O}_X)$ defined by setting some regular functions f_i, i.e., local sections of $\mathcal{O}_X$, equal to zero". This leads to the following definition.

01C2 Definition 17.13.1. A closed immersion of ringed spaces[1] is a morphism $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ with the following properties:

(1) The map i is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

[1]This is nonstandard notation; see discussion above.

Actually, this definition still does not guarantee that i_* of a quasi-coherent $\mathcal{O}_Z$-module is a quasi-coherent $\mathcal{O}_X$-module. The problem is that it is not clear how to convert a local presentation of a quasi-coherent $\mathcal{O}_Z$-module into a local presentation for the pushforward. However, the following is trivial.

01C3 Lemma 17.13.2. Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a closed immersion of ringed spaces. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is locally on X the cokernel of a map of quasi-coherent $\mathcal{O}_X$-modules.

Proof. This is true because $i_*\mathcal{O}_Z$ is quasi-coherent by definition. And locally on Z the sheaf $\mathcal{F}$ is a cokernel of a map between direct sums of copies of $\mathcal{O}_Z$. Moreover, any direct sum of copies of the the same quasi-coherent sheaf is quasi-coherent. And finally, i_* commutes with arbitrary colimits, see Lemma 17.6.3. Some details omitted. □

01C4 Lemma 17.13.3. Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Let $\mathcal{F}$ be an $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.

Proof. Suppose that $\mathcal{F}$ is of finite type. Pick $x \in X$. If $x \notin Z$, then $i_*\mathcal{F}$ is zero in a neighbourhood of x and hence finitely generated in a neighbourhood of x. If $x = i(z)$, then choose an open neighbourhood $z \in V \subset Z$ and sections $s_1, \ldots, s_n \in \mathcal{F}(V)$ which generate $\mathcal{F}$ over V. Write $V = Z \cap U$ for some open $U \subset X$. Note that U is a neighbourhood of x. Clearly the sections s_i give sections s_i of $i_*\mathcal{F}$ over U. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \longrightarrow i_*\mathcal{F}|_U$$

is surjective by inspection of what it does on stalks (here we use that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective). Hence $i_*\mathcal{F}$ is of finite type.

Conversely, suppose that $i_*\mathcal{F}$ is of finite type. Choose $z \in Z$. Set $x = i(z)$. By assumption there exists an open neighbourhood $U \subset X$ of x, and sections $s_1, \ldots, s_n \in (i_*\mathcal{F})(U)$ which generate $i_*\mathcal{F}$ over U. Set $V = Z \cap U$. By definition of i_* the sections s_i correspond to sections s_i of $\mathcal{F}$ over V. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \longrightarrow \mathcal{F}|_V$$

is surjective by inspection of what it does on stalks. Hence $\mathcal{F}$ is of finite type. □

08KS Lemma 17.13.4. Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}_X$ the kernel of $i^\sharp$. The functor

$$i_* : \text{Mod}(\mathcal{O}_Z) \longrightarrow \text{Mod}(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. We claim that for an $\mathcal{O}_Z$-module $\mathcal{F}$ the canonical map

$$i^*i_*\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. We check this on stalks. Say $z \in Z$ and $x = i(z)$. We have

$$(i^*i_*\mathcal{F})_z = (i_*\mathcal{F})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z$$

by Sheaves, Lemma 6.26.4, the fact that $\mathcal{O}_{Z,z}$ is a quotient of $\mathcal{O}_{X,x}$, and Sheaves, Lemma 6.32.1. It follows that i_* is fully faithful.

Let $\mathcal{G}$ be a $\mathcal{O}_X$-module with $\mathcal{I}\mathcal{G} = 0$. We will prove the canonical map

$$\mathcal{G} \longrightarrow i_*i^*\mathcal{G}$$

is an isomorphism. This proves that $\mathcal{G} = i_*\mathcal{F}$ with $\mathcal{F} = i^*\mathcal{G}$ which finishes the proof. We check the displayed map induces an isomorphism on stalks. If $x \in X$, $x \notin i(Z)$, then $\mathcal{G}_x = 0$ because $\mathcal{I}_x = \mathcal{O}_{X,x}$ in this case. As above $(i_*i^*\mathcal{G})_x = 0$ by Sheaves, Lemma 6.32.1. On the other hand, if $x \in Z$, then we obtain the map

$$\mathcal{G}_x \longrightarrow \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,x}$$

by Sheaves, Lemmas 6.26.4 and 6.32.1. This map is an isomorphism because $\mathcal{O}_{Z,x} = \mathcal{O}_{X,x}/\mathcal{I}_x$ and because $\mathcal{G}_x$ is annihilated by $\mathcal{I}_x$ by assumption. □

0G6N Remark 17.13.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. For an $\mathcal{O}_X$-module $\mathcal{F}$ we can consider the submodule of sections with support in Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \cap Z\}$$

Observe that $\mathcal{H}_Z(\mathcal{F})(U)$ is a module over $\mathcal{O}_X(U)$, i.e., $\mathcal{H}_Z(\mathcal{F})$ is an $\mathcal{O}_X$-module. By construction $\mathcal{H}_Z(\mathcal{F})$ is the largest $\mathcal{O}_X$-submodule of $\mathcal{F}$ whose support is contained in Z. Applying Lemma 17.13.4 to the morphism of ringed spaces $(Z, \mathcal{O}_X|_Z) \to (X, \mathcal{O}_X)$ we may (and we do) view $\mathcal{H}_Z(\mathcal{F})$ as an $\mathcal{O}_X|_Z$-module on Z. Thus we obtain a functor

$$\text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_X|_Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as an } \mathcal{O}_X|_Z\text{-module on } Z$$

This functor is left exact, but in general not exact. All of the statements made above follow directly from Lemma 17.5.2. Clearly the construction is compatible with the construction in Remark 17.6.2.

0G6P Lemma 17.13.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor $\mathcal{H}_Z : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X|_Z)$ of Remark 17.13.5 is right adjoint to $i_* : \text{Mod}(\mathcal{O}_X|_Z) \to \text{Mod}(\mathcal{O}_X)$.

Proof. We have to show that for any $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_X|_Z$-module $\mathcal{G}$ we have

$$\text{Hom}_{\mathcal{O}_X|_Z}(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \text{Hom}_{\mathcal{O}_X}(i_*\mathcal{G}, \mathcal{F})$$

This is clear because after all any section of $i_*\mathcal{G}$ has support in Z. Details omitted. □

17.14. Locally free sheaves

01C5 Let $(X, \mathcal{O}_X)$ be a ringed space. Our conventions allow (some of) the stalks $\mathcal{O}_{X,x}$ to be the zero ring. This means we have to be a little careful when defining the rank of a locally free sheaf.

01C6 Definition 17.14.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say $\mathcal{F}$ is locally free if for every point $x \in X$ there exist a set I and an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module.
(2) We say $\mathcal{F}$ is finite locally free if we may choose the index sets I to be finite.

(3) We say $\mathcal{F}$ is finite locally free of rank r if we may choose the index sets I to have cardinality r.

A finite direct sum of (finite) locally free sheaves is (finite) locally free. However, it may not be the case that an infinite direct sum of locally free sheaves is locally free.

01C7 Lemma 17.14.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is locally free then it is quasi-coherent.

Proof. Omitted. $\square$

01C8 Lemma 17.14.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a locally free $\mathcal{O}_X$-module.

Proof. Omitted. $\square$

01C9 Lemma 17.14.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that the support of $\mathcal{O}_X$ is X, i.e., all stalks of $\mathcal{O}_X$ are nonzero rings. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules. There exists a locally constant function

$$\text{rank}_{\mathcal{F}} : X \longrightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any point $x \in X$ the cardinality of any set I such that $\mathcal{F}$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ in a neighbourhood of x is $\text{rank}_{\mathcal{F}}(x)$.

Proof. Under the assumption of the lemma the cardinality of I can be read off from the rank of the free module $\mathcal{F}_x$ over the nonzero ring $\mathcal{O}_{X,x}$, and it is constant in a neighbourhood of x. $\square$

089Q Lemma 17.14.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $r \geq 0$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of finite locally free $\mathcal{O}_X$-modules of rank r. Then φ is an isomorphism if and only if φ is surjective.

Proof. Assume φ is surjective. Pick $x \in X$. There exists an open neighbourhood U of x such that both $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are isomorphic to $\mathcal{O}_U^{\oplus r}$. Pick lifts of the free generators of $\mathcal{G}|_U$ to obtain a map $\psi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi|_U \circ \psi = \text{id}$. Hence we conclude that the map $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ induced by φ is surjective. Since both $\Gamma(U, \mathcal{F})$ and $\Gamma(U, \mathcal{G})$ are isomorphic to $\Gamma(U, \mathcal{O}_U)^{\oplus r}$ as an $\Gamma(U, \mathcal{O}_U)$-module we may apply Algebra, Lemma 10.16.4 to see that $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ is injective. This finishes the proof. $\square$

0BCI Lemma 17.14.6. Let $(X, \mathcal{O}_X)$ be a ringed space. If all stalks $\mathcal{O}_{X,x}$ are local rings, then any direct summand of a finite locally free $\mathcal{O}_X$-module is finite locally free.

Proof. Assume $\mathcal{F}$ is a direct summand of the finite locally free $\mathcal{O}_X$-module $\mathcal{H}$. Let $x \in X$ be a point. Then $\mathcal{H}_x$ is a finite free $\mathcal{O}_{X,x}$-module. Because $\mathcal{O}_{X,x}$ is local, we see that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$ for some r, see Algebra, Lemma 10.78.2. By Lemma 17.11.6 we see that $\mathcal{F}$ is free of rank r in an open neighbourhood of x. (Note that $\mathcal{F}$ is of finite presentation as a summand of $\mathcal{H}$.) $\square$

17.15. Bilinear maps

0GIG Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be $\mathcal{O}_X$-modules. A bilinear map $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ of sheaves of $\mathcal{O}_X$-modules is a map of sheaves of sets as indicated such that for every open $U \subset X$ the induced map

$$\mathcal{F}(U) \times \mathcal{G}(U) \to \mathcal{H}(U)$$

is an $\mathcal{O}_X(U)$-bilinear map of modules. Equivalently you can ask certain diagrams of maps of sheaves of sets commute, immitating the usual axioms for bilinear maps of modules. For example, the axiom $f(x + y, z) = f(x, z) + f(y, z)$ is represented by the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F} \times \mathcal{F} \times \mathcal{G} \arrow[r, "(f \circ \mathrm{pr}_{13}, f \circ \mathrm{pr}_{23})"'] \arrow[d, "(+ \circ \mathrm{pr}_{12}, \mathrm{pr}_3)"'] & \mathcal{H} \times \mathcal{H} \arrow[d, "+"] \\
\mathcal{F} \times \mathcal{G} \arrow[r, "f"] & \mathcal{H}
\end{tikzcd}$$

Another characterization is this: if $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is a map of sheaves of sets and it induces a bilinar map of modules on stalks for all points of X, then f is a bilinear map of sheaves of modules. This is true as you can test whether local sections are equal by checking on stalks.

Let $\mathrm{Mor}(-, -)$ denote morphisms in the category of sheaves of sets on X. Another characterization of a bilinear map is this: a map of sheaves of sets $f : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is bilinear if given any sheaf of sets $\mathcal{S}$ the rule

$$\mathrm{Mor}(\mathcal{S}, \mathcal{F}) \times \mathrm{Mor}(\mathcal{S}, \mathcal{G}) \to \mathrm{Mor}(\mathcal{S}, \mathcal{H}), \quad (a, b) \mapsto f \circ (a \times b)$$

is a bilinear map of modules over the ring $\mathrm{Mor}(\mathcal{S}, \mathcal{O}_X)$. We don't usually take this point of view as it is easier to think about sets of local sections and it is clearly equivalent.

Finally, here is yet another way to say the definition: $\mathcal{O}_X$ is a ring object in the category of sheaves of sets and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are module objects over this ring. Then a bilinear map can be defined for module objects over a ring object in any category. To formulate what is a ring object and what is a module object over a ring object, and what is a bilinear map of such in a category it is pleasant (but not strictly necessary) to assume the category has finite products; and this is true for the category of sheaves of sets.

17.16. Tensor product

01CA We have already briefly discussed the tensor product in the setting of change of rings in Sheaves, Sections 6.6 and 6.20. Let us generalize this to tensor products of modules.

Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. We define first the tensor product presheaf

$$\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G}$$

as the rule which assigns to $U \subset X$ open the $\mathcal{O}_X(U)$-module $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Having defined this we define the tensor product sheaf as the sheafification of the above:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G})^\#$$

This can be characterized as the sheaf of $\mathcal{O}_X$-modules such that for any third sheaf of $\mathcal{O}_X$-modules $\mathcal{H}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Bilin}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

Here the right hand side indicates the set of bilinear maps of sheaves of $\mathcal{O}_X$-modules as defined in Section 17.15.

The tensor product of modules M, N over a ring R satisfies symmetry, namely $M \otimes_R N = N \otimes_R M$, hence the same holds for tensor products of sheaves of modules, i.e., we have

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

functorial in $\mathcal{F}$, $\mathcal{G}$. And since tensor product of modules satisfies associativity we also get canonical functorial isomorphisms

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$$

functorial in $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.

01CB Lemma 17.16.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. There is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$$

functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Omitted. □

05NA Lemma 17.16.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}'$, $\mathcal{G}'$ be presheaves of $\mathcal{O}_X$-modules with sheafifications $\mathcal{F}$, $\mathcal{G}$. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F}' \otimes_{p,\mathcal{O}_X} \mathcal{G}')^\#$.

Proof. Omitted. □

01CC Lemma 17.16.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules then the induced sequence

$$\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact.

Proof. This follows from the fact that exactness may be checked at stalks (Lemma 17.3.1), the description of stalks (Lemma 17.16.1) and the corresponding result for tensor products of modules (Algebra, Lemma 10.12.10). □

01CD Lemma 17.16.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.

Proof. Omitted. □

05NB Lemma 17.16.5. Let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-module $\mathcal{F}$ the functor

$$\mathrm{Mod}(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\mathcal{O}_X), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$$

commutes with arbitrary colimits.

Proof. Let I be a preordered set and let $\{\mathcal{G}_i\}$ be a system over I. Set $\mathcal{G} = \mathrm{colim}_i\, \mathcal{G}_i$. Recall that $\mathcal{G}$ is the sheaf associated to the presheaf $\mathcal{G}' : U \mapsto \mathrm{colim}_i\, \mathcal{G}_i(U)$, see Sheaves, Section 6.29. By Lemma 17.16.2 the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathrm{colim}_i\, \mathcal{G}_i(U) = \mathrm{colim}_i\, \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}_i(U)$$

where the equality sign is Algebra, Lemma 10.12.9. Hence the lemma follows from the description of colimits in $\text{Mod}(\mathcal{O}_X)$, see Lemma 17.3.2. $\square$

01CE Lemma 17.16.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.

(1) If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.
(2) If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.
(3) If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.
(4) If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.
(5) If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent.
(6) If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.
(7) If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Proof. We first prove that the tensor product of locally free $\mathcal{O}_X$-modules is locally free. This follows if we show that $(\bigoplus_{i \in I} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (\bigoplus_{j \in J} \mathcal{O}_X) \cong \bigoplus_{(i,j) \in I \times J} \mathcal{O}_X$. The sheaf $\bigoplus_{i \in I} \mathcal{O}_X$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{i \in I} \mathcal{O}_X(U)$. Hence the tensor product is the sheaf associated to the presheaf

$$U \longmapsto (\bigoplus\nolimits_{i \in I} \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} (\bigoplus\nolimits_{j \in J} \mathcal{O}_X(U)).$$

We deduce what we want since for any ring R we have $(\bigoplus_{i \in I} R) \otimes_R (\bigoplus_{j \in J} R) = \bigoplus_{(i,j) \in I \times J} R$.

If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is exact, then by Lemma 17.16.3 the complex $\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$ is exact. Using this we can prove (5). Namely, in this case there exists locally such an exact sequence with $\mathcal{F}_i$, $i = 1, 2$ finite free. Hence the two terms $\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}$ are isomorphic to finite direct sums of $\mathcal{G}$ (for example by Lemma 17.16.5). Since finite direct sums are coherent sheaves, these are coherent and so is the cokernel of the map, see Lemma 17.12.4.

And if also $\mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G} \to 0$ is exact, then we see that

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \oplus \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_2 \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact. Using this we can for example prove (3). Namely, the assumption means that we can locally find presentations as above with $\mathcal{F}_i$ and $\mathcal{G}_i$ free $\mathcal{O}_X$-modules. Hence the displayed presentation is a presentation of the tensor product by free sheaves as well.

The proof of the other statements is omitted. $\square$

17.17. Flat modules

05NC We can define flat modules exactly as in the case of modules over rings.

05ND Definition 17.17.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat if the functor

$$\text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_X), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$

is exact.

We can characterize flatness by looking at the stalks.

05NE Lemma 17.17.2. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat if and only if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$.

Proof. Assume $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$. In this case, if $\mathcal{G} \to \mathcal{H} \to \mathcal{K}$ is exact, then also $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F}$ is exact because we can check exactness at stalks and because tensor product commutes with taking stalks, see Lemma 17.16.1. Conversely, suppose that $\mathcal{F}$ is flat, and let $x \in X$. Consider the skyscraper sheaves $i_{x,*}M$ where M is a $\mathcal{O}_{X,x}$-module. Note that

$$M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x = (i_{x,*}M \otimes_{\mathcal{O}_X} \mathcal{F})_x$$

again by Lemma 17.16.1. Since $i_{x,*}$ is exact, we see that the fact that $\mathcal{F}$ is flat implies that $M \mapsto M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$ is exact. Hence $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. □

Thus the following definition makes sense.

05NF Definition 17.17.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat at x if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module.

Hence we see that $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if it is flat at every point.

05NG Lemma 17.17.4. Let $(X, \mathcal{O}_X)$ be a ringed space. A filtered colimit of flat $\mathcal{O}_X$-modules is flat. A direct sum of flat $\mathcal{O}_X$-modules is flat.

Proof. This follows from Lemma 17.16.5, Lemma 17.16.1, Algebra, Lemma 10.8.8, and the fact that we can check exactness at stalks. □

05NH Lemma 17.17.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be open. The sheaf $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}_X$-modules.

Proof. The stalks of $j_{U!}\mathcal{O}_U$ are either zero or equal to $\mathcal{O}_{X,x}$. Apply Lemma 17.17.2. □

05NI Lemma 17.17.6. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) Any sheaf of $\mathcal{O}_X$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.
(2) Any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For every open $U \subset X$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective, and the source is flat by combining Lemmas 17.17.4 and 17.17.5. □

05NJ Lemma 17.17.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is flat. Then for any $\mathcal{O}_X$-module $\mathcal{G}$ the sequence

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

is exact.

Proof. Using that $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for every $x \in X$ and that exactness can be checked on stalks, this follows from Algebra, Lemma 10.39.12. □

05NK Lemma 17.17.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules.

(1) If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.
(2) If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.

Proof. Since exactness and flatness may be checked at the level of stalks this follows from Algebra, Lemma 10.39.13. □

05NL Lemma 17.17.9. Let $(X, \mathcal{O}_X)$ be a ringed space. Let

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of $\mathcal{O}_X$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}_X$-modules, then for any $\mathcal{O}_X$-module $\mathcal{G}$ the complex

$$\ldots \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact also.

Proof. Follows from Lemma 17.17.7 by splitting the complex into short exact sequences and using Lemma 17.17.8 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. □

The following lemma gives one direction of the equational criterion of flatness (Algebra, Lemma 10.39.11).

08BK Lemma 17.17.10. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module. Let $U \subset X$ be open and let

$$\mathcal{O}_U \xrightarrow{(f_1, \ldots, f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. For every $x \in U$ there exists an open neighbourhood $V \subset U$ of x and a factorization

$$\mathcal{O}_V^{\oplus n} \xrightarrow{A} \mathcal{O}_V^{\oplus m} \xrightarrow{(t_1, \ldots, t_m)} \mathcal{F}|_V$$

of $(s_1, \ldots, s_n)|_V$ such that $A \circ (f_1, \ldots, f_n)|_V = 0$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1, \ldots, f_n$. Then $\sum f_i \otimes s_i$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists an open neighbourhood $V \subset U$ of x such that $\sum f_i|_V \otimes s_i|_V$ is zero in $\mathcal{I}(V) \otimes_{\mathcal{O}(V)} \mathcal{F}(V)$. Unwinding the definitions using Algebra, Lemma 10.107.10 we find $t_1, \ldots, t_m \in \mathcal{F}(V)$ and $a_{ij} \in \mathcal{O}(V)$ such that $\sum a_{ij} f_i|_V = 0$ and $s_i|_V = \sum a_{ij} t_j$. □

17.18. Duals

0FNU Let $(X, \mathcal{O}_X)$ be a ringed space. The category of $\mathcal{O}_X$-modules endowed with the tensor product constructed in Section 17.16 is a symmetric monoidal category. For an $\mathcal{O}_X$-module $\mathcal{F}$ the following are equivalent

(1) $\mathcal{F}$ has a left dual in the monoidal category of $\mathcal{O}_X$-modules,
(2) $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module, and
(3) $\mathcal{F}$ is of finite presentation and flat as an $\mathcal{O}_X$-module.

This is proved in Example 17.18.1 and Lemmas 17.18.2 and 17.18.3 of this section.

0FNV Example 17.18.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module which is locally a direct summand of a finite free $\mathcal{O}_X$-module. Then the map

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

is an isomorphism. Namely, this is a local question, it is true if $\mathcal{F}$ is finite free, and it holds for any summand of a module for which it is true. Denote

$$\eta : \mathcal{O}_X \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

the map sending 1 to the section corresponding to $\mathrm{id}_\mathcal{F}$ under the isomorphism above. Denote

$$\epsilon : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{O}_X$$

the evaluation map. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \eta, \epsilon$ is a left dual for $\mathcal{F}$ as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_\mathcal{F}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)}$.

0FNW Lemma 17.18.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{G}, \eta, \epsilon$ be a left dual of $\mathcal{F}$ in the monoidal category of $\mathcal{O}_X$-modules, see Categories, Definition 4.43.5. Then

(1) $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module,
(2) the map $e : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{G}$ sending a local section λ to $(\lambda \otimes 1)(\eta)$ is an isomorphism,
(3) we have $\epsilon(f, g) = e^{-1}(g)(f)$ for local sections f and g of $\mathcal{F}$ and $\mathcal{G}$.

Proof. The assumptions mean that

$$\mathcal{F} \xrightarrow{\eta \otimes 1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{1 \otimes \epsilon} \mathcal{F} \quad \text{and} \quad \mathcal{G} \xrightarrow{1 \otimes \eta} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \xrightarrow{\epsilon \otimes 1} \mathcal{G}$$

are the identity map. Let $x \in X$. We can find an open neighbourhood U of x, a finite number of sections $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ of $\mathcal{F}$ and $\mathcal{G}$ over U such that $\eta(1) = \sum f_i g_i$. Denote

$$\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$$

the map sending the ith basis vector to f_i. Then we can factor the map $\eta|_U$ over a map $\tilde{\eta} : \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \otimes_{\mathcal{O}_U} \mathcal{G}|_U$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F}|_U \arrow[r, "\eta \otimes 1"'] \arrow[dr, "\tilde{\eta} \otimes 1"'] & \mathcal{F}|_U \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U \arrow[r, "1 \otimes \epsilon"'] & \mathcal{F}|_U \\
 & \mathcal{O}_U^{\oplus n} \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U \arrow[u] \arrow[r, "1 \otimes \epsilon"] & \mathcal{O}_U^{\oplus n} \arrow[u]
\end{tikzcd}$$

This shows that the identity on $\mathcal{F}$ locally on X factors through a finite free module. This proves (1). Part (2) follows from Categories, Lemma 4.43.6 and its proof. Part (3) follows from the first equality of the proof. You can also deduce (2) and (3) from the uniqueness of left duals (Categories, Remark 4.43.7) and the construction of the left dual in Example 17.18.1. $\square$

08BL Lemma 17.18.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module of finite presentation. Then $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module.

Proof. After replacing X by the members of an open covering, we may assume there exists a presentation

$$\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$$

Let $x \in X$. By Lemma 17.17.10 we can, after shrinking X to an open neighbourhood of x, assume there exists a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1}$ annihilates the first summand of $\mathcal{O}_X^{\oplus r}$. Repeating this argument $r-1$ more times we obtain a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$ has a section and we win. $\square$

17.19. Constructible sheaves of sets

0CAG Let X be a topological space. Given a set S recall that $\underline{S}$ or $\underline{S}_X$ denotes the constant sheaf with value S, see Sheaves, Definition 6.7.4. Let $U \subset X$ be an open of a topological space X. We will denote j_U the inclusion morphism and we will denote $j_{U!} : Sh(U) \to Sh(X)$ the extension by the empty set described in Sheaves, Section 6.31.

0CAH **Lemma 17.19.1.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf of sets on X. There exists a set I and for each $i \in I$ an element $U_i \in \mathcal{B}$ and a finite set S_i such that there exists a surjection $\coprod_{i \in I} j_{U_i!}\underline{S_i} \to \mathcal{F}$.*

Proof. Let S be a singleton set. We will prove the result with $S_i = S$. For every $x \in X$ and element $s \in \mathcal{F}_x$ we can choose a $U(x,s) \in \mathcal{B}$ and $s(x,s) \in \mathcal{F}(U(x,s))$ which maps to s in $\mathcal{F}_x$. By Sheaves, Lemma 6.31.4 the section $s(x,s)$ corresponds to a map of sheaves $j_{U(x,s)!}\underline{S} \to \mathcal{F}$. Then

$$\coprod\nolimits_{(x,s)} j_{U(x,s)!}\underline{S} \to \mathcal{F}$$

is surjective on stalks and hence surjective. $\square$

0CAI **Lemma 17.19.2.** *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X and assume that each $U \in \mathcal{B}$ is quasi-compact. Then every sheaf of sets on X is a filtered colimit of sheaves of the form*

0CAJ (17.19.2.1) $$\text{Coequalizer}\left(\coprod\nolimits_{b=1,\ldots,m} j_{V_b!}\underline{S_b} \rightrightarrows \coprod\nolimits_{a=1,\ldots,n} j_{U_a!}\underline{S_a} \right)$$

with U_a and V_b in $\mathcal{B}$ and S_a and S_b finite sets.

Proof. By Lemma 17.19.1 every sheaf of sets $\mathcal{F}$ is the target of a surjection whose source $\mathcal{F}_0$ is a coproduct of sheaves the form $j_{U!}\underline{S}$ with $U \in \mathcal{B}$ and S finite. Applying this to $\mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ we find that $\mathcal{F}$ is a coequalizer of a pair of maps

$$\coprod\nolimits_{b \in B} j_{V_b!}\underline{S_b} \rightrightarrows \coprod\nolimits_{a \in A} j_{U_a!}\underline{S_a}$$

for some index sets A, B and V_b and U_a in $\mathcal{B}$ and S_a and S_b finite. For every finite subset $B' \subset B$ there is a finite subset $A' \subset A$ such that the coproduct over $b \in B'$ maps into the coproduct over $a \in A'$ via both maps. Namely, we can view the right hand side as a filtered colimit with injective transition maps. Hence taking sections over the quasi-compact opens V_b, $b \in B'$ commutes with this coproduct, see Sheaves, Lemma 6.29.1. Thus our sheaf is the colimit of the cokernels of these maps between finite coproducts. $\square$

0CAK Lemma 17.19.3. Let X be a spectral topological space. Let $\mathcal{B}$ be the set of quasi-compact open subsets of X. Let $\mathcal{F}$ be a sheaf of sets as in Equation (17.19.2.1). Then there exists a continuous spectral map $f : X \to Y$ to a finite sober topological space Y and a sheaf of sets $\mathcal{G}$ on Y with finite stalks such that $f^{-1}\mathcal{G} \cong \mathcal{F}$.

Proof. We can write $X = \lim X_i$ as a directed limit of finite sober spaces, see Topology, Lemma 5.23.14. Of course the transition maps $X_{i'} \to X_i$ are spectral and hence by Topology, Lemma 5.24.5 the maps $p_i : X \to X_i$ are spectral. For some i we can find opens $U_{a,i}$ and $V_{b,i}$ of X_i whose inverse images are U_a and V_b, see Topology, Lemma 5.24.6. The two maps

$$\beta, \gamma : \coprod\nolimits_{b \in B} j_{V_b!}\underline{S_b} \longrightarrow \coprod\nolimits_{a \in A} j_{U_a!}\underline{S_a}$$

whose coequalizer is $\mathcal{F}$ correspond by adjunction to two families

$$\beta_b, \gamma_b : S_b \longrightarrow \Gamma(V_b, \coprod\nolimits_{a \in A} j_{U_a!}\underline{S_a}), \quad b \in B$$

of maps of sets. Observe that $p_i^{-1}(j_{U_{a,i}!}\underline{S_a}) = j_{U_a!}\underline{S_a}$ and $(X_{i'} \to X_i)^{-1}(j_{U_{a,i}!}\underline{S_a}) = j_{U_{a,i'}!}\underline{S_a}$. It follows from Sheaves, Lemma 6.29.3 (and using that S_b and B are finite sets) that after increasing i we find maps

$$\beta_{b,i}, \gamma_{b,i} : S_b \longrightarrow \Gamma(V_{b,i}, \coprod\nolimits_{a \in A} j_{U_{a,i}!}\underline{S_a}), \quad b \in B$$

which give rise to the maps β_b and γ_b after pulling back by p_i. These maps correspond in turn to maps of sheaves

$$\beta_i, \gamma_i : \coprod\nolimits_{b \in B} j_{V_{b,i}!}\underline{S_b} \longrightarrow \coprod\nolimits_{a \in A} j_{U_{a,i}!}\underline{S_a}$$

on X_i. Then we can take $Y = X_i$ and

$$\mathcal{G} = \text{Coequalizer}\left(\coprod\nolimits_{b=1,\ldots,m} j_{V_{b,i}!}\underline{S_b} \rightrightarrows \coprod\nolimits_{a=1,\ldots,n} j_{U_{a,i}!}\underline{S_a} \right)$$

We omit some details. $\square$

0CAL Lemma 17.19.4. Let X be a spectral topological space. Let $\mathcal{B}$ be the set of quasi-compact open subsets of X. Let $\mathcal{F}$ be a sheaf of sets as in Equation (17.19.2.1). Then there exist finitely many constructible closed subsets $Z_1, \ldots, Z_n \subset X$ and finite sets S_i such that $\mathcal{F}$ is isomorphic to a subsheaf of $\prod (Z_i \to X)_*\underline{S_i}$.

Proof. By Lemma 17.19.3 we reduce to the case of a finite sober topological space and a sheaf with finite stalks. In this case $\mathcal{F} \subset \prod_{x \in X} i_{x,*}\mathcal{F}_x$ where $i_x : \overline{\{x\}} \to X$ is the embedding. We omit the proof that $i_{x,*}\mathcal{F}_x$ is a constant sheaf on $\overline{\{x\}}$. $\square$

17.20. Flat morphisms of ringed spaces

02N2 The pointwise definition is motivated by Lemma 17.17.2 and Definition 17.17.3 above.

02N3 Definition 17.20.1. Let $f : X \to Y$ be a morphism of ringed spaces. Let $x \in X$. We say f is flat at x if the map of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We say f is flat if f is flat at every $x \in X$.

Consider the map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We see that the stalk at x is the ring map $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Hence f is flat at x if and only if $\mathcal{O}_X$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module. And f is flat if and only if $\mathcal{O}_X$ is flat as an $f^{-1}\mathcal{O}_Y$-module. A very special case of a flat morphism is an open immersion.

02N4 Lemma 17.20.2. Let $f : X \to Y$ be a flat morphism of ringed spaces. Then the pullback functor $f^* : \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$ is exact.

Proof. The functor f^* is the composition of the exact functor $f^{-1} : \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(f^{-1}\mathcal{O}_Y)$ and the change of rings functor

$$\mathrm{Mod}(f^{-1}\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus the result follows from the discussion following Definition 17.20.1. $\square$

08KT Definition 17.20.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say that $\mathcal{F}$ is flat over Y at a point $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module.
(2) We say that $\mathcal{F}$ is flat over Y if $\mathcal{F}$ is flat over Y at every point x of X.

With this definition we see that $\mathcal{F}$ is flat over Y at x if and only if $\mathcal{F}$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module because $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)}$ by Sheaves, Lemma 6.21.5.

0GMU Lemma 17.20.4. Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module flat over Y. Then the functor

$$\mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X), \quad \mathcal{G} \longmapsto f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

Proof. This is true because $f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{F}$, the functor f^{-1} is exact, and $\mathcal{F}$ is a flat $f^{-1}\mathcal{O}_Y$-module. $\square$

17.21. Symmetric and exterior powers

01CF Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the tensor algebra of $\mathcal{F}$ to be the sheaf of noncommutative $\mathcal{O}_X$-algebras

$$\mathrm{T}(\mathcal{F}) = \mathrm{T}_{\mathcal{O}_X}(\mathcal{F}) = \bigoplus\nolimits_{n \geq 0} \mathrm{T}^n(\mathcal{F}).$$

Here $\mathrm{T}^0(\mathcal{F}) = \mathcal{O}_X$, $\mathrm{T}^1(\mathcal{F}) = \mathcal{F}$ and for $n \geq 2$ we have

$$\mathrm{T}^n(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{F} \quad (n \text{ factors})$$

We define $\wedge(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections $s \otimes s$ of $\mathrm{T}^2(\mathcal{F})$ where s is a local section of $\mathcal{F}$. This is called the exterior algebra of $\mathcal{F}$. Similarly, we define $\mathrm{Sym}(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections of the form $s \otimes t - t \otimes s$ of $\mathrm{T}^2(\mathcal{F})$.

Both $\wedge(\mathcal{F})$ and $\mathrm{Sym}(\mathcal{F})$ are graded $\mathcal{O}_X$-algebras, with grading inherited from $\mathrm{T}(\mathcal{F})$. Moreover $\mathrm{Sym}(\mathcal{F})$ is commutative, and $\wedge(\mathcal{F})$ is graded commutative.

01CG Lemma 17.21.1. In the situation described above. The sheaf $\wedge^n\mathcal{F}$ is the sheafification of the presheaf

$$U \longmapsto \wedge^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

See Algebra, Section 10.13. Similarly, the sheaf $\mathrm{Sym}^n\mathcal{F}$ is the sheafification of the presheaf

$$U \longmapsto \mathrm{Sym}^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

Proof. Omitted. It may be more efficient to define $\mathrm{Sym}(\mathcal{F})$ and $\wedge(\mathcal{F})$ in this way instead of the method given above. $\square$

01CH Lemma 17.21.2. In the situation described above. Let $x \in X$. There are canonical isomorphisms of $\mathcal{O}_{X,x}$-modules $\mathrm{T}(\mathcal{F})_x = \mathrm{T}(\mathcal{F}_x)$, $\mathrm{Sym}(\mathcal{F})_x = \mathrm{Sym}(\mathcal{F}_x)$, and $\wedge(\mathcal{F})_x = \wedge(\mathcal{F}_x)$.

Proof. Clear from Lemma 17.21.1 above, and Algebra, Lemma 10.13.5. $\square$

01CI Lemma 17.21.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^*\mathrm{T}(\mathcal{F}) = \mathrm{T}(f^*\mathcal{F})$, and similarly for the exterior and symmetric algebras associated to $\mathcal{F}$.

Proof. Omitted. $\square$

01CJ Lemma 17.21.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be an exact sequence of sheaves of $\mathcal{O}_X$-modules. For each $n \geq 1$ there is an exact sequence

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathrm{Sym}^{n-1}(\mathcal{F}_1) \to \mathrm{Sym}^n(\mathcal{F}_1) \to \mathrm{Sym}^n(\mathcal{F}) \to 0$$

and similarly an exact sequence

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \wedge^{n-1}(\mathcal{F}_1) \to \wedge^n(\mathcal{F}_1) \to \wedge^n(\mathcal{F}) \to 0$$

Proof. See Algebra, Lemma 10.13.2. $\square$

01CK Lemma 17.21.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) If $\mathcal{F}$ is locally generated by sections, then so is each $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.
(2) If $\mathcal{F}$ is of finite type, then so is each $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.
(3) If $\mathcal{F}$ is of finite presentation, then so is each $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.
(4) If $\mathcal{F}$ is coherent, then for $n > 0$ each $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$ is coherent.
(5) If $\mathcal{F}$ is quasi-coherent, then so is each $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.
(6) If $\mathcal{F}$ is locally free, then so is each $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $\mathrm{Sym}^n(\mathcal{F})$.

Proof. These statements for $\mathrm{T}^n(\mathcal{F})$ follow from Lemma 17.16.6.

Statements (1) and (2) follow from the fact that $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$ are quotients of $\mathrm{T}^n(\mathcal{F})$.

Statement (6) follows from Algebra, Lemma 10.13.1.

For (3) and (5) we will use Lemma 17.21.4 above. By locally choosing a presentation $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ with $\mathcal{F}_i$ free, or finite free and applying the lemma we see that $\mathrm{Sym}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ has a similar presentation; here we use (6) and Lemma 17.16.6.

To prove (4) we will use Algebra, Lemma 10.13.3. We may localize on X and assume that $\mathcal{F}$ is generated by a finite set $(s_i)_{i \in I}$ of global sections. The lemma mentioned above combined with Lemma 17.21.1 above implies that for $n \geq 2$ there exists an exact sequence

$$\bigoplus\nolimits_{j \in J} \mathrm{T}^{n-2}(\mathcal{F}) \to \mathrm{T}^n(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{F}) \to 0$$

where the index set J is finite. Now we know that $\mathrm{T}^{n-2}(\mathcal{F})$ is finitely generated and hence the image of the first arrow is a coherent subsheaf of $\mathrm{T}^n(\mathcal{F})$, see Lemma 17.12.4. By that same lemma we conclude that $\mathrm{Sym}^n(\mathcal{F})$ is coherent. $\square$

01CL Lemma 17.21.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) If $\mathcal{F}$ is quasi-coherent, then so is each $\mathrm{T}(\mathcal{F})$, $\wedge(\mathcal{F})$, and $\mathrm{Sym}(\mathcal{F})$.

(2) If $\mathcal{F}$ is locally free, then so is each $\mathrm{T}(\mathcal{F})$, $\wedge(\mathcal{F})$, and $\mathrm{Sym}(\mathcal{F})$.

Proof. It is not true that an infinite direct sum $\bigoplus \mathcal{G}_i$ of locally free modules is locally free, or that an infinite direct sum of quasi-coherent modules is quasi-coherent. The problem is that given a point $x \in X$ the open neighbourhoods U_i of x on which $\mathcal{G}_i$ becomes free (resp. has a suitable presentation) may have an intersection which is not an open neighbourhood of x. However, in the proof of Lemma 17.21.5 we saw that once a suitable open neighbourhood for $\mathcal{F}$ has been chosen, then this open neighbourhood works for each of the sheaves $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$. The lemma follows. $\square$

17.22. Internal Hom

01CM Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section 6.33 that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is rarely an isomorphism.

01CN Lemma 17.22.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be $\mathcal{O}_X$-modules. There is a canonical isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries (sheaf Hom in all three spots). In particular, to give a morphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}$ is the same as giving a morphism $\mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})$.

Proof. This is the analogue of Algebra, Lemma 10.12.8. The proof is the same, and is omitted. $\square$

01CO Lemma 17.22.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.

(1) If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, then

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G})$$

is exact.

(2) If $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ is an exact sequence of $\mathcal{O}_X$-modules, then

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_2)$$

is exact.

Proof. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be as in (1). For every $U \subset X$ open the sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_1|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_2|_U, \mathcal{G}|_U)$$

is exact by Homology, Lemma 12.5.8. This means that taking sections over U of the sequence of sheaves in (1) produces an exact sequence of abelian groups. Hence the sequence in (1) is exact by definition. The proof of (2) is exactly the same. $\square$

0A6F Lemma 17.22.3. Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Then we have

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{F}_{\mathcal{O}_1}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G}))$$

bifunctorially in $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_2)$ and $\mathcal{G} \in \mathrm{Mod}(\mathcal{O}_1)$.

Proof. Omitted. This is the analogue of Algebra, Lemma 10.14.4 and is proved in exactly the same way. $\square$

01CP Lemma 17.22.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then the canonical map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

is injective. If $\mathcal{F}$ is finitely presented, this canonical morphism is an isomorphism.

Proof. The map sends the equivalence class of (U, φ) in $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$, where $x \in U \subset X$ is open and $\varphi \in \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, to the the induced map on stalks at x, namely $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$.

Suppose $\mathcal{F}$ is of finite type. Pick a representative (U, φ) of an element σ in the kernel of the map, i.e., $\varphi_x = 0$. Shrinking U if necessary, choose sections $s^1, \ldots, s^n \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since $\varphi_x(s^i_x) = 0$ and we are dealing with a finite number of sections, we can find an open neighborhood $V \subset U$ of x such that $\varphi_V(s^i|_V) = 0$ for all $i = 1, \ldots, n$. Since $s^i|_V$, $i = 1, \ldots, n$ generate $\mathcal{F}|_V$ this means that $\varphi|_V = 0$. Since (U, φ) is equivalent to $(V, \varphi|_V)$ we conclude $\sigma = 0$ and injectivity of the map follows.

Next, assume $\mathcal{F}$ is finitely presented. By localizing on X we may assume that $\mathcal{F}$ has a presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0.$$

By Lemma 17.22.2 this gives an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus_{i=1,\ldots,n} \mathcal{G} \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}$. Taking stalks we get an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ and the result follows since $\mathcal{F}_x$ sits in an exact sequence $\bigoplus_{j=1,\ldots,m} \mathcal{O}_{X,x} \longrightarrow \bigoplus_{i=1,\ldots,n} \mathcal{O}_{X,x} \to \mathcal{F}_x \to 0$ which induces the exact sequence $0 \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ which is the same as the one above. $\square$

0C6I Lemma 17.22.5. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. If $\mathcal{F}$ is finitely presented and f is flat, then the canonical map

$$f^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})$$

is an isomorphism.

Proof. Note that $f^*\mathcal{F}$ is also finitely presented (Lemma 17.11.4). Let $x \in X$ map to $y \in Y$. Looking at the stalks at x we get an isomorphism by Lemma 17.22.4 and More on Algebra, Lemma 15.65.4 to see that in this case Hom commutes with base change by $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Second proof: use the exact same argument as given in the proof of Lemma 17.22.4. $\square$

01CQ Lemma 17.22.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is finitely presented then the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is locally a kernel of a map between finite direct sums of copies of $\mathcal{G}$. In particular, if $\mathcal{G}$ is coherent then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent too.

Proof. The first assertion we saw in the proof of Lemma 17.22.4. And the result for coherent sheaves then follows from Lemma 17.12.4. $\square$

0GMV Lemma 17.22.7. Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $\mathcal{G} = \mathrm{colim}_{\lambda \in \Lambda} \mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}_X$-modules. Then the canonical map

$$\mathrm{colim}_\lambda \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_\lambda) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. Taking colimits of sheaves of modules commutes with restriction to opens, see Sheaves, Section 6.29. Hence we may assume $\mathcal{F}$ has a global presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0$$

The functor $\mathcal{H}om_{\mathcal{O}_X}(-,-)$ commutes with finite direct sums in either variable and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, -)$ is the identity functor. By this and by Lemma 17.22.2 we obtain an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,m} \mathcal{G}$$

Since filtered colimits are exact in $\mathrm{Mod}(\mathcal{O}_X)$ also the top row in the following commutative diagram is exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{colim}_\lambda \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_\lambda) \arrow[r] \arrow[d] & \mathrm{colim}_\lambda \bigoplus_{i=1,\ldots,n} \mathcal{G}_\lambda \arrow[r] \arrow[d] & \mathrm{colim}_\lambda \bigoplus_{j=1,\ldots,m} \mathcal{G}_\lambda \arrow[d] \\
0 \arrow[r] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \arrow[r] & \bigoplus_{i=1,\ldots,n} \mathcal{G} \arrow[r] & \bigoplus_{j=1,\ldots,m} \mathcal{G}
\end{tikzcd}$$

Since the right two vertical arrows are isomorphisms we conclude. $\square$

01BS Lemma 17.22.8. Let X be a ringed space. Let I be a preordered set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 4.21). Assume

(1) I is directed,
(2) $\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation, and
(3) X has a cofinal system of open coverings $\mathcal{U} : X = \bigcup_{j \in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$.

Then we have

$$\mathrm{colim}_i \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \mathrm{Hom}_X(\mathcal{G}, \mathrm{colim}_i \mathcal{F}_i).$$

Proof. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \operatorname{colim} \mathcal{F}_i)$ and $\mathcal{H}_i = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}_i)$. Recall that

$$\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}) = \Gamma(X, \mathcal{H}) \quad \text{and} \quad \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \Gamma(X, \mathcal{H}_i)$$

by construction. By Lemma 17.22.7 we have $\mathcal{H} = \operatorname{colim} \mathcal{H}_i$. Thus the lemma follows from Sheaves, Lemma 6.29.1. $\square$

01BT Remark 17.22.9. In the lemma above some condition beyond the condition that X is quasi-compact is necessary. See Sheaves, Example 6.29.2.

17.23. The annihilator of a sheaf of modules

0H2G Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a canonical map of sheaves of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

which sends a local section $f \in \mathcal{O}_X(U)$ to the map $f : \mathcal{F}|_U \to \mathcal{F}|_U$ given by multiplication by f.

0H2H Definition 17.23.1. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The annihilator of $\mathcal{F}$, denoted $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$ is the kernel of the map $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ discussed above.

For each $x \in X$, there is an inclusion of ideals of $\mathcal{O}_{X,x}$:

0H2I (17.23.1.1)
$$(\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}))_x \subset \mathrm{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$$

since after all any section of $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$ will annihilate the stalks of $\mathcal{F}$ at all points at which it is defined. Here is a simple situation in which (??) becomes an equality.

0H2J Lemma 17.23.2. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type, then $(\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}))_x = \mathrm{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$.

Proof. By Lemma 17.22.4 the map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{F}_x)$$

is injective. Thus any section f of $\mathcal{O}_X$ over an open neighbourhood U of x which acts as zero on $\mathcal{F}_x$ will act as zero on $\mathcal{F}|_V$ for some $U \supset V \ni x$ open. Hence the inclusion (17.23.1.1) is an equality. $\square$

0H2K Lemma 17.23.3. Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. If $\mathcal{I} \subset \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$, then $\mathcal{F}$ has a natural $\mathcal{O}_X/\mathcal{I}$-module structure which agrees with the usual commutative algebra construction on stalks.

Proof. Applying the universal property of the cokernel of the inclusion $\mathcal{I} \to \mathcal{O}_X$, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \\ \mathcal{O}_X/\mathcal{I} \arrow[ur, dashed] & \end{tikzcd}$$

of $\mathcal{O}_X$-modules. By Lemma 17.22.1 the resulting map $\mathcal{O}_X/\mathcal{I} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ corresponds to a map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}$$

which means we have an $\mathcal{O}_X/\mathcal{I}$-module structure on $\mathcal{F}$ compatible with the given $\mathcal{O}_X$-module structure. We omit the verification of the statement on stalks. $\square$

0H2L Lemma 17.23.4. Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{O}_X$ and $\mathcal{F}$ are coherent, then so is $\text{Ann}_{\mathcal{O}_X}(\mathcal{F})$.

Proof. Since $\text{Ann}_{\mathcal{O}_X}(\mathcal{F})$ is the kernel of $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ by Lemma 17.12.4 it suffices to show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ is coherent. This follows from Lemma 17.22.6 and the fact that $\mathcal{F}$ is coherent and a fortiori finitely presented (Lemma 17.12.2). $\square$

17.24. Koszul complexes

062J We suggest first reading the section on Koszul complexes in More on Algebra, Section 15.28. We define the Koszul complex in the category of $\mathcal{O}_X$-modules as follows.

062K Definition 17.24.1. Let X be a ringed space. Let $\varphi : \mathcal{E} \to \mathcal{O}_X$ be an $\mathcal{O}_X$-module map. The Koszul complex $K_\bullet(\varphi)$ associated to φ is the sheaf of commutative differential graded algebras defined as follows:

(1) the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(\mathcal{E})$,
(2) the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all local sections e of $\mathcal{E} = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is a wedge product of local sections of $\mathcal{E}$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \wedge e$ and hence factors through the exterior algebra.

062L Definition 17.24.2. Let X be a ringed space and let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. The Koszul complex on $f_1, \ldots, f_r$ is the Koszul complex associated to the map $(f_1, \ldots, f_n) : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X$. Notation $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$, or $K_\bullet(\mathcal{O}_X, f_\bullet)$.

Of course, given an $\mathcal{O}_X$-module map $\varphi : \mathcal{E} \to \mathcal{O}_X$, if $\mathcal{E}$ is finite locally free, then $K_\bullet(\varphi)$ is locally on X isomorphic to a Koszul complex $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$.

17.25. Invertible modules

01CR Similarly to the case of modules over rings (More on Algebra, Section 15.117) we have the following definition.

01CS Definition 17.25.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An invertible $\mathcal{O}_X$-module is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that the functor

$$\text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an equivalence of categories. We say that $\mathcal{L}$ is trivial if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$.

Lemma 17.25.4 below explains the relationship with locally free modules of rank 1.

0B8K Lemma 17.25.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Equivalent are

(1) $\mathcal{L}$ is invertible, and
(2) there exists an $\mathcal{O}_X$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \cong \mathcal{O}_X$.

In this case $\mathcal{L}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module and the module $\mathcal{N}$ in (2) is isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Proof. Assume (1). Then the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ is essentially surjective, hence there exists an $\mathcal{O}_X$-module $\mathcal{N}$ as in (2). If (2) holds, then the functor $- \otimes_{\mathcal{O}_X} \mathcal{N}$ is a quasi-inverse to the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ and we see that (1) holds.

Assume (1) and (2) hold. Denote $\psi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{O}_X$ the given isomorphism. Let $x \in X$. Choose an open neighbourhood U an integer $n \geq 1$ and sections $s_i \in \mathcal{L}(U)$, $t_i \in \mathcal{N}(U)$ such that $\psi(\sum s_i \otimes t_i) = 1$. Consider the isomorphisms

$$\mathcal{L}|_U \to \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{N}|_U \to \mathcal{L}|_U$$

where the first arrow sends s to $\sum s_i \otimes s \otimes t_i$ and the second arrow sends $s \otimes s' \otimes t$ to $\psi(s' \otimes t)s$. We conclude that $s \mapsto \sum \psi(s \otimes t_i)s_i$ is an automorphism of $\mathcal{L}|_U$. This automorphism factors as

$$\mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{L}|_U$$

where the first arrow is given by $s \mapsto (\psi(s \otimes t_1), \ldots, \psi(s \otimes t_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i s_i$. In this way we conclude that $\mathcal{L}|_U$ is a direct summand of a finite free $\mathcal{O}_U$-module.

Assume (1) and (2) hold. Consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

To finish the proof of the lemma we will show this is an isomorphism by checking it induces isomorphisms on stalks. Let $x \in X$. Since we know (by the previous paragraph) that $\mathcal{L}$ is a finitely presented $\mathcal{O}_X$-module we can use Lemma 17.22.4 to see that it suffices to show that

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{L}_x, \mathcal{O}_{X,x}) \longrightarrow \mathcal{O}_{X,x}$$

is an isomorphism. Since $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x = (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N})_x = \mathcal{O}_{X,x}$ (Lemma 17.16.1) the desired result follows from More on Algebra, Lemma 15.117.2. □

0B8L **Lemma 17.25.3.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{L}$ of an invertible $\mathcal{O}_Y$-module is invertible.

Proof. By Lemma 17.25.2 there exists an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{N} \cong \mathcal{O}_Y$. Pulling back we get $f^*\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N} \cong \mathcal{O}_X$ by Lemma 17.16.4. Thus $f^*\mathcal{L}$ is invertible by Lemma 17.25.2. □

0B8M **Lemma 17.25.4.** Let $(X, \mathcal{O}_X)$ be a ringed space. Any locally free $\mathcal{O}_X$-module of rank 1 is invertible. If all stalks $\mathcal{O}_{X,x}$ are local rings, then the converse holds as well (but in general this is not the case).

Proof. The parenthetical statement follows by considering a one point space X with sheaf of rings $\mathcal{O}_X$ given by a ring R. Then invertible $\mathcal{O}_X$-modules correspond to invertible R-modules, hence as soon as $\mathrm{Pic}(R)$ is not the trivial group, then we get an example.

Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

Looking over an open covering trivialization $\mathcal{L}$, we see that this map is an isomorphism. Hence $\mathcal{L}$ is invertible by Lemma 17.25.2.

Assume all stalks $\mathcal{O}_{X,x}$ are local rings and $\mathcal{L}$ invertible. In the proof of Lemma 17.25.2 we have seen that $\mathcal{L}_x$ is an invertible $\mathcal{O}_{X,x}$-module for all $x \in X$. Since $\mathcal{O}_{X,x}$ is local, we see that $\mathcal{L}_x \cong \mathcal{O}_{X,x}$ (More on Algebra, Section 15.117). Since $\mathcal{L}$ is of finite presentation by Lemma 17.25.2 we conclude that $\mathcal{L}$ is locally free of rank 1 by Lemma 17.11.6. $\square$

01CT Lemma 17.25.5. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$.
(2) If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then so is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ and the evaluation map $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$ is an isomorphism.

Proof. Part (1) is clear from the definition and part (2) follows from Lemma 17.25.2 and its proof. $\square$

01CU Definition 17.25.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X and $n \in \mathbf{Z}$ we define the nth tensor power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ as the image of $\mathcal{O}_X$ under applying the equivalence $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ exactly n times.

This makes sense also for negative n as we've defined an invertible $\mathcal{O}_X$-module as one for which tensoring is an equivalence. More explicitly, we have

$$\mathcal{L}^{\otimes n} = \begin{cases} \mathcal{O}_X & \text{if} \quad n = 0 \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) & \text{if} \quad n = -1 \\ \mathcal{L} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L} & \text{if} \quad n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} & \text{if} \quad n < -1 \end{cases}$$

see Lemma 17.25.5. With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms satisfy a commutativity and an associativity constraint (formulation omitted).

Let $(X, \mathcal{O}_X)$ be a ringed space. We can define a $\mathbf{Z}$-graded ring structure on $\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ by mapping $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ to the section corresponding to $s \otimes t$ in $\Gamma(X, \mathcal{L}^{\otimes n+m})$. We omit the verification that this defines a commutative and associative ring with 1. However, by our conventions in Algebra, Section 10.56 a graded ring has no nonzero elements in negative degrees. This leads to the following definition.

01CV Definition 17.25.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X we define the associated graded ring to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

We often write simply $\Gamma_*(\mathcal{L})$ and $\Gamma_*(\mathcal{F})$ (although this is ambiguous if $\mathcal{F}$ is invertible). The multiplication of $\Gamma_*(\mathcal{L})$ on $\Gamma_*(\mathcal{F})$ is defined using the isomorphisms above. If $\gamma : \mathcal{F} \to \mathcal{G}$ is a $\mathcal{O}_X$-module map, then we get an $\Gamma_*(\mathcal{L})$-module homomorphism $\gamma : \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$. If $\alpha : \mathcal{L} \to \mathcal{N}$ is an $\mathcal{O}_X$-module map between invertible $\mathcal{O}_X$-modules, then we obtain a graded ring homomorphism $\Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{N})$. If $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces and if $\mathcal{L}$ is invertible on X, then we get an invertible sheaf $f^*\mathcal{L}$ on Y (Lemma 17.25.3) and an induced homomorphism of graded rings

$$f^* : \Gamma_*(X, \mathcal{L}) \longrightarrow \Gamma_*(Y, f^*\mathcal{L})$$

Furthermore, there are some compatibilities between the constructions above whose statements we omit.

01CW Lemma 17.25.8. Let $(X, \mathcal{O}_X)$ be a ringed space. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on X is isomorphic to exactly one of the $\mathcal{L}_i$.

Proof. Recall that any invertible $\mathcal{O}_X$-module is locally a direct summand of a finite free $\mathcal{O}_X$-module, see Lemma 17.25.2. For each open covering $\mathcal{U} : X = \bigcup_{j \in J} U_j$ and map $r : J \to \mathbf{N}$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{F}_j = \mathcal{F}|_{U_j}$ is a direct summand of $\mathcal{O}_{U_j}^{\oplus r(j)}$. The collection of isomorphism classes of $\mathcal{F}_j$ is a set, because $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}, \mathcal{O}_U^{\oplus r})$ is a set. The sheaf $\mathcal{F}$ is gotten by glueing $\mathcal{F}_j$, see Sheaves, Section 6.33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. For each covering there is a set of maps $r : J \to \mathbf{N}$. Hence the collection of all $\mathcal{F}$ forms a set. $\square$

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 17.25.5 says that tensor product defines the structure of an abelian group on this set.

01CX Definition 17.25.9. Let $(X, \mathcal{O}_X)$ be a ringed space. The Picard group $\mathrm{Pic}(X)$ of X is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}_X$-modules, with addition corresponding to tensor product.

01CY Lemma 17.25.10. Let X be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set

$$X_s = \{x \in X \mid \text{image } s \notin \mathfrak{m}_x \mathcal{L}_x\}$$

is open in X. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathcal{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

Proof. Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma 17.25.5. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma 10.20.1 that s_x is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood U of x such that t_x comes from a section t of $\mathcal{L}^{\otimes -1}$ over U and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that s generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that X_s is open. Moreover, the section t constructed over U above is unique, and hence these glue to give the section s' of the lemma. $\square$

It is also true that, given a morphism of locally ringed spaces $f : Y \to X$ (see Schemes, Definition 26.2.1) that the inverse image $f^{-1}(X_s)$ is equal to Y_{f^*s}, where $f^*s \in \Gamma(Y, f^*\mathcal{L})$ is the pullback of s.

17.26. Rank and determinant

0B37 Let $(X, \mathcal{O}_X)$ be a ringed space. Consider the category $\text{Vect}(X)$ of finite locally free $\mathcal{O}_X$-modules. This is an exact category (see Injectives, Remark 19.9.6) whose admissible epimorphisms are surjections and whose admissible monomorphisms are kernels of surjections. Moreover, there is a set of isomorphism classes of objects of $\text{Vect}(X)$ (proof omitted). Thus we can form the zeroth Grothendieck K-group $K_0(\text{Vect}(X))$. Explicitly, in this case $K_0(\text{Vect}(X))$ is the abelian group generated by $[\mathcal{E}]$ for $\mathcal{E}$ a finite locally free $\mathcal{O}_X$-module, subject to the relations

$$[\mathcal{E}'] = [\mathcal{E}] + [\mathcal{E}'']$$

whenever there is a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules.

Ranks. Assume all stalks $\mathcal{O}_{X,x}$ are nonzero rings. Given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$, the rank is a locally constant function

$$\text{rank}_\mathcal{E} : X \longrightarrow \mathbf{Z}_{\geq 0}, \quad x \longmapsto \text{rank}_{\mathcal{O}_{X,x}} \mathcal{E}_x$$

See Lemma 17.14.4. By definition of locally free modules the function $\text{rank}_\mathcal{E}$ is locally constant. If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of finite locally free $\mathcal{O}_X$-modules, then $\text{rank}_\mathcal{E} = \text{rank}_{\mathcal{E}'} + \text{rank}_{\mathcal{E}''}$, Thus the rank defines a homomorphism

$$K_0(\text{Vect}(X)) \longrightarrow \text{Map}_{cont}(X, \mathbf{Z}), \quad [\mathcal{E}] \longmapsto \text{rank}_\mathcal{E}$$

Determinants. Given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ we obtain a disjoint union decomposition

$$X = X_0 \amalg X_1 \amalg X_2 \amalg \ldots$$

with X_i open and closed, such that $\mathcal{E}$ is finite locally free of rank i on X_i (this is exactly the same as saying the $\text{rank}_\mathcal{E}$ is locally constant). In this case we define $\det(\mathcal{E})$ as the invertible sheaf on X which is equal to $\wedge^i(\mathcal{E}|_{X_i})$ on X_i for all $i \geq 0$. Since the decomposition above is disjoint, there are no glueing conditions to check. By Lemma 17.26.1 below this defines a homomorphism

$$\det : K_0(\text{Vect}(X)) \longrightarrow \text{Pic}(X), \quad [\mathcal{E}] \longmapsto \det(\mathcal{E})$$

of abelian groups. The elements of $\text{Pic}(X)$ we get in this manner are locally free of rank 1 (see below the lemma for a generalization).

0B38 Lemma 17.26.1. Let X be a ringed space. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. Then there is a canonical isomorphism

$$\det(\mathcal{E}') \otimes_{\mathcal{O}_X} \det(\mathcal{E}'') \longrightarrow \det(\mathcal{E})$$

of $\mathcal{O}_X$-modules.

Proof. We can decompose X into disjoint open and closed subsets such that both $\mathcal{E}'$ and $\mathcal{E}''$ have constant rank on them. Thus we reduce to the case where $\mathcal{E}'$ and $\mathcal{E}''$ have constant rank, say r' and r''. In this situation we define

$$\wedge^{r'}(\mathcal{E}') \otimes_{\mathcal{O}_X} \wedge^{r''}(\mathcal{E}'') \longrightarrow \wedge^{r'+r''}(\mathcal{E})$$

as follows. Given local sections $s'_1, \ldots, s'_{r'}$ of $\mathcal{E}'$ and local sections $s''_1, \ldots, s''_{r''}$ of $\mathcal{E}''$ we map

$$s'_1 \wedge \ldots \wedge s'_{r'} \otimes s''_1 \wedge \ldots \wedge s''_{r''} \quad \text{to} \quad s'_1 \wedge \ldots \wedge s'_{r'} \wedge \tilde{s}''_1 \wedge \ldots \wedge \tilde{s}''_{r''}$$

where $\tilde{s}''_i$ is a local lift of the section s''_i to a section of $\mathcal{E}$. We omit the details. $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space. Instead of looking at finite locally free $\mathcal{O}_X$-modules we can look at those $\mathcal{O}_X$-modules $\mathcal{F}$ which are locally on X a direct summand of a finite free $\mathcal{O}_X$-module. This is the same thing as asking $\mathcal{F}$ to be a flat $\mathcal{O}_X$-module of finite presentation, see Lemma 17.18.3. If all the stalks $\mathcal{O}_{X,x}$ are local, then such a module $\mathcal{F}$ is finite locally free, see Lemma 17.14.6. In general however this will not be the case; for example X could be a point and $\Gamma(X, \mathcal{O}_X)$ could be the product $A \times B$ of two nonzero rings and $\mathcal{F}$ could correspond to $A \times 0$. Thus for such a module the rank function is undefined. However, it turns out we can still define $\det(\mathcal{F})$ and this will be an invertible $\mathcal{O}_X$-module in the sense of Definition 17.25.1 (not necessarily locally free of rank 1). Our construction will agree with the one above in the case that $\mathcal{F}$ is finite locally free. We urge the reader to skip the rest of this section.

0FJN Lemma 17.26.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat and finitely presented $\mathcal{O}_X$-module. Denote

$$\det(\mathcal{F}) \subset \wedge^*_{\mathcal{O}_X}(\mathcal{F})$$

the annihilator of $\mathcal{F} \subset \wedge^*_{\mathcal{O}_X}(\mathcal{F})$. Then $\det(\mathcal{F})$ is an invertible $\mathcal{O}_X$-module.

Proof. To prove this we may work locally on X. Hence we may assume $\mathcal{F}$ is a direct summand of a finite free module, see Lemma 17.18.3. Say $\mathcal{F} \oplus \mathcal{G} = \mathcal{O}_X^{\oplus n}$. Set $R = \mathcal{O}_X(X)$. Then we see $\mathcal{F}(X) \oplus \mathcal{G}(X) = R^{\oplus n}$ and correspondingly $\mathcal{F}(U) \oplus \mathcal{G}(U) = \mathcal{O}_X(U)^{\oplus n}$ for all opens $U \subset X$. We conclude that $\mathcal{F} = \mathcal{F}_M$ as in Lemma 17.10.5 with $M = \mathcal{F}(X)$ a finite projective R-module. In other words, we have $\mathcal{F}(U) = M \otimes_R \mathcal{O}_X(U)$. This implies that $\det(M) \otimes_R \mathcal{O}_X(U) = \det(\mathcal{F}(U))$ for all open $U \subset X$ with det as in More on Algebra, Section 15.118. By More on Algebra, Remark 15.118.1 we see that

$$\det(M) \otimes_R \mathcal{O}_X(U) = \det(\mathcal{F}(U)) \subset \wedge^*_{\mathcal{O}_X(U)}(\mathcal{F}(U))$$

is the annihilator of $\mathcal{F}(U)$. We conclude that $\det(\mathcal{F})$ as defined in the statement of the lemma is equal to $\mathcal{F}_{\det(M)}$. Some details omitted; one has to be careful as annihilators cannot be defined as the sheafification of taking annihilators on sections over opens. Thus $\det(\mathcal{F})$ is the pullback of an invertible module and we conclude. $\square$

17.27. Localizing sheaves of rings

01CZ Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset, see Algebra, Definition 10.9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O}_X : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U).$$

The restriction mapping sends the section f/s, $f \in \mathcal{O}_X(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ if $V \subset U$ are opens of X.

01D0 Lemma 17.27.1. Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset.

(1) There is a map of presheaves of rings $\mathcal{O}_X \to \mathcal{S}^{-1}\mathcal{O}_X$ such that every local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{O}_X$.
(2) For any homomorphism of presheaves of rings $\mathcal{O}_X \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $\mathcal{S}^{-1}\mathcal{O}_X \to \mathcal{A}$.
(3) For any $x \in X$ we have
$$(\mathcal{S}^{-1}\mathcal{O}_X)_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$
(4) The sheafification $(\mathcal{S}^{-1}\mathcal{O}_X)^\#$ is a sheaf of rings with a map of sheaves of rings $(\mathcal{O}_X)^\# \to (\mathcal{S}^{-1}\mathcal{O}_X)^\#$ which is universal for maps of $(\mathcal{O}_X)^\#$ into sheaves of rings such that each local section of $\mathcal{S}$ maps to an invertible section.
(5) For any $x \in X$ we have
$$(\mathcal{S}^{-1}\mathcal{O}_X)^\#_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$

Proof. Omitted. $\square$

Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}_X$-modules
$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$
The restriction mapping sends the section t/s, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \subset U$ are opens of X.

01D1 Lemma 17.27.2. Let X be a topological space. Let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. For any presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have
$$\mathcal{S}^{-1}\mathcal{F} = \mathcal{S}^{-1}\mathcal{O}_X \otimes_{p,\mathcal{O}_X} \mathcal{F}$$
(see Sheaves, Section 6.6 for notation) and if $\mathcal{F}$ and $\mathcal{O}_X$ are sheaves then
$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O}_X)^\# \otimes_{\mathcal{O}_X} \mathcal{F}$$
(see Sheaves, Section 6.20 for notation).

Proof. Omitted. $\square$

17.28. Modules of differentials

08RL In this section we briefly explain how to define the module of relative differentials for a morphism of ringed spaces. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.131).

01UN Definition 17.28.1. Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. An $\mathcal{O}_1$-derivation or more precisely a φ-derivation into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the Leibniz rule
$$D(ab) = aD(b) + D(a)b$$
for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 10.131.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(X, \mathcal{O}_2) \longrightarrow \Gamma(X, \mathcal{F})$$

is a $\Gamma(X, \mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

08RM Lemma 17.28.2. Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor

$$\text{Mod}(\mathcal{O}_2) \longrightarrow \text{Ab}, \quad \mathcal{F} \longmapsto \text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on X, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_2(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over U. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

08RN (17.28.2.1)
$$\begin{matrix} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a, b)] \oplus [(f, g)] \oplus [h] & \longmapsto & [a + b] - [a] - [b] + \\ & & [fg] - g[f] - f[g] + \\ & & [\varphi(h)] \end{matrix}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends $[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (17.28.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. $\square$

08RP Definition 17.28.3. Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. The module of differentials of φ is the object representing the functor $\mathcal{F} \mapsto \text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 17.28.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the universal φ-derivation is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.28.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

08TD **Lemma 17.28.4.** Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Then $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.

Proof. Consider the map (17.28.2.1). There is a similar map of presheaves whose value on the open U is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over U by the construction of the module of differentials in Algebra, Definition 10.131.2. On the other hand, the sheaves in (17.28.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. □

08RQ **Lemma 17.28.5.** Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For $U \subset X$ open there is a canonical isomorphism

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. Holds because $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.28.2.1). □

08RR **Lemma 17.28.6.** Let $f : Y \to X$ be a continuous map of topological spaces. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.

Proof. This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.28.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor f^{-1} is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. □

08TE **Lemma 17.28.7.** Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $x \in X$. Then we have $\Omega_{\mathcal{O}_2/\mathcal{O}_1,x} = \Omega_{\mathcal{O}_{2,x}/\mathcal{O}_{1,x}}$.

Proof. This is a special case of Lemma 17.28.6 for the inclusion map $\{x\} \to X$. An alternative proof is to use Lemma 17.28.4, Sheaves, Lemma 6.17.2, and Algebra, Lemma 10.131.5 □

08RS **Lemma 17.28.8.** Let X be a topological space. Let

$$\begin{tikzcd} \mathcal{O}_2 \arrow[r, "\varphi"'] & \mathcal{O}_2' \\ \mathcal{O}_1 \arrow[u] \arrow[r] & \mathcal{O}_1' \arrow[u] \end{tikzcd}$$

be a commutative diagram of sheaves of rings on X. The map $\mathcal{O}_2 \to \mathcal{O}_2'$ composed with the map $\mathrm{d} : \mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$. It is uniquely characterized by the property that $\mathrm{d}(f) \mapsto \mathrm{d}(\varphi(f))$ for any local section f of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.

Proof. This lemma proves itself. □

08TF Lemma 17.28.9. In Lemma 17.28.8 suppose that $\mathcal{O}_2 \to \mathcal{O}_2'$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}_1'$. Then there is a canonical exact sequence of $\mathcal{O}_2'$-modules

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}_2' \longrightarrow \Omega_{\mathcal{O}_2'/\mathcal{O}_1} \longrightarrow 0$$

The leftmost map is characterized by the rule that a local section f of $\mathcal{I}$ maps to $\mathrm{d}f \otimes 1$.

Proof. For a local section f of $\mathcal{I}$ denote $\overline{f}$ the image of f in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1f_2 \otimes 1 = 0$ if f_1, f_2 are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1f_2 \otimes 1 = (f_1\mathrm{d}f_2 + f_2\mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_1 \otimes f_2 = 0$. A similar computation show this map is $\mathcal{O}_2' = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 17.28.8. To see that the sequence is exact, we can check on stalks (Lemma 17.3.1). By Lemma 17.28.7 this follows from Algebra, Lemma 10.131.9. □

08RT Definition 17.28.10. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An S-derivation into $\mathcal{F}$ is a $f^{-1}\mathcal{O}_S$-derivation, or more precisely a $f^\sharp$-derivation in the sense of Definition 17.28.1. We denote $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$ the set of S-derivations into $\mathcal{F}$.
(2) The sheaf of differentials $\Omega_{X/S}$ of X over S is the module of differentials $\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$ endowed with its universal S-derivation $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$.

Here is a particular situation where derivations come up naturally.

01UP Lemma 17.28.11. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

Here $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I} = \mathrm{Ker}(\pi)$ is its kernel. Assume $\mathcal{I}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{I}$ has a natural structure of an $\mathcal{O}_X$-module. A section $s : \mathcal{O}_X \to \mathcal{A}$ of π is a $f^{-1}\mathcal{O}_S$-algebra map such that $\pi \circ s = \mathrm{id}$. Given any section $s : \mathcal{O}_X \to \mathcal{A}$ of π and any S-derivation $D : \mathcal{O}_X \to \mathcal{I}$ the map

$$s + D : \mathcal{O}_X \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique S-derivation D.

Proof. Recall that the $\mathcal{O}_X$-module structure on $\mathcal{I}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_X$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{I}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned} (s + D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b)) \end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $f^{-1}\mathcal{O}_S$-algebra map because D is an S-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. □

08RU **Lemma 17.28.12.** Let

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "h'"] & X \arrow[d, "h"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a commutative diagram of ringed spaces.

(1) The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with $f_*\mathrm{d}_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation and we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$.
(2) The commutative diagram

$$\begin{tikzcd} f^{-1}\mathcal{O}_X \arrow[r] & \mathcal{O}_{X'} \\ f^{-1}h^{-1}\mathcal{O}_S \arrow[r] \arrow[u] & (h')^{-1}\mathcal{O}_{S'} \arrow[u] \end{tikzcd}$$

induces by Lemmas 17.28.6 and 17.28.8 a canonical map $f^{-1}\Omega_{X/S} \to \Omega_{X'/S'}$.

These two maps correspond (via adjointness of f_* and f^* and via $f^*\Omega_{X/S} = f^{-1}\Omega_{X/S} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}$ and Sheaves, Lemma 6.20.2) to the same $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}$$

which is uniquely characterized by the property that $f^*\mathrm{d}_{X/S}(a)$ maps to $\mathrm{d}_{X'/S'}(f^*a)$ for any local section a of $\mathcal{O}_X$.

Proof. Omitted. □

01UW **Lemma 17.28.13.** Let

$$\begin{tikzcd} X'' \arrow[r, "g"] \arrow[d] & X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

be a commutative diagram of ringed spaces. With notation as in Lemma 17.28.12 we have

$$c_{f \circ g} = c_g \circ g^*c_f$$

as maps $(f \circ g)^*\Omega_{X/S} \to \Omega_{X''/S''}$.

Proof. Omitted. □

17.29. Finite order differential operators

0G3P In this section we introduce differential operators of finite order. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.133).

0G3Q **Definition 17.29.1.** Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $k \geq 0$ be an integer. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. A differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k is an is an $\mathcal{O}_1$-linear map such that for all local sections g of $\mathcal{O}_2$ the map $s \mapsto D(gs) - gD(s)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an $\mathcal{O}_2$-linear map.

If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of order k, then for all local sections g of $\mathcal{O}_2$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\text{Diff}^k(\mathcal{F}, \mathcal{G}) = \text{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G})$$

is a $\Gamma(X, \mathcal{O}_2)$-module. We have

$$\text{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

The rule which maps $U \subset X$ open to the module of differential operators $D : \mathcal{F}|_U \to \mathcal{G}|_U$ of order k is a sheaf of $\mathcal{O}_2$-modules on X. Thus we obtain a sheaf of differential operators (if we ever need this we will add a definition here).

0G3R Lemma 17.29.2. Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. If $D : \mathcal{E} \to \mathcal{F}$ and $D' : \mathcal{F} \to \mathcal{G}$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.

Proof. Let g be a local section of $\mathcal{O}_2$. Then the map which sends a local section x of $\mathcal{E}$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. □

0G3S Lemma 17.29.3. Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. Let $k \geq 0$. There exists a sheaf of $\mathcal{O}_2$-modules $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ and a canonical isomorphism

$$\text{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}), \mathcal{G})$$

functorial in the $\mathcal{O}_2$-module $\mathcal{G}$.

Proof. The existence follows from general category theoretic arguments (insert future reference here), but we will also give a direct construction as this construction will be useful in the future proofs. We will freely use the notation introduced in the proof of Lemma 17.28.2. Given any differential operator $D : \mathcal{F} \to \mathcal{G}$ we obtain an $\mathcal{O}_2$-linear map $L_D : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ sending $[m]$ to $D(m)$. If D has order 0 then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where g_0 is a local section of $\mathcal{O}_2$ and m, m' are local sections of $\mathcal{F}$. If D has order 1, then L_D annihilates the local sections

$$[m + m' - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where f is a local section of $\mathcal{O}_1$, g_0, g_1 are local sections of $\mathcal{O}_2$, and m, m' are local sections of $\mathcal{F}$. If D has order k, then L_D annihilates the local sections $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the local sections

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ is an $\mathcal{O}_2$-linear map annihilating all the local sections listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ is the quotient of $\mathcal{O}_2[\mathcal{F}]$ by the $\mathcal{O}_2$-submodule generated by these local sections. □

0G3T Definition 17.29.4. Let X be a topoological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. The module $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ constructed in Lemma 17.29.3 is called the module of principal parts of order k of $\mathcal{F}$.

Note that the inclusions

$$\mathrm{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 4.3.5) to surjections

$$\ldots \to \mathcal{P}^2_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^0_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) = \mathcal{F}$$

0G3U Lemma 17.29.5. Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_2$-modules. Then $\mathcal{P}^k_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}(\mathcal{F}^\#)$ is the sheaf associated to the presheaf $U \mapsto P^k_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}(\mathcal{F}(U))$.

Proof. This can be proved in exactly the same way as is done for the sheaf of differentials in Lemma 17.28.4. Perhaps a more pleasing approach is to use the universal property of Lemma 17.29.3 directly to see the equality. We omit the details. □

0G3V Lemma 17.29.6. Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. There is a canonical short exact sequence

$$0 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{F} \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{F} \to 0$$

functorial in $\mathcal{F}$ called the sequence of principal parts.

Proof. Follows from the commutative algebra version (Algebra, Lemma 10.133.6) and Lemmas 17.28.4 and 17.29.5. □

0G3W Remark 17.29.7. Let X be a topological space. Suppose given a commutative diagram of sheaves of rings

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

on X, a $\mathcal{B}$-module $\mathcal{F}$, a $\mathcal{B}'$-module $\mathcal{F}'$, and a $\mathcal{B}$-linear map $\mathcal{F} \to \mathcal{F}'$. Then we get a compatible system of module maps

$$\begin{tikzcd} \ldots \arrow[r] & \mathcal{P}^2_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \arrow[r] & \mathcal{P}^1_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \arrow[r] & \mathcal{P}^0_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \\ \ldots \arrow[r] & \mathcal{P}^2_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[r] \arrow[u] & \mathcal{P}^1_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[r] \arrow[u] & \mathcal{P}^0_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[u] \end{tikzcd}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $\mathcal{P}^k_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ in terms of (local) generators and relations in the proof of Lemma 17.29.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 17.29.6.

Next, we extend our definition to morphisms of ringed spaces.

0G3X Definition 17.29.8. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $k \geq 0$ be an integer. A differential operator of order k on X/S is a differential operator $D : \mathcal{F} \to \mathcal{G}$ with respect to $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ We denote $\text{Diff}^k_{X/S}(\mathcal{F}, \mathcal{G})$ the set of these differential operators.

17.30. The de Rham complex

0FKL The section is the analogue of Algebra, Section 10.132 for morphisms of ringed spaces. We urge the reader to read that section first.

Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Denote $\text{d} : \mathcal{B} \to \Omega_{\mathcal{B}/\mathcal{A}}$ the module of differentials with its universal $\mathcal{A}$-derivation constructed in Section 17.28. Let

$$\Omega^i_{\mathcal{B}/\mathcal{A}} = \wedge^i_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{A}})$$

for $i \geq 0$ be the ith exterior power as in Section 17.21.

0FKM Definition 17.30.1. In the situation above, the de Rham complex of $\mathcal{B}$ over $\mathcal{A}$ is the unique complex

$$\Omega^0_{\mathcal{B}/\mathcal{A}} \to \Omega^1_{\mathcal{B}/\mathcal{A}} \to \Omega^2_{\mathcal{B}/\mathcal{A}} \to \ldots$$

of sheaves of $\mathcal{A}$-modules whose differential in degree 0 is given by $\text{d} : \mathcal{B} \to \Omega_{\mathcal{B}/\mathcal{A}}$ and whose differentials in higher degrees have the following property

0FKN (17.30.1.1) $$\text{d}\left(b_0 \text{d}b_1 \wedge \ldots \wedge \text{d}b_p\right) = \text{d}b_0 \wedge \text{d}b_1 \wedge \ldots \wedge \text{d}b_p$$

where $b_0, \ldots, b_p \in \mathcal{B}(U)$ are sections over a common open $U \subset X$.

We could construct this complex by repeating the cumbersome arguments given in Algebra, Section 10.132. Instead we recall that $\Omega_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the presheaf $U \mapsto \Omega_{\mathcal{B}(U)/\mathcal{A}(U)}$, see Lemma 17.28.4. Thus $\Omega^i_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the presheaf $U \mapsto \Omega^i_{\mathcal{B}(U)/\mathcal{A}(U)}$, see Lemma 17.21.1. Therefore we can define the de Rham complex as the sheafification of the rule

$$U \longmapsto \Omega^\bullet_{\mathcal{B}(U)/\mathcal{A}(U)}$$

0FKP Lemma 17.30.2. Let $f : Y \to X$ be a continuous map of topological spaces. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. Then there is a canonical identification $f^{-1}\Omega^\bullet_{\mathcal{B}/\mathcal{A}} = \Omega^\bullet_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$ of de Rham complexes.

Proof. Omitted. Hint: compare with Lemma 17.28.6. ☐

0G3Y Lemma 17.30.3. Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. The differentials $\text{d} : \Omega^i_{\mathcal{B}/\mathcal{A}} \to \Omega^{i+1}_{\mathcal{B}/\mathcal{A}}$ are differential operators of order 1.

Proof. Via our construction of the de Rham complex above as the sheafification of the rule $U \mapsto \Omega^\bullet_{\mathcal{B}(U)/\mathcal{A}(U)}$ this follows from Algebra, Lemma 10.133.8. ☐

Let X be a topological space. Let

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

be a commutative diagram of sheaves of rings on X. There is a natural map of de Rham complexes

$$\Omega^\bullet_{\mathcal{B}/\mathcal{A}} \longrightarrow \Omega^\bullet_{\mathcal{B}'/\mathcal{A}'}$$

Namely, in degree 0 this is the map $\mathcal{B} \to \mathcal{B}'$, in degree 1 this is the map $\Omega_{\mathcal{B}/\mathcal{A}} \to \Omega_{\mathcal{B}'/\mathcal{A}'}$ constructed in Section 17.28, and for $p \geq 2$ it is the induced map $\Omega^p_{\mathcal{B}/\mathcal{A}} = \wedge^p_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{A}}) \to \wedge^p_{\mathcal{B}'}(\Omega_{\mathcal{B}'/\mathcal{A}'}) = \Omega^p_{\mathcal{B}'/\mathcal{A}'}$. The compatibility with differentials follows from the characterization of the differentials by the formula (17.30.1.1).

0FKQ Definition 17.30.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The de Rham complex of f or of X over Y is the complex

$$\Omega^\bullet_{X/Y} = \Omega^\bullet_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$$

Consider a commutative diagram of ringed spaces

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d, "h'"'] & X \arrow[d, "h"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then we obtain a canonical map

$$\Omega^\bullet_{X/S} \to f_*\Omega^\bullet_{X'/S'}$$

of de Rham complexes. Namely, the commutative diagram of sheaves of rings

$$\begin{tikzcd} f^{-1}\mathcal{O}_X \arrow[r] & \mathcal{O}_{X'} \\ f^{-1}h^{-1}\mathcal{O}_S \arrow[u] \arrow[r] & (h')^{-1}\mathcal{O}_{S'} \arrow[u] \end{tikzcd}$$

on X' produces a map of complexes (see above)

$$f^{-1}\Omega^\bullet_{X/S} = \Omega^\bullet_{f^{-1}\mathcal{O}_X/f^{-1}h^{-1}\mathcal{O}_S} \longrightarrow \Omega^\bullet_{\mathcal{O}_{X'}/(h')^{-1}\mathcal{O}_{S'}} = \Omega^\bullet_{X'/S'}$$

(using Lemma 17.30.2 for the first equality) and then we can use adjunction.

0G3Z Lemma 17.30.5. Let $f : X \to Y$ be a morphism of ringed spaces. The differentials $\mathrm{d} : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$ are differential operators of order 1 on X/Y.

Proof. Immediate from Lemma 17.30.3 and the definition. $\square$

17.31. The naive cotangent complex

08TG This section is the analogue of Algebra, Section 10.134 for morphisms of ringed spaces. We urge the reader to read that section first.

Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. In this section, for any sheaf of sets $\mathcal{E}$ on X we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$ whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$ the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

08TH (17.31.0.1)
$$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 17.28.9 there is a canonical map

08TI (17.31.0.2)
$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

08TJ Definition 17.31.1. Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. The *naive cotangent complex* $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (17.31.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in degree -1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 17.28.8 for modules of differentials. Namely, given a commutative diagram

08TK (17.31.1.1)
$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

of sheaves of rings on X there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$ and a map between modules of differentials, which together give the desired map between the naive cotangent complexes. The map is compatible with compositions in the following sense: given a commutative diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \arrow[r] & \mathcal{B}'' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \arrow[r] & \mathcal{A}'' \arrow[u] \end{tikzcd}$$

of sheaves of rings then the composition

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'} \longrightarrow NL_{\mathcal{B}''/\mathcal{A}''}$$

is the map for the outer rectangle.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that $\mathcal{E}$ is a sheaves of sets on X and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. If this map is surjective, i.e., if $\alpha(\mathcal{E})$ generates $\mathcal{B}$ as an $\mathcal{A}$-algebra, then we set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

where $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ is the kernel of the surjection $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Here is the result.

08TL Lemma 17.31.2. In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.

Proof. Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\text{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \text{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \text{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this it suffices to check on stalks (Lemma 17.3.1). If $x \in X$ then the stalk of $NL(\alpha)$ is the complex $NL(\alpha_x)$ of Algebra, Section 10.134 associated to the presentation $\mathcal{A}_x[\mathcal{E}_x] \to \mathcal{B}_x$ coming from the map $\alpha_x : \mathcal{E}_x \to \mathcal{B}_x$. (Some details omitted; use Lemma 17.28.7 to see compatibility of forming differentials and taking stalks.) We conclude the result holds by Algebra, Lemma 10.134.2. □

08TM Lemma 17.31.3. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on Y. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.

Proof. Omitted. Hint: Use Lemma 17.28.6. □

0D09 Lemma 17.31.4. Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on X. Let $x \in X$. Then we have $NL_{\mathcal{B}/\mathcal{A},x} = NL_{\mathcal{B}_x/\mathcal{A}_x}$.

Proof. This is a special case of Lemma 17.31.3 for the inclusion map $\{x\} \to X$. □

0E1Y Lemma 17.31.5. Let X be a topological space. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be maps of sheaves of rings. Let C be the cone (Derived Categories, Definition 13.9.1) of the map of complexes $NL_{\mathcal{C}/\mathcal{A}} \to NL_{\mathcal{C}/\mathcal{B}}$. There is a canonical map

$$c : NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \longrightarrow C[-1]$$

of complexes of $\mathcal{C}$-modules which produces a canonical six term exact sequence

$$\begin{tikzcd} H^0(NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C}) \arrow[r] & H^0(NL_{\mathcal{C}/\mathcal{A}}) \arrow[r] & H^0(NL_{\mathcal{C}/\mathcal{B}}) \arrow[r] & 0 \\ H^{-1}(NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C}) \arrow[r] & H^{-1}(NL_{\mathcal{C}/\mathcal{A}}) \arrow[r] & H^{-1}(NL_{\mathcal{C}/\mathcal{B}}) \arrow[ull] \end{tikzcd}$$

of cohomology sheaves.

Proof. To give the map c we have to give a map $c_1 : NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \to NL_{\mathcal{C}/\mathcal{A}}$ and an explicit homotopy between the composition

$$NL_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \to NL_{\mathcal{C}/\mathcal{A}} \to NL_{\mathcal{C}/\mathcal{B}}$$

and the zero map, see Derived Categories, Lemma 13.9.3. For c_1 we use the functoriality described above for the obvious diagram. For the homotopy we use the map

$$NL^0_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \longrightarrow NL^{-1}_{\mathcal{C}/\mathcal{B}}, \quad \text{d}[b] \otimes 1 \longmapsto [\varphi(b)] - b[1]$$

where $\varphi : \mathcal{B} \to \mathcal{C}$ is the given map. Please compare with Algebra, Remark 10.134.5. To see the consequence for cohomology sheaves, it suffices to show that $H^0(c)$ is an isomorphism and $H^{-1}(c)$ surjective. To see this we can look at stalks, see Lemma

17.31.4, and then we can use the corresponding result in commutative algebra, see Algebra, Lemma 10.134.4. Some details omitted. □

The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

08TN **Definition 17.31.6.** The naive cotangent complex $NL_f = NL_{X/Y}$ of a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is $NL_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$.

Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

of ringed spaces, there is a canonical map $c : g^* NL_{X/Y} \to NL_{X'/Y'}$. Namely, it is the map

$$g^* NL_{X/Y} = \mathcal{O}_{X'} \otimes_{g^{-1}\mathcal{O}_X} NL_{g^{-1}\mathcal{O}_X/g^{-1}f^{-1}\mathcal{O}_Y} \longrightarrow NL_{\mathcal{O}_{X'}/(f')^{-1}\mathcal{O}_{Y'}} = NL_{X'/Y'}$$

where the arrow comes from the commutative diagram of sheaves of rings

$$\begin{tikzcd} g^{-1}\mathcal{O}_X \arrow[r, "g^\sharp"'] & \mathcal{O}_{X'} \\ g^{-1}f^{-1}\mathcal{O}_Y \arrow[u, "g^{-1}f^\sharp"] \arrow[r, "g^{-1}h^\sharp"] & (f')^{-1}\mathcal{O}_{Y'} \arrow[u, "(f')^\sharp"'] \end{tikzcd}$$

as in (17.31.1.1) above. Given a second such diagram

$$\begin{tikzcd} X'' \arrow[r, "g'"'] \arrow[d] & X' \arrow[d] \\ Y'' \arrow[r] & Y' \end{tikzcd}$$

the composition of $(g')^*c$ and the map $c' : (g')^* NL_{X'/Y'} \to NL_{X''/Y''}$ is the map $(g \circ g')^* NL_{X''/Y''} \to NL_{X/Y}$.

0E1Z **Lemma 17.31.7.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let C be the cone of the map $NL_{X/Z} \to NL_{X/Y}$ of complexes of $\mathcal{O}_X$-modules. There is a canonical map

$$f^* NL_{Y/Z} \to C[-1]$$

which produces a canonical six term exact sequence

$$\begin{tikzcd} H^0(f^* NL_{Y/Z}) \arrow[r] & H^0(NL_{X/Z}) \arrow[r] & H^0(NL_{X/Y}) \arrow[r] & 0 \\ H^{-1}(f^* NL_{Y/Z}) \arrow[r] & H^{-1}(NL_{X/Z}) \arrow[r] & H^{-1}(NL_{X/Y}) \arrow[ull] & \end{tikzcd}$$

of cohomology sheaves.

Proof. Consider the maps of sheaves rings

$$(g \circ f)^{-1}\mathcal{O}_Z \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

and apply Lemma 17.31.5. □

17.32. Other chapters

CHAPTER 18

Modules on Sites

03A4 18.1. Introduction

03A5 In this document we work out basic notions of sheaves of modules on ringed topoi or ringed sites. We first work out some basic facts on abelian sheaves. After this we introduce ringed sites and ringed topoi. We work through some of the very basic notions on (pre)sheaves of $\mathcal{O}$-modules, analogous to the material on (pre)sheaves of $\mathcal{O}$-modules in the chapter on sheaves on spaces. Having done this, we duplicate much of the discussion in the chapter on sheaves of modules (see Modules, Section 17.1). Basic references are [Ser55b], [DG67] and [AGV71].

18.2. Abelian presheaves

03A6 Let $\mathcal{C}$ be a category. Abelian presheaves were introduced in Sites, Sections 7.2 and 7.7 and discussed a bit more in Sites, Section 7.44. We will follow the convention of this last reference, in that we think of an abelian presheaf as a presheaf of sets endowed with addition rules on all sets of sections compatible with the restriction mappings. Recall that the category of abelian presheaves on $\mathcal{C}$ is denoted $\textit{PAb}(\mathcal{C})$.

The category $\textit{PAb}(\mathcal{C})$ is abelian as defined in Homology, Definition 12.5.1. Given a map of presheaves $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ the kernel of φ is the abelian presheaf $U \mapsto \text{Ker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$ and the cokernel of φ is the presheaf $U \mapsto \text{Coker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$. Since the category of abelian groups is abelian it follows that $\text{Coim} = \text{Im}$ because this holds over each U. A sequence of abelian presheaves

$$\mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3$$

is exact if and only if $\mathcal{G}_1(U) \to \mathcal{G}_2(U) \to \mathcal{G}_3(U)$ is an exact sequence of abelian groups for all $U \in \text{Ob}(\mathcal{C})$. We leave the verifications to the reader.

03CL **Lemma 18.2.1.** *Let $\mathcal{C}$ be a category.*

1. *All limits and colimits exist in $\textit{PAb}(\mathcal{C})$.*
2. *All limits and colimits commute with taking sections over objects of $\mathcal{C}$.*

Proof. Let $\mathcal{I} \to \textit{PAb}(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. We can simply define abelian presheaves L and C by the rules

$$L : U \longmapsto \lim_i \mathcal{F}_i(U)$$

and

$$C : U \longmapsto \text{colim}_i \mathcal{F}_i(U).$$

It is clear that there are maps of abelian presheaves $L \to \mathcal{F}_i$ and $\mathcal{F}_i \to C$, by using the corresponding maps on groups of sections over each U. It is straightforward to check that L and C endowed with these maps are the limit and colimit of the diagram in $\textit{PAb}(\mathcal{C})$. This proves (1) and (2). Details omitted. $\square$

18.3. Abelian sheaves

03CM Let $\mathcal{C}$ be a site. The category of abelian sheaves on $\mathcal{C}$ is denoted $\text{Ab}(\mathcal{C})$. It is the full subcategory of $\text{PAb}(\mathcal{C})$ consisting of those abelian presheaves whose underlying presheaves of sets are sheaves. Properties (α) – (ζ) of Sites, Section 7.44 hold, see Sites, Proposition 7.44.3. In particular the inclusion functor $\text{Ab}(\mathcal{C}) \to \text{PAb}(\mathcal{C})$ has a left adjoint, namely the sheafification functor $\mathcal{G} \mapsto \mathcal{G}^\#$.

We suggest the reader prove the lemma on a piece of scratch paper rather than reading the proof.

03CN **Lemma 18.3.1.** Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on $\mathcal{C}$.

(1) The category $\text{Ab}(\mathcal{C})$ is an abelian category.
(2) The kernel $\text{Ker}(\varphi)$ of φ is the same as the kernel of φ as a morphism of presheaves.
(3) The morphism φ is injective (Homology, Definition 12.5.3) if and only if φ is injective as a map of presheaves (Sites, Definition 7.3.1), if and only if φ is injective as a map of sheaves (Sites, Definition 7.11.1).
(4) The cokernel $\text{Coker}(\varphi)$ of φ is the sheafification of the cokernel of φ as a morphism of presheaves.
(5) The morphism φ is surjective (Homology, Definition 12.5.3) if and only if φ is surjective as a map of sheaves (Sites, Definition 7.11.1).
(6) A complex of abelian sheaves
$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$
is exact at $\mathcal{G}$ if and only if for all $U \in \text{Ob}(\mathcal{C})$ and all $s \in \mathcal{G}(U)$ mapping to zero in $\mathcal{H}(U)$ there exists a covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ such that each $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Proof. We claim that Homology, Lemma 12.7.4 applies to the categories $\mathcal{A} = \text{Ab}(\mathcal{C})$ and $\mathcal{B} = \text{PAb}(\mathcal{C})$, and the functors $a : \mathcal{A} \to \mathcal{B}$ (inclusion), and $b : \mathcal{B} \to \mathcal{A}$ (sheafification). Let us check the assumptions of Homology, Lemma 12.7.4. Assumption (1) is that $\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b. The first two statements are clear and adjointness is Sites, Section 7.44 (ϵ). Assumption (2) says that $\text{PAb}(\mathcal{C})$ is abelian which we saw in Section 18.2 and that sheafification is left exact, which is Sites, Section 7.44 (ζ). The final assumption is that $ba \cong \text{id}_\mathcal{A}$ which is Sites, Section 7.44 (δ). Hence Homology, Lemma 12.7.4 applies and we conclude that $\text{Ab}(\mathcal{C})$ is abelian.

In the proof of Homology, Lemma 12.7.4 it is shown that $\text{Ker}(\varphi)$ and $\text{Coker}(\varphi)$ are equal to the sheafification of the kernel and cokernel of φ as a morphism of abelian presheaves. This proves (4). Since the kernel is a equalizer (i.e., a limit) and since sheafification commutes with finite limits, we conclude that (2) holds.

Statement (2) implies (3). Statement (4) implies (5) by our description of sheafification. The characterization of exactness in (6) follows from (2) and (5), and the fact that the sequence is exact if and only if $\text{Im}(\mathcal{F} \to \mathcal{G}) = \text{Ker}(\mathcal{G} \to \mathcal{H})$. $\square$

Another way to say part (6) of the lemma is that a sequence of abelian sheaves
$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$

is exact if and only if the sheafification of $U \mapsto \mathrm{Im}(\mathcal{F}_1(U) \to \mathcal{F}_2(U))$ is equal to the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$.

03CO Lemma 18.3.2. Let $\mathcal{C}$ be a site.

(1) All limits and colimits exist in $\mathrm{Ab}(\mathcal{C})$.
(2) Limits are the same as the corresponding limits of abelian presheaves over $\mathcal{C}$ (i.e., commute with taking sections over objects of $\mathcal{C}$).
(3) Finite direct sums are the same as the corresponding finite direct sums in the category of abelian pre-sheaves over $\mathcal{C}$.
(4) A colimit is the sheafification of the corresponding colimit in the category of abelian presheaves.
(5) Filtered colimits are exact.

Proof. By Lemma 18.2.1 limits and colimits of abelian presheaves exist, and are described by taking limits and colimits on the level of sections over objects.

Let $\mathcal{I} \to \mathrm{Ab}(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $\lim_i \mathcal{F}_i$ be the limit of the diagram as an abelian presheaf. By Sites, Lemma 7.10.1 this is an abelian sheaf. Then it is quite easy to see that $\lim_i \mathcal{F}_i$ is the limit of the diagram in $\mathrm{Ab}(\mathcal{C})$. This proves limits exist and (2) holds.

By Categories, Lemma 4.24.5, and because sheafification is left adjoint to the inclusion functor we see that $\mathrm{colim}_i \mathcal{F}$ exists and is the sheafification of the colimit in $\mathrm{PAb}(\mathcal{C})$. This proves colimits exist and (4) holds.

Finite direct sums are the same thing as finite products in any abelian category. Hence (3) follows from (2).

Proof of (5). The statement means that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of abelian sheaves over a directed set I the sequence $0 \to \mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i \to \mathrm{colim}\, \mathcal{H}_i \to 0$ is exact as well. A formal argument using Homology, Lemma 12.5.8 and the definition of colimits shows that the sequence $\mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i \to \mathrm{colim}\, \mathcal{H}_i \to 0$ is exact. Note that $\mathrm{colim}\, \mathcal{F}_i \to \mathrm{colim}\, \mathcal{G}_i$ is the sheafification of the map of presheaf colimits which is injective as each of the maps $\mathcal{F}_i \to \mathcal{G}_i$ is injective. Since sheafification is exact we conclude. □

18.4. Free abelian presheaves

03CP In order to prepare notation for the following definition, let us agree to denote the free abelian group on a set S as[1] $\mathbf{Z}[S] = \bigoplus_{s \in S} \mathbf{Z}$. It is characterized by the property

$$\mathrm{Mor}_{\mathrm{Ab}}(\mathbf{Z}[S], A) = \mathrm{Mor}_{\mathrm{Sets}}(S, A)$$

In other words the construction $S \mapsto \mathbf{Z}[S]$ is a left adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{Sets}$.

03A7 Definition 18.4.1. Let $\mathcal{C}$ be a category. Let $\mathcal{G}$ be a presheaf of sets. The free abelian presheaf $\mathbf{Z}_\mathcal{G}$ on $\mathcal{G}$ is the abelian presheaf defined by the rule

$$U \longmapsto \mathbf{Z}[\mathcal{G}(U)].$$

In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}_X = \mathbf{Z}_{h_X}$. In other words

$$\mathbf{Z}_X(U) = \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(U, X)].$$

[1] In other chapters the notation $\mathbf{Z}[S]$ sometimes indicates the polynomial ring over $\mathbf{Z}$ on S.

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it is adjoint to the forgetful functor $\text{PAb}(\mathcal{C}) \to \text{PSh}(\mathcal{C})$. Here is the precise statement.

03A8 Lemma 18.4.2. Let $\mathcal{C}$ be a category. Let $\mathcal{G}$, $\mathcal{F}$ be a presheaves of sets. Let $\mathcal{A}$ be an abelian presheaf. Let U be an object of $\mathcal{C}$. Then we have

$$\begin{aligned} \text{Mor}_{\text{PSh}(\mathcal{C})}(h_U, \mathcal{F}) &= \mathcal{F}(U), \\ \text{Mor}_{\text{PAb}(\mathcal{C})}(\mathbf{Z}_\mathcal{G}, \mathcal{A}) &= \text{Mor}_{\text{PSh}(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ \text{Mor}_{\text{PAb}(\mathcal{C})}(\mathbf{Z}_U, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. $\square$

03A9 Lemma 18.4.3. Let $\mathcal{C}$ be a category. Let I be a set. For each $i \in I$ let $\mathcal{G}_i$ be a presheaf of sets. Then

$$\mathbf{Z}_{\coprod_i \mathcal{G}_i} = \bigoplus\nolimits_{i \in I} \mathbf{Z}_{\mathcal{G}_i}$$

in $\text{PAb}(\mathcal{C})$.

Proof. Omitted. $\square$

18.5. Free abelian sheaves

03CQ Here is the notion of a free abelian sheaf on a sheaf of sets.

03AA Definition 18.5.1. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. The free abelian sheaf $\mathbf{Z}_\mathcal{G}^\#$ on $\mathcal{G}$ is the abelian sheaf $\mathbf{Z}_\mathcal{G}^\#$ which is the sheafification of the free abelian presheaf on $\mathcal{G}$. In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}_X^\#$.

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it provides an adjoint to the forgetful functor $\text{Ab}(\mathcal{C}) \to Sh(\mathcal{C})$. Here is the precise statement.

03AB Lemma 18.5.2. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$, $\mathcal{F}$ be a sheaves of sets. Let $\mathcal{A}$ be an abelian sheaf. Let U be an object of $\mathcal{C}$. Then we have

$$\begin{aligned} \text{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) &= \mathcal{F}(U), \\ \text{Mor}_{\text{Ab}(\mathcal{C})}(\mathbf{Z}_\mathcal{G}^\#, \mathcal{A}) &= \text{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ \text{Mor}_{\text{Ab}(\mathcal{C})}(\mathbf{Z}_U^\#, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. $\square$

03AC Lemma 18.5.3. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. Then $\mathbf{Z}_\mathcal{G}^\# = (\mathbf{Z}_{\mathcal{G}^\#})^\#$.

Proof. Omitted. $\square$

18.6. Ringed sites

04KQ In this chapter we mainly work with sheaves of modules on a ringed site. Hence we need to define this notion.

03AD Definition 18.6.1. Ringed sites.

(1) A ringed site is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the structure sheaf of the ringed site.

(2) Let $(\mathcal{C}, \mathcal{O})$, $(\mathcal{C}', \mathcal{O}')$ be ringed sites. A morphism of ringed sites

$$(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{C}', \mathcal{O}')$$

is given by a morphism of sites $f : \mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 7.14.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.

(3) Let $(f, f^\sharp) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$ and $(g, g^\sharp) : (\mathcal{C}_2, \mathcal{O}_2) \to (\mathcal{C}_3, \mathcal{O}_3)$ be morphisms of ringed sites. Then we define the composition of morphisms of ringed sites by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of morphisms of sites defined in Sites, Definition 7.14.5 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings

$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_*f^\sharp} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

18.7. Ringed topoi

01D2 A ringed topos is just a ringed site, except that the notion of a morphism of ringed topoi is different from the notion of a morphism of ringed sites.

01D3 **Definition 18.7.1.** Ringed topoi.

(1) A ringed topos is a pair $(Sh(\mathcal{C}), \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed topos.

(2) Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. A *morphism of ringed topoi*

$$(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$$

is given by a morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ (see Sites, Definition 7.15.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.

(3) Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. Then we define the composition of morphisms of ringed topoi by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of morphisms of topoi defined in Sites, Definition 7.15.1 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings

$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_*f^\sharp} g_*f_*\mathcal{O}_1 = (g \circ f)_*\mathcal{O}_1$$

Every morphism of ringed topoi is the composition of an equivalence of ringed topoi with a morphism of ringed topoi associated to a morphism of ringed sites. Here is the precise statement.

03CR **Lemma 18.7.2.** *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. There exists a factorization*

$$\begin{tikzcd}
(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[r, "{(f,f^\sharp)}"'] \arrow[d, "{(g,g^\sharp)}"'] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \arrow[d, "{(e,e^\sharp)}"] \\
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "{(h,h^\sharp)}"] & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'})
\end{tikzcd}$$

where

(1) $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ is an equivalence of topoi induced by a special cocontinuous functor $\mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 7.29.2),
(2) $e : Sh(\mathcal{D}) \to Sh(\mathcal{D}')$ is an equivalence of topoi induced by a special cocontinuous functor $\mathcal{D} \to \mathcal{D}'$ (see Sites, Definition 7.29.2),
(3) $\mathcal{O}_{\mathcal{C}'} = g_*\mathcal{O}_\mathcal{C}$ and $g^\sharp$ is the obvious map,
(4) $\mathcal{O}_{\mathcal{D}'} = e_*\mathcal{O}_\mathcal{D}$ and $e^\sharp$ is the obvious map,
(5) the sites $\mathcal{C}'$ and $\mathcal{D}'$ have final objects and fibre products (i.e., all finite limits),
(6) h is a morphism of sites induced by a continuous functor $u : \mathcal{D}' \to \mathcal{C}'$ which commutes with all finite limits (i.e., it satisfies the assumptions of Sites, Proposition 7.14.7), and
(7) given any set of sheaves $\mathcal{F}_i$ (resp. $\mathcal{G}_j$) on $\mathcal{C}$ (resp. $\mathcal{D}$) we may assume each of these is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).

Moreover, if $(f, f^\sharp)$ is an equivalence of ringed topoi, then we can choose the diagram such that $\mathcal{C}' = \mathcal{D}'$, $\mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{D}'}$ and $(h, h^\sharp)$ is the identity.

Proof. This follows from Sites, Lemma 7.29.6, and Sites, Remarks 7.29.7 and 7.29.8. You just have to carry along the sheaves of rings. Some details omitted. □

18.8. 2-morphisms of ringed topoi

04IB This is a brief section concerning the notion of a 2-morphism of ringed topoi.

04IC Definition 18.8.1. Let $f, g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be two morphisms of ringed topoi. A 2-morphism from f to g is given by a transformation of functors $t : f_* \to g_*$ such that

$$\begin{tikzcd} & \mathcal{O}_\mathcal{D} \arrow[dl, "f^\sharp"'] \arrow[dr, "g^\sharp"] & \\ f_*\mathcal{O}_\mathcal{C} \arrow[rr, "t"] & & g_*\mathcal{O}_\mathcal{C} \end{tikzcd}$$

is commutative.

Pictorially we sometimes represent t as follows:

$$\begin{tikzcd} (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow t"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

As in Sites, Section 7.36 giving a 2-morphism $t : f_* \to g_*$ is equivalent to giving $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) such that the diagram

$$\begin{tikzcd} f^{-1}\mathcal{O}_\mathcal{D} \arrow[dr, "f^\sharp"'] & & g^{-1}\mathcal{O}_\mathcal{D} \arrow[ll, "t"'] \arrow[dl, "g^\sharp"] \\ & \mathcal{O}_\mathcal{C} & \end{tikzcd}$$

is commutative. As in Sites, Section 7.36 the axioms of a strict 2-category hold with horizontal and vertical compositions defined as explained in loc. cit.

18.9. Presheaves of modules

03CS Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. At this point we have not yet defined a presheaf of $\mathcal{O}$-modules. Thus we do so right now.

03CT Definition 18.9.1. Let $\mathcal{C}$ be a category, and let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$.

(1) A presheaf of $\mathcal{O}$-modules is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets

$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$

such that for every object U of $\mathcal{C}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.

(2) A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id} \times \varphi"'] & \mathcal{F} \arrow[d, "\varphi"] \\ \mathcal{O} \times \mathcal{G} \arrow[r] & \mathcal{G} \end{tikzcd}$$

commutes.

(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G})$.

(4) The category of presheaves of $\mathcal{O}$-modules is denoted $\mathrm{PMod}(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on the category $\mathcal{C}$. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the restriction of $\mathcal{F}$. We obtain the restriction functor

$$\mathrm{PMod}(\mathcal{O}_2) \longrightarrow \mathrm{PMod}(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$U \longmapsto (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

where $U \in \mathrm{Ob}(\mathcal{C})$, with obvious restriction mappings. The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the change of rings functor

$$\mathrm{PMod}(\mathcal{O}_1) \longrightarrow \mathrm{PMod}(\mathcal{O}_2)$$

03CU Lemma 18.9.2. With $\mathcal{C}$, $\mathcal{O}_1 \to \mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors defined above are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. $\square$

18.10. Sheaves of modules

03CV

03CW **Definition 18.10.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$.

(1) A *sheaf of $\mathcal{O}$-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 18.9.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
(2) A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of presheaves of $\mathcal{O}$-modules.
(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
(4) The category of sheaves of $\mathcal{O}$-modules is denoted $\mathrm{Mod}(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology "sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

18.11. Sheafification of presheaves of modules

03CX

03CY **Lemma 18.11.1.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$ as a presheaf of rings, see Sites, Section 7.44. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a unique map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O}^\# \times \mathcal{F}^\# \arrow[r] & \mathcal{F}^\# \end{tikzcd}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. $\square$

This actually means that the functor $i : \mathrm{Mod}(\mathcal{O}^\#) \to \mathrm{PMod}(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# : \mathrm{PMod}(\mathcal{O}) \to \mathrm{Mod}(\mathcal{O}^\#)$ are adjoint. In a formula

$$\mathrm{Mor}_{\mathrm{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

An important case happens when $\mathcal{O}$ is already a sheaf of rings. In this case the formula reads

$$\mathrm{Mor}_{\mathrm{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{F}^\#, \mathcal{G})$$

because $\mathcal{O} = \mathcal{O}^\#$ in this case.

03EI **Lemma 18.11.2.** *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$ The sheafification functor*

$$\mathrm{PMod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O}^\#), \quad \mathcal{F} \longmapsto \mathcal{F}^\#$$

is exact.

Proof. This is true because it holds for sheafification $\textit{PAb}(\mathcal{C}) \to \textit{Ab}(\mathcal{C})$. See the discussion in Section 18.3. □

Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on $\mathcal{C}$. In Section 18.9 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$\textit{Mod}(\mathcal{O}_2) \longrightarrow \textit{Mod}(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the tensor product sheaf $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the change of rings functor

$$\textit{Mod}(\mathcal{O}_1) \longrightarrow \textit{Mod}(\mathcal{O}_2)$$

03CZ Lemma 18.11.3. With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 18.9.2 and the fact that $\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. □

0930 Lemma 18.11.4. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be an epimorphism of sheaves of rings. Let $\mathcal{G}_1, \mathcal{G}_2$ be $\mathcal{O}'$-modules. Then

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}_1, \mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}_1, \mathcal{G}_2).$$

In other words, the restriction functor $\textit{Mod}(\mathcal{O}') \to \textit{Mod}(\mathcal{O})$ is fully faithful.

Proof. This is the sheaf version of Algebra, Lemma 10.107.14 and is proved in exactly the same way. □

18.12. Morphisms of topoi and sheaves of modules

03D0 All of this material is completely straightforward. We formulate everything in the case of morphisms of topoi, but of course the results also hold in the case of morphisms of sites.

03D1 Lemma 18.12.1. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a sheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{F} \to \mathcal{F}$ the multiplication map. Recall that f_* (on sheaves of sets) is left exact and hence commutes with products. Hence $f_*\mu$ is a map as indicated. This proves the lemma. □

03D2 **Lemma 18.12.2.** Let $\mathcal{C}, \mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{G} \to \mathcal{G}$ the multiplication map. Recall that f^{-1} (on sheaves of sets) is exact and hence commutes with products. Hence $f^{-1}\mu$ is a map as indicated. This proves the lemma. $\square$

03D3 **Lemma 18.12.3.** Let $\mathcal{C}, \mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then

$$\mathrm{Mor}_{\mathrm{Mod}(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 18.12.2 and 18.12.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. First we note that we have

$$\mathrm{Mor}_{\mathrm{Ab}(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{Ab}(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}).$$

by Sites, Proposition 7.44.3. Suppose that $\alpha : f^{-1}\mathcal{G} \to \mathcal{F}$ and $\beta : \mathcal{G} \to f_*\mathcal{F}$ are morphisms of abelian sheaves which correspond via the formula above. We have to show that α is $f^{-1}\mathcal{O}$-linear if and only if β is $\mathcal{O}$-linear. For example, suppose α is $f^{-1}\mathcal{O}$-linear, then clearly $f_*\alpha$ is $f_*f^{-1}\mathcal{O}$-linear, and hence (as restriction is a functor) is $\mathcal{O}$-linear. Hence it suffices to prove that the adjunction map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ is $\mathcal{O}$-linear. Using that both f_* and f^{-1} commute with products (on sheaves of sets) this comes down to showing that

$$\begin{tikzcd}
\mathcal{O} \times \mathcal{G} \arrow[r] \arrow[d] & f_*f^{-1}(\mathcal{O} \times \mathcal{G}) \arrow[d] \\
\mathcal{G} \arrow[r] & f_*f^{-1}\mathcal{G}
\end{tikzcd}$$

is commutative. This holds because the adjunction mapping $\mathrm{id}_{Sh(\mathcal{D})} \to f_*f^{-1}$ is a transformation of functors. We omit the proof of the implication β linear $\Rightarrow$ α linear. $\square$

03D4 **Lemma 18.12.4.** Let $\mathcal{C}, \mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then

$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 18.12.2 and 18.12.1, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. Note that we have

$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{Mod}(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}})$$

by Lemma 18.11.3. Hence the result follows from Lemma 18.12.3. $\square$

18.13. Morphisms of ringed topoi and modules

03D5 We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed topoi.

03D6 Definition 18.13.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. We define the pushforward of $\mathcal{F}$ as the sheaf of $\mathcal{O}_\mathcal{D}$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_\mathcal{D} \to f_*\mathcal{O}_\mathcal{C}$ of the module structure
$$f_*\mathcal{O}_\mathcal{C} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$
from Lemma 18.12.1.

(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. We define the pullback $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_\mathcal{C}$-modules defined by the formula
$$f^*\mathcal{G} = \mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}$$
where the ring map $f^{-1}\mathcal{O}_\mathcal{D} \to \mathcal{O}_\mathcal{C}$ is $f^\sharp$, and where the module structure is given by Lemma 18.12.2.

Thus we have defined functors
$$\begin{matrix} f_* : \text{Mod}(\mathcal{O}_\mathcal{C}) & \longrightarrow & \text{Mod}(\mathcal{O}_\mathcal{D}) \\ f^* : \text{Mod}(\mathcal{O}_\mathcal{D}) & \longrightarrow & \text{Mod}(\mathcal{O}_\mathcal{C}) \end{matrix}$$

The final result on these functors is that they are indeed adjoint as expected.

03D7 Lemma 18.13.2. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. There is a canonical bijection
$$\text{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) = \text{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}).$$

In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:
$$\begin{aligned} \text{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) &= \text{Mor}_{\text{Mod}(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}, \mathcal{F}) \\ &= \text{Mor}_{\text{Mod}(f^{-1}\mathcal{O}_\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_\mathcal{D}}) \\ &= \text{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

Here we use Lemmas 18.11.3 and 18.12.3. □

03D8 Lemma 18.13.3. $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. There are canonical isomorphisms of functors $(g \circ f)_* \cong g_* \circ f_*$ and $(g \circ f)^* \cong f^* \circ g^*$.

Proof. This is clear from the definitions. □

18.14. The abelian category of sheaves of modules

03D9 Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules, see Sheaves, Definition 6.10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the sum of φ and ψ as morphisms of abelian sheaves. This is clearly again a map of $\mathcal{O}$-modules. It is also clear that composition of maps of $\mathcal{O}$-modules is bilinear with respect to this addition. Thus $\text{Mod}(\mathcal{O})$ is a pre-additive category, see Homology, Definition 12.3.1.

We will denote 0 the sheaf of $\mathcal{O}$-modules which has constant value $\{0\}$ for all objects U of $\mathcal{C}$. Clearly this is both a final and an initial object of $\text{Mod}(\mathcal{O})$. Given a morphism of $\mathcal{O}$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each object U.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 12.3.5. Thus $\text{Mod}(\mathcal{O})$ is an additive category, see Homology, Definition 12.3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}$-modules. We may define $\text{Ker}(\varphi)$ to be the kernel of φ as a map of abelian sheaves. By Section 18.3 this is the subsheaf of $\mathcal{F}$ with sections

$$\text{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all objects U of $\mathcal{C}$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\text{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$.

Similarly, we define $\text{Coker}(\varphi)$ as the cokernel of φ as a map of abelian sheaves. There is a unique multiplication map

$$\mathcal{O} \times \text{Coker}(\varphi) \longrightarrow \text{Coker}(\varphi)$$

such that the map $\mathcal{G} \to \text{Coker}(\varphi)$ becomes a morphism of $\mathcal{O}$-modules (verification omitted). The map $\mathcal{G} \to \text{Coker}(\varphi)$ is surjective (as a map of sheaves of sets, see Section 18.3). To show that $\text{Coker}(\varphi)$ is a cokernel in $\text{Mod}(\mathcal{O})$, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}$-modules such that $\beta \circ \varphi$ is zero, then you get for every object U of $\mathcal{C}$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 6.20.1) we obtain a canonical map $\text{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \text{Coker}(\varphi) \to \mathcal{H}$. The morphism $\text{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

03DA **Lemma 18.14.1.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. The category $\text{Mod}(\mathcal{O})$ is an abelian category. The forgetful functor $\text{Mod}(\mathcal{O}) \to \text{Ab}(\mathcal{C})$ is exact, hence kernels, cokernels and exactness of $\mathcal{O}$-modules, correspond to the corresponding notions for abelian sheaves.

Proof. Above we have seen that $\text{Mod}(\mathcal{O})$ is an additive category, with kernels and cokernels and that $\text{Mod}(\mathcal{O}) \to \text{Ab}(\mathcal{C})$ preserves kernels and cokernels. By Homology, Definition 12.5.1 we have to show that image and coimage agree. This is clear because it is true in $\text{Ab}(\mathcal{C})$. The lemma follows. $\square$

03DB **Lemma 18.14.2.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. All limits and colimits exist in $\text{Mod}(\mathcal{O})$ and the forgetful functor $\text{Mod}(\mathcal{O}) \to \text{Ab}(\mathcal{C})$ commutes with them. Moreover, filtered colimits are exact.

Proof. The final statement follows from the first as filtered colimits are exact in $\text{Ab}(\mathcal{C})$ by Lemma 18.3.2. Let $\mathcal{I} \to \text{Mod}(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $\lim_i \mathcal{F}_i$ be the limit of the diagram in $\text{Ab}(\mathcal{C})$. By the description of this limit in Lemma 18.3.2 we see immediately that there exists a multiplication

$$\mathcal{O} \times \lim_i \mathcal{F}_i \longrightarrow \lim_i \mathcal{F}_i$$

which turns $\lim_i \mathcal{F}_i$ into a sheaf of $\mathcal{O}$-modules. It is easy to see that this is the limit of the diagram in $\text{Mod}(\mathcal{C})$. Let $\text{colim}_i \mathcal{F}_i$ be the colimit of the diagram in $\text{PAb}(\mathcal{C})$. By the description of this colimit in the proof of Lemma 18.2.1 we see immediately that there exists a multiplication

$$\mathcal{O} \times \text{colim}_i \mathcal{F}_i \longrightarrow \text{colim}_i \mathcal{F}_i$$

which turns $\text{colim}_i \mathcal{F}_i$ into a presheaf of $\mathcal{O}$-modules. Applying sheafification we get a sheaf of $\mathcal{O}$-modules $(\text{colim}_i \mathcal{F}_i)^\#$, see Lemma 18.11.1. It is easy to see that $(\text{colim}_i \mathcal{F}_i)^\#$ is the colimit of the diagram in $\text{Mod}(\mathcal{O})$, and by Lemma 18.3.2 forgetting the $\mathcal{O}$-module structure is the colimit in $\text{Ab}(\mathcal{C})$. $\square$

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 4.23. See Homology, Lemma 12.7.2 for a description of exactness properties in terms of short exact sequences.

03DC **Lemma 18.14.3.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi.

(1) The functor f_* is left exact. In fact it commutes with all limits.
(2) The functor f^* is right exact. In fact it commutes with all colimits.

Proof. This is true because (f^*, f_*) is an adjoint pair of functors, see Lemma 18.13.2. See Categories, Section 4.24. $\square$

05V3 **Lemma 18.14.4.** Let $\mathcal{C}$ be a site. If $\{p_i\}_{i \in I}$ is a conservative family of points, then we may check exactness of a sequence of abelian sheaves on the stalks at the points p_i, $i \in I$. If $\mathcal{C}$ has enough points, then exactness of a sequence of abelian sheaves may be checked on stalks.

Proof. This is immediate from Sites, Lemma 7.38.2. $\square$

18.15. Exactness of pushforward

04BC Some technical lemmas concerning exactness properties of pushforward.

04DA **Lemma 18.15.1.** Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. The following are equivalent:

(1) $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $\text{Ab}(\mathcal{C})$, and
(2) $f_* : \text{Ab}(\mathcal{C}) \to \text{Ab}(\mathcal{D})$ reflects surjections.

In this case the functor $f_* : \text{Ab}(\mathcal{C}) \to \text{Ab}(\mathcal{D})$ is faithful.

Proof. Assume (1). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of abelian sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (1) this implies that a is surjective. This means that (2) holds.

Assume (2). Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. By (2) it suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

We omit the proof of the final assertion. $\square$

04DB Lemma 18.15.2. Let $f : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{D})$ be a morphism of topoi. Assume at least one of the following properties holds

(1) f_* transforms surjections of sheaves of sets into surjections,
(2) f_* transforms surjections of abelian sheaves into surjections,
(3) f_* commutes with coequalizers on sheaves of sets,
(4) f_* commutes with pushouts on sheaves of sets,

Then $f_* : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{D})$ is exact.

Proof. Since $f_* : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{D})$ is a right adjoint we already know that it transforms a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of abelian sheaves on $\mathcal{C}$ into an exact sequence

$$0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$$

see Categories, Sections 4.23 and 4.24 and Homology, Section 12.7. Hence it suffices to prove that the map $f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ is surjective. If (1), (2) holds, then this is clear from the definitions. By Sites, Lemma 7.41.1 we see that either (3) or (4) formally implies (1), hence in these cases we are done also. $\square$

04BD Lemma 18.15.3. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume u is almost cocontinuous. Then

(1) $f_* : \mathrm{Ab}(\mathcal{D}) \to \mathrm{Ab}(\mathcal{C})$ is exact.
(2) if $f^\sharp : f^{-1}\mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D}$ is given so that f becomes a morphism of ringed sites, then $f_* : \mathrm{Mod}(\mathcal{O}_\mathcal{D}) \to \mathrm{Mod}(\mathcal{O}_\mathcal{C})$ is exact.

Proof. Part (2) follows from part (1) by Lemma 18.14.2. Part (1) follows from Sites, Lemmas 7.42.6 and 7.41.1. $\square$

18.16. Exactness of lower shriek

04BE Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between sites. Assume that

(a) u is cocontinuous, and
(b) u is continuous.

Let $g : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{D})$ be the morphism of topoi associated with u, see Sites, Lemma 7.21.1. Recall that $g^{-1} = u^p$, i.e., g^{-1} is given by the simple formula $(g^{-1}\mathcal{G})(U) = \mathcal{G}(u(U))$, see Sites, Lemma 7.21.5. We would like to show that $g^{-1} : \mathrm{Ab}(\mathcal{D}) \to \mathrm{Ab}(\mathcal{C})$ has a left adjoint $g_!$. By Sites, Lemma 7.21.5 the functor $g_!^{Sh} = (u_p\ )^\#$ is a left adjoint on sheaves of sets. Moreover, we know that $g_!^{Sh}\mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto \mathrm{colim}_{V \to u(U)}\, \mathcal{F}(U)$$

where the colimit is over $(\mathcal{I}_V^u)^{opp}$ and is taken in the category of sets. Hence the following definition is natural.

04BF Definition 18.16.1. With $u : \mathcal{C} \to \mathcal{D}$ satisfying (a), (b) above. For $\mathcal{F} \in \textit{PAb}(\mathcal{C})$ we define $g_{p!}\mathcal{F}$ as the presheaf

$$V \longmapsto \mathrm{colim}_{V \to u(U)} \mathcal{F}(U)$$

with colimits over $(\mathcal{I}_V^u)^{opp}$ taken in $\textit{Ab}$. For $\mathcal{F} \in \textit{PAb}(\mathcal{C})$ we set $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$.

The reason for being so explicit with this is that the functors $g_!^{Sh}$ and $g_!$ are different. Whenever we use both we have to be careful to make the distinction clear.

04BG Lemma 18.16.2. The functor $g_{p!}$ is a left adjoint to the functor u^p. The functor $g_!$ is a left adjoint to the functor g^{-1}. In other words the formulas

$$\mathrm{Mor}_{\textit{PAb}(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = \mathrm{Mor}_{\textit{PAb}(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}),$$
$$\mathrm{Mor}_{\textit{Ab}(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) = \mathrm{Mor}_{\textit{Ab}(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G})$$

hold bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. The second formula follows formally from the first, since if $\mathcal{F}$ and $\mathcal{G}$ are abelian sheaves then

$$\begin{aligned}\mathrm{Mor}_{\textit{Ab}(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \mathrm{Mor}_{\textit{PAb}(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}) \\ &= \mathrm{Mor}_{\textit{Ab}(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G})\end{aligned}$$

by the universal property of sheafification.

To prove the first formula, let $\mathcal{F}$, $\mathcal{G}$ be abelian presheaves. To prove the lemma we will construct maps from the group on the left to the group on the right and omit the verification that these are mutually inverse.

Note that there is a canonical map of abelian presheaves $\mathcal{F} \to u^p g_{p!}\mathcal{F}$ which on sections over U is the natural map $\mathcal{F}(U) \to \mathrm{colim}_{u(U) \to u(U')} \mathcal{F}(U')$, see Sites, Lemma 7.5.3. Given a map $\alpha : g_{p!}\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^p g_{p!}\mathcal{F} \to u^p\mathcal{G}$. which we can precompose by the map $\mathcal{F} \to u^p g_{p!}\mathcal{F}$.

Note that there is a canonical map of abelian presheaves $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ which on sections over V is the natural map $\mathrm{colim}_{V \to u(U)} \mathcal{G}(u(U)) \to \mathcal{G}(V)$. It maps a section $s \in u(U)$ in the summand corresponding to $t : V \to u(U)$ to $t^*s \in \mathcal{G}(V)$. Hence, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $g_{p!}\beta : g_{p!}\mathcal{F} \to g_{p!}u^p\mathcal{G}$ which we can postcompose with the map $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ above. $\square$

04BH Lemma 18.16.3. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(a) u is cocontinuous,
(b) u is continuous, and
(c) fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.

In this case the functor $g_! : \textit{Ab}(\mathcal{C}) \to \textit{Ab}(\mathcal{D})$ is exact.

Proof. Compare with Sites, Lemma 7.21.6. Assume (a), (b), and (c). We already know that $g_!$ is right exact as it is a left adjoint, see Categories, Lemma 4.24.6 and Homology, Section 12.7. We have $g_! = (g_{p!}\ )^\#$. We have to show that $g_!$ transforms injective maps of abelian sheaves into injective maps of abelian presheaves. Recall that sheafification of abelian presheaves is exact, see Lemma 18.3.2. Thus it suffices to show that $g_{p!}$ transforms injective maps of abelian presheaves into injective maps of abelian presheaves. To do this it suffices that colimits over the categories $(\mathcal{I}_V^u)^{opp}$ of Sites, Section 7.5 transform injective maps between diagrams into injections. This follows from Sites, Lemma 7.5.1 and Algebra, Lemma 10.8.10. □

077I Lemma 18.16.4. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that

(a) u is cocontinuous,
(b) u is continuous, and
(c) u is fully faithful.

For $g_!, g^{-1}, g_*$ as above the canonical maps $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ and $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ are isomorphisms for all abelian sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. The map $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ is an isomorphism by Sites, Lemma 7.21.7 and the fact that pullback and pushforward of abelian sheaves agrees with pullback and pushforward on the underlying sheaves of sets.

Pick $U \in \mathrm{Ob}(\mathcal{C})$. We will show that $g^{-1}g_!\mathcal{F}(U) = \mathcal{F}(U)$. First, note that $g^{-1}g_!\mathcal{F}(U) = g_!\mathcal{F}(u(U))$. Hence it suffices to show that $g_!\mathcal{F}(u(U)) = \mathcal{F}(U)$. We know that $g_!\mathcal{F}$ is the (abelian) sheaf associated to the presheaf $g_{p!}\mathcal{F}$ which is defined by the rule

$$V \longmapsto \mathrm{colim}_{V \to u(U')} \mathcal{F}(U')$$

with colimit taken in Ab. If $V = u(U)$, then, as u is fully faithful this colimit is over $U \to U'$. Hence we conclude that $g_{p!}\mathcal{F}(u(U) = \mathcal{F}(U)$. Since u is cocontinuous and continuous any covering of $u(U)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(U_i) \to u(U)\}$ of $\mathcal{D}$ where $\{U_i \to U\}$ is a covering in $\mathcal{C}$. This implies that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ also, since in the colimit defining the value of $(g_{p!}\mathcal{F})^+$ on $u(U)$ we may restrict to the cofinal system of coverings $\{u(U_i) \to u(U)\}$ as above. Hence we see that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(g_{p!}\mathcal{F})^\#(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

04BI Remark 18.16.5. In general the functor $g_!$ cannot be extended to categories of modules in case g is (part of) a morphism of ringed topoi. Namely, given any ring map $A \to B$ the functor $M \mapsto B \otimes_A M$ has a right adjoint (restriction) but not in general a left adjoint (because its existence would imply that $A \to B$ is flat). We will see in Section 18.19 below that it is possible to define $j_!$ on sheaves of modules in the case of a localization of sites. We will discuss this in greater generality in Section 18.41 below.

08P3 Lemma 18.16.6. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to a continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$.

(1) If u has a left adjoint w, then $g_!$ agrees with $g_!^{Sh}$ on underlying sheaves of sets and $g_!$ is exact.
(2) If in addition w is cocontinuous, then $g_! = h^{-1}$ and $g^{-1} = h_*$ where $h : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to w.

Proof. This Lemma is the analogue of Sites, Lemma 7.23.1. From Sites, Lemma 7.19.3 we see that the categories $\mathcal{I}_V^u$ have an initial object. Thus the underlying set of a colimit of a system of abelian groups over $(\mathcal{I}_V^u)^{opp}$ is the colimit of the underlying sets. Whence the agreement of $g_!^{Sh}$ and $g_!$ by our construction of $g_!$ in Definition 18.16.1. The exactness and (2) follow immediately from the corresponding statements of Sites, Lemma 7.23.1. □

18.17. Global types of modules

03DD

03DE Definition 18.17.1. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules.

(1) We say $\mathcal{F}$ is a free $\mathcal{O}$-module if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i \in I} \mathcal{O}$.
(2) We say $\mathcal{F}$ is finite free if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i \in I} \mathcal{O}$ with a finite index set I.
(3) We say $\mathcal{F}$ is generated by global sections if there exists a surjection
$$\bigoplus\nolimits_{i \in I} \mathcal{O} \longrightarrow \mathcal{F}$$
from a free $\mathcal{O}$-module onto $\mathcal{F}$.
(4) Given $r \geq 0$ we say $\mathcal{F}$ is generated by r global sections if there exists a surjection $\mathcal{O}^{\oplus r} \to \mathcal{F}$.
(5) We say $\mathcal{F}$ is generated by finitely many global sections if it is generated by r global sections for some $r \geq 0$.
(6) We say $\mathcal{F}$ has a global presentation if there exists an exact sequence
$$\bigoplus\nolimits_{j \in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$
of $\mathcal{O}$-modules.
(7) We say $\mathcal{F}$ has a global finite presentation if there exists an exact sequence
$$\bigoplus\nolimits_{j \in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$
of $\mathcal{O}$-modules with I and J finite sets.

Note that for any set I the direct sum $\bigoplus_{i \in I} \mathcal{O}$ exists (Lemma 18.14.2) and is the sheafification of the presheaf $U \mapsto \bigoplus_{i \in I} \mathcal{O}(U)$. This module is called the free $\mathcal{O}$-module on the set I.

03DF Lemma 18.17.2. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.

(1) If $\mathcal{F}$ is free then $f^*\mathcal{F}$ is free.
(2) If $\mathcal{F}$ is finite free then $f^*\mathcal{F}$ is finite free.
(3) If $\mathcal{F}$ is generated by global sections then $f^*\mathcal{F}$ is generated by global sections.
(4) Given $r \geq 0$ if $\mathcal{F}$ is generated by r global sections, then $f^*\mathcal{F}$ is generated by r global sections.
(5) If $\mathcal{F}$ is generated by finitely many global sections then $f^*\mathcal{F}$ is generated by finitely many global sections.
(6) If $\mathcal{F}$ has a global presentation then $f^*\mathcal{F}$ has a global presentation.
(7) If $\mathcal{F}$ has a finite global presentation then $f^*\mathcal{F}$ has a finite global presentation.

Proof. This is true because f^* commutes with arbitrary colimits (Lemma 18.14.3) and $f^*\mathcal{O}_\mathcal{D} = \mathcal{O}_\mathcal{C}$. $\square$

18.18. Intrinsic properties of modules

03DG Let $\mathcal{P}$ be a property of sheaves of modules on ringed topoi. We say $\mathcal{P}$ is an intrinsic property if we have $\mathcal{P}(\mathcal{F}) \Leftrightarrow \mathcal{P}(f^*\mathcal{F})$ whenever $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi. For example, the property of being free is intrinsic. Indeed, the free $\mathcal{O}$-module on the set I is characterized by the property that

$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\bigoplus\nolimits_{i \in I} \mathcal{O}, \mathcal{F}) = \prod\nolimits_{i \in I} \mathrm{Mor}_{Sh(\mathcal{C})}(\{*\}, \mathcal{F})$$

for a variable $\mathcal{F}$ in $\mathrm{Mod}(\mathcal{O})$. Alternatively, we can also use Lemma 18.17.2 to see that being free is intrinsic. In fact, each of the properties defined in Definition 18.17.1 is intrinsic for the same reason. How will we go about defining other intrinsic properties of $\mathcal{O}$-modules?

The upshot of Lemma 18.7.2 is the following: Suppose you want to define an intrinsic property $\mathcal{P}$ of an $\mathcal{O}$-module on a topos. Then you can proceed as follows:

(1) Given any site $\mathcal{C}$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ and any $\mathcal{O}$-module $\mathcal{F}$ define the corresponding property $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$.
(2) For any pair of sites $\mathcal{C}$, $\mathcal{C}'$, any special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ any $\mathcal{O}$-module $\mathcal{F}$, show that

$$\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F}) \Leftrightarrow \mathcal{P}(\mathcal{C}', g_*\mathcal{O}, g_*\mathcal{F})$$

where $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ is the equivalence of topoi associated to u.

In this case, given any ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ and any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we simply say that $\mathcal{F}$ has property $\mathcal{P}$ if $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$ is true. And Lemma 18.7.2 combined with (2) above guarantees that this is well defined.

Moreover, the same Lemma 18.7.2 also guarantees that if in addition

(3) For any morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that f is given by a functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Sites, Proposition 7.14.7, and any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have

$$\mathcal{P}(\mathcal{D}, \mathcal{O}_\mathcal{D}, \mathcal{F}) \Rightarrow \mathcal{P}(\mathcal{C}, \mathcal{O}_\mathcal{C}, f^*\mathcal{F})$$

then it is true that $\mathcal{P}$ is preserved under pullback of modules w.r.t. arbitrary morphisms of ringed topoi.

We will use this method in the following sections to see that: locally free, locally generated by sections, locally generated by r sections, finite type, finite presentation, quasi-coherent, and coherent are intrinsic properties of modules.

Perhaps a more satisfying method would be to find an intrinsic definition of these notions, rather than the laborious process sketched here. On the other hand, in many geometric situations where we want to apply these definitions we are given a definite ringed site, and a definite sheaf of modules, and it is nice to have a definition already adapted to this language.

18.19. Localization of ringed sites

03DH Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. We explain the counterparts of the results in Sites, Section 7.25 in this setting.

Denote $\mathcal{O}_U = j_U^{-1}\mathcal{O}$ the restriction of $\mathcal{O}$ to the site $\mathcal{C}/U$. It is described by the simple rule $\mathcal{O}_U(V/U) = \mathcal{O}(V)$. With this notation the localization morphism j_U becomes a morphism of ringed topoi

$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

namely, we take $j_U^\sharp : j_U^{-1}\mathcal{O} \to \mathcal{O}_U$ the identity map. Moreover, we obtain the following descriptions for pushforward and pullback of modules.

04IX Definition 18.19.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$.

(1) The ringed site $(\mathcal{C}/U, \mathcal{O}_U)$ is called the localization of the ringed site $(\mathcal{C}, \mathcal{O})$ at the object U.
(2) The morphism of ringed topoi $(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ is called the localization morphism.
(3) The functor $j_{U*} : \mathrm{Mod}(\mathcal{O}_U) \to \mathrm{Mod}(\mathcal{O})$ is called the direct image functor.
(4) For a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^*\mathcal{F}$ is called the restriction of $\mathcal{F}$ to $\mathcal{C}/U$. We will sometimes denote it by $\mathcal{F}|_{\mathcal{C}/U}$ or even $\mathcal{F}|_U$. It is described by the simple rule $j_U^*(\mathcal{F})(X/U) = \mathcal{F}(X)$.
(5) The left adjoint $j_{U!} : \mathrm{Mod}(\mathcal{O}_U) \to \mathrm{Mod}(\mathcal{O})$ of restriction is called extension by zero. It exists and is exact by Lemmas 18.19.2 and 18.19.3.

As in the topological case, see Sheaves, Section 6.31, the extension by zero $j_{U!}$ functor is different from extension by the empty set $j_{U!}$ defined on sheaves of sets. Here is the lemma defining extension by zero.

03DI Lemma 18.19.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The restriction functor $j_U^* : \mathrm{Mod}(\mathcal{O}) \to \mathrm{Mod}(\mathcal{O}_U)$ has a left adjoint $j_{U!} : \mathrm{Mod}(\mathcal{O}_U) \to \mathrm{Mod}(\mathcal{O})$. So

$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O}_U)}(\mathcal{G}, j_U^*\mathcal{F}) = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(j_{U!}\mathcal{G}, \mathcal{F})$$

for $\mathcal{F} \in \mathrm{Ob}(\mathrm{Mod}(\mathcal{O}))$ and $\mathcal{G} \in \mathrm{Ob}(\mathrm{Mod}(\mathcal{O}_U))$. Moreover, the extension by zero $j_{U!}\mathcal{G}$ of $\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings and an obvious $\mathcal{O}$-module structure.

Proof. The $\mathcal{O}$-module structure on the presheaf is defined as follows. If $f \in \mathcal{O}(V)$ and $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$, then we define $f \cdot s = fs$ where $f \in \mathcal{O}_U(\varphi : V \to U) = \mathcal{O}(V)$ (because $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ to $\mathcal{C}/U$).

Similarly, let $\alpha : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. In this case we can define a map from the presheaf of the lemma into $\mathcal{F}$ by mapping

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{F}(V)$$

by the rule that $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$ maps to $\alpha(s) \in \mathcal{F}(V)$. It is clear that this is $\mathcal{O}$-linear, and hence induces a morphism of $\mathcal{O}$-modules $\alpha' : j_{U!}\mathcal{G} \to \mathcal{F}$ by the properties of sheafification of modules (Lemma 18.11.1).

Conversely, let $\beta : j_{U!}\mathcal{G} \to \mathcal{F}$ by a map of $\mathcal{O}$-modules. Recall from Sites, Section 7.25 that there exists an extension by the empty set $j_{U!}^{Sh} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ on sheaves of sets which is left adjoint to j_U^{-1}. Moreover, $j_{U!}^{Sh}\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

Hence there is a natural map $j_{U!}^{Sh}\mathcal{G} \to j_{U!}\mathcal{G}$ of sheaves of sets. Hence precomposing β by this map we get a map of sheaves of sets $j_{U!}^{Sh}\mathcal{G} \to \mathcal{F}$ which by adjunction corresponds to a map of sheaves of sets $\beta' : \mathcal{G} \to \mathcal{F}|_U$. We claim that β' is $\mathcal{O}_U$-linear. Namely, suppose that $\varphi : V \to U$ is an object of $\mathcal{C}/U$ and that $s, s' \in \mathcal{G}(\varphi : V \to U)$, and $f \in \mathcal{O}(V) = \mathcal{O}_U(\varphi : V \to U)$. Then by the discussion above we see that $\beta'(s + s')$, resp. $\beta'(fs)$ in $\mathcal{F}|_U(\varphi : V \to U)$ correspond to $\beta(s + s')$, resp. $\beta(fs)$ in $\mathcal{F}(V)$. Since β is a homomorphism we conclude.

To conclude the proof of the lemma we have to show that the constructions $\alpha \mapsto \alpha'$ and $\beta \mapsto \beta'$ are mutually inverse. We omit the verifications. $\square$

Note that we have in the situation of Definition 18.19.1 we have

0G1V (18.19.2.1)
$$\mathrm{Hom}_{\mathcal{O}}(j_{U!}\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j_U^*\mathcal{F}) = \mathcal{F}(U)$$

for every $\mathcal{O}$-module $\mathcal{F}$. Namely, the first equality holds by the adjointness of $j_{U!}$ and j_U^* and the second because $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j_U^*\mathcal{F}) = j_U^*\mathcal{F}(U/U) = \mathcal{F}|_U(U/U) = \mathcal{F}(U)$.

03DJ **Lemma 18.19.3.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!} : \mathrm{Mod}(\mathcal{O}_U) \to \mathrm{Mod}(\mathcal{O})$ is exact.

Proof. Since $j_{U!}$ is a left adjoint to j_U^* we see that it is right exact (see Categories, Lemma 4.24.6 and Homology, Section 12.7). Hence it suffices to show that if $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of $\mathcal{O}_U$-modules, then $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective. The map on sections of presheaves over an object V (as in Lemma 18.19.2) is the map

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}_1(V \xrightarrow{\varphi} U) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}_2(V \xrightarrow{\varphi} U)$$

which is injective by assumption. Since sheafification is exact by Lemma 18.11.2 we conclude $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective and we win. $\square$

0E8G **Lemma 18.19.4.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. A complex of $\mathcal{O}_U$-modules $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ is exact if and only if $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2 \to j_{U!}\mathcal{G}_3$ is exact as a sequence of $\mathcal{O}$-modules.

Proof. We already know that $j_{U!}$ is exact, see Lemma 18.19.3. Thus it suffices to show that $j_{U!} : \mathrm{Mod}(\mathcal{O}_U) \to \mathrm{Mod}(\mathcal{O})$ reflects injections and surjections.

For every $\mathcal{G}$ in $\mathrm{Mod}(\mathcal{O}_U)$ we have the unit $\mathcal{G} \to j_U^*j_{U!}\mathcal{G}$ of the adjunction. We claim this map is an injection of sheaves. Namely, looking at the construction of Lemma 18.19.2 we see that this map is the sheafification of the rule sending the object V/U of $\mathcal{C}/U$ to the injective map

$$\mathcal{G}(V/U) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

given by the inclusion of the summand corresponding to the structure morphism $V \to U$. Since sheafification is exact the claim follows. Some details omitted.

If $\mathcal{G} \to \mathcal{G}'$ is a map of $\mathcal{O}_U$-modules with $j_{U!}\mathcal{G} \to j_{U!}\mathcal{G}'$ injective, then $j_U^*j_{U!}\mathcal{G} \to j_U^*j_{U!}\mathcal{G}'$ is injective (restriction is exact), hence $\mathcal{G} \to j_U^*j_{U!}\mathcal{G}'$ is injective, hence $\mathcal{G} \to \mathcal{G}'$ is injective. We conclude that $j_{U!}$ reflects injections.

Let $a : \mathcal{G} \to \mathcal{G}'$ be a map of $\mathcal{O}_U$-modules such that $j_{U!}\mathcal{G} \to j_{U!}\mathcal{G}'$ is surjective. Let $\mathcal{H}$ be the cokernel of a. Then $j_{U!}\mathcal{H} = 0$ as $j_{U!}$ is exact. By the above the map $\mathcal{H} \to j_U^*j_{U!}\mathcal{H}$ is injective. Hence $\mathcal{H} = 0$ as desired. $\square$

04IY Lemma 18.19.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram

$$\begin{tikzcd} (Sh(\mathcal{C}/V), \mathcal{O}_V) \arrow[rr, "{(j,j^\sharp)}"'] \arrow[dr, "{(j_V, j_V^\sharp)}"'] & & (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[dl, "{(j_U, j_U^\sharp)}"] \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{tikzcd}$$

of ringed topoi. Here $(j, j^\sharp)$ is the localization morphism associated to the object V/U of the ringed site $(\mathcal{C}/V, \mathcal{O}_V)$.

Proof. The only thing to check is that $j_V^\sharp = j^\sharp \circ j^{-1}(j_U^\sharp)$, since everything else follows directly from Sites, Lemma 7.25.8 and Sites, Equation (7.25.8.1). We omit the verification of the equality. $\square$

08P4 Remark 18.19.6. In the situation of Lemma 18.19.2 the diagram

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{O}_U) \arrow[r, "j_{U!}"'] \arrow[d, "forget"'] & \mathrm{Mod}(\mathcal{O}_\mathcal{C}) \arrow[d, "forget"] \\ \mathrm{Ab}(\mathcal{C}/U) \arrow[r, "j_{U!}^{Ab}"] & \mathrm{Ab}(\mathcal{C}) \end{tikzcd}$$

commutes. This is clear from the explicit description of the functor $j_{U!}$ in the lemma.

03EJ Remark 18.19.7. Localization and presheaves of modules; see Sites, Remark 7.25.10. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^*, j_{U*} and $j_{U!}$ have not been defined for presheaves of $\mathcal{O}$-modules. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Sites, Examples 7.6.6). Hence we also obtain a functor

$$j_U^* : \mathrm{PMod}(\mathcal{O}) \longrightarrow \mathrm{PMod}(\mathcal{O}_U)$$

and functors

$$j_{U*}, j_{U!} : \mathrm{PMod}(\mathcal{O}_U) \longrightarrow \mathrm{PMod}(\mathcal{O})$$

which are right, left adjoint to j_U^*. Inspecting the proof of Lemma 18.19.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ is exact (by Lemma 18.19.3 in the case of the discrete topologies). Moreover, if $\mathcal{C}$ is actually a site, and $\mathcal{O}$ is actually a sheaf of rings, then the diagram

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{O}_U) \arrow[r, "j_{U!}"'] \arrow[d, "forget"'] & \mathrm{Mod}(\mathcal{O}) \\ \mathrm{PMod}(\mathcal{O}_U) \arrow[r, "j_{U!}"] & \mathrm{PMod}(\mathcal{O}) \arrow[u, "(\ )^\#"'] \end{tikzcd}$$

commutes.

0F6Z Lemma 18.19.8. Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume that every X in $\mathcal{C}$ has at most one morphism to U. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}/U$. The canonical maps $\mathcal{F} \to j_U^{-1} j_{U!}\mathcal{F}$ and $j_U^{-1} j_{U*}\mathcal{F} \to \mathcal{F}$ are isomorphisms.

Proof. This is a special case of Lemma 18.16.4 because the assumption on U is equivalent to the fully faithfulness of the localization functor $\mathcal{C}/U \to \mathcal{C}$. $\square$

18.20. Localization of morphisms of ringed sites

04IZ This section is the analogue of Sites, Section 7.28.

04J0 Lemma 18.20.1. Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and set $U = u(V)$. Then there is a canonical map of sheaves of rings $(f')^\sharp$ such that the diagram of Sites, Lemma 7.28.1 is turned into a commutative diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[r, "(j_U, j_U^\sharp)"'] \arrow[d, "(f', (f')^\sharp)"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "(f, f^\sharp)"] \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) \arrow[r, "(j_V, j_V^\sharp)"] & (Sh(\mathcal{D}), \mathcal{O}'). \end{tikzcd}$$

Moreover, in this situation we have $f'_* j_U^{-1} = j_V^{-1} f_*$ and $f'_* j_U^* = j_V^* f_*$.

Proof. Just take $(f')^\sharp$ to be

$$(f')^{-1}\mathcal{O}'_V = (f')^{-1} j_V^{-1} \mathcal{O}' = j_U^{-1} f^{-1} \mathcal{O}' \xrightarrow{j_U^{-1} f^\sharp} j_U^{-1}\mathcal{O} = \mathcal{O}_U$$

and everything else follows from Sites, Lemma 7.28.1. (Note that $j^{-1} = j^*$ on sheaves of modules if j is a localization morphism, hence the first equality of functors implies the second.) $\square$

04J1 Lemma 18.20.2. Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[r, "(j_U, j_U^\sharp)"'] \arrow[d, "(f_c, f_c^\sharp)"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "(f, f^\sharp)"] \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) \arrow[r, "(j_V, j_V^\sharp)"] & (Sh(\mathcal{D}), \mathcal{O}'). \end{tikzcd}$$

The morphism $(f_c, f_c^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)}) \longrightarrow (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 18.20.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)})$$

of Lemma 18.19.5. Given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{tikzcd} U' \arrow[r, "c'"'] \arrow[d, "a"'] & u(V') \arrow[d, "u(b)"] \\ U \arrow[r, "c"] & u(V) \end{tikzcd}$$

commutes, then the following diagram of ringed topoi

$$\begin{tikzcd}
(Sh(\mathcal{C}/U'), \mathcal{O}_{U'}) \arrow[r, "(j_{U'/U}, j^\sharp_{U'/U})"'] \arrow[d, "(f_{c'}, f^\sharp_{c'})"'] & (Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[d, "(f_c, f^\sharp_c)"] \\
(Sh(\mathcal{D}/V'), \mathcal{O}'_{V'}) \arrow[r, "(j_{V'/V}, j^\sharp_{V'/V})"] & (Sh(\mathcal{D}/V), \mathcal{O}'_{V'})
\end{tikzcd}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 7.28.3. To check that the diagrams commute as morphisms of ringed topoi use Lemmas 18.19.5 and 18.20.1 exactly as in the proof of Sites, Lemma 7.28.3. □

18.21. Localization of ringed topoi

04ID This section is the analogue of Sites, Section 7.30 in the setting of ringed topoi.

04IE Lemma 18.21.1. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$ be a sheaf. For a sheaf $\mathcal{H}$ on $\mathcal{C}$ denote $\mathcal{H}_\mathcal{F}$ the sheaf $\mathcal{H} \times \mathcal{F}$ seen as an object of the category $Sh(\mathcal{C})/\mathcal{F}$. The pair $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a ringed topos and there is a canonical morphism of ringed topoi

$$(j_\mathcal{F}, j^\sharp_\mathcal{F}) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

which is a localization as in Section 18.19 such that

(1) the functor $j_\mathcal{F}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,
(2) the functor $j_\mathcal{F}^*$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,
(3) the functor $j_{\mathcal{F}!}$ on sheaves of sets is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$,
(4) the functor $j_{\mathcal{F}!}$ on sheaves of modules associates to the $\mathcal{O}_\mathcal{F}$-module $\varphi : \mathcal{G} \to \mathcal{F}$ the $\mathcal{O}$-module which is the sheafification of the presheaf

$$V \longmapsto \bigoplus\nolimits_{s \in \mathcal{F}(V)} \{\sigma \in \mathcal{G}(V) \mid \varphi(\sigma) = s\}$$

for $V \in \mathrm{Ob}(\mathcal{C})$.

Proof. By Sites, Lemma 7.30.1 we see that $Sh(\mathcal{C})/\mathcal{F}$ is a topos and that (1) and (3) are true. In particular this shows that $j_\mathcal{F}^{-1}\mathcal{O} = \mathcal{O}_\mathcal{F}$ and shows that $\mathcal{O}_\mathcal{F}$ is a sheaf of rings. Thus we may choose the map $j^\sharp_\mathcal{F}$ to be the identity, in particular we see that (2) is true. Moreover, the proof of Sites, Lemma 7.30.1 shows that we may assume $\mathcal{C}$ is a site with all finite limits and a subcanonical topology and that $\mathcal{F} = h_U$ for some object U of $\mathcal{C}$. Then (4) follows from the description of $j_{\mathcal{F}!}$ in Lemma 18.19.2. Alternatively one could show directly that the functor described in (4) is a left adjoint to $j^*_\mathcal{F}$. □

04J2 Definition 18.21.2. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$.

(1) The ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is called the localization of the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ at $\mathcal{F}$.
(2) The morphism of ringed topoi $(j_\mathcal{F}, j^\sharp_\mathcal{F}) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$ of Lemma 18.21.1 is called the localization morphism.

We continue the tradition, established in the chapter on sites, that we check the localization constructions on topoi are compatible with the constructions of localization on sites, whenever this makes sense.

04J3 Lemma 18.21.3. With $(Sh(\mathcal{C}), \mathcal{O})$ and $\mathcal{F} \in Sh(\mathcal{C})$ as in Lemma 18.21.1. If $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$ then via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Sites, Lemma 7.25.4 we have

(1) canonically $\mathcal{O}_U = \mathcal{O}_\mathcal{F}$, and
(2) with these identifications we have $(j_\mathcal{F}, j_\mathcal{F}^\sharp) = (j_U, j_U^\sharp)$.

Proof. The assertion for underlying topoi is Sites, Lemma 7.30.5. Note that $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ which by Sites, Lemma 7.25.7 corresponds to $\mathcal{O} \times h_U^\#$ under the equivalence of Sites, Lemma 7.25.4. By definition of $\mathcal{O}_\mathcal{F}$ we get (1). What's left is to prove that $j_\mathcal{F}^\sharp = j_U^\sharp$ under this identification. We omit the verification. □

Localization is functorial in the following two ways: We can "relocalize" a localization (see Lemma 18.21.4) or we can given a morphism of ringed topoi, localize upstairs at the inverse image of a sheaf downstairs and get a commutative diagram of ringed topoi (see Lemma 18.22.1).

04J4 Lemma 18.21.4. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) \arrow[rr, "{(j,j^\sharp)}"'] \arrow[dr, "{(j_\mathcal{G}, j_\mathcal{G}^\sharp)}"'] & & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[dl, "{(j_\mathcal{F}, j_\mathcal{F}^\sharp)}"] \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{tikzcd}$$

where $(j, j^\sharp)$ is the localization morphism of the ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ at the object $\mathcal{G}/\mathcal{F}$.

Proof. All assertions follow from Sites, Lemma 7.30.6 except the assertion that $j_\mathcal{G}^\sharp = j^\sharp \circ j^{-1}(j_\mathcal{F}^\sharp)$. We omit the verification. □

04J5 Lemma 18.21.5. With $(Sh(\mathcal{C}), \mathcal{O})$, $s : \mathcal{G} \to \mathcal{F}$ as in Lemma 18.21.4. If there exist a morphism $f : V \to U$ of $\mathcal{C}$ such that $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and s is induced by f, then the diagrams of Lemma 18.19.5 and Lemma 18.21.4 agree via the identifications $(j_\mathcal{F}, j_\mathcal{F}^\sharp) = (j_U, j_U^\sharp)$ and $(j_\mathcal{G}, j_\mathcal{G}^\sharp) = (j_V, j_V^\sharp)$ of Lemma 18.21.3.

Proof. All assertions follow from Sites, Lemma 7.30.7 except for the assertion that the two maps $j^\sharp$ agree. This holds since in both cases the map $j^\sharp$ is simply the identity. Some details omitted. □

18.22. Localization of morphisms of ringed topoi

04J6 This section is the analogue of Sites, Section 7.31.

04IF Lemma 18.22.1. Let

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[r, "{(j_\mathcal{F}, j_\mathcal{F}^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "{(f, f^\sharp)}"] \\ (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) \arrow[r, "{(j_\mathcal{G}, j_\mathcal{G}^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}') \end{tikzcd}$$

We have $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$ and $f'_*j_\mathcal{F}^* = j_\mathcal{G}^*f_*$. Moreover, the morphism f' is characterized by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F}).$$

Proof. By Sites, Lemma 7.31.1 we have the diagram of underlying topoi, the equality $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$, and the description of $(f')^{-1}$. To define $(f')^\sharp$ we use the map

$$(f')^\sharp : \mathcal{O}'_\mathcal{G} = j_\mathcal{G}^{-1}\mathcal{O}' \xrightarrow{j_\mathcal{G}^{-1}f^\sharp} j_\mathcal{G}^{-1}f_*\mathcal{O} = f'_*j_\mathcal{F}^{-1}\mathcal{O} = f'_*\mathcal{O}_\mathcal{F}$$

or equivalently the map

$$(f')^\sharp : (f')^{-1}\mathcal{O}'_\mathcal{G} = (f')^{-1}j_\mathcal{G}^{-1}\mathcal{O}' = j_\mathcal{F}^{-1}f^{-1}\mathcal{O}' \xrightarrow{j_\mathcal{F}^{-1}f^\sharp} j_\mathcal{F}^{-1}\mathcal{O} = \mathcal{O}_\mathcal{F}.$$

We omit the verification that these two maps are indeed adjoint to each other. The second construction of $(f')^\sharp$ shows that the diagram commutes in the 2-category of ringed topoi (as the maps $j_\mathcal{F}^\sharp$ and $j_\mathcal{G}^\sharp$ are identities). Finally, the equality $f'_*j_\mathcal{F}^* = j_\mathcal{G}^*f_*$ follows from the equality $f'_*j_\mathcal{F}^{-1} = j_\mathcal{G}^{-1}f_*$ and the fact that pullbacks of sheaves of modules and sheaves of sets agree, see Lemma 18.21.1. □

04J7 Lemma 18.22.2. Let

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h_V^\#$, then the commutative diagrams of Lemma 18.20.1 and Lemma 18.22.1 agree via the identifications of Lemma 18.21.3.

Proof. At the level of morphisms of topoi this is Sites, Lemma 7.31.2. This works also on the level of morphisms of ringed topoi since the formulas defining $(f')^\sharp$ in the proofs of Lemma 18.20.1 and Lemma 18.22.1 agree. □

04J8 Lemma 18.22.3. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of ringed topoi

$$\begin{tikzcd}
(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \arrow[r, "(j_\mathcal{F}, j_\mathcal{F}^\sharp)"'] \arrow[d, "(f_c, f_c^\sharp)"'] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d, "(f, f^\sharp)"] \\
(Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) \arrow[r, "(j_\mathcal{G}, j_\mathcal{G}^\sharp)"] & (Sh(\mathcal{D}), \mathcal{O}').
\end{tikzcd}$$

The morphism $(f_s, f_s^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 18.22.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}})$$

of Lemma 18.21.4. Given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$, and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{tikzcd}
\mathcal{F}' \arrow[r, "s'"'] \arrow[d, "a"'] & f^{-1}\mathcal{G}' \arrow[d, "f^{-1}b"] \\
\mathcal{F} \arrow[r, "s"] & f^{-1}\mathcal{G}
\end{tikzcd}$$

commutes, then the following diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C})/\mathcal{F}', \mathcal{O}_{\mathcal{F}'}) \arrow[r, "(j_{\mathcal{F}'/\mathcal{F}}, j^\sharp_{\mathcal{F}'/\mathcal{F}})"'] \arrow[d, "(f_{s'}, f^\sharp_{s'})"'] & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \arrow[d, "(f_s, f^\sharp_s)"] \\ (Sh(\mathcal{D})/\mathcal{G}', \mathcal{O}'_{\mathcal{G}'}) \arrow[r, "(j_{\mathcal{G}'/\mathcal{G}}, j^\sharp_{\mathcal{G}'/\mathcal{G}})"] & (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}'}) \end{tikzcd}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 7.31.3. To check that the diagrams commute as morphisms of ringed topoi use the commutative diagrams of Lemmas 18.21.4 and 18.22.1. □

04J9 Lemma 18.22.4. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$, $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be as in Lemma 18.22.3. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h_V^\#$, $\mathcal{F} = h_U^\#$ and s comes from a morphism $c : U \to u(V)$, then the commutative diagrams of Lemma 18.20.2 and Lemma 18.22.3 agree via the identifications of Lemma 18.21.3.

Proof. This is formal using Lemmas 18.21.5 and 18.22.2. □

18.23. Local types of modules

03DK According to our general strategy explained in Section 18.18 we first define the local types for sheaves of modules on a ringed site, and then we immediately show that these types are intrinsic, hence make sense for sheaves of modules on ringed topoi.

03DL Definition 18.23.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We will freely use the notions defined in Definition 18.17.1.

(1) We say $\mathcal{F}$ is locally free if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a free $\mathcal{O}_{U_i}$-module.
(2) We say $\mathcal{F}$ is finite locally free if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a finite free $\mathcal{O}_{U_i}$-module.
(3) We say $\mathcal{F}$ is locally generated by sections if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by global sections.
(4) Given $r \geq 0$ we sat $\mathcal{F}$ is locally generated by r sections if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by r global sections.
(5) We say $\mathcal{F}$ is of finite type if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by finitely many global sections.
(6) We say $\mathcal{F}$ is quasi-coherent if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a global presentation.
(7) We say $\mathcal{F}$ is of finite presentation if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a finite global presentation.
(8) We say $\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is of finite type, and for every object U of $\mathcal{C}$ and any $s_1, \ldots, s_n \in \mathcal{F}(U)$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $(\mathcal{C}/U, \mathcal{O}_U)$.

03DM **Lemma 18.23.2.** Any of the properties (1) – (8) of Definition 18.23.1 is intrinsic (see discussion in Section 18.18).

Proof. Let $\mathcal{C}, \mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the equivalence of topoi associated to u. Set $\mathcal{O}' = g_*\mathcal{O}$, and let $g^\sharp : \mathcal{O}' \to g_*\mathcal{O}$ be the identity. Finally, set $\mathcal{F}' = g_*\mathcal{F}$. Let $\mathcal{P}_l$ be one of the properties (1) – (7) listed in Definition 18.23.1. (We will discuss the coherent case at the end of the proof.) Let $\mathcal{P}_g$ denote the corresponding property listed in Definition 18.17.1. We have already seen that $\mathcal{P}_g$ is intrinsic. We have to show that $\mathcal{P}_l(\mathcal{C}, \mathcal{O}, \mathcal{F})$ holds if and only if $\mathcal{P}_l(\mathcal{D}, \mathcal{O}', \mathcal{F}')$ holds.

Assume that $\mathcal{F}$ has $\mathcal{P}_l$. Let V be an object of $\mathcal{D}$. One of the properties of a special cocontinuous functor is that there exists a covering $\{u(U_i) \to V\}_{i \in I}$ in the site $\mathcal{D}$. By assumption, for each i there exists a covering $\{U_{ij} \to U_i\}_{j \in J_i}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{U_{ij}}$ is $\mathcal{P}_g$. By Sites, Lemma 7.29.3 we have commutative diagrams of ringed topoi

$$\begin{tikzcd}
(Sh(\mathcal{C}/U_{ij}), \mathcal{O}_{U_{ij}}) \arrow[r] \arrow[d] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d] \\
(Sh(\mathcal{D}/u(U_{ij})), \mathcal{O}'_{u(U_{ij})}) \arrow[r] & (Sh(\mathcal{D}), \mathcal{O}')
\end{tikzcd}$$

where the vertical arrows are equivalences. Hence we conclude that $\mathcal{F}'|_{u(U_{ij})}$ has property $\mathcal{P}_g$ also. And moreover, $\{u(U_{ij}) \to V\}_{i \in I, j \in J_i}$ is a covering of the site $\mathcal{D}$. Hence $\mathcal{F}'$ has property $\mathcal{P}_l$.

Assume that $\mathcal{F}'$ has $\mathcal{P}_l$. Let U be an object of $\mathcal{C}$. By assumption, there exists a covering $\{V_i \to u(U)\}_{i \in I}$ such that $\mathcal{F}'|_{V_i}$ has property $\mathcal{P}_g$. Because u is cocontinuous we can refine this covering by a family $\{u(U_j) \to u(U)\}_{j \in J}$ where $\{U_j \to U\}_{j \in J}$ is a covering in $\mathcal{C}$. Say the refinement is given by $\alpha : J \to I$ and $u(U_j) \to V_{\alpha(j)}$. Restricting is transitive, i.e., $(\mathcal{F}'|_{V_{\alpha(j)}})|_{u(U_j)} = \mathcal{F}'|_{u(U_j)}$. Hence by Lemma 18.17.2 we see that $\mathcal{F}'|_{u(U_j)}$ has property $\mathcal{P}_g$. Hence the diagram

$$\begin{tikzcd}
(Sh(\mathcal{C}/U_j), \mathcal{O}_{U_j}) \arrow[r] \arrow[d] & (Sh(\mathcal{C}), \mathcal{O}) \arrow[d] \\
(Sh(\mathcal{D}/u(U_j)), \mathcal{O}'_{u(U_j)}) \arrow[r] & (Sh(\mathcal{D}), \mathcal{O}')
\end{tikzcd}$$

where the vertical arrows are equivalences shows that $\mathcal{F}|_{U_j}$ has property $\mathcal{P}_g$ also. Thus $\mathcal{F}$ has property $\mathcal{P}_l$ as desired.

Finally, we prove the lemma in case $\mathcal{P}_l = coherent$[2]. Assume $\mathcal{F}$ is coherent. This implies that $\mathcal{F}$ is of finite type and hence $\mathcal{F}'$ is of finite type also by the first part of the proof. Let V be an object of $\mathcal{D}$ and let $s_1, \ldots, s_n \in \mathcal{F}'(V)$. We have to show that the kernel $\mathcal{K}'$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}'|_V$ is of finite type on $\mathcal{D}/V$. This means we have to show that for any V'/V there exists a covering $\{V_i' \to V'\}$ such that $\mathcal{F}'|_{V_i'}$ is generated by finitely many sections. Replacing V by V' (and restricting the sections s_j to V') we reduce to the case where $V' = V$. Since u is a special

[2]The mechanics of this are a bit awkward, and we suggest the reader skip this part of the proof.

cocontinuous functor, there exists a covering $\{u(U_i) \to V\}_{i \in I}$ in the site $\mathcal{D}$. Using the isomorphism of topoi $Sh(\mathcal{C}/U_i) = Sh(\mathcal{D}/u(U_i))$ we see that $\mathcal{K}'|_{u(U_i)}$ corresponds to the kernel $\mathcal{K}_i$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{U_i} \to \mathcal{F}|_{U_i}$. Since $\mathcal{F}$ is coherent we see that $\mathcal{K}_i$ is of finite type. Hence we conclude (by the first part of the proof again) that $\mathcal{K}|_{u(U_i)}$ is of finite type. Thus there exist coverings $\{V_{il} \to u(U_i)\}$ such that $\mathcal{K}|_{V_{il}}$ is generated by finitely many global sections. Since $\{V_{il} \to V\}$ is a covering of $\mathcal{D}$ we conclude that $\mathcal{K}$ is of finite type as desired.

Assume $\mathcal{F}'$ is coherent. This implies that $\mathcal{F}'$ is of finite type and hence $\mathcal{F}$ is of finite type also by the first part of the proof. Let U be an object of $\mathcal{C}$, and let $s_1, \ldots, s_n \in \mathcal{F}(U)$. We have to show that the kernel $\mathcal{K}$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $\mathcal{C}/U$. Using the isomorphism of topoi $Sh(\mathcal{C}/U) = Sh(\mathcal{D}/u(U))$ we see that $\mathcal{K}|_U$ corresponds to the kernel $\mathcal{K}'$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{u(U)} \to \mathcal{F}'|_{u(U)}$. As $\mathcal{F}'$ is coherent, we see that $\mathcal{K}'$ is of finite type. Hence, by the first part of the proof again, we conclude that $\mathcal{K}$ is of finite type. $\square$

Hence from now on we may refer to the properties of $\mathcal{O}$-modules defined in Definition 18.23.1 without specifying a site.

03DN Lemma 18.23.3. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that the site $\mathcal{C}$ has a final object X. Then

(1) The following are equivalent
  (a) $\mathcal{F}$ is locally free,
  (b) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a locally free $\mathcal{O}_{X_i}$-module, and
  (c) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a free $\mathcal{O}_{X_i}$-module.
(2) The following are equivalent
  (a) $\mathcal{F}$ is finite locally free,
  (b) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a finite locally free $\mathcal{O}_{X_i}$-module, and
  (c) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a finite free $\mathcal{O}_{X_i}$-module.
(3) The following are equivalent
  (a) $\mathcal{F}$ is locally generated by sections,
  (b) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module locally generated by sections, and
  (c) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by sections.
(4) Given $r \geq 0$, the following are equivalent
  (a) $\mathcal{F}$ is locally generated by r sections,
  (b) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module locally generated by r sections, and
  (c) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by r sections.
(5) The following are equivalent
  (a) $\mathcal{F}$ is of finite type,
  (b) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module of finite type, and

   (c) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module globally generated by finitely many sections.
(6) The following are equivalent
   (a) $\mathcal{F}$ is quasi-coherent,
   (b) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a quasi-coherent $\mathcal{O}_{X_i}$-module, and
   (c) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module which has a global presentation.
(7) The following are equivalent
   (a) $\mathcal{F}$ is of finite presentation,
   (b) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module of finite presentation, and
   (c) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is an $\mathcal{O}_{X_i}$-module has a finite global presentation.
(8) The following are equivalent
   (a) $\mathcal{F}$ is coherent, and
   (b) there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a coherent $\mathcal{O}_{X_i}$-module.

Proof. In each case we have (a) $\Rightarrow (b)$. In each of the cases (1) - (6) condition (b) implies condition (c) by axiom (2) of a site (see Sites, Definition 7.6.2) and the definition of the local types of modules. Suppose $\{X_i \to X\}$ is a covering. Then for every object U of $\mathcal{C}$ we get an induced covering $\{X_i \times_X U \to U\}$. Moreover, the global property for $\mathcal{F}|_{\mathcal{C}/X_i}$ in part (c) implies the corresponding global property for $\mathcal{F}|_{\mathcal{C}/X_i \times_X U}$ by Lemma 18.17.2, hence the sheaf has property (a) by definition. We omit the proof of (b) $\Rightarrow$ (a) in case (7). □

03DO Lemma 18.23.4. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.

(1) If $\mathcal{F}$ is locally free then $f^*\mathcal{F}$ is locally free.
(2) If $\mathcal{F}$ is finite locally free then $f^*\mathcal{F}$ is finite locally free.
(3) If $\mathcal{F}$ is locally generated by sections then $f^*\mathcal{F}$ is locally generated by sections.
(4) If $\mathcal{F}$ is locally generated by r sections then $f^*\mathcal{F}$ is locally generated by r sections.
(5) If $\mathcal{F}$ is of finite type then $f^*\mathcal{F}$ is of finite type.
(6) If $\mathcal{F}$ is quasi-coherent then $f^*\mathcal{F}$ is quasi-coherent.
(7) If $\mathcal{F}$ is of finite presentation then $f^*\mathcal{F}$ is of finite presentation.

Proof. According to the discussion in Section 18.18 we need only check preservation under pullback for a morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that f is given by a left exact, continuous functor $u : \mathcal{D} \to \mathcal{C}$ between sites which have all finite limits. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules which has one of the properties (1) – (6) of Definition 18.23.1. We know $\mathcal{D}$ has a final object Y and $X = u(Y)$ is a final object for $\mathcal{C}$. By assumption we have a covering $\{Y_i \to Y\}$ such that $\mathcal{G}|_{\mathcal{D}/Y_i}$ has the corresponding global property. Set $X_i = u(Y_i)$ so that $\{X_i \to X\}$ is a covering in $\mathcal{C}$. We get a commutative diagram of morphisms ringed sites

$$\begin{tikzcd} (\mathcal{C}/X_i, \mathcal{O}_\mathcal{C}|_{X_i}) \arrow[r] \arrow[d] & (\mathcal{C}, \mathcal{O}_\mathcal{C}) \arrow[d] \\ (\mathcal{D}/Y_i, \mathcal{O}_\mathcal{D}|_{Y_i}) \arrow[r] & (\mathcal{D}, \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

by Sites, Lemma 7.28.2. Hence by Lemma 18.17.2 that $f^*\mathcal{G}|_{X_i}$ has the corresponding global property. Hence we conclude that $\mathcal{G}$ has the local property we started out with by Lemma 18.23.3. □

18.24. Basic results on local types of modules

082S Basic lemmas related to the definitions made above.

082T **Lemma 18.24.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\theta : \mathcal{G} \to \mathcal{F}$ be a surjective $\mathcal{O}$-module map with $\mathcal{F}$ of finite presentation and $\mathcal{G}$ of finite type. Then $\mathrm{Ker}(\theta)$ is of finite type.

Proof. Omitted. Hint: See Modules, Lemma 17.11.3. □

0GZN **Lemma 18.24.2.** Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology, see Sites, Example 7.6.6. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$ and let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if for all $U \to V$ in $\mathcal{C}$ the canonical map

$$\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow \mathcal{F}(U)$$

is an isomorphism.

Proof. Assume $\mathcal{F}$ is quasi-coherent and let $U \to V$ be a morphism of $\mathcal{C}$. Since every covering of V is given by an isomorphism we conclude from Definition 18.23.1 that there exists a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_V \longrightarrow \mathcal{F}|_{\mathcal{C}/V} \longrightarrow 0$$

Since the topology on $\mathcal{C}$ is chaotic, taking sections over any object of $\mathcal{C}$ is exact. We conclude that we obtain a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}(V) \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}(V) \longrightarrow \mathcal{F}(V) \longrightarrow 0$$

of $\mathcal{F}(V)$ as an $\mathcal{O}(V)$-module and similarly for $\mathcal{F}(U)$. This easily shows that the displayed map in the statement of the lemma is an isomorphism.

Assume the displayed map in the statement of the lemma is an isomorphism for every morphism $U \to V$ in $\mathcal{C}$. Fix V and choose a presentation

$$\bigoplus\nolimits_{j \in J} \mathcal{O}(V) \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}(V) \longrightarrow \mathcal{F}(V) \longrightarrow 0$$

of $\mathcal{F}(V)$ as an $\mathcal{O}(V)$-module. Then the assumption on $\mathcal{F}$ exactly means that the corresponding sequence

$$\bigoplus\nolimits_{j \in J} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_V \longrightarrow \mathcal{F}|_{\mathcal{C}/V} \longrightarrow 0$$

is exact and we conclude that $\mathcal{F}$ is quasi-coherent. □

0GZP **Lemma 18.24.3.** Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology, see Sites, Example 7.6.6. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Assume for all $U \to V$ in $\mathcal{C}$ the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is a flat ring map. Then the category of quasi-coherent $\mathcal{O}$-modules is a weak Serre subcategory of $\mathrm{Mod}(\mathcal{O})$.

Proof. We will check the definition of a weak Serre subcategory, see Homology, Definition 12.10.1. To do this we will use the characterization of quasi-coherent modules given in Lemma 18.24.2. Consider an exact sequence

$$\mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \mathcal{F}_4$$

in $\mathrm{Mod}(\mathcal{O})$ with $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_3$, and $\mathcal{F}_4$ quasi-coherent. Let $U \to V$ be a morphism of $\mathcal{C}$ and consider the commutative diagram

$$\begin{tikzcd}
\mathcal{F}_0(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{F}_1(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{F}_2(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{F}_3(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{F}_4(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \arrow[d] \\
\mathcal{F}_0(U) \arrow[r] & \mathcal{F}_1(U) \arrow[r] & \mathcal{F}_2(U) \arrow[r] & \mathcal{F}_3(U) \arrow[r] & \mathcal{F}_4(U)
\end{tikzcd}$$

By assumption the vertical arrows with indices 0, 1, 3, 4 are isomorphisms. Since the topology on $\mathcal{C}$ is chaotic taking sections over an object of $\mathcal{C}$ is exact and hence the lower row is exact. Since $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat also the upper row is exact. Thus we conclude that the middle arrow is an isomorphism by the 5 lemma (Homology, Lemma 12.5.20). $\square$

18.25. Closed immersions of ringed topoi

08M2 When do we declare a morphism of ringed topoi $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ to be a closed immersion? By analogy with the discussion in Modules, Section 17.13 it seems natural to assume at least:

(1) The functor i is a closed immersion of topoi (Sites, Definition 7.43.7).
(2) The associated map $\mathcal{O}' \to i_*\mathcal{O}$ is surjective.

These conditions already imply a number of pleasing results which we discuss in this section. However, it seems prudent to not actually define the notion of a closed immersion of ringed topoi as there are many different definitions we could use.

08M3 **Lemma 18.25.1.** Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume i is a closed immersion of topoi and $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}'$ the kernel of $i^\sharp$. The functor

$$i_* : \mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O}')$$

is exact, fully faithful, with essential image those $\mathcal{O}'$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. By Lemma 18.15.2 and Sites, Lemma 7.43.8 we see that i_* is exact. From the fact that i_* is fully faithful on sheaves of sets, and the fact that $i^\sharp$ is surjective it follows that i_* is fully faithful as a functor $\mathrm{Mod}(\mathcal{O}) \to \mathrm{Mod}(\mathcal{O}')$. Namely, suppose that $\alpha : i_*\mathcal{F}_1 \to i_*\mathcal{F}_2$ is an $\mathcal{O}'$-module map. By the fully faithfulness of i_* we obtain a map $\beta : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves of sets. To prove β is a map of modules we have to show that

$$\begin{tikzcd}
\mathcal{O} \times \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[d] \\
\mathcal{O} \times \mathcal{F}_2 \arrow[r] & \mathcal{F}_2
\end{tikzcd}$$

commutes. It suffices to prove commutativity after applying i_*. Consider

$$\begin{array}{ccccc} \mathcal{O}' \times i_*\mathcal{F}_1 & \longrightarrow & i_*\mathcal{O} \times i_*\mathcal{F}_1 & \longrightarrow & i_*\mathcal{F}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}' \times i_*\mathcal{F}_2 & \longrightarrow & i_*\mathcal{O} \times i_*\mathcal{F}_2 & \longrightarrow & i_*\mathcal{F}_2 \end{array}$$

We know the outer rectangle commutes. Since $i^\sharp$ is surjective we conclude.

To finish the proof we have to prove the statement on the essential image of i_*. It is clear that $i_*\mathcal{F}$ is annihilated by $\mathcal{I}$ for any $\mathcal{O}$-module $\mathcal{F}$. Conversely, let $\mathcal{G}$ be a $\mathcal{O}'$-module with $\mathcal{I}\mathcal{G} = 0$. By definition of a closed subtopos there exists a subsheaf $\mathcal{U}$ of the final object of $\mathcal{D}$ such that the essential image of i_* on sheaves of sets is the class of sheaves of sets $\mathcal{H}$ such that $\mathcal{H} \times \mathcal{U} \to \mathcal{U}$ is an isomorphism. In particular, $i_*\mathcal{O} \times \mathcal{U} = \mathcal{U}$. This implies that $\mathcal{I} \times \mathcal{U} = \mathcal{O} \times \mathcal{U}$. Hence our module $\mathcal{G}$ satisfies $\mathcal{G} \times \mathcal{U} = \{0\} \times \mathcal{U} = \mathcal{U}$ (because the zero module is isomorphic to the final object of sheaves of sets). Thus there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ with $i_*\mathcal{F} = \mathcal{G}$. Since i_* is fully faithful on sheaves of sets, we see that in order to define the addition $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and the multiplication $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ it suffices to use the addition

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$

(given to us as $\mathcal{G}$ is a $\mathcal{O}'$-module) and the multiplication

$$i_*\mathcal{O} \times \mathcal{G} \to \mathcal{G}$$

which is given to us as we have the multiplication by $\mathcal{O}'$ which annihilates $\mathcal{I}$ by assumption and $i_*\mathcal{O} = \mathcal{O}'/\mathcal{I}$. By construction $\mathcal{G}$ is isomorphic to the pushforward of the $\mathcal{O}$-module $\mathcal{F}$ so constructed. ☐

18.26. Tensor product

03EK In Sections 18.9 and 18.11 we defined the change of rings functor by a tensor product construction. To be sure this construction makes sense also to define the tensor product of presheaves of $\mathcal{O}$-modules. To be precise, suppose $\mathcal{C}$ is a category, $\mathcal{O}$ is a presheaf of rings, and $\mathcal{F}$, $\mathcal{G}$ are presheaves of $\mathcal{O}$-modules. In this case we define $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$ to be the presheaf

$$U \longmapsto (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$$

If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}$-modules then we define

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^\#$$

to be the sheaf of $\mathcal{O}$-modules associated to the presheaf $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$.

Here are some formulas which we will use below without further mention:

$$(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}) \otimes_{p,\mathcal{O}} \mathcal{H} = \mathcal{F} \otimes_{p,\mathcal{O}} (\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{H}),$$

and similarly for sheaves. If $\mathcal{O}_1 \to \mathcal{O}_2$ is a map of presheaves of rings, then

$$(\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{G}) \otimes_{p,\mathcal{O}_1} \mathcal{O}_2 = (\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2) \otimes_{p,\mathcal{O}_2} (\mathcal{G} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2),$$

and similarly for sheaves. These follow from their algebraic counterparts and sheafification.

0GMW **Lemma 18.26.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$, $\mathcal{G}$ be presheaves of $\mathcal{O}$-modules. Then $\mathcal{F}^\# \otimes_{\mathcal{O}^\#} \mathcal{G}^\#$ is equal to $(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^\#$.

Proof. Omitted. Hint: use the characterization of tensor product in terms of bilinear maps below and use the universal property of sheafification. $\square$

Let $\mathcal{C}$ be a site, let $\mathcal{O}$ be a sheaf of rings and let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be sheaves of $\mathcal{O}$-modules. In this case we define

$$\mathrm{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \{\varphi \in \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \mid \varphi \text{ is } \mathcal{O}\text{-bilinear}\}.$$

With this definition we have

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \mathrm{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

In other words $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ represents the functor which associates to $\mathcal{H}$ the set of bilinear maps $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$. In particular, since the notion of a bilinear map makes sense for a pair of modules on a ringed topos, we see that the tensor product of sheaves of modules is intrinsic to the topos (compare the discussion in Section 18.18). In fact we have the following.

03EL Lemma 18.26.2. Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_{\mathcal{D}}$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{D}}} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{C}}} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.

Proof. For a sheaf $\mathcal{H}$ of $\mathcal{O}_{\mathcal{C}}$ modules we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_{\mathcal{C}}}(f^*(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}), \mathcal{H}) &= \mathrm{Hom}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, f_*\mathcal{H}) \\
&= \mathrm{Bilin}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F} \times \mathcal{G}, f_*\mathcal{H}) \\
&= \mathrm{Bilin}_{f^{-1}\mathcal{O}_{\mathcal{D}}}(f^{-1}\mathcal{F} \times f^{-1}\mathcal{G}, \mathcal{H}) \\
&= \mathrm{Hom}_{f^{-1}\mathcal{O}_{\mathcal{D}}}(f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_{\mathcal{D}}} f^{-1}\mathcal{G}, \mathcal{H}) \\
&= \mathrm{Hom}_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{F} \otimes_{f^*\mathcal{O}_{\mathcal{D}}} f^*\mathcal{G}, \mathcal{H})
\end{aligned}$$

The interesting "=" in this sequence of equalities is the third equality. It follows from the definition and adjointness of f_* and f^{-1} (as discussed in previous sections) in a straightforward manner. $\square$

03L6 Lemma 18.26.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules.

(1) If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.
(2) If $\mathcal{F}$, $\mathcal{G}$ are finite locally free, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.
(3) If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.
(4) If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.
(5) If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.
(6) If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.
(7) If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is coherent.
(8) If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.

Proof. Omitted. Hint: Compare with Sheaves of Modules, Lemma 17.16.6. $\square$

18.27. Internal Hom

04TT Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$, $\mathcal{G}$ be presheaves of $\mathcal{O}$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For $\varphi : V \to U$ in $\mathcal{C}$ we define a restriction mapping

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V)$$

by restricting via the relocalization morphism $j : \mathcal{C}/V \to \mathcal{C}/U$, see Sites, Lemma 7.25.8. Hence this defines a presheaf $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a presheaf of $\mathcal{O}$-modules. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

03EM Lemma 18.27.1. If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $\mathcal{F}$ is a presheaf of $\mathcal{O}$-modules, and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathcal{O}$-modules.

Proof. Omitted. Hints: Note first that $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_\mathcal{O}(\mathcal{F}^\#, \mathcal{G})$, which reduces the question to the case where both $\mathcal{F}$ and $\mathcal{G}$ are sheaves. The result for sheaves of sets is Sites, Lemma 7.26.1. $\square$

0E8H Lemma 18.27.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}$-modules. Then formation of $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$ commutes with restriction to U for $U \in \mathrm{Ob}(\mathcal{C})$.

Proof. Immediate from the definition. $\square$

0GMX Remark 18.27.3. Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_\mathcal{D}$-modules. There is a canonical map

$$f^* \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})$$

Namely, this map is adjoint to the map

$$\mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \longrightarrow f_* \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})$$

defined as follows. Say f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For sections over $V \in \mathrm{Ob}(\mathcal{D})$ we use the map

$$\begin{aligned} \Gamma(V, \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G})) &= \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V) \\ &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{u(V)}}(f^*\mathcal{F}|_{u(V)}, \mathcal{G}|_{u(V)}) \\ &= \Gamma(u(V), \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})) \\ &= \Gamma(V, f_* \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})) \end{aligned}$$

where for the arrow we use pullback by the morphism $(\mathcal{C}/u(V), \mathcal{O}_{u(V)}) \to (\mathcal{D}/V, \mathcal{O}_V)$ induced by f.

In the situation of Lemma 18.27.1 the "evaluation" morphism factors through the tensor product of sheaves of modules

$$\mathcal{F} \otimes_\mathcal{O} \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

03EN Lemma 18.27.4. Internal hom and (co)limits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.

(1) For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor

$$\mathrm{PMod}(\mathcal{O}) \longrightarrow \mathrm{PMod}(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$$

commutes with arbitrary limits.

(2) For any presheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor

$$\mathrm{PMod}(\mathcal{O}) \longrightarrow \mathrm{PMod}(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$$

commutes with arbitrary colimits, in a formula

$$\mathcal{H}om_\mathcal{O}(\mathrm{colim}_i\, \mathcal{F}_i, \mathcal{G}) = \lim_i \mathcal{H}om_\mathcal{O}(\mathcal{F}_i, \mathcal{G}).$$

Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings.

(3) For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor
$$\text{Mod}(\mathcal{O}) \longrightarrow \text{Mod}(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary limits.

(4) For any sheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor
$$\text{Mod}(\mathcal{O}) \longrightarrow \text{Mod}(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary colimits, in a formula
$$\mathcal{H}om_{\mathcal{O}}(\text{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

Proof. Let $\mathcal{I} \to \text{PMod}(\mathcal{O})$, $i \mapsto \mathcal{G}_i$ be a diagram. Let U be an object of the category $\mathcal{C}$. As j_U^* is both a left and a right adjoint we see that $\lim_i j_U^*\mathcal{G}_i = j_U^* \lim_i \mathcal{G}_i$. Hence we have
$$\begin{aligned} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \lim_i \mathcal{G}_i)(U) &= \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \lim_i \mathcal{G}_i|_U) \\ &= \lim_i \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}_i|_U) \\ &= \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_i)(U) \end{aligned}$$
by definition of a limit. This proves (1). Part (2) is proved in exactly the same way. Part (3) follows from (1) because the limit of a diagram of sheaves is the same as the limit in the category of presheaves. Finally, (4) follow because, in the formula we have
$$\text{Mor}_{\text{Mod}(\mathcal{O})}(\text{colim}_i \mathcal{F}_i, \mathcal{G}) = \text{Mor}_{\text{PMod}(\mathcal{O})}(\text{colim}_i^{PSh} \mathcal{F}_i, \mathcal{G})$$
as the colimit $\text{colim}_i \mathcal{F}_i$ is the sheafification of the colimit $\text{colim}_i^{PSh} \mathcal{F}_i$ in $\text{PMod}(\mathcal{O})$. Hence (4) follows from (2) (by the remark on limits above again). $\square$

0GMY Lemma 18.27.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules.

(1) If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is an exact sequence of $\mathcal{O}$-modules, then
$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_2, \mathcal{G})$$
is exact.

(2) If $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ is an exact sequence of $\mathcal{O}$-modules, then
$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_2)$$
is exact.

Proof. Follows from Lemma 18.27.4 and Homology, Lemma 12.7.2. $\square$

03EO Lemma 18.27.6. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.

(1) Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be presheaves of $\mathcal{O}$-modules. There is a canonical isomorphism
$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$
which is functorial in all three entries (sheaf Hom in all three spots). In particular,
$$\text{Mor}_{\text{PMod}(\mathcal{O})}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) = \text{Mor}_{\text{PMod}(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

(2) Suppose that $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are sheaves of $\mathcal{O}$-modules. There is a canonical isomorphism

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries (sheaf Hom in all three spots). In particular,

$$\mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \mathrm{Mor}_{\mathrm{Mod}(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

Proof. This is the analogue of Algebra, Lemma 10.12.8. The proof is the same, and is omitted. □

03EP Lemma 18.27.7. Tensor product and colimits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.

(1) For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor

$$\mathrm{PMod}(\mathcal{O}) \longrightarrow \mathrm{PMod}(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$$

commutes with arbitrary colimits.

(2) Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings. For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor

$$\mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$

commutes with arbitrary colimits.

Proof. This is because tensor product is adjoint to internal hom according to Lemma 18.27.6. See Categories, Lemma 4.24.5. □

0932 Lemma 18.27.8. Let $\mathcal{C}$ be a category, resp. a site Let $\mathcal{O} \to \mathcal{O}'$ be a map of presheaves, resp. sheaves of rings. Then

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}', \mathcal{F}))$$

for any $\mathcal{O}'$-module $\mathcal{G}$ and $\mathcal{O}$-module $\mathcal{F}$.

Proof. This is the analogue of Algebra, Lemma 10.14.4. The proof is the same, and is omitted. □

0E8I Lemma 18.27.9. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For $\mathcal{G}$ in $\mathrm{Mod}(\mathcal{O}_U)$ and $\mathcal{F}$ in $\mathrm{Mod}(\mathcal{O})$ we have $j_{U!}\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F} = j_{U!}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U)$.

Proof. Let $\mathcal{H}$ be an object of $\mathrm{Mod}(\mathcal{O})$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}}(j_{U!}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U), \mathcal{H}) &= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G} \otimes_{\mathcal{O}_U} \mathcal{F}|_U, \mathcal{H}|_U) \\
&= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{H}|_U)) \\
&= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H})|_U) \\
&= \mathrm{Hom}_{\mathcal{O}}(j_{U!}\mathcal{G}, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H})) \\
&= \mathrm{Hom}_{\mathcal{O}}(j_{U!}\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}, \mathcal{H})
\end{aligned}$$

The first equality because $j_{U!}$ is a left adjoint to restriction of modules. The second by Lemma 18.27.6. The third by Lemma 18.27.2. The fourth because $j_{U!}$ is a left adjoint to restriction of modules. The fifth by Lemma 18.27.6. The lemma follows from this and the Yoneda lemma. □

0EYY Remark 18.27.10. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{C}$ and consider the localization morphism $j : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$. See Sites, Definition 7.30.4. We claim that (a) $j_!\mathbf{Z} = \mathbf{Z}^\#_\mathcal{F}$ and (b) $j_!(j^{-1}\mathcal{H}) = j_!\mathbf{Z} \otimes_\mathbf{Z} \mathcal{H}$ for any abelian sheaf $\mathcal{H}$ on $\mathcal{C}$. Let $\mathcal{G}$ be an abelian on $\mathcal{C}$. Part (a) follows from the Yoneda lemma because

$$\mathrm{Hom}(j_!\mathbf{Z}, \mathcal{G}) = \mathrm{Hom}(\mathbf{Z}, j^{-1}\mathcal{G}) = \mathrm{Hom}(\mathbf{Z}^\#_\mathcal{F}, \mathcal{G})$$

where the second equality holds because both sides of the equality evaluate to the set of maps from $\mathcal{F} \to \mathcal{G}$ viewed as an abelian group. For (b) we use the Yoneda lemma and

$$\begin{aligned}\mathrm{Hom}(j_!(j^{-1}\mathcal{H}), \mathcal{G}) &= \mathrm{Hom}(j^{-1}\mathcal{H}, j^{-1}\mathcal{G}) \\ &= \mathrm{Hom}(\mathbf{Z}, \mathcal{H}om(j^{-1}\mathcal{H}, j^{-1}\mathcal{G})) \\ &= \mathrm{Hom}(\mathbf{Z}, j^{-1}\,\mathcal{H}om(\mathcal{H}, \mathcal{G})) \\ &= \mathrm{Hom}(j_!\mathbf{Z}, \mathcal{H}om(\mathcal{H}, \mathcal{G})) \\ &= \mathrm{Hom}(j_!\mathbf{Z} \otimes_\mathbf{Z} \mathcal{H}, \mathcal{G})\end{aligned}$$

Here we use adjunction, the fact that taking $\mathcal{H}om$ commutes with localization, and Lemma 18.27.6.

0GMZ Lemma 18.27.11. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module of finite presentation. Let $\mathcal{G} = \mathrm{colim}_{\lambda\in\Lambda}\,\mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}$-modules. Then the canonical map

$$\mathrm{colim}_\lambda\,\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda) \longrightarrow \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. It suffices to show the arrow is an isomorphism after restriction to U for all U in $\mathcal{C}$. Both taking colimits of sheaves of modules and taking internal hom commute with restriction to U. See for example Lemmas 18.14.3 and 18.27.2. Fix U. Given a covering $\{U_i \to U\}_{i\in I}$, then it suffices to prove the restriction to each U_i is an isomorphism. Hence we may assume $\mathcal{F}$ has a global presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O} \to \mathcal{F} \to 0$$

The functor $\mathcal{H}om_\mathcal{O}(-,-)$ commutes with finite direct sums in either variable and $\mathcal{H}om_\mathcal{O}(\mathcal{O}, -)$ is the identity functor. By this and by Lemma 18.27.5 we obtain an exact sequence

$$0 \to \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,m} \mathcal{G}$$

Since filtered colimits are exact in $\mathrm{Mod}(\mathcal{O})$ by Lemma 18.14.2 also the top row in the following commutative diagram is exact

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{colim}_\lambda\,\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda) \arrow[r] \arrow[d] & \mathrm{colim}_\lambda \bigoplus_{i=1,\ldots,n} \mathcal{G}_\lambda \arrow[r] \arrow[d] & \mathrm{colim}_\lambda \bigoplus_{j=1,\ldots,m} \mathcal{G}_\lambda \arrow[d] \\ 0 \arrow[r] & \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) \arrow[r] & \bigoplus_{i=1,\ldots,n} \mathcal{G} \arrow[r] & \bigoplus_{j=1,\ldots,m} \mathcal{G} \end{tikzcd}$$

Since the right two vertical arrows are isomorphisms we conclude. □

0GN0 Lemma 18.27.12. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{G} = \mathrm{colim}_{\lambda\in\Lambda}\,\mathcal{G}_\lambda$ be a filtered colimit of $\mathcal{O}$-modules. Let $\mathcal{F}$ be an $\mathcal{O}$-module of finite presentation. Then we have

$$\mathrm{colim}_\lambda\,\mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda) = \mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}).$$

if the hypotheses of Sites, Lemma 7.17.8 part (4) are satisfied for the site $\mathcal{C}$; please see Sites, Remark 7.17.9.

Proof. Set $\mathcal{H} = \mathcal{H}om_\mathcal{O}(\mathcal{F}, \operatorname{colim} \mathcal{G}_\lambda)$ and $\mathcal{H}_\lambda = \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda)$. Recall that

$$\mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}) = \Gamma(\mathcal{C}, \mathcal{H}) \quad \text{and} \quad \mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}_\lambda) = \Gamma(\mathcal{C}, \mathcal{H}_\lambda)$$

by construction. By Lemma 18.27.11 we have $\mathcal{H} = \operatorname{colim} \mathcal{H}_\lambda$. Thus the lemma follows from Sites, Lemma 7.17.8. □

18.28. Flat modules

03EQ We can define flat modules exactly as in the case of modules over rings.

03ER Definition 18.28.1. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.

(1) A presheaf $\mathcal{F}$ of $\mathcal{O}$-modules is called flat if the functor

$$\mathrm{PMod}(\mathcal{O}) \longrightarrow \mathrm{PMod}(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$$

is exact.

(2) A map $\mathcal{O} \to \mathcal{O}'$ of presheaves of rings is called flat if $\mathcal{O}'$ is flat as a presheaf of $\mathcal{O}$-modules.

(3) If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then we say $\mathcal{F}$ is flat if the functor

$$\mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$

is exact.

(4) A map $\mathcal{O} \to \mathcal{O}'$ of sheaves of rings on a site is called flat if $\mathcal{O}'$ is flat as a sheaf of $\mathcal{O}$-modules.

The notion of a flat module or flat ring map is intrinsic (Section 18.18).

03ES Lemma 18.28.2. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If each $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module, then $\mathcal{F}$ is flat.

Proof. This is immediate from the definitions. □

03ET Lemma 18.28.3. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}^\#$ is a flat $\mathcal{O}^\#$-module.

Proof. Omitted. (Hint: Sheafification is exact.) □

0GN1 Lemma 18.28.4. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Assume that every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}_{i \in I}$ such that $\mathcal{F}(U_i)$ is a flat $\mathcal{O}(U_i)$-module. Then $\mathcal{F}^\#$ is a flat $\mathcal{O}^\#$-module.

Proof. Let $\mathcal{G} \subset \mathcal{G}'$ be an inclusion of $\mathcal{O}^\#$-modules. We have to show that

$$\mathcal{G} \otimes_{\mathcal{O}^\#} \mathcal{F}^\# \longrightarrow \mathcal{G}' \otimes_{\mathcal{O}^\#} \mathcal{F}^\#$$

is injective. By Lemma 18.26.1 the source of this arrow is the sheafification of the presheaf $\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$ and similarly for the target. If U is an object of $\mathcal{C}$ such that $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module, then

$$(\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F})(U) = \mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \longrightarrow \mathcal{G}'(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) = (\mathcal{G}' \otimes_{p,\mathcal{O}} \mathcal{F})(U)$$

is injective. Hence we reduce to showing: given a map of presheaves $f : \mathcal{H} \to \mathcal{H}'$ on $\mathcal{C}$ such that every U in $\mathcal{C}$ has a covering $\{U_i \to U\}_{i \in I}$ with $\mathcal{H}(U_i) \to \mathcal{H}'(U_i)$ injective, then $f^\#$ is injective. This we leave to the reader as an exercise. □

03EU Lemma 18.28.5. Colimits and tensor product.

(1) A filtered colimit of flat presheaves of modules is flat. A direct sum of flat presheaves of modules is flat.
(2) A filtered colimit of flat sheaves of modules is flat. A direct sum of flat sheaves of modules is flat.

Proof. Part (1) follows from Lemma 18.27.7 and Algebra, Lemma 10.8.8 by looking at sections over objects. To see part (2), use Lemma 18.27.7 and the fact that a filtered colimit of exact complexes is an exact complex (this uses that sheafification is exact and commutes with colimits). Some details omitted. □

0E8J Lemma 18.28.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}|_U$ is a flat $\mathcal{O}_U$-module.

Proof. Let $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ be an exact complex of $\mathcal{O}_U$-modules. Since $j_{U!}$ is exact (Lemma 18.19.3) and $\mathcal{F}$ is flat as an $\mathcal{O}$-modules then we see that the complex made up of the modules

$$j_{U!}(\mathcal{G}_i \otimes_{\mathcal{O}_U} \mathcal{F}|_U) = j_{U!}\mathcal{G}_i \otimes_{\mathcal{O}} \mathcal{F}$$

(Lemma 18.27.9) is exact. We conclude that $\mathcal{G}_1 \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{G}_2 \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{G}_3 \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is exact by Lemma 18.19.4. □

03EV Lemma 18.28.7. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Consider the functor $j_U : \mathcal{C}/U \to \mathcal{C}$.

(1) The presheaf of $\mathcal{O}$-modules $j_{U!}\mathcal{O}_U$ (see Remark 18.19.7) is flat.
(2) If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}$-modules.

Proof. Proof of (1). By the discussion in Remark 18.19.7 we see that

$$j_{U!}\mathcal{O}_U(V) = \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_\mathcal{C}(V, U)} \mathcal{O}(V)$$

which is a flat $\mathcal{O}(V)$-module. Hence (1) follows from Lemma 18.28.2. Then (2) follows as $j_{U!}\mathcal{O}_U = (j_{U!}\mathcal{O}_U)^\#$ (the first $j_{U!}$ on sheaves, the second on presheaves) and Lemma 18.28.3. □

03EW Lemma 18.28.8. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.

(1) Any presheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.
(2) Any presheaf of $\mathcal{O}$-modules is a quotient of a flat presheaf of $\mathcal{O}$-modules.
(3) If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.
(4) If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a flat sheaf of $\mathcal{O}$-modules.

Proof. Proof of (1). For every object U of $\mathcal{C}$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective. The source is flat by combining Lemmas 18.28.5 and 18.28.7 which proves (2). The sheaf case follows from this either by sheafifying or repeating the same argument. □

03EX Lemma 18.28.9. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules.

(1) If $\mathcal{F}$ is a flat presheaf of modules, then the sequence

$$0 \to \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$

is exact.

(2) If $\mathcal{C}$ is a site, $\mathcal{O}$, $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{G}$ are sheaves, and $\mathcal{F}$ is flat as a sheaf of modules, then the sequence

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

is exact.

Proof. Choose a flat presheaf of $\mathcal{O}$-modules $\mathcal{G}'$ which surjects onto $\mathcal{G}$. This is possible by Lemma 18.28.8. Let $\mathcal{G}'' = \mathrm{Ker}(\mathcal{G}' \to \mathcal{G})$. The lemma follows by applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the flat module $\mathcal{G}'$ is exact. The proof in the case of sheaves is exactly the same. □

03EY Lemma 18.28.10. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules.

(1) If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.
(2) If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.

If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds in $\mathrm{Mod}(\mathcal{O})$.

Proof. Let $\mathcal{G}^\bullet$ be an arbitrary exact complex of presheaves of $\mathcal{O}$-modules. Assume that $\mathcal{F}_0$ is flat. By Lemma 18.28.9 we see that

$$0 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_0 \to 0$$

is a short exact sequence of complexes of presheaves of $\mathcal{O}$-modules. Hence (1) and (2) follow from the snake lemma. The case of sheaves of modules is proved in the same way. □

03EZ Lemma 18.28.11. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of presheaves of $\mathcal{O}$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}$-modules, then for any presheaf $\mathcal{G}$ of $\mathcal{O}$-modules the complex

$$\ldots \to \mathcal{F}_2 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_1 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_0 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{Q} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$

is exact also. If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds $\textit{Mod}(\mathcal{O})$.

Proof. Follows from Lemma 18.28.9 by splitting the complex into short exact sequences and using Lemma 18.28.10 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. □

0G6Q Lemma 18.28.12. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{G}$ and $\mathcal{F}$ are flat $\mathcal{O}$-modules, then $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is a flat $\mathcal{O}$-module.

Proof. This is true because

$$(\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}) \otimes_{\mathcal{O}} \mathcal{H} = \mathcal{G} \otimes_{\mathcal{O}} (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H})$$

and a composition of exact functors is exact. □

05V4 Lemma 18.28.13. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on a site $\mathcal{C}$. If $\mathcal{G}$ is a flat $\mathcal{O}_1$-module, then $\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2$ is a flat $\mathcal{O}_2$-module.

Proof. This is true because

$$(\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2) \otimes_{\mathcal{O}_2} \mathcal{H} = \mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{F}$$

(as sheaves of abelian groups for example). □

The following lemma is the analogue of the equational criterion of flatness (Algebra, Lemma 10.39.11).

08FC Lemma 18.28.14. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module. The following are equivalent

(1) $\mathcal{F}$ is a flat $\mathcal{O}$-module.
(2) Let U be an object of $\mathcal{C}$ and let

$$\mathcal{O}_U \xrightarrow{(f_1, \ldots, f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. Then there exists a covering $\{U_i \to U\}$ and for each i a factorization

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{B_i} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1}, \ldots, t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1, \ldots, s_n)|_{U_i}$ such that $B_i \circ (f_1, \ldots, f_n)|_{U_i} = 0$.
(3) Let U be an object of $\mathcal{C}$ and let

$$\mathcal{O}_U^{\oplus m} \xrightarrow{A} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. Then there exists a covering $\{U_i \to U\}$ and for each i a factorization

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{B_i} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1}, \ldots, t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1, \ldots, s_n)|_{U_i}$ such that $B_i \circ A|_{U_i} = 0$.

Proof. Assume (1). Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1, \ldots, f_n$. Then $\sum f_j \otimes s_j$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat (Lemma 18.28.6) the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists a covering $\{U_i \to U\}$ such that $\sum f_j|_{U_i} \otimes s_j|_{U_i}$ is zero in $\mathcal{I}(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i)$. Unwinding the definitions using Algebra, Lemma 10.107.10 we find $t_{i1}, \ldots, t_{il_i} \in \mathcal{F}(U_i)$ and $a_{ijk} \in \mathcal{O}(U_i)$ such that $\sum_j a_{ijk} f_j|_{U_i} = 0$ and $s_j|_{U_i} = \sum_k a_{ijk} t_{ik}$. Thus (2) holds.

Assume (2). Let U, n, m, A and $s_1, \ldots, s_n$ as in (3) be given. Observe that A has m columns. We will prove the assertion of (3) is true by induction on m. For the base case $m = 0$ we can use the factorization through the zero sheaf (in other words $l_i = 0$). Let $(f_1, \ldots, f_n)$ be the last column of A and apply (2). This gives new diagrams

$$\mathcal{O}_{U_i}^{\oplus m} \xrightarrow{B_i \circ A|_{U_i}} \mathcal{O}_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1}, \ldots, t_{il_i})} \mathcal{F}|_{U_i}$$

but the first column of $A_i = B_i \circ A|_{U_i}$ is zero. Hence we can apply the induction hypothesis to U_i, l_i, $m-1$, the matrix consisting of the first $m-1$ columns of A_i, and $t_{i1}, \ldots, t_{il_i}$ to get coverings $\{U_{ij} \to U_j\}$ and factorizations

$$\mathcal{O}_{U_{ij}}^{\oplus l_i} \xrightarrow{C_{ij}} \mathcal{O}_{U_{ij}}^{\oplus k_{ij}} \xrightarrow{(v_{ij1}, \ldots, v_{ijk_{ij}})} \mathcal{F}|_{U_{ij}}$$

of $(t_{i1}, \ldots, t_{il_i})|_{U_{ij}}$ such that $C_i \circ B_i|_{U_{ij}} \circ A|_{U_{ij}} = 0$. Then $\{U_{ij} \to U\}$ is a covering and we get the desired factorizations using $B_{ij} = C_i \circ B_i|_{U_{ij}}$ and v_{ija}. In this way we see that (2) implies (3).

Assume (3). Let $\mathcal{G} \to \mathcal{H}$ be an injective homomorphism of $\mathcal{O}$-modules. We have to show that $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$ is injective. Let U be an object of $\mathcal{C}$ and let $s \in (\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F})(U)$ be a section which maps to zero in $\mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$. We have to show that s is zero. Since $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is a sheaf, it suffices to find a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that $s|_{U_i}$ is zero for all $i \in I$. Hence we may always replace U by the members of a covering. In particular, since $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is the sheafification of $\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$ we may assume that s is the image of $s' \in \mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U)$. Arguing similarly for $\mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}$ we may assume that s' maps to zero in $\mathcal{H}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U)$. Write $\mathcal{F}(U) = \operatorname{colim} M_\alpha$ as a filtered colimit of finitely presented $\mathcal{O}(U)$-modules M_α (Algebra, Lemma 10.11.3). Since tensor product commutes with filtered colimits (Algebra, Lemma 10.12.9) we can choose an α such that s' comes from some $s'' \in \mathcal{G}(U) \otimes_{\mathcal{O}(U)} M_\alpha$ and such that s'' maps to zero in $\mathcal{H}(U) \otimes_{\mathcal{O}(U)} M_\alpha$. Fix α and s''. Choose a presentation

$$\mathcal{O}(U)^{\oplus m} \xrightarrow{A} \mathcal{O}(U)^{\oplus n} \to M_\alpha \to 0$$

We apply (3) to the corresponding complex of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^{\oplus m} \xrightarrow{A} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

After replacing U by the members of the covering U_i we find that the map

$$M_\alpha \to \mathcal{F}(U)$$

factors through a free module $\mathcal{O}(U)^{\oplus l}$ for some l. Since $\mathcal{G}(U) \to \mathcal{H}(U)$ is injective we conclude that

$$\mathcal{G}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U)^{\oplus l} \to \mathcal{H}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U)^{\oplus l}$$

is injective too. Hence as s'' maps to zero in the module on the right, it also maps to zero in the module on the left, i.e., s is zero as desired. $\square$

08M4 Lemma 18.28.15. Let $\mathcal{C}$ be a site. Let $\mathcal{O}' \to \mathcal{O}$ be a surjection of sheaves of rings whose kernel $\mathcal{I}$ is an ideal of square zero. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = \mathcal{F}'/\mathcal{I}\mathcal{F}'$. The following are equivalent

(1) $\mathcal{F}'$ is a flat $\mathcal{O}'$-module, and
(2) $\mathcal{F}$ is a flat $\mathcal{O}$-module and $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}'$ is injective.

Proof. If (1) holds, then $\mathcal{F} = \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{O}$ is flat over $\mathcal{O}$ by Lemma 18.28.13 and we see the map $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}'$ is injective by applying $- \otimes_{\mathcal{O}'} \mathcal{F}'$ to the exact sequence $0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$, see Lemma 18.28.9. Assume (2). In the rest of the proof we will use without further mention that $\mathcal{K} \otimes_{\mathcal{O}'} \mathcal{F}' = \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$ for any $\mathcal{O}'$-module $\mathcal{K}$ annihilated by $\mathcal{I}$. Let $\alpha : \mathcal{G}' \to \mathcal{H}'$ be an injective map of $\mathcal{O}'$-modules. Let $\mathcal{G} \subset \mathcal{G}'$, resp. $\mathcal{H} \subset \mathcal{H}'$ be the subsheaf of sections annihilated by $\mathcal{I}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{G}' \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{G}'/\mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] \arrow[d] & 0 \\
\mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] & \mathcal{H}' \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] & \mathcal{H}'/\mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' \arrow[r] & 0
\end{tikzcd}$$

Note that $\mathcal{G}'/\mathcal{G}$ and $\mathcal{H}'/\mathcal{H}$ are annihilated by $\mathcal{I}$ and that $\mathcal{G}'/\mathcal{G} \to \mathcal{H}'/\mathcal{H}$ is injective. Thus the right vertical arrow is injective as $\mathcal{F}$ is flat over $\mathcal{O}$. The same is true for the left vertical arrow. Hence the middle vertical arrow is injective and $\mathcal{F}'$ is flat. □

0GLY Lemma 18.28.16. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a flat homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals such that the induced map $\mathcal{O}/\mathcal{I} \to \mathcal{O}'/\mathcal{I}\mathcal{O}'$ is an isomorphism. For any $\mathcal{O}$-module $\mathcal{F}$ annihilated by $\mathcal{I}^n$ for some $n \geq 0$ the map $\text{id} \otimes 1 : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}'$ is an isomorphism.

Proof. Omitted. Hint: See More on Algebra, Lemma 15.89.2. □

18.29. Duals

0FNX Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category of $\mathcal{O}$-modules endowed with the tensor product constructed in Section 18.26 is a symmetric monoidal category. For an $\mathcal{O}$-module $\mathcal{F}$ the following are equivalent

(1) $\mathcal{F}$ has a left dual in the monoidal category of $\mathcal{O}$-modules,
(2) for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module, and
(3) $\mathcal{F}$ is of finite presentation and flat as an $\mathcal{O}$-module.

This is proved in Example 18.29.1 and Lemmas 18.29.2 and 18.29.3 of this section.

0FNY Example 18.29.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module such that for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module. Then the map

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{F})$$

is an isomorphism. Namely, this is a local question, it is true if $\mathcal{F}$ is finite free, and it holds for any summand of a module for which it is true (details omitted). Denote

$$\eta : \mathcal{O} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$$

the map sending 1 to the section corresponding to $\text{id}_\mathcal{F}$ under the isomorphism above. Denote

$$\epsilon : \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{O}) \otimes_\mathcal{O} \mathcal{F} \longrightarrow \mathcal{O}$$

the evaluation map. Then we see that $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{O}), \eta, \epsilon$ is a left dual for $\mathcal{F}$ as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \text{id}_\mathcal{F}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}_{\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{O})}$.

0FNZ Lemma 18.29.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a $\mathcal{O}$-module. Let $\mathcal{G}, \eta, \epsilon$ be a left dual of $\mathcal{F}$ in the monoidal category of $\mathcal{O}$-modules, see Categories, Definition 4.43.5. Then

(1) for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}|_{U_i}$-module,
(2) the map $e : \mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{O}) \to \mathcal{G}$ sending a local section λ to $(\lambda \otimes 1)(\eta)$ is an isomorphism,
(3) we have $\epsilon(f, g) = e^{-1}(g)(f)$ for local sections f and g of $\mathcal{F}$ and $\mathcal{G}$.

Proof. The assumptions mean that

$$\mathcal{F} \xrightarrow{\eta \otimes 1} \mathcal{F} \otimes_\mathcal{O} \mathcal{G} \otimes_\mathcal{O} \mathcal{F} \xrightarrow{1 \otimes \epsilon} \mathcal{F} \quad \text{and} \quad \mathcal{G} \xrightarrow{1 \otimes \eta} \mathcal{G} \otimes_\mathcal{O} \mathcal{F} \otimes_\mathcal{O} \mathcal{G} \xrightarrow{\epsilon \otimes 1} \mathcal{G}$$

are the identity map. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering of U, we can find a finite number of sections $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ of $\mathcal{F}$ and $\mathcal{G}$ over U such that $\eta(1) = \sum f_i g_i$. Denote

$$\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$$

the map sending the ith basis vector to f_i. Then we can factor the map $\eta|_U$ over a map $\tilde{\eta} : \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \otimes_{\mathcal{O}_U} \mathcal{G}|_U$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F}|_U \arrow[r, "\eta \otimes 1"'] \arrow[dr, "\tilde{\eta} \otimes 1"'] & \mathcal{F}|_U \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U \arrow[r, "1 \otimes \epsilon"'] & \mathcal{F}|_U \\
& \mathcal{O}_U^{\oplus n} \otimes \mathcal{G}|_U \otimes \mathcal{F}|_U \arrow[r, "1 \otimes \epsilon"] \arrow[u] & \mathcal{O}_U^{\oplus n} \arrow[u]
\end{tikzcd}$$

This shows that the identity on $\mathcal{F}|_U$ factors through a finite free $\mathcal{O}_U$-module. This proves (1). Part (2) follows from Categories, Lemma 4.43.6 and its proof. Part (3) follows from the first equality of the proof. You can also deduce (2) and (3) from the uniqueness of left duals (Categories, Remark 4.43.7) and the construction of the left dual in Example 18.29.1. □

08FD Lemma 18.29.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be locally of finite presentation and flat. Then given an object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}_{U_i}$-module.

Proof. Choose an object U of $\mathcal{C}$. After replacing U by the members of a covering, we may assume there exists a presentation

$$\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

By Lemma 18.28.14 we may, after replacing U by the members of a covering, assume there exists a factorization

$$\mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_1} \to \mathcal{F}|_U$$

such that the composition $\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_U^{\oplus n_r} \to \mathcal{F}|_U$ has a section and we win. □

18.30. Towards constructible modules

0933 Recall that a quasi-compact object of a site is roughly an object such that every covering of it can be refined by a finite covering (the actual definition is slightly more involved, see Sites, Section 7.17). It turns out that if every object of a site has a covering by quasi-compact objects, then the modules $j_!\mathcal{O}_U$ with U quasi-compact form a particularly nice set of generators for the category of all modules.

0934 **Lemma 18.30.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then the sequence

$$\bigoplus j_{U_i \times_U U_j!}\mathcal{O}_{U_i \times_U U_j} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i} \to j_!\mathcal{O}_U \to 0$$

is exact.

Proof. For any $\mathcal{O}$-module $\mathcal{F}$ the functor $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ turns our sequence into the exact sequence $0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \times_U U_j)$, see (18.19.2.1). The lemma follows from this and Homology, Lemma 12.5.8. $\square$

0G1W **Lemma 18.30.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be covering of $\mathcal{C}$. If U is quasi-compact, then there exist a finite subset $I' \subset I$ such that the sequence

$$\bigoplus\nolimits_{i,i' \in I'} j_{U_i \times_U U_{i'}!}\mathcal{O}_{U_i \times_U U_{i'}} \to \bigoplus\nolimits_{i \in I'} j_{U_i!}\mathcal{O}_{U_i} \to j_!\mathcal{O}_U \to 0$$

is exact.

Proof. This lemma is immediate from Lemma 18.30.1 if U satisfies condition (3) of Sites, Lemma 7.17.2. We urge the reader to skip the proof in the general case. By definition there exists a covering $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and a morphism $\mathcal{V} \to \mathcal{U}$ of families of maps with fixed target given by $\mathrm{id} : U \to U$, $\alpha : J \to I$, and $f_j : V_j \to U_{\alpha(j)}$ (see Sites, Definition 7.8.1) such that the image $I' \subset I$ of α is finite. By Homology, Lemma 12.5.8 it suffices to show that for any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor $\mathrm{Hom}_\mathcal{O}(-, \mathcal{F})$ turns the sequence of the lemma into an exact sequence. By (18.19.2.1) we obtain the usual sequence

$$0 \to \mathcal{F}(U) \to \prod\nolimits_{i \in I'} \mathcal{F}(U_i) \to \prod\nolimits_{i,i' \in I'} \mathcal{F}(U_i \times_U U_{i'})$$

This is an exact sequence by Sites, Lemma 7.8.6 applied to the family of maps $\{U_i \to U\}_{i \in I'}$ which is refined by the covering $\mathcal{V}$. $\square$

0935 **Lemma 18.30.3.** Let $\mathcal{C}$ be a site. Let W be a quasi-compact object of $\mathcal{C}$.

(1) The functor $Sh(\mathcal{C}) \to \mathrm{Sets}$, $\mathcal{F} \mapsto \mathcal{F}(W)$ commutes with coproducts.
(2) Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The functor $\mathrm{Mod}(\mathcal{O}) \to \mathrm{Ab}$, $\mathcal{F} \mapsto \mathcal{F}(W)$ commutes with direct sums.

Proof. Proof of (1). Taking sections over W commutes with filtered colimits with injective transition maps by Sites, Lemma 7.17.7. If $\mathcal{F}_i$ is a family of sheaves of sets indexed by a set I. Then $\coprod \mathcal{F}_i$ is the filtered colimit over the partially ordered set of finite subsets $E \subset I$ of the coproducts $\mathcal{F}_E = \coprod_{i \in E} \mathcal{F}_i$. Since the transition maps are injective we conclude.

Proof of (2). Let $\mathcal{F}_i$ be a family of sheaves of $\mathcal{O}$-modules indexed by a set I. Then $\bigoplus \mathcal{F}_i$ is the filtered colimit over the partially ordered set of finite subsets $E \subset I$ of the direct sums $\mathcal{F}_E = \bigoplus_{i \in E} \mathcal{F}_i$. A filtered colimit of abelian sheaves can be computed in the category of sheaves of sets. Moreover, for $E \subset E'$ the transition map $\mathcal{F}_E \to \mathcal{F}_{E'}$ is injective (as sheafification is exact and the injectivity is clear on underlying presheaves). Hence it suffices to show the result for a finite index set by Sites, Lemma 7.17.7. The finite case is dealt with in Lemma 18.3.2 (it holds over any object of $\mathcal{C}$). □

0936 Lemma 18.30.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be a quasi-compact object of $\mathcal{C}$. Then the functor $\mathrm{Hom}_\mathcal{O}(j_!\mathcal{O}_U, -)$ commutes with direct sums.

Proof. This is true because $\mathrm{Hom}_\mathcal{O}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$ by (18.19.2.1) and because the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with direct sums by Lemma 18.30.3. □

In order to state the sharpest possible results in the following we introduce some notation.

0937 Situation 18.30.5. Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a set of objects. We consider the following conditions

0938 (1) Every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$.
0939 (2) Every $U \in \mathcal{B}$ is quasi-compact (Sites, Section 7.17).
093A (3) For a covering $\{U_i \to U\}$ with $U_i, U \in \mathcal{B}$ the fibre products $U_i \times_U U_j$ are quasi-compact.

093B Lemma 18.30.6. In Situation 18.30.5 assume (1) holds.

(1) Every sheaf of sets is the target of a surjective map whose source is a coproduct $\coprod h_{U_i}^\#$ with U_i in $\mathcal{B}$.
(2) If $\mathcal{O}$ is a sheaf of rings, then every $\mathcal{O}$-module is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$ with U_i in $\mathcal{B}$.

Proof. Part (1) follows from Sites, Lemmas 7.12.5 and 7.12.4. Part (2) follows from Lemmas 18.28.8 and 18.30.1. □

093C Lemma 18.30.7. In Situation 18.30.5 assume (1) and (2) hold.

(1) Every sheaf of sets is a filtered colimit of sheaves of the form

09Y7 (18.30.7.1)
$$\text{Coequalizer}\left(\coprod\nolimits_{j=1,\ldots,m} h_{V_j}^\# \rightrightarrows \coprod\nolimits_{i=1,\ldots,n} h_{U_i}^\#\right)$$

with U_i and V_j in $\mathcal{B}$.

(2) If $\mathcal{O}$ is a sheaf of rings, then every $\mathcal{O}$-module is a filtered colimit of sheaves of the form

093D (18.30.7.2)
$$\mathrm{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}\right)$$

with U_i and V_j in $\mathcal{B}$.

Proof. Proof of (1). By Lemma 18.30.6 every sheaf of sets $\mathcal{F}$ is the target of a surjection whose source is a coprod $\mathcal{F}_0$ of sheaves the form $h_U^\#$ with $U \in \mathcal{B}$. Applying this to $\mathcal{F}_0 \times_\mathcal{F} \mathcal{F}_0$ we find that $\mathcal{F}$ is a coequalizer of a pair of maps

$$\coprod\nolimits_{j \in J} h_{V_j}^\# \rightrightarrows \coprod\nolimits_{i \in I} h_{U_i}^\#$$

for some index sets I, J and V_j and U_i in $\mathcal{B}$. For every finite subset $J' \subset J$ there is a finite subset $I' \subset I$ such that the coproduct over $j \in J'$ maps into the coprod over $i \in I'$ via both maps, see Sites, Lemma 7.17.7. (Details omitted; hint: an infinite coproduct is the filtered colimit of the finite sub-coproducts.) Thus our sheaf is the colimit of the cokernels of these maps between finite coproducts.

Proof of (2). By Lemma 18.30.6 every module is a quotient of a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ with $U \in \mathcal{B}$. Thus every module is a cokernel

$$\mathrm{Coker}\left(\bigoplus\nolimits_{j \in J} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i \in I} j_{U_i!}\mathcal{O}_{U_i}\right)$$

for some index sets I, J and V_j and U_i in $\mathcal{B}$. For every finite subset $J' \subset J$ there is a finite subset $I' \subset I$ such that the direct sum over $j \in J'$ maps into the direct sum over $i \in I'$, see Lemma 18.30.4. Thus our module is the colimit of the cokernels of these maps between finite direct sums. □

093E Lemma 18.30.8. In Situation 18.30.5 assume (1) and (2) hold. Let $\mathcal{O}$ be a sheaf of rings. Then a cokernel of a map between modules as in (18.30.7.2) is another module as in (18.30.7.2).

Proof. Let $\mathcal{F} = \mathrm{Coker}(\bigoplus j_{V_j!}\mathcal{O}_{V_j} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i})$ as in (18.30.7.2). It suffices to show that the cokernel of a map $\varphi : j_{W!}\mathcal{O}_W \to \mathcal{F}$ with $W \in \mathcal{B}$ is another module of the same type. The map φ corresponds to $s \in \mathcal{F}(W)$. Since $\bigoplus j_{U_i!}\mathcal{O}_{U_i} \to \mathcal{F}$ is surjective, by (1) we may choose a covering $\{W_k \to W\}_{k \in K}$ with $W_k \in \mathcal{B}$ such that $s|_{W_k}$ is the image of some section s_k of $\bigoplus j_{U_i!}\mathcal{O}_{U_i})$. By (2) the object W is quasi-compact. By Lemma 18.30.2 there is a finite subset $K' \subset K$ such that $\bigoplus_{k \in K'} j_{W_k!}\mathcal{O}_{W_k} \to j_{W!}\mathcal{O}_W$ is surjective. We conclude that $\mathrm{Coker}(\varphi)$ is equal to

$$\mathrm{Coker}\left(\bigoplus\nolimits_{k \in K'} j_{W_k!}\mathcal{O}_{W_k} \oplus \bigoplus j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus j_{U_i!}\mathcal{O}_{U_i}\right)$$

where the map $\bigoplus_{k \in K'} j_{W_k!}\mathcal{O}_{W_k} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i}$ corresponds to $\sum_{k \in K'} s_k$. This finishes the proof. □

093F Lemma 18.30.9. In Situation 18.30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Assume given a map

$$\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$$

with U_i and V_j in $\mathcal{B}$, and coverings $\{U_{ik} \to U_i\}_{k \in K_i}$ with $U_{ik} \in \mathcal{B}$. Then there exist finite subsets $K'_i \subset K_i$ and a finite set L of $W_l \in \mathcal{B}$ and a commutative diagram

$$\begin{tikzcd}
\bigoplus_{l \in L} j_{W_l!}\mathcal{O}_{W_l} \arrow[r] \arrow[d] & \bigoplus_{i=1,\ldots,n} \bigoplus_{k \in K'_i} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \arrow[d] \\
\bigoplus_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \arrow[r] & \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}
\end{tikzcd}$$

inducing an isomorphism on cokernels of the horizontal maps.

Proof. Since U_i is quasi-compact, we may choose finite subsets $K'_i \subset K_i$ as in Lemma 18.30.2. Then since $\bigoplus_{i=1,\ldots,n} \bigoplus_{k \in K'_i} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \to \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$ is surjective, we can find coverings $\{V_{jm} \to V_j\}_{m \in M_j}$ with $V_{jm} \in \mathcal{B}$ such that we can find a commutative diagram

$$\begin{tikzcd}
\bigoplus_{j=1,\ldots,m} \bigoplus_{m \in M_j} j_{V_{jm}!}\mathcal{O}_{V_{jm}} \arrow[r] \arrow[d] & \bigoplus_{i=1,\ldots n} \bigoplus_{k \in K'_i} j_{U_{ik}!}\mathcal{O}_{U_{ik}} \arrow[d] \\
\bigoplus_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \arrow[r] & \bigoplus_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}
\end{tikzcd}$$

Since V_j is quasi-compact, we can choose finite subsets $M'_j \subset M_j$ as in Lemma 18.30.2. Set

$$L = \left(\coprod\nolimits_{i=1,\ldots,n} K'_i \times K'_i\right) \coprod \left(\coprod\nolimits_{j=1,\ldots,m} M'_j\right)$$

and for $l = (k, k') \in K'_i \times K'_i \subset L$ set $W_l = U_{ik} \times_{U_i} U_{ik'}$ and for $l = m \in M'_j \subset L$ set $W_l = V_{jm}$. Since we have the exact sequences of Lemma 18.30.2 for the families $\{U_{ik} \to U_i\}_{k \in K'_i}$ we conclude that we get a diagram as in the statement of the lemma (details omitted), except that it is not yet clear that $W_l \in \mathcal{B}$. However, since W_l is quasi-compact for all $l \in L$ we do another application of Lemma 18.30.2 and find finite families of maps $\{W_{lt} \to W_l\}_{t \in T_l}$ with $W_{lt} \in \mathcal{B}$ such that $\bigoplus j_{W_{lt}!}\mathcal{O}_{W_{lt}} \to j_{W_l!}\mathcal{O}_{W_l}$ is surjective. Then we replace L by $\coprod_{l \in L} T_l$ and everything is clear. $\square$

093G Lemma 18.30.10. In Situation 18.30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Then an extension of modules as in (18.30.7.2) is another module as in (18.30.7.2).

Proof. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_3$ as in (18.30.7.2). Choose presentations

$$\bigoplus A_{V_j} \to \bigoplus A_{U_i} \to \mathcal{F}_1 \to 0 \quad \text{and} \quad \bigoplus A_{T_j} \to \bigoplus A_{W_i} \to \mathcal{F}_3 \to 0$$

In this proof the direct sums are always finite, and we write $A_U = j_{U!}\mathcal{O}_U$ for $U \in \mathcal{B}$. Since $\mathcal{F}_2 \to \mathcal{F}_3$ is surjective, we can choose coverings $\{W_{ik} \to W_i\}$ with $W_{ik} \in \mathcal{B}$ such that $A_{W_{ik}} \to \mathcal{F}_3$ lifts to a map $A_{W_{ik}} \to \mathcal{F}_2$. By Lemma 18.30.9 we may replace our collection $\{W_i\}$ by a finite subcollection of the collection $\{W_{ik}\}$ and assume the map $\bigoplus A_{W_i} \to \mathcal{F}_3$ lifts to a map into $\mathcal{F}_2$. Consider the kernel

$$\mathcal{K}_2 = \mathrm{Ker}(\bigoplus A_{U_i} \oplus \bigoplus A_{W_i} \longrightarrow \mathcal{F}_2)$$

By the snake lemma this kernel surjects onto $\mathcal{K}_3 = \mathrm{Ker}(\bigoplus A_{W_i} \to \mathcal{F}_3)$. Thus, arguing as above, after replacing each T_j by a finite family of elements of $\mathcal{B}$ (permissible by Lemma 18.30.2) we may assume there is a map $\bigoplus A_{T_j} \to \mathcal{K}_2$ lifting the given map $\bigoplus A_{T_j} \to \mathcal{K}_3$. Then $\bigoplus A_{V_j} \oplus \bigoplus A_{T_j} \to \mathcal{K}_2$ is surjective which finishes the proof. $\square$

093H Lemma 18.30.11. In Situation 18.30.5 assume (1), (2), and (3) hold. Let $\mathcal{O}$ be a sheaf of rings. Let $\mathcal{A} \subset \mathrm{Mod}(\mathcal{O})$ be the full subcategory of modules isomorphic to a cokernel as in (18.30.7.2). If the kernel of every map of $\mathcal{O}$-modules of the form

$$\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$$

with U_i and V_j in $\mathcal{B}$, is in $\mathcal{A}$, then $\mathcal{A}$ is weak Serre subcategory of $\mathrm{Mod}(\mathcal{O})$.

Proof. We will use the criterion of Homology, Lemma 12.10.3. By the results of Lemmas 18.30.8 and 18.30.10 it suffices to see that the kernel of a map $\mathcal{F} \to \mathcal{G}$ between objects of $\mathcal{A}$ is in $\mathcal{A}$. To prove this choose presentations

$$\bigoplus A_{V_j} \to \bigoplus A_{U_i} \to \mathcal{F} \to 0 \quad \text{and} \quad \bigoplus A_{T_j} \to \bigoplus A_{W_i} \to \mathcal{G} \to 0$$

In this proof the direct sums are always finite, and we write $A_U = j_{U!}\mathcal{O}_U$ for $U \in \mathcal{B}$. Using Lemmas 18.30.1 and 18.30.9 and arguing as in the proof of Lemma 18.30.10

we may assume that the map $\mathcal{F} \to \mathcal{G}$ lifts to a map of presentations

$$\begin{tikzcd} \bigoplus A_{V_j} \arrow[r] \arrow[d] & \bigoplus A_{U_i} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ \bigoplus A_{T_j} \arrow[r] & \bigoplus A_{W_i} \arrow[r] & \mathcal{G} \arrow[r] & 0 \end{tikzcd}$$

Then we see that

$$\mathrm{Ker}(\mathcal{F} \to \mathcal{G}) = \mathrm{Coker}\left(\bigoplus A_{V_j} \to \mathrm{Ker}\left(\bigoplus A_{T_j} \oplus \bigoplus A_{U_i} \to \bigoplus A_{W_i}\right)\right)$$

and the lemma follows from the assumption and Lemma 18.30.8. □

18.31. Flat morphisms

04JA

04JB Definition 18.31.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We say $(f, f^\sharp)$ is flat if the ring map $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ is flat. We say a morphism of ringed sites is flat if the associated morphism of ringed topoi is flat.

04JC Lemma 18.31.2. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be a morphism of ringed topoi. Then

$$f^{-1} : \mathrm{Ab}(\mathcal{C}') \longrightarrow \mathrm{Ab}(\mathcal{C}), \quad \mathcal{F} \longmapsto f^{-1}\mathcal{F}$$

is exact. If $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a flat morphism of ringed topoi then

$$f^* : \mathrm{Mod}(\mathcal{O}') \longrightarrow \mathrm{Mod}(\mathcal{O}), \quad \mathcal{F} \longmapsto f^*\mathcal{F}$$

is exact.

Proof. Given an abelian sheaf $\mathcal{G}$ on $\mathcal{C}'$ the underlying sheaf of sets of $f^{-1}\mathcal{G}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{G}$, see Sites, Section 7.44. Hence the exactness of f^{-1} for sheaves of sets (required in the definition of a morphism of topoi, see Sites, Definition 7.15.1) implies the exactness of f^{-1} as a functor on abelian sheaves.

To see the statement on modules recall that $f^*\mathcal{F}$ is defined as the tensor product $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}', f^\sharp} \mathcal{O}$. Hence f^* is a composition of functors both of which are exact. □

08M5 Definition 18.31.3. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We say that $\mathcal{F}$ is flat over $(Sh(\mathcal{D}), \mathcal{O}')$ if $\mathcal{F}$ is flat as an $f^{-1}\mathcal{O}'$-module.

This is compatible with the notion as defined for morphisms of ringed spaces, see Modules, Definition 17.20.3 and the discussion following.

0GN2 Lemma 18.31.4. Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_\mathcal{D}$-modules. If $\mathcal{F}$ is finitely presented and f is flat, then the canonical map

$$f^* \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G})$$

of Remark 18.27.3 is an isomorphism.

Proof. Say f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. We have to show that the restriction of the map to $\mathcal{C}/U$ for any $U \in \mathrm{Ob}(\mathcal{C})$ is an isomorphism. We may replace U by the members of a covering of U. Hence by Sites, Lemma 7.14.10 we may assume there exists a morphism $U \to u(V)$ for some $V \in \mathrm{Ob}(\mathcal{C})$. Of course, then we may replace U by $u(V)$. Then since u is continuous, we may replace V by a covering and assume there is a presentation $\mathcal{O}_V^{\oplus m} \to \mathcal{O}_V^{\oplus n} \to \mathcal{F}|_V \to 0$ over $\mathcal{D}/V$. Since formation of $\mathcal{H}om$ commutes with localization (Lemma 18.27.2) we may replace f by the morphism $(\mathcal{C}/u(V), \mathcal{O}_{u(V)}) \to (\mathcal{D}/V, \mathcal{O}_V)$ induced by f. Hence we reduce to the case where $\mathcal{F}$ has a global presentation $\mathcal{O}_\mathcal{D}^{\oplus m} \to \mathcal{O}_\mathcal{D}^{\oplus n} \to \mathcal{F} \to 0$. Since f is flat and $f^*\mathcal{O}_\mathcal{D} = \mathcal{O}_\mathcal{C}$ we obtain a corresponding presentation $\mathcal{O}_\mathcal{C}^{\oplus m} \to \mathcal{O}_\mathcal{C}^{\oplus n} \to f^*\mathcal{F} \to 0$, see Lemma 18.31.2. Using that $\mathcal{H}om$ commutes with finite direct sums in the first variable, using that both $\mathcal{H}om_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_\mathcal{C}, -)$ and $\mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{O}_\mathcal{D}, -)$ are the identity functor, and using the functoriality of the construction of Remark 18.27.3 we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f^* \mathcal{H}om_{\mathcal{O}_\mathcal{D}}(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] & f^*\mathcal{G}^{\oplus n} \arrow[r] \arrow[d] & f^*\mathcal{G}^{\oplus n} \arrow[d] \\
0 \arrow[r] & \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{F}, f^*\mathcal{G}) \arrow[r] & f^*\mathcal{G}^{\oplus n} \arrow[r] & f^*\mathcal{G}^{\oplus n}
\end{tikzcd}$$

where the right two vertical arrows are isomorphisms. By Lemma 18.27.5 the rows are exact. We conclude by the 5 lemma. $\square$

18.32. Invertible modules

0408 Here is the definition.

0409 Definition 18.32.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site.

1. A finite locally free $\mathcal{O}$-module $\mathcal{F}$ is said to have rank r if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ of U such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}^{\oplus r}$ as an $\mathcal{O}_{U_i}$-module.
2. An $\mathcal{O}$-module $\mathcal{L}$ is invertible if the functor
$$\mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\mathcal{O}), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_\mathcal{O} \mathcal{L}$$
is an equivalence.
3. The sheaf $\mathcal{O}^*$ is the subsheaf of $\mathcal{O}$ defined by the rule
$$U \longmapsto \mathcal{O}^*(U) = \{f \in \mathcal{O}(U) \mid \exists g \in \mathcal{O}(U) \text{ such that } fg = 1\}$$
It is a sheaf of abelian groups with multiplication as the group law.

Lemma 18.40.7 below explains the relationship with locally free modules of rank 1.

0B8N Lemma 18.32.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{L}$ be an $\mathcal{O}$-module. The following are equivalent:

1. $\mathcal{L}$ is invertible, and
2. there exists an $\mathcal{O}$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_\mathcal{O} \mathcal{N} \cong \mathcal{O}$.

In this case we have

(a) $\mathcal{L}$ is a flat $\mathcal{O}$-module of finite presentation,
(b) for every object U of $\mathcal{C}$ there exists a covering $U\{U_i \to U\}$ such that $\mathcal{L}|_{U_i}$ is a direct summand of a finite free module, and
(c) the module $\mathcal{N}$ in (2) is isomorphic to $\mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O})$.

Proof. Assume (1). Then the functor $- \otimes_{\mathcal{O}} \mathcal{L}$ is essentially surjective, hence there exists an $\mathcal{O}$-module $\mathcal{N}$ as in (2). If (2) holds, then the functor $- \otimes_{\mathcal{O}} \mathcal{N}$ is a quasi-inverse to the functor $- \otimes_{\mathcal{O}} \mathcal{L}$ and we see that (1) holds.

Assume (1) and (2) hold. Since $- \otimes_{\mathcal{O}} \mathcal{L}$ is an equivalence, it is exact, and hence $\mathcal{L}$ is flat. Denote $\psi : \mathcal{L} \otimes_{\mathcal{O}} \mathcal{N} \to \mathcal{O}$ the given isomorphism. Let U be an object of $\mathcal{C}$. We will show that the restriction $\mathcal{L}$ to the members of a covering of U is a direct summand of a free module, which will certainly imply that $\mathcal{L}$ is of finite presentation. By construction of $\otimes$ we may assume (after replacing U by the members of a covering) that there exists an integer $n \geq 1$ and sections $x_i \in \mathcal{L}(U)$, $y_i \in \mathcal{N}(U)$ such that $\psi(\sum x_i \otimes y_i) = 1$. Consider the isomorphisms

$$\mathcal{L}|_U \to \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{N}|_U \to \mathcal{L}|_U$$

where the first arrow sends x to $\sum x_i \otimes x \otimes y_i$ and the second arrow sends $x \otimes x' \otimes y$ to $\psi(x' \otimes y)x$. We conclude that $x \mapsto \sum \psi(x \otimes y_i)x_i$ is an automorphism of $\mathcal{L}|_U$. This automorphism factors as

$$\mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{L}|_U$$

where the first arrow is given by $x \mapsto (\psi(x \otimes y_1), \ldots, \psi(x \otimes y_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. In this way we conclude that $\mathcal{L}|_U$ is a direct summand of a finite free $\mathcal{O}_U$-module.

Assume (1) and (2) hold. Consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

To finish the proof of the lemma we will show this is an isomorphism. By Lemma 18.27.6 we have

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{N} \otimes_{\mathcal{O}} \mathcal{L}, \mathcal{O}) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{N}, \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}))$$

The image of 1 gives a morphism $\mathcal{N} \to \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$. Tensoring with $\mathcal{L}$ we obtain

$$\mathcal{O} = \mathcal{L} \otimes_{\mathcal{O}} \mathcal{N} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$$

This map is the inverse to the evaluation map; computation omitted. $\square$

0B8P **Lemma 18.32.3.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. The pullback $f^*\mathcal{L}$ of an invertible $\mathcal{O}_{\mathcal{D}}$-module is invertible.

Proof. By Lemma 18.32.2 there exists an $\mathcal{O}_{\mathcal{D}}$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{D}}} \mathcal{N} \cong \mathcal{O}_{\mathcal{D}}$. Pulling back we get $f^*\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{C}}} f^*\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$ by Lemma 18.26.2. Thus $f^*\mathcal{L}$ is invertible by Lemma 18.32.2. $\square$

040A **Lemma 18.32.4.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed space.

(1) If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{N}$.
(2) If $\mathcal{L}$ is an invertible $\mathcal{O}$-module, then so is $\mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$ and the evaluation map $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}) \to \mathcal{O}$ is an isomorphism.

Proof. Part (1) is clear from the definition and part (2) follows from Lemma 18.32.2 and its proof. $\square$

040B **Lemma 18.32.5.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on $(\mathcal{C}, \mathcal{O})$ is isomorphic to exactly one of the $\mathcal{L}_i$.

Proof. Omitted, but see Sheaves of Modules, Lemma 17.25.8. $\square$

Lemma 18.32.5 says that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 18.32.4 says that tensor product defines the structure of an abelian group on this set with inverse of $\mathcal{L}$ given by $\mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O})$.

In fact, given an invertible $\mathcal{O}$-module $\mathcal{L}$ and $n \in \mathbf{Z}$ we define the nth tensor power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ as the image of $\mathcal{O}$ under applying the equivalence $\mathcal{F} \mapsto \mathcal{F} \otimes_\mathcal{O} \mathcal{L}$ exactly n times. This makes sense also for negative n as we've defined an invertible $\mathcal{O}$-module as one for which tensoring is an equivalence. More explicitly, we have

$$\mathcal{L}^{\otimes n} = \left\{ \begin{matrix} \mathcal{O} & \text{if} & n = 0 \\ \mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O}) & \text{if} & n = -1 \\ \mathcal{L} \otimes_\mathcal{O} \ldots \otimes_\mathcal{O} \mathcal{L} & \text{if} & n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_\mathcal{O} \ldots \otimes_\mathcal{O} \mathcal{L}^{\otimes -1} & \text{if} & n < -1 \end{matrix} \right.$$

see Lemma 18.32.4. With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_\mathcal{O} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms satisfy a commutativity and an associativity constraint (formulation omitted).

040C Definition 18.32.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The Picard group $\text{Pic}(\mathcal{O})$ of the ringed site is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}$-modules, with addition corresponding to tensor product.

18.33. Modules of differentials

04BJ In this section we briefly explain how to define the module of relative differentials for a morphism of ringed topoi. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.131).

04BK Definition 18.33.1. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. A $\mathcal{O}_1$-derivation or more precisely a φ-derivation into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the Leibniz rule

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 18.33.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(\mathcal{O}_2) \longrightarrow \Gamma(\mathcal{F})$$

clearly is a $\Gamma(\mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

04BL Lemma 18.33.2. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor

$$\text{Mod}(\mathcal{O}_2) \longrightarrow \text{Ab}, \quad \mathcal{F} \longmapsto \text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on $\mathcal{C}$, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_2(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over U. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

04BM (18.33.2.1)
$$\begin{matrix} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a, b)] \oplus [(f, g)] \oplus [h] & \longmapsto & [a + b] - [a] - [b]+ \\ & & [fg] - g[f] - f[g]+ \\ & & [\varphi(h)] \end{matrix}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends $[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (18.33.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. □

04BN Definition 18.33.3. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The module of differentials of the ring map φ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 18.33.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the universal φ-derivation is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Since this module and the derivation form the universal object representing a functor, this notion is clearly intrinsic (i.e., does not depend on the choice of the site underlying the ringed topos, see Section 18.18). Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (18.33.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

08TP Lemma 18.33.4. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Then $\Omega_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.

Proof. Consider the map (18.33.2.1). There is a similar map of presheaves whose value on $U \in \mathrm{Ob}(\mathcal{C})$ is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U) \times \mathcal{O}_2(U)] \oplus \mathcal{O}_2(U)[\mathcal{O}_2(U) \times \mathcal{O}_2(U)] \oplus \mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over U by the construction of the module of differentials in Algebra, Definition 10.131.2. On the other hand, the sheaves in (18.33.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. □

08TQ **Lemma 18.33.5.** Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.

Proof. This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (18.33.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor f^{-1} is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. $\square$

04BO **Lemma 18.33.6.** Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For any object U of $\mathcal{C}$ there is a canonical isomorphism

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. This is a special case of Lemma 18.33.5. $\square$

08TR **Lemma 18.33.7.** Let $\mathcal{C}$ be a site. Let

$$\begin{tikzcd} \mathcal{O}_2 \arrow[r, "\varphi"'] & \mathcal{O}_2' \\ \mathcal{O}_1 \arrow[u] \arrow[r] & \mathcal{O}_1' \arrow[u] \end{tikzcd}$$

be a commutative diagram of sheaves of rings on $\mathcal{C}$. The map $\mathcal{O}_2 \to \mathcal{O}_2'$ composed with the map $\mathrm{d} : \mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$. It is uniquely characterized by the property that $\mathrm{d}(f)$ mapsto $\mathrm{d}(\varphi(f))$ for any local section f of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.

Proof. This lemma proves itself. $\square$

08TS **Lemma 18.33.8.** In Lemma 18.33.7 suppose that $\mathcal{O}_2 \to \mathcal{O}_2'$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}_1'$. Then there is a canonical exact sequence of $\mathcal{O}_2'$-modules

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}_2' \longrightarrow \Omega_{\mathcal{O}_2'/\mathcal{O}_1} \longrightarrow 0$$

The leftmost map is characterized by the rule that a local section f of $\mathcal{I}$ maps to $\mathrm{d}f \otimes 1$.

Proof. For a local section f of $\mathcal{I}$ denote $\overline{f}$ the image of f in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1f_2 \otimes 1 = 0$ if f_1, f_2 are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1f_2 \otimes 1 = (f_1\mathrm{d}f_2 + f_2\mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_2 \otimes f_1 = 0$. A similar computation show this map is $\mathcal{O}_2' = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 18.33.7.

To see that the sequence is exact, we argue as follows. Let $\mathcal{O}_2'' \subset \mathcal{O}_2'$ be the presheaf of $\mathcal{O}_1$-algebras whose value on U is the image of $\mathcal{O}_2(U) \to \mathcal{O}_2'(U)$. By Algebra, Lemma 10.131.9 the sequences

$$\mathcal{I}(U)/\mathcal{I}(U)^2 \longrightarrow \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \otimes_{\mathcal{O}_2(U)} \mathcal{O}_2''(U) \longrightarrow \Omega_{\mathcal{O}_2''(U)/\mathcal{O}_1(U)} \longrightarrow 0$$

are exact for all objects U of $\mathcal{C}$. Since sheafification is exact this gives an exact sequence of sheaves of $(\mathcal{O}_2')^\#$-modules. By Lemma 18.33.4 and the fact that $(\mathcal{O}_2'')^\# = \mathcal{O}_2'$ we conclude. $\square$

Here is a particular situation where derivations come up naturally.

04BP Lemma 18.33.9. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Consider a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{O}_2 \to 0$$

Here $\mathcal{A}$ is a sheaf of $\mathcal{O}_1$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_2$ is a surjection of sheaves of $\mathcal{O}_1$-algebras, and $\mathcal{F} = \mathrm{Ker}(\pi)$ is its kernel. Assume $\mathcal{F}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{F}$ has a natural structure of an $\mathcal{O}_2$-module. A section $s : \mathcal{O}_2 \to \mathcal{A}$ of π is a $\mathcal{O}_1$-algebra map such that $\pi \circ s = \mathrm{id}$. Given any section $s : \mathcal{O}_2 \to \mathcal{F}$ of π and any φ-derivation $D : \mathcal{O}_1 \to \mathcal{F}$ the map

$$s + D : \mathcal{O}_1 \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique φ-derivation D.

Proof. Recall that the $\mathcal{O}_2$-module structure on $\mathcal{F}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_2$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{F}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned} (s + D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b)) \end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $\mathcal{O}_1$-algebra map because D is an $\mathcal{O}_1$-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. □

04BQ Definition 18.33.10. Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. In this situation

(1) for a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules a Y-derivation $D : \mathcal{O} \to \mathcal{F}$ is just a $f^\sharp$-derivation, and
(2) the sheaf of differentials $\Omega_{X/Y}$ of X over Y is the module of differentials of $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, see Definition 18.33.3.

Thus $\Omega_{X/Y}$ comes equipped with a universal Y-derivation $\mathrm{d}_{X/Y} : \mathcal{O} \longrightarrow \Omega_{X/Y}$. We sometimes write $\Omega_{X/Y} = \Omega_f$.

Recall that $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ so that this definition makes sense.

04BR Lemma 18.33.11. Let $X = (Sh(\mathcal{C}_X), \mathcal{O}_X)$, $Y = (Sh(\mathcal{C}_Y), \mathcal{O}_Y)$, $X' = (Sh(\mathcal{C}_{X'}), \mathcal{O}_{X'})$, and $Y' = (Sh(\mathcal{C}_{Y'}), \mathcal{O}_{Y'})$ be ringed topoi. Let

$$\begin{tikzcd} X' \arrow[r, "f"'] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram of morphisms of ringed topoi. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*\mathrm{d}_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation.

Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*\mathrm{d}_{X/Y}(t)$ mapsto $\mathrm{d}_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is clear except for the last assertion. Let us explain the meaning of this. Let $U \in \mathrm{Ob}(\mathcal{C}_X)$ and let $t \in \mathcal{O}_X(U)$. This is what it means for t to be a local section of $\mathcal{O}_X$. Now, we may think of t as a map of sheaves of sets $t : h_U^\# \to \mathcal{O}_X$. Then $f^{-1}t : f^{-1}h_U^\# \to f^{-1}\mathcal{O}_X$. By f^*t we mean the composition

$$\begin{tikzcd} f^{-1}h_U^\# \arrow[r, "f^{-1}t"] \arrow[rr, bend left, "f^*t"] & f^{-1}\mathcal{O}_X \arrow[r, "f^\sharp"] & \mathcal{O}_{X'} \end{tikzcd}$$

Note that $\mathrm{d}_{X/Y}(t) \in \Omega_{X/Y}(U)$. Hence we may think of $\mathrm{d}_{X/Y}(t)$ as a map $\mathrm{d}_{X/Y}(t) : h_U^\# \to \Omega_{X/Y}$. Then $f^{-1}\mathrm{d}_{X/Y}(t) : f^{-1}h_U^\# \to f^{-1}\Omega_{X/Y}$. By $f^*\mathrm{d}_{X/Y}(t)$ we mean the composition

$$\begin{tikzcd} f^{-1}h_U^\# \arrow[r, "f^{-1}\mathrm{d}_{X/Y}(t)"] \arrow[rr, bend left, "f^*\mathrm{d}_{X/Y}(t)"] & f^{-1}\Omega_{X/Y} \arrow[r, "1\otimes\mathrm{id}"] & f^*\Omega_{X/Y} \end{tikzcd}$$

OK, and now the statement of the lemma means that we have

$$c_f \circ f^*t = f^*\mathrm{d}_{X/Y}(t)$$

as maps from $f^{-1}h_U^\#$ to $\Omega_{X'/Y'}$. We omit the verification that this property holds for c_f as defined in the lemma. (Hint: The first map $c'_f : \Omega_{X/Y} \to f_*\Omega_{X'/Y'}$ satisfies $c'_f(\mathrm{d}_{X/Y}(t)) = f_*\mathrm{d}_{X'/Y'}(f^\sharp(t))$ as sections of $f_*\Omega_{X'/Y'}$ over U, and you have to turn this into the equality above by using adjunction.) The reason that this uniquely characterizes c_f is that the images of $f^*\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_{X'}$-module $f^*\Omega_{X/Y}$ simply because the local sections $\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_X$-module $\Omega_{X/Y}$. □

18.34. Finite order differential operators

09CQ In this section we introduce differential operators of finite order. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.133).

09CR Definition 18.34.1. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $k \geq 0$ be an integer. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. A differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k is an is an $\mathcal{O}_1$-linear map such that for all local sections g of $\mathcal{O}_2$ the map $s \mapsto D(gs) - gD(s)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an $\mathcal{O}_2$-linear map.

If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of order k, then for all local sections g of $\mathcal{O}_2$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\mathrm{Diff}^k(\mathcal{F}, \mathcal{G}) = \mathrm{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G})$$

is a $\Gamma(\mathcal{C}, \mathcal{O}_2)$-module. We have

$$\text{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \text{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

The rule which maps $U \in \text{Ob}(\mathcal{C})$ to the module of differential operators $D : \mathcal{F}|_U \to \mathcal{G}|_U$ of order k is a sheaf of $\mathcal{O}_2$-modules on the site $\mathcal{C}$. Thus we obtain a sheaf of differential operators (if we ever need this we will add a definition here).

09CS Lemma 18.34.2. Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. If $D : \mathcal{E} \to \mathcal{F}$ and $D' : \mathcal{F} \to \mathcal{G}$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.

Proof. Let g be a local section of $\mathcal{O}_2$. Then the map which sends a local section x of $\mathcal{E}$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. $\square$

09CT Lemma 18.34.3. Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. Let $k \geq 0$. There exists a sheaf of $\mathcal{O}_2$-modules $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ and a canonical isomorphism

$$\text{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}), \mathcal{G})$$

functorial in the $\mathcal{O}_2$-module $\mathcal{G}$.

Proof. The existence follows from general category theoretic arguments (insert future reference here), but we will also give a direct construction as this construction will be useful in the future proofs. We will freely use the notation introduced in the proof of Lemma 18.33.2. Given any differential operator $D : \mathcal{F} \to \mathcal{G}$ we obtain an $\mathcal{O}_2$-linear map $L_D : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ sending $[m]$ to $D(m)$. If D has order 0 then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where g_0 is a local section of $\mathcal{O}_2$ and m, m' are local sections of $\mathcal{F}$. If D has order 1, then L_D annihilates the local sections

$$[m + m' - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where f is a local section of $\mathcal{O}_1$, g_0, g_1 are local sections of $\mathcal{O}_2$, and m, m' are local sections of $\mathcal{F}$. If D has order k, then L_D annihilates the local sections $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the local sections

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ is an $\mathcal{O}_2$-linear map annihilating all the local sections listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ is the quotient of $\mathcal{O}_2[\mathcal{F}]$ by the $\mathcal{O}_2$-submodule generated by these local sections. $\square$

09CU Definition 18.34.4. Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. The module $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ constructed in Lemma 18.34.3 is called the module of principal parts of order k of $\mathcal{F}$.

Note that the inclusions

$$\mathrm{Diff}^0(\mathcal{F},\mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F},\mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F},\mathcal{G}) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 4.3.5) to surjections

$$\ldots \to \mathcal{P}^2_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^0_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) = \mathcal{F}$$

09CV Lemma 18.34.5. Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_2$-modules. Then $\mathcal{P}^k_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}(\mathcal{F}^\#)$ is the sheaf associated to the presheaf $U \mapsto P^k_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}(\mathcal{F}(U))$.

Proof. This can be proved in exactly the same way as is done for the sheaf of differentials in Lemma 18.33.4. Perhaps a more pleasing approach is to use the universal property of Lemma 18.34.3 directly to see the equality. We omit the details. $\square$

09CW Lemma 18.34.6. Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. There is a canonical short exact sequence

$$0 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{F} \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{F} \to 0$$

functorial in $\mathcal{F}$ called the sequence of principal parts.

Proof. Follows from the commutative algebra version (Algebra, Lemma 10.133.6) and Lemmas 18.33.4 and 18.34.5. $\square$

09CX Remark 18.34.7. Let $\mathcal{C}$ be a site. Suppose given a commutative diagram of sheaves of rings

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

a $\mathcal{B}$-module $\mathcal{F}$, a $\mathcal{B}'$-module $\mathcal{F}'$, and a $\mathcal{B}$-linear map $\mathcal{F} \to \mathcal{F}'$. Then we get a compatible system of module maps

$$\begin{tikzcd} \ldots \arrow[r] & \mathcal{P}^2_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \arrow[r] & \mathcal{P}^1_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \arrow[r] & \mathcal{P}^0_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \\ \ldots \arrow[r] & \mathcal{P}^2_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[u] \arrow[r] & \mathcal{P}^1_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[u] \arrow[r] & \mathcal{P}^0_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \arrow[u] \end{tikzcd}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $\mathcal{P}^k_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ in terms of (local) generators and relations in the proof of Lemma 18.34.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 18.34.6.

18.35. The naive cotangent complex

08TT This section is the analogue of Algebra, Section 10.134 and Modules, Section 17.31. We advise the reader to read those sections first.

Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. In this section, for any sheaf of sets $\mathcal{E}$ on $\mathcal{C}$ we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$

whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$ the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

08TU (18.35.0.1)
$$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 18.33.8 there is a canonical map

08TV (18.35.0.2)
$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

08TW Definition 18.35.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The naive cotangent complex $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (18.35.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in degree -1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 18.33.7 for modules of differentials. Namely, given a commutative diagram

08TX (18.35.1.1)
$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{B}' \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}' \arrow[u] \end{tikzcd}$$

of sheaves of rings on $\mathcal{C}$ there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$ and a map between modules of differentials, which together give the desired map between the naive cotangent complexes.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that $\mathcal{E}$ is a sheaves of sets on $\mathcal{C}$ and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Assume this map is surjective, and let $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ be the kernel. Set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

Here is the result.

08TY Lemma 18.35.2. In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.

Proof. Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\mathrm{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \mathrm{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \mathrm{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this we argue as follows. First, observe that $H^0(NL(\alpha_i)) = \Omega_{\mathcal{B}/\mathcal{A}}$ and $H^0(NL(\alpha)) = \Omega_{\mathcal{B}/\mathcal{A}}$ by Lemma 18.33.8 hence the map is an isomorphism on cohomology sheaves in degree 0. Similarly, we claim that $H^{-1}(NL(\alpha_i))$ and $H^{-1}(NL(\alpha))$ are the sheaves associated to the presheaf $U \mapsto H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ where $H_1(L_{-/-})$ is as in Algebra, Definition 10.134.1. If the claim holds, then the proof is finished.

Proof of the claim. Let $\alpha : \mathcal{E} \to \mathcal{B}$ be as above. Let $\mathcal{B}' \subset \mathcal{B}$ be the subpresheaf of $\mathcal{A}$-algebras whose value on U is the image of $\mathcal{A}(U)[\mathcal{E}(U)] \to \mathcal{B}(U)$. Let $\mathcal{I}'$ be the presheaf whose value on U is the kernel of $\mathcal{A}(U)[\mathcal{E}(U)] \to \mathcal{B}(U)$. Then $\mathcal{I}$ is the sheafification of $\mathcal{I}'$ and $\mathcal{B}$ is the sheafification of $\mathcal{B}'$. Similarly, $H^{-1}(NL(\alpha))$ is the sheafification of the presheaf

$$U \longmapsto \mathrm{Ker}(\mathcal{I}'(U)/\mathcal{I}'(U)^2 \to \Omega_{\mathcal{A}(U)[\mathcal{E}(U)]/\mathcal{A}(U)} \otimes_{\mathcal{A}(U)[\mathcal{E}(U)]} \mathcal{B}'(U))$$

by Lemma 18.33.4. By Algebra, Lemma 10.134.2 we conclude $H^{-1}(NL(\alpha))$ is the sheaf associated to the presheaf $U \mapsto H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$. Thus we have to show that the maps $H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)}) \to H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ induce an isomorphism $\mathcal{H}'_1 \to \mathcal{H}_1$ of sheafifications.

Injectivity of $\mathcal{H}'_1 \to \mathcal{H}_1$. Let $f \in H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$ map to zero in $\mathcal{H}_1(U)$. To show: f maps to zero in $\mathcal{H}'_1(U)$. The assumption means there is a covering $\{U_i \to U\}$ such that f maps to zero in $H_1(L_{\mathcal{B}(U_i)/\mathcal{A}(U_i)})$ for all i. Replace U by U_i to get to the point where f maps to zero in $H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$. By Algebra, Lemma 10.134.9 we can find a finitely generated subalgebra $\mathcal{B}'(U) \subset B \subset \mathcal{B}(U)$ such that f maps to zero in $H_1(L_{B/\mathcal{A}(U)})$. Since $\mathcal{B} = (\mathcal{B}')^\#$ we can find a covering $\{U_i \to U\}$ such that $B \to \mathcal{B}(U_i)$ factors through $\mathcal{B}'(U_i)$. Hence f maps to zero in $H_1(L_{\mathcal{B}'(U_i)/\mathcal{A}(U_i)})$ as desired.

The surjectivity of $\mathcal{H}'_1 \to \mathcal{H}_1$ is proved in exactly the same way. $\square$

08TZ Lemma 18.35.3. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be morphism of topoi. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{D}$. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.

Proof. Omitted. Hint: Use Lemma 18.33.5. $\square$

The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

08U0 Definition 18.35.4. Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. The naive cotangent complex $NL_f = NL_{X/Y}$ of the given morphism of ringed topoi is $NL_{\mathcal{O}/f^{-1}\mathcal{O}'}$. We sometimes write $NL_{X/Y} = NL_{\mathcal{O}/\mathcal{O}'}$.

18.36. Stalks of modules

04EM We have to be a bit careful when taking stalks at points, since the colimit defining a stalk (see Sites, Equation 7.32.1.1) may not be filtered[3]. On the other hand, by definition of a point of a site the stalk functor is exact and commutes with arbitrary colimits. In other words, it behaves exactly as if the colimit were filtered.

04EN Lemma 18.36.1. Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.

[3]Of course in almost any naturally occurring case the colimit is filtered and some of the discussion in this section may be simplified.

(1) We have $(\mathcal{F}^\#)_p = \mathcal{F}_p$ for any presheaf of sets on $\mathcal{C}$.
(2) The stalk functor $Sh(\mathcal{C}) \to \text{Sets}$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact (see Categories, Definition 4.23.1) and commutes with arbitrary colimits.
(3) The stalk functor $\text{PSh}(\mathcal{C}) \to \text{Sets}$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact (see Categories, Definition 4.23.1) and commutes with arbitrary colimits.

Proof. By Sites, Lemma 7.32.5 we have (1). By Sites, Lemmas 7.32.4 we see that $\text{PSh}(\mathcal{C}) \to \text{Sets}$, $\mathcal{F} \mapsto \mathcal{F}_p$ is a left adjoint, and by Sites, Lemma 7.32.5 we see the same thing for $Sh(\mathcal{C}) \to \text{Sets}$, $\mathcal{F} \mapsto \mathcal{F}_p$. Hence the stalk functor commutes with arbitrary colimits (see Categories, Lemma 4.24.5). It follows from the definition of a point of a site, see Sites, Definition 7.32.2 that $Sh(\mathcal{C}) \to \text{Sets}$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. Since sheafification is exact (Sites, Lemma 7.10.14) it follows that $\text{PSh}(\mathcal{C}) \to \text{Sets}$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. □

In particular, since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ on presheaves commutes with all finite limits and colimits we may apply the reasoning of the proof of Sites, Proposition 7.44.3. The result of such an argument is that if $\mathcal{F}$ is a (pre)sheaf of algebraic structures listed in Sites, Proposition 7.44.3 then the stalk $\mathcal{F}_p$ is naturally an algebraic structure of the same kind. Let us explain this in detail when $\mathcal{F}$ is an abelian presheaf. In this case the addition map $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ induces a map

$$+ : \mathcal{F}_p \times \mathcal{F}_p = (\mathcal{F} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

where the equal sign uses that stalk functor on presheaves of sets commutes with finite limits. This defines a group structure on the stalk $\mathcal{F}_p$. In this way we obtain our stalk functor

$$\text{PAb}(\mathcal{C}) \longrightarrow \text{Ab}, \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of abelian groups by precomposing with the inclusion $\text{Ab}(\mathcal{C}) \subset \text{PAb}(\mathcal{C})$.

04EP Lemma 18.36.2. Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.

(1) The functor $\text{Ab}(\mathcal{C}) \to \text{Ab}$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.
(2) The stalk functor $\text{PAb}(\mathcal{C}) \to \text{Ab}$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.
(3) For $\mathcal{F} \in \text{Ob}(\text{PAb}(\mathcal{C}))$ we have $\mathcal{F}_p = \mathcal{F}_p^\#$.

Proof. This is formal from the results of Lemma 18.36.1 and the construction of the stalk functor above. □

Next, we turn to the case of sheaves of modules. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. (It suffices for the discussion that $\mathcal{O}$ be a presheaf of rings.) Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let p be a point of $\mathcal{C}$. In this case we get a map

$$\cdot : \mathcal{O}_p \times \mathcal{O}_p = (\mathcal{O} \times \mathcal{O})_p \longrightarrow \mathcal{O}_p$$

which is the stalk of the multiplication map and

$$\cdot : \mathcal{O}_p \times \mathcal{F}_p = (\mathcal{O} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

which is the stalk of the multiplication map. We omit the verification that this defines a ring structure on $\mathcal{O}_p$ and an $\mathcal{O}_p$-module structure on $\mathcal{F}_p$. In this way we obtain a functor

$$\text{PMod}(\mathcal{O}) \longrightarrow \text{Mod}(\mathcal{O}_p), \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of $\mathcal{O}$-modules by precomposing with the inclusion $\text{Mod}(\mathcal{O}) \subset \text{PMod}(\mathcal{O})$.

04EQ Lemma 18.36.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$.

(1) The functor $\text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.
(2) The stalk functor $\text{PMod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.
(3) For $\mathcal{F} \in \text{Ob}(\text{PMod}(\mathcal{O}))$ we have $\mathcal{F}_p = \mathcal{F}^\#_p$.

Proof. This is formal from the results of Lemma 18.36.2, the construction of the stalk functor above, and Lemma 18.14.1. □

05V5 Lemma 18.36.4. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let p be a point of $\mathcal{C}$ or $Sh(\mathcal{C})$ and set $q = f \circ p$. Then

$$(f^*\mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D},q}} \mathcal{O}_{\mathcal{C},p}$$

for any $\mathcal{O}_\mathcal{D}$-module $\mathcal{F}$.

Proof. We have

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_\mathcal{C}$$

by definition. Since taking stalks at p (i.e., applying p^{-1}) commutes with $\otimes$ by Lemma 18.26.2 we win by the relation between the stalk of pullbacks at p and stalks at q explained in Sites, Lemma 7.34.2 or Sites, Lemma 7.34.3. □

18.37. Skyscraper sheaves

05V6 Let p be a point of a site $\mathcal{C}$ or a topos $Sh(\mathcal{C})$. In this section we study the exactness properties of the functor which associates to an abelian group A the skyscraper sheaf p_*A. First, recall that $p_* : \text{Sets} \to Sh(\mathcal{C})$ has a lot of exactness properties, see Sites, Lemmas 7.32.9 and 7.32.10.

05V7 Lemma 18.37.1. Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ or of its associated topos.

(1) The functor $p_* : \text{Ab} \to \text{Ab}(\mathcal{C})$, $A \mapsto p_*A$ is exact.
(2) There is a functorial direct sum decomposition
$$p^{-1}p_*A = A \oplus I(A)$$
for $A \in \text{Ob}(\text{Ab})$.

Proof. By Sites, Lemma 7.32.9 there are functorial maps $A \to p^{-1}p_*A \to A$ whose composition equals id_A. Hence a functorial direct sum decomposition as in (2) with $I(A)$ the kernel of the adjunction map $p^{-1}p_*A \to A$. The functor p_* is left exact by Lemma 18.14.3. The functor p_* transforms surjections into surjections by Sites, Lemma 7.32.10. Hence (1) holds. □

To do the same thing for sheaves of modules, suppose given a point p of a ringed topos $(Sh(\mathcal{C}), \mathcal{O})$. Recall that p^{-1} is just the stalk functor. Hence we can think of p as a morphism of ringed topoi

$$(p, \text{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

Thus we get a pullback functor $p^* : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_p)$ which equals the stalk functor, and which we discussed in Lemma 18.36.3. In this section we consider the functor $p_* : \text{Mod}(\mathcal{O}_p) \to \text{Mod}(\mathcal{O})$.

05V8 **Lemma 18.37.2.** Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let p be a point of the topos $Sh(\mathcal{C})$.

(1) The functor $p_* : \text{Mod}(\mathcal{O}_p) \to \text{Mod}(\mathcal{O})$, $M \mapsto p_*M$ is exact.
(2) The canonical surjection $p^{-1}p_*M \to M$ is $\mathcal{O}_p$-linear.
(3) The functorial direct sum decomposition $p^{-1}p_*M = M \oplus I(M)$ of Lemma 18.37.1 is not $\mathcal{O}_p$-linear in general.

Proof. Part (1) and surjectivity in (2) follow immediately from the corresponding result for abelian sheaves in Lemma 18.37.1. Since $p^{-1}\mathcal{O} = \mathcal{O}_p$ we have $p^{-1} = p^*$ and hence $p^{-1}p_*M \to M$ is the same as the counit $p^*p_*M \to M$ of the adjunction for modules, whence linear.

Proof of (3). Suppose that G is a group. Consider the topos $G\text{-Sets} = Sh(\mathcal{T}_G)$ and the point $p : \text{Sets} \to G\text{-Sets}$. See Sites, Section 7.9 and Example 7.33.7. Here p^{-1} is the functor forgetting about the G-action. And p_* is the right adjoint of the forgetful functor, sending M to $\text{Map}(G, M)$. The maps in the direct sum decomposition are the maps

$$M \to \text{Map}(G, M) \to M$$

where the first sends $m \in M$ to the constant map with value m and where the second map is evaluation at the identity element 1 of G. Next, suppose that R is a ring endowed with an action of G. This determines a sheaf of rings $\mathcal{O}$ on $\mathcal{T}_G$. The category of $\mathcal{O}$-modules is the category of R-modules M endowed with an action of G compatible with the action on R. The R-module structure on $\text{Map}(G, M)$ is given by

$$(rf)(\sigma) = \sigma(r)f(\sigma)$$

for $r \in R$ and $f \in \text{Map}(G, M)$. This is true because it is the unique G-invariant R-module strucure compatible with evaluation at 1. The reader observes that in general the image of $M \to \text{Map}(G, M)$ is not an R-submodule (for example take $M = R$ and assume the G-action is nontrivial), which concludes the proof. $\square$

05V9 **Example 18.37.3.** Let G be a group. Consider the site $\mathcal{T}_G$ and its point p, see Sites, Example 7.33.7. Let R be a ring with a G-action which corresponds to a sheaf of rings $\mathcal{O}$ on $\mathcal{T}_G$. Then $\mathcal{O}_p = R$ where we forget the G-action. In this case $p^{-1}p_*M = \text{Map}(G, M)$ and $I(M) = \{f : G \to M \mid f(1_G) = 0\}$ and $M \to \text{Map}(G, M)$ assigns to $m \in M$ the constant function with value m.

18.38. Localization and points

070Z

0710 **Lemma 18.38.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. Let U be an object of $\mathcal{C}$. For $\mathcal{G}$ in $\text{Mod}(\mathcal{O}_U)$ we have

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ lying over p, see Sites, Lemma 7.35.2.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \bigoplus_{\varphi \in \text{Mor}_\mathcal{C}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 18.19.2. The stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 18.36.3. Let $u : \mathcal{C} \to \textit{Sets}$ be the functor corresponding to p (see Sites, Section 7.32). Hence we see that

$$(j_{U!}\mathcal{G})_p = \mathrm{colim}_{(V,y)} \bigoplus\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

where the colimit is taken in the category of abelian groups. To a quadruple (V, y, φ, s) occurring in this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_{x \in u(U)} \mathrm{colim}_{(\varphi : V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by the description of the points q lying over x in Sites, Lemma 7.35.2. ☐

0711 Remark 18.38.2. Warning: The result of Lemma 18.38.1 has no analogue for $j_{U,*}$.

18.39. Pullbacks of flat modules

05VA The pullback of a flat module along a morphism of ringed topoi is flat. This is a bit tricky to prove.

05VD Lemma 18.39.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Then $f^*\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{C}$-module whenever $\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{D}$-module. [AGV71, Exposé V, Corollary 1.7.1]

Proof. Choose a diagram as in Lemma 18.7.2. Recall that being a flat module is intrinsic (see Section 18.18 and Definition 18.28.1). Hence it suffices to prove the lemma for the morphism $(h, h^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \to (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'})$. In other words, we may assume that our sites $\mathcal{C}$ and $\mathcal{D}$ have all finite limits and that f is a morphism of sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ which commutes with finite limits.

Recall that $f^*\mathcal{F} = \mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{F}$ (Definition 18.13.1). By Lemma 18.28.13 it suffices to prove that $f^{-1}\mathcal{F}$ is a flat $f^{-1}\mathcal{O}_\mathcal{D}$-module. Combined with the previous paragraph this reduces us to the situation of the next paragraph.

Assume $\mathcal{C}$ and $\mathcal{D}$ are sites which have all finite limits and that $u : \mathcal{D} \to \mathcal{C}$ is a continuous functor which commutes with finite limits. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$ and let $\mathcal{F}$ be a flat $\mathcal{O}$-module. Then u defines a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ (Sites, Proposition 7.14.7). To show: $f^{-1}\mathcal{F}$ is a flat $f^{-1}\mathcal{O}$-module. Let U be an object of $\mathcal{C}$ and let

$$f^{-1}\mathcal{O}|_U \xrightarrow{(f_1, \ldots, f_n)} f^{-1}\mathcal{O}|_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} f^{-1}\mathcal{F}|_U$$

be a complex of $f^{-1}\mathcal{O}|_U$-modules. Our goal is to construct a factorization of $(s_1, \ldots, s_n)$ on the members of a covering of U as in Lemma 18.28.14 part (2). Consider the elements $s_a \in f^{-1}\mathcal{F}(U)$ and $f_a \in f^{-1}\mathcal{O}(U)$. Since $f^{-1}\mathcal{F}$, resp. $f^{-1}\mathcal{O}$ is the sheafification of $u_p\mathcal{F}$ we may, after replacing U by the members of a covering, assume that s_a is the image of an element $s'_a \in u_p\mathcal{F}(U)$ and f_a is the image of an element $f'_a \in u_p\mathcal{O}(U)$. Then after another replacement of U by the members of a covering we may assume that $\sum f'_a s'_a$ is zero in $u_p\mathcal{F}(U)$. Recall that the category $(\mathcal{I}_U^u)^{opp}$ is directed (Sites, Lemma 7.5.2) and that $u_p\mathcal{F}(U) = \mathrm{colim}_{(\mathcal{I}_U^u)^{opp}} \mathcal{F}(V)$ and $u_p\mathcal{O}(U) = \mathrm{colim}_{(\mathcal{I}_U^u)^{opp}} \mathcal{O}(V)$. Hence we may assume there is a pair $(V, \phi) \in \mathrm{Ob}(\mathcal{I}_U^u)$ where V is an object of $\mathcal{D}$ and ϕ is a morphism $\phi : U \to u(V)$ of $\mathcal{D}$ and elements

$s''_a \in \mathcal{F}(V)$ and $f''_a \in \mathcal{O}(V)$ whose images in $u_p\mathcal{F}(U)$ and $u_p\mathcal{O}(U)$ are equal to s'_a and f'_a and such that $\sum f''_a s''_a = 0$ in $\mathcal{F}(V)$. Then we obtain a complex

$$\mathcal{O}|_V \xrightarrow{(f''_1, \ldots, f''_n)} \mathcal{O}|_V^{\oplus n} \xrightarrow{(s''_1, \ldots, s''_n)} \mathcal{F}|_V$$

and we can apply the other direction of Lemma 18.28.14 to see there exists a covering $\{V_i \to V\}$ of $\mathcal{D}$ and for each i a factorization

$$\mathcal{O}|_{V_i}^{\oplus n} \xrightarrow{B''_i} \mathcal{O}|_{V_i}^{\oplus l_i} \xrightarrow{(t''_{i1}, \ldots, t''_{il_i})} \mathcal{F}|_{V_i}$$

of $(s''_1, \ldots, s''_n)|_{V_i}$ such that $B_i \circ (f''_1, \ldots, f''_n)|_{V_i} = 0$. Set $U_i = U \times_{\phi, u(V)} u(V_i)$, denote $B_i \in \text{Mat}(l_i \times n, f^{-1}\mathcal{O}(U_i))$ the image of B''_i, and denote $t_{ij} \in f^{-1}\mathcal{F}(U_i)$ the image of t''_{ij}. Then we get a factorization

$$f^{-1}\mathcal{O}|_{U_i}^{\oplus n} \xrightarrow{B_i} f^{-1}\mathcal{O}|_{U_i}^{\oplus l_i} \xrightarrow{(t_{i1}, \ldots, t_{il_i})} \mathcal{F}|_{U_i}$$

of $(s_1, \ldots, s_n)|_{U_i}$ such that $B_i \circ (f_1, \ldots, f_n)|_{U_i} = 0$. This finishes the proof. □

05VB Lemma 18.39.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}_p$ is a flat $\mathcal{O}_p$-module.

Proof. In Section 18.37 we have seen that we can think of p as a morphism of ringed topoi

$$(p, \text{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

such that the pullback functor $p^* : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}_p)$ equals the stalk functor. Thus the lemma follows from Lemma 18.39.1. □

05VC Lemma 18.39.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\{p_i\}_{i \in I}$ be a conservative family of points of $\mathcal{C}$. Then $\mathcal{F}$ is flat if and only if $\mathcal{F}_{p_i}$ is a flat $\mathcal{O}_{p_i}$-module for all $i \in I$.

Proof. By Lemma 18.39.2 we see one of the implications. For the converse, use that $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p = \mathcal{F}_p \otimes_{\mathcal{O}_p} \mathcal{G}_p$ by Lemma 18.26.2 (as taking stalks at p is given by p^{-1}) and Lemma 18.14.4. □

0G6R Lemma 18.39.4. Let $f : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}'), \mathcal{O})$ be a morphism of ringed topoi. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{H}$ a flat $\mathcal{O}$-module. Then the sequence $0 \to f^*\mathcal{F} \to f^*\mathcal{G} \to f^*\mathcal{H} \to 0$ is exact as well.

Proof. Since f^{-1} is exact we have the short exact sequence $0 \to f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to 0$ of $f^{-1}\mathcal{O}$-modules. By Lemma 18.39.1 the $f^{-1}\mathcal{O}$-module $f^{-1}\mathcal{H}$ is flat. By Lemma 18.28.9 this implies that tensoring the sequence over $f^{-1}\mathcal{O}$ with $\mathcal{O}'$ the sequence remains exact. Since $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}} \mathcal{O}'$ and similarly for $\mathcal{G}$ and $\mathcal{H}$ we conclude. □

18.40. Locally ringed topoi

04ER A reference for this section is [AGV71, Exposé IV, Exercice 13.9].

04ES Lemma 18.40.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The following are equivalent

(1) For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1 - f)|_{U_j}$ is invertible.
(2) For $U \in \text{Ob}(\mathcal{C})$, $n \geq 1$, and $f_1, \ldots, f_n \in \mathcal{O}(U)$ which generate the unit ideal in $\mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j there exists an i such that $f_i|_{U_j}$ is invertible.

(3) The map of sheaves of sets
$$(\mathcal{O} \times \mathcal{O}) \amalg (\mathcal{O} \times \mathcal{O}) \longrightarrow \mathcal{O} \times \mathcal{O}$$
which maps (f, a) in the first component to (f, af) and (f, b) in the second component to $(f, b(1-f))$ is surjective.

Proof. It is clear that (2) implies (1). To show that (1) implies (2) we argue by induction on n. The first case is $n = 2$ (since $n = 1$ is trivial). In this case we have $a_1 f_1 + a_2 f_2 = 1$ for some $a_1, a_2 \in \mathcal{O}(U)$. By assumption we can find a covering $\{U_j \to U\}$ such that for each j either $a_1 f_1|_{U_j}$ is invertible or $a_2 f_2|_{U_j}$ is invertible. Hence either $f_1|_{U_j}$ is invertible or $f_2|_{U_j}$ is invertible as desired. For $n > 2$ we have $a_1 f_1 + \ldots + a_n f_n = 1$ for some $a_1, \ldots, a_n \in \mathcal{O}(U)$. By the case $n = 2$ we see that we have some covering $\{U_j \to U\}_{j \in J}$ such that for each j either $f_n|_{U_j}$ is invertible or $a_1 f_1 + \ldots + a_{n-1} f_{n-1}|_{U_j}$ is invertible. Say the first case happens for $j \in J_n$. Set $J' = J \setminus J_n$. By induction hypothesis, for each $j \in J'$ we can find a covering $\{U_{jk} \to U_j\}_{k \in K_j}$ such that for each $k \in K_j$ there exists an $i \in \{1, \ldots, n-1\}$ such that $f_i|_{U_{jk}}$ is invertible. By the axioms of a site the family of morphisms $\{U_j \to U\}_{j \in J_n} \cup \{U_{jk} \to U\}_{j \in J', k \in K_j}$ is a covering which has the desired property.

Assume (1). To see that the map in (3) is surjective, let (f, c) be a section of $\mathcal{O} \times \mathcal{O}$ over U. By assumption there exists a covering $\{U_j \to U\}$ such that for each j either f or $1 - f$ restricts to an invertible section. In the first case we can take $a = c|_{U_j}(f|_{U_j})^{-1}$, and in the second case we can take $b = c|_{U_j}(1 - f|_{U_j})^{-1}$. Hence (f, c) is in the image of the map on each of the members. Conversely, assume (3) holds. For any U and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ of U such that the section $(f, 1)|_{U_j}$ is in the image of the map in (3) on sections over U_j. This means precisely that either f or $1 - f$ restricts to an invertible section over U_j, and we see that (1) holds. $\square$

04ET Lemma 18.40.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the following conditions

(1) For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1 - f)|_{U_j}$ is invertible.
(2) For every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is either the zero ring or a local ring.

We always have (1) $\Rightarrow$ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent.

Proof. Assume (1). Let p be a point of $\mathcal{C}$ given by a functor $u : \mathcal{C} \to \textit{Sets}$. Let $f_p \in \mathcal{O}_p$. Since $\mathcal{O}_p$ is computed by Sites, Equation (7.32.1.1) we may represent f_p by a triple (U, x, f) where $x \in U(U)$ and $f \in \mathcal{O}(U)$. By assumption there exists a covering $\{U_i \to U\}$ such that for each i either f or $1 - f$ is invertible on U_i. Because u defines a point of the site we see that for some i there exists an $x_i \in u(U_i)$ which maps to $x \in u(U)$. By the discussion surrounding Sites, Equation (7.32.1.1) we see that (U, x, f) and $(U_i, x_i, f|_{U_i})$ define the same element of $\mathcal{O}_p$. Hence we conclude that either f_p or $1 - f_p$ is invertible. Thus $\mathcal{O}_p$ is a ring such that for every element a either a or $1 - a$ is invertible. This means that $\mathcal{O}_p$ is either zero or a local ring, see Algebra, Lemma 10.18.2.

Assume (2) and assume that $\mathcal{C}$ has enough points. Consider the map of sheaves of sets
$$\mathcal{O} \times \mathcal{O} \amalg \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O} \times \mathcal{O}$$
of Lemma 18.40.1 part (3). For any local ring R the corresponding map $(R \times R) \amalg (R \times R) \to R \times R$ is surjective, see for example Algebra, Lemma 10.18.2. Since each

$\mathcal{O}_p$ is a local ring or zero the map is surjective on stalks. Hence, by our assumption that $\mathcal{C}$ has enough points it is surjective and we win. $\square$

In Modules, Section 17.2 we pointed out how in a ringed space $(X, \mathcal{O}_X)$ there can be an open subspace over which the structure sheaf is zero. To prevent this we can require the sections 1 and 0 to have different values in every stalk of the space X. In the setting of ringed topoi and ringed sites the condition is that

05D7 (18.40.2.1)
$$\emptyset^\# \longrightarrow \text{Equalizer}(0, 1 : * \longrightarrow \mathcal{O})$$

is an isomorphism of sheaves. Here $*$ is the singleton sheaf, resp. $\emptyset^\#$ is the "empty sheaf", i.e., the final, resp. initial object in the category of sheaves, see Sites, Example 7.10.2, resp. Section 7.42. In other words, the condition is that whenever $U \in \text{Ob}(\mathcal{C})$ is not sheaf theoretically empty, then $1, 0 \in \mathcal{O}(U)$ are not equal. Let us state the obligatory lemma.

05D8 **Lemma 18.40.3.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the statements

1. (18.40.2.1) is an isomorphism, and
2. for every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is not the zero ring.

We always have (1) $\Rightarrow$ (2) and if $\mathcal{C}$ has enough points then (1) $\Leftrightarrow$ (2).

Proof. Omitted. $\square$

Lemmas 18.40.1, 18.40.2, and 18.40.3 motivate the following definition.

04EU **Definition 18.40.4.** A ringed site $(\mathcal{C}, \mathcal{O})$ is said to be *locally ringed site* if (18.40.2.1) is an isomorphism, and the equivalent properties of Lemma 18.40.1 are satisfied.

In [AGV71, Exposé IV, Exercice 13.9] the condition that (18.40.2.1) be an isomorphism is missing leading to a slightly different notion of a locally ringed site and locally ringed topos. As we are motivated by the notion of a locally ringed space we decided to add this condition (see explanation above).

04H7 **Lemma 18.40.5.** Being a locally ringed site is an intrinsic property. More precisely,

1. if $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ is a morphism of topoi and $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site, and
2. if $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi, then $(\mathcal{C}, \mathcal{O})$ is locally ringed if and only if $(\mathcal{C}', \mathcal{O}')$ is locally ringed.

Proof. It is clear that (2) follows from (1). To prove (1) note that as f^{-1} is exact we have $f^{-1}* = *$, $f^{-1}\emptyset^\# = \emptyset^\#$, and f^{-1} commutes with products, equalizers and transforms isomorphisms and surjections into isomorphisms and surjections. Thus f^{-1} transforms the isomorphism (18.40.2.1) into its analogue for $f^{-1}\mathcal{O}$ and transforms the surjection of Lemma 18.40.1 part (3) into the corresponding surjection for $f^{-1}\mathcal{O}$. $\square$

In fact Lemma 18.40.5 part (2) is the analogue of Schemes, Lemma 26.2.2. It assures us that the following definition makes sense.

04H8 **Definition 18.40.6.** A ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is said to be *locally ringed* if the underlying ringed site $(\mathcal{C}, \mathcal{O})$ is locally ringed.

Here is an example of a consequence of being locally ringed.

0B8Q Lemma 18.40.7. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Any locally free $\mathcal{O}$-module of rank 1 is invertible. If $(\mathcal{C}, \mathcal{O})$ is locally ringed, then the converse holds as well (but in general this is not the case).

Proof. Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O}) \longrightarrow \mathcal{O}$$

Given any object U of $\mathcal{C}$ and restricting to the members of a covering trivializing $\mathcal{L}$, we see that this map is an isomorphism (details omitted). Hence $\mathcal{L}$ is invertible by Lemma 18.32.2.

Assume $(Sh(\mathcal{C}), \mathcal{O})$ is locally ringed. Let U be an object of $\mathcal{C}$. In the proof of Lemma 18.32.2 we have seen that there exists a covering $\{U_i \to U\}$ such that $\mathcal{L}|_{\mathcal{C}/U_i}$ is a direct summand of a finite free $\mathcal{O}_{U_i}$-module. After replacing U by U_i, let $p : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus r}$ be a projector whose image is isomorphic to $\mathcal{L}|_{\mathcal{C}/U}$. Then p corresponds to a matrix

$$P = (p_{ij}) \in \text{Mat}(r \times r, \mathcal{O}(U))$$

which is a projector: $P^2 = P$. Set $A = \mathcal{O}(U)$ so that $P \in \text{Mat}(r \times r, A)$. By Algebra, Lemma 10.78.2 the image of P is a finite locally free module M over A. Hence there are $f_1, \ldots, f_t \in A$ generating the unit ideal, such that M_{f_i} is finite free. By Lemma 18.40.1 after replacing U by the members of an open covering, we may assume that M is free. This means that $\mathcal{L}|_U$ is free (details omitted). Of course, since $\mathcal{L}$ is invertible, this is only possible if the rank of $\mathcal{L}|_U$ is 1 and the proof is complete. $\square$

Next, we want to work out what it means to have a morphism of locally ringed spaces. In order to do this we have the following lemma.

04H9 Lemma 18.40.8. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the following conditions

(1) The diagram of sheaves

$$\begin{tikzcd} f^{-1}(\mathcal{O}^*_\mathcal{D}) \arrow[r, "f^\sharp"'] \arrow[d] & \mathcal{O}^*_\mathcal{C} \arrow[d] \\ f^{-1}(\mathcal{O}_\mathcal{D}) \arrow[r, "f^\sharp"] & \mathcal{O}_\mathcal{C} \end{tikzcd}$$

is cartesian.

(2) For any point p of $\mathcal{C}$, setting $q = f \circ p$, the diagram

$$\begin{tikzcd} \mathcal{O}^*_{\mathcal{D},q} \arrow[r] \arrow[d] & \mathcal{O}^*_{\mathcal{C},p} \arrow[d] \\ \mathcal{O}_{\mathcal{D},q} \arrow[r] & \mathcal{O}_{\mathcal{C},p} \end{tikzcd}$$

of sets is cartesian.

We always have (1) $\Rightarrow$ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent. If $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi then (2) is equivalent to

(3) For any point p of $\mathcal{C}$, setting $q = f \circ p$, the ring map $\mathcal{O}_{\mathcal{D},q} \to \mathcal{O}_{\mathcal{C},p}$ is a local ring map.

In fact, properties (2), or (3) for a conservative family of points implies (1).

Proof. This lemma proves itself, in other words, it follows by unwinding the definitions. □

04HA Definition 18.40.9. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Assume $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi. We say that $(f, f^\sharp)$ is a morphism of locally ringed topoi if and only if the diagram of sheaves

$$\begin{tikzcd} f^{-1}(\mathcal{O}^*_\mathcal{D}) \arrow[r, "f^\sharp"'] \arrow[d] & \mathcal{O}^*_\mathcal{C} \arrow[d] \\ f^{-1}(\mathcal{O}_\mathcal{D}) \arrow[r, "f^\sharp"] & \mathcal{O}_\mathcal{C} \end{tikzcd}$$

(see Lemma 18.40.8) is cartesian. If $(f, f^\sharp)$ is a morphism of ringed sites, then we say that it is a morphism of locally ringed sites if the associated morphism of ringed topoi is a morphism of locally ringed topoi.

It is clear that an isomorphism of ringed topoi between locally ringed topoi is automatically an isomorphism of locally ringed topoi.

04IG Lemma 18.40.10. Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of locally ringed topoi. Then the composition $(g, g^\sharp) \circ (f, f^\sharp)$ (see Definition 18.7.1) is also a morphism of locally ringed topoi.

Proof. Omitted. □

04KR Lemma 18.40.11. If $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ is a morphism of topoi. If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$, then

$$f^{-1}(\mathcal{O}^*) = (f^{-1}\mathcal{O})^*.$$

In particular, if $\mathcal{O}$ turns $\mathcal{C}$ into a locally ringed site, then setting $f^\sharp = \text{id}$ the morphism of ringed topoi

$$(f, f^\sharp) : (Sh(\mathcal{C}'), f^{-1}\mathcal{O}) \to (Sh(\mathcal{C}, \mathcal{O})$$

is a morphism of locally ringed topoi.

Proof. Note that the diagram

$$\begin{tikzcd} \mathcal{O}^* \arrow[r] \arrow[d, "u \mapsto (u, u^{-1})"'] & * \arrow[d, "1"] \\ \mathcal{O} \times \mathcal{O} \arrow[r, "(a,b) \mapsto ab"] & \mathcal{O} \end{tikzcd}$$

is cartesian. Since f^{-1} is exact we conclude that

$$\begin{tikzcd} f^{-1}(\mathcal{O}^*) \arrow[r] \arrow[d, "u \mapsto (u, u^{-1})"'] & * \arrow[d, "1"] \\ f^{-1}\mathcal{O} \times f^{-1}\mathcal{O} \arrow[r, "(a,b) \mapsto ab"] & f^{-1}\mathcal{O} \end{tikzcd}$$

is cartesian which implies the first assertion. For the second, note that $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site by Lemma 18.40.5 so that the assertion makes sense. Now the first part implies that the morphism is a morphism of locally ringed topoi. □

04IH Lemma 18.40.12. Localization of locally ringed sites and topoi.

(1) Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let U be an object of $\mathcal{C}$. Then the localization $(\mathcal{C}/U, \mathcal{O}_U)$ is a locally ringed site, and the localization morphism
$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$$
is a morphism of locally ringed topoi.

(2) Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then the morphism
$$(j, j^\sharp) : (Sh(\mathcal{C}/V), \mathcal{O}_V) \to (Sh(\mathcal{C}/U), \mathcal{O}_U)$$
of Lemma 18.19.5 is a morphism of locally ringed topoi.

(3) Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and let $U = u(V)$. Then the morphism
$$(f', (f')^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$
of Lemma 18.20.1 is a morphism of locally ringed sites.

(4) Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$, and $c : U \to u(V)$. Then the morphism
$$(f_c, (f_c)^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$
of Lemma 18.20.2 is a morphism of locally ringed topoi.

(5) Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the localization $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a locally ringed topos and the localization morphism
$$(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$$
is a morphism of locally ringed topoi.

(6) Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $s : \mathcal{G} \to \mathcal{F}$ be a map of sheaves on $\mathcal{C}$. Then the morphism
$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) \longrightarrow (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$$
of Lemma 18.21.4 is a morphism of locally ringed topoi.

(7) Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then the morphism
$$(f', (f')^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$
of Lemma 18.22.1 is a morphism of locally ringed topoi.

(8) Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be a morphism of sheaves. Then the morphism
$$(f_s, (f_s)^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$
of Lemma 18.22.3 is a morphism of locally ringed topoi.

Proof. Part (1) is clear since $\mathcal{O}_U$ is just the restriction of $\mathcal{O}$, so Lemmas 18.40.5 and 18.40.11 apply. Part (2) is clear as the morphism $(j, j^\sharp)$ is actually a localization of a locally ringed site so (1) applies. Part (3) is clear also since $(f')^\sharp$ is just the restriction of $f^\sharp$ to the topos $Sh(\mathcal{C})/\mathcal{F}$, see proof of Lemma 18.22.1 (hence the diagram of Definition 18.40.9 for the morphism f' is just the restriction of the corresponding diagram for f, and restriction is an exact functor). Part (4) follows formally on combining (2) and (3). Parts (5), (6), (7), and (8) follow from their counterparts (1), (2), (3), and (4) by enlarging the sites as in Lemma 18.7.2 and translating everything in terms of sites and morphisms of sites using the comparisons of Lemmas 18.21.3, 18.21.5, 18.22.2, and 18.22.4. (Alternatively one could use the same arguments as in the proofs of (1), (2), (3), and (4) to prove (5), (6), (7), and (8) directly.) □

18.41. Lower shriek for modules

0796 In this section we extend the construction of $g_!$ discussed in Section 18.16 to the case of sheaves of modules.

0797 Lemma 18.41.1. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor between sites. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ the associated morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings on $\mathcal{D}$. Set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. Hence g becomes a morphism of ringed topoi with $g^* = g^{-1}$. In this case there exists a functor

$$g_! : \text{Mod}(\mathcal{O}_\mathcal{C}) \longrightarrow \text{Mod}(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*.

Proof. Let U be an object of $\mathcal{C}$. For any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have

$$\begin{aligned} \text{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{G}) &= g^{-1}\mathcal{G}(U) \\ &= \mathcal{G}(u(U)) \\ &= \text{Hom}_{\mathcal{O}_\mathcal{D}}(j_{u(U)!}\mathcal{O}_{u(U)}, \mathcal{G}) \end{aligned}$$

because g^{-1} is described by restriction, see Sites, Lemma 7.21.5. Of course a similar formula holds a direct sum of modules of the form $j_{U!}\mathcal{O}_U$. By Homology, Lemma 12.29.6 and Lemma 18.28.8 we see that $g_!$ exists. □

0798 Remark 18.41.2. Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 18.41.1. In general it is not the case that the diagram

$$\begin{tikzcd} \text{Mod}(\mathcal{O}_\mathcal{C}) \arrow[r, "g_!"'] \arrow[d, "forget"'] & \text{Mod}(\mathcal{O}_\mathcal{D}) \arrow[d, "forget"] \\ \text{Ab}(\mathcal{C}) \arrow[r, "g_!^{Ab}"] & \text{Ab}(\mathcal{D}) \end{tikzcd}$$

commutes (here $g_!^{Ab}$ is the one from Lemma 18.16.2). There is a transformation of functors

$$g_!^{Ab} \circ forget \longrightarrow forget \circ g_!$$

From the proof of Lemma 18.41.1 we see that this is an isomorphism if and only if $g_!^{Ab} j_{U!}\mathcal{O}_U \to g_! j_{U!}\mathcal{O}_U$ is an isomorphism for all objects U of $\mathcal{C}$. Since we have $g_! j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$ this holds if and only if

$$g_!^{Ab} j_{U!}\mathcal{O}_U \longrightarrow j_{u(U)!}\mathcal{O}_{u(U)}$$

is an isomorphism for all objects U of $\mathcal{C}$. Note that for such a U we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{C}/U \arrow[r, "u'"'] \arrow[d, "j_U"'] & \mathcal{D}/u(U) \arrow[d, "j_{u(U)}"] \\ \mathcal{C} \arrow[r, "u"] & \mathcal{D} \end{tikzcd}$$

of cocontinuous functors of sites, see Sites, Lemma 7.28.4 and therefore $g_!^{Ab} j_{U!} = j_{u(U)!}(g')_!^{Ab}$ where $g' : Sh(\mathcal{C}/U) \to Sh(\mathcal{D}/u(U))$ is the morphism of topoi induced by the cocontinuous functor u'. Hence we see that $g_! = g_!^{Ab}$ if the canonical map

0799 (18.41.2.1) $$(g')_!^{Ab}\mathcal{O}_U \longrightarrow \mathcal{O}_{u(U)}$$

is an isomorphism for all objects U of $\mathcal{C}$.

The following two results are of a slightly different nature.

0FN3 **Lemma 18.41.3.** Assume given a commutative diagram

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "{(g', (g')^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "{(f, f^\sharp)}"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "{(g, g^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})
\end{tikzcd}$$

of ringed topoi. Assume

(1) f, f', g, and g' correspond to cocontinuous functors u, u', v, and v' as in Sites, Lemma 7.21.1,
(2) $v \circ u' = u \circ v'$,
(3) v and v' are continuous as well as cocontinuous,
(4) for any object V' of $\mathcal{D}'$ the functor ${}^{u'}_{V'}\mathcal{I} \to {}^{u}_{v(V')}\mathcal{I}$ given by v is cofinal, and
(5) $g^{-1}\mathcal{O}_{\mathcal{D}} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}'}$.

Then we have $f'_* \circ (g')^* = g^* \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ on modules.

Proof. We have $(g')^*\mathcal{F} = (g')^{-1}\mathcal{F}$ and $g^*\mathcal{G} = g^{-1}\mathcal{G}$ because of condition (5). Thus the first equality follows immediately from the corresponding equality in Sites, Lemma 7.28.6. Since the left adjoint functors $g_!$ and $g'_!$ to g^* and $(g')^*$ exist by Lemma 18.41.1 we see that the second equality follows by uniqueness of adjoint functors. $\square$

0FN4 **Lemma 18.41.4.** Consider a commutative diagram

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "{(g', (g')^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "{(f, f^\sharp)}"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "{(g, g^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})
\end{tikzcd}$$

of ringed topoi and suppose we have functors

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "v'"'] & \mathcal{C} \\
\mathcal{D}' \arrow[u, "u'"] \arrow[r, "v"] & \mathcal{D} \arrow[u, "u"']
\end{tikzcd}$$

such that (with notation as in Sites, Sections 7.14 and 7.21) we have

(1) u and u' are continuous and give rise to the morphisms f and f',
(2) v and v' are cocontinuous giving rise to the morphisms g and g',
(3) $u \circ v = v' \circ u'$,
(4) v and v' are continuous as well as cocontinuous, and
(5) $g^{-1}\mathcal{O}_{\mathcal{D}} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}'}$.

Then $f'_* \circ (g')^* = g^* \circ f_*$ and $g'_! \circ (f')^{-1} = f^{-1} \circ g_!$ on modules.

Proof. We have $(g')^*\mathcal{F} = (g')^{-1}\mathcal{F}$ and $g^*\mathcal{G} = g^{-1}\mathcal{G}$ because of condition (5). Thus the first equality follows immediately from the corresponding equality in Sites, Lemma 7.28.7. Since the left adjoint functors $g_!$ and $g'_!$ to g^* and $(g')^*$ exist by Lemma 18.41.1 we see that the second equality follows by uniqueness of adjoint functors. $\square$

18.42. Constant sheaves

093I Let E be a set and let $\mathcal{C}$ be a site. We will denote $\underline{E}$ the constant sheaf with value E on $\mathcal{C}$. If E is an abelian group, ring, module, etc, then $\underline{E}$ is a sheaf of abelian groups, rings, modules, etc.

093J Lemma 18.42.1. Let $\mathcal{C}$ be a site. If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then $0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$ is an exact sequence of abelian sheaves and in fact it is even exact as a sequence of abelian presheaves.

Proof. Since sheafification is exact it is clear that $0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$ is an exact sequence of abelian sheaves. Thus $0 \to \underline{A} \to \underline{B} \to \underline{C}$ is an exact sequence of abelian presheaves. To see that $\underline{B} \to \underline{C}$ is surjective, pick a set theoretical section $s : C \to B$. This induces a section $\underline{s} : \underline{C} \to \underline{B}$ of sheaves of sets left inverse to the surjection $\underline{B} \to \underline{C}$. $\square$

093K Lemma 18.42.2. Let $\mathcal{C}$ be a site. Let Λ be a ring and let M and Q be Λ-modules. If Q is a finitely presented Λ-module, then we have $\underline{M \otimes_\Lambda Q}(U) = \underline{M}(U) \otimes_\Lambda Q$ for all $U \in \text{Ob}(\mathcal{C})$.

Proof. Choose a presentation $\Lambda^{\oplus m} \to \Lambda^{\oplus n} \to Q \to 0$. This gives an exact sequence $M^{\oplus m} \to M^{\oplus n} \to M \otimes Q \to 0$. By Lemma 18.42.1 we obtain an exact sequence

$$\underline{M}(U)^{\oplus m} \to \underline{M}(U)^{\oplus n} \to \underline{M \otimes Q}(U) \to 0$$

which proves the lemma. (Note that taking sections over U always commutes with finite direct sums, but not arbitrary direct sums.) $\square$

093L Lemma 18.42.3. Let $\mathcal{C}$ be a site. Let Λ be a coherent ring. Let M be a flat Λ-module. For $U \in \text{Ob}(\mathcal{C})$ the module $\underline{M}(U)$ is a flat Λ-module.

Proof. Let $I \subset \Lambda$ be a finitely generated ideal. By Algebra, Lemma 10.39.5 it suffices to show that $\underline{M}(U) \otimes_\Lambda I \to \underline{M}(U)$ is injective. As Λ is coherent I is finitely presented as a Λ-module. By Lemma 18.42.2 we see that $\underline{M}(U) \otimes I = \underline{M \otimes I}$. Since M is flat the map $M \otimes I \to M$ is injective, whence $\underline{M \otimes I} \to \underline{M}$ is injective. $\square$

093M Lemma 18.42.4. Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. The sheaf $\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$ is a flat $\underline{\Lambda}$-algebra. Moreover we have canonical identifications

$$\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda}^\wedge/I\underline{\Lambda}^\wedge = \underline{\Lambda}^\wedge/\underline{I} \cdot \underline{\Lambda}^\wedge = \underline{\Lambda}^\wedge/\underline{I}^\wedge = \underline{\Lambda/I}$$

where $\underline{I}^\wedge = \lim \underline{I/I^n}$.

Proof. To prove $\underline{\Lambda}^\wedge$ is flat, it suffices to show that $\underline{\Lambda}^\wedge(U)$ is flat as a Λ-module for each $U \in \text{Ob}(\mathcal{C})$, see Lemmas 18.28.2 and 18.28.3. By Lemma 18.42.3 we see that

$$\underline{\Lambda}^\wedge(U) = \lim \underline{\Lambda/I^n}(U)$$

is a limit of a system of flat Λ/I^n-modules. By Lemma 18.42.1 we see that the transition maps are surjective. We conclude by More on Algebra, Lemma 15.27.4.

To see the equalities, note that $\underline{\Lambda}(U)/I\underline{\Lambda}(U) = \underline{\Lambda/I}(U)$ by Lemma 18.42.2. It follows that $\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda/I}$. The system of short exact sequences

$$0 \to \underline{I/I^n}(U) \to \underline{\Lambda/I^n}(U) \to \underline{\Lambda/I}(U) \to 0$$

has surjective transition maps, hence gives a short exact sequence

$$0 \to \lim \underline{I/I^n}(U) \to \lim \underline{\Lambda/I^n}(U) \to \lim \underline{\Lambda/I}(U) \to 0$$

see Homology, Lemma 12.31.3. Thus we see that $\underline{\Lambda}^\wedge/\underline{I}^\wedge = \underline{\Lambda/I}$. Since

$$I\underline{\Lambda}^\wedge \subset \underline{I} \cdot \underline{\Lambda}^\wedge \subset \underline{I}^\wedge$$

it suffices to show that $I\underline{\Lambda}^\wedge(U) = \underline{I}^\wedge(U)$ for all U. Choose generators $I = (f_1, \ldots, f_r)$. For every n we obtain a short exact sequence

$$0 \to K_n/(I^n)^{\oplus r} \to (\Lambda/I^n)^{\oplus r} \xrightarrow{(f_1,\ldots,f_r)} I/I^{n+1} \to 0$$

where $K_n = \{(x_1, \ldots, x_r) \in \Lambda^{\oplus r} \mid \sum x_i f_i \in I^{n+1}\}$. We obtain short exact sequences

$$0 \to \underline{K_n/(I^n)^{\oplus r}}(U) \to \underline{(\Lambda/I^n)^{\oplus r}}(U) \to \underline{I/I^{n+1}}(U) \to 0$$

A calculation shows $K_n = K + (I^n)^{\oplus r}$, hence the transition maps $K_{n+1}/(I^{n+1})^{\oplus r} \to K_n/(I^n)^{\oplus r}$ are surjective. Hence the system of modules on the left hand side has surjective transition maps and a fortiori has ML. Thus we see that $(f_1, \ldots, f_r) : (\underline{\Lambda}^\wedge)^{\oplus r}(U) \to \underline{I}^\wedge(U)$ is surjective by Homology, Lemma 12.31.3 which is what we wanted to show. $\square$

093N Lemma 18.42.5. Let $\mathcal{C}$ be a site. Let Λ be a ring and let M be a Λ-module. Assume $Sh(\mathcal{C})$ is not the empty topos. Then

(1) $\underline{M}$ is a finite type sheaf of $\underline{\Lambda}$-modules if and only if M is a finite Λ-module, and
(2) $\underline{M}$ is a finitely presented sheaf of $\underline{\Lambda}$-modules if and only if M is a finitely presented Λ-module.

Proof. Proof of (1). If M is generated by $x_1, \ldots, x_r$ then $x_1, \ldots, x_r$ define global sections of $\underline{M}$ which generate it, hence $\underline{M}$ is of finite type. Conversely, assume $\underline{M}$ is of finite type. Let $U \in \mathcal{C}$ be an object which is not sheaf theoretically empty (Sites, Definition 7.42.1). Such an object exists as we assumed $Sh(\mathcal{C})$ is not the empty topos. Then there exists a covering $\{U_i \to U\}$ and finitely many sections $s_{ij} \in \underline{M}(U_i)$ generating $\underline{M}|_{U_i}$. After refining the covering we may assume that s_{ij} come from elements x_{ij} of M. Then x_{ij} define global sections of $\underline{M}$ whose restriction to U generate $\underline{M}$.

Assume there exist elements $x_1, \ldots, x_r$ of M which define global sections of $\underline{M}$ generating $\underline{M}$ as a sheaf of $\underline{\Lambda}$-modules. We will show that $x_1, \ldots, x_r$ generate M as a Λ-module. Let $x \in M$. We can find a covering $\{U_i \to U\}_{i \in I}$ and $f_{i,j} \in \underline{\Lambda}(U_i)$ such that $x|_{U_i} = \sum f_{i,j} x_j|_{U_i}$. After refining the covering we may assume $f_{i,j} \in \Lambda$. Since U is not sheaf theoretically empty, there is at least one $i \in I$ such that U_i is not sheaf theoretically empty. Then the map $M \to \underline{M}(U_i)$ is injective (details omitted). We conclude that $x = \sum f_{i,j} x_j$ in M as desired.

Proof of (2). Assume $\underline{M}$ is a $\underline{\Lambda}$-module of finite presentation. By (1) we see that M is of finite type. Choose generators $x_1, \ldots, x_r$ of M as a Λ-module. This determines a short exact sequence $0 \to K \to \Lambda^{\oplus r} \to M \to 0$ which turns into a short exact sequence

$$0 \to \underline{K} \to \underline{\Lambda}^{\oplus r} \to \underline{M} \to 0$$

by Lemma 18.42.1. By Lemma 18.24.1 we see that $\underline{K}$ is of finite type. Hence K is a finite Λ-module by (1). Thus M is a Λ-module of finite presentation. □

18.43. Locally constant sheaves

093P Here is the general definition.

093Q Definition 18.43.1. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc.

(1) We say $\mathcal{F}$ is a constant sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc if it is isomorphic as a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc to a constant sheaf $\underline{E}$ as in Section 18.42.
(2) We say $\mathcal{F}$ is locally constant if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(3) If $\mathcal{F}$ is a sheaf of sets or groups, then we say $\mathcal{F}$ is finite locally constant if the constant values are finite sets or finite groups.

093R Lemma 18.43.2. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. If $\mathcal{G}$ is a locally constant sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc on $\mathcal{D}$, the same is true for $f^{-1}\mathcal{G}$ on $\mathcal{C}$.

Proof. Omitted. □

093S Lemma 18.43.3. Let $\mathcal{C}$ be a site with a final object X.

(1) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $\mathcal{C}$. If $\mathcal{F}$ is finite locally constant, there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.
(2) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of abelian groups on $\mathcal{C}$. If $\mathcal{F}$ is finite locally constant, there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant abelian sheaves associated to a map of abelian groups.
(3) Let Λ be a ring. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of Λ-modules on $\mathcal{C}$. If $\mathcal{F}$ is of finite type, then there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves of Λ-modules associated to a map of Λ-modules.

Proof. Proof omitted. □

093T Lemma 18.43.4. Let $\mathcal{C}$ be a site. Let Λ be a ring. Let M, N be Λ-modules. Let $\mathcal{F}, \mathcal{G}$ be a locally constant sheaves of Λ-modules.

(1) If M is of finite presentation, then
$$\underline{\mathrm{Hom}_\Lambda(M, N)} = \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N})$$
(2) If M and N are both of finite presentation, then
$$\underline{\mathrm{Isom}_\Lambda(M, N)} = \mathit{Isom}_{\underline{\Lambda}}(\underline{M}, \underline{N})$$
(3) If $\mathcal{F}$ is of finite presentation, then $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ is a locally constant sheaf of Λ-modules.

(4) If $\mathcal{F}$ and $\mathcal{G}$ are both of finite presentation, then $Isom_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ is a locally constant sheaf of sets.

Proof. Proof of (1). Set $E = \mathrm{Hom}_\Lambda(M, N)$. We want to show the canonical map

$$\underline{E} \longrightarrow \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N})$$

is an isomorphism. The module M has a presentation $\Lambda^{\oplus s} \to \Lambda^{\oplus t} \to M \to 0$. Then E sits in an exact sequence

$$0 \to E \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus t}, N) \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus s}, N)$$

and we have similarly

$$0 \to \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N}) \to \mathcal{H}om_{\underline{\Lambda}}(\underline{\Lambda^{\oplus t}}, \underline{N}) \to \mathcal{H}om_{\underline{\Lambda}}(\underline{\Lambda^{\oplus s}}, \underline{N})$$

This reduces the question to the case where M is a finite free module where the result is clear.

Proof of (3). The question is local on $\mathcal{C}$, hence we may assume $\mathcal{F} = \underline{M}$ and $\mathcal{G} = \underline{N}$ for some Λ-modules M and N. By Lemma 18.42.5 the module M is of finite presentation. Thus the result follows from (1).

Parts (2) and (4) follow from parts (1) and (3) and the fact that $Isom$ can be viewed as the subsheaf of sections of $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ which have an inverse in $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{G}, \mathcal{F})$. $\square$

093U Lemma 18.43.5. Let $\mathcal{C}$ be a site.

(1) The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside $Sh(\mathcal{C})$.
(2) The category of finite locally constant abelian sheaves is a weak Serre subcategory of $\mathrm{Ab}(\mathcal{C})$.
(3) Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $\mathcal{C}$ is a weak Serre subcategory of $\mathrm{Mod}(\mathcal{C}, \Lambda)$.

Proof. Proof of (1). We may work locally on $\mathcal{C}$. Hence by Lemma 18.43.3 we may assume we are given a finite diagram of finite sets such that our diagram of sheaves is the associated diagram of constant sheaves. Then we just take the limit or colimit in the category of sets and take the associated constant sheaf. Some details omitted.

To prove (2) and (3) we use the criterion of Homology, Lemma 12.10.3. Existence of kernels and cokernels is argued in the same way as above. Of course, the reason for using a Noetherian ring in (3) is to assure us that the kernel of a map of finite Λ-modules is a finite Λ-module. To see that the category is closed under extensions (in the case of sheaves Λ-modules), assume given an extension of sheaves of Λ-modules

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

on $\mathcal{C}$ with $\mathcal{F}$, $\mathcal{G}$ finite type and locally constant. Localizing on $\mathcal{C}$ we may assume $\mathcal{F}$ and $\mathcal{G}$ are constant, i.e., we get

$$0 \to \underline{M} \to \mathcal{E} \to \underline{N} \to 0$$

for some Λ-modules M, N. Choose generators $y_1, \ldots, y_m$ of N, so that we get a short exact sequence $0 \to K \to \Lambda^{\oplus m} \to N \to 0$ of Λ-modules. Localizing further we may assume y_j lifts to a section s_j of $\mathcal{E}$. Thus we see that $\mathcal{E}$ is a pushout as in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{K} \arrow[r] \arrow[d] & \underline{\Lambda^{\oplus m}} \arrow[r] \arrow[d] & \underline{N} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{M} \arrow[r] & \mathcal{E} \arrow[r] & \underline{N} \arrow[r] & 0 \end{tikzcd}$$

By Lemma 18.43.3 again (and the fact that K is a finite Λ-module as Λ is Noetherian) we see that the map $\underline{K} \to \underline{M}$ is locally constant, hence we conclude. $\square$

093V Lemma 18.43.6. Let $\mathcal{C}$ be a site. Let Λ be a ring. The tensor product of two locally constant sheaves of Λ-modules on $\mathcal{C}$ is a locally constant sheaf of Λ-modules.

Proof. Omitted. $\square$

18.44. Localizing sheaves of rings

0EMB Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{S} \subset \mathcal{O}$ be a sub-presheaf of sets such that for all $U \in \mathrm{Ob}(\mathcal{C})$ the set $\mathcal{S}(U) \subset \mathcal{O}(U)$ is a multiplicative subset, see Algebra, Definition 10.9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}(U).$$

The restriction mapping sends the section f/s, $f \in \mathcal{O}(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ for $V \to U$ in $\mathcal{C}$.

0EMC Lemma 18.44.1. In the situation above the map to the sheafification

$$\mathcal{O} \longrightarrow (\mathcal{S}^{-1}\mathcal{O})^\#$$

is a homomorphism of sheaves of rings with the following universal property: for any homomorphism of sheaves of rings $\mathcal{O} \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $(\mathcal{S}^{-1}\mathcal{O})^\# \to \mathcal{A}$.

Proof. Omitted. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{S} \subset \mathcal{O}$ be a sub-presheaf of sets such that for all $U \in \mathcal{C}$ the set $\mathcal{S}(U) \subset \mathcal{O}(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}$-modules

$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$

The restriction mapping sends the section t/s, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \to U$ is a morphism of $\mathcal{C}$. Then $\mathcal{S}^{-1}\mathcal{F}$ is a presheaf of $\mathcal{S}^{-1}\mathcal{O}$-modules.

0EMD Lemma 18.44.2. In the situation above the map to the sheafification

$$\mathcal{F} \longrightarrow (\mathcal{S}^{-1}\mathcal{F})^\#$$

has the following universal property: for any homomorphism of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ such that each local section of $\mathcal{S}$ acts invertibly on $\mathcal{G}$ there exists a unique factorization $(\mathcal{S}^{-1}\mathcal{F})^\# \to \mathcal{G}$. Moreover we have

$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O})^\# \otimes_\mathcal{O} \mathcal{F}$$

as sheaves of $(\mathcal{S}^{-1}\mathcal{O})^\#$-modules.

Proof. Omitted. $\square$

18.45. Sheaves of pointed sets

0F4H In this section we collect some facts about sheaves of pointed sets which we've previously mentioned only for abelian sheaves.

A pointed set is a pair $(S, 0)$ where S is a set and $0 \in S$ is an element of S. A morphism $(S, 0) \to (S', 0')$ of pointed sets is simply a map of sets $S \to S'$ sending 0 to $0'$. We'll abuse notation and say "let S be a pointed set" to mean S is endowed with a marked element $0 \in S$. A sheaf of pointed sets is the same thing as a sheaf of sets $\mathcal{F}$ endowed with a "marking" $0 : * \to \mathcal{F}$ where $*$ is the final sheaf (Sites, Example 7.10.2).

Given a morphism of sites or of topoi, there are pushforward and pullback functors on the categories of sheaves of pointed sets, see Sites, Section 7.44. These are constructed by taking the pushforward, resp. pullback of the underlying sheaf of sets and suitably marking it (using that the pullback of the final sheaf is the final sheaf).

Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor between sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated with u, see Sites, Lemma 7.21.1. Then g^{-1} on sheaves of pointed sets has an left adjoint $g_!$ as well. The construction of this functor is entirely analogous to the construction of $g_!$ on abelian sheaves in Section 18.16.

Similarly, if $j : \mathcal{C}/U \to \mathcal{C}$ is as in Section 18.19 then there is a left adjoint $j_!$ to the functor j^{-1} on sheaves of pointed sets

If we ever need these facts and constructions we will precisely state and prove here the corresponding lemmas.

18.46. Other chapters

CHAPTER 19

Injectives

01D4 ## 19.1. Introduction

01D5 In future chapters we will use the existence of injectives and K-injective complexes to do cohomology of sheaves of modules on ringed sites. In this chapter we explain how to produce injectives and K-injective complexes first for modules on sites and later more generally for Grothendieck abelian categories.

We observe that we already know that the category of abelian groups and the category of modules over a ring have enough injectives, see More on Algebra, Sections 15.54 and 15.55

19.2. Baer's argument for modules

05NM There is another, more set-theoretic approach to showing that any R-module M can be imbedded in an injective module. This approach constructs the injective module by a transfinite colimit of push-outs. While this method is somewhat abstract and more complicated than the one of More on Algebra, Section 15.55, it is also more general. Apparently this method originates with Baer, and was revisited by Cartan and Eilenberg in [CE56] and by Grothendieck in [Gro57]. There Grothendieck uses it to show that many other abelian categories have enough injectives. We will get back to the general case later (Section 19.11).

We begin with a few set theoretic remarks. Let $\{B_\beta\}_{\beta \in \alpha}$ be an inductive system of objects in some category $\mathcal{C}$, indexed by an ordinal α. Assume that $\mathrm{colim}_{\beta \in \alpha} B_\beta$ exists in $\mathcal{C}$. If A is an object of $\mathcal{C}$, then there is a natural map

05NN (19.2.0.1) $$\mathrm{colim}_{\beta \in \alpha} \mathrm{Mor}_\mathcal{C}(A, B_\beta) \longrightarrow \mathrm{Mor}_\mathcal{C}(A, \mathrm{colim}_{\beta \in \alpha} B_\beta).$$

because if one is given a map $A \to B_\beta$ for some β, one naturally gets a map from A into the colimit by composing with $B_\beta \to \mathrm{colim}_{\beta \in \alpha} B_\alpha$. Note that the left colimit is one of sets! In general, (19.2.0.1) is neither injective or surjective.

05NP Example 19.2.1. Consider the category of sets. Let $A = \mathbf{N}$ and $B_n = \{1, \ldots, n\}$ be the inductive system indexed by the natural numbers where $B_n \to B_m$ for $n \leq m$ is the obvious map. Then $\mathrm{colim}\, B_n = \mathbf{N}$, so there is a map $A \to \mathrm{colim}\, B_n$, which does not factor as $A \to B_m$ for any m. Consequently, $\mathrm{colim}\,\mathrm{Mor}(A, B_n) \to \mathrm{Mor}(A, \mathrm{colim}\, B_n)$ is not surjective.

05NQ Example 19.2.2. Next we give an example where the map fails to be injective. Let $B_n = \mathbf{N}/\{1, 2, \ldots, n\}$, that is, the quotient set of $\mathbf{N}$ with the first n elements collapsed to one element. There are natural maps $B_n \to B_m$ for $n \leq m$, so the $\{B_n\}$ form a system of sets over $\mathbf{N}$. It is easy to see that $\mathrm{colim}\, B_n = \{*\}$: it is the one-point set. So it follows that $\mathrm{Mor}(A, \mathrm{colim}\, B_n)$ is a one-element set for every set A. However, $\mathrm{colim}\,\mathrm{Mor}(A, B_n)$ is not a one-element set. Consider the family of maps $A \to B_n$ which are just the natural projections $\mathbf{N} \to \mathbf{N}/\{1, 2, \ldots, n\}$ and the family of maps $A \to B_n$ which map the whole of A to the class of 1. These two families of maps are distinct at each step and thus are distinct in $\mathrm{colim}\,\mathrm{Mor}(A, B_n)$, but they induce the same map $A \to \mathrm{colim}\, B_n$.

Nonetheless, if we map out of a finite set then (19.2.0.1) is an isomorphism always.

05NR Lemma 19.2.3. Suppose that, in (19.2.0.1), $\mathcal{C}$ is the category of sets and A is a finite set, then the map is a bijection.

Proof. Let $f : A \to \mathrm{colim}\, B_\beta$. The range of f is finite, containing say elements $c_1, \ldots, c_r \in \mathrm{colim}\, B_\beta$. These all come from some elements in B_β for $\beta \in \alpha$ large by definition of the colimit. Thus we can define $\widetilde{f} : A \to B_\beta$ lifting f at a finite stage. This proves that (19.2.0.1) is surjective. Next, suppose two maps $f : A \to B_\gamma, f' : A \to B_{\gamma'}$ define the same map $A \to \mathrm{colim}\, B_\beta$. Then each of the finitely many elements of A gets sent to the same point in the colimit. By definition of the colimit for sets, there is $\beta \geq \gamma, \gamma'$ such that the finitely many elements of A get sent to the same points in B_β under f and f'. This proves that (19.2.0.1) is injective. $\square$

The most interesting case of the lemma is when $\alpha = \omega$, i.e., when the system $\{B_\beta\}$ is a system $\{B_n\}_{n \in \mathbf{N}}$ over the natural numbers as in Examples 19.2.1 and 19.2.2. The essential idea is that A is "small" relative to the long chain of compositions $B_1 \to B_2 \to \ldots$, so that it has to factor through a finite step. A more general version of this lemma can be found in Sets, Lemma 3.7.1. Next, we generalize this to the category of modules.

05NS Definition 19.2.4. Let $\mathcal{C}$ be a category, let $I \subset \mathrm{Arrows}(\mathcal{C})$, and let α be an ordinal. An object A of $\mathcal{C}$ is said to be α-small with respect to I if whenever $\{B_\beta\}$ is a system over α with transition maps in I, then the map (19.2.0.1) is an isomorphism.

In the rest of this section we shall restrict ourselves to the category of R-modules for a fixed commutative ring R. We shall also take I to be the collection of injective maps, i.e., the monomorphisms in the category of modules over R. In this case, for any system $\{B_\beta\}$ as in the definition each of the maps

$$B_\beta \to \mathrm{colim}_{\beta \in \alpha} B_\beta$$

is an injection. It follows that the map (19.2.0.1) is an injection. We can in fact interpret the B_β's as submodules of the module $B = \mathrm{colim}_{\beta \in \alpha} B_\beta$, and then we have $B = \bigcup_{\beta \in \alpha} B_\beta$. This is not an abuse of notation if we identify B_α with the image in the colimit. We now want to show that modules are always small for "large" ordinals α.

05NT Proposition 19.2.5. Let R be a ring. Let M be an R-module. Let κ the cardinality of the set of submodules of M. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.

Proof. The proof is straightforward, but let us first think about a special case. If M is finite, then the claim is that for any inductive system $\{B_\beta\}$ with injections between them, parametrized by a limit ordinal, any map $M \to \mathrm{colim}\, B_\beta$ factors through one of the B_β. And this we proved in Lemma 19.2.3.

Now we start the proof in the general case. We need only show that the map (19.2.0.1) is a surjection. Let $f : M \to \mathrm{colim}\, B_\beta$ be a map. Consider the subobjects

$\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, we know that

$$\bigcup f^{-1}(B_\beta) = f^{-1}\left(\bigcup B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over all the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\widetilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\widetilde{\alpha}})$. It follows that $f^{-1}(B_{\widetilde{\alpha}}) = M$. In particular, the map f factors through $B_{\widetilde{\alpha}}$. □

From this lemma we will be able to deduce the existence of lots of injectives. Let us recall Baer's criterion.

05NU Lemma 19.2.6 (Baer's criterion). Let R be a ring. An R-module Q is injective if and only if in every commutative diagram [Bae40, Theorem 1]

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d] & Q \\ R \arrow[ur, dashed] & \end{tikzcd}$$

for $\mathfrak{a} \subset R$ an ideal, the dotted arrow exists.

Proof. This is the equivalence of (1) and (3) in More on Algebra, Lemma 15.55.4; please observe that the proof given there is elementary (and does not use Ext groups or the existence of injectives or projectives in the category of R-modules). □

If M is an R-module, then in general we may have a semi-complete diagram as in Lemma 19.2.6. In it, we can form the push-out

$$\begin{tikzcd} \mathfrak{a} \arrow[r] \arrow[d] & M \arrow[d] \\ R \arrow[r] & R \oplus_{\mathfrak{a}} M. \end{tikzcd}$$

Here the vertical map is injective, and the diagram commutes. The point is that we can extend $\mathfrak{a} \to M$ to R if we extend M to the larger module $R \oplus_{\mathfrak{a}} M$.

The key point of Baer's argument is to repeat this procedure transfinitely many times. To do this we first define, given an R-module M the following (huge) pushout

05NV (19.2.6.1)
$$\begin{tikzcd} \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} \arrow[r] \arrow[d] & M \arrow[d] \\ \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} R \arrow[r] & \mathbf{M}(M). \end{tikzcd}$$

Here the top horizontal arrow maps the element $a \in \mathfrak{a}$ in the summand corresponding to φ to the element $\varphi(a) \in M$. The left vertical arrow maps $a \in \mathfrak{a}$ in the summand corresponding to φ simply to the element $a \in R$ in the summand corresponding to φ. The fundamental properties of this construction are formulated in the following lemma.

05NW **Lemma 19.2.7.** Let R be a ring.

(1) The construction $M \mapsto (M \to \mathbf{M}(M))$ is functorial in M.
(2) The map $M \to \mathbf{M}(M)$ is injective.
(3) For any ideal $\mathfrak{a}$ and any R-module map $\varphi : \mathfrak{a} \to M$ there is an R-module map $\varphi' : R \to \mathbf{M}(M)$ such that

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "\varphi"'] \arrow[d] & M \arrow[d] \\ R \arrow[r, "\varphi'"] & \mathbf{M}(M) \end{tikzcd}$$

commutes.

Proof. Parts (2) and (3) are immediate from the construction. To see (1), let $\chi : M \to N$ be an R-module map. We claim there exists a canonical commutative diagram

$$\begin{tikzcd} \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} \arrow[r] \arrow[d] \arrow[drr] & M \arrow[drr, "\chi"] & & \\ \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} R \arrow[drr] & & \bigoplus_{\mathfrak{a}} \bigoplus_{\psi \in \mathrm{Hom}_R(\mathfrak{a}, N)} \mathfrak{a} \arrow[r] \arrow[d] & N \\ & & \bigoplus_{\mathfrak{a}} \bigoplus_{\psi \in \mathrm{Hom}_R(\mathfrak{a}, N)} R & \end{tikzcd}$$

which induces the desired map $\mathbf{M}(M) \to \mathbf{M}(N)$. The middle east-south-east arrow maps the summand $\mathfrak{a}$ corresponding to φ via $\mathrm{id}_{\mathfrak{a}}$ to the summand $\mathfrak{a}$ corresponding to $\psi = \chi \circ \varphi$. Similarly for the lower east-south-east arrow. Details omitted. $\square$

The idea will now be to apply the functor $\mathbf{M}$ a transfinite number of times. We define for each ordinal α a functor $\mathbf{M}_\alpha$ on the category of R-modules, together with a natural injection $N \to \mathbf{M}_\alpha(N)$. We do this by transfinite recursion. First, $\mathbf{M}_1 = \mathbf{M}$ is the functor defined above. Now, suppose given an ordinal α, and suppose $\mathbf{M}_{\alpha'}$ is defined for $\alpha' < \alpha$. If α has an immediate predecessor $\widetilde{\alpha}$, we let

$$\mathbf{M}_\alpha = \mathbf{M} \circ \mathbf{M}_{\widetilde{\alpha}}.$$

If not, i.e., if α is a limit ordinal, we let

$$\mathbf{M}_\alpha(N) = \mathrm{colim}_{\alpha' < \alpha} \mathbf{M}_{\alpha'}(N).$$

It is clear (e.g., inductively) that the $\mathbf{M}_\alpha(N)$ form an inductive system over ordinals, so this is reasonable.

05NX **Theorem 19.2.8.** Let κ be the cardinality of the set of ideals in R, and let α be an ordinal whose cofinality is greater than κ. Then $\mathbf{M}_\alpha(N)$ is an injective R-module, and $N \to \mathbf{M}_\alpha(N)$ is a functorial injective embedding.

Proof. By Baer's criterion Lemma 19.2.6, it suffices to show that if $\mathfrak{a} \subset R$ is an ideal, then any map $f : \mathfrak{a} \to \mathbf{M}_\alpha(N)$ extends to $R \to \mathbf{M}_\alpha(N)$. However, we know since α is a limit ordinal that

$$\mathbf{M}_\alpha(N) = \operatorname{colim}_{\beta<\alpha} \mathbf{M}_\beta(N),$$

so by Proposition 19.2.5, we find that

$$\operatorname{Hom}_R(\mathfrak{a}, \mathbf{M}_\alpha(N)) = \operatorname{colim}_{\beta<\alpha} \operatorname{Hom}_R(\mathfrak{a}, \mathbf{M}_\beta(N)).$$

This means in particular that there is some $\beta' < \alpha$ such that f factors through the submodule $\mathbf{M}_{\beta'}(N)$, as

$$f : \mathfrak{a} \to \mathbf{M}_{\beta'}(N) \to \mathbf{M}_\alpha(N).$$

However, by the fundamental property of the functor $\mathbf{M}$, see Lemma 19.2.7 part (3), we know that the map $\mathfrak{a} \to \mathbf{M}_{\beta'}(N)$ can be extended to

$$R \to \mathbf{M}(\mathbf{M}_{\beta'}(N)) = \mathbf{M}_{\beta'+1}(N),$$

and the last object imbeds in $\mathbf{M}_\alpha(N)$ (as $\beta' + 1 < \alpha$ since α is a limit ordinal). In particular, f can be extended to $\mathbf{M}_\alpha(N)$. $\square$

19.3. G-modules

04JE We will see later (Differential Graded Algebra, Section 22.17) that the category of modules over an algebra has functorial injective embeddings. The construction is exactly the same as the construction in More on Algebra, Section 15.55.

04JF Lemma 19.3.1. Let G be a topological group. Let R be a ring. The category $\text{Mod}_{R,G}$ of R-G-modules, see Étale Cohomology, Definition 59.57.1, has functorial injective hulls. In particular this holds for the category of discrete G-modules.

Proof. By the remark above the lemma the category $\text{Mod}_{R[G]}$ has functorial injective embeddings. Consider the forgetful functor $v : \text{Mod}_{R,G} \to \text{Mod}_{R[G]}$. This functor is fully faithful, transforms injective maps into injective maps and has a right adjoint, namely

$$u : M \mapsto u(M) = \{x \in M \mid \text{stabilizer of } x \text{ is open}\}$$

Since $v(M) = 0 \Rightarrow M = 0$ we conclude by Homology, Lemma 12.29.5. $\square$

19.4. Abelian sheaves on a space

01DF

01DG Lemma 19.4.1. Let X be a topological space. The category of abelian sheaves on X has enough injectives. In fact it has functorial injective embeddings.

Proof. For an abelian group A we denote $j : A \to J(A)$ the functorial injective embedding constructed in More on Algebra, Section 15.55. Let $\mathcal{F}$ be an abelian sheaf on X. By Sheaves, Example 6.7.5 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod\nolimits_{x\in U} J(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x\in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 6.11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective abelian group the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod\nolimits_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves $i_{x,*} I_x$ (see Sheaves, Section 6.27 for notation.) We have

$$\mathrm{Mor}_{\mathrm{Ab}}(\mathcal{F}_x, I_x) = \mathrm{Mor}_{\mathrm{Ab}(X)}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 6.27.3. Hence it is clear that each $i_{x,*} I_x$ is injective. Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 12.27.3. □

19.5. Sheaves of modules on a ringed space

01DH

01DI Lemma 19.5.1. Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Section 6.25. The category of sheaves of $\mathcal{O}_X$-modules on X has enough injectives. In fact it has functorial injective embeddings.

Proof. For any ring R and any R-module M we denote $j : M \to J_R(M)$ the functorial injective embedding constructed in More on Algebra, Section 15.55. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. By Sheaves, Examples 6.7.5 and 6.15.6 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod\nolimits_{x \in U} J_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x \in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 6.11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective $\mathcal{O}_{X,x}$-module the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod\nolimits_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves $i_{x,*} I_x$ (see Sheaves, Section 6.27 for notation.) We have

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 6.27.3. Hence it is clear that each $i_{x,*} I_x$ is an injective $\mathcal{O}_X$-module (see Homology, Lemma 12.29.1 or argue directly). Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 12.27.3. □

19.6. Abelian presheaves on a category

01DJ Let $\mathcal{C}$ be a category. Recall that this means that $\mathrm{Ob}(\mathcal{C})$ is a set. On the one hand, consider abelian presheaves on $\mathcal{C}$, see Sites, Section 7.2. On the other hand, consider families of abelian groups indexed by elements of $\mathrm{Ob}(\mathcal{C})$; in other words presheaves on the discrete category with underlying set of objects $\mathrm{Ob}(\mathcal{C})$. Let us denote this discrete category simply $\mathrm{Ob}(\mathcal{C})$. There is a natural functor

$$i : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathcal{C}$$

and hence there is a natural restriction or forgetful functor

$$v = i^p : \mathrm{PAb}(\mathcal{C}) \longrightarrow \mathrm{PAb}(\mathrm{Ob}(\mathcal{C}))$$

compare Sites, Section 7.5. We will denote presheaves on $\mathcal{C}$ by B and presheaves on $\mathrm{Ob}(\mathcal{C})$ by A.

There are also two functors, namely i_p and ${}_pi$ which assign an abelian presheaf on $\mathcal{C}$ to an abelian presheaf on $\mathrm{Ob}(\mathcal{C})$, see Sites, Sections 7.5 and 7.19. Here we will use $u = {}_pi$ which is defined (in the case at hand) as follows:

$$uA(U) = \prod\nolimits_{U' \to U} A(U').$$

So an element is a family $(a_\phi)_\phi$ with ϕ ranging through all morphisms in $\mathcal{C}$ with target U. The restriction map on uA corresponding to $g : V \to U$ maps our element $(a_\phi)_\phi$ to the element $(a_{g\circ\psi})_\psi$.

There is a canonical surjective map $vuA \to A$ and a canonical injective map $B \to uvB$. We leave it to the reader to show that

$$\mathrm{Mor}_{\mathrm{PAb}(\mathcal{C})}(B, uA) = \mathrm{Mor}_{\mathrm{PAb}(\mathrm{Ob}(\mathcal{C}))}(vB, A).$$

in this simple case; the general case is in Sites, Section 7.5. Thus the pair (u, v) is an example of a pair of adjoint functors, see Categories, Section 4.24.

At this point we can list the following facts about the situation above.

(1) The functors u and v are exact. This follows from the explicit description of these functors given above.
(2) In particular the functor v transforms injective maps into injective maps.
(3) The category $\mathrm{PAb}(\mathrm{Ob}(\mathcal{C}))$ has enough injectives.
(4) In fact there is a functorial injective embedding $A \mapsto \big(A \to J(A)\big)$ as in Homology, Definition 12.27.5. Namely, we can take $J(A)$ to be the presheaf $U \mapsto J(A(U))$, where $J(-)$ is the functor constructed in More on Algebra, Section 15.55 for the ring $\mathbf{Z}$.

Putting all of this together gives us the following procedure for embedding objects B of $\mathrm{PAb}(\mathcal{C}))$ into an injective object: $B \to uJ(vB)$. See Homology, Lemma 12.29.5.

01DK Proposition 19.6.1. For abelian presheaves on a category there is a functorial injective embedding.

Proof. See discussion above. □

19.7. Abelian Sheaves on a site

01DL Let $\mathcal{C}$ be a site. In this section we prove that there are enough injectives for abelian sheaves on $\mathcal{C}$.

Denote $i : \mathrm{Ab}(\mathcal{C}) \longrightarrow \mathrm{PAb}(\mathcal{C})$ the forgetful functor from abelian sheaves to abelian presheaves. Let ${}^\# : \mathrm{PAb}(\mathcal{C}) \longrightarrow \mathrm{Ab}(\mathcal{C})$ denote the sheafification functor. Recall that ${}^\#$ is a left adjoint to i, that ${}^\#$ is exact, and that $i\mathcal{F}^\# = \mathcal{F}$ for any abelian sheaf $\mathcal{F}$. Finally, let $\mathcal{G} \to J(\mathcal{G})$ denote the canonical embedding into an injective presheaf we found in Section 19.6.

For any sheaf $\mathcal{F}$ in $\mathrm{Ab}(\mathcal{C})$ and any ordinal β we define a sheaf $J_\beta(\mathcal{F})$ by transfinite recursion. We set $J_0(\mathcal{F}) = \mathcal{F}$. We define $J_1(\mathcal{F}) = J(i\mathcal{F})^\#$. Sheafification of the canonical map $i\mathcal{F} \to J(i\mathcal{F})$ gives a functorial map

$$\mathcal{F} \longrightarrow J_1(\mathcal{F})$$

which is injective as $\#$ is exact. We set $J_{\alpha+1}(\mathcal{F}) = J_1(J_\alpha(\mathcal{F}))$. So that there are canonical injective maps $J_\alpha(\mathcal{F}) \to J_{\alpha+1}(\mathcal{F})$. For a limit ordinal β, we define

$$J_\beta(\mathcal{F}) = \mathrm{colim}_{\alpha<\beta} J_\alpha(\mathcal{F}).$$

Note that this is a directed colimit. Hence for any ordinals $\alpha < \beta$ we have an injective map $J_\alpha(\mathcal{F}) \to J_\beta(\mathcal{F})$.

01DM Lemma 19.7.1. With notation as above. Suppose that $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of abelian sheaves on $\mathcal{C}$. Let α be an ordinal and let $\mathcal{G}_1 \to J_\alpha(\mathcal{F})$ be a morphism of sheaves. There exists a morphism $\mathcal{G}_2 \to J_{\alpha+1}(\mathcal{F})$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[d] \\ J_\alpha(\mathcal{F}) \arrow[r] & J_{\alpha+1}(\mathcal{F}) \end{tikzcd}$$

Proof. This is because the map $i\mathcal{G}_1 \to i\mathcal{G}_2$ is injective and hence $i\mathcal{G}_1 \to iJ_\alpha(\mathcal{F})$ extends to $i\mathcal{G}_2 \to J(iJ_\alpha(\mathcal{F}))$ which gives the desired map after applying the sheafification functor. $\square$

This lemma says that somehow the system $\{J_\alpha(\mathcal{F})\}$ is an injective embedding of $\mathcal{F}$. Of course we cannot take the limit over all α because they form a class and not a set. However, the idea is now that you don't have to check injectivity on all injections $\mathcal{G}_1 \to \mathcal{G}_2$, plus the following lemma.

01DN Lemma 19.7.2. Suppose that $\mathcal{G}_i$, $i \in I$ is set of abelian sheaves on $\mathcal{C}$. There exists an ordinal β such that for any sheaf $\mathcal{F}$, any $i \in I$, and any map $\varphi : \mathcal{G}_i \to J_\beta(\mathcal{F})$ there exists an $\alpha < \beta$ such that φ factors through $J_\alpha(\mathcal{F})$.

Proof. This reduces to the case of a single sheaf $\mathcal{G}$ by taking the direct sum of all the $\mathcal{G}_i$.

Consider the sets

$$S = \coprod\nolimits_{U \in \mathrm{Ob}(\mathcal{C})} \mathcal{G}(U).$$

and

$$T_\beta = \coprod\nolimits_{U \in \mathrm{Ob}(\mathcal{C})} J_\beta(\mathcal{F})(U)$$

The transition maps between the sets T_β are injective. If the cofinality of β is large enough, then $T_\beta = \mathrm{colim}_{\alpha<\beta} T_\alpha$, see Sites, Lemma 7.17.10. A morphism $\mathcal{G} \to J_\beta(\mathcal{F})$ factors through $J_\alpha(\mathcal{F})$ if and only if the associated map $S \to T_\beta$ factors through T_α. By Sets, Lemma 3.7.1 if the cofinality of β is bigger than the cardinality of S, then the result of the lemma is true. Hence the lemma follows from the fact that there are ordinals with arbitrarily large cofinality, see Sets, Proposition 3.7.2. $\square$

Recall that for an object X of $\mathcal{C}$ we denote $\mathbf{Z}_X$ the presheaf of abelian groups $\Gamma(U, \mathbf{Z}_X) = \oplus_{U \to X} \mathbf{Z}$, see Modules on Sites, Section 18.4. The sheaf associated to this presheaf is denoted $\mathbf{Z}_X^\#$, see Modules on Sites, Section 18.5. It can be characterized by the property

05NY (19.7.2.1) $$\mathrm{Mor}_{\mathrm{Ab}(\mathcal{C})}(\mathbf{Z}_X^\#, \mathcal{G}) = \mathcal{G}(X)$$

where the element φ of the left hand side is mapped to $\varphi(1 \cdot \mathrm{id}_X)$ in the right hand side. We can use these sheaves to characterize injective abelian sheaves.

01DO **Lemma 19.7.3.** Suppose $\mathcal{J}$ is a sheaf of abelian groups with the following property: For all $X \in \text{Ob}(\mathcal{C})$, for any abelian subsheaf $\mathcal{S} \subset \mathbf{Z}_X^\#$ and any morphism $\varphi : \mathcal{S} \to \mathcal{J}$, there exists a morphism $\mathbf{Z}_X^\# \to \mathcal{J}$ extending φ. Then $\mathcal{J}$ is an injective sheaf of abelian groups.

Proof. Let $\mathcal{F} \to \mathcal{G}$ be an injective map of abelian sheaves. Suppose $\varphi : \mathcal{F} \to \mathcal{J}$ is a morphism. Arguing as in the proof of More on Algebra, Lemma 15.54.1 we see that it suffices to prove that if $\mathcal{F} \neq \mathcal{G}$, then we can find an abelian sheaf $\mathcal{F}'$, $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{G}$ such that (a) the inclusion $\mathcal{F} \subset \mathcal{F}'$ is strict, and (b) φ can be extended to $\mathcal{F}'$. To find $\mathcal{F}'$, let X be an object of $\mathcal{C}$ such that the inclusion $\mathcal{F}(X) \subset \mathcal{G}(X)$ is strict. Pick $s \in \mathcal{G}(X)$, $s \notin \mathcal{F}(X)$. Let $\psi : \mathbf{Z}_X^\# \to \mathcal{G}$ be the morphism corresponding to the section s via (19.7.2.1). Set $\mathcal{S} = \psi^{-1}(\mathcal{F})$. By assumption the morphism

$$\mathcal{S} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{J}$$

can be extended to a morphism $\varphi' : \mathbf{Z}_X^\# \to \mathcal{J}$. Note that φ' annihilates the kernel of ψ (as this is true for φ). Thus φ' gives rise to a morphism $\varphi'' : \text{Im}(\psi) \to \mathcal{J}$ which agrees with φ on the intersection $\mathcal{F} \cap \text{Im}(\psi)$ by construction. Thus φ and φ'' glue to give an extension of φ to the strictly bigger subsheaf $\mathcal{F}' = \mathcal{F} + \text{Im}(\psi)$. $\square$

01DP **Theorem 19.7.4.** The category of sheaves of abelian groups on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 12.27.5.

Proof. Let $\mathcal{G}_i$, $i \in I$ be a set of abelian sheaves such that every subsheaf of every $\mathbf{Z}_X^\#$ occurs as one of the $\mathcal{G}_i$. Apply Lemma 19.7.2 to this collection to get an ordinal β. We claim that for any sheaf of abelian groups $\mathcal{F}$ the map $\mathcal{F} \to J_\beta(\mathcal{F})$ is an injection of $\mathcal{F}$ into an injective. Note that by construction the assignment $\mathcal{F} \mapsto \big(\mathcal{F} \to J_\beta(\mathcal{F})\big)$ is indeed functorial.

The proof of the claim comes from the fact that by Lemma 19.7.3 it suffices to extend any morphism $\gamma : \mathcal{G} \to J_\beta(\mathcal{F})$ from a subsheaf $\mathcal{G}$ of some $\mathbf{Z}_X^\#$ to all of $\mathbf{Z}_X^\#$. Then by Lemma 19.7.2 the map γ lifts into $J_\alpha(\mathcal{F})$ for some $\alpha < \beta$. Finally, we apply Lemma 19.7.1 to get the desired extension of γ to a morphism into $J_{\alpha+1}(\mathcal{F}) \to J_\beta(\mathcal{F})$. $\square$

19.8. Modules on a ringed site

01DQ Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. By analogy with More on Algebra, Section 15.55 let us try to prove that there are enough injective $\mathcal{O}$-modules. First of all, we pick an injective embedding

$$\bigoplus\nolimits_{U,\mathcal{I}} j_{U!}\mathcal{O}_U/\mathcal{I} \longrightarrow \mathcal{J}$$

where $\mathcal{J}$ is an injective abelian sheaf (which exists by the previous section). Here the direct sum is over all objects U of $\mathcal{C}$ and over all $\mathcal{O}$-submodules $\mathcal{I} \subset j_{U!}\mathcal{O}_U$. Please see Modules on Sites, Section 18.19 to read about the functors restriction and extension by 0 for the localization functor $j_U : \mathcal{C}/U \to \mathcal{C}$.

For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ denote

$$\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{J})$$

with its natural $\mathcal{O}$-module structure. Insert here future reference to internal hom. We will also need a canonical flat resolution of a sheaf of $\mathcal{O}$-modules. This we can do as follows: For any $\mathcal{O}$-module $\mathcal{F}$ we denote

$$F(\mathcal{F}) = \bigoplus\nolimits_{U \in \mathrm{Ob}(\mathcal{C}), s \in \mathcal{F}(U)} j_{U!}\mathcal{O}_U.$$

This is a flat sheaf of $\mathcal{O}$-modules which comes equipped with a canonical surjection $F(\mathcal{F}) \to \mathcal{F}$, see Modules on Sites, Lemma 18.28.8. Moreover the construction $\mathcal{F} \mapsto F(\mathcal{F})$ is functorial in $\mathcal{F}$.

01DR Lemma 19.8.1. The functor $\mathcal{F} \mapsto \mathcal{F}^\vee$ is exact.

Proof. This because $\mathcal{J}$ is an injective abelian sheaf. □

There is a canonical map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ given by evaluation: given $x \in \mathcal{F}(U)$ we let $ev(x) \in (\mathcal{F}^\vee)^\vee = \mathcal{H}om(\mathcal{F}^\vee, \mathcal{J})$ be the map $\varphi \mapsto \varphi(x)$.

01DS Lemma 19.8.2. For any $\mathcal{O}$-module $\mathcal{F}$ the evaluation map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is injective.

Proof. You can check this using the definition of $\mathcal{J}$. Namely, if $s \in \mathcal{F}(U)$ is not zero, then let $j_{U!}\mathcal{O}_U \to \mathcal{F}$ be the map of $\mathcal{O}$-modules it corresponds to via adjunction. Let $\mathcal{I}$ be the kernel of this map. There exists a nonzero map $\mathcal{F} \supset j_{U!}\mathcal{O}_U/\mathcal{I} \to \mathcal{J}$ which does not annihilate s. As $\mathcal{J}$ is an injective $\mathcal{O}$-module, this extends to a map $\varphi : \mathcal{F} \to \mathcal{J}$. Then $ev(s)(\varphi) = \varphi(s) \neq 0$ which is what we had to prove. □

The canonical surjection $F(\mathcal{F}) \to \mathcal{F}$ of $\mathcal{O}$-modules turns into a canonical injection, see above, of $\mathcal{O}$-modules

$$(\mathcal{F}^\vee)^\vee \longrightarrow (F(\mathcal{F}^\vee))^\vee.$$

Set $J(\mathcal{F}) = (F(\mathcal{F}^\vee))^\vee$. The composition of ev with this the displayed map gives $\mathcal{F} \to J(\mathcal{F})$ functorially in $\mathcal{F}$.

01DT Lemma 19.8.3. Let $\mathcal{O}$ be a sheaf of rings. For every $\mathcal{O}$-module $\mathcal{F}$ the $\mathcal{O}$-module $J(\mathcal{F})$ is injective.

Proof. We have to show that the functor $\mathrm{Hom}_\mathcal{O}(\mathcal{G}, J(\mathcal{F}))$ is exact. Note that

$$\begin{aligned} \mathrm{Hom}_\mathcal{O}(\mathcal{G}, J(\mathcal{F})) &= \mathrm{Hom}_\mathcal{O}(\mathcal{G}, (F(\mathcal{F}^\vee))^\vee) \\ &= \mathrm{Hom}_\mathcal{O}(\mathcal{G}, \mathcal{H}om(F(\mathcal{F}^\vee), \mathcal{J})) \\ &= \mathrm{Hom}(\mathcal{G} \otimes_\mathcal{O} F(\mathcal{F}^\vee), \mathcal{J}) \end{aligned}$$

Thus what we want follows from the fact that $F(\mathcal{F}^\vee)$ is flat and $\mathcal{J}$ is injective. □

01DU Theorem 19.8.4. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The category of sheaves of $\mathcal{O}$-modules on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 12.27.5.

Proof. From the discussion in this section. □

01DV Proposition 19.8.5. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The category $\mathrm{PMod}(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules has functorial injective embeddings.

Proof. We could prove this along the lines of the discussion in Section 19.6. But instead we argue using the theorem above. Endow $\mathcal{C}$ with the structure of a site by letting the set of coverings of an object U consist of all singletons $\{f : V \to U\}$ where f is an isomorphism. We omit the verification that this defines a site. A sheaf for this topology is the same as a presheaf (proof omitted). Hence the theorem applies. □

19.9. Embedding abelian categories

05PL In this section we show that an abelian category embeds in the category of abelian sheaves on a site having enough points. The site will be the one described in the following lemma.

05PM **Lemma 19.9.1.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\text{Cov} = \{\{f : V \to U\} \mid f \text{ is surjective}\}.$$

Then $(\mathcal{A}, \text{Cov})$ is a site, see Sites, Definition 7.6.2.

Proof. Note that $\text{Ob}(\mathcal{A})$ is a set by our conventions about categories. An isomorphism is a surjective morphism. The composition of surjective morphisms is surjective. And the base change of a surjective morphism in $\mathcal{A}$ is surjective, see Homology, Lemma 12.5.14. □

Let $\mathcal{A}$ be a pre-additive category. In this case the Yoneda embedding $\mathcal{A} \to \text{PSh}(\mathcal{A})$, $X \mapsto h_X$ factors through a functor $\mathcal{A} \to \text{PAb}(\mathcal{A})$.

05PN **Lemma 19.9.2.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} = (\mathcal{A}, \text{Cov})$ be the site defined in Lemma 19.9.1. Then $X \mapsto h_X$ defines a fully faithful, exact functor*

$$\mathcal{A} \longrightarrow \text{Ab}(\mathcal{C}).$$

Moreover, the site $\mathcal{C}$ has enough points.

Proof. Suppose that $f : V \to U$ is a surjective morphism of $\mathcal{A}$. Let $K = \text{Ker}(f)$. Recall that $V \times_U V = \text{Ker}((f, -f) : V \oplus V \to U)$, see Homology, Example 12.5.6. In particular there exists an injection $K \oplus K \to V \times_U V$. Let $p, q : V \times_U V \to V$ be the two projection morphisms. Note that $p - q : V \times_U V \to V$ is a morphism such that $f \circ (p - q) = 0$. Hence $p - q$ factors through $K \to V$. Let us denote this morphism by $c : V \times_U V \to K$. And since the composition $K \oplus K \to V \times_U V \to K$ is surjective, we conclude that c is surjective. It follows that

$$V \times_U V \xrightarrow{p-q} V \to U \to 0$$

is an exact sequence of $\mathcal{A}$. Hence for an object X of $\mathcal{A}$ the sequence

$$0 \to \text{Hom}_{\mathcal{A}}(U, X) \to \text{Hom}_{\mathcal{A}}(V, X) \to \text{Hom}_{\mathcal{A}}(V \times_U V, X)$$

is an exact sequence of abelian groups, see Homology, Lemma 12.5.8. This means that h_X satisfies the sheaf condition on $\mathcal{C}$.

The functor is fully faithful by Categories, Lemma 4.3.5. The functor is a left exact functor between abelian categories by Homology, Lemma 12.5.8. To show that it is right exact, let $X \to Y$ be a surjective morphism of $\mathcal{A}$. Let U be an object of $\mathcal{A}$, and let $s \in h_Y(U) = \text{Mor}_{\mathcal{A}}(U, Y)$ be a section of h_Y over U. By Homology, Lemma 12.5.14 the projection $U \times_Y X \to U$ is surjective. Hence $\{V = U \times_Y X \to U\}$ is a covering of U such that $s|_V$ lifts to a section of h_X. This proves that $h_X \to h_Y$ is a surjection of abelian sheaves, see Sites, Lemma 7.11.2.

The site $\mathcal{C}$ has enough points by Sites, Proposition 7.39.3. □

05PP Remark 19.9.3. The Freyd-Mitchell embedding theorem says there exists a fully faithful exact functor from any abelian category $\mathcal{A}$ to the category of modules over a ring. Lemma 19.9.2 is not quite as strong. But the result is suitable for the Stacks project as we have to understand sheaves of abelian groups on sites in detail anyway. Moreover, "diagram chasing" works in the category of abelian sheaves on $\mathcal{C}$, for example by working with sections over objects, or by working on the level of stalks using that $\mathcal{C}$ has enough points. To see how to deduce the Freyd-Mitchell embedding theorem from Lemma 19.9.2 see Remark 19.9.5.

05PQ Remark 19.9.4. If $\mathcal{A}$ is a "big" abelian category, i.e., if $\mathcal{A}$ has a class of objects, then Lemma 19.9.2 does not work. In this case, given any set of objects $E \subset \mathrm{Ob}(\mathcal{A})$ there exists an abelian full subcategory $\mathcal{A}' \subset \mathcal{A}$ such that $\mathrm{Ob}(\mathcal{A}')$ is a set and $E \subset \mathrm{Ob}(\mathcal{A}')$. Then one can apply Lemma 19.9.2 to $\mathcal{A}'$. One can use this to prove that results depending on a diagram chase hold in $\mathcal{A}$.

05PR Remark 19.9.5. Let $\mathcal{C}$ be a site. Note that $\mathrm{Ab}(\mathcal{C})$ has enough injectives, see Theorem 19.7.4. (In the case that $\mathcal{C}$ has enough points this is straightforward because p_*I is an injective sheaf if I is an injective $\mathbf{Z}$-module and p is a point.) Also, $\mathrm{Ab}(\mathcal{C})$ has a cogenerator (details omitted). Hence Lemma 19.9.2 proves that we have a fully faithful, exact embedding $\mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ has a cogenerator and enough injectives. We can apply this to $\mathcal{A}^{opp}$ and we get a fully faithful exact functor $i : \mathcal{A} \to \mathcal{D} = \mathcal{B}^{opp}$ where $\mathcal{D}$ has enough projectives and a generator. Hence $\mathcal{D}$ has a projective generator P. Set $R = \mathrm{Mor}_\mathcal{D}(P, P)$. Then

$$\mathcal{A} \longrightarrow \mathrm{Mod}_R, \quad X \longmapsto \mathrm{Hom}_\mathcal{D}(P, X).$$

One can check this is a fully faithful, exact functor. In other words, one retrieves the Freyd-Mitchell theorem mentioned in Remark 19.9.3 above.

05SF Remark 19.9.6. The arguments proving Lemmas 19.9.1 and 19.9.2 work also for exact categories, see [Büh10, Appendix A] and [BBD82, 1.1.4]. We quickly review this here and we add more details if we ever need it in the Stacks project.

Let $\mathcal{A}$ be an additive category. A kernel-cokernel pair is a pair (i, p) of morphisms of $\mathcal{A}$ with $i : A \to B$, $p : B \to C$ such that i is the kernel of p and p is the cokernel of i. Given a set $\mathcal{E}$ of kernel-cokernel pairs we say $i : A \to B$ is an admissible monomorphism if $(i, p) \in \mathcal{E}$ for some morphism p. Similarly we say a morphism $p : B \to C$ is an admissible epimorphism if $(i, p) \in \mathcal{E}$ for some morphism i. The pair $(\mathcal{A}, \mathcal{E})$ is said to be an exact category if the following axioms hold

(1) $\mathcal{E}$ is closed under isomorphisms of kernel-cokernel pairs,
(2) for any object A the morphism 1_A is both an admissible epimorphism and an admissible monomorphism,
(3) admissible monomorphisms are stable under composition,
(4) admissible epimorphisms are stable under composition,
(5) the push-out of an admissible monomorphism $i : A \to B$ via any morphism $A \to A'$ exist and the induced morphism $i' : A' \to B'$ is an admissible monomorphism, and
(6) the base change of an admissible epimorphism $p : B \to C$ via any morphism $C' \to C$ exist and the induced morphism $p' : B' \to C'$ is an admissible epimorphism.

Given such a structure let $\mathcal{C} = (\mathcal{A}, \mathrm{Cov})$ where coverings (i.e., elements of Cov) are given by admissible epimorphisms. The axioms listed above immediately imply that this is a site. Consider the functor

$$F : \mathcal{A} \longrightarrow \mathrm{Ab}(\mathcal{C}), \quad X \longmapsto h_X$$

exactly as in Lemma 19.9.2. It turns out that this functor is fully faithful, exact, and reflects exactness. Moreover, any extension of objects in the essential image of F is in the essential image of F.

19.10. Grothendieck's AB conditions

079A This and the next few sections are mostly interesting for "big" abelian categories, i.e., those categories listed in Categories, Remark 4.2.2. A good case to keep in mind is the category of sheaves of modules on a ringed site.

Grothendieck proved the existence of injectives in great generality in the paper [Gro57]. He used the following conditions to single out abelian categories with special properties.

079B Definition 19.10.1. Let $\mathcal{A}$ be an abelian category. We name some conditions

- AB3 $\mathcal{A}$ has direct sums,
- AB4 $\mathcal{A}$ has AB3 and direct sums are exact,
- AB5 $\mathcal{A}$ has AB3 and filtered colimits are exact.

Here are the dual notions

- AB3* $\mathcal{A}$ has products,
- AB4* $\mathcal{A}$ has AB3* and products are exact,
- AB5* $\mathcal{A}$ has AB3* and cofiltered limits are exact.

We say an object U of $\mathcal{A}$ is a generator if for every $N \subset M$, $N \neq M$ in $\mathcal{A}$ there exists a morphism $U \to M$ which does not factor through N. We say $\mathcal{A}$ is a Grothendieck abelian category if it has AB5 and a generator.

Discussion: A direct sum in an abelian category is a coproduct. If an abelian category has direct sums (i.e., AB3), then it has colimits, see Categories, Lemma 4.14.12. Similarly if $\mathcal{A}$ has AB3* then it has limits, see Categories, Lemma 4.14.11. Exactness of direct sums means the following: given an index set I and short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0, \quad i \in I$$

in $\mathcal{A}$ then the sequence

$$0 \to \bigoplus\nolimits_{i \in I} A_i \to \bigoplus\nolimits_{i \in I} B_i \to \bigoplus\nolimits_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB4 it is only true in general that the sequence is exact on the right (i.e., taking direct sums is a right exact functor if direct sums exist). Similarly, exactness of filtered colimits means the following: given a directed set I and a system of short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

over I in $\mathcal{A}$ then the sequence

$$0 \to \mathrm{colim}_{i \in I} A_i \to \mathrm{colim}_{i \in I} B_i \to \mathrm{colim}_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB5 it is only true in general that the sequence is exact on the right (i.e., taking colimits is a right exact functor if colimits exist). A similar explanation holds for AB4* and AB5*.

19.11. Injectives in Grothendieck categories

05AB The existence of a generator implies that given an object M of a Grothendieck abelian category $\mathcal{A}$ there is a set of subobjects. (This may not be true for a general "big" abelian category.)

0E8N Lemma 19.11.1. Let $\mathcal{A}$ be an abelian category with a generator U and X and object of $\mathcal{A}$. If κ is the cardinality of $\mathrm{Mor}(U, X)$ then

(1) There does not exist a strictly increasing (or strictly decreasing) chain of subobjects of X indexed by a cardinal bigger than κ.
(2) If α is an ordinal of cofinality $> \kappa$ then any increasing (or decreasing) sequence of subobjects of X indexed by α is eventually constant.
(3) The cardinality of the set of subobjects of X is $\leq 2^\kappa$.

Proof. For (1) assume $\kappa' > \kappa$ is a cardinal and assume X_i, $i \in \kappa'$ is strictly increasing. Then take for each i a $\phi_i \in \mathrm{Mor}(U, X)$ such that ϕ_i factors through X_{i+1} but not through X_i. Then the morphisms ϕ_i are distinct, which contradicts the definition of κ.

Part (2) follows from the definition of cofinality and (1).

Proof of (3). For any subobject $Y \subset X$ define $S_Y \in \mathcal{P}(\mathrm{Mor}(U, X))$ (power set) as $S_Y = \{\phi \in \mathrm{Mor}(U, X) : \phi)$ factors through $Y\}$. Then $Y = Y'$ if and only if $S_Y = S_{Y'}$. Hence the cardinality of the set of subobjects is at most the cardinality of this power set. $\square$

By Lemma 19.11.1 the following definition makes sense.

079C Definition 19.11.2. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. The size $|M|$ of M is the cardinality of the set of subobjects of M.

079D Lemma 19.11.3. Let $\mathcal{A}$ be a Grothendieck abelian category. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $\mathcal{A}$, then $|M'|, |M''| \leq |M|$.

Proof. Immediate from the definitions. $\square$

079E Lemma 19.11.4. Let $\mathcal{A}$ be a Grothendieck abelian category with generator U.

(1) If $|M| \leq \kappa$, then M is the quotient of a direct sum of at most κ copies of U.
(2) For every cardinal κ there exists a set of isomorphism classes of objects M with $|M| \leq \kappa$.

Proof. For (1) choose for every proper subobject $M' \subset M$ a morphism $\varphi_{M'} : U \to M$ whose image is not contained in M'. Then $\bigoplus_{M' \subset M} \varphi_{M'} : \bigoplus_{M' \subset M} U \to M$ is surjective. It is clear that (1) implies (2). $\square$

079F Proposition 19.11.5. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. Let $\kappa = |M|$. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.

Proof. Please compare with Proposition 19.2.5. We need only show that the map (19.2.0.1) is a surjection. Let $f : M \to \mathrm{colim}\, B_\beta$ be a map. Consider the subobjects $\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, because $\mathcal{A}$ has AB5 we have

$$\operatorname{colim} f^{-1}(B_\beta) = f^{-1}\left(\operatorname{colim} B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over all the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\widetilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\widetilde{\alpha}})$. It follows that $f^{-1}(B_{\widetilde{\alpha}}) = M$. In particular, the map f factors through $B_{\widetilde{\alpha}}$. $\square$

079G Lemma 19.11.6. Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. An object I of $\mathcal{A}$ is injective if and only if in every commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & I \\ U \arrow[ur, dashed] & \end{tikzcd}$$

for $M \subset U$ a subobject, the dotted arrow exists.

Proof. Please see Lemma 19.2.6 for the case of modules. Choose an injection $A \subset B$ and a morphism $\varphi : A \to I$. Consider the set S of pairs (A', φ') consisting of subobjects $A \subset A' \subset B$ and a morphism $\varphi' : A' \to I$ extending φ. Define a partial ordering on this set in the obvious manner. Choose a totally ordered subset $T \subset S$. Then

$$A' = \operatorname{colim}_{t \in T} A_t \xrightarrow{\operatorname{colim}_{t\in T} \varphi_t} I$$

is an upper bound. Hence by Zorn's lemma the set S has a maximal element (A', φ'). We claim that $A' = B$. If not, then choose a morphism $\psi : U \to B$ which does not factor through A'. Set $N = A' \cap \psi(U)$. Set $M = \psi^{-1}(N)$. Then the map

$$M \to N \to A' \xrightarrow{\varphi'} I$$

can be extended to a morphism $\chi : U \to I$. Since $\chi|_{\operatorname{Ker}(\psi)} = 0$ we see that χ factors as

$$U \to \operatorname{Im}(\psi) \xrightarrow{\varphi''} I$$

Since φ' and φ'' agree on $N = A' \cap \operatorname{Im}(\psi)$ we see that combined the define a morphism $A' + \operatorname{Im}(\psi) \to I$ contradicting the assumed maximality of A'. $\square$

079H Theorem 19.11.7. Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathcal{A}$ has functorial injective embeddings.

Proof. Please compare with the proof of Theorem 19.2.8. Choose a generator U of $\mathcal{A}$. For an object M we define $\mathbf{M}(M)$ by the following pushout diagram

$$\begin{tikzcd} \bigoplus_{N \subset U} \bigoplus_{\varphi \in \operatorname{Hom}(N,M)} N \arrow[r] \arrow[d] & M \arrow[d] \\ \bigoplus_{N \subset U} \bigoplus_{\varphi \in \operatorname{Hom}(N,M)} U \arrow[r] & \mathbf{M}(M). \end{tikzcd}$$

Note that $M \to \mathbf{M}(N)$ is a functor and that there exist functorial injective maps $M \to \mathbf{M}(M)$. By transfinite induction we define functors $\mathbf{M}_\alpha(M)$ for every ordinal

α. Namely, set $\mathbf{M}_0(M) = M$. Given $\mathbf{M}_\alpha(M)$ set $\mathbf{M}_{\alpha+1}(M) = \mathbf{M}(\mathbf{M}_\alpha(M))$. For a limit ordinal β set

$$\mathbf{M}_\beta(M) = \mathrm{colim}_{\alpha<\beta}\,\mathbf{M}_\alpha(M).$$

Finally, pick any ordinal α whose cofinality is greater than $|U|$. Such an ordinal exists by Sets, Proposition 3.7.2. We claim that $M \to \mathbf{M}_\alpha(M)$ is the desired functorial injective embedding. Namely, if $N \subset U$ is a subobject and $\varphi : N \to \mathbf{M}_\alpha(M)$ is a morphism, then we see that φ factors through $\mathbf{M}_{\alpha'}(M)$ for some $\alpha' < \alpha$ by Proposition 19.11.5. By construction of $\mathbf{M}(-)$ we see that φ extends to a morphism from U into $\mathbf{M}_{\alpha'+1}(M)$ and hence into $\mathbf{M}_\alpha(M)$. By Lemma 19.11.6 we conclude that $\mathbf{M}_\alpha(M)$ is injective. $\square$

19.12. K-injectives in Grothendieck categories

079I The material in this section is taken from the paper [Ser03] authored by Serpé. This paper generalizes some of the results of [Spa88] by Spaltenstein to general Grothendieck abelian categories. Our Lemma 19.12.3 is only implicit in the paper by Serpé. Our approach is to mimic Grothendieck's proof of Theorem 19.11.7.

079J **Lemma 19.12.1.** Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. Let c be the function on cardinals defined by $c(\kappa) = |\bigoplus_{\alpha\in\kappa} U|$. If $\pi : M \to N$ is a surjection then there exists a subobject $M' \subset M$ which surjects onto N with $|M'| \leq c(|N|)$.

Proof. For every proper subobject $N' \subset N$ choose a morphism $\varphi_{N'} : U \to M$ such that $U \to M \to N$ does not factor through N'. Set

$$M' = \mathrm{Im}\left(\bigoplus\nolimits_{N'\subset N} \varphi_{N'} : \bigoplus\nolimits_{N'\subset N} U \longrightarrow M\right)$$

Then M' works. $\square$

079K **Lemma 19.12.2.** Let $\mathcal{A}$ be a Grothendieck abelian category. There exists a cardinal κ such that given any acyclic complex $M^\bullet$ we have

(1) if $M^\bullet$ is nonzero, there is a nonzero subcomplex $N^\bullet$ which is bounded above, acyclic, and $|N^n| \leq \kappa$,
(2) there exists a surjection of complexes

$$\bigoplus\nolimits_{i\in I} M_i^\bullet \longrightarrow M^\bullet$$

where $M_i^\bullet$ is bounded above, acyclic, and $|M_i^n| \leq \kappa$.

Proof. Choose a generator U of $\mathcal{A}$. Denote c the function of Lemma 19.12.1. Set $\kappa = \sup\{c^n(|U|), n = 1, 2, 3, \ldots\}$. Let $n \in \mathbf{Z}$ and let $\psi : U \to M^n$ be a morphism. In order to prove (1) and (2) it suffices to prove there exists a subcomplex $N^\bullet \subset M^\bullet$ which is bounded above, acyclic, and $|N^m| \leq \kappa$, such that ψ factors through N^n. To do this set $N^n = \mathrm{Im}(\psi)$, $N^{n+1} = \mathrm{Im}(U \to M^n \to M^{n+1})$, and $N^m = 0$ for $m \geq n+2$. Suppose we have constructed $N^m \subset M^m$ for all $m \geq k$ such that

(1) $\mathrm{d}(N^m) \subset N^{m+1}$, $m \geq k$,
(2) $\mathrm{Im}(N^{m-1} \to N^m) = \mathrm{Ker}(N^m \to N^{m+1})$ for all $m \geq k+1$, and
(3) $|N^m| \leq c^{\max\{n-m,0\}}(|U|)$.

for some $k \leq n$. Because $M^\bullet$ is acyclic, we see that the subobject $\mathrm{d}^{-1}(\mathrm{Ker}(N^k \to N^{k+1})) \subset M^{k-1}$ surjects onto $\mathrm{Ker}(N^k \to N^{k+1})$. Thus we can choose $N^{k-1} \subset M^{k-1}$ surjecting onto $\mathrm{Ker}(N^k \to N^{k+1})$ with $|N^{k-1}| \leq c^{n-k+1}(|U|)$ by Lemma 19.12.1. The proof is finished by induction on k. $\square$

079L Lemma 19.12.3. Let $\mathcal{A}$ be a Grothendieck abelian category. Let κ be a cardinal as in Lemma 19.12.2. Suppose that $I^\bullet$ is a complex such that

(1) each I^j is injective, and
(2) for every bounded above acyclic complex $M^\bullet$ such that $|M^n| \leq \kappa$ we have $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.

Then $I^\bullet$ is an K-injective complex.

Proof. Let $M^\bullet$ be an acyclic complex. We are going to construct by induction on the ordinal α an acyclic subcomplex $K_\alpha^\bullet \subset M^\bullet$ as follows. For $\alpha = 0$ we set $K_0^\bullet = 0$. For $\alpha > 0$ we proceed as follows:

(1) If $\alpha = \beta + 1$ and $K_\beta^\bullet = M^\bullet$ then we choose $K_\alpha^\bullet = K_\beta^\bullet$.
(2) If $\alpha = \beta + 1$ and $K_\beta^\bullet \neq M^\bullet$ then $M^\bullet / K_\beta^\bullet$ is a nonzero acyclic complex. We choose a subcomplex $N_\alpha^\bullet \subset M^\bullet / K_\beta^\bullet$ as in Lemma 19.12.2. Finally, we let $K_\alpha^\bullet \subset M^\bullet$ be the inverse image of $N_\alpha^\bullet$.
(3) If α is a limit ordinal we set $K_\beta^\bullet = \mathrm{colim}\, K_\alpha^\bullet$.

It is clear that $M^\bullet = K_\alpha^\bullet$ for a suitably large ordinal α. We will prove that

$$\mathrm{Hom}_{K(\mathcal{A})}(K_\alpha^\bullet, I^\bullet)$$

is zero by transfinite induction on α. It holds for $\alpha = 0$ since $K_0^\bullet$ is zero. Suppose it holds for β and $\alpha = \beta + 1$. In case (1) of the list above the result is clear. In case (2) there is a short exact sequence of complexes

$$0 \to K_\beta^\bullet \to K_\alpha^\bullet \to N_\alpha^\bullet \to 0$$

Since each component of $I^\bullet$ is injective we see that we obtain an exact sequence

$$\mathrm{Hom}_{K(\mathcal{A})}(K_\beta^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(K_\alpha^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(N_\alpha^\bullet, I^\bullet)$$

By induction the term on the left is zero and by assumption on $I^\bullet$ the term on the right is zero. Thus the middle group is zero too. Finally, suppose that α is a limit ordinal. Then we see that

$$\mathrm{Hom}^\bullet(K_\alpha^\bullet, I^\bullet) = \lim_{\beta < \alpha} \mathrm{Hom}^\bullet(K_\beta^\bullet, I^\bullet)$$

with notation as in More on Algebra, Section 15.71. These complexes compute morphisms in $K(\mathcal{A})$ by More on Algebra, Equation (15.71.0.1). Note that the transition maps in the system are surjective because I^j is surjective for each j. Moreover, for a limit ordinal α we have equality of limit and value (see displayed formula above). Thus we may apply Homology, Lemma 12.31.8 to conclude. □

079M Lemma 19.12.4. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $(K_i^\bullet)_{i \in I}$ be a set of acyclic complexes. There exists a functor $M^\bullet \mapsto \mathbf{M}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ such

(1) $j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and
(2) for every $i \in I$ and $w : K_i^\bullet \to M^\bullet$ the morphism $j_{M^\bullet} \circ w$ is homotopic to zero.

Proof. For every $i \in I$ choose a (termwise) injective map of complexes $K_i^\bullet \to L_i^\bullet$ which is homotopic to zero with $L_i^\bullet$ quasi-isomorphic to zero. For example, take $L_i^\bullet$ to be the cone on the identity of $K_i^\bullet$. We define $\mathbf{M}^\bullet(M^\bullet)$ by the following pushout

diagram

$$\begin{tikzcd}
\bigoplus_{i\in I} \bigoplus_{w:K_i^\bullet \to M^\bullet} K_i^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[d] \\
\bigoplus_{i\in I} \bigoplus_{w:K_i^\bullet \to M^\bullet} L_i^\bullet \arrow[r] & \mathbf{M}^\bullet(M^\bullet).
\end{tikzcd}$$

Then $M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ is a functor. The right vertical arrow defines the functorial injective map $j_{M^\bullet}$. The cokernel of $j_{M^\bullet}$ is isomorphic to the direct sum of the cokernels of the maps $K_i^\bullet \to L_i^\bullet$ hence acyclic. Thus $j_{M^\bullet}$ is a quasi-isomorphism. Part (2) holds by construction. □

079N Lemma 19.12.5. Let $\mathcal{A}$ be a Grothendieck abelian category. There exists a functor $M^\bullet \mapsto \mathbf{N}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ such

(1) $j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and
(2) for every $n \in \mathbf{Z}$ the map $M^n \to \mathbf{N}^n(M^\bullet)$ factors through a subobject $I^n \subset \mathbf{N}^n(M^\bullet)$ where I^n is an injective object of $\mathcal{A}$.

Proof. Choose a functorial injective embeddings $i_M : M \to I(M)$, see Theorem 19.11.7. For every complex $M^\bullet$ denote $J^\bullet(M^\bullet)$ the complex with terms $J^n(M^\bullet) = I(M^n) \oplus I(M^{n+1})$ and differential

$$d_{J^\bullet(M^\bullet)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There exists a canonical injective map of complexes $u_{M^\bullet} : M^\bullet \to J^\bullet(M^\bullet)$ by mapping M^n to $I(M^n)\oplus I(M^{n+1})$ via the maps $i_{M^n} : M^n \to I(M^n)$ and $i_{M^{n+1}} \circ d : M^n \to M^{n+1} \to I(M^{n+1})$. Hence a short exact sequence of complexes

$$0 \to M^\bullet \xrightarrow{u_{M^\bullet}} J^\bullet(M^\bullet) \xrightarrow{v_{M^\bullet}} Q^\bullet(M^\bullet) \to 0$$

functorial in $M^\bullet$. Set

$$\mathbf{N}^\bullet(M^\bullet) = C(v_{M^\bullet})^\bullet[-1].$$

Note that

$$\mathbf{N}^n(M^\bullet) = Q^{n-1}(M^\bullet) \oplus J^n(M^\bullet)$$

with differential

$$\begin{pmatrix} -d^{n-1}_{Q^\bullet(M^\bullet)} & -v^n_{M^\bullet} \\ 0 & d^n_{J^\bullet(M)} \end{pmatrix}$$

Hence we see that there is a map of complexes $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ induced by u. It is injective and factors through an injective subobject by construction. The map $j_{M^\bullet}$ is a quasi-isomorphism as one can prove by looking at the long exact sequence of cohomology associated to the short exact sequences of complexes above. □

079P Theorem 19.12.6. Let $\mathcal{A}$ be a Grothendieck abelian category. For every complex $M^\bullet$ there exists a quasi-isomorphism $M^\bullet \to I^\bullet$ such that $M^n \to I^n$ is injective and I^n is an injective object of $\mathcal{A}$ for all n and $I^\bullet$ is a K-injective complex. Moreover, the construction is functorial in $M^\bullet$.

Proof. Please compare with the proof of Theorem 19.2.8 and Theorem 19.11.7. Choose a cardinal κ as in Lemmas 19.12.2 and 19.12.3. Choose a set $(K_i^\bullet)_{i\in I}$ of bounded above, acyclic complexes such that every bounded above acyclic complex $K^\bullet$ such that $|K^n| \leq \kappa$ is isomorphic to $K_i^\bullet$ for some $i \in I$. This is possible by Lemma 19.11.4. Denote $\mathbf{M}^\bullet(-)$ the functor constructed in Lemma 19.12.4. Denote

$\mathbf{N}^\bullet(-)$ the functor constructed in Lemma 19.12.5. Both of these functors come with injective transformations $\text{id} \to \mathbf{M}$ and $\text{id} \to \mathbf{N}$.

Using transfinite recursion we define a sequence of functors $\mathbf{T}_\alpha(-)$ and corresponding transformations $\text{id} \to \mathbf{T}_\alpha$. Namely we set $\mathbf{T}_0(M^\bullet) = M^\bullet$. If $\mathbf{T}_\alpha$ is given then we set

$$\mathbf{T}_{\alpha+1}(M^\bullet) = \mathbf{N}^\bullet(\mathbf{M}^\bullet(\mathbf{T}_\alpha(M^\bullet)))$$

If β is a limit ordinal we set

$$\mathbf{T}_\beta(M^\bullet) = \text{colim}_{\alpha<\beta} \mathbf{T}_\alpha(M^\bullet)$$

The transition maps of the system are injective quasi-isomorphisms. By AB5 we see that the colimit is still quasi-isomorphic to $M^\bullet$. We claim that $M^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ does the job if the cofinality of α is larger than $\max(\kappa, |U|)$ where U is a generator of $\mathcal{A}$. Namely, it suffices to check conditions (1) and (2) of Lemma 19.12.3.

For (1) we use the criterion of Lemma 19.11.6. Suppose that $M \subset U$ and $\varphi : M \to \mathbf{T}^n_\alpha(M^\bullet)$ is a morphism for some $n \in \mathbf{Z}$. By Proposition 19.11.5 we see that φ factor through $\mathbf{T}^n_{\alpha'}(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{N}^\bullet(-)$ we see that φ factors through an injective object of $\mathcal{A}$ which shows that φ lifts to a morphism on U.

For (2) let $w : K^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ be a morphism of complexes where $K^\bullet$ is a bounded above acyclic complex such that $|K^n| \leq \kappa$. Then $K^\bullet \cong K_i^\bullet$ for some $i \in I$. Moreover, by Proposition 19.11.5 once again we see that w factor through $\mathbf{T}^n_{\alpha'}(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{M}^\bullet(-)$ we see that w is homotopic to zero. This finishes the proof. $\square$

19.13. Additional remarks on Grothendieck abelian categories

07D6 In this section we put some results on Grothendieck abelian categories which are folklore.

07D7 **Lemma 19.13.1.** Let $\mathcal{A}$ be a Grothendieck abelian category. Let $F : \mathcal{A}^{opp} \to \textit{Sets}$ be a functor. Then F is representable if and only if F commutes with colimits, i.e.,

$$F(\text{colim}_i N_i) = \lim F(N_i)$$

for any diagram $\mathcal{I} \to \mathcal{A}$, $i \in \mathcal{I}$.

Proof. If F is representable, then it commutes with colimits by definition of colimits.

Assume that F commutes with colimits. Then $F(M \oplus N) = F(M) \times F(N)$ and we can use this to define a group structure on $F(M)$. Hence we get $F : \mathcal{A} \to \textit{Ab}$ which is additive and right exact, i.e., transforms a short exact sequence $0 \to K \to L \to M \to 0$ into an exact sequence $F(K) \leftarrow F(L) \leftarrow F(M) \leftarrow 0$ (compare with Homology, Section 12.7).

Let U be a generator for $\mathcal{A}$. Set $A = \bigoplus_{s \in F(U)} U$. Let $s_{univ} = (s)_{s \in F(U)} \in F(A) = \prod_{s \in F(U)} F(U)$. Let $A' \subset A$ be the largest subobject such that s_{univ} restricts to zero on A'. This exists because $\mathcal{A}$ is a Grothendieck category and because F commutes with colimits. Because F commutes with colimits there exists a unique element $\overline{s}_{univ} \in F(A/A')$ which maps to s_{univ} in $F(A)$. We claim that A/A' represents F, in other words, the Yoneda map

$$\overline{s}_{univ} : h_{A/A'} \longrightarrow F$$

is an isomorphism. Let $M \in \text{Ob}(\mathcal{A})$ and $s \in F(M)$. Consider the surjection

$$c_M : A_M = \bigoplus\nolimits_{\varphi \in \text{Hom}_\mathcal{A}(U, M)} U \longrightarrow M.$$

This gives $F(c_M)(s) = (s_\varphi) \in \prod_\varphi F(U)$. Consider the map

$$\psi : A_M = \bigoplus\nolimits_{\varphi \in \text{Hom}_\mathcal{A}(U, M)} U \longrightarrow \bigoplus\nolimits_{s \in F(U)} U = A$$

which maps the summand corresponding to φ to the summand corresponding to s_φ by the identity map on U. Then s_{univ} maps to $(s_\varphi)_\varphi$ by construction. in other words the right square in the diagram

$$\begin{tikzcd} A' \arrow[r] & A \arrow[r, dotted, "s_{univ}"'] & F \\ K \arrow[u, "?"] \arrow[r] & A_M \arrow[u, "\psi"] \arrow[r] & M \arrow[u, dotted, "s"'] \end{tikzcd}$$

commutes. Let $K = \text{Ker}(A_M \to M)$. Since s restricts to zero on K we see that $\psi(K) \subset A'$ by definition of A'. Hence there is an induced morphism $M \to A/A'$. This construction gives an inverse to the map $h_{A/A'}(M) \to F(M)$ (details omitted). $\square$

07D8 Lemma 19.13.2. A Grothendieck abelian category has Ab3*.

Proof. Let M_i, $i \in I$ be a family of objects of $\mathcal{A}$ indexed by a set I. The functor $F = \prod_{i \in I} h_{M_i}$ commutes with colimits. Hence Lemma 19.13.1 applies. $\square$

079Q Remark 19.13.3. In the chapter on derived categories we consistently work with "small" abelian categories (as is the convention in the Stacks project). For a "big" abelian category $\mathcal{A}$ it isn't clear that the derived category $D(\mathcal{A})$ exists because it isn't clear that morphisms in the derived category are sets. In general this isn't true, see Examples, Lemma 110.61.1. However, if $\mathcal{A}$ is a Grothendieck abelian category, and given $K^\bullet, L^\bullet$ in $K(\mathcal{A})$, then by Theorem 19.12.6 there exists a quasi-isomorphism $L^\bullet \to I^\bullet$ to a K-injective complex $I^\bullet$ and Derived Categories, Lemma 13.31.2 shows that

$$\text{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \text{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

07D9 Lemma 19.13.4. Let $\mathcal{A}$ be a Grothendieck abelian category. Then

(1) $D(\mathcal{A})$ has both direct sums and products,
(2) direct sums are obtained by taking termwise direct sums of any complexes,
(3) products are obtained by taking termwise products of K-injective complexes.

Proof. Let $K_i^\bullet$, $i \in I$ be a family of objects of $D(\mathcal{A})$ indexed by a set I. We claim that the termwise direct sum $\bigoplus_{i \in I} K_i^\bullet$ is a direct sum in $D(\mathcal{A})$. Namely, let $I^\bullet$ be a K-injective complex. Then we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(\bigoplus\nolimits_{i\in I} K_i^\bullet, I^\bullet) &= \mathrm{Hom}_{K(\mathcal{A})}(\bigoplus\nolimits_{i\in I} K_i^\bullet, I^\bullet) \\ &= \prod\nolimits_{i\in I} \mathrm{Hom}_{K(\mathcal{A})}(K_i^\bullet, I^\bullet) \\ &= \prod\nolimits_{i\in I} \mathrm{Hom}_{D(\mathcal{A})}(K_i^\bullet, I^\bullet)\end{aligned}$$

as desired. This is sufficient since any complex can be represented by a K-injective complex by Theorem 19.12.6. To construct the product, choose a K-injective resolution $K_i^\bullet \to I_i^\bullet$ for each i. Then we claim that $\prod_{i\in I} I_i^\bullet$ is a product in $D(\mathcal{A})$. This follows from Derived Categories, Lemma 13.31.5. □

07DA Remark 19.13.5. Let R be a ring. Suppose that M_n, $n \in \mathbf{Z}$ are R-modules. Denote $E_n = M_n[-n] \in D(R)$. We claim that $E = \bigoplus M_n[-n]$ is both the direct sum and the product of the objects E_n in $D(R)$. To see that it is the direct sum, take a look at the proof of Lemma 19.13.4. To see that it is the direct product, take injective resolutions $M_n \to I_n^\bullet$. By the proof of Lemma 19.13.4 we have

$$\prod E_n = \prod I_n^\bullet[-n]$$

in $D(R)$. Since products in Mod_R are exact, we see that $\prod I_n^\bullet[-n]$ is quasi-isomorphic to E. This works more generally in $D(\mathcal{A})$ where $\mathcal{A}$ is a Grothendieck abelian category with Ab4*.

08U1 Lemma 19.13.6. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Assume

(1) $\mathcal{A}$ is a Grothendieck abelian category,
(2) $\mathcal{B}$ has exact countable products, and
(3) F commutes with countable products.

Then $RF : D(\mathcal{A}) \to D(\mathcal{B})$ commutes with derived limits.

Proof. Observe that RF exists as $\mathcal{A}$ has enough K-injectives (Theorem 19.12.6 and Derived Categories, Lemma 13.31.6). The statement means that if $K = R\lim K_n$, then $RF(K) = R\lim RF(K_n)$. See Derived Categories, Definition 13.34.1 for notation. Since RF is an exact functor of triangulated categories it suffices to see that RF commutes with countable products of objects of $D(\mathcal{A})$. In the proof of Lemma 19.13.4 we have seen that products in $D(\mathcal{A})$ are computed by taking products of K-injective complexes and moreover that a product of K-injective complexes is K-injective. Moreover, in Derived Categories, Lemma 13.34.2 we have seen that products in $D(\mathcal{B})$ are computed by taking termwise products. Since RF is computed by applying F to a K-injective representative and since we've assumed F commutes with countable products, the lemma follows. □

The following lemma is some kind of generalization of the existence of Cartan-Eilenberg resolutions (Derived Categories, Section 13.21).

0BKI Lemma 19.13.7. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 12.24.1. Then there exists a morphism $j : K^\bullet \to J^\bullet$ of filtered complexes of $\mathcal{A}$ such that

(1) J^n, F^pJ^n, J^n/F^pJ^n and $F^pJ^n/F^{p'}J^n$ are injective objects of $\mathcal{A}$,
(2) $J^\bullet$, $F^pJ^\bullet$, $J^\bullet/F^pJ^\bullet$, and $F^pJ^\bullet/F^{p'}J^\bullet$ are K-injective complexes,

(3) j induces quasi-isomorphisms $K^\bullet \to J^\bullet$, $F^pK^\bullet \to F^pJ^\bullet$, $K^\bullet/F^pK^\bullet \to J^\bullet/F^pJ^\bullet$, and $F^pK^\bullet/F^{p'}K^\bullet \to F^pJ^\bullet/F^{p'}J^\bullet$.

Proof. By Theorem 19.12.6 we obtain quasi-isomorphisms $i : K^\bullet \to I^\bullet$ and $i^p : F^pK^\bullet \to I^{p,\bullet}$ as well as commutative diagrams

$$\begin{tikzcd} K^\bullet \arrow[d, "i"'] & F^pK^\bullet \arrow[l] \arrow[d, "i^p"'] \\ I^\bullet & I^{p,\bullet} \arrow[l, "\alpha^p"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F^{p'}K^\bullet \arrow[d, "i^{p'}"'] & F^pK^\bullet \arrow[l] \arrow[d, "i^p"'] \\ I^{p',\bullet} & I^{p,\bullet} \arrow[l, "\alpha^{pp'}"'] \end{tikzcd} \quad \text{for } p' \leq p$$

such that $\alpha^p \circ \alpha^{p'p} = \alpha^{p'}$ and $\alpha^{p'p''} \circ \alpha^{pp'} = \alpha^{pp''}$. The problem is that the maps $\alpha^p : I^{p,\bullet} \to I^\bullet$ need not be injective. For each p we choose an injection $t^p : I^{p,\bullet} \to J^{p,\bullet}$ into an acyclic K-injective complex $J^{p,\bullet}$ whose terms are injective objects of $\mathcal{A}$ (first map to the cone on the identity and then use the theorem). Choose a map of complexes $s^p : I^\bullet \to J^{p,\bullet}$ such that the following diagram commutes

$$\begin{tikzcd} K^\bullet \arrow[d, "i"'] & F^pK^\bullet \arrow[l] \arrow[d, "i^p"'] \\ I^\bullet \arrow[dr, "s^p"'] & I^{p,\bullet} \arrow[d, "t^p"] \\ & J^{p,\bullet} \end{tikzcd}$$

This is possible: the composition $F^pK^\bullet \to J^{p,\bullet}$ is homotopic to zero because $J^{p,\bullet}$ is acyclic and K-injective (Derived Categories, Lemma 13.31.2). Since the objects $J^{p,n-1}$ are injective and since $F^pK^n \to K^n \to I^n$ are injective morphisms, we can lift the maps $F^pK^n \to J^{p,n-1}$ giving the homotopy to a map $h^n : I^n \to J^{p,n-1}$. Then we set s^p equal to $h \circ \text{d} + \text{d} \circ h$. (Warning: It will not be the case that $t^p = s^p \circ \alpha^p$, so we have to be careful not to use this below.)

Consider

$$J^\bullet = I^\bullet \times \prod\nolimits_p J^{p,\bullet}$$

Because products in $D(\mathcal{A})$ are given by taking products of K-injective complexes (Lemma 19.13.4) and since $J^{p,\bullet}$ is isomorphic to 0 in $D(\mathcal{A})$ we see that $J^\bullet \to I^\bullet$ is an isomorphism in $D(\mathcal{A})$. Consider the map

$$j = i \times (s^p \circ i)_{p \in \mathbf{Z}} : K^\bullet \longrightarrow I^\bullet \times \prod\nolimits_p J^{p,\bullet} = J^\bullet$$

By our remarks above this is a quasi-isomorphism. It is also injective. For $p \in \mathbf{Z}$ we let $F^pJ^\bullet \subset J^\bullet$ be

$$\mathrm{Im}\left(\alpha^p \times (t^{p'} \circ \alpha^{pp'})_{p' \leq p} : I^{p,\bullet} \to I^\bullet \times \prod\nolimits_{p' \leq p} J^{p',\bullet}\right) \times \prod\nolimits_{p' > p} J^{p',\bullet}$$

This complex is isomorphic to the complex $I^{p,\bullet} \times \prod_{p' > p} J^{p,\bullet}$ as $\alpha^{pp} = \text{id}$ and t^p is injective. Hence $F^pJ^\bullet$ is quasi-isomorphic to $I^{p,\bullet}$ (argue as above). We have $j(F^pK^\bullet) \subset F^pJ^\bullet$ because of the commutativity of the diagram above. The corresponding map of complexes $F^pK^\bullet \to F^pJ^\bullet$ is a quasi-isomorphism by what we just said. Finally, to see that $F^{p+1}J^\bullet \subset F^pJ^\bullet$ use that $\alpha^{p+1p} \circ \alpha^{pp'} = \alpha^{p+1p'}$ and the commutativity of the first displayed diagram in the first paragraph of the proof.

We claim that $j : K^\bullet \to J^\bullet$ is a solution to the problem posed by the lemma. Namely, F^pJ^n is an injective object of $\mathcal{A}$ because it is isomorphic to $I^{p,n} \times \prod_{p' > p} J^{p',n}$ and products of injectives are injective. Then the injective map $F^pJ^n \to J^n$ splits and hence the quotient J^n/F^pJ^n is injective as well as a direct summand of the injective object J^n. Similarly for $F^pJ^n/F^{p'}J^n$. This in particular means that $0 \to F^pJ^\bullet \to J^\bullet \to J^\bullet/F^pJ^\bullet \to 0$ is a termwise split short exact sequence of complexes, hence defines a distinguished triangle in $K(\mathcal{A})$ by fiat. Since $J^\bullet$ and $F^pJ^\bullet$ are K-injective complexes we see that the same is true for $J^\bullet/F^pJ^\bullet$ by Derived Categories, Lemma 13.31.3. A similar argument shows that $F^pJ^\bullet/F^{p'}J^\bullet$ is K-injective. By construction $j : K^\bullet \to J^\bullet$ and the induced maps $F^pK^\bullet \to F^pJ^\bullet$ are quasi-isomorphisms. Using the long exact cohomology sequences of the complexes in play we find that the same holds for $K^\bullet/F^pK^\bullet \to J^\bullet/F^pJ^\bullet$ and $F^pK^\bullet/F^{p'}K^\bullet \to F^pJ^\bullet/F^{p'}J^\bullet$. □

0G1X Remark 19.13.8. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 12.24.1. For ease of notation denote K, F^pK, gr^pK the object of $D(\mathcal{A})$ represented by $K^\bullet$, $F^pK^\bullet$, $\text{gr}^pK^\bullet$. Let $M \in D(\mathcal{A})$. Using Lemma 19.13.7 we can construct a spectral sequence $(E_r, d_r)_{r \geq 1}$ of bigraded objects of $\mathcal{A}$ with d_r of bidgree $(r, -r+1)$ and with

$$E_1^{p,q} = \text{Ext}^{p+q}(M, \text{gr}^pK)$$

If for every n we have

$$\text{Ext}^n(M, F^pK) = 0 \text{ for } p \gg 0 \quad \text{and} \quad \text{Ext}^n(M, F^pK) = \text{Ext}^n(M, K) \text{ for } p \ll 0$$

then the spectral sequence is bounded and converges to $\text{Ext}^{p+q}(M, K)$. Namely, choose any complex $M^\bullet$ representing M, choose $j : K^\bullet \to J^\bullet$ as in the lemma, and consider the complex

$$\text{Hom}^\bullet(M^\bullet, I^\bullet)$$

defined exactly as in More on Algebra, Section 15.71. Setting $F^p \text{Hom}^\bullet(M^\bullet, I^\bullet) = \text{Hom}^\bullet(M^\bullet, F^pI^\bullet)$ we obtain a filtered complex. The spectral sequence of Homology, Section 12.24 has differentials and terms as described above; details omitted. The boundedness and convergence follows from Homology, Lemma 12.24.13.

0G1Y Remark 19.13.9. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M, K be objects of $D(\mathcal{A})$. For any choice of complex $K^\bullet$ representing K we can use the filtration $F^pK^\bullet = \tau_{\leq -p}K^\bullet$ and the discussion in Remark 19.13.8 to get a spectral sequence with

$$E_1^{p,q} = \text{Ext}^{2p+q}(M, H^{-p}(K))$$

This spectral sequence is independent of the choice of complex $K^\bullet$ representing K. After renumbering $p = -j$ and $q = i + 2j$ we find a spectral sequence $(E'_r, d'_r)_{r \geq 2}$ with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = \text{Ext}^i(M, H^j(K))$$

If $M \in D^-(\mathcal{A})$ and $K \in D^+(\mathcal{A})$ then both E_r and E'_r are bounded and converge to $\text{Ext}^{p+q}(M, K)$. If we use the filtration $F^pK^\bullet = \sigma_{\geq p}K^\bullet$ then we get

$$E_1^{p,q} = \text{Ext}^q(M, K^p)$$

If $M \in D^-(\mathcal{A})$ and $K^\bullet$ is bounded below, then this spectral sequence is bounded and converges to $\text{Ext}^{p+q}(M, K)$.

0G1Z Remark 19.13.10. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $K \in D(\mathcal{A})$. Let $M^\bullet$ be a filtered complex of $\mathcal{A}$, see Homology, Definition 12.24.1. For ease of notation denote M, M/F^pM, gr^pM the object of $D(\mathcal{A})$ represented by $M^\bullet$, $M^\bullet/F^pM^\bullet$, $\text{gr}^pM^\bullet$. Dually to Remark 19.13.8 we can construct a spectral sequence $(E_r, d_r)_{r\geq 1}$ of bigraded objects of $\mathcal{A}$ with d_r of bidgree $(r, -r+1)$ and with

$$E_1^{p,q} = \text{Ext}^{p+q}(\text{gr}^{-p}M, K)$$

If for every n we have

$$\text{Ext}^n(M/F^pM, K) = 0 \text{ for } p \ll 0 \quad \text{and} \quad \text{Ext}^n(M/F^pM, K) = \text{Ext}^n(M, K) \text{ for } p \gg 0$$

then the spectral sequence is bounded and converges to $\text{Ext}^{p+q}(M, K)$. Namely, choose a K-injective complex $I^\bullet$ with injective terms representing K, see Theorem 19.12.6. Consider the complex

$$\text{Hom}^\bullet(M^\bullet, I^\bullet)$$

defined exactly as in More on Algebra, Section 15.71. Setting

$$F^p\,\text{Hom}^\bullet(M^\bullet, I^\bullet) = \text{Hom}^\bullet(M^\bullet/F^{-p+1}M^\bullet, I^\bullet)$$

we obtain a filtered complex (note sign and shift in filtration). The spectral sequence of Homology, Section 12.24 has differentials and terms as described above; details omitted. The boundedness and convergence follows from Homology, Lemma 12.24.13.

0G20 Remark 19.13.11. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M, K be objects of $D(\mathcal{A})$. For any choice of complex $M^\bullet$ representing M we can use the filtration $F^pM^\bullet = \tau_{\leq -p}M^\bullet$ and the discussion in Remark 19.13.8 to get a spectral sequence with

$$E_1^{p,q} = \text{Ext}^{2p+q}(H^p(M), K)$$

This spectral sequence is independent of the choice of complex $M^\bullet$ representing M. After renumbering $p = -j$ and $q = i + 2j$ we find a spectral sequence $(E'_r, d'_r)_{r\geq 2}$ with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = \text{Ext}^i(H^{-j}(M), K)$$

If $M \in D^-(\mathcal{A})$ and $K \in D^+(\mathcal{A})$ then E_r and E'_r are bounded and converge to $\text{Ext}^{p+q}(M, K)$. If we use the filtration $F^pM^\bullet = \sigma_{\geq p}M^\bullet$ then we get

$$E_1^{p,q} = \text{Ext}^q(M^{-p}, K)$$

If $K \in D^+(\mathcal{A})$ and $M^\bullet$ is bounded above, then this spectral sequence is bounded and converges to $\text{Ext}^{p+q}(M, K)$.

0ESJ Lemma 19.13.12. Let $\mathcal{A}$ be a Grothendieck abelian category. Suppose given an object $E \in D(\mathcal{A})$ and an inverse system $\{E^i\}_{i\in\mathbf{Z}}$ of objects of $D(\mathcal{A})$ over $\mathbf{Z}$ together with a compatible system of maps $E^i \to E$. Picture:

$$\ldots \to E^{i+1} \to E^i \to E^{i-1} \to \ldots \to E$$

Then there exists a filtered complex $K^\bullet$ of $\mathcal{A}$ (Homology, Definition 12.24.1) such that $K^\bullet$ represents E and $F^iK^\bullet$ represents E^i compatibly with the given maps.

Proof. By Theorem 19.12.6 we can choose a K-injective complex $I^\bullet$ representing E all of whose terms I^n are injective objects of $\mathcal{A}$. Choose a complex $G^{0,\bullet}$ representing E^0. Choose a map of complexes $\varphi^0 : G^{0,\bullet} \to I^\bullet$ representing $E^0 \to E$. For $i > 0$ we inductively represent $E^i \to E^{i-1}$ by a map of complexes $\delta : G^{i,\bullet} \to G^{i-1,\bullet}$ and we set $\varphi^i = \delta \circ \varphi^{i-1}$. For $i < 0$ we inductively represent $E^{i+1} \to E^i$ by a termwise injective map of complexes $\delta : G^{i+1,\bullet} \to G^{i,\bullet}$ (for example you can use Derived Categories, Lemma 13.9.6). Claim: we can find a map of complexes $\varphi^i : G^{i,\bullet} \to I^\bullet$ representing the map $E^i \to E$ and fitting into the commutative diagram

$$\begin{tikzcd} G^{i+1,\bullet} \arrow[r, "\delta"'] \arrow[d, "\varphi^{i+1}"'] & G^{i,\bullet} \arrow[dl, "\varphi^i"] \\ I^\bullet & \end{tikzcd}$$

Namely, we first choose any map of complexes $\varphi : G^{i,\bullet} \to I^\bullet$ representing the map $E^i \to E$. Then we see that $\varphi \circ \delta$ and φ^{i+1} are homotopic by some homotopy $h^p : G^{i+1,p} \to I^{p-1}$. Since the terms of $I^\bullet$ are injective and since δ is termwise injective, we can lift h^p to $(h')^p : G^{i,p} \to I^{p-1}$. Then we set $\varphi^i = \varphi + h' \circ d + d \circ h'$ and we get what we claimed.

Next, we choose for every i a termwise injective map of complexes $a^i : G^{i,\bullet} \to J^{i,\bullet}$ with $J^{i,\bullet}$ acyclic, K-injective, with $J^{i,p}$ injective objects of $\mathcal{A}$. To do this first map $G^{i,\bullet}$ to the cone on the identity and then apply the theorem cited above. Arguing as above we can find maps of complexes $\delta' : J^{i,\bullet} \to J^{i-1,\bullet}$ such that the diagrams

$$\begin{tikzcd} G^{i,\bullet} \arrow[r, "\delta"'] \arrow[d, "a^i"'] & G^{i-1,\bullet} \arrow[d, "a^{i-1}"] \\ J^{i,\bullet} \arrow[r, "\delta'"] & J^{i-1,\bullet} \end{tikzcd}$$

commute. (You could also use the functoriality of cones plus the functoriality in the theorem to get this.) Then we consider the maps

$$\begin{tikzcd} G^{i+1,\bullet} \times \prod_{p > i+1} J^{p,\bullet} \arrow[r] \arrow[dr] & G^{i,\bullet} \times \prod_{p > i} J^{p,\bullet} \arrow[r] \arrow[d] & G^{i-1,\bullet} \times \prod_{p > i-1} J^{p,\bullet} \arrow[dl] \\ & I^\bullet \times \prod_p J^{p,\bullet} & \end{tikzcd}$$

Here the arrows on $J^{p,\bullet}$ are the obvious ones (identity or zero). On the factor $G^{i,\bullet}$ we use $\delta : G^{i,\bullet} \to G^{i-1,\bullet}$, the map $\varphi^i : G^{i,\bullet} \to I^\bullet$, the zero map $0 : G^{i,\bullet} \to J^{p,\bullet}$ for $p > i$, the map $a^i : G^{i,\bullet} \to J^{p,\bullet}$ for $p = i$, and $(\delta')^{i-p} \circ a^i = a^p \circ \delta^{i-p} : G^{i,\bullet} \to J^{p,\bullet}$ for $p < i$. We omit the verification that all the arrows in the diagram are termwise injective. Thus we obtain a filtered complex. Because products in $D(\mathcal{A})$ are given by taking products of K-injective complexes (Lemma 19.13.4) and because $J^{p,\bullet}$ is zero in $D(\mathcal{A})$ we conclude this diagram represents the given diagram in the derived category. This finishes the proof. $\square$

0ESK Lemma 19.13.13. In the situation of Lemma 19.13.12 assume we have a second inverse system $\{(E')^i\}_{i \in \mathbf{Z}}$ and a compatible system of maps $(E')^i \to E$. Then there exists a bi-filtered complex $K^\bullet$ of $\mathcal{A}$ such that $K^\bullet$ represents E, $F^iK^\bullet$ represents E^i, and $(F')^iK^\bullet$ represents $(E')^i$ compatibly with the given maps.

Proof. Using the lemma we can first choose $K^\bullet$ and F. Then we can choose $(K')^\bullet$ and F' which work for $\{(E')^i\}_{i \in \mathbf{Z}}$ and the maps $(E')^i \to E$. Using Lemma 19.13.7 we can assume $K^\bullet$ is a K-injective complex. Then we can choose a map of complexes $(K')^\bullet \to K^\bullet$ corresponding to the given identifications $(K')^\bullet \cong E \cong K^\bullet$. We can additionally choose a termwise injective map $(K')^\bullet \to J^\bullet$ with $J^\bullet$ acyclic and K-injective. (To do this first map $(K')^\bullet$ to the cone on the identity and then apply Theorem 19.12.6.) Then $(K')^\bullet \to K^\bullet \times J^\bullet$ and $K^\bullet \to K^\bullet \times J^\bullet$ are both termwise injective and quasi-isomorphisms (as the product represents E by Lemma 19.13.4). Then we can simply take the images of the filtrations on $K^\bullet$ and $(K')^\bullet$ under these maps to conclude. $\square$

19.14. The Gabriel-Popescu theorem

0F5R In this section we discuss the main theorem of [PG64]. The method of proof follows a write-up by Jacob Lurie and another by Akhil Mathew who in turn follow the presentation by Kuhn in [Kuh94]. See also [Tak71].

Let $\mathcal{A}$ be a Grothendieck abelian category and let U be a generator for $\mathcal{A}$, see Definition 19.10.1. Let $R = \mathrm{Hom}_\mathcal{A}(U, U)$. Consider the functor $G : \mathcal{A} \to \mathrm{Mod}_R$ given by

$$G(A) = \mathrm{Hom}_\mathcal{A}(U, A)$$

endowed with its canonical right R-module structure.

0F5S Lemma 19.14.1. The functor G above has a left adjoint $F : \mathrm{Mod}_R \to \mathcal{A}$.

Proof. We will give two proofs of this lemma.

The first proof will use the adjoint functor theorem, see Categories, Theorem 4.25.3. Observe that that $G : \mathcal{A} \to \mathrm{Mod}_R$ is left exact and sends products to products. Hence G commutes with limits. To check the set theoretical condition in the theorem, suppose that M is an object of Mod_R. Choose a suitably large cardinal κ and denote E a set of objects of $\mathcal{A}$ such that every object A with $|A| \leq \kappa$ is isomorphic to an element of E. This is possible by Lemma 19.11.4. Set $I = \coprod_{A \in E} \mathrm{Hom}_R(M, G(A))$. We think of an element $i \in I$ as a pair (A_i, f_i). Finally, let A be an arbitrary object of $\mathcal{A}$ and $f : M \to G(A)$ arbitrary. We are going to think of elements of $\mathrm{Im}(f) \subset G(A) = \mathrm{Hom}_\mathcal{A}(U, A)$ as maps $u : U \to A$. Set

$$A' = \mathrm{Im}(\bigoplus\nolimits_{u \in \mathrm{Im}(f)} U \xrightarrow{u} A)$$

Since G is left exact, we see that $G(A') \subset G(A)$ contains $\mathrm{Im}(f)$ and we get $f' : M \to G(A')$ factoring f. On the other hand, the object A' is the quotient of a direct sum of at most $|M|$ copies of U. Hence if $\kappa = |\bigoplus_{|M|} U|$, then we see that (A', f') is isomorphic to an element (A_i, f_i) of E and we conclude that f factors as $M \xrightarrow{f_i} G(A_i) \to G(A)$ as desired.

The second proof will give a construction of F which will show that "$F(M) = M \otimes_R U$" in some sense. Namely, for any R-module M we can choose a resolution

$$\bigoplus\nolimits_{j \in J} R \to \bigoplus\nolimits_{i \in I} R \to M \to 0$$

Then we define $F(M)$ by the corresponding exact sequence

$$\bigoplus\nolimits_{j \in J} U \to \bigoplus\nolimits_{i \in I} U \to F(M) \to 0$$

This construction is independent of the choice of the resolution and is functorial; we omit the details. For any A in $\mathcal{A}$ we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(F(M), A) \to \prod\nolimits_{i \in I} G(A) \to \prod\nolimits_{j \in J} G(A)$$

which is isomorphic to the sequence

$$0 \to \mathrm{Hom}_R(M, G(A)) \to \mathrm{Hom}_R(\bigoplus\nolimits_{i \in I} R, G(A)) \to \mathrm{Hom}_R(\bigoplus\nolimits_{j \in J} R, G(A))$$

which shows that F is the left adjoint to G. □

0F5T Lemma 19.14.2. Let $f : M \to G(A)$ be an injective map in Mod_R. Then the adjoint map $f' : F(M) \to A$ is injective too.

Proof. Choose a map $R^{\oplus n} \to M$ and consider the corresponding map $U^{\oplus n} \to F(M)$. Consider a map $v : U \to U^{\oplus n}$ such that the composition $U \to U^{\oplus n} \to F(M) \to A$ is 0. Then this arrow $v : U \to U^{\oplus n}$ is an element v of $R^{\oplus n}$ mapping to zero in $G(A)$. Since f is injective, we conclude that v maps to zero in M which means that $U \to U^{\oplus n} \to F(M)$ is zero by construction of $F(M)$ in the proof of Lemma 19.14.1. Since U is a generator we conclude that

$$\mathrm{Ker}(U^{\oplus n} \to F(M) \to A) = \mathrm{Ker}(U^{\oplus n} \to F(M))$$

To finish the proof we choose a surjection $\bigoplus_{i \in I} R \to M$ and we consider the corresponding surjection

$$\pi : \bigoplus\nolimits_{i \in I} U \longrightarrow F(M)$$

To prove f' is injective it suffices to show that $\mathrm{Ker}(\pi) = \mathrm{Ker}(f' \circ \pi)$ as subobjects of $\bigoplus_{i \in I} U$. However, now we can write $\bigoplus_{i \in I} U$ as the filtered colimit of its subobjects $\bigoplus_{i \in I'} U$ where $I' \subset I$ ranges over the finite subsets. Since filtered colimits are exact by AB5 for $\mathcal{A}$, we see that

$$\mathrm{Ker}(\pi) = \mathrm{colim}_{I' \subset I \text{ finite}} \left(\bigoplus\nolimits_{i \in I'} U\right) \bigcap \mathrm{Ker}(\pi)$$

and

$$\mathrm{Ker}(f' \circ \pi) = \mathrm{colim}_{I' \subset I \text{ finite}} \left(\bigoplus\nolimits_{i \in I'} U\right) \bigcap \mathrm{Ker}(f' \circ \pi)$$

and we get equality because the same is true for each I' by the first displayed equality above. □

0F5U Theorem 19.14.3. Let $\mathcal{A}$ be a Grothendieck abelian category. Then there exists a (noncommutative) ring R and functors $G : \mathcal{A} \to \mathrm{Mod}_R$ and $F : \mathrm{Mod}_R \to \mathcal{A}$ such that

(1) F is the left adjoint to G,
(2) G is fully faithful, and
(3) F is exact.

Moreover, the functors are the ones constructed above.

Proof. We first prove G is fully faithful, or equivalently that $F \circ G \to \mathrm{id}$ is an isomorphism, see Categories, Lemma 4.24.4. First, given an object A the map $F(G(A)) \to A$ is surjective, because every map of $U \to A$ factors through $F(G(A))$ by construction. On the other hand, the map $F(G(A)) \to A$ is the adjoint of the map $\mathrm{id} : G(A) \to G(A)$ and hence injective by Lemma 19.14.2.

The functor F is right exact as it is a left adjoint. Since Mod_R has enough projectives, to show that F is exact, it is enough to show that the first left derived functor L_1F is zero. To prove $L_1F(M) = 0$ for some R-module M choose an exact sequence $0 \to K \to P \to M \to 0$ of R-modules with P free. It suffices to show $F(K) \to F(P)$ is injective. Now we can write this sequence as a filtered colimit of sequences $0 \to K_i \to P_i \to M_i \to 0$ with P_i a finite free R-module: just write P in this manner and set $K_i = K \cap P_i$ and $M_i = \mathrm{Im}(P_i \to M)$. Because F is a left adjoint it commutes with colimits and because $\mathcal{A}$ is a Grothendieck abelian category, we find that $F(K) \to F(P)$ is injective if each $F(K_i) \to F(P_i)$ is injective. Thus it suffices to check $F(K) \to F(P)$ is injective when $K \subset P = R^{\oplus n}$. Thus $F(K) \to U^{\oplus n}$ is injective by an application of Lemma 19.14.2. □

0F5V Lemma 19.14.4. Let $\mathcal{A}$ be a Grothendieck abelian category. Let R, F, G be as in the Gabriel-Popescu theorem (Theorem 19.14.3). Then we obtain derived functors [Ser03, Corollary 4.1]

$$RG : D(\mathcal{A}) \to D(\mathrm{Mod}_R) \quad \text{and} \quad F : D(\mathrm{Mod}_R) \to D(\mathcal{A})$$

such that F is left adjoint to RG, RG is fully faithful, and $F \circ RG = \mathrm{id}$.

Proof. The existence and adjointness of the functors follows from Theorems 19.14.3 and 19.12.6 and Derived Categories, Lemmas 13.31.6, 13.16.9, and 13.30.3. The statement $F \circ RG = \mathrm{id}$ follows because we can compute RG on an object of $D(\mathcal{A})$ by applying G to a suitable representative complex $I^\bullet$ (for example a K-injective one) and then $F(G(I^\bullet)) = I^\bullet$ because $F \circ G = \mathrm{id}$. Fully faithfulness of RG follows from this by Categories, Lemma 4.24.4. □

19.15. Brown representability and Grothendieck abelian categories

0F5W In this section we quickly prove a representability theorem for derived categories of Grothendieck abelian categories. The reader should first read the case of compactly generated triangulated categories in Derived Categories, Section 13.38. After that, instead of reading this section, it makes sense to consult the literature for more general results of this nature, for example see [Fra01], [Nee01], [Kra02], or take a look at Derived Categories, Section 13.39.

0F5X Lemma 19.15.1. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $H : D(\mathcal{A}) \to \mathrm{Ab}$ be a contravariant cohomological functor which transforms direct sums into products. Then H is representable.

Proof. Let R, F, G, RG be as in Lemma 19.14.4 and consider the functor $H \circ F : D(\mathrm{Mod}_R) \to \mathrm{Ab}$. Observe that since F is a left adjoint it sends direct sums to direct sums and hence $H \circ F$ transforms direct sums into products. On the other hand, the derived category $D(\mathrm{Mod}_R)$ is generated by a single compact object, namely R. By Derived Categories, Lemma 13.38.1 we see that $H \circ F$ is representable, say by $L \in D(\mathrm{Mod}_R)$. Choose a distinguished triangle

$$M \to L \to RG(F(L)) \to M[1]$$

in $D(\mathrm{Mod}_R)$. Then $F(M) = 0$ because $F \circ RG = \mathrm{id}$. Hence $H(F(M)) = 0$ hence $\mathrm{Hom}(M, L) = 0$. It follows that $L \to RG(F(L))$ is the inclusion of a direct summand, see Derived Categories, Lemma 13.4.11. For A in $D(\mathcal{A})$ we obtain

$$\begin{aligned} H(A) &= H(F(RG(A))) \\ &= \mathrm{Hom}(RG(A), L) \\ &\to \mathrm{Hom}(RG(A), RG(F(L))) \\ &= \mathrm{Hom}(F(RG(A)), F(L)) \\ &= \mathrm{Hom}(A, F(L)) \end{aligned}$$

where the arrow has a left inverse functorial in A. In other words, we find that H is the direct summand of a representable functor. Since $D(\mathcal{A})$ is Karoubian (Derived Categories, Lemma 13.4.14) we conclude. □

0F5Y Proposition 19.15.2. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $\mathcal{D}$ be a triangulated category. Let $F : D(\mathcal{A}) \to \mathcal{D}$ be an exact functor of triangulated categories which transforms direct sums into direct sums. Then F has an exact right adjoint.

Proof. For an object Y of $\mathcal{D}$ consider the contravariant functor

$$D(\mathcal{A}) \to \mathrm{Ab}, \quad W \mapsto \mathrm{Hom}_{\mathcal{D}}(F(W), Y)$$

This is a cohomological functor as F is exact and transforms direct sums into products as F transforms direct sums into direct sums. Thus by Lemma 19.15.1 we find an object X of $D(\mathcal{A})$ such that $\mathrm{Hom}_{D(\mathcal{A})}(W, X) = \mathrm{Hom}_{\mathcal{D}}(F(W), Y)$. The existence of the adjoint follows from Categories, Lemma 4.24.2. Exactness follows from Derived Categories, Lemma 13.7.1. □

19.16. Other chapters

CHAPTER 20

Cohomology of Sheaves

01DW 20.1. Introduction

01DX In this document we work out some topics on cohomology of sheaves on topological spaces. We mostly work in the generality of modules over a sheaf of rings and we work with morphisms of ringed spaces. To see what happens for sheaves on sites take a look at the chapter Cohomology on Sites, Section 21.1. Basic references are [God73] and [Ive86].

20.2. Cohomology of sheaves

01DZ Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf. We know that the category of abelian sheaves on X has enough injectives, see Injectives, Lemma 19.4.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

0712 (20.2.0.1)
$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

to be the ith cohomology group of the abelian sheaf $\mathcal{F}$. The family of functors $H^i(X, -)$ forms a universal δ-functor from $\textit{Ab}(X) \to \textit{Ab}$.

Let $f : X \to Y$ be a continuous map of topological spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

0713 (20.2.0.2)
$$R^i f_*\mathcal{F} = H^i(f_*\mathcal{I}^\bullet)$$

to be the ith higher direct image of $\mathcal{F}$. The family of functors $R^i f_*$ forms a universal δ-functor from $\textit{Ab}(X) \to \textit{Ab}(Y)$.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We know that the category of $\mathcal{O}_X$-modules on X has enough injectives, see Injectives, Lemma 19.5.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

0714 (20.2.0.3)
$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

to be the ith cohomology group of $\mathcal{F}$. The family of functors $H^i(X, -)$ forms a universal δ-functor from $\textit{Mod}(\mathcal{O}_X) \to \text{Mod}_{\mathcal{O}_X(X)}$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

0715 (20.2.0.4)
$$R^i f_*\mathcal{F} = H^i(f_*\mathcal{I}^\bullet)$$

to be the ith higher direct image of $\mathcal{F}$. The family of functors $R^i f_*$ forms a universal δ-functor from $\textit{Mod}(\mathcal{O}_X) \to \textit{Mod}(\mathcal{O}_Y)$.

20.3. Derived functors

0716 We briefly explain how to get right derived functors using resolution functors. For the unbounded derived functors, please see Section 20.28.

Let $(X, \mathcal{O}_X)$ be a ringed space. The category $\text{Mod}(\mathcal{O}_X)$ is abelian, see Modules, Lemma 17.3.1. In this chapter we will write

$$K(\mathcal{O}_X) = K(\text{Mod}(\mathcal{O}_X)) \quad \text{and} \quad D(\mathcal{O}_X) = D(\text{Mod}(\mathcal{O}_X)).$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 13.8.1 and Definition 13.11.3. By Derived Categories, Remark 13.24.3 there exists a resolution functor

$$j = j_X : K^+(\text{Mod}(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $\text{Mod}(\mathcal{O}_X)$ consisting of injective sheaves. For any left exact functor $F : \text{Mod}(\mathcal{O}_X) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor described in Derived Categories, Section 13.20 and constructed using the resolution functor j_X just described:

05U3 (20.3.0.1)
$$RF = F \circ j'_X : D^+(X) \longrightarrow D^+(\mathcal{B})$$

see Derived Categories, Lemma 13.25.1 for notation. Note that we may think of RF as defined on $\text{Mod}(\mathcal{O}_X)$, $\text{Comp}^+(\text{Mod}(\mathcal{O}_X))$, $K^+(X)$, or $D^+(X)$ depending on the situation. According to Derived Categories, Definition 13.16.2 we obtain the ith right derived functor

05U4 (20.3.0.2)
$$R^iF = H^i \circ RF : \text{Mod}(\mathcal{O}_X) \longrightarrow \mathcal{B}$$

so that $R^0F = F$ and $\{R^iF, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 13.20.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\text{Mod}_R)$ and $D(R) = D(\text{Mod}_R)$ and similarly for bounded versions. For any open $U \subset X$ we have a left exact functor $\Gamma(U, -) : \text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}_{\mathcal{O}_X(U)}$ which gives rise to

0717 (20.3.0.3)
$$R\Gamma(U, -) : D^+(X) \longrightarrow D^+(\mathcal{O}_X(U))$$

by the discussion above. We set $H^i(U, -) = R^i\Gamma(U, -)$. If $U = X$ we recover (20.2.0.3). If $f : X \to Y$ is a morphism of ringed spaces, then we have the left exact functor $f_* : \text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_Y)$ which gives rise to the derived pushforward

0718 (20.3.0.4)
$$Rf_* : D^+(X) \longrightarrow D^+(Y)$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^if_*\mathcal{F}^\bullet$ and called the ith higher direct image in accordance with (20.2.0.4). The two displayed functors above are exact functors of derived categories.

Abuse of notation: When the functor Rf_*, or any other derived functor, is applied to a sheaf $\mathcal{F}$ on X or a complex of sheaves it is understood that $\mathcal{F}$ has been replaced by a suitable resolution of $\mathcal{F}$. To facilitate this kind of operation we will say, given an object $\mathcal{F}^\bullet \in D(\mathcal{O}_X)$, that a bounded below complex $\mathcal{I}^\bullet$ of injectives of $\text{Mod}(\mathcal{O}_X)$ represents $\mathcal{F}^\bullet$ in the derived category if there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. In the same vein the phrase "let $\alpha : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a morphism of $D(\mathcal{O}_X)$" does not mean that α is represented by a morphism of complexes. If we have an actual morphism of complexes we will say so.

20.4. First cohomology and torsors

02FN

02FO **Definition 20.4.1.** Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on X endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive, and
(2) for every $x \in X$ the stalk $\mathcal{F}_x$ is nonempty.

A *morphism of $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

02FP **Lemma 20.4.2.** Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\mathcal{F}(X) \neq \emptyset$.

Proof. Omitted. $\square$

02FQ **Lemma 20.4.3.** Let X be a topological space. Let $\mathcal{H}$ be an abelian sheaf on X. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(X, \mathcal{H})$.

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \subset X$ open the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by the local section of the form $[s] - [s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\sigma) \arrow[r] \arrow[d, "a"] & \mathbf{Z}[\mathcal{F}] \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & 0
\end{tikzcd}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(X, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(X, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(X, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(X, \mathcal{Q})$ because $H^1(X, \mathcal{I}) = 0$ (see Derived Categories, Lemma 13.20.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a torsor.

We omit the verification that the two constructions given above are mutually inverse. $\square$

20.5. First cohomology and extensions

0B39

0B3A **Lemma 20.5.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. There is a canonical bijection

$$\mathrm{Ext}^1_{\mathit{Mod}(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F})$$

which associates to the extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_X \to 0$$

the image of $1 \in \Gamma(X, \mathcal{O}_X)$ in $H^1(X, \mathcal{F})$.

Proof. Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(X, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $\mathit{Mod}(\mathcal{O}_X)$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that ξ is the image of a section $\tilde{\xi} \in \Gamma(X, \mathcal{Q}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{Q})$. Now, we just form the pullback

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "\tilde{\xi}"] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

see Homology, Section 12.6. $\square$

20.6. First cohomology and invertible sheaves

09NT The Picard group of a ringed space is defined in Modules, Section 17.25.

09NU **Lemma 20.6.1.** Let $(X, \mathcal{O}_X)$ be a locally ringed space. There is a canonical isomorphism

$$H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s \cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}_X^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}_X^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by Modules, Lemma 17.25.4. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}_X^*$-torsor. Thus we get a map

$$\begin{matrix} \text{invertible sheaves on } (X, \mathcal{O}_X) \\ \text{up to isomorphism} \end{matrix} \longrightarrow \begin{matrix} \mathcal{O}_X^*\text{-torsors} \\ \text{up to isomorphism} \end{matrix}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 20.4.3 the right hand side is canonically bijective to $H^1(X, \mathcal{O}_X^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 20.4.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}_X$ is the neutral element in $\mathrm{Pic}(X)$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}_X^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}_X(U))/\mathcal{O}_X^*(U)$$

where the action of $f \in \mathcal{O}_X^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}_X$-modules by setting $(s, g) + (s', g') = (s, g + (s'/s)g')$ where s'/s is the local section f of $\mathcal{O}_X^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}_X$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}_X$-module whose associated $\mathcal{O}_X^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. □

20.7. Locality of cohomology

01E0 The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an open.

01E1 **Lemma 20.7.1.** Let X be a ringed space. Let $U \subset X$ be an open subspace.

(1) If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.
(2) For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}|_U)$.

Proof. Denote $j : U \to X$ the open immersion. Recall that the functor j^{-1} of restriction to U is a right adjoint to the functor $j_!$ of extension by 0, see Sheaves, Lemma 6.31.8. Moreover, $j_!$ is exact. Hence (1) follows from Homology, Lemma 12.29.1.

By definition $H^p(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\text{Mod}(\mathcal{O}_X)$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $\text{Mod}(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\Gamma(U, \mathcal{I}^\bullet|_U))$. Of course $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X. Hence the equality in (2). □

Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset V \subset X$ be open subsets. Then there is a canonical restriction mapping

01E2 (20.7.1.1) $$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}_X$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 20.11.4.

01E3 **Lemma 20.7.2.** Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subspace. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\xi|_{U_i} = 0$ for all $i \in I$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\text{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\text{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \text{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$.

By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. $\square$

01E4 **Lemma 20.7.3.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $R^if_*\mathcal{F}$ are the sheaves associated to the presheaves

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F})$$

with restriction mappings as in Equation (20.7.1.1). There is a similar statement for R^if_* applied to a bounded below complex $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^if_*\mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V)))}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. $\square$

01E5 **Lemma 20.7.4.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Denote $g : f^{-1}(V) \to V$ the restriction of f. Then we have

$$R^pg_*(\mathcal{F}|_{f^{-1}(V)}) = (R^pf_*\mathcal{F})|_V$$

There is a similar statement for the derived image $Rf_*\mathcal{F}^\bullet$ where $\mathcal{F}^\bullet$ is a bounded below complex of $\mathcal{O}_X$-modules.

Proof. First proof. Apply Lemmas 20.7.3 and 20.7.1 to see the displayed equality. Second proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and use that $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}^\bullet|_{f^{-1}(V)}$ is an injective resolution also. $\square$

03BA **Remark 20.7.5.** Here is a different approach to the proofs of Lemmas 20.7.2 and 20.7.3 above. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i_X : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{PMod}(\mathcal{O}_X)$ be the inclusion functor and let $\#$ be the sheafification functor. Recall that i_X is left exact and $\#$ is exact.

(1) First prove Lemma 20.11.4 below which says that the right derived functors of i_X are given by $R^pi_X\mathcal{F} = \underline{H}^p(\mathcal{F})$. Here is another proof: The equality is clear for $p = 0$. Both $(R^pi_X)_{p\geq 0}$ and $(\underline{H}^p)_{p\geq 0}$ are delta functors vanishing on injectives, hence both are universal, hence they are isomorphic. See Homology, Section 12.12.
(2) A restatement of Lemma 20.7.2 is that $(\underline{H}^p(\mathcal{F}))^\# = 0$, $p > 0$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. To see this is true, use that $^\#$ is exact so

$$(\underline{H}^p(\mathcal{F}))^\# = (R^pi_X\mathcal{F})^\# = R^p(\# \circ i_X)(\mathcal{F}) = 0$$

because $\# \circ i_X$ is the identity functor.

(3) Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The presheaf $V \mapsto H^p(f^{-1}V, \mathcal{F})$ is equal to $R^p(i_Y \circ f_*)\mathcal{F}$. You can prove this by noticing that both give universal delta functors as in the argument of (1) above. Hence Lemma 20.7.3 says that $R^p f_*\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$. Again using that $\#$ is exact a that $\# \circ i_Y$ is the identity functor we see that

$$R^p f_*\mathcal{F} = R^p(\# \circ i_Y \circ f_*)\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$$

as desired.

20.8. Mayer-Vietoris

01E9 Below will construct the Čech-to-cohomology spectral sequence, see Lemma 20.11.5. A special case of that spectral sequence is the Mayer-Vietoris long exact sequence. Since it is such a basic, useful and easy to understand variant of the spectral sequence we treat it here separately.

01EA **Lemma 20.8.1.** Let X be a ringed space. Let $U' \subset U \subset X$ be open subspaces. For any injective $\mathcal{O}_X$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.

Proof. Let $j : U \to X$ and $j' : U' \to X$ be the open immersions. Recall that $j_!\mathcal{O}_U$ is the extension by zero of $\mathcal{O}_U = \mathcal{O}_X|_U$, see Sheaves, Section 6.31. Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

see Sheaves, Lemma 6.31.8. Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover there is an obvious canonical map of $\mathcal{O}_X$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 4.3.5). By the description of the stalks of the sheaves $j'_!\mathcal{O}_{U'}$, $j_!\mathcal{O}_U$ we see that the displayed map above is injective (see lemma cited above). Hence if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then the map

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{I}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(j'_!\mathcal{O}_{U'}, \mathcal{I})$$

is surjective, see Homology, Lemma 12.27.2. Putting everything together we obtain the lemma. □

01EB **Lemma 20.8.2** (Mayer-Vietoris)**.** Let X be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact cohomology sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. The sheaf condition says that the kernel of $(1, -1) : \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is equal to the image of $\mathcal{F}(X)$ by the first map for any abelian sheaf $\mathcal{F}$. Lemma 20.8.1 above implies that the map $(1, -1) : \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is surjective whenever $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Hence if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, then we get a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

Taking cohomology gives the result (use Homology, Lemma 12.13.12). We omit the proof of the functoriality of the sequence. □

01EC Lemma 20.8.3 (Relative Mayer-Vietoris). Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U\cap V} : U \cap V \to Y$. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U\cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

Namely, for any open $W \subset Y$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(f^{-1}(W)) \to \mathcal{I}^n(U\cap f^{-1}(W))\oplus\mathcal{I}^n(V\cap f^{-1}(W)) \to \mathcal{I}^n(U\cap V\cap f^{-1}(W)) \to 0$$

was shown to be short exact in the proof of Lemma 20.8.2. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U\cap V}$ see Lemma 20.7.1. □

20.9. The Čech complex and Čech cohomology

01ED Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering, see Topology, Basic notion (13). As is customary we denote $U_{i_0\ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ for the $(p+1)$-fold intersection of members of $\mathcal{U}$. Let $\mathcal{F}$ be an abelian presheaf on X. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0,\ldots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0\ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0\ldots i_p}$ its value in $\mathcal{F}(U_{i_0\ldots i_p})$. Note that if $s \in \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F})$ and $i, j \in I$ then s_{ij} and s_{ji} are both elements of $\mathcal{F}(U_i\cap U_j)$ but there is no imposed relation between s_{ij} and s_{ji}. In other words, we are not working with alternating cochains (these will be defined in Section 20.23). We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

01EE (20.9.0.1)
$$d(s)_{i_0\ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0\ldots\hat{i}_j\ldots i_{p+1}}|_{U_{i_0\ldots i_{p+1}}}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

01EF Definition 20.9.1. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the Čech complex associated to $\mathcal{F}$ and the open covering $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the Čech cohomology groups associated to $\mathcal{F}$ and the covering $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

01EG Lemma 20.9.2. Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. The following are equivalent

(1) $\mathcal{F}$ is an abelian sheaf and

(2) for every open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ the natural map
$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$
is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. □

0G6S Lemma 20.9.3. Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. If $U_i = U$ for some $i \in I$, then the extended Čech complex
$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$
obtained by putting $\mathcal{F}(U)$ in degree -1 with differential given by the canonical map of $\mathcal{F}(U)$ into $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F})$ is homotopy equivalent to 0.

Proof. Fix an element $i \in I$ with $U = U_i$. Observe that $U_{i_0 \ldots i_p} = U_{i_0 \ldots \hat{i}_j \ldots i_p}$ if $i_j = i$. Let us define a homotopy
$$h : \prod\nolimits_{i_0 \ldots i_{p+1}} \mathcal{F}(U_{i_0 \ldots i_{p+1}}) \longrightarrow \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}(U_{i_0 \ldots i_p})$$
by the rule
$$h(s)_{i_0 \ldots i_p} = s_{i i_0 \ldots i_p}$$
In other words, $h : \prod_{i_0} \mathcal{F}(U_{i_0}) \to \mathcal{F}(U)$ is projection onto the factor $\mathcal{F}(U_i) = \mathcal{F}(U)$ and in general the map h equals the projection onto the factors $\mathcal{F}(U_{i i_1 \ldots i_{p+1}}) = \mathcal{F}(U_{i_1 \ldots i_{p+1}})$. We compute
$$\begin{aligned}(dh + hd)(s)_{i_0 \ldots i_p} &= \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i i_0 \ldots i_p} \\ &= \sum\nolimits_{j=0}^{p} (-1)^j s_{i i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum\nolimits_{j=0}^{p} (-1)^{j+1} s_{i i_0 \ldots \hat{i}_j \ldots i_p} \\ &= s_{i_0 \ldots i_p}\end{aligned}$$
This proves the identity map is homotopic to zero as desired. □

20.10. Čech cohomology as a functor on presheaves

01EH Warning: In this section we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. We have the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ just by thinking of $\mathcal{F}$ as a presheaf of abelian groups. However, each term $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ has a natural structure of a $\mathcal{O}_X(U)$-module and the differential is given by $\mathcal{O}_X(U)$-module maps. Moreover, it is clear that the construction
$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$
is functorial in $\mathcal{F}$. In fact, it is a functor

01EI (20.10.0.1)
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : \textit{PMod}(\mathcal{O}_X) \longrightarrow \text{Comp}^+(\text{Mod}_{\mathcal{O}_X(U)})$$

see Derived Categories, Definition 13.8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.13.9.

01EJ **Lemma 20.10.1.** The functor given by Equation (20.10.0.1) is an exact functor (see Homology, Lemma 12.7.2).

Proof. For any open $W \subset U$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $\text{PMod}(\mathcal{O}_X)$ to $\text{Mod}_{\mathcal{O}_X(U)}$. The terms $\check{C}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. $\square$

01EK **Lemma 20.10.2.** Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules (see Homology, Definition 12.12.1).

Proof. By Lemma 20.10.1 a short exact sequence of presheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathcal{O}_X(U)$-modules. Hence we can use Homology, Lemma 12.13.12 to get the boundary maps $\delta_{\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. $\square$

In the formulation of the following lemma we use the functor $j_{p!}$ of extension by 0 for presheaves of modules relative to an open immersion $j : U \to X$. See Sheaves, Section 6.31. For any open $W \subset X$ and any presheaf $\mathcal{G}$ of $\mathcal{O}_X|_U$-modules we have

$$(j_{p!}\mathcal{G})(W) = \begin{cases} \mathcal{G}(W) & \text{if } W \subset U \\ 0 & \text{else.} \end{cases}$$

Moreover, the functor $j_{p!}$ is a left adjoint to the restriction functor see Sheaves, Lemma 6.31.8. In particular we have the following formula

$$\text{Hom}_{\mathcal{O}_X}(j_{p!}\mathcal{O}_U, \mathcal{F}) = \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U).$$

Since the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor on the category of presheaves we conclude that the presheaf $j_{p!}\mathcal{O}_U$ is a projective object in the category $\text{PMod}(\mathcal{O}_X)$, see Homology, Lemma 12.28.2.

Note that if we are given open subsets $U \subset V \subset X$ with associated open immersions j_U, j_V, then we have a canonical map $(j_U)_{p!}\mathcal{O}_U \to (j_V)_{p!}\mathcal{O}_V$. It is the identity on sections over any open $W \subset U$ and 0 else. In terms of the identification $\text{Hom}_{\mathcal{O}_X}((j_U)_{p!}\mathcal{O}_U, (j_V)_{p!}\mathcal{O}_V) = (j_V)_{p!}\mathcal{O}_V(U) = \mathcal{O}_V(U)$ it corresponds to the element $1 \in \mathcal{O}_V(U)$.

01EL **Lemma 20.10.3.** Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. Denote $j_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to X$ the open immersion. Consider the chain complex $K(\mathcal{U})_\bullet$ of presheaves of $\mathcal{O}_X$-modules

$$\ldots \to \bigoplus_{i_0i_1i_2} (j_{i_0i_1i_2})_{p!}\mathcal{O}_{U_{i_0i_1i_2}} \to \bigoplus_{i_0i_1} (j_{i_0i_1})_{p!}\mathcal{O}_{U_{i_0i_1}} \to \bigoplus_{i_0} (j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$(j_{i_0\ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0\ldots i_{p+1}}} \longrightarrow (j_{i_0\ldots \hat{i}_j \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0\ldots \hat{i}_j \ldots i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\text{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{F}) = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \text{Ob}(\text{PMod}(\mathcal{O}_X))$.

Proof. We saw in the discussion just above the lemma that

$$\text{Hom}_{\mathcal{O}_X}((j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}, \mathcal{F}) = \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence we see that it is indeed the case that the direct sum

$$\bigoplus\nolimits_{i_0\ldots i_p} (j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$$

represents the functor

$$\mathcal{F} \longmapsto \prod\nolimits_{i_0\ldots i_p} \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence by Categories, Yoneda Lemma 4.3.5 we see that there is a complex $K(\mathcal{U})_\bullet$ with terms as given. It is a simple matter to see that the maps are as given in the lemma. □

01EM Lemma 20.10.4. Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. Let $\mathcal{O}_\mathcal{U} \subset \mathcal{O}_X$ be the image presheaf of the map $\bigoplus j_{p!}\mathcal{O}_{U_i} \to \mathcal{O}_X$. The chain complex $K(\mathcal{U})_\bullet$ of presheaves of Lemma 20.10.3 above has homology presheaves

$$H_i(K(\mathcal{U})_\bullet) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ \mathcal{O}_\mathcal{U} & \text{if} \quad i = 0 \end{cases}$$

Proof. Consider the extended complex $K^{ext}_\bullet$ one gets by putting $\mathcal{O}_\mathcal{U}$ in degree -1 with the obvious map $K(\mathcal{U})_0 = \bigoplus_{i_0}(j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to \mathcal{O}_\mathcal{U}$. It suffices to show that taking sections of this extended complex over any open $W \subset X$ leads to an acyclic complex. In fact, we claim that for every $W \subset X$ the complex $K^{ext}_\bullet(W)$ is homotopy equivalent to the zero complex. Write $I = I_1 \amalg I_2$ where $W \subset U_i$ if and only if $i \in I_1$.

If $I_1 = \emptyset$, then the complex $K^{ext}_\bullet(W) = 0$ so there is nothing to prove.

If $I_1 \neq \emptyset$, then $\mathcal{O}_\mathcal{U}(W) = \mathcal{O}_X(W)$ and

$$K^{ext}_p(W) = \bigoplus\nolimits_{i_0\ldots i_p \in I_1} \mathcal{O}_X(W).$$

This is true because of the simple description of the presheaves $(j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$. Moreover, the differential of the complex $K^{ext}_\bullet(W)$ is given by

$$d(s)_{i_0\ldots i_p} = \sum\nolimits_{j=0,\ldots,p+1} \sum\nolimits_{i\in I_1} (-1)^j s_{i_0\ldots i_{j-1} i i_j \ldots i_p}.$$

The sum is finite as the element s has finite support. Fix an element $i_{\text{fix}} \in I_1$. Define a map

$$h : K^{ext}_p(W) \longrightarrow K^{ext}_{p+1}(W)$$

by the rule

$$h(s)_{i_0\ldots i_{p+1}} = \begin{cases} 0 & \text{if} \quad i_0 \neq i_{\text{fix}} \\ s_{i_1\ldots i_{p+1}} & \text{if} \quad i_0 = i_{\text{fix}} \end{cases}$$

We will use the shorthand $h(s)_{i_0\ldots i_{p+1}} = (i_0 = i_{\text{fix}})s_{i_1\ldots i_p}$ for this. Then we compute

$$\begin{aligned} & (dh + hd)(s)_{i_0\ldots i_p} \\ = \quad & \sum_j \sum_{i\in I_1} (-1)^j h(s)_{i_0\ldots i_{j-1} i i_j \ldots i_p} + (i = i_0) d(s)_{i_1\ldots i_p} \\ = \quad & s_{i_0\ldots i_p} + \sum_{j\geq 1} \sum_{i\in I_1} (-1)^j (i_0 = i_{\text{fix}}) s_{i_1\ldots i_{j-1} i i_j \ldots i_p} + (i_0 = i_{\text{fix}}) d(s)_{i_1\ldots i_p} \end{aligned}$$

which is equal to $s_{i_0 \ldots i_p}$ as desired. $\square$

01EN Lemma 20.10.5. Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering of $U \subset X$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor

$$\check{H}^0(\mathcal{U}, -) : \text{PMod}(\mathcal{O}_X) \longrightarrow \text{Mod}_{\mathcal{O}_X(U)}.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the right derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(\text{PMod}(\mathcal{O}_X)) \longrightarrow D^+(\mathcal{O}_X(U))$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of presheaves of $\mathcal{O}_X$-modules has enough injectives, see Injectives, Proposition 19.8.5. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 13.20.

Let $\mathcal{I}$ be a injective presheaf of $\mathcal{O}_X$-modules. In this case the functor $\text{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact on $\text{PMod}(\mathcal{O}_X)$. By Lemma 20.10.3 we have

$$\text{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 20.10.4 we have that $K(\mathcal{U})_\bullet$ is quasi-isomorphic to $\mathcal{O}_\mathcal{U}[0]$. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 20.10.2) satisfies the assumptions of Homology, Lemma 12.12.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 13.20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 12.12.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any presheaf of $\mathcal{O}_X$-modules. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $\text{PMod}(\mathcal{O}_X)$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Consider the associated total complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$, see Homology, Definition 12.18.3. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 12.25.4. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 20.10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathcal{O}_X(U))$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. $\square$

20.11. Čech cohomology and cohomology

01EO

01EP Lemma 20.11.1. Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. An injective $\mathcal{O}_X$-module is also injective as an object in the category $\text{PMod}(\mathcal{O}_X)$ (for example since sheafification is an exact left adjoint to the inclusion functor, using Homology, Lemma 12.29.1). Hence we can apply Lemma 20.10.5 (or its proof) to see the result. $\square$

01EQ Lemma 20.11.2. Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. There is a transformation

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $\text{Mod}(\mathcal{O}_X) \to D^+(\mathcal{O}_X(U))$. In particular this provides canonical maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $\text{Mod}(\mathcal{O}_X)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 12.25.4 to see that α is a quasi-isomorphism. Namely, Lemma 20.11.1 implies that the qth row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathcal{O}_X(U))$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. $\square$

0B8R Lemma 20.11.3. Let X be a topological space. Let $\mathcal{H}$ be an abelian sheaf on X. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. The map

$$\check{H}^1(\mathcal{U}, \mathcal{H}) \longrightarrow H^1(X, \mathcal{H})$$

is injective and identifies $\check{H}^1(\mathcal{U}, \mathcal{H})$ via the bijection of Lemma 20.4.3 with the set of isomorphism classes of $\mathcal{H}$-torsors which restrict to trivial torsors over each U_i.

Proof. To see this we construct an inverse map. Namely, let $\mathcal{F}$ be a $\mathcal{H}$-torsor whose restriction to U_i is trivial. By Lemma 20.4.2 this means there exists a section $s_i \in \mathcal{F}(U_i)$. On $U_{i_0} \cap U_{i_1}$ there is a unique section $s_{i_0i_1}$ of $\mathcal{H}$ such that $s_{i_0i_1} \cdot s_{i_0}|_{U_{i_0} \cap U_{i_1}} = s_{i_1}|_{U_{i_0} \cap U_{i_1}}$. A computation shows that $s_{i_0i_1}$ is a Čech cocycle and that its class is well defined (i.e., does not depend on the choice of the sections s_i). The inverse maps the isomorphism class of $\mathcal{F}$ to the cohomology class of the cocycle $(s_{i_0i_1})$. We omit the verification that this map is indeed an inverse. $\square$

01ER Lemma 20.11.4. Let X be a ringed space. Consider the functor $i : \text{Mod}(\mathcal{O}_X) \to \text{PMod}(\mathcal{O}_X)$. It is a left exact functor with right derived functors given by

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 20.7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology presheaf of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^p(U) \to \mathcal{I}^{p+1}(U))}{\mathrm{Im}(\mathcal{I}^{p-1}(U) \to \mathcal{I}^p(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

01ES Lemma 20.11.5. Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 13.22.2) for the functors

$$i : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{PMod}(\mathcal{O}_X) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : \mathrm{PMod}(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(U)}.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 20.9.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 20.11.1. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $\mathrm{PMod}(\mathcal{O}_X)$ by Lemma 20.10.5. Putting everything together gives the lemma. □

01ET Lemma 20.11.6. Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that $H^i(U_{i_0 \ldots i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$ as $\mathcal{O}_X(U)$-modules.

Proof. We will use the spectral sequence of Lemma 20.11.5. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

01EU Lemma 20.11.7. Let X be a ringed space. Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose an open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0 i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. □

01EV Lemma 20.11.8. Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$$

for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open of X. Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any open $U \subset X$.

Proof. Let $\mathcal{F}$ be a sheaf satisfying the assumption of the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any open covering". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 20.11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 20.11.7 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Lemma 20.10.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\ & H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\ & \ldots & \ldots & \ldots \end{tikzcd}$$

for any open $U \subset X$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

01EW Lemma 20.11.9. (Variant of Lemma 20.11.8.) Let X be a ringed space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings Cov with the following properties:

(1) For every $\mathcal{U} \in \text{Cov}$ with $\mathcal{U} : U = \bigcup_{i \in I} U_i$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0 \ldots i_p} \in \mathcal{B}$.
(2) For every $U \in \mathcal{B}$ the open coverings of U occurring in Cov is a cofinal system of open coverings of U.
(3) For every $\mathcal{U} \in \text{Cov}$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \text{Cov}$". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 20.11.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \text{Cov}$. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 20.11.7 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in \text{Cov}$ we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in \text{Cov}$. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\
 & H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\
 & \ldots & \ldots & \ldots
\end{tikzcd}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

01EX Lemma 20.11.10. Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then

(1) $\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$ and any open covering $\mathcal{V} : V = \bigcup_{j \in J} V_j$ of Y.
(2) $H^p(V, f_*\mathcal{I}) = 0$ for all $p > 0$ and every open $V \subset Y$.

In other words, $f_*\mathcal{I}$ is right acyclic for $\Gamma(V, -)$ (see Derived Categories, Definition 13.15.3) for any $V \subset Y$ open.

Proof. Set $\mathcal{U} : f^{-1}(V) = \bigcup_{j \in J} f^{-1}(V_j)$. It is an open covering of X and

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This is true because

$$f_*\mathcal{I}(V_{j_0 \ldots j_p}) = \mathcal{I}(f^{-1}(V_{j_0 \ldots j_p})) = \mathcal{I}(f^{-1}(V_{j_0}) \cap \ldots \cap f^{-1}(V_{j_p})) = \mathcal{I}(U_{j_0 \ldots j_p}).$$

Thus the first statement of the lemma follows from Lemma 20.11.1. The second statement follows from the first and Lemma 20.11.8. □

The following lemma implies in particular that $f_* : \textit{Ab}(X) \to \textit{Ab}(Y)$ transforms injective abelian sheaves into injective abelian sheaves.

02N5 Lemma 20.11.11. Let $f : X \to Y$ be a morphism of ringed spaces. Assume f is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$.

Proof. In this case the functor f^* transforms injections into injections (Modules, Lemma 17.20.2). Hence the result follows from Homology, Lemma 12.29.1. $\square$

0D0A Lemma 20.11.12. Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$ let $\mathcal{F}_i$ be an $\mathcal{O}_X$-module. Let $U \subset X$ be open. The canonical map

$$H^p(U, \prod\nolimits_{i \in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i \in I} H^p(U, \mathcal{F}_i)$$

is an isomorphism for $p = 0$ and injective for $p = 1$.

Proof. The statement for $p = 0$ is true because the product of sheaves is equal to the product of the underlying presheaves, see Sheaves, Section 6.29. Proof for $p = 1$. Set $\mathcal{F} = \prod \mathcal{F}_i$. Let $\xi \in H^1(U, \mathcal{F})$ map to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 20.7.2, there exists an open covering $\mathcal{U} : U = \bigcup U_j$ such that $\xi|_{U_j} = 0$ for all j. By Lemma 20.11.3 this means ξ comes from an element $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$. Since the maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i (by Lemma 20.11.3), and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 12.32.1 we conclude that $\check{\xi} = 0$ as desired. $\square$

20.12. Flasque sheaves

09SV Here is the definition.

09SW Definition 20.12.1. Let X be a topological space. We say a presheaf of sets $\mathcal{F}$ is flasque or flabby if for every $U \subset V$ open in X the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

We will use this terminology also for abelian sheaves and sheaves of modules if X is a ringed space. Clearly it suffices to assume the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for every open $U \subset X$.

09SX Lemma 20.12.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective $\mathcal{O}_X$-module is flasque.

Proof. This is a reformulation of Lemma 20.8.1. $\square$

09SY Lemma 20.12.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Any flasque $\mathcal{O}_X$-module is acyclic for $R\Gamma(X, -)$ as well as $R\Gamma(U, -)$ for any open U of X.

Proof. We will prove this using Derived Categories, Lemma 13.15.6. Since every injective module is flasque we see that we can embed every $\mathcal{O}_X$-module into a flasque module, see Injectives, Lemma 19.4.1. Thus it suffices to show that given a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{F}$, $\mathcal{G}$ flasque, then $\mathcal{H}$ is flasque and the sequence remains short exact after taking sections on any open of X. In fact, the second statement implies the first. Thus, let $U \subset X$ be an open subspace. Let $s \in \mathcal{H}(U)$. We will show that we can lift s to a section of $\mathcal{G}$ over U. To do this consider the set T of pairs (V, t) where $V \subset U$ is open and $t \in \mathcal{G}(V)$ is a section mapping to $s|_V$ in $\mathcal{H}$. We put a partial ordering on T by setting $(V, t) \leq (V', t')$ if and only if $V \subset V'$ and $t'|_V = t$. If (V_α, t_α), $\alpha \in A$ is a totally ordered subset of T, then $V = \bigcup V_\alpha$ is open and there is a unique section $t \in \mathcal{G}(V)$ restricting to t_α over V_α by the sheaf condition on $\mathcal{G}$.

Thus by Zorn's lemma there exists a maximal element (V, t) in T. We will show that $V = U$ thereby finishing the proof. Namely, pick any $x \in U$. We can find a small open neighbourhood $W \subset U$ of x and $t' \in \mathcal{G}(W)$ mapping to $s|_W$ in $\mathcal{H}$. Then $t'|_{W \cap V} - t|_{W \cap V}$ maps to zero in $\mathcal{H}$, hence comes from some section $r' \in \mathcal{F}(W \cap V)$. Using that $\mathcal{F}$ is flasque we find a section $r \in \mathcal{F}(W)$ restricting to r' over $W \cap V$. Modifying t' by the image of r we may assume that t and t' restrict to the same section over $W \cap V$. By the sheaf condition of $\mathcal{G}$ we can find a section $\tilde{t}$ of $\mathcal{G}$ over $W \cup V$ restricting to t and t'. By maximality of (V, t) we see that $V \cup W = V$. Thus $x \in V$ and we are done. □

The following lemma does not hold for flasque presheaves.

09SZ Lemma 20.12.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{U} : U = \bigcup U_i$ be an open covering. If $\mathcal{F}$ is flasque, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.

Proof. The presheaves $\underline{H}^q(\mathcal{F})$ used in the statement of Lemma 20.11.5 are zero by Lemma 20.12.3. Hence $\check{H}^p(U, \mathcal{F}) = H^p(U, \mathcal{F}) = 0$ by Lemma 20.12.3 again. □

09T0 Lemma 20.12.5. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is flasque, then $R^p f_* \mathcal{F} = 0$ for $p > 0$.

Proof. Immediate from Lemma 20.7.3 and Lemma 20.12.3. □

The following lemma can be proved by an elementary induction argument for finite coverings, compare with the discussion of Čech cohomology in [Vak].

0A36 Lemma 20.12.6. Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume the restriction mappings $\mathcal{F}(U) \to \mathcal{F}(U')$ are surjective for U' an arbitrary union of opens of the form $U_{i_0 \ldots i_p}$. Then $\check{H}^p(\mathcal{U}, \mathcal{F})$ vanishes for $p > 0$.

Proof. Let Y be the set of nonempty subsets of I. We will use the letters $A, B, C, \ldots$ to denote elements of Y, i.e., nonempty subsets of I. For a finite nonempty subset $J \subset I$ let

$$V_J = \{A \in Y \mid J \subset A\}$$

This means that $V_{\{i\}} = \{A \in Y \mid i \in A\}$ and $V_J = \bigcap_{j \in J} V_{\{j\}}$. Then $V_J \subset V_K$ if and only if $J \supset K$. There is a unique topology on Y such that the collection of subsets V_J is a basis for the topology on Y. Any open is of the form

$$V = \bigcup\nolimits_{t \in T} V_{J_t}$$

for some family of finite subsets J_t. If $J_t \subset J_{t'}$ then we may remove $J_{t'}$ from the family without changing V. Thus we may assume there are no inclusions among the J_t. In this case the minimal elements of V are the sets $A = J_t$. Hence we can read off the family $(J_t)_{t \in T}$ from the open V.

We can completely understand open coverings in Y. First, because the elements $A \in Y$ are nonempty subsets of I we have

$$Y = \bigcup\nolimits_{i \in I} V_{\{i\}}$$

To understand other coverings, let V be as above and let $V_s \subset Y$ be an open corresponding to the family $(J_{s,t})_{t \in T_s}$. Then

$$V = \bigcup\nolimits_{s \in S} V_s$$

if and only if for each $t \in T$ there exists an $s \in S$ and $t_s \in T_s$ such that $J_t = J_{s,t_s}$. Namely, as the family $(J_t)_{t \in T}$ is minimal, the minimal element $A = J_t$ has to be in V_s for some s, hence $A \in V_{J_{t_s}}$ for some $t_s \in T_s$. But since A is also minimal in V_s we conclude that $J_{t_s} = J_t$.

Next we map the set of opens of Y to opens of X. Namely, we send Y to U, we use the rule

$$V_J \mapsto U_J = \bigcap\nolimits_{i \in J} U_i$$

on the opens V_J, and we extend it to arbitrary opens V by the rule

$$V = \bigcup\nolimits_{t \in T} V_{J_t} \mapsto \bigcup\nolimits_{t \in T} U_{J_t}$$

The classification of open coverings of Y given above shows that this rule transforms open coverings into open coverings. Thus we obtain an abelian sheaf $\mathcal{G}$ on Y by setting $\mathcal{G}(Y) = \mathcal{F}(U)$ and for $V = \bigcup_{t \in T} V_{J_t}$ setting

$$\mathcal{G}(V) = \mathcal{F}\left(\bigcup\nolimits_{t \in T} U_{J_t}\right)$$

and using the restriction maps of $\mathcal{F}$.

With these preliminaries out of the way we can prove our lemma as follows. We have an open covering $\mathcal{V} : Y = \bigcup_{i \in I} V_{\{i\}}$ of Y. By construction we have an equality

$$\check{C}^\bullet(\mathcal{V}, \mathcal{G}) = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$$

of Čech complexes. Since the sheaf $\mathcal{G}$ is flasque on Y (by our assumption on $\mathcal{F}$ in the statement of the lemma) the vanishing follows from Lemma 20.12.4. □

20.13. The Leray spectral sequence

01EY

01EZ **Lemma 20.13.1.** Let $f : X \to Y$ be a morphism of ringed spaces. There is a commutative diagram

$$\begin{tikzcd} D^+(X) \arrow[r, "R\Gamma(X,-)"'] \arrow[d, "Rf_*"'] & D^+(\mathcal{O}_X(X)) \arrow[d, "\text{restriction}"] \\ D^+(Y) \arrow[r, "R\Gamma(Y,-)"] & D^+(\mathcal{O}_Y(Y)) \end{tikzcd}$$

More generally for any $V \subset Y$ open and $U = f^{-1}(V)$ there is a commutative diagram

$$\begin{tikzcd} D^+(X) \arrow[r, "R\Gamma(U,-)"'] \arrow[d, "Rf_*"'] & D^+(\mathcal{O}_X(U)) \arrow[d, "\text{restriction}"] \\ D^+(Y) \arrow[r, "R\Gamma(V,-)"] & D^+(\mathcal{O}_Y(V)) \end{tikzcd}$$

See also Remark 20.13.2 for more explanation.

Proof. Let $\Gamma_{res} : \text{Mod}(\mathcal{O}_X) \to \text{Mod}_{\mathcal{O}_Y(Y)}$ be the functor which associates to an $\mathcal{O}_X$-module $\mathcal{F}$ the global sections of $\mathcal{F}$ viewed as an $\mathcal{O}_Y(Y)$-module via the map $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Let *restriction* : $\text{Mod}_{\mathcal{O}_X(X)} \to \text{Mod}_{\mathcal{O}_Y(Y)}$ be the restriction functor induced by $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Note that *restriction* is exact so that its right derived functor is computed by simply applying the restriction functor, see Derived Categories, Lemma 13.16.9. It is clear that

$$\Gamma_{res} = restriction \circ \Gamma(X, -) = \Gamma(Y, -) \circ f_*$$

We claim that Derived Categories, Lemma 13.22.1 applies to both compositions. For the first this is clear by our remarks above. For the second, it follows from Lemma 20.11.10 which implies that injective $\mathcal{O}_X$-modules are mapped to $\Gamma(Y, -)$-acyclic sheaves on Y. □

01F0 **Remark 20.13.2.** Here is a down-to-earth explanation of the meaning of Lemma 20.13.1. It says that given $f : X \to Y$ and $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ and given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ we have

$$\begin{matrix} R\Gamma(X, \mathcal{F}) & \text{is represented by} & \Gamma(X, \mathcal{I}^\bullet) \\ Rf_*\mathcal{F} & \text{is represented by} & f_*\mathcal{I}^\bullet \\ R\Gamma(Y, Rf_*\mathcal{F}) & \text{is represented by} & \Gamma(Y, f_*\mathcal{I}^\bullet) \end{matrix}$$

the last fact coming from Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and Lemma 20.11.10. Finally, it combines this with the trivial observation that

$$\Gamma(X, \mathcal{I}^\bullet) = \Gamma(Y, f_*\mathcal{I}^\bullet).$$

to arrive at the commutativity of the diagram of the lemma.

01F1 **Lemma 20.13.3.** Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.

(1) The cohomology groups $H^i(U, \mathcal{F})$ for $U \subset X$ open of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.
(2) Let $f : X \to Y$ be a morphism of ringed spaces. The higher direct images $R^if_*\mathcal{F}$ of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.

There are similar statements in the case of bounded below complexes of $\mathcal{O}_X$-modules.

Proof. Consider the morphism of ringed spaces $(X, \mathcal{O}_X) \to (X, \underline{\mathbf{Z}}_X)$ given by the identity on the underlying topological space and by the unique map of sheaves of rings $\underline{\mathbf{Z}}_X \to \mathcal{O}_X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{ab}$ the same sheaf seen as an $\underline{\mathbf{Z}}_X$-module, i.e., seen as a sheaf of abelian groups. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. By Remark 20.13.2 we see that $\Gamma(X, \mathcal{I}^\bullet)$ computes both $R\Gamma(X, \mathcal{F})$ and $R\Gamma(X, \mathcal{F}_{ab})$. This proves (1).

To prove (2) we use (1) and Lemma 20.7.3. The result follows immediately. □

01F2 **Lemma 20.13.4** (Leray spectral sequence)**.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There is a spectral sequence

$$E_2^{p,q} = H^p(Y, R^qf_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 13.22.2 coming from the composition of functors $\Gamma_{res} = \Gamma(Y, -) \circ f_*$ where Γ_{res} is as in the proof of Lemma 20.13.1. To see that the assumptions of Derived Categories, Lemma 13.22.2 are satisfied, see the proof of Lemma 20.13.1 or Remark 20.13.2. $\square$

01F3 Remark 20.13.5. The Leray spectral sequence, the way we proved it in Lemma 20.13.4 is a spectral sequence of $\Gamma(Y, \mathcal{O}_Y)$-modules. However, it is quite easy to see that it is in fact a spectral sequence of $\Gamma(X, \mathcal{O}_X)$-modules. For example f gives rise to a morphism of ringed spaces $f' : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$. By Lemma 20.13.3 the terms $E_r^{p,q}$ of the Leray spectral sequence for an $\mathcal{O}_X$-module $\mathcal{F}$ and f are identical with those for $\mathcal{F}$ and f' at least for $r \geq 2$. Namely, they both agree with the terms of the Leray spectral sequence for $\mathcal{F}$ as an abelian sheaf. And since $(f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$ we see the result. It is often the case that the Leray spectral sequence carries additional structure.

01F4 Lemma 20.13.6. Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.

(1) If $R^q f_*\mathcal{F} = 0$ for $q > 0$, then $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$ for all p.
(2) If $H^p(Y, R^q f_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(X, \mathcal{F}) = H^0(Y, R^q f_*\mathcal{F})$ for all q.

Proof. These are two simple conditions that force the Leray spectral sequence to degenerate at E_2. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. $\square$

01F5 Lemma 20.13.7. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. In this case $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors from $D^+(X) \to D^+(Z)$.

Proof. We are going to apply Derived Categories, Lemma 13.22.1. It is clear that $g_* \circ f_* = (g \circ f)_*$, see Sheaves, Lemma 6.21.2. It remains to show that $f_*\mathcal{I}$ is g_*-acyclic. This follows from Lemma 20.11.10 and the description of the higher direct images $R^i g_*$ in Lemma 20.7.3. $\square$

01F6 Lemma 20.13.8 (Relative Leray spectral sequence). Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a spectral sequence with

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_*\mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_X$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors and follows from Lemma 20.13.7 and Derived Categories, Lemma 13.22.2. $\square$

20.14. Functoriality of cohomology

01F7

01F8 Lemma 20.14.1. Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}^\bullet$, resp. $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_Y$-modules, resp. $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet$ be a morphism of complexes. There is a canonical morphism

$$\mathcal{G}^\bullet \longrightarrow Rf_*(\mathcal{F}^\bullet)$$

in $D^+(Y)$. Moreover this construction is functorial in the triple $(\mathcal{G}^\bullet, \mathcal{F}^\bullet, \varphi)$.

Proof. Choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. By definition $Rf_*(\mathcal{F}^\bullet)$ is represented by $f_*\mathcal{I}^\bullet$ in $K^+(\mathcal{O}_Y)$. The composition

$$\mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet \to f_*\mathcal{I}^\bullet$$

is a morphism in $K^+(Y)$ which turns into the morphism of the lemma upon applying the localization functor $j_Y : K^+(Y) \to D^+(Y)$. □

Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Recall that an f-map φ from $\mathcal{G}$ to $\mathcal{F}$ is a map $\varphi : \mathcal{G} \to f_*\mathcal{F}$, or what is the same thing, a map $\varphi : f^*\mathcal{G} \to \mathcal{F}$. See Sheaves, Definition 6.21.7. Such an f-map gives rise to a morphism of complexes

01F9 (20.14.1.1) $$\varphi : R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D^+(\mathcal{O}_Y(Y))$. Namely, we use the morphism $\mathcal{G} \to Rf_*\mathcal{F}$ in $D^+(Y)$ of Lemma 20.14.1, and we apply $R\Gamma(Y, -)$. By Lemma 20.13.1 we see that $R\Gamma(X, \mathcal{F}) = R\Gamma(Y, Rf_*\mathcal{F})$ and we get the displayed arrow. We spell this out completely in Remark 20.14.2 below. In particular it gives rise to maps on cohomology

01FA (20.14.1.2) $$\varphi : H^i(Y, \mathcal{G}) \longrightarrow H^i(X, \mathcal{F}).$$

01FB Remark 20.14.2. Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let φ be an f-map from $\mathcal{G}$ to $\mathcal{F}$. Choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by a complex of injective $\mathcal{O}_X$-modules. Choose resolutions $\mathcal{G} \to \mathcal{J}^\bullet$ and $f_*\mathcal{I}^\bullet \to (\mathcal{J}')^\bullet$ by complexes of injective $\mathcal{O}_Y$-modules. By Derived Categories, Lemma 13.18.6 there exists a map of complexes β such that the diagram

01FC (20.14.2.1) $$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & f_*\mathcal{F} \arrow[r] & f_*\mathcal{I}^\bullet \arrow[d] \\ \mathcal{J}^\bullet \arrow[rr, "\beta"] & & (\mathcal{J}')^\bullet \end{tikzcd}$$

commutes. Applying global section functors we see that we get a diagram

$$\begin{tikzcd} & \Gamma(Y, f_*\mathcal{I}^\bullet) \arrow[r, equal] \arrow[d, "qis"'] & \Gamma(X, \mathcal{I}^\bullet) \\ \Gamma(Y, \mathcal{J}^\bullet) \arrow[r, "\beta"] & \Gamma(Y, (\mathcal{J}')^\bullet) & \end{tikzcd}$$

The complex on the bottom left represents $R\Gamma(Y, \mathcal{G})$ and the complex on the top right represents $R\Gamma(X, \mathcal{F})$. The vertical arrow is a quasi-isomorphism by Lemma 20.13.1 which becomes invertible after applying the localization functor $K^+(\mathcal{O}_Y(Y)) \to D^+(\mathcal{O}_Y(Y))$. The arrow (20.14.1.1) is given by the composition of the horizontal map by the inverse of the vertical map.

20.15. Refinements and Čech cohomology

09UY Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : X = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $\mathcal{V}$. Choose a map $c : I \to J$ such that $U_i \subset V_{c(i)}$ for all $i \in I$. This induces a map of Čech complexes

$$\gamma : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}), \quad (\xi_{j_0 \ldots j_p}) \longmapsto (\xi_{c(i_0) \ldots c(i_p)}|_{U_{i_0 \ldots i_p}})$$

functorial in the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Suppose that $c' : I \to J$ is a second map such that $U_i \subset V_{c'(i)}$ for all $i \in I$. Then the corresponding maps γ and γ' are homotopic. Namely, $\gamma - \gamma' = \text{d} \circ h + h \circ \text{d}$ with $h : \check{\mathcal{C}}^{p+1}(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ given by the rule

$$h(\alpha)_{i_0 \ldots i_p} = \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{c(i_0) \ldots c(i_a) c'(i_a) \ldots c'(i_p)}$$

We omit the computation showing this works; please see the discussion following (20.25.0.2) for the proof in a more general case. In particular, the map on Čech cohomology groups is independent of the choice of c. Moreover, it is clear that if $\mathcal{W} : X = \bigcup_{k \in K} W_k$ is a third open covering and $\mathcal{V}$ is a refinement of $\mathcal{W}$, then the composition of the maps

$$\check{\mathcal{C}}^\bullet(\mathcal{W}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

associated to maps $I \to J$ and $J \to K$ is the map associated to the composition $I \to K$. In particular, we can define the Čech cohomology groups

$$\check{H}^p(X, \mathcal{F}) = \text{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

where the colimit is over all open coverings of X preordered by refinement.

It turns out that the maps γ defined above are compatible with the map to cohomology, in other words, the composition

$$\check{H}^p(\mathcal{V}, \mathcal{F}) \to \check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{Lemma 20.11.2}} H^p(X, \mathcal{F})$$

is the canonical map from the first group to cohomology of Lemma 20.11.2. In the lemma below we will prove this in a slightly more general setting. A consequence is that we obtain a well defined map

09UZ (20.15.0.1)
$$\check{H}^p(X, \mathcal{F}) = \text{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

from Čech cohomology to cohomology.

01FD **Lemma 20.15.1.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $\varphi : f^*\mathcal{G} \to \mathcal{F}$ be an f-map from an $\mathcal{O}_Y$-module $\mathcal{G}$ to an $\mathcal{O}_X$-module $\mathcal{F}$. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $f^{-1}\mathcal{V} : X = \bigcup_{j \in J} f^{-1}(V_j)$. In this case there exists a commutative diagram

$$\begin{tikzcd}
\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] & R\Gamma(X, \mathcal{F}) \\
\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \arrow[r] \arrow[u, "\gamma"] & R\Gamma(Y, \mathcal{G}) \arrow[u]
\end{tikzcd}$$

in $D^+(\mathcal{O}_X(X))$ with horizontal arrows given by Lemma 20.11.2 and right vertical arrow by (20.14.1.1). In particular we get commutative diagrams of cohomology groups

$$\begin{tikzcd}
\check{H}^p(\mathcal{U}, \mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}) \\
\check{H}^p(\mathcal{V}, \mathcal{G}) \arrow[r] \arrow[u, "\gamma"] & H^p(Y, \mathcal{G}) \arrow[u]
\end{tikzcd}$$

where the right vertical arrow is (20.14.1.2)

Proof. We first define the left vertical arrow. Namely, choose a map $c : I \to J$ such that $U_i \subset f^{-1}(V_{c(i)})$ for all $i \in I$. In degree p we define the map by the rule

$$\gamma(s)_{i_0\ldots i_p} = \varphi(s)_{c(i_0)\ldots c(i_p)}$$

This makes sense because φ does indeed induce maps $\mathcal{G}(V_{c(i_0)\ldots c(i_p)}) \to \mathcal{F}(U_{i_0\ldots i_p})$ by assumption. It is also clear that this defines a morphism of complexes. Choose injective resolutions $\mathcal{F} \to \mathcal{I}^\bullet$ on X and $\mathcal{G} \to J^\bullet$ on Y. According to the proof of Lemma 20.11.2 we introduce the double complexes $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ with terms

$$B^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, \mathcal{J}^q) \quad \text{and} \quad A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

As in Remark 20.14.2 above we also choose an injective resolution $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ on Y and a morphism of complexes $\beta : \mathcal{J} \to (\mathcal{J}')^\bullet$ making (20.14.2.1) commutes. We introduce some more double complexes, namely $(B')^{\bullet,\bullet}$ and $(B'')^{\bullet,\bullet}$ with

$$(B')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, (\mathcal{J}')^q) \quad \text{and} \quad (B'')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V}, f_*\mathcal{I}^q).$$

Note that there is an f-map of complexes from $f_*\mathcal{I}^\bullet$ to $\mathcal{I}^\bullet$. Hence it is clear that the same rule as above defines a morphism of double complexes

$$\gamma : (B'')^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}.$$

Consider the diagram of complexes

$$\begin{tikzcd}
\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] & \mathrm{Tot}(A^{\bullet,\bullet}) & & & \Gamma(X, \mathcal{I}^\bullet) \arrow[llll, "qis"'] \\
\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \arrow[u, "\gamma"] \arrow[r] & \mathrm{Tot}(B^{\bullet,\bullet}) \arrow[r, "\beta"] & \mathrm{Tot}((B')^{\bullet,\bullet}) & \mathrm{Tot}((B'')^{\bullet,\bullet}) \arrow[l] \arrow[ull, "s\gamma"'] & \\
& \Gamma(Y, \mathcal{J}^\bullet) \arrow[u, "qis"] \arrow[r, "\beta"] & \Gamma(Y, (\mathcal{J}')^\bullet) \arrow[u] & \Gamma(Y, f_*\mathcal{I}^\bullet) \arrow[l, "qis"'] \arrow[u] \arrow[uur, equal] &
\end{tikzcd}$$

The two horizontal arrows with targets $\mathrm{Tot}(A^{\bullet,\bullet})$ and $\mathrm{Tot}(B^{\bullet,\bullet})$ are the ones explained in Lemma 20.11.2. The left upper shape (a pentagon) is commutative simply because (20.14.2.1) is commutative. The two lower squares are trivially commutative. It is also immediate from the definitions that the right upper shape (a square) is commutative. The result of the lemma now follows from the definitions and the fact that going around the diagram on the outer sides from $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$ to $\Gamma(X, \mathcal{I}^\bullet)$ either on top or on bottom is the same (where you have to invert any quasi-isomorphisms along the way). □

20.16. Cohomology on Hausdorff quasi-compact spaces

09V0 For such a space Čech cohomology agrees with cohomology.

09V1 Lemma 20.16.1. Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf. Then the map $\check{H}^1(X, \mathcal{F}) \to H^1(X, \mathcal{F})$ defined in (20.15.0.1) is an isomorphism.

Proof. Let $\mathcal{U}$ be an open covering of X. By Lemma 20.11.5 there is an exact sequence

$$0 \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$$

Thus the map is injective. To show surjectivity it suffices to show that any element of $\check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$ maps to zero after replacing $\mathcal{U}$ by a refinement. This is immediate from the definitions and the fact that $\underline{H}^1(\mathcal{F})$ is a presheaf of abelian groups whose sheafification is zero by locality of cohomology, see Lemma 20.7.2. $\square$

09V2 **Lemma 20.16.2.** Let X be a Hausdorff and quasi-compact topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Then the map $\check{H}^n(X, \mathcal{F}) \to H^n(X, \mathcal{F})$ defined in (20.15.0.1) is an isomorphism for all n.

Proof. We already know that $\check{H}^n(X, -) \to H^n(X, -)$ is an isomorphism of functors for $n = 0, 1$, see Lemma 20.16.1. The functors $H^n(X, -)$ form a universal δ-functor, see Derived Categories, Lemma 13.20.4. If we show that $\check{H}^n(X, -)$ forms a universal δ-functor and that $\check{H}^n(X, -) \to H^n(X, -)$ is compatible with boundary maps, then the map will automatically be an isomorphism by uniqueness of universal δ-functors, see Homology, Lemma 12.12.5.

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of abelian sheaves on X. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. This gives a complex of complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

which is in general not exact on the right. The sequence defines the maps

$$\check{H}^n(\mathcal{U}, \mathcal{F}) \to \check{H}^n(\mathcal{U}, \mathcal{G}) \to \check{H}^n(\mathcal{U}, \mathcal{H})$$

but isn't good enough to define a boundary operator $\delta : \check{H}^n(\mathcal{U}, \mathcal{H}) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F})$. Indeed such a thing will not exist in general. However, given an element $\overline{h} \in \check{H}^n(\mathcal{U}, \mathcal{H})$ which is the cohomology class of a cocycle $h = (h_{i_0 \ldots i_n})$ we can choose open coverings

$$U_{i_0 \ldots i_n} = \bigcup W_{i_0 \ldots i_n, k}$$

such that $h_{i_0 \ldots i_n}|_{W_{i_0 \ldots i_n, k}}$ lifts to a section of $\mathcal{G}$ over $W_{i_0 \ldots i_n, k}$. By Topology, Lemma 5.13.5 (this is where we use the assumption that X is hausdorff and quasi-compact) we can choose an open covering $\mathcal{V} : X = \bigcup_{j \in J} V_j$ and $\alpha : J \to I$ such that $V_j \subset U_{\alpha(j)}$ (it is a refinement) and such that for all $j_0, \ldots, j_n \in J$ there is a k such that $V_{j_0 \ldots j_n} \subset W_{\alpha(j_0) \ldots \alpha(j_n), k}$. We obtain maps of complexes

$$\begin{tikzcd}
0 \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{H}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{H}) \arrow[r] & 0
\end{tikzcd}$$

In fact, the vertical arrows are the maps of complexes used to define the transition maps between the Čech cohomology groups. Our choice of refinement shows that we may choose

$$g_{j_0 \ldots j_n} \in \mathcal{G}(V_{j_0 \ldots j_n}), \quad g_{j_0 \ldots j_n} \longmapsto h_{\alpha(j_0) \ldots \alpha(j_n)}|_{V_{j_0 \ldots j_n}}$$

The cochain $g = (g_{j_0 \ldots j_n})$ is not a cocycle in general but we know that its Čech boundary $\mathrm{d}(g)$ maps to zero in $\check{\mathcal{C}}^{n+1}(\mathcal{V}, \mathcal{H})$ (by the commutative diagram above and the fact that h is a cocycle). Hence $\mathrm{d}(g)$ is a cocycle in $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$. This allows us to define

$$\delta(\overline{h}) = \text{class of } \mathrm{d}(g) \text{ in } \check{H}^{n+1}(\mathcal{V}, \mathcal{F})$$

Now, given an element $\xi \in \check{H}^n(X, \mathcal{G})$ we choose an open covering $\mathcal{U}$ and an element $\overline{h} \in \check{H}^n(\mathcal{U}, \mathcal{G})$ mapping to ξ in the colimit defining Čech cohomology. Then we choose $\mathcal{V}$ and g as above and set $\delta(\xi)$ equal to the image of $\delta(\overline{h})$ in $\check{H}^n(X, \mathcal{F})$. At this point a lot of properties have to be checked, all of which are straightforward. For example, we need to check that our construction is independent of the choice of $\mathcal{U}, \overline{h}, \mathcal{V}, \alpha : J \to I, g$. The class of $\mathrm{d}(g)$ is independent of the choice of the lifts $g_{i_0 \ldots i_n}$ because the difference will be a coboundary. Independence of α holds[1] because a different choice of α determines homotopic vertical maps of complexes in the diagram above, see Section 20.15. For the other choices we use that given a finite collection of coverings of X we can always find a covering refining all of them. We also need to check additivity which is shown in the same manner. Finally, we need to check that the maps $\check{H}^n(X, -) \to H^n(X, -)$ are compatible with boundary maps. To do this we choose injective resolutions

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}_1^\bullet \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_3^\bullet \arrow[r] & 0 \end{tikzcd}$$

as in Derived Categories, Lemma 13.18.9. This will give a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_1^\bullet)) \arrow[r] & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_2^\bullet)) \arrow[r] & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_3^\bullet)) \arrow[r] & 0 \end{tikzcd}$$

Here $\mathcal{U}$ is an open covering as above and the vertical maps are those used to define the maps $\check{H}^n(\mathcal{U}, -) \to H^n(X, -)$, see Lemma 20.11.2. The bottom complex is exact as the sequence of complexes of injectives is termwise split exact. Hence the boundary map in cohomology is computed by the usual procedure for this lower exact sequence, see Homology, Lemma 12.13.12. The same will be true after passing to the refinement $\mathcal{V}$ where the boundary map for Čech cohomology was defined. Hence the boundary maps agree because they use the same construction (whenever the first one is defined on an element in Čech cohomology on a given covering). This finishes our discussion of the construction of the structure of a δ-functor on Čech cohomology and why this structure is compatible with the given δ-functor structure on usual cohomology.

Finally, we may apply Lemma 20.11.1 to see that higher Čech cohomology is trivial on injective sheaves. Hence we see that Čech cohomology is a universal δ-functor by Homology, Lemma 12.12.4. □

09V3 **Lemma 20.16.3.** Let X be a topological space. Let $Z \subset X$ be a quasi-compact subset such that any two points of Z have disjoint open neighbourhoods in X. For every abelian sheaf $\mathcal{F}$ on X the canonical map [AGV71, Expose V bis, 4.1.3]

$$\mathrm{colim}\, H^p(U, \mathcal{F}) \longrightarrow H^p(Z, \mathcal{F}|_Z)$$

where the colimit is over open neighbourhoods U of Z in X is an isomorphism.

[1]This is an important check because the nonuniqueness of α is the only thing preventing us from taking the colimit of Čech complexes over all open coverings of X to get a short exact sequence of complexes computing Čech cohomology.

Proof. We first prove this for $p = 0$. Injectivity follows from the definition of $\mathcal{F}|_Z$ and holds in general (for any subset of any topological space X). Next, suppose that $s \in H^0(Z, \mathcal{F}|_Z)$. Then we can find opens $U_i \subset X$ such that $Z \subset \bigcup U_i$ and such that $s|_{Z \cap U_i}$ comes from $s_i \in \mathcal{F}(U_i)$. It follows that there exist opens $W_{ij} \subset U_i \cap U_j$ with $W_{ij} \cap Z = U_i \cap U_j \cap Z$ such that $s_i|_{W_{ij}} = s_j|_{W_{ij}}$. Applying Topology, Lemma 5.13.7 we find opens V_i of X such that $V_i \subset U_i$ and such that $V_i \cap V_j \subset W_{ij}$. Hence we see that $s_i|_{V_i}$ glue to a section of $\mathcal{F}$ over the open neighbourhood $\bigcup V_i$ of Z.

To finish the proof, it suffices to show that if $\mathcal{I}$ is an injective abelian sheaf on X, then $H^p(Z, \mathcal{I}|_Z) = 0$ for $p > 0$. This follows using short exact sequences and dimension shifting; details omitted. Thus, suppose $\overline{\xi}$ is an element of $H^p(Z, \mathcal{I}|_Z)$ for some $p > 0$. By Lemma 20.16.2 the element $\overline{\xi}$ comes from $\check{H}^p(\mathcal{V}, \mathcal{I}|_Z)$ for some open covering $\mathcal{V} : Z = \bigcup V_i$ of Z. Say $\overline{\xi}$ is the image of the class of a cocycle $\xi = (\xi_{i_0 \ldots i_p})$ in $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{I}|_Z)$.

Let $\mathcal{I}' \subset \mathcal{I}|_Z$ be the subpresheaf defined by the rule

$$\mathcal{I}'(V) = \{s \in \mathcal{I}|_Z(V) \mid \exists (U, t),\ U \subset X \text{ open},\ t \in \mathcal{I}(U),\ V = Z \cap U,\ s = t|_{Z \cap U}\}$$

Then $\mathcal{I}|_Z$ is the sheafification of $\mathcal{I}'$. Thus for every $(p+1)$-tuple $i_0 \ldots i_p$ we can find an open covering $V_{i_0 \ldots i_p} = \bigcup W_{i_0 \ldots i_p, k}$ such that $\xi_{i_0 \ldots i_p}|_{W_{i_0 \ldots i_p, k}}$ is a section of $\mathcal{I}'$. Applying Topology, Lemma 5.13.5 we may after refining $\mathcal{V}$ assume that each $\xi_{i_0 \ldots i_p}$ is a section of the presheaf $\mathcal{I}'$.

Write $V_i = Z \cap U_i$ for some opens $U_i \subset X$. Since $\mathcal{I}$ is flasque (Lemma 20.12.2) and since $\xi_{i_0 \ldots i_p}$ is a section of $\mathcal{I}'$ for every $(p+1)$-tuple $i_0 \ldots i_p$ we can choose a section $s_{i_0 \ldots i_p} \in \mathcal{I}(U_{i_0 \ldots i_p})$ which restricts to $\xi_{i_0 \ldots i_p}$ on $V_{i_0 \ldots i_p} = Z \cap U_{i_0 \ldots i_p}$. (This appeal to injectives being flasque can be avoided by an additional application of Topology, Lemma 5.13.7.) Let $s = (s_{i_0 \ldots i_p})$ be the corresponding cochain for the open covering $U = \bigcup U_i$. Since $\mathrm{d}(\xi) = 0$ we see that the sections $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}$ restrict to zero on $Z \cap U_{i_0 \ldots i_{p+1}}$. Hence, by the initial remarks of the proof, there exists open subsets $W_{i_0 \ldots i_{p+1}} \subset U_{i_0 \ldots i_{p+1}}$ with $Z \cap W_{i_0 \ldots i_{p+1}} = Z \cap U_{i_0 \ldots i_{p+1}}$ such that $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}|_{W_{i_0 \ldots i_{p+1}}} = 0$. By Topology, Lemma 5.13.7 we can find $U'_i \subset U_i$ such that $Z \subset \bigcup U'_i$ and such that $U'_{i_0 \ldots i_{p+1}} \subset W_{i_0 \ldots i_{p+1}}$. Then $s' = (s'_{i_0 \ldots i_p})$ with $s'_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}|_{U'_{i_0 \ldots i_p}}$ is a cocycle for $\mathcal{I}$ for the open covering $U' = \bigcup U'_i$ of an open neighbourhood of Z. Since $\mathcal{I}$ has trivial higher Čech cohomology groups (Lemma 20.11.1) we conclude that s' is a coboundary. It follows that the image of ξ in the Čech complex for the open covering $Z = \bigcup Z \cap U'_i$ is a coboundary and we are done. $\square$

20.17. The base change map

02N6 We will need to know how to construct the base change map in some cases. Since we have not yet discussed derived pullback we only discuss this in the case of a base change by a flat morphism of ringed spaces. Before we state the result, let us discuss flat pullback on the derived category. Namely, suppose that $g : X \to Y$ is a flat morphism of ringed spaces. By Modules, Lemma 17.20.2 the functor $g^* : \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$ is exact. Hence it has a derived functor

$$g^* : D^+(Y) \to D^+(X)$$

which is computed by simply pulling back an representative of a given object in $D^+(Y)$, see Derived Categories, Lemma 13.16.9. Hence as indicated we indicate this functor by g^* rather than Lg^*.

02N7 Lemma 20.17.1. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a commutative diagram of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*(g')^*\mathcal{F}^\bullet$$

in $D^+(S')$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 20.11.11 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 13.18.6 and 13.18.7 the arrow β in the diagram

$$\begin{tikzcd} (g')_*(g')^*\mathcal{F}^\bullet \arrow[r] & (g')_*\mathcal{J}^\bullet \\ \mathcal{F}^\bullet \arrow[r] \arrow[u, "adjunction"] & \mathcal{I}^\bullet \arrow[u, "\beta"'] \end{tikzcd}$$

exists and is unique up to homotopy. Pushing down to S we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. □

02N8 Remark 20.17.2. The "correct" version of the base change map is the map

$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet.$$

The construction of this map involves unbounded complexes, see Remark 20.28.3.

20.18. Proper base change in topology

09V4 In this section we prove a very general version of the proper base change theorem in topology. It tells us that the stalks of the higher direct images R^pf_* can be computed on the fibre.

09V5 Lemma 20.18.1. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $y \in Y$. Assume that

(1) f is closed,
(2) f is separated, and
(3) $f^{-1}(y)$ is quasi-compact.

Then for E in $D^+(\mathcal{O}_X)$ we have $(Rf_*E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$ in $D^+(\mathcal{O}_{Y,y})$.

Proof. The base change map of Lemma 20.17.1 gives a canonical map $(Rf_*E)_y \to R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$. To prove this map is an isomorphism, we represent E by a bounded below complex of injectives $\mathcal{I}^\bullet$. Set $Z = f^{-1}(\{y\})$. The assumptions of Lemma 20.16.3 are satisfied, see Topology, Lemma 5.4.2. Hence the restrictions $\mathcal{I}^n|_Z$ are acyclic for $\Gamma(Z, -)$. Thus $R\Gamma(Z, E|_Z)$ is represented by the complex $\Gamma(Z, \mathcal{I}^\bullet|_Z)$, see Derived Categories, Lemma 13.16.7. In other words, we have to show the map

$$\mathrm{colim}_V \mathcal{I}^\bullet(f^{-1}(V)) \longrightarrow \Gamma(Z, \mathcal{I}^\bullet|_Z)$$

is an isomorphism. Using Lemma 20.16.3 we see that it suffices to show that the collection of open neighbourhoods $f^{-1}(V)$ of $Z = f^{-1}(\{y\})$ is cofinal in the system of all open neighbourhoods. If $f^{-1}(\{y\}) \subset U$ is an open neighbourhood, then as f is closed the set $V = Y \setminus f(X \setminus U)$ is an open neighbourhood of y with $f^{-1}(V) \subset U$. This proves the lemma. □

09V6 **Theorem 20.18.2** (Proper base change). Consider a cartesian square of topological spaces [AGV71, Expose V bis, 4.1.1]

$$\begin{tikzcd} X' = Y' \times_Y X \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Assume that f is proper. Let E be an object of $D^+(X)$. Then the base change map

$$g^{-1}Rf_*E \longrightarrow Rf'_*(g')^{-1}E$$

of Lemma 20.17.1 is an isomorphism in $D^+(Y')$.

Proof. Let $y' \in Y'$ be a point with image $y \in Y$. It suffices to show that the base change map induces an isomorphism on stalks at y'. As f is proper it follows that f' is proper, the fibres of f and f' are quasi-compact and f and f' are closed, see Topology, Theorem 5.17.5 and Lemma 5.4.4. Thus we can apply Lemma 20.18.1 twice to see that

$$(Rf'_*(g')^{-1}E)_{y'} = R\Gamma((f')^{-1}(y'), (g')^{-1}E|_{(f')^{-1}(y')})$$

and

$$(Rf_*E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$$

The induced map of fibres $(f')^{-1}(y') \to f^{-1}(y)$ is a homeomorphism of topological spaces and the pull back of $E|_{f^{-1}(y)}$ is $(g')^{-1}E|_{(f')^{-1}(y')}$. The desired result follows. □

0D90 **Lemma 20.18.3** (Proper base change for sheaves of sets). Consider a cartesian square of topological spaces

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Assume that f is proper. Then $g^{-1}f_*\mathcal{F} = f'_*(g')^{-1}\mathcal{F}$ for any sheaf of sets $\mathcal{F}$ on X.

Proof. We argue exactly as in the proof of Theorem 20.18.2 and we find it suffices to show $(f_*\mathcal{F})_y = \Gamma(X_y, \mathcal{F}|_{X_y})$. Then we argue as in Lemma 20.18.1 to reduce this to the $p = 0$ case of Lemma 20.16.3 for sheaves of sets. The first part of the proof of Lemma 20.16.3 works for sheaves of sets and this finishes the proof. Some details omitted. □

20.19. Cohomology and colimits

01FE Let X be a ringed space. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of sheaves of $\mathcal{O}_X$-modules over the directed set I, see Categories, Section 4.21. Since for each i there is a canonical map $\mathcal{F}_i \to \operatorname{colim}_i \mathcal{F}_i$ we get a canonical map

$$\operatorname{colim}_i H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \operatorname{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$. Of course there is a similar map for every open $U \subset X$. These maps are in general not isomorphisms, even for $p = 0$. In this section we generalize the results of Sheaves, Lemma 6.29.1. See also Modules, Lemma 17.22.8 (in the special case $\mathcal{G} = \mathcal{O}_X$).

01FF Lemma 20.19.1. Let X be a ringed space. Assume that the underlying topological space of X has the following properties:

(1) there exists a basis of quasi-compact open subsets, and
(2) the intersection of any two quasi-compact opens is quasi-compact.

Then for any directed system $(\mathcal{F}_i, \varphi_{ii'})$ of sheaves of $\mathcal{O}_X$-modules and for any quasi-compact open $U \subset X$ the canonical map

$$\operatorname{colim}_i H^q(U, \mathcal{F}_i) \longrightarrow H^q(U, \operatorname{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $q \geq 0$.

Proof. It is important in this proof to argue for all quasi-compact opens $U \subset X$ at the same time. The result is true for $q = 0$ and any quasi-compact open $U \subset X$ by Sheaves, Lemma 6.29.1 (combined with Topology, Lemma 5.27.1). Assume that we have proved the result for all $q \leq q_0$ and let us prove the result for $q = q_0 + 1$.

By our conventions on directed systems the index set I is directed, and any system of $\mathcal{O}_X$-modules $(\mathcal{F}_i, \varphi_{ii'})$ over I is directed. By Injectives, Lemma 19.5.1 the category of $\mathcal{O}_X$-modules has functorial injective embeddings. Thus for any system $(\mathcal{F}_i, \varphi_{ii'})$ there exists a system $(\mathcal{I}_i, \varphi_{ii'})$ with each $\mathcal{I}_i$ an injective $\mathcal{O}_X$-module and a morphism of systems given by injective $\mathcal{O}_X$-module maps $\mathcal{F}_i \to \mathcal{I}_i$. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

We claim that the sequence

$$0 \to \operatorname{colim}_i \mathcal{F}_i \to \operatorname{colim}_i \mathcal{I}_i \to \operatorname{colim}_i \mathcal{Q}_i \to 0.$$

is also a short exact sequence of $\mathcal{O}_X$-modules. We may check this on stalks. By Sheaves, Sections 6.28 and 6.29 taking stalks commutes with colimits. Since a directed colimit of short exact sequences of abelian groups is short exact (see Algebra, Lemma 10.8.8) we deduce the result. We claim that $H^q(U, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all quasi-compact open $U \subset X$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{tikzcd} \mathrm{colim}_i H^{q_0}(U, \mathcal{I}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i H^{q_0}(U, \mathcal{Q}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i H^{q_0+1}(U, \mathcal{F}_i) \arrow[r] \arrow[d] & 0 \arrow[d] \\ H^{q_0}(U, \mathrm{colim}_i \mathcal{I}_i) \arrow[r] & H^{q_0}(U, \mathrm{colim}_i \mathcal{Q}_i) \arrow[r] & H^{q_0+1}(U, \mathrm{colim}_i \mathcal{F}_i) \arrow[r] & 0 \end{tikzcd}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 10.8.8. Hence by the snake lemma we deduce the result for $q = q_0 + 1$.

It remains to show that the claim is true. We will use Lemma 20.11.9. Let $\mathcal{B}$ be the collection of all quasi-compact open subsets of X. This is a basis for the topology on X by assumption. Let Cov be the collection of finite open coverings $\mathcal{U} : U = \bigcup_{j=1,\ldots,m} U_j$ with each of U, U_j quasi-compact open in X. By the result for $q = 0$ we see that for $\mathcal{U} \in \text{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{colim}_i \mathcal{I}_i) = \mathrm{colim}_i \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the multiple intersections $U_{j_0\ldots j_p}$ are quasi-compact. By Lemma 20.11.1 each of the complexes in the colimit of Čech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 10.8.8 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{colim}_i \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \mathrm{colim}_i \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 20.11.9 are satisfied and the claim follows. $\square$

Next we formulate the analogy of Sheaves, Lemma 6.29.4 for cohomology. Let X be a spectral space which is written as a cofiltered limit of spectral spaces X_i for a diagram with spectral transition morphisms as in Topology, Lemma 5.24.5. Assume given

(1) an abelian sheaf $\mathcal{F}_i$ on X_i for all $i \in \mathrm{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ an f_a-map $\varphi_a : \mathcal{F}_i \to \mathcal{F}_j$ of abelian sheaves (see Sheaves, Definition 6.21.7)

such that $\varphi_c = \varphi_b \circ \varphi_a$ whenever $c = a \circ b$. Set $\mathcal{F} = \mathrm{colim}\, p_i^{-1}\mathcal{F}_i$ on X.

0A37 Lemma 20.19.2. In the situation discussed above. Let $i \in \mathrm{Ob}(\mathcal{I})$ and let $U_i \subset X_i$ be quasi-compact open. Then

$$\mathrm{colim}_{a:j\to i} H^p(f_a^{-1}(U_i), \mathcal{F}_j) = H^p(p_i^{-1}(U_i), \mathcal{F})$$

for all $p \geq 0$. In particular we have $H^p(X, \mathcal{F}) = \mathrm{colim}\, H^p(X_i, \mathcal{F}_i)$.

Proof. The case $p = 0$ is Sheaves, Lemma 6.29.4.

In this paragraph we show that we can find a map of systems $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ with $\mathcal{G}_i$ an injective abelian sheaf and γ_i injective. For each i we pick an injection $\mathcal{F}_i \to \mathcal{I}_i$ where $\mathcal{I}_i$ is an injective abelian sheaf on X_i. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod\nolimits_{b:k\to i} f_{b,*}\mathcal{I}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_b^{-1}\mathcal{F}_i \to \mathcal{F}_k \to \mathcal{I}_k$. For $a : j \to i$ in $\mathcal{I}$ there is a canonical map

$$\psi_a : f_a^{-1}\mathcal{G}_i \to \mathcal{G}_j$$

whose components are the canonical maps $f_b^{-1}f_{a\circ b,*}\mathcal{I}_k \to f_{b,*}\mathcal{I}_k$ for $b : k \to j$. Thus we find an injection $\{\gamma_i\} : \{\mathcal{F}_i, \varphi_a\} \to (\mathcal{G}_i, \psi_a)$ of systems of abelian sheaves. Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on X_i, see Lemma 20.11.11 and Homology, Lemma 12.27.3. This finishes the construction.

Arguing exactly as in the proof of Lemma 20.19.1 we see that it suffices to prove that $H^p(X, \mathrm{colim}\, f_i^{-1}\mathcal{G}_i) = 0$ for $p > 0$.

Set $\mathcal{G} = \mathrm{colim}\, f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every quasi-compact open of X, it suffices to show that the Čech cohomology of $\mathcal{G}$ for any covering $\mathcal{U}$ of a quasi-compact open of X by finitely many quasi-compact opens is zero, see Lemma 20.11.9. Such a covering is the inverse by p_i of such a covering $\mathcal{U}_i$ on the space X_i for some i by Topology, Lemma 5.24.6. We have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) = \mathrm{colim}_{a:j\to i}\, \check{\mathcal{C}}^\bullet(f_a^{-1}(\mathcal{U}_i), \mathcal{G}_j)$$

by the case $p = 0$. The right hand side is a filtered colimit of complexes each of which is acyclic in positive degrees by Lemma 20.11.1. Thus we conclude by Algebra, Lemma 10.8.8. □

20.20. Vanishing on Noetherian topological spaces

02UU The aim is to prove a theorem of Grothendieck namely Proposition 20.20.7. See [Gro57].

02UV Lemma 20.20.1. Let $i : Z \to X$ be a closed immersion of topological spaces. For any abelian sheaf $\mathcal{F}$ on Z we have $H^p(Z, \mathcal{F}) = H^p(X, i_*\mathcal{F})$.

Proof. This is true because i_* is exact (see Modules, Lemma 17.6.1), and hence $R^pi_* = 0$ as a functor (Derived Categories, Lemma 13.16.9). Thus we may apply Lemma 20.13.6. □

02UW Lemma 20.20.2. Let X be an irreducible topological space. Then $H^p(X, \underline{A}) = 0$ for all $p > 0$ and any abelian group A.

Proof. Recall that $\underline{A}$ is the constant sheaf as defined in Sheaves, Definition 6.7.4. Since X is irreducible, any nonempty open U is irreducible and a fortiori connected. Hence for $U \subset X$ nonempty open we have $\underline{A}(U) = A$. We have $\underline{A}(\emptyset) = 0$. Thus $\underline{A}$ is a flasque abelian sheaf on X. The vanishing follows from Lemma 20.12.3. □

0A38 Lemma 20.20.3. Let X be a topological space such that the intersection of any two quasi-compact opens is quasi-compact. Let $\mathcal{F} \subset \underline{\mathbf{Z}}$ be a subsheaf generated by finitely many sections over quasi-compact opens. Then there exists a finite filtration [Gro57, Page 168].

$$(0) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

by abelian subsheaves such that for each $0 < i \leq n$ there exists a short exact sequence

$$0 \to j'_!\underline{\mathbf{Z}}_V \to j_!\underline{\mathbf{Z}}_U \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0$$

with $j : U \to X$ and $j' : V \to X$ the inclusion of quasi-compact opens into X.

Proof. Say $\mathcal{F}$ is generated by the sections $s_1, \ldots, s_t$ over the quasi-compact opens $U_1, \ldots, U_t$. Since U_i is quasi-compact and s_i a locally constant function to $\mathbf{Z}$ we may assume, after possibly replacing U_i by the parts of a finite decomposition into open and closed subsets, that s_i is a constant section. Say $s_i = n_i$ with $n_i \in \mathbf{Z}$. Of course we can remove (U_i, n_i) from the list if $n_i = 0$. Flipping signs if necessary we may also assume $n_i > 0$. Next, for any subset $I \subset \{1, \ldots, t\}$ we may add $\bigcap_{i \in I} U_i$ and $\gcd(n_i, i \in I)$ to the list. After doing this we see that our list $(U_1, n_1), \ldots, (U_t, n_t)$ satisfies the following property: For $x \in X$ set $I_x = \{i \in \{1, \ldots, t\} \mid x \in U_i\}$. Then $\gcd(n_i, i \in I_x)$ is attained by n_i for some $i \in I_x$.

As our filtration we take $\mathcal{F}_0 = (0)$ and $\mathcal{F}_n$ generated by the sections n_i over U_i for those i such that $n_i \leq n$. It is clear that $\mathcal{F}_n = \mathcal{F}$ for $n \gg 0$. Moreover, the quotient $\mathcal{F}_n/\mathcal{F}_{n-1}$ is generated by the section n over $U = \bigcup_{n_i \leq n} U_i$ and the kernel of the map $j_!\underline{\mathbf{Z}}_U \to \mathcal{F}_n/\mathcal{F}_{n-1}$ is generated by the section n over $V = \bigcup_{n_i \leq n-1} U_i$. Thus a short exact sequence as in the statement of the lemma. $\square$

02UX Lemma 20.20.4. Let X be a topological space. Let $d \geq 0$ be an integer. Assume

(1) X is quasi-compact,
(2) the quasi-compact opens form a basis for X, and
(3) the intersection of two quasi-compact opens is quasi-compact.
(4) $H^p(X, j_!\underline{\mathbf{Z}}_U) = 0$ for all $p > d$ and any quasi-compact open $j : U \to X$.

Then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and any abelian sheaf $\mathcal{F}$ on X.

This is a special case of [Gro57, Proposition 3.6.1].

Proof. Let $S = \coprod_{U \subset X} \mathcal{F}(U)$ where U runs over the quasi-compact opens of X. For any finite subset $A = \{s_1, \ldots, s_n\} \subset S$, let $\mathcal{F}_A$ be the subsheaf of $\mathcal{F}$ generated by all s_i (see Modules, Definition 17.4.5). Note that if $A \subset A'$, then $\mathcal{F}_A \subset \mathcal{F}_{A'}$. Hence $\{\mathcal{F}_A\}$ forms a system over the directed partially ordered set of finite subsets of S. By Modules, Lemma 17.4.6 it is clear that

$$\operatorname{colim}_A \mathcal{F}_A = \mathcal{F}$$

by looking at stalks. By Lemma 20.19.1 we have

$$H^p(X, \mathcal{F}) = \operatorname{colim}_A H^p(X, \mathcal{F}_A)$$

Hence it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}_A$. In other words, it suffices to prove the result when $\mathcal{F}$ is generated by finitely many local sections over quasi-compact opens of X.

Suppose that $\mathcal{F}$ is generated by the local sections $s_1, \ldots, s_n$. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf generated by $s_1, \ldots, s_{n-1}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

From the long exact sequence of cohomology we see that it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ which are generated by fewer than n local sections. Hence it suffices to prove the vanishing for sheaves generated by at most one local section. These sheaves are exactly the quotients of the sheaves $j_!\underline{\mathbf{Z}}_U$ where U is a quasi-compact open of X.

Assume now that we have a short exact sequence

$$0 \to \mathcal{K} \to j_!\underline{\mathbf{Z}}_U \to \mathcal{F} \to 0$$

with U quasi-compact open in X. It suffices to show that $H^q(X, \mathcal{K})$ is zero for $q \geq d+1$. As above we can write $\mathcal{K}$ as the filtered colimit of subsheaves $\mathcal{K}'$ generated by finitely many sections over quasi-compact opens. Then $\mathcal{F}$ is the filtered colimit of the sheaves $j_!\underline{\mathbf{Z}}_U/\mathcal{K}'$. In this way we reduce to the case that $\mathcal{K}$ is generated by finitely many sections over quasi-compact opens. Note that $\mathcal{K}$ is a subsheaf of $\underline{\mathbf{Z}}_X$. Thus by Lemma 20.20.3 there exists a finite filtration of $\mathcal{K}$ whose successive quotients $\mathcal{Q}$ fit into a short exact sequence

$$0 \to j''_!\underline{\mathbf{Z}}_W \to j'_!\underline{\mathbf{Z}}_V \to \mathcal{Q} \to 0$$

with $j'' : W \to X$ and $j' : V \to X$ the inclusions of quasi-compact opens. Hence the vanishing of $H^p(X, \mathcal{Q})$ for $p > d$ follows from our assumption (in the lemma) on the vanishing of the cohomology groups of $j''_!\underline{\mathbf{Z}}_W$ and $j'_!\underline{\mathbf{Z}}_V$. Returning to $\mathcal{K}$ this, via an induction argument using the long exact cohomology sequence, implies the desired vanishing for it as well. □

0BX0 Example 20.20.5. Let $X = \mathbf{N}$ endowed with the topology whose opens are $\emptyset$, X, and $U_n = \{i \mid i \leq n\}$ for $n \geq 1$. An abelian sheaf $\mathcal{F}$ on X is the same as an inverse system of abelian groups $A_n = \mathcal{F}(U_n)$ and $\Gamma(X, \mathcal{F}) = \lim A_n$. Since the inverse limit functor is not an exact functor on the category of inverse systems, we see that there is an abelian sheaf with nonzero H^1. Finally, the reader can check that $H^p(X, j_!\mathbf{Z}_U) = 0$, $p \geq 1$ if $j : U = U_n \to X$ is the inclusion. Thus we see that X is an example of a space satisfying conditions (2), (3), and (4) of Lemma 20.20.4 for $d = 0$ but not the conclusion.

02UY Lemma 20.20.6. Let X be an irreducible topological space. Let $\mathcal{H} \subset \underline{\mathbf{Z}}$ be an abelian subsheaf of the constant sheaf. Then there exists a nonempty open $U \subset X$ such that $\mathcal{H}|_U = \underline{d\mathbf{Z}}_U$ for some $d \in \mathbf{Z}$.

Proof. Recall that $\underline{\mathbf{Z}}(V) = \mathbf{Z}$ for any nonempty open V of X (see proof of Lemma 20.20.2). If $\mathcal{H} = 0$, then the lemma holds with $d = 0$. If $\mathcal{H} \neq 0$, then there exists a nonempty open $U \subset X$ such that $\mathcal{H}(U) \neq 0$. Say $\mathcal{H}(U) = n\mathbf{Z}$ for some $n \geq 1$. Hence we see that $\underline{n\mathbf{Z}}_U \subset \mathcal{H}|_U \subset \underline{\mathbf{Z}}_U$. If the first inclusion is strict we can find a nonempty $U' \subset U$ and an integer $1 \leq n' < n$ such that $\underline{n'\mathbf{Z}}_{U'} \subset \mathcal{H}|_{U'} \subset \underline{\mathbf{Z}}_{U'}$. This process has to stop after a finite number of steps, and hence we get the lemma. □

02UZ Proposition 20.20.7 (Grothendieck). Let X be a Noetherian topological space. If $\dim(X) \leq d$, then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and any abelian sheaf $\mathcal{F}$ on X. [Gro57, Theorem 3.6.5].

Proof. We prove this lemma by induction on d. So fix d and assume the lemma holds for all Noetherian topological spaces of dimension $< d$.

Let $\mathcal{F}$ be an abelian sheaf on X. Suppose $U \subset X$ is an open. Let $Z \subset X$ denote the closed complement. Denote $j : U \to X$ and $i : Z \to X$ the inclusion maps. Then there is a short exact sequence

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

see Modules, Lemma 17.7.1. Note that $j_!j^*\mathcal{F}$ is supported on the topological closure Z' of U, i.e., it is of the form $i'_*\mathcal{F}'$ for some abelian sheaf $\mathcal{F}'$ on Z', where $i' : Z' \to X$ is the inclusion.

We can use this to reduce to the case where X is irreducible. Namely, according to Topology, Lemma 5.9.2 X has finitely many irreducible components. If X has more than one irreducible component, then let $Z \subset X$ be an irreducible component of X and set $U = X \setminus Z$. By the above, and the long exact sequence of cohomology, it suffices to prove the vanishing of $H^p(X, i_*i^*\mathcal{F})$ and $H^p(X, i'_*\mathcal{F}')$ for $p > d$. By

Lemma 20.20.1 it suffices to prove $H^p(Z, i^*\mathcal{F})$ and $H^p(Z', \mathcal{F}')$ vanish for $p > d$. Since Z' and Z have fewer irreducible components we indeed reduce to the case of an irreducible X.

If $d = 0$ and X is irreducible, then X is the only nonempty open subset of X. Hence every sheaf is constant and higher cohomology groups vanish (for example by Lemma 20.20.2).

Suppose X is irreducible of dimension $d > 0$. By Lemma 20.20.4 we reduce to the case where $\mathcal{F} = j_!\underline{\mathbf{Z}}_U$ for some open $U \subset X$. In this case we look at the short exact sequence

$$0 \to j_!(\underline{\mathbf{Z}}_U) \to \underline{\mathbf{Z}}_X \to i_*\underline{\mathbf{Z}}_Z \to 0$$

where $Z = X \setminus U$. By Lemma 20.20.2 we have the vanishing of $H^p(X, \underline{\mathbf{Z}}_X)$ for all $p \geq 1$. By induction we have $H^p(X, i_*\underline{\mathbf{Z}}_Z) = H^p(Z, \underline{\mathbf{Z}}_Z) = 0$ for $p \geq d$. Hence we win by the long exact cohomology sequence. □

20.21. Cohomology with support in a closed subset

0A39 This section just discusses the bare minimum – the discussion will be continued in Section 20.34.

Let X be a topological space and let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be an abelian sheaf on X. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \mathrm{Supp}(s) \subset Z\}$$

be the subset of sections whose support is contained in Z. The support of a section is defined in Modules, Definition 17.5.1. Modules, Lemma 17.5.2 implies that $\Gamma_Z(X, \mathcal{F})$ is a subgroup of $\Gamma(X, \mathcal{F})$. The same lemma guarantees that the assignment $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$ is a functor in $\mathcal{F}$. This functor is left exact but not exact in general.

Since the category of abelian sheaves has enough injectives (Injectives, Lemma 19.4.1) we we obtain a right derived functor

$$R\Gamma_Z(X, -) : D^+(X) \longrightarrow D^+(\mathrm{Ab})$$

by Derived Categories, Lemma 13.20.2. The value of $R\Gamma_Z(X, -)$ on an object K is computed by representing K by a bounded below complex $\mathcal{I}^\bullet$ of injective abelian sheaves and taking $\Gamma_Z(X, \mathcal{I}^\bullet)$, see Derived Categories, Lemma 13.20.1. The cohomology groups of an abelian sheaf $\mathcal{F}$ with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on X. Let $U = X \setminus Z$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Lemma 20.8.1) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D^+(X)$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D^+(\mathrm{Ab})$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D^+(X)$.

For an abelian sheaf $\mathcal{F}$ on X we can consider the subsheaf of sections with support in Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \mathrm{Supp}(s) \subset U \cap Z\} = \Gamma_{Z \cap U}(U, \mathcal{F}|_U)$$

Using the equivalence of Modules, Lemma 17.6.1 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on Z, see Modules, Remark 17.6.2. Thus we obtain a functor

$$\mathrm{Ab}(X) \longrightarrow \mathrm{Ab}(Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as a sheaf on } Z$$

This functor is left exact, but in general not exact. Exactly as above we obtain a right derived functor

$$R\mathcal{H}_Z : D^+(X) \longrightarrow D^+(Z)$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$.

Observe that we have $\Gamma_Z(X, \mathcal{F}) = \Gamma(Z, \mathcal{H}_Z(\mathcal{F}))$ for any abelian sheaf $\mathcal{F}$. By Lemma 20.21.1 below the functor $\mathcal{H}_Z$ transforms injective abelian sheaves into sheaves right acyclic for $\Gamma(Z, -)$. Thus by Derived Categories, Lemma 13.22.2 we obtain a convergent Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(K)) \Rightarrow H^{p+q}_Z(X, K)$$

functorial in K in $D^+(X)$.

0A3A Lemma 20.21.1. Let $i : Z \to X$ be the inclusion of a closed subset. Let $\mathcal{I}$ be an injective abelian sheaf on X. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on Z.

Proof. This follows from Homology, Lemma 12.29.1 as $\mathcal{H}_Z(-)$ is right adjoint to the exact functor i_*. See Modules, Lemmas 17.6.1 and 17.6.3. □

20.22. Cohomology on spectral spaces

0A3C A key result on the cohomology of spectral spaces is Lemma 20.19.2 which loosely speaking says that cohomology commutes with cofiltered limits in the category of spectral spaces as defined in Topology, Definition 5.23.1. This can be applied to give analogues of Lemmas 20.16.3 and 20.18.1 as follows.

0A3D Lemma 20.22.1. Let X be a spectral space. Let $\mathcal{F}$ be an abelian sheaf on X. Let $E \subset X$ be a quasi-compact subset. Let $W \subset X$ be the set of points of X which specialize to a point of E.

(1) $H^p(W, \mathcal{F}|_W) = \operatorname{colim} H^p(U, \mathcal{F})$ where the colimit is over quasi-compact open neighbourhoods of E,
(2) $H^p(W \setminus E, \mathcal{F}|_{W\setminus E}) = \operatorname{colim} H^p(U \setminus E, \mathcal{F}|_{U\setminus E})$ if E is a constructible subset.

Proof. From Topology, Lemma 5.24.7 we see that $W = \lim U$ where the limit is over the quasi-compact opens containing E. Each U is a spectral space by Topology, Lemma 5.23.5. Thus we may apply Lemma 20.19.2 to conclude that (1) holds. The same proof works for part (2) except we use Topology, Lemma 5.24.8. □

0A3E Lemma 20.22.2. Let $f : X \to Y$ be a spectral map of spectral spaces. Let $y \in Y$. Let $E \subset Y$ be the set of points specializing to y. Let $\mathcal{F}$ be an abelian sheaf on X. Then $(R^pf_*\mathcal{F})_y = H^p(f^{-1}(E), \mathcal{F}|_{f^{-1}(E)})$.

Proof. Observe that $E = \bigcap V$ where V runs over the quasi-compact open neighbourhoods of y in Y. Hence $f^{-1}(E) = \bigcap f^{-1}(V)$. This implies that $f^{-1}(E) = \lim f^{-1}(V)$ as topological spaces. Since f is spectral, each $f^{-1}(V)$ is a spectral space too (Topology, Lemma 5.23.5). We conclude that $f^{-1}(E)$ is a spectral space and that

$$H^p(f^{-1}(E), \mathcal{F}|_{f^{-1}(E)}) = \operatorname{colim} H^p(f^{-1}(V), \mathcal{F})$$

by Lemma 20.19.2. On the other hand, the stalk of $R^pf_*\mathcal{F}$ at y is given by the colimit on the right. □

0A3F Lemma 20.22.3. Let X be a profinite topological space. Then $H^q(X, \mathcal{F}) = 0$ for all $q > 0$ and all abelian sheaves $\mathcal{F}$.

Proof. Any open covering of X can be refined by a finite disjoint union decomposition with open parts, see Topology, Lemma 5.22.4. Hence if $\mathcal{F} \to \mathcal{G}$ is a surjection of abelian sheaves on X, then $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective. In other words, the global sections functor is an exact functor. Therefore its higher derived functors are zero, see Derived Categories, Lemma 13.16.9. □

The following result on cohomological vanishing improves Grothendieck's result (Proposition 20.20.7) and can be found in [Sch92].

0A3G Proposition 20.22.4. Let X be a spectral space of Krull dimension d. Let $\mathcal{F}$ be an abelian sheaf on X.

Part (1) is the main theorem of [Sch92].

(1) $H^q(X, \mathcal{F}) = 0$ for $q > d$,
(2) $H^d(X, \mathcal{F}) \to H^d(U, \mathcal{F})$ is surjective for every quasi-compact open $U \subset X$,
(3) $H^q_Z(X, \mathcal{F}) = 0$ for $q > d$ and any constructible closed subset $Z \subset X$.

Proof. We prove this result by induction on d.

If $d = 0$, then X is a profinite space, see Topology, Lemma 5.23.8. Thus (1) holds by Lemma 20.22.3. If $U \subset X$ is quasi-compact open, then U is also closed as a quasi-compact subset of a Hausdorff space. Hence $X = U \amalg (X \setminus U)$ as a topological space and we see that (2) holds. Given Z as in (3) we consider the long exact sequence

$$H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus Z, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \to H^q(X, \mathcal{F})$$

Since X and $U = X \setminus Z$ are profinite (namely U is quasi-compact because Z is constructible) and since we have (2) and (1) we obtain the desired vanishing of the cohomology groups with support in Z.

Induction step. Assume $d \geq 1$ and assume the proposition is valid for all spectral spaces of dimension $< d$. We first prove part (2) for X. Let U be a quasi-compact open. Let $\xi \in H^d(U, \mathcal{F})$. Set $Z = X \setminus U$. Let $W \subset X$ be the set of points specializing to Z. By Lemma 20.22.1 we have

$$H^d(W \setminus Z, \mathcal{F}|_{W \setminus Z}) = \operatorname{colim}_{Z \subset V} H^d(V \setminus Z, \mathcal{F})$$

where the colimit is over the quasi-compact open neighbourhoods V of Z in X. By Topology, Lemma 5.24.7 we see that $W \setminus Z$ is a spectral space. Since every point of W specializes to a point of Z, we see that $W \setminus Z$ is a spectral space of Krull dimension $< d$. By induction hypothesis we see that the image of ξ in $H^d(W \setminus Z, \mathcal{F}|_{W \setminus Z})$ is zero. By the displayed formula, there exists a $Z \subset V \subset X$ quasi-compact open such that $\xi|_{V \setminus Z} = 0$. Since $V \setminus Z = V \cap U$ we conclude by the Mayer-Vietoris (Lemma 20.8.2) for the covering $X = U \cup V$ that there exists a $\tilde{\xi} \in H^d(X, \mathcal{F})$ which restricts to ξ on U and to zero on V. In other words, part (2) is true.

Proof of part (1) assuming (2). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Set

$$\mathcal{G} = \operatorname{Im}(\mathcal{I}^{d-1} \to \mathcal{I}^d) = \operatorname{Ker}(\mathcal{I}^d \to \mathcal{I}^{d+1})$$

For $U \subset X$ quasi-compact open we have a map of exact sequences as follows

$$\begin{tikzcd} \mathcal{I}^{d-1}(X) \arrow[r] \arrow[d] & \mathcal{G}(X) \arrow[r] \arrow[d] & H^d(X, \mathcal{F}) \arrow[r] \arrow[d] & 0 \\ \mathcal{I}^{d-1}(U) \arrow[r] & \mathcal{G}(U) \arrow[r] & H^d(U, \mathcal{F}) \arrow[r] & 0 \end{tikzcd}$$

The sheaf $\mathcal{I}^{d-1}$ is flasque by Lemma 20.12.2 and the fact that $d \geq 1$. By part (2) we see that the right vertical arrow is surjective. We conclude by a diagram chase that the map $\mathcal{G}(X) \to \mathcal{G}(U)$ is surjective. By Lemma 20.12.6 we conclude that $\check{H}^q(\mathcal{U}, \mathcal{G}) = 0$ for $q > 0$ and any finite covering $\mathcal{U} : U = U_1 \cup \ldots \cup U_n$ of a quasi-compact open by quasi-compact opens. Applying Lemma 20.11.9 we find that $H^q(U, \mathcal{G}) = 0$ for all $q > 0$ and all quasi-compact opens U of X. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we conclude that

$$H^q(X, \mathcal{F}) = H^q\left(\Gamma(X, \mathcal{I}^0) \to \ldots \to \Gamma(X, \mathcal{I}^{d-1}) \to \Gamma(X, \mathcal{G})\right)$$

In particular the cohomology group vanishes if $q > d$.

Proof of (3). Given Z as in (3) we consider the long exact sequence

$$H^{q-1}(X, \mathcal{F}) \to H^{q-1}(X \setminus Z, \mathcal{F}) \to H^q_Z(X, \mathcal{F}) \to H^q(X, \mathcal{F})$$

Since X and $U = X \setminus Z$ are spectral spaces (Topology, Lemma 5.23.5) of dimension $\leq d$ and since we have (2) and (1) we obtain the desired vanishing. □

20.23. The alternating Čech complex

01FG This section compares the Čech complex with the alternating Čech complex and some related complexes.

Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. For $p \geq 0$ set

$$\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = \left\{ \begin{matrix} s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \text{ such that } s_{i_0 \ldots i_p} = 0 \text{ if } i_n = i_m \text{ for some } n \neq m \\ \text{and } s_{i_0 \ldots i_n \ldots i_m \ldots i_p} = -s_{i_0 \ldots i_m \ldots i_n \ldots i_p} \text{ in any case.} \end{matrix} \right\}$$

We omit the verification that the differential d of Equation (20.9.0.1) maps $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F})$ into $\check{\mathcal{C}}^{p+1}_{alt}(\mathcal{U}, \mathcal{F})$.

01FH Definition 20.23.1. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is the alternating Čech complex associated to $\mathcal{F}$ and the open covering $\mathcal{U}$.

Hence there is a canonical morphism of complexes

$$\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

namely the inclusion of the alternating Čech complex into the usual Čech complex.

Suppose our covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ comes equipped with a total ordering $<$ on I. In this case, set

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}, i_0 < \ldots < i_p} \mathcal{F}(U_{i_0 \ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in $\mathcal{F}(U_{i_0 \ldots i_p})$. We define

$$d : \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}_{ord}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0\ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1}(-1)^j s_{i_0\ldots\hat{i}_j\ldots i_{p+1}}|_{U_{i_0\ldots i_{p+1}}}$$

for any $i_0 < \ldots < i_{p+1}$. Note that this formula is identical to Equation (20.9.0.1). It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a complex.

01FI Definition 20.23.2. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume given a total ordering on I. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is the ordered Čech complex associated to $\mathcal{F}$, the open covering $\mathcal{U}$ and the given total ordering on I.

This complex is sometimes called the alternating Čech complex. The reason is that there is an obvious comparison map between the ordered Čech complex and the alternating Čech complex. Namely, consider the map

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

given by the rule

$$c(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_n = i_m \text{ for some } n \neq m \\ \mathrm{sgn}(\sigma) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}} & \text{if} \quad i_{\sigma(0)} < i_{\sigma(1)} < \ldots < i_{\sigma(p)} \end{cases}$$

Here σ denotes a permutation of $\{0, \ldots, p\}$ and $\mathrm{sgn}(\sigma)$ denotes its sign. The alternating and ordered Čech complexes are often identified in the literature via the map c. Namely we have the following easy lemma.

01FJ Lemma 20.23.3. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map c is a morphism of complexes. In fact it induces an isomorphism

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

of complexes.

Proof. Omitted. □

There is also a map

$$\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

which is described by the rule

$$\pi(s)_{i_0\ldots i_p} = s_{i_0\ldots i_p}$$

whenever $i_0 < i_1 < \ldots < i_p$.

01FK Lemma 20.23.4. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a morphism of complexes. It induces an isomorphism

$$\pi : \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

of complexes which is a left inverse to the morphism c.

Proof. Omitted. □

01FL Remark 20.23.5. This means that if we have two total orderings $<_1$ and $<_2$ on the index set I, then we get an isomorphism of complexes $\tau = \pi_2 \circ c_1 : \check{\mathcal{C}}_{ord\text{-}1}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{ord\text{-}2}(\mathcal{U}, \mathcal{F})$. It is clear that

$$\tau(s)_{i_0 \ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $i_0 <_1 i_1 <_1 \ldots <_1 i_p$ and $i_{\sigma(0)} <_2 i_{\sigma(1)} <_2 \ldots <_2 i_{\sigma(p)}$. This is the sense in which the ordered Čech complex is independent of the chosen total ordering.

01FM Lemma 20.23.6. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $c \circ \pi$ is homotopic to the identity on $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$. In particular the inclusion map $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a homotopy equivalence.

Proof. For any multi-index $(i_0, \ldots, i_p) \in I^{p+1}$ there exists a unique permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ such that

$$i_{\sigma(0)} \leq i_{\sigma(1)} \leq \ldots \leq i_{\sigma(p)} \quad \text{and} \quad \sigma(j) < \sigma(j+1) \quad \text{if} \quad i_{\sigma(j)} = i_{\sigma(j+1)}.$$

We denote this permutation $\sigma = \sigma^{i_0 \ldots i_p}$.

For any permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ and any a, $0 \leq a \leq p$ we denote σ_a the permutation of $\{0, \ldots, p\}$ such that

$$\sigma_a(j) = \begin{cases} \sigma(j) & \text{if} \quad 0 \leq j < a, \\ \min\{j' \mid j' > \sigma_a(j-1), j' \neq \sigma(k), \forall k < a\} & \text{if} \quad a \leq j \end{cases}$$

So if $p = 3$ and σ, τ are given by

$$\begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \sigma & 3 & 2 & 1 & 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \tau & 3 & 0 & 2 & 1 \end{matrix}$$

then we have

$$\begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \sigma_0 & 0 & 1 & 2 & 3 \\ \sigma_1 & 3 & 0 & 1 & 2 \\ \sigma_2 & 3 & 2 & 0 & 1 \\ \sigma_3 & 3 & 2 & 1 & 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \text{id} & 0 & 1 & 2 & 3 \\ \tau_0 & 0 & 1 & 2 & 3 \\ \tau_1 & 3 & 0 & 1 & 2 \\ \tau_2 & 3 & 0 & 1 & 2 \\ \tau_3 & 3 & 0 & 2 & 1 \end{matrix}$$

It is clear that always $\sigma_0 = \text{id}$ and $\sigma_p = \sigma$.

Having introduced this notation we define for $s \in \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$ the element $h(s) \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ to be the element with components

01FN (20.23.6.1) $$h(s)_{i_0 \ldots i_p} = \sum\nolimits_{0 \leq a \leq p} (-1)^a \text{sign}(\sigma_a) s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. The index $i_{\sigma(a)}$ occurs twice in $i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}$ once in the first group of $a + 1$ indices and once in the second group of $p - a + 1$ indices since $\sigma_a(j) = \sigma(a)$ for some $j \geq a$ by definition of σ_a. Hence the sum makes sense since each of the elements $s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}}$ is defined over the open $U_{i_0 \ldots i_p}$. Note also that for $a = 0$ we get $s_{i_0 \ldots i_p}$ and for $a = p$ we get $(-1)^p \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$.

We claim that

$$(dh + hd)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p} - \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. We omit the verification of this claim. (There is a PARI/gp script called first-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) Write

$$\kappa : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

for the operator given by the rule

$$\kappa(s)_{i_0\ldots i_p} = \text{sign}(\sigma^{i_0\ldots i_p}) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}.$$

The claim above implies that κ is a morphism of complexes and that κ is homotopic to the identity map of the Čech complex. This does not immediately imply the lemma since the image of the operator κ is not the alternating subcomplex. Namely, the image of κ is the "semi-alternating" complex $\check{\mathcal{C}}^p_{semi\text{-}alt}(\mathcal{U}, \mathcal{F})$ where s is a p-cochain of this complex if and only if

$$s_{i_0\ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$$

for any $(i_0, \ldots, i_p) \in I^{p+1}$ with $\sigma = \sigma^{i_0\ldots i_p}$. We introduce yet another variant Čech complex, namely the semi-ordered Čech complex defined by

$$\check{\mathcal{C}}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{i_0 \leq i_1 \leq \ldots \leq i_p} \mathcal{F}(U_{i_0\ldots i_p})$$

It is easy to see that Equation (20.9.0.1) also defines a differential and hence that we get a complex. It is also clear (analogous to Lemma 20.23.4) that the projection map

$$\check{\mathcal{C}}^\bullet_{semi\text{-}alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Hence the Lemma follows if we can show that the obvious inclusion map

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is a homotopy equivalence. To see this we use the homotopy

01FO (20.23.6.2)
$$h(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ (-1)^a s_{i_0\ldots i_{a-1} i_a i_a i_{a+1}\ldots i_p} & \text{if} \quad i_0 < i_1 < \ldots < i_{a-1} < i_a = i_{a+1} \end{cases}$$

We claim that

$$(dh + hd)(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ s_{i_0\ldots i_p} & \text{else} \end{cases}$$

We omit the verification. (There is a PARI/gp script called second-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) The claim clearly shows that the composition

$$\check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

of the projection with the natural inclusion is homotopic to the identity map as desired. □

0G6T **Lemma 20.23.7.** *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. If $U_i = U$ for some $i \in I$, then the extended alternating Čech complex*

$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

obtained by putting $\mathcal{F}(U)$ in degree -1 with differential given by the canonical map of $\mathcal{F}(U)$ into $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F})$ is homotopy equivalent to 0. Similarly, for any total ordering on I the extended ordered Čech complex

$$\mathcal{F}(U) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

is homotopy equivalent to 0.

First proof. Combine Lemmas 20.9.3 and 20.23.6. □

Second proof. Since the alternating and ordered Čech complexes are isomorphic it suffices to prove this for the ordered one. We will use standard notation: a cochain s of degree p in the extended ordered Čech complex has the form $s = (s_{i_0 \ldots i_p})$ where $s_{i_0 \ldots i_p}$ is in $\mathcal{F}(U_{i_0 \ldots i_p})$ and $i_0 < \ldots < i_p$. With this notation we have

$$d(x)_{i_0 \ldots i_{p+1}} = \sum\nolimits_j (-1)^j x_{i_0 \ldots \hat{i}_j \ldots i_p}$$

Fix an index $i \in I$ with $U = U_i$. As homotopy we use the maps

$$h : \text{cochains of degree } p+1 \to \text{cochains of degree } p$$

given by the rule

$$h(s)_{i_0 \ldots i_p} = 0 \text{ if } i \in \{i_0, \ldots, i_p\} \text{ and } h(s)_{i_0 \ldots i_p} = (-1)^j s_{i_0 \ldots i_j i i_{j+1} \ldots i_p} \text{ if not}$$

Here j is the unique index such that $i_j < i < i_{j+1}$ in the second case; also, since $U = U_i$ we have the equality

$$\mathcal{F}(U_{i_0 \ldots i_p}) = \mathcal{F}(U_{i_0 \ldots i_j i i_{j+1} \ldots i_p})$$

which we can use to make sense of thinking of $(-1)^j s_{i_0 \ldots i_j i i_{j+1} \ldots i_p}$ as an element of $\mathcal{F}(U_{i_0 \ldots i_p})$. We will show by a computation that $dh + hd$ equals the negative of the identity map which finishes the proof. To do this fix s a cochain of degree p and let $i_0 < \ldots < i_p$ be elements of I.

Case I: $i \in \{i_0, \ldots, i_p\}$. Say $i = i_t$. Then we have $h(d(s))_{i_0 \ldots i_p} = 0$. On the other hand we have

$$d(h(s))_{i_0 \ldots i_p} = \sum (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} = (-1)^t h(s)_{i_0 \ldots \hat{i} \ldots i_p} = (-1)^t (-1)^{t-1} s_{i_0 \ldots i_p}$$

Thus $(dh + hd)(s)_{i_0 \ldots i_p} = -s_{i_0 \ldots i_p}$ as desired.

Case II: $i \notin \{i_0, \ldots, i_p\}$. Let j be such that $i_j < i < i_{j+1}$. Then we see that

$$\begin{aligned} h(d(s))_{i_0 \ldots i_p} &= (-1)^j d(s)_{i_0 \ldots i_j i i_{j+1} \ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j+j'} s_{i_0 \ldots \hat{i}_{j'} \ldots i_j i i_{j+1} \ldots i_p} - s_{i_0 \ldots i_p} \\ &+ \sum\nolimits_{j' > j} (-1)^{j+j'+1} s_{i_0 \ldots i_j i i_{j+1} \ldots \hat{i}_{j'} \ldots i_p} \end{aligned}$$

On the other hand we have

$$\begin{aligned} d(h(s))_{i_0 \ldots i_p} &= \sum\nolimits_{j'} (-1)^{j'} h(s)_{i_0 \ldots \hat{i}_{j'} \ldots i_p} \\ &= \sum\nolimits_{j' \leq j} (-1)^{j'+j-1} s_{i_0 \ldots \hat{i}_{j'} \ldots i_j i i_{j+1} \ldots i_p} \\ &+ \sum\nolimits_{j' > j} (-1)^{j'+j} s_{i_0 \ldots i_j i i_{j+1} \ldots \hat{i}_{j'} \ldots i_p} \end{aligned}$$

Adding these up we obtain $(dh + hd)(s)_{i_0 \ldots i_p} = -s_{i_0 \ldots i_p}$ as desired. □

20.24. Alternative view of the Čech complex

02FR In this section we discuss an alternative way to establish the relationship between the Čech complex and cohomology.

02FU Lemma 20.24.1. Let X be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{i_0 \ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0 \ldots i_p}$. There exists a complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{O}_X$-modules with

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_* \mathcal{F}_{i_0 \ldots i_p}$$

and differential $d : \mathfrak{C}^p(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^{p+1}(\mathcal{U}, \mathcal{F})$ as in Equation (20.9.0.1). Moreover, there exists a canonical map

$$\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$$

which is a quasi-isomorphism, i.e., $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$.

Proof. We check

$$0 \to \mathcal{F} \to \mathfrak{C}^0(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^1(\mathcal{U}, \mathcal{F}) \to \ldots$$

is exact on stalks. Let $x \in X$ and choose $i_{\text{fix}} \in I$ such that $x \in U_{i_{\text{fix}}}$. Then define

$$h : \mathfrak{C}^p(\mathcal{U}, \mathcal{F})_x \to \mathfrak{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$$

as follows: If $s \in \mathfrak{C}^p(\mathcal{U}, \mathcal{F})_x$, take a representative

$$\widetilde{s} \in \mathfrak{C}^p(\mathcal{U}, \mathcal{F})(V) = \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}(V \cap U_{i_0} \cap \ldots \cap U_{i_p})$$

defined on some neighborhood V of x, and set

$$h(s)_{i_0 \ldots i_{p-1}} = \widetilde{s}_{i_{\text{fix}} i_0 \ldots i_{p-1}, x}.$$

By the same formula (for $p = 0$) we get a map $\mathfrak{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathcal{F}_x$. We compute formally as follows:

$$\begin{aligned}(dh + hd)(s)_{i_0 \ldots i_p} &= \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}} i_0 \ldots i_p} \\ &= \sum\nolimits_{j=0}^{p} (-1)^j s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum\nolimits_{j=0}^{p} (-1)^{j+1} s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} \\ &= s_{i_0 \ldots i_p}\end{aligned}$$

This shows h is a homotopy from the identity map of the extended complex

$$0 \to \mathcal{F}_x \to \mathfrak{C}^0(\mathcal{U}, \mathcal{F})_x \to \mathfrak{C}^1(\mathcal{U}, \mathcal{F})_x \to \ldots$$

to zero and we conclude. □

With this lemma it is easy to reprove the Čech to cohomology spectral sequence of Lemma 20.11.5. Namely, let X, $\mathcal{U}$, $\mathcal{F}$ as in Lemma 20.24.1 and let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then we may consider the double complex

$$A^{\bullet, \bullet} = \Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

By construction we have

$$A^{p,q} = \prod\nolimits_{i_0 \ldots i_p} \mathcal{I}^q(U_{i_0 \ldots i_p})$$

Consider the two spectral sequences of Homology, Section 12.25 associated to this double complex, see especially Homology, Lemma 12.25.1. For the spectral sequence $('E_r, 'd_r)_{r \geq 0}$ we get $'E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ because taking products is exact (Homology, Lemma 12.32.1). For the spectral sequence $(''E_r, ''d_r)_{r \geq 0}$ we get $''E_2^{p,q} = 0$

if $p > 0$ and ${}''E_2^{0,q} = H^q(X, \mathcal{F})$. Namely, for fixed q the complex of sheaves $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^q)$ is a resolution (Lemma 20.24.1) of the injective sheaf $\mathcal{I}^q$ by injective sheaves (by Lemmas 20.7.1 and 20.11.11 and Homology, Lemma 12.27.3). Hence the cohomology of $\Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^q))$ is zero in positive degrees and equal to $\Gamma(X, \mathcal{I}^q)$ in degree 0. Taking cohomology of the next differential we get our claim about the spectral sequence $({}''E_r, {}''d_r)_{r \geq 0}$. Whence the result since both spectral sequences converge to the cohomology of the associated total complex of $A^{\bullet,\bullet}$.

02FS Definition 20.24.2. Let X be a topological space. An open covering $X = \bigcup_{i \in I} U_i$ is said to be locally finite if for every $x \in X$ there exists an open neighbourhood W of x such that $\{i \in I \mid W \cap U_i \neq \emptyset\}$ is finite.

02FT Remark 20.24.3. Let $X = \bigcup_{i \in I} U_i$ be a locally finite open covering. Denote $j_i : U_i \to X$ the inclusion map. Suppose that for each i we are given an abelian sheaf $\mathcal{F}_i$ on U_i. Consider the abelian sheaf $\mathcal{G} = \bigoplus_{i \in I} (j_i)_*\mathcal{F}_i$. Then for $V \subset X$ open we actually have

$$\Gamma(V, \mathcal{G}) = \prod\nolimits_{i \in I} \mathcal{F}_i(V \cap U_i).$$

In other words we have

$$\bigoplus\nolimits_{i \in I} (j_i)_*\mathcal{F}_i = \prod\nolimits_{i \in I} (j_i)_*\mathcal{F}_i$$

This seems strange until you realize that the direct sum of a collection of sheaves is the sheafification of what you think it should be. See discussion in Modules, Section 17.3. Thus we conclude that in this case the complex of Lemma 20.24.1 has terms

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \bigoplus\nolimits_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_*\mathcal{F}_{i_0 \ldots i_p}$$

which is sometimes useful.

20.25. Čech cohomology of complexes

01FP In general for sheaves of abelian groups $\mathcal{F}$ and $\mathcal{G}$ on X there is a cup product map

$$H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes_{\mathbf{Z}} \mathcal{G}).$$

In this section we define it using Čech cocycles by an explicit formula for the cup product. If you are worried about the fact that cohomology may not equal Čech cohomology, then you can use hypercoverings and still use the cocycle notation. This also has the advantage that it works to define the cup product for hypercohomology on any topos (insert future reference here).

Let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups on X. We can often compute $H^n(X, \mathcal{F}^\bullet)$ using Čech cocycles. Namely, let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Since the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ (Definition 20.9.1) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$. The associated total complex to $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ is the complex with degree n term

$$\mathrm{Tot}^n(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) = \bigoplus\nolimits_{p+q=n} \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}^q(U_{i_0 \ldots i_p})$$

see Homology, Definition 12.18.3. A typical element in Tot^n will be denoted $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ where $\alpha_{i_0 \ldots i_p} \in \mathcal{F}^q(U_{i_0 \ldots i_p})$. In other words the $\mathcal{F}$-degree of $\alpha_{i_0 \ldots i_p}$ is $q = n - p$. This notation requires us to be aware of the degree α lives in at all times. We indicate this situation by the formula $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_p}) = q$. According to our conventions in Homology, Definition 12.18.3 the differential of an element α of degree n is given by

$$d(\alpha)_{i_0\ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1}(-1)^j\alpha_{i_0\ldots\hat{i}_j\ldots i_{p+1}} + (-1)^{p+1}d_{\mathcal{F}}(\alpha_{i_0\ldots i_{p+1}})$$

where $d_{\mathcal{F}}$ denotes the differential on the complex $\mathcal{F}^\bullet$. The expression $\alpha_{i_0\ldots\hat{i}_j\ldots i_{p+1}}$ means the restriction of $\alpha_{i_0\ldots\hat{i}_j\ldots i_{p+1}} \in \mathcal{F}(U_{i_0\ldots\hat{i}_j\ldots i_{p+1}})$ to $U_{i_0\ldots i_{p+1}}$.

The construction of $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

07M9 (20.25.0.1)
$$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0\ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0\ldots i_p} = 0$ for $p > 0$.

Refinements. Let $\mathcal{V} = \{V_j\}_{j\in J}$ be a refinement of $\mathcal{U}$. This means there is a map $t : J \to I$ such that $V_j \subset U_{t(j)}$ for all $j \in J$. This gives rise to a functorial transformation

08BM (20.25.0.2)
$$T_t : \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)).$$

defined by the rule

$$T_t(\alpha)_{j_0\ldots j_p} = \alpha_{t(j_0)\ldots t(j_p)}|_{V_{j_0\ldots j_p}}.$$

Given two maps $t, t' : J \to I$ as above the maps T_t and $T_{t'}$ constructed above are homotopic. The homotopy is given by

$$h(\alpha)_{j_0\ldots j_p} = \sum\nolimits_{a=0}^{p}(-1)^a\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}$$

for an element α of degree n. This works because of the following computation, again with α an element of degree n (so $d(\alpha)$ has degree $n+1$ and $h(\alpha)$ has degree $n-1$):

$$\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{j_0\ldots j_p} = & \sum\nolimits_{k=0}^{p}(-1)^k h(\alpha)_{j_0\ldots\hat{j}_k\ldots j_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{j_0\ldots j_p}) + \\
& \sum\nolimits_{a=0}^{p}(-1)^a d(\alpha)_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} \\
= & \sum\nolimits_{k=0}^{p}\sum\nolimits_{a=0}^{k-1}(-1)^{k+a}\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(\hat{j}_k)\ldots t'(j_p)} + \\
& \sum\nolimits_{k=0}^{p}\sum\nolimits_{a=k+1}^{p}(-1)^{k+a-1}\alpha_{t(j_0)\ldots t(\hat{j}_k)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p}(-1)^{p+a}d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}) + \\
& \sum\nolimits_{a=0}^{p}\sum\nolimits_{k=0}^{a}(-1)^{a+k}\alpha_{t(j_0)\ldots t(\hat{j}_k)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p}\sum\nolimits_{k=a}^{p}(-1)^{a+k+1}\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(\hat{j}_k)\ldots t'(j_p)} + \\
& \sum\nolimits_{a=0}^{p}(-1)^{a+p+1}d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}) \\
= & \alpha_{t'(j_0)\ldots t'(j_p)} + (-1)^{2p+1}\alpha_{t(j_0)\ldots t(j_p)} \\
= & T_{t'}(\alpha)_{j_0\ldots j_p} - T_t(\alpha)_{j_0\ldots j_p}
\end{aligned}$$

We leave it to the reader to verify the cancellations. (Note that the terms having both k and a in the 1st, 2nd and 4th, 5th summands cancel, except the ones where

$a = k$ which only occur in the 4th and 5th and these cancel against each other except for the two desired terms.) It follows that the induced map

$$H^n(T_t) : H^n(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to H^n(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)))$$

is independent of the choice of t. We define Čech hypercohomology as the limit of the Čech cohomology groups over all refinements via the maps $H^\bullet(T_t)$.

In the limit (over all open coverings of X) the following lemma provides a map of Čech hypercohomology into cohomology, which is often an isomorphism and is always an isomorphism if we use hypercoverings.

08BN Lemma 20.25.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. For a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and compatible with (20.25.0.1) and (20.25.0.2). There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = H^p(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^q(\mathcal{F}^\bullet)))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives. The map (20.25.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 12.25.4 applied to the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemma 20.11.1. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of $\mathcal{F}^\bullet$ into a bounded below complex of injectives. Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. To construct the spectral sequence of the lemma, choose a Cartan-Eilenberg resolution $\mathcal{F}^\bullet \to \mathcal{I}^{\bullet,\bullet}$, see Derived Categories, Lemma 13.21.2. In this case $\mathcal{F}^\bullet \to \text{Tot}(\mathcal{I}^{\bullet,\bullet})$ is an injective resolution and hence

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{I}^{\bullet,\bullet})))$$

computes $R\Gamma(X, \mathcal{F}^\bullet)$ as we've seen above. By Homology, Remark 12.18.4 we can view this as the total complex associated to the triple complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^{\bullet,\bullet})$ hence, using the same remark we can view it as the total complex associate to the double complex $A^{\bullet,\bullet}$ with terms

$$A^{n,m} = \bigoplus\nolimits_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^{q,m})$$

Since $\mathcal{I}^{q,\bullet}$ is an injective resolution of $\mathcal{F}^q$ we can apply the first spectral sequence associated to $A^{\bullet,\bullet}$ (Homology, Lemma 12.25.1) to get a spectral sequence with

$$E_1^{n,m} = \bigoplus\nolimits_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \underline{H}^m(\mathcal{F}^q))$$

which is the nth term of the complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^m(\mathcal{F}^\bullet))$. Hence we obtain E_2 terms as described in the lemma. Convergence by Homology, Lemma 12.25.3. $\square$

0FLH Lemma 20.25.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. If $H^i(U_{i_0 \ldots i_p}, \mathcal{F}^q) = 0$ for all $i > 0$ and all $p, i_0, \ldots, i_p, q$, then the map $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \to R\Gamma(X, \mathcal{F}^\bullet)$ of Lemma 20.25.1 is an isomorphism.

Proof. Immediate from the spectral sequence of Lemma 20.25.1. $\square$

0FLI Remark 20.25.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Let b be an integer. We claim there is a commutative diagram

$$\begin{tikzcd}
\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))[b] \arrow[r] \arrow[d, "\gamma"'] & R\Gamma(X, \mathcal{F}^\bullet)[b] \arrow[d] \\
\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet[b])) \arrow[r] & R\Gamma(X, \mathcal{F}^\bullet[b])
\end{tikzcd}$$

in the derived category where the map γ is the map on complexes constructed in Homology, Remark 12.18.5. This makes sense because the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet[b])$ is clearly the same as the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)[0, b]$ introduced in Homology, Remark 12.18.5. To check that the diagram commutes, we may choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ as in the proof of Lemma 20.25.1. Chasing diagrams, we see that it suffices to check the diagram commutes when we replace $\mathcal{F}^\bullet$ by $\mathcal{I}^\bullet$. Then we consider the extended diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{I}^\bullet)[b] \arrow[r] \arrow[d] & \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))[b] \arrow[r] \arrow[d, "\gamma"'] & R\Gamma(X, \mathcal{I}^\bullet)[b] \arrow[d] \\
\Gamma(X, \mathcal{I}^\bullet[b]) \arrow[r] & \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet[b])) \arrow[r] & R\Gamma(X, \mathcal{I}^\bullet[b])
\end{tikzcd}$$

where the left horizontal arrows are (20.25.0.1). Since in this case the horizonal arrows are isomorphisms in the derived category (see proof of Lemma 20.25.1) it suffices to show that the left square commutes. This is true because the map γ uses the sign 1 on the summands $\check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^{q+b})$, see formula in Homology, Remark 12.18.5.

Let X be a topological space, let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering, and let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups. Consider the map $\tau : \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ defined by

$$\tau(\alpha)_{i_0 \ldots i_p} = (-1)^{p(p+1)/2} \alpha_{i_p \ldots i_0}.$$

Then we have for an element α of degree n that

$$\begin{aligned}
& d(\tau(\alpha))_{i_0 \ldots i_{p+1}} \\
& = \sum\nolimits_{j=0}^{p+1} (-1)^j \tau(\alpha)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\tau(\alpha)_{i_0 \ldots i_{p+1}}) \\
& = \sum\nolimits_{j=0}^{p+1} (-1)^{j + \frac{p(p+1)}{2}} \alpha_{i_{p+1} \ldots \hat{i}_j \ldots i_0} + (-1)^{p+1+\frac{(p+1)(p+2)}{2}} d_{\mathcal{F}}(\alpha_{i_{p+1} \ldots i_0})
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\tau(d(\alpha))_{i_0\ldots i_{p+1}}\\
&= (-1)^{\frac{(p+1)(p+2)}{2}} d(\alpha)_{i_{p+1}\ldots i_0}\\
&= (-1)^{\frac{(p+1)(p+2)}{2}} \left(\sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_{p+1}\ldots \hat i_{p+1-j}\ldots i_0} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_{p+1}\ldots i_0})\right)
\end{aligned}$$

Thus we conclude that $d(\tau(\alpha)) = \tau(d(\alpha))$ because $p(p+1)/2 \equiv (p+1)(p+2)/2 + p+1 \bmod 2$. In other words τ is an endomorphism of the complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. Note that the diagram

$$\begin{matrix}
\Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \\
\downarrow \text{id} & & \downarrow \tau \\
\Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))
\end{matrix}$$

commutes. In addition τ is clearly compatible with refinements. This suggests that τ acts as the identity on Čech cohomology (i.e., in the limit – provided Čech hypercohomology agrees with hypercohomology, which is always the case if we use hypercoverings). We claim that τ actually is homotopic to the identity on the total Čech complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. To prove this, we use as homotopy

$$h(\alpha)_{i_0\ldots i_p} = \sum\nolimits_{a=0}^{p} \epsilon_p(a) \alpha_{i_0\ldots i_a i_p \ldots i_a} \quad \text{with} \quad \epsilon_p(a) = (-1)^{\frac{(p-a)(p-a-1)}{2} + p}$$

for α of degree n. As usual we omit writing $|_{U_{i_0\ldots i_p}}$. This works because of the following computation, again with α an element of degree n:

$$\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{i_0\ldots i_p} = & \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{i_0\ldots \hat i_k \ldots i_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{i_0\ldots i_p}) + \\
& \sum\nolimits_{a=0}^{p} \epsilon_p(a) d(\alpha)_{i_0\ldots i_a i_p\ldots i_a} \\
= & \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=0}^{k-1} (-1)^k \epsilon_{p-1}(a) \alpha_{i_0\ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\
& \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=k+1}^{p} (-1)^k \epsilon_{p-1}(a-1) \alpha_{i_0\ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} (-1)^p \epsilon_p(a) d_{\mathcal{F}}(\alpha_{i_0\ldots i_a i_p\ldots i_a}) + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=0}^{a} \epsilon_p(a) (-1)^k \alpha_{i_0\ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=a}^{p} \epsilon_p(a) (-1)^{p+a+1-k} \alpha_{i_0\ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} \epsilon_p(a) (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0\ldots i_a i_p \ldots i_a}) \\
= & \epsilon_p(0) \alpha_{i_p\ldots i_0} + \epsilon_p(p)(-1)^{p+1} \alpha_{i_0\ldots i_p} \\
= & (-1)^{\frac{p(p+1)}{2}} \alpha_{i_p\ldots i_0} - \alpha_{i_0\ldots i_p}
\end{aligned}$$

The cancellations follow because

$$(-1)^k \epsilon_{p-1}(a) + \epsilon_p(a)(-1)^{p+a+1-k} = 0 \quad \text{and} \quad (-1)^k \epsilon_{p-1}(a-1) + \epsilon_p(a)(-1)^k = 0$$

We leave it to the reader to verify the cancellations.

Suppose we have two bounded below complexes of abelian sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. We define the complex $\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet)$ to be to complex with terms $\bigoplus_{p+q=n} \mathcal{F}^p \otimes \mathcal{G}^q$

and differential according to the rule

07MA (20.25.3.1)
$$d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^{\deg(\alpha)}\alpha \otimes d(\beta)$$

when α and β are homogeneous, see Homology, Definition 12.18.3.

Suppose that $M^\bullet$ and $N^\bullet$ are two bounded below complexes of abelian groups. Then if m, resp. n is a cocycle for $M^\bullet$, resp. $N^\bullet$, it is immediate that $m \otimes n$ is a cocycle for $\text{Tot}(M^\bullet \otimes N^\bullet)$. Hence a cup product

$$H^i(M^\bullet) \times H^j(N^\bullet) \longrightarrow H^{i+j}(Tot(M^\bullet \otimes N^\bullet)).$$

This is discussed also in More on Algebra, Section 15.63.

So the construction of the cup product in hypercohomology of complexes rests on a construction of a map of complexes

(20.25.3.2)

07MB
$$\text{Tot}\left(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet))\right) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)))$$

This map is denoted $\cup$ and is given by the rule

$$(\alpha \cup \beta)_{i_0\ldots i_p} = \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r)\alpha_{i_0\ldots i_r} \otimes \beta_{i_r\ldots i_p}.$$

where α has degree n and β has degree m and with

$$\epsilon(n, m, p, r) = (-1)^{(p+r)n+rp+r}.$$

Note that $\epsilon(n, m, p, n) = 1$. Hence if $\mathcal{F}^\bullet = \mathcal{F}[0]$ is the complex consisting in a single abelian sheaf $\mathcal{F}$ placed in degree 0, then there no signs in the formula for $\cup$ (as in that case $\alpha_{i_0\ldots i_r} = 0$ unless $r = n$). For an explanation of why there has to be a sign and how to compute it see [AGV71, Exposee XVII] by Deligne. To check (20.25.3.2) is a map of complexes we have to show that

$$d(\alpha \cup \beta) = d(\alpha) \cup \beta + (-1)^{\deg(\alpha)}\alpha \cup d(\beta)$$

by the definition of the differential on $\text{Tot}(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet)))$ as given in Homology, Definition 12.18.3. We compute first

$$\begin{aligned} d(\alpha \cup \beta)_{i_0\ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1} (-1)^j (\alpha \cup \beta)_{i_0\ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}\otimes\mathcal{G}}((\alpha \cup \beta)_{i_0\ldots i_{p+1}}) \\ &= \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=0}^{j-1} (-1)^j \epsilon(n, m, p, r)\alpha_{i_0\ldots i_r} \otimes \beta_{i_r\ldots \hat{i}_j \ldots i_{p+1}} + \\ &\quad \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=j+1}^{p+1} (-1)^j \epsilon(n, m, p, r-1)\alpha_{i_0\ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r\ldots i_{p+1}} + \\ &\quad \sum\nolimits_{r=0}^{p+1} (-1)^{p+1} \epsilon(n, m, p+1, r) d_{\mathcal{F}\otimes\mathcal{G}}(\alpha_{i_0\ldots i_r} \otimes \beta_{i_r\ldots i_{p+1}}) \end{aligned}$$

and note that the summands in the last term equal

$$(-1)^{p+1}\epsilon(n, m, p+1, r)\left(d_{\mathcal{F}}(\alpha_{i_0\ldots i_r}) \otimes \beta_{i_r\ldots i_{p+1}} + (-1)^{n-r}\alpha_{i_0\ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r\ldots i_{p+1}})\right).$$

because $\deg_{\mathcal{F}}(\alpha_{i_0\ldots i_r}) = n - r$. On the other hand

$$\begin{aligned} (d(\alpha) \cup \beta)_{i_0\ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r) d(\alpha)_{i_0\ldots i_r} \otimes \beta_{i_r\ldots i_{p+1}} \\ &= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=0}^{r} \epsilon(n+1, m, p+1, r)(-1)^j \alpha_{i_0\ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r\ldots i_{p+1}} + \\ &\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r)(-1)^r d_{\mathcal{F}}(\alpha_{i_0\ldots i_r}) \otimes \beta_{i_r\ldots i_{p+1}} \end{aligned}$$

and

$$\begin{aligned}
(\alpha \cup d(\beta))_{i_0\ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r)\alpha_{i_0\ldots i_r} \otimes d(\beta)_{i_r\ldots i_{p+1}} \\
&= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=r}^{p+1} \epsilon(n, m+1, p+1, r)(-1)^{j-r}\alpha_{i_0\ldots i_r} \otimes \beta_{i_r\ldots \hat{i_j}\ldots i_{p+1}} + \\
&\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r)(-1)^{p+1-r}\alpha_{i_0\ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r\ldots i_{p+1}})
\end{aligned}$$

The desired equality holds if we have

$$\begin{aligned}
(-1)^{p+1}\epsilon(n, m, p+1, r) &= \epsilon(n+1, m, p+1, r)(-1)^r \\
(-1)^{p+1}\epsilon(n, m, p+1, r)(-1)^{n-r} &= (-1)^n\epsilon(n, m+1, p+1, r)(-1)^{p+1-r} \\
\epsilon(n+1, m, p+1, r)(-1)^r &= (-1)^{1+n}\epsilon(n, m+1, p+1, r-1) \\
(-1)^j\epsilon(n, m, p, r) &= (-1)^n\epsilon(n, m+1, p+1, r)(-1)^{j-r} \\
(-1)^j\epsilon(n, m, p, r-1) &= \epsilon(n+1, m, p+1, r)(-1)^j
\end{aligned}$$

(The third equality is necessary to get the terms with $r = j$ from $d(\alpha) \cup \beta$ and $(-1)^n\alpha \cup d(\beta)$ to cancel each other.) We leave the verifications to the reader. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Associativity of the cup product. Suppose that $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ and $\mathcal{H}^\bullet$ are bounded below complexes of abelian groups on X. The obvious map (without the intervention of signs) is an isomorphism of complexes

$$\mathrm{Tot}(\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet) \otimes_{\mathbf{Z}} \mathcal{H}^\bullet) \longrightarrow \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathbf{Z}} \mathcal{H}^\bullet)).$$

Another way to say this is that the triple complex $\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet \otimes_{\mathbf{Z}} \mathcal{H}^\bullet$ gives rise to a well defined total complex with differential satisfying

$$d(\alpha\otimes\beta\otimes\gamma) = d(\alpha)\otimes\beta\otimes\gamma + (-1)^{\deg(\alpha)}\alpha\otimes d(\beta)\otimes\gamma + (-1)^{\deg(\alpha)+\deg(\beta)}\alpha\otimes\beta\otimes d(\gamma)$$

for homogeneous elements. Using this map it is easy to verify that

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

namely, if α has degree a, β has degree b and γ has degree c, then

$$\begin{aligned}
((\alpha \cup \beta) \cup \gamma)_{i_0\ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(a+b, c, p, r)(\alpha \cup \beta)_{i_0\ldots i_r} \otimes \gamma_{i_r\ldots i_p} \\
&= \sum\nolimits_{r=0}^{p} \sum\nolimits_{s=0}^{r} \epsilon(a+b, c, p, r)\epsilon(a, b, r, s)\alpha_{i_0\ldots i_s} \otimes \beta_{i_s\ldots i_r} \otimes \gamma_{i_r\ldots i_p}
\end{aligned}$$

and

$$\begin{aligned}
(\alpha \cup (\beta \cup \gamma)_{i_0\ldots i_p} &= \sum\nolimits_{s=0}^{p} \epsilon(a, b+c, p, s)\alpha_{i_0\ldots i_s} \otimes (\beta \cup \gamma)_{i_s\ldots i_p} \\
&= \sum\nolimits_{s=0}^{p} \sum\nolimits_{r=s}^{p} \epsilon(a, b+c, p, s)\epsilon(b, c, p-s, r-s)\alpha_{i_0\ldots i_s} \otimes \beta_{i_s\ldots i_r} \otimes \gamma_{i_r\ldots i_p}
\end{aligned}$$

and a trivial mod 2 calculation shows the signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Finally, we indicate why the cup product preserves a graded commutative structure, at least on a cohomological level. For this we use the operator τ introduced above. Let $\mathcal{F}^\bullet$ be a bounded below complexes of abelian groups, and assume we are given a graded commutative multiplication

$$\wedge^\bullet : \mathrm{Tot}(\mathcal{F}^\bullet \otimes \mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet.$$

This means the following: For s a local section of $\mathcal{F}^a$, and t a local section of $\mathcal{F}^b$ we have $s \wedge t$ a local section of $\mathcal{F}^{a+b}$. Graded commutative means we have $s \wedge t = (-1)^{ab} t \wedge s$. Since $\wedge$ is a map of complexes we have $d(s \wedge t) = d(s) \wedge t + (-1)^a s \wedge d(t)$. The composition

$$\text{Tot}(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{F}^\bullet))) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

induces a cup product on cohomology

$$H^n(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \times H^m(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \longrightarrow H^{n+m}(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)))$$

and so in the limit also a product on Čech cohomology and therefore (using hypercoverings if needed) a product in cohomology of $\mathcal{F}^\bullet$. We claim this product (on cohomology) is graded commutative as well. To prove this we first consider an element α of degree n in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and an element β of degree m in $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and we compute

$$\begin{aligned} \wedge^\bullet(\alpha \cup \beta)_{i_0 \ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \wedge \beta_{i_r \ldots i_p} \\ &= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r)(-1)^{\deg(\alpha_{i_0 \ldots i_r}) \deg(\beta_{i_r \ldots i_p})} \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r} \end{aligned}$$

because $\wedge$ is graded commutative. On the other hand we have

$$\begin{aligned} \tau(\wedge^\bullet(\tau(\beta) \cup \tau(\alpha)))_{i_0 \ldots i_p} &= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \tau(\beta)_{i_p \ldots i_{p-r}} \wedge \tau(\alpha)_{i_{p-r} \ldots i_0} \\ &= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \chi(r) \chi(p-r) \beta_{i_{p-r} \ldots i_p} \wedge \alpha_{i_0 \ldots i_{p-r}} \\ &= \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, p-r) \chi(r) \chi(p-r) \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r} \end{aligned}$$

where $\chi(t) = (-1)^{\frac{t(t+1)}{2}}$. Since we proved earlier that τ acts as the identity on cohomology we have to verify that

$$\epsilon(n, m, p, r)(-1)^{(n-r)(m-(p-r))} = (-1)^{nm} \chi(p) \epsilon(m, n, p, p-r) \chi(r) \chi(p-r)$$

A trivial mod 2 calculation shows these signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the Stacks project.)

Finally, we study the compatibility of cup product with boundary maps. Suppose that

$$0 \to \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to 0 \quad \text{and} \quad 0 \leftarrow \mathcal{G}_1^\bullet \leftarrow \mathcal{G}_2^\bullet \leftarrow \mathcal{G}_3^\bullet \leftarrow 0$$

are short exact sequences of bounded below complexes of abelian sheaves on X. Let $\mathcal{H}^\bullet$ be another bounded below complex of abelian sheaves, and suppose we have maps of complexes

$$\gamma_i : \text{Tot}(\mathcal{F}_i^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_i^\bullet) \longrightarrow \mathcal{H}^\bullet$$

which are compatible with the maps between the complexes, namely such that the diagrams

$$\begin{tikzcd} \text{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_1^\bullet) \arrow[d, "\gamma_1"'] & \text{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \arrow[l] \arrow[d] \\ \mathcal{H}^\bullet & \text{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \arrow[l, "\gamma_2"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \arrow[d, "\gamma_2"'] & \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet) \arrow[l] \arrow[d] \\ \mathcal{H}^\bullet & \mathrm{Tot}(\mathcal{F}_3^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet) \arrow[l, "\gamma_3"'] \end{tikzcd}$$

are commutative.

07MC Lemma 20.25.4. In the situation above, assume Čech cohomology agrees with cohomology for the sheaves $\mathcal{F}_i^p$ and $\mathcal{G}_j^q$. Let $a_3 \in H^n(X, \mathcal{F}_3^\bullet)$ and $b_1 \in H^m(X, \mathcal{G}_1^\bullet)$. Then we have

$$\gamma_1(\partial a_3 \cup b_1) = (-1)^{n+1}\gamma_3(a_3 \cup \partial b_1)$$

in $H^{n+m}(X, \mathcal{H}^\bullet)$ where ∂ indicates the boundary map on cohomology associated to the short exact sequences of complexes above.

Proof. We will use the following conventions and notation. We think of $\mathcal{F}_1^p$ as a subsheaf of $\mathcal{F}_2^p$ and we think of $\mathcal{G}_3^q$ as a subsheaf of $\mathcal{G}_2^q$. Hence if s is a local section of $\mathcal{F}_1^p$ we use s to denote the corresponding section of $\mathcal{F}_2^p$ as well. Similarly for local sections of $\mathcal{G}_3^q$. Furthermore, if s is a local section of $\mathcal{F}_2^p$ then we denote $\bar{s}$ its image in $\mathcal{F}_3^p$. Similarly for the map $\mathcal{G}_2^q \to \mathcal{G}_1^q$. In particular if s is a local section of $\mathcal{F}_2^p$ and $\bar{s} = 0$ then s is a local section of $\mathcal{F}_1^p$. The commutativity of the diagrams above implies, for local sections s of $\mathcal{F}_2^p$ and t of $\mathcal{G}_3^q$ that $\gamma_2(s \otimes t) = \gamma_3(\bar{s} \otimes t)$ as sections of $\mathcal{H}^{p+q}$.

Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Suppose that α_3, resp. β_1 is a degree n, resp. m cocycle of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3^\bullet))$, resp. $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_1^\bullet))$ representing a_3, resp. b_1. After refining $\mathcal{U}$ if necessary, we can find cochains α_2, resp. β_2 of degree n, resp. m in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2^\bullet))$, resp. $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_2^\bullet))$ mapping to α_3, resp. β_1. Then we see that

$$\overline{d(\alpha_2)} = d(\bar{\alpha}_2) = 0 \quad \text{and} \quad \overline{d(\beta_2)} = d(\bar{\beta}_2) = 0.$$

This means that $\alpha_1 = d(\alpha_2)$ is a degree $n+1$ cocycle in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1^\bullet))$ representing ∂a_3. Similarly, $\beta_3 = d(\beta_2)$ is a degree $m+1$ cocycle in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_3^\bullet))$ representing ∂b_1. Thus we may compute

$$\begin{aligned} d(\gamma_2(\alpha_2 \cup \beta_2)) &= \gamma_2(d(\alpha_2 \cup \beta_2)) \\ &= \gamma_2(d(\alpha_2) \cup \beta_2 + (-1)^n \alpha_2 \cup d(\beta_2)) \\ &= \gamma_2(\alpha_1 \cup \beta_2) + (-1)^n \gamma_2(\alpha_2 \cup \beta_3) \\ &= \gamma_1(\alpha_1 \cup \beta_1) + (-1)^n \gamma_3(\alpha_3 \cup \beta_3) \end{aligned}$$

So this even tells us that the sign is $(-1)^{n+1}$ as indicated in the lemma[2]. □

0B8S Lemma 20.25.5. Let X be a topological space. Let $\mathcal{O}' \to \mathcal{O}$ be a surjection of sheaves of rings whose kernel $\mathcal{I} \subset \mathcal{O}'$ has square zero. Then $M = H^1(X, \mathcal{I})$ is a $R = H^0(X, \mathcal{O})$-module and the boundary map $\partial : R \to M$ associated to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

[2] The sign depends on the convention for the signs in the long exact sequence in cohomology associated to a triangle in $D(X)$. The conventions in the Stacks project are (a) distinguished triangles correspond to termwise split exact sequences and (b) the boundary maps in the long exact sequence are given by the maps in the snake lemma without the intervention of signs. See Derived Categories, Section 13.10.

is a derivation (Algebra, Definition 10.131.1).

Proof. The map $\mathcal{O}' \to \mathcal{H}om(\mathcal{I}, \mathcal{I})$ factors through $\mathcal{O}$ as $\mathcal{I} \cdot \mathcal{I} = 0$ by assumption. Hence $\mathcal{I}$ is a sheaf of $\mathcal{O}$-modules and this defines the R-module structure on M. The boundary map is additive hence it suffices to prove the Leibniz rule. Let $f \in R$. Choose an open covering $\mathcal{U} : X = \bigcup U_i$ such that there exist $f_i \in \mathcal{O}'(U_i)$ lifting $f|_{U_i} \in \mathcal{O}(U_i)$. Observe that $f_i - f_j$ is an element of $\mathcal{I}(U_i \cap U_j)$. Then $\partial(f)$ corresponds to the Čech cohomology class of the 1-cocycle α with $\alpha_{i_0 i_1} = f_{i_0} - f_{i_1}$. (Observe that by Lemma 20.11.3 the first Čech cohomology group with respect to $\mathcal{U}$ is a submodule of M.) Next, let $g \in R$ be a second element and assume (after possibly refining the open covering) that $g_i \in \mathcal{O}'(U_i)$ lifts $g|_{U_i} \in \mathcal{O}(U_i)$. Then we see that $\partial(g)$ is given by the cocycle β with $\beta_{i_0 i_1} = g_{i_0} - g_{i_1}$. Since $f_i g_i \in \mathcal{O}'(U_i)$ lifts $fg|_{U_i}$ we see that $\partial(fg)$ is given by the cocycle γ with

$$\gamma_{i_0 i_1} = f_{i_0} g_{i_0} - f_{i_1} g_{i_1} = (f_{i_0} - f_{i_1}) g_{i_0} + f_{i_1}(g_{i_0} - g_{i_1}) = \alpha_{i_0 i_1} g + f \beta_{i_0 i_1}$$

by our definition of the $\mathcal{O}$-module structure on $\mathcal{I}$. This proves the Leibniz rule and the proof is complete. □

20.26. Flat resolutions

06Y7 A reference for the material in this section is [Spa88]. Let $(X, \mathcal{O}_X)$ be a ringed space. By Modules, Lemma 17.17.6 any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module. By Derived Categories, Lemma 13.15.4 any bounded above complex of $\mathcal{O}_X$-modules has a left resolution by a bounded above complex of flat $\mathcal{O}_X$-modules. However, for unbounded complexes, it turns out that flat resolutions aren't good enough.

06Y8 **Lemma 20.26.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The functors*

$$K(\text{Mod}(\mathcal{O}_X)) \longrightarrow K(\text{Mod}(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto \text{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$$

and

$$K(\text{Mod}(\mathcal{O}_X)) \longrightarrow K(\text{Mod}(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

are exact functors of triangulated categories.

Proof. This follows from Derived Categories, Remark 13.10.9. □

06Y9 **Definition 20.26.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the complex

$$\text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

is acyclic.

06YA **Lemma 20.26.3.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(\text{Mod}(\mathcal{O}_X)) \longrightarrow K(\text{Mod}(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto \text{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 20.26.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

06YB Lemma 20.26.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then $\mathcal{K}^\bullet$ is K-flat if and only if for all $x \in X$ the complex $\mathcal{K}_x^\bullet$ of $\mathcal{O}_{X,x}$-modules is K-flat (More on Algebra, Definition 15.59.1).

Proof. If $\mathcal{K}_x^\bullet$ is K-flat for all $x \in X$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules, Lemma 17.3.1. Conversely, assume $\mathcal{K}^\bullet$ is K-flat. Pick $x \in X$ $M^\bullet$ be an acyclic complex of $\mathcal{O}_{X,x}$-modules. Then $i_{x,*}M^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules. Thus $\text{Tot}(i_{x,*}M^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$ is acyclic. Taking stalks at x shows that $\text{Tot}(M^\bullet \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_x^\bullet)$ is acyclic. $\square$

079R Lemma 20.26.5. Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}_X$-modules, then $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}_X$-modules.

Proof. Follows from the isomorphism

$$\text{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) = \text{Tot}(\text{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$$

and the definition. $\square$

079S Lemma 20.26.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(\text{Mod}(\mathcal{O}_X))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.

Proof. Follows from Lemma 20.26.1 and the fact that in a distinguished triangle in $K(\text{Mod}(\mathcal{O}_X))$ if two out of three are acyclic, so is the third. $\square$

0G6U Lemma 20.26.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $0 \to \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \mathcal{K}_3^\bullet \to 0$ be a short exact sequence of complexes such that the terms of $\mathcal{K}_3^\bullet$ are flat $\mathcal{O}_X$-modules. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.

Proof. By Modules, Lemma 17.17.7 for every complex $\mathcal{L}^\bullet$ we obtain a short exact sequence

$$0 \to \text{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to \text{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to \text{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_1^\bullet) \to 0$$

of complexes. Hence the lemma follows from the long exact sequence of cohomology sheaves and the definition of K-flat complexes. $\square$

06YC Lemma 20.26.8. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback of a K-flat complex of $\mathcal{O}_Y$-modules is a K-flat complex of $\mathcal{O}_X$-modules.

Proof. We can check this on stalks, see Lemma 20.26.4. Hence this follows from Sheaves, Lemma 6.26.4 and More on Algebra, Lemma 15.59.3. $\square$

06YD Lemma 20.26.9. Let $(X, \mathcal{O}_X)$ be a ringed space. A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat.

Proof. We can check this on stalks, see Lemma 20.26.4. Thus this lemma follows from Modules, Lemma 17.17.2 and More on Algebra, Lemma 15.59.7. $\square$

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

06YE Lemma 20.26.10. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\text{colim}_i \mathcal{K}_i^\bullet$ is K-flat.

Proof. Because we are taking termwise colimits it is clear that

$$\mathrm{colim}_i \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_i^\bullet) = \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathrm{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

079T **Lemma 20.26.11.** Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules there exists a commutative diagram of complexes of $\mathcal{O}_X$-modules

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms and termwise surjective, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\mathrm{colim}\,\mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules, Lemma 17.17.6 and Derived Categories, Lemma 13.29.1. The induced map $\mathrm{colim}\,\mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. □

06YF **Lemma 20.26.12.** Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ whose terms are flat $\mathcal{O}_X$-modules and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ which is termwise surjective.

Proof. Choose a diagram as in Lemma 20.26.11. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules, Lemma 17.17.5. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 20.26.9. Thus $\mathrm{colim}\,\mathcal{K}_n^\bullet$ is K-flat by Lemma 20.26.10. The induced map $\mathrm{colim}\,\mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism and termwise surjective by construction. Property (3) of Lemma 20.26.11 shows that $\mathrm{colim}\,\mathcal{K}_n^m$ is a direct sum of flat modules and hence flat which proves the final assertion. □

06YG **Lemma 20.26.13.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}_X$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the induced map

$$\operatorname{Tot}(\mathrm{id}_{\mathcal{F}^\bullet} \otimes \alpha) : \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \longrightarrow \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 20.26.12. Consider the commutative diagram

$$\begin{tikzcd} \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \arrow[r] \arrow[d] & \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \arrow[d] \\ \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \arrow[r] & \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \end{tikzcd}$$

The result follows as by Lemma 20.26.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 20.26.12. By Lemma 20.26.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}_X) \longrightarrow K(\mathcal{O}_X), \quad \mathcal{G}^\bullet \longmapsto \text{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

By Lemma 20.26.3 this functor induces a functor $D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ simply because $D(\mathcal{O}_X)$ is the localization of $K(\mathcal{O}_X)$ at quasi-isomorphisms. By Lemma 20.26.13 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

06YH Definition 20.26.14. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. The derived tensor product

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$. Here we use sign rules as given in More on Algebra, Section 15.72. Hence when we write $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

08BP Definition 20.26.15. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. The Tor's of $\mathcal{F}$ and $\mathcal{G}$ are define by the formula

$$\text{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{tikzcd}
\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r] & \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r] & \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[r] & 0 \\
\text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \arrow[r] & \text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G}) \arrow[r] & \text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{G}) \arrow[ull]
\end{tikzcd}$$

for every $\mathcal{O}_X$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

08BQ Lemma 20.26.16. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is a flat $\mathcal{O}_X$-module, and
(2) $\text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}_X$-module $\mathcal{G}$.

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}_X} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}$ is a quotient of $\text{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. $\square$

0G6V Lemma 20.26.17. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $a : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ be a map of complexes of $\mathcal{O}_X$-modules. If $\mathcal{K}^\bullet$ is K-flat, then there exist a complex $\mathcal{N}^\bullet$ and maps of complexes $b : \mathcal{K}^\bullet \to \mathcal{N}^\bullet$ and $c : \mathcal{N}^\bullet \to \mathcal{L}^\bullet$ such that

(1) $\mathcal{N}^\bullet$ is K-flat,

(2) c is a quasi-isomorphism,
(3) a is homotopic to $c \circ b$.

If the terms of $\mathcal{K}^\bullet$ are flat, then we may choose $\mathcal{N}^\bullet$, b, and c such that the same is true for $\mathcal{N}^\bullet$.

Proof. We will use that the homotopy category $K(\text{Mod}(\mathcal{O}_X))$ is a triangulated category, see Derived Categories, Proposition 13.10.3. Choose a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{L}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$. Choose a quasi-isomorphism $\mathcal{M}^\bullet \to \mathcal{C}^\bullet$ with $\mathcal{M}^\bullet$ K-flat with flat terms, see Lemma 20.26.12. By the axioms of triangulated categories, we may fit the composition $\mathcal{M}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$ into a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{N}^\bullet \to \mathcal{M}^\bullet \to \mathcal{K}^\bullet[1]$. By Lemma 20.26.6 we see that $\mathcal{N}^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ fitting into the following morphism of distinghuised triangles

$$\begin{tikzcd} \mathcal{K}^\bullet \arrow[r] \arrow[d] & \mathcal{N}^\bullet \arrow[r] \arrow[d] & \mathcal{M}^\bullet \arrow[r] \arrow[d] & \mathcal{K}^\bullet[1] \arrow[d] \\ \mathcal{K}^\bullet \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{C}^\bullet \arrow[r] & \mathcal{K}^\bullet[1] \end{tikzcd}$$

Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(\mathcal{O}_X)$ and use Derived Categories, Lemma 13.4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $\mathcal{N}^\bullet$ such that $\mathcal{N}^n \cong \mathcal{M}^n \oplus \mathcal{K}^n$, see Derived Categories, Lemma 13.10.7. Hence we get the desired flatness if the terms of $\mathcal{K}^\bullet$ are flat. $\square$

20.27. Derived pullback

06YI Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

Namely, for every complex of $\mathcal{O}_Y$-modules $\mathcal{G}^\bullet$ we can choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ and set $Lf^*\mathcal{G}^\bullet = f^*\mathcal{K}^\bullet$. You can use Lemmas 20.26.8, 20.26.12, and 20.26.13 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

06YJ Lemma 20.27.1. The construction above is independent of choices and defines an exact functor of triangulated categories $Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$.

Proof. To see this we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(\mathcal{O}_Y)$ and $\mathcal{D}' = D(\mathcal{O}_X)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}_Y)$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \text{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 20.26.12 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between K-flat complexes of $\mathcal{O}_Y$-modules the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. By Lemma 20.26.8 applied to the morphism $(X, f^{-1}\mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ the complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}_Y$-modules. Hence Lemma 20.26.13 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}_Y) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O}_X)$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is K-flat, i.e., $Lf^* = LF$ is indeed computed in the way described above. □

0D5S Lemma 20.27.2. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then $Lf^* \circ Lg^* = L(g \circ f)^*$ as functors $D(\mathcal{O}_Z) \to D(\mathcal{O}_X)$.

Proof. Let E be an object of $D(\mathcal{O}_Z)$. By construction Lg^*E is computed by choosing a K-flat complex $\mathcal{K}^\bullet$ representing E on Z and setting $Lg^*E = g^*\mathcal{K}^\bullet$. By Lemma 20.26.8 we see that $g^*\mathcal{K}^\bullet$ is K-flat on Y. Then Lf^*Lg^*E is given by $f^*g^*\mathcal{K}^\bullet = (g \circ f)^*\mathcal{K}^\bullet$ which also represents $L(g \circ f)^*E$. □

079U Lemma 20.27.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}_Y))$.

Proof. We may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet$. By Lemma 20.26.5 $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\mathrm{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{G}^\bullet) \longrightarrow f^*\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}_X} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}_Y} \mathcal{G}^q)$ of Modules, Lemma 17.16.4. □

08DE Lemma 20.27.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$ and $\mathcal{G}^\bullet$ in $D(\mathcal{O}_Y)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

0FP0 **Lemma 20.27.5.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_Y$-modules. The diagram

$$\begin{tikzcd}
Lf^*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{M}^\bullet) \arrow[r] \arrow[d] & Lf^*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \arrow[d] \\
Lf^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{M}^\bullet \arrow[d] & f^*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \arrow[d] \\
f^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^*\mathcal{M}^\bullet \arrow[r] & \text{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{M}^\bullet)
\end{tikzcd}$$

commutes.

Proof. We will use the existence of K-flat resolutions as in Lemma 20.26.8. If we choose such resolutions $\mathcal{P}^\bullet \to \mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet \to \mathcal{M}^\bullet$, then we see that

$$\begin{tikzcd}
Lf^*\text{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{Q}^\bullet) \arrow[r] \arrow[d] & Lf^*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \arrow[d] \\
f^*\text{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{Q}^\bullet) \arrow[r] \arrow[d] & f^*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{M}^\bullet) \arrow[d] \\
\text{Tot}(f^*\mathcal{P}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{Q}^\bullet) \arrow[r] & \text{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{M}^\bullet)
\end{tikzcd}$$

commutes. However, now the left hand side of the diagram is the left hand side of the diagram by our choice of $\mathcal{P}^\bullet$ and $\mathcal{Q}^\bullet$ and Lemma 20.26.5. □

20.28. Cohomology of unbounded complexes

079V Let $(X, \mathcal{O}_X)$ be a ringed space. The category $\text{Mod}(\mathcal{O}_X)$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \subset X \text{ open}} j_{U!}\mathcal{O}_U,$$

see Modules, Section 17.3 and Lemmas 17.17.5 and 17.17.6. By Injectives, Theorem 19.12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}_X$-modules all of whose terms are injective $\mathcal{O}_X$-modules and moreover this embedding can be chosen functorial in the complex $\mathcal{F}^\bullet$. It follows from Derived Categories, Lemma 13.31.7 that

(1) any exact functor $F : K(\text{Mod}(\mathcal{O}_X)) \to \mathcal{D}$ into a trianguated category $\mathcal{D}$ has a right derived functor $RF : D(\mathcal{O}_X) \to \mathcal{D}$,
(2) for any additive functor $F : \text{Mod}(\mathcal{O}_X) \to \mathcal{A}$ into an abelian category $\mathcal{A}$ we consider the exact functor $F : K(\text{Mod}(\mathcal{O}_X)) \to D(\mathcal{A})$ induced by F and we obtain a right derived functor $RF : D(\mathcal{O}_X) \to K(\mathcal{A})$.

By construction we have $RF(\mathcal{F}^\bullet) = F(\mathcal{I}^\bullet)$ where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is as above.

Here are some examples of the above:

(1) The functor $\Gamma(X, -) : \text{Mod}(\mathcal{O}_X) \to \text{Mod}_{\Gamma(X,\mathcal{O}_X)}$ gives rise to
$$R\Gamma(X, -) : D(\mathcal{O}_X) \to D(\Gamma(X, \mathcal{O}_X))$$
We shall use the notation $H^i(X, K) = H^i(R\Gamma(X, K))$ for cohomology.

(2) For an open $U \subset X$ we consider the functor $\Gamma(U, -) : \text{Mod}(\mathcal{O}_X) \to \text{Mod}_{\Gamma(U,\mathcal{O}_X)}$. This gives rise to

$$R\Gamma(U, -) : D(\mathcal{O}_X) \to D(\Gamma(U, \mathcal{O}_X))$$

We shall use the notation $H^i(U, K) = H^i(R\Gamma(U, K))$ for cohomology.

(3) For a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we consider the functor $f_* : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_Y)$ which gives rise to the total direct image

$$Rf_* : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_Y)$$

on unbounded derived categories.

079W Lemma 20.28.1. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The functor Rf_* defined above and the functor Lf^* defined in Lemma 20.27.1 are adjoint:

$$\text{Hom}_{D(\mathcal{O}_X)}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \text{Hom}_{D(\mathcal{O}_Y)}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \text{Ob}(D(\mathcal{O}_X))$ and $\mathcal{G}^\bullet \in \text{Ob}(D(\mathcal{O}_Y))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 13.30.3. □

0D5T Lemma 20.28.2. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_Z)$.

Proof. By Lemma 20.28.1 we see that $Rg_* \circ Rf_*$ is adjoint to $Lf^* \circ Lg^*$. We have $Lf^* \circ Lg^* = L(g \circ f)^*$ by Lemma 20.27.2 and hence by uniqueness of adjoint functors we have $Rg_* \circ Rf_* = R(g \circ f)_*$. □

08HY Remark 20.28.3. The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

is a commutative diagram of ringed spaces. Let K be an object of $D(\mathcal{O}_X)$. Then there exists a canonical base change map

$$Lg^*Rf_*K \longrightarrow R(f')_*L(g')^*K$$

in $D(\mathcal{O}_{S'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*K \to L(g')^*K$ Since $L(f')^*Lg^* = L(g')^*Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*K \to L(g')^*K$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*K \to K$.

0ATL Remark 20.28.4. Consider a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "k"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "l"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z' \arrow[r, "m"] & Z \end{tikzcd}$$

of ringed spaces. Then the base change maps of Remark 20.28.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} Lm^* \circ R(g \circ f)_* &= Lm^* \circ Rg_* \circ Rf_* \\ &\to Rg'_* \circ Ll^* \circ Rf_* \\ &\to Rg'_* \circ Rf'_* \circ Lk^* \\ &= R(g' \circ f')_* \circ Lk^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0ATM Remark 20.28.5. Consider a commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "g'"'] \arrow[d, "f''"'] & X' \arrow[r, "g"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h'"] & Y' \arrow[r, "h"] & Y \end{tikzcd}$$

of ringed spaces. Then the base change maps of Remark 20.28.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} L(h \circ h')^* \circ Rf_* &= L(h')^* \circ Lh_* \circ Rf_* \\ &\to L(h')^* \circ Rf'_* \circ Lg^* \\ &\to Rf''_* \circ L(g')^* \circ Lg^* \\ &= Rf''_* \circ L(g \circ g')^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0FP1 Lemma 20.28.6. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The diagram

$$\begin{tikzcd} Lf^* f_* \mathcal{K}^\bullet \arrow[r] \arrow[d] & f^* f_* \mathcal{K}^\bullet \arrow[d] \\ Lf^* Rf_* \mathcal{K}^\bullet \arrow[r] & \mathcal{K}^\bullet \end{tikzcd}$$

coming from $Lf^* \to f^*$ on complexes, $f_* \to Rf_*$ on complexes, and adjunction $Lf^* \circ Rf_* \to \mathrm{id}$ commutes in $D(\mathcal{O}_X)$.

Proof. We will use the existence of K-flat resolutions and K-injective resolutions, see Lemma 20.26.8 and the discussion above. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_X$-modules. Choose a quasi-isomorphism $\mathcal{Q}^\bullet \to f_*\mathcal{I}^\bullet$ where $\mathcal{Q}^\bullet$ is K-flat as a complex of $\mathcal{O}_Y$-modules. We can choose a K-flat complex of $\mathcal{O}_Y$-modules $\mathcal{P}^\bullet$ and a diagram of morphisms of complexes

$$\begin{tikzcd} \mathcal{P}^\bullet \arrow[r] \arrow[d] & f_*\mathcal{K}^\bullet \arrow[d] \\ \mathcal{Q}^\bullet \arrow[r] & f_*\mathcal{I}^\bullet \end{tikzcd}$$

commutative up to homotopy where the top horizontal arrow is a quasi-isomorphism. Namely, we can first choose such a diagram for some complex $\mathcal{P}^\bullet$ because the quasi-isomorphisms form a multiplicative system in the homotopy category of complexes and then we can replace $\mathcal{P}^\bullet$ by a K-flat complex. Taking pullbacks we obtain a diagram of morphisms of complexes

$$\begin{tikzcd} f^*\mathcal{P}^\bullet \arrow[r] \arrow[d] & f^*f_*\mathcal{K}^\bullet \arrow[r] \arrow[d] & \mathcal{K}^\bullet \arrow[d] \\ f^*\mathcal{Q}^\bullet \arrow[r] & f^*f_*\mathcal{I}^\bullet \arrow[r] & \mathcal{I}^\bullet \end{tikzcd}$$

commutative up to homotopy. The outer rectangle witnesses the truth of the statement in the lemma. ☐

0B68 Remark 20.28.7. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The adjointness of Lf^* and Rf_* allows us to construct a relative cup product

$$Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \longrightarrow Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

in $D(\mathcal{O}_Y)$ for all K, L in $D(\mathcal{O}_X)$. Namely, this map is adjoint to a map $Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L) \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ for which we can take the composition of the isomorphism $Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L) = Lf^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*Rf_*L$ (Lemma 20.27.3) with the map $Lf^*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*Rf_*L \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ coming from the counit $Lf^* \circ Rf_* \to \mathrm{id}$.

20.29. Cohomology of filtered complexes

0FLJ Filtered complexes of sheaves frequently come up in a natural fashion when studying cohomology of algebraic varieties, for example the de Rham complex comes with its Hodge filtration. In this sectionwe use the very general Injectives, Lemma 19.13.7 to find construct spectral sequences on cohomology and we relate these to previously constructed spectral sequences.

0BKK Lemma 20.29.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. There exists a canonical spectral sequence $(E_r, \mathrm{d}_r)_{r\geq 1}$ of bigraded $\Gamma(X, \mathcal{O}_X)$-modules with d_r of bidegree $(r, -r+1)$ and

$$E_1^{p,q} = H^{p+q}(X, \mathrm{gr}^p\mathcal{F}^\bullet)$$

If for every n we have

$$H^n(X, F^p\mathcal{F}^\bullet) = 0 \text{ for } p \gg 0 \quad \text{and} \quad H^n(X, F^p\mathcal{F}^\bullet) = H^n(X, \mathcal{F}^\bullet) \text{ for } p \ll 0$$

then the spectral sequence is bounded and converges to $H^*(X, \mathcal{F}^\bullet)$.

Proof. (For a proof in case the complex is a bounded below complex of modules with finite filtrations, see the remark below.) Choose an map of filtered complexes $j : \mathcal{F}^\bullet \to \mathcal{J}^\bullet$ as in Injectives, Lemma 19.13.7. The spectral sequence is the spectral sequence of Homology, Section 12.24 associated to the filtered complex

$$\Gamma(X, \mathcal{J}^\bullet) \quad \text{with} \quad F^p\Gamma(X, \mathcal{J}^\bullet) = \Gamma(X, F^p\mathcal{J}^\bullet)$$

Since cohomology is computed by evaluating on K-injective representatives we see that the E_1 page is as stated in the lemma. The convergence and boundedness under the stated conditions follows from Homology, Lemma 12.24.13. ☐

0BKL Remark 20.29.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. If $\mathcal{F}^\bullet$ is bounded from below and for each n the filtration on $\mathcal{F}^n$ is finite, then there is a construction of the spectral sequence in Lemma 20.29.1 avoiding Injectives, Lemma 19.13.7. Namely, by Derived Categories, Lemma 13.26.9 there is a filtered quasi-isomorphism $i : \mathcal{F}^\bullet \to \mathcal{I}^\bullet$ of filtered complexes with $\mathcal{I}^\bullet$

bounded below, the filtration on $\mathcal{I}^n$ is finite for all n, and with each $\text{gr}^p\mathcal{I}^n$ an injective $\mathcal{O}_X$-module. Then we take the spectral sequence associated to

$$\Gamma(X, \mathcal{I}^\bullet) \quad \text{with} \quad F^p\Gamma(X, \mathcal{I}^\bullet) = \Gamma(X, F^p\mathcal{I}^\bullet)$$

Since cohomology can be computed by evaluating on bounded below complexes of injectives we see that the E_1 page is as stated in the lemma. The convergence and boundedness under the stated conditions follows from Homology, Lemma 12.24.11. In fact, this is a special case of the spectral sequence in Derived Categories, Lemma 13.26.14.

0BKM Example 20.29.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We can apply Lemma 20.29.1 with $F^p\mathcal{F}^\bullet = \tau_{\leq -p}\mathcal{F}^\bullet$. (If $\mathcal{F}^\bullet$ is bounded below we can use Remark 20.29.2.) Then we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(X, H^{-p}(\mathcal{F}^\bullet)[p]) = H^{2p+q}(X, H^{-p}(\mathcal{F}^\bullet))$$

After renumbering $p = -j$ and $q = i + 2j$ we find that for any $K \in D(\mathcal{O}_X)$ there is a spectral sequence $(E'_r, d'_r)_{r\geq 2}$ of bigraded modules with d'_r of bidegree $(r, -r+1)$, with

$$(E'_2)^{i,j} = H^i(X, H^j(K))$$

If K is bounded below (for example), then this spectral sequence is bounded and converges to $H^{i+j}(X, K)$. In the bounded below case this spectral sequence is an example of the second spectral sequence of Derived Categories, Lemma 13.21.3 (constructed using Cartan-Eilenberg resolutions).

0FLK Example 20.29.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We can apply Lemma 20.29.1 with $F^p\mathcal{F}^\bullet = \sigma_{\geq p}\mathcal{F}^\bullet$. Then we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(X, \mathcal{F}^p[-p]) = H^q(X, \mathcal{F}^p)$$

If $\mathcal{F}^\bullet$ is bounded below, then

(1) we can use Remark 20.29.2 to construct this spectral sequence,
(2) the spectral sequence is bounded and converges to $H^{i+j}(X, \mathcal{F}^\bullet)$, and
(3) the spectral sequence is equal to the first spectral sequence of Derived Categories, Lemma 13.21.3 (constructed using Cartan-Eilenberg resolutions).

0FLL Lemma 20.29.5. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a filtered complex of $\mathcal{O}_X$-modules. There exists a canonical spectral sequence $(E_r, \text{d}_r)_{r\geq 1}$ of bigraded $\mathcal{O}_Y$-modules with d_r of bidegree $(r, -r+1)$ and

$$E_1^{p,q} = R^{p+q}f_*\text{gr}^p\mathcal{F}^\bullet$$

If for every n we have

$$R^nf_*F^p\mathcal{F}^\bullet = 0 \text{ for } p \gg 0 \quad \text{and} \quad R^nf_*F^p\mathcal{F}^\bullet = R^nf_*\mathcal{F}^\bullet \text{ for } p \ll 0$$

then the spectral sequence is bounded and converges to $Rf_*\mathcal{F}^\bullet$.

Proof. The proof is exactly the same as the proof of Lemma 20.29.1. $\square$

20.30. Godement resolution

0FKR A reference is [God73].

Let $(X, \mathcal{O}_X)$ be a ringed space. Denote X_{disc} the discrete topological space with the same points as X. Denote $f : X_{disc} \to X$ the obvious continuous map. Set $\mathcal{O}_{X_{disc}} = f^{-1}\mathcal{O}_X$. Then $f : (X_{disc}, \mathcal{O}_{X_{disc}}) \to (X, \mathcal{O}_X)$ is a flat morphism of ringed spaces. We can apply the dual of the material in Simplicial, Section 14.34 to the adjoint pair of functors f^*, f_* on sheaves of modules. Thus we obtain an augmented cosimplicial object

$$\mathrm{id} \longrightarrow f_*f^* \rightleftarrows f_*f^*f_*f^* \ \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\ f_*f^*f_*f^*f_*f^*$$

in the category of functors from $\mathrm{Mod}(\mathcal{O}_X)$ to itself, see Simplicial, Lemma 14.34.2. Moreover, the augmentation

$$f^* \longrightarrow f^*f_*f^* \rightleftarrows f^*f_*f^*f_*f^* \ \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\ f^*f_*f^*f_*f^*f_*f^*$$

is a homotopy equivalence, see Simplicial, Lemma 14.34.3.

0FKS **Lemma 20.30.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a resolution

$$0 \to \mathcal{F} \to f_*f^*\mathcal{F} \to f_*f^*f_*f^*\mathcal{F} \to f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots$$

functorial in $\mathcal{F}$ such that each term $f_*f^* \ldots f_*f^*\mathcal{F}$ is a flasque $\mathcal{O}_X$-module and such that for all $x \in X$ the map

$$\mathcal{F}_x[0] \to \Big((f_*f^*\mathcal{F})_x \to (f_*f^*f_*f^*\mathcal{F})_x \to (f_*f^*f_*f^*f_*f^*\mathcal{F})_x \to \ldots\Big)$$

is a homotopy equivalence in the category of complexes of $\mathcal{O}_{X,x}$-modules.

Proof. The complex $f_*f^*\mathcal{F} \to f_*f^*f_*f^*\mathcal{F} \to f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots$ is the complex associated to the cosimplicial object with terms $f_*f^*\mathcal{F}, f_*f^*f_*f^*\mathcal{F}, f_*f^*f_*f^*f_*f^*\mathcal{F}, \ldots$ described above, see Simplicial, Section 14.25. The augmentation gives rise to the map $\mathcal{F} \to f_*f^*\mathcal{F}$ as indicated. For any abelian sheaf $\mathcal{H}$ on X_{disc} the pushforward $f_*\mathcal{H}$ is flasque because X_{disc} is a discrete space and the pushforward of a flasque sheaf is flasque. Hence the terms of the complex are flasque $\mathcal{O}_X$-modules.

If $x \in X_{disc} = X$ is a point, then $(f^*\mathcal{G})_x = \mathcal{G}_x$ for any $\mathcal{O}_X$-module $\mathcal{G}$. Hence f^* is an exact functor and a complex of $\mathcal{O}_X$-modules $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3$ is exact if and only if $f^*\mathcal{G}_1 \to f^*\mathcal{G}_2 \to f^*\mathcal{G}_3$ is exact (see Modules, Lemma 17.3.1). The result mentioned in the introduction to this section proves the pullback by f^* gives a homotopy equivalence from the constant cosimplicial object $f^*\mathcal{F}$ to the cosimplicial object with terms $f_*f^*\mathcal{F}, f_*f^*f_*f^*\mathcal{F}, f_*f^*f_*f^*f_*f^*\mathcal{F}, \ldots$. By Simplicial, Lemma 14.28.7 we obtain that

$$f^*\mathcal{F}[0] \to \Big(f^*f_*f^*\mathcal{F} \to f^*f_*f^*f_*f^*\mathcal{F} \to f^*f_*f^*f_*f^*f_*f^*\mathcal{F} \to \ldots\Big)$$

is a homotopy equivalence. This immediately implies the two remaining statements of the lemma. $\square$

0FKT **Lemma 20.30.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ where $\mathcal{G}^\bullet$ be a bounded below complex of flasque $\mathcal{O}_X$-modules and for all $x \in X$ the map $\mathcal{F}^\bullet_x \to \mathcal{G}^\bullet_x$ is a homotopy equivalence in the category of complexes of $\mathcal{O}_{X,x}$-modules.

Proof. Let $\mathcal{A}$ be the category of complexes of $\mathcal{O}_X$-modules and let $\mathcal{B}$ be the category of complexes of $\mathcal{O}_X$-modules. Then we can apply the discussion above to the adjoint functors f^* and f_* between $\mathcal{A}$ and $\mathcal{B}$. Arguing exactly as in the proof of Lemma 20.30.1 we get a resolution

$$0 \to \mathcal{F}^\bullet \to f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet \to \ldots$$

in the abelian category $\mathcal{A}$ such that each term of each $f_*f^* \ldots f_*f^*\mathcal{F}^\bullet$ is a flasque $\mathcal{O}_X$-module and such that for all $x \in X$ the map

$$\mathcal{F}^\bullet_x[0] \to \Big((f_*f^*\mathcal{F}^\bullet)_x \to (f_*f^*f_*f^*\mathcal{F}^\bullet)_x \to (f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet)_x \to \ldots\Big)$$

is a homotopy equivalence in the category of complexes of complexes of $\mathcal{O}_{X,x}$-modules. Since a complex of complexes is the same thing as a double complex, we can consider the induced map

$$\mathcal{F}^\bullet \to \mathcal{G}^\bullet = \mathrm{Tot}(f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*\mathcal{F}^\bullet \to f_*f^*f_*f^*f_*f^*\mathcal{F}^\bullet \to \ldots)$$

Since the complex $\mathcal{F}^\bullet$ is bounded below, the same is true for $\mathcal{G}^\bullet$ and in fact each term of $\mathcal{G}^\bullet$ is a finite direct sum of terms of the complexes $f_*f^* \ldots f_*f^*\mathcal{F}^\bullet$ and hence is flasque. The final assertion of the lemma now follows from Homology, Lemma 12.25.5. Since this in particular shows that $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism, the proof is complete. □

20.31. Cup product

0FKU Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, M be objects of $D(\mathcal{O}_X)$. Set $A = \Gamma(X, \mathcal{O}_X)$. The (global) cup product in this setting is a map

$$\mu : R\Gamma(X, K) \otimes_A^{\mathbf{L}} R\Gamma(X, M) \longrightarrow R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

in $D(A)$. We define it as the relative cup product for the morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (pt, A)$ as in Remark 20.28.7 via $D(pt, A) = D(A)$. This map in particular defines pairings

$$\cup : H^i(X, K) \times H^j(X, M) \longrightarrow H^{i+j}(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

Namely, given $\xi \in H^i(X, K) = H^i(R\Gamma(X, K))$ and $\eta \in H^j(X, M) = H^j(R\Gamma(X, M))$ we can first "tensor" them to get an element $\xi \otimes \eta$ in $H^{i+j}(R\Gamma(X, K) \otimes_A^{\mathbf{L}} R\Gamma(X, M))$, see More on Algebra, Section 15.63. Then we can apply μ to get the desired element $\xi \cup \eta = \mu(\xi \otimes \eta)$ of $H^{i+j}(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$.

Here is another way to think of the cup product of ξ and η. Namely, we can write

$$R\Gamma(X, K) = R\operatorname{Hom}_X(\mathcal{O}_X, K) \quad \text{and} \quad R\Gamma(X, M) = R\operatorname{Hom}_X(\mathcal{O}_X, M)$$

because $\operatorname{Hom}(\mathcal{O}_X, -) = \Gamma(X, -)$. Thus ξ and η are the "same" thing as maps

$$\tilde{\xi} : \mathcal{O}_X[-i] \to K \quad \text{and} \quad \tilde{\eta} : \mathcal{O}_X[-j] \to M$$

Combining this with the functoriality of the derived tensor product we obtain

$$\mathcal{O}_X[-i-j] = \mathcal{O}_X[-i] \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_X[-j] \xrightarrow{\tilde{\xi} \otimes \tilde{\eta}} K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M$$

which by the same token as above is an element of $H^{i+j}(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$.

0FP2 Lemma 20.31.1. This construction gives the cup product.

Proof. With $f : (X, \mathcal{O}_X) \to (pt, A)$ as above we have $Rf_*(-) = R\Gamma(X, -)$ and our map μ is adjoint to the map

$$Lf^*(Rf_*K \otimes_A^{\mathbf{L}} Rf_*M) = Lf^*Rf_*K \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*Rf_*M \xrightarrow{\epsilon_K \otimes \epsilon_M} K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M$$

where ϵ is the counit of the adjunction between Lf^* and Rf_*. If we think of ξ and η as maps $\xi : A[-i] \to R\Gamma(X, K)$ and $\eta : A[-j] \to R\Gamma(X, M)$, then the tensor $\xi \otimes \eta$ corresponds to the map[3]

$$A[-i-j] = A[-i] \otimes_A^{\mathbf{L}} A[-j] \xrightarrow{\xi \otimes \eta} R\Gamma(X, K) \otimes_A^{\mathbf{L}} R\Gamma(X, M)$$

By definition the cup product $\xi \cup \eta$ is the map $A[-i-j] \to R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$ which is adjoint to

$$(\epsilon_K \otimes \epsilon_M) \circ Lf^*(\xi \otimes \eta) = (\epsilon_K \circ Lf^*\xi) \otimes (\epsilon_M \circ Lf^*\eta)$$

However, it is easy to see that $\epsilon_K \circ Lf^*\xi = \tilde{\xi}$ and $\epsilon_M \circ Lf^*\eta = \tilde{\eta}$. We conclude that $\widetilde{\xi \cup \eta} = \tilde{\xi} \otimes \tilde{\eta}$ which means we have the desired agreement. $\square$

0G6W **Remark 20.31.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, M be objects of $D(\mathcal{O}_X)$. Set $A = \Gamma(X, \mathcal{O}_X)$. Given $\xi \in H^i(X, K)$ we get an associated map

$$\xi = \text{“}\xi \cup -\text{''} : R\Gamma(X, M)[-i] \to R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

by representing ξ as a map $\xi : A[-i] \to R\Gamma(X, K)$ as in the proof of Lemma 20.31.1 and then using the composition

$$R\Gamma(X, M)[-i] = A[-i] \otimes_A^{\mathbf{L}} R\Gamma(X, M) \xrightarrow{\xi \otimes 1} R\Gamma(X, K) \otimes_A^{\mathbf{L}} R\Gamma(X, M) \to R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)$$

where the second arrow is the global cup product μ above. On cohomology this recovers the cup product by ξ as is clear from Lemma 20.31.1 and its proof.

Let us formulate and prove a natural compatibility of the relative cup product. Namely, suppose that we have a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_X$-modules. There is a naive cup product

$$\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \longrightarrow f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

We claim that this is related to the relative cup product.

0FP3 **Lemma 20.31.3.** *In the situation above the following diagram commutes*

$$\begin{tikzcd}
f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_*\mathcal{M}^\bullet \arrow[r] \arrow[d] & Rf_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*\mathcal{M}^\bullet \arrow[d, "\text{Remark 20.28.7}"] \\
\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \arrow[d, "\text{naive cup product}"'] & Rf_*(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{M}^\bullet) \arrow[d] \\
f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \arrow[r] & Rf_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)
\end{tikzcd}$$

[3] There is a sign hidden here, namely, the equality is defined by the composition

$$A[-i-j] \to (A \otimes_A^{\mathbf{L}} A)[-i-j] \to A[-i] \otimes_A^{\mathbf{L}} A[-j]$$

where in the second step we use the identification of More on Algebra, Item (7) which uses a sign in principle. Except, in this case the sign is $+1$ by our convention and even if it wasn't $+1$ it wouldn't matter since we used the same sign in the identification $\mathcal{O}_X[-i-j] = \mathcal{O}_X[-i] \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_X[-j]$.

Proof. By the construction in Remark 20.28.7 we see that going around the diagram clockwise the map

$$f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_*\mathcal{M}^\bullet \longrightarrow Rf_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)$$

is adjoint to the map

$$\begin{aligned} Lf^*(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lf^*f_*\mathcal{M}^\bullet \\ &\to Lf^*Rf_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lf^*Rf_*\mathcal{M}^\bullet \\ &\to \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{M}^\bullet \\ &\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \end{aligned}$$

By Lemma 20.28.6 this is also equal to

$$\begin{aligned} Lf^*(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Lf^*f_*\mathcal{M}^\bullet \\ &\to f^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f^*f_*\mathcal{M}^\bullet \\ &\to \mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{M}^\bullet \\ &\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \end{aligned}$$

Going around anti-clockwise we obtain the map adjoint to the map

$$\begin{aligned} Lf^*(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_*\mathcal{M}^\bullet) &\to Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \\ &\to Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\ &\to Lf^*Rf_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\ &\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \end{aligned}$$

By Lemma 20.28.6 this is also equal to

$$\begin{aligned} Lf^*(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_*\mathcal{M}^\bullet) &\to Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \\ &\to Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\ &\to f^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \\ &\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \end{aligned}$$

Now the proof is finished by a contemplation of the diagram

$$\begin{tikzcd}
Lf^*(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y}^{\mathbf{L}} f_*\mathcal{M}^\bullet) \arrow[rr] \arrow[d] & & Lf^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*f_*\mathcal{M}^\bullet \arrow[d] \\
Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \arrow[r] \arrow[d, "naive"'] & f^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_Y} f_*\mathcal{M}^\bullet) \arrow[ddl, "naive"] \arrow[dd] & f^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} f^*f_*\mathcal{M}^\bullet \arrow[ddl] \arrow[dd] \\
Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \arrow[d] & & \\
f^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \arrow[dr] & \text{Tot}(f^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} f^*f_*\mathcal{M}^\bullet) \arrow[d] & \mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{M}^\bullet \arrow[dl] \\
& \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) &
\end{tikzcd}$$

All of the polygons in this diagram commute. The top one commutes by Lemma 20.27.5. The square with the two naive cup products commutes because $Lf^* \to f^*$ is functorial in the complex of modules. Similarly with the square involving the two maps $\mathcal{A}^\bullet \otimes^{\mathbf{L}} \mathcal{B}^\bullet \to \text{Tot}(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet)$. Finally, the commutativity of the remaining square is true on the level of complexes and may be viewed as the definiton of the naive cup product (by the adjointness of f^* and f_*). The proof is finished because going around the diagram on the outside are the two maps given above. $\square$

Let $(X, \mathcal{O}_X)$ be a ring space. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_X$-modules. Then we have a "naive" cup product

$$\mu' : \text{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) \longrightarrow \Gamma(X, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))$$

By Lemma 20.31.3 applied to the morphism $(X, \mathcal{O}_X) \to (pt, A)$ this naive cup product is related to the cup product μ defined in the first paragraph of this section by the following commutative diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{K}^\bullet) \otimes_A^{\mathbf{L}} \Gamma(X, \mathcal{M}^\bullet) \arrow[r] \arrow[d] & R\Gamma(X, \mathcal{K}^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(X, \mathcal{M}^\bullet) \arrow[d, "\mu"] \\
\text{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) \arrow[d, "\mu'"'] & R\Gamma(X, \mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{M}^\bullet) \arrow[d] \\
\Gamma(X, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) \arrow[r] & R\Gamma(X, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))
\end{tikzcd}$$

in $D(A)$. On cohomology we obtain the commutative diagram

$$\begin{tikzcd}
H^i(\Gamma(X, \mathcal{K}^\bullet)) \times H^j(\Gamma(X, \mathcal{M}^\bullet)) \arrow[r] \arrow[d] & H^{i+j}(X, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) \\
H^i(X, \mathcal{K}^\bullet) \times H^j(X, \mathcal{M}^\bullet) \arrow[r, "\cup"] & H^{i+j}(X, \mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{M}^\bullet) \arrow[u]
\end{tikzcd}$$

relating the naive cup product with the actual cuproduct.

0FKV **Lemma 20.31.4.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be bounded below complexes of $\mathcal{O}_X$-modules. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering Then

$$\begin{tikzcd}
\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A^{\mathbf{L}} \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet)) \arrow[r] \arrow[d] & R\Gamma(X, \mathcal{K}^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(X, \mathcal{M}^\bullet) \arrow[d, "\mu"] \\
\text{Tot}(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet))) \arrow[d, "(20.25.3.2)"] & R\Gamma(X, \mathcal{K}^\bullet \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{M}^\bullet) \arrow[d] \\
\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))) \arrow[r] & R\Gamma(X, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet))
\end{tikzcd}$$

where the horizontal arrows are the ones in Lemma 20.25.1 commutes in $D(A)$.

Proof. Choose quasi-isomorphisms of complexes $a : \mathcal{K}^\bullet \to \mathcal{K}_1^\bullet$ and $b : \mathcal{M}^\bullet \to \mathcal{M}_1^\bullet$ as in Lemma 20.30.2. Since the maps a and b on stalks are homotopy equivalences we see that the induced map

$$\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet) \to \text{Tot}(\mathcal{K}_1^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}_1^\bullet)$$

is a homotopy equivalence on stalks too (More on Algebra, Lemma 15.58.2) and hence a quasi-isomorphism. Thus the targets

$$R\Gamma(X, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) = R\Gamma(X, \text{Tot}(\mathcal{K}_1^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}_1^\bullet))$$

of the two diagrams are the same in $D(A)$. It follows that it suffices to prove the diagram commutes for $\mathcal{K}$ and $\mathcal{M}$ replaced by $\mathcal{K}_1$ and $\mathcal{M}_1$. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ are bounded below complexes of flasque $\mathcal{O}_X$-modules and consider the diagram relating the cup product with the cup product (20.25.3.2) on Čech complexes. Then we can consider the commutative diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{K}^\bullet) \otimes_A^{\mathbf{L}} \Gamma(X, \mathcal{M}^\bullet) \arrow[r] \arrow[d] & \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A^{\mathbf{L}} \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet)) \arrow[d] \\
\text{Tot}(\Gamma(X, \mathcal{K}^\bullet) \otimes_A \Gamma(X, \mathcal{M}^\bullet)) \arrow[r] \arrow[d] & \text{Tot}(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) \otimes_A \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{M}^\bullet))) \arrow[d, "(20.25.3.2)"] \\
\Gamma(X, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)) \arrow[r] & \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet)))
\end{tikzcd}$$

In this diagram the horizontal arrows are isomorphisms in $D(A)$ because for a bounded below complex of flasque modules such as $\mathcal{K}^\bullet$ we have

$$\Gamma(X, \mathcal{K}^\bullet) = \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet)) = R\Gamma(X, \mathcal{K}^\bullet)$$

in $D(A)$. This follows from Lemma 20.12.3, Derived Categories, Lemma 13.16.7, and Lemma 20.25.2. Hence the commutativity of the diagram of the lemma involving (20.25.3.2) follows from the already proven commutativity of Lemma 20.31.3 where f is the morphism to a point (see discussion following Lemma 20.31.3). $\square$

0FP4 Lemma 20.31.5. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The relative cup product of Remark 20.28.7 is associative in the sense that the diagram

$$\begin{tikzcd}
Rf_*K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*L \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*M \arrow[r] \arrow[d] & Rf_*(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*M \arrow[d] \\
Rf_*K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*(L \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) \arrow[r] & Rf_*(K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)
\end{tikzcd}$$

is commutative in $D(\mathcal{O}_Y)$ for all K, L, M in $D(\mathcal{O}_X)$.

Proof. Going around either side we obtain the map adjoint to the obvious map

$$\begin{aligned} Lf^*(Rf_*K \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*L \otimes_{\mathcal{O}_Y}^{\mathbf{L}} Rf_*M) &= Lf^*(Rf_*K) \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*(Rf_*L) \otimes_{\mathcal{O}_X}^{\mathbf{L}} Lf^*(Rf_*M) \\ &\to K \otimes_{\mathcal{O}_X}^{\mathbf{L}} L \otimes_{\mathcal{O}_X}^{\mathbf{L}} M \end{aligned}$$

in $D(\mathcal{O}_X)$. $\square$

0FP5 Lemma 20.31.6. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The relative cup product of Remark 20.28.7 is commutative in the sense that the

diagram

$$\begin{tikzcd} Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \arrow[r] \arrow[d, "\psi"'] & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \arrow[d, "Rf_*\psi"] \\ Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*K \arrow[r] & Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}_X} K) \end{tikzcd}$$

is commutative in $D(\mathcal{O}_Y)$ for all K, L in $D(\mathcal{O}_X)$. Here ψ is the commutativity constraint on the derived category (Lemma 20.50.6).

Proof. Omitted. □

0FP6 Lemma 20.31.7. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. The relative cup product of Remark 20.28.7 is compatible with compositions in the sense that the diagram

$$\begin{tikzcd} R(g \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} R(g \circ f)_*L \arrow[rr, equal] \arrow[d] & & Rg_*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} Rg_*Rf_*L \arrow[d] \\ R(g \circ f)_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \arrow[r, equal] & Rg_*Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) & Rg_*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L) \arrow[l] \end{tikzcd}$$

is commutative in $D(\mathcal{O}_Z)$ for all K, L in $D(\mathcal{O}_X)$.

Proof. This is true because going around the diagram either way we obtain the map adjoint to the map

$$\begin{aligned} & L(g \circ f)^* \left(R(g \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_Z} R(g \circ f)_*L\right) \\ & = L(g \circ f)^*R(g \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L(g \circ f)^*R(g \circ f)_*L) \\ & \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \end{aligned}$$

in $D(\mathcal{O}_X)$. To see this one uses that the composition of the counits like so

$$L(g \circ f)^*R(g \circ f)_* = Lf^*Lg^*Rg_*Rf_* \to Lf^*Rf_* \to \mathrm{id}$$

is the counit for $L(g \circ f)^*$ and $R(g \circ f)_*$. See Categories, Lemma 4.24.9. □

20.32. Some properties of K-injective complexes

0D5U Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}_X) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : \mathrm{Mod}(\mathcal{O}_U) \to \mathrm{Mod}(\mathcal{O}_X)$ (see Sheaves, Section 6.31) is an exact functor (Modules, Lemma 17.3.4). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

08BS Lemma 20.32.1. Let X be a ringed space. Let $U \subset X$ be an open subspace. The restriction of a K-injective complex of $\mathcal{O}_X$-modules to U is a K-injective complex of $\mathcal{O}_U$-modules.

Proof. Follows from Derived Categories, Lemma 13.31.9 and the fact that the restriction functor has the exact left adjoint $j_!$. For the construction of $j_!$ see Sheaves, Section 6.31 and for exactness see Modules, Lemma 17.3.4. $\square$

0D5V Lemma 20.32.2. Let X be a ringed space. Let $U \subset X$ be an open subspace. For K in $D(\mathcal{O}_X)$ we have $H^p(U, K) = H^p(U, K|_U)$.

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules representing K. Then

$$H^q(U, K) = H^q(\Gamma(U, \mathcal{I}^\bullet)) = H^q(\Gamma(U, \mathcal{I}^\bullet|_U))$$

by construction of cohomology. By Lemma 20.32.1 the complex $\mathcal{I}^\bullet|_U$ is a K-injective complex representing $K|_U$ and the lemma follows. $\square$

0BKJ Lemma 20.32.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be an object of $D(\mathcal{O}_X)$. The sheafification of

$$U \mapsto H^q(U, K) = H^q(U, K|_U)$$

is the qth cohomology sheaf $H^q(K)$ of K.

Proof. The equality $H^q(U, K) = H^q(U, K|_U)$ holds by Lemma 20.32.2. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Then

$$H^q(U, K) = \frac{\mathrm{Ker}(\mathcal{I}^q(U) \to \mathcal{I}^{q+1}(U))}{\mathrm{Im}(\mathcal{I}^{q-1}(U) \to \mathcal{I}^q(U))}.$$

by our construction of cohomology. Since $H^q(K) = \mathrm{Ker}(\mathcal{I}^q \to \mathcal{I}^{q+1}) / \mathrm{Im}(\mathcal{I}^{q-1} \to \mathcal{I}^q)$ the result is clear. $\square$

08FE Lemma 20.32.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Given an open subspace $V \subset Y$, set $U = f^{-1}(V)$ and denote $g : U \to V$ the induced morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for E in $D(\mathcal{O}_X)$.

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Lemma 20.32.1. Since it is clear that $(f_*\mathcal{F})|_V = g_*(\mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X the result follows. $\square$

0D5W Lemma 20.32.5. Let $f : X \to Y$ be a morphism of ringed spaces. Then $R\Gamma(Y, -) \circ Rf_* = R\Gamma(X, -)$ as functors $D(\mathcal{O}_X) \to D(\Gamma(Y, \mathcal{O}_Y))$. More generally for $V \subset Y$ open and $U = f^{-1}(V)$ we have $R\Gamma(U, -) = R\Gamma(V, -) \circ Rf_*$.

Proof. Let Z be the ringed space consisting of a singleton space with $\Gamma(Z, \mathcal{O}_Z) = \Gamma(Y, \mathcal{O}_Y)$. There is a canonical morphism $Y \to Z$ of ringed spaces inducing the identification on global sections of structure sheaves. Then $D(\mathcal{O}_Z) = D(\Gamma(Y, \mathcal{O}_Y))$. Hence the assertion $R\Gamma(Y, -) \circ Rf_* = R\Gamma(X, -)$ follows from Lemma 20.28.2 applied to $X \to Y \to Z$.

The second (more general) statement follows from the first statement after applying Lemma 20.32.4. $\square$

0D5X Lemma 20.32.6. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let K be in $D(\mathcal{O}_X)$. Then $H^i(Rf_*K)$ is the sheaf associated to the presheaf

$$V \mapsto H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$$

Proof. The equality $H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$ follows upon taking cohomology from the second statement in Lemma 20.32.5. Then the statement on sheafification follows from Lemma 20.32.3. $\square$

0D5Y Lemma 20.32.7. Let X be a ringed space. Let K be an object of $D(\mathcal{O}_X)$ and denote K_{ab} its image in $D(\underline{\mathbf{Z}}_X)$.

(1) For any open $U \subset X$ there is a canonical map $R\Gamma(U, K) \to R\Gamma(U, K_{ab})$ which is an isomorphism in $D(\text{Ab})$.
(2) Let $f : X \to Y$ be a morphism of ringed spaces. There is a canonical map $Rf_*K \to Rf_*(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_Y)$.

Proof. The map is constructed as follows. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by $\Gamma(U, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{J}^\bullet)$ and the map in (2) is given by $f_*\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$. To show that these maps are isomorphisms, it suffices to prove they induce isomorphisms on cohomology groups and cohomology sheaves. By Lemmas 20.32.2 and 20.32.6 it suffices to show that the map

$$H^0(X, K) \longrightarrow H^0(X, K_{ab})$$

is an isomorphism. Observe that

$$H^0(X, K) = \text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, K)$$

and similarly for the other group. Choose any complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules representing K. By construction of the derived category as a localization we have

$$\text{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X, K) = \text{colim}_{s:\mathcal{F}^\bullet \to \mathcal{O}_X} \text{Hom}_{K(\mathcal{O}_X)}(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

where the colimit is over quasi-isomorphisms s of complexes of $\mathcal{O}_X$-modules. Similarly, we have

$$\text{Hom}_{D(\underline{\mathbf{Z}}_X)}(\underline{\mathbf{Z}}_X, K) = \text{colim}_{s:\mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X} \text{Hom}_{K(\underline{\mathbf{Z}}_X)}(\mathcal{G}^\bullet, \mathcal{K}^\bullet)$$

Next, we observe that the quasi-isomorphisms $s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X$ with $\mathcal{G}^\bullet$ bounded above complex of flat $\underline{\mathbf{Z}}_X$-modules is cofinal in the system. (This follows from Modules, Lemma 17.17.6 and Derived Categories, Lemma 13.15.4; see discussion in Section 20.26.) Hence we can construct an inverse to the map $H^0(X, K) \longrightarrow H^0(X, K_{ab})$ by representing an element $\xi \in H^0(X, K_{ab})$ by a pair

$$(s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_X, a : \mathcal{G}^\bullet \to \mathcal{K}^\bullet)$$

with $\mathcal{G}^\bullet$ a bounded above complex of flat $\underline{\mathbf{Z}}_X$-modules and sending this to

$$(\mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_X} \mathcal{O}_X \to \mathcal{O}_X, \mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_X} \mathcal{O}_X \to \mathcal{K}^\bullet)$$

The only thing to note here is that the first arrow is a quasi-isomorphism by Lemmas 20.26.13 and 20.26.9. We omit the detailed verification that this construction is indeed an inverse. $\square$

08BT Lemma 20.32.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The restriction functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X)$.

Proof. This follows formally from the fact that $j_!$ and j^* are adjoint and exact (and hence $Lj_! = j_!$ and $Rj^* = j^*$ exist), see Derived Categories, Lemma 13.30.3. $\square$

0D5Z Lemma 20.32.9. Let $f : X \to Y$ be a flat morphism of ringed spaces. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $f_*\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_Y$-modules.

Proof. This is true because

$$\mathrm{Hom}_{K(\mathcal{O}_Y)}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}_X)}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

by Sheaves, Lemma 6.26.2 and the fact that f^* is exact as f is assumed to be flat. $\square$

20.33. Unbounded Mayer-Vietoris

08BR There is a Mayer-Vietoris sequence for unbounded cohomology as well.

08BU **Lemma 20.33.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle

$$j_{U\cap V!}E|_{U\cap V} \to j_{U!}E|_U \oplus j_{V!}E|_V \to E \to j_{U\cap V!}E|_{U\cap V}[1]$$

in $D(\mathcal{O}_X)$.

Proof. We have seen in Section 20.32 that the restriction functors and the extension by zero functors are computed by just applying the functors to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing E. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules

$$0 \to j_{U\cap V!}\mathcal{E}^\bullet|_{U\cap V} \to j_{U!}\mathcal{E}^\bullet|_U \oplus j_{V!}\mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$

To see this sequence is exact one checks on stalks using Sheaves, Lemma 6.31.8 (computation omitted). $\square$

08BV **Lemma 20.33.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\cap V,*}E|_{U\cap V} \to E[1]$$

in $D(\mathcal{O}_X)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $\mathrm{Mod}(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. We have seen that $\mathcal{I}^\bullet|U$ is a K-injective complex as well (Lemma 20.32.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for V and $U \cap V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U\cap V,*}\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

This sequence is exact because for any $W \subset X$ open and any n the sequence

$$0 \to \mathcal{I}^n(W) \to \mathcal{I}^n(W \cap U) \oplus \mathcal{I}^n(W \cap V) \to \mathcal{I}^n(W \cap U \cap V) \to 0$$

is exact (see proof of Lemma 20.8.2). $\square$

08BW **Lemma 20.33.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of X. For objects E, F of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence

$$\begin{tikzcd} & \ldots \arrow[r] & \mathrm{Ext}^{-1}(E_{U\cap V}, F_{U\cap V}) \arrow[dll] \\ \mathrm{Hom}(E, F) \arrow[r] & \mathrm{Hom}(E_U, F_U) \oplus \mathrm{Hom}(E_V, F_V) \arrow[r] & \mathrm{Hom}(E_{U\cap V}, F_{U\cap V}) \end{tikzcd}$$

where the subscripts denote restrictions to the relevant opens and the Hom's and Ext's are taken in the relevant derived categories.

Proof. Use the distinguished triangle of Lemma 20.33.1 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 13.4.2) and use that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(j_{U!}E|_U, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E|_U, F|_U)$$

by Lemma 20.32.8. □

08BX Lemma 20.33.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle

$$R\Gamma(X, E) \to R\Gamma(U, E) \oplus R\Gamma(V, E) \to R\Gamma(U \cap V, E) \to R\Gamma(X, E)[1]$$

and in particular a long exact cohomology sequence

$$\ldots \to H^n(X, E) \to H^n(U, E) \oplus H^0(V, E) \to H^n(U \cap V, E) \to H^{n+1}(X, E) \to \ldots$$

The construction of the distinguished triangle and the long exact sequence is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $\mathrm{Mod}(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then $R\Gamma(X, E)$ is computed by $\Gamma(X, \mathcal{I}^\bullet)$. Similarly for U, V, and $U \cap V$ by Lemma 20.32.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

We have seen this is a short exact sequence in the proof of Lemma 20.8.2. The final statement follows from the functoriality of the construction in Injectives, Theorem 19.12.6. □

08HZ Lemma 20.33.5. Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every object E of $D(\mathcal{O}_X)$ there exists a distinguished triangle

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U \cap V}) \to Rf_*E[1]$$

This triangle is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $\mathrm{Mod}(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then Rf_*E is computed by $f_*\mathcal{I}^\bullet$. Similarly for U, V, and $U \cap V$ by Lemma 20.32.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.12 and especially Lemma 13.12.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

This is a short exact sequence of complexes by Lemma 20.8.3 and the fact that $R^1f_*\mathcal{I} = 0$ for an injective object $\mathcal{I}$ of $\mathrm{Mod}(\mathcal{O}_X)$. The final statement follows from the functoriality of the construction in Injectives, Theorem 19.12.6. □

08DF **Lemma 20.33.6.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let $T \subset X$ be a closed subset contained in U.

(1) If E is an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T, then $E \to Rj_*(E|_U)$ is an isomorphism.
(2) If F is an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on T, then $j_!F \to Rj_*F$ is an isomorphism.

Proof. Let $V = X \setminus T$ and $W = U \cap V$. Note that $X = U \cup V$ is an open covering of X. Denote $j_W : W \to V$ the open immersion. Let E be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T. By Lemma 20.32.4 we have $(Rj_*E|_U)|_V = Rj_{W,*}(E|_W) = 0$ because $E|_W = 0$ by our assumption. On the other hand, $Rj_*(E|_U)|_U = E|_U$. Thus (1) is clear. Let F be an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on T. By Lemma 20.32.4 we have $(Rj_*F)|_V = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. We also have $(j_!F)|_V = j_{W!}(F|_W) = 0$ (the first equality is immediate from the definition of extension by zero). Since both $(Rj_*F)|_U = F$ and $(j_!F)|_U = F$ we see that (2) holds. $\square$

0G6X **Lemma 20.33.7.** Let $(X, \mathcal{O}_X)$ be a ringed space. Set $A = \Gamma(X, \mathcal{O}_X)$. Suppose that $X = U \cup V$ is a union of two open subsets. For objects K and M of $D(\mathcal{O}_X)$ we have a map of distinguished triangles

$$\begin{tikzcd}
R\Gamma(X,K) \otimes_A^{\mathbf{L}} R\Gamma(X,M) \arrow[r] \arrow[d] & R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) \arrow[d] \\
R\Gamma(X,K) \otimes_A^{\mathbf{L}} (R\Gamma(U,M) \oplus R\Gamma(V,M)) \arrow[r] \arrow[d] & R\Gamma(U, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) \oplus R\Gamma(V, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)) \arrow[d] \\
R\Gamma(X,K) \otimes_A^{\mathbf{L}} R\Gamma(U \cap V,M) \arrow[r] \arrow[d] & R\Gamma(U \cap V, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M) \arrow[d] \\
R\Gamma(X,K) \otimes_A^{\mathbf{L}} R\Gamma(X,M)[1] \arrow[r] & R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M)[1]
\end{tikzcd}$$

where

(1) the horizontal arrows are given by cup product,
(2) on the right hand side we have the distinguished triangle of Lemma 20.33.4 for $K \otimes_{\mathcal{O}_X}^{\mathbf{L}} M$, and
(3) on the left hand side we have the exact functor $R\Gamma(X,K) \otimes_A^{\mathbf{L}} -$ applied to the distinguished triangle of Lemma 20.33.4 for M.

Proof. Choose a K-flat complex $T^\bullet$ of flat A-modules representing $R\Gamma(X,K)$, see More on Algebra, Lemma 15.59.10. Denote $T^\bullet \otimes_A \mathcal{O}_X$ the pullback of $T^\bullet$ by the morphism of ringed spaces $(X, \mathcal{O}_X) \to (pt, A)$. There is a natural adjunction map $\epsilon : T^\bullet \otimes_A \mathcal{O}_X \to K$ in $D(\mathcal{O}_X)$. Observe that $T^\bullet \otimes_A \mathcal{O}_X$ is a K-flat complex of $\mathcal{O}_X$-modules with flat terms, see Lemma 20.26.8 and Modules, Lemma 17.20.2. By Lemma 20.26.17 we can find a morphism of complexes

$$T^\bullet \otimes_A \mathcal{O}_X \longrightarrow \mathcal{K}^\bullet$$

of $\mathcal{O}_X$-modules representing ϵ such that $\mathcal{K}^\bullet$ is a K-flat complex with flat terms. Namely, by the construction of $D(\mathcal{O}_X)$ we can first represent ϵ by some map of complexes $e : T^\bullet \otimes_A \mathcal{O}_X \to \mathcal{L}^\bullet$ of $\mathcal{O}_X$-modules representing ϵ and then we can apply the lemma to e. Choose a K-injective complex $\mathcal{I}^\bullet$ whose terms are injective $\mathcal{O}_X$-modules representing M. Finally, choose a quasi-isomorphism

$$\text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{I}^\bullet) \longrightarrow \mathcal{J}^\bullet$$

into a K-injective complex whose terms are injective $\mathcal{O}_X$-modules. Observe that source and target of this arrow represent $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$ in $D(\mathcal{O}_X)$. At this point, for any open $W \subset X$ we obtain a map of complexes

$$\text{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(W)) \to \text{Tot}(\mathcal{K}^\bullet(W) \otimes_A \mathcal{I}^\bullet(W)) \to \mathcal{J}^\bullet(W)$$

of A-modules whose composition represents the map

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(W, M) \longrightarrow R\Gamma(W, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D(A)$. Clearly, these maps are compatible with restriction mappings. OK, so now we can consider the following commutative(!) diagram of complexes of A-modules

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\text{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(X)) \arrow[r] \arrow[d] & \mathcal{J}^\bullet(X) \arrow[d] \\
\text{Tot}(T^\bullet \otimes_A (\mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V)) \arrow[r] \arrow[d] & \mathcal{J}^\bullet(U) \oplus \mathcal{J}^\bullet(V) \arrow[d] \\
\text{Tot}(T^\bullet \otimes_A \mathcal{I}^\bullet(U \cap V)) \arrow[r] \arrow[d] & \mathcal{J}^\bullet(U \cap V) \arrow[d] \\
0 & 0
\end{tikzcd}$$

By the proof of Lemma 20.8.2 the columns are exact sequences of complexes of A-modules (this also uses that $\text{Tot}(T^\bullet \otimes_A -)$ transforms short exact sequences of complexes of A-modules into short exact sequences as the terms of $T^\bullet$ are flat A-modules). Since the distinguished triangles of Lemma 20.33.4 are the distinguished triangles associated to these short exact sequences of complexes, the desired result follows from the functoriality of "taking the associated distinguished triangle" discussed in Derived Categories, Section 13.12. □

20.34. Cohomology with support in a closed subset, II

0G6Y We continue the discussion started in Section 20.21.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. In this situation we can consider the functor $\text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X(X))$ given by $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$. See Modules, Definition 17.5.1 and Modules, Lemma 17.5.2. Using K-injective resolutions, see Section 20.28, we obtain the right derived functor

$$R\Gamma_Z(X, -) : D(\mathcal{O}_X) \to D(\mathcal{O}_X(X))$$

Given an object K in $D(\mathcal{O}_X)$ we denote $H^q_Z(X, K) = H^q(R\Gamma_Z(X, K))$ the cohomology module with support in Z. We will see later (Lemma 20.34.8) that this agrees with the construction in Section 20.21.

For an $\mathcal{O}_X$-module $\mathcal{F}$ we can consider the subsheaf of sections with support in Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \cap Z\} = \Gamma_{Z\cap U}(U, \mathcal{F}|_U)$$

As discussed in Modules, Remark 17.13.5 we may view $\mathcal{H}_Z(\mathcal{F})$ as an $\mathcal{O}_X|_Z$-module on Z and we obtain a functor

$$\text{Mod}(\mathcal{O}_X) \longrightarrow \text{Mod}(\mathcal{O}_X|_Z), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F}) \text{ viewed as an } \mathcal{O}_X|_Z\text{-module on } Z$$

This functor is left exact, but in general not exact. Exactly as above we obtain a right derived functor

$$R\mathcal{H}_Z : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X|_Z)$$

We set $\mathcal{H}^q_Z(K) = H^q(R\mathcal{H}_Z(K))$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

0A3B Lemma 20.34.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset.

(1) $R\mathcal{H}_Z : D(\mathcal{O}_X) \to D(\mathcal{O}_X|_Z)$ is right adjoint to $i_* : D(\mathcal{O}_X|_Z) \to D(\mathcal{O}_X)$.
(2) For K in $D(\mathcal{O}_X|_Z)$ we have $R\mathcal{H}_Z(i_*K) = K$.
(3) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X|_Z$-modules on Z. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.

Proof. The functor i_* is exact, so $i_* = Ri_* = Li_*$. Hence part (1) of the lemma follows from Modules, Lemma 17.13.6 and Derived Categories, Lemma 13.30.3. Let K be as in (2). We can represent K by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X|_Z$-modules. By Lemma 20.32.9 the complex $i_*\mathcal{I}^\bullet$, which represents i_*K, is a K-injective complex of $\mathcal{O}_X$-modules. Thus $R\mathcal{H}_Z(i_*K)$ is computed by $\mathcal{H}_Z(i_*\mathcal{I}^\bullet) = \mathcal{I}^\bullet$ which proves (2). Part (3) is a special case of (2). □

Let $(X, \mathcal{O}_X)$ be a ringed space and let $Z \subset X$ be a closed subset. The category of $\mathcal{O}_X$-modules whose support is contained in Z is a Serre subcategory of the category of all $\mathcal{O}_X$-modules, see Homology, Definition 12.10.1 and Modules, Lemma 17.5.2. We denote $D_Z(\mathcal{O}_X)$ the strictly full saturated triangulated subcategory of $D(\mathcal{O}_X)$ consisting of complexes whose cohomology sheaves are supported on Z, see Derived Categories, Section 13.17.

0AEF Lemma 20.34.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset.

(1) For K in $D(\mathcal{O}_X|_Z)$ we have i_*K in $D_Z(\mathcal{O}_X)$.
(2) The functor $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is an equivalence with quasi-inverse $i^{-1}|_{D_Z(\mathcal{O}_X)} = R\mathcal{H}_Z|_{D_Z(\mathcal{O}_X)}$.
(3) The functor $i_* \circ R\mathcal{H}_Z : D(\mathcal{O}_X) \to D_Z(\mathcal{O}_X)$ is right adjoint to the inclusion functor $D_Z(\mathcal{O}_X) \to D(\mathcal{O}_X)$.

Proof. Part (1) is immediate from the definitions. Part (3) is a formal consequence of part (2) and Lemma 20.34.1. In the rest of the proof we prove part (2).

Let us think of i as the morphism of ringed spaces $i : (Z, \mathcal{O}_X|_Z) \to (X, \mathcal{O}_X)$. Recall that i^* and i_* is an adjoint pair of functors. Since i is a closed immersion, i_* is exact. Since $i^{-1}\mathcal{O}_X = \mathcal{O}_X|_Z$ is the structure sheaf of $(Z, \mathcal{O}_X|_Z)$ we see that $i^* = i^{-1}$

is exact and we see that that $i^*i_* = i^{-1}i_*$ is isomorphic to the identify functor. See Modules, Lemmas 17.3.3 and 17.6.1. Thus $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is fully faithful and i^{-1} determines a left inverse. On the other hand, suppose that K is an object of $D_Z(\mathcal{O}_X)$ and consider the adjunction map $K \to i_*i^{-1}K$. Using exactness of i_* and i^{-1} this induces the adjunction maps $H^n(K) \to i_*i^{-1}H^n(K)$ on cohomology sheaves. Since these cohomology sheaves are supported on Z we see these adjunction maps are isomorphisms and we conclude that $i_* : D(\mathcal{O}_X|_Z) \to D_Z(\mathcal{O}_X)$ is an equivalence.

To finish the proof it suffices to show that $R\mathcal{H}_Z(K) = i^{-1}K$ if K is an object of $D_Z(\mathcal{O}_X)$. To do this we can use that $K = i_*i^{-1}K$ as we've just proved this is the case. Then Lemma 20.34.1 tells us what we want. □

0G6Z Lemma 20.34.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules, then $\mathcal{H}_Z(\mathcal{I}^\bullet)$ is K-injective complex of $\mathcal{O}_X|_Z$-modules.

Proof. Since $i_* : \text{Mod}(\mathcal{O}_X|_Z) \to \text{Mod}(\mathcal{O}_X)$ is exact and left adjoint to $\mathcal{H}_Z$ (Modules, Lemma 17.13.6) this follows from Derived Categories, Lemma 13.31.9. □

0G70 Lemma 20.34.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Then $R\Gamma(Z, -) \circ R\mathcal{H}_Z = R\Gamma_Z(X, -)$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_X(X))$.

Proof. Follows from the construction of right derived functors using K-injective resolutions, Lemma 20.34.3, and the fact that $\Gamma_Z(X, -) = \Gamma(Z, -) \circ \mathcal{H}_Z$. □

0G71 Lemma 20.34.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Let $U = X \setminus Z$. There is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(\mathcal{O}_X(X))$ functorial for K in $D(\mathcal{O}_X)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ all of whose terms are injective $\mathcal{O}_X$-modules representing K. See Section 20.28. Recall that $\mathcal{I}^\bullet|_U$ is a K-injective complex of $\mathcal{O}_U$-modules, see Lemma 20.32.1. Hence each of the derived functors in the distinguished triangle is gotten by applying the underlying functor to $\mathcal{I}^\bullet$. Hence we find that it suffices to prove that for an injective $\mathcal{O}_X$-module $\mathcal{I}$ we have a short exact sequence

$$0 \to \Gamma_Z(X, \mathcal{I}) \to \Gamma(X, \mathcal{I}) \to \Gamma(U, \mathcal{I}) \to 0$$

This follows from Lemma 20.8.1 and the definitions. □

0G72 Lemma 20.34.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Denote $j : U = X \setminus Z \to X$ the inclusion of the complement. There is a distinguished triangle

$$i_*R\mathcal{H}_Z(K) \to K \to Rj_*(K|_U) \to i_*R\mathcal{H}_Z(K)[1]$$

in $D(\mathcal{O}_X)$ functorial for K in $D(\mathcal{O}_X)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ all of whose terms are injective $\mathcal{O}_X$-modules representing K. See Section 20.28. Recall that $\mathcal{I}^\bullet|_U$ is a K-injective complex of $\mathcal{O}_U$-modules, see Lemma 20.32.1. Hence each of the derived functors in the distinguished triangle is gotten by applying the underlying functor to $\mathcal{I}^\bullet$.

Hence it suffices to prove that for an injective $\mathcal{O}_X$-module $\mathcal{I}$ we have a short exact sequence

$$0 \to i_*\mathcal{H}_Z(\mathcal{I}) \to \mathcal{I} \to j_*(\mathcal{I}|_U) \to 0$$

This follows from Lemma 20.8.1 and the definitions. □

0G73 Lemma 20.34.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Let $j : U \to X$ be the inclusion of an open subset with $U \cap Z = \emptyset$. Then $R\mathcal{H}_Z(Rj_*K) = 0$ for all K in $D(\mathcal{O}_U)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_U$-modules representing K. Then $j_*\mathcal{I}^\bullet$ represents Rj_*K. By Lemma 20.32.9 the complex $j_*\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules. Hence $\mathcal{H}_Z(j_*\mathcal{I}^\bullet)$ represents $R\mathcal{H}_Z(Rj_*K)$. Thus it suffices to show that $\mathcal{H}_Z(j_*\mathcal{G}) = 0$ for any abelian sheaf $\mathcal{G}$ on U. Thus we have to show that a section s of $j_*\mathcal{G}$ over some open W which is supported on $W \cap Z$ is zero. The support condition means that $s|_{W \setminus W \cap Z} = 0$. Since $j_*\mathcal{G}(W) = \mathcal{G}(U \cap W) = j_*\mathcal{G}(W \setminus W \cap Z)$ this implies that s is zero as desired. □

0G74 Lemma 20.34.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Let K be an object of $D(\mathcal{O}_X)$ and denote K_{ab} its image in $D(\underline{\mathbf{Z}}_X)$.

(1) There is a canonical map $R\Gamma_Z(X, K) \to R\Gamma_Z(X, K_{ab})$ which is an isomorphism in $D(\text{Ab})$.
(2) There is a canonical map $R\mathcal{H}_Z(K) \to R\mathcal{H}_Z(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_Z)$.

Proof. Proof of (1). The map is constructed as follows. Choose a K-injective complex of $\mathcal{O}_X$-modules $\mathcal{I}^\bullet$ representing K. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by

$$\Gamma_Z(X, \mathcal{I}^\bullet) \to \Gamma_Z(X, \mathcal{J}^\bullet)$$

determined by the fact that Γ_Z is a functor on abelian sheaves. An easy check shows that the resulting map combined with the canonical maps of Lemma 20.32.7 fit into a morphism of distinguished triangles

$$\begin{tikzcd} R\Gamma_Z(X, K) \arrow[r] \arrow[d] & R\Gamma(X, K) \arrow[r] \arrow[d] & R\Gamma(U, K) \arrow[d] \\ R\Gamma_Z(X, K_{ab}) \arrow[r] & R\Gamma(X, K_{ab}) \arrow[r] & R\Gamma(U, K_{ab}) \end{tikzcd}$$

of Lemma 20.34.5. Since two of the three arrows are isomorphisms by the lemma cited, we conclude by Derived Categories, Lemma 13.4.3.

The proof of (2) is omitted. Hint: use the same argument with Lemma 20.34.6 for the distinguished triangle. □

0G75 Remark 20.34.9. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i : Z \to X$ be the inclusion of a closed subset. Given K and M in $D(\mathcal{O}_X)$ there is a canonical map

$$K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M) \longrightarrow R\mathcal{H}_Z(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

in $D(\mathcal{O}_X|_Z)$. Here $K|_Z = i^{-1}K$ is the restriction of K to Z viewed as an object of $D(\mathcal{O}_X|_Z)$. By adjointness of i_* and $R\mathcal{H}_Z$ of Lemma 20.34.1 to construct this map it suffices to produce a canonical map

$$i_*\left(K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M)\right) \longrightarrow K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$$

To construct this map, we choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing M and a K-flat complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules representing K. Observe that $\mathcal{K}^\bullet|_Z$ is a K-flat complex of $\mathcal{O}_X|_Z$-modules representing $K|_Z$, see Lemma 20.26.8. Hence we need to produce a map of complexes

$$i_*\mathrm{Tot}\left(\mathcal{K}^\bullet|_Z \otimes_{\mathcal{O}_X|_Z} \mathcal{H}_Z(\mathcal{I}^\bullet)\right) \longrightarrow \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet)$$

of $\mathcal{O}_X$-modules. For this it suffices to produce maps

$$i_*(\mathcal{K}^a|_Z \otimes_{\mathcal{O}_X|_Z} \mathcal{H}_Z(\mathcal{I}^b)) \longrightarrow \mathcal{K}^a \otimes_{\mathcal{O}_X} \mathcal{I}^b$$

Looking at stalks (for example), we see that the left hand side of this formula is equal to $\mathcal{K}^a \otimes_{\mathcal{O}_X} i_*\mathcal{H}_Z(\mathcal{I}^b)$ and we can use the inclusion $\mathcal{H}_Z(\mathcal{I}^b) \to \mathcal{I}^b$ to get our map.

0G76 Remark 20.34.10. With notation as in Remark 20.34.9 we obtain a canonical cup product

$$\begin{aligned} H^a(X, K) \times H^b_Z(X, M) &= H^a(X, K) \times H^b(Z, R\mathcal{H}_Z(M)) \\ &\to H^a(Z, K|_Z) \times H^b(Z, R\mathcal{H}_Z(M)) \\ &\to H^{a+b}(Z, K|_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X|_Z} R\mathcal{H}_Z(M)) \\ &\to H^{a+b}(Z, R\mathcal{H}_Z(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)) \\ &= H^{a+b}_Z(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \end{aligned}$$

Here the equal signs are given by Lemma 20.34.4, the first arrow is restriction to Z, the second arrow is the cup product (Section 20.31), and the third arrow is the map from Remark 20.34.9.

0G77 Lemma 20.34.11. With notation as in Remark 20.34.9 the diagram

$$\begin{tikzcd} H^i(X, K) \times H^j_Z(X, M) \arrow[r] \arrow[d] & H^{i+j}_Z(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \arrow[d] \\ H^i(X, K) \times H^j(X, M) \arrow[r] & H^{i+j}(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \end{tikzcd}$$

commutes where the top horizontal arrow is the cup product of Remark 20.34.10.

Proof. Omitted. □

0G78 Remark 20.34.12. Let $f : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Let $Z \subset X$ be a closed subset and $Z' = f^{-1}(Z)$. Denote $f|_{Z'} : (Z', \mathcal{O}_{X'}|_{Z'}) \to (Z, \mathcal{O}_X|Z)$ be the induced morphism of ringed spaces. For any K in $D(\mathcal{O}_X)$ there is a canonical map

$$L(f|_{Z'})^* R\mathcal{H}_Z(K) \longrightarrow R\mathcal{H}_{Z'}(Lf^*K)$$

in $D(\mathcal{O}_{X'}|_{Z'})$. Denote $i : Z \to X$ and $i' : Z' \to X'$ the inclusion maps. By Lemma 20.34.2 part (2) applied to i' it is the same thing to give a map

$$i'_* L(f|_{Z'})^* R\mathcal{H}_Z(K) \longrightarrow i'_* R\mathcal{H}_{Z'}(Lf^*K)$$

in $D_{Z'}(\mathcal{O}_{X'})$. The map of functors $Lf^* \circ i_* \to i'_* \circ L(f|_{Z'})^*$ of Remark 20.28.3 is an isomorphism in this case (follows by checking what happens on stalks using that i_* and i'_* are exact and that $\mathcal{O}_{Z,z} = \mathcal{O}_{X,z}$ and similarly for Z'). Hence it suffices to construct a the top horizonal arrow in the following diagram

$$\begin{tikzcd} Lf^*i_*R\mathcal{H}_Z(K) \arrow[rr] \arrow[dr] & & i'_*R\mathcal{H}_{Z'}(Lf^*K) \arrow[dl] \\ & Lf^*K & \end{tikzcd}$$

The complex $Lf^*i_*R\mathcal{H}_Z(K)$ is supported on Z'. The south-east arrow comes from the adjunction mapping $i_*R\mathcal{H}_Z(K) \to K$ (Lemma 20.34.1). Since the adjunction mapping $i'_*R\mathcal{H}_{Z'}(Lf^*K) \to Lf^*K$ is universal by Lemma 20.34.2 part (3), we find that the south-east arrow factors uniquely over the south-west arrow and we obtain the desired arrow.

0G79 Lemma 20.34.13. With notation and assumptions as in Remark 20.34.12 the diagram

$$\begin{tikzcd} H^p_Z(X, K) \arrow[r] \arrow[d] & H^p_{Z'}(X, Lf^*K) \arrow[d] \\ H^p(X, K) \arrow[r] & H^p(X', Lf^*K) \end{tikzcd}$$

commutes. Here the top horizontal arrow comes from the identifications $H^p_Z(X, K) = H^p(Z, R\mathcal{H}_Z(K))$ and $H^p_{Z'}(X', Lf^*K) = H^p(Z', R\mathcal{H}_{Z'}(K'))$, the pullback map $H^p(Z, R\mathcal{H}_Z(K)) \to H^p(Z', L(f|_{Z'})^*R\mathcal{H}_Z(K))$, and the map constructed in Remark 20.34.12.

Proof. Omitted. Hints: Using that $H^p(Z, R\mathcal{H}_Z(K)) = H^p(X, i_*R\mathcal{H}_Z(K))$ and similarly for $R\mathcal{H}_{Z'}(Lf^*K)$ this follows from the functoriality of the pullback maps and the commutative diagram used to define the map of Remark 20.34.12. $\square$

20.35. Inverse systems and cohomology, I

0GYJ Let A be a ring and let $I \subset A$ be an ideal. We prove some results on inverse systems of sheaves of A/I^n-modules.

0GYK Lemma 20.35.1. Let I be an ideal of a ring A. Let X be a topological space. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on X such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n \geq 0} H^{p+1}(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[4].

Proof. Set $N_n = H^{p+1}(X, I^n\mathcal{F}_{n+1})$ and let $\delta_n : M_n \to N_n$ be the boundary map on cohomology coming from the short exact sequence $0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$.

[4]In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

Then $\bigoplus \mathrm{Im}(\delta_n) \subset \bigoplus N_n$ is a graded submodule. Namely, if $s \in M_n$ and $f \in I^m$, then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I^n\mathcal{F}_{n+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_{n+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_n \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I^{n+m}\mathcal{F}_{n+m+1} \arrow[r] & \mathcal{F}_{n+m+1} \arrow[r] & \mathcal{F}_{n+m} \arrow[r] & 0 \end{tikzcd}$$

The middle vertical map is given by lifting a local section of $\mathcal{F}_{n+1}$ to a section of $\mathcal{F}_{n+m+1}$ and then multiplying by f; similarly for the other vertical arrows. We conclude that $\delta_{n+m}(fs) = f\delta_n(s)$. By assumption we can find $s_j \in M_{n_j}$, $j = 1, \ldots, N$ such that $\delta_{n_j}(s_j)$ generate $\bigoplus \mathrm{Im}(\delta_n)$ as a graded module. Let $n > c = \max(n_j)$. Let $s \in M_n$. Then we can find $f_j \in I^{n-n_j}$ such that $\delta_n(s) = \sum f_j\delta_{n_j}(s_j)$. We conclude that $\delta(s - \sum f_js_j) = 0$, i.e., we can find $s' \in M_{n+1}$ mapping to $s - \sum f_js_j$ in M_n. It follows that

$$\mathrm{Im}(M_{n+1} \to M_{n-c}) = \mathrm{Im}(M_n \to M_{n-c})$$

Namely, the elements f_js_j map to zero in M_{n-c}. This proves the lemma. □

0GYL Lemma 20.35.2. Let I be an ideal of a ring A. Let X be a topological space. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of A-modules on X such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m\geq n} \mathrm{Im}\left(H^{p+1}(X, I^n\mathcal{F}_{m+1}) \to H^{p+1}(X, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[5].

Proof. The proof is exactly the same as the proof of Lemma 20.35.1. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(X, I^n\mathcal{F}_{n+1})$ and that the boundary maps $\delta_n : M_n \to H^{p+1}(X, I^n\mathcal{F}_{n+1})$ have image contained in N_n.

Suppose that $\xi \in N_n$ and $f \in I^k$. Choose $m \gg n+k$. Choose $\xi' \in H^{p+1}(X, I^n\mathcal{F}_{m+1})$ lifting ξ. We consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & I^n\mathcal{F}_{m+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_{m+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_n \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I^{n+k}\mathcal{F}_{m+1} \arrow[r] & \mathcal{F}_{m+1} \arrow[r] & \mathcal{F}_{n+k} \arrow[r] & 0 \end{tikzcd}$$

constructed as in the proof of Lemma 20.35.1. We get an induced map on cohomology and we see that $f\xi' \in H^{p+1}(X, I^{n+k}\mathcal{F}_{m+1})$ maps to $f\xi$. Since this is true for all $m \gg n + k$ we see that $f\xi$ is in N_{n+k} as desired.

[5]In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

To see the boundary maps δ_n have image contained in N_n we consider the diagrams

$$\begin{tikzcd} 0 \arrow[r] & I^n\mathcal{F}_{m+1} \arrow[r] \arrow[d] & \mathcal{F}_{m+1} \arrow[r] \arrow[d] & \mathcal{F}_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^n\mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_n \arrow[r] & 0 \end{tikzcd}$$

for $m \geq n$. Looking at the induced maps on cohomology we conclude. □

0GYM Lemma 20.35.3. Let I be an ideal of a ring A. Let X be a topological space. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on X such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n \geq 0} H^p(X, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim H^p(X, \mathcal{F}_n)$ is the I-adic topology.

Proof. Set $F^n = \mathrm{Ker}(M \to H^p(X, \mathcal{F}_n))$ for $n \geq 1$ and $F^0 = M$. Observe that $IF^n \subset F^{n+1}$. In particular $I^nM \subset F^n$. Hence the I-adic topology is finer than the limit topology. For the converse, we will show that given n there exists an $m \geq n$ such that $F^m \subset I^nM$[6]. We have injective maps

$$F^n/F^{n+1} \longrightarrow H^p(X, \mathcal{F}_{n+1})$$

whose image is contained in the image of $H^p(X, I^n\mathcal{F}_{n+1}) \to H^p(X, \mathcal{F}_{n+1})$. Denote

$$E_n \subset H^p(X, I^n\mathcal{F}_{n+1})$$

the inverse image of F^n/F^{n+1}. Then $\bigoplus E_n$ is a graded $\bigoplus I^n/I^{n+1}$-submodule of $\bigoplus H^p(X, I^n\mathcal{F}_{n+1})$ and $\bigoplus E_n \to \bigoplus F^n/F^{n+1}$ is a homomorphism of graded modules; details omitted. By assumption $\bigoplus E_n$ is generated by finitely many homogeneous elements over $\bigoplus I^n/I^{n+1}$. Since $E_n \to F^n/F^{n+1}$ is surjective, we see that the same thing is true of $\bigoplus F^n/F^{n+1}$. Hence we can find r and $c_1, \ldots, c_r \geq 0$ and $a_i \in F^{c_i}$ whose images in $\bigoplus F^n/F^{n+1}$ generate. Set $c = \max(c_i)$.

For $n \geq c$ we claim that $IF^n = F^{n+1}$. The claim shows that $F^{n+c} = I^nF^c \subset I^nM$ as desired. To prove the claim suppose $a \in F^{n+1}$. The image of a in F^{n+1}/F^{n+2} is a linear combination of our a_i. Therefore $a - \sum f_ia_i \in F^{n+2}$ for some $f_i \in I^{n+1-c_i}$. Since $I^{n+1-c_i} = I \cdot I^{n-c_i}$ as $n \geq c_i$ we can write $f_i = \sum g_{i,j}h_{i,j}$ with $g_{i,j} \in I$ and $h_{i,j}a_i \in F^n$. Thus we see that $F^{n+1} = F^{n+2} + IF^n$. A simple induction argument gives $F^{n+1} = F^{n+e} + IF^n$ for all $e > 0$. It follows that IF^n is dense in F^{n+1}. Choose generators $k_1, \ldots, k_r$ of I and consider the continuous map

$$u : (F^n)^{\oplus r} \longrightarrow F^{n+1}, \quad (x_1, \ldots, x_r) \mapsto \sum k_ix_i$$

(in the limit topology). By the above the image of $(F^m)^{\oplus r}$ under u is dense in F^{m+1} for all $m \geq n$. By the open mapping lemma (More on Algebra, Lemma 15.36.5) we find that u is open. Hence u is surjective. Hence $IF^n = F^{n+1}$ for $n \geq c$. This concludes the proof. □

[6]In fact, there exist a $c \geq 0$ such that $F^{n+c} \subset I^nM$ for all n.

0GYN Lemma 20.35.4. Let I be an ideal of a ring A. Let X be a topological space. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on X such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\,(H^p(X, I^n\mathcal{F}_{m+1}) \to H^p(X, I^n\mathcal{F}_{n+1}))$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module, then the limit topology on $M = \lim H^p(X, \mathcal{F}_n)$ is the I-adic topology.

Proof. The proof is exactly the same as the proof of Lemma 20.35.3. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(X, I^n\mathcal{F}_{n+1})$ and that $F^n/F^{n+1} \subset H^p(X, \mathcal{F}_{n+1})$ is contained in the image of $N_n \to H^p(X, \mathcal{F}_{n+1})$. In the proof of Lemma 20.35.2 we have seen the statement on the module structure.

Let $t \in F^n$. Choose an element $s \in H^p(X, I^n\mathcal{F}_{n+1})$ which maps to the image of t in $H^p(X, \mathcal{F}_{n+1})$. We have to show that s is in N_n. Now F^n is the kernel of the map from $M \to H^p(X, \mathcal{F}_n)$ hence for all $m \geq n$ we can map t to an element $t_m \in H^p(X, \mathcal{F}_{m+1})$ which maps to zero in $H^p(X, \mathcal{F}_n)$. Consider the cohomology sequence

$$H^{p-1}(X, \mathcal{F}_n) \to H^p(X, I^n\mathcal{F}_{m+1}) \to H^p(X, \mathcal{F}_{m+1}) \to H^p(X, \mathcal{F}_n)$$

coming from the short exact sequence $0 \to I^n\mathcal{F}_{m+1} \to \mathcal{F}_{m+1} \to \mathcal{F}_n \to 0$. We can choose $s_m \in H^p(X, I^n\mathcal{F}_{m+1})$ mapping to t_m. Comparing the sequence above with the one for $m = n$ we see that s_m maps to s up to an element in the image of $H^{p-1}(X, \mathcal{F}_n) \to H^p(X, I^n\mathcal{F}_{n+1})$. However, this map factors through the map $H^p(X, I^n\mathcal{F}_{m+1}) \to H^p(X, I^n\mathcal{F}_{n+1})$ and we see that s is in the image as desired. $\square$

20.36. Inverse systems and cohomology, II

0H38 This section continues the discussion in Section 20.35 in the setting where the ideal is principal.

0H39 Lemma 20.36.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be inverse system of $\mathcal{O}_X$-modules. Consider the conditions

(1) for all $n \geq 1$ the map $f : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_n$ to give a short exact sequence $0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_1 \to 0$,
(2) for all $n \geq 1$ the map $f^n : \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}$ factors through $\mathcal{F}_{n+1} \to \mathcal{F}_1$ to give a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$
(3) there exists an $\mathcal{O}_X$-module $\mathcal{G}$ which is f-divisible such that $\mathcal{F}_n = \mathcal{G}[f^n]$, and
(4) there exists an $\mathcal{O}_X$-module $\mathcal{F}$ which is f-torsion free such that $\mathcal{F}_n = \mathcal{F}/f^n\mathcal{F}$.

Then (4) $\Rightarrow$ (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1).

Proof. We omit the proof of the equivalence of (1) and (2). We omit the proof that (3) implies (1). Given $\mathcal{F}_n$ as in (1) to prove (3) we set $\mathcal{G} = \mathrm{colim}\, \mathcal{F}_n$ where the maps $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \ldots$ are as in (1). The map $f : \mathcal{G} \to \mathcal{G}$ is surjective as the image of $\mathcal{F}_{n+1} \subset \mathcal{G}$ is $\mathcal{F}_n \subset \mathcal{G}$ by the short exact sequence (1). Thus $\mathcal{G}$ is an f-divisible $\mathcal{O}_X$-module with $\mathcal{F}_n = \mathcal{G}[f^n]$.

Assume given $\mathcal{F}$ as in (4). The map $\mathcal{F}/f^{n+1}\mathcal{F} \to \mathcal{F}/f^n\mathcal{F}$ is always surjective with kernel the image of the map $\mathcal{F}/f\mathcal{F} \to \mathcal{F}/f^{n+1}\mathcal{F}$ induced by multiplication with f^n. To verify (2) it suffices to see that the kernel of $f^n : \mathcal{F} \to \mathcal{F}/f^{n+1}\mathcal{F}$ is $f\mathcal{F}$. To see this it suffices to show that given sections s, t of $\mathcal{F}$ over an open $U \subset X$ with $f^n s = f^{n+1} t$ we have $s = ft$. This is clear because $f : \mathcal{F} \to \mathcal{F}$ is injective as $\mathcal{F}$ is f-torsion free. □

0EHA Lemma 20.36.2. Suppose X, f, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 20.36.1. Let $p \geq 0$ and set $H^p = \lim H^p(X, \mathcal{F}_n)$. Then $f^c H^p$ is the kernel of $H^p \to H^p(X, \mathcal{F}_c)$ for all $c \geq 1$. Thus the limit topology on H^p is the f-adic topology.

Proof. Let $c \geq 1$. It is clear that $f^c H^p$ maps to zero in $H^p(X, \mathcal{F}_c)$. If $\xi = (\xi_n) \in H^p$ is small in the limit topology, then $\xi_c = 0$, and hence ξ_n maps to zero in $H^p(X, \mathcal{F}_c)$ for $n \geq c$. Consider the inverse system of short exact sequences

$$0 \to \mathcal{F}_{n-c} \xrightarrow{f^c} \mathcal{F}_n \to \mathcal{F}_c \to 0$$

and the corresponding inverse system of long exact cohomology sequences

$$H^{p-1}(X, \mathcal{F}_c) \to H^p(X, \mathcal{F}_{n-c}) \to H^p(X, \mathcal{F}_n) \to H^p(X, \mathcal{F}_c)$$

Since the term $H^{p-1}(X, \mathcal{F}_c)$ is independent of n we can choose a compatible sequence of elements $\xi'_n \in H^p(X, \mathcal{F}_{n-c})$ lifting ξ_n. Setting $\xi' = (\xi'_n)$ we see that $\xi = f^c \xi'$ as desired. □

0BLB Lemma 20.36.3. Let A be a Noetherian ring complete with respect to a principal ideal (f). Let X be a topological space. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules. Assume

(1) $\Gamma(X, \mathcal{F}_1)$ is a finite A-module,
(2) X, f, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 20.36.1.

Then

$$M = \lim \Gamma(X, \mathcal{F}_n)$$

is a finite A-module, f is a nonzerodivisor on M, and M/fM is the image of M in $\Gamma(X, \mathcal{F}_1)$.

Proof. By Lemma 20.36.2 we have $M/fM \subset H^0(X, \mathcal{F}_1)$. From (1) and the Noetherian property of A we get that M/fM is a finite A-module. Observe that $\bigcap f^n M = 0$ as $f^n M$ maps to zero in $H^0(X, \mathcal{F}_n)$. By Algebra, Lemma 10.96.12 we conclude that M is finite over A. Finally, suppose $s = (s_n) \in M = \lim H^0(X, \mathcal{F}_n)$ satisfies $fs = 0$. Then s_{n+1} is in the kernel of $\mathcal{F}_{n+1} \to \mathcal{F}_n$ by condition (1) of Lemma 20.36.1. Hence $s_n = 0$. Since n was arbitrary, we see $s = 0$. Thus f is a nonzerodivisor on M. □

0BLC Lemma 20.36.4. Let A be a ring. Let $f \in A$. Let X be a topological space. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules. Let $p \geq 0$. Assume

(1) either $H^{p+1}(X, \mathcal{F}_1)$ is an A-module of finite length or A is Noetherian and $H^{p+1}(X, \mathcal{F}_1)$ is a finite A-module,

(2) X, f, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 20.36.1.

Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. Set $I = (f)$. We will use the criterion of Lemma 20.35.1. Observe that $f^n : \mathcal{F}_1 \to I^n\mathcal{F}_{n+1}$ is an isomorphism for all $n \geq 0$. Thus it suffices to show that

$$\bigoplus\nolimits_{n\geq 1} H^{p+1}(X, \mathcal{F}_1) \cdot f^{n+1}$$

is a graded $S = \bigoplus_{n\geq 0} A/(f) \cdot f^n$-module satisfying the ascending chain condition. If A is not Noetherian, then $H^{p+1}(X, \mathcal{F}_1)$ has finite length and the result holds. If A is Noetherian, then S is a Noetherian ring and the result holds as the module is finite over S by the assumed finiteness of $H^{p+1}(X, \mathcal{F}_1)$. Some details omitted. $\square$

0DXG Lemma 20.36.5. Let A be a ring. Let $f \in A$. Let X be a topological space. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules. Let $p \geq 0$. Assume

(1) either there is an $m \geq 1$ such that the image of $H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)$ is an A-module of finite length or A is Noetherian and the intersection of the images of $H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)$ is a finite A-module,
(2) X, f, $(\mathcal{F}_n)$ satisfy condition (1) of Lemma 20.36.1.

Then the inverse system $M_n = H^p(X, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition.

Proof. Set $I = (f)$. We will use the criterion of Lemma 20.35.2 involving the modules N_n. For $m \geq n$ we have $I^n\mathcal{F}_{m+1} = \mathcal{F}_{m+1-n}$. Thus we see that

$$N_n = \bigcap\nolimits_{m\geq 1} \text{Im}\left(H^{p+1}(X, \mathcal{F}_m) \to H^{p+1}(X, \mathcal{F}_1)\right)$$

is independent of n and $\bigoplus N_n = \bigoplus N_1 \cdot f^{n+1}$. Thus we conclude exactly as in the proof of Lemma 20.36.4. $\square$

0H3A Remark 20.36.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. Let $\mathcal{F}$ be $\mathcal{O}_X$-module. If $\mathcal{F}$ is f-torsion free, then for every $p \geq 0$ we have a short exact sequence of inverse systems

$$0 \to \{H^p(X, \mathcal{F})/f^nH^p(X, \mathcal{F})\} \to \{H^p(X, \mathcal{F}/f^n\mathcal{F})\} \to \{H^{p+1}(X, \mathcal{F})[f^n]\} \to 0$$

Since the first inverse system has the Mittag-Leffler condition (ML) we learn three things from this:

(1) There is a short exact sequence

$$0 \to \widehat{H^p(X, \mathcal{F})} \to \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) \to T_f(H^{p+1}(X, \mathcal{F})) \to 0$$

where $\widehat{\ }$ denotes the usual f-adic completion and $T_f(-)$ denotes the f-adic Tate module from More on Algebra, Example 15.93.5.
(2) We have $R^1 \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) = R^1 \lim H^{p+1}(X, \mathcal{F})[f^n]$.
(3) The system $\{H^{p+1}(X, \mathcal{F})[f^n]\}$ is ML if and only if $\{H^p(X, \mathcal{F}/f^n\mathcal{F})\}$ is ML.

See Homology, Lemma 12.31.3 and More on Algebra, Lemmas 15.86.2 and 15.86.13.

20.37. Derived limits

0BKN Let $(X, \mathcal{O}_X)$ be a ringed space. Since the triangulated category $D(\mathcal{O}_X)$ has products (Injectives, Lemma 19.13.4) it follows that $D(\mathcal{O}_X)$ has derived limits, see Derived Categories, Definition 13.34.1. If (K_n) is an inverse system in $D(\mathcal{O}_X)$ then we denote $R\lim K_n$ the derived limit.

0D60 Lemma 20.37.1. Let $(X, \mathcal{O}_X)$ be a ringed space. For $U \subset X$ open the functor $R\Gamma(U, -)$ commutes with $R\lim$. Moreover, there are short exact sequences

$$0 \to R^1\lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

for any inverse system (K_n) in $D(\mathcal{O}_X)$ and any $m \in \mathbf{Z}$.

Proof. The first statement follows from Injectives, Lemma 19.13.6. Then we may apply More on Algebra, Remark 15.86.10 to $R\lim R\Gamma(U, K_n) = R\Gamma(U, R\lim K_n)$ to get the short exact sequences. $\square$

0BKP Lemma 20.37.2. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then Rf_* commutes with $R\lim$, i.e., Rf_* commutes with derived limits.

Proof. Let (K_n) be an inverse system in $D(\mathcal{O}_X)$. Consider the defining distinguished triangle

$$R\lim K_n \to \prod K_n \to \prod K_n$$

in $D(\mathcal{O}_X)$. Applying the exact functor Rf_* we obtain the distinguished triangle

$$Rf_*(R\lim K_n) \to Rf_*\left(\prod K_n\right) \to Rf_*\left(\prod K_n\right)$$

in $D(\mathcal{O}_Y)$. Thus we see that it suffices to prove that Rf_* commutes with products in the derived category (which are not just given by products of complexes, see Injectives, Lemma 19.13.4). However, since Rf_* is a right adjoint by Lemma 20.28.1 this follows formally (see Categories, Lemma 4.24.5). Caution: Note that we cannot apply Categories, Lemma 4.24.5 directly as $R\lim K_n$ is not a limit in $D(\mathcal{O}_X)$. $\square$

0BKQ Remark 20.37.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K_n) be an inverse system in $D(\mathcal{O}_X)$. Set $K = R\lim K_n$. For each n and m let $\mathcal{H}^m_n = H^m(K_n)$ be the mth cohomology sheaf of K_n and similarly set $\mathcal{H}^m = H^m(K)$. Let us denote $\underline{\mathcal{H}}^m_n$ the presheaf

$$U \longmapsto \underline{\mathcal{H}}^m_n(U) = H^m(U, K_n)$$

Similarly we set $\underline{\mathcal{H}}^m(U) = H^m(U, K)$. By Lemma 20.32.3 we see that $\mathcal{H}^m_n$ is the sheafification of $\underline{\mathcal{H}}^m_n$ and $\mathcal{H}^m$ is the sheafification of $\underline{\mathcal{H}}^m$. Here is a diagram

$$\begin{tikzcd} K \arrow[d, equal] & \underline{\mathcal{H}}^m \arrow[r] \arrow[d] & \mathcal{H}^m \arrow[d] \\ R\lim K_n & \lim \underline{\mathcal{H}}^m_n \arrow[r] & \lim \mathcal{H}^m_n \end{tikzcd}$$

In general it may not be the case that $\lim \mathcal{H}^m_n$ is the sheafification of $\lim \underline{\mathcal{H}}^m_n$. If $U \subset X$ is an open, then we have short exact sequences

0BKR (20.37.3.1) $$0 \to R^1\lim \underline{\mathcal{H}}^{m-1}_n(U) \to \underline{\mathcal{H}}^m(U) \to \lim \underline{\mathcal{H}}^m_n(U) \to 0$$

by Lemma 20.37.1.

The following lemma applies to an inverse system of quasi-coherent modules with surjective transition maps on a scheme.

0BKS **Lemma 20.37.4.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{F}_n)$ be an inverse system of $\mathcal{O}_X$-modules. Let $\mathcal{B}$ be a set of opens of X. Assume

(1) every open of X has a covering whose members are elements of $\mathcal{B}$,
(2) $H^p(U, \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$,
(3) the inverse system $\mathcal{F}_n(U)$ has vanishing $R^1 \lim$ for $U \in \mathcal{B}$.

Then $R\lim \mathcal{F}_n = \lim \mathcal{F}_n$ and we have $H^p(U, \lim \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$.

Proof. Set $K_n = \mathcal{F}_n$ and $K = R\lim \mathcal{F}_n$. Using the notation of Remark 20.37.3 and assumption (2) we see that for $U \in \mathcal{B}$ we have $\underline{\mathcal{H}}_n^m(U) = 0$ when $m \neq 0$ and $\underline{\mathcal{H}}_n^0(U) = \mathcal{F}_n(U)$. From Equation (20.37.3.1) and assumption (3) we see that $\underline{\mathcal{H}}^m(U) = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n(U)$ when $m = 0$. Sheafifying using (1) we find that $\mathcal{H}^m = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n$ when $m = 0$. Hence $K = \lim \mathcal{F}_n$. Since $H^m(U, K) = \underline{\mathcal{H}}^m(U) = 0$ for $m > 0$ (see above) we see that the second assertion holds. $\square$

0D61 **Lemma 20.37.5.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K_n) be an inverse system in $D(\mathcal{O}_X)$. Let $x \in X$ and $m \in \mathbf{Z}$. Assume there exist an integer $n(x)$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of x such that for $U \in \mathfrak{U}_x$

(1) $R^1 \lim H^{m-1}(U, K_n) = 0$, and
(2) $H^m(U, K_n) \to H^m(U, K_{n(x)})$ is injective for $n \geq n(x)$.

Then the map on stalks $H^m(R\lim K_n)_x \to H^m(K_{n(x)})_x$ is injective.

Proof. Let γ be an element of $H^m(R\lim K_n)_x$ which maps to zero in $H^m(K_{n(x)})_x$. Since $H^m(R\lim K_n)$ is the sheafification of $U \mapsto H^m(U, R\lim K_n)$ (by Lemma 20.32.3) we can choose $U \in \mathfrak{U}_x$ and an element $\tilde{\gamma} \in H^m(U, R\lim K_n)$ mapping to γ. Then $\tilde{\gamma}$ maps to $\tilde{\gamma}_{n(x)} \in H^m(U, K_{n(x)})$. Using that $H^m(K_{n(x)})$ is the sheafification of $U \mapsto H^m(U, K_{n(x)})$ (by Lemma 20.32.3 again) we see that after shrinking U we may assume that $\tilde{\gamma}_{n(x)} = 0$. For this U we consider the short exact sequence

$$0 \to R^1 \lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

of Lemma 20.37.1. By assumption (1) the group on the left is zero and by assumption (2) the group on the right maps injectively into $H^m(U, K_{n(x)})$. We conclude $\tilde{\gamma} = 0$ and hence $\gamma = 0$ as desired. $\square$

0D62 **Lemma 20.37.6.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume that for every $x \in X$ there exist a function $p(x, -) : \mathbf{Z} \to \mathbf{Z}$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of x such that

$$H^p(U, H^{m-p}(E)) = 0 \text{ for } U \in \mathfrak{U}_x \text{ and } p > p(x, m)$$

Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O}_X)$.

Proof. Set $K_n = \tau_{\geq -n} E$ and $K = R\lim K_n$. The canonical map $E \to K$ comes from the canonical maps $E \to K_n = \tau_{\geq -n} E$. We have to show that $E \to K$ induces an isomorphism $H^m(E) \to H^m(K)$ of cohomology sheaves. In the rest of the proof we fix m. If $n \geq -m$, then the map $E \to \tau_{\geq -n} E = K_n$ induces an isomorphism $H^m(E) \to H^m(K_n)$. To finish the proof it suffices to show that for every $x \in X$ there exists an integer $n(x) \geq -m$ such that the map $H^m(K)_x \to H^m(K_{n(x)})_x$ is injective. Namely, then the composition

$$H^m(E)_x \to H^m(K)_x \to H^m(K_{n(x)})_x$$

is a bijection and the second arrow is injective, hence the first arrow is bijective. Set

$$n(x) = 1 + \max\{-m, p(x, m-1) - m, -1 + p(x, m) - m, -2 + p(x, m+1) - m\}.$$

so that in any case $n(x) \geq -m$. Claim: the maps

$$H^{m-1}(U, K_{n+1}) \to H^{m-1}(U, K_n) \quad \text{and} \quad H^m(U, K_{n+1}) \to H^m(U, K_n)$$

are isomorphisms for $n \geq n(x)$ and $U \in \mathfrak{U}_x$. The claim implies conditions (1) and (2) of Lemma 20.37.5 are satisfied and hence implies the desired injectivity. Recall (Derived Categories, Remark 13.12.4) that we have distinguished triangles

$$H^{-n-1}(E)[n+1] \to K_{n+1} \to K_n \to H^{-n-1}(E)[n+2]$$

Looking at the asssociated long exact cohomology sequence the claim follows if

$$H^{m+n}(U, H^{-n-1}(E)), \quad H^{m+n+1}(U, H^{-n-1}(E)), \quad H^{m+n+2}(U, H^{-n-1}(E))$$

are zero for $n \geq n(x)$ and $U \in \mathfrak{U}_x$. This follows from our choice of $n(x)$ and the assumption in the lemma. □

0D63 Lemma 20.37.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume that for every $x \in X$ there exist an integer $d_x \geq 0$ and a fundamental system $\mathfrak{U}_x$ of open neighbourhoods of x such that [Spa88, Proposition 3.13]

$$H^p(U, H^q(E)) = 0 \text{ for } U \in \mathfrak{U}_x,\ p > d_x, \text{ and } q < 0$$

Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O}_X)$.

Proof. This follows from Lemma 20.37.6 with $p(x, m) = d_x + \max(0, m)$. □

08U2 Lemma 20.37.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume there exist a function $p(-) : \mathbf{Z} \to \mathbf{Z}$ and a set $\mathcal{B}$ of opens of X such that

(1) every open in X has a covering whose members are elements of $\mathcal{B}$, and
(2) $H^p(U, H^{m-p}(E)) = 0$ for $p > p(m)$ and $U \in \mathcal{B}$.

Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O}_X)$.

Proof. Apply Lemma 20.37.6 with $p(x, m) = p(m)$ and $\mathfrak{U}_x = \{U \in \mathcal{B} \mid x \in U\}$. □

0D64 Lemma 20.37.9. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $E \in D(\mathcal{O}_X)$. Assume there exist an integer $d \geq 0$ and a basis $\mathcal{B}$ for the topology of X such that

$$H^p(U, H^q(E)) = 0 \text{ for } U \in \mathcal{B},\ p > d, \text{ and } q < 0$$

Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O}_X)$.

Proof. Apply Lemma 20.37.7 with $d_x = d$ and $\mathfrak{U}_x = \{U \in \mathcal{B} \mid x \in U\}$. □

The lemmas above can be used to compute cohomology in certain situations.

0BKT Lemma 20.37.10. Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be an object of $D(\mathcal{O}_X)$. Let $\mathcal{B}$ be a set of opens of X. Assume

(1) every open of X has a covering whose members are elements of $\mathcal{B}$,
(2) $H^p(U, H^q(K)) = 0$ for all $p > 0$, $q \in \mathbf{Z}$, and $U \in \mathcal{B}$.

Then $H^q(U, K) = H^0(U, H^q(K))$ for $q \in \mathbf{Z}$ and $U \in \mathcal{B}$.

Proof. Observe that $K = R\lim \tau_{\geq -n}K$ by Lemma 20.37.9 with $d = 0$. Let $U \in \mathcal{B}$. By Equation (20.37.3.1) we get a short exact sequence

$$0 \to R^1 \lim H^{q-1}(U, \tau_{\geq -n}K) \to H^q(U, K) \to \lim H^q(U, \tau_{\geq -n}K) \to 0$$

Condition (2) implies $H^q(U, \tau_{\geq -n}K) = H^0(U, H^q(\tau_{\geq -n}K))$ for all q by using the spectral sequence of Example 20.29.3. The spectral sequence converges because $\tau_{\geq -n}K$ is bounded below. If $n > -q$ then we have $H^q(\tau_{\geq -n}K) = H^q(K)$. Thus the systems on the left and the right of the displayed short exact sequence are eventually constant with values $H^0(U, H^{q-1}(K))$ and $H^0(U, H^q(K))$. The lemma follows. $\square$

Here is another case where we can describe the derived limit.

0BKU **Lemma 20.37.11.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K_n) be an inverse system of objects of $D(\mathcal{O}_X)$. Let $\mathcal{B}$ be a set of opens of X. Assume

1. every open of X has a covering whose members are elements of $\mathcal{B}$,
2. for all $U \in \mathcal{B}$ and all $q \in \mathbf{Z}$ we have
   - (a) $H^p(U, H^q(K_n)) = 0$ for $p > 0$,
   - (b) the inverse system $H^0(U, H^q(K_n))$ has vanishing $R^1 \lim$.

Then $H^q(R\lim K_n) = \lim H^q(K_n)$ for $q \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. We will use notation as in Remark 20.37.3. Let $U \in \mathcal{B}$. By Lemma 20.37.10 and (2)(a) we have $H^q(U, K_n) = H^0(U, H^q(K_n))$. Using that the functor $R\Gamma(U, -)$ commutes with derived limits we have

$$H^q(U, K) = H^q(R\lim R\Gamma(U, K_n)) = \lim H^0(U, H^q(K_n))$$

where the final equality follows from More on Algebra, Remark 15.86.10 and assumption (2)(b). Thus $H^q(U, K)$ is the inverse limit the sections of the sheaves $H^q(K_n)$ over U. Since $\lim H^q(K_n)$ is a sheaf we find using assumption (1) that $H^q(K)$, which is the sheafification of the presheaf $U \mapsto H^q(U, K)$, is equal to $\lim H^q(K_n)$. This proves the lemma. $\square$

20.38. Producing K-injective resolutions

0719 Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The category $\textit{Mod}(\mathcal{O}_X)$ has enough injectives, hence we can use Derived Categories, Lemma 13.29.3 produce a diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \tau_{\geq -2}\mathcal{F}^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}\mathcal{F}^\bullet \arrow[d] \\
\ldots \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_1^\bullet
\end{tikzcd}$$

in the category of complexes of $\mathcal{O}_X$-modules such that

1. the vertical arrows are quasi-isomorphisms,
2. $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
3. the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}_X$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 13.31.4 and 13.31.8 this is a K-injective complex. In general the canonical map

071A (20.38.0.1) $$\mathcal{F}^\bullet \to \mathcal{I}^\bullet$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

071B Lemma 20.38.1. In the situation described above. Denote $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$ the mth cohomology sheaf. Let $\mathcal{B}$ be a set of open subsets of X. Let $d \in \mathbf{N}$. Assume

(1) every open in X has a covering whose members are elements of $\mathcal{B}$,
(2) for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$ and $q < 0$[7].

Then (20.38.0.1) is a quasi-isomorphism.

Proof. By Derived Categories, Lemma 13.34.4 it suffices to show that the canonical map $\mathcal{F}^\bullet \to R\lim \tau_{\geq -n}\mathcal{F}^\bullet$ is an isomorphism. This is Lemma 20.37.9. □

Here is a technical lemma about the cohomology sheaves of the inverse limit of a system of complexes of sheaves. In some sense this lemma is the wrong thing to try to prove as one should take derived limits and not actual inverse limits.

08BY Lemma 20.38.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$. Assume there exist a set $\mathcal{B}$ of open subsets of X and an integer n_0 such that

(1) every open in X has a covering whose members are elements of $\mathcal{B}$,
(2) for every $U \in \mathcal{B}$
  (a) the systems of abelian groups $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ have vanishing $R^1\lim$ (for example these have the Mittag-Leffler condition),
  (b) the system of abelian groups $H^{m-1}(\mathcal{F}_n^\bullet(U))$ has vanishing $R^1\lim$ (for example it has the Mittag-Leffler condition), and
  (c) we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.

Then the maps $H^m(\mathcal{F}^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves where $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$ is the termwise inverse limit.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^{m}(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 15.86.3 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. □

[7]It suffices if $\forall m$, $\exists p(m)$, $H^p(U.\mathcal{H}^{m-p}) = 0$ for $p > p(m)$, see Lemma 20.37.8.

20.39. Inverse systems and cohomology, III

0H3B This section continues the discussion in Section 20.36 using derived limits.

0H3C **Lemma 20.39.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $A \to \Gamma(X, \mathcal{O}_X)$ be a ring map and let $f \in A$. Let E be an object of $D(\mathcal{O}_X)$. Denote

$$E_n = E \otimes_{\mathcal{O}_X} (\mathcal{O}_X \xrightarrow{f^n} \mathcal{O}_X)$$

and set $E^\wedge = R\lim E_n$. For $p \in \mathbf{Z}$ is a canonical commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \widehat{H^p(X, E)} \arrow[r] \arrow[u] & \lim H^p(X, E_n) \arrow[r] \arrow[u] & T_f(H^{p+1}(X, E)) \arrow[r] & 0 \\
0 \arrow[r] & H^0(H^p(X, E)^\wedge) \arrow[r] \arrow[u] & H^p(X, E^\wedge) \arrow[r] \arrow[u] & T_f(H^{p+1}(X, E)) \arrow[r] \arrow[u, equal] & 0 \\
 & R^1 \lim H^p(X, E)[f^n] \arrow[r, "\cong"] \arrow[u] & R^1 \lim H^{p-1}(X, E_n) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

with exact rows and columns where $\widehat{H^p(X, E)} = \lim H^p(X, E)/f^n H^p(X, E)$ is the usual f-adic completion, $H^p(X, E)^\wedge$ is the derived f-adic completion, and $T_f(H^{p+1}(X, E))$ is the f-adic Tate module, see More on Algebra, Example 15.93.5. Finally, we have $H^p(X, E^\wedge) = H^p(R\Gamma(X, E)^\wedge)$.

Proof. Observe that $R\Gamma(X, E^\wedge) = R\lim R\Gamma(X, E_n)$ by Lemma 20.37.2. On the other hand, we have

$$R\Gamma(X, E_n) = R\Gamma(X, E) \otimes_A^{\mathbf{L}} (A \xrightarrow{f^n} A)$$

(details omitted). We find that $R\Gamma(X, E^\wedge)$ is the derived f-adic completion $R\Gamma(X, E)^\wedge$. Whence the diagram by More on Algebra, Lemma 15.93.6. $\square$

0H3D **Lemma 20.39.2.** Let $\mathcal{A}$ be an abelian category. Let $f : M \to M$ be a morphism of $\mathcal{A}$. If $M[f^n] = \mathrm{Ker}(f^n : M \to M)$ stabilizes, then the inverse systems

$$(M \xrightarrow{f^n} M) \quad \text{and} \quad \mathrm{Coker}(f^n : M \to M)$$

are pro-isomorphic in $D(\mathcal{A})$.

Proof. There is clearly a map from the first inverse system to the second. Suppose that $M[f^c] = M[f^{c+1}] = M[f^{c+2}] = \ldots$. Then we can define an arrow of inverse systems in $D(\mathcal{A})$ in the other direction by the diagrams

$$\begin{tikzcd}
M/M[f^c] \arrow[r, "f^{n+c}"'] \arrow[d, "f^c"'] & M \arrow[d, "1"] \\
M \arrow[r, "f^n"] & M
\end{tikzcd}$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $\mathrm{Coker}(f^{n+c} : M \to M)$. Some details omitted. □

0H3E Example 20.39.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $A \to \Gamma(X, \mathcal{O}_X)$ be a ring map and let $f \in A$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there is a c such that $\mathcal{F}[f^c] = \mathcal{F}[f^n]$ for all $n \geq c$. We are going to apply Lemma 20.39.1 with $E = \mathcal{F}$. By Lemma 20.39.2 we see that the inverse system (E_n) is pro-isomorphic to the inverse system $(\mathcal{F}/f^n\mathcal{F})$. We conclude that for $p \in \mathbf{Z}$ we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \widehat{H^p(X, \mathcal{F})} \arrow[r] \arrow[u] & \lim H^p(X, \mathcal{F}/f^n\mathcal{F}) \arrow[r] \arrow[u] & T_f(H^{p+1}(X, \mathcal{F})) \arrow[r] & 0 \\
0 \arrow[r] & H^0(H^p(X, \mathcal{F})^\wedge) \arrow[r] \arrow[u] & H^p(R\Gamma(X, \mathcal{F})^\wedge) \arrow[r] \arrow[u] & T_f(H^{p+1}(X, \mathcal{F})) \arrow[r] \arrow[u, equal] & 0 \\
 & R^1 \lim H^p(X, \mathcal{F})[f^n] \arrow[r, "\cong"] \arrow[u] & R^1 \lim H^{p-1}(X, \mathcal{F}/f^n\mathcal{F}) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

with exact rows and columns where $\widehat{H^p(X, \mathcal{F})} = \lim H^p(X, \mathcal{F})/f^n H^p(X, \mathcal{F})$ is the usual f-adic completion and $M^\wedge$ denotes derived f-adic completion for M in $D(A)$.

20.40. Čech cohomology of unbounded complexes

08BZ The construction of Section 20.25 isn't the "correct" one for unbounded complexes. The problem is that in the Stacks project we use direct sums in the totalization of a double complex and we would have to replace this by a product. Instead of doing so in this section we assume the covering is finite and we use the alternating Čech complex.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of presheaves of $\mathcal{O}_X$-modules. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite open covering of X. Since the alternating Čech complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ (Section 20.23) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)$. In this section we work with the associated total complex. The construction of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

08C0 (20.40.0.1)
$$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0 \ldots i_p} = 0$ for $p > 0$.

08C1 Lemma 20.40.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite open covering. For a complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and compatible with (20.40.0.1).

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex whose terms are injective $\mathcal{O}_X$-modules. The map (20.40.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 12.25.4 applied to the double complex $\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemmas 20.11.1 and 20.23.6. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of $\mathcal{F}^\bullet$ into a K-injective complex whose terms are injectives (Injectives, Theorem 19.12.6). Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. □

08C2 Lemma 20.40.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite open covering. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $\mathcal{B}$ be a set of open subsets of X. Assume

(1) every open in X has a covering whose members are elements of $\mathcal{B}$,
(2) we have $U_{i_0 \ldots i_p} \in \mathcal{B}$ for all $i_0, \ldots, i_p \in I$,
(3) for every $U \in \mathcal{B}$ and $p > 0$ we have
   (a) $H^p(U, \mathcal{F}^q) = 0$,
   (b) $H^p(U, \text{Coker}(\mathcal{F}^{q-1} \to \mathcal{F}^q)) = 0$, and
   (c) $H^p(U, H^q(\mathcal{F})) = 0$.

Then the map

$$\text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

of Lemma 20.40.1 is an isomorphism in $D(\text{Ab})$.

Proof. First assume $\mathcal{F}^\bullet$ is bounded below. In this case the map

$$\text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

is a quasi-isomorphism by Lemma 20.23.6. Namely, the map of double complexes $\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \mathcal{F}^\bullet) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ induces an isomorphism between the first pages of the second spectral sequences associated to these complexes (by Homology, Lemma 12.25.1) and these spectral sequences converge (Homology, Lemma 12.25.3). Thus the conclusion in this case by Lemma 20.25.2 and assumption (3)(a).

In general, by assumption (3)(c) we may choose a resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in Lemma 20.38.1. Then the map of the lemma becomes

$$\lim_n \text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \Gamma(X, \mathcal{I}^\bullet) = \lim_n \Gamma(X, \mathcal{I}_n^\bullet)$$

Here the arrow is in the derived category, but the equality on the right holds on the level of complexes. Note that (3)(b) shows that $\tau_{\geq -n}\mathcal{F}^\bullet$ is a bounded below complex satisfying the hypothesis of the lemma. Thus the case of bounded below complexes shows each of the maps

$$\text{Tot}(\check{\mathcal{C}}_{alt}^\bullet(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \Gamma(X, \mathcal{I}_n^\bullet)$$

is a quasi-isomorphism. The cohomologies of the complexes on the left hand side in given degree are eventually constant (as the alternating Čech complex is finite). Hence the same is true on the right hand side. Thus the cohomology of the limit on the right hand side is this constant value by Homology, Lemma 12.31.7 (or the stronger More on Algebra, Lemma 15.86.3) and we win. □

20.41. Hom complexes

0A8K Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of $\mathcal{O}_X$-modules. We construct a complex of $\mathcal{O}_X$-modules $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. Namely, for each n we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of $\mathcal{H}om^n$ as the sheaf of $\mathcal{O}_X$-modules of all $\mathcal{O}_X$-linear maps from $\mathcal{L}^\bullet$ to $\mathcal{M}^\bullet$ (viewed as graded $\mathcal{O}_X$-modules) which are homogenous of degree n. In this terminology, we define the differential by the rule

$$\mathrm{d}(f) = \mathrm{d}_\mathcal{M} \circ f - (-1)^n f \circ \mathrm{d}_\mathcal{L}$$

for $f \in \mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. We omit the verification that $\mathrm{d}^2 = 0$. This construction is a special case of Differential Graded Algebra, Example 22.26.6. It follows immediately from the construction that we have

0A8L (20.41.0.1) $$H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n])$$

for all $n \in \mathbf{Z}$ and every open $U \subset X$.

0A8M Lemma 20.41.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is an isomorphism

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules functorial in $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.1. $\square$

0A8N Lemma 20.41.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.3. $\square$

0BYR Lemma 20.41.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism

$$\mathrm{Tot}\left(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{L}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}_X$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.4. $\square$

0A8Q Lemma 20.41.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}_X$-modules functorial in both complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.5. $\square$

0A8P Lemma 20.41.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical morphism

$$\text{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}_X$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.6. □

0A8R Lemma 20.41.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules. Let $\mathcal{L}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \text{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

for all $U \subset X$ open.

Proof. We have

$$\begin{aligned} H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) &= \text{Hom}_{K(\mathcal{O}_U)}(L|_U, M|_U) \\ &= \text{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U) \end{aligned}$$

The first equality is (20.41.0.1). The second equality is true because $\mathcal{I}^\bullet|_U$ is K-injective by Lemma 20.32.1. □

0A8S Lemma 20.41.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{I}')^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of K-injective complexes of $\mathcal{O}_X$-modules. Let $(\mathcal{L}')^\bullet \to \mathcal{L}^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}_X$-modules. Then

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

Proof. Let M be the object of $D(\mathcal{O}_X)$ represented by $\mathcal{I}^\bullet$ and $(\mathcal{I}')^\bullet$. Let L be the object of $D(\mathcal{O}_X)$ represented by $\mathcal{L}^\bullet$ and $(\mathcal{L}')^\bullet$. By Lemma 20.41.6 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \text{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism. □

0A8T Lemma 20.41.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}_X$-modules. Let $\mathcal{L}^\bullet$ be a K-flat complex of $\mathcal{O}_X$-modules. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_X$-modules.

Proof. Namely, if $\mathcal{K}^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules, then

$$\begin{aligned} \text{Hom}_{K(\mathcal{O}_X)}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(X, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\ &= H^0(\Gamma(X, \mathcal{H}om^\bullet(\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\ &= \text{Hom}_{K(\mathcal{O}_X)}(\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\ &= 0 \end{aligned}$$

The first equality by (20.41.0.1). The second equality by Lemma 20.41.1. The third equality by (20.41.0.1). The final equality because $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is acyclic because $\mathcal{L}^\bullet$ is K-flat (Definition 20.26.2) and because $\mathcal{I}^\bullet$ is K-injective. □

20.42. Internal hom in the derived category

08DH Let $(X, \mathcal{O}_X)$ be a ringed space. Let L, M be objects of $D(\mathcal{O}_X)$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O}_X)$ such that for every third object K of $D(\mathcal{O}_X)$ there exists a canonical bijection

08DI (20.42.0.1)
$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M)$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 4.3.5).

To construct such an object, choose a K-injective complex $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Then we set

$$R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the right hand side is the complex of $\mathcal{O}_X$-modules constructed in Section 20.41. This is well defined by Lemma 20.41.7. We get a functor

$$D(\mathcal{O}_X)^{opp} \times D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)$$

As a prelude to proving (20.42.0.1) we compute the cohomology groups of $R\mathcal{H}om(K, L)$.

08DK **Lemma 20.42.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let L, M be objects of $D(\mathcal{O}_X)$. For every open U we have

$$H^0(U, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

and in particular $H^0(X, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(L, M)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing M and a K-flat complex $\mathcal{L}^\bullet$ representing L. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is K-injective by Lemma 20.41.8. Hence we can compute cohomology over U by simply taking sections over U and the result follows from Lemma 20.41.6. □

08DJ **Lemma 20.42.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L, M be objects of $D(\mathcal{O}_X)$. With the construction as described above there is a canonical isomorphism

$$R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M which recovers (20.42.0.1) by taking $H^0(X, -)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Let $\mathcal{K}^\bullet$ be any complex of $\mathcal{O}_X$-modules representing K. Then we have

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

by Lemma 20.41.1. Note that the left hand side represents $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$ (use Lemma 20.41.8) and that the right hand side represents $R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M)$. This proves the displayed formula of the lemma. Taking global sections and using Lemma 20.42.1 we obtain (20.42.0.1). □

08DL **Lemma 20.42.3.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions to opens, i.e., for every open U we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.

Proof. This is clear from the construction and Lemma 20.32.1. □

08I0 **Lemma 20.42.4.** Let $(X, \mathcal{O}_X)$ be a ringed space. The bifunctor $R\mathcal{H}om(-, -)$ transforms distinguished triangles into distinguished triangles in both variables.

Proof. This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted. $\square$

0A8V Lemma 20.42.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Given K, L, M in $D(\mathcal{O}_X)$ there is a canonical morphism

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and any complex of $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$ representing K. By Lemma 20.41.2 there is a map of complexes

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

The complexes of $\mathcal{O}_X$-modules $\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet)$, $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, and $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ represent $R\mathcal{H}om(L, M)$, $R\mathcal{H}om(K, L)$, and $R\mathcal{H}om(K, M)$. If we choose a K-flat complex $\mathcal{H}^\bullet$ and a quasi-isomorphism $\mathcal{H}^\bullet \to \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, then there is a map

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}^\bullet\right) \longrightarrow \mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right)$$

whose source represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. $\square$

0BYS Lemma 20.42.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Given K, L, M in $D(\mathcal{O}_X)$ there is a canonical morphism

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K, and a K-injective complex $\mathcal{I}^\bullet$ representing L, and choose any complex of $\mathcal{O}_X$-modules $\mathcal{M}^\bullet$ representing M. Choose a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is K-injective. Then we use the map

$$\mathrm{Tot}\left(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{I}^\bullet)\right) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$$

where the first map is the map from Lemma 20.41.3. $\square$

0A8W Lemma 20.42.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Given K, L in $D(\mathcal{O}_X)$ there is a canonical morphism

$$K \longrightarrow R\mathcal{H}om(L, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

in $D(\mathcal{O}_X)$ functorial in both K and L.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K and any complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Choose a K-injective complex $\mathcal{J}^\bullet$ and a quasi-isomorphism $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to \mathcal{J}^\bullet$. Then we use

$$\mathcal{K}^\bullet \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet)$$

where the first map comes from Lemma 20.41.4. $\square$

08I1 **Lemma 20.42.8.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let L be an object of $D(\mathcal{O}_X)$. Set $L^\vee = R\mathcal{H}om(L, \mathcal{O}_X)$. For M in $D(\mathcal{O}_X)$ there is a canonical map

08I2 (20.42.8.1) $$M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L^\vee \longrightarrow R\mathcal{H}om(L, M)$$

which induces a canonical map

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} L^\vee) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(L, M)$$

functorial in M in $D(\mathcal{O}_X)$.

Proof. The map (20.42.8.1) is a special case of Lemma 20.42.5 using the identification $M = R\mathcal{H}om(\mathcal{O}_X, M)$. $\square$

0A8U **Lemma 20.42.9.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L, M be objects of $D(\mathcal{O}_X)$. There is a canonical morphism

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

in $D(\mathcal{O}_X)$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and a K-flat complex $\mathcal{K}^\bullet$ representing K. The map is defined using the map

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 20.41.5. By our particular choice of complexes the left hand side represents $R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K$ and the right hand side represents $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$. We omit the proof that this is functorial in all three objects of $D(\mathcal{O}_X)$. $\square$

0FXP **Remark 20.42.10.** Let $(X, \mathcal{O}_X)$ be a ringed space. For K, K', M, M' in $D(\mathcal{O}_X)$ there is a canonical map

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, M') \longrightarrow R\mathcal{H}om(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M, K' \otimes^{\mathbf{L}}_{\mathcal{O}_X} M')$$

Namely, by (20.42.0.1) is the same thing as a map

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, M') \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow K' \otimes^{\mathbf{L}}_{\mathcal{O}_X} M'$$

For this we can first flip the middle two factors (with sign rules as in More on Algebra, Section 15.72) and use the maps

$$R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to K' \quad \text{and} \quad R\mathcal{H}om(M, M') \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \to M'$$

from Lemma 20.42.5 when thinking of $K = R\mathcal{H}om(\mathcal{O}_X, K)$ and similarly for K', M, and M'.

0B69 **Remark 20.42.11.** Let $f : X \to Y$ be a morphism of ringed spaces. Let K, L be objects of $D(\mathcal{O}_X)$. We claim there is a canonical map

$$Rf_*R\mathcal{H}om(L, K) \longrightarrow R\mathcal{H}om(Rf_*L, Rf_*K)$$

Namely, by (20.42.0.1) this is the same thing as a map $Rf_*R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \to Rf_*K$. For this we can use the composition

$$Rf_*R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*L \to Rf_*(R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) \to Rf_*K$$

where the first arrow is the relative cup product (Remark 20.28.7) and the second arrow is Rf_* applied to the canonical map $R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \to K$ coming from Lemma 20.42.5 (with $\mathcal{O}_X$ in one of the spots).

0G7A Remark 20.42.12. Let $h : X \to Y$ be a morphism of ringed spaces. Let K, L, M be objects of $D(\mathcal{O}_Y)$. The diagram

$$\begin{tikzcd} Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K,M) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M \arrow[r] \arrow[d] & Rf_*\left(R\mathcal{H}om_{\mathcal{O}_X}(K,M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} M\right) \arrow[d] \\ R\mathcal{H}om_{\mathcal{O}_Y}(Rf_*K, Rf_*M) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_*M \arrow[r] & Rf_*M \end{tikzcd}$$

is commutative. Here the left vertical arrow comes from Remark 20.42.11. The top horizontal arrow is Remark 20.28.7. The other two arrows are instances of the map in Lemma 20.42.5 (with one of the entries replaced with $\mathcal{O}_X$ or $\mathcal{O}_Y$).

08I3 Remark 20.42.13. Let $h : X \to Y$ be a morphism of ringed spaces. Let K, L be objects of $D(\mathcal{O}_Y)$. We claim there is a canonical map

$$Lh^*R\mathcal{H}om(K,L) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_X)$. Namely, by (20.42.0.1) proved in Lemma 20.42.2 such a map is the same thing as a map

$$Lh^*R\mathcal{H}om(K,L) \otimes^{\mathbf{L}} Lh^*K \longrightarrow Lh^*L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K,L) \otimes^{\mathbf{L}} K)$ by Lemma 20.27.3 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K,L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\mathrm{id} : R\mathcal{H}om(K,L) \longrightarrow R\mathcal{H}om(K,L)$$

via (20.42.0.1).

08I4 Remark 20.42.14. Suppose that

$$\begin{tikzcd} X' \arrow[r, "h"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

is a commutative diagram of ringed spaces. Let K, L be objects of $D(\mathcal{O}_X)$. We claim there exists a canonical base change map

$$Lg^*Rf_*R\mathcal{H}om(K,L) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_{S'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^*Lg^*Rf_*R\mathcal{H}om(K,L) &= Lh^*Lf^*Rf_*R\mathcal{H}om(K,L) \\ &\to Lh^*R\mathcal{H}om(K,L) \\ &\to R\mathcal{H}om(Lh^*K, Lh^*L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^*Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 20.42.13.

20.43. Ext sheaves

0BQP Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L \in D(\mathcal{O}_X)$. Using the construction of the internal hom in the derived category we obtain a well defined sheaves of $\mathcal{O}_X$-modules

$$\mathcal{E}xt^n(K, L) = H^n(R\mathcal{H}om(K, L))$$

by taking the nth cohomology sheaf of the object $R\mathcal{H}om(K, L)$ of $D(\mathcal{O}_X)$. We will sometimes write $\mathcal{E}xt^n_{\mathcal{O}_X}(K, L)$ for this object. By Lemma 20.42.1 we see that this $\mathcal{E}xt^n$-sheaf is the sheafification of the rule

$$U \longmapsto \mathrm{Ext}^n_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

By Example 20.29.3 there is always a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(K, L))$$

converging to $\mathrm{Ext}^{p+q}_{D(\mathcal{O}_X)}(K, L)$ in favorable situations (for example if L is bounded below and K is bounded above).

20.44. Global derived hom

0B6A Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L \in D(\mathcal{O}_X)$. Using the construction of the internal hom in the derived category we obtain a well defined object

$$R\mathrm{Hom}_X(K, L) = R\Gamma(X, R\mathcal{H}om(K, L))$$

in $D(\Gamma(X, \mathcal{O}_X))$. We will sometimes write $R\mathrm{Hom}_{\mathcal{O}_X}(K, L)$ for this object. By Lemma 20.42.1 we have

$$H^0(R\mathrm{Hom}_X(K, L)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L), \quad H^p(R\mathrm{Hom}_X(K, L)) = \mathrm{Ext}^p_{D(\mathcal{O}_X)}(K, L)$$

If $f : Y \to X$ is a morphism of ringed spaces, then there is a canonical map

$$R\mathrm{Hom}_X(K, L) \longrightarrow R\mathrm{Hom}_Y(Lf^*K, Lf^*L)$$

in $D(\Gamma(X, \mathcal{O}_X))$ by taking global sections of the map defined in Remark 20.42.13.

20.45. Glueing complexes

0D65 We can glue complexes! More precisely, in certain circumstances we can glue locally given objects of the derived category to a global object. We first prove some easy cases and then we'll prove the very general [BBD82, Theorem 3.2.4] in the setting of topological spaces and open coverings.

08DG Lemma 20.45.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of X. Suppose given

(1) an object A of $D(\mathcal{O}_U)$,
(2) an object B of $D(\mathcal{O}_V)$, and
(3) an isomorphism $c : A|_{U\cap V} \to B|_{U\cap V}$.

Then there exists an object F of $D(\mathcal{O}_X)$ and isomorphisms $f : F|_U \to A$, $g : F|_V \to B$ such that $c = g|_{U\cap V} \circ f^{-1}|_{U\cap V}$. Moreover, given

(1) an object E of $D(\mathcal{O}_X)$,
(2) a morphism $a : A \to E|_U$ of $D(\mathcal{O}_U)$,
(3) a morphism $b : B \to E|_V$ of $D(\mathcal{O}_V)$,

such that

$$a|_{U \cap V} = b|_{U \cap V} \circ c.$$

Then there exists a morphism $F \to E$ in $D(\mathcal{O}_X)$ whose restriction to U is $a \circ f$ and whose restriction to V is $b \circ g$.

Proof. Denote j_U, j_V, $j_{U \cap V}$ the corresponding open immersions. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U \cap V,*}(B|_{U \cap V}) \to F[1]$$

where the map $Rj_{V,*}B \to Rj_{U \cap V,*}(B|_{U \cap V})$ is the obvious one and where $Rj_{U,*}A \to Rj_{U \cap V,*}(B|_{U \cap V})$ is the composition of $Rj_{U,*}A \to Rj_{U \cap V,*}(A|_{U \cap V})$ with $Rj_{U \cap V,*}c$. Restricting to U we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U \cap V,*}(B|_{U \cap V}))|_U \to F|_U[1]$$

Denote $j : U \cap V \to U$. Compatibility of restriction to opens and cohomology shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U \cap V,*}(B|_{U \cap V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U \cap V})$. Hence the second arrow of the last displayed diagram has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism. Similarly, the morphism $F|_V \to B$ is an isomorphism. The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 20.33.3. $\square$

0D66 Lemma 20.45.2. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{B}$ be a basis for the topology on Y.

(1) Assume K is in $D(\mathcal{O}_X)$ such that for $V \in \mathcal{B}$ we have $H^i(f^{-1}(V), K) = 0$ for $i < 0$. Then Rf_*K has vanishing cohomology sheaves in negative degrees, $H^i(f^{-1}(V), K) = 0$ for $i < 0$ for all opens $V \subset Y$, and the rule $V \mapsto H^0(f^{-1}V, K)$ is a sheaf on Y.
(2) Assume K, L are in $D(\mathcal{O}_X)$ such that for $V \in \mathcal{B}$ we have $\mathrm{Ext}^i(K|_{f^{-1}V}, L|_{f^{-1}V}) = 0$ for $i < 0$. Then $\mathrm{Ext}^i(K|_{f^{-1}V}, L|_{f^{-1}V}) = 0$ for $i < 0$ for all opens $V \subset Y$ and the rule $V \mapsto \mathrm{Hom}(K|_{f^{-1}V}, L|_{f^{-1}V})$ is a sheaf on Y.

Proof. Lemma 20.32.6 tells us $H^i(Rf_*K)$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), K) = H^i(V, Rf_*K)$. The assumptions in (1) imply that Rf_*K has vanishing cohomology sheaves in degrees < 0. We conclude that for any open $V \subset Y$ the cohomology group $H^i(V, Rf_*K)$ is zero for $i < 0$ and is equal to $H^0(V, H^0(Rf_*K))$ for $i = 0$. This proves (1).

To prove (2) apply (1) to the complex $R\mathcal{H}om(K, L)$ using Lemma 20.42.1 to do the translation. $\square$

0D67 Situation 20.45.3. Let $(X, \mathcal{O}_X)$ be a ringed space. We are given

(1) a collection of opens $\mathcal{B}$ of X,
(2) for $U \in \mathcal{B}$ an object K_U in $D(\mathcal{O}_U)$,
(3) for $V \subset U$ with $V, U \in \mathcal{B}$ an isomorphism $\rho^U_V : K_U|_V \to K_V$ in $D(\mathcal{O}_V)$,

such that whenever we have $W \subset V \subset U$ with U, V, W in $\mathcal{B}$, then $\rho^U_W = \rho^V_W \circ \rho^U_V|_W$.

We won't be able to prove anything about this without making more assumptions. An interesting case is where $\mathcal{B}$ is a basis for the topology on X. Another is the case where we have a morphism $f : X \to Y$ of topological spaces and the elements of $\mathcal{B}$ are the inverse images of the elements of a basis for the topology of Y.

In Situation 20.45.3 a solution will be a pair (K, ρ_U) where K is an object of $D(\mathcal{O}_X)$ and $\rho_U : K|_U \to K_U$, $U \in \mathcal{B}$ are isomorphisms such that we have $\rho_V^U \circ \rho_U|_V = \rho_V$ for all $V \subset U$, $U, V \in \mathcal{B}$. In certain cases solutions are unique.

0D68 Lemma 20.45.4. In Situation 20.45.3 assume

(1) $X = \bigcup_{U \in \mathcal{B}} U$ and for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$,
(2) for any $U \in \mathcal{B}$ we have $\text{Ext}^i(K_U, K_U) = 0$ for $i < 0$.

If a solution (K, ρ_U) exists, then it is unique up to unique isomorphism and moreover $\text{Ext}^i(K, K) = 0$ for $i < 0$.

Proof. Let (K, ρ_U) and (K', ρ'_U) be a pair of solutions. Let $f : X \to Y$ be the continuous map constructed in Topology, Lemma 5.5.6. Set $\mathcal{O}_Y = f_*\mathcal{O}_X$. Then K, K' and $\mathcal{B}$ are as in Lemma 20.45.2 part (2). Hence we obtain the vanishing of negative exts for K and we see that the rule

$$V \longmapsto \text{Hom}(K|_{f^{-1}V}, K'|_{f^{-1}V})$$

is a sheaf on Y. As both (K, ρ_U) and (K', ρ'_U) are solutions the maps

$$(\rho'_U)^{-1} \circ \rho_U : K|_U \longrightarrow K'|_U$$

over $U = f^{-1}(f(U))$ agree on overlaps. Hence we get a unique global section of the sheaf above which defines the desired isomorphism $K \to K'$ compatible with all structure available. $\square$

0D69 Remark 20.45.5. With notation and assumptions as in Lemma 20.45.4. Suppose that $U, V \in \mathcal{B}$. Let $\mathcal{B}'$ be the set of elements of $\mathcal{B}$ contained in $U \cap V$. Then

$$(\{K_{U'}\}_{U' \in \mathcal{B}'}, \{\rho_{V'}^{U'}\}_{V' \subset U' \text{ with } U', V' \in \mathcal{B}'})$$

is a system on the ringed space $U \cap V$ satisfying the assumptions of Lemma 20.45.4. Moreover, both $(K_U|_{U \cap V}, \rho_{U'}^U)$ and $(K_V|_{U \cap V}, \rho_{U'}^V)$ are solutions to this system. By the lemma we find a unique isomorphism

$$\rho_{U,V} : K_U|_{U \cap V} \longrightarrow K_V|_{U \cap V}$$

such that for every $U' \subset U \cap V$, $U' \in \mathcal{B}$ the diagram

$$\begin{tikzcd} K_U|_{U'} \arrow[rr, "\rho_{U,V}|_{U'}"'] \arrow[dr, "\rho_{U'}^U"'] & & K_V|_{U'} \arrow[dl, "\rho_{U'}^V"] \\ & K_{U'} & \end{tikzcd}$$

commutes. Pick a third element $W \in \mathcal{B}$. We obtain isomorphisms $\rho_{U,W} : K_U|_{U \cap W} \to K_W|_{U \cap W}$ and $\rho_{V,W} : K_U|_{V \cap W} \to K_W|_{V \cap W}$ satisfying similar properties to those of $\rho_{U,V}$. Finally, we have

$$\rho_{U,W}|_{U \cap V \cap W} = \rho_{V,W}|_{U \cap V \cap W} \circ \rho_{U,V}|_{U \cap V \cap W}$$

This is true by the uniqueness in the lemma because both sides of the equality are the unique isomorphism compatible with the maps $\rho_{U''}^U$ and $\rho_{U''}^W$ for $U'' \subset U \cap V \cap W$, $U'' \in \mathcal{B}$. Some minor details omitted. The collection $(K_U, \rho_{U,V})$ is a descent datum in the derived category for the open covering $\mathcal{U} : X = \bigcup_{U \in \mathcal{B}} U$ of X. In this language we are looking for "effectiveness of the descent datum" when we look for the existence of a solution.

0D6A Lemma 20.45.6. In Situation 20.45.3 assume

(1) $X = U_1 \cup \ldots \cup U_n$ with $U_i \in \mathcal{B}$,
(2) for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W \in \mathcal{B}, W \subset U \cap V} W$,
(3) for any $U \in \mathcal{B}$ we have $\mathrm{Ext}^i(K_U, K_U) = 0$ for $i < 0$.

Then a solution exists and is unique up to unique isomorphism.

Proof. Uniqueness was seen in Lemma 20.45.4. We may prove the lemma by induction on n. The case $n = 1$ is immediate.

The case $n = 2$. Consider the isomorphism $\rho_{U_1,U_2} : K_{U_1}|_{U_1 \cap U_2} \to K_{U_2}|_{U_1 \cap U_2}$ constructed in Remark 20.45.5. By Lemma 20.45.1 we obtain an object K in $D(\mathcal{O}_X)$ and isomorphisms $\rho_{U_1} : K|_{U_1} \to K_{U_1}$ and $\rho_{U_2} : K|_{U_2} \to K_{U_2}$ compatible with ρ_{U_1,U_2}. Take $U \in \mathcal{B}$. We will construct an isomorphism $\rho_U : K|_U \to K_U$ and we will leave it to the reader to verify that (K, ρ_U) is a solution. Consider the set $\mathcal{B}'$ of elements of $\mathcal{B}$ contained in either $U \cap U_1$ or contained in $U \cap U_2$. Then $(K_U, \rho^U_{U'})$ is a solution for the system $(\{K_{U'}\}_{U' \in \mathcal{B}'}, \{\rho^{U'}_{V'}\}_{V' \subset U' \text{ with } U', V' \in \mathcal{B}'})$ on the ringed space U. We claim that $(K|_U, \tau_{U'})$ is another solution where $\tau_{U'}$ for $U' \in \mathcal{B}'$ is chosen as follows: if $U' \subset U_1$ then we take the composition

$$K|_{U'} \xrightarrow{\rho_{U_1}|_{U'}} K_{U_1}|_{U'} \xrightarrow{\rho^{U_1}_{U'}} K_{U'}$$

and if $U' \subset U_2$ then we take the composition

$$K|_{U'} \xrightarrow{\rho_{U_2}|_{U'}} K_{U_2}|_{U'} \xrightarrow{\rho^{U_2}_{U'}} K_{U'}.$$

To verify this is a solution use the property of the map ρ_{U_1,U_2} described in Remark 20.45.5 and the compatibility of ρ_{U_1} and ρ_{U_2} with ρ_{U_1,U_2}. Having said this we apply Lemma 20.45.4 to see that we obtain a unique isomorphism $K|_{U'} \to K_{U'}$ compatible with the maps $\tau_{U'}$ and $\rho^U_{U'}$ for $U' \in \mathcal{B}'$.

The case $n > 2$. Consider the open subspace $X' = U_1 \cup \ldots \cup U_{n-1}$ and let $\mathcal{B}'$ be the set of elements of $\mathcal{B}$ contained in X'. Then we find a system $(\{K_U\}_{U \in \mathcal{B}'}, \{\rho^U_V\}_{U,V \in \mathcal{B}'})$ on the ringed space X' to which we may apply our induction hypothesis. We find a solution $(K_{X'}, \rho^{X'}_U)$. Then we can consider the collection $\mathcal{B}^* = \mathcal{B} \cup \{X'\}$ of opens of X and we see that we obtain a system $(\{K_U\}_{U \in \mathcal{B}^*}, \{\rho^U_V\}_{V \subset U \text{ with } U,V \in \mathcal{B}^*})$. Note that this new system also satisfies condition (3) by Lemma 20.45.4 applied to the solution $K_{X'}$. For this system we have $X = X' \cup U_n$. This reduces us to the case $n = 2$ we worked out above. □

0D6B Lemma 20.45.7. Let X be a ringed space. Let E be a well ordered set and let

$$X = \bigcup\nolimits_{\alpha \in E} W_\alpha$$

be an open covering with $W_\alpha \subset W_{\alpha+1}$ and $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$ if α is not a successor. Let K_α be an object of $D(\mathcal{O}_{W_\alpha})$ with $\mathrm{Ext}^i(K_\alpha, K_\alpha) = 0$ for $i < 0$. Assume given isomorphisms $\rho^\alpha_\beta : K_\alpha|_{W_\beta} \to K_\beta$ in $D(\mathcal{O}_{W_\beta})$ for all $\beta < \alpha$ with $\rho^\alpha_\gamma = \rho^\beta_\gamma \circ \rho^\alpha_\beta|_{W_\gamma}$ for $\gamma < \beta < \alpha$. Then there exists an object K in $D(\mathcal{O}_X)$ and isomorphisms $K|_{W_\alpha} \to K_\alpha$ for $\alpha \in E$ compatible with the isomorphisms ρ^α_β.

Proof. In this proof $\alpha, \beta, \gamma, \ldots$ represent elements of E. Choose a K-injective complex $I^\bullet_\alpha$ on W_α representing K_α. For $\beta < \alpha$ denote $j_{\beta,\alpha} : W_\beta \to W_\alpha$ the inclusion morphism. Using transfinite recursion we will construct for all $\beta < \alpha$ a map of complexes

$$\tau_{\beta,\alpha} : (j_{\beta,\alpha})_! I^\bullet_\beta \longrightarrow I^\bullet_\alpha$$

representing the adjoint to the inverse of the isomorphism $\rho^\alpha_\beta : K_\alpha|_{W_\beta} \to K_\beta$. Moreover, we will do this in such that for $\gamma < \beta < \alpha$ we have

$$\tau_{\gamma,\alpha} = \tau_{\beta,\alpha} \circ (j_{\beta,\alpha})_!\tau_{\gamma,\beta}$$

as maps of complexes. Namely, suppose already given $\tau_{\gamma,\beta}$ composing correctly for all $\gamma < \beta < \alpha$. If $\alpha = \alpha' + 1$ is a successor, then we choose any map of complexes

$$(j_{\alpha',\alpha})_! I^\bullet_{\alpha'} \to I^\bullet_\alpha$$

which is adjoint to the inverse of the isomorphism $\rho^\alpha_{\alpha'} : K_\alpha|_{W_{\alpha'}} \to K_{\alpha'}$ (possible because $I^\bullet_\alpha$ is K-injective) and for any $\beta < \alpha'$ we set

$$\tau_{\beta,\alpha} = \tau_{\alpha',\alpha} \circ (j_{\alpha',\alpha})_!\tau_{\beta,\alpha'}$$

If α is not a successor, then we can consider the complex on W_α given by

$$C^\bullet = \mathrm{colim}_{\beta<\alpha}(j_{\beta,\alpha})_! I^\bullet_\beta$$

(termwise colimit) where the transition maps of the sequence are given by the maps $\tau_{\beta',\beta}$ for $\beta' < \beta < \alpha$. We claim that $C^\bullet$ represents K_α. Namely, for $\beta < \alpha$ the restriction of the coprojection $(j_{\beta,\alpha})_! I^\bullet_\beta \to C^\bullet$ gives a map

$$\sigma_\beta : I^\bullet_\beta \longrightarrow C^\bullet|_{W_\beta}$$

which is a quasi-isomorphism: if $x \in W_\beta$ then looking at stalks we get

$$(C^\bullet)_x = \mathrm{colim}_{\beta'<\alpha}\left((j_{\beta',\alpha})_! I^\bullet_{\beta'}\right)_x = \mathrm{colim}_{\beta\leq\beta'<\alpha}(I^\bullet_{\beta'})_x \longleftarrow (I^\bullet_\beta)_x$$

which is a quasi-isomorphism. Here we used that taking stalks commutes with colimits, that filtered colimits are exact, and that the maps $(I^\bullet_\beta)_x \to (I^\bullet_{\beta'})_x$ are quasi-isomorphisms for $\beta \leq \beta' < \alpha$. Hence $(C^\bullet, \sigma_\beta^{-1})$ is a solution to the system $(\{K_\beta\}_{\beta<\alpha}, \{\rho^\beta_{\beta'}\}_{\beta'<\beta<\alpha})$. Since $(K_\alpha, \rho^\alpha_\beta)$ is another solution we obtain a unique isomorphism $\sigma : K_\alpha \to C^\bullet$ in $D(\mathcal{O}_{W_\alpha})$ compatible with all our maps, see Lemma 20.45.6 (this is where we use the vanishing of negative ext groups). Choose a morphism $\tau : C^\bullet \to I^\bullet_\alpha$ of complexes representing σ. Then we set

$$\tau_{\beta,\alpha} = \tau|_{W_\beta} \circ \sigma_\beta$$

to get the desired maps. Finally, we take K to be the object of the derived category represented by the complex

$$K^\bullet = \mathrm{colim}_{\alpha\in E}(W_\alpha \to X)_! I^\bullet_\alpha$$

where the transition maps are given by our carefully constructed maps $\tau_{\beta,\alpha}$ for $\beta < \alpha$. Arguing exactly as above we see that for all α the restriction of the coprojection determines an isomorphism

$$K|_{W_\alpha} \longrightarrow K_\alpha$$

compatible with the given maps ρ^α_β. □

Using transfinite induction we can prove the result in the general case.

0D6C **Theorem 20.45.8** (BBD gluing lemma). In Situation 20.45.3 assume

Special case of [BBD82, Theorem 3.2.4] without boundedness assumption.

(1) $X = \bigcup_{U\in\mathcal{B}} U$,
(2) for $U, V \in \mathcal{B}$ we have $U \cap V = \bigcup_{W\in\mathcal{B}, W\subset U\cap V} W$,
(3) for any $U \in \mathcal{B}$ we have $\mathrm{Ext}^i(K_U, K_U) = 0$ for $i < 0$.

Then there exists an object K of $D(\mathcal{O}_X)$ and isomorphisms $\rho_U : K|_U \to K_U$ in $D(\mathcal{O}_U)$ for $U \in \mathcal{B}$ such that $\rho_V^U \circ \rho_U|_V = \rho_V$ for all $V \subset U$ with $U, V \in \mathcal{B}$. The pair (K, ρ_U) is unique up to unique isomorphism.

Proof. A pair (K, ρ_U) is called a solution in the text above. The uniqueness follows from Lemma 20.45.4. If X has a finite covering by elements of $\mathcal{B}$ (for example if X is quasi-compact), then the theorem is a consequence of Lemma 20.45.6. In the general case we argue in exactly the same manner, using transfinite induction and Lemma 20.45.7.

First we use transfinite recursion to choose opens $W_\alpha \subset X$ for any ordinal α. Namely, we set $W_0 = \emptyset$. If $\alpha = \beta + 1$ is a successor, then either $W_\beta = X$ and we set $W_\alpha = X$ or $W_\beta \neq X$ and we set $W_\alpha = W_\beta \cup U_\alpha$ where $U_\alpha \in \mathcal{B}$ is not contained in W_β. If α is a limit ordinal we set $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$. Then for large enough α we have $W_\alpha = X$. Observe that for every α the open W_α is a union of elements of $\mathcal{B}$. Hence if $\mathcal{B}_\alpha = \{U \in \mathcal{B}, U \subset W_\alpha\}$, then

$$S_\alpha = (\{K_U\}_{U \in \mathcal{B}_\alpha}, \{\rho_V^U\}_{V \subset U \text{ with } U, V \in \mathcal{B}_\alpha})$$

is a system as in Lemma 20.45.4 on the ringed space W_α.

We will show by transfinite induction that for every α the system S_α has a solution. This will prove the theorem as this system is the system given in the theorem for large α.

The case where $\alpha = \beta + 1$ is a successor ordinal. (This case was already treated in the proof of the lemma above but for clarity we repeat the argument.) Recall that $W_\alpha = W_\beta \cup U_\alpha$ for some $U_\alpha \in \mathcal{B}$ in this case. By induction hypothesis we have a solution $(K_{W_\beta}, \{\rho_U^{W_\beta}\}_{U \in \mathcal{B}_\beta})$ for the system S_β. Then we can consider the collection $\mathcal{B}_\alpha^* = \mathcal{B}_\alpha \cup \{W_\beta\}$ of opens of W_α and we see that we obtain a system $(\{K_U\}_{U \in \mathcal{B}_\alpha^*}, \{\rho_V^U\}_{V \subset U \text{ with } U, V \in \mathcal{B}_\alpha^*})$. Note that this new system also satisfies condition (3) by Lemma 20.45.4 applied to the solution K_{W_β}. For this system we have $W_\alpha = W_\beta \cup U_\alpha$. This reduces us to the case handled in Lemma 20.45.6.

The case where α is a limit ordinal. Recall that $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$ in this case. For $\beta < \alpha$ let $(K_{W_\beta}, \{\rho_U^{W_\beta}\}_{U \in \mathcal{B}_\beta})$ be the solution for S_β. For $\gamma < \beta < \alpha$ the restriction $K_{W_\beta}|_{W_\gamma}$ endowed with the maps $\rho_U^{W_\beta}$, $U \in \mathcal{B}_\gamma$ is a solution for S_γ. By uniqueness we get unique isomorphisms $\rho_{W_\gamma}^{W_\beta} : K_{W_\beta}|_{W_\gamma} \to K_{W_\gamma}$ compatible with the maps $\rho_U^{W_\beta}$ and $\rho_U^{W_\gamma}$ for $U \in \mathcal{B}_\gamma$. These maps compose in the correct manner, i.e., $\rho_{W_\delta}^{W_\gamma} \circ \rho_{W_\gamma}^{W_\beta}|_{W_\delta} = \rho_{W_\beta}^{W_\delta}$ for $\delta < \gamma < \beta < \alpha$. Thus we may apply Lemma 20.45.7 (note that the vanishing of negative exts is true for K_{W_β} by Lemma 20.45.4 applied to the solution K_{W_β}) to obtain K_{W_α} and isomorphisms

$$\rho_{W_\beta}^{W_\alpha} : K_{W_\alpha}|_{W_\beta} \longrightarrow K_{W_\beta}$$

compatible with the maps $\rho_{W_\gamma}^{W_\beta}$ for $\gamma < \beta < \alpha$.

To show that K_{W_α} is a solution we still need to construct the isomorphisms $\rho_U^{W_\alpha} : K_{W_\alpha}|_U \to K_U$ for $U \in \mathcal{B}_\alpha$ satisfying certain compatibilities. We choose $\rho_U^{W_\alpha}$ to be the unique map such that for any $\beta < \alpha$ and any $V \in \mathcal{B}_\beta$ with $V \subset U$ the diagram

$$\begin{tikzcd} K_{W_\alpha}|_V \arrow[r, "\rho^{W_\alpha}_U|_V"'] \arrow[d, "\rho^{W_\alpha}_{W_\beta}|_V"'] & K_U|_V \arrow[d, "\rho^V_U"] \\ K_{W_\beta} \arrow[r, "\rho^{W_\beta}_V"] & K_V \end{tikzcd}$$

commutes. This makes sense because

$$(\{K_V\}_{V \subset U, V \in \mathcal{B}_\beta \text{ for some } \beta < \alpha}, \{\rho_V^{V'}\}_{V \subset V' \text{ with } V, V' \subset U \text{ and } V, V' \in \mathcal{B}_\beta \text{ for some } \beta < \alpha})$$

is a system as in Lemma 20.45.4 on the ringed space U and because (K_U, ρ^U_V) and $(K_{W_\alpha}|_U, \rho^{W_\beta}_V \circ \rho^{W_\alpha}_{W_\beta}|_V)$ are both solutions for this system. This gives existence and uniqueness. We omit the proof that these maps satisfy the desired compatibilities (it is just bookkeeping). □

20.46. Strictly perfect complexes

08C3 Strictly perfect complexes of modules are used to define the notions of pseudo-coherent and perfect complexes later on. They are defined as follows.

08C4 **Definition 20.46.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}_X$-module for all i.

Warning: Since we do not assume that X is a locally ringed space, it may not be true that a direct summand of a finite free $\mathcal{O}_X$-module is finite locally free.

08C5 **Lemma 20.46.2.** *The cone on a morphism of strictly perfect complexes is strictly perfect.*

Proof. This is immediate from the definitions. □

09J2 **Lemma 20.46.3.** *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

Proof. Omitted. □

09U6 **Lemma 20.46.4.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_Y$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_X$-modules.*

Proof. The pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. □

08C6 **Lemma 20.46.5.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of $\mathcal{O}_X$-modules*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr, dotted] & \mathcal{F} \\ & \mathcal{G} \arrow[u, "p"'] \end{tikzcd}$$

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_X$-module and p surjective, then a dotted arrow making the diagram commute exists locally on X.

Proof. We may assume $\mathcal{E} = \mathcal{O}_X^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(X, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sheaves, Section 6.16). $\square$

08C7 Lemma 20.46.6. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then α is locally on X homotopic to zero.
(2) Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then α is locally on X homotopic to zero.

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}_X$-module and integer $n \geq a$, then the result follows from Lemma 20.46.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_X)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_X)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is locally homotopic to zero, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is locally zero by induction hypothesis. $\square$

08C8 Lemma 20.46.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of complexes of $\mathcal{O}_X$-modules

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r, "\alpha"'] \arrow[dr, dotted] & \mathcal{F}^\bullet \\ & \mathcal{G}^\bullet \arrow[u, "f"'] \end{tikzcd}$$

with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then a dotted arrow making the diagram commute up to homotopy exists locally on X.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 20.46.6 guarantees there is an open covering $X = \bigcup U_i$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. $\square$

08C9 Lemma 20.46.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect.

(1) For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.

(2) Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is homotopic to zero.

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 20.46.7. We omit the proof of (2). □

08DM Lemma 20.46.9. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 20.41.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 20.42. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an open $U \subset X$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 20.46.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. □

0GM5 Lemma 20.46.10. In the situation of Lemma 20.46.9 if $\mathcal{F}^\bullet$ is K-flat, then $\mathcal{H}^\bullet$ is K-flat.

Proof. Observe that $\mathcal{H}^\bullet$ is simply the hom complex $\mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ since the boundedness of the strictly prefect complex $\mathcal{E}^\bullet$ insures that the products in the definition of the hom complex turn into direct sums. Let $\mathcal{K}^\bullet$ be an acyclic complex of $\mathcal{O}_X$-modules. Consider the map

$$\gamma : \mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)) \longrightarrow \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{F}^\bullet))$$

of Lemma 20.41.3. Since $\mathcal{F}^\bullet$ is K-flat, the complex $\mathrm{Tot}(\mathcal{K}^\bullet \otimes \mathcal{F}^\bullet)$ is acyclic, and hence by Lemma 20.46.8 (or Lemma 20.46.9 if you like) the target of γ is acyclic too. Hence to prove the lemma it suffices to show that γ is an isomorphism of complexes. To see this, we may argue by induction on the length of the complex $\mathcal{E}^\bullet$. If the length is ≤ 1 then the $\mathcal{E}^\bullet$ is a direct summand of $\mathcal{O}_X^{\oplus n}[k]$ for some $n \geq 0$ and $k \in \mathbf{Z}$ and in this case the result follows by inspection. If the length is > 1, then we reduce to smaller length by considering the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq a}\mathcal{E}^\bullet \to 0$$

for a suitable $a \in \mathbf{Z}$, see Homology, Section 12.15. Then γ fits into a morphism of termwise split short exact sequences of complexes. By induction γ is an isomorphism for $\sigma_{\geq a+1}\mathcal{E}^\bullet$ and $\sigma_{\leq a}\mathcal{E}^\bullet$ and hence the result for $\mathcal{E}^\bullet$ follows. Some details omitted. $\square$

08I5 Lemma 20.46.11. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with

(1) $\mathcal{F}^n = 0$ for $n \ll 0$,
(2) $\mathcal{E}^n = 0$ for $n \gg 0$, and
(3) $\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}_X$-module.

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 20.42.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.31.4). Hence the construction in Section 20.42 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 12.25.3)

$$'E_1^{p,q} = H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^p(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. $\square$

20.47. Pseudo-coherent modules

08CA In this section we discuss pseudo-coherent complexes.

08CB Definition 20.47.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$.

(1) We say $\mathcal{E}^\bullet$ is m-pseudo-coherent if there exists an open covering $X = \bigcup U_i$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i$ is strictly perfect on U_i and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
(2) We say $\mathcal{E}^\bullet$ is pseudo-coherent if it is m-pseudo-coherent for all m.
(3) We say an object E of $D(\mathcal{O}_X)$ is m-pseudo-coherent (resp. pseudo-coherent) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}_X$-modules.

If X is quasi-compact, then an m-pseudo-coherent object of $D(\mathcal{O}_X)$ is in $D^-(\mathcal{O}_X)$. But this need not be the case if X is not quasi-compact.

08CC Lemma 20.47.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$.

(1) If there exists an open covering $X = \bigcup U_i$, strictly perfect complexes $\mathcal{E}_i^\bullet$ on U_i, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.
(2) If E is m-pseudo-coherent, then any complex representing E is m-pseudo-coherent.

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let $X = \bigcup U_i$ and $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) simultaneously. By Lemma 20.46.8 we can after refining the open covering $X = \bigcup U_i$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent. □

09U7 Lemma 20.47.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.

Proof. Represent E by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules and choose an open covering $Y = \bigcup V_i$ and $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i}$ as in Definition 20.47.1. Set $U_i = f^{-1}(V_i)$. By Lemma 20.47.2 it suffices to show that $Lf^*\mathcal{E}^\bullet|_{U_i}$ is m-pseudo-coherent. Choose a distinguished triangle

$$\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i} \to C \to \mathcal{E}_i^\bullet[1]$$

The assumption on α_i means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Denote $f_i : U_i \to V_i$ the restriction of f. Note that $Lf^*\mathcal{E}^\bullet|_{U_i} = Lf_i^*(\mathcal{E}|_{V_i})$. Applying Lf_i^* we obtain the distinguished triangle

$$Lf_i^*\mathcal{E}_i^\bullet \to Lf_i^*\mathcal{E}|_{V_i} \to Lf_i^*C \to Lf_i^*\mathcal{E}_i^\bullet[1]$$

By the construction of Lf_i^* as a left derived functor we see that $H^j(Lf_i^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 13.16.1). Hence $H^j(Lf_i^*\alpha_i)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha_i)$ is surjective. On the other hand, $Lf_i^*\mathcal{E}_i^\bullet = f_i^*\mathcal{E}_i^\bullet$. is strictly perfect by Lemma 20.46.4. Thus we conclude. □

08CD Lemma 20.47.4. Let $(X, \mathcal{O}_X)$ be a ringed space and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$.

(1) If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.
(2) If K and M are m-pseudo-coherent, then L is m-pseudo-coherent.
(3) If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.

Proof. Proof of (1). Choose an open covering $X = \bigcup U_i$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the open covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 20.46.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{tikzcd} K|_{U_i} \arrow[r] & L|_{U_i} \\ \mathcal{K}_i^\bullet \arrow[u, "\alpha_i"] \arrow[r, "\gamma_i"] & \mathcal{L}_i^\bullet \arrow[u, "\beta_i"'] \end{tikzcd}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 20.46.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 20.47.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

09J3 Lemma 20.47.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$.

(1) If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.
(2) If K and L are pseudo-coherent, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is pseudo-coherent.

Proof. Proof of (1). By replacing X by the members of an open covering we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). We may first replace X by the members of an open covering to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). □

08CE Lemma 20.47.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O}_X)$ so are K and L.

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. After replacing X by the members of an open covering we may assume $K \oplus L \in D^-(\mathcal{O}_X)$, hence $L \in D^-(\mathcal{O}_X)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.10. By Lemma 20.47.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is

m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. □

09V7 Lemma 20.47.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent for all i. Then $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof. Omitted. Hint: use Lemma 20.47.4 and truncations as in the proof of More on Algebra, Lemma 15.64.9. □

09V8 Lemma 20.47.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. If E is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent for all i, then E is m-pseudo-coherent.

Proof. Omitted. Hint: use Lemma 20.47.4 and truncations as in the proof of More on Algebra, Lemma 15.64.10. □

08DN Lemma 20.47.9. Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.

(1) If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}_X$-module.
(2) If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m+1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}_X$-module.

Proof. Proof of (1). We may work locally on X. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 20.46.5 we can locally find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). We may work locally on X. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \text{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

09V9 Lemma 20.47.10. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is 0-pseudo-coherent if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and
(2) $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is (-1)-pseudo-coherent if and only if $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.

Proof. Use Lemma 20.47.9 to prove the implications in one direction and Lemma 20.47.8 for the other. □

20.48. Tor dimension

08CF In this section we take a closer look at resolutions by flat modules.

08CG Definition 20.48.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

(1) We say E has tor-amplitude in $[a, b]$ if $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$ and all $i \notin [a, b]$.
(2) We say E has finite tor dimension if it has tor-amplitude in $[a, b]$ for some a, b.
(3) We say E locally has finite tor dimension if there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ has finite tor dimension for all i.

An $\mathcal{O}_X$-module $\mathcal{F}$ has tor dimension $\leq d$ if $\mathcal{F}[0]$ viewed as an object of $D(\mathcal{O}_X)$ has tor-amplitude in $[-d, 0]$.

Note that if E as in the definition has finite tor dimension, then E is an object of $D^b(\mathcal{O}_X)$ as can be seen by taking $\mathcal{F} = \mathcal{O}_X$ in the definition above.

08CH Lemma 20.48.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}_X$-modules with tor-amplitude in $[a, b]$. Then $\text{Coker}(d^{a-1}_{\mathcal{E}^\bullet})$ is a flat $\mathcal{O}_X$-module.

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$. Hence for every $\mathcal{O}_X$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \text{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\text{Tor}_1^{\mathcal{O}_X}(\text{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$. This means that $\text{Coker}(d^{a-1})$ is flat, see Lemma 20.26.16. $\square$

08CI Lemma 20.48.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent

(1) E has tor-amplitude in $[a, b]$.
(2) E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}_X$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.

Proof. If (2) holds, then we may compute $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$, see Section 20.26. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\text{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules, Lemma 17.17.8). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}_X$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 20.48.2 and the definitions. $\square$

09U8 Lemma 20.48.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 20.48.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \not\in [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules, Lemma 17.20.2 the modules $f^*\mathcal{E}^i$ are flat. Thus by Lemma 20.48.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. $\square$

09U9 Lemma 20.48.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent

(1) E has tor-amplitude in $[a, b]$.
(2) for every $x \in X$ the object E_x of $D(\mathcal{O}_{X,x})$ has tor-amplitude in $[a, b]$.

Proof. Taking stalks at x is the same thing as pulling back by the morphism of ringed spaces $(x, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 20.48.4. For the converse, note that taking stalks commutes with tensor products (Modules, Lemma 17.16.1). Hence

$$(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x$$

On the other hand, taking stalks is exact, so

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = H^i((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x) = H^i(E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x)$$

and we can check whether $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$ is zero by checking whether all of its stalks are zero (Modules, Lemma 17.3.1). Thus (2) implies (1). $\square$

08CJ Lemma 20.48.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$.

(1) If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.
(2) If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.
(3) If L has tor-amplitude in $[a + 1, b + 1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a + 1, b + 1]$.

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. $\square$

09J4 Lemma 20.48.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ has tor amplitude in $[a + c, b + d]$.

Proof. Omitted. Hint: use the spectral sequence for tors. $\square$

08CK Lemma 20.48.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O}_X)$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.

Proof. Clear from the fact that the Tor functors are additive. $\square$

20.49. Perfect complexes

08CL In this section we discuss properties of perfect complexes on ringed spaces.

08CM Definition 20.49.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is perfect if there exists an open covering $X = \bigcup U_i$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ a strictly perfect complex of $\mathcal{O}_{U_i}$-modules. An object E of $D(\mathcal{O}_X)$ is perfect if it can be represented by a perfect complex of $\mathcal{O}_X$-modules.

08CN Lemma 20.49.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$.

(1) If there exists an open covering $X = \bigcup U_i$ and strictly perfect complexes $\mathcal{E}_i^\bullet$ on U_i such that $\mathcal{E}_i^\bullet$ represents $E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.
(2) If E is perfect, then any complex representing E is perfect.

Proof. Identical to the proof of Lemma 20.47.2. □

0BCJ Lemma 20.49.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Then the following are equivalent

(1) E is perfect,
(2) there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ can be represented by a finite complex of finite locally free $\mathcal{O}_{U_i}$-modules, and
(3) there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ can be represented by a finite complex of finite free $\mathcal{O}_{U_i}$-modules.

Proof. This follows from Lemma 20.49.2 and the fact that on X every direct summand of a finite free module is finite locally free. See Modules, Lemma 17.14.6. □

08CP Lemma 20.49.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.

Proof. After replacing X by the members of an open covering we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[a-2]$ with $\mathcal{K} = \mathrm{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Applying $- \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F}$ has image $\mathrm{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F})$. It follows that $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \mathrm{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 20.26.16). Thus $\mathcal{E}'$ is locally a direct summand of a finite free module by Modules, Lemma 17.18.3. Thus locally the complex

$$\mathcal{E}' \to \mathcal{E}^{a-1} \to \ldots \to \mathcal{E}^b$$

is quasi-isomorphic to E and E is perfect. □

08CQ Lemma 20.49.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent

(1) E is perfect, and
(2) E is pseudo-coherent and locally has finite tor dimension.

Proof. Assume (1). By definition this means there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 20.47.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 20.48.3. Thus (2) holds.

Assume (2). After replacing X by the members of an open covering we may assume there exist integers $a \leq b$ such that E has tor amplitude in $[a, b]$. Since E is m-pseudo-coherent for all m we conclude using Lemma 20.49.4. $\square$

09UA Lemma 20.49.6. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E is perfect in $D(\mathcal{O}_Y)$, then Lf^*E is perfect in $D(\mathcal{O}_X)$.

Proof. This follows from Lemma 20.49.5, 20.48.4, and 20.47.3. (An alternative proof is to copy the proof of Lemma 20.47.3.) $\square$

08CR Lemma 20.49.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$. If two out of three of K, L, M are perfect then the third is also perfect.

Proof. First proof: Combine Lemmas 20.49.5, 20.47.4, and 20.48.6. Second proof (sketch): Say K and L are perfect. After replacing X by the members of an open covering we may assume that K and L are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing X by the members of an open covering we may assume the map $K \to L$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 20.46.8. Then M is isomorphic to the cone of α which is strictly perfect by Lemma 20.46.2. $\square$

09J5 Lemma 20.49.8. Let $(X, \mathcal{O}_X)$ be a ringed space. If K, L are perfect objects of $D(\mathcal{O}_X)$, then so is $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$.

Proof. Follows from Lemmas 20.49.5, 20.47.5, and 20.48.7. $\square$

08CS Lemma 20.49.9. Let $(X, \mathcal{O}_X)$ be a ringed space. If $K \oplus L$ is a perfect object of $D(\mathcal{O}_X)$, then so are K and L.

Proof. Follows from Lemmas 20.49.5, 20.47.6, and 20.48.8. $\square$

08DP Lemma 20.49.10. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let E be a perfect object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on a closed subset $T \subset U$ with $j(T)$ closed in X. Then Rj_*E is a perfect object of $D(\mathcal{O}_X)$.

Proof. Being a perfect complex is local on X. Thus it suffices to check that Rj_*E is perfect when restricted to U and $V = X \setminus j(T)$. We have $Rj_*E|_U = E$ which is perfect. We have $Rj_*E|_V = 0$ because $E|_{U \setminus T} = 0$. $\square$

0GT1 Lemma 20.49.11. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E in $D(\mathcal{O}_X)$ be perfect. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Then the set

$$U = \{x \in X \mid H^i(E)_x \text{ is a finite free } \mathcal{O}_{X,x}\text{-module for all } i \in \mathbf{Z}\}$$

is open in X and is the maximal open set $U \subset X$ such that $H^i(E)|_U$ is finite locally free for all $i \in \mathbf{Z}$.

Proof. Note that if $V \subset X$ is some open such that $H^i(E)|_V$ is finite locally free for all $i \in \mathbf{Z}$ then $V \subset U$. Let $x \in U$. We will show that an open neighbourhood of x is contained in U and that $H^i(E)$ is finite locally free on this neighbourhood for all i. This will finish the proof. During the proof we may (finitely many times) replace X by an open neighbourhood of x. Hence we may assume E is represented by a strictly perfect complex $\mathcal{E}^\bullet$. Say $\mathcal{E}^i = 0$ for $i \not\in [a, b]$. We will prove the result by induction on $b - a$. The module $H^b(E) = \mathrm{Coker}(d^{b-1} : \mathcal{E}^{b-1} \to \mathcal{E}^b)$ is of finite presentation. Since $H^b(E)_x$ is finite free, we conclude $H^b(E)$ is finite free in an open neighbourhood of x by Modules, Lemma 17.11.6. Thus after replacing X by a (possibly smaller) open neighbourhood we may assume we have a direct sum decomposition $\mathcal{E}^b = \mathrm{Im}(d^{b-1}) \oplus H^b(E)$ and $H^b(E)$ is finite free, see Lemma 20.46.5. Doing the same argument again, we see that we may assume $\mathcal{E}^{b-1} = \mathrm{Ker}(d^{b-1}) \oplus \mathrm{Im}(d^{b-1})$. The complex $\mathcal{E}^a \to \ldots \to \mathcal{E}^{b-2} \to \mathrm{Ker}(d^{b-1})$ is a strictly perfect complex representing a perfect object E' with $H^i(E) = H^i(E')$ for $i \neq b$. Hence we conclude by our induction hypothesis. □

20.50. Duals

0FP7 In this section we characterize the dualizable objects of the category of complexes and of the derived category. In particular, we will see that an object of $D(\mathcal{O}_X)$ has a dual if and only if it is perfect (this follows from Example 20.50.7 and Lemma 20.50.8).

0FP8 Lemma 20.50.1. Let $(X, \mathcal{O}_X)$ be a ringed space. The category of complexes of $\mathcal{O}_X$-modules with tensor product defined by $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet = \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$ is a symmetric monoidal category (for sign rules, see More on Algebra, Section 15.72).

Proof. Omitted. Hints: as unit $\mathbf{1}$ we take the complex having $\mathcal{O}_X$ in degree 0 and zero in other degrees with obvious isomorphisms $\mathrm{Tot}(\mathbf{1} \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet) = \mathcal{G}^\bullet$ and $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathbf{1}) = \mathcal{F}^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 4.43.1 and 4.43.9. The verifications are straightforward in each case. □

0FP9 Example 20.50.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a locally bounded complex of $\mathcal{O}_X$-modules such that each $\mathcal{F}^n$ is locally a direct summand of a finite free $\mathcal{O}_X$-module. In other words, there is an open covering $X = \bigcup U_i$ such that $\mathcal{F}^\bullet|_{U_i}$ is a strictly perfect complex. Consider the complex

$$\mathcal{G}^\bullet = \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O}_X)$$

as in Section 20.41. Let

$$\eta : \mathcal{O}_X \to \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet) \quad \text{and} \quad \epsilon : \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \mathcal{O}_X$$

be $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O}_X \to \mathcal{F}^n \otimes_{\mathcal{O}_X} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_{\mathcal{O}_X} \mathcal{F}^n \to \mathcal{O}_X$ are as in Modules, Example 17.18.1. Then $\mathcal{G}^\bullet, \eta, \epsilon$ is a left dual for $\mathcal{F}^\bullet$ as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{\mathcal{G}^\bullet}$. Please compare with More on Algebra, Lemma 15.72.2.

0FPA Lemma 20.50.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. If $\mathcal{F}^\bullet$ has a left dual in the monoidal category of complexes of $\mathcal{O}_X$-modules (Categories, Definition 4.43.5) then $\mathcal{F}^\bullet$ is a locally bounded complex whose terms are locally direct summands of finite free $\mathcal{O}_X$-modules and the left dual is as constructed in Example 20.50.2.

Proof. By uniqueness of left duals (Categories, Remark 4.43.7) we get the final statement provided we show that $\mathcal{F}^\bullet$ is as stated. Let $\mathcal{G}^\bullet, \eta, \epsilon$ be a left dual. Write $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O}_X \to \mathcal{F}^n \otimes_{\mathcal{O}_X} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_{\mathcal{O}_X} \mathcal{F}^n \to$

$\mathcal{O}_X$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \text{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}_{\mathcal{G}^\bullet}$ by Categories, Definition 4.43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \text{id}_{\mathcal{F}^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \text{id}_{\mathcal{G}^{-n}}$. Hence we see that $\mathcal{F}^n$ is locally a direct summand of a finite free $\mathcal{O}_X$-module by Modules, Lemma 17.18.2. Since the sum $\eta = \sum \eta_n$ is locally finite, we conclude that $\mathcal{F}^\bullet$ is locally bounded. $\square$

0G40 Lemma 20.50.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $K, L, M \in D(\mathcal{O}_X)$. If K is perfect, then the map

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

of Lemma 20.42.9 is an isomorphism.

Proof. Since the map is globally defined and since formation of the right and left hand side commute with localization (see Lemma 20.42.3), to prove this we may work locally on X. Thus we may assume K is represented by a strictly perfect complex $\mathcal{E}^\bullet$.

If $K_1 \to K_2 \to K_3$ is a distinguished triangle in $D(\mathcal{O}_X)$, then we get distinguished triangles

$$R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_1 \to R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_2 \to R\mathcal{H}om(L, M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_3$$

and

$$R\mathcal{H}om(R\mathcal{H}om(K_1, L), M) \to R\mathcal{H}om(R\mathcal{H}om(K_2, L), M) R\mathcal{H}om(R\mathcal{H}om(K_3, L), M)$$

See Section 20.26 and Lemma 20.42.4. The arrow of Lemma 20.42.9 is functorial in K hence we get a morphism between these distinguished triangles. Thus, if the result holds for K_1 and K_3, then the result holds for K_2 by Derived Categories, Lemma 13.4.3.

Combining the remarks above with the distinguished triangles

$$\sigma_{\geq n}\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to \sigma_{\leq n-1}\mathcal{E}^\bullet$$

of stupid trunctions, we reduce to the case where K consists of a direct summand of a finite free $\mathcal{O}_X$-module placed in some degree. By an obvious compatibility of the problem with direct sums (similar to what was said above) and shifts this reduces us to the case where $K = \mathcal{O}_X^{\oplus n}$ for some integer n. This case is clear. $\square$

08DQ Lemma 20.50.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be a perfect object of $D(\mathcal{O}_X)$. Then $K^\vee = R\mathcal{H}om(K, \mathcal{O}_X)$ is a perfect object too and $(K^\vee)^\vee \cong K$. There are functorial isomorphisms

$$M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee = R\mathcal{H}om(K, M)$$

and

$$H^0(X, M \otimes^{\mathbf{L}}_{\mathcal{O}_X} K^\vee) = \text{Hom}_{D(\mathcal{O}_X)}(K, M)$$

for M in $D(\mathcal{O}_X)$.

Proof. By Lemma 20.42.9 there is a canonical map

$$K = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, \mathcal{O}_X), \mathcal{O}_X) = (K^\vee)^\vee$$

which is an isomorphism by Lemma 20.50.4. To check the other statements we will use without further mention that formation of internal hom commutes with restriction to opens (Lemma 20.42.3). We may check $K^\vee$ is perfect locally on X.

By Lemma 20.42.8 to see the final statement it suffices to check that the map (20.42.8.1)

$$M \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is local on X as well. Hence it suffices to prove these two statements K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 20.46.9 that $K^\vee$ is represented by the complex whose terms are $(\mathcal{E}^{-n})^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-n}, \mathcal{O}_X)$ in degree n. Since $\mathcal{E}^{-n}$ is a direct summand of a finite free $\mathcal{O}_X$-module, so is $(\mathcal{E}^{-n})^\vee$. Hence $K^\vee$ is represented by a strictly perfect complex too and we see that $K^\vee$ is perfect. To see that (20.42.8.1) is an isomorphism, represent M by a complex $\mathcal{F}^\bullet$. By Lemma 20.46.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

On the other hand, the object $M \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{F}^p \otimes_{\mathcal{O}_X} (\mathcal{E}^{-q})^\vee$$

Thus the assertion that (20.42.8.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module and $\mathcal{F}$ is any $\mathcal{O}_X$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. □

0FPB Lemma 20.50.6. Let $(X, \mathcal{O}_X)$ be a ringed space. The derived category $D(\mathcal{O}_X)$ is a symmetric monoidal category with tensor product given by derived tensor product with usual associativity and commutativity constraints (for sign rules, see More on Algebra, Section 15.72).

Proof. Omitted. Compare with Lemma 20.50.1. □

0FPC Example 20.50.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be a perfect object of $D(\mathcal{O}_X)$. Set $K^\vee = R\mathcal{H}om(K, \mathcal{O}_X)$ as in Lemma 20.50.5. Then the map

$$K \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee \longrightarrow R\mathcal{H}om(K, K)$$

is an isomorphism (by the lemma). Denote

$$\eta : \mathcal{O}_X \longrightarrow K \otimes_{\mathcal{O}_X}^{\mathbf{L}} K^\vee$$

the map sending 1 to the section corresponding to id_K under the isomorphism above. Denote

$$\epsilon : K^\vee \otimes_{\mathcal{O}_X}^{\mathbf{L}} K \longrightarrow \mathcal{O}_X$$

the evaluation map (to construct it you can use Lemma 20.42.5 for example). Then $K^\vee, \eta, \epsilon$ is a left dual for K as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \text{id}_K$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}_{K^\vee}$.

0FPD Lemma 20.50.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Let M be an object of $D(\mathcal{O}_X)$. If M has a left dual in the monoidal category $D(\mathcal{O}_X)$ (Categories, Definition 4.43.5) then M is perfect and the left dual is as constructed in Example 20.50.7.

Proof. Let $x \in X$. It suffices to find an open neighbourhood U of x such that M restricts to a perfect complex over U. Hence during the proof we can (finitely often) replace X by an open neighbourhood of x. Let N, η, ϵ be a left dual.

We are going to use the following argument several times. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules representing M. Choose a K-flat complex $\mathcal{N}^\bullet$ representing N whose terms are flat $\mathcal{O}_X$-modules, see Lemma 20.26.12. Consider the map

$$\eta : \mathcal{O}_X \to \text{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$$

After shrinking X we can find an integer N and for $i = 1, \ldots, N$ integers $n_i \in \mathbf{Z}$ and sections f_i and g_i of $\mathcal{M}^{n_i}$ and $\mathcal{N}^{-n_i}$ such that

$$\eta(1) = \sum\nolimits_i f_i \otimes g_i$$

Let $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ be any subcomplex of $\mathcal{O}_X$-modules containing the sections f_i for $i = 1, \ldots, N$. Since $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet) \subset \text{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$ by flatness of the modules $\mathcal{N}^n$, we see that η factors through

$$\tilde{\eta} : \mathcal{O}_X \to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet)$$

Denoting K the object of $D(\mathcal{O}_X)$ represented by $\mathcal{K}^\bullet$ we find a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\eta\otimes 1"'] \arrow[dr, "\tilde{\eta}\otimes 1"'] & M \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M \arrow[r, "1\otimes\epsilon"'] & M \\ & K \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M \arrow[r, "1\otimes\epsilon"] \arrow[u] & K \arrow[u] \end{tikzcd}$$

Since the composition of the upper row is the identity on M we conclude that M is a direct summand of K in $D(\mathcal{O}_X)$.

As a first use of the argument above, we can choose the subcomplex $\mathcal{K}^\bullet = \sigma_{\geq a}\tau_{\leq b}\mathcal{M}^\bullet$ with $a < n_i < b$ for $i = 1, \ldots, N$. Thus M is a direct summand in $D(\mathcal{O}_X)$ of a bounded complex and we conclude we may assume M is in $D^b(\mathcal{O}_X)$. (Recall that the process above involves shrinking X.)

Since M is in $D^b(\mathcal{O}_X)$ we may choose $\mathcal{M}^\bullet$ to be a bounded above complex of flat modules (by Modules, Lemma 17.17.6 and Derived Categories, Lemma 13.15.4). Then we can choose $\mathcal{K}^\bullet = \sigma_{\geq a}\mathcal{M}^\bullet$ with $a < n_i$ for $i = 1, \ldots, N$ in the argument above. Thus we find that we may assume M is a direct summand in $D(\mathcal{O}_X)$ of a bounded complex of flat modules. In particular, M has finite tor amplitude.

Say M has tor amplitude in $[a, b]$. Assuming M is m-pseudo-coherent we are going to show that (after shrinking X) we may assume M is $(m-1)$-pseudo-coherent. This will finish the proof by Lemma 20.49.4 and the fact that M is $(b+1)$-pseudo-coherent in any case. After shrinking X we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to M$ in $D(\mathcal{O}_X)$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and surjective for $i = m$. We may and do assume that $\mathcal{E}^i = 0$ for $i < m$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to M \to L \to \mathcal{E}^\bullet[1]$$

Observe that $H^i(L) = 0$ for $i \geq m$. Thus we may represent L by a complex $\mathcal{L}^\bullet$ with $\mathcal{L}^i = 0$ for $i \geq m$. The map $L \to \mathcal{E}^\bullet[1]$ is given by a map of complexes $\mathcal{L}^\bullet \to \mathcal{E}^\bullet[1]$ which is zero in all degrees except in degree $m-1$ where we obtain a map $\mathcal{L}^{m-1} \to \mathcal{E}^m$, see Derived Categories, Lemma 13.27.3. Then M is represented by the complex

$$\mathcal{M}^\bullet : \ldots \to \mathcal{L}^{m-2} \to \mathcal{L}^{m-1} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots$$

Apply the discussion in the second paragraph to this complex to get sections f_i of $\mathcal{M}^{n_i}$ for $i = 1, \ldots, N$. For $n < m$ let $\mathcal{K}^n \subset \mathcal{L}^n$ be the $\mathcal{O}_X$-submodule generated by the sections f_i for $n_i = n$ and $d(f_i)$ for $n_i = n - 1$. For $n \geq m$ set $\mathcal{K}^n = \mathcal{E}^n$. Clearly, we have a morphism of distinguished triangles

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[r] & \mathcal{M}^\bullet \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{E}^\bullet[1] \\
\mathcal{E}^\bullet \arrow[r] \arrow[u] & \mathcal{K}^\bullet \arrow[r] \arrow[u] & \sigma_{\leq m-1}\mathcal{K}^\bullet \arrow[r] \arrow[u] & \mathcal{E}^\bullet[1] \arrow[u]
\end{tikzcd}$$

where all the morphisms are as indicated above. Denote K the object of $D(\mathcal{O}_X)$ corresponding to the complex $\mathcal{K}^\bullet$. By the arguments in the second paragraph of the proof we obtain a morphism $s : M \to K$ in $D(\mathcal{O}_X)$ such that the composition $M \to K \to M$ is the identity on M. We don't know that the diagram

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[r] & \mathcal{K}^\bullet \arrow[r, equal] & K \\
\mathcal{E}^\bullet \arrow[r, "i"] \arrow[u, "\mathrm{id}"] & \mathcal{M}^\bullet \arrow[r, equal] & M \arrow[u, "s"']
\end{tikzcd}$$

commutes, but we do know it commutes after composing with the map $K \to M$. By Lemma 20.46.8 after shrinking X we may assume that $s \circ i$ is given by a map of complexes $\sigma : \mathcal{E}^\bullet \to \mathcal{K}^\bullet$. By the same lemma we may assume the composition of σ with the inclusion $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ is homotopic to zero by some homotopy $\{h^i : \mathcal{E}^i \to \mathcal{M}^{i-1}\}$. Thus, after replacing $\mathcal{K}^{m-1}$ by $\mathcal{K}^{m-1} + \mathrm{Im}(h^m)$ (note that after doing this it is still the case that $\mathcal{K}^{m-1}$ is generated by finitely many global sections), we see that σ itself is homotopic to zero! This means that we have a commutative solid diagram

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[r] & M \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{E}^\bullet[1] \\
\mathcal{E}^\bullet \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & \sigma_{\leq m-1}\mathcal{K}^\bullet \arrow[r] \arrow[u] & \mathcal{E}^\bullet[1] \arrow[u] \\
\mathcal{E}^\bullet \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "s"] & \mathcal{L}^\bullet \arrow[r] \arrow[u, dotted] & \mathcal{E}^\bullet[1] \arrow[u]
\end{tikzcd}$$

By the axioms of triangulated categories we obtain a dotted arrow fitting into the diagram. Looking at cohomology sheaves in degree $m - 1$ we see that we obtain

$$\begin{tikzcd}
H^{m-1}(M) \arrow[r] & H^{m-1}(\mathcal{L}^\bullet) \arrow[r] & H^m(\mathcal{E}^\bullet) \\
H^{m-1}(K) \arrow[r] \arrow[u] & H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \arrow[r] \arrow[u] & H^m(\mathcal{E}^\bullet) \arrow[u] \\
H^{m-1}(M) \arrow[r] \arrow[u] & H^{m-1}(\mathcal{L}^\bullet) \arrow[r] \arrow[u] & H^m(\mathcal{E}^\bullet) \arrow[u]
\end{tikzcd}$$

Since the vertical compositions are the identity in both the left and right column, we conclude the vertical composition $H^{m-1}(\mathcal{L}^\bullet) \to H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ in the middle is surjective! In particular $H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ is surjective. Using the induced map of long exact sequences of cohomology sheaves from the morphism of triangles above, a diagram chase shows this implies $H^i(K) \to H^i(M)$ is an isomorphism for $i \geq m$ and surjective for $i = m-1$. By construction we can choose an $r \geq 0$ and a surjection $\mathcal{O}_X^{\oplus r} \to \mathcal{K}^{m-1}$. Then the composition

$$(\mathcal{O}_X^{\oplus r} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots) \longrightarrow K \longrightarrow M$$

induces an isomorphism on cohomology sheaves in degrees $\geq m$ and a surjection in degree $m-1$ and the proof is complete. $\square$

20.51. Miscellany

0GM6 Some results which do not fit anywhere else.

0DJI **Lemma 20.51.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(K_n)_{n\in\mathbf{N}}$ be a system of perfect objects of $D(\mathcal{O}_X)$. Let $K = \text{hocolim}K_n$ be the derived colimit (Derived Categories, Definition 13.33.1). Then for any object E of $D(\mathcal{O}_X)$ we have*

$$R\mathcal{H}om(K, E) = R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_n^\vee$$

where $(K_n^\vee)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 20.50.5 we have $R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_n^\vee = R\lim R\mathcal{H}om(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\mathcal{H}om(K_n, E) = R\mathcal{H}om(\bigoplus K_n, E)$. This is a formal consequence of (20.42.0.1) and the fact that derived tensor product commutes with direct sums. $\square$

0FVB **Lemma 20.51.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K and E be objects of $D(\mathcal{O}_X)$ with E perfect. The diagram*

$$\begin{tikzcd}
H^0(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee) \times H^0(X, E) \arrow[r] \arrow[d] & H^0(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) \arrow[d] \\
\text{Hom}_X(E, K) \times H^0(X, E) \arrow[r] & H^0(X, K)
\end{tikzcd}$$

commutes where the top horizontal arrow is the cup product, the right vertical arrow uses $\epsilon : E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E \to \mathcal{O}_X$ (Example 20.50.7), the left vertical arrow uses Lemma 20.50.5, and the bottom horizontal arrow is the obvious one.

Proof. We will abbreviate $\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X}$ and $\mathcal{O} = \mathcal{O}_X$. We will identify E and K with $R\mathcal{H}om(\mathcal{O}, E)$ and $R\mathcal{H}om(\mathcal{O}, K)$ and we will identify $E^\vee$ with $R\mathcal{H}om(E, \mathcal{O})$.

Let $\xi \in H^0(X, K \otimes E^\vee)$ and $\eta \in H^0(X, E)$. Denote $\tilde{\xi} : \mathcal{O} \to K \otimes E^\vee$ and $\tilde{\eta} : \mathcal{O} \to E$ the corresponding maps in $D(\mathcal{O})$. By Lemma 20.31.1 the cup product $\xi \cup \eta$ corresponds to $\tilde{\xi} \otimes \tilde{\eta} : \mathcal{O} \to K \otimes E^\vee \otimes E$.

We claim the map $\xi' : E \to K$ corresponding to ξ by Lemma 20.50.5 is the composition

$$E = \mathcal{O} \otimes E \xrightarrow{\tilde{\xi} \otimes 1_E} K \otimes E^\vee \otimes E \xrightarrow{1_K \otimes \epsilon} K$$

The construction in Lemma 20.50.5 uses the evaluation map (20.42.8.1) which in turn is constructed using the identification of E with $R\mathcal{H}om(\mathcal{O}, E)$ and the composition $\underline{\circ}$ constructed in Lemma 20.42.5. Hence ξ' is the composition

$$\begin{aligned} E = \mathcal{O} \otimes R\mathcal{H}om(\mathcal{O}, E) &\xrightarrow{\tilde{\xi}\otimes 1} R\mathcal{H}om(\mathcal{O}, K) \otimes R\mathcal{H}om(E, \mathcal{O}) \otimes R\mathcal{H}om(\mathcal{O}, E) \\ &\xrightarrow{\underline{\circ}\otimes 1} R\mathcal{H}om(E, K) \otimes R\mathcal{H}om(\mathcal{O}, E) \\ &\xrightarrow{\underline{\circ}} R\mathcal{H}om(\mathcal{O}, K) = K \end{aligned}$$

The claim follows immediately from this and the fact that the composition $\underline{\circ}$ constructed in Lemma 20.42.5 is associative (insert future reference here) and the fact that ϵ is defined as the composition $\underline{\circ} : E^\vee \otimes E \to \mathcal{O}$ in Example 20.50.7.

Using the results from the previous two paragraphs, we find the statement of the lemma is that $(1_K \otimes \epsilon) \circ (\tilde{\xi} \otimes \tilde{\eta})$ is equal to $(1_K \otimes \epsilon) \circ (\tilde{\xi} \otimes 1_E) \circ (1_\mathcal{O} \otimes \tilde{\eta})$ which is immediate. $\square$

0GM7 Lemma 20.51.3. Let $h : X \to Y$ be a morphism of ringed spaces. Let K, M be objects of $D(\mathcal{O}_Y)$. The canonical map

$$Lh^* R\mathcal{H}om(K, M) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*M)$$

of Remark 20.42.13 is an isomorphism in the following cases

(1) K is perfect,
(2) h is flat, K is pseudo-coherent, and M is (locally) bounded below,
(3) $\mathcal{O}_X$ has finite tor dimension over $h^{-1}\mathcal{O}_Y$, K is pseudo-coherent, and M is (locally) bounded below,

Proof. Proof of (1). The question is local on Y, hence we may assume that K is represented by a strictly perfect complex $\mathcal{E}^\bullet$, see Section 20.49. Choose a K-flat complex $\mathcal{F}^\bullet$ representing M. Apply Lemma 20.46.9 to see that $R\mathcal{H}om(K, L)$ is represented by the complex $\mathcal{H}^\bullet = \mathcal{H}om^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ with terms $\mathcal{H}^n = \bigoplus_{n=p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-q}, \mathcal{F}^p)$. By the construction of Lh^* in Section 20.27 we see that Lh^*K is represented by the strictly perfect complex $h^*\mathcal{E}^\bullet$ (Lemma 20.46.4). Similarly, the object Lh^*M is represented by the complex $h^*\mathcal{F}^\bullet$. Finally, the object $Lh^*R\mathcal{H}om(K, M)$ is represented by $h^*\mathcal{H}^\bullet$ as $\mathcal{H}^\bullet$ is K-flat by Lemma 20.46.10. Thus to finish the proof it suffices to show that $h^*\mathcal{H}^\bullet = \mathcal{H}om^\bullet(h^*\mathcal{E}^\bullet, h^*\mathcal{F}^\bullet)$. For this it suffices to note that $h^*\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{H}om(h^*\mathcal{E}, \mathcal{F})$ whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module.

Proof of (2). Since h is flat, we can compute Lh^* by simply using h^* on any complex of $\mathcal{O}_Y$-modules. In particular we have $H^i(Lh^*K) = h^*H^i(K)$ for all $i \in \mathbf{Z}$. Say $H^i(M) = 0$ for $i < a$. Let $K' \to K$ be a morphism of $D(\mathcal{O}_Y)$ which defines an isomorphism $H^i(K') \to H^i(K)$ for all $i \geq b$. Then the corresponding maps

$$R\mathcal{H}om(K, M) \to R\mathcal{H}om(K', M)$$

and

$$R\mathcal{H}om(Lh^*K, Lh^*M) \to R\mathcal{H}om(Lh^*K', Lh^*M)$$

are isomorphisms on cohomology sheaves in degrees $< a - b$ (details omitted). Thus to prove the map in the statement of the lemma induces an isomorphism on cohomology sheaves in degrees $< a - b$ it suffices to prove the result for K' in those degrees. Also, as in the proof of part (1) the question is local on Y. Thus we may assume K is represented by a strictly perfect complex, see Section 20.47. This reduces us to case (1).

Proof of (3). The proof is the same as the proof of (2) except one uses that Lh^* has bounded cohomological dimension to get the desired vanishing. We omit the details. $\square$

0GM8 Lemma 20.51.4. Let X be a ringed space. Let K, M be objects of $D(\mathcal{O}_X)$. Let $x \in X$. The canonical map

$$R\mathcal{H}om(K, M)_x \longrightarrow R\operatorname{Hom}_{\mathcal{O}_{X,x}}(K_x, M_x)$$

is an isomorphism in the following cases

1. K is perfect,
2. K is pseudo-coherent and M is (locally) bounded below.

Proof. Let $Y = \{x\}$ be the singleton ringed space with structure sheaf given by $\mathcal{O}_{X,x}$. Then apply Lemma 20.51.3 to the flat inclusion morphism $Y \to X$. $\square$

20.52. Invertible objects in the derived category

0FPE We characterize invertible objects in the derived category of a ringed space (both in the case where the stalks of the structure sheaf are local and where not).

0FPF Lemma 20.52.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. The category of $\mathcal{O}_X$-modules which are summands of finite free $\mathcal{O}_X$-modules is equivalent to the category of finite projective R-modules.

Proof. Observe that a finite projective R-module is the same thing as a summand of a finite free R-module. The equivalence is given by the functor $\mathcal{E} \mapsto \Gamma(X, \mathcal{E})$. The inverse functor is given by the construction of Modules, Lemma 17.10.5. $\square$

0FPG Lemma 20.52.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let M be an object of $D(\mathcal{O}_X)$. The following are equivalent

1. M is invertible in $D(\mathcal{O}_X)$, see Categories, Definition 4.43.4, and
2. there is a locally finite direct product decomposition
$$\mathcal{O}_X = \prod\nolimits_{n \in \mathbf{Z}} \mathcal{O}_n$$
and for each n there is an invertible $\mathcal{O}_n$-module $\mathcal{H}^n$ (Modules, Definition 17.25.1) and $M = \bigoplus \mathcal{H}^n[-n]$ in $D(\mathcal{O}_X)$.

If (1) and (2) hold, then M is a perfect object of $D(\mathcal{O}_X)$. If $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$ these condition are also equivalent to

3. there exists an open covering $X = \bigcup U_i$ and for each i an integer n_i such that $M|_{U_i}$ is represented by an invertible $\mathcal{O}_{U_i}$-module placed in degree n_i.

Proof. Assume (2). Consider the object $R\mathcal{H}om(M, \mathcal{O}_X)$ and the composition map

$$R\mathcal{H}om(M, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \to \mathcal{O}_X$$

To prove this is an isomorphism, we may work locally. Thus we may assume $\mathcal{O}_X = \prod_{a \leq n \leq b} \mathcal{O}_n$ and $M = \bigoplus_{a \leq n \leq b} \mathcal{H}^n[-n]$. Then it suffices to show that

$$R\mathcal{H}om(\mathcal{H}^m, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{H}^n$$

is zero if $n \neq m$ and equal to $\mathcal{O}_n$ if $n = m$. The case $n \neq m$ follows from the fact that $\mathcal{O}_n$ and $\mathcal{O}_m$ are flat $\mathcal{O}_X$-algebras with $\mathcal{O}_n \otimes_{\mathcal{O}_X} \mathcal{O}_m = 0$. Using the local structure of invertible $\mathcal{O}_X$-modules (Modules, Lemma 17.25.2) and working locally the isomorphism in case $n = m$ follows in a straightforward manner; we omit the details. Because $D(\mathcal{O}_X)$ is symmetric monoidal, we conclude that M is invertible.

Assume (1). The description in (2) shows that we have a candidate for $\mathcal{O}_n$, namely, $\mathcal{H}om_{\mathcal{O}_X}(H^n(M), H^n(M))$. If this is a locally finite family of sheaves of rings and if $\mathcal{O}_X = \prod \mathcal{O}_n$, then we immediately obtain the direct sum decomposition $M = \bigoplus H^n(M)[-n]$ using the idempotents in $\mathcal{O}_X$ coming from the product decomposition. This shows that in order to prove (2) we may work locally on X.

Choose an object N of $D(\mathcal{O}_X)$ and an isomorphism $M \otimes_{\mathcal{O}_X}^{\mathbf{L}} N \cong \mathcal{O}_X$. Let $x \in X$. Then N is a left dual for M in the monoidal category $D(\mathcal{O}_X)$ and we conclude that M is perfect by Lemma 20.50.8. By symmetry we see that N is perfect. After replacing X by an open neighbourhood of x, we may assume M and N are represented by a strictly perfect complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$. Then $M \otimes_{\mathcal{O}_X}^{\mathbf{L}} N$ is represented by $\text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$. After another shinking of X we may assume the mutually inverse isomorphisms $\mathcal{O}_X \to M \otimes_{\mathcal{O}_X}^{\mathbf{L}} N$ and $M \otimes_{\mathcal{O}_X}^{\mathbf{L}} N \to \mathcal{O}_X$ are given by maps of complexes

$$\alpha : \mathcal{O}_X \to \text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \quad \text{and} \quad \beta : \text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \mathcal{O}_X$$

See Lemma 20.46.8. Then $\beta \circ \alpha = 1$ as maps of complexes and $\alpha \circ \beta = 1$ as a morphism in $D(\mathcal{O}_X)$. After shrinking X we may assume the composition $\alpha \circ \beta$ is homotopic to 1 by some homotopy θ with components

$$\theta^n : \text{Tot}^n(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \to \text{Tot}^{n-1}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$$

by the same lemma as before. Set $R = \Gamma(X, \mathcal{O}_X)$. By Lemma 20.52.1 we find that we obtain

(1) $M^\bullet = \Gamma(X, \mathcal{E}^\bullet)$ is a bounded complex of finite projective R-modules,
(2) $N^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ is a bounded complex of finite projective R-modules,
(3) α and β correspond to maps of complexes $a : R \to \text{Tot}(M^\bullet \otimes_R N^\bullet)$ and $b : \text{Tot}(M^\bullet \otimes_R N^\bullet) \to R$,
(4) θ^n corresponds to a map $h^n : \text{Tot}^n(M^\bullet \otimes_R N^\bullet) \to \text{Tot}^{n-1}(M^\bullet \otimes_R N^\bullet)$, and
(5) $b \circ a = 1$ and $b \circ a - 1 = dh + hd$,

It follows that $M^\bullet$ and $N^\bullet$ define mutually inverse objects of $D(R)$. By More on Algebra, Lemma 15.126.4 we find a product decomposition $R = \prod_{a \leq n \leq b} R_n$ and invertible R_n-modules H^n such that $M^\bullet \cong \bigoplus_{a \leq n \leq b} H^n[-n]$. This isomorphism in $D(R)$ can be lifted to an morphism

$$\bigoplus H^n[-n] \longrightarrow M^\bullet$$

of complexes because each H^n is projective as an R-module. Correspondingly, using Lemma 20.52.1 again, we obtain an morphism

$$\bigoplus H^n \otimes_R \mathcal{O}_X[-n] \to \mathcal{E}^\bullet$$

which is an isomorphism in $D(\mathcal{O}_X)$. Setting $\mathcal{O}_n = R_n \otimes_R \mathcal{O}_X$ we conclude (2) is true.

If all stalks of $\mathcal{O}_X$ are local, then it is straightforward to prove the equivalence of (2) and (3). We omit the details. $\square$

20.53. Compact objects

09J6 In this section we study compact objects in the derived category of modules on a ringed space. We recall that compact objects are defined in Derived Categories, Definition 13.37.1. On suitable ringed spaces the perfect objects are compact.

0F5Z Lemma 20.53.1. Let X be a ringed space. Let $j : U \to X$ be the inclusion of an open. The $\mathcal{O}_X$-module $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O}_X)$ if there exists an integer d such that

(1) $H^p(U, \mathcal{F}) = 0$ for all $p > d$, and
(2) the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums.

Proof. Assume (1) and (2). Since $\mathrm{Hom}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$ by Sheaves, Lemma 6.31.8 we have $\mathrm{Hom}(j_!\mathcal{O}_U, K) = R\Gamma(U, K)$ for K in $D(\mathcal{O}_X)$. Thus we have to show that $R\Gamma(U, -)$ commutes with direct sums. The first assumption means that the functor $F = H^0(U, -)$ has finite cohomological dimension. Moreover, the second assumption implies any direct sum of injective modules is acyclic for F. Let K_i be a family of objects of $D(\mathcal{O}_X)$. Choose K-injective representatives $I_i^\bullet$ with injective terms representing K_i, see Injectives, Theorem 19.12.6. Since we may compute RF by applying F to any complex of acyclics (Derived Categories, Lemma 13.32.2) and since $\bigoplus K_i$ is represented by $\bigoplus I_i^\bullet$ (Injectives, Lemma 19.13.4) we conclude that $R\Gamma(U, \bigoplus K_i)$ is represented by $\bigoplus H^0(U, I_i^\bullet)$. Hence $R\Gamma(U, -)$ commutes with direct sums as desired. □

09J7 Lemma 20.53.2. Let X be a ringed space. Assume that the underlying topological space of X has the following properties:

(1) X is quasi-compact,
(2) there exists a basis of quasi-compact open subsets, and
(3) the intersection of any two quasi-compact opens is quasi-compact.

Let K be a perfect object of $D(\mathcal{O}_X)$. Then

(a) K is a compact object of $D^+(\mathcal{O}_X)$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ is bounded below, then $\mathrm{Hom}(K, M) = \bigoplus_{i \in I} \mathrm{Hom}(K, M_i)$.
(b) If X has finite cohomological dimension, i.e., if there exists a d such that $H^i(X, \mathcal{F}) = 0$ for $i > d$, then K is a compact object of $D(\mathcal{O}_X)$.

Proof. Let $K^\vee$ be the dual of K, see Lemma 20.50.5. Then we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M in $D(\mathcal{O}_X)$. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums it suffices to show that $R\Gamma(X, -)$ commutes with the relevant direct sums.

Proof of (b). Since $R\Gamma(X, K) = R\,\mathrm{Hom}(\mathcal{O}_X, K)$ and since $H^p(X, -)$ commutes with direct sums by Lemma 20.19.1 this is a special case of Lemma 20.53.1

Proof of (a). Let $\mathcal{I}_i$, $i \in I$ be a collection of injective $\mathcal{O}_X$-modules. By Lemma 20.19.1 we see that

$$H^p(X, \bigoplus\nolimits_{i \in I} \mathcal{I}_i) = \bigoplus\nolimits_{i \in I} H^p(X, \mathcal{I}_i) = 0$$

for all p. Now if $M = \bigoplus M_i$ is as in (a), then we see that there exists an $a \in \mathbf{Z}$ such that $H^n(M_i) = 0$ for $n < a$. Thus we can choose complexes of injective $\mathcal{O}_X$-modules $\mathcal{I}_i^\bullet$ representing M_i with $\mathcal{I}_i^n = 0$ for $n < a$, see Derived Categories, Lemma 13.18.3. By Injectives, Lemma 19.13.4 we see that the direct sum complex

$\bigoplus \mathcal{I}_i^\bullet$ represents M. By Leray acyclicity (Derived Categories, Lemma 13.16.7) we see that

$$R\Gamma(X, M) = \Gamma(X, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus \Gamma(X, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus R\Gamma(X, M_i)$$

as desired. □

20.54. Projection formula

01E6 In this section we collect variants of the projection formula. The most basic version is Lemma 20.54.2. After we state and prove it, we discuss a more general version involving perfect complexes.

01E7 **Lemma 20.54.1.** Let X be a ringed space. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{E}$ is finite locally free on X, see Modules, Definition 17.14.1. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}$ is an injective $\mathcal{O}_X$-module.

Proof. This is true because under the assumptions of the lemma we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee, \mathcal{I})$$

where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual of $\mathcal{E}$ which is finite locally free also. Since tensoring with a finite locally free sheaf is an exact functor we win by Homology, Lemma 12.27.2. □

01E8 **Lemma 20.54.2.** Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Assume $\mathcal{E}$ is finite locally free on Y, see Modules, Definition 17.14.1. Then there exist isomorphisms

$$\mathcal{E} \otimes_{\mathcal{O}_Y} R^q f_*\mathcal{F} \longrightarrow R^q f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

for all $q \geq 0$. In fact there exists an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Y} Rf_*\mathcal{F} \longrightarrow Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

in $D^+(Y)$ functorial in $\mathcal{F}$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on X. Note that $f^*\mathcal{E}$ is finite locally free also, hence we get a resolution

$$f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$$

which is an injective resolution by Lemma 20.54.1. Apply f_* to see that

$$Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet).$$

Hence the lemma follows if we can show that $f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$ functorially in the $\mathcal{O}_X$-module $\mathcal{F}$. This is clear when $\mathcal{E} = \mathcal{O}_Y^{\oplus n}$, and follows in general by working locally on Y. Details omitted. □

Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. Without any further assumptions there is a map

0B53 (20.54.2.1) $$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) = Lf^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$ and Lemmas 20.27.3 and 20.28.1. A reasonably general version of the projection formula is the following.

0B54 Lemma 20.54.3. Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. If K is perfect, then

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

in $D(\mathcal{O}_Y)$.

Proof. To check (20.54.2.1) is an isomorphism we may work locally on Y, i.e., we have to find a covering $\{V_j \to Y\}$ such that the map restricts to an isomorphism on V_j. By definition of perfect objects, this means we may assume K is represented by a strictly perfect complex of $\mathcal{O}_Y$-modules. Note that, completely generally, the statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for K_1 and K_2. Hence we may assume K is a finite complex of finite free $\mathcal{O}_Y$-modules. In this case a simple argument involving stupid truncations reduces the statement to the case where K is represented by a finite free $\mathcal{O}_Y$-module. Since the statement is invariant under finite direct summands in the K variable, we conclude it suffices to prove it for $K = \mathcal{O}_Y[n]$ in which case it is trivial. $\square$

Here is a case where the projection formula is true in complete generality.

0B55 Lemma 20.54.4. Let $f : X \to Y$ be a morphism of ringed spaces such that f is a homeomorphism onto a closed subset. Then (20.54.2.1) is an isomorphism always.

Proof. Since f is a homeomorphism onto a closed subset, the functor f_* is exact (Modules, Lemma 17.6.1). Hence Rf_* is computed by applying f_* to any representative complex. Choose a K-flat complex $\mathcal{K}^\bullet$ of $\mathcal{O}_Y$-modules representing K and choose any complex $\mathcal{E}^\bullet$ of $\mathcal{O}_X$-modules representing E. Then Lf^*K is represented by $f^*\mathcal{K}^\bullet$ which is a K-flat complex of $\mathcal{O}_X$-modules (Lemma 20.26.8). Thus the right hand side of (20.54.2.1) is represented by

$$f_*\text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{K}^\bullet)$$

By the same reasoning we see that the left hand side is represented by

$$\text{Tot}(f_*\mathcal{E}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{K}^\bullet)$$

Since f_* commutes with direct sums (Modules, Lemma 17.6.3) it suffices to show that

$$f_*(\mathcal{E} \otimes_{\mathcal{O}_X} f^*\mathcal{K}) = f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{K}$$

for any $\mathcal{O}_X$-module $\mathcal{E}$ and $\mathcal{O}_Y$-module $\mathcal{K}$. We will check this by checking on stalks. Let $y \in Y$. If $y \notin f(X)$, then the stalks of both sides are zero. If $y = f(x)$, then we see that we have to show

$$\mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y) = \mathcal{E}_x \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y$$

(using Sheaves, Lemma 6.32.1 and Lemma 6.26.4). This equality holds and therefore the lemma has been proved. $\square$

0B6B Remark 20.54.5. The map (20.54.2.1) is compatible with the base change map of Remark 20.28.3 in the following sense. Namely, suppose that

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a commutative diagram of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. Then the diagram

$$\begin{tikzcd}
Lg^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) \arrow[r, "p"'] \arrow[d, "t"'] & Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K) \arrow[d, "b"'] \\
Lg^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} Lg^*K \arrow[d, "b"'] & Rf'_*L(g')^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K) \arrow[d, "t"'] \\
Rf'_*L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} Lg^*K \arrow[dr, "p"'] & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} L(g')^*Lf^*K) \arrow[d, "c"'] \\
 & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{Y'}} L(f')^*Lg^*K)
\end{tikzcd}$$

is commutative. Here arrows labeled t are gotten by an application of Lemma 20.27.3, arrows labeled b by an application of Remark 20.28.3, arrows labeled p by an application of (20.54.2.1), and c comes from $L(g')^* \circ Lf^* = L(f')^* \circ Lg^*$. We omit the verification.

20.55. An operator introduced by Berthelot and Ogus

0GT2 This section continuous the discussion started in More on Algebra, Section 15.95. We strongly encourage the reader to read that section first.

0GT3 **Lemma 20.55.1.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. Consider the following two conditions

(1) for every $x \in X$ there exists an open neighbourhood $U \subset X$ of x and $f \in \mathcal{I}(U)$ such that $\mathcal{I}|_U = \mathcal{O}_U \cdot f$ and $f : \mathcal{O}_U \to \mathcal{O}_U$ is injective, and
(2) $\mathcal{I}$ is invertible as an $\mathcal{O}_X$-module.

Then (1) implies (2) and the converse is true if all stalks $\mathcal{O}_{X,x}$ of the structure sheaf are local rings.

Proof. Omitted. Hint: Use Modules, Lemma 17.25.4. □

0GT4 **Situation 20.55.2.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals satisfying condition (1) of Lemma 20.55.1[8].

0GT5 **Lemma 20.55.3.** In Situation 20.55.2 let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent

(1) the subsheaf $\mathcal{F}[\mathcal{I}] \subset \mathcal{F}$ of sections annihilated by $\mathcal{I}$ is zero,
(2) the subsheaf $\mathcal{F}[\mathcal{I}^n]$ is zero for all $n \geq 1$,
(3) the multiplication map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$ is injective,
(4) for every open $U \subset X$ such that $\mathcal{I}|_U = \mathcal{O}_U \cdot f$ for some $f \in \mathcal{I}(U)$ the map $f : \mathcal{F}|_U \to \mathcal{F}|_U$ is injective,
(5) for every $x \in X$ and generator f of the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ the element f is a nonzerodivisor on the stalk $\mathcal{F}_x$.

Proof. Omitted. □

[8]The discussion in this section can be generalized to the case where all we require is that $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module as defined in Modules, Section 17.25.

In Situation 20.55.2 let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If the equivalent conditions of Lemma 20.55.3 hold, then we will say that $\mathcal{F}$ is $\mathcal{I}$-torsion free. If so, then for any $i \in \mathbf{Z}$ we will denote

$$\mathcal{I}^i\mathcal{F} = \mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{F}$$

so that we have inclusions

$$\ldots \subset \mathcal{I}^{i+1}\mathcal{F} \subset \mathcal{I}^i\mathcal{F} \subset \mathcal{I}^{i-1}\mathcal{F} \subset \ldots$$

The modules $\mathcal{I}^i\mathcal{F}$ are locally isomorphic to $\mathcal{F}$ as $\mathcal{O}_X$-modules, but not globally.

Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules with differentials $d^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$. In this case we define $\eta_\mathcal{I}\mathcal{F}^\bullet$ to be the complex with terms

$$\begin{aligned}(\eta_\mathcal{I}\mathcal{F})^i &= \text{Ker}\left(d^i, -1 : \mathcal{I}^i\mathcal{F}^i \oplus \mathcal{I}^{i+1}\mathcal{F}^{i+1} \to \mathcal{I}^i\mathcal{F}^{i+1}\right) \\ &= \text{Ker}\left(d^i : \mathcal{I}^i\mathcal{F}^i \to \mathcal{I}^i\mathcal{F}^{i+1}/\mathcal{I}^{i+1}\mathcal{F}^{i+1}\right)\end{aligned}$$

and differential induced by d^i. In other words, a local section s of $(\eta_\mathcal{I}\mathcal{F})^i$ is the same thing as a local section s of $\mathcal{I}^i\mathcal{F}^i$ such that its image $d^i(s)$ in $\mathcal{I}^i\mathcal{F}^{i+1}$ is in the subsheaf $\mathcal{I}^{i+1}\mathcal{F}^{i+1}$. Observe that $\eta_\mathcal{I}\mathcal{F}^\bullet$ is another complex of $\mathcal{I}$-torsion free modules.

Let $a^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a map of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Then we obtain a map of complexes

$$\eta_\mathcal{I}a^\bullet : \eta_\mathcal{I}\mathcal{F}^\bullet \longrightarrow \eta_\mathcal{I}\mathcal{G}^\bullet$$

induced by the maps $\mathcal{I}^i\mathcal{F}^i \to \mathcal{I}^i\mathcal{G}^i$. The reader checks that we obtain an endofunctor on the category of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules.

If $a^\bullet, b^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ are two maps of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules and $h = \{h^i : \mathcal{F}^i \to \mathcal{G}^{i-1}\}$ is a homotopy between $a^\bullet$ and $b^\bullet$, then we define $\eta_\mathcal{I}h$ to be the family of maps $(\eta_\mathcal{I}h)^i : (\eta_\mathcal{I}\mathcal{F})^i \to (\eta_\mathcal{I}\mathcal{G})^{i-1}$ which sends x to $h^i(x)$; this makes sense as x a local section of $\mathcal{I}^i\mathcal{F}^i$ implies $h^i(x)$ is a local section of $\mathcal{I}^i\mathcal{G}^{i-1}$ which is certainly contained in $(\eta_\mathcal{I}\mathcal{G})^{i-1}$. The reader checks that $\eta_\mathcal{I}h$ is a homotopy between $\eta_\mathcal{I}a^\bullet$ and $\eta_\mathcal{I}b^\bullet$. All in all we see that we obtain a functor

$$\eta_f : K(\mathcal{I}\text{-torsion free } \mathcal{O}_X\text{-modules}) \longrightarrow K(\mathcal{I}\text{-torsion free } \mathcal{O}_X\text{-modules})$$

on the homotopy category (Derived Categories, Section 13.8) of the additive category of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. There is no sense in which $\eta_\mathcal{I}$ is an exact functor of triangulated categories; compare with More on Algebra, Example 15.95.1.

0GT6 **Lemma 20.55.4.** In Situation 20.55.2 let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. For $x \in X$ choose a generator $f \in \mathcal{I}_x$. Then the stalk $(\eta_\mathcal{I}\mathcal{F}^\bullet)_x$ is canonically isomorphic to the complex $\eta_f\mathcal{F}^\bullet_x$ constructed in More on Algebra, Section 15.95.

Proof. Omitted. □

0F8N **Lemma 20.55.5.** In Situation 20.55.2 let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. There is a canonical isomorphism

$$\mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} \left(H^i(\mathcal{F}^\bullet)/H^i(\mathcal{F}^\bullet)[\mathcal{I}]\right) \longrightarrow H^i(\eta_\mathcal{I}\mathcal{F}^\bullet)$$

of cohomology sheaves.

Proof. We define a map

$$\mathcal{I}^{\otimes i} \otimes_{\mathcal{O}_X} H^i(\mathcal{F}^\bullet) \longrightarrow H^i(\eta_\mathcal{I}\mathcal{F}^\bullet)$$

as follows. Let g be a local section of $\mathcal{I}^{\otimes i}$ and let $\overline{s}$ be a local section of $H^i(\mathcal{F}^\bullet)$. Then $\overline{s}$ is (locally) the class of a local section s of $\mathrm{Ker}(d^i : \mathcal{F}^i \to \mathcal{F}^{i+1})$. Then we send $g \otimes \overline{s}$ to the local section gs of $(\eta_\mathcal{I}\mathcal{F})^i \subset \mathcal{I}^i\mathcal{F}$. Of course gs is in the kernel of d^i on $\eta_\mathcal{I}\mathcal{F}^\bullet$ and hence defines a local section of $H^i(\eta_\mathcal{I}\mathcal{F}^\bullet)$. Checking that this is well defined is without problems. We claim that this map factors through an isomorphism as given in the lemma. This we my check on stalks and hence via Lemma 20.55.4 this translates into the result of More on Algebra, Lemma 15.95.2. □

0F8P Lemma 20.55.6. In Situation 20.55.2 let $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a map of complexes of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Then the induced map $\eta_\mathcal{I}\mathcal{F}^\bullet \to \eta_\mathcal{I}\mathcal{G}^\bullet$ is a quasi-isomorphism too.

Proof. This is true because the isomorphisms of Lemma 20.55.5 are compatible with maps of complexes. □

0F8Q Lemma 20.55.7. In Situation 20.55.2 there is an additive functor[9] $L\eta_\mathcal{I} : D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ such that if M in $D(\mathcal{O}_X)$ is represented by a complex $\mathcal{F}^\bullet$ of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules, then $L\eta_\mathcal{I} M = \eta_\mathcal{I}\mathcal{F}^\bullet$. Similarly for morphisms.

Proof. Denote $\mathcal{T} \subset \mathrm{Mod}(\mathcal{O}_X)$ the full subcategory of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. We have a corresponding inclusion

$$K(\mathcal{T}) \quad \subset \quad K(\mathrm{Mod}(\mathcal{O}_X)) = K(\mathcal{O}_X)$$

of $K(\mathcal{T})$ as a full triangulated subcategory of $K(\mathcal{O}_X)$. Let $S \subset \mathrm{Arrows}(K(\mathcal{T}))$ be the quasi-isomorphisms. We will apply Derived Categories, Lemma 13.5.8 to show that the map

$$S^{-1}K(\mathcal{T}) \longrightarrow D(\mathcal{O}_X)$$

is an equivalence of triangulated categories. The lemma shows that it suffices to prove: given a complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules, there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ with $\mathcal{F}^\bullet$ a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. By Lemma 20.26.12 we can find a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ such that the complex $\mathcal{F}^\bullet$ is K-flat (we won't use this) and consists of flat $\mathcal{O}_X$-modules $\mathcal{F}^i$. By the third characterization of Lemma 20.55.3 we see that a flat $\mathcal{O}_X$-module is an $\mathcal{I}$-torsion free $\mathcal{O}_X$-module and we are done.

With these preliminaries out of the way we can define $L\eta_f$. Namely, by the discussion following Lemma 20.55.3 this section we have already a well defined functor

$$K(\mathcal{T}) \xrightarrow{\eta_f} K(\mathcal{T}) \to K(\mathcal{O}_X) \to D(\mathcal{O}_X)$$

which according to Lemma 20.55.6 sends quasi-isomorphisms to quasi-isomorphisms. Hence this functor factors over $S^{-1}K(\mathcal{T}) = D(\mathcal{O}_X)$ by Categories, Lemma 4.27.8. □

[9]Beware that this functor isn't exact, i.e., does not transform distinguished triangles into distinguished triangles.

In Situation 20.55.2 let us construct the Bockstein operators. First we observe that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}^{i+1} \arrow[r] \arrow[d] & \mathcal{I}^i \arrow[r] \arrow[d] & \mathcal{I}^i/\mathcal{I}^{i+1} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{I}^{i+1}/\mathcal{I}^{i+2} \arrow[r] & \mathcal{I}^i/\mathcal{I}^{i+2} \arrow[r] & \mathcal{I}^i/\mathcal{I}^{i+1} \arrow[r] & 0 \end{tikzcd}$$

whose rows are short exact sequences of $\mathcal{O}_X$-modules. Let M be an object of $D(\mathcal{O}_X)$. Tensoring the above diagram with M gives a morphism

$$\begin{tikzcd} M \otimes^{\mathbf{L}} \mathcal{I}^{i+1} \arrow[r] \arrow[d] & M \otimes^{\mathbf{L}} \mathcal{I}^i \arrow[r] \arrow[d] & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1} \arrow[d, "\mathrm{id}"] \\ M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2} \arrow[r] & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+2} \arrow[r] & M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1} \end{tikzcd}$$

of distinguished triangles. The long exact sequence of cohomology sheaves associated the bottom triangle in particular determines the Bockstein operator

$$\beta = \beta^i : H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) \longrightarrow H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2})$$

for all $i \in \mathbf{Z}$. For later use we record here that by the commutative diagram above there is a factorization

0GT7 (20.55.7.1)
$$\begin{tikzcd} H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) \arrow[r, "\delta"'] \arrow[rd, "\beta"'] & H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}) \arrow[d] \\ & H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2}) \end{tikzcd}$$

of the Bockstein operator where δ is the boundary operator coming from the top distinguished triangle in the commutative diagram above. We obtain a complex

0GT8 (20.55.7.2)
$$H^\bullet(M/\mathcal{I}) = \left[\begin{matrix} \ldots \\ \downarrow \\ H^{i-1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i-1}/\mathcal{I}^i) \\ \downarrow \beta \\ H^i(M \otimes^{\mathbf{L}} \mathcal{I}^i/\mathcal{I}^{i+1}) \\ \downarrow \beta \\ H^{i+1}(M \otimes^{\mathbf{L}} \mathcal{I}^{i+1}/\mathcal{I}^{i+2}) \\ \downarrow \\ \ldots \end{matrix} \right]$$

i.e., that $\beta \circ \beta = 0$. Namely, we can check this on stalks and in this case we can deduce it from the corresponding result in algebra shown in More on Algebra, Section 15.95. Alternative proof: the short exact sequences $0 \to \mathcal{I}^{i+1}/\mathcal{I}^{i+2} \to \mathcal{I}^i/\mathcal{I}^{i+2} \to \mathcal{I}^i/\mathcal{I}^{i+1} \to 0$ define maps $b^i : \mathcal{I}^i/\mathcal{I}^{i+1} \to (\mathcal{I}^{i+1}/\mathcal{I}^{i+2})[1]$ in $D(\mathcal{O}_X)$ which induce the maps β above by tensoring with M and taking cohomology sheaves. Then one shows that the composition $b^{i+1}[1] \circ b^i : \mathcal{I}^i/\mathcal{I}^{i+1} \to (\mathcal{I}^{i+1}/\mathcal{I}^{i+2})[1] \to (\mathcal{I}^{i+2}/\mathcal{I}^{i+3})[2]$ is zero in $D(\mathcal{O}_X)$ by using the criterion in Derived Categories, Lemma 13.27.7 using that the module $\mathcal{I}^i/\mathcal{I}^{i+3}$ is an extension of $\mathcal{I}^{i+1}/\mathcal{I}^{i+3}$ by $\mathcal{I}^i/\mathcal{I}^{i+1}$.

0GT9 **Lemma 20.55.8.** In Situation 20.55.2 let M be an object of $D(\mathcal{O}_X)$. There is a canonical isomorphism

$$L\eta_\mathcal{I} M \otimes^{\mathbf{L}} \mathcal{O}_X/\mathcal{I} \longrightarrow H^\bullet(M/\mathcal{I})$$

in $D(\mathcal{O}_X)$ where the right hand side is the complex (20.55.7.2).

Proof. By the construction of $L\eta_\mathcal{I}$ in Lemma 20.55.6 we may assume M is represented by a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$. Then $L\eta_\mathcal{I} M$ is represented by the complex $\eta_\mathcal{I}\mathcal{F}^\bullet$ which is a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules as well. Thus $L\eta_\mathcal{I} M \otimes^{\mathbf{L}} \mathcal{O}_X/\mathcal{I}$ is represented by the complex $\eta_\mathcal{I}\mathcal{F}^\bullet \otimes \mathcal{O}_X/\mathcal{I}$. Similarly, the complex $H^\bullet(M/\mathcal{I})$ has terms $H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$.

Let f be a local generator for $\mathcal{I}$. Let s be a local section of $(\eta_\mathcal{I}\mathcal{F})^i$. Then we can write $s = f^i s'$ for a local section s' of $\mathcal{F}^i$ and similarly $d^i(s) = f^{i+1}t$ for a local section t of $\mathcal{F}^{i+1}$. Thus d^i maps $f^i s'$ to zero in $\mathcal{F}^{i+1} \otimes \mathcal{I}^i/\mathcal{I}^{i+1}$. Hence we may map s to the class of $f^i s'$ in $H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$. This rule defines a map

$$(\eta_\mathcal{I}\mathcal{F})^i \otimes \mathcal{O}_X/\mathcal{I} \longrightarrow H^i(\mathcal{F}^\bullet \otimes \mathcal{I}^i/\mathcal{I}^{i+1})$$

of $\mathcal{O}_X$-modules. A calculation shows that these maps are compatible with differentials (essentially because β sends the class of $f^i s'$ to the class of $f^{i+1}t$), whence a map of complexes representing the arrow in the statement of the lemma.

To finish the proof, we observe that the construction given in the previous paragraph agrees on stalks with the maps constructed in More on Algebra, Lemma 15.95.6 hence we conclude. □

0F9W **Lemma 20.55.9.** In Situation 20.55.2 let $\mathcal{F}^\bullet$ be a complex of $\mathcal{I}$-torsion free $\mathcal{O}_X$-modules. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\eta_\mathcal{I}(\mathcal{F}^\bullet \otimes \mathcal{L}) = (\eta_\mathcal{I}\mathcal{F}^\bullet) \otimes \mathcal{L}$.

Proof. Immediate from the construction. □

0GTA **Lemma 20.55.10.** In Situation 20.55.2 let M be an object of $D(\mathcal{O}_X)$. Let $x \in X$ with $\mathcal{O}_{X,x}$ nonzero. If $H^i(M)_x$ is finite free over $\mathcal{O}_{X,x}$, then $H^i(L\eta_\mathcal{I} M)_x$ is finite free over $\mathcal{O}_{X,x}$ of the same rank.

Proof. Namely, say $f \in \mathcal{O}_{X,x}$ generates the stalk $\mathcal{I}_x$. Then f is a nonzerodivisor in $\mathcal{O}_{X,x}$ and hence $H^i(M)_x[f] = 0$. Thus by Lemma 20.55.5 we see that $H^i(L\eta_\mathcal{I} M)_x$ is isomorphic to $\mathcal{I}_x^i \otimes_{\mathcal{O}_{X,x}} H^i(M)_x$ which is free of the same rank as desired. □

20.56. Other chapters

CHAPTER 21

Cohomology on Sites

01FQ

21.1. Introduction

01FR In this document we work out some topics on cohomology of sheaves. We work out what happens for sheaves on sites, although often we will simply duplicate the discussion, the constructions, and the proofs from the topological case in the case. Basic references are [AGV71], [God73] and [Ive86].

21.2. Cohomology of sheaves

01FT Let $\mathcal{C}$ be a site, see Sites, Definition 7.6.2. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. We know that the category of abelian sheaves on $\mathcal{C}$ has enough injectives, see Injectives, Theorem 19.7.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

071C (21.2.0.1)
$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet))$$

to be the ith cohomology group of the abelian sheaf $\mathcal{F}$ over the object U. In other words, these are the right derived functors of the functor $\mathcal{F} \mapsto \mathcal{F}(U)$. The family of functors $H^i(U, -)$ forms a universal δ-functor $\textit{Ab}(\mathcal{C}) \to \textit{Ab}$.

It sometimes happens that the site $\mathcal{C}$ does not have a final object. In this case we define the global sections of a presheaf of sets $\mathcal{F}$ over $\mathcal{C}$ to be the set

071D (21.2.0.2)
$$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(e, \mathcal{F})$$

where e is a final object in the category of presheaves on $\mathcal{C}$. In this case, given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$, we define the ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$ as follows

071E (21.2.0.3)
$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet))$$

in other words, it is the ith right derived functor of the global sections functor. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $\textit{Ab}(\mathcal{C}) \to \textit{Ab}$.

Let $f : \mathit{Sh}(\mathcal{C}) \to \mathit{Sh}(\mathcal{D})$ be a morphism of topoi, see Sites, Definition 7.15.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

071F (21.2.0.4)
$$R^i f_*\mathcal{F} = H^i(f_*\mathcal{I}^\bullet)$$

to be the ith higher direct image of $\mathcal{F}$. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $\textit{Ab}(\mathcal{C}) \to \textit{Ab}(\mathcal{D})$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, see Modules on Sites, Definition 18.6.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. We know that the category of $\mathcal{O}$-modules has enough injectives, see Injectives, Theorem 19.8.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

071G (21.2.0.5)
$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet))$$

to be the the ith cohomology group of $\mathcal{F}$ over U. The family of functors $H^i(U, -)$ forms a universal δ-functor $\text{Mod}(\mathcal{O}) \to \text{Mod}_{\mathcal{O}(U)}$. Similarly

071H (21.2.0.6) $$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet))$$

it the ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $\text{Mod}(\mathcal{C}) \to \text{Mod}_{\Gamma(\mathcal{C},\mathcal{O})}$.

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi, see Modules on Sites, Definition 18.7.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

071I (21.2.0.7) $$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet)$$

to be the ith higher direct image of $\mathcal{F}$. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $\text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}')$.

21.3. Derived functors

071J We briefly explain an approach to right derived functors using resolution functors. Namely, suppose that $(\mathcal{C}, \mathcal{O})$ is a ringed site. In this chapter we will write

$$K(\mathcal{O}) = K(\text{Mod}(\mathcal{O})) \quad \text{and} \quad D(\mathcal{O}) = D(\text{Mod}(\mathcal{O}))$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 13.8.1 and Definition 13.11.3. By Derived Categories, Remark 13.24.3 there exists a resolution functor

$$j = j_{(\mathcal{C},\mathcal{O})} : K^+(\text{Mod}(\mathcal{O})) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $\text{Mod}(\mathcal{O})$ which consists of injective $\mathcal{O}$-modules. For any left exact functor $F : \text{Mod}(\mathcal{O}) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor of Derived Categories, Section 13.20 constructed using the resolution functor j just described:

05U5 (21.3.0.1) $$RF = F \circ j' : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{B})$$

see Derived Categories, Lemma 13.25.1 for notation. Note that we may think of RF as defined on $\text{Mod}(\mathcal{O})$, $\text{Comp}^+(\text{Mod}(\mathcal{O}))$, or $K^+(\mathcal{O})$ depending on the situation. According to Derived Categories, Definition 13.16.2 we obtain the $i$the right derived functor

05U6 (21.3.0.2) $$R^i F = H^i \circ RF : \text{Mod}(\mathcal{O}) \longrightarrow \mathcal{B}$$

so that $R^0 F = F$ and $\{R^i F, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 13.20.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\text{Mod}_R)$ and $D(R) = D(\text{Mod}_R)$ and similarly for the bounded versions. For any object U of $\mathcal{C}$ have a left exact functor $\Gamma(U, -) : \text{Mod}(\mathcal{O}) \longrightarrow \text{Mod}_{\mathcal{O}(U)}$ which gives rise to

$$R\Gamma(U, -) : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{O}(U))$$

by the discussion above. Note that $H^i(U, -) = R^i\Gamma(U, -)$ is compatible with (21.2.0.5) above. We similarly have

$$R\Gamma(\mathcal{C}, -) : D^+(\mathcal{O}) \longrightarrow D^+(\Gamma(\mathcal{C}, \mathcal{O}))$$

compatible with (21.2.0.6). If $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is a morphism of ringed topoi then we get a left exact functor $f_* : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}')$ which gives rise to derived pushforward

$$Rf_* : D^+(\mathcal{O}) \to D^+(\mathcal{O}')$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^if_*\mathcal{F}^\bullet$ and called the ith higher direct image in accordance with (21.2.0.7). The displayed functors above are exact functor of derived categories.

21.4. First cohomology and torsors

03AG

03AH Definition 21.4.1. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A pseudo torsor, or more precisely a pseudo $\mathcal{G}$-torsor, is a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive.

A morphism of pseudo $\mathcal{G}$-torsors $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. A torsor, or more precisely a $\mathcal{G}$-torsor, is a pseudo $\mathcal{G}$-torsor such that in addition

(2) for every $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i \in I}$ of U such that $\mathcal{F}(U_i)$ is nonempty for all $i \in I$.

A morphism of $\mathcal{G}$-torsors is simply a morphism of pseudo $\mathcal{G}$-torsors. The trivial $\mathcal{G}$-torsor is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

03AI Lemma 21.4.2. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\Gamma(\mathcal{C}, \mathcal{F}) \neq \emptyset$.

Proof. Omitted. $\square$

03AJ Lemma 21.4.3. Let $\mathcal{C}$ be a site. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{C}$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(\mathcal{C}, \mathcal{H})$.

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \in \text{Ob}(\mathcal{C})$ the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by sections of the form $[s] - [s']$. There is a canonical map $a : \text{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \text{Ker}(\sigma) \arrow[r] \arrow[d, "a"] & \mathbf{Z}[\mathcal{F}] \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & 0
\end{tikzcd}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(\mathcal{C}, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(\mathcal{C}, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(\mathcal{C}, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(\mathcal{C}, \mathcal{Q})$ because $H^1(\mathcal{C}, \mathcal{I}) = 0$ (see Derived Categories, Lemma 13.20.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a $\mathcal{H}$-torsor.

We omit the verification that the two constructions given above are mutually inverse. □

21.5. First cohomology and extensions

03F0

03F1 **Lemma 21.5.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. There is a canonical bijection

$$\mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{O})}(\mathcal{O}, \mathcal{F}) \longrightarrow H^1(\mathcal{C}, \mathcal{F})$$

which associates to the extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$$

the image of $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ in $H^1(\mathcal{C}, \mathcal{F})$.

Proof. Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(\mathcal{C}, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $\mathrm{Mod}(\mathcal{O})$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that ξ is the image of a section $\tilde{\xi} \in \Gamma(\mathcal{C}, \mathcal{Q}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{Q})$. Now, we just form the pullback

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, "\tilde{\xi}"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{Q} \arrow[r] & 0
\end{tikzcd}$$

see Homology, Section 12.6. □

The following lemma will be superseded by the more general Lemma 21.12.4.

03F2 **Lemma 21.5.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $\mathcal{F}_{ab}$ denote the underlying sheaf of abelian groups. Then there is a functorial isomorphism

$$H^1(\mathcal{C}, \mathcal{F}_{ab}) = H^1(\mathcal{C}, \mathcal{F})$$

where the left hand side is cohomology computed in $\mathrm{Ab}(\mathcal{C})$ and the right hand side is cohomology computed in $\mathrm{Mod}(\mathcal{O})$.

Proof. Let $\underline{\mathbf{Z}}$ denote the constant sheaf $\mathbf{Z}$. As $\mathrm{Ab}(\mathcal{C}) = \mathrm{Mod}(\underline{\mathbf{Z}})$ we may apply Lemma 21.5.1 twice, and it follows that we have to show

$$\mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{O})}(\mathcal{O}, \mathcal{F}) = \mathrm{Ext}^1_{\mathrm{Mod}(\underline{\mathbf{Z}})}(\underline{\mathbf{Z}}, \mathcal{F}_{ab}).$$

Suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$ is an extension in $\mathrm{Mod}(\mathcal{O})$. Then we can use the obvious map of abelian sheaves $1 : \underline{\mathbf{Z}} \to \mathcal{O}$ and pullback to obtain an extension

$\mathcal{E}_{ab}$, like so:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_{ab} \arrow[r] \arrow[d, equal] & \mathcal{E}_{ab} \arrow[r] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0 \end{tikzcd}$$

The converse is a little more fun. Suppose that $0 \to \mathcal{F}_{ab} \to \mathcal{E}_{ab} \to \underline{\mathbf{Z}} \to 0$ is an extension in $\text{Mod}(\underline{\mathbf{Z}})$. Since $\underline{\mathbf{Z}}$ is a flat $\underline{\mathbf{Z}}$-module we see that the sequence

$$0 \to \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to 0$$

is exact, see Modules on Sites, Lemma 18.28.9. Of course $\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} = \mathcal{O}$. Hence we can form the pushout via the ($\mathcal{O}$-linear) multiplication map $\mu : \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{F}$ to get an extension of $\mathcal{O}$ by $\mathcal{F}$, like this

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \arrow[r] \arrow[d, "\mu"] & \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0 \end{tikzcd}$$

which is the desired extension. We omit the verification that these constructions are mutually inverse. □

21.6. First cohomology and invertible sheaves

040D The Picard group of a ringed site is defined in Modules on Sites, Section 18.32.

040E **Lemma 21.6.1.** Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. There is a canonical isomorphism

$$H^1(\mathcal{C}, \mathcal{O}^*) = \text{Pic}(\mathcal{O}).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s\cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by Modules on Sites, Lemma 18.40.7. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}^*$-torsor. Thus we get a map

$$\begin{matrix} \text{set of invertible sheaves on } (\mathcal{C}, \mathcal{O}) \\ \text{up to isomorphism} \end{matrix} \longrightarrow \begin{matrix} \text{set of } \mathcal{O}^*\text{-torsors} \\ \text{up to isomorphism} \end{matrix}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 21.4.3 the right hand side is canonically bijective to $H^1(\mathcal{C}, \mathcal{O}^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 21.4.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}$ is the neutral element in $\text{Pic}(\mathcal{O})$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}(U))/\mathcal{O}^*(U)$$

where the action of $f \in \mathcal{O}^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}$-modules by setting $(s, g)+(s', g') = (s, g+(s'/s)g')$ where s'/s is the local section f of $\mathcal{O}^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}$-module whose associated $\mathcal{O}^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. □

21.7. Locality of cohomology

01FU The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an object of the site.

03F3 Lemma 21.7.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.

(1) If $\mathcal{I}$ is an injective $\mathcal{O}$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.
(2) For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(\mathcal{C}/U, \mathcal{F}|_U)$.

Proof. Recall that the functor j_U^{-1} of restriction to U is a right adjoint to the functor $j_{U!}$ of extension by 0, see Modules on Sites, Section 18.19. Moreover, $j_{U!}$ is exact. Hence (1) follows from Homology, Lemma 12.29.1.

By definition $H^p(U, \mathcal{F}) = H^p(\mathcal{I}^\bullet(U))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\text{Mod}(\mathcal{O})$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $\text{Mod}(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\mathcal{I}^\bullet|_U(U))$. Of course $\mathcal{F}(U) = \mathcal{F}|_U(U)$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$. Hence the equality in (2). □

The following lemma will be use to see what happens if we change a partial universe, or to compare cohomology of the small and big étale sites.

03YU Lemma 21.7.2. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume u satisfies the hypotheses of Sites, Lemma 7.21.8. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. For any abelian sheaf $\mathcal{F}$ on $\mathcal{D}$ we have isomorphisms

$$R\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = R\Gamma(\mathcal{D}, \mathcal{F}),$$

in particular $H^p(\mathcal{C}, g^{-1}\mathcal{F}) = H^p(\mathcal{D}, \mathcal{F})$ and for any $U \in \text{Ob}(\mathcal{C})$ we have isomorphisms

$$R\Gamma(U, g^{-1}\mathcal{F}) = R\Gamma(u(U), \mathcal{F}),$$

in particular $H^p(U, g^{-1}\mathcal{F}) = H^p(u(U), \mathcal{F})$. All of these isomorphisms are functorial in $\mathcal{F}$.

Proof. Since it is clear that $\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = \Gamma(\mathcal{D}, \mathcal{F})$ by hypothesis (e), it suffices to show that g^{-1} transforms injective abelian sheaves into injective abelian sheaves. As usual we use Homology, Lemma 12.29.1 to see this. The left adjoint to g^{-1} is $g_! = f^{-1}$ with the notation of Sites, Lemma 7.21.8 which is an exact functor. Hence the lemma does indeed apply. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\varphi : U \to V$ be a morphism of $\mathcal{O}$. Then there is a canonical restriction mapping

01FV (21.7.2.1) $$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 21.10.5.

The following lemma says that it is possible to kill higher cohomology classes by going to a covering.

01FW Lemma 21.7.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let U be an object of $\mathcal{C}$. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists a covering $\{U_i \to U\}$ of $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists a covering $\{U_i \to U\}$ of the site such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. □

072W Lemma 21.7.4. Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For any $\mathcal{F} \in \mathrm{Ob}(\mathrm{Mod}(\mathcal{O}_\mathcal{C}))$ the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto H^i(u(V), \mathcal{F})$$

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_\mathcal{D}$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(u(V)) \to \mathcal{I}^{i+1}(u(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(u(V)) \to \mathcal{I}^i(u(V)))}$$

which is equal to $H^i(u(V), \mathcal{F})$ and we win. □

21.8. The Čech complex and Čech cohomology

03AK Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target, see Sites, Definition 7.6.1. Assume that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. Set

$$\check{C}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in the factor $\mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})$. We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

03AL (21.8.0.1)
$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}|_{U_{i_0} \times_U \ldots \times_U U_{i_{p+1}}}$$

where the restriction is via the projection map

$$U_{i_0} \times_U \ldots \times_U U_{i_{p+1}} \longrightarrow U_{i_0} \times_U \ldots \times_U \widehat{U_{i_j}} \times_U \ldots \times_U U_{i_{p+1}}.$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

03AM Definition 21.8.1. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the Čech complex associated to $\mathcal{F}$ and the family $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the Čech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

We observe that any covering $\{U_i \to U\}$ of a site $\mathcal{C}$ is a family of morphisms with fixed target to which the definition applies.

03AN Lemma 21.8.2. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The following are equivalent

(1) $\mathcal{F}$ is an abelian sheaf on $\mathcal{C}$ and
(2) for every covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$ the natural map
$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$
(see Sites, Section 7.10) is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every covering of $\mathcal{C}$. □

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another. Let $f : U \to V$, $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ be a morphism of families of morphisms with fixed target, see Sites, Section 7.8. In this case we get a map of Čech complexes

03F4 (21.8.2.1)
$$\varphi : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

which in degree p is given by

$$\varphi(s)_{i_0 \ldots i_p} = (f_{i_0} \times \ldots \times f_{i_p})^* s_{\alpha(i_0) \ldots \alpha(i_p)}$$

21.9. Čech cohomology as a functor on presheaves

03AO Warning: In this section we work exclusively with abelian presheaves on a category. The results are completely wrong in the setting of sheaves and categories of sheaves!

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

03AP (21.9.0.1)
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : \mathrm{PAb}(\mathcal{C}) \longrightarrow \mathrm{Comp}^+(\mathrm{Ab})$$

see Derived Categories, Definition 13.8.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.13.9.

03AQ **Lemma 21.9.1.** The functor given by Equation (21.9.0.1) is an exact functor (see Homology, Lemma 12.7.2).

Proof. For any object W of $\mathcal{C}$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $\mathrm{PAb}(\mathcal{C})$ to Ab. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. $\square$

03AR **Lemma 21.9.2.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category $\mathrm{PAb}(\mathcal{C})$ to the category of $\mathbf{Z}$-modules (see Homology, Definition 12.12.1).

Proof. By Lemma 21.9.1 a short exact sequence of abelian presheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathbf{Z}$-modules. Hence we can use Homology, Lemma 12.13.12 to get the boundary maps $\delta_{\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. $\square$

03AS **Lemma 21.9.3.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Consider the chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of abelian presheaves

$$\ldots \to \bigoplus_{i_0 i_1 i_2} \mathbf{Z}_{U_{i_0}\times_U U_{i_1}\times_U U_{i_2}} \to \bigoplus_{i_0 i_1} \mathbf{Z}_{U_{i_0}\times_U U_{i_1}} \to \bigoplus_{i_0} \mathbf{Z}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$\mathbf{Z}_{U_{i_0}\times_U \ldots \times_u U_{i_{p+1}}} \longrightarrow \mathbf{Z}_{U_{i_0}\times_U \ldots \widehat{U_{i_j}} \ldots \times_U U_{i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \mathrm{Ob}(\mathrm{PAb}(\mathcal{C}))$.

Proof. This is a tautology based on the fact that

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\bigoplus_{i_0\ldots i_p} \mathbf{Z}_{U_{i_0}\times_U\ldots\times_U U_{i_p}}, \mathcal{F}) &= \prod_{i_0\ldots i_p} \mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbf{Z}_{U_{i_0}\times_U\ldots\times_U U_{i_p}}, \mathcal{F}) \\
&= \prod_{i_0\ldots i_p} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})
\end{aligned}$$

see Modules on Sites, Lemma 18.4.2. $\square$

03AT **Lemma 21.9.4.** Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of presheaves of Lemma 21.9.3 above is exact in positive degrees, i.e., the homology presheaves $H_i(\mathbf{Z}_{\mathcal{U},\bullet})$ are zero for $i > 0$.

Proof. Let V be an object of $\mathcal{C}$. We have to show that the chain complex of abelian groups $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is exact in degrees > 0. This is the complex

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_\mathcal{C}(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

For any morphism $\varphi : V \to U$ denote $\mathrm{Mor}_\varphi(V, U_i) = \{\varphi_i : V \to U_i \mid f_i \circ \varphi_i = \varphi\}$. We will use a similar notation for $\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p})$. Note that composing with the various projection maps between the fibred products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ preserves these morphism sets. Hence we see that the complex above is the same as the complex

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0 i_1 i_2} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0} \times_U U_{i_1} \times_U U_{i_2})] \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0 i_1} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0} \times_U U_{i_1})] \\ \downarrow \\ \bigoplus_\varphi \bigoplus_{i_0} \mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0})] \\ \downarrow \\ 0 \end{array}$$

Next, we make the remark that we have

$$\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p}) = \mathrm{Mor}_\varphi(V, U_{i_0}) \times \ldots \times \mathrm{Mor}_\varphi(V, U_{i_p})$$

Using this and the fact that $\mathbf{Z}[A] \oplus \mathbf{Z}[B] = \mathbf{Z}[A \amalg B]$ we see that the complex becomes

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1 i_2} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1}) \times \mathrm{Mor}_\varphi(V, U_{i_2})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0} \mathrm{Mor}_\varphi(V, U_{i_0})\right] \\ \downarrow \\ 0 \end{array}$$

Finally, on setting $S_\varphi = \coprod_{i \in I} \mathrm{Mor}_\varphi(V, U_i)$ we see that we get

$$\bigoplus\nolimits_\varphi (\ldots \to \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi] \to 0 \to \ldots)$$

Thus we have simplified our task. Namely, it suffices to show that for any nonempty set S the (extended) complex of free abelian groups

$$\ldots \to \mathbf{Z}[S \times S \times S] \to \mathbf{Z}[S \times S] \to \mathbf{Z}[S] \xrightarrow{\Sigma} \mathbf{Z} \to 0 \to \ldots$$

is exact in all degrees. To see this fix an element $s \in S$, and use the homotopy

$$n_{(s_0, \ldots, s_p)} \longmapsto n_{(s, s_0, \ldots, s_p)}$$

with obvious notations. □

03F5 Lemma 21.9.5. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The chain complex

$$\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}$$

is exact in positive degrees. Here $\mathbf{Z}_{\mathcal{U},\bullet}$ is the chain complex of Lemma 21.9.3, and the tensor product is over the constant presheaf of rings with value $\mathbf{Z}$.

Proof. Let V be an object of $\mathcal{C}$. In the proof of Lemma 21.9.4 we saw that $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathbf{Z}$ placed in degree zero. Hence also $\mathbf{Z}_{\mathcal{U},\bullet}(V) \otimes_{\mathbf{Z}} \mathcal{O}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathcal{O}(V)$ placed in degree zero. Or you can use Modules on Sites, Lemma 18.28.11, which applies since the presheaves $\mathbf{Z}_{\mathcal{U},i}$ are flat, and the proof of Lemma 21.9.4 shows that $H_0(\mathbf{Z}_{\mathcal{U},\bullet})$ is a flat presheaf also. □

03AU Lemma 21.9.6. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor

$$\check{H}^0(\mathcal{U}, -) : \mathrm{PAb}(\mathcal{C}) \longrightarrow \mathrm{Ab}.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(\mathrm{PAb}(\mathcal{C})) \longrightarrow D^+(\mathbf{Z})$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of abelian presheaves has enough injectives, see Injectives, Proposition 19.6.1. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of abelian presheaves to the category of $\mathbf{Z}$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 13.20.

Let $\mathcal{I}$ be a injective abelian presheaf. In this case the functor $\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(-, \mathcal{I})$ is exact on $\mathrm{PAb}(\mathcal{C})$. By Lemma 21.9.3 we have

$$\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 21.9.4 we have that $\mathbf{Z}_{\mathcal{U},\bullet}$ is exact in positive degrees. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 21.9.2) satisfies the assumptions of Homology, Lemma 12.12.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 13.20.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 12.12.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any abelian presheaf on $\mathcal{C}$. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $\mathrm{PAb}(\mathcal{C})$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Next, consider the total complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$ associated to this double complex, see Homology, Section 12.18. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 12.25.4. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 21.9.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathbf{Z})$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. □

21.10. Čech cohomology and cohomology

03AV The relationship between cohomology and Čech cohomology comes from the fact that the Čech cohomology of an injective abelian sheaf is zero. To see this we note that an injective abelian sheaf is an injective abelian presheaf and then we apply results in Čech cohomology in the preceding section.

03F6 **Lemma 21.10.1.** Let $\mathcal{C}$ be a site. An injective abelian sheaf is also injective as an object in the category $\textit{PAb}(\mathcal{C})$.

Proof. Apply Homology, Lemma 12.29.1 to the categories $\mathcal{A} = \textit{Ab}(\mathcal{C})$, $\mathcal{B} = \textit{PAb}(\mathcal{C})$, the inclusion functor and sheafification. (See Modules on Sites, Section 18.3 to see that all assumptions of the lemma are satisfied.) □

03AW **Lemma 21.10.2.** Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective abelian sheaf, i.e., an injective object of $\textit{Ab}(\mathcal{C})$. Then

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. By Lemma 21.10.1 we see that $\mathcal{I}$ is an injective object in $\textit{PAb}(\mathcal{C})$. Hence we can apply Lemma 21.9.6 (or its proof) to see the vanishing of higher Čech cohomology group. For the zeroth see Lemma 21.8.2. □

03AX **Lemma 21.10.3.** Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. There is a transformation

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $\textit{Ab}(\mathcal{C}) \to D^+(\mathbf{Z})$. In particular this gives a transformation of functors $\check{H}^p(U, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $\textit{Ab}(\mathcal{C})$.

Proof. Let $\mathcal{F}$ be an abelian sheaf. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Next, consider the associated total complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$, see Homology, Definition 12.18.3. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 12.25.4 to see that α is a quasi-isomorphism. Namely, Lemma 21.10.2 implies that the qth row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathbf{Z})$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

0A6G **Lemma 21.10.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be an abelian sheaf on $\mathcal{C}$. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. The map

$$\check{H}^1(\mathcal{U}, \mathcal{G}) \longrightarrow H^1(U, \mathcal{G})$$

is injective and identifies $\check{H}^1(\mathcal{U}, \mathcal{G})$ via the bijection of Lemma 21.4.3 with the set of isomorphism classes of $\mathcal{G}|_U$-torsors which restrict to trivial torsors over each U_i.

Proof. To see this we construct an inverse map. Namely, let $\mathcal{F}$ be a $\mathcal{G}|_U$-torsor on $\mathcal{C}/U$ whose restriction to $\mathcal{C}/U_i$ is trivial. By Lemma 21.4.2 this means there exists a section $s_i \in \mathcal{F}(U_i)$. On $U_{i_0} \times_U U_{i_1}$ there is a unique section $s_{i_0i_1}$ of $\mathcal{G}$ such that $s_{i_0i_1} \cdot s_{i_0}|_{U_{i_0} \times_U U_{i_1}} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}}$. An easy computation shows that $s_{i_0i_1}$ is a Čech cocycle and that its class is well defined (i.e., does not depend on the choice of the sections s_i). The inverse maps the isomorphism class of $\mathcal{F}$ to the cohomology class of the cocycle $(s_{i_0i_1})$. We omit the verification that this map is indeed an inverse. □

03AY Lemma 21.10.5. Let $\mathcal{C}$ be a site. Consider the functor $i : \text{Ab}(\mathcal{C}) \to \text{PAb}(\mathcal{C})$. It is a left exact functor with right derived functors given by

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 21.7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology presheaf of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an object U of $\mathcal{C}$ are given by

$$\frac{\text{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. $\square$

03AZ Lemma 21.10.6. Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. For any abelian sheaf $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 13.22.2) for the functors

$$i : \text{Ab}(\mathcal{C}) \to \text{PAb}(\mathcal{C}) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : \text{PAb}(\mathcal{C}) \to \text{Ab}.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 21.8.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 21.10.2. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $\text{PAb}(\mathcal{C})$ by Lemma 21.9.6. Putting everything together gives the lemma. $\square$

03F7 Lemma 21.10.7. Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering. Let $\mathcal{F} \in \text{Ob}(\text{Ab}(\mathcal{C}))$. Assume that $H^i(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$.

Proof. We will use the spectral sequence of Lemma 21.10.6. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. $\square$

03F8 Lemma 21.10.8. Let $\mathcal{C}$ be a site. Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let U be an object of $\mathcal{C}$. If there exists a cofinal system of coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}} - s_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0} \times_U U_{i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0} \times_U U_{i_1}} - t_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. $\square$

03F9 **Lemma 21.10.9.** (Variant of Cohomology, Lemma 20.11.8.) Let $\mathcal{C}$ be a site. Let $\text{Cov}_\mathcal{C}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 7.6.2). Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$, and $\text{Cov} \subset \text{Cov}_\mathcal{C}$ be subsets. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Assume that

(1) For every $\mathcal{U} \in \text{Cov}$, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.
(2) For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.
(3) For every $\mathcal{U} \in \text{Cov}$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \text{Cov}$". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective abelian sheaf. By Lemma 21.10.2 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \text{Cov}$. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 21.10.8 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in \text{Cov}$ we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any covering $\mathcal{U} \in \text{Cov}$. This implies that $\mathcal{Q}$ is also an abelian sheaf with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$.

Next, we look at the long exact cohomology sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(U, \mathcal{F}) \arrow[r] & H^0(U, \mathcal{I}) \arrow[r] & H^0(U, \mathcal{Q}) \arrow[dll] \\
& H^1(U, \mathcal{F}) \arrow[r] & H^1(U, \mathcal{I}) \arrow[r] & H^1(U, \mathcal{Q}) \arrow[dll] \\
& \ldots & \ldots & \ldots
\end{tikzcd}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.20.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary abelian sheaf with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. $\square$

21.11. Second cohomology and gerbes

0CJZ Let $p : \mathcal{S} \to \mathcal{C}$ be a gerbe over a site all of whose automorphism groups are commutative. In this situation the first and second cohomology groups of the sheaf of automorphisms (Stacks, Lemma 8.11.8) controls the existence of objects.

The following lemma will be made obsolete by a more complete discussion of this relationship we will add in the future.

0CK0 **Lemma 21.11.1.** Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a gerbe over a site whose automorphism sheaves are abelian. Let $\mathcal{G}$ be the sheaf of abelian groups constructed in Stacks, Lemma 8.11.8. Let U be an object of $\mathcal{C}$ such that

(1) there exists a cofinal system of coverings $\{U_i \to U\}$ of U in $\mathcal{C}$ such that $H^1(U_i, \mathcal{G}) = 0$ and $H^1(U_i \times_U U_j, \mathcal{G}) = 0$ for all i, j, and
(2) $H^2(U, \mathcal{G}) = 0$.

Then there exists an object of $\mathcal{S}$ lying over U.

Proof. By Stacks, Definition 8.11.1 there exists a covering $\mathcal{U} = \{U_i \to U\}$ and x_i in $\mathcal{S}$ lying over U_i. Write $U_{ij} = U_i \times_U U_j$. By (1) after refining the covering we may assume that $H^1(U_i, \mathcal{G}) = 0$ and $H^1(U_{ij}, \mathcal{G}) = 0$. Consider the sheaf

$$\mathcal{F}_{ij} = \mathit{Isom}(x_i|_{U_{ij}}, x_j|_{U_{ij}})$$

on $\mathcal{C}/U_{ij}$. Since $\mathcal{G}|_{U_{ij}} = \mathit{Aut}(x_i|_{U_{ij}})$ we see that there is an action

$$\mathcal{G}|_{U_{ij}} \times \mathcal{F}_{ij} \to \mathcal{F}_{ij}$$

by precomposition. It is clear that $\mathcal{F}_{ij}$ is a pseudo $\mathcal{G}|_{U_{ij}}$-torsor and in fact a torsor because any two objects of a gerbe are locally isomorphic. By our choice of the covering and by Lemma 21.4.3 these torsors are trivial (and hence have global sections by Lemma 21.4.2). In other words, we can choose isomorphisms

$$\varphi_{ij} : x_i|_{U_{ij}} \longrightarrow x_j|_{U_{ij}}$$

To find an object x over U we are going to massage our choice of these φ_{ij} to get a descent datum (which is necessarily effective as $p : \mathcal{S} \to \mathcal{C}$ is a stack). Namely, the obstruction to being a descent datum is that the cocycle condition may not hold. Namely, set $U_{ijk} = U_i \times_U U_j \times_U U_k$. Then we can consider

$$g_{ijk} = \varphi_{ik}^{-1}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$$

which is an automorphism of x_i over U_{ijk}. Thus we may and do consider g_{ijk} as a section of $\mathcal{G}$ over U_{ijk}. A computation (omitted) shows that $(g_{i_0 i_1 i_2})$ is a 2-cocycle in the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{G})$ of $\mathcal{G}$ with respect to the covering $\mathcal{U}$. By the spectral sequence of Lemma 21.10.6 and since $H^1(U_i, \mathcal{G}) = 0$ for all i we see that $\check{H}^2(\mathcal{U}, \mathcal{G}) \to H^2(U, \mathcal{G})$ is injective. Hence $(g_{i_0 i_1 i_2})$ is a coboundary by our assumption that $H^2(U, \mathcal{G}) = 0$. Thus we can find sections $g_{ij} \in \mathcal{G}(U_{ij})$ such that $g_{ik}^{-1}|_{U_{ijk}} g_{jk}|_{U_{ijk}} g_{ij}|_{U_{ijk}} = g_{ijk}$ for all i, j, k. After replacing φ_{ij} by $\varphi_{ij} g_{ij}^{-1}$ we see that φ_{ij} gives a descent datum on the objects x_i over U_i and the proof is complete. $\square$

21.12. Cohomology of modules

03FA Everything that was said for cohomology of abelian sheaves goes for cohomology of modules, since the two agree.

03FB **Lemma 21.12.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. An injective sheaf of modules is also injective as an object in the category $\mathrm{PMod}(\mathcal{O})$.

Proof. Apply Homology, Lemma 12.29.1 to the categories $\mathcal{A} = \text{Mod}(\mathcal{O})$, $\mathcal{B} = \text{PMod}(\mathcal{O})$, the inclusion functor and sheafification. (See Modules on Sites, Section 18.11 to see that all assumptions of the lemma are satisfied.) □

06YK Lemma 21.12.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the functor $i : \text{Mod}(\mathcal{C}) \to \text{PMod}(\mathcal{C})$. It is a left exact functor with right derived functors given by

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 21.7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $\text{Mod}(\mathcal{O})$. By definition $R^p i$ is the pth cohomology presheaf of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an object U of $\mathcal{C}$ are given by

$$\frac{\text{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\text{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

03FC Lemma 21.12.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective $\mathcal{O}$-module, i.e., an injective object of $\text{Mod}(\mathcal{O})$. Then

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. Lemma 21.9.3 gives the first equality in the following sequence of equalities

$$\begin{aligned} \check{C}^\bullet(\mathcal{U}, \mathcal{I}) &= \text{Mor}_{\text{PAb}(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \text{Mor}_{\text{PMod}(\mathbf{Z})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \text{Mor}_{\text{PMod}(\mathcal{O})}(\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}, \mathcal{I}) \end{aligned}$$

The third equality by Modules on Sites, Lemma 18.9.2. By Lemma 21.12.1 we see that $\mathcal{I}$ is an injective object in $\text{PMod}(\mathcal{O})$. Hence $\text{Hom}_{\text{PMod}(\mathcal{O})}(-, \mathcal{I})$ is an exact functor. By Lemma 21.9.5 we see the vanishing of higher Čech cohomology groups. For the zeroth see Lemma 21.8.2. □

03FD Lemma 21.12.4. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module, and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then we have

$$H^i(\mathcal{C}, \mathcal{F}_{ab}) = H^i(\mathcal{C}, \mathcal{F})$$

and for any object U of $\mathcal{C}$ we also have

$$H^i(U, \mathcal{F}_{ab}) = H^i(U, \mathcal{F}).$$

Here the left hand side is cohomology computed in $\text{Ab}(\mathcal{C})$ and the right hand side is cohomology computed in $\text{Mod}(\mathcal{O})$.

Proof. By Derived Categories, Lemma 13.20.4 the δ-functor $(\mathcal{F} \mapsto H^p(U, \mathcal{F}))_{p \geq 0}$ is universal. The functor $\text{Mod}(\mathcal{O}) \to \text{Ab}(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}_{ab}$ is exact. Hence $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is a δ-functor also. Suppose we show that $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is also universal. This will imply the second statement of the lemma by uniqueness of universal δ-functors, see Homology, Lemma 12.12.5. Since $\text{Mod}(\mathcal{O})$ has enough injectives, it suffices to show that $H^i(U, \mathcal{I}_{ab}) = 0$ for any injective object $\mathcal{I}$ in $\text{Mod}(\mathcal{O})$, see Homology, Lemma 12.12.4.

Let $\mathcal{I}$ be an injective object of $\text{Mod}(\mathcal{O})$. Apply Lemma 21.10.9 with $\mathcal{F} = \mathcal{I}$, $\mathcal{B} = \mathcal{C}$ and $\text{Cov} = \text{Cov}_\mathcal{C}$. Assumption (3) of that lemma holds by Lemma 21.12.3. Hence we see that $H^i(U, \mathcal{I}_{ab}) = 0$ for every object U of $\mathcal{C}$.

If $\mathcal{C}$ has a final object then this also implies the first equality. If not, then according to Sites, Lemma 7.29.5 we see that the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to a ringed topos where the underlying site does have a final object. Hence the lemma follows. $\square$

060L Lemma 21.12.5. Let $\mathcal{C}$ be a site. Let I be a set. For $i \in I$ let $\mathcal{F}_i$ be an abelian sheaf on $\mathcal{C}$. Let $U \in \text{Ob}(\mathcal{C})$. The canonical map

$$H^p(U, \prod\nolimits_{i \in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i \in I} H^p(U, \mathcal{F}_i)$$

is an isomorphism for $p = 0$ and injective for $p = 1$.

Proof. The statement for $p = 0$ is true because the product of sheaves is equal to the product of the underlying presheaves, see Sites, Lemma 7.10.1. Proof for $p = 1$. Set $\mathcal{F} = \prod \mathcal{F}_i$. Let $\xi \in H^1(U, \mathcal{F})$ map to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 21.7.3, there exists a covering $\mathcal{U} = \{U_j \to U\}$ such that $\xi|_{U_j} = 0$ for all j. By Lemma 21.10.4 this means ξ comes from an element $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$. Since the maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i (by Lemma 21.10.4), and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 12.32.1 we conclude that $\check{\xi} = 0$ as desired. $\square$

093X Lemma 21.12.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a : U' \to U$ be a monomorphism in $\mathcal{C}$. Then for any injective $\mathcal{O}$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.

Proof. Let $j : \mathcal{C}/U \to \mathcal{C}$ and $j' : \mathcal{C}/U' \to \mathcal{C}$ be the localization morphisms (Modules on Sites, Section 18.19). Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}$-modules

$$\text{Hom}_\mathcal{O}(j_!\mathcal{O}_U, \mathcal{F}) = \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover below we describe a canonical map of $\mathcal{O}$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 4.3.5). It suffices to prove the displayed map of modules is injective, see Homology, Lemma 12.27.2.

To construct our map it suffices to construct a map between the presheaves which assign to an object V of $\mathcal{C}$ the $\mathcal{O}(V)$-module

$$\bigoplus\nolimits_{\varphi' \in \text{Mor}_\mathcal{C}(V, U')} \mathcal{O}(V) \quad \text{and} \quad \bigoplus\nolimits_{\varphi \in \text{Mor}_\mathcal{C}(V, U)} \mathcal{O}(V)$$

see Modules on Sites, Lemma 18.19.2. We take the map which maps the summand corresponding to φ' to the summand corresponding to $\varphi = a \circ \varphi'$ by the identity map on $\mathcal{O}(V)$. As a is a monomorphism, this map is injective. As sheafification is exact, the result follows. $\square$

21.13. Totally acyclic sheaves

079X Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a presheaf of sets on $\mathcal{C}$ (we intentionally use a roman capital here to distinguish from abelian sheaves). Given a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we set

$$\mathcal{F}(K) = \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(K, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(K^\#, \mathcal{F})$$

The functor $\mathcal{F} \mapsto \mathcal{F}(K)$ is a left exact functor $\mathrm{Mod}(\mathcal{O}) \to \mathrm{Ab}$ hence we have its right derived functors. We will denote these $H^p(K, \mathcal{F})$ so that $H^0(K, \mathcal{F}) = \mathcal{F}(K)$.

Here are some observations:

(1) Since $\mathcal{F}(K) = \mathcal{F}(K^\#)$, we have $H^p(K, \mathcal{F}) = H^p(K^\#, \mathcal{F})$. Allowing K to be a presheaf in the definition above is a purely notational convenience.
(2) Suppose that $K = h_U$ or $K = h_U^\#$ for some object U of $\mathcal{C}$. Then $H^p(K, \mathcal{F}) = H^p(U, \mathcal{F})$, because $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$, see Sites, Section 7.12.
(3) If $\mathcal{O} = \mathbf{Z}$ (the constant sheaf), then the cohomology groups are functors $H^p(K, -) : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}$ since $\mathrm{Mod}(\mathcal{O}) = \mathrm{Ab}(\mathcal{C})$ in this case.

We can translate some of our already proven results using this language.

079Y Lemma 21.13.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a presheaf of sets on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then $H^p(K, \mathcal{F}) = H^p(K, \mathcal{F}_{ab})$.

Proof. We may replace K by its sheafification and assume K is a sheaf. Note that both $H^p(K, \mathcal{F})$ and $H^p(K, \mathcal{F}_{ab})$ depend only on the topos, not on the underlying site. Hence by Sites, Lemma 7.29.5 we may replace $\mathcal{C}$ by a "larger" site such that $K = h_U$ for some object U of $\mathcal{C}$. In this case the result follows from Lemma 21.12.4. □

079Z Lemma 21.13.2. Let $\mathcal{C}$ be a site. Let $K' \to K$ be a map of presheaves of sets on $\mathcal{C}$ whose sheafification is surjective. Set $K'_p = K' \times_K \ldots \times_K K'$ ($p + 1$-factors). For every abelian sheaf $\mathcal{F}$ there is a spectral sequence with $E_1^{p,q} = H^q(K'_p, \mathcal{F})$ converging to $H^{p+q}(K, \mathcal{F})$.

Proof. Since sheafification is exact, we see that $(K'_p)^\#$ is equal to $(K')^\# \times_{K^\#} \ldots \times_{K^\#} (K')^\#$ ($p+1$-factors). Thus we may replace K and K' by their sheafifications and assume $K \to K'$ is a surjective map of sheaves. After replacing $\mathcal{C}$ by a "larger" site as in Sites, Lemma 7.29.5 we may assume that K, K' are objects of $\mathcal{C}$ and that $\mathcal{U} = \{K' \to K\}$ is a covering. Then we have the Čech to cohomology spectral sequence of Lemma 21.10.6 whose E_1 page is as indicated in the statement of the lemma. □

07A0 Lemma 21.13.3. Let $\mathcal{C}$ be a site. Let K be a sheaf of sets on $\mathcal{C}$. Consider the morphism of topoi $j : Sh(\mathcal{C}/K) \to Sh(\mathcal{C})$, see Sites, Lemma 7.30.3. Then j^{-1} preserves injectives and $H^p(K, \mathcal{F}) = H^p(\mathcal{C}/K, j^{-1}\mathcal{F})$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$.

Proof. By Sites, Lemmas 7.30.1 and 7.30.3 the morphism of topoi j is equivalent to a localization. Hence this follows from Lemma 21.7.1. □

Keeping in mind Lemma 21.13.1 we see that the following definition is the "correct one" also for sheaves of modules on ringed sites.

072Y Definition 21.13.4. Let $\mathcal{C}$ be a site. We say an abelian sheaf $\mathcal{F}$ is totally acyclic[1] if for every sheaf of sets K we have $H^p(K, \mathcal{F}) = 0$ for all $p \geq 1$.

It is clear that being totally acyclic is an intrinsic property, i.e., preserved under equivalences of topoi. A totally acyclic sheaf has vanishing higher cohomology on all objects of the site, but in general the condition of being totally acyclic is strictly stronger. Here is a characterization of totally acyclic sheaves which is sometimes useful.

07A1 Lemma 21.13.5. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian sheaf. If

(1) $H^p(U, \mathcal{F}) = 0$ for $p > 0$ and $U \in \text{Ob}(\mathcal{C})$, and
(2) for every surjection $K' \to K$ of sheaves of sets the extended Čech complex
$$0 \to H^0(K, \mathcal{F}) \to H^0(K', \mathcal{F}) \to H^0(K' \times_K K', \mathcal{F}) \to \ldots$$
is exact,

then $\mathcal{F}$ is totally acyclic (and the converse holds too).

Proof. By assumption (1) we have $H^p(h_U^\#, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and all objects U of $\mathcal{C}$. Note that if $K = \coprod K_i$ is a coproduct of sheaves of sets on $\mathcal{C}$ then $H^p(K, g^{-1}\mathcal{I}) = \prod H^p(K_i, g^{-1}\mathcal{I})$. For any sheaf of sets K there exists a surjection
$$K' = \coprod h_{U_i}^\# \longrightarrow K$$
see Sites, Lemma 7.12.5. Thus we conclude that: (*) for every sheaf of sets K there exists a surjection $K' \to K$ of sheaves of sets such that $H^p(K', \mathcal{F}) = 0$ for $p > 0$. We claim that (*) and condition (2) imply that $\mathcal{F}$ is totally acyclic. Note that conditions (*) and (2) only depend on $\mathcal{F}$ as an object of the topos $Sh(\mathcal{C})$ and not on the underlying site. (We will not use property (1) in the rest of the proof.)

We are going to prove by induction on $n \geq 0$ that (*) and (2) imply the following induction hypothesis IH_n: $H^p(K, \mathcal{F}) = 0$ for all $0 < p \leq n$ and all sheaves of sets K. Note that IH_0 holds. Assume IH_n. Pick a sheaf of sets K. Pick a surjection $K' \to K$ such that $H^p(K', \mathcal{F}) = 0$ for all $p > 0$. We have a spectral sequence with
$$E_1^{p,q} = H^q(K'_p, \mathcal{F})$$
covering to $H^{p+q}(K, \mathcal{F})$, see Lemma 21.13.2. By IH_n we see that $E_1^{p,q} = 0$ for $0 < q \leq n$ and by assumption (2) we see that $E_2^{p,0} = 0$ for $p > 0$. Finally, we have $E_1^{0,q} = 0$ for $q > 0$ because $H^q(K', \mathcal{F}) = 0$ by choice of K'. Hence we conclude that $H^{n+1}(K, \mathcal{F}) = 0$ because all the terms $E_2^{p,q}$ with $p + q = n + 1$ are zero. □

21.14. The Leray spectral sequence

072X The key to proving the existence of the Leray spectral sequence is the following lemma.

072Z Lemma 21.14.1. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Then for any injective object $\mathcal{I}$ in $\text{Mod}(\mathcal{O}_\mathcal{C})$ the pushforward $f_*\mathcal{I}$ is totally acyclic.

[1]Although this terminology is is used in [AGV71, Vbis, Proposition 1.3.10] this is probably nonstandard notation. In [AGV71, V, Definition 4.1] this property is dubbed “flasque”, but we cannot use this because it would clash with our definition of flasque sheaves on topological spaces. Please email stacks.project@gmail.com if you have a better suggestion.

Proof. Let K be a sheaf of sets on $\mathcal{D}$. By Modules on Sites, Lemma 18.7.2 we may replace $\mathcal{C}$, $\mathcal{D}$ by "larger" sites such that f comes from a morphism of ringed sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that $K = h_V$ for some object V of $\mathcal{D}$.

Thus we have to show that $H^q(V, f_*\mathcal{I})$ is zero for $q > 0$ and all objects V of $\mathcal{D}$ when f is given by a morphism of ringed sites. Let $\mathcal{V} = \{V_j \to V\}$ be any covering of $\mathcal{D}$. Since u is continuous we see that $\mathcal{U} = \{u(V_j) \to u(V)\}$ is a covering of $\mathcal{C}$. Then we have an equality of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I})$$

by the definition of f_*. By Lemma 21.12.3 we see that the cohomology of this complex is zero in positive degrees. We win by Lemma 21.10.9. $\square$

For flat morphisms the functor f_* preserves injective modules. In particular the functor $f_* : \text{Ab}(\mathcal{C}) \to \text{Ab}(\mathcal{D})$ always transforms injective abelian sheaves into injective abelian sheaves.

0730 **Lemma 21.14.2.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If f is flat, then $f_*\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module for any injective $\mathcal{O}_\mathcal{C}$-module $\mathcal{I}$.

Proof. In this case the functor f^* is exact, see Modules on Sites, Lemma 18.31.2. Hence the result follows from Homology, Lemma 12.29.1. $\square$

0731 **Lemma 21.14.3.** Let $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be a ringed topos. A totally acyclic sheaf is right acyclic for the following functors:

(1) the functor $H^0(U, -)$ for any object U of $\mathcal{C}$,
(2) the functor $\mathcal{F} \mapsto \mathcal{F}(K)$ for any presheaf of sets K,
(3) the functor $\Gamma(\mathcal{C}, -)$ of global sections,
(4) the functor f_* for any morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi.

Proof. Part (2) is the definition of a totally acyclic sheaf. Part (1) is a consequence of (2) as pointed out in the discussion following the definition of totally acyclic sheaves. Part (3) is a special case of (2) where $K = e$ is the final object of $Sh(\mathcal{C})$.

To prove (4) we may assume, by Modules on Sites, Lemma 18.7.2 that f is given by a morphism of sites. In this case we see that $R^i f_*$, $i > 0$ of a totally acyclic sheaf are zero by the description of higher direct images in Lemma 21.7.4. $\square$

08J6 **Remark 21.14.4.** As a consequence of the results above we find that Derived Categories, Lemma 13.22.1 applies to a number of situations. For example, given a morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi we have

$$R\Gamma(\mathcal{D}, Rf_*\mathcal{F}) = R\Gamma(\mathcal{C}, \mathcal{F})$$

for any sheaf of $\mathcal{O}_\mathcal{C}$-modules $\mathcal{F}$. Namely, for an injective $\mathcal{O}_\mathcal{X}$-module $\mathcal{I}$ the $\mathcal{O}_\mathcal{D}$-module $f_*\mathcal{I}$ is totally acyclic by Lemma 21.14.1 and a totally acyclic sheaf is acyclic for $\Gamma(\mathcal{D}, -)$ by Lemma 21.14.3.

0732 **Lemma 21.14.5** (Leray spectral sequence). Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. There is a spectral sequence

$$E_2^{p,q} = H^p(\mathcal{D}, R^q f_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{C}, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 13.22.2 coming from the composition of functors $\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{D}, -) \circ f_*$. To see that the assumptions of Derived Categories, Lemma 13.22.2 are satisfied, see Lemmas 21.14.1 and 21.14.3. $\square$

0733 Lemma 21.14.6. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module.

(1) If $R^qf_*\mathcal{F} = 0$ for $q > 0$, then $H^p(\mathcal{C}, \mathcal{F}) = H^p(\mathcal{D}, f_*\mathcal{F})$ for all p.
(2) If $H^p(\mathcal{D}, R^qf_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(\mathcal{C}, \mathcal{F}) = H^0(\mathcal{D}, R^qf_*\mathcal{F})$ for all q.

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. $\square$

0734 Lemma 21.14.7 (Relative Leray spectral sequence). Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$ be morphisms of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module. There is a spectral sequence with

$$E_2^{p,q} = R^pg_*(R^qf_*\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_*\mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_\mathcal{C}$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors, see Derived Categories, Lemma 13.22.2 and Lemmas 21.14.1 and 21.14.3. $\square$

21.15. The base change map

0735 In this section we construct the base change map in some cases; the general case is treated in Remark 21.19.3. The discussion in this section avoids using derived pullback by restricting to the case of a base change by a flat morphism of ringed sites. Before we state the result, let us discuss flat pullback on the derived category. Suppose $g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is a flat morphism of ringed topoi. By Modules on Sites, Lemma 18.31.2 the functor $g^* : \textit{Mod}(\mathcal{O}_\mathcal{D}) \to \textit{Mod}(\mathcal{O}_\mathcal{C})$ is exact. Hence it has a derived functor

$$g^* : D(\mathcal{O}_\mathcal{D}) \to D(\mathcal{O}_\mathcal{C})$$

which is computed by simply pulling back an representative of a given object in $D(\mathcal{O}_\mathcal{D})$, see Derived Categories, Lemma 13.16.9. It preserved the bounded (above, below) subcategories. Hence as indicated we indicate this functor by g^* rather than Lg^*.

0736 Lemma 21.15.1. Let

$$\begin{tikzcd}
(Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "f"] \\
(Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})
\end{tikzcd}$$

be a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*(g')^*\mathcal{F}^\bullet$$

in $D^+(\mathcal{O}_{\mathcal{D}'})$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 21.14.2 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 13.18.6 and 13.18.7 the arrow β in the diagram

$$\begin{tikzcd}
(g')_*(g')^*\mathcal{F}^\bullet \arrow[r] & (g')_*\mathcal{J}^\bullet \\
\mathcal{F}^\bullet \arrow[u, "adjunction"] \arrow[r] & \mathcal{I}^\bullet \arrow[u, "\beta"']
\end{tikzcd}$$

exists and is unique up to homotopy. Pushing down to $\mathcal{D}$ we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. $\square$

21.16. Cohomology and colimits

0737 Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \to \text{Mod}(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram over the index category $\mathcal{I}$, see Categories, Section 4.14. For each i there is a canonical map $\mathcal{F}_i \to \text{colim}_i \mathcal{F}_i$ which induces a map on cohomology. Hence we get a canonical map

$$\text{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \text{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$ and every object U of $\mathcal{C}$. These maps are in general not isomorphisms, even for $p = 0$.

The following lemma is the analogue of Sites, Lemma 7.17.7 for cohomology.

0739 Lemma 21.16.1. Let $\mathcal{C}$ be a site. Let $\text{Cov}_\mathcal{C}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 7.6.2). Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$, and $\text{Cov} \subset \text{Cov}_\mathcal{C}$ be subsets. Assume that

(1) For every $\mathcal{U} \in \text{Cov}$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.
(2) For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.

Then the map

$$\text{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \text{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $p \geq 0$, every $U \in \mathcal{B}$, and every filtered diagram $\mathcal{I} \to \text{Ab}(\mathcal{C})$.

Proof. To prove the lemma we will argue by induction on p. Note that we require in (1) the coverings $\mathcal{U} \in \text{Cov}$ to be finite, so that all the elements of $\mathcal{B}$ are quasi-compact. Hence (2) and (1) imply that any $U \in \mathcal{B}$ satisfies the hypothesis of Sites, Lemma 7.17.7 (4). Thus we see that the result holds for $p = 0$. Now we assume the lemma holds for p and prove it for $p + 1$.

Choose a filtered diagram $\mathcal{F} : \mathcal{I} \to \text{Ab}(\mathcal{C})$, $i \mapsto \mathcal{F}_i$. Since $\text{Ab}(\mathcal{C})$ has functorial injective embeddings, see Injectives, Theorem 19.7.4, we can find a morphism of filtered diagrams $\mathcal{F} \to \mathcal{I}$ such that each $\mathcal{F}_i \to \mathcal{I}_i$ is an injective map of abelian sheaves into an injective abelian sheaf. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

Since colimits of sheaves are the sheafification of colimits on the level of presheaves, since sheafification is exact, and since filtered colimits of abelian groups are exact (see Algebra, Lemma 10.8.8), we see the sequence

$$0 \to \operatorname{colim}_i \mathcal{F}_i \to \operatorname{colim}_i \mathcal{I}_i \to \operatorname{colim}_i \mathcal{Q}_i \to 0.$$

is also a short exact sequence. We claim that $H^q(U, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all $U \in \mathcal{B}$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{tikzcd} \operatorname{colim}_i H^p(U, \mathcal{I}_i) \arrow[r] \arrow[d] & \operatorname{colim}_i H^p(U, \mathcal{Q}_i) \arrow[r] \arrow[d] & \operatorname{colim}_i H^{p+1}(U, \mathcal{F}_i) \arrow[r] \arrow[d] & 0 \arrow[d] \\ H^p(U, \operatorname{colim}_i \mathcal{I}_i) \arrow[r] & H^p(U, \operatorname{colim}_i \mathcal{Q}_i) \arrow[r] & H^{p+1}(U, \operatorname{colim}_i \mathcal{F}_i) \arrow[r] & 0 \end{tikzcd}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 10.8.8. Hence by the snake lemma we deduce the result for $p + 1$.

It remains to show that the claim is true. We will use Lemma 21.10.9. By the result for $p = 0$ we see that for $\mathcal{U} \in \text{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i) = \operatorname{colim}_i \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the $U_{j_0} \times_U \ldots \times_U U_{j_p}$ are in $\mathcal{B}$. By Lemma 21.10.2 each of the complexes in the colimit of Čech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 10.8.8 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 21.10.9. are satisfied and the claim follows. □

0GN3 **Lemma 21.16.2.** Let $\mathcal{C}$ be a site. Let $S \subset \text{Ob}(Sh(\mathcal{C}))$ be a subset. Denote $*$ the final object of $Sh(\mathcal{C})$. Assume

(1) for some $K \in S$ the map $K \to *$ is surjective,
(2) given a surjective map of sheaves $\mathcal{F} \to K$ with $K \in S$ there exists a $K' \in S$ and a map $K' \to \mathcal{F}$ such that the composition $K' \to K$ is surjective,
(3) given $K, K' \in S$ there is a surjection $K'' \to K \times K'$ with $K'' \in S$,
(4) given $a, b : K \to K'$ with $K, K' \in S$ there exists a surjection $K'' \to \text{Equalizer}(a, b)$ with $K'' \in S$, and
(5) every $K \in S$ is quasi-compact (Sites, Definition 7.17.4).

Then for all $p \geq 0$ the map

$$\operatorname{colim}_\lambda H^p(\mathcal{C}, \mathcal{F}_\lambda) \longrightarrow H^p(\mathcal{C}, \operatorname{colim}_\lambda \mathcal{F}_\lambda)$$

is an isomorphism for every filtered diagram $\Lambda \to \text{Ab}(\mathcal{C})$, $\lambda \mapsto \mathcal{F}_\lambda$.

Proof. We will prove this by induction on p. The base case $p = 0$ follows from Sites, Lemma 7.17.8 part (4). We check the assumptions hold, but we urge the reader to skip this part. Suppose $\mathcal{F} \to *$ is surjective. Choose $K \in S$ and $K \to *$ surjective as in (1). Then $\mathcal{F} \times K \to K$ is surjective. Choose $K' \to \mathcal{F} \times K$ with $K' \in S$ and $K' \to K$ surjective as in (2). Then there is a map $K' \to \mathcal{F}$ and $K' \to *$

is surjective. Hence Sites, Lemma 7.17.8 assumption (4)(a) is satisfied. By Sites, Lemma 7.17.5, assumptions (3) and (5) we see that $K \times K$ is quasi-compact for all $K \in S$. Hence Sites, Lemma 7.17.8 assumption (4)(b) is satisfied. This finishes the proof of the base case.

Induction step. Assume the result holds for H^p for $p \leq p_0$ and for all topoi $Sh(\mathcal{C})$ such that a set $S \subset \text{Ob}(Sh(\mathcal{C}))$ can be found satisfying (1) – (5). Arguing exactly as in the proof of Lemma 21.16.1 we see that it suffices to show: given a filtered colimit $\mathcal{I} = \text{colim}\, \mathcal{I}_\lambda$ with $\mathcal{I}_\lambda$ injective abelian sheaves, we have $H^{p_0+1}(\mathcal{C}, \mathcal{I}) = 0$. Choose $K \to *$ surjective with $K \in S$ as in (1). Denote K^n the n-fold self product of K. Consider the spectral sequence

$$E_1^{p,q} = H^q(K^{p+1}, \mathcal{I}) \Rightarrow H^{p+q}(*, \mathcal{I}) = H^{p+q}(\mathcal{C}, \mathcal{I})$$

of Lemma 21.13.2. Recall that $H^q(K^{p+1}, \mathcal{F}) = H^q(\mathcal{C}/K^{p+1}, j^{-1}\mathcal{F})$, for any abelian sheaf on $\mathcal{C}$, see Lemma 21.13.3. We have $j^{-1}\mathcal{I} = \text{colim}\, j^{-1}\mathcal{I}_\lambda$ as j^{-1} commutes with colimits. The restrictions $j^{-1}\mathcal{I}_\lambda$ are injective abelian sheaves on $\mathcal{C}/K^{p+1}$ by Lemma 21.7.1. Below we will show that the induction hypothesis applies to $\mathcal{C}/K^{p+1}$ and hence we see that $H^q(K^{p+1}, \mathcal{I}) = \text{colim}\, H^q(K^{p+1}, \mathcal{I}_\lambda) = 0$ for $q < p_0 + 1$ (vanishing as $\mathcal{I}_\lambda$ is injective). It follows that

$$H^{p_0+1}(\mathcal{C}, \mathcal{I}) = H^{p_0+1}\left(\ldots \to H^0(K^{p_0}, \mathcal{I}) \to H^0(K^{p_0+1}, \mathcal{I}) \to H^0(K^{p_0+2}, \mathcal{I}) \to \ldots\right)$$

Again using the induction hypothesis, the complex depicted on the right hand side is the colimit over Λ of the complexes

$$\ldots \to H^0(K^{p_0}, \mathcal{I}_\lambda) \to H^0(K^{p_0+1}, \mathcal{I}_\lambda) \to H^0(K^{p_0+2}, \mathcal{I}_\lambda) \to \ldots$$

These complexes are exact as $\mathcal{I}_\lambda$ is an injective abelian sheaf (follows from the spectral sequence for example). Since filtered colimits are exact in the category of abelian groups we obtain the desired vanishing.

We still have to show that the induction hypothesis applies to the site $\mathcal{C}/K^n$ for all $n \geq 1$. Recall that $Sh(\mathcal{C}/K^n) = Sh(\mathcal{C})/K^n$, see Sites, Lemma 7.30.3. Thus we may work in $Sh(\mathcal{C})/K^n$. Denote $S_n \subset \text{Ob}(Sh(\mathcal{C}/K^n)$ the set of objects of the form $K' \to K^n$. We check each property in turn:

(1) By (3) and induction there exists a surjection $K' \to K^n$ with $K' \in S$. Then $(K' \to K^n) \to (K^n \to K^n)$ is a surjection in $Sh(\mathcal{C})/K^n$ and $K^n \to K^n$ is the final object of $Sh(\mathcal{C})/K^n$. Hence (1) holds for S_n,
(2) Property (2) for S_n is an immediate consquence of (2) for S.
(3) Let $a : K_1 \to K^n$ and $b : K_2 \to K^n$ be in S_n. Then $(K_1 \to K^n) \times (K_2 \to K^n)$ is the object $K_1 \times_{K^n} K_2 \to K^n$ of $Sh(\mathcal{C})/K^n$. The subsheaf $K_1 \times_{K^n} K_2 \subset K_1 \times K_2$ is the equalizer of $a \circ \text{pr}_1$ and $b \circ \text{pr}_2$. Write $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Pick $K_3 \to K_1 \times K_2$ surjective with $K_3 \in S$; this is possibly by assumption (3) for $\mathcal{C}$. Pick

$$K_4 \longrightarrow \text{Equalizer}(K_3 \to K_1 \times K_2 \xrightarrow{a_1, b_1} K)$$

surjective with $K_4 \in S$. This is possible by assumption (4) for $\mathcal{C}$. Pick

$$K_5 \longrightarrow \text{Equalizer}(K_4 \to K_1 \times K_2 \xrightarrow{a_2, b_2} K)$$

surjective with $K_5 \in S$. Again this is possible. Continue in this fashion until we get

$$K_{3+n} \longrightarrow \text{Equalizer}(K_{2+n} \to K_1 \times K_2 \xrightarrow{a_n, b_n} K)$$

surjective with $K_{3+n} \in S$. By construction $K_{3+n} \to K_1 \times_{K^n} K_2$ is surjective. Hence $(K_{3+n} \to K^n)$ is in S_n and surjects onto the product $(K_1 \to K^n) \times (K_2 \to K^n)$. Thus (3) holds for S_n.
(4) Property (4) for S_n is an immediate consequence of property (4) for S.
(5) Property (5) for S_n is a consequence of property (5) for S. Namely, an object $\mathcal{F} \to K^n$ of $Sh(\mathcal{C})/K^n$ corresponds to a quasi-compact object of $Sh(\mathcal{C}/K^n)$ if and only if $\mathcal{F}$ is a quasi-compact object of $Sh(\mathcal{C})$.

This finishes the proof of the lemma. □

0GN4 Remark 21.16.3. Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $S \subset \mathrm{Ob}(Sh(\mathcal{C}))$ be the set of sheaves K which have the form

$$K = \coprod\nolimits_{i=1,\ldots,n} h_{U_i}^\#$$

with $U_1, \ldots, U_n \in \mathcal{B}$. Then we can ask: when does this set satisfy the assumptions of Lemma 21.16.2? One answer is that it suffices if

(1) for some $n \geq 0$, $U_1, \ldots, U_n \in \mathcal{B}$ the map $\coprod_{i=1,\ldots,n} h_{U_i}^\# \to *$ is surjective,
(2) every covering of $U \in \mathcal{B}$ can be refined by a covering of the form $\{U_i \to U\}_{i=1,\ldots,n}$ with $U_i \in \mathcal{B}$,
(3) given $U, U' \in \mathcal{B}$ there exist $n \geq 0$, $U_1, \ldots, U_n \in \mathcal{B}$, maps $U_i \to U$ and $U_i \to U'$ such that $\coprod_{i=1,\ldots,n} h_{U_i}^\# \to h_U^\# \times h_{U'}^\#$ is surjective,
(4) given morphisms $a, b : U \to U'$ in $\mathcal{C}$ with $U, U' \in \mathcal{B}$, there exist $U_1, \ldots, U_n \in \mathcal{B}$, maps $U_i \to U$ equalizing a, b such that $\coprod_{i=1,\ldots,n} h_{U_i}^\# \to \text{Equalizer}(h_a^\#, h_b^\# : h_U^\# \to h_{U'}^\#)$ is surjective.

We omit the detailed verification, except to mention that part (2) above insures that every element of $\mathcal{B}$ is quasi-compact and hence every $K \in S$ is quasi-compact as well by Sites, Lemma 7.17.6.

0EXZ Lemma 21.16.4. Let $\mathcal{I}$ be a cofiltered index category and let $(\mathcal{C}_i, f_a)$ be an inverse system of sites over $\mathcal{I}$ as in Sites, Situation 7.18.1. Set $\mathcal{C} = \mathrm{colim}\, \mathcal{C}_i$ as in Sites, Lemmas 7.18.2 and 7.18.3. Moreover, assume given

(1) an abelian sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$,
(2) for $a : j \to i$ a map $\varphi_a : f_a^{-1}\mathcal{F}_i \to \mathcal{F}_j$ of abelian sheaves on $\mathcal{C}_j$

such that $\varphi_c = \varphi_b \circ f_b^{-1}\varphi_a$ whenever $c = a \circ b$. Then there exists a map of systems $(\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ such that $\mathcal{F}_i \to \mathcal{G}_i$ is injective and $\mathcal{G}_i$ is an injective abelian sheaf.

Proof. For each i we pick an injection $\mathcal{F}_i \to \mathcal{A}_i$ where $\mathcal{A}_i$ is an injective abelian sheaf on $\mathcal{C}_i$. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod\nolimits_{b:k \to i} f_{b,*}\mathcal{A}_k = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_b^{-1}\mathcal{F}_i \to \mathcal{F}_k \to \mathcal{A}_k$. For $a : j \to i$ in $\mathcal{I}$ there is a canonical map

$$\psi_a : f_a^{-1}\mathcal{G}_i \to \mathcal{G}_j$$

whose components are the canonical maps $f_b^{-1} f_{a \circ b,*}\mathcal{A}_k \to f_{b,*}\mathcal{A}_k$ for $b : k \to j$. Thus we find an injection $(\gamma_i) : (\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ of systems of abelian sheaves. Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on $\mathcal{C}_i$, see Lemma 21.14.2 and Homology, Lemma 12.27.3. This finishes the construction. □

09YP Lemma 21.16.5. In the situation of Lemma 21.16.4 set $\mathcal{F} = \text{colim}\, f_i^{-1}\mathcal{F}_i$. Let $i \in \text{Ob}(\mathcal{I})$, $X_i \in \text{Ob}(\mathcal{C}_i)$. Then

$$\text{colim}_{a:j\to i}\, H^p(u_a(X_i), \mathcal{F}_j) = H^p(u_i(X_i), \mathcal{F})$$

for all $p \geq 0$.

Proof. The case $p = 0$ is Sites, Lemma 7.18.4.

Choose $(\mathcal{F}_i, \varphi_a) \to (\mathcal{G}_i, \psi_a)$ as in Lemma 21.16.4. Arguing exactly as in the proof of Lemma 21.16.1 we see that it suffices to prove that $H^p(X, \text{colim}\, f_i^{-1}\mathcal{G}_i) = 0$ for $p > 0$.

Set $\mathcal{G} = \text{colim}\, f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every object of $\mathcal{C}$ we show that the Čech cohomology of $\mathcal{G}$ for any covering $\mathcal{U}$ of $\mathcal{C}$ is zero (Lemma 21.10.9). The covering $\mathcal{U}$ comes from a covering $\mathcal{U}_i$ of $\mathcal{C}_i$ for some i. We have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) = \text{colim}_{a:j\to i}\, \check{\mathcal{C}}^\bullet(u_a(\mathcal{U}_i), \mathcal{G}_j)$$

by the case $p = 0$. The right hand side is acyclic in positive degrees as a filtered colimit of acyclic complexes by Lemma 21.10.2. See Algebra, Lemma 10.8.8. ☐

21.17. Flat resolutions

06YL In this section we redo the arguments of Cohomology, Section 20.26 in the setting of ringed sites and ringed topoi.

06YM Lemma 21.17.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}$-modules. The functors

$$K(\text{Mod}(\mathcal{O})) \longrightarrow K(\text{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \text{Tot}(\mathcal{G}^\bullet \otimes_\mathcal{O} \mathcal{F}^\bullet)$$

and

$$K(\text{Mod}(\mathcal{O})) \longrightarrow K(\text{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{G}^\bullet)$$

are exact functors of triangulated categories.

Proof. This follows from Derived Categories, Remark 13.10.9. ☐

06YN Definition 21.17.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}$-modules is called K-flat if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the complex

$$\text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

is acyclic.

06YP Lemma 21.17.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor

$$K(\text{Mod}(\mathcal{O})) \longrightarrow K(\text{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 21.17.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. ☐

0E8K Lemma 21.17.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. If $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}$-modules, then $\mathcal{K}^\bullet|_U$ is a K-flat complex of $\mathcal{O}_U$-modules.

Proof. Let $\mathcal{G}^\bullet$ be an exact complex of $\mathcal{O}_U$-modules. Since $j_{U!}$ is exact (Modules on Sites, Lemma 18.19.3) and $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}$-modules we see that the complex

$$j_{U!}(\text{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_U} \mathcal{K}^\bullet|_U)) = \text{Tot}(j_{U!}\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

is exact. Here the equality comes from Modules on Sites, Lemma 18.27.9 and the fact that $j_{U!}$ commutes with direct sums (as a left adjoint). We conclude because $j_{U!}$ reflects exactness by Modules on Sites, Lemma 18.19.4. □

07A2 **Lemma 21.17.5.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules, then $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}$-modules.

Proof. Follows from the isomorphism

$$\text{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) = \text{Tot}(\text{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$$

and the definition. □

07A3 **Lemma 21.17.6.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(\text{Mod}(\mathcal{O}))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.

Proof. Follows from Lemma 21.17.1 and the fact that in a distinguished triangle in $K(\text{Mod}(\mathcal{O}))$ if two out of three are acyclic, so is the third. □

0G7B **Lemma 21.17.7.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $0 \to \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \mathcal{K}_3^\bullet \to 0$ be a short exact sequence of complexes such that the terms of $\mathcal{K}_3^\bullet$ are flat $\mathcal{O}$-modules. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.

Proof. By Modules on Sites, Lemma 18.28.9 for every complex $\mathcal{L}^\bullet$ we obtain a short exact sequence

$$0 \to \text{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_1^\bullet) \to \text{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_1^\bullet) \to \text{Tot}(\mathcal{L}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_1^\bullet) \to 0$$

of complexes. Hence the lemma follows from the long exact sequence of cohomology sheaves and the definition of K-flat complexes. □

06YQ **Lemma 21.17.8.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A bounded above complex of flat $\mathcal{O}$-modules is K-flat.

Proof. Let $\mathcal{K}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be an acyclic complex of $\mathcal{O}$-modules. Note that $\mathcal{L}^\bullet = \text{colim}_m \tau_{\leq m}\mathcal{L}^\bullet$ where we take termwise colimits. Hence also

$$\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) = \text{colim}_m \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \tau_{\leq m}\mathcal{L}^\bullet)$$

termwise. Hence to prove the complex on the left is acyclic it suffices to show each of the complexes on the right is acyclic. Since $\tau_{\leq m}\mathcal{L}^\bullet$ is acyclic this reduces us to the case where $\mathcal{L}^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 12.25.3 has

$${}'E_1^{p,q} = H^p(\mathcal{L}^\bullet \otimes_R \mathcal{K}^q)$$

which is zero as $\mathcal{K}^q$ is flat and $\mathcal{L}^\bullet$ acyclic. Hence we win. □

06YR **Lemma 21.17.9.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\text{colim}_i \mathcal{K}_i^\bullet$ is K-flat.

Proof. Because we are taking termwise colimits it is clear that

$$\mathrm{colim}_i \mathrm{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{K}_i^\bullet) = \mathrm{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathrm{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. $\square$

077J Lemma 21.17.10. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a commutative diagram of complexes of $\mathcal{O}$-modules

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{G}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{G}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms and termwise surjective, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\mathrm{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules on Sites, Lemma 18.28.8 and Derived Categories, Lemma 13.29.1. The induced map $\mathrm{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. $\square$

06YS Lemma 21.17.11. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ whose terms are flat $\mathcal{O}$-modules and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ which is termwise surjective.

Proof. Choose a diagram as in Lemma 21.17.10. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 18.28.7. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 21.17.8. Thus $\mathrm{colim}\, \mathcal{K}_n^\bullet$ is K-flat by Lemma 21.17.9. The induced map $\mathrm{colim}\, \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism and termwise surjective by construction. Property (3) of Lemma 21.17.10 shows that $\mathrm{colim}\, \mathcal{K}_n^m$ is a direct sum of flat modules and hence flat which proves the final assertion. $\square$

06YT Lemma 21.17.12. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the induced map

$$\mathrm{Tot}(\mathrm{id}_{\mathcal{F}^\bullet} \otimes \alpha) : \mathrm{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \longrightarrow \mathrm{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 21.17.11. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \arrow[r] \arrow[d] & \mathrm{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet) \arrow[d] \\ \mathrm{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{P}^\bullet) \arrow[r] & \mathrm{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{Q}^\bullet) \end{tikzcd}$$

The result follows as by Lemma 21.17.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 21.17.11. By Lemma 21.17.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}) \longrightarrow K(\mathcal{O}), \quad \mathcal{G}^\bullet \longmapsto \mathrm{Tot}(\mathcal{G}^\bullet \otimes_\mathcal{O} \mathcal{K}^\bullet)$$

By Lemma 21.17.3 this functor induces a functor $D(\mathcal{O}) \to D(\mathcal{O})$ simply because $D(\mathcal{O})$ is the localization of $K(\mathcal{O})$ at quasi-isomorphisms. By Lemma 21.17.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

06YU Definition 21.17.13. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. The derived tensor product

$$- \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}^\bullet : D(\mathcal{O}) \longrightarrow D(\mathcal{O})$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes_\mathcal{O}^\mathbf{L} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O})$. Hence when we write $\mathcal{F}^\bullet \otimes_\mathcal{O}^\mathbf{L} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

08FF Definition 21.17.14. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. The Tor's of $\mathcal{F}$ and $\mathcal{G}$ are defined by the formula

$$\mathrm{Tor}_p^\mathcal{O}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes_\mathcal{O}^\mathbf{L} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{tikzcd}
\mathcal{F}_1 \otimes_\mathcal{O} \mathcal{G} \arrow[r] & \mathcal{F}_2 \otimes_\mathcal{O} \mathcal{G} \arrow[r] & \mathcal{F}_3 \otimes_\mathcal{O} \mathcal{G} \arrow[r] & 0 \\
\mathrm{Tor}_1^\mathcal{O}(\mathcal{F}_1, \mathcal{G}) \arrow[r] & \mathrm{Tor}_1^\mathcal{O}(\mathcal{F}_2, \mathcal{G}) \arrow[r] & \mathrm{Tor}_1^\mathcal{O}(\mathcal{F}_3, \mathcal{G}) \arrow[llu]
\end{tikzcd}$$

for every $\mathcal{O}$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

08FG Lemma 21.17.15. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module. The following are equivalent

(1) $\mathcal{F}$ is a flat $\mathcal{O}$-module, and
(2) $\mathrm{Tor}_1^\mathcal{O}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}$-module $\mathcal{G}$.

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_\mathcal{O} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_\mathcal{O} \mathcal{G} \to \mathcal{F} \otimes_\mathcal{O} \mathcal{H}$ is a quotient of $\mathrm{Tor}_1^\mathcal{O}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. $\square$

0G7C Lemma 21.17.16. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat, acyclic complex with flat terms. Then $\mathcal{F} = \mathrm{Ker}(\mathcal{K}^n \to \mathcal{K}^{n+1})$ is a flat $\mathcal{O}$-module.

Proof. Observe that

$$\ldots \to \mathcal{K}^{n-2} \to \mathcal{K}^{n-1} \to \mathcal{F} \to 0$$

is a flat resolution of our module $\mathcal{F}$. Since a bounded above complex of flat modules is K-flat (Lemma 21.17.8) we may use this resolution to compute $\text{Tor}_i(\mathcal{F}, \mathcal{G})$ for any $\mathcal{O}$-module $\mathcal{G}$. On the one hand $\mathcal{K}^\bullet \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{G}$ is zero in $D(\mathcal{O})$ because $\mathcal{K}^\bullet$ is acyclic and on the other hand it is represented by $\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{G}$. Hence we see that

$$\mathcal{K}^{n-3} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{K}^{n-2} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{K}^{n-1} \otimes_{\mathcal{O}} \mathcal{G}$$

is exact. Thus $\text{Tor}_1(\mathcal{F}, \mathcal{G}) = 0$ and we conclude by Lemma 21.17.15. □

0G7D **Lemma 21.17.17.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. Let $a : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ be a map of complexes of $\mathcal{O}$-modules. If $\mathcal{K}^\bullet$ is K-flat, then there exist a complex $\mathcal{N}^\bullet$ and maps of complexes $b : \mathcal{K}^\bullet \to \mathcal{N}^\bullet$ and $c : \mathcal{N}^\bullet \to \mathcal{L}^\bullet$ such that

(1) $\mathcal{N}^\bullet$ is K-flat,
(2) c is a quasi-isomorphism,
(3) a is homotopic to $c \circ b$.

If the terms of $\mathcal{K}^\bullet$ are flat, then we may choose $\mathcal{N}^\bullet$, b, and c such that the same is true for $\mathcal{N}^\bullet$.

Proof. We will use that the homotopy category $K(\text{Mod}(\mathcal{O}))$ is a triangulated category, see Derived Categories, Proposition 13.10.3. Choose a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{L}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$. Choose a quasi-isomorphism $\mathcal{M}^\bullet \to \mathcal{C}^\bullet$ with $\mathcal{M}^\bullet$ K-flat with flat terms, see Lemma 21.17.11. By the axioms of triangulated categories, we may fit the composition $\mathcal{M}^\bullet \to \mathcal{C}^\bullet \to \mathcal{K}^\bullet[1]$ into a distinguished triangle $\mathcal{K}^\bullet \to \mathcal{N}^\bullet \to \mathcal{M}^\bullet \to \mathcal{K}^\bullet[1]$. By Lemma 21.17.6 we see that $\mathcal{N}^\bullet$ is K-flat. Again using the axioms of triangulated categories, we can choose a map $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ fitting into the following morphism of distinghuised triangles

$$\begin{tikzcd}
\mathcal{K}^\bullet \arrow[r] \arrow[d] & \mathcal{N}^\bullet \arrow[r] \arrow[d] & \mathcal{M}^\bullet \arrow[r] \arrow[d] & \mathcal{K}^\bullet[1] \arrow[d] \\
\mathcal{K}^\bullet \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{C}^\bullet \arrow[r] & \mathcal{K}^\bullet[1]
\end{tikzcd}$$

Since two out of three of the arrows are quasi-isomorphisms, so is the third arrow $\mathcal{N}^\bullet \to \mathcal{L}^\bullet$ by the long exact sequences of cohomology associated to these distinguished triangles (or you can look at the image of this diagram in $D(\mathcal{O})$ and use Derived Categories, Lemma 13.4.3 if you like). This finishes the proof of (1), (2), and (3). To prove the final assertion, we may choose $\mathcal{N}^\bullet$ such that $\mathcal{N}^n \cong \mathcal{M}^n \oplus \mathcal{K}^n$, see Derived Categories, Lemma 13.10.7. Hence we get the desired flatness if the terms of $\mathcal{K}^\bullet$ are flat. □

21.18. Derived pullback

06YV Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}') \to D(\mathcal{O})$$

0G7E **Lemma 21.18.1.** Let $f : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ be a K-flat complex of $\mathcal{O}$-modules whose terms are flat $\mathcal{O}$-modules. Then $f^*\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}'$-modules whose terms are flat $\mathcal{O}'$-modules.

Proof. The terms $f^*\mathcal{K}^n$ are flat $\mathcal{O}'$-modules by Modules on Sites, Lemma 18.39.1. Choose a diagram

$$\begin{tikzcd} \mathcal{K}_1^\bullet \arrow[r] \arrow[d] & \mathcal{K}_2^\bullet \arrow[r] \arrow[d] & \ldots \\ \tau_{\leq 1}\mathcal{K}^\bullet \arrow[r] & \tau_{\leq 2}\mathcal{K}^\bullet \arrow[r] & \ldots \end{tikzcd}$$

as in Lemma 21.17.10. We will use all of the properties stated in the lemma without further mention. Each $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 18.28.7. Consider the short exact sequence of complexes

$$0 \to \mathcal{M}^\bullet \to \operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{K}^\bullet \to 0$$

defining $\mathcal{M}^\bullet$. By Lemmas 21.17.8 and 21.17.9 the complex $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat and by Modules on Sites, Lemma 18.28.5 it has flat terms. By Modules on Sites, Lemma 18.28.10 $\mathcal{M}^\bullet$ has flat terms, by Lemma 21.17.7 $\mathcal{M}^\bullet$ is K-flat, and by the long exact cohomology sequence $\mathcal{M}^\bullet$ is acyclic (because the second arrow is a quasi-isomorphism). The pullback $f^*(\operatorname{colim} \mathcal{K}_n^\bullet) = \operatorname{colim} f^*\mathcal{K}_n^\bullet$ is a colimit of bounded below complexes of flat $\mathcal{O}'$-modules and hence is K-flat (by the same lemmas as above). The pullback of our short exact sequence

$$0 \to f^*\mathcal{M}^\bullet \to f^*(\operatorname{colim} \mathcal{K}_n^\bullet) \to f^*\mathcal{K}^\bullet \to 0$$

is a short exact sequence of complexes by Modules on Sites, Lemma 18.39.4. Hence by Lemma 21.17.7 it suffices to show that $f^*\mathcal{M}^\bullet$ is K-flat. This reduces us to the case discussed in the next paragraph.

Assume $\mathcal{K}^\bullet$ is acyclic as well as K-flat and with flat terms. Then Lemma 21.17.16 guarantees that all terms of $\tau_{\leq n}\mathcal{K}^\bullet$ are flat $\mathcal{O}$-modules. We choose a diagram as above and we will use all the properties proven above for this diagram. Denote $\mathcal{M}_n^\bullet$ the kernel of the map of complexes $\mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{K}^\bullet$ so that we have short exact sequences of complexes

$$0 \to \mathcal{M}_n^\bullet \to \mathcal{K}_n^\bullet \to \tau_{\leq n}\mathcal{K}^\bullet \to 0$$

By Modules on Sites, Lemma 18.28.10 we see that the terms of the complex $\mathcal{M}_n^\bullet$ are flat. Hence we see that $\mathcal{M} = \operatorname{colim} \mathcal{M}_n^\bullet$ is a filtered colimit of bounded below complexes of flat modules in this case. Thus $f^*\mathcal{M}^\bullet$ is K-flat (same argument as above) and we win. $\square$

06YY **Lemma 21.18.2.** Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. There exists an exact functor

$$Lf^* : D(\mathcal{O}') \longrightarrow D(\mathcal{O})$$

of triangulated categories so that $Lf^*\mathcal{K}^\bullet = f^*\mathcal{K}^\bullet$ for any K-flat complex $\mathcal{K}^\bullet$ with flat terms and in particular for any bounded above complex of flat $\mathcal{O}'$-modules.

Proof. To see this we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(\mathcal{O}')$ and $\mathcal{D}' = D(\mathcal{O})$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}')$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \operatorname{Ob}(\mathcal{D})$ the collection of K-flat complexes $\mathcal{K}^\bullet$ with flat terms. Namely, (1) follows from Lemma 21.17.11 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between elements of $\mathcal{P}$ the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O} \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O}$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. The complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}'$-modules by Lemma 21.18.1 because we can consider the morphism of ringed topoi $(Sh(\mathcal{C}), f^{-1}\mathcal{O}') \to (Sh(\mathcal{C}'), \mathcal{O}')$. Hence Lemma 21.17.12 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}') = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O})$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is in $\mathcal{P}$. The proof is finished by observing that bounded above complexes of flat modules are in $\mathcal{P}$ by Lemma 21.17.8. □

0D6D Lemma 21.18.3. Consider morphisms of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$. Then $Lf^* \circ Lg^* = L(g \circ f)^*$ as functors $D(\mathcal{O}_\mathcal{E}) \to D(\mathcal{O}_\mathcal{C})$.

Proof. Let E be an object of $D(\mathcal{O}_\mathcal{E})$. We may represent E by a K-flat complex $\mathcal{K}^\bullet$ with flat terms, see Lemma 21.17.11. By construction Lg^*E is computed by $g^*\mathcal{K}^\bullet$, see Lemma 21.18.2. By Lemma 21.18.1 the complex $g^*\mathcal{K}^\bullet$ is K-flat with flat terms. Hence Lf^*Lg^*E is represented by $f^*g^*\mathcal{K}^\bullet$. Since also $L(g \circ f)^*E$ is represented by $(g \circ f)^*\mathcal{K}^\bullet = f^*g^*\mathcal{K}^\bullet$ we conclude. □

07A4 Lemma 21.18.4. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}'))$.

Proof. By our construction of derived pullback in Lemma 21.18.2. and the existence of resolutions in Lemma 21.17.11 we may replace $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ by complexes of $\mathcal{O}'$-modules which are K-flat and have flat terms. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet$. The complex $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$ is K-flat with flat terms by Lemma 21.17.5 and Modules on Sites, Lemma 18.28.12. Hence the isomorphism of the lemma comes from the isomorphism

$$\mathrm{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}} f^*\mathcal{G}^\bullet) \longrightarrow f^*\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}'} \mathcal{G}^q)$ of Modules on Sites, Lemma 18.26.2. □

08I6 Lemma 21.18.5. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(\mathcal{O})$ and $\mathcal{G}^\bullet$ in $D(\mathcal{O}')$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}$-module and let $\mathcal{G}$ be an $\mathcal{O}'$-module. Then $\mathcal{F} \otimes_{\mathcal{O}} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

0DEN Lemma 21.18.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}$-modules.

(1) If $\mathcal{K}^\bullet$ is K-flat, then for every point p of the site $\mathcal{C}$ the complex of $\mathcal{O}_p$-modules $\mathcal{K}_p^\bullet$ is K-flat in the sense of More on Algebra, Definition 15.59.1
(2) If $\mathcal{C}$ has enough points, then the converse is true.

Proof. Proof of (2). If $\mathcal{C}$ has enough points and $\mathcal{K}_p^\bullet$ is K-flat for all points p of $\mathcal{C}$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules on Sites, Lemma 18.14.4.

Proof of (1). Assume $\mathcal{K}^\bullet$ is K-flat. Choose a quasi-isomorphism $a : \mathcal{L}^\bullet \to \mathcal{K}^\bullet$ such that $\mathcal{L}^\bullet$ is K-flat with flat terms, see Lemma 21.17.11. Any pullback of $\mathcal{L}^\bullet$ is K-flat, see Lemma 21.18.1. In particular the stalk $\mathcal{L}_p^\bullet$ is a K-flat complex of $\mathcal{O}_p$-modules. Thus the cone $C(a)$ on a is a K-flat (Lemma 21.17.6) acyclic complex of $\mathcal{O}$-modules and it suffuces to show the stalk of $C(a)$ is K-flat (by More on Algebra, Lemma 15.59.5). Thus we may assume that $\mathcal{K}^\bullet$ is K-flat and acyclic.

Assume $\mathcal{K}^\bullet$ is acyclic and K-flat. Before continuing we replace the site $\mathcal{C}$ by another one as in Sites, Lemma 7.29.5 to insure that $\mathcal{C}$ has all finite limits. This implies the category of neighbourhoods of p is filtered (Sites, Lemma 7.33.2) and the colimit defining the stalk of a sheaf is filtered. Let M be a finitely presented $\mathcal{O}_p$-module. It suffices to show that $\mathcal{K}^\bullet \otimes_{\mathcal{O}_p} M$ is acyclic, see More on Algebra, Lemma 15.59.9. Since $\mathcal{O}_p$ is the filtered colimit of $\mathcal{O}(U)$ where U runs over the neighbourhoods of p, we can find a neighbourhood (U, x) of p and a finitely presented $\mathcal{O}(U)$-module M' whose base change to $\mathcal{O}_p$ is M, see Algebra, Lemma 10.127.6. By Lemma 21.17.4 we may replace $\mathcal{C}, \mathcal{O}, \mathcal{K}^\bullet$ by $\mathcal{C}/U, \mathcal{O}_U, \mathcal{K}^\bullet|_U$. We conclude that we may assume there exists an $\mathcal{O}$-module $\mathcal{F}$ such that $M \cong \mathcal{F}_p$. Since $\mathcal{K}^\bullet$ is K-flat and acyclic, we see that $\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{F}$ is acyclic (as it computes the derived tensor product by definition). Taking stalks is an exact functor, hence we get that $\mathcal{K}^\bullet \otimes_{\mathcal{O}_p} M$ is acyclic as desired. $\square$

0DEP Lemma 21.18.7. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. If $\mathcal{C}$ has enough points, then the pullback of a K-flat complex of $\mathcal{O}'$-modules is a K-flat complex of $\mathcal{O}$-modules.

Proof. This follows from Lemma 21.18.6, Modules on Sites, Lemma 18.36.4, and More on Algebra, Lemma 15.59.3. $\square$

0FPH Lemma 21.18.8. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_\mathcal{D}$-modules. The diagram

$$\begin{tikzcd}
Lf^*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \arrow[r] \arrow[d] & Lf^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \arrow[d] \\
Lf^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*\mathcal{M}^\bullet \arrow[d] & f^*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \arrow[d] \\
f^*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet \arrow[r] & \mathrm{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet)
\end{tikzcd}$$

commutes.

Proof. We will use the existence of K-flat resolutions with flat terms (Lemma 21.17.11), we will use that derived pullback is computed by such complexes (Lemma

21.18.2), and that pullbacks preserve these properties (Lemma 21.18.1). If we choose such resolutions $\mathcal{P}^\bullet \to \mathcal{K}^\bullet$ and $\mathcal{Q}^\bullet \to \mathcal{M}^\bullet$, then we see that

$$\begin{array}{ccc} Lf^*\text{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{Q}^\bullet) & \longrightarrow & Lf^*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \\ \downarrow & & \downarrow \\ f^*\text{Tot}(\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{Q}^\bullet) & \longrightarrow & f^*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet) \\ \downarrow & & \downarrow \\ \text{Tot}(f^*\mathcal{P}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{Q}^\bullet) & \longrightarrow & \text{Tot}(f^*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*\mathcal{M}^\bullet) \end{array}$$

commutes. However, now the left hand side of the diagram is the left hand side of the diagram by our choice of $\mathcal{P}^\bullet$ and $\mathcal{Q}^\bullet$ and Lemma 21.17.5. $\square$

21.19. Cohomology of unbounded complexes

07A5 Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category $\text{Mod}(\mathcal{O})$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \in \text{Ob}(\mathcal{C})} j_{U!}\mathcal{O}_U,$$

see Modules on Sites, Section 18.14 and Lemmas 18.28.7 and 18.28.8. By Injectives, Theorem 19.12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}$-modules and moreover this embedding can be chosen functorial in $\mathcal{F}^\bullet$. It follows from Derived Categories, Lemma 13.31.7 that

(1) any exact functor $F : K(\text{Mod}(\mathcal{O})) \to \mathcal{D}$ into a trianguated category $\mathcal{D}$ has a right derived functor $RF : D(\mathcal{O}) \to \mathcal{D}$,
(2) for any additive functor $F : \text{Mod}(\mathcal{O}) \to \mathcal{A}$ into an abelian category $\mathcal{A}$ we consider the exact functor $F : K(\text{Mod}(\mathcal{O})) \to D(\mathcal{A})$ induced by F and we obtain a right derived functor $RF : D(\mathcal{O}) \to K(\mathcal{A})$.

By construction we have $RF(\mathcal{F}^\bullet) = F(\mathcal{I}^\bullet)$ where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is as above.

Here are some examples of the above:

(1) The functor $\Gamma(\mathcal{C}, -) : \text{Mod}(\mathcal{O}) \to \text{Mod}_{\Gamma(\mathcal{C},\mathcal{O})}$ gives rise to
$$R\Gamma(\mathcal{C}, -) : D(\mathcal{O}) \longrightarrow D(\Gamma(\mathcal{C}, \mathcal{O}))$$
We shall use the notation $H^i(\mathcal{C}, K) = H^i(R\Gamma(\mathcal{C}, K))$ for cohomology.
(2) For an object U of $\mathcal{C}$ we consider the functor $\Gamma(U, -) : \text{Mod}(\mathcal{O}) \to \text{Mod}_{\Gamma(U,\mathcal{O})}$. This gives rise to
$$R\Gamma(U, -) : D(\mathcal{O}) \to D(\Gamma(U, \mathcal{O}))$$
We shall use the notation $H^i(U, K) = H^i(R\Gamma(U, K))$ for cohomology.
(3) For a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ we consider the functor $f_* : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}')$ which gives rise to the total direct image
$$Rf_* : D(\mathcal{O}) \longrightarrow D(\mathcal{O}')$$
on unbounded derived categories.

07A6 Lemma 21.19.1. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. The functor Rf_* defined above and the functor Lf^* defined in Lemma 21.18.2 are adjoint:

$$\mathrm{Hom}_{D(\mathcal{O})}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}')}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \mathrm{Ob}(D(\mathcal{O}))$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}'))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 13.30.3. □

0D6E Lemma 21.19.2. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$ be morphisms of ringed topoi. Then $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_\mathcal{E})$.

Proof. By Lemma 21.19.1 we see that $Rg_* \circ Rf_*$ is adjoint to $Lf^* \circ Lg^*$. We have $Lf^* \circ Lg^* = L(g \circ f)^*$ by Lemma 21.18.3 and hence by uniqueness of adjoint functors we have $Rg_* \circ Rf_* = R(g \circ f)_*$. □

07A7 Remark 21.19.3. The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "f"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

is a commutative diagram of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{C})$. Then there exists a canonical base change map

$$Lg^*Rf_*K \longrightarrow R(f')_*L(g')^*K$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*K \to L(g')^*K$. Since $L(f')^* \circ Lg^* = L(g')^* \circ Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*K \to L(g')^*K$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*K \to K$.

0E46 Remark 21.19.4. Consider a commutative diagram

$$\begin{tikzcd} (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) \arrow[r, "k"'] \arrow[d, "f'"'] & (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \arrow[d, "f"] \\ (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "l"] \arrow[d, "g'"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "g"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "m"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

of ringed topoi. Then the base change maps of Remark 21.19.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned} Lm^* \circ R(g \circ f)_* &= Lm^* \circ Rg_* \circ Rf_* \\ &\to Rg'_* \circ Ll^* \circ Rf_* \\ &\to Rg'_* \circ Rf'_* \circ Lk^* \\ &= R(g' \circ f')_* \circ Lk^* \end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0E47 Remark 21.19.5. Consider a commutative diagram

$$\begin{tikzcd}
(Sh(\mathcal{C}''), \mathcal{O}_{\mathcal{C}''}) \arrow[r, "g'"'] \arrow[d, "f''"'] & (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "f"] \\
(Sh(\mathcal{D}''), \mathcal{O}_{\mathcal{D}''}) \arrow[r, "h'"] & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "h"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})
\end{tikzcd}$$

of ringed topoi. Then the base change maps of Remark 21.19.3 for the two squares compose to give the base change map for the outer rectangle. More precisely, the composition

$$\begin{aligned}
L(h \circ h')^* \circ Rf_* &= L(h')^* \circ Lh^* \circ Rf_* \\
&\to L(h')^* \circ Rf'_* \circ Lg^* \\
&\to Rf''_* \circ L(g')^* \circ Lg^* \\
&= Rf''_* \circ L(g \circ g')^*
\end{aligned}$$

is the base change map for the rectangle. We omit the verification.

0FPI Lemma 21.19.6. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_\mathcal{C}$-modules. The diagram

$$\begin{tikzcd}
Lf^*f_*\mathcal{K}^\bullet \arrow[r] \arrow[d] & f^*f_*\mathcal{K}^\bullet \arrow[d] \\
Lf^*Rf_*\mathcal{K}^\bullet \arrow[r] & \mathcal{K}^\bullet
\end{tikzcd}$$

coming from $Lf^* \to f^*$ on complexes, $f_* \to Rf_*$ on complexes, and adjunction $Lf^* \circ Rf_* \to \text{id}$ commutes in $D(\mathcal{O}_\mathcal{C})$.

Proof. We will use the existence of K-flat resolutions and K-injective resolutions, see Lemmas 21.17.11, 21.18.2, and 21.18.1 and the discussion above. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_\mathcal{C}$-modules. Choose a quasi-isomorphism $\mathcal{Q}^\bullet \to f_*\mathcal{I}^\bullet$ where $\mathcal{Q}^\bullet$ is a K-flat complex of $\mathcal{O}_\mathcal{D}$-modules with flat terms. We can choose a K-flat complex of $\mathcal{O}_\mathcal{D}$-modules $\mathcal{P}^\bullet$ with flat terms and a diagram of morphisms of complexes

$$\begin{tikzcd}
\mathcal{P}^\bullet \arrow[r] \arrow[d] & f_*\mathcal{K}^\bullet \arrow[d] \\
\mathcal{Q}^\bullet \arrow[r] & f_*\mathcal{I}^\bullet
\end{tikzcd}$$

commutative up to homotopy where the top horizontal arrow is a quasi-isomorphism. Namely, we can first choose such a diagram for some complex $\mathcal{P}^\bullet$ because the quasi-isomorphisms form a multiplicative system in the homotopy category of complexes and then we can choose a resolution of $\mathcal{P}^\bullet$ by a K-flat complex with flat terms. Taking pullbacks we obtain a diagram of morphisms of complexes

$$\begin{tikzcd}
f^*\mathcal{P}^\bullet \arrow[r] \arrow[d] & f^*f_*\mathcal{K}^\bullet \arrow[r] \arrow[d] & \mathcal{K}^\bullet \arrow[d] \\
f^*\mathcal{Q}^\bullet \arrow[r] & f^*f_*\mathcal{I}^\bullet \arrow[r] & \mathcal{I}^\bullet
\end{tikzcd}$$

commutative up to homotopy. The outer rectangle witnesses the truth of the statement in the lemma. □

0B6C Remark 21.19.7. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. The adjointness of Lf^* and Rf_* allows us to construct a relative cup product

$$Rf_*K \otimes_{\mathcal{O}_\mathcal{D}}^{\mathbf{L}} Rf_*L \longrightarrow Rf_*(K \otimes_{\mathcal{O}_\mathcal{C}}^{\mathbf{L}} L)$$

in $D(\mathcal{O}_\mathcal{D})$ for all K, L in $D(\mathcal{O}_\mathcal{C})$. Namely, this map is adjoint to a map $Lf^*(Rf_*K \otimes_{\mathcal{O}_\mathcal{D}}^{\mathbf{L}} Rf_*L) \to K \otimes_{\mathcal{O}_\mathcal{C}}^{\mathbf{L}} L$ for which we can take the composition of the isomorphism $Lf^*(Rf_*K \otimes_{\mathcal{O}_\mathcal{D}}^{\mathbf{L}} Rf_*L) = Lf^*Rf_*K \otimes_{\mathcal{O}_\mathcal{C}}^{\mathbf{L}} Lf^*Rf_*L$ (Lemma 21.18.4) with the map $Lf^*Rf_*K \otimes_{\mathcal{O}_\mathcal{C}}^{\mathbf{L}} Lf^*Rf_*L \to K \otimes_{\mathcal{O}_\mathcal{C}}^{\mathbf{L}} L$ coming from the counit $Lf^* \circ Rf_* \to \text{id}$.

0DD7 Lemma 21.19.8. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \subset \text{Ab}(\mathcal{C})$ denote the Serre subcategory consisting of torsion abelian sheaves. Then the functor $D(\mathcal{A}) \to D_\mathcal{A}(\mathcal{C})$ is an equivalence.

Proof. A key observation is that an injective abelian sheaf $\mathcal{I}$ is divisible. Namely, if $s \in \mathcal{I}(U)$ is a local section, then we interpret s as a map $s : j_{U!}\mathbf{Z} \to \mathcal{I}$ and we apply the defining property of an injective object to the injective map of sheaves $n : j_{U!}\mathbf{Z} \to j_{U!}\mathbf{Z}$ to see that there exists an $s' \in \mathcal{I}(U)$ with $ns' = s$.

For a sheaf $\mathcal{F}$ denote $\mathcal{F}_{tor}$ its torsion subsheaf. We claim that if $\mathcal{I}^\bullet$ is a complex of injective abelian sheaves whose cohomology sheaves are torsion, then

$$\mathcal{I}^\bullet_{tor} \to \mathcal{I}^\bullet$$

is a quasi-isomorphism. Namely, by flatness of $\mathbf{Q}$ over $\mathbf{Z}$ we have

$$H^p(\mathcal{I}^\bullet) \otimes_\mathbf{Z} \mathbf{Q} = H^p(\mathcal{I}^\bullet \otimes_\mathbf{Z} \mathbf{Q})$$

which is zero because the cohomology sheaves are torsion. By divisibility (shown above) we see that $\mathcal{I}^\bullet \to \mathcal{I}^\bullet \otimes_\mathbf{Z} \mathbf{Q}$ is surjective with kernel $\mathcal{I}^\bullet_{tor}$. The claim follows from the long exact sequence of cohomology sheaves associated to the short exact sequence you get.

To prove the lemma we will construct right adjoint $T : D(\mathcal{C}) \to D(\mathcal{A})$. Namely, given K in $D(\mathcal{C})$ we can represent K by a K-injective complex $\mathcal{I}^\bullet$ whose cohomology sheaves are injective, see Injectives, Theorem 19.12.6. Then we set $T(K) = \mathcal{I}^\bullet_{tor}$, in other words, T is the right derived functor of taking torsion. The functor T is a right adjoint to $i : D(\mathcal{A}) \to D_\mathcal{A}(\mathcal{C})$. This readily follows from the observation that if $\mathcal{F}^\bullet$ is a complex of torsion sheaves, then

$$\text{Hom}_{K(\mathcal{A})}(\mathcal{F}^\bullet, I^\bullet_{tor}) = \text{Hom}_{K(\text{Ab}(\mathcal{C}))}(\mathcal{F}^\bullet, I^\bullet)$$

in particular $\mathcal{I}^\bullet_{tor}$ is a K-injective complex of $\mathcal{A}$. Some details omitted; in case of doubt, it also follows from the more general Derived Categories, Lemma 13.30.3. Our claim above gives that $L = T(i(L))$ for L in $D(\mathcal{A})$ and $i(T(K)) = K$ if K is in $D_\mathcal{A}(\mathcal{C})$. Using Categories, Lemma 4.24.4 the result follows. □

21.20. Some properties of K-injective complexes

08FH Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : \text{Mod}(\mathcal{O}_U) \to \text{Mod}(\mathcal{O})$ (see Modules on Sites, Definition 18.19.1) is an exact functor (Modules on Sites, Lemma 18.19.3). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

08FI Lemma 21.20.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The restriction of a K-injective complex of $\mathcal{O}$-modules to $\mathcal{C}/U$ is a K-injective complex of $\mathcal{O}_U$-modules.

Proof. Follows immediately from Derived Categories, Lemma 13.31.9 and the fact that the restriction functor has the exact left adjoint $j_!$. See discussion above. $\square$

0D6F Lemma 21.20.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \text{Ob}(\mathcal{C})$. For K in $D(\mathcal{O})$ we have $H^p(U, K) = H^p(\mathcal{C}/U, K|_{\mathcal{C}/U})$.

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then

$$H^q(U, K) = H^q(\Gamma(U, \mathcal{I}^\bullet)) = H^q(\Gamma(\mathcal{C}/U, \mathcal{I}^\bullet|_{\mathcal{C}/U}))$$

by construction of cohomology. By Lemma 21.20.1 the complex $\mathcal{I}^\bullet|_{\mathcal{C}/U}$ is a K-injective complex representing $K|_{\mathcal{C}/U}$ and the lemma follows. $\square$

0BKV Lemma 21.20.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. The sheafification of

$$U \mapsto H^q(U, K) = H^q(\mathcal{C}/U, K|_{\mathcal{C}/U})$$

is the qth cohomology sheaf $H^q(K)$ of K.

Proof. The equality $H^q(U, K) = H^q(\mathcal{C}/U, K|_{\mathcal{C}/U})$ holds by Lemma 21.20.2. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Then

$$H^q(U, K) = \frac{\text{Ker}(\mathcal{I}^q(U) \to \mathcal{I}^{q+1}(U))}{\text{Im}(\mathcal{I}^{q-1}(U) \to \mathcal{I}^q(U))}.$$

by our construction of cohomology. Since $H^q(K) = \text{Ker}(\mathcal{I}^q \to \mathcal{I}^{q+1})/\text{Im}(\mathcal{I}^{q-1} \to \mathcal{I}^q)$ the result is clear. $\square$

0D6G Lemma 21.20.4. Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Given $V \in \mathcal{D}$, set $U = u(V)$ and denote $g : (\mathcal{C}/U, \mathcal{O}_U) \to (\mathcal{D}/V, \mathcal{O}_V)$ the induced morphism of ringed sites (Modules on Sites, Lemma 18.20.1). Then $(Rf_*E)|_{\mathcal{D}/V} = Rg_*(E|_{\mathcal{C}/U})$ for E in $D(\mathcal{O}_\mathcal{C})$.

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_\mathcal{C}$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_{\mathcal{C}/U}) = g_*(\mathcal{I}^\bullet|_{\mathcal{C}/U})$ by Lemma 21.20.1. Since it is clear that $(f_*\mathcal{F})|_{\mathcal{D}/V} = g_*(\mathcal{F}|_{\mathcal{C}/U})$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$ (see Modules on Sites, Lemma 18.20.1 or the more basic Sites, Lemma 7.28.1) the result follows. $\square$

0D6H Lemma 21.20.5. Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Then $R\Gamma(\mathcal{D}, -) \circ Rf_* = R\Gamma(\mathcal{C}, -)$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\Gamma(\mathcal{O}_\mathcal{D}))$. More generally, for $V \in \mathcal{D}$ with $U = u(V)$ we have $R\Gamma(U, -) = R\Gamma(V, -) \circ Rf_*$.

Proof. Consider the punctual topos pt endowed with $\mathcal{O}_{pt}$ given by the ring $\Gamma(\mathcal{O}_\mathcal{D})$. There is a canonical morphism $(\mathcal{D}, \mathcal{O}_\mathcal{D}) \to (pt, \mathcal{O}_{pt})$ of ringed topoi inducing the identification on global sections of structure sheaves. Then $D(\mathcal{O}_{pt}) = D(\Gamma(\mathcal{O}_\mathcal{D}))$. The assertion $R\Gamma(\mathcal{D}, -) \circ Rf_* = R\Gamma(\mathcal{C}, -)$ follows from Lemma 21.19.2 applied to

$$(\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D}) \to (pt, \mathcal{O}_{pt})$$

The second (more general) statement follows from the first statement after applying Lemma 21.20.4. □

0D6I Lemma 21.20.6. Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let K be in $D(\mathcal{O}_\mathcal{C})$. Then $H^i(Rf_*K)$ is the sheaf associated to the presheaf

$$V \mapsto H^i(u(V), K) = H^i(V, Rf_*K)$$

Proof. The equality $H^i(u(V), K) = H^i(V, Rf_*K)$ follows upon taking cohomology from the second statement in Lemma 21.20.5. Then the statement on sheafification follows from Lemma 21.20.3. □

0D6J Lemma 21.20.7. Let $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ be a ringed site. Let K be an object of $D(\mathcal{O}_\mathcal{C})$ and denote K_{ab} its image in $D(\underline{\mathbf{Z}}_\mathcal{C})$.

(1) There is a canonical map $R\Gamma(\mathcal{C}, K) \to R\Gamma(\mathcal{C}, K_{ab})$ which is an isomorphism in $D(\text{Ab})$.
(2) For any $U \in \mathcal{C}$ there is a canonical map $R\Gamma(U, K) \to R\Gamma(U, K_{ab})$ which is an isomorphism in $D(\text{Ab})$.
(3) Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. There is a canonical map $Rf_*K \to Rf_*(K_{ab})$ which is an isomorphism in $D(\underline{\mathbf{Z}}_\mathcal{D})$.

Proof. The map is constructed as follows. Choose a K-injective complex $\mathcal{I}^\bullet$ representing K. Choose a quasi-isomorpism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is a K-injective complex of abelian groups. Then the map in (1) is given by $\Gamma(\mathcal{C}, \mathcal{I}^\bullet) \to \Gamma(\mathcal{C}, \mathcal{J}^\bullet)$ (2) is given by $\Gamma(U, \mathcal{I}^\bullet) \to \Gamma(U, \mathcal{J}^\bullet)$ and the map in (3) is given by $f_*\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$. To show that these maps are isomorphisms, it suffices to prove they induce isomorphisms on cohomology groups and cohomology sheaves. By Lemmas 21.20.2 and 21.20.6 it suffices to show that the map

$$H^0(\mathcal{C}, K) \longrightarrow H^0(\mathcal{C}, K_{ab})$$

is an isomorphism. Observe that

$$H^0(\mathcal{C}, K) = \text{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C}, K)$$

and similarly for the other group. Choose any complex $\mathcal{K}^\bullet$ of $\mathcal{O}_\mathcal{C}$-modules representing K. By construction of the derived category as a localization we have

$$\text{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C}, K) = \text{colim}_{s : \mathcal{F}^\bullet \to \mathcal{O}_\mathcal{C}} \text{Hom}_{K(\mathcal{O}_\mathcal{C})}(\mathcal{F}^\bullet, \mathcal{K}^\bullet)$$

where the colimit is over quasi-isomorphisms s of complexes of $\mathcal{O}_\mathcal{C}$-modules. Similarly, we have

$$\text{Hom}_{D(\underline{\mathbf{Z}}_\mathcal{C})}(\underline{\mathbf{Z}}_\mathcal{C}, K) = \text{colim}_{s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}} \text{Hom}_{K(\underline{\mathbf{Z}}_\mathcal{C})}(\mathcal{G}^\bullet, \mathcal{K}^\bullet)$$

Next, we observe that the quasi-isomorphisms $s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}$ with $\mathcal{G}^\bullet$ bounded above complex of flat $\underline{\mathbf{Z}}_\mathcal{C}$-modules is cofinal in the system. (This follows from Modules on Sites, Lemma 18.28.8 and Derived Categories, Lemma 13.15.4; see discussion in Section 21.17.) Hence we can construct an inverse to the map $H^0(\mathcal{C}, K) \longrightarrow H^0(\mathcal{C}, K_{ab})$ by representing an element $\xi \in H^0(\mathcal{C}, K_{ab})$ by a pair

$$(s : \mathcal{G}^\bullet \to \underline{\mathbf{Z}}_\mathcal{C}, a : \mathcal{G}^\bullet \to \mathcal{K}^\bullet)$$

with $\mathcal{G}^\bullet$ a bounded above complex of flat $\underline{\mathbf{Z}}_\mathcal{C}$-modules and sending this to

$$(\mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_\mathcal{C}} \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{C}, \mathcal{G}^\bullet \otimes_{\underline{\mathbf{Z}}_\mathcal{C}} \mathcal{O}_\mathcal{C} \to \mathcal{K}^\bullet)$$

The only thing to note here is that the first arrow is a quasi-isomorphism by Lemmas 21.17.12 and 21.17.8. We omit the detailed verification that this construction is indeed an inverse. □

08FJ Lemma 21.20.8. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The restriction functor $D(\mathcal{O}) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O})$.

Proof. We have to show that

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!E, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E, F|_U)$$

Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_U$-modules representing E and choose a K-injective complex $\mathcal{I}^\bullet$ representing F. By Lemma 21.20.1 the complex $\mathcal{I}^\bullet|_U$ is K-injective as well. Hence we see that the formula above becomes

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!\mathcal{E}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{I}^\bullet|_U)$$

which holds as $|_U$ and $j_!$ are adjoint functors (Modules on Sites, Lemma 18.19.2) and Derived Categories, Lemma 13.31.2. □

0GL1 Lemma 21.20.9. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For L in $D(\mathcal{O}_U)$ and K in $D(\mathcal{O})$ we have $j_!L \otimes^{\mathbf{L}}_\mathcal{O} K = j_!(L \otimes^{\mathbf{L}}_{\mathcal{O}_U} K|_U)$.

Proof. Represent L by a complex of $\mathcal{O}_U$-modules and K by a K-flat complexe of $\mathcal{O}$-modules and apply Modules on Sites, Lemma 18.27.9. Details omitted. □

093Y Lemma 21.20.10. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a flat morphism of ringed topoi. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_\mathcal{C}$-modules, then $f_*\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}_\mathcal{D}$-modules.

Proof. This is true because

$$\mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}_\mathcal{C})}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

by Modules on Sites, Lemma 18.13.2 and the fact that f^* is exact as f is assumed to be flat. □

093Z Lemma 21.20.11. Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules, then $\mathcal{H}om_\mathcal{O}(\mathcal{O}', \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}'$-modules.

Proof. This is true because $\mathrm{Hom}_{K(\mathcal{O}')}(\mathcal{G}^\bullet, \mathrm{Hom}_\mathcal{O}(\mathcal{O}', \mathcal{I}^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)$ by Modules on Sites, Lemma 18.27.8. □

21.21. Localization and cohomology

0EYZ Let $\mathcal{C}$ be a site. Let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then we obtain a morphism of topoi

$$j_{X/Y} : Sh(\mathcal{C}/X) \longrightarrow Sh(\mathcal{C}/Y)$$

See Sites, Sections 7.25 and 7.27. Some questions about cohomology are easier for this type of morphisms of topoi. Here is an example where we get a trivial type of base change theorem.

0EZ0 Lemma 21.21.1. Let $\mathcal{C}$ be a site. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a cartesian diagram of $\mathcal{C}$. Then we have $j_{Y'/Y}^{-1} \circ Rj_{X/Y,*} = Rj_{X'/Y',*} \circ j_{X'/X}^{-1}$ as functors $D(\mathcal{C}/X) \to D(\mathcal{C}/Y')$.

Proof. Let $E \in D(\mathcal{C}/X)$. Choose a K-injective complex $\mathcal{I}^\bullet$ of abelian sheaves on $\mathcal{C}/X$ representing E. By Lemma 21.20.1 we see that $j_{X'/X}^{-1}\mathcal{I}^\bullet$ is K-injective too. Hence we may compute $Rj_{X'/Y'}(j_{X'/X}^{-1}E)$ by $j_{X'/Y',*}j_{X'/X}^{-1}\mathcal{I}^\bullet$. Thus we see that the equality holds by Sites, Lemma 7.27.5. □

If we have a ringed site $(\mathcal{C}, \mathcal{O})$ and a morphism $f : X \to Y$ of $\mathcal{C}$, then $j_{X/Y}$ becomes a morphism of ringed topoi

$$j_{X/Y} : (Sh(\mathcal{C}/X), \mathcal{O}_X) \longrightarrow (Sh(\mathcal{C}/Y), \mathcal{O}_Y)$$

See Modules on Sites, Lemma 18.19.5.

0FN5 Lemma 21.21.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a cartesian diagram of $\mathcal{C}$. Then we have $j_{Y'/Y}^* \circ Rj_{X/Y,*} = Rj_{X'/Y',*} \circ j_{X'/X}^*$ as functors $D(\mathcal{O}_X) \to D(\mathcal{O}_{Y'})$.

Proof. Since $j_{Y'/Y}^{-1}\mathcal{O}_Y = \mathcal{O}_{Y'}$ we have $j_{Y'/Y}^* = Lj_{Y'/Y}^* = j_{Y'/Y}^{-1}$. Similarly we have $j_{X'/X}^* = Lj_{X'/X}^* = j_{X'/X}^{-1}$. Thus by Lemma 21.20.7 it suffices to prove the result on derived categories of abelian sheaves which we did in Lemma 21.21.1. □

21.22. Inverse systems and cohomology

0GYP We prove some results on inverse systems of sheaves of modules.

0GYQ Lemma 21.22.1. Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n \geq 0} H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module. Then the inverse system $M_n = H^p(\mathcal{C}, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[2].

Proof. Set $N_n = H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and let $\delta_n : M_n \to N_n$ be the boundary map on cohomology coming from the short exact sequence $0 \to I^n\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$. Then $\bigoplus \mathrm{Im}(\delta_n) \subset \bigoplus N_n$ is a graded submodule. Namely, if $s \in M_n$ and $f \in I^m$, then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I^n\mathcal{F}_{n+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_{n+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_n \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I^{n+m}\mathcal{F}_{n+m+1} \arrow[r] & \mathcal{F}_{n+m+1} \arrow[r] & \mathcal{F}_{n+m} \arrow[r] & 0 \end{tikzcd}$$

The middle vertical map is given by lifting a local section of $\mathcal{F}_{n+1}$ to a section of $\mathcal{F}_{n+m+1}$ and then multiplying by f; similarly for the other vertical arrows. We conclude that $\delta_{n+m}(fs) = f\delta_n(s)$. By assumption we can find $s_j \in M_{n_j}$, $j = 1, \ldots, N$ such that $\delta_{n_j}(s_j)$ generate $\bigoplus \mathrm{Im}(\delta_n)$ as a graded module. Let $n > c = \max(n_j)$. Let $s \in M_n$. Then we can find $f_j \in I^{n-n_j}$ such that $\delta_n(s) = \sum f_j \delta_{n_j}(s_j)$. We conclude that $\delta(s - \sum f_j s_j) = 0$, i.e., we can find $s' \in M_{n+1}$ mapping to $s - \sum f_j s_j$ in M_n. It follows that

$$\mathrm{Im}(M_{n+1} \to M_{n-c}) = \mathrm{Im}(M_n \to M_{n-c})$$

Namely, the elements $f_j s_j$ map to zero in M_{n-c}. This proves the lemma. $\square$

0GYR Lemma 21.22.2. Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module, then the inverse system $M_n = H^p(\mathcal{C}, \mathcal{F}_n)$ satisfies the Mittag-Leffler condition[3].

Proof. The proof is exactly the same as the proof of Lemma 21.22.1. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and that the boundary maps $\delta_n : M_n \to H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ have image contained in N_n.

Suppose that $\xi \in N_n$ and $f \in I^k$. Choose $m \gg n+k$. Choose $\xi' \in H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{m+1})$ lifting ξ. We consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & I^n\mathcal{F}_{m+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_{m+1} \arrow[r] \arrow[d, "f"] & \mathcal{F}_n \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & I^{n+k}\mathcal{F}_{m+1} \arrow[r] & \mathcal{F}_{m+1} \arrow[r] & \mathcal{F}_{n+k} \arrow[r] & 0 \end{tikzcd}$$

constructed as in the proof of Lemma 21.22.1. We get an induced map on cohomology and we see that $f\xi' \in H^{p+1}(\mathcal{C}, I^{n+k}\mathcal{F}_{m+1})$ maps to $f\xi$. Since this is true for all $m \gg n+k$ we see that $f\xi$ is in N_{n+k} as desired.

[2] In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.
[3] In fact, there exists a $c \geq 0$ such that $\mathrm{Im}(M_n \to M_{n-c})$ is the stable image for all $n \geq c$.

To see the boundary maps δ_n have image contained in N_n we consider the diagrams

$$\begin{tikzcd} 0 \arrow[r] & I^n\mathcal{F}_{m+1} \arrow[r] \arrow[d] & \mathcal{F}_{m+1} \arrow[r] \arrow[d] & \mathcal{F}_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^n\mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_n \arrow[r] & 0 \end{tikzcd}$$

for $m \geq n$. Looking at the induced maps on cohomology we conclude. □

0GYS Lemma 21.22.3. Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Assume

$$\bigoplus\nolimits_{n \geq 0} H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

satisfies the ascending chain condition as a graded $\bigoplus_{n\geq 0} I^n/I^{n+1}$-module. Then the limit topology on $M = \lim H^p(\mathcal{C}, \mathcal{F}_n)$ is the I-adic topology.

Proof. Set $F^n = \text{Ker}(M \to H^p(\mathcal{C}, \mathcal{F}_n))$ for $n \geq 1$ and $F^0 = M$. Observe that $IF^n \subset F^{n+1}$. In particular $I^nM \subset F^n$. Hence the I-adic topology is finer than the limit topology. For the converse, we will show that given n there exists an $m \geq n$ such that $F^m \subset I^nM$[4]. We have injective maps

$$F^n/F^{n+1} \longrightarrow H^p(\mathcal{C}, \mathcal{F}_{n+1})$$

whose image is contained in the image of $H^p(\mathcal{C}, I^n\mathcal{F}_{n+1}) \to H^p(\mathcal{C}, \mathcal{F}_{n+1})$. Denote

$$E_n \subset H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$$

the inverse image of F^n/F^{n+1}. Then $\bigoplus E_n$ is a graded $\bigoplus I^n/I^{n+1}$-submodule of $\bigoplus H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and $\bigoplus E_n \to \bigoplus F^n/F^{n+1}$ is a homomorphism of graded modules; details omitted. By assumption $\bigoplus E_n$ is generated by finitely many homogeneous elements over $\bigoplus I^n/I^{n+1}$. Since $E_n \to F^n/F^{n+1}$ is surjective, we see that the same thing is true of $\bigoplus F^n/F^{n+1}$. Hence we can find r and $c_1, \ldots, c_r \geq 0$ and $a_i \in F^{c_i}$ whose images in $\bigoplus F^n/F^{n+1}$ generate. Set $c = \max(c_i)$.

For $n \geq c$ we claim that $IF^n = F^{n+1}$. The claim shows that $F^{n+c} = I^nF^c \subset I^nM$ as desired. To prove the claim suppose $a \in F^{n+1}$. The image of a in F^{n+1}/F^{n+2} is a linear combination of our a_i. Therefore $a - \sum f_ia_i \in F^{n+2}$ for some $f_i \in I^{n+1-c_i}$. Since $I^{n+1-c_i} = I \cdot I^{n-c_i}$ as $n \geq c_i$ we can write $f_i = \sum g_{i,j}h_{i,j}$ with $g_{i,j} \in I$ and $h_{i,j}a_i \in F^n$. Thus we see that $F^{n+1} = F^{n+2} + IF^n$. A simple induction argument gives $F^{n+1} = F^{n+e} + IF^n$ for all $e > 0$. It follows that IF^n is dense in F^{n+1}. Choose generators $k_1, \ldots, k_r$ of I and consider the continuous map

$$u : (F^n)^{\oplus r} \longrightarrow F^{n+1}, \quad (x_1, \ldots, x_r) \mapsto \sum k_ix_i$$

(in the limit topology). By the above the image of $(F^m)^{\oplus r}$ under u is dense in F^{m+1} for all $m \geq n$. By the open mapping lemma (More on Algebra, Lemma 15.36.5) we find that u is open. Hence u is surjective. Hence $IF^n = F^{n+1}$ for $n \geq c$. This concludes the proof. □

[4]In fact, there exist a $c \geq 0$ such that $F^{n+c} \subset I^nM$ for all n.

0GYT Lemma 21.22.4. Let I be an ideal of a ring A. Let $\mathcal{C}$ be a site. Let

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$$

be an inverse system of sheaves of A-modules on $\mathcal{C}$ such that $\mathcal{F}_n = \mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1}$. Let $p \geq 0$. Given n define

$$N_n = \bigcap\nolimits_{m \geq n} \mathrm{Im}\left(H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})\right)$$

If $\bigoplus N_n$ satisfies the ascending chain condition as a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-module, then the limit topology on $M = \lim H^p(\mathcal{C}, \mathcal{F}_n)$ is the I-adic topology.

Proof. The proof is exactly the same as the proof of Lemma 21.22.3. In fact, the result will follow from the arguments given there as soon as we show that $\bigoplus N_n$ is a graded $\bigoplus_{n \geq 0} I^n/I^{n+1}$-submodule of $\bigoplus H^{p+1}(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and that $F^n/F^{n+1} \subset H^p(\mathcal{C}, \mathcal{F}_{n+1})$ is contained in the image of $N_n \to H^p(\mathcal{C}, \mathcal{F}_{n+1})$. In the proof of Lemma 21.22.2 we have seen the statement on the module structure.

Let $t \in F^n$. Choose an element $s \in H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ which maps to the image of t in $H^p(\mathcal{C}, \mathcal{F}_{n+1})$. We have to show that s is in N_n. Now F^n is the kernel of the map from $M \to H^p(\mathcal{C}, \mathcal{F}_n)$ hence for all $m \geq n$ we can map t to an element $t_m \in H^p(\mathcal{C}, \mathcal{F}_{m+1})$ which maps to zero in $H^p(\mathcal{C}, \mathcal{F}_n)$. Consider the cohomology sequence

$$H^{p-1}(\mathcal{C}, \mathcal{F}_n) \to H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, \mathcal{F}_{m+1}) \to H^p(\mathcal{C}, \mathcal{F}_n)$$

coming from the short exact sequence $0 \to I^n\mathcal{F}_{m+1} \to \mathcal{F}_{m+1} \to \mathcal{F}_n \to 0$. We can choose $s_m \in H^p(\mathcal{C}, I^n\mathcal{F}_{m+1})$ mapping to t_m. Comparing the sequence above with the one for $m = n$ we see that s_m maps to s up to an element in the image of $H^{p-1}(\mathcal{C}, \mathcal{F}_n) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$. However, this map factors through the map $H^p(\mathcal{C}, I^n\mathcal{F}_{m+1}) \to H^p(\mathcal{C}, I^n\mathcal{F}_{n+1})$ and we see that s is in the image as desired. □

21.23. Derived and homotopy limits

0940 Let $\mathcal{C}$ be a site. Consider the category $\mathcal{C} \times \mathbf{N}$ with $\mathrm{Mor}((U, n), (V, m)) = \emptyset$ if $n > m$ and $\mathrm{Mor}((U, n), (V, m)) = \mathrm{Mor}(U, V)$ else. We endow this with the structure of a site by letting coverings be families $\{(U_i, n) \to (U, n)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then the reader verifies immediately that sheaves on $\mathcal{C} \times \mathbf{N}$ are the same thing as inverse systems of sheaves on $\mathcal{C}$. In particular $\mathrm{Ab}(\mathcal{C} \times \mathbf{N})$ is inverse systems of abelian sheaves on $\mathcal{C}$. Consider now the functor

$$\lim : \mathrm{Ab}(\mathcal{C} \times \mathbf{N}) \to \mathrm{Ab}(\mathcal{C})$$

which takes an inverse system to its limit. This is nothing but g_* where $g : Sh(\mathcal{C} \times \mathbf{N}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to the continuous and cocontinuous functor $\mathcal{C} \times \mathbf{N} \to \mathcal{C}$. (Observe that g^{-1} assigns to a sheaf on $\mathcal{C}$ the corresponding constant inverse system.)

By the general machinery explained above we obtain a derived functor

$$R\lim = Rg_* : D(\mathcal{C} \times \mathbf{N}) \to D(\mathcal{C}).$$

As indicated this functor is often denoted $R\lim$.

On the other hand, the continuous and cocontinuous functors $\mathcal{C} \to \mathcal{C} \times \mathbf{N}$, $U \mapsto (U, n)$ define morphisms of topoi $i_n : Sh(\mathcal{C}) \to Sh(\mathcal{C} \times \mathbf{N})$. Of course i_n^{-1} is the functor which picks the nth term of the inverse system. Thus there are transformations of functors $i_{n+1}^{-1} \to i_n^{-1}$. Hence given $K \in D(\mathcal{C} \times \mathbf{N})$ we get $K_n = i_n^{-1}K \in D(\mathcal{C})$ and maps $K_{n+1} \to K_n$. In Derived Categories, Definition 13.34.1 we have defined the notion of a homotopy limit

$$R\lim K_n \in D(\mathcal{C})$$

We claim the two notions agree (as far as it makes sense).

0941 Lemma 21.23.1. Let $\mathcal{C}$ be a site. Let K be an object of $D(\mathcal{C} \times \mathbf{N})$. Set $K_n = i_n^{-1}K$ as above. Then

$$R\lim K \cong R\lim K_n$$

in $D(\mathcal{C})$.

Proof. To calculate $R\lim$ on an object K of $D(\mathcal{C} \times \mathbf{N})$ we choose a K-injective representative $\mathcal{I}^\bullet$ whose terms are injective objects of $\text{Ab}(\mathcal{C} \times \mathbf{N})$, see Injectives, Theorem 19.12.6. We may and do think of $\mathcal{I}^\bullet$ as an inverse system of complexes $(\mathcal{I}_n^\bullet)$ and then we see that

$$R\lim K = \lim \mathcal{I}_n^\bullet$$

where the right hand side is the termwise inverse limit.

Let $\mathcal{J} = (\mathcal{J}_n)$ be an injective object of $\text{Ab}(\mathcal{C} \times \mathbf{N})$. The morphisms $(U, n) \to (U, n+1)$ are monomorphisms of $\mathcal{C} \times \mathbf{N}$, hence $\mathcal{J}(U, n+1) \to \mathcal{J}(U, n)$ is surjective (Lemma 21.12.6). It follows that $\mathcal{J}_{n+1} \to \mathcal{J}_n$ is surjective as a map of presheaves.

Note that the functor i_n^{-1} has an exact left adjoint $i_{n,!}$. Namely, $i_{n,!}\mathcal{F}$ is the inverse system $\ldots 0 \to 0 \to \mathcal{F} \to \ldots \to \mathcal{F}$. Thus the complexes $i_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$ are K-injective by Derived Categories, Lemma 13.31.9.

Because we chose our K-injective complex to have injective terms we conclude that

$$0 \to \lim \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

is a short exact sequence of complexes of abelian sheaves as it is a short exact sequence of complexes of abelian presheaves. Moreover, the products in the middle and the right represent the products in $D(\mathcal{C})$, see Injectives, Lemma 19.13.4 and its proof (this is where we use that $\mathcal{I}_n^\bullet$ is K-injective). Thus $R\lim K$ is a homotopy limit of the inverse system (K_n) by definition of homotopy limits in triangulated categories. $\square$

0D6K Lemma 21.23.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The functors $R\Gamma(\mathcal{C}, -)$ and $R\Gamma(U, -)$ for $U \in \text{Ob}(\mathcal{C})$ commute with $R\lim$. Moreover, there are short exact sequences

$$0 \to R^1\lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

for any inverse system (K_n) in $D(\mathcal{O})$ and $m \in \mathbf{Z}$. Similar for $H^m(\mathcal{C}, R\lim K_n)$.

Proof. The first statement follows from Injectives, Lemma 19.13.6. Then we may apply More on Algebra, Remark 15.86.10 to $R\lim R\Gamma(U, K_n) = R\Gamma(U, R\lim K_n)$ to get the short exact sequences. $\square$

0A07 Lemma 21.23.3. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Then Rf_* commutes with $R\lim$, i.e., Rf_* commutes with derived limits.

Proof. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. By induction on n we may choose actual complexes $\mathcal{K}_n^\bullet$ of $\mathcal{O}$-modules and maps of complexes $\mathcal{K}_{n+1}^\bullet \to \mathcal{K}_n^\bullet$ representing the maps $K_{n+1} \to K_n$ in $D(\mathcal{O})$. In other words, there exists an object K in $D(\mathcal{C} \times \mathbf{N})$ whose associated inverse system is the given one. Next, consider the commutative diagram

$$\begin{tikzcd} Sh(\mathcal{C} \times \mathbf{N}) \arrow[r, "g"'] \arrow[d, "f \times 1"'] & Sh(\mathcal{C}) \arrow[d, "f"'] \\ Sh(\mathcal{C}' \times \mathbf{N}) \arrow[r, "g'"] & Sh(\mathcal{C}') \end{tikzcd}$$

of morphisms of topoi. It follows that $R\lim R(f \times 1)_*K = Rf_*R\lim K$. Working through the definitions and using Lemma 21.23.1 we obtain that $R\lim(Rf_*K_n) = Rf_*(R\lim K_n)$.

Alternate proof in case $\mathcal{C}$ has enough points. Consider the defining distinguished triangle

$$R\lim K_n \to \prod K_n \to \prod K_n$$

in $D(\mathcal{O})$. Applying the exact functor Rf_* we obtain the distinguished triangle

$$Rf_*(R\lim K_n) \to Rf_*\left(\prod K_n\right) \to Rf_*\left(\prod K_n\right)$$

in $D(\mathcal{O}')$. Thus we see that it suffices to prove that Rf_* commutes with products in the derived category (which are not just given by products of complexes, see Injectives, Lemma 19.13.4). However, since Rf_* is a right adjoint by Lemma 21.19.1 this follows formally (see Categories, Lemma 4.24.5). Caution: Note that we cannot apply Categories, Lemma 4.24.5 directly as $R\lim K_n$ is not a limit in $D(\mathcal{O})$. □

0BKW Remark 21.23.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system in $D(\mathcal{O})$. Set $K = R\lim K_n$. For each n and m let $\mathcal{H}_n^m = H^m(K_n)$ be the mth cohomology sheaf of K_n and similarly set $\mathcal{H}^m = H^m(K)$. Let us denote $\underline{\mathcal{H}}_n^m$ the presheaf

$$U \longmapsto \underline{\mathcal{H}}_n^m(U) = H^m(U, K_n)$$

Similarly we set $\underline{\mathcal{H}}^m(U) = H^m(U, K)$. By Lemma 21.20.3 we see that $\mathcal{H}_n^m$ is the sheafification of $\underline{\mathcal{H}}_n^m$ and $\mathcal{H}^m$ is the sheafification of $\underline{\mathcal{H}}^m$. Here is a diagram

$$\begin{tikzcd} K \arrow[d, equal] & \underline{\mathcal{H}}^m \arrow[r] \arrow[d] & \mathcal{H}^m \arrow[d] \\ R\lim K_n & \lim \underline{\mathcal{H}}_n^m \arrow[r] & \lim \mathcal{H}_n^m \end{tikzcd}$$

In general it may not be the case that $\lim \mathcal{H}_n^m$ is the sheafification of $\lim \underline{\mathcal{H}}_n^m$. If $U \in \mathcal{C}$, then we have short exact sequences

0BKX (21.23.4.1) $$0 \to R^1\lim \underline{\mathcal{H}}_n^{m-1}(U) \to \underline{\mathcal{H}}^m(U) \to \lim \underline{\mathcal{H}}_n^m(U) \to 0$$

by Lemma 21.23.2.

The following lemma applies to an inverse system of quasi-coherent modules with surjective transition maps on an algebraic space or an algebraic stack.

0BKY Lemma 21.23.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{F}_n)$ be an inverse system of $\mathcal{O}$-modules. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,
(2) $H^p(U, \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$,
(3) the inverse system $\mathcal{F}_n(U)$ has vanishing $R^1 \lim$ for $U \in \mathcal{B}$.

Then $R\lim \mathcal{F}_n = \lim \mathcal{F}_n$ and we have $H^p(U, \lim \mathcal{F}_n) = 0$ for $p > 0$ and $U \in \mathcal{B}$.

Proof. Set $K_n = \mathcal{F}_n$ and $K = R\lim \mathcal{F}_n$. Using the notation of Remark 21.23.4 and assumption (2) we see that for $U \in \mathcal{B}$ we have $\underline{\mathcal{H}}_n^m(U) = 0$ when $m \neq 0$ and $\underline{\mathcal{H}}_n^0(U) = \mathcal{F}_n(U)$. From Equation (21.23.4.1) and assumption (3) we see that $\underline{\mathcal{H}}^m(U) = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n(U)$ when $m = 0$. Sheafifying using (1) we find that $\mathcal{H}^m = 0$ when $m \neq 0$ and equal to $\lim \mathcal{F}_n$ when $m = 0$. Hence $K = \lim \mathcal{F}_n$. Since $H^m(U, K) = \underline{\mathcal{H}}^m(U) = 0$ for $m > 0$ (see above) we see that the second assertion holds. ☐

0D6L Lemma 21.23.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system in $D(\mathcal{O})$. Let $V \in \text{Ob}(\mathcal{C})$ and $m \in \mathbf{Z}$. Assume there exist an integer $n(V)$ and a cofinal system Cov_V of coverings of V such that for $\{V_i \to V\} \in \text{Cov}_V$

(1) $R^1 \lim H^{m-1}(V_i, K_n) = 0$, and
(2) $H^m(V_i, K_n) \to H^m(V_i, K_{n(V)})$ is injective for $n \geq n(V)$.

Then the map on sections $H^m(R\lim K_n)(V) \to H^m(K_{n(V)})(V)$ is injective.

Proof. Let $\gamma \in H^m(R\lim K_n)(V)$ map to zero in $H^m(K_{n(V)})(V)$. Since $H^m(R\lim K_n)$ is the sheafification of $U \mapsto H^m(U, R\lim K_n)$ (by Lemma 21.20.3) we can choose $\{V_i \to V\} \in \text{Cov}_V$ and elements $\tilde{\gamma}_i \in H^m(V_i, R\lim K_n)$ mapping to $\gamma|_{V_i}$. Then $\tilde{\gamma}_i$ maps to $\tilde{\gamma}_{i,n(V)} \in H^m(V_i, K_{n(V)})$. Using that $H^m(K_{n(V)})$ is the sheafification of $U \mapsto H^m(U, K_{n(V)})$ (by Lemma 21.20.3 again) we see that after replacing $\{V_i \to V\}$ by a refinement we may assume that $\tilde{\gamma}_{i,n(V)} = 0$ for all i. For this covering we consider the short exact sequences

$$0 \to R^1 \lim H^{m-1}(V_i, K_n) \to H^m(V_i, R\lim K_n) \to \lim H^m(V_i, K_n) \to 0$$

of Lemma 21.23.2. By assumption (1) the group on the left is zero and by assumption (2) the group on the right maps injectively into $H^m(V_i, K_{n(V)})$. We conclude $\tilde{\gamma}_i = 0$ and hence $\gamma = 0$ as desired. ☐

0D6M Lemma 21.23.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Assume

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$, and
(2) for every $V \in \mathcal{B}$ there exist a function $p(V, -) : \mathbf{Z} \to \mathbf{Z}$ and a cofinal system Cov_V of coverings of V such that
$$H^p(V_i, H^{m-p}(E)) = 0$$
for all $\{V_i \to V\} \in \text{Cov}_V$ and all integers p, m satisfying $p > p(V, m)$.

Then the canonical map $E \to R\lim \tau_{\geq -n}E$ is an isomorphism in $D(\mathcal{O})$.

Proof. Set $K_n = \tau_{\geq -n}E$ and $K = R\lim K_n$. The canonical map $E \to K$ comes from the canonical maps $E \to K_n = \tau_{\geq -n}E$. We have to show that $E \to K$ induces an isomorphism $H^m(E) \to H^m(K)$ of cohomology sheaves. In the rest of the proof we fix m. If $n \geq -m$, then the map $E \to \tau_{\geq -n}E = K_n$ induces an isomorphism $H^m(E) \to H^m(K_n)$. To finish the proof it suffices to show that for every $V \in \mathcal{B}$ there exists an integer $n(V) \geq -m$ such that the map $H^m(K)(V) \to H^m(K_{n(V)})(V)$ is injective. Namely, then the composition

$$H^m(E)(V) \to H^m(K)(V) \to H^m(K_{n(V)})(V)$$

is a bijection and the second arrow is injective, hence the first arrow is bijective. By property (1) this will imply $H^m(E) \to H^m(K)$ is an isomorphism. Set

$$n(V) = 1 + \max\{-m, p(V, m-1) - m, -1 + p(V, m) - m, -2 + p(V, m+1) - m\}.$$

so that in any case $n(V) \geq -m$. Claim: the maps

$$H^{m-1}(V_i, K_{n+1}) \to H^{m-1}(V_i, K_n) \quad \text{and} \quad H^m(V_i, K_{n+1}) \to H^m(V_i, K_n)$$

are isomorphisms for $n \geq n(V)$ and $\{V_i \to V\} \in \text{Cov}_V$. The claim implies conditions (1) and (2) of Lemma 21.23.6 are satisfied and hence implies the desired injectivity. Recall (Derived Categories, Remark 13.12.4) that we have distinguished triangles

$$H^{-n-1}(E)[n+1] \to K_{n+1} \to K_n \to H^{-n-1}(E)[n+2]$$

Looking at the asssociated long exact cohomology sequence the claim follows if

$$H^{m+n}(V_i, H^{-n-1}(E)), \quad H^{m+n+1}(V_i, H^{-n-1}(E)), \quad H^{m+n+2}(V_i, H^{-n-1}(E))$$

are zero for $n \geq n(V)$ and $\{V_i \to V\} \in \text{Cov}_V$. This follows from our choice of $n(V)$ and the assumption in the lemma. $\square$

0D6N Lemma 21.23.8. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Assume

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$, and
(2) for every $V \in \mathcal{B}$ there exist an integer $d_V \geq 0$ and a cofinal system Cov_V of coverings of V such that
$$H^p(V_i, H^q(E)) = 0 \text{ for } \{V_i \to V\} \in \text{Cov}_V,\ p > d_V, \text{ and } q < 0$$

Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O})$.

Proof. This follows from Lemma 21.23.7 with $p(V, m) = d_V + \max(0, m)$. $\square$

08U3 Lemma 21.23.9. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Assume there exists a function $p(-) : \mathbf{Z} \to \mathbf{Z}$ and a subset $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ such that

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,
(2) $H^p(V, H^{m-p}(E)) = 0$ for $p > p(m)$ and $V \in \mathcal{B}$.

Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O})$.

Proof. Apply Lemma 21.23.7 with $p(V, m) = p(m)$ and Cov_V equal to the set of coverings $\{V_i \to V\}$ with $V_i \in \mathcal{B}$ for all i. $\square$

0D6P Lemma 21.23.10. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $E \in D(\mathcal{O})$. Assume there exists an integer $d \geq 0$ and a subset $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ such that

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,
(2) $H^p(V, H^q(E)) = 0$ for $p > d$, $q < 0$, and $V \in \mathcal{B}$.

Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O})$.

Proof. Apply Lemma 21.23.8 with $d_V = d$ and Cov_V equal to the set of coverings $\{V_i \to V\}$ with $V_i \in \mathcal{B}$ for all i. $\square$

The lemmas above can be used to compute cohomology in certain situations.

0BKZ Lemma 21.23.11. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Assume

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,

(2) $H^p(U, H^q(K)) = 0$ for all $p > 0$, $q \in \mathbf{Z}$, and $U \in \mathcal{B}$.

Then $H^q(U, K) = H^0(U, H^q(K))$ for $q \in \mathbf{Z}$ and $U \in \mathcal{B}$.

Proof. Observe that $K = R\lim \tau_{\geq -n} K$ by Lemma 21.23.10 with $d = 0$. Let $U \in \mathcal{B}$. By Equation (21.23.4.1) we get a short exact sequence

$$0 \to R^1 \lim H^{q-1}(U, \tau_{\geq -n} K) \to H^q(U, K) \to \lim H^q(U, \tau_{\geq -n} K) \to 0$$

Condition (2) implies $H^q(U, \tau_{\geq -n} K) = H^0(U, H^q(\tau_{\geq -n} K))$ for all q by using the spectral sequence of Derived Categories, Lemma 13.21.3. The spectral sequence converges because $\tau_{\geq -n} K$ is bounded below. If $n > -q$ then we have $H^q(\tau_{\geq -n} K) = H^q(K)$. Thus the systems on the left and the right of the displayed short exact sequence are eventually constant with values $H^0(U, H^{q-1}(K))$ and $H^0(U, H^q(K))$ and the lemma follows. $\square$

Here is another case where we can describe the derived limit.

0A09 Lemma 21.23.12. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K_n) be an inverse system of objects of $D(\mathcal{O})$. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,
(2) for all $U \in \mathcal{B}$ and all $q \in \mathbf{Z}$ we have
  (a) $H^p(U, H^q(K_n)) = 0$ for $p > 0$,
  (b) the inverse system $H^0(U, H^q(K_n))$ has vanishing $R^1 \lim$.

Then $H^q(R\lim K_n) = \lim H^q(K_n)$ for $q \in \mathbf{Z}$.

Proof. Set $K = R\lim K_n$. We will use notation as in Remark 21.23.4. Let $U \in \mathcal{B}$. By Lemma 21.23.11 and (2)(a) we have $H^q(U, K_n) = H^0(U, H^q(K_n))$. Using that the functor $R\Gamma(U, -)$ commutes with derived limits we have

$$H^q(U, K) = H^q(R\lim R\Gamma(U, K_n)) = \lim H^0(U, H^q(K_n))$$

where the final equality follows from More on Algebra, Remark 15.86.10 and assumption (2)(b). Thus $H^q(U, K)$ is the inverse limit the sections of the sheaves $H^q(K_n)$ over U. Since $\lim H^q(K_n)$ is a sheaf we find using assumption (1) that $H^q(K)$, which is the sheafification of the presheaf $U \mapsto H^q(U, K)$, is equal to $\lim H^q(K_n)$. This proves the lemma. $\square$

21.24. Producing K-injective resolutions

070N Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. The category $\mathrm{Mod}(\mathcal{O})$ has enough injectives, hence we can use Derived Categories, Lemma 13.29.3 produce a diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \tau_{\geq -2}\mathcal{F}^\bullet \arrow[r] \arrow[d] & \tau_{\geq -1}\mathcal{F}^\bullet \arrow[d] \\
\ldots \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_1^\bullet
\end{tikzcd}$$

in the category of complexes of $\mathcal{O}$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 13.31.4 and 13.31.8 this is a K-injective complex. In general the canonical map

070P (21.24.0.1) $$\mathcal{F}^\bullet \to \mathcal{I}^\bullet$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

070Q Lemma 21.24.1. In the situation described above. Denote $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$ the mth cohomology sheaf. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,
(2) for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$ and $q < 0$[5].

Then (21.24.0.1) is a quasi-isomorphism.

Proof. By Derived Categories, Lemma 13.34.4 it suffices to show that the canonical map $\mathcal{F}^\bullet \to R\lim \tau_{\geq -n}\mathcal{F}^\bullet$ is an isomorphism. This follows from Lemma 21.23.10. □

Here is a technical lemma about cohomology sheaves of termwise limits of inverse systems of complexes of modules. We should avoid using this lemma as much as possible and instead use arguments with derived inverse limits.

08CT Lemma 21.24.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$. Suppose given $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ and an integer n_0 such that

(1) every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,
(2) for every $U \in \mathcal{B}$
    (a) the systems of abelian groups $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ have vanishing $R^1\lim$ (for example these have the Mittag-Leffler property),
    (b) the system of abelian groups $H^{m-1}(\mathcal{F}_n^\bullet(U))$ has vanishing $R^1\lim$ (for example it has the Mittag-Leffler property), and
    (c) we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.

Then the maps $H^m(\mathcal{F}^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves where $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$ is the termwise inverse limit.

Proof. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^{m}(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply More on Algebra, Lemma 15.86.3 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. □

[5]It suffices if $\forall m$, $\exists p(m)$, $H^p(U.\mathcal{H}^{m-p}) = 0$ for $p > p(m)$, see Lemma 21.23.9.

21.25. Bounded cohomological dimension

0D6Q In this section we ask when a functor Rf_* has bounded cohomological dimension. This is a rather subtle question when we consider unbounded complexes.

0D6R Situation 21.25.1. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{A} \subset \text{Mod}(\mathcal{O})$ be a weak Serre subcategory. We assume the following is true: there exists a subset $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ such that

(1) every object of $\mathcal{C}$ has a covering whose members are in $\mathcal{B}$, and
(2) for every $V \in \mathcal{B}$ there exists an integer d_V and a cofinal system Cov_V of coverings of V such that

$$H^p(V_i, \mathcal{F}) = 0 \text{ for } \{V_i \to V\} \in \text{Cov}_V,\ p > d_V,\text{ and } \mathcal{F} \in \text{Ob}(\mathcal{A})$$

0D6S Lemma 21.25.2. In Situation 21.25.1 for any $E \in D_\mathcal{A}(\mathcal{O})$ the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism in $D(\mathcal{O})$.

This is [LO08a, Proposition 2.1.4] with slightly changed hypotheses; it is the analogue of [Spa88, Proposition 3.13] for sites.

Proof. Follows immediately from Lemma 21.23.8. $\square$

0D6T Lemma 21.25.3. In Situation 21.25.1 let (K_n) be an inverse system in $D^+_\mathcal{A}(\mathcal{O})$. Assume that for every j the inverse system $(H^j(K_n))$ in $\mathcal{A}$ is eventually constant with value $\mathcal{H}^j$. Then $H^j(R\lim K_n) = \mathcal{H}^j$ for all j.

Proof. Let $V \in \mathcal{B}$. Let $\{V_i \to V\}$ be in the set Cov_V of Situation 21.25.1. Because K_n is bounded below there is a spectral sequence

$$E_2^{p,q} = H^p(V_i, H^q(K_n))$$

converging to $H^{p+q}(V_i, K_n)$. See Derived Categories, Lemma 13.21.3. Observe that $E_2^{p,q} = 0$ for $p > d_V$ by assumption. Pick n_0 such that

$$\begin{matrix} \mathcal{H}^{j+1} & = & H^{j+1}(K_n), \\ \mathcal{H}^j & = & H^j(K_n), \\ \ldots, \\ \mathcal{H}^{j-d_V-2} & = & H^{j-d_V-2}(K_n) \end{matrix}$$

for all $n \geq n_0$. Comparing the spectral sequences above for K_n and K_{n_0}, we see that for $n \geq n_0$ the cohomology groups $H^{j-1}(V_i, K_n)$ and $H^j(V_i, K_n)$ are independent of n. It follows that the map on sections $H^j(R\lim K_n)(V) \to H^j(K_n)(V)$ is injective for n large enough (depending on V), see Lemma 21.23.6. Since every object of $\mathcal{C}$ can be covered by elements of $\mathcal{B}$, we conclude that the map $H^j(R\lim K_n) \to \mathcal{H}^j$ is injective.

Surjectivity is shown in a similar manner. Namely, pick $U \in \text{Ob}(\mathcal{C})$ and $\gamma \in \mathcal{H}^j(U)$. We want to lift γ to a section of $H^j(R\lim K_n)$ after replacing U by the members of a covering. Hence we may assume $U = V \in \mathcal{B}$ by property (1) of Situation 21.25.1. Pick n_0 such that

$$\begin{matrix} \mathcal{H}^{j+1} & = & H^{j+1}(K_n), \\ \mathcal{H}^j & = & H^j(K_n), \\ \ldots, \\ \mathcal{H}^{j-d_V-2} & = & H^{j-d_V-2}(K_n) \end{matrix}$$

for all $n \geq n_0$. Choose an element $\{V_i \to V\}$ of Cov_V such that $\gamma|_{V_i} \in \mathcal{H}^j(V_i) = H^j(K_{n_0})(V_i)$ lifts to an element $\gamma_{n_0,i} \in H^j(V_i, K_{n_0})$. This is possible because $H^j(K_{n_0})$ is the sheafification of $U \mapsto H^j(U, K_{n_0})$ by Lemma 21.20.3. By the discussion in the first paragraph of the proof we have that $H^{j-1}(V_i, K_n)$ and $H^j(V_i, K_n)$

are independent of $n \geq n_0$. Hence $\gamma_{n_0,i}$ lifts to an element $\gamma_i \in H^j(V_i, R\lim K_n)$ by Lemma 21.23.2. This finishes the proof. $\square$

0D6U Lemma 21.25.4. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset \text{Mod}(\mathcal{O})$ and $\mathcal{A}' \subset \text{Mod}(\mathcal{O}')$ be weak Serre subcategories. Assume there is an integer N such that

(1) $\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 21.25.1,
(2) $\mathcal{C}', \mathcal{O}', \mathcal{A}'$ satisfy the assumption of Situation 21.25.1,
(3) $R^pf_*\mathcal{F} \in \text{Ob}(\mathcal{A}')$ for $p \geq 0$ and $\mathcal{F} \in \text{Ob}(\mathcal{A})$,
(4) $R^pf_*\mathcal{F} = 0$ for $p > N$ and $\mathcal{F} \in \text{Ob}(\mathcal{A})$,

Then for K in $D_{\mathcal{A}}(\mathcal{O})$ we have

(a) Rf_*K is in $D_{\mathcal{A}'}(\mathcal{O}')$,
(b) the map $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism for $j \geq N-n$.

This is a version of [LO08a, Lemma 2.1.10] with slightly changed hypotheses.

Proof. By Lemma 21.25.2 we have $K = R\lim \tau_{\geq -n}K$. By Lemma 21.23.3 we have $Rf_*K = R\lim Rf_*\tau_{\geq -n}K$. The complexes $Rf_*\tau_{\geq -n}K$ are bounded below. The spectral sequence

$$E_2^{p,q} = R^pf_*H^q(\tau_{\geq -n}K)$$

converging to $H^{p+q}(Rf_*\tau_{\geq -n}K)$ (Derived Categories, Lemma 13.21.3) and assumption (3) show that $Rf_*\tau_{\geq -n}K$ lies in $D^+_{\mathcal{A}'}(\mathcal{O}')$, see Homology, Lemma 12.24.11. Observe that for $m \geq n$ the map

$$Rf_*(\tau_{\geq -m}K) \longrightarrow Rf_*(\tau_{\geq -n}K)$$

induces an isomorphism on cohomology sheaves in degrees $j \geq -n + N$ by the spectral sequences above. Hence we may apply Lemma 21.25.3 to conclude. $\square$

It turns out that we sometimes need a variant of the lemma above where the assumptions are sligthly different.

0D6V Situation 21.25.5. Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. Let $u : \mathcal{C}' \to \mathcal{C}$ be the corresponding continuous functor of sites. Let $\mathcal{A} \subset \text{Mod}(\mathcal{O})$ be a weak Serre subcategory. We assume the following is true: there exists a subset $\mathcal{B}' \subset \text{Ob}(\mathcal{C}')$ such that

(1) every object of $\mathcal{C}'$ has a covering whose members are in $\mathcal{B}'$, and
(2) for every $V' \in \mathcal{B}'$ there exists an integer $d_{V'}$ and a cofinal system $\text{Cov}_{V'}$ of coverings of V' such that

$$H^p(u(V'_i), \mathcal{F}) = 0 \text{ for } \{V'_i \to V'\} \in \text{Cov}_{V'},\ p > d_{V'},\text{ and } \mathcal{F} \in \text{Ob}(\mathcal{A})$$

0D6W Lemma 21.25.6. Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. assume moreover there is an integer N such that

(1) $\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 21.25.1,
(2) $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ and $\mathcal{A}$ satisfy the assumption of Situation 21.25.5,
(3) $R^pf_*\mathcal{F} = 0$ for $p > N$ and $\mathcal{F} \in \text{Ob}(\mathcal{A})$,

Then for K in $D_{\mathcal{A}}(\mathcal{O})$ the map $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism for $j \geq N - n$.

This is a version of [LO08a, Lemma 2.1.10] with slightly changed hypotheses.

Proof. Let K be in $D_{\mathcal{A}}(\mathcal{O})$. By Lemma 21.25.2 we have $K = R\lim \tau_{\geq -n}K$. By Lemma 21.23.3 we have $Rf_*K = R\lim Rf_*(\tau_{\geq -n}K)$. Let $V' \in \mathcal{B}'$ and let $\{V'_i \to V'\}$ be an element of $\text{Cov}_{V'}$. Then we consider

$$H^j(V'_i, Rf_*K) = H^j(u(V'_i), K) \quad \text{and} \quad H^j(V'_i, Rf_*(\tau_{\geq -n}K)) = H^j(u(V'_i), \tau_{\geq -n}K)$$

The assumption in Situation 21.25.5 implies that the last group is independent of n for n large enough depending on j and $d_{V'}$. Some details omitted. We apply this for j and $j-1$ and via Lemma 21.23.2 this gives that

$$H^j(V_i', Rf_*K) = \lim H^j(V_i', Rf_*(\tau_{\geq -n}K))$$

and the system on the right is constant for n larger than a constant depending only on $d_{V'}$ and j. Thus Lemma 21.23.6 implies that

$$H^j(Rf_*K)(V') \longrightarrow \left(\lim H^j(Rf_*(\tau_{\geq -n}K))\right)(V')$$

is injective. Since the elements $V' \in \mathcal{B}'$ cover every object of $\mathcal{C}'$ we conclude that the map $H^j(Rf_*K) \to \lim H^j(Rf_*(\tau_{\geq -n}K))$ is injective. The spectral sequence

$$E_2^{p,q} = R^p f_* H^q(\tau_{\geq -n}K)$$

converging to $H^{p+q}(Rf_*(\tau_{\geq -n}K))$ (Derived Categories, Lemma 13.21.3) and assumption (3) show that $H^j(Rf_*(\tau_{\geq -n}K))$ is constant for $n \geq N - j$. Hence $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is injective for $j \geq N - n$.

Thus we proved the lemma with "isomorphism" in the last line of the lemma replaced by "injective". However, now choose j and n with $j \geq N - n$. Then consider the distinguished triangle

$$\tau_{\leq -n-1}K \to K \to \tau_{\geq -n}K \to (\tau_{\leq -n-1}K)[1]$$

See Derived Categories, Remark 13.12.4. Since $\tau_{\geq -n}\tau_{\leq -n-1}K = 0$, the injectivity already proven for $\tau_{-n-1}K$ implies

$$0 = H^j(Rf_*(\tau_{\leq -n-1}K)) = H^{j+1}(Rf_*(\tau_{\leq -n-1}K)) = H^{j+2}(Rf_*(\tau_{\leq -n-1}K)) = \ldots$$

By the long exact cohomology sequence associated to the distinguished triangle

$$Rf_*(\tau_{\leq -n-1}K) \to Rf_*K \to Rf_*(\tau_{\geq -n}K) \to Rf_*(\tau_{\leq -n-1}K)[1]$$

this implies that $H^j(Rf_*K) \to H^j(Rf_*(\tau_{\geq -n}K))$ is an isomorphism. $\square$

21.26. Mayer-Vietoris

0EVX For the usual statement and proof of Mayer-Vietoris, please see Cohomology, Section 20.8.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider a commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

in the category $\mathcal{C}$. In this situation, given an object K of $D(\mathcal{O})$ we get what looks like the beginning of a distinguished triangle

$$R\Gamma(X, K) \to R\Gamma(Z, K) \oplus R\Gamma(Y, K) \to R\Gamma(E, K)$$

In the following lemma we make this more precise.

0F16 **Lemma 21.26.1.** In the situation above, choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}$-modules representing K. Using -1 times the canonical map for one of the four arrows we get maps of complexes

$$\mathcal{I}^\bullet(X) \xrightarrow{\alpha} \mathcal{I}^\bullet(Z) \oplus \mathcal{I}^\bullet(Y) \xrightarrow{\beta} \mathcal{I}^\bullet(E)$$

with $\beta \circ \alpha = 0$. Thus a canonical map

$$c^K_{X,Z,Y,E} : \mathcal{I}^\bullet(X) \longrightarrow C(\beta)^\bullet[-1]$$

This map is canonical in the sense that a different choice of K-injective complex representing K determines an isomorphic arrow in the derived category of abelian groups. If $c^K_{X,Z,Y,E}$ is an isomorphism, then using its inverse we obtain a canonical distinguished triangle

$$R\Gamma(X, K) \to R\Gamma(Z, K) \oplus R\Gamma(Y, K) \to R\Gamma(E, K) \to R\Gamma(X, K)[1]$$

All of these constructions are functorial in K.

Proof. This lemma proves itself. For example, if $\mathcal{J}^\bullet$ is a second K-injective complex representing K, then we can choose a quasi-isomorphism $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ which determines quasi-isomorphisms between all the complexes in sight. Details omitted. For the construction of cones and the relationship with distinguished triangles see Derived Categories, Sections 13.9 and 13.10. □

0EWP Lemma 21.26.2. In the situation above, let $K_1 \to K_2 \to K_3 \to K_1[1]$ be a distinguished triangle in $D(\mathcal{O})$. If $c^{K_i}_{X,Z,Y,E}$ is a quasi-isomorphism for two i out of $\{1, 2, 3\}$, then it is a quasi-isomorphism for the third i.

Proof. By rotating the triangle we may assume $c^{K_1}_{X,Z,Y,E}$ and $c^{K_2}_{X,Z,Y,E}$ are quasi-isomorphisms. Choose a map $f : \mathcal{I}_1^\bullet \to \mathcal{I}_2^\bullet$ of K-injective complexes of $\mathcal{O}$-modules representing $K_1 \to K_2$. Then K_3 is represented by the K-injective complex $C(f)^\bullet$, see Derived Categories, Lemma 13.31.3. Then the morphism $c^{K_3}_{X,Z,Y,E}$ is an isomorphism as it is the third leg in a map of distinguished triangles in $K(\text{Ab})$ whose other two legs are quasi-isomorphisms. Some details omitted; use Derived Categories, Lemma 13.4.3. □

Let us give a criterion for when this does produce a distinguished triangle.

0EVY Lemma 21.26.3. In the situation above assume

(1) $h_X^\# = h_Y^\# \amalg_{h_E^\#} h_Z^\#$, and
(2) $h_E^\# \to h_Y^\#$ is injective.

Then the construction of Lemma 21.26.1 produces a distinguished triangle

$$R\Gamma(X, K) \to R\Gamma(Z, K) \oplus R\Gamma(Y, K) \to R\Gamma(E, K) \to R\Gamma(X, K)[1]$$

functorial for K in $D(\mathcal{C})$.

Proof. We can represent K by a K-injective complex whose terms are injective abelian sheaves, see Section 21.19. Thus it suffices to show: if $\mathcal{I}$ is an injective abelian sheaf, then

$$0 \to \mathcal{I}(X) \to \mathcal{I}(Z) \oplus \mathcal{I}(Y) \to \mathcal{I}(E) \to 0$$

is a short exact sequence. The first arrow is injective because by condition (1) the map $h_Y \amalg h_Z \to h_X$ becomes surjective after sheafification, which means that $\{Y \to X, Z \to X\}$ can be refined by a covering of X. The last arrow is surjective because $\mathcal{I}(Y) \to \mathcal{I}(E)$ is surjective. Namely, we have $\mathcal{I}(E) = \text{Hom}(\mathbf{Z}_E^\#, \mathcal{I})$, $\mathcal{I}(Y) = \text{Hom}(\mathbf{Z}_Y^\#, \mathcal{I})$, the map $\mathbf{Z}_E^\# \to \mathbf{Z}_Y^\#$ is injective by (2), and $\mathcal{I}$ is an injective abelian sheaf. Please compare with Modules on Sites, Section 18.5. Finally, suppose we have $s \in \mathcal{I}(Y)$ and $t \in \mathcal{F}(Z)$ mapping to the same element of $\mathcal{I}(E)$. Then s and t define a map

$$s \amalg t : h_Y^\# \amalg h_Z^\# \longrightarrow \mathcal{I}$$

which by assumption factors through $h_Y^\# \amalg_{h_E^\#} h_Z^\#$. Thus by assumption (1) we obtain a unique map $h_X^\# \to \mathcal{I}$ which corresponds to an element of $\mathcal{I}(X)$ restricting to s on Y and t on Z. $\square$

0EVZ Lemma 21.26.4. Let $\mathcal{C}$ be a site. Consider a commutative diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{G} \end{tikzcd}$$

of presheaves of sets on $\mathcal{C}$ and assume that

(1) $\mathcal{G}^\# = \mathcal{E}^\# \amalg_{\mathcal{D}^\#} \mathcal{F}^\#$, and
(2) $\mathcal{D}^\# \to \mathcal{F}^\#$ is injective.

Then there is a canonical distinguished triangle

$$R\Gamma(\mathcal{G}, K) \to R\Gamma(\mathcal{E}, K) \oplus R\Gamma(\mathcal{F}, K) \to R\Gamma(\mathcal{D}, K) \to R\Gamma(\mathcal{G}, K)[1]$$

functorial in $K \in D(\mathcal{C})$ where $R\Gamma(\mathcal{G}, -)$ is the cohomology discussed in Section 21.13.

Proof. Since sheafification is exact and since $R\Gamma(\mathcal{G}, -) = R\Gamma(\mathcal{G}^\#, -)$ we may assume $\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}$ are sheaves of sets. Moreover, the cohomology $R\Gamma(\mathcal{G}, -)$ only depends on the topos, not on the underlying site. Hence by Sites, Lemma 7.29.5 we may replace $\mathcal{C}$ by a "larger" site with a subcanonical topology such that $\mathcal{G} = h_X$, $\mathcal{F} = h_Y$, $\mathcal{E} = h_Z$, and $\mathcal{D} = h_E$ for some objects X, Y, Z, E of $\mathcal{C}$. In this case the result follows from Lemma 21.26.3. $\square$

21.27. Comparing two topologies

0EWK Let $\mathcal{C}$ be a category. Let $\text{Cov}(\mathcal{C}) \supset \text{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\text{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\text{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf. See Sites, Examples 7.14.3 and 7.22.3. In the situation above we have the following

(1) $\epsilon_* : Sh(\mathcal{C}_\tau) \to Sh(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ \epsilon_* = \text{id}$,
(2) $\epsilon_* : \text{Ab}(\mathcal{C}_\tau) \to \text{Ab}(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ \epsilon_* = \text{id}$,
(3) $R\epsilon_* : D(\mathcal{C}_\tau) \to D(\mathcal{C}_{\tau'})$ is fully faithful and $\epsilon^{-1} \circ R\epsilon_* = \text{id}$,
(4) if $\mathcal{O}$ is a sheaf of rings for the τ-topology, then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a flat morphism of ringed sites

$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$

(5) $\epsilon_* : \text{Mod}(\mathcal{O}_\tau) \to \text{Mod}(\mathcal{O}_{\tau'})$ is fully faithful and $\epsilon^* \circ \epsilon_* = \text{id}$
(6) $R\epsilon_* : D(\mathcal{O}_\tau) \to D(\mathcal{O}_{\tau'})$ is fully faithful and $\epsilon^* \circ R\epsilon_* = \text{id}$.

Here are some explanations.

Ad (1). Let $\mathcal{F}$ be a sheaf of sets in the τ-topology. Then $\epsilon_*\mathcal{F}$ is just $\mathcal{F}$ viewed as a sheaf in the τ'-topology. Applying ϵ^{-1} means taking the τ-sheafification of $\mathcal{F}$, which doesn't do anything as $\mathcal{F}$ is already a τ-sheaf. Thus $\epsilon^{-1}(\epsilon_*\mathcal{F})) = \mathcal{F}$. The fully faithfulness follows by Categories, Lemma 4.24.4.

Ad (2). This is a consequence of (1) since pullback and pushforward of abelian sheaves is the same as doing those operations on the underlying sheaves of sets.

Ad (3). Let K be an object of $D(\mathcal{C}_\tau)$. To compute $R\epsilon_*K$ we choose a K-injective complex $\mathcal{I}^\bullet$ representing K and we set $R\epsilon_*K = \epsilon_*\mathcal{I}^\bullet$. Since $\epsilon^{-1} : D(\mathcal{C}_{\tau'}) \to D(\mathcal{C}_\tau)$ is computed on an object L by applying the exact functor ϵ^{-1} to any complex of abelian sheaves representing L, we find that $\epsilon^{-1}R\epsilon_*K$ is represented by $\epsilon^{-1}\epsilon_*\mathcal{I}^\bullet$. By Part (1) we have $\mathcal{I}^\bullet = \epsilon^{-1}\epsilon_*\mathcal{I}^\bullet$. In other words, we have $\epsilon^{-1} \circ R\epsilon_* = \text{id}$ and we conclude as before.

Ad (4). Observe that $\epsilon^{-1}\mathcal{O}_{\tau'} = \mathcal{O}_\tau$, see discussion in part (1). Hence ϵ is a flat morphism of ringed sites, see Modules on Sites, Definition 18.31.1. Not only that, it is moreover clear that $\epsilon^* = \epsilon^{-1}$ on $\mathcal{O}_{\tau'}$-modules (the pullback as a module has the same underlying abelian sheaf as the pullback of the underlying abelian sheaf).

Ad (5). This is clear from (2) and what we said in (4).

Ad (6). This is analogous to (3). We omit the details.

21.28. Formalities on cohomological descent

0D7N In this section we discuss only to what extent a morphism of ringed topoi determines an embedding from the derived category downstairs to the derived category upstairs. Here is a typical result.

0D7Q **Lemma 21.28.1.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the full subcategory $D' \subset D(\mathcal{O}_\mathcal{D})$ consisting of objects K such that

$$K \longrightarrow Rf_*Lf^*K$$

is an isomorphism. Then D' is a saturated triangulated strictly full subcategory of $D(\mathcal{O}_\mathcal{D})$ and the functor $Lf^* : D' \to D(\mathcal{O}_\mathcal{C})$ is fully faithful.

Proof. See Derived Categories, Definition 13.6.1 for the definition of saturated in this setting. See Derived Categories, Lemma 13.4.16 for a discussion of triangulated subcategories. The canonical map of the lemma is the unit of the adjoint pair of functors (Lf^*, Rf_*), see Lemma 21.19.1. Having said this the proof that D' is a saturated triangulated subcategory is omitted; it follows formally from the fact that Lf^* and Rf_* are exact functors of triangulated categories. The final part follows formally from fact that Lf^* and Rf_* are adjoint; compare with Categories, Lemma 4.24.4. □

0D7R **Lemma 21.28.2.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the full subcategory $D' \subset D(\mathcal{O}_\mathcal{C})$ consisting of objects K such that

$$Lf^*Rf_*K \longrightarrow K$$

is an isomorphism. Then D' is a saturated triangulated strictly full subcategory of $D(\mathcal{O}_\mathcal{C})$ and the functor $Rf_* : D' \to D(\mathcal{O}_\mathcal{D})$ is fully faithful.

Proof. See Derived Categories, Definition 13.6.1 for the definition of saturated in this setting. See Derived Categories, Lemma 13.4.16 for a discussion of triangulated subcategories. The canonical map of the lemma is the counit of the adjoint pair of functors (Lf^*, Rf_*), see Lemma 21.19.1. Having said this the proof that D' is a saturated triangulated subcategory is omitted; it follows formally from the fact that Lf^* and Rf_* are exact functors of triangulated categories. The final part follows formally from fact that Lf^* and Rf_* are adjoint; compare with Categories, Lemma 4.24.4. □

0D7S Lemma 21.28.3. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{C})$. Assume

(1) f is flat,
(2) K is bounded below,
(3) $f^*Rf_*H^q(K) \to H^q(K)$ is an isomorphism.

Then $f^*Rf_*K \to K$ is an isomorphism.

Proof. Observe that $f^*Rf_*K \to K$ is an isomorphism if and only if it is an isomorphism on cohomology sheaves H^j. Observe that $H^j(f^*Rf_*K) = f^*H^j(Rf_*K) = f^*H^j(Rf_*\tau_{\leq j}K) = H^j(f^*Rf_*\tau_{\leq j}K)$. Hence we may assume that K is bounded. Then property (3) tells us the cohomology sheaves are in the triangulated subcategory $D' \subset D(\mathcal{O}_\mathcal{C})$ of Lemma 21.28.2. Hence K is in it too. □

0D7T Lemma 21.28.4. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K be an object of $D(\mathcal{O}_\mathcal{D})$. Assume

(1) f is flat,
(2) K is bounded below,
(3) $H^q(K) \to Rf_*f^*H^q(K)$ is an isomorphism.

Then $K \to Rf_*f^*K$ is an isomorphism.

Proof. Observe that $K \to Rf_*f^*K$ is an isomorphism if and only if it is an isomorphism on cohomology sheaves H^j. Observe that $H^j(Rf_*f^*K) = H^j(Rf_*\tau_{\leq j}f^*K) = H^j(Rf_*f^*\tau_{\leq j}K)$. Hence we may assume that K is bounded. Then property (3) tells us the cohomology sheaves are in the triangulated subcategory $D' \subset D(\mathcal{O}_\mathcal{D})$ of Lemma 21.28.1. Hence K is in it too. □

0D7U Lemma 21.28.5. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset \text{Mod}(\mathcal{O})$ and $\mathcal{A}' \subset \text{Mod}(\mathcal{O}')$ be weak Serre subcategories. Assume

(1) f is flat,
(2) f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,
(3) $\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \text{Ob}(\mathcal{A}')$.

Then $f^* : D^+_{\mathcal{A}'}(\mathcal{O}') \to D^+_\mathcal{A}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D^+_\mathcal{A}(\mathcal{O}) \to D^+_{\mathcal{A}'}(\mathcal{O}')$.

Proof. By assumptions (2) and (3) and Lemmas 21.28.4 and 21.28.1 we see that $f^* : D^+_{\mathcal{A}'}(\mathcal{O}') \to D^+_\mathcal{A}(\mathcal{O})$ is fully faithful. Let $\mathcal{F} \in \text{Ob}(\mathcal{A})$. Then we can write $\mathcal{F} = f^*\mathcal{F}'$. Then $Rf_*\mathcal{F} = Rf_*f^*\mathcal{F}' = \mathcal{F}'$. In particular, we have $R^pf_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F} \in \text{Ob}(\mathcal{A}')$. Thus for any $K \in D^+_\mathcal{A}(\mathcal{O})$ we see, using the spectral sequence $E_2^{p,q} = R^pf_*H^q(K)$ converging to $R^{p+q}f_*K$, that Rf_*K is in $D^+_{\mathcal{A}'}(\mathcal{O}')$. Of course, it also follows from Lemmas 21.28.3 and 21.28.2 that $Rf_* : D^+_\mathcal{A}(\mathcal{O}) \to D^+_{\mathcal{A}'}(\mathcal{O}')$ is fully faithful. Since f^* and Rf_* are adjoint we then get the result of the lemma, for example by Categories, Lemma 4.24.4. □

0D7V Lemma 21.28.6. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A} \subset \textit{Mod}(\mathcal{O})$ and $\mathcal{A}' \subset \textit{Mod}(\mathcal{O}')$ be weak Serre subcategories. Assume

This is analogous to [LO08a, Theorem 2.2.3].

(1) f is flat,
(2) f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,
(3) $\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \text{Ob}(\mathcal{A}')$,
(4) $\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 21.25.1,
(5) $\mathcal{C}', \mathcal{O}', \mathcal{A}'$ satisfy the assumption of Situation 21.25.1.

Then $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_{\mathcal{A}}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D_{\mathcal{A}}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$.

Proof. Since f^* is exact, it is clear that f^* defines a functor $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_{\mathcal{A}}(\mathcal{O})$ as in the statement of the lemma and that moreover this functor commutes with the truncation functors $\tau_{\geq -n}$. We already know that f^* and Rf_* are quasi-inverse equivalence on the corresponding bounded below categories, see Lemma 21.28.5. By Lemma 21.25.4 with $N = 0$ we see that Rf_* indeed defines a functor $Rf_* : D_{\mathcal{A}}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$ and that moreover this functor commutes with the truncation functors $\tau_{\geq -n}$. Thus for K in $D_{\mathcal{A}}(\mathcal{O})$ the map $f^*Rf_*K \to K$ is an isomorphism as this is true on trunctions. Similarly, for K' in $D_{\mathcal{A}'}(\mathcal{O}')$ the map $K' \to Rf_*f^*K'$ is an isomorphism as this is true on trunctions. This finishes the proof. $\square$

0D7W Lemma 21.28.7. Let $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ be a morphism of ringed sites. Let $\mathcal{A} \subset \textit{Mod}(\mathcal{O})$ and $\mathcal{A}' \subset \textit{Mod}(\mathcal{O}')$ be weak Serre subcategories. Assume

This is analogous to [LO08a, Theorem 2.2.3].

(1) f is flat,
(2) f^* induces an equivalence of categories $\mathcal{A}' \to \mathcal{A}$,
(3) $\mathcal{F}' \to Rf_*f^*\mathcal{F}'$ is an isomorphism for $\mathcal{F}' \in \text{Ob}(\mathcal{A}')$,
(4) $\mathcal{C}, \mathcal{O}, \mathcal{A}$ satisfy the assumption of Situation 21.25.1,
(5) $f : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ and $\mathcal{A}$ satisfy the assumption of Situation 21.25.5.

Then $f^* : D_{\mathcal{A}'}(\mathcal{O}') \to D_{\mathcal{A}}(\mathcal{O})$ is an equivalence of categories with quasi-inverse given by $Rf_* : D_{\mathcal{A}}(\mathcal{O}) \to D_{\mathcal{A}'}(\mathcal{O}')$.

Proof. The proof of this lemma is exactly the same as the proof of Lemma 21.28.6 except the reference to Lemma 21.25.4 is replaced by a reference to Lemma 21.25.6. $\square$

21.29. Comparing two topologies, II

0F17 Let $\mathcal{C}$ be a category. Let $\text{Cov}(\mathcal{C}) \supset \text{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\text{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\text{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf (hence clearly exact). Let $\mathcal{O}$ be a sheaf of rings for the τ-topology. Then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a morphism of ringed sites

$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$

For more discussion, see Section 21.27.

07A8 Lemma 21.29.1. With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Let $\mathcal{A} \subset \textit{PMod}(\mathcal{O})$ be a full subcategory. Assume

(1) every object of $\mathcal{A}$ is a sheaf for the τ-topology,
(2) $\mathcal{A}$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_\tau)$,
(3) every object of $\mathcal{C}$ has a τ'-covering whose members are elements of $\mathcal{B}$, and
(4) for every $U \in \mathcal{B}$ we have $H^p_\tau(U, \mathcal{F}) = 0$, $p > 0$ for all $\mathcal{F} \in \mathcal{A}$.

Then $\mathcal{A}$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\tau'})$ and there is an equivalence of triangulated categories $D_\mathcal{A}(\mathcal{O}_\tau) = D_\mathcal{A}(\mathcal{O}_{\tau'})$ given by ϵ^* and $R\epsilon_*$.

Proof. Since $\epsilon^{-1}\mathcal{O}_{\tau'} = \mathcal{O}_\tau$ we see that ϵ is a flat morphism of ringed sites and that in fact $\epsilon^{-1} = \epsilon^*$ on sheaves of modules. By property (1) we can think of every object of $\mathcal{A}$ as a sheaf of $\mathcal{O}_\tau$-modules and as a sheaf of $\mathcal{O}_{\tau'}$-modules. In other words, we have fully faithful inclusion functors

$$\mathcal{A} \to \text{Mod}(\mathcal{O}_\tau) \to \text{Mod}(\mathcal{O}_{\tau'})$$

To avoid confusion we will denote $\mathcal{A}' \subset \text{Mod}(\mathcal{O}_{\tau'})$ the image of $\mathcal{A}$. Then it is clear that $\epsilon_* : \mathcal{A} \to \mathcal{A}'$ and $\epsilon^* : \mathcal{A}' \to \mathcal{A}$ are quasi-inverse equivalences (see discussion preceding the lemma and use that objects of $\mathcal{A}'$ are sheaves in the τ topology).

Conditions (3) and (4) imply that $R^p\epsilon_*\mathcal{F} = 0$ for $p > 0$ and $\mathcal{F} \in \text{Ob}(\mathcal{A})$. This is true because $R^p\epsilon_*$ is the sheaf associated to the presheave $U \mapsto H^p_\tau(U, \mathcal{F})$, see Lemma 21.7.4. Thus any exact complex in $\mathcal{A}$ (which is the same thing as an exact complex in $\text{Mod}(\mathcal{O}_\tau)$ whose terms are in $\mathcal{A}$, see Homology, Lemma 12.10.3) remains exact upon applying the functor ϵ_*.

Consider an exact sequence

$$\mathcal{F}'_0 \to \mathcal{F}'_1 \to \mathcal{F}'_2 \to \mathcal{F}'_3 \to \mathcal{F}'_4$$

in $\text{Mod}(\mathcal{O}_{\tau'})$ with $\mathcal{F}'_0, \mathcal{F}'_1, \mathcal{F}'_3, \mathcal{F}'_4$ in $\mathcal{A}'$. Apply the exact functor ϵ^* to get an exact sequence

$$\epsilon^*\mathcal{F}'_0 \to \epsilon^*\mathcal{F}'_1 \to \epsilon^*\mathcal{F}'_2 \to \epsilon^*\mathcal{F}'_3 \to \epsilon^*\mathcal{F}'_4$$

in $\text{Mod}(\mathcal{O}_\tau)$. Since $\mathcal{A}$ is a weak Serre subcategory and since $\epsilon^*\mathcal{F}'_0, \epsilon^*\mathcal{F}'_1, \epsilon^*\mathcal{F}'_3, \epsilon^*\mathcal{F}'_4$ are in $\mathcal{A}$, we conclude that $\epsilon^*\mathcal{F}_2$ is in $\mathcal{A}$ by Homology, Definition 12.10.1. Consider the map of sequences

$$\begin{tikzcd}
\mathcal{F}'_0 \arrow[r] \arrow[d] & \mathcal{F}'_1 \arrow[r] \arrow[d] & \mathcal{F}'_2 \arrow[r] \arrow[d] & \mathcal{F}'_3 \arrow[r] \arrow[d] & \mathcal{F}'_4 \arrow[d] \\
\epsilon_*\epsilon^*\mathcal{F}'_0 \arrow[r] & \epsilon_*\epsilon^*\mathcal{F}'_1 \arrow[r] & \epsilon_*\epsilon^*\mathcal{F}'_2 \arrow[r] & \epsilon_*\epsilon^*\mathcal{F}'_3 \arrow[r] & \epsilon_*\epsilon^*\mathcal{F}'_4
\end{tikzcd}$$

The lower row is exact by the discussion in the preceding paragraph. The vertical arrows with index 0, 1, 3, 4 are isomorphisms by the discussion in the first paragraph. By the 5 lemma (Homology, Lemma 12.5.20) we find that $\mathcal{F}'_2 \cong \epsilon_*\epsilon^*\mathcal{F}'_2$ and hence $\mathcal{F}'_2$ is in $\mathcal{A}'$. In this way we see that $\mathcal{A}'$ is a weak Serre subcategory of $\text{Mod}(\mathcal{O}_{\tau'})$, see Homology, Definition 12.10.1.

At this point it makes sense to talk about the derived categories $D_\mathcal{A}(\mathcal{O}_\tau)$ and $D_{\mathcal{A}'}(\mathcal{O}_{\tau'})$, see Derived Categories, Section 13.17. To finish the proof we show that conditions (1) – (5) of Lemma 21.28.7 apply. We have already seen (1), (2), (3) above. Note that since every object has a τ'-covering by objects of $\mathcal{B}$, a fortiori every object has a τ-covering by objects of $\mathcal{B}$. Hence condition (4) of Lemma 21.28.7 is satisfied. Similarly, condition (5) is satisfied as well. $\square$

0F18 **Lemma 21.29.2.** With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let A be a set and for $\alpha \in A$ let

$$\begin{tikzcd} E_\alpha \arrow[r] \arrow[d] & Y_\alpha \arrow[d] \\ Z_\alpha \arrow[r] & X_\alpha \end{tikzcd}$$

be a commutative diagram in the category $\mathcal{C}$. Assume that

(1) a τ'-sheaf $\mathcal{F}'$ is a τ-sheaf if $\mathcal{F}'(X_\alpha) = \mathcal{F}'(Z_\alpha) \times_{\mathcal{F}'(E_\alpha)} \mathcal{F}'(Y_\alpha)$ for all α,
(2) for K' in $D(\mathcal{O}_{\tau'})$ in the essential image of $R\epsilon_*$ the maps $c^{K'}_{X_\alpha, Z_\alpha, Y_\alpha, E_\alpha}$ of Lemma 21.26.1 are isomorphisms for all α.

Then $K' \in D^+(\mathcal{O}_{\tau'})$ is in the essential image of $R\epsilon_*$ if and only if the maps $c^{K'}_{X_\alpha, Z_\alpha, Y_\alpha, E_\alpha}$ are isomorphisms for all α.

Proof. The "only if" direction is implied by assumption (2). On the other hand, if K' has a unique nonzero cohomology sheaf, then the "if" direction follows from assumption (1). In general we will use an induction argument to prove the "if" direction. Let us say an object K' of $D^+(\mathcal{O}_{\tau'})$ satisfies (P) if the maps $c^{K'}_{X_\alpha, Z_\alpha, Y_\alpha, E_\alpha}$ are isomorphisms for all $\alpha \in A$.

Namely, let K' be an object of $D^+(\mathcal{O}_{\tau'})$ satisfying (P). Choose a distinguished triangle

$$K' \to R\epsilon_*\epsilon^{-1}K' \to M' \to K'[1]$$

in $D^+(\mathcal{O}_{\tau'})$ where the first arrow is the adjuntion map. By (2) and Lemma 21.26.2 we see that M' has (P). On the other hand, applying ϵ^{-1} and using that $\epsilon^{-1}R\epsilon_* = \text{id}$ by Section 21.27 we find that $\epsilon^{-1}M' = 0$. In the next paragraph we will show $M' = 0$ which finishes the proof.

Let K' be an object of $D^+(\mathcal{O}_{\tau'})$ satisfying (P) with $\epsilon^{-1}K' = 0$. We will show $K' = 0$. Namely, given $n \in \mathbf{Z}$ such that $H^i(K') = 0$ for $i < n$ we will show that $H^n(K') = 0$. For $\alpha \in A$ we have a distinguished triangle

$$R\Gamma_{\tau'}(X_\alpha, K') \to R\Gamma_{\tau'}(Z_\alpha, K') \oplus R\Gamma_{\tau'}(Y_\alpha, K') \to R\Gamma_{\tau'}(E_\alpha, K') \to R\Gamma_{\tau'}(X_\alpha, K')[1]$$

by Lemma 21.26.1. Taking cohomology in degree n and using the assumed vanishing of cohomology sheaves of K' we obtain an exact sequence

$$0 \to H^n_{\tau'}(X_\alpha, K') \to H^n_{\tau'}(Z_\alpha, K') \oplus H^n_{\tau'}(Y_\alpha, K') \to H^n_{\tau'}(E_\alpha, K')$$

which is the same as the exact sequence

$$0 \to \Gamma(X_\alpha, H^n(K')) \to \Gamma(Z_\alpha, H^n(K')) \oplus \Gamma(Y_\alpha, H^n(K')) \to \Gamma(E_\alpha, H^n(K'))$$

We conclude that $H^n(K')$ is a a τ-sheaf by assumption (1). However, since the τ-sheafification $\epsilon^{-1}H^n(K') = H^n(\epsilon^{-1}K')$ is 0 as $\epsilon^{-1}K' = 0$ we conclude that $H^n(K') = 0$ as desired. $\square$

0F19 **Lemma 21.29.3.** With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

be a commutative diagram in the category $\mathcal{C}$ such that

(1) $h_X^\# = h_Y^\# \amalg_{h_E^\#} h_Z^\#$, and
(2) $h_E^\# \to h_Y^\#$ is injective

where $^\#$ denotes τ-sheafification. Then for $K' \in D(\mathcal{O}_{\tau'})$ in the essential image of $R\epsilon_*$ the map $c^{K'}_{X,Z,Y,E}$ of Lemma 21.26.1 (using the τ'-topology) is an isomorphism.

Proof. This helper lemma is an almost immediate consequence of Lemma 21.26.3 and we strongly urge the reader skip the proof. Say $K' = R\epsilon_* K$. Choose a K-injective complex of $\mathcal{O}_\tau$-modules $\mathcal{J}^\bullet$ representing K. Then $\epsilon_*\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}_{\tau'}$-modules representing K', see Lemma 21.20.10. Next,

$$0 \to \mathcal{J}^\bullet(X) \xrightarrow{\alpha} \mathcal{J}^\bullet(Z) \oplus \mathcal{J}^\bullet(Y) \xrightarrow{\beta} \mathcal{J}^\bullet(E) \to 0$$

is a short exact sequence of complexes of abelian groups, see Lemma 21.26.3 and its proof. Since this is the same as the sequence of complexes of abelian groups which is used to define $c^{K'}_{X,Z,Y,E}$, we conclude. $\square$

21.30. Comparing cohomology

0EZ1 We develop some general theory which will help us compare cohomology in different topologies. Given $\mathcal{C}$, τ, and τ' as in Section 21.27 and a morphism $f : X \to Y$ in $\mathcal{C}$ we obtain a commutative diagram of morphisms of topoi

0EZ2 (21.30.0.1)
$$\begin{tikzcd} Sh(\mathcal{C}_\tau/X) \arrow[r, "f_\tau"'] \arrow[d, "\epsilon_X"'] & Sh(\mathcal{C}_\tau/Y) \arrow[d, "\epsilon_Y"] \\ Sh(\mathcal{C}_{\tau'}/X) \arrow[r, "f_{\tau'}"] & Sh(\mathcal{C}_{\tau'}/X) \end{tikzcd}$$

Here the morphism ϵ_X, resp. ϵ_Y is the comparison morphism of Section 21.27 for the category $\mathcal{C}/X$ endowed with the two topologies τ and τ'. The morphisms f_τ and $f_{\tau'}$ are "relocalization" morphisms (Sites, Lemma 7.25.8). The commutativity of the diagram is a special case of Sites, Lemma 7.28.1 (applied with $\mathcal{C} = \mathcal{C}_\tau/Y$, $\mathcal{D} = \mathcal{C}_{\tau'}/Y$, $u = \mathrm{id}$, $U = X$, and $V = X$). We also get $\epsilon_{X,*} \circ f_\tau^{-1} = f_{\tau'}^{-1} \circ \epsilon_{Y,*}$ either from the lemma or because it is obvious.

0EZ3 Situation 21.30.1. With $\mathcal{C}$, τ, and τ' as in Section 21.27. Assume we are given a subset $\mathcal{P} \subset \mathrm{Arrows}(\mathcal{C})$ and for every object X of $\mathcal{C}$ we are given a weak Serre subcategory $\mathcal{A}'_X \subset \mathrm{Ab}(\mathcal{C}_{\tau'}/X)$. We make the following assumption:

0EZ4 (1) given $f : X \to Y$ in $\mathcal{P}$ and $Y' \to Y$ general, then $X \times_Y Y'$ exists and $X \times_Y Y' \to Y'$ is in $\mathcal{P}$,
0EZ5 (2) $f_{\tau'}^{-1}$ sends $\mathcal{A}'_Y$ into $\mathcal{A}'_X$ for any morphism $f : X \to Y$ of $\mathcal{C}$,
0EZ6 (3) given X in $\mathcal{C}$ and $\mathcal{F}'$ in $\mathcal{A}'_X$, then $\mathcal{F}'$ satisfies the sheaf condition for τ-coverings, i.e., $\mathcal{F}' = \epsilon_{X,*}\epsilon_X^{-1}\mathcal{F}'$,
0EZ7 (4) if $f : X \to Y$ in $\mathcal{P}$ and $\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}'_X)$, then $R^i f_{\tau',*}\mathcal{F}' \in \mathrm{Ob}(\mathcal{A}'_Y)$ for $i \geq 0$.
0EZ8 (5) if $\{U_i \to U\}_{i \in I}$ is a τ-covering, then there exist
   (a) a τ'-covering $\{V_j \to U\}_{j \in J}$,
   (b) a τ-covering $\{f_j : W_j \to V_j\}$ consisting of a single $f_j \in \mathcal{P}$, and
   (c) a τ'-covering $\{W_{jk} \to W_j\}_{k \in K_j}$
   such that $\{W_{jk} \to U\}_{j \in J, k \in K_j}$ is a refinement of $\{U_i \to U\}_{i \in I}$.

0EZ9 Lemma 21.30.2. In Situation 21.30.1 for X in $\mathcal{C}$ denote $\mathcal{A}_X$ the objects of $\mathrm{Ab}(\mathcal{C}_\tau/X)$ of the form $\epsilon_X^{-1}\mathcal{F}'$ with $\mathcal{F}'$ in $\mathcal{A}'_X$. Then

(1) for $\mathcal{F}$ in $\textit{Ab}(\mathcal{C}_\tau/X)$ we have $\mathcal{F} \in \mathcal{A}_X \Leftrightarrow \epsilon_{X,*}\mathcal{F} \in \mathcal{A}'_X$, and
(2) f_τ^{-1} sends $\mathcal{A}_Y$ into $\mathcal{A}_X$ for any morphism $f : X \to Y$ of $\mathcal{C}$.

Proof. Part (1) follows from (3) and part (2) follows from (2) and the commutativity of (21.30.0.1) which gives $\epsilon_X^{-1} \circ f_{\tau'}^{-1} = f_\tau^{-1} \circ \epsilon_Y^{-1}$. $\square$

Our next goal is to prove Lemmas 21.30.10 and 21.30.9. We will do this by an induction argument using the following induction hypothesis.

(V_n) For X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$ we have $R^i\epsilon_{X,*}\mathcal{F} = 0$ for $1 \leq i \leq n$.

0EZA Lemma 21.30.3. In Situation 21.30.1 assume (V_n) holds. For $f : X \to Y$ in $\mathcal{P}$ and $\mathcal{F}$ in $\mathcal{A}_X$ we have $R^if_{\tau',*}\epsilon_{X,*}\mathcal{F} = \epsilon_{Y,*}R^if_{\tau,*}\mathcal{F}$ for $i \leq n$.

Proof. We will use the commutative diagram (21.30.0.1) without further mention. In particular have

$$Rf_{\tau',*}R\epsilon_{X,*}\mathcal{F} = R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}$$

Assumption (V_n) tells us that $\epsilon_{X,*}\mathcal{F} \to R\epsilon_{X,*}\mathcal{F}$ is an isomorphism in degrees $\leq n$. Hence $Rf_{\tau',*}\epsilon_{X,*}\mathcal{F} \to Rf_{\tau',*}R\epsilon_{X,*}\mathcal{F}$ is an isomorphism in degrees $\leq n$. We conclude that

$$R^if_{\tau',*}\epsilon_{X,*}\mathcal{F} \to H^i(R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F})$$

is an isomorphism for $i \leq n$. We will prove the lemma by looking at the second page of the spectral sequence of Lemma 21.14.7 for $R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}$. Here is a picture:

$$\begin{matrix} \ldots & \ldots & \ldots & \ldots \\ \epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}R^2f_{\tau,*}\mathcal{F} & \ldots \\ \epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}R^1f_{\tau,*}\mathcal{F} & \ldots \\ \epsilon_{Y,*}f_{\tau,*}\mathcal{F} & R^1\epsilon_{Y,*}f_{\tau,*}\mathcal{F} & R^2\epsilon_{Y,*}f_{\tau,*}\mathcal{F} & \ldots \end{matrix}$$

Let (C_m) be the hypothesis: $R^if_{\tau',*}\epsilon_{X,*}\mathcal{F} = \epsilon_{Y,*}R^if_{\tau,*}\mathcal{F}$ for $i \leq m$. Observe that (C_0) holds. We will show that (C_{m-1}) $\Rightarrow$ (C_m) for $m < n$. Namely, if (C_{m-1}) holds, then for $n \geq p > 0$ and $q \leq m-1$ we have

$$\begin{aligned} R^p\epsilon_{Y,*}R^qf_{\tau,*}\mathcal{F} &= R^p\epsilon_{Y,*}\epsilon_Y^{-1}\epsilon_{Y,*}R^qf_{\tau,*}\mathcal{F} \\ &= R^p\epsilon_{Y,*}\epsilon_Y^{-1}R^qf_{\tau',*}\epsilon_{X,*}\mathcal{F} = 0 \end{aligned}$$

First equality as $\epsilon_Y^{-1}\epsilon_{Y,*} = \text{id}$, the second by ($C_{m-1}$), and the final by by ($V_n$) because $\epsilon_Y^{-1}R^qf_{\tau',*}\epsilon_{X,*}\mathcal{F}$ is in $\mathcal{A}_Y$ by (4). Looking at the spectral sequence we see that $E_2^{0,m} = \epsilon_{Y,*}R^mf_{\tau,*}\mathcal{F}$ is the only nonzero term $E_2^{p,q}$ with $p + q = m$. Recall that $\text{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. Hence there are no nonzero differentials $\text{d}_r^{p,q}$, $r \geq 2$ either emanating or entering this spot. We conclude that $H^m(R\epsilon_{Y,*}Rf_{\tau,*}\mathcal{F}) = \epsilon_{Y,*}R^mf_{\tau,*}\mathcal{F}$ which implies (C_m) by the discussion above.

Finally, assume (C_{n-1}). The same analysis shows that $E_2^{0,n} = \epsilon_{Y,*}R^nf_{\tau,*}\mathcal{F}$ is the only nonzero term $E_2^{p,q}$ with $p + q = n$. We do still have no nonzero differentials entering this spot, but there can be a nonzero differential emanating it. Namely, the map $d_{n+1}^{0,n} : \epsilon_{Y,*}R^nf_{\tau,*}\mathcal{F} \to R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$. We conclude that there is an exact sequence

$$0 \to R^nf_{\tau',*}\epsilon_{X,*}\mathcal{F} \to \epsilon_{Y,*}R^nf_{\tau,*}\mathcal{F} \to R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$$

By (4) and (3) the sheaf $R^nf_{\tau',*}\epsilon_{X,*}\mathcal{F}$ satisfies the sheaf property for τ-coverings as does $\epsilon_{Y,*}R^nf_{\tau,*}\mathcal{F}$ (use the description of ϵ_* in Section 21.27). However, the τ-sheafification of the τ'-sheaf $R^{n+1}\epsilon_{Y,*}f_{\tau,*}\mathcal{F}$ is zero (by locality of cohomology;

use Lemmas 21.7.3 and 21.7.4). Thus $R^n f_{\tau',*}\epsilon_{X,*}\mathcal{F} \to \epsilon_{Y,*}R^n f_{\tau,*}\mathcal{F}$ has to be an isomorphism and the proof is complete. $\square$

If E', resp. E is an object of $D(\mathcal{C}_{\tau'}/X)$, resp. $D(\mathcal{C}_\tau/X)$ then we will write $H^n_{\tau'}(U, E')$, resp. $H^n_\tau(U, E)$ for the cohomology of E', resp. E over an object U of $\mathcal{C}/X$.

0EZB Lemma 21.30.4. In Situation 21.30.1 if (V_n) holds, then for X in $\mathcal{C}$ and $L \in D(\mathcal{C}_{\tau'}/X)$ with $H^i(L) = 0$ for $i < 0$ and $H^i(L)$ in $\mathcal{A}'_X$ for $0 \leq i \leq n$ we have $H^n_{\tau'}(X, L) = H^n_\tau(X, \epsilon_X^{-1}L)$.

Proof. By Lemma 21.20.5 we have $H^n_\tau(X, \epsilon_X^{-1}L) = H^n_{\tau'}(X, R\epsilon_{X,*}\epsilon_X^{-1}L)$. There is a spectral sequence

$$E_2^{p,q} = R^p\epsilon_{X,*}\epsilon_X^{-1}H^q(L)$$

converging to $H^{p+q}(R\epsilon_{X,*}\epsilon_X^{-1}L)$. By (V_n) we have the vanishing of $E_2^{p,q}$ for $0 < p \leq n$ and $0 \leq q \leq n$. Thus $E_2^{0,q} = \epsilon_{X,*}\epsilon_X^{-1}H^q(L) = H^q(L)$ are the only nonzero terms $E_2^{p,q}$ with $p + q \leq n$. It follows that the map

$$L \longrightarrow R\epsilon_{X,*}\epsilon_X^{-1}L$$

is an isomorphism in degrees $\leq n$ (small detail omitted). Hence we find that $H^i_{\tau'}(X, L) = H^i_{\tau'}(X, R\epsilon_{X,*}\epsilon_X^{-1}L)$ for $i \leq n$. Thus the lemma is proved. $\square$

0EZC Lemma 21.30.5. In Situation 21.30.1 if (V_n) holds, then for X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$ the map $H^{n+1}_{\tau'}(X, \epsilon_{X,*}\mathcal{F}) \to H^{n+1}_\tau(X, \mathcal{F})$ is injective with image those classes which become trivial on a τ'-covering of X.

Proof. Recall that $\epsilon_X^{-1}\epsilon_{X,*}\mathcal{F} = \mathcal{F}$ hence the map is given by pulling back cohomology classes by ϵ_X. The Leray spectral sequence (Lemma 21.14.5)

$$E_2^{p,q} = H^p_{\tau'}(X, R^q\epsilon_{X,*}\mathcal{F}) \Rightarrow H^{p+q}_\tau(X, \mathcal{F})$$

combined with the assumed vanishing gives an exact sequence

$$0 \to H^{n+1}_{\tau'}(X, \epsilon_{X,*}\mathcal{F}) \to H^{n+1}_\tau(X, \mathcal{F}) \to H^0_{\tau'}(X, R^{n+1}\epsilon_{X,*}\mathcal{F})$$

This is a restatement of the lemma. $\square$

0EZD Lemma 21.30.6. In Situation 21.30.1 let $f : X \to Y$ be in $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering. Let $\mathcal{F}'$ be in $\mathcal{A}'_Y$. If $n \geq 0$ and

$$\theta \in \text{Equalizer}\left(H^{n+1}_{\tau'}(X, \mathcal{F}') \rightrightarrows H^{n+1}_{\tau'}(X \times_Y X, \mathcal{F}') \right)$$

then there exists a τ'-covering $\{Y_i \to Y\}$ such that θ restricts to zero in $H^{n+1}_{\tau'}(Y_i \times_Y X, \mathcal{F}')$.

Proof. Observe that $X \times_Y X$ exists by (1). For Z in $\mathcal{C}/Y$ denote $\mathcal{F}'|_Z$ the restriction of $\mathcal{F}'$ to $\mathcal{C}_{\tau'}/Z$. Recall that $H^{n+1}_{\tau'}(X, \mathcal{F}') = H^{n+1}(\mathcal{C}_{\tau'}/X, \mathcal{F}'|_X)$, see Lemma 21.7.1. The lemma asserts that the image $\overline{\theta} \in H^0(Y, R^{n+1}f_{\tau',*}\mathcal{F}'|_X)$ of θ is zero. Consider the cartesian diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

By trivial base change (Lemma 21.21.1) we have

$$f_{\tau'}^{-1}R^{n+1}f_{\tau',*}(\mathcal{F}'|_X) = R^{n+1}\mathrm{pr}_{1,\tau',*}(\mathcal{F}'|_{X \times_Y X})$$

If $\text{pr}_1^{-1}\theta = \text{pr}_2^{-1}\theta$, then the section $f_{\tau'}^{-1}\overline{\theta}$ of $f_{\tau'}^{-1}R^{n+1}f_{\tau',*}(\mathcal{F}'|_X)$ is zero, because it is clear that $\text{pr}_1^{-1}\theta$ maps to the zero element in $H^0(X, R^{n+1}\text{pr}_{1,\tau',*}(\mathcal{F}'|_{X \times_Y X}))$. By (2) we have $\mathcal{F}'|_X$ in $\mathcal{A}'_X$. Thus $\mathcal{G}' = R^{n+1}f_{\tau',*}(\mathcal{F}'|_X)$ is an object of $\mathcal{A}'_Y$ by (4). Thus $\mathcal{G}'$ satisfies the sheaf property for τ-coverings by (3). Since $\{X \to Y\}$ is a τ-covering we conclude that restriction $\mathcal{G}'(Y) \to \mathcal{G}'(X)$ is injective. It follows that $\overline{\theta}$ is zero. □

0EZE Lemma 21.30.7. In Situation 21.30.1 we have $(V_n) \Rightarrow (V_{n+1})$.

Proof. Let X in $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{A}_X$. Let $\xi \in H^{n+1}_\tau(U, \mathcal{F})$ for some U/X. We have to show that ξ restricts to zero on the members of a τ'-covering of U. See Lemma 21.7.4. It follows from this that we may replace U by the members of a τ'-covering of U.

By locality of cohomology (Lemma 21.7.3) we can choose a τ-covering $\{U_i \to U\}$ such that ξ restricts to zero on U_i. Choose $\{V_j \to V\}$, $\{f_j : W_j \to V_j\}$, and $\{W_{jk} \to W_j\}$ as in (5). After replacing both U by V_j and $\mathcal{F}$ by its restriction to $\mathcal{C}_\tau/V_j$, which is allowed by (1), we reduce to the case discussed in the next paragraph.

Here $f : X \to Y$ is an element of $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering, $\mathcal{F}$ is an object of $\mathcal{A}_Y$, and $\xi \in H^{n+1}_\tau(Y, \mathcal{F})$ is such that there exists a τ'-covering $\{X_i \to X\}_{i \in I}$ such that ξ restricts to zero on X_i for all $i \in I$. Problem: show that ξ restricts to zero on a τ'-covering of Y.

By Lemma 21.30.5 there exists a unique τ'-cohomology class $\theta \in H^{n+1}_{\tau'}(X, \epsilon_{X,*}\mathcal{F})$ whose image is $\xi|_X$. Since $\xi|_X$ pulls back to the same class on $X \times_Y X$ via the two projections, we find that the same is true for θ (by uniqueness). By Lemma 21.30.6 we see that after replacing Y by the members of a τ'-covering, we may assume that $\theta = 0$. Consequently, we may assume that $\xi|_X$ is zero.

Let $f : X \to Y$ be an element of $\mathcal{P}$ such that $\{X \to Y\}$ is a τ-covering, $\mathcal{F}$ is an object of $\mathcal{A}_Y$, and $\xi \in H^{n+1}_\tau(Y, \mathcal{F})$ maps to zero in $H^{n+1}_\tau(X, \mathcal{F})$. Problem: show that ξ restricts to zero on a τ'-covering of Y.

The assumptions tell us ξ maps to zero under the map

$$\mathcal{F} \longrightarrow Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

Use Lemma 21.20.5. A simple argument using the distinguished triangle of truncations (Derived Categories, Remark 13.12.4) shows that ξ maps to zero under the map

$$\mathcal{F} \longrightarrow \tau_{\leq n}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

We will compare this with the map $\epsilon_{Y,*}\mathcal{F} \to K$ where

$$K = \tau_{\leq n}Rf_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F} = \tau_{\leq n}Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F}$$

The equality $\epsilon_{X,*}f_\tau^{-1} = f_{\tau'}^{-1}\epsilon_{Y,*}$ is a property of (21.30.0.1). Consider the map

$$Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F} \longrightarrow Rf_{\tau',*}R\epsilon_{X,*}f_\tau^{-1}\mathcal{F} = R\epsilon_{Y,*}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

used in the proof of Lemma 21.30.3 which induces by adjunction a map

$$\epsilon_Y^{-1}Rf_{\tau',*}\epsilon_{X,*}f_\tau^{-1}\mathcal{F} \to Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

Taking trunctions we find a map

$$\epsilon_Y^{-1}K \longrightarrow \tau_{\leq n}Rf_{\tau,*}f_\tau^{-1}\mathcal{F}$$

which is an isomorphism by Lemma 21.30.3; the lemma applies because $f_\tau^{-1}\mathcal{F}$ is in $\mathcal{A}_X$ by Lemma 21.30.2. Choose a distinguished triangle

$$\epsilon_{Y,*}\mathcal{F} \to K \to L \to \epsilon_{Y,*}\mathcal{F}[1]$$

The map $\mathcal{F} \to f_{\tau,*}f_\tau^{-1}\mathcal{F}$ is injective as $\{X \to Y\}$ is a τ-covering. Thus $\epsilon_{Y,*}\mathcal{F} \to \epsilon_{Y,*}f_{\tau,*}f_\tau^{-1}\mathcal{F} = f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F}$ is injective too. Hence L only has nonzero cohomology sheaves in degrees $0, \ldots, n$. As $f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F}$ is in $\mathcal{A}'_Y$ by (2) and (4) we conclude that

$$H^0(L) = \mathrm{Coker}(\epsilon_{Y,*}\mathcal{F} \to f_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F})$$

is in the weak Serre subcategory $\mathcal{A}'_Y$. For $1 \leq i \leq n$ we see that $H^i(L) = R^if_{\tau',*}f_{\tau'}^{-1}\epsilon_{Y,*}\mathcal{F}$ is in $\mathcal{A}'_Y$ by (2) and (4). Pulling back the distinguished triangle above by ϵ_Y we get the distinguished triangle

$$\mathcal{F} \to \tau_{\leq n}Rf_{\tau,*}f_\tau^{-1}\mathcal{F} \to \epsilon_Y^{-1}L \to \mathcal{F}[1]$$

Since ξ maps to zero in the middle term we find that ξ is the image of an element $\xi' \in H^n_\tau(Y, \epsilon_Y^{-1}L)$. By Lemma 21.30.4 we have

$$H^n_{\tau'}(Y, L) = H^n_\tau(Y, \epsilon_Y^{-1}L),$$

Thus we may lift ξ' to an element of $H^n_{\tau'}(Y, L)$ and take the boundary into $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F})$ to see that ξ is in the image of the canonical map $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F}) \to H^{n+1}_\tau(Y, \mathcal{F})$. By locality of cohomology for $H^{n+1}_{\tau'}(Y, \epsilon_{Y,*}\mathcal{F})$, see Lemma 21.7.3, we conclude. □

0EZF Lemma 21.30.8. In Situation 21.30.1 we have that (V_n) is true for all n. Moreover:

(1) For X in $\mathcal{C}$ and $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ the map $K' \to R\epsilon_{X,*}(\epsilon_X^{-1}K')$ is an isomorphism.
(2) For $f : X \to Y$ in $\mathcal{P}$ and $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ we have $Rf_{\tau',*}K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/Y)$ and $\epsilon_Y^{-1}(Rf_{\tau',*}K') = Rf_{\tau,*}(\epsilon_X^{-1}K')$.

Proof. Observe that (V_0) holds as it is the empty condition. Then we get (V_n) for all n by Lemma 21.30.7.

Proof of (1). The object $K = \epsilon_X^{-1}K'$ has cohomology sheaves $H^i(K) = \epsilon_X^{-1}H^i(K')$ in $\mathcal{A}_X$. Hence the spectral sequence

$$E_2^{p,q} = R^p\epsilon_{X,*}H^q(K) \Rightarrow H^{p+q}(R\epsilon_{X,*}K)$$

degenerates by (V_n) for all n and we find

$$H^n(R\epsilon_{X,*}K) = \epsilon_{X,*}H^n(K) = \epsilon_{X,*}\epsilon_X^{-1}H^i(K') = H^i(K').$$

again because $H^i(K')$ is in $\mathcal{A}'_X$. Thus the canonical map $K' \to R\epsilon_{X,*}(\epsilon_X^{-1}K')$ is an isomorphism.

Proof of (2). Using the spectral sequence

$$E_2^{p,q} = R^pf_{\tau',*}H^q(K') \Rightarrow R^{p+q}f_{\tau',*}K'$$

the fact that $R^pf_{\tau',*}H^q(K')$ is in $\mathcal{A}'_Y$ by (4), the fact that $\mathcal{A}'_Y$ is a weak Serre subcategory of $\mathrm{Ab}(\mathcal{C}_{\tau'}/Y)$, and Homology, Lemma 12.24.11 we conclude that $Rf_{\tau',*}K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$. To finish the proof we have to show the base change map

$$\epsilon_Y^{-1}(Rf_{\tau',*}K') \longrightarrow Rf_{\tau,*}(\epsilon_X^{-1}K')$$

is an isomorphism. Comparing the spectral sequence above to the spectral sequence

$$E_2^{p,q} = R^p f_{\tau,*} H^q(\epsilon_X^{-1} K') \Rightarrow R^{p+q} f_{\tau,*} \epsilon_X^{-1} K'$$

we reduce this to the case where K' has a single nonzero cohomology sheaf $\mathcal{F}'$ in $\mathcal{A}'_X$; details omitted. Then Lemma 21.30.3 gives $\epsilon_Y^{-1} R^i f_{\tau',*}\mathcal{F}' = R^i f_{\tau,*}\epsilon_X^{-1}\mathcal{F}'$ for all i and the proof is complete. □

0EZG Lemma 21.30.9. In Situation 21.30.1. For any X in $\mathcal{C}$ the category $\mathcal{A}_X \subset \text{Ab}(\mathcal{C}_\tau/X)$ is a weak Serre subcategory and the functor

$$R\epsilon_{X,*} : D^+_{\mathcal{A}_X}(\mathcal{C}_\tau/X) \longrightarrow D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$$

is an equivalence with quasi-inverse given by ϵ_X^{-1}.

Proof. We need to check the conditions listed in Homology, Lemma 12.10.3 for $\mathcal{A}_X$. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map in $\mathcal{A}_X$, then $\epsilon_{X,*}\varphi : \epsilon_{X,*}\mathcal{F} \to \epsilon_{X,*}\mathcal{G}$ is a map in $\mathcal{A}'_X$. Hence $\text{Ker}(\epsilon_{X,*}\varphi)$ and $\text{Coker}(\epsilon_{X,*}\varphi)$ are objects of $\mathcal{A}'_X$ as this is a weak Serre subcategory of $\text{Ab}(\mathcal{C}_{\tau'}/X)$. Applying ϵ_X^{-1} we obtain an exact sequence

$$0 \to \epsilon_X^{-1}\,\text{Ker}(\epsilon_{X,*}\varphi) \to \mathcal{F} \to \mathcal{G} \to \epsilon_X^{-1}\,\text{Coker}(\epsilon_{X,*}\varphi) \to 0$$

and we see that $\text{Ker}(\varphi)$ and $\text{Coker}(\varphi)$ are in $\mathcal{A}_X$. Finally, suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence in $\text{Ab}(\mathcal{C}_\tau/X)$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ in $\mathcal{A}_X$. Then applying $\epsilon_{X,*}$ we obtain an exact sequence

$$0 \to \epsilon_{X,*}\mathcal{F}_1 \to \epsilon_{X,*}\mathcal{F}_2 \to \epsilon_{X,*}\mathcal{F}_3 \to R^1\epsilon_{X,*}\mathcal{F}_1 = 0$$

Vanishing by Lemma 21.30.8. Hence $\epsilon_{X,*}\mathcal{F}_2$ is in $\mathcal{A}'_X$ as this is a weak Serre subcategory of $\text{Ab}(\mathcal{C}_{\tau'}/X)$. Pulling back by ϵ_X we conclude that $\mathcal{F}_2$ is in $\mathcal{A}_X$.

Thus $\mathcal{A}_X$ is a weak Serre subcategory of $\text{Ab}(\mathcal{C}_\tau/X)$ and it makes sense to consider the category $D^+_{\mathcal{A}_X}(\mathcal{C}_\tau/X)$. Observe that $\epsilon_X^{-1} : \mathcal{A}'_X \to \mathcal{A}_X$ is an equivalence and that $\mathcal{F}' \to R\epsilon_{X,*}\epsilon_X^{-1}\mathcal{F}'$ is an isomorphism for $\mathcal{F}'$ in $\mathcal{A}'_X$ since we have (V_n) for all n by Lemma 21.30.8. Thus we conclude by Lemma 21.28.5. □

0EZH Lemma 21.30.10. In Situation 21.30.1. Let X be in $\mathcal{C}$.

(1) for $\mathcal{F}'$ in $\mathcal{A}'_X$ we have $H^n_{\tau'}(X, \mathcal{F}') = H^n_\tau(X, \epsilon_X^{-1}\mathcal{F}')$,
(2) for $K' \in D^+_{\mathcal{A}'_X}(\mathcal{C}_{\tau'}/X)$ we have $H^n_{\tau'}(X, K') = H^n_\tau(X, \epsilon_X^{-1}K')$.

Proof. This follows from Lemma 21.30.8 by Remark 21.14.4. □

21.31. Cohomology on Hausdorff and locally quasi-compact spaces

09WY We continue our convention to say “Hausdorff and locally quasi-compact” instead of saying “locally compact” as is often done in the literature. Let LC denote the category whose objects are Hausdorff and locally quasi-compact topological spaces and whose morphisms are continuous maps.

09WZ Lemma 21.31.1. The category LC has fibre products and a final object and hence has arbitrary finite limits. Given morphisms $X \to Z$ and $Y \to Z$ in LC with X and Y quasi-compact, then $X \times_Z Y$ is quasi-compact.

Proof. The final object is the singleton space. Given morphisms $X \to Z$ and $Y \to Z$ of LC the fibre product $X \times_Z Y$ is a subspace of $X \times Y$. Hence $X \times_Z Y$ is Hausdorff as $X \times Y$ is Hausdorff by Topology, Section 5.3.

If X and Y are quasi-compact, then $X \times Y$ is quasi-compact by Topology, Theorem 5.14.4. Since $X \times_Z Y$ is a closed subset of $X \times Y$ (Topology, Lemma 5.3.4) we find that $X \times_Z Y$ is quasi-compact by Topology, Lemma 5.12.3.

Finally, returning to the general case, if $x \in X$ and $y \in Y$ we can pick quasi-compact neighbourhoods $x \in E \subset X$ and $y \in F \subset Y$ and we find that $E \times_Z F$ is a quasi-compact neighbourhood of (x, y) by the result above. Thus $X \times_Z Y$ is an object of LC by Topology, Lemma 5.13.2. $\square$

We can endow LC with a stronger topology than the usual one.

09X0 Definition 21.31.2. Let $\{f_i : X_i \to X\}$ be a family of morphisms with fixed target in the category LC. We say this family is a qc covering[6] if for every $x \in X$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x.

Observe that an open covering $X = \bigcup U_i$ of an object of LC gives a qc covering $\{U_i \to X\}$ because X is locally quasi-compact. We start with the obligatory lemma.

09X1 Lemma 21.31.3. Let X be a Hausdorff and locally quasi-compact space, in other words, an object of LC.

(1) If $X' \to X$ is an isomorphism in LC then $\{X' \to X\}$ is a qc covering.
(2) If $\{f_i : X_i \to X\}_{i \in I}$ is a qc covering and for each i we have a qc covering $\{g_{ij} : X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a qc covering.
(3) If $\{X_i \to X\}_{i \in I}$ is a qc covering and $X' \to X$ is a morphism of LC then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a qc covering.

Proof. Part (1) holds by the remark above that open coverings are qc coverings.

Proof of (2). Let $x \in X$. Choose $i_1, \ldots, i_n \in I$ and $E_a \subset X_{i_a}$ quasi-compact such that $\bigcup f_{i_a}(E_a)$ is a neighbourhood of x. For every $e \in E_a$ we can find a finite subset $J_e \subset J_{i_a}$ and quasi-compact $F_{e,j} \subset X_{ij}$, $j \in J_e$ such that $\bigcup g_{ij}(F_{e,j})$ is a neighbourhood of e. Since E_a is quasi-compact we find a finite collection $e_1, \ldots, e_{m_a}$ such that

$$E_a \subset \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} g_{ij}(F_{e_k,j})$$

Then we find that

$$\bigcup\nolimits_{a=1,\ldots,n} \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} f_i(g_{ij}(F_{e_k,j}))$$

is a neighbourhood of x.

Proof of (3). Let $x' \in X'$ be a point. Let $x \in X$ be its image. Choose $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x. Choose a quasi-compact neighbourhood $F \subset X'$ of x' which maps into the quasi-compact neighbourhood $\bigcup f_{i_j}(E_j)$ of x. Then $F \times_X E_j \subset X' \times_X X_{i_j}$ is a quasi-compact subset and F is the image of the map $\coprod F \times_X E_j \to F$. Hence the base change is a qc covering and the proof is finished. $\square$

[6]This is nonstandard notation. We chose it to remind the reader of fpqc coverings of schemes.

Since all objects of LC are Hausdorff any morphism $f : X \to Y$ of LC is a separated continuous map of topological spaces. Hence f is a proper map of topological spaces if and only if f is universally closed. See discussion in Topology, Section 5.17.

09X5 Lemma 21.31.4. Let $f : X \to Y$ be a morphism of LC. If f is proper and surjective, then $\{f : X \to Y\}$ is a qc covering.

Proof. Let $y \in Y$ be a point. For each $x \in X_y$ choose a quasi-compact neighbourhood $E_x \subset X$. Choose $x \in U_x \subset E_x$ open. Since f is proper the fibre X_y is quasi-compact and we find $x_1, \ldots, x_n \in X_y$ such that $X_y \subset U_{x_1} \cup \ldots \cup U_{x_n}$. We claim that $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ is a neighbourhood of y. Namely, as f is closed (Topology, Theorem 5.17.5) we see that $Z = f(X \setminus U_{x_1} \cup \ldots \cup U_{x_n})$ is a closed subset of Y not containing y. As f is surjective we see that $Y \setminus Z$ is contained in $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ as desired. $\square$

Besides some set theoretic issues Lemma 21.31.3 shows that LC with the collection of qc coverings forms a site. We will denote this site (suitably modified to overcome the set theoretical issues) LC_{qc}.

09X2 Remark 21.31.5 (Set theoretic issues). The category LC is a "big" category as its objects form a proper class. Similarly, the coverings form a proper class. Let us define the size of a topological space X to be the cardinality of the set of points of X. Choose a function *Bound* on cardinals, for example as in Sets, Equation (3.9.1.1). Finally, let S_0 be an initial set of objects of LC, for example $S_0 = \{(\mathbf{R}, \text{euclidean topology})\}$. Exactly as in Sets, Lemma 3.9.2 we can choose a limit ordinal α such that $\mathrm{LC}_\alpha = \mathrm{LC} \cap V_\alpha$ contains S_0 and is preserved under all countable limits and colimits which exist in LC. Moreover, if $X \in \mathrm{LC}_\alpha$ and if $Y \in \mathrm{LC}$ and $\text{size}(Y) \leq \mathit{Bound}(\text{size}(X))$, then Y is isomorphic to an object of LC_α. Next, we apply Sets, Lemma 3.11.1 to choose set Cov of qc covering on LC_α such that every qc covering in LC_α is combinatorially equivalent to a covering this set. In this way we obtain a site $(\mathrm{LC}_\alpha, \mathrm{Cov})$ which we will denote LC_{qc}.

There is a second topology on the site LC_{qc} of Remark 21.31.5. Namely, given an object X we can consider all coverings $\{X_i \to X\}$ of LC_{qc} such that $X_i \to X$ is an open immersion. We denote this site LC_{Zar}. The identity functor $\mathrm{LC}_{Zar} \to \mathrm{LC}_{qc}$ is continuous and defines a morphism of sites

$$\epsilon : \mathrm{LC}_{qc} \longrightarrow \mathrm{LC}_{Zar}$$

See Section 21.27. For a Hausdorff and locally quasi-compact topological space X, more precisely for $X \in \mathrm{Ob}(\mathrm{LC}_{qc})$, we denote the induced morphism

$$\epsilon_X : \mathrm{LC}_{qc}/X \longrightarrow \mathrm{LC}_{Zar}/X$$

(see Sites, Lemma 7.28.1). Let X_{Zar} be the site whose objects are opens of X, see Sites, Example 7.6.4. There is a morphism of sites

$$\pi_X : \mathrm{LC}_{Zar}/X \longrightarrow X_{Zar}$$

given by the continuous functor $X_{Zar} \to \mathrm{LC}_{Zar}/X$, $U \mapsto U$. Namely, X_{Zar} has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.14.7 applies. We often think of π as a morphism of topoi

$$\pi_X : \mathit{Sh}(\mathrm{LC}_{Zar}/X) \longrightarrow \mathit{Sh}(X)$$

using the equality $\mathit{Sh}(X_{Zar}) = \mathit{Sh}(X)$.

09X3 Lemma 21.31.6. Let X be an object of LC_{qc}. Let $\mathcal{F}$ be a sheaf on X. The rule

$$\mathrm{LC}_{qc}/X \longrightarrow \text{Sets}, \quad (f : Y \to X) \longmapsto \Gamma(Y, f^{-1}\mathcal{F})$$

is a sheaf and a fortiori also a sheaf on LC_{Zar}/X. This sheaf is equal to $\pi_X^{-1}\mathcal{F}$ on LC_{Zar}/X and $\epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$ on LC_{qc}/X.

Proof. Denote $\mathcal{G}$ the presheaf given by the formula in the lemma. Of course the pullback f^{-1} in the formula denotes usual pullback of sheaves on topological spaces. It is immediate from the definitions that $\mathcal{G}$ is a sheaf for the Zar topology.

Let $Y \to X$ be a morphism in LC_{qc}. Let $\mathcal{V} = \{g_i : Y_i \to Y\}_{i \in I}$ be a qc covering. To prove $\mathcal{G}$ is a sheaf for the qc topology it suffices to show that $\mathcal{G}(Y) \to H^0(\mathcal{V}, \mathcal{G})$ is an isomorphism, see Sites, Section 7.10. We first point out that the map is injective as a qc covering is surjective and we can detect equality of sections at stalks (use Sheaves, Lemmas 6.11.1 and 6.21.4). Thus $\mathcal{G}$ is a separated presheaf on LC_{qc} hence it suffices to show that any element $(s_i) \in H^0(\mathcal{V}, \mathcal{G})$ maps to an element in the image of $\mathcal{G}(Y)$ after replacing $\mathcal{V}$ by a refinement (Sites, Theorem 7.10.10).

Identifying sheaves on $Y_{i,Zar}$ and sheaves on Y_i we find that $\mathcal{G}|_{Y_{i,Zar}}$ is the pullback of $f^{-1}\mathcal{F}$ under the continuous map $g_i : Y_i \to Y$. Thus we can choose an open covering $Y_i = \bigcup V_{ij}$ such that for each j there is an open $W_{ij} \subset Y$ and a section $t_{ij} \in \mathcal{G}(W_{ij})$ such that V_{ij} maps into W_{ij} and such that $s|_{V_{ij}}$ is the pullback of t_{ij}. In other words, after refining the covering $\{Y_i \to Y\}$ we may assume there are opens $W_i \subset Y$ such that $Y_i \to Y$ factors through W_i and sections t_i of $\mathcal{G}$ over W_i which restrict to the given sections s_i. Moreover, if $y \in Y$ is in the image of both $Y_i \to Y$ and $Y_j \to Y$, then the images $t_{i,y}$ and $t_{j,y}$ in the stalk $f^{-1}\mathcal{F}_y$ agree (because s_i and s_j agree over $Y_i \times_Y Y_j$). Thus for $y \in Y$ there is a well defined element t_y of $f^{-1}\mathcal{F}_y$ agreeing with $t_{i,y}$ whenever y is in the image of $Y_i \to Y$. We will show that the element (t_y) comes from a global section of $f^{-1}\mathcal{F}$ over Y which will finish the proof of the lemma.

It suffices to show that this is true locally on Y, see Sheaves, Section 6.17. Let $y_0 \in Y$. Pick $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset Y_{i_j}$ such that $\bigcup g_{i_j}(E_j)$ is a neighbourhood of y_0. Let $V \subset Y$ be an open neighbourhood of y_0 contained in $\bigcup g_{i_j}(E_j)$ and contained in $W_{i_1} \cap \ldots \cap W_{i_n}$. Since $t_{i_1,y_0} = \ldots = t_{i_n,y_0}$, after shrinking V we may assume the sections $t_{i_j}|_V$, $j = 1, \ldots, n$ of $f^{-1}\mathcal{F}$ agree. As $V \subset \bigcup g_{i_j}(E_j)$ we see that $(t_y)_{y \in V}$ comes from this section.

We still have to show that $\mathcal{G}$ is equal to $\epsilon_X^{-1}\pi_X^{-1}\mathcal{F}$ on LC_{qc}, resp. $\pi_X^{-1}\mathcal{F}$ on LC_{Zar}. In both cases the pullback is defined by taking the presheaf

$$(f : Y \to X) \longmapsto \mathrm{colim}_{f(Y) \subset U \subset X}\, \mathcal{F}(U)$$

and then sheafifying. Sheafifying in the Zar topology exactly produces our sheaf $\mathcal{G}$ and the fact that $\mathcal{G}$ is a qc sheaf, shows that it works as well in the qc topology. □

Let $X \in \mathrm{Ob}(\mathrm{LC}_{Zar})$ and let $\mathcal{H}$ be an abelian sheaf on LC_{Zar}/X. Then we will write $H^n_{Zar}(U, \mathcal{H})$ for the cohomology of $\mathcal{H}$ over an object U of LC_{Zar}/X.

0DCU Lemma 21.31.7. Let X be an object of LC_{Zar}. Then

(1) for $\mathcal{F} \in \mathrm{Ab}(X)$ we have $H^n_{Zar}(X, \pi_X^{-1}\mathcal{F}) = H^n(X, \mathcal{F})$,
(2) $\pi_{X,*} : \mathrm{Ab}(\mathrm{LC}_{Zar}/X) \to \mathrm{Ab}(X)$ is exact,
(3) the unit $\mathrm{id} \to \pi_{X,*} \circ \pi_X^{-1}$ of the adjunction is an isomorphism, and

(4) for $K \in D(X)$ the canonical map $K \to R\pi_{X,*}\pi_X^{-1}K$ is an isomorphism.

Let $f : X \to Y$ be a morphism of LC_{Zar}. Then

(5) there is a commutative diagram

$$\begin{tikzcd} Sh(\mathrm{LC}_{Zar}/X) \arrow[r, "f_{Zar}"'] \arrow[d, "\pi_X"'] & Sh(\mathrm{LC}_{Zar}/Y) \arrow[d, "\pi_Y"] \\ Sh(X_{Zar}) \arrow[r, "f"] & Sh(Y_{Zar}) \end{tikzcd}$$

of topoi,

(6) for $L \in D^+(Y)$ we have $H^n_{Zar}(X, \pi_Y^{-1}L) = H^n(X, f^{-1}L)$,

(7) if f is proper, then we have
   (a) $\pi_Y^{-1} \circ f_* = f_{Zar,*} \circ \pi_X^{-1}$ as functors $Sh(X) \to Sh(\mathrm{LC}_{Zar}/Y)$,
   (b) $\pi_Y^{-1} \circ Rf_* = Rf_{Zar,*} \circ \pi_X^{-1}$ as functors $D^+(X) \to D^+(\mathrm{LC}_{Zar}/Y)$.

Proof. Proof of (1). The equality $H^n_{Zar}(X, \pi_X^{-1}\mathcal{F}) = H^n(X, \mathcal{F})$ is a general fact coming from the trivial observation that coverings of X in LC_{Zar} are the same thing as open coverings of X. The reader who wishes to see a detailed proof should apply Lemma 21.7.2 to the functor $X_{Zar} \to \mathrm{LC}_{Zar}$.

Proof of (2). This is true because $\pi_{X,*} = \tau_X^{-1}$ for some morphism of topoi $\tau_X : Sh(X_{Zar}) \to Sh(\mathrm{LC}_{Zar})$ as follows from Sites, Lemma 7.21.8 applied to the functor $X_{Zar} \to \mathrm{LC}_{Zar}/X$ used to define π_X.

Proof of (3). This is true because $\tau_X^{-1} \circ \pi_X^{-1}$ is the identity functor by Sites, Lemma 7.21.8. Or you can deduce it from the explicit description of π_X^{-1} in Lemma 21.31.6.

Proof of (4). Apply (3) to an complex of abelian sheaves representing K.

Proof of (5). The morphism of topoi f_{Zar} comes from an application of Sites, Lemma 7.25.8 and in our case comes from the continuous functor $Z/Y \mapsto Z \times_Y X/X$ by Sites, Lemma 7.27.3. The diagram commutes simply because the corresponding continuous functors compose correctly (see Sites, Lemma 7.14.4).

Proof of (6). We have $H^n_{Zar}(X, \pi_Y^{-1}\mathcal{G}) = H^n_{Zar}(X, f_{Zar}^{-1}\pi_Y^{-1}\mathcal{G})$ for $\mathcal{G}$ in $\mathrm{Ab}(Y)$, see Lemma 21.7.1. This is equal to $H^n_{Zar}(X, \pi_X^{-1}f^{-1}\mathcal{G})$ by the commutativity of the diagram in (5). Hence we conclude by (1) in the case L consists of a single sheaf in degree 0. The general case follows by representing L by a bounded below complex of abelian sheaves.

Proof of (7a). Let $\mathcal{F}$ be a sheaf on X. Let $g : Z \to Y$ be an object of LC_{Zar}/Y. Consider the fibre product

$$\begin{tikzcd} Z' \arrow[r, "f'"'] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then we have

$$(f_{Zar,*}\pi_X^{-1}\mathcal{F})(Z/Y) = (\pi_X^{-1}\mathcal{F})(Z'/X) = \Gamma(Z', (g')^{-1}\mathcal{F}) = \Gamma(Z, f'_*(g')^{-1}\mathcal{F})$$

the second equality by Lemma 21.31.6. On the other hand

$$(\pi_Y^{-1}f_*\mathcal{F})(Z/Y) = \Gamma(Z, g^{-1}f_*\mathcal{F})$$

again by Lemma 21.31.6. Hence by proper base change for sheaves of sets (Cohomology, Lemma 20.18.3) we conclude the two sets are canonically isomorphic. The isomorphism is compatible with restriction mappings and defines an isomorphism $\pi_Y^{-1} f_*\mathcal{F} = f_{Zar,*}\pi_X^{-1}\mathcal{F}$. Thus an isomorphism of functors $\pi_Y^{-1} \circ f_* = f_{Zar,*} \circ \pi_X^{-1}$.

Proof of (7b). Let $K \in D^+(X)$. By Lemma 21.20.6 the nth cohomology sheaf of $Rf_{Zar,*}\pi_X^{-1}K$ is the sheaf associated to the presheaf

$$(g : Z \to Y) \longmapsto H^n_{Zar}(Z', \pi_X^{-1}K)$$

with notation as above. Observe that

$$\begin{aligned} H^n_{Zar}(Z', \pi_X^{-1}K) &= H^n(Z', (g')^{-1}K) \\ &= H^n(Z, Rf'_*(g')^{-1}K) \\ &= H^n(Z, g^{-1}Rf_*K) \\ &= H^n_{Zar}(Z, \pi_Y^{-1}Rf_*K) \end{aligned}$$

The first equality is (6) applied to K and $g' : Z' \to X$. The second equality is Leray for $f' : Z' \to Z$ (Cohomology, Lemma 20.13.1). The third equality is the proper base change theorem (Cohomology, Theorem 20.18.2). The fourth equality is (6) applied to $g : Z \to Y$ and Rf_*K. Thus $Rf_{Zar,*}\pi_X^{-1}K$ and $\pi_Y^{-1}Rf_*K$ have the same cohomology sheaves. We omit the verification that the canonical base change map $\pi_Y^{-1}Rf_*K \to Rf_{Zar,*}\pi_X^{-1}K$ induces this isomorphism. □

In the situation of Lemma 21.31.6 the composition of ϵ and π and the equality $Sh(X) = Sh(X_{Zar})$ determine a morphism of topoi

$$a_X : Sh(\mathrm{LC}_{qc}/X) \longrightarrow Sh(X)$$

0D92 Lemma 21.31.8. Let $f : X \to Y$ be a morphism of LC_{qc}. Then there are commutative diagrams of topoi

$$\begin{tikzcd} Sh(\mathrm{LC}_{qc}/X) \arrow[r, "f_{qc}"'] \arrow[d, "\epsilon_X"'] & Sh(\mathrm{LC}_{qc}/Y) \arrow[d, "\epsilon_Y"] \\ Sh(\mathrm{LC}_{Zar}/X) \arrow[r, "f_{Zar}"] & Sh(\mathrm{LC}_{Zar}/Y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Sh(\mathrm{LC}_{qc}/X) \arrow[r, "f_{qc}"'] \arrow[d, "a_X"'] & Sh(\mathrm{LC}_{qc}/Y) \arrow[d, "a_Y"] \\ Sh(X) \arrow[r, "f"] & Sh(Y) \end{tikzcd}$$

with $a_X = \pi_X \circ \epsilon_X$, $a_Y = \pi_X \circ \epsilon_X$. If f is proper, then $a_Y^{-1} \circ f_* = f_{qc,*} \circ a_X^{-1}$.

Proof. The morphism of topoi f_{qc} is the one from Sites, Lemma 7.25.8 which in our case comes from the continuous functor $Z/Y \mapsto Z \times_Y X/X$, see Sites, Lemma 7.27.3. The diagram on the left commutes because the corresponding continuous functors compose correctly (see Sites, Lemma 7.14.4). The diagram on the right commutes because the one on the left does and because of part (5) of Lemma 21.31.7.

Proof of the final assertion. The reader may repeat the proof of part (7a) of Lemma 21.31.7; we will instead deduce this from it. As $\epsilon_{Y,*}$ is the identity functor on underlying presheaves, it reflects isomorphisms. The description in Lemma 21.31.6 shows that $\epsilon_{Y,*} \circ a_Y^{-1} = \pi_Y^{-1}$ and similarly for X. To show that the canonical map $a_Y^{-1}f_*\mathcal{F} \to f_{qc,*}a_X^{-1}\mathcal{F}$ is an isomorphism, it suffices to show that

$$\pi_Y^{-1}f_*\mathcal{F} = \epsilon_{Y,*}a_Y^{-1}f_*\mathcal{F} \to \epsilon_{Y,*}f_{qc,*}a_X^{-1}\mathcal{F} = f_{Zar,*}\epsilon_{X,*}a_X^{-1}\mathcal{F} = f_{Zar,*}\pi_X^{-1}\mathcal{F}$$

is an isomorphism. This is part (7a) of Lemma 21.31.7. □

0EZI Lemma 21.31.9. Consider the comparison morphism $\epsilon : \mathrm{LC}_{qc} \to \mathrm{LC}_{Zar}$. Let $\mathcal{P}$ denote the class of proper maps of topological spaces. For X in LC_{Zar} denote $\mathcal{A}'_X \subset \mathrm{Ab}(\mathrm{LC}_{Zar}/X)$ the full subcategory consisting of sheaves of the form $\pi_X^{-1}\mathcal{F}$ with $\mathcal{F}$ in $\mathrm{Ab}(X)$. Then (1), (2), (3), (4), and (5) of Situation 21.30.1 hold.

Proof. We first show that $\mathcal{A}'_X \subset \mathrm{Ab}(\mathrm{LC}_{Zar}/X)$ is a weak Serre subcategory by checking conditions (1), (2), (3), and (4) of Homology, Lemma 12.10.3. Parts (1), (2), (3) are immediate as π_X^{-1} is exact and fully faithful by Lemma 21.31.7 part (3). If $0 \to \pi_X^{-1}\mathcal{F} \to \mathcal{G} \to \pi_X^{-1}\mathcal{F}' \to 0$ is a short exact sequence in $\mathrm{Ab}(\mathrm{LC}_{Zar}/X)$ then $0 \to \mathcal{F} \to \pi_{X,*}\mathcal{G} \to \mathcal{F}' \to 0$ is exact by Lemma 21.31.7 part (2). Hence $\mathcal{G} = \pi_X^{-1}\pi_{X,*}\mathcal{G}$ is in $\mathcal{A}'_X$ which checks the final condition.

Property (1) holds by Lemma 21.31.1 and the fact that the base change of a proper map is a proper map (see Topology, Theorem 5.17.5 and Lemma 5.4.4).

Property (2) follows from the commutative diagram (5) in Lemma 21.31.7.

Property (3) is Lemma 21.31.6.

Property (4) is Lemma 21.31.7 part (7)(b).

Proof of (5). Suppose given a qc covering $\{U_i \to U\}$. For $u \in U$ pick $i_1, \ldots, i_m \in I$ and quasi-compact subsets $E_j \subset U_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of u. Observe that $Y = \coprod_{j=1,\ldots,m} E_j \to U$ is proper as a continuous map between Hausdorff quasi-compact spaces (Topology, Lemma 5.17.7). Choose an open neighbourhood $u \in V$ contained in $\bigcup f_{i_j}(E_j)$. Then $Y \times_U V \to V$ is a surjective proper morphism and hence a *qc* covering by Lemma 21.31.4. Since we can do this for every $u \in U$ we see that (5) holds. □

0DCY Lemma 21.31.10. With notation as above.

1. For $X \in \mathrm{Ob}(\mathrm{LC}_{qc})$ and an abelian sheaf $\mathcal{F}$ on X we have $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$ and $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$.
2. For a proper morphism $f : X \to Y$ in LC_{qc} and abelian sheaf $\mathcal{F}$ on X we have $a_Y^{-1}(R^if_*\mathcal{F}) = R^if_{qc,*}(a_X^{-1}\mathcal{F})$ for all i.
3. For $X \in \mathrm{Ob}(\mathrm{LC}_{qc})$ and K in $D^+(X)$ the map $\pi_X^{-1}K \to R\epsilon_{X,*}(a_X^{-1}K)$ is an isomorphism.
4. For a proper morphism $f : X \to Y$ in LC_{qc} and K in $D^+(X)$ we have $a_Y^{-1}(Rf_*K) = Rf_{qc,*}(a_X^{-1}K)$.

Proof. By Lemma 21.31.9 the lemmas in Section 21.30 all apply to our current setting. To translate the results observe that the category $\mathcal{A}_X$ of Lemma 21.30.2 is the essential image of $a_X^{-1} : \mathrm{Ab}(X) \to \mathrm{Ab}(\mathrm{LC}_{qc}/X)$.

Part (1) is equivalent to (V_n) for all n which holds by Lemma 21.30.8.

Part (2) follows by applying ϵ_Y^{-1} to the conclusion of Lemma 21.30.3.

Part (3) follows from Lemma 21.30.8 part (1) because $\pi_X^{-1}K$ is in $D^+_{\mathcal{A}'_X}(\mathrm{LC}_{Zar}/X)$ and $a_X^{-1} = \epsilon_X^{-1} \circ a_X^{-1}$.

Part (4) follows from Lemma 21.30.8 part (2) for the same reason. □

0D91 Lemma 21.31.11. Let X be an object of LC_{qc}. For $K \in D^+(X)$ the map

$$K \longrightarrow Ra_{X,*}a_X^{-1}K$$

is an isomorphism with $a_X : Sh(\mathrm{LC}_{qc}/X) \to Sh(X)$ as above.

Proof. We first reduce the statement to the case where K is given by a single abelian sheaf. Namely, represent K by a bounded below complex $\mathcal{F}^\bullet$. By the case of a sheaf we see that $\mathcal{F}^n = a_{X,*}a_X^{-1}\mathcal{F}^n$ and that the sheaves $R^q a_{X,*}a_X^{-1}\mathcal{F}^n$ are zero for $q > 0$. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) applied to $a_X^{-1}\mathcal{F}^\bullet$ and the functor $a_{X,*}$ we conclude. From now on assume $K = \mathcal{F}$.

By Lemma 21.31.6 we have $a_{X,*}a_X^{-1}\mathcal{F} = \mathcal{F}$. Thus it suffices to show that $R^q a_{X,*}a_X^{-1}\mathcal{F} = 0$ for $q > 0$. For this we can use $a_X = \epsilon_X \circ \pi_X$ and the Leray spectral sequence Lemma 21.14.7. By Lemma 21.31.10 we have $R^i\epsilon_{X,*}(a_X^{-1}\mathcal{F}) = 0$ for $i > 0$ and $\epsilon_{X,*}a_X^{-1}\mathcal{F} = \pi_X^{-1}\mathcal{F}$. By Lemma 21.31.7 we have $R^j\pi_{X,*}(\pi_X^{-1}\mathcal{F}) = 0$ for $j > 0$. This concludes the proof. $\square$

09X4 Lemma 21.31.12. With $X \in \mathrm{Ob}(\mathrm{LC}_{qc})$ and $a_X : Sh(\mathrm{LC}_{qc}/X) \to Sh(X)$ as above:

(1) for an abelian sheaf $\mathcal{F}$ on X we have $H^n(X, \mathcal{F}) = H^n_{qc}(X, a_X^{-1}\mathcal{F})$,
(2) for $K \in D^+(X)$ we have $H^n(X, K) = H^n_{qc}(X, a_X^{-1}K)$.

For example, if A is an abelian group, then we have $H^n(X, \underline{A}) = H^n_{qc}(X, \underline{A})$.

Proof. This follows from Lemma 21.31.11 by Remark 21.14.4. $\square$

21.32. Spectral sequences for Ext

07A9 In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of complexes $\mathcal{G}^\bullet, \mathcal{F}^\bullet$ of complexes of modules on a ringed site $(\mathcal{C}, \mathcal{O})$ we denote

$$\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{F}^\bullet[n])$$

according to our general conventions in Derived Categories, Section 13.27.

07AA Example 21.32.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a bounded above complex of $\mathcal{O}$-modules. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Then there is a spectral sequence with E_2-page

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{O}}(H^{-j}(\mathcal{K}^\bullet), \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{F})$$

and another spectral sequence with E_1-page

$$E_1^{i,j} = \mathrm{Ext}^j_{\mathcal{O}}(\mathcal{K}^{-i}, \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{F}).$$

To construct these spectral sequences choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and consider the two spectral sequences coming from the double complex $\mathrm{Hom}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$, see Homology, Section 12.25.

21.33. Cup product

0FPJ Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, M be objects of $D(\mathcal{O})$. Set $A = \Gamma(\mathcal{C}, \mathcal{O})$. The (global) cup product in this setting is a map

$$R\Gamma(\mathcal{C}, K) \otimes_A^{\mathbf{L}} R\Gamma(\mathcal{C}, M) \longrightarrow R\Gamma(\mathcal{C}, K \otimes_{\mathcal{O}}^{\mathbf{L}} M)$$

in $D(A)$. We define it as the relative cup product for the morphism of ringed topoi $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(pt), A)$ as in Remark 21.19.7.

Let us formulate and prove a natural compatibility of the relative cup product. Namely, suppose that we have a morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi. Let $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ be complexes of $\mathcal{O}_\mathcal{C}$-modules. There is a naive cup product

$$\mathrm{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \longrightarrow f_*\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)$$

We claim that this is related to the relative cup product.

0FPK Lemma 21.33.1. In the situation above the following diagram commutes

$$\begin{tikzcd}
f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet \arrow[r] \arrow[d] & Rf_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*\mathcal{M}^\bullet \arrow[d, "\text{Remark 21.19.7}"] \\
\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \arrow[d, "\text{naive cup product}"'] & Rf_*(\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \arrow[d] \\
f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \arrow[r] & Rf_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{tikzcd}$$

Proof. By the construction in Remark 21.19.7 we see that going around the diagram clockwise the map

$$f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet \longrightarrow Rf_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)$$

is adjoint to the map

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*f_*\mathcal{M}^\bullet \\
&\to Lf^*Rf_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*Rf_*\mathcal{M}^\bullet \\
&\to \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet \\
&\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

By Lemma 21.19.6 this is also equal to

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &= Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Lf^*f_*\mathcal{M}^\bullet \\
&\to f^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f^*f_*\mathcal{M}^\bullet \\
&\to \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} \mathcal{M}^\bullet \\
&\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

Going around anti-clockwise we obtain the map adjoint to the map

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &\to Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \\
&\to Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to Lf^*Rf_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

By Lemma 21.19.6 this is also equal to

$$\begin{aligned}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) &\to Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \\
&\to Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to f^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \\
&\to \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet)
\end{aligned}$$

Now the proof is finished by a contemplation of the diagram

$$
\begin{tikzcd}
Lf^*(f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \arrow[rr] \arrow[d] & & Lf^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*f_*\mathcal{M}^\bullet \arrow[d] \\
Lf^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \arrow[r] \arrow[d, "naive"'] & f^*\text{Tot}(f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{D}} f_*\mathcal{M}^\bullet) \arrow[ddl, "naive"] \arrow[dd] & f^*f_*\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} f^*f_*\mathcal{M}^\bullet \arrow[ddl] \arrow[dd] \\
Lf^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \arrow[d] & & \\
f^*f_*\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) \arrow[dr] & \text{Tot}(f^*f_*\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} f^*f_*\mathcal{M}^\bullet) \arrow[d] & \mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet \arrow[dl] \\
& \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{M}^\bullet) &
\end{tikzcd}
$$

All of the polygons in this diagram commute. The top one commutes by Lemma 21.18.8. The square with the two naive cup products commutes because $Lf^* \to f^*$ is functorial in the complex of modules. Similarly with the square involving the two maps $\mathcal{A}^\bullet \otimes^{\mathbf{L}} \mathcal{B}^\bullet \to \text{Tot}(\mathcal{A}^\bullet \otimes \mathcal{B}^\bullet)$. Finally, the commutativity of the remaining square is true on the level of complexes and may be viewed as the definiton of the naive cup product (by the adjointness of f^* and f_*). The proof is finished because going around the diagram on the outside are the two maps given above. $\square$

0FPL Lemma 21.33.2. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. The relative cup product of Remark 21.19.7 is associative in the sense that the diagram

$$
\begin{tikzcd}
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M \arrow[r] \arrow[d] & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M \arrow[d] \\
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}} M) \arrow[r] & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L \otimes^{\mathbf{L}}_{\mathcal{O}} M)
\end{tikzcd}
$$

is commutative in $D(\mathcal{O}')$ for all K, L, M in $D(\mathcal{O})$.

Proof. Going around either side we obtain the map adjoint to the obvious map

$$
\begin{aligned}
Lf^*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*M) &= Lf^*(Rf_*K) \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*(Rf_*L) \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*(Rf_*M) \\
&\to K \otimes^{\mathbf{L}}_{\mathcal{O}} L \otimes^{\mathbf{L}}_{\mathcal{O}} M
\end{aligned}
$$

in $D(\mathcal{O})$. $\square$

0FPM Lemma 21.33.3. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. The relative cup product of Remark 21.19.7 is commutative in the sense that the diagram

$$
\begin{tikzcd}
Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L \arrow[r] \arrow[d, "\psi"'] & Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) \arrow[d, "Rf_*\psi"] \\
Rf_*L \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*K \arrow[r] & Rf_*(L \otimes^{\mathbf{L}}_{\mathcal{O}} K)
\end{tikzcd}
$$

is commutative in $D(\mathcal{O}')$ for all K, L in $D(\mathcal{O})$. Here ψ is the commutativity constraint on the derived category (Lemma 21.48.5).

Proof. Omitted. $\square$

0FPN Lemma 21.33.4. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and $f' : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}''), \mathcal{O}'')$ be morphisms of ringed topoi. The relative cup product of Remark 21.19.7 is compatible with compositions in the sense that the diagram

$$\begin{tikzcd} R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} R(f' \circ f)_*L \arrow[rr, equal] \arrow[d] & & Rf'_*Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} Rf'_*Rf_*L \arrow[d] \\ R(f' \circ f)_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) \arrow[r, equal] & Rf'_*Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}} L) & Rf'_*(Rf_*K \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*L) \arrow[l] \end{tikzcd}$$

is commutative in $D(\mathcal{O}'')$ for all K, L in $D(\mathcal{O})$.

Proof. This is true because going around the diagram either way we obtain the map adjoint to the map

$$\begin{aligned} & L(f' \circ f)^* \left(R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}''} R(f' \circ f)_*L\right) \\ & = L(f' \circ f)^*R(f' \circ f)_*K \otimes^{\mathbf{L}}_{\mathcal{O}} L(f' \circ f)^*R(f' \circ f)_*L) \\ & \to K \otimes^{\mathbf{L}}_{\mathcal{O}} L \end{aligned}$$

in $D(\mathcal{O})$. To see this one uses that the composition of the counits like so

$$L(f' \circ f)^*R(f' \circ f)_* = Lf^*L(f')^*Rf'_*Rf_* \to Lf^*Rf_* \to \mathrm{id}$$

is the counit for $L(f' \circ f)^*$ and $R(f' \circ f)_*$. See Categories, Lemma 4.24.9. $\square$

21.34. Hom complexes

0A8X Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of $\mathcal{O}$-modules. We construct a complex of $\mathcal{O}$-modules $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. Namely, for each n we set

$$\mathcal{H}om^n(\mathcal{L}^\bullet, \mathcal{M}^\bullet) = \prod\nolimits_{n=p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

It is a good idea to think of $\mathcal{H}om^n$ as the sheaf of $\mathcal{O}$-modules of all $\mathcal{O}$-linear maps from $\mathcal{L}^\bullet$ to $\mathcal{M}^\bullet$ (viewed as graded $\mathcal{O}$-modules) which are homogenous of degree n. In this terminology, we define the differential by the rule

$$\mathrm{d}(f) = \mathrm{d}_{\mathcal{M}} \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}}$$

for $f \in \mathcal{H}om^n_{\mathcal{O}}(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$. We omit the verification that $\mathrm{d}^2 = 0$. This construction is a special case of Differential Graded Algebra, Example 22.26.6. It follows immediately from the construction that we have

0A8Y (21.34.0.1) $$H^n(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{M}^\bullet[n]|_U)$$

for all $n \in \mathbf{Z}$ and every $U \in \mathrm{Ob}(\mathcal{C})$. Similarly, we have

0A8Z (21.34.0.2) $$H^n(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet))) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{L}^\bullet, \mathcal{M}^\bullet[n])$$

for the complex of global sections.

0A90 Lemma 21.34.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is an isomorphism

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules functorial in $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.1. ☐

0A91 Lemma 21.34.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism

$$\mathrm{Tot}\left(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_\mathcal{O} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.3. ☐

0BYT Lemma 21.34.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism

$$\mathrm{Tot}\left(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{L}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.4. ☐

0A93 Lemma 21.34.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism

$$\mathcal{K}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathrm{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet))$$

of complexes of $\mathcal{O}$-modules functorial in both complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.5. ☐

0A92 Lemma 21.34.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given complexes $\mathcal{K}^\bullet, \mathcal{L}^\bullet, \mathcal{M}^\bullet$ of $\mathcal{O}$-modules there is a canonical morphism

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \otimes_\mathcal{O} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet), \mathcal{M}^\bullet)$$

of complexes of $\mathcal{O}$-modules functorial in all three complexes.

Proof. Omitted. Hint: This is proved in exactly the same way as More on Algebra, Lemma 15.71.6. ☐

0A94 Lemma 21.34.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L and M be objects of $D(\mathcal{O})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing M. Let $\mathcal{L}^\bullet$ be a complex of $\mathcal{O}$-modules representing L. Then

$$H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

for all $U \in \mathrm{Ob}(\mathcal{C})$. Similarly, $H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) = \mathrm{Hom}_{D(\mathcal{O})}(L, M)$.

Proof. We have

$$\begin{aligned} H^0(\Gamma(U, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet))) &= \mathrm{Hom}_{K(\mathcal{O}_U)}(L|_U, M|_U) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U) \end{aligned}$$

The first equality is (21.34.0.1). The second equality is true because $\mathcal{I}^\bullet|_U$ is K-injective by Lemma 21.20.1. The proof of the last equation is similar except that it uses (21.34.0.2). ☐

0A95 **Lemma 21.34.7.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{I}')^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of K-injective complexes of $\mathcal{O}$-modules. Let $(\mathcal{L}')^\bullet \to \mathcal{L}^\bullet$ be a quasi-isomorphism of complexes of $\mathcal{O}$-modules. Then

$$\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet) \longrightarrow \mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

Proof. Let M be the object of $D(\mathcal{O})$ represented by $\mathcal{I}^\bullet$ and $(\mathcal{I}')^\bullet$. Let L be the object of $D(\mathcal{O})$ represented by $\mathcal{L}^\bullet$ and $(\mathcal{L}')^\bullet$. By Lemma 21.34.6 we see that the sheaves

$$H^0(\mathcal{H}om^\bullet(\mathcal{L}^\bullet, (\mathcal{I}')^\bullet)) \quad \text{and} \quad H^0(\mathcal{H}om^\bullet((\mathcal{L}')^\bullet, \mathcal{I}^\bullet))$$

are both equal to the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

Thus the map is a quasi-isomorphism. □

0A96 **Lemma 21.34.8.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be a K-flat complex of $\mathcal{O}$-modules. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}$-modules.

Proof. Namely, if $\mathcal{K}^\bullet$ is an acyclic complex of $\mathcal{O}$-modules, then

$$\begin{aligned}
\mathrm{Hom}_{K(\mathcal{O})}(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) &= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)))) \\
&= H^0(\Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet))) \\
&= \mathrm{Hom}_{K(\mathcal{O})}(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet) \\
&= 0
\end{aligned}$$

The first equality by (21.34.0.2). The second equality by Lemma 21.34.1. The third equality by (21.34.0.2). The final equality because $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ is acyclic because $\mathcal{L}^\bullet$ is K-flat (Definition 21.17.2) and because $\mathcal{I}^\bullet$ is K-injective. □

21.35. Internal hom in the derived category

08J7 Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L, M be objects of $D(\mathcal{O})$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O})$ such that for every third object K of $D(\mathcal{O})$ there exists a canonical bijection

08J8 (21.35.0.1) $$\mathrm{Hom}_{D(\mathcal{O})}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O})}(K \otimes_{\mathcal{O}}^{\mathbf{L}} L, M)$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 4.3.5).

To construct such an object, choose a K-injective complex of $\mathcal{O}$-modules $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Then we set Then we set

$$R\mathcal{H}om(L, M) = \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$$

where the right hand side is the complex of $\mathcal{O}$-modules constructed in Section 21.34. This is well defined by Lemma 21.34.7. We get a functor

$$D(\mathcal{O})^{opp} \times D(\mathcal{O}) \longrightarrow D(\mathcal{O}), \quad (K, L) \longmapsto R\mathcal{H}om(K, L)$$

As a prelude to proving (21.35.0.1) we compute the cohomology groups of $R\mathcal{H}om(K, L)$.

08JA Lemma 21.35.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. For every object U of $\mathcal{C}$ we have

$$H^0(U, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(L|_U, M|_U)$$

and we have $H^0(\mathcal{C}, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O})}(L, M)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}$-modules representing M and a K-flat complex $\mathcal{L}^\bullet$ representing L. Then $\mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)$ is K-injective by Lemma 21.34.8. Hence we can compute cohomology over U by simply taking sections over U and the result follows from Lemma 21.34.6. $\square$

08J9 Lemma 21.35.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L, M be objects of $D(\mathcal{O})$. With the construction as described above there is a canonical isomorphism

$$R\mathcal{H}om(K, R\mathcal{H}om(L, M)) = R\mathcal{H}om(K \otimes_{\mathcal{O}}^{\mathbf{L}} L, M)$$

in $D(\mathcal{O})$ functorial in K, L, M which recovers (21.35.0.1) on taking $H^0(\mathcal{C}, -)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. For any complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ we have

$$\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{I}^\bullet)) = \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

by Lemma 21.34.1. Note that the left hand side represents $R\mathcal{H}om(K, R\mathcal{H}om(L, M))$ (use Lemma 21.34.8) and that the right hand side represents $R\mathcal{H}om(K \otimes_{\mathcal{O}}^{\mathbf{L}} L, M)$. This proves the displayed formula of the lemma. Taking global sections and using Lemma 21.35.1 we obtain (21.35.0.1). $\square$

08JB Lemma 21.35.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions, i.e., for every object U of $\mathcal{C}$ we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.

Proof. This is clear from the construction and Lemma 21.20.1. $\square$

08JC Lemma 21.35.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The bifunctor $R\mathcal{H}om(-, -)$ transforms distinguished triangles into distinguished triangles in both variables.

Proof. This follows from the observation that the assignment

$$(\mathcal{L}^\bullet, \mathcal{M}^\bullet) \longmapsto \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{M}^\bullet)$$

transforms a termwise split short exact sequences of complexes in either variable into a termwise split short exact sequence. Details omitted. $\square$

0A97 Lemma 21.35.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L, M be objects of $D(\mathcal{O})$. There is a canonical morphism

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, L), M)$$

in $D(\mathcal{O})$ functorial in K, L, M.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and a K-flat complex $\mathcal{K}^\bullet$ representing K. The map is defined using the map

$$\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet), \mathcal{I}^\bullet)$$

of Lemma 21.34.5. By our particular choice of complexes the left hand side represents $R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} K$ and the right hand side represents $R\mathcal{H}om(R\mathcal{H}om(K, L), M)$. We omit the proof that this is functorial in all three objects of $D(\mathcal{O})$. $\square$

0A98 **Lemma 21.35.6.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L, M in $D(\mathcal{O})$ there is a canonical morphism

$$R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, M)$$

in $D(\mathcal{O})$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M, a K-injective complex $\mathcal{J}^\bullet$ representing L, and any complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ representing K. By Lemma 21.34.2 there is a map of complexes

$$\text{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right) \longrightarrow \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

The complexes of $\mathcal{O}$-modules $\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet)$, $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, and $\mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ represent $R\mathcal{H}om(L, M)$, $R\mathcal{H}om(K, L)$, and $R\mathcal{H}om(K, M)$. If we choose a K-flat complex $\mathcal{H}^\bullet$ and a quasi-isomorphism $\mathcal{H}^\bullet \to \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)$, then there is a map

$$\text{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}^\bullet\right) \longrightarrow \text{Tot}\left(\mathcal{H}om^\bullet(\mathcal{J}^\bullet, \mathcal{I}^\bullet) \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{K}^\bullet, \mathcal{J}^\bullet)\right)$$

whose source represents $R\mathcal{H}om(L, M) \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(K, L)$. Composing the two displayed arrows gives the desired map. We omit the proof that the construction is functorial. $\square$

0BYU **Lemma 21.35.7.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L, M in $D(\mathcal{O})$ there is a canonical morphism

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes_{\mathcal{O}}^{\mathbf{L}} L)$$

in $D(\mathcal{O})$ functorial in K, L, M.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K, and a K-injective complex $\mathcal{I}^\bullet$ representing L, and choose any complex of $\mathcal{O}$-modules $\mathcal{M}^\bullet$ representing M. Choose a quasi-isomorphism $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet) \to \mathcal{J}^\bullet$ where $\mathcal{J}^\bullet$ is K-injective. Then we use the map

$$\text{Tot}\left(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{I}^\bullet)\right) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{I}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$$

where the first map is the map from Lemma 21.34.3. $\square$

0A99 **Lemma 21.35.8.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Given K, L in $D(\mathcal{O})$ there is a canonical morphism

$$K \longrightarrow R\mathcal{H}om(L, K \otimes_{\mathcal{O}}^{\mathbf{L}} L)$$

in $D(\mathcal{O})$ functorial in both K and L.

Proof. Choose a K-flat complex $\mathcal{K}^\bullet$ representing K and any complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Choose a K-injective complex $\mathcal{J}^\bullet$ and a quasi-isomorphism $\text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \to \mathcal{J}^\bullet$. Then we use

$$\mathcal{K}^\bullet \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \text{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{L}^\bullet, \mathcal{J}^\bullet)$$

where the first map comes from Lemma 21.34.4. $\square$

08JD **Lemma 21.35.9.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L be an object of $D(\mathcal{O})$. Set $L^\vee = R\mathcal{H}om(L, \mathcal{O})$. For M in $D(\mathcal{O})$ there is a canonical map

08JE (21.35.9.1)
$$M \otimes_{\mathcal{O}}^{\mathbf{L}} L^\vee \longrightarrow R\mathcal{H}om(L, M)$$

which induces a canonical map

$$H^0(\mathcal{C}, M \otimes_{\mathcal{O}}^{\mathbf{L}} L^\vee) \longrightarrow \text{Hom}_{D(\mathcal{O})}(L, M)$$

functorial in M in $D(\mathcal{O})$.

Proof. The map (21.35.9.1) is a special case of Lemma 21.35.6 using the identification $M = R\mathcal{H}om(\mathcal{O}, M)$. $\square$

0B6D Remark 21.35.10. Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let K, L be objects of $D(\mathcal{O}_\mathcal{C})$. We claim there is a canonical map

$$Rf_*R\mathcal{H}om(L, K) \longrightarrow R\mathcal{H}om(Rf_*L, Rf_*K)$$

Namely, by (21.35.0.1) this is the same thing as a map $Rf_*R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*L \to Rf_*K$. For this we can use the composition

$$Rf_*R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} Rf_*L \to Rf_*(R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L) \to Rf_*K$$

where the first arrow is the relative cup product (Remark 21.19.7) and the second arrow is Rf_* applied to the canonical map $R\mathcal{H}om(L, K) \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} L \to K$ coming from Lemma 21.35.6 (with $\mathcal{O}_\mathcal{C}$ in one of the spots).

08JF Remark 21.35.11. Let $h : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let K, L be objects of $D(\mathcal{O}')$. We claim there is a canonical map

$$Lh^*R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O})$. Namely, by (21.35.0.1) proved in Lemma 21.35.2 such a map is the same thing as a map

$$Lh^*R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^*K \longrightarrow Lh^*L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 21.18.4 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\mathrm{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (21.35.0.1).

08JG Remark 21.35.12. Suppose that

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "h"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "f"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

is a commutative diagram of ringed topoi. Let K, L be objects of $D(\mathcal{O}_\mathcal{C})$. We claim there exists a canonical base change map

$$Lg^*Rf_*R\mathcal{H}om(K, L) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^*Lg^*Rf_*R\mathcal{H}om(K, L) &= Lh^*Lf^*Rf_*R\mathcal{H}om(K, L) \\ &\to Lh^*R\mathcal{H}om(K, L) \\ &\to R\mathcal{H}om(Lh^*K, Lh^*L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^*Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 21.35.11.

21.36. Global derived hom

0B6E Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $K, L \in D(\mathcal{O})$. Using the construction of the internal hom in the derived category we obtain a well defined object

$$R\operatorname{Hom}_{\mathcal{O}}(K, L) = R\Gamma(\mathcal{C}, R\mathcal{H}om(K, L))$$

in $D(\Gamma(\mathcal{C}, \mathcal{O}))$. By Lemma 21.35.1 we have

$$H^0(R\operatorname{Hom}_{\mathcal{O}}(K, L)) = \operatorname{Hom}_{D(\mathcal{O})}(K, L)$$

and

$$H^p(R\operatorname{Hom}_{\mathcal{O}}(K, L)) = \operatorname{Ext}^p_{D(\mathcal{O})}(K, L)$$

If $f : (\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})$ is a morphism of ringed topoi, then there is a canonical map

$$R\operatorname{Hom}_{\mathcal{O}}(K, L) \longrightarrow R\operatorname{Hom}_{\mathcal{O}'}(Lf^*K, Lf^*L)$$

in $D(\Gamma(\mathcal{O}))$ by taking global sections of the map defined in Remark 21.35.11.

21.37. Derived lower shriek

07AB In this section we study morphisms g of ringed topoi where besides Lg^* and Rg_* there also exists a derived functor $Lg_!$.

0D6X Lemma 21.37.1. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and let $\mathcal{I}$ be an injective $\mathcal{O}_\mathcal{D}$-module. Then $H^p(U, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and $U \in \operatorname{Ob}(\mathcal{C})$.

Proof. The vanishing of the lemma follows from Lemma 21.10.9 if we can prove vanishing of all higher Čech cohomology groups $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I})$ for any covering $\mathcal{U} = \{U_i \to U\}$ of $\mathcal{C}$. Since u is continuous, $u(\mathcal{U}) = \{u(U_i) \to u(U)\}$ is a covering of $\mathcal{D}$, and $u(U_{i_0} \times_U \ldots \times_U U_{i_n}) = u(U_{i_0}) \times_{u(U)} \ldots \times_{u(U)} u(U_{i_n})$. Thus we have

$$\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I}) = \check{H}^p(u(\mathcal{U}), \mathcal{I})$$

because $g^{-1} = u^p$ by Sites, Lemma 7.21.5. Since $\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module these Čech cohomology groups vanish, see Lemma 21.12.3. $\square$

07AC Lemma 21.37.2. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. The functor $g_! : \operatorname{Mod}(\mathcal{O}_\mathcal{C}) \to \operatorname{Mod}(\mathcal{O}_\mathcal{D})$ (see Modules on Sites, Lemma 18.41.1) has a left derived functor

$$Lg_! : D(\mathcal{O}_\mathcal{C}) \longrightarrow D(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*. Moreover, for $U \in \operatorname{Ob}(\mathcal{C})$ we have

$$Lg_!(j_{U!}\mathcal{O}_U) = g_!j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}.$$

where $j_{U!}$ and $j_{u(U)!}$ are extension by zero associated to the localization morphism $j_U : \mathcal{C}/U \to \mathcal{C}$ and $j_{u(U)} : \mathcal{D}/u(U) \to \mathcal{D}$.

Proof. We are going to use Derived Categories, Proposition 13.29.2 to construct $Lg_!$. To do this we have to verify assumptions (1), (2), (3), (4), and (5) of that proposition. First, since $g_!$ is a left adjoint we see that it is right exact and commutes with all colimits, so (5) holds. Conditions (3) and (4) hold because the category of modules on a ringed site is a Grothendieck abelian category. Let $\mathcal{P} \subset \operatorname{Ob}(\operatorname{Mod}(\mathcal{O}_\mathcal{C}))$ be the collection of $\mathcal{O}_\mathcal{C}$-modules which are direct sums of modules of the form $j_{U!}\mathcal{O}_U$.

Note that $g_! j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$, see proof of Modules on Sites, Lemma 18.41.1. Every $\mathcal{O}_\mathcal{C}$-module is a quotient of an object of $\mathcal{P}$, see Modules on Sites, Lemma 18.28.8. Thus (1) holds. Finally, we have to prove (2). Let $\mathcal{K}^\bullet$ be a bounded above acyclic complex of $\mathcal{O}_\mathcal{C}$-modules with $\mathcal{K}^n \in \mathcal{P}$ for all n. We have to show that $g_!\mathcal{K}^\bullet$ is exact. To do this it suffices to show, for every injective $\mathcal{O}_\mathcal{D}$-module $\mathcal{I}$ that

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) = 0$$

for all $n \in \mathbf{Z}$. Since $\mathcal{I}$ is injective we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) &= \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(g_!\mathcal{K}^\bullet, \mathcal{I})) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}))\end{aligned}$$

the last equality by the adjointness of $g_!$ and g^{-1}.

The vanishing of this group would be clear if $g^{-1}\mathcal{I}$ were an injective $\mathcal{O}_\mathcal{C}$-module. But $g^{-1}\mathcal{I}$ isn't necessarily an injective $\mathcal{O}_\mathcal{C}$-module as $g_!$ isn't exact in general. We do know that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{I}) = H^p(U, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1$$

Here the first equality follows from $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, \mathcal{H}) = \mathcal{H}(U)$ and taking derived functors and the vanishing of $H^p(U, g^{-1}\mathcal{I})$ for $p > 0$ and $U \in \mathrm{Ob}(\mathcal{C})$ follows from Lemma 21.37.1. Since each $\mathcal{K}^{-q}$ is a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ we see that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1 \text{ and all } q$$

Let us use the spectral sequence (see Example 21.32.1)

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}) = 0.$$

Note that the spectral sequence abuts to zero as $\mathcal{K}^\bullet$ is acyclic (hence vanishes in the derived category, hence produces vanishing ext groups). By the vanishing of higher exts proved above the only nonzero terms on the E_1 page are the terms $E_1^{0,q} = \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I})$. We conclude that the complex $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})$ is acyclic as desired.

Thus the left derived functor $Lg_!$ exists. It is left adjoint to $g^{-1} = g^* = Rg^* = Lg^*$, i.e., we have

07AD (21.37.2.1) $$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(K, g^*L) = \mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(Lg_!K, L)$$

by Derived Categories, Lemma 13.30.3. This finishes the proof. □

07AE Remark 21.37.3. Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 21.37.2. In general it is not the case that the diagram

$$\begin{tikzcd} D(\mathcal{O}_\mathcal{C}) \arrow[r, "Lg_!"'] \arrow[d, "forget"'] & D(\mathcal{O}_\mathcal{D}) \arrow[d, "forget"] \\ D(\mathcal{C}) \arrow[r, "Lg_!^{Ab}"] & D(\mathcal{D}) \end{tikzcd}$$

commutes where the functor $Lg_!^{Ab}$ is the one constructed in Lemma 21.37.2 but using the constant sheaf $\mathbf{Z}$ as the structure sheaf on both $\mathcal{C}$ and $\mathcal{D}$. In general it isn't even the case that $g_! = g_!^{Ab}$ (see Modules on Sites, Remark 18.41.2), but this phenomenon can occur even if $g_! = g_!^{Ab}$! Namely, the construction of $Lg_!$ in the proof of Lemma 21.37.2 shows that $Lg_!$ agrees with $Lg_!^{\text{Ab}}$ if and only if the canonical maps

$$Lg_!^{Ab} j_{U!}\mathcal{O}_U \longrightarrow j_{u(U)!}\mathcal{O}_{u(U)}$$

are isomorphisms in $D(\mathcal{D})$ for all objects U in $\mathcal{C}$. In general all we can say is that there exists a natural transformation

$$Lg_!^{Ab} \circ forget \longrightarrow forget \circ Lg_!$$

0D6Y Lemma 21.37.4. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and let $\mathcal{I}$ be an injective $\mathcal{O}_\mathcal{D}$-module. If $g_!^{Sh} : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ commutes with fibre products[7], then $g^{-1}\mathcal{I}$ is totally acyclic.

Proof. We will use the criterion of Lemma 21.13.5. Condition (1) holds by Lemma 21.37.1. Let $K' \to K$ be a surjective map of sheaves of sets on $\mathcal{C}$. Since $g_!^{Sh}$ is a left adjoint, we see that $g_!^{Sh}K' \to g_!^{Sh}K$ is surjective. Observe that

$$\begin{aligned} H^0(K' \times_K \ldots \times_K K', g^{-1}\mathcal{I}) &= H^0(g_!^{Sh}(K' \times_K \ldots \times_K K'), \mathcal{I}) \\ &= H^0(g_!^{Sh}K' \times_{g_!^{Sh}K} \ldots \times_{g_!^{Sh}K} g_!^{Sh}K', \mathcal{I}) \end{aligned}$$

by our assumption on $g_!^{Sh}$. Since $\mathcal{I}$ is an injective module it is totally acyclic by Lemma 21.14.1 (applied to the identity). Hence we can use the converse of Lemma 21.13.5 to see that the complex

$$0 \to H^0(K, g^{-1}\mathcal{I}) \to H^0(K', g^{-1}\mathcal{I}) \to H^0(K' \times_K K', g^{-1}\mathcal{I}) \to \ldots$$

is exact as desired. □

0DD8 Lemma 21.37.5. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the corresponding morphism of topoi. Let $U \in \text{Ob}(\mathcal{C})$.

(1) For M in $D(\mathcal{D})$ we have $R\Gamma(U, g^{-1}M) = R\Gamma(u(U), M)$.
(2) If $\mathcal{O}_\mathcal{D}$ is a sheaf of rings and $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$, then for M in $D(\mathcal{O}_\mathcal{D})$ we have $R\Gamma(U, g^*M) = R\Gamma(u(U), M)$.

Proof. In the bounded below case (1) and (2) can be seen by representing K by a bounded below complex of injectives and using Lemma 21.37.1 as well as Leray's acyclicity lemma. In the unbounded case, first note that (1) is a special case of (2). For (2) we can use

$$R\Gamma(U, g^*M) = R\operatorname{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^*M) = R\operatorname{Hom}_{\mathcal{O}_\mathcal{D}}(j_{u(U)!}\mathcal{O}_{u(U)}, M) = R\Gamma(u(U), M)$$

where the middle equality is a consequence of Lemma 21.37.2. □

0FN6 Lemma 21.37.6. Assume given a commutative diagram

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "(g', (g')^\sharp)"'] \arrow[d, "(f', (f')^\sharp)"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "(f, f^\sharp)"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "(g, g^\sharp)"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

of ringed topoi. Assume

(1) f, f', g, and g' correspond to cocontinuous functors u, u', v, and v' as in Sites, Lemma 7.21.1,

[7] Holds if $\mathcal{C}$ has finite connected limits and u commutes with them, see Sites, Lemma 7.21.6.

(2) $v \circ u' = u \circ v'$,
(3) v and v' are continuous as well as cocontinuous,
(4) for any object V' of $\mathcal{D}'$ the functor ${}^{u'}_{V'}\mathcal{I} \to {}^{\ u}_{v(V')}\mathcal{I}$ given by v is cofinal,
(5) $g^{-1}\mathcal{O}_\mathcal{D} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$, and
(6) $g'_! : \text{Ab}(\mathcal{C}') \to \text{Ab}(\mathcal{C})$ is exact[8].

Then we have $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_{\mathcal{D}'})$.

Proof. We have $g^* = Lg^* = g^{-1}$ and $(g')^* = L(g')^* = (g')^{-1}$ by condition (5). By Lemma 21.20.7 it suffices to prove the result on the derived category $D(\mathcal{C})$ of abelian sheaves. Choose an object $K \in D(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of abelian sheaves on $\mathcal{C}$ representing K. By Derived Categories, Lemma 13.31.9 and assumption (6) we find that $(g')^{-1}\mathcal{I}^\bullet$ is a K-injective complex of abelian sheaves on $\mathcal{C}'$. By Modules on Sites, Lemma 18.41.3 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $f_*\mathcal{I}^\bullet$ represents Rf_*K and since $f'_*(g')^{-1}\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$ we conclude. □

0FN7 Lemma 21.37.7. Consider a commutative diagram

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "{(g',(g')^\sharp)}"'] \arrow[d, "{(f',(f')^\sharp)}"'] & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[d, "{(f,f^\sharp)}"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "{(g,g^\sharp)}"] & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{tikzcd}$$

of ringed topoi and suppose we have functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "v'"'] & \mathcal{C} \\ \mathcal{D}' \arrow[u, "u'"] \arrow[r, "v"] & \mathcal{D} \arrow[u, "u"'] \end{tikzcd}$$

such that (with notation as in Sites, Sections 7.14 and 7.21) we have
(1) u and u' are continuous and give rise to the morphisms f and f',
(2) v and v' are cocontinuous giving rise to the morphisms g and g',
(3) $u \circ v = v' \circ u'$,
(4) v and v' are continuous as well as cocontinuous, and
(5) $g^{-1}\mathcal{O}_\mathcal{D} = \mathcal{O}_{\mathcal{D}'}$ and $(g')^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$.

Then $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D^+(\mathcal{O}_\mathcal{C}) \to D^+(\mathcal{O}_{\mathcal{D}'})$. If in addition
(6) $g'_! : \text{Ab}(\mathcal{C}') \to \text{Ab}(\mathcal{C})$ is exact[9],

then $Rf'_* \circ (g')^* = g^* \circ Rf_*$ as functors $D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_{\mathcal{D}'})$.

Proof. We have $g^* = Lg^* = g^{-1}$ and $(g')^* = L(g')^* = (g')^{-1}$ by condition (5). By Lemma 21.20.7 it suffices to prove the result on the derived category $D^+(\mathcal{C})$ or $D(\mathcal{C})$ of abelian sheaves.

Choose an object $K \in D^+(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a bounded below complex of injective abelian sheaves on $\mathcal{C}$ representing K. By Lemma 21.37.1 we see that $H^p(U', (g')^{-1}\mathcal{I}^q) = 0$ for all $p > 0$ and any q and any $U' \in \text{Ob}(\mathcal{C}')$. Recall that $R^pf'_*(g')^{-1}\mathcal{I}^q$ is the sheaf

[8] Holds if fibre products and equalizers exist in $\mathcal{C}'$ and v' commutes with them, see Modules on Sites, Lemma 18.16.3.

[9] Holds if fibre products and equalizers exist in $\mathcal{C}'$ and v' commutes with them, see Modules on Sites, Lemma 18.16.3.

associated to the presheaf $V' \mapsto H^p(u'(V'), (g')^{-1}\mathcal{I}^q)$, see Lemma 21.7.4. Thus we see that $(g')^{-1}\mathcal{I}^q$ is right acyclic for the functor f'_*. By Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) we find that $f'_*(g')^*\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$. By Modules on Sites, Lemma 18.41.4 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $g^{-1}f_*\mathcal{I}^\bullet$ represents $g^{-1}Rf_*K$ we conclude.

Choose an object $K \in D(\mathcal{C})$. Let $\mathcal{I}^\bullet$ be a K-injective complex of abelian sheaves on $\mathcal{C}$ representing K. By Derived Categories, Lemma 13.31.9 and assumption (6) we find that $(g')^{-1}\mathcal{I}^\bullet$ is a K-injective complex of abelian sheaves on $\mathcal{C}'$. By Modules on Sites, Lemma 18.41.4 we find that $f'_*(g')^{-1}\mathcal{I}^\bullet = g^{-1}f_*\mathcal{I}^\bullet$. Since $f_*\mathcal{I}^\bullet$ represents Rf_*K and since $f'_*(g')^{-1}\mathcal{I}^\bullet$ represents $Rf'_*(g')^{-1}K$ we conclude. □

21.38. Derived lower shriek for fibred categories

08RV In this section we work out some special cases of the situation discussed in Section 21.37. We make sure that we have equality between lower shriek on modules and sheaves of abelian groups. We encourage the reader to skip this section on a first reading.

08P8 Situation 21.38.1. Here $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site and $p : \mathcal{C} \to \mathcal{D}$ is a fibred category. We endow $\mathcal{C}$ with the topology inherited from $\mathcal{D}$ (Stacks, Section 8.10). We denote $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ the morphism of topoi associated to p (Stacks, Lemma 8.10.3). We set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ so that we obtain a morphism of ringed topoi

$$\pi : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

08P9 Lemma 21.38.2. Assumptions and notation as in Situation 21.38.1. For $U \in \text{Ob}(\mathcal{C})$ consider the induced morphism of topoi

$$\pi_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{D}/p(U))$$

Then there exists a morphism of topoi

$$\sigma : Sh(\mathcal{D}/p(U)) \to Sh(\mathcal{C}/U)$$

such that $\pi_U \circ \sigma = \text{id}$ and $\sigma^{-1} = \pi_{U,*}$.

Proof. Observe that π_U is the restriction of π to the localizations, see Sites, Lemma 7.28.4. For an object $V \to p(U)$ of $\mathcal{D}/p(U)$ denote $V \times_{p(U)} U \to U$ the strongly cartesian morphism of $\mathcal{C}$ over $\mathcal{D}$ which exists as p is a fibred category. The functor

$$v : \mathcal{D}/p(U) \to \mathcal{C}/U, \quad V/p(U) \mapsto V \times_{p(U)} U/U$$

is continuous by the definition of the topology on $\mathcal{C}$. Moreover, it is a right adjoint to p by the definition of strongly cartesian morphisms. Hence we are in the situation discussed in Sites, Section 7.22 and we see that the sheaf $\pi_{U,*}\mathcal{F}$ is equal to $V \mapsto \mathcal{F}(V \times_{p(U)} U)$ (see especially Sites, Lemma 7.22.2).

But here we have more. Namely, the functor v is also cocontinuous (as all morphisms in coverings of $\mathcal{C}$ are strongly cartesian). Hence v defines a morphism σ as indicated in the lemma. The equality $\sigma^{-1} = \pi_{U,*}$ is immediate from the definition. Since $\pi_U^{-1}\mathcal{G}$ is given by the rule $U'/U \mapsto \mathcal{G}(p(U')/p(U))$ it follows that $\sigma^{-1} \circ \pi_U^{-1} = \text{id}$ which proves the equality $\pi_U \circ \sigma = \text{id}$. □

08PA Situation 21.38.3. Let $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site. Let $u : \mathcal{C}' \to \mathcal{C}$ be a 1-morphism of fibred categories over $\mathcal{D}$ (Categories, Definition 4.33.9). Endow $\mathcal{C}$ and $\mathcal{C}'$ with their inherited topologies (Stacks, Definition 8.10.2) and let $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$,

$\pi' : Sh(\mathcal{C}') \to Sh(\mathcal{D})$, and $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ be the corresponding morphisms of topoi (Stacks, Lemma 8.10.3). Set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ and $\mathcal{O}_{\mathcal{C}'} = (\pi')^{-1}\mathcal{O}_\mathcal{D}$. Observe that $g^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$ so that

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[rr, "g"'] \arrow[dr, "\pi'"'] & & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \arrow[dl, "\pi"] \\ & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) & \end{tikzcd}$$

is a commutative diagram of morphisms of ringed topoi.

08PB Lemma 21.38.4. Assumptions and notation as in Situation 21.38.3. For $U' \in \mathrm{Ob}(\mathcal{C}')$ set $U = u(U')$ and $V = p'(U')$ and consider the induced morphisms of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C}'/U'), \mathcal{O}_{U'}) \arrow[rr, "g'"'] \arrow[dr, "\pi'_{U'}"'] & & (Sh(\mathcal{C}), \mathcal{O}_U) \arrow[dl, "\pi_U"] \\ & (Sh(\mathcal{D}/V), \mathcal{O}_V) & \end{tikzcd}$$

Then there exists a morphism of topoi

$$\sigma' : Sh(\mathcal{D}/V) \to Sh(\mathcal{C}'/U'),$$

such that setting $\sigma = g' \circ \sigma'$ we have $\pi'_{U'} \circ \sigma' = \mathrm{id}$, $\pi_U \circ \sigma = \mathrm{id}$, $(\sigma')^{-1} = \pi'_{U',*}$, and $\sigma^{-1} = \pi_{U,*}$.

Proof. Let $v' : \mathcal{D}/V \to \mathcal{C}'/U'$ be the functor constructed in the proof of Lemma 21.38.2 starting with $p' : \mathcal{C}' \to \mathcal{D}'$ and the object U'. Since u is a 1-morphism of fibred categories over $\mathcal{D}$ it transforms strongly cartesian morphisms into strongly cartesian morphisms, hence the functor $v = u \circ v'$ is the functor of the proof of Lemma 21.38.2 relative to $p : \mathcal{C} \to \mathcal{D}$ and U. Thus our lemma follows from that lemma. $\square$

08PC Lemma 21.38.5. Assumption and notation as in Situation 21.38.3.

(1) There are left adjoints $g_! : \mathrm{Mod}(\mathcal{O}_{\mathcal{C}'}) \to \mathrm{Mod}(\mathcal{O}_\mathcal{C})$ and $g_!^{\mathrm{Ab}} : \mathrm{Ab}(\mathcal{C}') \to \mathrm{Ab}(\mathcal{C})$ to $g^* = g^{-1}$ on modules and on abelian sheaves.
(2) The diagram

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{O}_{\mathcal{C}'}) \arrow[r, "g_!"'] \arrow[d] & \mathrm{Mod}(\mathcal{O}_\mathcal{C}) \arrow[d] \\ \mathrm{Ab}(\mathcal{C}') \arrow[r, "g_!^{\mathrm{Ab}}"] & \mathrm{Ab}(\mathcal{C}) \end{tikzcd}$$

commutes.
(3) There are left adjoints $Lg_! : D(\mathcal{O}_{\mathcal{C}'}) \to D(\mathcal{O}_\mathcal{C})$ and $Lg_!^{\mathrm{Ab}} : D(\mathcal{C}') \to D(\mathcal{C})$ to $g^* = g^{-1}$ on derived categories of modules and abelian sheaves.
(4) The diagram

$$\begin{tikzcd} D(\mathcal{O}_{\mathcal{C}'}) \arrow[r, "Lg_!"'] \arrow[d] & D(\mathcal{O}_\mathcal{C}) \arrow[d] \\ D(\mathcal{C}') \arrow[r, "Lg_!^{\mathrm{Ab}}"] & D(\mathcal{C}) \end{tikzcd}$$

commutes.

Proof. The functor u is continuous and cocontinuous Stacks, Lemma 8.10.3. Hence the existence of the functors $g_!$, $g_!^{\text{Ab}}$, $Lg_!$, and $Lg_!^{\text{Ab}}$ can be found in Modules on Sites, Sections 18.16 and 18.41 and Section 21.37.

To prove (2) it suffices to show that the canonical map

$$g_!^{\text{Ab}} j_{U'!}\mathcal{O}_{U'} \to j_{u(U')!}\mathcal{O}_{u(U')}$$

is an isomorphism for all objects U' of $\mathcal{C}'$, see Modules on Sites, Remark 18.41.2. Similarly, to prove (4) it suffices to show that the canonical map

$$Lg_!^{\text{Ab}} j_{U'!}\mathcal{O}_{U'} \to j_{u(U')!}\mathcal{O}_{u(U')}$$

is an isomorphism in $D(\mathcal{C})$ for all objects U' of $\mathcal{C}'$, see Remark 21.37.3. This will also imply the previous formula hence this is what we will show.

We will use that for a localization morphism j the functors $j_!$ and $j_!^{\text{Ab}}$ agree (see Modules on Sites, Remark 18.19.6) and that $j_!$ is exact (Modules on Sites, Lemma 18.19.3). Let us adopt the notation of Lemma 21.38.4. Since $Lg_!^{\text{Ab}} \circ j_{U'!} = j_{U!} \circ L(g')_!^{\text{Ab}}$ (by commutativity of Sites, Lemma 7.28.4 and uniqueness of adjoint functors) it suffices to prove that $L(g')_!^{\text{Ab}}\mathcal{O}_{U'} = \mathcal{O}_U$. Using the results of Lemma 21.38.4 we have for any object E of $D(\mathcal{C}/u(U'))$ the following sequence of equalities

$$\begin{aligned}
\text{Hom}_{D(\mathcal{C}/U)}(L(g')_!^{\text{Ab}}\mathcal{O}_{U'}, E) &= \text{Hom}_{D(\mathcal{C}'/U')}(\mathcal{O}_{U'}, (g')^{-1}E) \\
&= \text{Hom}_{D(\mathcal{C}'/U')}((\pi'_{U'})^{-1}\mathcal{O}_V, (g')^{-1}E) \\
&= \text{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, R\pi'_{U',*}(g')^{-1}E) \\
&= \text{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, (\sigma')^{-1}(g')^{-1}E) \\
&= \text{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \sigma^{-1}E) \\
&= \text{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \pi_{U,*}E) \\
&= \text{Hom}_{D(\mathcal{C}/U)}(\pi_U^{-1}\mathcal{O}_V, E) \\
&= \text{Hom}_{D(\mathcal{C}/U)}(\mathcal{O}_U, E)
\end{aligned}$$

By Yoneda's lemma we conclude. □

09CY Remark 21.38.6. Assumptions and notation as in Situation 21.38.1. Note that setting $\mathcal{C}' = \mathcal{D}$ and u equal to the structure functor of $\mathcal{C}$ gives a situation as in Situation 21.38.3. Hence Lemma 21.38.5 tells us we have functors $\pi_!$, $\pi_!^{\text{Ab}}$, $L\pi_!$, and $L\pi_!^{\text{Ab}}$ such that $forget \circ \pi_! = \pi_!^{\text{Ab}} \circ forget$ and $forget \circ L\pi_! = L\pi_!^{\text{Ab}} \circ forget$.

08PD Remark 21.38.7. Assumptions and notation as in Situation 21.38.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$, let $\mathcal{F}'$ be an abelian sheaf on $\mathcal{C}'$, and let $t : \mathcal{F}' \to g^{-1}\mathcal{F}$ be a map. Then we obtain a canonical map

$$L\pi'_!(\mathcal{F}') \longrightarrow L\pi_!(\mathcal{F})$$

by using the adjoint $g_!\mathcal{F}' \to \mathcal{F}$ of t, the map $Lg_!(\mathcal{F}') \to g_!\mathcal{F}'$, and the equality $L\pi'_! = L\pi_! \circ Lg_!$.

08PE Lemma 21.38.8. Assumptions and notation as in Situation 21.38.1. For $\mathcal{F}$ in $\text{Ab}(\mathcal{C})$ the sheaf $\pi_!\mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto \text{colim}_{\mathcal{C}_V^{opp}} \mathcal{F}|_{\mathcal{C}_V}$$

with restriction maps as indicated in the proof.

Proof. Denote $\mathcal{H}$ be the rule of the lemma. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 4.33.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

Hence a map $\mathcal{H}(V) \to \mathcal{H}(V')$ between colimits, see Categories, Lemma 4.14.8.

To prove the lemma we show that

$$\mathrm{Mor}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{H}, \mathcal{G}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \pi^{-1}\mathcal{G})$$

for every sheaf $\mathcal{G}$ on $\mathcal{C}$. An element of the left hand side is a compatible system of maps $\mathcal{F}(U) \to \mathcal{G}(p(U))$ for all U in $\mathcal{C}$. Since $\pi^{-1}\mathcal{G}(U) = \mathcal{G}(p(U))$ by our choice of topology on $\mathcal{C}$ we see the same thing is true for the right hand side and we win. □

21.39. Homology on a category

08RW In the case of a category over a point we will baptize the left derived lower shriek functors the homology functors.

08PF Example 21.39.1 (Category over point). Let $\mathcal{C}$ be a category. Endow $\mathcal{C}$ with the chaotic topology (Sites, Example 7.6.6). Thus presheaves and sheaves agree on $\mathcal{C}$. The functor $p : \mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism is cocontinuous and continuous. Let $\pi : Sh(\mathcal{C}) \to Sh(*)$ be the corresponding morphism of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}$ with $\mathcal{O}_\mathcal{C} = \pi^{-1}B$ which we will denote $\underline{B}$. In this way

$$\pi : (Sh(\mathcal{C}), \underline{B}) \to (Sh(*), B)$$

is an example of Situation 21.38.1. By Remark 21.38.6 we do not need to distinguish between $\pi_!$ on modules or abelian sheaves. By Lemma 21.38.8 we see that $\pi_!\mathcal{F} = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F}$. Thus $L_n\pi_!$ is the nth left derived functor of taking colimits. In the following, we write

$$H_n(\mathcal{C}, \mathcal{F}) = L_n\pi_!(\mathcal{F})$$

and we will name this the nth homology group of $\mathcal{F}$ on $\mathcal{C}$.

08PG Example 21.39.2 (Computing homology). In Example 21.39.1 we can compute the functors $H_n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(\mathrm{Ab}(\mathcal{C}))$. Consider the chain complex

$$K_\bullet(\mathcal{F}) : \ \ldots \to \bigoplus\nolimits_{U_2 \to U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_0} \mathcal{F}(U_0)$$

where the transition maps are given by

$$(U_2 \to U_1 \to U_0, s) \longmapsto (U_1 \to U_0, s) - (U_2 \to U_0, s) + (U_2 \to U_1, s|_{U_1})$$

and similarly in other degrees. By construction

$$H_0(\mathcal{C}, \mathcal{F}) = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F} = H_0(K_\bullet(\mathcal{F})),$$

see Categories, Lemma 4.14.12. The construction of $K_\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $\mathrm{Ab}(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H_n(K_\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 12.12.1 and Lemma 12.13.12. For $\mathcal{F} = j_{U!}\mathbf{Z}_U$ the complex $K_\bullet(\mathcal{F})$ is the complex associated to the free $\mathbf{Z}$-module on the simplicial set $X_\bullet$ with terms

$$X_n = \coprod\nolimits_{U_n \to \ldots \to U_1 \to U_0} \mathrm{Mor}_\mathcal{C}(U_0, U)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \mathrm{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 14.26.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_n \to \ldots \to U_0, f) \longmapsto (U_n \to \ldots \to U_i \to U \to \ldots \to U, \mathrm{id})$$

for $i > 0$ and $h_{n,0} = \mathrm{id}$. Verifications omitted. This implies that $K_\bullet(j_{U!}\mathbf{Z}_U)$ has trivial cohomology in negative degrees (by the functoriality of Simplicial, Remark 14.26.4 and the result of Simplicial, Lemma 14.27.1). Thus $K_\bullet(\mathcal{F})$ computes the left derived functors $H_n(\mathcal{C}, -)$ of $H_0(\mathcal{C}, -)$ for example by (the duals of) Homology, Lemma 12.12.4 and Derived Categories, Lemma 13.16.6.

08PH Example 21.39.3. Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor. Endow $\mathcal{C}'$ and $\mathcal{C}$ with the chaotic topology as in Example 21.39.1. The functors u, $\mathcal{C}' \to *$, and $\mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism are cocontinuous and continuous. Let $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$, $\pi' : Sh(\mathcal{C}') \to Sh(*)$, and $\pi : Sh(\mathcal{C}) \to Sh(*)$, be the corresponding morphisms of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}'$, $\mathcal{C}$ with the constant sheaf $\underline{B}$. In this way

$$\begin{tikzcd} (Sh(\mathcal{C}'), \underline{B}) \arrow[rr, "g"] \arrow[dr, "\pi'"'] & & (Sh(\mathcal{C}), \underline{B}) \arrow[dl, "\pi"] \\ & (Sh(*), B) & \end{tikzcd}$$

is an example of Situation 21.38.3. Thus Lemma 21.38.5 applies to g so we do not need to distinguish between $g_!$ on modules or abelian sheaves. In particular Remark 21.38.7 produces canonical maps

$$H_n(\mathcal{C}', \mathcal{F}') \longrightarrow H_n(\mathcal{C}, \mathcal{F})$$

whenever we have $\mathcal{F}$ in $\mathrm{Ab}(\mathcal{C})$, $\mathcal{F}'$ in $\mathrm{Ab}(\mathcal{C}')$, and a map $t : \mathcal{F}' \to g^{-1}\mathcal{F}$. In terms of the computation of homology given in Example 21.39.2 we see that these maps come from a map of complexes

$$K_\bullet(\mathcal{F}') \longrightarrow K_\bullet(\mathcal{F})$$

given by the rule

$$(U'_n \to \ldots \to U'_0, s') \longmapsto (u(U'_n) \to \ldots \to u(U'_0), t(s'))$$

with obvious notation.

08Q6 Remark 21.39.4. Notation and assumptions as in Example 21.39.1. Let $\mathcal{F}^\bullet$ be a bounded complex of abelian sheaves on $\mathcal{C}$. For any object U of $\mathcal{C}$ there is a canonical map

$$\mathcal{F}^\bullet(U) \longrightarrow L\pi_!(\mathcal{F}^\bullet)$$

in $D(\mathrm{Ab})$. If $\mathcal{F}^\bullet$ is a complex of $\underline{B}$-modules then this map is in $D(B)$. To prove this, note that we compute $L\pi_!(\mathcal{F}^\bullet)$ by taking a quasi-isomorphism $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$ where $\mathcal{P}^\bullet$ is a complex of projectives. However, since the topology is chaotic this means that $\mathcal{P}^\bullet(U) \to \mathcal{F}^\bullet(U)$ is a quasi-isomorphism hence can be inverted in $D(\mathrm{Ab})$, resp. $D(B)$. Composing with the canonical map $\mathcal{P}^\bullet(U) \to \pi_!(\mathcal{P}^\bullet)$ coming from the computation of $\pi_!$ as a colimit we obtain the desired arrow.

08Q7 **Lemma 21.39.5.** Notation and assumptions as in Example 21.39.1. If $\mathcal{C}$ has either an initial or a final object, then $L\pi_! \circ \pi^{-1} = \text{id}$ on $D(\text{Ab})$, resp. $D(B)$.

Proof. If $\mathcal{C}$ has an initial object, then $\pi_!$ is computed by evaluating on this object and the statement is clear. If $\mathcal{C}$ has a final object, then $R\pi_*$ is computed by evaluating on this object, hence $R\pi_* \circ \pi^{-1} \cong \text{id}$ on $D(\text{Ab})$, resp. $D(B)$. This implies that $\pi^{-1} : D(\text{Ab}) \to D(\mathcal{C})$, resp. $\pi^{-1} : D(B) \to D(\underline{B})$ is fully faithful, see Categories, Lemma 4.24.4. Then the same lemma implies that $L\pi_! \circ \pi^{-1} = \text{id}$ as desired. $\square$

08Q8 **Lemma 21.39.6.** Notation and assumptions as in Example 21.39.1. Let $B \to B'$ be a ring map. Consider the commutative diagram of ringed topoi

$$\begin{tikzcd} (Sh(\mathcal{C}), \underline{B}) \arrow[d, "\pi"'] & (Sh(\mathcal{C}), \underline{B'}) \arrow[l, "h"] \arrow[d, "\pi'"] \\ (*, B) & (*, B') \arrow[l, "f"'] \end{tikzcd}$$

Then $L\pi_! \circ Lh^* = Lf^* \circ L\pi'_!$.

Proof. Both functors are right adjoint to the obvious functor $D(B') \to D(\underline{B})$. $\square$

08Q9 **Lemma 21.39.7.** Notation and assumptions as in Example 21.39.1. Let $U_\bullet$ be a cosimplicial object in $\mathcal{C}$ such that for every $U \in \text{Ob}(\mathcal{C})$ the simplicial set $\text{Mor}_\mathcal{C}(U_\bullet, U)$ is homotopy equivalent to the constant simplicial set on a singleton. Then

$$L\pi_!(\mathcal{F}) = \mathcal{F}(U_\bullet)$$

in $D(\text{Ab})$, resp. $D(B)$ functorially in $\mathcal{F}$ in $\text{Ab}(\mathcal{C})$, resp. $\text{Mod}(\underline{B})$.

Proof. As $L\pi_!$ agrees for modules and abelian sheaves by Lemma 21.38.5 it suffices to prove this when $\mathcal{F}$ is an abelian sheaf. For $U \in \text{Ob}(\mathcal{C})$ the abelian sheaf $j_{U!}\mathbf{Z}_U$ is a projective object of $\text{Ab}(\mathcal{C})$ since $\text{Hom}(j_{U!}\mathbf{Z}_U, \mathcal{F}) = \mathcal{F}(U)$ and taking sections is an exact functor as the topology is chaotic. Every abelian sheaf is a quotient of a direct sum of $j_{U!}\mathbf{Z}_U$ by Modules on Sites, Lemma 18.28.8. Thus we can compute $L\pi_!(\mathcal{F})$ by choosing a resolution

$$\ldots \to \mathcal{G}^{-1} \to \mathcal{G}^0 \to \mathcal{F} \to 0$$

whose terms are direct sums of sheaves of the form above and taking $L\pi_!(\mathcal{F}) = \pi_!(\mathcal{G}^\bullet)$. Consider the double complex $A^{\bullet,\bullet} = \mathcal{G}^\bullet(U_\bullet)$. The map $\mathcal{G}^0 \to \mathcal{F}$ gives a map of complexes $A^{0,\bullet} \to \mathcal{F}(U_\bullet)$. Since $\pi_!$ is computed by taking the colimit over $\mathcal{C}^{opp}$ (Lemma 21.38.8) we see that the two compositions $\mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m$ are equal. Thus we obtain a canonical map of complexes

$$\text{Tot}(A^{\bullet,\bullet}) \longrightarrow \pi_!(\mathcal{G}^\bullet) = L\pi_!(\mathcal{F})$$

To prove the lemma it suffices to show that the complexes

$$\ldots \to \mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m \to 0$$

are exact, see Homology, Lemma 12.25.4. Since the sheaves $\mathcal{G}^m$ are direct sums of the sheaves $j_{U!}\mathbf{Z}_U$ we reduce to $\mathcal{G} = j_{U!}\mathbf{Z}_U$. The complex $j_{U!}\mathbf{Z}_U(U_\bullet)$ is the complex of abelian groups associated to the free $\mathbf{Z}$-module on the simplicial set $\text{Mor}_\mathcal{C}(U_\bullet, U)$ which we assumed to be homotopy equivalent to a singleton. We conclude that

$$j_{U!}\mathbf{Z}_U(U_\bullet) \to \mathbf{Z}$$

is a homotopy equivalence of abelian groups hence a quasi-isomorphism (Simplicial, Remark 14.26.4 and Lemma 14.27.1). This finishes the proof since $\pi_! j_{U!}\mathbf{Z}_U = \mathbf{Z}$ as was shown in the proof of Lemma 21.38.5. $\square$

08QA Lemma 21.39.8. Notation and assumptions as in Example 21.39.3. If there exists a cosimplicial object $U'_\bullet$ of $\mathcal{C}'$ such that Lemma 21.39.7 applies to both $U'_\bullet$ in $\mathcal{C}'$ and $u(U'_\bullet)$ in $\mathcal{C}$, then we have $L\pi'_! \circ g^{-1} = L\pi_!$ as functors $D(\mathcal{C}) \to D(\text{Ab})$, resp. $D(\mathcal{C}, \underline{B}) \to D(B)$.

Proof. Follows immediately from Lemma 21.39.7 and the fact that g^{-1} is given by precomposing with u. $\square$

08QB Lemma 21.39.9. Let $\mathcal{C}_i$, $i = 1, 2$ be categories. Let $u_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$ be the projection functors. Let B be a ring. Let $g_i : (Sh(\mathcal{C}_1 \times \mathcal{C}_2), \underline{B}) \to (Sh(\mathcal{C}_i), \underline{B})$ be the corresponding morphisms of ringed topoi, see Example 21.39.3. For $K_i \in D(\mathcal{C}_i, B)$ we have

$$L(\pi_1 \times \pi_2)_!(g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2) = L\pi_{1,!}(K_1) \otimes_B^{\mathbf{L}} L\pi_{2,!}(K_2)$$

in $D(B)$ with obvious notation.

Proof. As both sides commute with colimits, it suffices to prove this for $K_1 = j_{U!}\underline{B}_U$ and $K_2 = j_{V!}\underline{B}_V$ for $U \in \text{Ob}(\mathcal{C}_1)$ and $V \in \text{Ob}(\mathcal{C}_2)$. See construction of $L\pi_!$ in Lemma 21.37.2. In this case

$$g_1^{-1}K_1 \otimes_{\underline{B}}^{\mathbf{L}} g_2^{-1}K_2 = g_1^{-1}K_1 \otimes_{\underline{B}} g_2^{-1}K_2 = j_{(U,V)!}\underline{B}_{(U,V)}$$

Verification omitted. Hence the result follows as both the left and the right hand side of the formula of the lemma evaluate to B, see construction of $L\pi_!$ in Lemma 21.37.2. $\square$

08QC Lemma 21.39.10. Notation and assumptions as in Example 21.39.1. If there exists a cosimplicial object $U_\bullet$ of $\mathcal{C}$ such that Lemma 21.39.7 applies, then

$$L\pi_!(K_1 \otimes_{\underline{B}}^{\mathbf{L}} K_2) = L\pi_!(K_1) \otimes_B^{\mathbf{L}} L\pi_!(K_2)$$

for all $K_i \in D(\underline{B})$.

Proof. Consider the diagram of categories and functors

$$\begin{tikzcd} & & \mathcal{C} \\ \mathcal{C} \arrow[r, "u"] & \mathcal{C} \times \mathcal{C} \arrow[ur, "u_1"] \arrow[dr, "u_2"] & \\ & & \mathcal{C} \end{tikzcd}$$

where u is the diagonal functor and u_i are the projection functors. This gives morphisms of ringed topoi g, g_1, g_2. For any object (U_1, U_2) of $\mathcal{C}$ we have

$$\text{Mor}_{\mathcal{C}\times\mathcal{C}}(u(U_\bullet), (U_1, U_2)) = \text{Mor}_{\mathcal{C}}(U_\bullet, U_1) \times \text{Mor}_{\mathcal{C}}(U_\bullet, U_2)$$

which is homotopy equivalent to a point by Simplicial, Lemma 14.26.10. Thus Lemma 21.39.8 gives $L\pi_!(g^{-1}K) = L(\pi \times \pi)_!(K)$ for any K in $D(\mathcal{C} \times \mathcal{C}, B)$. Take $K = g_1^{-1}K_1 \otimes_B^{\mathbf{L}} g_2^{-1}K_2$. Then $g^{-1}K = K_1 \otimes_{\underline{B}}^{\mathbf{L}} K_2$ because $g^{-1} = g^* = Lg^*$ commutes with derived tensor product (Lemma 21.18.4). To finish we apply Lemma 21.39.9. $\square$

08QD Remark 21.39.11 (Simplicial modules). Let $\mathcal{C} = \Delta$ and let B be any ring. This is a special case of Example 21.39.1 where the assumptions of Lemma 21.39.7 hold. Namely, let $U_\bullet$ be the cosimplicial object of Δ given by the identity functor. To verify the condition we have to show that for $[m] \in \text{Ob}(\Delta)$ the simplicial set $\Delta[m] : n \mapsto \text{Mor}_\Delta([n], [m])$ is homotopy equivalent to a point. This is explained in Simplicial, Example 14.26.7.

In this situation the category $\text{Mod}(\underline{B})$ is just the category of simplicial B-modules and the functor $L\pi_!$ sends a simplicial B-module $M_\bullet$ to its associated complex $s(M_\bullet)$ of B-modules. Thus the results above can be reinterpreted in terms of results on simplicial modules. For example a special case of Lemma 21.39.10 is: if $M_\bullet$, $M'_\bullet$ are flat simplicial B-modules, then the complex $s(M_\bullet \otimes_B M'_\bullet)$ is quasi-isomorphic to the total complex associated to the double complex $s(M_\bullet) \otimes_B s(M'_\bullet)$. (Hint: use flatness to convert from derived tensor products to usual tensor products.) This is a special case of the Eilenberg-Zilber theorem which can be found in [EZ53].

08RX Lemma 21.39.12. Let $\mathcal{C}$ be a category (endowed with chaotic topology). Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings on $\mathcal{C}$. Assume

(1) there exists a cosimplicial object $U_\bullet$ in $\mathcal{C}$ as in Lemma 21.39.7, and
(2) $L\pi_!\mathcal{O} \to L\pi_!\mathcal{O}'$ is an isomorphism.

For K in $D(\mathcal{O})$ we have
$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}')$$
in $D(\text{Ab})$.

Proof. Note: in this proof $L\pi_!$ denotes the left derived functor of $\pi_!$ on abelian sheaves. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 13.29.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are direct sums of $j_{U!}\mathcal{O}_U$ with $U \in \text{Ob}(\mathcal{C})$, see Modules on Sites, Lemma 18.28.8. Thus it suffices to prove the lemma for $j_{U!}\mathcal{O}_U$. By assumption
$$S_\bullet = \text{Mor}_{\mathcal{C}}(U_\bullet, U)$$
is a simplicial set homotopy equivalent to the constant simplicial set on a singleton. Set $P_n = \mathcal{O}(U_n)$ and $P'_n = \mathcal{O}'(U_n)$. Observe that the complex associated to the simplicial abelian group
$$X_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P_n$$
computes $L\pi_!(j_{U!}\mathcal{O}_U)$ by Lemma 21.39.7. Since $j_{U!}\mathcal{O}_U$ is a flat $\mathcal{O}$-module we have $j_{U!}\mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}' = j_{U!}\mathcal{O}'_U$ and $L\pi_!$ of this is computed by the complex associated to the simplicial abelian group
$$X'_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P'_n$$
As the rule which to a simplicial set $T_\bullet$ associated the simplicial abelian group with terms $\bigoplus_{t \in T_n} P_n$ is a functor, we see that $X_\bullet \to P_\bullet$ is a homotopy equivalence of simplicial abelian groups. Similarly, the rule which to a simplicial set $T_\bullet$ associates the simplicial abelian group with terms $\bigoplus_{t \in T_n} P'_n$ is a functor. Hence $X'_\bullet \to P'_\bullet$ is a homotopy equivalence of simplicial abelian groups. By assumption $P_\bullet \to P'_\bullet$ is a quasi-isomorphism (since $P_\bullet$, resp. $P'_\bullet$ computes $L\pi_!\mathcal{O}$, resp. $L\pi_!\mathcal{O}'$ by Lemma 21.39.7). We conclude that $X_\bullet$ and $X'_\bullet$ are quasi-isomorphic as desired. $\square$

09CZ Remark 21.39.13. Let $\mathcal{C}$ and B be as in Example 21.39.1. Assume there exists a cosimplicial object as in Lemma 21.39.7. Let $\mathcal{O} \to \underline{B}$ be a map sheaf of rings on $\mathcal{C}$ which induces an isomorphism $L\pi_!\mathcal{O} \to L\pi_!\underline{B}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{O}) \longrightarrow D(B)$$

Namely, for any object K of $D(\mathcal{O})$ we have $L\pi_!^{\text{Ab}}(K) = L\pi_!^{\text{Ab}}(K \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B})$ by Lemma 21.39.12. Thus we can define the displayed functor as the composition of $- \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B}$ with the functor $L\pi_! : D(\underline{B}) \to D(B)$. In other words, we obtain a B-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes_{\mathcal{O}}^{\mathbf{L}} \underline{B})$ of the lemma.

21.40. Calculating derived lower shriek

08P7 In this section we apply the results from Section 21.39 to compute $L\pi_!$ in Situation 21.38.1 and $Lg_!$ in Situation 21.38.3.

08PI Lemma 21.40.1. Assumptions and notation as in Situation 21.38.1. For $\mathcal{F}$ in $\textit{PAb}(\mathcal{C})$ and $n \geq 0$ consider the abelian sheaf $L_n(\mathcal{F})$ on $\mathcal{D}$ which is the sheaf associated to the presheaf

$$V \longmapsto H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V})$$

with restriction maps as indicated in the proof. Then $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$.

Proof. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 4.33.6). Moreover for $U \in \text{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

By Example 21.39.3 we obtain the desired restriction map

$$H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V}) \longrightarrow H_n(\mathcal{C}_{V'}, \mathcal{F}|_{\mathcal{C}_{V'}})$$

Let us denote $L_{n,p}(\mathcal{F})$ this presheaf, so that $L_n(\mathcal{F}) = L_{n,p}(\mathcal{F})^\#$. The canonical map $\gamma : \mathcal{F} \to \mathcal{F}^+$ (Sites, Theorem 7.10.10) defines a canonical map $L_{n,p}(\mathcal{F}) \to L_{n,p}(\mathcal{F}^+)$. We have to prove this map becomes an isomorphism after sheafification.

Let us use the computation of homology given in Example 21.39.2. Denote $K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$ the complex associated to the restriction of $\mathcal{F}$ to the fibre category $\mathcal{C}_V$. By the remarks above we obtain a presheaf $K_\bullet(\mathcal{F})$ of complexes

$$V \longmapsto K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$$

whose cohomology presheaves are the presheaves $L_{n,p}(\mathcal{F})$. Thus it suffices to show that

$$K_\bullet(\mathcal{F}) \longrightarrow K_\bullet(\mathcal{F}^+)$$

becomes an isomorphism on sheafification.

Injectivity. Let V be an object of $\mathcal{D}$ and let $\xi \in K_n(\mathcal{F})(V)$ be an element which maps to zero in $K_n(\mathcal{F}^+)(V)$. We have to show there exists a covering $\{V_j \to V\}$ such that $\xi|_{V_j}$ is zero in $K_n(\mathcal{F})(V_j)$. We write

$$\xi = \sum (U_{i,n+1} \to \ldots \to U_{i,0}, \sigma_i)$$

with $\sigma_i \in \mathcal{F}(U_{i,0})$. We arrange it so that each sequence of morphisms $U_n \to \ldots \to U_0$ of $\mathcal{C}_V$ occurs are most once. Since the sums in the definition of the complex $K_\bullet$ are direct sums, the only way this can map to zero in $K_\bullet(\mathcal{F}^+)(V)$ is if all σ_i map to zero in $\mathcal{F}^+(U_{i,0})$. By construction of $\mathcal{F}^+$ there exist coverings $\{U_{i,0,j} \to U_{i,0}\}$ such that $\sigma_i|_{U_{i,0,j}}$ is zero. By our construction of the topology on $\mathcal{C}$ we can write $U_{i,0,j} \to U_{i,0}$ as the pullback (Categories, Definition 4.33.6) of some morphisms $V_{i,j} \to V$ and moreover each $\{V_{i,j} \to V\}$ is a covering. Choose a covering $\{V_j \to V\}$ dominating each of the coverings $\{V_{i,j} \to V\}$. Then it is clear that $\xi|_{V_j} = 0$.

Surjectivity. Proof omitted. Hint: Argue as in the proof of injectivity. □

08PJ Lemma 21.40.2. Assumptions and notation as in Situation 21.38.1. For $\mathcal{F}$ in $\textit{Ab}(\mathcal{C})$ and $n \geq 0$ the sheaf $L_n\pi_!(\mathcal{F})$ is equal to the sheaf $L_n(\mathcal{F})$ constructed in Lemma 21.40.1.

Proof. Consider the sequence of functors $\mathcal{F} \mapsto L_n(\mathcal{F})$ from $\textit{PAb}(\mathcal{C}) \to \textit{Ab}(\mathcal{C})$. Since for each $V \in \text{Ob}(\mathcal{D})$ the sequence of functors $H_n(\mathcal{C}_V, -)$ forms a δ-functor so do the functors $\mathcal{F} \mapsto L_n(\mathcal{F})$. Our goal is to show these form a universal δ-functor. In order to do this we construct some abelian presheaves on which these functors vanish.

For $U' \in \text{Ob}(\mathcal{C})$ consider the abelian presheaf $\mathcal{F}_{U'} = j_{U'!}^{PAb}\mathbf{Z}_{U'}$ (Modules on Sites, Remark 18.19.7). Recall that

$$\mathcal{F}_{U'}(U) = \bigoplus\nolimits_{\text{Mor}_\mathcal{C}(U,U')} \mathbf{Z}$$

If U lies over $V = p(U)$ in $\mathcal{D}$) and U' lies over $V' = p(U')$ then any morphism $a : U \to U'$ factors uniquely as $U \to h^*U' \to U'$ where $h = p(a) : V \to V'$ (see Categories, Definition 4.33.6). Hence we see that

$$\mathcal{F}_{U'}|_{\mathcal{C}_V} = \bigoplus\nolimits_{h \in \text{Mor}_\mathcal{D}(V,V')} j_{h^*U'!}\mathbf{Z}_{h^*U'}$$

where $j_{h^*U'} : \textit{Sh}(\mathcal{C}_V/h^*U') \to \textit{Sh}(\mathcal{C}_V)$ is the localization morphism. The sheaves $j_{h^*U'!}\mathbf{Z}_{h^*U'}$ have vanishing higher homology groups (see Example 21.39.2). We conclude that $L_n(\mathcal{F}_{U'}) = 0$ for all $n > 0$ and all U'. It follows that any abelian presheaf $\mathcal{F}$ is a quotient of an abelian presheaf $\mathcal{G}$ with $L_n(\mathcal{G}) = 0$ for all $n > 0$ (Modules on Sites, Lemma 18.28.8). Since $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$ we see that the same thing is true for abelian sheaves. Thus the sequence of functors $L_n(-)$ is a universal delta functor on $\textit{Ab}(\mathcal{C})$ (Homology, Lemma 12.12.4). Since we have agreement with $H^{-n}(L\pi_!(-))$ for $n = 0$ by Lemma 21.38.8 we conclude by uniqueness of universal δ-functors (Homology, Lemma 12.12.5) and Derived Categories, Lemma 13.16.6. □

08PK Lemma 21.40.3. Assumptions and notation as in Situation 21.38.3. For an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}'$ the sheaf $L_ng_!(\mathcal{F}')$ is the sheaf associated to the presheaf

$$U \longmapsto H_n(\mathcal{I}_U, \mathcal{F}'_U)$$

For notation and restriction maps see proof.

Proof. Say $p(U) = V$. The category $\mathcal{I}_U$ is the category of pairs (U', φ) where $\varphi : U \to u(U')$ is a morphism of $\mathcal{C}$ with $p(\varphi) = \text{id}_V$, i.e., φ is a morphism of the fibre category $\mathcal{C}_V$. Morphisms $(U'_1, \varphi_1) \to (U'_2, \varphi_2)$ are given by morphisms $a : U'_1 \to U'_2$ of the fibre category $\mathcal{C}'_V$ such that $\varphi_2 = u(a) \circ \varphi_1$. The presheaf $\mathcal{F}'_U$ sends (U', φ) to $\mathcal{F}'(U')$. We will construct the restriction mappings below.

Choose a factorization

$$\begin{tikzcd} \mathcal{C}' \arrow[r, shift left, "u'"] & \mathcal{C}'' \arrow[l, shift left, "w"] \arrow[r, "u''"] & \mathcal{C} \end{tikzcd}$$

of u as in Categories, Lemma 4.33.14. Then $g_! = g''_! \circ g'_!$ and similarly for derived functors. On the other hand, the functor $g'_!$ is exact, see Modules on Sites, Lemma 18.16.6. Thus we get $Lg_!(\mathcal{F}') = Lg''_!(\mathcal{F}'')$ where $\mathcal{F}'' = g'_!\mathcal{F}'$. Note that $\mathcal{F}'' = h^{-1}\mathcal{F}'$ where $h : Sh(\mathcal{C}'') \to Sh(\mathcal{C}')$ is the morphism of topoi associated to w, see Sites, Lemma 7.23.1. The functor u'' turns $\mathcal{C}''$ into a fibred category over $\mathcal{C}$, hence Lemma 21.40.2 applies to the computation of $L_n g''_!$. The result follows as the construction of $\mathcal{C}''$ in the proof of Categories, Lemma 4.33.14 shows that the fibre category $\mathcal{C}''_U$ is equal to $\mathcal{I}_U$. Moreover, $h^{-1}\mathcal{F}'|_{\mathcal{C}''_U}$ is given by the rule described above (as w is continuous and cocontinuous by Stacks, Lemma 8.10.3 so we may apply Sites, Lemma 7.21.5). $\square$

21.41. Simplicial modules

09D0 Let $A_\bullet$ be a simplicial ring. Recall that we may think of $A_\bullet$ as a sheaf on Δ (endowed with the chaotic topology), see Simplicial, Section 14.4. Then a simplicial module $M_\bullet$ over $A_\bullet$ is just a sheaf of $A_\bullet$-modules on Δ. In other words, for every $n \geq 0$ we have an A_n-module M_n and for every map $\varphi : [n] \to [m]$ we have a corresponding map

$$M_\bullet(\varphi) : M_m \longrightarrow M_n$$

which is $A_\bullet(\varphi)$-linear such that these maps compose in the usual manner.

Let $\mathcal{C}$ be a site. A simplicial sheaf of rings $\mathcal{A}_\bullet$ on $\mathcal{C}$ is a simplicial object in the category of sheaves of rings on $\mathcal{C}$. In this case the assignment $U \mapsto \mathcal{A}_\bullet(U)$ is a sheaf of simplicial rings and in fact the two notions are equivalent. A similar discussion holds for simplicial abelian sheaves, simplicial sheaves of Lie algebras, and so on.

However, as in the case of simplicial rings above, there is another way to think about simplicial sheaves. Namely, consider the projection

$$p : \Delta \times \mathcal{C} \longrightarrow \mathcal{C}$$

This defines a fibred category with strongly cartesian morphisms exactly the morphisms of the form $([n], U) \to ([n], V)$. We endow the category $\Delta \times \mathcal{C}$ with the topology inherited from $\mathcal{C}$ (see Stacks, Section 8.10). The simple description of the coverings in $\Delta \times \mathcal{C}$ (Stacks, Lemma 8.10.1) immediately implies that a simplicial sheaf of rings on $\mathcal{C}$ is the same thing as a sheaf of rings on $\Delta \times \mathcal{C}$.

By analogy with the case of simplicial modules over a simplicial ring, we define simplicial modules over simplicial sheaves of rings as follows.

09D1 **Definition 21.41.1.** Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet$ be a simplicial sheaf of rings on $\mathcal{C}$. A simplicial $\mathcal{A}_\bullet$-module $\mathcal{F}_\bullet$ (sometimes called a simplicial sheaf of $\mathcal{A}_\bullet$-modules) is a sheaf of modules over the sheaf of rings on $\Delta \times \mathcal{C}$ associated to $\mathcal{A}_\bullet$.

We obtain a category $\text{Mod}(\mathcal{A}_\bullet)$ of simplicial modules and a corresponding derived category $D(\mathcal{A}_\bullet)$. Given a map $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ of simplicial sheaves of rings we obtain a functor

$$- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{B}_\bullet)$$

Moreover, the material of the preceding sections determines a functor

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{C})$$

Given a simplicial module $\mathcal{F}_\bullet$ the object $L\pi_!(\mathcal{F}_\bullet)$ is represented by the associated chain complex $s(\mathcal{F}_\bullet)$ (Simplicial, Section 14.23). This follows from Lemmas 21.40.2 and 21.39.7.

09D2 Lemma 21.41.2. Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ be a homomorphism of simplicial sheaves of rings on $\mathcal{C}$. If $L\pi_!\mathcal{A}_\bullet \to L\pi_!\mathcal{B}_\bullet$ is an isomorphism in $D(\mathcal{C})$, then we have

$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet)$$

for all K in $D(\mathcal{A}_\bullet)$.

Proof. Let $([n], U)$ be an object of $\Delta \times \mathcal{C}$. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 13.29.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are flat modules, see Modules on Sites, Lemma 18.28.8. Thus it suffices to prove the lemma for a flat $\mathcal{A}_\bullet$-module $\mathcal{F}$. In this case the derived tensor product is the usual tensor product and is a sheaf also. Hence by Lemma 21.40.2 we can compute the cohomology sheaves of both sides of the equation by the procedure of Lemma 21.40.1. Thus it suffices to prove the result for the restriction of $\mathcal{F}$ to the fibre categories (i.e., to $\Delta \times U$). In this case the result follows from Lemma 21.39.12. $\square$

09D3 Remark 21.41.3. Let $\mathcal{C}$ be a site. Let $\epsilon : \mathcal{A}_\bullet \to \mathcal{O}$ be an augmentation (Simplicial, Definition 14.20.1) in the category of sheaves of rings. Assume ϵ induces a quasi-isomorphism $s(\mathcal{A}_\bullet) \to \mathcal{O}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{O})$$

Namely, for any object K of $D(\mathcal{A}_\bullet)$ we have $L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ by Lemma 21.41.2. Thus we can define the displayed functor as the composition of $- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O}$ with the functor $L\pi_! : D(\Delta \times \mathcal{C}, \pi^{-1}\mathcal{O}) \to D(\mathcal{O})$ of Remark 21.38.6. In other words, we obtain a $\mathcal{O}$-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ of the lemma.

21.42. Cohomology on a category

08RY In the situation of Example 21.39.1 in addition to the derived functor $L\pi_!$, we also have the functor $R\pi_*$. For an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $H_n(\mathcal{C}, \mathcal{F}) = H^{-n}(L\pi_!\mathcal{F})$ and $H^n(\mathcal{C}, \mathcal{F}) = H^n(R\pi_*\mathcal{F})$.

08RZ Example 21.42.1 (Computing cohomology). In Example 21.39.1 we can compute the functors $H^n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(\mathrm{Ab}(\mathcal{C}))$. Consider the cochain complex

$$K^\bullet(\mathcal{F}) : \prod\nolimits_{U_0} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1 \to U_2} \mathcal{F}(U_0) \to \ldots$$

where the transition maps are given by

$$(s_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto s_{U_0 \to U_1} - s_{U_0 \to U_2} + s_{U_1 \to U_2}|_{U_0})$$

and similarly in other degrees. By construction

$$H^0(\mathcal{C}, \mathcal{F}) = \lim\nolimits_{\mathcal{C}^{opp}} \mathcal{F} = H^0(K^\bullet(\mathcal{F})),$$

see Categories, Lemma 4.14.11. The construction of $K^\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $\text{Ab}(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H^n(K^\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 12.12.1 and Lemma 12.13.12. For an object U of $\mathcal{C}$ denote $p_U : Sh(*) \to Sh(\mathcal{C})$ the corresponding point with p_U^{-1} equal to evaluation at U, see Sites, Example 7.33.8. Let A be an abelian group and set $\mathcal{F} = p_{U,*}A$. In this case the complex $K^\bullet(\mathcal{F})$ is the complex with terms $\text{Map}(X_n, A)$ where

$$X_n = \coprod\nolimits_{U_0 \to \ldots \to U_{n-1} \to U_n} \text{Mor}_\mathcal{C}(U, U_0)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \text{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 14.26.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_0 \to \ldots \to U_n, f) \longmapsto (U \to \ldots \to U \to U_i \to \ldots \to U_n, \text{id})$$

for $i > 0$ and $h_{n,0} = \text{id}$. Verifications omitted. Since $\text{Map}(-, A)$ is a contravariant functor, implies that $K^\bullet(p_{U,*}A)$ has trivial cohomology in positive degrees (by the functoriality of Simplicial, Remark 14.26.4 and the result of Simplicial, Lemma 14.28.6). This implies that $K^\bullet(\mathcal{F})$ is acyclic in positive degrees also if $\mathcal{F}$ is a product of sheaves of the form $p_{U,*}A$. As every abelian sheaf on $\mathcal{C}$ embeds into such a product we conclude that $K^\bullet(\mathcal{F})$ computes the left derived functors $H^n(\mathcal{C}, -)$ of $H^0(\mathcal{C}, -)$ for example by Homology, Lemma 12.12.4 and Derived Categories, Lemma 13.16.6.

08S0 Example 21.42.2 (Computing Exts). In Example 21.39.1 assume we are moreover given a sheaf of rings $\mathcal{O}$ on $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. Consider the complex $K^\bullet(\mathcal{G}, \mathcal{F})$ with degree n term

$$\prod\nolimits_{U_0 \to U_1 \to \ldots \to U_n} \text{Hom}_{\mathcal{O}(U_n)}(\mathcal{G}(U_n), \mathcal{F}(U_0))$$

and transition map given by

$$(\varphi_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto \varphi_{U_0 \to U_1} \circ \rho^{U_2}_{U_1} - \varphi_{U_0 \to U_2} + \rho^{U_1}_{U_0} \circ \varphi_{U_1 \to U_2}$$

and similarly in other degrees. Here the ρ's indicate restriction maps. By construction

$$\text{Hom}_\mathcal{O}(\mathcal{G}, \mathcal{F}) = H^0(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for all pairs of $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$. The assignment $(\mathcal{G}, \mathcal{F}) \mapsto K^\bullet(\mathcal{G}, \mathcal{F})$ is a bifunctor which transforms direct sums in the first variable into products and commutes with products in the second variable. We claim that

$$\text{Ext}^i_\mathcal{O}(\mathcal{G}, \mathcal{F}) = H^i(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for $i \geq 0$ provided either

(1) $\mathcal{G}(U)$ is a projective $\mathcal{O}(U)$-module for all $U \in \text{Ob}(\mathcal{C})$, or
(2) $\mathcal{F}(U)$ is an injective $\mathcal{O}(U)$-module for all $U \in \text{Ob}(\mathcal{C})$.

Namely, case (1) the functor $K^\bullet(\mathcal{G}, -)$ is an exact functor from the category of $\mathcal{O}$-modules to the category of cochain complexes of abelian groups. Thus, arguing as in Example 21.42.1, it suffices to show that $K^\bullet(\mathcal{G}, \mathcal{F})$ is acyclic in positive degrees when $\mathcal{F}$ is $p_{U,*}A$ for an $\mathcal{O}(U)$-module A. Choose a short exact sequence

08S1 (21.42.2.1) $$0 \to \mathcal{G}' \to \bigoplus j_{U_i!}\mathcal{O}_{U_i} \to \mathcal{G} \to 0$$

see Modules on Sites, Lemma 18.28.8. Since (1) holds for the middle and right sheaves, it also holds for $\mathcal{G}'$ and evaluating (21.42.2.1) on an object of $\mathcal{C}$ gives a split exact sequence of modules. We obtain a short exact sequence of complexes

$$0 \to K^\bullet(\mathcal{G}, \mathcal{F}) \to \prod K^\bullet(j_{U_i!}\mathcal{O}_{U_i}, \mathcal{F}) \to K^\bullet(\mathcal{G}', \mathcal{F}) \to 0$$

for any $\mathcal{F}$, in particular $\mathcal{F} = p_{U,*}A$. On H^0 we obtain

$$0 \to \mathrm{Hom}(\mathcal{G}, p_{U,*}A) \to \mathrm{Hom}(\prod j_{U_i!}\mathcal{O}_{U_i}, p_{U,*}A) \to \mathrm{Hom}(\mathcal{G}', p_{U,*}A) \to 0$$

which is exact as $\mathrm{Hom}(\mathcal{H}, p_{U,*}A) = \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{H}(U), A)$ and the sequence of sections of (21.42.2.1) over U is split exact. Thus we can use dimension shifting to see that it suffices to prove $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is acyclic in positive degrees for all $U, U' \in \mathrm{Ob}(\mathcal{C})$. In this case $K^n(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is equal to

$$\prod\nolimits_{U \to U_0 \to U_1 \to \ldots \to U_n \to U'} A$$

In other words, $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is the complex with terms $\mathrm{Map}(X_\bullet, A)$ where

$$X_n = \coprod\nolimits_{U_0 \to \ldots \to U_{n-1} \to U_n} \mathrm{Mor}_\mathcal{C}(U, U_0) \times \mathrm{Mor}_\mathcal{C}(U_n, U')$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$ as can be proved in exactly the same way as the corresponding statement in Example 21.42.1. This finishes the proof of the claim.

The argument in case (2) is similar (but dual).

21.43. Modules on a category

0GYU The material in this section will be used to define a variant of the derived category of quasi-coherent modules on a stack in groupoids over the category of schemes. See Sheaves on Stacks, Section 96.26.

Let $\mathcal{C}$ be a category. We think of $\mathcal{C}$ as a site with the chaotic topology. As in Example 21.42.2 we let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. In other words, $\mathcal{O}$ is a presheaf of rings on the category $\mathcal{C}$, see Categories, Definition 4.3.3.

0GYV **Definition 21.43.1.** In the situation above, we denote $QC(\mathcal{C}, \mathcal{O})$ or simply $QC(\mathcal{O})$ the full subcategory of $D(\mathcal{O}) = D(\mathcal{C}, \mathcal{O})$ consisting of objects K such that for all $U \to V$ in $\mathcal{C}$ the canonical map

$$R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \longrightarrow R\Gamma(U, K)$$

is an isomorphism in $D(\mathcal{O}(U))$.

0GYW **Lemma 21.43.2.** *In the situation above, the subcategory $QC(\mathcal{O})$ is a strictly full, saturated, triangulated subcategory of $D(\mathcal{O})$ preserved by arbitrary direct sums.*

Proof. Let U be an object of $\mathcal{C}$. Since the topology on $\mathcal{C}$ is chaotic, the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is exact and commutes with direct sums. Hence the exact functor $K \mapsto R\Gamma(U, K)$ is computed by representing K by any complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules and taking $\mathcal{F}^\bullet(U)$. Thus $R\Gamma(U, -)$ commutes with direct sums, see Injectives, Lemma 19.13.4. Similarly, given a morphism $U \to V$ of $\mathcal{C}$ the derived tensor product functor $- \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) : D(\mathcal{O}(V)) \to D(\mathcal{O}(U))$ is exact and commutes with direct sums. The lemma follows from these observations in a straightforward manner; details omitted. ☐

0GZQ Lemma 21.43.3. In the situation above, suppose that M is an object of $QC(\mathcal{O})$ and $b \in \mathbf{Z}$ such that $H^i(M) = 0$ for all $i > b$. Then $H^b(M)$ is a quasi-coherent module on $(\mathcal{C}, \mathcal{O})$ in the sense of Modules on Sites, Definition 18.23.1.

Proof. By Modules on Sites, Lemma 18.24.2 it suffices to show that for every morphism $U \to V$ of $\mathcal{C}$ the map

$$H^p(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to H^b(M)(U)$$

is an isomorphism. We are given that the map

$$R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, M)$$

is an isomorphism. Thus the result by the Tor spectral sequence for example. Details omitted. ☐

0H0R Lemma 21.43.4. In the situation above, suppose that $\mathcal{C}$ has a final object X. Set $R = \mathcal{O}(X)$ and denote $f : (\mathcal{C}, \mathcal{O}) \to (pt, R)$ the obvious morphism of sites. Then $QC(\mathcal{O}) = D(R)$ given by Lf^* and Rf_*.

Proof. Omitted. ☐

0H0S Lemma 21.43.5. In the situation above, suppose that K is an object of $QC(\mathcal{O})$ and M arbitrary in $D(\mathcal{O})$. For every object U of $\mathcal{C}$ we have

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, M|_U) = R\,\mathrm{Hom}_{\mathcal{O}(U)}(R\Gamma(U, K), R\Gamma(U, M))$$

Proof. We may replace $\mathcal{C}$ by $\mathcal{C}/U$. Thus we may assume $U = X$ is a final object of $\mathcal{C}$. By Lemma 21.43.4 we see that $K = Lf^*P$ where $P = R\Gamma(U, K) = R\Gamma(X, K) = Rf_*K$. Thus the result because Lf^* is the left adjoint to $Rf_*(-) = R\Gamma(U, -)$. ☐

Let $(\mathcal{C}, \mathcal{O})$ be as above. For a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules we define the size $|\mathcal{F}^\bullet|$ of $\mathcal{F}^\bullet$ as

$$|\mathcal{F}^\bullet| = \left|\coprod\nolimits_{i \in \mathbf{Z},\ U \in \mathrm{Ob}(\mathcal{C})} \mathcal{F}^i(U)\right|$$

For an object K of $D(\mathcal{O})$ we define the size $|K|$ of K to be the cardinal

$$|K| = \min\{|\mathcal{F}^\bullet| \text{ where } \mathcal{F}^\bullet \text{ represents } K\}$$

By properties of cardinals the minimum exists.

0GYX Lemma 21.43.6. In the situation above, there exists a cardinal κ with the following property: given a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules and subsets $\Omega^i_U \subset \mathcal{F}^i(U)$ there exists a subcomplex $\mathcal{H}^\bullet \subset \mathcal{F}^\bullet$ with $\Omega^i_U \subset \mathcal{H}^i(U)$ and $|\mathcal{H}^\bullet| \leq \max(\kappa, |\bigcup \Omega^i_U|)$.

Proof. Define $\mathcal{H}^i(U)$ to be the $\mathcal{O}(U)$-submodule of $\mathcal{F}^i(U)$ generated by the images of Ω^i_V and $\mathrm{d}(\Omega^{i-1}_U)$ by restriction along any morphism $f : U \to V$. The cardinality of $\mathcal{H}^i(U)$ is bounded by the maximum of $\aleph_0$, the cardinality of the $\mathcal{O}(U)$, the cardinality of $\mathrm{Arrows}(\mathcal{C})$, and $|\bigcup \Omega^i_U|$. Details omitted. ☐

0GYY Lemma 21.43.7. In the situation above, there exists a cardinal κ with the following property: given a complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules representing an object K of $D(\mathcal{O})$ there exists a subcomplex $\mathcal{H}^\bullet \subset \mathcal{F}^\bullet$ such that $\mathcal{H}^\bullet$ represents K and such that $|\mathcal{H}^\bullet| \leq \max(\kappa, |K|)$.

Proof. First, for every i and U we choose a subset $\Omega_U^i \subset \mathrm{Ker}(\mathrm{d} : \mathcal{F}^i(U) \to \mathcal{F}^{i+1}(U))$ mapping bijectively onto $H^i(K)(U) = H^i(\mathcal{F}^\bullet(U))$. Hence $|\Omega_U^i| \leq |K|$ as we may represent K by a complex whose size is $|K|$. Applying Lemma 21.43.6 we find a subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ of size at most $\max(\kappa, |K|)$ containing Ω_U^i and hence such that $H^i(\mathcal{S}^\bullet) \to H^i(\mathcal{F}^\bullet)$ is a surjection of sheaves.

We are going to inductively construct subcomplexes

$$\mathcal{S}^\bullet = \mathcal{S}_0^\bullet \subset \mathcal{S}_1^\bullet \subset \mathcal{S}_2^\bullet \subset \ldots \subset \mathcal{F}^\bullet$$

of size $\leq \max(\kappa, |K|)$ such that the kernel of $H^i(\mathcal{S}_n^\bullet) \to H^i(\mathcal{F}^\bullet)$ is the same as the kernel of $H^i(\mathcal{S}_n^\bullet) \to H^i(\mathcal{S}_{n+1}^\bullet)$. Once this is done we can take $\mathcal{H}^\bullet = \bigcup \mathcal{S}_n^\bullet$ as our solution.

Construction of $\mathcal{S}_{n+1}^\bullet$ given $\mathcal{S}_n^\bullet$. For ever U and i let $\Omega_U^{i-1} \subset \mathcal{F}^{i-1}(U)$ be a subset such that $\mathrm{d} : \mathcal{F}^{i-1}(U) \to \mathcal{F}^i(U)$ maps Ω_U^{i-1} bijectively onto

$$\mathcal{S}_n^i(U) \cap \mathrm{Im}(\mathrm{d} : \mathcal{F}^{i-1}(U) \to \mathcal{F}^i(U))$$

Observe that $|\Omega_U^i| \leq |K|$ because $\mathcal{S}_n^i(U)$ is so bounded. Then we get $\mathcal{S}_{n+1}^\bullet$ by an application of Lemma 21.43.6 to the subsets

$$\mathcal{S}^i(U) \cup \Omega_U^i \subset \mathcal{F}^i(U)$$

and everything is clear. $\square$

0GYZ Lemma 21.43.8. In the situation above, there exists a cardinal κ with the following properties:

(1) for every nonzero object K of $QC(\mathcal{O})$ there exists a nonzero morphism $E \to K$ of $QC(\mathcal{O})$ such that $|E| \leq \kappa$,
(2) for every morphism $\alpha : E \to \bigoplus_n K_n$ of $QC(\mathcal{O})$ such that $|E| \leq \kappa$, there exist morphisms $E_n \to K_n$ in $QC(\mathcal{O})$ with $|E_n| \leq \kappa$ such that α factors through $\bigoplus E_n \to \bigoplus K_n$.

Proof. Let κ be an upper bound for the following set of cardinals:

(1) $|\coprod_V j_{U!}\mathcal{O}_U(V)|$ for all $U \in \mathrm{Ob}(\mathcal{C})$,
(2) the cardinals $\kappa(\mathcal{O}(V) \to \mathcal{O}(U))$ found in More on Algebra, Lemma 15.102.5 for all morphisms $U \to V$ in $\mathcal{C}$,
(3) the cardinal found in Lemma 21.43.7.

We claim that for any complex $\mathcal{F}^\bullet$ representing an object of $QC(\mathcal{O})$ and any subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ with $|\mathcal{S}^\bullet| \leq \kappa$ there exists a subcomplex $\mathcal{H}^\bullet$ of $\mathcal{F}^\bullet$ containing $\mathcal{S}^\bullet$ such that $\mathcal{H}^\bullet$ represents an object of $QC(\mathcal{O})$ and such that $|\mathcal{H}^\bullet| \leq \kappa$. In the next two paragraphs we show that the claim implies the lemma.

As in (1) let K be a nonzero object of $QC(\mathcal{O})$. Say K is represented by the complex of $\mathcal{O}$-modules $\mathcal{F}^\bullet$. Then $H^i(\mathcal{F}^\bullet)$ is nonzero for some i. Hence there exists an object U of $\mathcal{C}$ and a section $s \in \mathcal{F}^i(U)$ with $d(s) = 0$ which determines a nonzero section of $H^i(\mathcal{F}^\bullet)$ over U. Then the image of $s : j_{U!}\mathcal{O}_U[-i] \to \mathcal{F}^\bullet$ is a subcomplex $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ with $|\mathcal{S}^\bullet| \leq \kappa$. Applying the claim we get $\mathcal{H}^\bullet \to \mathcal{F}^\bullet$ in $QC(\mathcal{O})$ nonzero with $|\mathcal{H}^\bullet| \leq \kappa$. Thus (1) holds.

Let $\alpha : E \to \bigoplus K_n$ be as in (2). Choose any complexes $\mathcal{K}_n^\bullet$ representing K_n. Then $\bigoplus \mathcal{K}_n^\bullet$ represents $\bigoplus K_n$. By the construction of the derived category we can represent E by a complex $\mathcal{E}^\bullet$ such that α is represented by a morphism $a : \mathcal{E}^\bullet \to \bigoplus \mathcal{K}_n^\bullet$ of complexes. By Lemma 21.43.7 and our choice of κ above we may assume

$|\mathcal{E}^\bullet| \leq \kappa$. By the claim we get subcomplexes $\mathcal{E}_n^\bullet \subset \mathcal{K}_n^\bullet$ representing objects E_n of $QC(\mathcal{O})$ with $|E_n| \leq \kappa$ containing the image of $a_n : \mathcal{E}^\bullet \to \mathcal{K}_n^\bullet$ as desired.

Proof of the claim. Let $\mathcal{F}^\bullet$ be a complex representing an object of $QC(\mathcal{O})$ and let $\mathcal{S}^\bullet \subset \mathcal{F}^\bullet$ be a subcomplex of size $\leq \kappa$. We are going to inductively construct subcomplexes

$$\mathcal{S}^\bullet = \mathcal{S}_0^\bullet \subset \mathcal{S}_1^\bullet \subset \mathcal{S}_2^\bullet \subset \ldots \subset \mathcal{F}^\bullet$$

of size $\leq \kappa$ such that for every morphism $f : U \to V$ of $\mathcal{C}$ and every $i \in \mathbf{Z}$

(1) the kernel of the arrow $H^i(\mathcal{S}_n^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U)) \to H^i(\mathcal{S}_n^\bullet(U))$ maps to zero in $H^i(\mathcal{S}_{n+1}^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U))$,
(2) the image of the arrow $H^i(\mathcal{S}_n^\bullet(U)) \to H^i(\mathcal{S}_{n+1}^\bullet(U))$ is contained in the image of $H^i(\mathcal{S}_{n+1}^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U)) \to H^i(\mathcal{S}_{n+1}^\bullet(U))$,

Once this is done we can set $\mathcal{H}^\bullet = \bigcup \mathcal{S}_n^\bullet$. Namely, since derived tensor product and taking cohomology of complexes of modules over rings commute with filtered colimits, the conditions (1) and (2) together will guarantee that

$$\mathcal{H}^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U) \longrightarrow \mathcal{H}^\bullet(U)$$

is an isomorphism on cohomology in all degrees and hence an isomorphism in $D(\mathcal{O}(U))$ for all $f : U \to V$ in $\mathcal{C}$. Hence $\mathcal{H}^\bullet$ represents an object of $QC(\mathcal{O})$ as desired.

Construction of $\mathcal{S}_{n+1}$ given $\mathcal{S}_n$. For every morphism $f : U \to V$ of $\mathcal{C}$ we consider the commutative diagram

$$\begin{tikzcd} \mathcal{S}_n^\bullet(V) \arrow[r] \arrow[d] & \mathcal{S}_n^\bullet(U) \arrow[d] \\ \mathcal{F}^\bullet(V) \arrow[r] & \mathcal{F}^\bullet(U) \end{tikzcd}$$

This is a diagram as in More on Algebra, Lemma 15.102.5 for the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$, i.e., the bottom row induces an isomorphism

$$\mathcal{F}^\bullet(V) \otimes_{\mathcal{O}(V)}^{\mathbf{L}} \mathcal{O}(U) \longrightarrow \mathcal{F}^\bullet(U)$$

in $D(\mathcal{O}(U))$. Thus we may choose subcomplexes

$$\mathcal{S}_n^\bullet(V) \subset M_f^\bullet \subset \mathcal{F}^\bullet(V) \quad \text{and} \quad \mathcal{S}_n^\bullet(U) \subset N_f^\bullet \subset \mathcal{F}^\bullet(U)$$

as in More on Algebra, Lemma 15.102.5 and in particular we see that $|N_f^i|, |M_f^i| \leq \kappa$. Next, we apply Lemma 21.43.6 using the subsets

$$\mathcal{S}_n^i(U) \amalg \coprod\nolimits_{f:U \to V} N_f^i \amalg \coprod\nolimits_{g:W \to U} M_g^i \subset \mathcal{F}^i(U)$$

to find a subcomplex

$$\mathcal{S}_n^\bullet \subset \mathcal{S}_{n+1}^\bullet \subset \mathcal{F}^\bullet$$

with containing those subsets and such that $|\mathcal{S}_{n+1}^\bullet| \leq \kappa$. Conditions (1) and (2) hold because the corresponding statements hold for $\mathcal{S}_n^\bullet(V) \subset M_f^\bullet$ and $\mathcal{S}_n^\bullet(U) \subset N_f^\bullet$ by the construction in More on Algebra, Lemma 15.102.5. Thus the proof is complete. $\square$

0GZ0 Proposition 21.43.9. Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. With $QC(\mathcal{O})$ as in Definition 21.43.1 we have

(1) $QC(\mathcal{O})$ is a strictly full, saturated, triangulated subcategory of $D(\mathcal{O})$ preserved by arbitrary direct sums,

(2) any contravariant cohomological functor $H : QC(\mathcal{O}) \to \text{Ab}$ which transforms direct sums into products is representable,
(3) any exact functor $F : QC(\mathcal{O}) \to \mathcal{D}$ of triangulated categories which transforms direct sums into direct sums has an exact right adjoint, and
(4) the inclusion functor $QC(\mathcal{O}) \to D(\mathcal{O})$ has an exact right adjoint.

Proof. Part (1) is Lemma 21.43.2. Part (2) follows from Lemma 21.43.8 and Derived Categories, Lemma 13.39.1. Part (3) follows from Lemma 21.43.8 and Derived Categories, Proposition 13.39.2. Part (4) is a special case of (3). $\square$

Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor between categories. If we view $\mathcal{C}$ and $\mathcal{C}'$ as sites with the chaotic topology, then u is a continuous and cocontinuous functor. Hence we obtain a morphism $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ of topoi, see Sites, Lemma 7.21.1. Additionally, suppose given sheaves of rings $\mathcal{O}$ on $\mathcal{C}$ and $\mathcal{O}'$ on $\mathcal{C}'$ and a map $g^\sharp : g^{-1}\mathcal{O} \to \mathcal{O}'$. We denote the corresponding morphism of ringed topoi simply $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$, see Modules on Sites, Section 18.7.

0GZ1 Lemma 21.43.10. Let $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be as above. Then the functor $Lg^* : D(\mathcal{O}) \to D(\mathcal{O}')$ maps $QC(\mathcal{O})$ into $QC(\mathcal{O}')$.

Proof. Let $U' \in \text{Ob}(\mathcal{C}')$ with image $U = u(U')$ in $\mathcal{C}$. Let pt denote the category with a single object and a single morphism. Denote $(Sh(pt), \mathcal{O}'(U'))$ and $(Sh(pt), \mathcal{O}(U))$ the ringed topoi as indicated. Of course we identify the derived category of modules on these ringed topoi with $D(\mathcal{O}'(U'))$ and $D(\mathcal{O}(U))$. Then we have a commutative diagram of ringed topoi

$$\begin{tikzcd} (Sh(pt), \mathcal{O}'(U')) \arrow[r, "U'"'] \arrow[d] & (Sh(\mathcal{C}'), \mathcal{O}') \arrow[d, "g"] \\ (Sh(pt), \mathcal{O}(U)) \arrow[r, "U"] & (Sh(\mathcal{C}), \mathcal{O}) \end{tikzcd}$$

Pullback along the lower horizontal morphism sends K in $D(\mathcal{O})$ to $R\Gamma(U, K)$. Pullback by the left vertical arrow sends M to $M \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U')$. Going around the diagram either direction produces the same result (Lemma 21.18.3) and hence we conclude

$$R\Gamma(U', Lg^*K) = R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U')$$

Finally, let $f' : U' \to V'$ be a morphism in $\mathcal{C}'$ and denote $f = u(f') : U = u(U') \to V = u(V')$ the image in $\mathcal{C}$. If K is in $QC(\mathcal{O})$ then we have

$$\begin{aligned} R\Gamma(V', Lg^*K) \otimes^{\mathbf{L}}_{\mathcal{O}'(V')} \mathcal{O}'(U') &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}'(V') \otimes^{\mathbf{L}}_{\mathcal{O}'(V')} \mathcal{O}'(U') \\ &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}'(U') \\ &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U') \\ &= R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}(U)} \mathcal{O}'(U') \\ &= R\Gamma(U', Lg^*K) \end{aligned}$$

as desired. Here we have used the observation above both for U' and V'. $\square$

0GZR Lemma 21.43.11. Let $\mathcal{C}$ be a category viewed as a site with the chaotic topology. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Assume for all $U \to V$ in $\mathcal{C}$ the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is a flat ring map. Then $QC(\mathcal{O})$ agrees with the subcategory $D_{QCoh}(\mathcal{O}) \subset D(\mathcal{O})$ of complexes whose cohomology sheaves are quasi-coherent.

Proof. Recall that $QCoh(\mathcal{O}) \subset \text{Mod}(\mathcal{O})$ is a weak Serre subcategory under our assumptions, see Modules on Sites, Lemma 18.24.3. Thus taking the full subcategory

$$D_{QCoh}(\mathcal{O}) = D_{QCoh(\mathcal{O})}(\text{Mod}(\mathcal{O}))$$

of $D(\mathcal{O})$ makes sense, see Derived Categories, Section 13.17. (Strictly speaking we don't need this in the proof of the lemma.)

Let M be an object of $QC(\mathcal{O})$. Since for every morphism $U \to V$ in $\mathcal{C}$ the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat, we see that

$$\begin{aligned} H^i(M)(U) &= H^i(R\Gamma(U, M)) \\ &= H^i(R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U)) \\ &= H^i(R\Gamma(V, M)) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \\ &= H^i(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \end{aligned}$$

and hence $H^i(M)$ is quasi-coherent by Modules on Sites, Lemma 18.24.2. The first and last equality above follow from the fact that taking sections over an object of $\mathcal{C}$ is an exact functor due to the fact that the topology on $\mathcal{C}$ is chaotic.

Conversely, if M is an object of $D_{QCoh}(\mathcal{O})$, then due to Modules on Sites, Lemma 18.24.2 we see that the map $R\Gamma(V, M) \to R\Gamma(U, M)$ induces isomorphisms $H^i(M)(U) \to H^i(M)(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$. Whence $R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, K)$ is an isomorphism in $D(\mathcal{O}(U))$ by the flatness of $\mathcal{O}(V) \to \mathcal{O}(U)$ and we conclude that M is in $QC(\mathcal{O})$. □

0GZS Lemma 21.43.12. Let $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \to (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ be as in Section 21.27. Assume

(1) τ' is the chaotic topology on the category $\mathcal{C}$,
(2) for all $U \in \text{Ob}(\mathcal{C})$ and all K-flat complexes of $\mathcal{O}(U)$-modules $M^\bullet$ the map

$$M^\bullet \longrightarrow R\Gamma((\mathcal{C}/U)_\tau, (M^\bullet \otimes_{\mathcal{O}(U)} \mathcal{O}_U)^\#)$$

is a quasi-isomorphism (see proof for an explanation).

Then ϵ^* and $R\epsilon_*$ define mutually quasi-inverse equivalences between $QC(\mathcal{O})$ and the full subcategory of $D(\mathcal{C}_\tau, \mathcal{O}_\tau)$ consisting of objects K such that $R\epsilon_* K$ is in $QC(\mathcal{O})$[10].

Proof. We will use the observations made in Section 21.27 without further mention. Since $R\epsilon_*$ is fully faithful and $\epsilon^* \circ R\epsilon_* = \text{id}$, to prove the lemma it suffices to show that for M in $QC(\mathcal{O})$ we have $R\epsilon_*(\epsilon^* M) = M$. Condition (2) is exactly the condition needed to see this. Namely, we choose a K-flat complex $\mathcal{M}^\bullet$ of $\mathcal{O}$-modules with flat terms representing M. Then we see that $\epsilon^* M$ is represented by the τ-sheafification $(\mathcal{M}^\bullet)^\#$ of $\mathcal{M}^\bullet$. Let $U \in \text{Ob}(\mathcal{C})$. By Leray we get

$$R\Gamma(U, R\epsilon_*(\epsilon^* M)) = R\Gamma((\mathcal{C}/U)_\tau, (\mathcal{M}^\bullet)^\#|_{\mathcal{C}/U}) = R\Gamma((\mathcal{C}/U)_\tau, (\mathcal{M}^\bullet|_{\mathcal{C}/U})^\#)$$

The last equality since sheafification commutes with restriction to $\mathcal{C}/U$. As usual, denote $\mathcal{O}_U$ the restriction of $\mathcal{O}$ to $\mathcal{C}/U$. Consider the map

$$\mathcal{M}^\bullet(U) \otimes_{\mathcal{O}(U)} \mathcal{O}_U \longrightarrow \mathcal{M}^\bullet|_{\mathcal{C}/U}$$

of complexes of $\mathcal{O}_U$-modules (in τ'-topology). By our choice of $\mathcal{M}^\bullet$ the complex $\mathcal{M}^\bullet(U)$ is a K-flat complex of $\mathcal{O}(U)$-modules; see Lemma 21.18.1 and use that

[10]This means that $R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{O}(V)} \mathcal{O}(U) \to R\Gamma(U, K)$ is an isomorphism for all $U \to V$ in $\mathcal{C}$.

the inclusion of U into $\mathcal{C}$ defines a morphism of ringed topoi $(Sh(pt), \mathcal{O}(U)) \to (Sh(\mathcal{C}_{\tau'}), \mathcal{O})$. Since M is in $QC(\mathcal{O})$ we conclude that the displayed arrow is a quasi-isomorphism. Since sheafification is exact, we see that the same remains true after sheafification. Hence

$$R\Gamma(U, R\epsilon_*(\epsilon^* M)) = R\Gamma((\mathcal{C}/U)_\tau, (M^\bullet \otimes_{\mathcal{O}(U)} \mathcal{O}_U)^\#)$$

and assumption (2) tells us this is equal to $R\Gamma(U, M) = \mathcal{M}^\bullet(U)$ as desired. □

0H0T Lemma 21.43.13. Notation and assumptions as in Lemma 21.43.12. Suppose that K is an object of $QC(\mathcal{O})$ and M arbitrary in $D(\mathcal{O}_\tau)$. For every object U of $\mathcal{C}$ we have

$$\mathrm{Hom}_{D((\mathcal{O}_U)_\tau)}(\epsilon^* K|_U, M|_U) = R\,\mathrm{Hom}_{\mathcal{O}(U)}(R\Gamma(U, K), R\Gamma(U, M))$$

Proof. We have

$$\mathrm{Hom}_{D((\mathcal{O}_U)_\tau)}(\epsilon^* K|_U, M|_U) = \mathrm{Hom}_{D((\mathcal{O}_U)_{\tau'})}(K|_U, R\epsilon_* M|_U)$$

by adjunction. Hence the result by Lemma 21.43.5 and the fact that

$$R\Gamma(U, M) = R\Gamma(U, R\epsilon_* M)$$

by Leray. □

21.44. Strictly perfect complexes

08FK This section is the analogue of Cohomology, Section 20.46.

08FL Definition 21.44.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is strictly perfect if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}$-module for all i.

Let U be an object of $\mathcal{C}$. We will often say "Let $\mathcal{E}^\bullet$ be a strictly perfect complex of $\mathcal{O}_U$-modules" to mean $\mathcal{E}^\bullet$ is a strictly perfect complex of modules on the ringed site $(\mathcal{C}/U, \mathcal{O}_U)$, see Modules on Sites, Definition 18.19.1.

08FM Lemma 21.44.2. The cone on a morphism of strictly perfect complexes is strictly perfect.

Proof. This is immediate from the definitions. □

09J8 Lemma 21.44.3. The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.

Proof. Omitted. □

08H3 Lemma 21.44.4. Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{C}$-modules.

Proof. We have seen in Modules on Sites, Lemma 18.17.2 that the pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. □

08FN Lemma 21.44.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of $\mathcal{O}_U$-modules

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr, dotted] & \mathcal{F} \\ & \mathcal{G} \arrow[u, "p"'] \end{tikzcd}$$

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_U$-module and p surjective, then there exists a covering $\{U_i \to U\}$ such that a dotted arrow making the diagram commute exists over each U_i.

Proof. We may assume $\mathcal{E} = \mathcal{O}_U^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(U, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sites, Section 7.11). $\square$

08FP Lemma 21.44.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.

(1) Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.
(2) Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}$-module and integer $n \geq a$, then the result follows from Lemma 21.44.5 and the fact that $\mathcal{F}^{n-1} \to \text{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_U)$. Hence an exact sequence

$$\text{Hom}_{K(\mathcal{O}_U)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \text{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \text{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is homotopic to zero on the members of a covering of U by the above, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is zero on the members of a covering of U by induction hypothesis. $\square$

08FQ Lemma 21.44.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of complexes of $\mathcal{O}_U$-modules

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r, "\alpha"'] \arrow[dr, dotted] & \mathcal{F}^\bullet \\ & \mathcal{G}^\bullet \arrow[u, "f"'] \end{tikzcd}$$

with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then there exists a covering $\{U_i \to U\}$ and for each i a dotted arrow over U_i making the diagram commute up to homotopy.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 21.44.6 guarantees there is a covering $\{U_i \to U\}$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. □

08FR Lemma 21.44.8. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect.

(1) For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.
(2) Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is homotopic to zero.

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 21.44.7. We omit the proof of (2). □

08JH Lemma 21.44.9. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 21.35.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{L}^{-q}, \mathcal{I}^p)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 21.35. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an object U of $\mathcal{C}$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 21.44.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. □

08JI Lemma 21.44.10. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with

(1) $\mathcal{F}^n = 0$ for $n \ll 0$,
(2) $\mathcal{E}^n = 0$ for $n \gg 0$, and
(3) $\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}$-module.

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

and differential as described in Section 21.35.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.31.4). Hence the construction in Section 21.35 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{I}^p) = \bigoplus\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{I}^p)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 12.25.3)

$${}'E_1^{p,q} = H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^p(\mathcal{H}om_\mathcal{O}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. $\square$

21.45. Pseudo-coherent modules

08FS In this section we discuss pseudo-coherent complexes.

08FT Definition 21.45.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$.

(1) We say $\mathcal{E}^\bullet$ is m-pseudo-coherent if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i$ is a strictly perfect complex of $\mathcal{O}_{U_i}$-modules and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
(2) We say $\mathcal{E}^\bullet$ is pseudo-coherent if it is m-pseudo-coherent for all m.
(3) We say an object E of $D(\mathcal{O})$ is m-pseudo-coherent (resp. pseudo-coherent) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}$-modules.

If $\mathcal{C}$ has a final object X which is quasi-compact (for example if every covering of X can be refined by a finite covering), then an m-pseudo-coherent object of $D(\mathcal{O})$ is in $D^-(\mathcal{O})$. But this need not be the case in general.

08FU Lemma 21.45.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.

(1) If $\mathcal{C}$ has a final object X and if there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.
(2) If E is m-pseudo-coherent, then any complex of $\mathcal{O}$-modules representing E is m-pseudo-coherent.

(3) If for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is m-pseudo-coherent, then E is m-pseudo-coherent.

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let X, $\{U_i \to X\}$, and $\alpha_i : \mathcal{E}_i \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) in case $\mathcal{C}$ has a final object. By Lemma 21.44.8 we can after refining the covering $\{U_i \to X\}$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (2). By the above we see that $\mathcal{F}^\bullet|_U$ is m-pseudo-coherent as a complex of $\mathcal{O}_U$-modules for all objects U of $\mathcal{C}$. It is a formal consequence of the definitions that $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (3). Follows from the definitions and Sites, Definition 7.6.2 part (2). □

08H4 Lemma 21.45.3. Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{C})$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.

Proof. Say f is given by the functor $u : \mathcal{D} \to \mathcal{C}$. Let U be an object of $\mathcal{C}$. By Sites, Lemma 7.14.10 we can find a covering $\{U_i \to U\}$ and for each i a morphism $U_i \to u(V_i)$ for some object V_i of $\mathcal{D}$. By Lemma 21.45.2 it suffices to show that $Lf^*E|_{U_i}$ is m-pseudo-coherent. To do this it is enough to show that $Lf^*E|_{u(V_i)}$ is m-pseudo-coherent, since $Lf^*E|_{U_i}$ is the restriction of $Lf^*E|_{u(V_i)}$ to $\mathcal{C}/U_i$ (via Modules on Sites, Lemma 18.19.5). By the commutative diagram of Modules on Sites, Lemma 18.20.1 it suffices to prove the lemma for the morphism of ringed sites $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Thus we may assume $\mathcal{D}$ has a final object Y such that $X = u(Y)$ is a final object of $\mathcal{C}$.

Let $\{V_i \to Y\}$ be a covering such that for each i there exists a strictly perfect complex $\mathcal{F}_i^\bullet$ of $\mathcal{O}_{V_i}$-modules and a morphism $\alpha_i : \mathcal{F}_i^\bullet \to E|_{V_i}$ of $D(\mathcal{O}_{V_i})$ such that $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective. Arguing as above it suffices to prove the result for $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Hence we may assume that there exists a strictly perfect complex $\mathcal{F}^\bullet$ of $\mathcal{O}_\mathcal{D}$-modules and a morphism $\alpha : \mathcal{F}^\bullet \to E$ of $D(\mathcal{O}_\mathcal{D})$ such that $H^j(\alpha)$ is an isomorphism for $j > m$ and $H^m(\alpha)$ is surjective. In this case, choose a distinguished triangle

$$\mathcal{F}^\bullet \to E \to C \to \mathcal{F}^\bullet[1]$$

The assumption on α means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Applying Lf^* we obtain the distinguished triangle

$$Lf^*\mathcal{F}^\bullet \to Lf^*E \to Lf^*C \to Lf^*\mathcal{F}^\bullet[1]$$

By the construction of Lf^* as a left derived functor we see that $H^j(Lf^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 13.16.1). Hence $H^j(Lf^*\alpha)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha)$ is surjective. On the other hand, since $\mathcal{F}^\bullet$ is a bounded above complex of flat $\mathcal{O}_\mathcal{D}$-modules we see that $Lf^*\mathcal{F}^\bullet = f^*\mathcal{F}^\bullet$. Applying Lemma 21.44.4 we conclude. □

08FV Lemma 21.45.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$.

(1) If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.

(2) If K and M are m-pseudo-coherent, then L is m-pseudo-coherent.
(3) If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. Choose a covering $\{U_i \to U\}$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 21.44.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{tikzcd} K|_{U_i} \arrow[r] & L|_{U_i} \\ \mathcal{K}_i^\bullet \arrow[u, "\alpha_i"] \arrow[r, "\gamma_i"] & \mathcal{L}_i^\bullet \arrow[u, "\beta_i"'] \end{tikzcd}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 21.44.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 21.45.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. $\square$

09J9 Lemma 21.45.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$.

(1) If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes_\mathcal{O}^\mathbf{L} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.
(2) If K and L are pseudo-coherent, then $K \otimes_\mathcal{O}^\mathbf{L} L$ is pseudo-coherent.

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. By replacing U by the members of a covering and replacing $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^\mathbf{L} \beta : \text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{L}^\bullet) \to K \otimes_\mathcal{O}^\mathbf{L} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). Let U be an object of $\mathcal{C}$. We may first replace U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). $\square$

08FW Lemma 21.45.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O})$ so are K and L.

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume $K \oplus L \in D^-(\mathcal{O}_U)$, hence $L \in D^-(\mathcal{O}_U)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.10. By Lemma 21.45.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. □

08FX Lemma 21.45.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. Let $m \in \mathbf{Z}$.

(1) If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}$-module.
(2) If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m+1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}$-module.

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. We have to show that $H^m(K)$ is can be generated by finitely many sections over the members of a covering of U (see Modules on Sites, Definition 18.23.1). Thus during the proof we may (finitely often) choose a covering $\{U_i \to U\}$ and replace $\mathcal{C}$ by $\mathcal{C}/U_i$ and U by U_i. In particular, by our definitions we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 21.44.5 we can (after replacing U by the members of a covering) find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Pick an object U of $\mathcal{C}$. As in the proof of (1) we may work locally on U. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \text{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

21.46. Tor dimension

08FY In this section we take a closer look at resolutions by flat modules.

08FZ Definition 21.46.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

(1) We say E has tor-amplitude in $[a, b]$ if $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$ and all $i \notin [a, b]$.
(2) We say E has finite tor dimension if it has tor-amplitude in $[a, b]$ for some a, b.

(3) We say E locally has finite tor dimension if for any object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ has finite tor dimension for all i.

An $\mathcal{O}$-module $\mathcal{F}$ has tor dimension $\leq d$ if $\mathcal{F}[0]$ viewed as an object of $D(\mathcal{O})$ has tor-amplitude in $[-d, 0]$.

Note that if E as in the definition has finite tor dimension, then E is an object of $D^b(\mathcal{O})$ as can be seen by taking $\mathcal{F} = \mathcal{O}$ in the definition above.

08G0 Lemma 21.46.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules with tor-amplitude in $[a, b]$. Then $\mathrm{Coker}(d_{\mathcal{E}^\bullet}^{a-1})$ is a flat $\mathcal{O}$-module.

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_\mathcal{O} \mathcal{F} = \mathcal{E}^\bullet \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$. Hence for every $\mathcal{O}$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{E}^{a} \otimes_\mathcal{O} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \mathrm{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\mathrm{Tor}_1^\mathcal{O}(\mathrm{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$. This means that $\mathrm{Coker}(d^{a-1})$ is flat, see Lemma 21.17.15. $\square$

08G1 Lemma 21.46.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent

(1) E has tor-amplitude in $[a, b]$.
(2) E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.

Proof. If (2) holds, then we may compute $E \otimes_\mathcal{O}^\mathbf{L} \mathcal{F} = \mathcal{E}^\bullet \otimes_\mathcal{O} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}$-modules $\mathcal{K}^\bullet$, see Section 21.17. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\mathrm{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules on Sites, Lemma 18.28.10). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 21.46.2 and the definitions. $\square$

0F1M Lemma 21.46.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a \in \mathbf{Z}$. The following are equivalent

(1) E has tor-amplitude in $[a, \infty]$.
(2) E can be represented by a K-flat complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, \infty]$.

Moreover, we can choose $\mathcal{E}^\bullet$ such that any pullback by a morphism of ringed sites is a K-flat complex with flat terms.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1) holds. First we choose a K-flat complex $\mathcal{K}^\bullet$ with flat terms representing E, see Lemma 21.17.11. For any $\mathcal{O}$-module $\mathcal{M}$ the cohomology of

$$\mathcal{K}^{n-1} \otimes_\mathcal{O} \mathcal{M} \to \mathcal{K}^n \otimes_\mathcal{O} \mathcal{M} \to \mathcal{K}^{n+1} \otimes_\mathcal{O} \mathcal{M}$$

computes $H^n(E \otimes_\mathcal{O}^\mathbf{L} \mathcal{M})$. This is always zero for $n < a$. Hence if we apply Lemma 21.46.2 to the complex $\ldots \to \mathcal{K}^{a-1} \to \mathcal{K}^a \to \mathcal{K}^{a+1}$ we conclude that $\mathcal{N} = \text{Coker}(\mathcal{K}^{a-1} \to \mathcal{K}^a)$ is a flat $\mathcal{O}$-module. We set

$$\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet = (\ldots \to 0 \to \mathcal{N} \to \mathcal{K}^{a+1} \to \ldots)$$

The kernel $\mathcal{L}^\bullet$ of $\mathcal{K}^\bullet \to \mathcal{E}^\bullet$ is the complex

$$\mathcal{L}^\bullet = (\ldots \to \mathcal{K}^{a-1} \to \mathcal{I} \to 0 \to \ldots)$$

where $\mathcal{I} \subset \mathcal{K}^a$ is the image of $\mathcal{K}^{a-1} \to \mathcal{K}^a$. Since we have the short exact sequence $0 \to \mathcal{I} \to \mathcal{K}^a \to \mathcal{N} \to 0$ we see that $\mathcal{I}$ is a flat $\mathcal{O}$-module. Thus $\mathcal{L}^\bullet$ is a bounded above complex of flat modules, hence K-flat by Lemma 21.17.8. It follows that $\mathcal{E}^\bullet$ is K-flat by Lemma 21.17.7.

Proof of the final assertion. Let $f : (\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})$ be a morphism of ringed sites. By Lemma 21.18.1 the complex $f^*\mathcal{K}^\bullet$ is K-flat with flat terms. The complex $f^*\mathcal{L}^\bullet$ is K-flat as it is a bounded above complex of flat $\mathcal{O}'$-modules. We have a short exact sequence of complexes of $\mathcal{O}'$-modules

$$0 \to f^*\mathcal{L}^\bullet \to f^*\mathcal{K}^\bullet \to f^*\mathcal{E}^\bullet \to 0$$

because the short exact sequence $0 \to \mathcal{I} \to \mathcal{K}^a \to \mathcal{N} \to 0$ of flat modules pulls back to a short exact sequence. By Lemma 21.17.7. the complex $f^*\mathcal{E}^\bullet$ is K-flat and the proof is complete. □

08H5 Lemma 21.46.5. Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 21.46.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules on Sites, Lemma 18.39.1 the module $f^*\mathcal{E}^i$ are flat. Thus by Lemma 21.46.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. □

08G2 Lemma 21.46.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$.

(1) If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.
(2) If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.
(3) If L has tor-amplitude in $[a + 1, b + 1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a + 1, b + 1]$.

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes_\mathcal{O}^\mathbf{L} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

09JA Lemma 21.46.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes_\mathcal{O}^\mathbf{L} L$ has tor amplitude in $[a + c, b + d]$.

Proof. Omitted. Hint: use the spectral sequence for tors. □

08G3 Lemma 21.46.8. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O})$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.

Proof. Clear from the fact that the Tor functors are additive. $\square$

0942 Lemma 21.46.9. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let K be an object of $D(\mathcal{O})$.

(1) If $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$ is bounded above, then $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$ is uniformly bounded above for all n.
(2) If $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$, then $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$ as an object of $D(\mathcal{O}/\mathcal{I}^n)$ has tor amplitude in $[a, b]$ for all n.

Proof. Proof of (1). Assume that $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$ is bounded above, say $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}) = 0$ for $i > b$. Note that we have distinguished triangles

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1} \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^{n+1} \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n \to K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1}[1]$$

and that

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1} = \left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}\right) \otimes_{\mathcal{O}/\mathcal{I}}^{\mathbf{L}} \mathcal{I}^n/\mathcal{I}^{n+1}$$

By induction we conclude that $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n) = 0$ for $i > b$ for all n.

Proof of (2). Assume $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}/\mathcal{I}^n$-modules. Then we have a finite filtration

$$0 \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

whose successive quotients are sheaves of $\mathcal{O}/\mathcal{I}$-modules. Thus to prove that $K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n$ has tor amplitude in $[a, b]$ it suffices to show $H^i(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n \otimes_{\mathcal{O}/\mathcal{I}^n}^{\mathbf{L}} \mathcal{G})$ is zero for $i \notin [a, b]$ for all $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$. Since

$$\left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}^n\right) \otimes_{\mathcal{O}/\mathcal{I}^n}^{\mathbf{L}} \mathcal{G} = \left(K \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{O}/\mathcal{I}\right) \otimes_{\mathcal{O}/\mathcal{I}}^{\mathbf{L}} \mathcal{G}$$

for every sheaf of $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$ the result follows. $\square$

0DJJ Lemma 21.46.10. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$.

(1) If E has tor amplitude in $[a, b]$, then for every point p of the site $\mathcal{C}$ the object E_p of $D(\mathcal{O}_p)$ has tor amplitude in $[a, b]$.
(2) If $\mathcal{C}$ has enough points, then the converse is true.

Proof. Proof of (1). This follows because taking stalks at p is the same as pulling back by the morphism of ringed sites $(p, \mathcal{O}_p) \to (\mathcal{C}, \mathcal{O})$ and hence we can apply Lemma 21.46.5.

Proof of (2). If $\mathcal{C}$ has enough points, then we can check vanishing of $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F})$ at stalks, see Modules on Sites, Lemma 18.14.4. Since $H^i(E \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F})_p = H^i(E_p \otimes_{\mathcal{O}_p}^{\mathbf{L}} \mathcal{F}_p)$ we conclude. $\square$

21.47. Perfect complexes

08G4 In this section we discuss properties of perfect complexes on ringed sites.

08G5 Definition 21.47.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is perfect if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ strictly perfect. An object E of $D(\mathcal{O})$ is perfect if it can be represented by a perfect complex of $\mathcal{O}$-modules.

08G6 Lemma 21.47.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.

(1) If $\mathcal{C}$ has a final object X and there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and isomorphisms $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.
(2) If E is perfect, then any complex representing E is perfect.

Proof. Identical to the proof of Lemma 21.45.2. □

08G7 Lemma 21.47.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.

Proof. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[a-2]$ with $\mathcal{K} = \mathrm{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Applying $- \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F}$ has image $\mathrm{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F})$. It follows that $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \mathrm{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 21.17.15). Thus there exists a covering $\{U_i \to U\}$ such that $\mathcal{E}'|_{U_i}$ is a direct summand of a finite free module by Modules on Sites, Lemma 18.29.3. Thus the complex

$$\mathcal{E}'|_{U_i} \to \mathcal{E}^{a-1}|_{U_i} \to \ldots \to \mathcal{E}^b|_{U_i}$$

is quasi-isomorphic to $E|_{U_i}$ and E is perfect. □

08G8 Lemma 21.47.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. The following are equivalent

(1) E is perfect, and
(2) E is pseudo-coherent and locally has finite tor dimension.

Proof. Assume (1). Let U be an object of $\mathcal{C}$. By definition there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 21.45.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 21.46.3. Thus (2) holds.

Assume (2). Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume there exist integers $a \leq b$ such that $E|_U$ has tor amplitude in $[a, b]$. Since $E|_U$ is m-pseudo-coherent for all m we conclude using Lemma 21.47.3. □

08H6 Lemma 21.47.5. Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E is perfect in $D(\mathcal{O}_\mathcal{D})$, then Lf^*E is perfect in $D(\mathcal{O}_\mathcal{C})$.

Proof. This follows from Lemma 21.47.4, 21.46.5, and 21.45.3. □

08G9 Lemma 21.47.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. If two out of three of K, L, M are perfect then the third is also perfect.

Proof. First proof: Combine Lemmas 21.47.4, 21.45.4, and 21.46.6. Second proof (sketch): Say K and L are perfect. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume that $K|_U$ and $L|_U$ are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing U by the members of a covering we may assume the map $K|_U \to L|_U$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 21.44.8. Then $M|_U$ is isomorphic to the cone of α which is strictly perfect by Lemma 21.44.2. $\square$

09JB **Lemma 21.47.7.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If K, L are perfect objects of $D(\mathcal{O})$, then so is $K \otimes_\mathcal{O}^\mathbf{L} L$.

Proof. Follows from Lemmas 21.47.4, 21.45.5, and 21.46.7. $\square$

08GA **Lemma 21.47.8.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $K \oplus L$ is a perfect object of $D(\mathcal{O})$, then so are K and L.

Proof. Follows from Lemmas 21.47.4, 21.45.6, and 21.46.8. $\square$

21.48. Duals

0FPP In this section we characterize the dualizable objects of the category of complexes and of the derived category. In particular, we will see that an object of $D(\mathcal{O})$ has a dual if and only if it is perfect (this follows from Example 21.48.6 and Lemma 21.48.7).

0FPQ **Lemma 21.48.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. The category of complexes of $\mathcal{O}$-modules with tensor product defined by $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet = \text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{G}^\bullet)$ is a symmetric monoidal category.

Proof. Omitted. Hints: as unit $\mathbf{1}$ we take the complex having $\mathcal{O}$ in degree 0 and zero in other degrees with obvious isomorphisms $\text{Tot}(\mathbf{1} \otimes_\mathcal{O} \mathcal{G}^\bullet) = \mathcal{G}^\bullet$ and $\text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathbf{1}) = \mathcal{F}^\bullet$. to prove the lemma you have to check the commutativity of various diagrams, see Categories, Definitions 4.43.1 and 4.43.9. The verifications are straightforward in each case. $\square$

0FPR **Example 21.48.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules such that for every $U \in \text{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}^\bullet|_{U_i}$ is strictly perfect. Consider the complex

$$\mathcal{G}^\bullet = \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O})$$

as in Section 21.34. Let

$$\eta : \mathcal{O} \to \text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} \mathcal{G}^\bullet) \quad \text{and} \quad \epsilon : \text{Tot}(\mathcal{G}^\bullet \otimes_\mathcal{O} \mathcal{F}^\bullet) \to \mathcal{O}$$

be $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O} \to \mathcal{F}^n \otimes_\mathcal{O} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_\mathcal{O} \mathcal{F}^n \to \mathcal{O}$ are as in Modules on Sites, Example 18.29.1. Then $\mathcal{G}^\bullet, \eta, \epsilon$ is a left dual for $\mathcal{F}^\bullet$ as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \text{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}_{\mathcal{G}^\bullet}$. Please compare with More on Algebra, Lemma 15.72.2.

0FPS **Lemma 21.48.3.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. If $\mathcal{F}^\bullet$ has a left dual in the monoidal category of complexes of $\mathcal{O}$-modules (Categories, Definition 4.43.5) then for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}^\bullet|_{U_i}$ is strictly perfect and the left dual is as constructed in Example 21.48.2.

Proof. By uniqueness of left duals (Categories, Remark 4.43.7) we get the final statement provided we show that $\mathcal{F}^\bullet$ is as stated. Let $\mathcal{G}^\bullet, \eta, \epsilon$ be a left dual. Write $\eta = \sum \eta_n$ and $\epsilon = \sum \epsilon_n$ where $\eta_n : \mathcal{O} \to \mathcal{F}^n \otimes_\mathcal{O} \mathcal{G}^{-n}$ and $\epsilon_n : \mathcal{G}^{-n} \otimes_\mathcal{O} \mathcal{F}^n \to \mathcal{O}$. Since $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \text{id}_{\mathcal{F}^\bullet}$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \text{id}_{\mathcal{G}^\bullet}$ by Categories, Definition 4.43.5 we see immediately that we have $(1 \otimes \epsilon_n) \circ (\eta_n \otimes 1) = \text{id}_{\mathcal{F}^n}$ and $(\epsilon_n \otimes 1) \circ (1 \otimes \eta_n) = \text{id}_{\mathcal{G}^{-n}}$. In other words, we see that $\mathcal{G}^{-n}$ is a left dual of $\mathcal{F}^n$ and we see that Modules on Sites, Lemma 18.29.2 applies to each $\mathcal{F}^n$. Let U be an object of $\mathcal{C}$. There exists a covering $\{U_i \to U\}$ such that for every i only a finite number of $\eta_n|_{U_i}$ are nonzero. Thus after replacing U by U_i we may assume only a finite number of $\eta_n|_U$ are nonzero and by the lemma cited this implies only a finite number of $\mathcal{F}^n|_U$ are nonzero. Using the lemma again we can then find a covering $\{U_i \to U\}$ such that each $\mathcal{F}^n|_{U_i}$ is a direct summand of a finite free $\mathcal{O}$-module and the proof is complete. $\square$

08JJ Lemma 21.48.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a perfect object of $D(\mathcal{O})$. Then $K^\vee = R\mathcal{H}om(K, \mathcal{O})$ is a perfect object too and $(K^\vee)^\vee \cong K$. There are functorial isomorphisms

$$M \otimes_\mathcal{O}^\mathbf{L} K^\vee = R\mathcal{H}om_\mathcal{O}(K, M)$$

and

$$H^0(\mathcal{C}, M \otimes_\mathcal{O}^\mathbf{L} K^\vee) = \text{Hom}_{D(\mathcal{O})}(K, M)$$

for M in $D(\mathcal{O})$.

Proof. We will us without further mention that formation of internal hom commutes with restriction (Lemma 21.35.3). Let U be an arbitrary object of $\mathcal{C}$. To check that $K^\vee$ is perfect, it suffices to show that there exists a covering $\{U_i \to U\}$ such that $K^\vee|_{U_i}$ is perfect for all i. There is a canonical map

$$K = R\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X}^\mathbf{L} K \longrightarrow R\mathcal{H}om(R\mathcal{H}om(K, \mathcal{O}_X), \mathcal{O}_X) = (K^\vee)^\vee$$

see Lemma 21.35.5. It suffices to prove there is a covering $\{U_i \to U\}$ such that the restriction of this map to $\mathcal{C}/U_i$ is an isomorphism for all i. By Lemma 21.35.9 to see the final statement it suffices to check that the map (21.35.9.1)

$$M \otimes_\mathcal{O}^\mathbf{L} K^\vee \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is a local question as well (in the sense above). Hence it suffices to prove the lemma when K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 21.44.9 that $K^\vee$ is represented by the complex whose terms are $(\mathcal{E}^n)^\vee = \mathcal{H}om_\mathcal{O}(\mathcal{E}^n, \mathcal{O})$ in degree $-n$. Since $\mathcal{E}^n$ is a direct summand of a finite free $\mathcal{O}$-module, so is $(\mathcal{E}^n)^\vee$. Hence $K^\vee$ is represented by a strictly perfect complex too and we see that $K^\vee$ is perfect. The map $K \to (K^\vee)^\vee$ is an isomorphism as it is given up to sign by the evaluation maps $\mathcal{E}^n \to ((\mathcal{E}^n)^\vee)^\vee$ which are isomorphisms. To see that (21.35.9.1) is an isomorphism, represent M by a K-flat complex $\mathcal{F}^\bullet$. By Lemma 21.44.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{H}om_\mathcal{O}(\mathcal{E}^{-q}, \mathcal{F}^p)$$

On the other hand, the object $M \otimes_\mathcal{O}^\mathbf{L} K^\vee$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=p+q} \mathcal{F}^p \otimes_\mathcal{O} (\mathcal{E}^{-q})^\vee$$

Thus the assertion that (21.35.9.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module and $\mathcal{F}$ is any $\mathcal{O}$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. ☐

0FPT Lemma 21.48.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The derived category $D(\mathcal{O})$ is a symmetric monoidal category with tensor product given by derived tensor product with usual associativity and commutativity constraints (for sign rules, see More on Algebra, Section 15.72).

Proof. Omitted. Compare with Lemma 21.48.1. ☐

0FPU Example 21.48.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a perfect object of $D(\mathcal{O})$. Set $K^\vee = R\mathcal{H}om(K, \mathcal{O})$ as in Lemma 21.48.4. Then the map

$$K \otimes_{\mathcal{O}}^{\mathbf{L}} K^\vee \longrightarrow R\mathcal{H}om(K, K)$$

is an isomorphism (by the lemma). Denote

$$\eta : \mathcal{O} \longrightarrow K \otimes_{\mathcal{O}}^{\mathbf{L}} K^\vee$$

the map sending 1 to the section corresponding to id_K under the isomorphism above. Denote

$$\epsilon : K^\vee \otimes_{\mathcal{O}}^{\mathbf{L}} K \longrightarrow \mathcal{O}$$

the evaluation map (to construct it you can use Lemma 21.35.6 for example). Then $K^\vee, \eta, \epsilon$ is a left dual for K as in Categories, Definition 4.43.5. We omit the verification that $(1 \otimes \epsilon) \circ (\eta \otimes 1) = \mathrm{id}_K$ and $(\epsilon \otimes 1) \circ (1 \otimes \eta) = \mathrm{id}_{K^\vee}$.

0FPV Lemma 21.48.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let M be an object of $D(\mathcal{O})$. If M has a left dual in the monoidal category $D(\mathcal{O})$ (Categories, Definition 4.43.5) then M is perfect and the left dual is as constructed in Example 21.48.6.

Proof. Let N, η, ϵ be a left dual. Observe that for any object U of $\mathcal{C}$ the restriction $N|_U, \eta|_U, \epsilon|_U$ is a left dual for $M|_U$.

Let U be an object of $\mathcal{C}$. It suffices to find a covering $\{U_i \to U\}_{i \in I}$ fo $\mathcal{C}$ such that $M|_{U_i}$ is a perfect object of $D(\mathcal{O}_{U_i})$. Hence we may replace $\mathcal{C}, \mathcal{O}, M, N, \eta, \epsilon$ by $\mathcal{C}/U, \mathcal{O}_U, M|_U, N|_U, \eta|_U, \epsilon|_U$ and assume $\mathcal{C}$ has a final object X. Moreover, during the proof we can (finitely often) replace X by the members of a covering $\{U_i \to X\}$ of X.

We are going to use the following argument several times. Choose any complex $\mathcal{M}^\bullet$ of $\mathcal{O}$-modules representing M. Choose a K-flat complex $\mathcal{N}^\bullet$ representing N whose terms are flat $\mathcal{O}$-modules, see Lemma 21.17.11. Consider the map

$$\eta : \mathcal{O} \to \mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{N}^\bullet)$$

After replacing X by the members of a covering, we can find an integer N and for $i = 1, \ldots, N$ integers $n_i \in \mathbf{Z}$ and sections f_i and g_i of $\mathcal{M}^{n_i}$ and $\mathcal{N}^{-n_i}$ such that

$$\eta(1) = \sum\nolimits_i f_i \otimes g_i$$

Let $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ be any subcomplex of $\mathcal{O}$-modules containing the sections f_i for $i = 1, \ldots, N$. Since $\text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{N}^\bullet) \subset \text{Tot}(\mathcal{M}^\bullet \otimes_\mathcal{O} \mathcal{N}^\bullet)$ by flatness of the modules $\mathcal{N}^n$, we see that η factors through

$$\tilde{\eta} : \mathcal{O} \to \text{Tot}(\mathcal{K}^\bullet \otimes_\mathcal{O} \mathcal{N}^\bullet)$$

Denoting K the object of $D(\mathcal{O})$ represented by $\mathcal{K}^\bullet$ we find a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\eta\otimes 1"] \arrow[dr, "\tilde{\eta}\otimes 1"'] & M \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M \arrow[r, "1\otimes\epsilon"] & M \\
 & K \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} M \arrow[u] \arrow[r, "1\otimes\epsilon"] & K \arrow[u]
\end{tikzcd}$$

Since the composition of the upper row is the identity on M we conclude that M is a direct summand of K in $D(\mathcal{O})$.

As a first use of the argument above, we can choose the subcomplex $\mathcal{K}^\bullet = \sigma_{\geq a}\tau_{\leq b}\mathcal{M}^\bullet$ with $a < n_i < b$ for $i = 1, \ldots, N$. Thus M is a direct summand in $D(\mathcal{O})$ of a bounded complex and we conclude we may assume M is in $D^b(\mathcal{O})$. (Recall that the process above involves replacing X by the members of a covering.)

Since M is in $D^b(\mathcal{O})$ we may choose $\mathcal{M}^\bullet$ to be a bounded above complex of flat modules (by Modules, Lemma 17.17.6 and Derived Categories, Lemma 13.15.4). Then we can choose $\mathcal{K}^\bullet = \sigma_{\geq a}\mathcal{M}^\bullet$ with $a < n_i$ for $i = 1, \ldots, N$ in the argument above. Thus we find that we may assume M is a direct summand in $D(\mathcal{O})$ of a bounded complex of flat modules. In particular, we find M has finite tor amplitude.

Say M has tor amplitude in $[a, b]$. Assuming M is m-pseudo-coherent we are going to show that (after replacing X by the members of a covering) we may assume M is $(m-1)$-pseudo-coherent. This will finish the proof by Lemma 21.47.3 and the fact that M is $(b+1)$-pseudo-coherent in any case. After replacing X by the members of a covering we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to M$ in $D(\mathcal{O})$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and surjective for $i = m$. We may and do assume that $\mathcal{E}^i = 0$ for $i < m$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to M \to L \to \mathcal{E}^\bullet[1]$$

Observe that $H^i(L) = 0$ for $i \geq m$. Thus we may represent L by a complex $\mathcal{L}^\bullet$ with $\mathcal{L}^i = 0$ for $i \geq m$. The map $L \to \mathcal{E}^\bullet[1]$ is given by a map of complexes $\mathcal{L}^\bullet \to \mathcal{E}^\bullet[1]$ which is zero in all degrees except in degree $m-1$ where we obtain a map $\mathcal{L}^{m-1} \to \mathcal{E}^m$, see Derived Categories, Lemma 13.27.3. Then M is represented by the complex

$$\mathcal{M}^\bullet : \ldots \to \mathcal{L}^{m-2} \to \mathcal{L}^{m-1} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots$$

Apply the discussion in the second paragraph to this complex to get sections f_i of $\mathcal{M}^{n_i}$ for $i = 1, \ldots, N$. For $n < m$ let $\mathcal{K}^n \subset \mathcal{L}^n$ be the $\mathcal{O}$-submodule generated by the sections f_i for $n_i = n$ and $d(f_i)$ for $n_i = n - 1$. For $n \geq m$ set $\mathcal{K}^n = \mathcal{E}^n$. Clearly, we have a morphism of distinguished triangles

$$\begin{tikzcd}
\mathcal{E}^\bullet \arrow[r] & \mathcal{M}^\bullet \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{E}^\bullet[1] \\
\mathcal{E}^\bullet \arrow[r] \arrow[u] & \mathcal{K}^\bullet \arrow[r] \arrow[u] & \sigma_{\leq m-1}\mathcal{K}^\bullet \arrow[r] \arrow[u] & \mathcal{E}^\bullet[1] \arrow[u]
\end{tikzcd}$$

where all the morphisms are as indicated above. Denote K the object of $D(\mathcal{O})$ corresponding to the complex $\mathcal{K}^\bullet$. By the arguments in the second paragraph of the proof we obtain a morphism $s : M \to K$ in $D(\mathcal{O})$ such that the composition $M \to K \to M$ is the identity on M. We don't know that the diagram

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r] & \mathcal{K}^\bullet \arrow[r, equal] & K \\ \mathcal{E}^\bullet \arrow[u, "\mathrm{id}"] \arrow[r, "i"] & \mathcal{M}^\bullet \arrow[r, equal] & M \arrow[u, "s"'] \end{tikzcd}$$

commutes, but we do know it commutes after composing with the map $K \to M$. By Lemma 21.44.8 after replacing X by the members of a covering, we may assume that $s \circ i$ is given by a map of complexes $\sigma : \mathcal{E}^\bullet \to \mathcal{K}^\bullet$. By the same lemma we may assume the composition of σ with the inclusion $\mathcal{K}^\bullet \subset \mathcal{M}^\bullet$ is homotopic to zero by some homotopy $\{h^i : \mathcal{E}^i \to \mathcal{M}^{i-1}\}$. Thus, after replacing $\mathcal{K}^{m-1}$ by $\mathcal{K}^{m-1} + \mathrm{Im}(h^m)$ (note that after doing this it is still the case that $\mathcal{K}^{m-1}$ is generated by finitely many global sections), we see that σ itself is homotopic to zero! This means that we have a commutative solid diagram

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[r] & M \arrow[r] & \mathcal{L}^\bullet \arrow[r] & \mathcal{E}^\bullet[1] \\ \mathcal{E}^\bullet \arrow[u] \arrow[r] & K \arrow[u] \arrow[r] & \sigma_{\leq m-1}\mathcal{K}^\bullet \arrow[u] \arrow[r] & \mathcal{E}^\bullet[1] \arrow[u] \\ \mathcal{E}^\bullet \arrow[u] \arrow[r] & M \arrow[u, "s"] \arrow[r] & \mathcal{L}^\bullet \arrow[u, dotted] \arrow[r] & \mathcal{E}^\bullet[1] \arrow[u] \end{tikzcd}$$

By the axioms of triangulated categories we obtain a dotted arrow fitting into the diagram. Looking at cohomology sheaves in degree $m - 1$ we see that we obtain

$$\begin{tikzcd} H^{m-1}(M) \arrow[r] & H^{m-1}(\mathcal{L}^\bullet) \arrow[r] & H^m(\mathcal{E}^\bullet) \\ H^{m-1}(K) \arrow[u] \arrow[r] & H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \arrow[u] \arrow[r] & H^m(\mathcal{E}^\bullet) \arrow[u] \\ H^{m-1}(M) \arrow[u] \arrow[r] & H^{m-1}(\mathcal{L}^\bullet) \arrow[u] \arrow[r] & H^m(\mathcal{E}^\bullet) \arrow[u] \end{tikzcd}$$

Since the vertical compositions are the identity in both the left and right column, we conclude the vertical composition $H^{m-1}(\mathcal{L}^\bullet) \to H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ in the middle is surjective! In particular $H^{m-1}(\sigma_{\leq m-1}\mathcal{K}^\bullet) \to H^{m-1}(\mathcal{L}^\bullet)$ is surjective. Using the induced map of long exact sequences of cohomology sheaves from the morphism of triangles above, a diagram chase shows this implies $H^i(K) \to H^i(M)$ is an isomorphism for $i \geq m$ and surjective for $i = m - 1$. By construction we can choose an $r \geq 0$ and a surjection $\mathcal{O}^{\oplus r} \to \mathcal{K}^{m-1}$. Then the composition

$$(\mathcal{O}^{\oplus r} \to \mathcal{E}^m \to \mathcal{E}^{m+1} \to \ldots) \longrightarrow K \longrightarrow M$$

induces an isomorphism on cohomology sheaves in degrees $\geq m$ and a surjection in degree $m - 1$ and the proof is complete. $\square$

0A0A Lemma 21.48.8. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(K_n)_{n\in\mathbf{N}}$ be a system of perfect objects of $D(\mathcal{O})$. Let $K = \text{hocolim} K_n$ be the derived colimit (Derived Categories, Definition 13.33.1). Then for any object E of $D(\mathcal{O})$ we have

$$R\mathcal{H}om(K, E) = R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n^\vee$$

where $(K_n^\vee)$ is the inverse system of dual perfect complexes.

Proof. By Lemma 21.48.4 we have $R\lim E \otimes^{\mathbf{L}}_{\mathcal{O}} K_n^\vee = R\lim R\mathcal{H}om(K_n, E)$ which fits into the distinguished triangle

$$R\lim R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E) \to \prod R\mathcal{H}om(K_n, E)$$

Because K similarly fits into the distinguished triangle $\bigoplus K_n \to \bigoplus K_n \to K$ it suffices to show that $\prod R\mathcal{H}om(K_n, E) = R\mathcal{H}om(\bigoplus K_n, E)$. This is a formal consequence of (21.35.0.1) and the fact that derived tensor product commutes with direct sums. $\square$

21.49. Invertible objects in the derived category

0FPW We characterize invertible objects in the derived category of a ringed space (both in the case of a locally ringed topos and in the general case).

0FPX Lemma 21.49.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. Set $R = \Gamma(\mathcal{C}, \mathcal{O})$. The category of $\mathcal{O}$-modules which are summands of finite free $\mathcal{O}$-modules is equivalent to the category of finite projective R-modules.

Proof. Observe that a finite projective R-module is the same thing as a summand of a finite free R-module. The equivalence is given by the functor $\mathcal{E} \mapsto \Gamma(\mathcal{C}, \mathcal{E})$. The inverse functor is given by the following construction. Consider the morphism of topoi $f : Sh(\mathcal{C}) \to Sh(pt)$ with f_* given by taking global sections and f^{-1} by sending a set S, i.e., an object of $Sh(pt)$, to the constant sheaf with value S. We obtain a morphism $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(pt), R)$ of ringed topoi by using the identity map $R \to f_*\mathcal{O}$. Then the inverse functor is given by f^*. $\square$

0FPY Lemma 21.49.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let M be an object of $D(\mathcal{O})$. The following are equivalent

(1) M is invertible in $D(\mathcal{O})$, see Categories, Definition 4.43.4, and
(2) there is a locally finite[11] direct product decomposition

$$\mathcal{O} = \prod\nolimits_{n\in\mathbf{Z}} \mathcal{O}_n$$

and for each n there is an invertible $\mathcal{O}_n$-module $\mathcal{H}^n$ (Modules on Sites, Definition 18.32.1) and $M = \bigoplus \mathcal{H}^n[-n]$ in $D(\mathcal{O})$.

If (1) and (2) hold, then M is a perfect object of $D(\mathcal{O})$. If $(\mathcal{C}, \mathcal{O})$ is a locally ringed site these condition are also equivalent to

(3) for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ and for each i an integer n_i such that $M|_{U_i}$ is represented by an invertible $\mathcal{O}_{U_i}$-module placed in degree n_i.

[11] This means that for every object U of $\mathcal{C}$ there is a covering $\{U_i \to U\}$ such that for every i the sheaf $\mathcal{O}_n|_{U_i}$ is nonzero for only a finite number of n.

Proof. Assume (2). Consider the object $R\mathcal{H}om(M, \mathcal{O})$ and the composition map

$$R\mathcal{H}om(M, \mathcal{O}) \otimes_{\mathcal{O}}^{\mathbf{L}} M \to \mathcal{O}$$

To prove this is an isomorphism, we may work locally. Thus we may assume $\mathcal{O} = \prod_{a \leq n \leq b} \mathcal{O}_n$ and $M = \bigoplus_{a \leq n \leq b} \mathcal{H}^n[-n]$. Then it suffices to show that

$$R\mathcal{H}om(\mathcal{H}^m, \mathcal{O}) \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{H}^n$$

is zero if $n \neq m$ and equal to $\mathcal{O}_n$ if $n = m$. The case $n \neq m$ follows from the fact that $\mathcal{O}_n$ and $\mathcal{O}_m$ are flat $\mathcal{O}$-algebras with $\mathcal{O}_n \otimes_{\mathcal{O}} \mathcal{O}_m = 0$. Using the local structure of invertible $\mathcal{O}$-modules (Modules on Sites, Lemma 18.32.2) and working locally the isomorphism in case $n = m$ follows in a straightforward manner; we omit the details. Because $D(\mathcal{O})$ is symmetric monoidal, we conclude that M is invertible.

Assume (1). The description in (2) shows that we have a candidate for $\mathcal{O}_n$, namely, $\mathcal{H}om_{\mathcal{O}}(H^n(M), H^n(M))$. If this is a locally finite family of sheaves of rings and if $\mathcal{O} = \prod \mathcal{O}_n$, then we immediately obtain the direct sum decomposition $M = \bigoplus H^n(M)[-n]$ using the idempotents in $\mathcal{O}$ coming from the product decomposition. This shows that in order to prove (2) we may work locally in the following sense. Let U be an object of $\mathcal{C}$. We have to show there exists a covering $\{U_i \to U\}$ of U such that with $\mathcal{O}_n$ as above we have the statements above and those of (2) after restriction to $\mathcal{C}/U_i$. Thus we may assume $\mathcal{C}$ has a final object X and during the proof of (2) we may finitely many times replace X by the members of a covering of X.

Choose an object N of $D(\mathcal{O})$ and an isomorphism $M \otimes_{\mathcal{O}}^{\mathbf{L}} N \cong \mathcal{O}$. Then N is a left dual for M in the monoidal category $D(\mathcal{O})$ and we conclude that M is perfect by Lemma 21.48.7. By symmetry we see that N is perfect. After replacing X by the members of a covering, we may assume M and N are represented by a strictly perfect complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$. Then $M \otimes_{\mathcal{O}}^{\mathbf{L}} N$ is represented by $\text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$. After replacing X by the members of a covering of X we may assume the mutually inverse isomorphisms $\mathcal{O} \to M \otimes_{\mathcal{O}}^{\mathbf{L}} N$ and $M \otimes_{\mathcal{O}}^{\mathbf{L}} N \to \mathcal{O}$ are given by maps of complexes

$$\alpha : \mathcal{O} \to \text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \quad \text{and} \quad \beta : \text{Tot}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \to \mathcal{O}$$

See Lemma 21.44.8. Then $\beta \circ \alpha = 1$ as maps of complexes and $\alpha \circ \beta = 1$ as a morphism in $D(\mathcal{O})$. After replacing X by the members of a covering of X we may assume the composition $\alpha \circ \beta$ is homotopic to 1 by some homotopy θ with components

$$\theta^n : \text{Tot}^n(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet) \to \text{Tot}^{n-1}(\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}^\bullet)$$

by the same lemma as before. Set $R = \Gamma(\mathcal{C}, \mathcal{O})$. By Lemma 21.49.1 we find that we obtain

(1) $M^\bullet = \Gamma(X, \mathcal{E}^\bullet)$ is a bounded complex of finite projective R-modules,
(2) $N^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ is a bounded complex of finite projective R-modules,
(3) α and β correspond to maps of complexes $a : R \to \text{Tot}(M^\bullet \otimes_R N^\bullet)$ and $b : \text{Tot}(M^\bullet \otimes_R N^\bullet) \to R$,
(4) θ^n corresponds to a map $h^n : \text{Tot}^n(M^\bullet \otimes_R N^\bullet) \to \text{Tot}^{n-1}(M^\bullet \otimes_R N^\bullet)$, and
(5) $b \circ a = 1$ and $b \circ a - 1 = dh + hd$,

It follows that $M^\bullet$ and $N^\bullet$ define mutually inverse objects of $D(R)$. By More on Algebra, Lemma 15.126.4 we find a product decomposition $R = \prod_{a \leq n \leq b} R_n$ and invertible R_n-modules H^n such that $M^\bullet \cong \bigoplus_{a \leq n \leq b} H^n[-n]$. This isomorphism in $D(R)$ can be lifted to an morphism

$$\bigoplus H^n[-n] \longrightarrow M^\bullet$$

of complexes because each H^n is projective as an R-module. Correspondingly, using Lemma 21.49.1 again, we obtain an morphism

$$\bigoplus H^n \otimes_R \mathcal{O}[-n] \to \mathcal{E}^\bullet$$

which is an isomorphism in $D(\mathcal{O})$. Here $M \otimes_R \mathcal{O}$ denotes the functor from finite projective R-modules to $\mathcal{O}$-modules constructed in the proof of Lemma 21.49.1. Setting $\mathcal{O}_n = R_n \otimes_R \mathcal{O}$ we conclude (2) is true.

If $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then given an object U and a finite product decomposition $\mathcal{O}|_U = \prod_{a \leq n \leq b} \mathcal{O}_n|_U$ we can find a covering $\{U_i \to U\}$ such that for every i there is at most one n with $\mathcal{O}_n|_{U_i}$ nonzero. This follows readily from part (2) of Modules on Sites, Lemma 18.40.1 and the definition of locally ringed sites as given in Modules on Sites, Definition 18.40.4. From this the implication (2) $\Rightarrow$ (3) is easily seen. The implication (3) $\Rightarrow$ (2) holds without any assumptions on the ringed site. We omit the details. $\square$

21.50. Projection formula

0943 Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. Without any further assumptions there is a map

0B56
$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K) \quad (21.50.0.1)$$

Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K) = Lf^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$ and Lemmas 21.18.4 and 21.19.1. A reasonably general version of the projection formula is the following.

0944 **Lemma 21.50.1.** Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. If K is perfect, then

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K)$$

in $D(\mathcal{O}_\mathcal{D})$.

Proof. To check (21.50.0.1) is an isomorphism we may work locally on $\mathcal{D}$, i.e., for any object V of $\mathcal{D}$ we have to find a covering $\{V_j \to V\}$ such that the map restricts to an isomorphism on V_j. By definition of perfect objects, this means we may assume K is represented by a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules. Note that, completely generally, the statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for K_1 and K_2. Hence we may assume K is a finite complex of finite free $\mathcal{O}_\mathcal{D}$-modules. In this case a simple argument involving stupid truncations reduces the statement to the case where K is represented by a finite free $\mathcal{O}_\mathcal{D}$-module. Since the statement is invariant under finite direct summands in the K variable, we conclude it suffices to prove it for $K = \mathcal{O}_\mathcal{D}[n]$ in which case it is trivial. $\square$

0E48 Remark 21.50.2. The map (21.50.0.1) is compatible with the base change map of Remark 21.19.3 in the following sense. Namely, suppose that

$$\begin{tikzcd} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) \arrow[r, "g'"'] \arrow[d, "f'"'] & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \arrow[d, "f"] \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \arrow[r, "g"] & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{tikzcd}$$

is a commutative diagram of ringed topoi. Let $E \in D(\mathcal{O}_\mathcal{C})$ and $K \in D(\mathcal{O}_\mathcal{D})$. Then the diagram

$$\begin{tikzcd} Lg^*(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{D}} K) \arrow[r, "p"'] \arrow[d, "t"'] & Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K) \arrow[d, "b"] \\ Lg^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} Lg^*K \arrow[d, "b"'] & Rf'_*L(g')^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_\mathcal{C}} Lf^*K) \arrow[d, "t"] \\ Rf'_*L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} Lg^*K \arrow[rd, "p"'] & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} L(g')^*Lf^*K) \arrow[d, "c"] \\ & Rf'_*(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}'}} L(f')^*Lg^*K) \end{tikzcd}$$

is commutative. Here arrows labeled t are gotten by an application of Lemma 21.18.4, arrows labeled b by an application of Remark 21.19.3, arrows labeled p by an application of (21.50.0.1), and c comes from $L(g')^* \circ Lf^* = L(f')^* \circ Lg^*$. We omit the verification.

21.51. Weakly contractible objects

0945 An object U of a site is weakly contractible if every surjection $\mathcal{F} \to \mathcal{G}$ of sheaves of sets gives rise to a surjection $\mathcal{F}(U) \to \mathcal{G}(U)$, see Sites, Definition 7.40.2.

0946 Lemma 21.51.1. Let $\mathcal{C}$ be a site. Let U be a weakly contractible object of $\mathcal{C}$. Then

(1) the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor $\textit{Ab}(\mathcal{C}) \to \textit{Ab}$,
(2) $H^p(U, \mathcal{F}) = 0$ for every abelian sheaf $\mathcal{F}$ and all $p \geq 1$, and
(3) for any sheaf of groups $\mathcal{G}$ any $\mathcal{G}$-torsor has a section over U.

Proof. The first statement follows immediately from the definition (see also Homology, Section 12.7). The higher derived functors vanish by Derived Categories, Lemma 13.16.9. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor. Then $\mathcal{F} \to *$ is a surjective map of sheaves. Hence (3) follows from the definition as well. $\square$

It is convenient to list some consequences of having enough weakly contractible objects here.

0947 Proposition 21.51.2. Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Then

(1) A complex $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ of $\mathcal{O}$-modules is exact, if and only if $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$.

(2) Every object K of $D(\mathcal{O})$ is a derived limit of its canonical truncations: $K = R\lim \tau_{\geq -n} K$.
(3) Given an inverse system $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ with surjective transition maps, the projection $\lim \mathcal{F}_n \to \mathcal{F}_1$ is surjective.
(4) Products are exact on $\text{Mod}(\mathcal{O})$.
(5) Products on $D(\mathcal{O})$ can be computed by taking products of any representative complexes.
(6) If $(\mathcal{F}_n)$ is an inverse system of $\mathcal{O}$-modules, then $R^p \lim \mathcal{F}_n = 0$ for all $p > 1$ and
$$R^1 \lim \mathcal{F}_n = \text{Coker}(\prod \mathcal{F}_n \to \prod \mathcal{F}_n)$$
where the map is $(x_n) \mapsto (x_n - f(x_{n+1}))$.
(7) If (K_n) is an inverse system of objects of $D(\mathcal{O})$, then there are short exact sequences
$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(R\lim K_n) \to \lim H^p(K_n) \to 0$$

Proof. Proof of (1). If the sequence is exact, then evaluating at any weakly contractible element of $\mathcal{C}$ gives an exact sequence by Lemma 21.51.1. Conversely, assume that $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$. Let V be an object of $\mathcal{C}$ and let $s \in \mathcal{F}_2(V)$ be an element of the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$. By assumption there exists a covering $\{U_i \to V\}$ with $U_i \in \mathcal{B}$. Then $s|_{U_i}$ lifts to a section $s_i \in \mathcal{F}_1(U_i)$. Thus s is a section of the image sheaf $\text{Im}(\mathcal{F}_1 \to \mathcal{F}_2)$. In other words, the sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact.

Proof of (2). This holds by Lemma 21.23.10 with $d = 0$.

Proof of (3). Let $(\mathcal{F}_n)$ be a system as in (2) and set $\mathcal{F} = \lim \mathcal{F}_n$. If $U \in \mathcal{B}$, then $\mathcal{F}(U) = \lim \mathcal{F}_n(U)$ surjects onto $\mathcal{F}_1(U)$ as all the transition maps $\mathcal{F}_{n+1}(U) \to \mathcal{F}_n(U)$ are surjective. Thus $\mathcal{F} \to \mathcal{F}_1$ is surjective by Sites, Definition 7.11.1 and the assumption that every object has a covering by elements of $\mathcal{B}$.

Proof of (4). Let $\mathcal{F}_{i,1} \to \mathcal{F}_{i,2} \to \mathcal{F}_{i,3}$ be a family of exact sequences of $\mathcal{O}$-modules. We want to show that $\prod \mathcal{F}_{i,1} \to \prod \mathcal{F}_{i,2} \to \prod \mathcal{F}_{i,3}$ is exact. We use the criterion of (1). Let $U \in \mathcal{B}$. Then
$$(\prod \mathcal{F}_{i,1})(U) \to (\prod \mathcal{F}_{i,2})(U) \to (\prod \mathcal{F}_{i,3})(U)$$
is the same as
$$\prod \mathcal{F}_{i,1}(U) \to \prod \mathcal{F}_{i,2}(U) \to \prod \mathcal{F}_{i,3}(U)$$
Each of the sequences $\mathcal{F}_{i,1}(U) \to \mathcal{F}_{i,2}(U) \to \mathcal{F}_{i,3}(U)$ are exact by (1). Thus the displayed sequences are exact by Homology, Lemma 12.32.1. We conclude by (1) again.

Proof of (5). Follows from (4) and (slightly generalized) Derived Categories, Lemma 13.34.2.

Proof of (6) and (7). We refer to Section 21.23 for a discussion of derived and homotopy limits and their relationship. By Derived Categories, Definition 13.34.1 we have a distinguished triangle
$$R\lim K_n \to \prod K_n \to \prod K_n \to R\lim K_n[1]$$
Taking the long exact sequence of cohomology sheaves we obtain
$$H^{p-1}(\prod K_n) \to H^{p-1}(\prod K_n) \to H^p(R\lim K_n) \to H^p(\prod K_n) \to H^p(\prod K_n)$$

Since products are exact by (4) this becomes

$$\prod H^{p-1}(K_n) \to \prod H^{p-1}(K_n) \to H^p(R\lim K_n) \to \prod H^p(K_n) \to \prod H^p(K_n)$$

Now we first apply this to the case $K_n = \mathcal{F}_n[0]$ where $(\mathcal{F}_n)$ is as in (6). We conclude that (6) holds. Next we apply it to (K_n) as in (7) and we conclude (7) holds. $\square$

21.52. Compact objects

0948 In this section we study compact objects in the derived category of modules on a ringed site. We recall that compact objects are defined in Derived Categories, Definition 13.37.1.

094B Lemma 21.52.1. Let $\mathcal{A}$ be a Grothendieck abelian category. Let $S \subset \mathrm{Ob}(\mathcal{A})$ be a set of objects such that

(1) any object of $\mathcal{A}$ is a quotient of a direct sum of elements of S, and
(2) for any $E \in S$ the functor $\mathrm{Hom}_\mathcal{A}(E, -)$ commutes with direct sums.

Then every compact object of $D(\mathcal{A})$ is a direct summand in $D(\mathcal{A})$ of a finite complex of finite direct sums of elements of S.

Proof. Assume $K \in D(\mathcal{A})$ is a compact object. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 12.15. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

We may represent K by a bounded above complex $K^\bullet$ each of whose terms is a direct sum of objects from S, see Derived Categories, Lemma 13.15.4. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 12.15. Hence by Derived Categories, Lemmas 13.33.7 and 13.33.9 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(\mathcal{A})$ for some complex $L^\bullet$ which is bounded and whose terms are direct sums of elements of S. Say L^i is zero for $i \not\in [a, b]$. Let c be the largest integer $\leq b + 1$ such that L^i a finite direct sum of elements of S for $i < c$. Claim: if $c < b + 1$, then we can modify $L^\bullet$ to increase c. By induction this claim will show we have a factorization of 1_K as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(\mathcal{A})$ where L can be represented by a finite complex of finite direct sums of elements of S. Note that $e = \varphi \circ \psi \in \mathrm{End}_{D(\mathcal{A})}(L)$ is an idempotent. By Derived Categories, Lemma 13.4.14 we see that $L = \mathrm{Ker}(e) \oplus \mathrm{Ker}(1-e)$. The map $\varphi : K \to L$ induces an isomorphism with $\mathrm{Ker}(1 - e)$ in $D(R)$ and we conclude.

Proof of the claim. Write $L^c = \bigoplus_{\lambda \in \Lambda} E_\lambda$. Since L^{c-1} is a finite direct sum of elements of S we can by assumption (2) find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda \in \Lambda'} E_\lambda \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda \in \Lambda \setminus \Lambda'} E_\lambda)[-i]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(\mathcal{A})$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(\mathcal{A})$ (see Derived Categories, Lemma 13.12.1). Since $\mathrm{Hom}_{D(\mathcal{A})}(K, -)$ is homological (see Derived Categories, Lemma 13.4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(\mathcal{A})$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda \in \Lambda'} E_\lambda$ the claim is proved. □

094C Lemma 21.52.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume every object of $\mathcal{C}$ has a covering by quasi-compact objects. Then every compact object of $D(\mathcal{O})$ is a direct summand in $D(\mathcal{O})$ of a finite complex whose terms are finite direct sums of $\mathcal{O}$-modules of the form $j_!\mathcal{O}_U$ where U is a quasi-compact object of $\mathcal{C}$.

Proof. Apply Lemma 21.52.1 where $S \subset \mathrm{Ob}(\mathrm{Mod}(\mathcal{O}))$ is the set of modules of the form $j_!\mathcal{O}_U$ with $U \in \mathrm{Ob}(\mathcal{C})$ quasi-compact. Assumption (1) holds by Modules on Sites, Lemma 18.28.8 and the assumption that every U can be covered by quasi-compact objects. Assumption (2) follows as

$$\mathrm{Hom}_\mathcal{O}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$$

which commutes with direct sums by Sites, Lemma 7.17.7. □

In the situation of the lemma above it is not always true that the modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O})$ (even if U is a quasi-compact object of $\mathcal{C}$). Here are two lemmas addressing this issue.

0G21 Lemma 21.52.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Assume the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums. Then $\mathcal{O}$-module $j_!\mathcal{O}_U$ is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ in $D(\mathcal{O})$ is bounded below, then $\mathrm{Hom}(j_{U!}\mathcal{O}_U, M) = \bigoplus_{i \in I} \mathrm{Hom}(j_{U!}\mathcal{O}_U, M_i)$.

Proof. Since $\mathrm{Hom}(j_{U!}\mathcal{O}_U, -)$ is the same as the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ by Modules on Sites, Equation (18.19.2.1) it suffices to prove that $H^p(U, M) = \bigoplus H^p(U, M_i)$. Let $\mathcal{I}_i$, $i \in I$ be a collection of injective $\mathcal{O}$-modules. By assumption we have

$$H^p(U, \bigoplus\nolimits_{i \in I} \mathcal{I}_i) = \bigoplus\nolimits_{i \in I} H^p(U, \mathcal{I}_i) = 0$$

for all p. Since $M = \bigoplus M_i$ is bounded below, we see that there exists an $a \in \mathbf{Z}$ such that $H^n(M_i) = 0$ for $n < a$. Thus we can choose complexes of injective $\mathcal{O}$-modues $\mathcal{I}_i^\bullet$ representing M_i with $\mathcal{I}_i^n = 0$ for $n < a$, see Derived Categories, Lemma 13.18.3. By Injectives, Lemma 19.13.4 we see that the direct sum complex $\bigoplus \mathcal{I}_i^\bullet$ represents M. By Leray acyclicity (Derived Categories, Lemma 13.16.7) we see that

$$R\Gamma(U, M) = \Gamma(U, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus \Gamma(U, \bigoplus \mathcal{I}_i^\bullet) = \bigoplus R\Gamma(U, M_i)$$

as desired. □

0G22 **Lemma 21.52.4.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site with set of coverings $\text{Cov}_\mathcal{C}$. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$, and $\text{Cov} \subset \text{Cov}_\mathcal{C}$ be subsets. Assume that

(1) For every $\mathcal{U} \in \text{Cov}$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.
(2) For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.

Then for $U \in \mathcal{B}$ the object $j_{U!}\mathcal{O}_U$ is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ in $D(\mathcal{O})$ is bounded below, then $\text{Hom}(j_{U!}\mathcal{O}_U, M) = \bigoplus_{i \in I} \text{Hom}(j_{U!}\mathcal{O}_U, M_i)$.

Proof. This follows from Lemma 21.52.3 and Lemma 21.16.1. $\square$

094D **Lemma 21.52.5.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The $\mathcal{O}$-module $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$ if there exists an integer d such that

(1) $H^p(U, \mathcal{F}) = 0$ for all $p > d$, and
(2) the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums.

Proof. Assume (1) and (2). Recall that $\text{Hom}(j_!\mathcal{O}_U, K) = R\Gamma(U, K)$ for K in $D(\mathcal{O})$. Thus we have to show that $R\Gamma(U, -)$ commutes with direct sums. The first assumption means that the functor $F = H^0(U, -)$ has finite cohomological dimension. Moreover, the second assumption implies any direct sum of injective modules is acyclic for F. Let K_i be a family of objects of $D(\mathcal{O})$. Choose K-injective representatives $I_i^\bullet$ with injective terms representing K_i, see Injectives, Theorem 19.12.6. Since we may compute RF by applying F to any complex of acyclics (Derived Categories, Lemma 13.32.2) and since $\bigoplus K_i$ is represented by $\bigoplus I_i^\bullet$ (Injectives, Lemma 19.13.4) we conclude that $R\Gamma(U, \bigoplus K_i)$ is represented by $\bigoplus H^0(U, I_i^\bullet)$. Hence $R\Gamma(U, -)$ commutes with direct sums as desired. $\square$

094E **Lemma 21.52.6.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$ which is quasi-compact and weakly contractible. Then $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$.

Proof. Combine Lemmas 21.52.5 and 21.51.1 with Modules on Sites, Lemma 18.30.3. $\square$

09JC **Lemma 21.52.7.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume $\mathcal{C}$ has the following properties

(1) $\mathcal{C}$ has a quasi-compact final object X,
(2) every quasi-compact object of $\mathcal{C}$ has a cofinal system of coverings which are finite and consist of quasi-compact objects,
(3) for a finite covering $\{U_i \to U\}_{i \in I}$ with U, U_i quasi-compact the fibre products $U_i \times_U U_j$ are quasi-compact.

Let K be a perfect object of $D(\mathcal{O})$. Then

(a) K is a compact object of $D^+(\mathcal{O})$ in the following sense: if $M = \bigoplus_{i \in I} M_i$ is bounded below, then $\text{Hom}(K, M) = \bigoplus_{i \in I} \text{Hom}(K, M_i)$.
(b) If $(\mathcal{C}, \mathcal{O})$ has finite cohomological dimension, i.e., if there exists a d such that $H^i(X, \mathcal{F}) = 0$ for $i > d$ for any $\mathcal{O}$-module $\mathcal{F}$, then K is a compact object of $D(\mathcal{O})$.

Proof. Let $K^\vee$ be the dual of K, see Lemma 21.48.4. Then we have

$$\text{Hom}_{D(\mathcal{O})}(K, M) = H^0(X, K^\vee \otimes_\mathcal{O}^\mathbf{L} M)$$

functorially in M in $D(\mathcal{O})$. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}} -$ commutes with direct sums it suffices to show that $R\Gamma(X, -)$ commutes with the relevant direct sums.

Proof of (a). After reformulation as above this is a special case of Lemma 21.52.4 with $U = X$.

Proof of (b). Since $R\Gamma(X, K) = R\operatorname{Hom}(\mathcal{O}, K)$ and since $H^p(X, -)$ commutes with direct sums by Lemma 21.16.1 this is a special case of Lemma 21.52.5. □

21.53. Complexes with locally constant cohomology sheaves

094F Locally constant sheaves are introduced in Modules on Sites, Section 18.43. Let $\mathcal{C}$ be a site. Let Λ be a ring. We denote $D(\mathcal{C}, \Lambda)$ the derived category of the abelian category of $\underline{\Lambda}$-modules on $\mathcal{C}$.

094G **Lemma 21.53.1.** Let $\mathcal{C}$ be a site with final object X. Let Λ be a Noetherian ring. Let $K \in D^b(\mathcal{C}, \Lambda)$ with $H^i(K)$ locally constant sheaves of Λ-modules of finite type. Then there exists a covering $\{U_i \to X\}$ such that each $K|_{U_i}$ is represented by a complex of locally constant sheaves of Λ-modules of finite type.

Proof. Let $a \leq b$ be such that $H^i(K) = 0$ for $i \notin [a, b]$. By induction on $b - a$ we will prove there exists a covering $\{U_i \to X\}$ such that $K|_{U_i}$ can be represented by a complex $\underline{M^\bullet}_{U_i}$ with M^p a finite type Λ-module and $M^p = 0$ for $p \notin [a, b]$. If $b = a$, then this is clear. In general, we may replace X by the members of a covering and assume that $H^b(K)$ is constant, say $H^b(K) = \underline{M}$. By Modules on Sites, Lemma 18.42.5 the module M is a finite Λ-module. Choose a surjection $\Lambda^{\oplus r} \to M$ given by generators $x_1, \ldots, x_r$ of M.

By a slight generalization of Lemma 21.7.3 (details omitted) there exists a covering $\{U_i \to X\}$ such that $x_i \in H^0(X, H^b(K))$ lifts to an element of $H^b(U_i, K)$. Thus, after replacing X by the U_i we reach the situation where there is a map $\underline{\Lambda^{\oplus r}}[-b] \to K$ inducing a surjection on cohomology sheaves in degree b. Choose a distinguished triangle

$$\underline{\Lambda^{\oplus r}}[-b] \to K \to L \to \underline{\Lambda^{\oplus r}}[-b+1]$$

Now the cohomology sheaves of L are nonzero only in the interval $[a, b-1]$, agree with the cohomology sheaves of K in the interval $[a, b-2]$ and there is a short exact sequence

$$0 \to H^{b-1}(K) \to H^{b-1}(L) \to \underline{\operatorname{Ker}(\Lambda^{\oplus r} \to M)} \to 0$$

in degree $b-1$. By Modules on Sites, Lemma 18.43.5 we see that $H^{b-1}(L)$ is locally constant of finite type. By induction hypothesis we obtain an isomorphism $\underline{M^\bullet} \to L$ in $D(\mathcal{C}, \underline{\Lambda})$ with M^p a finite Λ-module and $M^p = 0$ for $p \notin [a, b-1]$. The map $L \to \Lambda^{\oplus r}[-b+1]$ gives a map $\underline{M^{b-1}} \to \underline{\Lambda^{\oplus r}}$ which locally is constant (Modules on Sites, Lemma 18.43.3). Thus we may assume it is given by a map $M^{b-1} \to \Lambda^{\oplus r}$. The distinguished triangle shows that the composition $M^{b-2} \to M^{b-1} \to \Lambda^{\oplus r}$ is zero and the axioms of triangulated categories produce an isomorphism

$$\underline{M^a \to \ldots \to M^{b-1} \to \Lambda^{\oplus r}} \longrightarrow K$$

in $D(\mathcal{C}, \Lambda)$. □

Let $\mathcal{C}$ be a site. Let Λ be a ring. Using the morphism $Sh(\mathcal{C}) \to Sh(pt)$ we see that there is a functor $D(\Lambda) \to D(\mathcal{C}, \Lambda)$, $K \mapsto \underline{K}$.

09BD **Lemma 21.53.2.** Let $\mathcal{C}$ be a site with final object X. Let Λ be a ring. Let

(1) K a perfect object of $D(\Lambda)$,
(2) a finite complex $K^\bullet$ of finite projective Λ-modules representing K,
(3) $\mathcal{L}^\bullet$ a complex of sheaves of Λ-modules, and
(4) $\varphi : \underline{K} \to \mathcal{L}^\bullet$ a map in $D(\mathcal{C}, \Lambda)$.

Then there exists a covering $\{U_i \to X\}$ and maps of complexes $\alpha_i : \underline{K}^\bullet|_{U_i} \to \mathcal{L}^\bullet|_{U_i}$ representing $\varphi|_{U_i}$.

Proof. Follows immediately from Lemma 21.44.8. $\square$

09BE Lemma 21.53.3. Let $\mathcal{C}$ be a site with final object X. Let Λ be a ring. Let K, L be objects of $D(\Lambda)$ with K perfect. Let $\varphi : \underline{K} \to \underline{L}$ be map in $D(\mathcal{C}, \Lambda)$. There exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is equal to $\underline{\alpha_i}$ for some map $\alpha_i : K \to L$ in $D(\Lambda)$.

Proof. Follows from Lemma 21.53.2 and Modules on Sites, Lemma 18.43.3. $\square$

094H Lemma 21.53.4. Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $K, L \in D^-(\mathcal{C}, \Lambda)$. If the cohomology sheaves of K and L are locally constant sheaves of Λ-modules of finite type, then the cohomology sheaves of $K \otimes_\Lambda^{\mathbf{L}} L$ are locally constant sheaves of Λ-modules of finite type.

Proof. We'll prove this as an application of Lemma 21.53.1. Note that $H^i(K \otimes_\Lambda^{\mathbf{L}} L)$ is the same as $H^i(\tau_{\geq i-1} K \otimes_\Lambda^{\mathbf{L}} \tau_{\geq i-1} L)$. Thus we may assume K and L are bounded. By Lemma 21.53.1 we may assume that K and L are represented by complexes of locally constant sheaves of Λ-modules of finite type. Then we can replace these complexes by bounded above complexes of finite free Λ-modules. In this case the result is clear. $\square$

094I Lemma 21.53.5. Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $K \in D^-(\mathcal{C}, \Lambda)$. If the cohomology sheaves of $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}$ are locally constant sheaves of Λ/I-modules of finite type, then the cohomology sheaves of $K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n}$ are locally constant sheaves of Λ/I^n-modules of finite type for all $n \geq 1$.

Proof. Recall that the locally constant sheaves of Λ-modules of finite type form a weak Serre subcategory of all $\underline{\Lambda}$-modules, see Modules on Sites, Lemma 18.43.5. Thus the subcategory of $D(\mathcal{C}, \Lambda)$ consisting of complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type forms a strictly full, saturated triangulated subcategory of $D(\mathcal{C}, \Lambda)$, see Derived Categories, Lemma 13.17.1. Next, consider the distinguished triangles

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} \to K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^{n+1}} \to K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I^n} \to K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes_\Lambda^{\mathbf{L}} \underline{I^n/I^{n+1}} = \left(K \otimes_\Lambda^{\mathbf{L}} \underline{\Lambda/I}\right) \otimes_{\Lambda/I}^{\mathbf{L}} \underline{I^n/I^{n+1}}$$

Combined with Lemma 21.53.4 we obtain the result. $\square$

21.54. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves

CHAPTER 22

Differential Graded Algebra

09JD 22.1. Introduction

09JE In this chapter we talk about differential graded algebras, modules, categories, etc. A basic reference is [Kel94]. A survey paper is [Kel06].

Since we do not worry about length of exposition in the Stacks project we first develop the material in the setting of categories of differential graded modules. After that we redo the constructions in the setting of differential graded modules over differential graded categories.

22.2. Conventions

09JF In this chapter we hold on to the convention that ring means commutative ring with 1. If R is a ring, then an R-algebra A will be an R-module A endowed with an R-bilinear map $A \times A \to A$ (multiplication) such that multiplication is associative and has a unit. In other words, these are unital associative R-algebras such that the structure map $R \to A$ maps into the center of A.

Sign rules. In this chapter we will work with graded algebras and graded modules often equipped with differentials. The sign rules on underlying complexes will always be (compatible with) those introduced in More on Algebra, Section 15.72. This will occasionally cause the multiplicative structure to be twisted in unexpected ways especially when considering left modules or the relationship between left and right modules.

22.3. Differential graded algebras

061U Just the definitions.

061V Definition 22.3.1. Let R be a commutative ring. A differential graded algebra over R is either

(1) a chain complex $A_\bullet$ of R-modules endowed with R-bilinear maps $A_n \times A_m \to A_{n+m}$, $(a, b) \mapsto ab$ such that

$$\mathrm{d}_{n+m}(ab) = \mathrm{d}_n(a)b + (-1)^n a\mathrm{d}_m(b)$$

and such that $\bigoplus A_n$ becomes an associative and unital R-algebra, or

(2) a cochain complex $A^\bullet$ of R-modules endowed with R-bilinear maps $A^n \times A^m \to A^{n+m}$, $(a, b) \mapsto ab$ such that

$$\mathrm{d}^{n+m}(ab) = \mathrm{d}^n(a)b + (-1)^n a\mathrm{d}^m(b)$$

and such that $\bigoplus A^n$ becomes an associative and unital R-algebra.

We often just write $A = \bigoplus A_n$ or $A = \bigoplus A^n$ and think of this as an associative unital R-algebra endowed with a $\mathbf{Z}$-grading and an R-linear operator d whose square is zero and which satisfies the Leibniz rule as explained above. In this case we often say "Let (A, d) be a differential graded algebra".

The Leibniz rule relating differentials and multiplication on a differential graded R-algebra A exactly means that the multiplication map defines a map of cochain complexes

$$\text{Tot}(A^\bullet \otimes_R A^\bullet) \to A^\bullet$$

Here $A^\bullet$ denote the underlying cochain complex of A.

061X Definition 22.3.2. A homomorphism of differential graded algebras $f : (A, \text{d}) \to (B, \text{d})$ is an algebra map $f : A \to B$ compatible with the gradings and d.

061W Definition 22.3.3. A differential graded algebra (A, d) is commutative if $ab = (-1)^{nm}ba$ for a in degree n and b in degree m. We say A is strictly commutative if in addition $a^2 = 0$ for $\deg(a)$ odd.

The following definition makes sense in general but is perhaps "correct" only when tensoring commutative differential graded algebras.

065W Definition 22.3.4. Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. The tensor product differential graded algebra of A and B is the algebra $A \otimes_R B$ with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a') \deg(b)} aa' \otimes bb'$$

endowed with differential d defined by the rule $\text{d}(a \otimes b) = \text{d}(a) \otimes b + (-1)^m a \otimes \text{d}(b)$ where $m = \deg(a)$.

065X Lemma 22.3.5. Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. Denote $A^\bullet$, $B^\bullet$ the underlying cochain complexes. As cochain complexes of R-modules we have

$$(A \otimes_R B)^\bullet = \text{Tot}(A^\bullet \otimes_R B^\bullet).$$

Proof. Recall that the differential of the total complex is given by $\text{d}_1^{p,q} + (-1)^p \text{d}_2^{p,q}$ on $A^p \otimes_R B^q$. And this is exactly the same as the rule for the differential on $A \otimes_R B$ in Definition 22.3.4. □

22.4. Differential graded modules

09JH Our default in this chapter is right modules; we discuss left modules in Section 22.11.

09JI Definition 22.4.1. Let R be a ring. Let (A, d) be a differential graded algebra over R. A (right) differential graded module M over A is a right A-module M which has a grading $M = \bigoplus M^n$ and a differential d such that $M^n A^m \subset M^{n+m}$, such that $\text{d}(M^n) \subset M^{n+1}$, and such that

$$\text{d}(ma) = \text{d}(m)a + (-1)^n m\text{d}(a)$$

for $a \in A$ and $m \in M^n$. A homomorphism of differential graded modules $f : M \to N$ is an A-module map compatible with gradings and differentials. The category of (right) differential graded A-modules is denoted $\text{Mod}_{(A,\text{d})}$.

Note that we can think of M as a cochain complex $M^\bullet$ of (right) R-modules. Namely, for $r \in R$ we have $\mathrm{d}(r) = 0$ and r maps to a degree 0 element of A, hence $\mathrm{d}(mr) = \mathrm{d}(m)r$.

The Leibniz rule relating differentials and multiplication on a differential graded R-module M over a differential graded R-algebra A exactly means that the multiplication map defines a map of cochain complexes

$$\mathrm{Tot}(M^\bullet \otimes_R A^\bullet) \to M^\bullet$$

Here $A^\bullet$ and $M^\bullet$ denote the underlying cochain complexes of A and M.

09JJ Lemma 22.4.2. Let (A, d) be a differential graded algebra. The category $\mathrm{Mod}_{(A,\mathrm{d})}$ is abelian and has arbitrary limits and colimits.

Proof. Kernels and cokernels commute with taking underlying A-modules. Similarly for direct sums and colimits. In other words, these operations in $\mathrm{Mod}_{(A,\mathrm{d})}$ commute with the forgetful functor to the category of A-modules. This is not the case for products and limits. Namely, if N_i, $i \in I$ is a family of differential graded A-modules, then the product $\prod N_i$ in $\mathrm{Mod}_{(A,\mathrm{d})}$ is given by setting $(\prod N_i)^n = \prod N_i^n$ and $\prod N_i = \bigoplus_n (\prod N_i)^n$. Thus we see that the product does commute with the forgetful functor to the category of graded A-modules. A category with products and equalizers has limits, see Categories, Lemma 4.14.11. □

Thus, if (A, d) is a differential graded algebra over R, then there is an exact functor

$$\mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Comp}(R)$$

of abelian categories. For a differential graded module M the cohomology groups $H^n(M)$ are defined as the cohomology of the corresponding complex of R-modules. Therefore, a short exact sequence $0 \to K \to L \to M \to 0$ of differential graded modules gives rise to a long exact sequence

09JK (22.4.2.1) $$H^n(K) \to H^n(L) \to H^n(M) \to H^{n+1}(K)$$

of cohomology modules, see Homology, Lemma 12.13.12.

Moreover, from now on we borrow all the terminology used for complexes of modules. For example, we say that a differential graded A-module M is acyclic if $H^k(M) = 0$ for all $k \in \mathbf{Z}$. We say that a homomorphism $M \to N$ of differential graded A-modules is a quasi-isomorphism if it induces isomorphisms $H^k(M) \to H^k(N)$ for all $k \in \mathbf{Z}$. And so on and so forth.

09JL Definition 22.4.3. Let (A, d) be a differential graded algebra. Let M be a differential graded module whose underlying complex of R-modules is $M^\bullet$. For any $k \in \mathbf{Z}$ we define the k-shifted module $M[k]$ as follows

(1) the underlying complex of R-modules of $M[k]$ is $M^\bullet[k]$, i.e., we have $M[k]^n = M^{n+k}$ and $\mathrm{d}_{M[k]} = (-1)^k \mathrm{d}_M$ and
(2) as A-module the multiplication

$$(M[k])^n \times A^m \longrightarrow (M[k])^{n+m}$$

is equal to the given multiplication $M^{n+k} \times A^m \to M^{n+k+m}$.

For a morphism $f : M \to N$ of differential graded A-modules we let $f[k] : M[k] \to N[k]$ be the map equal to f on underlying A-modules. This defines a functor $[k] : \mathrm{Mod}_{(A,\mathrm{d})} \to \mathrm{Mod}_{(A,\mathrm{d})}$.

Let us check that with this choice the Leibniz rule is satisfied. Let $x \in M[k]^n = M^{n+k}$ and $a \in A^m$ and denoting $\cdot_{M[k]}$ the product in $M[k]$ then we see

$$\begin{aligned} \mathrm{d}_{M[k]}(x \cdot_{M[k]} a) &= (-1)^k \mathrm{d}_M(xa) \\ &= (-1)^k \mathrm{d}_M(x)a + (-1)^{k+n+k} x\mathrm{d}(a) \\ &= \mathrm{d}_{M[k]}(x)a + (-1)^n x\mathrm{d}(a) \\ &= \mathrm{d}_{M[k]}(x) \cdot_{M[k]} a + (-1)^n x \cdot_{M[k]} \mathrm{d}(a) \end{aligned}$$

This is what we want as x has degree n as a homogeneous element of $M[k]$. We also observe that with these choices we may think of the multiplication map as the map of complexes

$$\mathrm{Tot}(M^\bullet[k] \otimes_R A^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R A^\bullet)[k] \to M^\bullet[k]$$

where the first arrow is More on Algebra, Section 15.72 (7) which in this case does not involve a sign. (In fact, we could have deduced that the Liebniz rule holds from this observation.)

The remarks in Homology, Section 12.14 apply. In particular, we will identify the cohomology groups of all shifts $M[k]$ without the intervention of signs.

At this point we have enough structure to talk about triangles, see Derived Categories, Definition 13.3.1. In fact, our next goal is to develop enough theory to be able to state and prove that the homotopy category of differential graded modules is a triangulated category. First we define the homotopy category.

22.5. The homotopy category

09JM Our homotopies take into account the A-module structure and the grading, but not the differential (of course).

09JN **Definition 22.5.1.** Let (A, d) be a differential graded algebra. Let $f, g : M \to N$ be homomorphisms of differential graded A-modules. A *homotopy between f and g* is an A-module map $h : M \to N$ such that

1. $h(M^n) \subset N^{n-1}$ for all n, and
2. $f(x) - g(x) = \mathrm{d}_N(h(x)) + h(\mathrm{d}_M(x))$ for all $x \in M$.

If a homotopy exists, then we say f and g are *homotopic*.

Thus h is compatible with the A-module structure and the grading but not with the differential. If $f = g$ and h is a homotopy as in the definition, then h defines a morphism $h : M \to N[-1]$ in $\mathrm{Mod}_{(A,\mathrm{d})}$.

09JP **Lemma 22.5.2.** Let (A, d) be a differential graded algebra. Let $f, g : L \to M$ be homomorphisms of differential graded A-modules. Suppose given further homomorphisms $a : K \to L$, and $c : M \to N$. If $h : L \to M$ is an A-module map which defines a homotopy between f and g, then $c \circ h \circ a$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.

Proof. Immediate from Homology, Lemma 12.13.7. ☐

This lemma allows us to define the homotopy category as follows.

09JQ Definition 22.5.3. Let (A, d) be a differential graded algebra. The homotopy category, denoted $K(\text{Mod}_{(A,\mathrm{d})})$, is the category whose objects are the objects of $\text{Mod}_{(A,\mathrm{d})}$ and whose morphisms are homotopy classes of homomorphisms of differential graded A-modules.

The notation $K(\text{Mod}_{(A,\mathrm{d})})$ is not standard but at least is consistent with the use of $K(-)$ in other places of the Stacks project.

09JR Lemma 22.5.4. Let (A, d) be a differential graded algebra. The homotopy category $K(\text{Mod}_{(A,\mathrm{d})})$ has direct sums and products.

Proof. Omitted. Hint: Just use the direct sums and products as in Lemma 22.4.2. This works because we saw that these functors commute with the forgetful functor to the category of graded A-modules and because $\prod$ is an exact functor on the category of families of abelian groups. □

22.6. Cones

09K9 We introduce cones for the category of differential graded modules.

09KA Definition 22.6.1. Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. The cone of f is the differential graded A-module $C(f)$ given by $C(f) = L \oplus K$ with grading $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d_{C(f)} = \begin{pmatrix} \mathrm{d}_L & f \\ 0 & -\mathrm{d}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L \to C(f)$ and $p : C(f) \to K[1]$ induced by the obvious maps $L \to C(f)$ and $C(f) \to K$.

The formation of the cone triangle is functorial in the following sense.

09KD Lemma 22.6.2. Let (A, d) be a differential graded algebra. Suppose that

$$\begin{tikzcd} K_1 \arrow[r, "f_1"'] \arrow[d, "a"'] & L_1 \arrow[d, "b"] \\ K_2 \arrow[r, "f_2"] & L_2 \end{tikzcd}$$

is a diagram of homomorphisms of differential graded A-modules which is commutative up to homotopy. Then there exists a morphism $c : C(f_1) \to C(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \to (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

in $K(\text{Mod}_{(A,\mathrm{d})})$.

Proof. Let $h : K_1 \to L_2$ be a homotopy between $f_2 \circ a$ and $b \circ f_1$. Define c by the matrix

$$c = \begin{pmatrix} b & h \\ 0 & a \end{pmatrix} : L_1 \oplus K_1 \to L_2 \oplus K_2$$

A matrix computation show that c is a morphism of differential graded modules. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. □

22.7. Admissible short exact sequences

09JS An admissible short exact sequence is the analogue of termwise split exact sequences in the setting of differential graded modules.

09JT Definition 22.7.1. Let (A, d) be a differential graded algebra.

(1) A homomorphism $K \to L$ of differential graded A-modules is an admissible monomorphism if there exists a graded A-module map $L \to K$ which is left inverse to $K \to L$.
(2) A homomorphism $L \to M$ of differential graded A-modules is an admissible epimorphism if there exists a graded A-module map $M \to L$ which is right inverse to $L \to M$.
(3) A short exact sequence $0 \to K \to L \to M \to 0$ of differential graded A-modules is an admissible short exact sequence if it is split as a sequence of graded A-modules.

Thus the splittings are compatible with all the data except for the differentials. Given an admissible short exact sequence we obtain a triangle; this is the reason that we require our splittings to be compatible with the A-module structure.

09JU Lemma 22.7.2. Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. Let $s : M \to L$ and $\pi : L \to K$ be splittings such that $\mathrm{Ker}(\pi) = \mathrm{Im}(s)$. Then we obtain a morphism

$$\delta = \pi \circ \mathrm{d}_L \circ s : M \to K[1]$$

of $\mathrm{Mod}_{(A,\mathrm{d})}$ which induces the boundary maps in the long exact sequence of cohomology (22.4.2.1).

Proof. The map $\pi \circ \mathrm{d}_L \circ s$ is compatible with the A-module structure and the gradings by construction. It is compatible with differentials by Homology, Lemmas 12.14.10. Let R be the ring that A is a differential graded algebra over. The equality of maps is a statement about R-modules. Hence this follows from Homology, Lemmas 12.14.10 and 12.14.11. $\square$

09JV Lemma 22.7.3. Let (A, d) be a differential graded algebra. Let

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "a"] & L \arrow[d, "b"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

be a diagram of homomorphisms of differential graded A-modules commuting up to homotopy.

(1) If f is an admissible monomorphism, then b is homotopic to a homomorphism which makes the diagram commute.
(2) If g is an admissible epimorphism, then a is homotopic to a morphism which makes the diagram commute.

Proof. Let $h : K \to N$ be a homotopy between bf and ga, i.e., $bf - ga = \mathrm{d}h + h\mathrm{d}$. Suppose that $\pi : L \to K$ is a graded A-module map left inverse to f. Take $b' = b - \mathrm{d}h\pi - h\pi\mathrm{d}$. Suppose $s : N \to M$ is a graded A-module map right inverse to g. Take $a' = a + \mathrm{d}sh + sh\mathrm{d}$. Computations omitted. $\square$

09JW **Lemma 22.7.4.** Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ be a homomorphism of differential graded A-modules. There exists a factorization

$$\begin{tikzcd} K \arrow[r, "\tilde{\alpha}"] \arrow[rr, bend right, "\alpha"'] & \tilde{L} \arrow[r, "\pi"] & L \end{tikzcd}$$

in $\text{Mod}_{(A,\mathrm{d})}$ such that

(1) $\tilde{\alpha}$ is an admissible monomorphism (see Definition 22.7.1),
(2) there is a morphism $s : L \to \tilde{L}$ such that $\pi \circ s = \mathrm{id}_L$ and such that $s \circ \pi$ is homotopic to $\mathrm{id}_{\tilde{L}}$.

Proof. The proof is identical to the proof of Derived Categories, Lemma 13.9.6. Namely, we set $\tilde{L} = L \oplus C(1_K)$ and we use elementary properties of the cone construction. $\square$

09JX **Lemma 22.7.5.** Let (A, d) be a differential graded algebra. Let $L_1 \to L_2 \to \ldots \to L_n$ be a sequence of composable homomorphisms of differential graded A-modules. There exists a commutative diagram

$$\begin{tikzcd} L_1 \arrow[r] & L_2 \arrow[r] & \ldots \arrow[r] & L_n \\ M_1 \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & \ldots \arrow[r] & M_n \arrow[u] \end{tikzcd}$$

in $\text{Mod}_{(A,\mathrm{d})}$ such that each $M_i \to M_{i+1}$ is an admissible monomorphism and each $M_i \to L_i$ is a homotopy equivalence.

Proof. The case $n = 1$ is without content. Lemma 22.7.4 is the case $n = 2$. Suppose we have constructed the diagram except for M_n. Apply Lemma 22.7.4 to the composition $M_{n-1} \to L_{n-1} \to L_n$. The result is a factorization $M_{n-1} \to M_n \to L_n$ as desired. $\square$

09JY **Lemma 22.7.6.** Let (A, d) be a differential graded algebra. Let $0 \to K_i \to L_i \to M_i \to 0$, $i = 1, 2, 3$ be admissible short exact sequence of differential graded A-modules. Let $b : L_1 \to L_2$ and $b' : L_2 \to L_3$ be homomorphisms of differential graded modules such that

$$\begin{tikzcd} K_1 \arrow[r] \arrow[d, "0"'] & L_1 \arrow[r] \arrow[d, "b"'] & M_1 \arrow[d, "0"'] \\ K_2 \arrow[r] & L_2 \arrow[r] & M_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K_2 \arrow[r] \arrow[d, "0"] & L_2 \arrow[r] \arrow[d, "b'"] & M_2 \arrow[d, "0"] \\ K_3 \arrow[r] & L_3 \arrow[r] & M_3 \end{tikzcd}$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 22.7.3 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\text{Im}(b) \subset \text{Im}(K_2 \to L_2)$ and $\text{Ker}((b')^n) \supset \text{Im}(K_2 \to L_2)$. Then $b \circ b' = 0$ as a map of modules. $\square$

22.8. Distinguished triangles

09K5 The following lemma produces our distinguished triangles.

09K6 Lemma 22.8.1. Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. The triangle

09K7 (22.8.1.1) $$K \to L \to M \xrightarrow{\delta} K[1]$$

with δ as in Lemma 22.7.2 is, up to canonical isomorphism in $K(\text{Mod}_{(A,\mathrm{d})})$, independent of the choices made in Lemma 22.7.2.

Proof. Namely, let (s', π') be a second choice of splittings as in Lemma 22.7.2. Then we claim that δ and δ' are homotopic. Namely, write $s' = s + \alpha \circ h$ and $\pi' = \pi + g \circ \beta$ for some unique homomorphisms of A-modules $h : M \to K$ and $g : M \to K$ of degree -1. Then $g = -h$ and g is a homotopy between δ and δ'. The computations are done in the proof of Homology, Lemma 12.14.12. □

09K8 Definition 22.8.2. Let (A, d) be a differential graded algebra.

(1) If $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded A-modules, then the triangle associated to $0 \to K \to L \to M \to 0$ is the triangle (22.8.1.1) of $K(\text{Mod}_{(A,\mathrm{d})})$.
(2) A triangle of $K(\text{Mod}_{(A,\mathrm{d})})$ is called a distinguished triangle if it is isomorphic to a triangle associated to an admissible short exact sequence of differential graded A-modules.

22.9. Cones and distinguished triangles

09P1 Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. Then $(K, L, C(f), f, i, p)$ forms a triangle:

$$K \to L \to C(f) \to K[1]$$

in $\text{Mod}_{(A,\mathrm{d})}$ and hence in $K(\text{Mod}_{(A,\mathrm{d})})$. Cones are not distinguished triangles in general, but the difference is a sign or a rotation (your choice). Here are two precise statements.

09KB Lemma 22.9.1. Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded modules. The triangle $(L, C(f), K[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence

$$0 \to L \to C(f) \to K[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. □

09KC Lemma 22.9.2. Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ and $\beta : L \to M$ define an admissible short exact sequence

$$0 \to K \to L \to M \to 0$$

of differential graded A-modules. Let $(K, L, M, \alpha, \beta, \delta)$ be the associated triangle. Then the triangles

$$(M[-1], K, L, \delta[-1], \alpha, \beta) \quad \text{and} \quad (M[-1], K, C(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. Using a choice of splittings we write $L = K \oplus M$ and we identify α and β with the natural inclusion and projection maps. By construction of δ we have

$$d_B = \begin{pmatrix} d_K & \delta \\ 0 & d_M \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : M[-1] \to K$ is given as $C(\delta[-1]) = K \oplus M$ with differential identical with the matrix above! Whence the lemma. $\square$

09KE Lemma 22.9.3. Let (A, d) be a differential graded algebra. Let $f_1 : K_1 \to L_1$ and $f_2 : K_2 \to L_2$ be homomorphisms of differential graded A-modules. Let

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \longrightarrow (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

be any morphism of triangles of $K(\text{Mod}_{(A,\text{d})})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2 \to K_1$ be a homomorphism of differential graded A-modules which is inverse to a in $K(\text{Mod}_{(A,\text{d})})$. Let $b^{-1} : L_2 \to L_1$ be a homomorphism of differential graded A-modules which is inverse to b in $K(\text{Mod}_{(A,\text{d})})$. Let $c' : C(f_2) \to C(f_1)$ be the morphism from Lemma 22.6.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\text{Mod}_{(A,\text{d})})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K, L, C(f), f, i, p)$ in $K(\text{Mod}_{(A,\text{d})})$ the morphism c is an isomorphism in $K(\text{Mod}_{(A,\text{d})})$. By assumption the two squares in the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "1"'] & C(f) \arrow[r] \arrow[d, "c"'] & K[1] \arrow[d, "1"'] \\ L \arrow[r] & C(f) \arrow[r] & K[1] \end{tikzcd}$$

commute up to homotopy. By construction of $C(f)$ the rows form admissible short exact sequences. Thus we see that $(c - 1)^2 = 0$ in $K(\text{Mod}_{(A,\text{d})})$ by Lemma 22.7.6. Hence c is an isomorphism in $K(\text{Mod}_{(A,\text{d})})$ with inverse $2 - c$. $\square$

The following lemma shows that the collection of triangles of the homotopy category given by cones and the distinguished triangles are the same up to isomorphisms, at least up to sign!

09KF Lemma 22.9.4. Let (A, d) be a differential graded algebra.

(1) Given an admissible short exact sequence $0 \to K \xrightarrow{\alpha} L \to M \to 0$ of differential graded A-modules there exists a homotopy equivalence $C(\alpha) \to M$ such that the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & C(\alpha) \arrow[r, "-p"'] \arrow[d] & K[1] \arrow[d] \\ K \arrow[r, "\alpha"] & L \arrow[r, "\beta"] & M \arrow[r, "\delta"] & K[1] \end{tikzcd}$$

defines an isomorphism of triangles in $K(\text{Mod}_{(A,\text{d})})$.

(2) Given a morphism of complexes $f : K \to L$ there exists an isomorphism of triangles

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \tilde{L} \arrow[r] \arrow[d] & M \arrow[r, "\delta"'] \arrow[d] & K[1] \arrow[d] \\ K \arrow[r] & L \arrow[r] & C(f) \arrow[r, "-p"] & K[1] \end{tikzcd}$$

where the upper triangle is the triangle associated to a admissible short exact sequence $K \to \tilde{L} \to M$.

Proof. Proof of (1). We have $C(\alpha) = L \oplus K$ and we simply define $C(\alpha) \to M$ via the projection onto L followed by β. This defines a morphism of differential graded modules because the compositions $K^{n+1} \to L^{n+1} \to M^{n+1}$ are zero. Choose splittings $s : M \to L$ and $\pi : L \to K$ with $\text{Ker}(\pi) = \text{Im}(s)$ and set $\delta = \pi \circ \text{d}_L \circ s$ as usual. To get a homotopy inverse we take $M \to C(\alpha)$ given by $(s, -\delta)$. This is compatible with differentials because δ^n can be characterized as the unique map $M^n \to K^{n+1}$ such that $\text{d} \circ s^n - s^{n+1} \circ \text{d} = \alpha \circ \delta^n$, see proof of Homology, Lemma 12.14.10. The composition $M \to C(f) \to M$ is the identity. The composition $C(f) \to M \to C(f)$ is equal to the morphism

$$\begin{pmatrix} s \circ \beta & 0 \\ -\delta \circ \beta & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h : C(\alpha) \to C(\alpha)$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} : C(\alpha) = L \oplus K \to L \oplus K = C(\alpha)$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s \\ -\delta \end{pmatrix} \begin{pmatrix} \beta & 0 \end{pmatrix} = \begin{pmatrix} \text{d} & \alpha \\ 0 & -\text{d} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \begin{pmatrix} \text{d} & \alpha \\ 0 & -\text{d} \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha) \to K[1]$ (see Definition 22.6.1) and $C(\alpha) \to M \to K[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : M \to C(\alpha)$ and check instead that the two maps $M \to K[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K \to \tilde{L}$, $s : L \to \tilde{L}$ and $\pi : L \to L$ be as in Lemma 22.7.4. By Lemmas 22.6.2 and 22.9.3 the triangles $(K, L, C(f), i, p)$ and $(K, \tilde{L}, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace L by $\tilde{L}$ and f by $\tilde{f}$. In other words we may assume that f is an admissible monomorphism. In this case the result follows from part (1). □

22.10. The homotopy category is triangulated

09KG We first prove that it is pre-triangulated.

09KH Lemma 22.10.1. Let (A, d) be a differential graded algebra. The homotopy category $K(\text{Mod}_{(A,\text{d})})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(K, K, 0, 1, 0, 0)$ is distinguished since $0 \to K \to K \to 0 \to 0$ is an admissible short exact sequence. Finally, given any homomorphism $f : K \to L$ of differential graded A-modules the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 22.9.4.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to K \to L \to M \to 0$ such that the associated triangle $(K, L, M, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, L, -\delta[-1], \alpha, \beta)$. It follows from Lemma 22.9.2 that the triangle $(M[-1], K, L, \delta[-1], \alpha, \beta)$ is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, -p)$. Hence it is distinguished by Lemma 22.9.4. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 22.9.4 this means that it is isomorphic to a triangle of the form $(K, L, C(f), f, i, -p)$ for some morphism f of $\text{Mod}_{(A,\text{d})}$. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L, C(f), K[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L, C(f), K[1], i, p, f[1])$. By Lemma 22.9.1 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ f$. By Lemma 22.9.4 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 22.6.2 to the commutative diagram given by f, f', a, b. $\square$

Before we prove TR4 in general we prove it in a special case.

09KI Lemma 22.10.2. Let (A, d) be a differential graded algebra. Suppose that $\alpha : K \to L$ and $\beta : L \to M$ are admissible monomorphisms of differential graded A-modules. Then there exist distinguished triangles $(K, L, Q_1, \alpha, p_1, d_1)$, $(K, M, Q_2, \beta \circ \alpha, p_2, d_2)$ and $(L, M, Q_3, \beta, p_3, d_3)$ for which TR4 holds.

Proof. Say $\pi_1 : L \to K$ and $\pi_3 : M \to L$ are homomorphisms of graded A-modules which are left inverse to α and β. Then also $K \to M$ is an admissible monomorphism with left inverse $\pi_2 = \pi_1 \circ \pi_3$. Let us write Q_1, Q_2 and Q_3 for the cokernels of $K \to L$, $K \to M$, and $L \to M$. Then we obtain identifications (as graded A-modules) $Q_1 = \text{Ker}(\pi_1)$, $Q_3 = \text{Ker}(\pi_3)$ and $Q_2 = \text{Ker}(\pi_2)$. Then $L = K \oplus Q_1$ and $M = L \oplus Q_3$ as graded A-modules. This implies $M = K \oplus Q_1 \oplus Q_3$. Note that $\pi_2 = \pi_1 \circ \pi_3$ is zero on both Q_1 and Q_3. Hence $Q_2 = Q_1 \oplus Q_3$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & K & \to & L & \to & Q_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & M & \to & Q_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M & \to & Q_3 & \to & 0
\end{array}
$$

The rows of this diagram are admissible short exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(K \to L \to Q_1 \to K[1]) \longrightarrow (K \to M \to Q_2 \to K[1])$$

and

$$(K \to M \to Q_2 \to K[1]) \longrightarrow (L \to M \to Q_3 \to L[1]).$$

Note that the splittings $Q_3 \to M$ of the bottom sequence in the diagram provides a splitting for the split sequence $0 \to Q_1 \to Q_2 \to Q_3 \to 0$ upon composing with $M \to Q_2$. It follows easily from this that the morphism $\delta : Q_3 \to Q_1[1]$ in the corresponding distinguished triangle

$$(Q_1 \to Q_2 \to Q_3 \to Q_1[1])$$

is equal to the composition $Q_3 \to L[1] \to Q_1[1]$. Hence we get a structure as in the conclusion of axiom TR4. □

Here is the final result.

09KJ Proposition 22.10.3. Let (A, d) be a differential graded algebra. The homotopy category $K(\text{Mod}_{(A,\text{d})})$ of differential graded A-modules with its natural translation functors and distinguished triangles is a triangulated category.

Proof. We know that $K(\text{Mod}_{(A,\text{d})})$ is a pre-triangulated category. Hence it suffices to prove TR4 and to prove it we can use Derived Categories, Lemma 13.4.15. Let $K \to L$ and $L \to M$ be composable morphisms of $K(\text{Mod}_{(A,\text{d})})$. By Lemma 22.7.5 we may assume that $K \to L$ and $L \to M$ are admissible monomorphisms. In this case the result follows from Lemma 22.10.2. □

22.11. Left modules

0FPZ Everything we have said sofar has an analogue in the setting of left differential graded modules, except that one has to take care with some sign rules.

Let (A, d) be a differential graded R-algebra. Exactly analogous to right modules, we define a left differential graded A-module M as a left A-module M which has a grading $M = \bigoplus M^n$ and a differential d, such that $A^n M^m \subset M^{n+m}$, such that $\text{d}(M^n) \subset M^{n+1}$, and such that

$$\text{d}(am) = \text{d}(a)m + (-1)^{\deg(a)} a\text{d}(m)$$

for homogeneous elements $a \in A$ and $m \in M$. As before this Leibniz rule exactly signifies that the multiplication defines a map of complexes

$$\text{Tot}(A^\bullet \otimes_R M^\bullet) \to M^\bullet$$

Here $A^\bullet$ and $M^\bullet$ denote the complexes of R-modules underlying A and M.

09JG Definition 22.11.1. Let R be a ring. Let (A, d) be a differential graded algebra over R. The opposite differential graded algebra is the differential graded algebra (A^{opp}, d) over R where $A^{opp} = A$ as a graded R-module, $\text{d} = \text{d}$, and multiplication is given by

$$a \cdot_{opp} b = (-1)^{\deg(a)\deg(b)} ba$$

for homogeneous elements $a, b \in A$.

This makes sense because

$$\begin{aligned} d(a \cdot_{opp} b) &= (-1)^{\deg(a)\deg(b)} d(ba) \\ &= (-1)^{\deg(a)\deg(b)} d(b)a + (-1)^{\deg(a)\deg(b)+\deg(b)} b d(a) \\ &= (-1)^{\deg(a)} a \cdot_{opp} d(b) + d(a) \cdot_{opp} b \end{aligned}$$

as desired. In terms of underlying complexes of R-modules this means that the diagram

$$\begin{tikzcd} \mathrm{Tot}(A^\bullet \otimes_R A^\bullet) \arrow[r, "\text{multiplication of } A^{opp}"'] \arrow[d, "\text{commutativity constraint}"'] & A^\bullet \arrow[d, "\mathrm{id}"] \\ \mathrm{Tot}(A^\bullet \otimes_R A^\bullet) \arrow[r, "\text{multiplication of } A"] & A^\bullet \end{tikzcd}$$

commutes. Here the commutativity constraint on the symmetric monoidal category of complexes of R-modules is given in More on Algebra, Section 15.72.

Let (A, d) be a differential graded algebra over R. Let M be a left differential graded A-module. We will denote M^{opp} the module M viewed as a right A^{opp}-module with multiplication $\cdot_{opp}$ defined by the rule

$$m \cdot_{opp} a = (-1)^{\deg(a)\deg(m)} am$$

for a and m homogeneous. This is compatible with differentials because we could have used the diagram

$$\begin{tikzcd} \mathrm{Tot}(M^\bullet \otimes_R A^\bullet) \arrow[r, "\text{multiplication on } M^{opp}"'] \arrow[d, "\text{commutativity constraint}"'] & M^\bullet \arrow[d, "\mathrm{id}"] \\ \mathrm{Tot}(A^\bullet \otimes_R M^\bullet) \arrow[r, "\text{multiplication on } M"] & M^\bullet \end{tikzcd}$$

to define the multiplication $\cdot_{opp}$ on M^{opp}. To see that it is an associative multiplication we compute for homogeneous elements $a, b \in A$ and $m \in M$ that

$$\begin{aligned} m \cdot_{opp} (a \cdot_{opp} b) &= (-1)^{\deg(a)\deg(b)} m \cdot_{opp} (ba) \\ &= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)} bam \\ &= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)+\deg(b)\deg(am)} (am) \cdot_{opp} b \\ &= (-1)^{\deg(a)\deg(b)+\deg(ab)\deg(m)+\deg(b)\deg(am)+\deg(a)\deg(m)} (m \cdot_{opp} a) \cdot_{opp} b \\ &= (m \cdot_{opp} a) \cdot_{opp} b \end{aligned}$$

Of course, we could have been shown this using the compatibility between the associativity and commutativity constraint on the symmetric monoidal category of complexes of R-modules as well.

0FQ0 **Lemma 22.11.2.** Let (A, d) be a differential graded R-algebra. The functor $M \mapsto M^{opp}$ from the category of left differential graded A-modules to the category of right differential graded A^{opp}-modules is an equivalence.

Proof. Omitted. $\square$

Mext, we come to shifts. Let (A, d) be a differential graded algebra. Let M be a left differential graded A-module whose underlying complex of R-modules is denoted $M^\bullet$. For any $k \in \mathbf{Z}$ we define the k-shifted module $M[k]$ as follows

(1) the underlying complex of R-modules of $M[k]$ is $M^\bullet[k]$

(2) as A-module the multiplication
$$A^n \times (M[k])^m \longrightarrow (M[k])^{n+m}$$
is equal to $(-1)^{nk}$ times the given multiplication $A^n \times M^{m+k} \to M^{n+m+k}$.

Let us check that with this choice the Leibniz rule is satisfied. Let $a \in A^n$ and $x \in M[k]^m = M^{m+k}$ and denoting $\cdot_{M[k]}$ the product in $M[k]$ then we see

$$\begin{aligned}
\mathrm{d}_{M[k]}(a \cdot_{M[k]} x) &= (-1)^{k+nk}\mathrm{d}_M(ax) \\
&= (-1)^{k+nk}\mathrm{d}(a)x + (-1)^{k+nk+n}a\mathrm{d}_M(x) \\
&= \mathrm{d}(a) \cdot_{M[k]} x + (-1)^{nk+n}a\mathrm{d}_{M[k]}(x) \\
&= \mathrm{d}(a) \cdot_{M[k]} x + (-1)^n a \cdot_{M[k]} \mathrm{d}_{M[k]}(x)
\end{aligned}$$

This is what we want as a has degree n as a homogeneous element of A. We also observe that with these choices we may think of the multiplication map as the map of complexes

$$\mathrm{Tot}(A^\bullet \otimes_R M^\bullet[k]) \to \mathrm{Tot}(A^\bullet \otimes_R M^\bullet)[k] \to M^\bullet[k]$$

where the first arrow is More on Algebra, Section 15.72 (7) which in this case involves exactly the sign we chose above. (In fact, we could have deduced that the Liebniz rule holds from this observation.)

With the rule above we have canonical identifications

$$(M[k])^{opp} = M^{opp}[k]$$

of right differential graded A^{opp}-modules defined without the intervention of signs, in other words, the equivalence of Lemma 22.11.2 is compatible with shift functors.

Our choice above necessitates the following definition.

0FQ1 **Definition 22.11.3.** Let R be a ring. Let A be a $\mathbf{Z}$-graded R-algebra.

(1) Given a right graded A-module M we define the kth shifted A-module $M[k]$ as the same as a right A-module but with grading $(M[k])^n = M^{n+k}$.
(2) Given a left graded A-module M we define the kth shifted A-module $M[k]$ as the module with grading $(M[k])^n = M^{n+k}$ and multiplication $A^n \times (M[k])^m \to (M[k])^{n+m}$ equal to $(-1)^{nk}$ times the given multiplication $A^n \times M^{m+k} \to M^{n+m+k}$.

Let (A, d) be a differential graded algebra. Let $f, g : M \to N$ be homomorphisms of left differential graded A-modules. A homotopy between f and g is a graded A-module map $h : M \to N[-1]$ (observe the shift!) such that

$$f(x) - g(x) = \mathrm{d}_N(h(x)) + h(\mathrm{d}_M(x))$$

for all $x \in M$. If a homotopy exists, then we say f and g are homotopic. Thus h is compatible with the A-module structure (with the shifted one on N) and the grading (with shifted grading on N) but not with the differential. If $f = g$ and h is a homotopy, then h defines a morphism $h : M \to N[-1]$ of left differential graded A-modules.

With the rule above we find that $f, g : M \to N$ are homotopic if and only if the induced morphisms $f^{opp}, g^{opp} : M^{opp} \to N^{opp}$ are homotopic as right differential graded A^{opp}-module homomorphisms (with the same homotopy).

The homotopy category, cones, admissible short exact sequences, distinguished triangles are all defined in exactly the same manner as for right differential graded modules (and everything agrees on underlying complexes of R-modules with the constructions for complexes of R-modules). In this manner we obtain the analogue of Proposition 22.10.3 for left modules as well, or we can deduce it by working with right modules over the opposite algebra.

22.12. Tensor product

09LL Let R be a ring. Let A be an R-algebra (see Section 22.2). Given a right A-module M and a left A-module N there is a tensor product

$$M \otimes_A N$$

This tensor product is a module over R. As an R-module $M \otimes_A N$ is generated by symbols $x \otimes y$ with $x \in M$ and $y \in N$ subject to the relations

$$\begin{gathered}(x_1 + x_2) \otimes y - x_1 \otimes y - x_2 \otimes y, \\ x \otimes (y_1 + y_2) - x \otimes y_1 - x \otimes y_2, \\ xa \otimes y - x \otimes ay\end{gathered}$$

for $a \in A$, $x, x_1, x_2 \in M$ and $y, y_1, y_2 \in N$. We list some properties of the tensor product

In each variable the tensor product is right exact, in fact commutes with direct sums and arbitrary colimits.

The tensor product $M \otimes_A N$ is the receptacle of the universal A-bilinear map $M \times N \to M \otimes_A N$, $(x, y) \mapsto x \otimes y$. In a formula

$$\text{Bilinear}_A(M \times N, Q) = \text{Hom}_R(M \otimes_A N, Q)$$

for any R-module Q.

If A is a $\mathbf{Z}$-graded algebra and M, N are graded A-modules then $M \otimes_A N$ is a graded R-module. Then nth graded piece $(M \otimes_A N)^n$ of $M \otimes_A N$ is equal to

$$\text{Coker}\left(\bigoplus\nolimits_{r+t+s=n} M^r \otimes_R A^t \otimes_R N^s \to \bigoplus\nolimits_{p+q=n} M^p \otimes_R N^q\right)$$

where the map sends $x \otimes a \otimes y$ to $x \otimes ay - xa \otimes y$ for $x \in M^r$, $y \in N^s$, and $a \in A^t$ with $r + s + t = n$. In this case the map $M \times N \to M \otimes_A N$ is A-bilinear and compatible with gradings and universal in the sense that

$$\text{GradedBilinear}_A(M \times N, Q) = \text{Hom}_{\text{graded } R\text{-modules}}(M \otimes_A N, Q)$$

for any graded R-module Q with an obvious notion of graded bilinar map.

If (A, d) is a differential graded algebra and M and N are left and right differential graded A-modules, then $M \otimes_A N$ is a differential graded R-module with differential

$$\text{d}(x \otimes y) = \text{d}(x) \otimes y + (-1)^{\deg(x)} x \otimes \text{d}(y)$$

for $x \in M$ and $y \in N$ homogeneous. In this case the map $M \times N \to M \otimes_A N$ is A-bilinear, compatible with gradings, and compatible with differentials and universal in the sense that

$$\text{DifferentialGradedBilinear}_A(M \times N, Q) = \text{Hom}_{\text{Comp}(R)}(M \otimes_A N, Q)$$

for any differential graded R-module Q with an obvious notion of differential graded bilinar map.

22.13. Hom complexes and differential graded modules

0FQ2 We urge the reader to skip this section.

Let R be a ring and let $M^\bullet$ be a complex of R-modules. Consider the complex of R-modules

$$E^\bullet = \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$$

introduced in More on Algebra, Section 15.71. By More on Algebra, Lemma 15.71.3 there is a canonical composition law

$$\mathrm{Tot}(E^\bullet \otimes_R E^\bullet) \to E^\bullet$$

which is a map of complexes. Thus we see that $E^\bullet$ with this multiplication is a differential graded R-algebra which we will denote (E, d). Moreover, viewing $M^\bullet$ as $\mathrm{Hom}^\bullet(R, M^\bullet)$ we see that composition defines a multiplication

$$\mathrm{Tot}(E^\bullet \otimes_R M^\bullet) \to M^\bullet$$

which turns $M^\bullet$ into a left differential graded E-module which we will denote M.

0FQ3 **Lemma 22.13.1.** *In the situation above, let A be a differential graded R-algebra. To give a left A-module structure on M is the same thing as giving a homomorphism $A \to E$ of differential graded R-algebras.*

Proof. Proof omitted. Observe that no signs intervene in this correspondence. $\square$

We continue with the discussion above and we assume given another complex $N^\bullet$ of R-modules. Consider the complex of R-modules $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ introduced in More on Algebra, Section 15.71. As above we see that composition

$$\mathrm{Tot}(\mathrm{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R E^\bullet) \to \mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$$

defines a multiplication which turns $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ into a right differential graded E-module. Using Lemma 22.13.1 we conclude that given a left differential graded A-module M and a complex of R-modules $N^\bullet$ there is a canonical right differential graded A-module whose underlying complex of R-modules is $\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)$ and where multiplication

$$\mathrm{Hom}^n(M^\bullet, N^\bullet) \times A^m \longrightarrow \mathrm{Hom}^{n+m}(M^\bullet, N^\bullet)$$

sends $f = (f_{p,q})_{p+q=n}$ with $f_{p,q} \in \mathrm{Hom}(M^{-q}, N^p)$ and $a \in A^m$ to the element $f \cdot a = (f_{p,q} \circ a)$ where $f_{p,q} \circ a$ is the map

$$M^{-q-m} \xrightarrow{a} M^{-q} \xrightarrow{f_{p,q}} N^p, \quad x \longmapsto f_{p,q}(ax)$$

without the intervention of signs. Let us use the notation $\mathrm{Hom}(M, N^\bullet)$ to denote this right differential graded A-module.

0FQ4 **Lemma 22.13.2.** *Let R be a ring. Let (A, d) be a differential graded R-algebra. Let M' be a right differential graded A-module and let M be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. Then we have*

$$\mathrm{Hom}_{\mathrm{Mod}_{(A,d)}}(M', \mathrm{Hom}(M, N^\bullet)) = \mathrm{Hom}_{\mathrm{Comp}(R)}(M' \otimes_A M, N^\bullet)$$

where $M \otimes_A M$ is viewed as a complex of R-modules as in Section 22.12.

Proof. Let us show that both sides correspond to graded A-bilinear maps

$$M' \times M \longrightarrow N^\bullet$$

compatible with differentials. We have seen this is true for the right hand side in Section 22.12. Given an element g of the left hand side, the equality of More on Algebra, Lemma 15.71.1 determines a map of complexes of R-modules $g' : \text{Tot}(M' \otimes_R M) \to N^\bullet$. In other words, we obtain a graded R-bilinear map $g'' : M' \times M \to N^\bullet$ compatible with differentials. The A-linearity of g translates immediately into A-bilinarity of g''. $\square$

Let R, $M^\bullet$, $E^\bullet$, E, and M be as above. However, now suppose given a differential graded R-algebra A and a right differential graded A-module structure on M. Then we can consider the map

$$\text{Tot}(A^\bullet \otimes_R M^\bullet) \xrightarrow{\psi} \text{Tot}(A^\bullet \otimes_R M^\bullet) \to M^\bullet$$

where the first arrow is the commutativity constraint on the differential graded category of complexes of R-modules. This corresponds to a map

$$\tau : A^\bullet \longrightarrow E^\bullet$$

of complexes of R-modules. Recall that $E^n = \prod_{p+q=n} \text{Hom}_R(M^{-q}, M^p)$ and write $\tau(a) = (\tau_{p,q}(a))_{p+q=n}$ for $a \in A^n$. Then we see

$$\tau_{p,q}(a) : M^{-q} \longrightarrow M^p, \quad x \longmapsto (-1)^{\deg(a)\deg(x)} xa = (-1)^{-nq} xa$$

This is not compatible with the product on A as the readed should expect from the discussion in Section 22.11. Namely, we have

$$\tau(aa') = (-1)^{\deg(a)\deg(a')} \tau(a')\tau(a)$$

We conclude the following lemma is true

0FQ5 Lemma 22.13.3. In the situation above, let A be a differential graded R-algebra. To give a right A-module structure on M is the same thing as giving a homomorphism $\tau : A \to E^{opp}$ of differential graded R-algebras.

Proof. See discussion above and note that the construction of τ from the multiplication map $M^n \times A^m \to M^{n+m}$ uses signs. $\square$

Let R, $M^\bullet$, $E^\bullet$, E, A and M be as above and let a right differential graded A-module structure on M be given as in the lemma. In this case there is a canonical left differential graded A-module whose underlying complex of R-modules is $\text{Hom}^\bullet(M^\bullet, N^\bullet)$. Namely, for multiplication we can use

$$\begin{aligned} \text{Tot}(A^\bullet \otimes_R \text{Hom}^\bullet(M^\bullet, N^\bullet)) &\xrightarrow{\psi} \text{Tot}(\text{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R A^\bullet) \\ &\xrightarrow{\tau} \text{Tot}(\text{Hom}^\bullet(M^\bullet, N^\bullet) \otimes_R \text{Hom}^\bullet(M^\bullet, M^\bullet)) \\ &\to \text{Tot}(\text{Hom}^\bullet(M^\bullet, N^\bullet) \end{aligned}$$

The first arrow uses the commutativity constraint on the category of complexes of R-modules, the second arrow is described above, and the third arrow is the composition law for the Hom complex. Each map is a map of complexes, hence the result is a map of complexes. In fact, this construction turns $\text{Hom}^\bullet(M^\bullet, N^\bullet)$ into a left differential graded A-module (associativity of the multiplication can be shown using the symmetric monoidal structure or by a direct calculation using the formulae below). Let us explicate the multiplication

$$A^n \times \mathrm{Hom}^m(M^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}^{n+m}(M^\bullet, N^\bullet)$$

It sends $a \in A^n$ and $f = (f_{p,q})_{p+q=m}$ with $f_{p,q} \in \mathrm{Hom}(M^{-q}, N^p)$ to the element $a \cdot f$ with constituents

$$(-1)^{nm} f_{p,q} \circ \tau_{-q,q+n}(a) = (-1)^{nm-n(q+n)} f_{p,q} \circ a = (-1)^{np+n} f_{p,q} \circ a$$

in $\mathrm{Hom}_R(M^{-q-n}, N^p)$ where $f_{p,q} \circ a$ is the map

$$M^{-q-n} \xrightarrow{a} M^{-q} \xrightarrow{f_{p,q}} N^p, \quad x \longmapsto f_{p,q}(xa)$$

Here a sign of $(-1)^{np+n}$ does intervene. Let us use the notation $\mathrm{Hom}(M, N^\bullet)$ to denote this left differential graded A-module.

0FQ6 Lemma 22.13.4. Let R be a ring. Let (A, d) be a differential graded R-algebra. Let M be a right differential graded A-module and let M' be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. Then we have

$$\mathrm{Hom}_{\text{left diff graded } A\text{-modules}}(M', \mathrm{Hom}(M, N^\bullet)) = \mathrm{Hom}_{\mathrm{Comp}(R)}(M \otimes_A M', N^\bullet)$$

where $M \otimes_A M'$ is viewed as a complex of R-modules as in Section 22.12.

Proof. Let us show that both sides correspond to graded A-bilinear maps

$$M \times M' \longrightarrow N^\bullet$$

compatible with differentials. We have seen this is true for the right hand side in Section 22.12. Given an element g of the left hand side, the equality of More on Algebra, Lemma 15.71.1 determines a map of complexes $g' : \mathrm{Tot}(M' \otimes_R M) \to N^\bullet$. We precompose with the commutativity constraint to get

$$\mathrm{Tot}(M \otimes_R M') \xrightarrow{\psi} \mathrm{Tot}(M' \otimes_R M) \xrightarrow{g'} N^\bullet$$

which corresponds to a graded R-bilinear map $g'' : M \times M' \to N^\bullet$ compatible with differentials. The A-linearity of g translates immediately into A-bilinarity of g''. Namely, say $x \in M^e$ and $x' \in (M')^{e'}$ and $a \in A^n$. Then on the one hand we have

$$\begin{aligned} g''(x, ax') &= (-1)^{e(n+e')} g'(ax' \otimes x) \\ &= (-1)^{e(n+e')} g(ax')(x) \\ &= (-1)^{e(n+e')} (a \cdot g(x'))(x) \\ &= (-1)^{e(n+e') + n(n+e+e') + n} g(x')(xa) \end{aligned}$$

and on the other hand we have

$$g''(xa, x') = (-1)^{(e+n)e'} g'(x' \otimes xa) = (-1)^{(e+n)e'} g(x')(xa)$$

which is the same thing by a trivial mod 2 calculation of the exponents. $\square$

0FQ7 Remark 22.13.5. Let R be a ring. Let A be a differential graded R-algebra. Let M be a left differential graded A-module. Let $N^\bullet$ be a complex of R-modules. The constructions above produce a right differential graded A-module $\mathrm{Hom}(M, N^\bullet)$ and then a leftt differential graded A-module $\mathrm{Hom}(\mathrm{Hom}(M, N^\bullet), N^\bullet)$. We claim there is an evaluation map

$$ev : M \longrightarrow \mathrm{Hom}(\mathrm{Hom}(M, N^\bullet), N^\bullet)$$

in the category of left differential graded A-modules. To define it, by Lemma 22.13.2 it suffices to construct an A-bilinear pairing

$$\mathrm{Hom}(M, N^\bullet) \times M \longrightarrow N^\bullet$$

compatible with grading and differentials. For this we take

$$(f, x) \longmapsto f(x)$$

We leave it to the reader to verify this is compatible with grading, differentials, and A-bilinear. The map ev on underlying complexes of R-modules is More on Algebra, Item (17).

0FQ8 Remark 22.13.6. Let R be a ring. Let A be a differential graded R-algebra. Let M be a right differential graded A-module. Let $N^\bullet$ be a complex of R-modules. The constructions above produce a left differential graded A-module $\mathrm{Hom}(M, N^\bullet)$ and then a right differential graded A-module $\mathrm{Hom}(\mathrm{Hom}(M, N^\bullet), N^\bullet)$. We claim there is an evaluation map

$$ev : M \longrightarrow \mathrm{Hom}(\mathrm{Hom}(M, N^\bullet), N^\bullet)$$

in the category of right differential graded A-modules. To define it, by Lemma 22.13.2 it suffices to construct an A-bilinear pairing

$$M \times \mathrm{Hom}(M, N^\bullet) \longrightarrow N^\bullet$$

compatible with grading and differentials. For this we take

$$(x, f) \longmapsto (-1)^{\deg(x)\deg(f)} f(x)$$

We leave it to the reader to verify this is compatible with grading, differentials, and A-bilinear. The map ev on underlying complexes of R-modules is More on Algebra, Item (17).

0FQ9 Remark 22.13.7. Let R be a ring. Let A be a differential graded R-algebra. Let $M^\bullet$ and $N^\bullet$ be complexes of R-modules. Let $k \in \mathbf{Z}$ and consider the isomorphism

$$\mathrm{Hom}^\bullet(M^\bullet, N^\bullet)[-k] \longrightarrow \mathrm{Hom}^\bullet(M^\bullet[k], N^\bullet)$$

of complexes of R-modules defined in More on Algebra, Item (18). If $M^\bullet$ has the structure of a left, resp. right differential graded A-module, then this is a map of right, resp. left differential graded A-modules (with the module structures as defined in this section). We omit the verification; we warn the reader that the A-module structure on the shift of a left graded A-module is defined using a sign, see Definition 22.11.3.

22.14. Projective modules over algebras

09JZ In this section we discuss projective modules over algebras analogous to Algebra, Section 10.77. This section should probably be moved somewhere else.

Let R be a ring and let A be an R-algebra, see Section 22.2 for our conventions. It is clear that A is a projective right A-module since $\mathrm{Hom}_A(A, M) = M$ for any right A-module M (and thus $\mathrm{Hom}_A(A, -)$ is exact). Conversely, let P be a projective right A-module. Then we can choose a surjection $\bigoplus_{i \in I} A \to P$ by choosing a set $\{p_i\}_{i \in I}$ of generators of P over A. Since P is projective there is a left inverse to the surjection, and we find that P is isomorphic to a direct summand of a free module, exactly as in the commutative case (Algebra, Lemma 10.77.2).

We conclude

(1) the category of A-modules has enough projectives,
(2) A is a projective A-module,
(3) every A-module is a quotient of a direct sum of copies of A,
(4) every projective A-module is a direct summand of a direct sum of copies of A.

22.15. Projective modules over graded algebras

0FQA In this section we discuss projective graded modules over graded algebras analogous to Algebra, Section 10.77.

Let R be a ring. Let A be a $\mathbf{Z}$-graded algebra over R. Section 22.2 for our conventions. Let Mod_A denote the category of graded right A-modules. For an integer k let $A[k]$ denote the shift of A. For a graded right A-module we have

$$\text{Hom}_{\text{Mod}_A}(A[k], M) = M^{-k}$$

As the functor $M \mapsto M^{-k}$ is exact on Mod_A we conclude that $A[k]$ is a projective object of Mod_A. Conversely, suppose that P is a projective object of Mod_A. By choosing a set of homogeneous generators of P as an A-module, we can find a surjection

$$\bigoplus\nolimits_{i \in I} A[k_i] \longrightarrow P$$

Thus we conclude that a projective object of Mod_A is a direct summand of a direct sum of the shifts $A[k]$.

We conclude

(1) the category of graded A-modules has enough projectives,
(2) $A[k]$ is a projective A-module for every $k \in \mathbf{Z}$,
(3) every graded A-module is a quotient of a direct sum of copies of the modules $A[k]$ for varying k,
(4) every projective A-module is a direct summand of a direct sum of copies of the modules $A[k]$ for varying k.

22.16. Projective modules and differential graded algebras

0FQB If (A, d) is a differential graded algebra and P is an object of $\text{Mod}_{(A,\text{d})}$ then we say P is projective as a graded A-module or sometimes P is graded projective to mean that P is a projective object of the abelian category Mod_A of graded A-modules as in Section 22.15.

09K0 **Lemma 22.16.1.** *Let (A, d) be a differential graded algebra. Let $M \to P$ be a surjective homomorphism of differential graded A-modules. If P is projective as a graded A-module, then $M \to P$ is an admissible epimorphism.*

Proof. This is immediate from the definitions. ☐

09K1 **Lemma 22.16.2.** *Let (A, d) be a differential graded algebra. Then we have*

$$\text{Hom}_{\text{Mod}_{(A,\text{d})}}(A[k], M) = \text{Ker}(\text{d} : M^{-k} \to M^{-k+1})$$

and

$$\text{Hom}_{K(\text{Mod}_{(A,\text{d})})}(A[k], M) = H^{-k}(M)$$

for any differential graded A-module M.

Proof. Immediate from the definitions. ☐

22.17. Injective modules over algebras

04JD In this section we discuss injective modules over algebras analogous to More on Algebra, Section 15.55. This section should probably be moved somewhere else.

Let R be a ring and let A be an R-algebra, see Section 22.2 for our conventions. For a right A-module M we set

$$M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$$

which we think of as a left A-module by the multiplication $(af)(x) = f(xa)$. Namely, $((ab)f)(x) = f(xab) = (bf)(xa) = (a(bf))(x)$. Conversely, if M is a left A-module, then $M^\vee$ is a right A-module. Since $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group (More on Algebra, Lemma 15.54.1), the functor $M \mapsto M^\vee$ is exact (More on Algebra, Lemma 15.55.6). Moreover, the evaluation map $M \to (M^\vee)^\vee$ is injective for all modules M (More on Algebra, Lemma 15.55.7).

We claim that $A^\vee$ is an injective right A-module. Namely, given a right A-module N we have

$$\mathrm{Hom}_A(N, A^\vee) = \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = N^\vee$$

and we conclude because the functor $N \mapsto N^\vee$ is exact. The second equality holds because

$$\mathrm{Hom}_{\mathbf{Z}}(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_{\mathbf{Z}} A, \mathbf{Q}/\mathbf{Z})$$

by Algebra, Lemma 10.12.8. Inside this module A-linearity exactly picks out the bilinear maps $\varphi : N \times A \to \mathbf{Q}/\mathbf{Z}$ which have the same value on $x \otimes a$ and $xa \otimes 1$, i.e., come from elements of $N^\vee$.

Finally, for every right A-module M we can choose a surjection $\bigoplus_{i \in I} A \to M^\vee$ to get an injection $M \to (M^\vee)^\vee \to \prod_{i \in I} A^\vee$.

We conclude

(1) the category of A-modules has enough injectives,
(2) $A^\vee$ is an injective A-module, and
(3) every A-module injects into a product of copies of $A^\vee$.

22.18. Injective modules over graded algebras

0FQC In this section we discuss injective graded modules over graded algebras analogous to More on Algebra, Section 15.55.

Let R be a ring. Let A be a $\mathbf{Z}$-graded algebra over R. Section 22.2 for our conventions. If M is a graded R-module we set

$$M^\vee = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(M^{-n}, \mathbf{Q}/\mathbf{Z}) = \bigoplus\nolimits_{n \in \mathbf{Z}} (M^{-n})^\vee$$

as a graded R-module (no signs in the actions of R on the homogeneous parts). If M has the structure of a left graded A-module, then we define a right graded A-module structure on $M^\vee$ by letting $a \in A^m$ act by

$$(M^{-n})^\vee \to (M^{-n-m})^\vee, \quad f \mapsto f \circ a$$

as in Section 22.13. If M has the structure of a right graded A-module, then we define a left graded A-module structure on $M^\vee$ by letting $a \in A^n$ act by

$$(M^{-m})^\vee \to (M^{-m-n})^\vee, \quad f \mapsto (-1)^{nm} f \circ a$$

as in Section 22.13 (the sign is forced on us because we want to use the same formula for the case when working with differential graded modules — if you only care about graded modules, then you can omit the sign here). On the category of (left or right) graded A-modules the functor $M \mapsto M^\vee$ is exact (check on graded pieces). Moreover, there is an injective evaluation map

$$ev : M \longrightarrow (M^\vee)^\vee, \quad ev^n = (-1)^n \text{ the evaluation map } M^n \to ((M^n)^\vee)^\vee$$

of graded R-modules, see More on Algebra, Item (17). This evaluation map is a left, resp. right A-module homomorphism if M is a left, resp. right A-module, see Remarks 22.13.5 and 22.13.6. Finally, given $k \in \mathbf{Z}$ there is a canonical isomorphism

$$M^\vee[-k] \longrightarrow (M[k])^\vee$$

of graded R-modules which uses a sign and which, if M is a left, resp. right A-module, is an isomorphism of right, resp. left A-modules. See Remark 22.13.7.

We claim that $A^\vee$ is an injective object of the category Mod_A of graded right A-modules. Namely, given a graded right A-module N we have

$$\text{Hom}_{\text{Mod}_A}(N, A^\vee) = \text{Hom}_{\text{Comp}(\mathbf{Z})}(N \otimes_A A, \mathbf{Q}/\mathbf{Z})) = (N^0)^\vee$$

by Lemma 22.13.2 (applied to the case where all the differentials are zero). We conclude because the functor $N \mapsto (N^0)^\vee = (N^\vee)^0$ is exact.

Finally, for every graded right A-module M we can choose a surjection of graded left A-modules

$$\bigoplus\nolimits_{i \in I} A[k_i] \to M^\vee$$

where $A[k_i]$ denotes the shift of A by $k_i \in \mathbf{Z}$. We do this by choosing homogeneous generators for $M^\vee$. In this way we get an injection

$$M \to (M^\vee)^\vee \to \prod A[k_i]^\vee = \prod A^\vee[-k_i]$$

Observe that the products in the formula above are products in the category of graded modules (in other words, take products in each degree and then take the direct sum of the pieces).

We conclude that

(1) the category of graded A-modules has enough injectives,
(2) for every $k \in \mathbf{Z}$ the module $A^\vee[k]$ is injective, and
(3) every A-module injects into a product in the category of graded modules of copies of shifts $A^\vee[k]$.

22.19. Injective modules and differential graded algebras

0FQD If (A, d) is a differential graded algebra and I is an object of $\text{Mod}_{(A,\text{d})}$ then we say I is injective as a graded A-module or sometimes I is graded injective to mean that I is a injective object of the abelian category Mod_A of graded A-modules.

09K2 **Lemma 22.19.1.** Let (A, d) be a differential graded algebra. Let $I \to M$ be an injective homomorphism of differential graded A-modules. If I is graded injective, then $I \to M$ is an admissible monomorphism.

Proof. This is immediate from the definitions. $\square$

Let (A, d) be a differential graded algebra. If M is a left, resp. right differential graded A-module, then

$$M^\vee = \mathrm{Hom}^\bullet(M^\bullet, \mathbf{Q}/\mathbf{Z})$$

with A-module structure constructed in Section 22.18 is a right, resp. left differential graded A-module by the discussion in Section 22.13. By Remarks 22.13.5 and 22.13.6 there evaluation map of Section 22.18

$$M \longrightarrow (M^\vee)^\vee$$

is a homomorphism of left, resp. right differential graded A-modules

09K3 Lemma 22.19.2. Let (A, d) be a differential graded algebra. If M is a left differential graded A-module and N is a right differential graded A-module, then

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mod}_{(A,\mathrm{d})}}(N, M^\vee) &= \mathrm{Hom}_{\mathrm{Comp}(\mathbf{Z})}(N \otimes_A M, \mathbf{Q}/\mathbf{Z}) \\ &= \mathrm{DifferentialGradedBilinear}_A(N \times M, \mathbf{Q}/\mathbf{Z}) \end{aligned}$$

Proof. The first equality is Lemma 22.13.2 and the second equality was shown in Section 22.12. $\square$

09K4 Lemma 22.19.3. Let (A, d) be a differential graded algebra. Then we have

$$\mathrm{Hom}_{\mathrm{Mod}_{(A,\mathrm{d})}}(M, A^\vee[k]) = \mathrm{Ker}(\mathrm{d} : (M^\vee)^k \to (M^\vee)^{k+1})$$

and

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(M, A^\vee[k]) = H^k(M^\vee)$$

as functors in the differential graded A-module M.

Proof. This is clear from the discussion above. $\square$

22.20. P-resolutions

09KK This section is the analogue of Derived Categories, Section 13.29.

Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. We say P has property (P) if it there exists a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded submodules such that

(1) $P = \bigcup F_pP$,
(2) the inclusions $F_iP \to F_{i+1}P$ are admissible monomorphisms,
(3) the quotients $F_{i+1}P/F_iP$ are isomorphic as differential graded A-modules to a direct sum of $A[k]$.

In fact, condition (2) is a consequence of condition (3), see Lemma 22.16.1. Moreover, the reader can verify that as a graded A-module P will be isomorphic to a direct sum of shifts of A.

09KL Lemma 22.20.1. Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. If $F_\bullet$ is a filtration as in property (P), then we obtain an admissible short exact sequence

$$0 \to \bigoplus F_iP \to \bigoplus F_iP \to P \to 0$$

of differential graded A-modules.

Proof. The second map is the direct sum of the inclusion maps. The first map on the summand F_iP of the source is the sum of the identity $F_iP \to F_iP$ and the negative of the inclusion map $F_iP \to F_{i+1}P$. Choose homomorphisms $s_i : F_{i+1}P \to F_iP$ of graded A-modules which are left inverse to the inclusion maps. Composing gives maps $s_{j,i} : F_jP \to F_iP$ for all $j > i$. Then a left inverse of the first arrow maps $x \in F_jP$ to $(s_{j,0}(x), s_{j,1}(x), \ldots, s_{j,j-1}(x), 0, \ldots)$ in $\bigoplus F_iP$. □

The following lemma shows that differential graded modules with property (P) are the dual notion to K-injective modules (i.e., they are K-projective in some sense). See Derived Categories, Definition 13.31.1.

09KM Lemma 22.20.2. Let (A, d) be a differential graded algebra. Let P be a differential graded A-module with property (P). Then

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\text{d})})}(P, N) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,\text{d})})$ is a triangulated category (Proposition 22.10.3). Let $F_\bullet$ be a filtration on P as in property (P). The short exact sequence of Lemma 22.20.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 13.4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\text{d})})}(F_iP, N) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules F_iP has a finite filtration by admissible monomorphisms, whose graded pieces are direct sums of shifts $A[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\text{d})})}(A[k], N) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 22.16.2. □

09KN Lemma 22.20.3. Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules with the following properties

(1) $P \to M$ is surjective,
(2) $\mathrm{Ker}(\text{d}_P) \to \mathrm{Ker}(\text{d}_M)$ is surjective, and
(3) P sits in an admissible short exact sequence $0 \to P' \to P \to P'' \to 0$ where P', P'' are direct sums of shifts of A.

Proof. Let P_k be the free A-module with generators x, y in degrees k and $k + 1$. Define the structure of a differential graded A-module on P_k by setting $\text{d}(x) = y$ and $\text{d}(y) = 0$. For every element $m \in M^k$ there is a homomorphism $P_k \to M$ sending x to m and y to $\text{d}(m)$. Thus we see that there is a surjection from a direct sum of copies of P_k to M. This clearly produces $P \to M$ having properties (1) and (3). To obtain property (2) note that if $m \in \mathrm{Ker}(\text{d}_M)$ has degree k, then there is a map $A[k] \to M$ mapping 1 to m. Hence we can achieve (2) by adding a direct sum of copies of shifts of A. □

09KP Lemma 22.20.4. Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules such that

(1) $P \to M$ is a quasi-isomorphism, and

(2) P has property (P).

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_{i+1} \to P_i \to M_i \to 0$$

where the maps $P_i \to M_i$ are chosen as in Lemma 22.20.3. This gives a "resolution"

$$\ldots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

Then we set

$$P = \bigoplus\nolimits_{i \geq 0} P_i$$

as an A-module with grading given by $P^n = \bigoplus_{a+b=n} P^b_{-a}$ and differential (as in the construction of the total complex associated to a double complex) by

$$\mathrm{d}_P(x) = f_{-a}(x) + (-1)^a \mathrm{d}_{P_{-a}}(x)$$

for $x \in P^b_{-a}$. With these conventions P is indeed a differential graded A-module. Recalling that each P_i has a two step filtration $0 \to P'_i \to P_i \to P''_i \to 0$ we set

$$F_{2i}P = \bigoplus\nolimits_{i \geq j \geq 0} P_j \subset \bigoplus\nolimits_{i \geq 0} P_i = P$$

and we add P'_{i+1} to $F_{2i}P$ to get F_{2i+1}. These are differential graded submodules and the successive quotients are direct sums of shifts of A. By Lemma 22.16.1 we see that the inclusions $F_iP \to F_{i+1}P$ are admissible monomorphisms. Finally, we have to show that the map $P \to M$ (given by the augmentation $P_0 \to M$) is a quasi-isomorphism. This follows from Homology, Lemma 12.26.2. □

22.21. I-resolutions

09KQ This section is the dual of the section on P-resolutions.

Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. We say I has property (I) if it there exists a filtration

$$I = F_0I \supset F_1I \supset F_2I \supset \ldots \supset 0$$

by differential graded submodules such that

(1) $I = \lim I/F_pI$,
(2) the maps $I/F_{i+1}I \to I/F_iI$ are admissible epimorphisms,
(3) the quotients $F_iI/F_{i+1}I$ are isomorphic as differential graded A-modules to products of the modules $A^\vee[k]$ constructed in Section 22.19.

In fact, condition (2) is a consequence of condition (3), see Lemma 22.19.1. The reader can verify that as a graded module I will be isomorphic to a product of $A^\vee[k]$.

09KR Lemma 22.21.1. Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. If $F_\bullet$ is a filtration as in property (I), then we obtain an admissible short exact sequence

$$0 \to I \to \prod I/F_iI \to \prod I/F_iI \to 0$$

of differential graded A-modules.

Proof. Omitted. Hint: This is dual to Lemma 22.20.1. □

The following lemma shows that differential graded modules with property (I) are the analogue of K-injective modules. See Derived Categories, Definition 13.31.1.

09KS **Lemma 22.21.2.** Let (A, d) be a differential graded algebra. Let I be a differential graded A-module with property (I). Then

$$\text{Hom}_{K(\text{Mod}_{(A,\text{d})})}(N, I) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\text{Mod}_{(A,\text{d})})$ is a triangulated category (Proposition 22.10.3). Let $F_\bullet$ be a filtration on I as in property (I). The short exact sequence of Lemma 22.21.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 13.4.2 it suffices to show that

$$\text{Hom}_{K(\text{Mod}_{(A,\text{d})})}(N, I/F_iI) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules I/F_iI has a finite filtration by admissible monomorphisms, whose graded pieces are products of $A^\vee[k]$. Thus it suffices to prove that

$$\text{Hom}_{K(\text{Mod}_{(A,\text{d})})}(N, A^\vee[k]) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 22.19.3 and the fact that $(-)^\vee$ is an exact functor. □

09KT **Lemma 22.21.3.** Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules with the following properties

(1) $M \to I$ is injective,
(2) $\text{Coker}(\text{d}_M) \to \text{Coker}(\text{d}_I)$ is injective, and
(3) I sits in an admissible short exact sequence $0 \to I' \to I \to I'' \to 0$ where I', I'' are products of shifts of $A^\vee$.

Proof. We will use the functors $N \mapsto N^\vee$ (from left to right differential graded modules and from right to left differential graded modules) constructed in Section 22.19 and all of their properties. For every $k \in \mathbf{Z}$ let Q_k be the free left A-module with generators x, y in degrees k and $k+1$. Define the structure of a left differential graded A-module on Q_k by setting $\text{d}(x) = y$ and $\text{d}(y) = 0$. Arguing exactly as in the proof of Lemma 22.20.3 we find a surjection

$$\bigoplus\nolimits_{i \in I} Q_{k_i} \longrightarrow M^\vee$$

of left differential graded A-modules. Then we can consider the injection

$$M \to (M^\vee)^\vee \to (\bigoplus\nolimits_{i \in I} Q_{k_i})^\vee = \prod\nolimits_{i \in I} I_{k_i}$$

where $I_k = Q_{-k}^\vee$ is the "dual" right differential graded A-module. Further, the short exact sequence $0 \to A[-k-1] \to Q_k \to A[-k] \to 0$ produces a short exact sequence $0 \to A^\vee[k] \to I_k \to A^\vee[k+1] \to 0$.

The result of the previous paragraph produces $M \to I$ having properties (1) and (3). To obtain property (2), suppose $\overline{m} \in \text{Coker}(\text{d}_M)$ is a nonzero element of degree k. Pick a map $\lambda : M^k \to \mathbf{Q}/\mathbf{Z}$ which vanishes on $\text{Im}(M^{k-1} \to M^k)$ but not on m. By Lemma 22.19.3 this corresponds to a homomorphism $M \to A^\vee[k]$ of differential graded A-modules which does not vanish on m. Hence we can achieve (2) by adding a product of copies of shifts of $A^\vee$. □

09KU Lemma 22.21.4. Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules such that

(1) $M \to I$ is a quasi-isomorphism, and
(2) I has property (I).

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_i \to I_i \to M_{i+1} \to 0$$

where the maps $M_i \to I_i$ are chosen as in Lemma 22.21.3. This gives a "resolution"

$$0 \to M \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_1 \to \ldots$$

Denote I the differential graded A-module with graded parts

$$I^n = \prod\nolimits_{i \geq 0} I_i^{n-i}$$

and differential defined by

$$\mathrm{d}_I(x) = f_i(x) + (-1)^i \mathrm{d}_{I_i}(x)$$

for $x \in I_i^{n-i}$. With these conventions I is indeed a differential graded A-module. Recalling that each I_i has a two step filtration $0 \to I_i' \to I_i \to I_i'' \to 0$ we set

$$F_{2i}I^n = \prod\nolimits_{j \geq i} I_j^{n-j} \subset \prod\nolimits_{i \geq 0} I_i^{n-i} = I^n$$

and we add a factor I_{i+1}' to $F_{2i}I$ to get $F_{2i+1}I$. These are differential graded submodules and the successive quotients are products of shifts of $A^\vee$. By Lemma 22.19.1 we see that the inclusions $F_{i+1}I \to F_iI$ are admissible monomorphisms. Finally, we have to show that the map $M \to I$ (given by the augmentation $M \to I_0$) is a quasi-isomorphism. This follows from Homology, Lemma 12.26.3. □

22.22. The derived category

09KV Recall that the notions of acyclic differential graded modules and quasi-isomorphism of differential graded modules make sense (see Section 22.4).

09KW Lemma 22.22.1. Let (A, d) be a differential graded algebra. The full subcategory Ac of $K(\mathrm{Mod}_{(A,\mathrm{d})})$ consisting of acyclic modules is a strictly full saturated triangulated subcategory of $K(\mathrm{Mod}_{(A,\mathrm{d})})$. The corresponding saturated multiplicative system (see Derived Categories, Lemma 13.6.10) of $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is the class Qis of quasi-isomorphisms. In particular, the kernel of the localization functor

$$Q : K(\mathrm{Mod}_{(A,\mathrm{d})}) \to \mathrm{Qis}^{-1}K(\mathrm{Mod}_{(A,\mathrm{d})})$$

is Ac. Moreover, the functor H^0 factors through Q.

Proof. We know that H^0 is a homological functor by the long exact sequence of homology (22.4.2.1). The kernel of H^0 is the subcategory of acyclic objects and the arrows with induce isomorphisms on all H^i are the quasi-isomorphisms. Thus this lemma is a special case of Derived Categories, Lemma 13.6.11.

Set theoretical remark. The construction of the localization in Derived Categories, Proposition 13.5.6 assumes the given triangulated category is "small", i.e., that the underlying collection of objects forms a set. Let V_α be a partial universe (as in Sets, Section 3.5) containing (A, d) and where the cofinality of α is bigger than $\aleph_0$ (see Sets, Proposition 3.7.2). Then we can consider the category $\mathrm{Mod}_{(A,\mathrm{d}),\alpha}$

of differential graded A-modules contained in V_α. A straightforward check shows that all the constructions used in the proof of Proposition 22.10.3 work inside of $\text{Mod}_{(A,\text{d}),\alpha}$ (because at worst we take finite direct sums of differential graded modules). Thus we obtain a triangulated category $\text{Qis}_\alpha^{-1}K(\text{Mod}_{(A,\text{d}),\alpha})$. We will see below that if $\beta > \alpha$, then the transition functors

$$\text{Qis}_\alpha^{-1}K(\text{Mod}_{(A,\text{d}),\alpha}) \longrightarrow \text{Qis}_\beta^{-1}K(\text{Mod}_{(A,\text{d}),\beta})$$

are fully faithful as the morphism sets in the quotient categories are computed by maps in the homotopy categories from P-resolutions (the construction of a P-resolution in the proof of Lemma 22.20.4 takes countable direct sums as well as direct sums indexed over subsets of the given module). The reader should therefore think of the category of the lemma as the union of these subcategories. □

Taking into account the set theoretical remark at the end of the proof of the preceding lemma we define the derived category as follows.

09KX Definition 22.22.2. Let (A, d) be a differential graded algebra. Let Ac and Qis be as in Lemma 22.22.1. The derived category of (A, d) is the triangulated category

$$D(A, \text{d}) = K(\text{Mod}_{(A,\text{d})})/\text{Ac} = \text{Qis}^{-1}K(\text{Mod}_{(A,\text{d})}).$$

We denote $H^0 : D(A, \text{d}) \to \text{Mod}_R$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above.

Here is the promised lemma computing morphism sets in the derived category.

09KY Lemma 22.22.3. Let (A, d) be a differential graded algebra. Let M and N be differential graded A-modules.

(1) Let $P \to M$ be a P-resolution as in Lemma 22.20.4. Then

$$\text{Hom}_{D(A,\text{d})}(M, N) = \text{Hom}_{K(\text{Mod}_{(A,\text{d})})}(P, N)$$

(2) Let $N \to I$ be an I-resolution as in Lemma 22.21.4. Then

$$\text{Hom}_{D(A,\text{d})}(M, N) = \text{Hom}_{K(\text{Mod}_{(A,\text{d})})}(M, I)$$

Proof. Let $P \to M$ be as in (1). Since $P \to M$ is a quasi-isomorphism we see that

$$\text{Hom}_{D(A,\text{d})}(P, N) = \text{Hom}_{D(A,\text{d})}(M, N)$$

by definition of the derived category. A morphism $f : P \to N$ in $D(A, \text{d})$ is equal to $s^{-1}f'$ where $f' : P \to N'$ is a morphism and $s : N \to N'$ is a quasi-isomorphism. Choose a distinguished triangle

$$N \to N' \to Q \to N[1]$$

As s is a quasi-isomorphism, we see that Q is acyclic. Thus $\text{Hom}_{K(\text{Mod}_{(A,\text{d})})}(P, Q[k]) = 0$ for all k by Lemma 22.20.2. Since $\text{Hom}_{K(\text{Mod}_{(A,\text{d})})}(P, -)$ is cohomological, we conclude that we can lift $f' : P \to N'$ uniquely to a morphism $f : P \to N$. This finishes the proof.

The proof of (2) is dual to that of (1) using Lemma 22.21.2 in stead of Lemma 22.20.2. □

09QI Lemma 22.22.4. Let (A, d) be a differential graded algebra. Then

(1) $D(A, \text{d})$ has both direct sums and products,

(2) direct sums are obtained by taking direct sums of differential graded modules,
(3) products are obtained by taking products of differential graded modules.

Proof. We will use that $\text{Mod}_{(A,\text{d})}$ is an abelian category with arbitrary direct sums and products, and that these give rise to direct sums and products in $K(\text{Mod}_{(A,\text{d})})$. See Lemmas 22.4.2 and 22.5.4.

Let M_j be a family of differential graded A-modules. Consider the graded direct sum $M = \bigoplus M_j$ which is a differential graded A-module with the obvious. For a differential graded A-module N choose a quasi-isomorphism $N \to I$ where I is a differential graded A-module with property (I). See Lemma 22.21.4. Using Lemma 22.22.3 we have

$$\begin{aligned} \text{Hom}_{D(A,\text{d})}(M, N) &= \text{Hom}_{K(A,\text{d})}(M, I) \\ &= \prod \text{Hom}_{K(A,\text{d})}(M_j, I) \\ &= \prod \text{Hom}_{D(A,\text{d})}(M_j, N) \end{aligned}$$

whence the existence of direct sums in $D(A, \text{d})$ as given in part (2) of the lemma.

Let M_j be a family of differential graded A-modules. Consider the product $M = \prod M_j$ of differential graded A-modules. For a differential graded A-module N choose a quasi-isomorphism $P \to N$ where P is a differential graded A-module with property (P). See Lemma 22.20.4. Using Lemma 22.22.3 we have

$$\begin{aligned} \text{Hom}_{D(A,\text{d})}(N, M) &= \text{Hom}_{K(A,\text{d})}(P, M) \\ &= \prod \text{Hom}_{K(A,\text{d})}(P, M_j) \\ &= \prod \text{Hom}_{D(A,\text{d})}(N, M_j) \end{aligned}$$

whence the existence of direct sums in $D(A, \text{d})$ as given in part (3) of the lemma. □

0FQE Remark 22.22.5. Let R be a ring. Let (A, d) be a differential graded R-algebra. Using P-resolutions we can sometimes reduce statements about general objects of $D(A, \text{d})$ to statements about $A[k]$. Namely, let T be a property of objects of $D(A, \text{d})$ and assume that

(1) if K_i, $i \in I$ is a family of objects of $D(A, \text{d})$ and $T(K_i)$ holds for all $i \in I$, then $T(\bigoplus K_i)$,
(2) if $K \to L \to M \to K[1]$ is a distinguished triangle of $D(A, \text{d})$ and T holds for two, then T holds for the third object, and
(3) $T(A[k])$ holds for all $k \in \mathbf{Z}$.

Then T holds for all objects of $D(A, \text{d})$. This is clear from Lemmas 22.20.1 and 22.20.4.

22.23. The canonical delta-functor

09KZ Let (A, d) be a differential graded algebra. Consider the functor $\text{Mod}_{(A,\text{d})} \to K(\text{Mod}_{(A,\text{d})})$. This functor is not a δ-functor in general. However, it turns out that the functor $\text{Mod}_{(A,\text{d})} \to D(A, \text{d})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to K \xrightarrow{a} L \xrightarrow{b} M \to 0$$

in the abelian category $\text{Mod}_{(A,\text{d})}$. Consider the cone $C(a)$ of the morphism a. We have $C(a) = L \oplus K$ and we define $q : C(a) \to M$ via the projection to L followed by b. Hence a homomorphism of differential graded A-modules

$$q : C(a) \longrightarrow M.$$

It is clear that $q \circ i = b$ where i is as in Definition 22.6.1. Note that, as a is injective, the kernel of q is identified with the cone of id_K which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 22.9.4 the triangle

$$(K, L, C(a), a, i, -p)$$

is a distinguished triangle in $K(\text{Mod}_{(A,\text{d})})$. As the localization functor $K(\text{Mod}_{(A,\text{d})}) \to D(A, \text{d})$ is exact we see that $(K, L, C(a), a, i, -p)$ is a distinguished triangle in $D(A, \text{d})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(A, \text{d})$. Hence we deduce that

$$(K, L, M, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(A, \text{d})$. This suggests the following lemma.

09L0 Lemma 22.23.1. Let (A, d) be a differential graded algebra. The functor $\text{Mod}_{(A,\text{d})} \to D(A, \text{d})$ defined has the natural structure of a δ-functor, with

$$\delta_{K \to L \to M} = -p \circ q^{-1}$$

with p and q as explained above.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show functoriality of this construction, see Derived Categories, Definition 13.3.6. This follows from Lemma 22.6.2 with a bit of work. Compare with Derived Categories, Lemma 13.12.1. □

0CRL Lemma 22.23.2. Let (A, d) be a differential graded algebra. Let M_n be a system of differential graded modules. Then the derived colimit $\text{hocolim} M_n$ in $D(A, \text{d})$ is represented by the differential graded module $\text{colim}\, M_n$.

Proof. Set $M = \text{colim}\, M_n$. We have an exact sequence of differential graded modules

$$0 \to \bigoplus M_n \to \bigoplus M_n \to M \to 0$$

by Derived Categories, Lemma 13.33.6 (applied the underlying complexes of abelian groups). The direct sums are direct sums in $D(\mathcal{A})$ by Lemma 22.22.4. Thus the result follows from the definition of derived colimits in Derived Categories, Definition 13.33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 22.23.1). □

22.24. Linear categories

09MI Just the definitions.

09MJ Definition 22.24.1. Let R be a ring. An R-linear category $\mathcal{A}$ is a category where every morphism set is given the structure of an R-module and where for $x, y, z \in \text{Ob}(\mathcal{A})$ composition law

$$\text{Hom}_{\mathcal{A}}(y, z) \times \text{Hom}_{\mathcal{A}}(x, y) \longrightarrow \text{Hom}_{\mathcal{A}}(x, z)$$

is R-bilinear.

Thus composition determines an R-linear map

$$\mathrm{Hom}_{\mathcal{A}}(y,z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x,z)$$

of R-modules. Note that we do not assume R-linear categories to be additive.

09MK Definition 22.24.2. Let R be a ring. A functor of R-linear categories, or an R-linear functor is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x,y) \to \mathrm{Hom}_{\mathcal{B}}(F(x), F(y))$ is a homomorphism of R-modules.

22.25. Graded categories

09L1 Just some definitions.

09L2 Definition 22.25.1. Let R be a ring. A graded category $\mathcal{A}$ over R is a category where every morphism set is given the structure of a graded R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(y,z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x,z)$$

of graded R-modules (i.e., preserving degrees).

In this situation we denote $\mathrm{Hom}^i_{\mathcal{A}}(x,y)$ the degree i part of the graded object $\mathrm{Hom}_{\mathcal{A}}(x,y)$, so that

$$\mathrm{Hom}_{\mathcal{A}}(x,y) = \bigoplus\nolimits_{i \in \mathbf{Z}} \mathrm{Hom}^i_{\mathcal{A}}(x,y)$$

is the direct sum decomposition into graded parts.

09L3 Definition 22.25.2. Let R be a ring. A functor of graded categories over R, or a graded functor is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x,y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of graded R-modules.

Given a graded category we are often interested in the corresponding "usual" category of maps of degree 0. Here is a formal definition.

09ML Definition 22.25.3. Let R be a ring. Let $\mathcal{A}$ be a graded category over R. We let $\mathcal{A}^0$ be the category with the same objects as $\mathcal{A}$ and with

$$\mathrm{Hom}_{\mathcal{A}^0}(x,y) = \mathrm{Hom}^0_{\mathcal{A}}(x,y)$$

the degree 0 graded piece of the graded module of morphisms of $\mathcal{A}$.

09P2 Definition 22.25.4. Let R be a ring. Let $\mathcal{A}$ be a graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 12.3.6) is a graded direct sum if i, j, p, q are homogeneous of degree 0.

09MM Example 22.25.5 (Graded category of graded objects). Let $\mathcal{B}$ be an additive category. Recall that we have defined the category $\mathrm{Gr}(\mathcal{B})$ of graded objects of $\mathcal{B}$ in Homology, Definition 12.16.1. In this example, we will construct a graded category $\mathrm{Gr}^{gr}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category $\mathrm{Gr}^{gr}(\mathcal{B})^0$ recovers $\mathrm{Gr}(\mathcal{B})$. As objects of $\mathrm{Gr}^{gr}(\mathcal{B})$ we take graded objects of $\mathcal{B}$. Then, given graded objects $A = (A^i)$ and $B = (B^i)$ of $\mathcal{B}$ we set

$$\mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathcal{B})}(A,B) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(A,B)$$

where the graded piece of degree n is the abelian group of homogeneous maps of degree n from A to B. Explicitly we have

$$\mathrm{Hom}^n(A,B) = \prod\nolimits_{p+q=n} \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

(observe reversal of indices and observe that we have a product here and not a direct sum). In other words, a degree n morphism f from A to B can be seen as a system $f = (f_{p,q})$ where $p, q \in \mathbf{Z}$, $p + q = n$ with $f_{p,q} : A^{-q} \to B^p$ a morphism of $\mathcal{B}$. Given graded objects A, B, C of $\mathcal{B}$ composition of morphisms in $\text{Gr}^{gr}(\mathcal{B})$ is defined via the maps

$$\text{Hom}^m(B, C) \times \text{Hom}^n(A, B) \longrightarrow \text{Hom}^{n+m}(A, C)$$

by simple composition $(g, f) \mapsto g \circ f$ of homogeneous maps of graded objects. In terms of components we have

$$(g \circ f)_{p,r} = g_{p,q} \circ f_{-q,r}$$

where q is such that $p + q = m$ and $-q + r = n$.

09MN Example 22.25.6 (Graded category of graded modules). Let A be a $\mathbf{Z}$-graded algebra over a ring R. We will construct a graded category Mod^{gr}_A over R whose associated category $(\text{Mod}^{gr}_A)^0$ is the category of graded A-modules. As objects of Mod^{gr}_A we take right graded A-modules (see Section 22.14). Given graded A-modules L and M we set

$$\text{Hom}_{\text{Mod}^{gr}_A}(L, M) = \bigoplus\nolimits_{n \in \mathbf{Z}} \text{Hom}^n(L, M)$$

where $\text{Hom}^n(L, M)$ is the set of right A-module maps $L \to M$ which are homogeneous of degree n, i.e., $f(L^i) \subset M^{i+n}$ for all $i \in \mathbf{Z}$. In terms of components, we have that

$$\text{Hom}^n(L, M) \subset \prod\nolimits_{p+q=n} \text{Hom}_R(L^{-q}, M^p)$$

(observe reversal of indices) is the subset consisting of those $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for $a \in A^i$ and $m \in L^{-q-i}$. For graded A-modules K, L, M we define composition in Mod^{gr}_A via the maps

$$\text{Hom}^m(L, M) \times \text{Hom}^n(K, L) \longrightarrow \text{Hom}^{n+m}(K, M)$$

by simple composition of right A-module maps: $(g, f) \mapsto g \circ f$.

09P3 Remark 22.25.7. Let R be a ring. Let $\mathcal{D}$ be an R-linear category endowed with a collection of R-linear functors $[n] : \mathcal{D} \to \mathcal{D}$, $x \mapsto x[n]$ indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n + m]$ and $[0] = \text{id}_\mathcal{D}$ (equality as functors). This allows us to construct a graded category $\mathcal{D}^{gr}$ over R with the same objects of $\mathcal{D}$ setting

$$\text{Hom}_{\mathcal{D}^{gr}}(x, y) = \bigoplus\nolimits_{n \in \mathbf{Z}} \text{Hom}_\mathcal{D}(x, y[n])$$

for x, y in $\mathcal{D}$. Observe that $(\mathcal{D}^{gr})^0 = \mathcal{D}$ (see Definition 22.25.3). Moreover, the graded category $\mathcal{D}^{gr}$ inherits R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n + m]$ and $[0] = \text{id}_{\mathcal{D}^{gr}}$ with the property that

$$\text{Hom}_{\mathcal{D}^{gr}}(x, y[n]) = \text{Hom}_{\mathcal{D}^{gr}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms.

Conversely, suppose given a graded category $\mathcal{A}$ over R endowed with a collection of R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n + m]$ and $[0] = \text{id}_\mathcal{A}$ which are moreover equipped with isomorphisms

$$\text{Hom}_\mathcal{A}(x, y[n]) = \text{Hom}_\mathcal{A}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms. Then the reader easily shows that $\mathcal{A} = (\mathcal{A}^0)^{gr}$.

Here are two examples of the relationship $\mathcal{D} \leftrightarrow \mathcal{A}$ we established above:

(1) Let $\mathcal{B}$ be an additive category. If $\mathcal{D} = \text{Gr}(\mathcal{B})$, then $\mathcal{A} = \text{Gr}^{gr}(\mathcal{B})$ as in Example 22.25.5.
(2) If A is a graded ring and $\mathcal{D} = \text{Mod}_A$ is the category of graded right A-modules, then $\mathcal{A} = \text{Mod}_A^{gr}$, see Example 22.25.6.

22.26. Differential graded categories

09L4 Note that if R is a ring, then R is a differential graded algebra over itself (with $R = R^0$ of course). In this case a differential graded R-module is the same thing as a complex of R-modules. In particular, given two differential graded R-modules M and N we denote $M \otimes_R N$ the differential graded R-module corresponding to the total complex associated to the double complex obtained by the tensor product of the complexes of R-modules associated to M and N.

09L5 **Definition 22.26.1.** Let R be a ring. A differential graded category $\mathcal{A}$ over R is a category where every morphism set is given the structure of a differential graded R-module and where for $x, y, z \in \text{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\text{Hom}_\mathcal{A}(y, z) \otimes_R \text{Hom}_\mathcal{A}(x, y) \longrightarrow \text{Hom}_\mathcal{A}(x, z)$$

of differential graded R-modules.

The final condition of the definition signifies the following: if $f \in \text{Hom}^n_\mathcal{A}(x, y)$ and $g \in \text{Hom}^m_\mathcal{A}(y, z)$ are homogeneous of degrees n and m, then

$$\text{d}(g \circ f) = \text{d}(g) \circ f + (-1)^m g \circ \text{d}(f)$$

in $\text{Hom}^{n+m+1}_\mathcal{A}(x, z)$. This follows from the sign rule for the differential on the total complex of a double complex, see Homology, Definition 12.18.3.

09L6 **Definition 22.26.2.** Let R be a ring. A functor of differential graded categories over R is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \text{Hom}_\mathcal{A}(x, y) \to \text{Hom}_\mathcal{A}(F(x), F(y))$ is a homomorphism of differential graded R-modules.

Given a differential graded category we are often interested in the corresponding categories of complexes and homotopy category. Here is a formal definition.

09L7 **Definition 22.26.3.** Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Then we let

(1) the category of complexes of $\mathcal{A}$[1] be the category $\text{Comp}(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with

$$\text{Hom}_{\text{Comp}(\mathcal{A})}(x, y) = \text{Ker}(d : \text{Hom}^0_\mathcal{A}(x, y) \to \text{Hom}^1_\mathcal{A}(x, y))$$

(2) the homotopy category of $\mathcal{A}$ be the category $K(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with

$$\text{Hom}_{K(\mathcal{A})}(x, y) = H^0(\text{Hom}_\mathcal{A}(x, y))$$

Our use of the symbol $K(\mathcal{A})$ is nonstandard, but at least is compatible with the use of $K(-)$ in other chapters of the Stacks project.

[1] This may be nonstandard terminology.

09P4 **Definition 22.26.4.** Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 12.3.6) is a *differential graded direct sum* if i, j, p, q are homogeneous of degree 0 and closed, i.e., $\mathrm{d}(i) = 0$, etc.

09L8 **Lemma 22.26.5.** Let R be a ring. A functor $F : \mathcal{A} \to \mathcal{B}$ of differential graded categories over R induces functors $\text{Comp}(\mathcal{A}) \to \text{Comp}(\mathcal{B})$ and $K(\mathcal{A}) \to K(\mathcal{B})$.

Proof. Omitted. $\square$

09L9 **Example 22.26.6** (Differential graded category of complexes)**.** Let $\mathcal{B}$ be an additive category. We will construct a differential graded category $\text{Comp}^{dg}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category of complexes is $\text{Comp}(\mathcal{B})$ and whose associated homotopy category is $K(\mathcal{B})$. As objects of $\text{Comp}^{dg}(\mathcal{B})$ we take complexes of $\mathcal{B}$. Given complexes $A^\bullet$ and $B^\bullet$ of $\mathcal{B}$, we sometimes also denote $A^\bullet$ and $B^\bullet$ the corresponding graded objects of $\mathcal{B}$ (i.e., forget about the differential). Using this abuse of notation, we set

$$\text{Hom}_{\text{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet) = \text{Hom}_{\text{Gr}^{gr}(\mathcal{B})}(A^\bullet, B^\bullet) = \bigoplus\nolimits_{n \in \mathbf{Z}} \text{Hom}^n(A, B)$$

as a graded $\mathbf{Z}$-module with notation and definitions as in Example 22.25.5. In other words, the nth graded piece is the abelian group of homogeneous morphism of degree n of graded objects

$$\text{Hom}^n(A^\bullet, B^\bullet) = \prod\nolimits_{p+q=n} \text{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

Observe reversal of indices and observe we have a direct product and not a direct sum. For an element $f \in \text{Hom}^n(A^\bullet, B^\bullet)$ of degree n we set

$$\mathrm{d}(f) = \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A$$

The sign is exactly as in More on Algebra, Section 15.72. To make sense of this we think of d_B and d_A as maps of graded objects of $\mathcal{B}$ homogeneous of degree 1 and we use composition in the category $\text{Gr}^{gr}(\mathcal{B})$ on the right hand side. In terms of components, if $f = (f_{p,q})$ with $f_{p,q} : A^{-q} \to B^p$ we have

09LA (22.26.6.1) $$\mathrm{d}(f_{p,q}) = \mathrm{d}_B \circ f_{p,q} - (-1)^{p+q} f_{p,q} \circ \mathrm{d}_A$$

Note that the first term of this expression is in $\text{Hom}_{\mathcal{B}}(A^{-q}, B^{p+1})$ and the second term is in $\text{Hom}_{\mathcal{B}}(A^{-q-1}, B^p)$. The reader checks that

(1) d has square zero,
(2) an element f in $\text{Hom}^n(A^\bullet, B^\bullet)$ has $\mathrm{d}(f) = 0$ if and only if the morphism $f : A^\bullet \to B^\bullet[n]$ of graded objects of $\mathcal{B}$ is actually a map of complexes,
(3) in particular, the category of complexes of $\text{Comp}^{dg}(\mathcal{B})$ is equal to $\text{Comp}(\mathcal{B})$,
(4) the morphism of complexes defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \text{Hom}^{n-1}(A^\bullet, B^\bullet)$.
(5) in particular, we obtain a canonical isomorphism

$$\text{Hom}_{K(\mathcal{B})}(A^\bullet, B^\bullet) \longrightarrow H^0(\text{Hom}_{\text{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet))$$

and the homotopy category of $\text{Comp}^{dg}(\mathcal{B})$ is equal to $K(\mathcal{B})$.

Given complexes $A^\bullet$, $B^\bullet$, $C^\bullet$ we define composition

$$\text{Hom}^m(B^\bullet, C^\bullet) \times \text{Hom}^n(A^\bullet, B^\bullet) \longrightarrow \text{Hom}^{n+m}(A^\bullet, C^\bullet)$$

by composition $(g, f) \mapsto g \circ f$ in the graded category $\text{Gr}^{gr}(\mathcal{B})$, see Example 22.25.5. This defines a map of differential graded modules

$$\text{Hom}_{\text{Comp}^{dg}(\mathcal{B})}(B^\bullet, C^\bullet) \otimes_R \text{Hom}_{\text{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet) \longrightarrow \text{Hom}_{\text{Comp}^{dg}(\mathcal{B})}(A^\bullet, C^\bullet)$$

as required in Definition 22.26.1 because

$$\begin{aligned} \text{d}(g \circ f) &= \text{d}_C \circ g \circ f - (-1)^{n+m} g \circ f \circ \text{d}_A \\ &= (\text{d}_C \circ g - (-1)^m g \circ \text{d}_B) \circ f + (-1)^m g \circ (\text{d}_B \circ f - (-1)^n f \circ \text{d}_A) \\ &= \text{d}(g) \circ f + (-1)^m g \circ \text{d}(f) \end{aligned}$$

as desired.

09LB Lemma 22.26.7. Let $F : \mathcal{B} \to \mathcal{B}'$ be an additive functor between additive categories. Then F induces a functor of differential graded categories

$$F : \text{Comp}^{dg}(\mathcal{B}) \to \text{Comp}^{dg}(\mathcal{B}')$$

of Example 22.26.6 inducing the usual functors on the category of complexes and the homotopy categories.

Proof. Omitted. $\square$

09LC Example 22.26.8 (Differential graded category of differential graded modules). Let (A, d) be a differential graded algebra over a ring R. We will construct a differential graded category $\text{Mod}^{dg}_{(A,\text{d})}$ over R whose category of complexes is $\text{Mod}_{(A,\text{d})}$ and whose homotopy category is $K(\text{Mod}_{(A,\text{d})})$. As objects of $\text{Mod}^{dg}_{(A,\text{d})}$ we take the differential graded A-modules. Given differential graded A-modules L and M we set

$$\text{Hom}_{\text{Mod}^{dg}_{(A,\text{d})}}(L, M) = \text{Hom}_{\text{Mod}^{gr}_A}(L, M) = \bigoplus \text{Hom}^n(L, M)$$

as a graded R-module where the right hand side is defined as in Example 22.25.6. In other words, the nth graded piece $\text{Hom}^n(L, M)$ is the R-module of right A-module maps homogeneous of degree n. For an element $f \in \text{Hom}^n(L, M)$ we set

$$\text{d}(f) = \text{d}_M \circ f - (-1)^n f \circ \text{d}_L$$

To make sense of this we think of d_M and d_L as graded R-module maps and we use composition of graded R-module maps. It is clear that $\text{d}(f)$ is homogeneous of degree $n + 1$ as a graded R-module map, and it is A-linear because

$$\begin{aligned} \text{d}(f)(xa) &= \text{d}_M(f(x)a) - (-1)^n f(\text{d}_L(xa)) \\ &= \text{d}_M(f(x))a + (-1)^{\deg(x)+n} f(x)\text{d}(a) - (-1)^n f(\text{d}_L(x))a - (-1)^{n+\deg(x)} f(x)\text{d}(a) \\ &= \text{d}(f)(x)a \end{aligned}$$

as desired (observe that this calculation would not work without the sign in the definition of our differential on Hom). Similar formulae to those of Example 22.26.6 hold for the differential of f in terms of components. The reader checks (in the same way as in Example 22.26.6) that

(1) d has square zero,
(2) an element f in $\text{Hom}^n(L, M)$ has $\text{d}(f) = 0$ if and only if $f : L \to M[n]$ is a homomorphism of differential graded A-modules,
(3) in particular, the category of complexes of $\text{Mod}^{dg}_{(A,\text{d})}$ is $\text{Mod}_{(A,\text{d})}$,
(4) the homomorphism defined by f as in (2) is homotopy equivalent to zero if and only if $f = \text{d}(g)$ for some $g \in \text{Hom}^{n-1}(L, M)$.

(5) in particular, we obtain a canonical isomorphism
$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(L,M) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L,M))$$
and the homotopy category of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Given differential graded A-modules K, L, M we define composition
$$\mathrm{Hom}^m(L,M) \times \mathrm{Hom}^n(K,L) \longrightarrow \mathrm{Hom}^{n+m}(K,M)$$
by composition of homogeneous right A-module maps $(g,f) \mapsto g \circ f$. This defines a map of differential graded modules
$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L,M) \otimes_R \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(K,L) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(K,M)$$
as required in Definition 22.26.1 because
$$\begin{aligned}
\mathrm{d}(g \circ f) &= \mathrm{d}_M \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_K \\
&= (\mathrm{d}_M \circ g - (-1)^m g \circ \mathrm{d}_L) \circ f + (-1)^m g \circ (\mathrm{d}_L \circ f - (-1)^n f \circ \mathrm{d}_K) \\
&= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)
\end{aligned}$$
as desired.

09LD Lemma 22.26.9. Let $\varphi : (A,\mathrm{d}) \to (E,\mathrm{d})$ be a homomorphism of differential graded algebras. Then φ induces a functor of differential graded categories
$$F : \mathrm{Mod}^{dg}_{(E,\mathrm{d})} \longrightarrow \mathrm{Mod}^{dg}_{(A,\mathrm{d})}$$
of Example 22.26.8 inducing obvious restriction functors on the categories of differential graded modules and homotopy categories.

Proof. Omitted. □

09LE Lemma 22.26.10. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Let x be an object of $\mathcal{A}$. Let
$$(E,\mathrm{d}) = \mathrm{Hom}_{\mathcal{A}}(x,x)$$
be the differential graded R-algebra of endomorphisms of x. We obtain a functor
$$\mathcal{A} \longrightarrow \mathrm{Mod}^{dg}_{(E,\mathrm{d})}, \quad y \longmapsto \mathrm{Hom}_{\mathcal{A}}(x,y)$$
of differential graded categories by letting E act on $\mathrm{Hom}_{\mathcal{A}}(x,y)$ via composition in $\mathcal{A}$. This functor induces functors
$$\mathrm{Comp}(\mathcal{A}) \to \mathrm{Mod}_{(A,\mathrm{d})} \quad \text{and} \quad K(\mathcal{A}) \to K(\mathrm{Mod}_{(A,\mathrm{d})})$$
by an application of Lemma 22.26.5.

Proof. This lemma proves itself. □

22.27. Obtaining triangulated categories

09P5 In this section we discuss the most general setup to which the arguments proving Derived Categories, Proposition 13.10.3 and Proposition 22.10.3 apply.

Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. To make our argument work, we impose some axioms on $\mathcal{A}$:

(A) $\mathcal{A}$ has a zero object and differential graded direct sums of two objects (as in Definition 22.26.4).

(B) there are functors $[n] : \mathcal{A} \longrightarrow \mathcal{A}$ of differential graded categories such that $[0] = \mathrm{id}_\mathcal{A}$ and $[n+m] = [n] \circ [m]$ and given isomorphisms

$$\mathrm{Hom}_\mathcal{A}(x, y[n]) = \mathrm{Hom}_\mathcal{A}(x, y)[n]$$

of differential graded R-modules compatible with composition.

Given our differential graded category $\mathcal{A}$ we say

(1) a sequence $x \to y \to z$ of morphisms of $\text{Comp}(\mathcal{A})$ is an admissible short exact sequence if there exists an isomorphism $y \cong x \oplus z$ in the underlying graded category such that $x \to z$ and $y \to z$ are (co)projections.
(2) a morphism $x \to y$ of $\text{Comp}(\mathcal{A})$ is an admissible monomorphism if it extends to an admissible short exact sequence $x \to y \to z$.
(3) a morphism $y \to z$ of $\text{Comp}(\mathcal{A})$ is an admissible epimorphism if it extends to an admissible short exact sequence $x \to y \to z$.

The next lemma tells us an admissible short exact sequence gives a triangle, provided we have axioms (A) and (B).

09P6 **Lemma 22.27.1.** Let $\mathcal{A}$ be a differential graded category satisfying axioms (A) and (B). Given an admissible short exact sequence $x \to y \to z$ we obtain (see proof) a triangle

$$x \to y \to z \to x[1]$$

in $\text{Comp}(\mathcal{A})$ with the property that any two compositions in $z[-1] \to x \to y \to z \to x[1]$ are zero in $K(\mathcal{A})$.

Proof. Choose a diagram

$$\begin{tikzcd} x \arrow[rr, "1"] \arrow[dr, "a"] & & x \\ & y \arrow[ur, "\pi"] \arrow[dr, "b"] & \\ z \arrow[rr, "1"] \arrow[ur, "s"] & & z \end{tikzcd}$$

giving the isomorphism of graded objects $y \cong x \oplus z$ as in the definition of an admissible short exact sequence. Here are some equations that hold in this situation

(1) $1 = \pi a$ and hence $\mathrm{d}(\pi)a = 0$,
(2) $1 = bs$ and hence $b\mathrm{d}(s) = 0$,
(3) $1 = a\pi + sb$ and hence $a\mathrm{d}(\pi) + \mathrm{d}(s)b = 0$,
(4) $\pi s = 0$ and hence $\mathrm{d}(\pi)s + \pi\mathrm{d}(s) = 0$,
(5) $\mathrm{d}(s) = a\pi\mathrm{d}(s)$ because $\mathrm{d}(s) = (a\pi + sb)\mathrm{d}(s)$ and $b\mathrm{d}(s) = 0$,
(6) $\mathrm{d}(\pi) = \mathrm{d}(\pi)sb$ because $\mathrm{d}(\pi) = \mathrm{d}(\pi)(a\pi + sb)$ and $\mathrm{d}(\pi)a = 0$,
(7) $\mathrm{d}(\pi\mathrm{d}(s)) = 0$ because if we postcompose it with the monomorphism a we get $\mathrm{d}(a\pi\mathrm{d}(s)) = \mathrm{d}(\mathrm{d}(s)) = 0$, and
(8) $\mathrm{d}(\mathrm{d}(\pi)s) = 0$ as by (4) it is the negative of $\mathrm{d}(\pi\mathrm{d}(s))$ which is 0 by (7).

We've used repeatedly that $\mathrm{d}(a) = 0$, $\mathrm{d}(b) = 0$, and that $\mathrm{d}(1) = 0$. By (7) we see that

$$\delta = \pi\mathrm{d}(s) = -\mathrm{d}(\pi)s : z \to x[1]$$

is a morphism in $\text{Comp}(\mathcal{A})$. By (5) we see that the composition $a\delta = a\pi\mathrm{d}(s) = \mathrm{d}(s)$ is homotopic to zero. By (6) we see that the composition $\delta b = -\mathrm{d}(\pi)sb = \mathrm{d}(-\pi)$ is homotopic to zero. $\square$

Besides axioms (A) and (B) we need an axiom concerning the existence of cones. We formalize everything as follows.

09QJ **Situation 22.27.2.** Here R is a ring and $\mathcal{A}$ is a differential graded category over R having axioms (A), (B), and

(C) given an arrow $f : x \to y$ of degree 0 with $\mathrm{d}(f) = 0$ there exists an admissible short exact sequence $y \to c(f) \to x[1]$ in $\text{Comp}(\mathcal{A})$ such that the map $x[1] \to y[1]$ of Lemma 22.27.1 is equal to $f[1]$.

We will call $c(f)$ a cone of the morphism f. If (A), (B), and (C) hold, then cones are functorial in a weak sense.

09P7 **Lemma 22.27.3.** *In Situation 22.27.2 suppose that*

$$\begin{tikzcd} x_1 \arrow[r, "f_1"] \arrow[d, "a"] & y_1 \arrow[d, "b"] \\ x_2 \arrow[r, "f_2"] & y_2 \end{tikzcd}$$

is a diagram of $\text{Comp}(\mathcal{A})$ commutative up to homotopy. Then there exists a morphism $c : c(f_1) \to c(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (x_1, y_1, c(f_1)) \to (x_1, y_1, c(f_1))$$

in $K(\mathcal{A})$.

Proof. The assumption means there exists a morphism $h : x_1 \to y_2$ of degree -1 such that $\mathrm{d}(h) = bf_1 - f_2a$. Choose isomorphisms $c(f_i) = y_i \oplus x_i[1]$ of graded objects compatible with the morphisms $y_i \to c(f_i) \to x_i[1]$. Let's denote $a_i : y_i \to c(f_i)$, $b_i : c(f_i) \to x_i[1]$, $s_i : x_i[1] \to c(f_i)$, and $\pi_i : c(f_i) \to y_i$ the given morphisms. Recall that $x_i[1] \to y_i[1]$ is given by $\pi_i \mathrm{d}(s_i)$. By axiom (C) this means that

$$f_i = \pi_i \mathrm{d}(s_i) = -\mathrm{d}(\pi_i) s_i$$

(we identify $\text{Hom}(x_i, y_i)$ with $\text{Hom}(x_i[1], y_i[1])$ using the shift functor [1]). Set $c = a_2 b \pi_1 + s_2 a b_1 + a_2 h b$. Then, using the equalities found in the proof of Lemma 22.27.1 we obtain

$$\begin{aligned} \mathrm{d}(c) &= a_2 b \mathrm{d}(\pi_1) + \mathrm{d}(s_2) a b_1 + a_2 \mathrm{d}(h) b_1 \\ &= -a_2 b f_1 b_1 + a_2 f_2 a b_1 + a_2 (b f_1 - f_2 a) b_1 \\ &= 0 \end{aligned}$$

(where we have used in particular that $\mathrm{d}(\pi_1) = \mathrm{d}(\pi_1) s_1 b_1 = f_1 b_1$ and $\mathrm{d}(s_2) = a_2 \pi_2 \mathrm{d}(s_2) = a_2 f_2$). Thus c is a degree 0 morphism $c : c(f_1) \to c(f_2)$ of $\mathcal{A}$ compatible with the given morphisms $y_i \to c(f_i) \to x_i[1]$. $\square$

In Situation 22.27.2 we say that a triangle (x, y, z, f, g, h) in $K(\mathcal{A})$ is a distinguished triangle if there exists an admissible short exact sequence $x' \to y' \to z'$ such that (x, y, z, f, g, h) is isomorphic as a triangle in $K(\mathcal{A})$ to the triangle $(x', y', z', x' \to y', y' \to z', \delta)$ constructed in Lemma 22.27.1. We will show below that

$$\boxed{K(\mathcal{A}) \text{ is a triangulated category}}$$

This result, although not as general as one might think, applies to a number of natural generalizations of the cases covered so far in the Stacks project. Here are some examples:

(1) Let $(X, \mathcal{O}_X)$ be a ringed space. Let (A, d) be a sheaf of differential graded $\mathcal{O}_X$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category.
(2) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (A, d) be a sheaf of differential graded $\mathcal{O}$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category. See Differential Graded Sheaves, Proposition 24.22.4.
(3) Two examples with a different flavor may be found in Examples, Section 110.69.

The following simple lemma is a key to the construction.

09QK Lemma 22.27.4. In Situation 22.27.2 given any object x of $\mathcal{A}$, and the cone $C(1_x)$ of the identity morphism $1_x : x \to x$, the identity morphism on $C(1_x)$ is homotopic to zero.

Proof. Consider the admissible short exact sequence given by axiom (C).

$$\begin{tikzcd} x \arrow[r, shift left, "a"] & C(1_x) \arrow[l, shift left, "\pi"] \arrow[r, shift left, "b"] & x[1] \arrow[l, shift left, "s"] \end{tikzcd}$$

Then by Lemma 22.27.1, identifying hom-sets under shifting, we have $1_x = \pi d(s) = -d(\pi)s$ where s is regarded as a morphism in $\mathrm{Hom}_{\mathcal{A}}^{-1}(x, C(1_x))$. Therefore $a = a\pi d(s) = d(s)$ using formula (5) of Lemma 22.27.1, and $b = -d(\pi)sb = -d(\pi)$ by formula (6) of Lemma 22.27.1. Hence

$$1_{C(1_x)} = a\pi + sb = d(s)\pi - sd(\pi) = d(s\pi)$$

since s is of degree -1. $\square$

A more general version of the above lemma will appear in Lemma 22.27.13. The following lemma is the analogue of Lemma 22.7.3.

09QL Lemma 22.27.5. In Situation 22.27.2 given a diagram

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "a"] & y \arrow[d, "b"] \\ z \arrow[r, "g"] & w \end{tikzcd}$$

in $\mathrm{Comp}(\mathcal{A})$ commuting up to homotopy. Then

(1) If f is an admissible monomorphism, then b is homotopic to a morphism b' which makes the diagram commute.
(2) If g is an admissible epimorphism, then a is homotopic to a morphism a' which makes the diagram commute.

Proof. To prove (1), observe that the hypothesis implies that there is some $h \in \mathrm{Hom}_{\mathcal{A}}(x, w)$ of degree -1 such that $bf - ga = d(h)$. Since f is an admissible monomorphism, there is a morphism $\pi : y \to x$ in the category $\mathcal{A}$ of degree 0. Let $b' = b - d(h\pi)$. Then

$$\begin{aligned} b'f = bf - d(h\pi)f =& bf - d(h\pi f) \quad (\text{since } d(f) = 0) \\ =& bf - d(h) \\ =& ga \end{aligned}$$

as desired. The proof for (2) is omitted. $\square$

The following lemma is the analogue of Lemma 22.7.4.

09QM Lemma 22.27.6. In Situation 22.27.2 let $\alpha : x \to y$ be a morphism in Comp($\mathcal{A}$). Then there exists a factorization in Comp($\mathcal{A}$):

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

such that

(1) $\tilde{\alpha}$ is an admissible monomorphism, and $\pi\tilde{\alpha} = \alpha$.
(2) There exists a morphism $s : y \to \tilde{y}$ in Comp($\mathcal{A}$) such that $\pi s = 1_y$ and $s\pi$ is homotopic to $1_{\tilde{y}}$.

Proof. By axiom (A), we may let $\tilde{y}$ be the differential graded direct sum of y and $C(1_x)$, i.e., there exists a diagram

$$y \underset{\pi}{\overset{s}{\rightleftarrows}} y \oplus C(1_x) \underset{t}{\overset{p}{\rightleftarrows}} C(1_x)$$

where all morphisms are of degree zero, and in Comp($\mathcal{A}$). Let $\tilde{y} = y \oplus C(1_x)$. Then $1_{\tilde{y}} = s\pi + tp$. Consider now the diagram

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

where $\tilde{\alpha}$ is induced by the morphism $x \xrightarrow{\alpha} y$ and the natural morphism $x \to C(1_x)$ fitting in the admissible short exact sequence

$$x \rightleftarrows C(1_x) \rightleftarrows x[1]$$

So the morphism $C(1_x) \to x$ of degree 0 in this diagram, together with the zero morphism $y \to x$, induces a degree-0 morphism $\beta : \tilde{y} \to x$. Then $\tilde{\alpha}$ is an admissible monomorphism since it fits into the admissible short exact sequence

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow x[1]$$

Furthermore, $\pi\tilde{\alpha} = \alpha$ by the construction of $\tilde{\alpha}$, and $\pi s = 1_y$ by the first diagram. It remains to show that $s\pi$ is homotopic to $1_{\tilde{y}}$. Write 1_x as $d(h)$ for some degree -1 map. Then, our last statement follows from

$$\begin{aligned} 1_{\tilde{y}} - s\pi =& tp \\ =& t(dh)p \quad \text{(by Lemma 22.27.4)} \\ =& d(thp) \end{aligned}$$

since $dt = dp = 0$, and t is of degree zero. □

The following lemma is the analogue of Lemma 22.7.5.

09QN Lemma 22.27.7. In Situation 22.27.2 let $x_1 \to x_2 \to \ldots \to x_n$ be a sequence of composable morphisms in Comp($\mathcal{A}$). Then there exists a commutative diagram in Comp($\mathcal{A}$):

$$\begin{tikzcd} x_1 \arrow[r] & x_2 \arrow[r] & \ldots \arrow[r] & x_n \\ y_1 \arrow[r] \arrow[u] & y_2 \arrow[r] \arrow[u] & \ldots \arrow[r] & y_n \arrow[u] \end{tikzcd}$$

such that each $y_i \to y_{i+1}$ is an admissible monomorphism and each $y_i \to x_i$ is a homotopy equivalence.

Proof. The case for $n = 1$ is trivial: one simply takes $y_1 = x_1$ and the identity morphism on x_1 is in particular a homotopy equivalence. The case $n = 2$ is given by Lemma 22.27.6. Suppose we have constructed the diagram up to x_{n-1}. We apply Lemma 22.27.6 to the composition $y_{n-1} \to x_{n-1} \to x_n$ to obtain y_n. Then $y_{n-1} \to y_n$ will be an admissible monomorphism, and $y_n \to x_n$ a homotopy equivalence. □

The following lemma is the analogue of Lemma 22.7.6.

09QP Lemma 22.27.8. In Situation 22.27.2 let $x_i \to y_i \to z_i$ be morphisms in $\mathcal{A}$ ($i = 1, 2, 3$) such that $x_2 \to y_2 \to z_2$ is an admissible short exact sequence. Let $b : y_1 \to y_2$ and $b' : y_2 \to y_3$ be morphisms in $\text{Comp}(\mathcal{A})$ such that

$$\begin{tikzcd} x_1 \arrow[r] \arrow[d, "0"'] & y_1 \arrow[r] \arrow[d, "b"'] & z_1 \arrow[d, "0"'] \\ x_2 \arrow[r] & y_2 \arrow[r] & z_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x_2 \arrow[r] \arrow[d, "0"] & y_2 \arrow[r] \arrow[d, "b'"] & z_2 \arrow[d, "0"] \\ x_3 \arrow[r] & y_3 \arrow[r] & z_3 \end{tikzcd}$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 22.27.5, we can replace b and b' by homotopic maps $\tilde{b}$ and $\tilde{b}'$, such that the right square of the left diagram commutes and the left square of the right diagram commutes. Say $b = \tilde{b} + d(h)$ and $b' = \tilde{b}' + d(h')$ for degree -1 morphisms h and h' in $\mathcal{A}$. Hence

$$b'b = \tilde{b}'\tilde{b} + d(\tilde{b}'h + h'\tilde{b} + h'd(h))$$

since $d(\tilde{b}) = d(\tilde{b}') = 0$, i.e. $b'b$ is homotopic to $\tilde{b}'\tilde{b}$. We now want to show that $\tilde{b}'\tilde{b} = 0$. Because $x_2 \xrightarrow{f} y_2 \xrightarrow{g} z_2$ is an admissible short exact sequence, there exist degree 0 morphisms $\pi : y_2 \to x_2$ and $s : z_2 \to y_2$ such that $\text{id}_{y_2} = f\pi + sg$. Therefore

$$\tilde{b}'\tilde{b} = \tilde{b}'(f\pi + sg)\tilde{b} = 0$$

since $g\tilde{b} = 0$ and $\tilde{b}'f = 0$ as consequences of the two commuting squares. □

The following lemma is the analogue of Lemma 22.8.1.

09QQ Lemma 22.27.9. In Situation 22.27.2 let $0 \to x \to y \to z \to 0$ be an admissible short exact sequence in $\text{Comp}(\mathcal{A})$. The triangle

$$x \longrightarrow y \longrightarrow z \xrightarrow{\delta} x[1]$$

with $\delta : z \to x[1]$ as defined in Lemma 22.27.1 is up to canonical isomorphism in $K(\mathcal{A})$, independent of the choices made in Lemma 22.27.1.

Proof. Suppose δ is defined by the splitting

$$\begin{tikzcd} x \arrow[r, shift left, "a"] & y \arrow[l, shift left, "\pi"] \arrow[r, shift left, "b"] & z \arrow[l, shift left, "s"] \end{tikzcd}$$

and δ' is defined by the splitting with π', s' in place of π, s. Then

$$s' - s = (a\pi + sb)(s' - s) = a\pi s'$$

since $bs' = bs = 1_z$ and $\pi s = 0$. Similarly,

$$\pi' - \pi = (\pi' - \pi)(a\pi + sb) = \pi' sb$$

Since $\delta = \pi d(s)$ and $\delta' = \pi' d(s')$ as constructed in Lemma 22.27.1, we may compute

$$\delta' = \pi' d(s') = (\pi + \pi' sb)d(s + a\pi s') = \delta + d(\pi s')$$

using $\pi a = 1_x$, $ba = 0$, and $\pi' sbd(s') = \pi' sba\pi d(s') = 0$ by formula (5) in Lemma 22.27.1. $\square$

The following lemma is the analogue of Lemma 22.9.1.

09QR **Lemma 22.27.10.** In Situation 22.27.2 let $f : x \to y$ be a morphism in $\text{Comp}(\mathcal{A})$. The triangle $(y, c(f), x[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence

$$y \longrightarrow c(f) \longrightarrow x[1]$$

where the cone $c(f)$ is defined as in Lemma 22.27.1.

Proof. This follows from axiom (C). $\square$

The following lemma is the analogue of Lemma 22.9.2.

09QS **Lemma 22.27.11.** In Situation 22.27.2 let $\alpha : x \to y$ and $\beta : y \to z$ define an admissible short exact sequence

$$x \longrightarrow y \longrightarrow z$$

in $\text{Comp}(\mathcal{A})$. Let $(x, y, z, \alpha, \beta, \delta)$ be the associated triangle in $K(\mathcal{A})$. Then, the triangles

$$(z[-1], x, y, \delta[-1], \alpha, \beta) \quad \text{and} \quad (z[-1], x, c(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. We have a diagram of the form

$$\begin{tikzcd}
z[-1] \arrow[r, "{\delta[-1]}"] \arrow[d, "1"] & x \arrow[r, shift left, "\alpha"] \arrow[d, "1"] & y \arrow[l, shift left, "\tilde{\alpha}"] \arrow[r, shift left, "\beta"] \arrow[d, dotted] & z \arrow[l, shift left, "\tilde{\beta}"] \arrow[d, "1"] \\
z[-1] \arrow[r, "{\delta[-1]}"] & x \arrow[r, shift left, "i"] & c(\delta[-1]) \arrow[l, shift left, "\tilde{i}"] \arrow[r, shift left, "p"] & z \arrow[l, shift left, "\tilde{p}"]
\end{tikzcd}$$

with splittings to α, β, i, and p given by $\tilde{\alpha}, \tilde{\beta}, \tilde{i}$, and $\tilde{p}$ respectively. Define a morphism $y \to c(\delta[-1])$ by $i\tilde{\alpha} + \tilde{p}\beta$ and a morphism $c(\delta[-1]) \to y$ by $\alpha\tilde{i} + \tilde{\beta}p$. Let us first check that these define morphisms in $\text{Comp}(\mathcal{A})$. We remark that by identities from Lemma 22.27.1, we have the relation $\delta[-1] = \tilde{\alpha}d(\tilde{\beta}) = -d(\tilde{\alpha})\tilde{\beta}$ and the relation $\delta[-1] = \tilde{i}d(\tilde{p})$. Then

$$\begin{aligned} d(\tilde{\alpha}) &= d(\tilde{\alpha})\tilde{\beta}\beta \\ &= -\delta[-1]\beta \end{aligned}$$

where we have used equation (6) of Lemma 22.27.1 for the first equality and the preceeding remark for the second. Similarly, we obtain $d(\tilde{p}) = i\delta[-1]$. Hence

$$\begin{aligned} d(i\tilde{\alpha} + \tilde{p}\beta) &= d(i)\tilde{\alpha} + id(\tilde{\alpha}) + d(\tilde{p})\beta + \tilde{p}d(\beta) \\ &= id(\tilde{\alpha}) + d(\tilde{p})\beta \\ &= -i\delta[-1]\beta + i\delta[-1]\beta \\ &= 0 \end{aligned}$$

so $i\tilde{\alpha} + \tilde{p}\beta$ is indeed a morphism of $\text{Comp}(\mathcal{A})$. By a similar calculation, $\alpha\tilde{i} + \tilde{\beta}p$ is also a morphism of $\text{Comp}(\mathcal{A})$. It is immediate that these morphisms fit in the commutative diagram. We compute:

$$\begin{aligned}(i\tilde{\alpha} + \tilde{p}\beta)(\alpha\tilde{i} + \tilde{\beta}p) &= i\tilde{\alpha}\alpha\tilde{i} + i\tilde{\alpha}\tilde{\beta}p + \tilde{p}\beta\alpha\tilde{i} + \tilde{p}\beta\tilde{\beta}p \\ &= i\tilde{i} + \tilde{p}p \\ &= 1_{c(\delta[-1])}\end{aligned}$$

where we have freely used the identities of Lemma 22.27.1. Similarly, we compute $(\alpha\tilde{i} + \tilde{\beta}p)(i\tilde{\alpha} + \tilde{p}\beta) = 1_y$, so we conclude $y \cong c(\delta[-1])$. Hence, the two triangles in question are isomorphic. $\square$

The following lemma is the analogue of Lemma 22.9.3.

09QT Lemma 22.27.12. In Situation 22.27.2 let $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$ be morphisms in $\text{Comp}(\mathcal{A})$. Let

$$(a, b, c) : (x_1, y_1, c(f_1), f_1, i_1, p_1) \to (x_2, y_2, c(f_2), f_2, i_1, p_1)$$

be any morphism of triangles in $K(\mathcal{A})$. If a and b are homotopy equivalences, then so is c.

Proof. Since a and b are homotopy equivalences, they are invertible in $K(\mathcal{A})$ so let a^{-1} and b^{-1} denote their inverses in $K(\mathcal{A})$, giving us a commutative diagram

$$\begin{tikzcd} x_2 \arrow[r, "f_2"] \arrow[d, "a^{-1}"] & y_2 \arrow[r, "i_2"] \arrow[d, "b^{-1}"] & c(f_2) \arrow[d, "c'"] \\ x_1 \arrow[r, "f_1"] & y_1 \arrow[r, "i_1"] & c(f_1) \end{tikzcd}$$

where the map c' is defined via Lemma 22.27.3 applied to the left commutative box of the above diagram. Since the diagram commutes in $K(\mathcal{A})$, it suffices by Lemma 22.27.8 to prove the following: given a morphism of triangle $(1, 1, c) : (x, y, c(f), f, i, p) \to (x, y, c(f), f, i, p)$ in $K(\mathcal{A})$, the map c is an isomorphism in $K(\mathcal{A})$. We have the commutative diagrams in $K(\mathcal{A})$:

$$\begin{tikzcd} y \arrow[r] \arrow[d, "1"] & c(f) \arrow[r] \arrow[d, "c"] & x[1] \arrow[d, "1"] \\ y \arrow[r] & c(f) \arrow[r] & x[1] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} y \arrow[r] \arrow[d, "0"] & c(f) \arrow[r] \arrow[d, "c-1"] & x[1] \arrow[d, "0"] \\ y \arrow[r] & c(f) \arrow[r] & x[1] \end{tikzcd}$$

Since the rows are admissible short exact sequences, we obtain the identity $(c-1)^2 = 0$ by Lemma 22.27.8, from which we conclude that $2 - c$ is inverse to c in $K(\mathcal{A})$ so that c is an isomorphism. $\square$

The following lemma is the analogue of Lemma 22.9.4.

09QU Lemma 22.27.13. In Situation 22.27.2.

(1) Given an admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$. Then there exists a homotopy equivalence $e : C(\alpha) \to z$ such that the diagram

09QV (22.27.13.1)
$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d] & y \arrow[r, "b"] \arrow[d] & C(\alpha) \arrow[r, "-c"] \arrow[d, dotted, "e"] & x[1] \arrow[d] \\ x \arrow[r, "\alpha"] & y \arrow[r, "\beta"] & z \arrow[r, "\delta"] & x[1] \end{tikzcd}$$

defines an isomorphism of triangles in $K(\mathcal{A})$. Here $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is the admissible short exact sequence given as in axiom (C).

(2) Given a morphism $\alpha : x \to y$ in $\text{Comp}(\mathcal{A})$, let $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ be the factorization given as in Lemma 22.27.6, where the admissible monomorphism $x \xrightarrow{\tilde{\alpha}} y$ extends to the admissible short exact sequence

$$\begin{tikzcd} x \arrow[r, "\tilde{\alpha}"] & \tilde{y} \arrow[r] & z \end{tikzcd}$$

Then there exists an isomorphism of triangles

$$\begin{tikzcd} x \arrow[r, "\tilde{\alpha}"] \arrow[d] & \tilde{y} \arrow[r] \arrow[d] & z \arrow[r, "\delta"] \arrow[d, dotted, "e"] & x[1] \arrow[d] \\ x \arrow[r, "\alpha"] & y \arrow[r] & C(\alpha) \arrow[r, "-c"] & x[1] \end{tikzcd}$$

where the upper triangle is the triangle associated to the sequence $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to z$.

Proof. For (1), we consider the more complete diagram, without the sign change on c:

$$\begin{tikzcd} x \arrow[r, shift left, "\alpha"] \arrow[d] & y \arrow[l, shift left, "\pi"] \arrow[r, shift left, "b"] \arrow[d] & C(\alpha) \arrow[l, shift left, "p"] \arrow[r, shift left, "c"] \arrow[d, dotted, shift left, "e"] & x[1] \arrow[l, shift left, "\sigma"] \arrow[r, shift left, "\alpha"] \arrow[d] & y[1] \arrow[l, shift left, "\pi"] \\ x \arrow[r, shift left, "\alpha"] & y \arrow[l, shift left, "\pi"] \arrow[r, shift left, "\beta"] & z \arrow[l, shift left, "s"] \arrow[u, dotted, shift left, "f"] \arrow[r, "\delta"] & x[1] \end{tikzcd}$$

where the admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ is given the splitting π, s, and the admissible short exact sequence $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is given the splitting p, σ. Note that (identifying hom-sets under shifting)

$$\alpha = pd(\sigma) = -d(p)\sigma, \quad \delta = \pi d(s) = -d(\pi)s$$

by the construction in Lemma 22.27.1.

We define $e = \beta p$ and $f = bs - \sigma\delta$. We first check that they are morphisms in $\text{Comp}(\mathcal{A})$. To show that $d(e) = \beta d(p)$ vanishes, it suffices to show that $\beta d(p)b$ and $\beta d(p)\sigma$ both vanish, whereas

$$\beta d(p)b = \beta d(pb) = \beta d(1_y) = 0, \quad \beta d(p)\sigma = -\beta\alpha = 0$$

Similarly, to check that $d(f) = bd(s) - d(\sigma)\delta$ vanishes, it suffices to check the post-compositions by p and c both vanish, whereas

$$\begin{aligned} pbd(s) - pd(\sigma)\delta =& d(s) - \alpha\delta = d(s) - \alpha\pi d(s) = 0 \\ cbd(s) - cd(\sigma)\delta =& - cd(\sigma)\delta = -d(c\sigma)\delta = 0 \end{aligned}$$

The commutativity of left two squares of the diagram 22.27.13.1 follows directly from definition. Before we prove the commutativity of the right square (up to homotopy), we first check that e is a homotopy equivalence. Clearly,

$$ef = \beta p(bs - \sigma\delta) = \beta s = 1_z$$

To check that fe is homotopic to $1_{C(\alpha)}$, we first observe

$$b\alpha = bpd(\alpha) = d(\sigma), \quad \alpha c = -d(p)\sigma c = -d(p), \quad d(\pi)p = d(\pi)s\beta p = -\delta\beta p$$

Using these identities, we compute

$$\begin{aligned} 1_{C(\alpha)} = & bp + \sigma c \quad (\text{from } y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]) \\ = & b(\alpha\pi + s\beta)p + \sigma(\pi\alpha)c \quad (\text{from } x \xrightarrow{\alpha} y \xrightarrow{\beta} z) \\ = & d(\sigma)\pi p + bs\beta p - \sigma\pi d(p) \quad (\text{by the first two identities above}) \\ = & d(\sigma)\pi p + bs\beta p - \sigma\delta\beta p + \sigma\delta\beta p - \sigma\pi d(p) \\ = & (bs - \sigma\delta)\beta p + d(\sigma)\pi p - \sigma d(\pi)p - \sigma\pi d(p) \quad (\text{by the third identity above}) \\ = & fe + d(\sigma\pi p) \end{aligned}$$

since $\sigma \in \mathrm{Hom}^{-1}(x, C(\alpha))$ (cf. proof of Lemma 22.27.4). Hence e and f are homotopy inverses. Finally, to check that the right square of diagram 22.27.13.1 commutes up to homotopy, it suffices to check that $-cf = \delta$. This follows from

$$-cf = -c(bs - \sigma\delta) = c\sigma\delta = \delta$$

since $cb = 0$.

For (2), consider the factorization $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ given as in Lemma 22.27.6, so the second morphism is a homotopy equivalence. By Lemmas 22.27.3 and 22.27.12, there exists an isomorphism of triangles between

$$x \xrightarrow{\alpha} y \to C(\alpha) \to x[1] \quad \text{and} \quad x \xrightarrow{\tilde{\alpha}} \tilde{y} \to C(\tilde{\alpha}) \to x[1]$$

Since we can compose isomorphisms of triangles, by replacing α by $\tilde{\alpha}$, y by $\tilde{y}$, and $C(\alpha)$ by $C(\tilde{\alpha})$, we may assume α is an admissible monomorphism. In this case, the result follows from (1). □

The following lemma is the analogue of Lemma 22.10.1.

09QW Lemma 22.27.14. In Situation 22.27.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.

Proof. We will verify each of TR1, TR2, and TR3.

Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Since

$$x \xrightarrow{1_x} x \longrightarrow 0$$

is an admissible short exact sequence, $(x, x, 0, 1_x, 0, 0)$ is a distinguished triangle. Moreover, given a morphism $\alpha : x \to y$ in $\mathrm{Comp}(\mathcal{A})$, the triangle given by $(x, y, c(\alpha), \alpha, i, -p)$ is distinguished by Lemma 22.27.13.

Proof of TR2. Let $(x, y, z, \alpha, \beta, \gamma)$ be a triangle and suppose $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to x' \to y' \to z' \to 0$ such that the associated triangle $(x', y', z', \alpha', \beta', \gamma')$ is isomorphic to

$(y, z, x[1], \beta, \gamma, -\alpha[1])$. After rotating, we conclude that $(x, y, z, \alpha, \beta, \gamma)$ is isomorphic to $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$. By Lemma 22.27.11, we deduce that $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$ is isomorphic to $(z'[-1], x', c(\gamma'[-1]), \gamma'[-1], i, p)$. Composing the two isomorphisms with sign changes as indicated in the following diagram:

$$\begin{tikzcd}
& x \arrow[r, "\alpha"] \arrow[d] & y \arrow[r, "\beta"] \arrow[d] & z \arrow[r, "\gamma"] \arrow[d] & x[1] \arrow[d] \\
& z'[-1] \arrow[r, "-\gamma'[-1]"] \arrow[d, "-1_{z'[-1]}"'] & x \arrow[r, "\alpha'"] \arrow[d, equal] & y' \arrow[r, "\beta'"] \arrow[d] & z' \arrow[d, "-1_{z'}"] \\
& z'[-1] \arrow[r, "\gamma'[-1]"] & x \arrow[r, "\alpha'"] & c(\gamma'[-1]) \arrow[r, "-p"] & z'
\end{tikzcd}$$

We conclude that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished by Lemma 22.27.13 (2). Conversely, suppose that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished, so that by Lemma 22.27.13 (1), it is isomorphic to a triangle of the form $(x', y', c(\alpha'), \alpha', i, -p)$ for some morphism $\alpha' : x' \to y'$ in $\text{Comp}(\mathcal{A})$. The rotated triangle $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is isomorphic to the triangle $(y', c(\alpha'), x'[1], i, -p, -\alpha[1])$ which is isomorphic to $(y', c(\alpha'), x'[1], i, p, \alpha[1])$. By Lemma 22.27.10, this triangle is distinguished, from which it follows that $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished.

Proof of TR3: Suppose $(x, y, z, \alpha, \beta, \gamma)$ and $(x', y', z', \alpha', \beta', \gamma')$ are distinguished triangles of $\text{Comp}(\mathcal{A})$ and let $f : x \to x'$ and $g : y \to y'$ be morphisms such that $\alpha' \circ f = g \circ \alpha$. By Lemma 22.27.13, we may assume that $(x, y, z, \alpha, \beta, \gamma) = (x, y, c(\alpha), \alpha, i, -p)$ and $(x', y', z', \alpha', \beta', \gamma') = (x', y', c(\alpha'), \alpha', i', -p')$. Now apply Lemma 22.27.3 and we are done. □

The following lemma is the analogue of Lemma 22.10.2.

09QX Lemma 22.27.15. In Situation 22.27.2 given admissible monomorphisms $x \xrightarrow{\alpha} y$, $y \xrightarrow{\beta} z$ in $\mathcal{A}$, there exist distinguished triangles $(x, y, q_1, \alpha, p_1, \delta_1)$, $(x, z, q_2, \beta\alpha, p_2, \delta_2)$ and $(y, z, q_3, \beta, p_3, \delta_3)$ for which TR4 holds.

Proof. Given admissible monomorphisms $x \xrightarrow{\alpha} y$ and $y \xrightarrow{\beta} z$, we can find distinguished triangles, via their extensions to admissible short exact sequences,

$$\begin{tikzcd} x \arrow[r, shift left, "\alpha"] & y \arrow[l, shift left, "\pi_1"] \arrow[r, shift left, "p_1"] & q_1 \arrow[l, shift left, "s_1"] \arrow[r, "\delta_1"] & x[1] \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, shift left, "\beta\alpha"] & z \arrow[l, shift left, "\pi_1\pi_3"] \arrow[r, shift left, "p_2"] & q_2 \arrow[l, shift left, "s_2"] \arrow[r, "\delta_2"] & x[1] \end{tikzcd}$$

$$\begin{tikzcd} y \arrow[r, shift left, "\beta"] & z \arrow[l, shift left, "\pi_3"] \arrow[r, shift left, "p_3"] & q_3 \arrow[l, shift left, "s_3"] \arrow[r, "\delta_3"] & x[1] \end{tikzcd}$$

In these diagrams, the maps δ_i are defined as $\delta_i = \pi_i d(s_i)$ analagous to the maps defined in Lemma 22.27.1. They fit in the following solid commutative diagram

$$
\begin{tikzcd}
x \arrow[r, shift left, "\alpha"] \arrow[dr, shift left, "\beta\alpha"] & y \arrow[l, shift left, "\pi_1"] \arrow[r, shift left, "p_1"] \arrow[d, shift left, "\beta"] & q_1 \arrow[l, shift left, "s_1"] \arrow[r, "\delta_1"] \arrow[dd, dotted, "p_2\beta s_1"] & x[1] \\
 & z \arrow[ul, shift left, "\pi_1\pi_3"] \arrow[u, shift left, "\pi_3"] \arrow[d, shift left, "p_3"] \arrow[dr, shift left, "p_2"] & & \\
 & q_3 \arrow[u, shift left, "s_3"] \arrow[d, "\delta_3"] & q_2 \arrow[ul, shift left, "s_2"] \arrow[l, dotted, "p_3 s_2"] \arrow[dr, "\delta_2"] & \\
 & y[1] & & x[1]
\end{tikzcd}
$$

where we have defined the dashed arrows as indicated. Clearly, their composition $p_3 s_2 p_2 \beta s_1 = 0$ since $s_2 p_2 = 0$. We claim that they both are morphisms of $\text{Comp}(\mathcal{A})$. We can check this using equations in Lemma 22.27.1:

$$d(p_2 \beta s_1) = p_2 \beta d(s_1) = p_2 \beta \alpha \pi_1 d(s_1) = 0$$

since $p_2 \beta \alpha = 0$, and

$$d(p_3 s_2) = p_3 d(s_2) = p_3 \beta \alpha \pi_1 \pi_3 d(s_2) = 0$$

since $p_3 \beta = 0$. To check that $q_1 \to q_2 \to q_3$ is an admissible short exact sequence, it remains to show that in the underlying graded category, $q_2 = q_1 \oplus q_3$ with the above two morphisms as coprojection and projection. To do this, observe that in the underlying graded category $\mathcal{C}$, there hold

$$y = x \oplus q_1, \quad z = y \oplus q_3 = x \oplus q_1 \oplus q_3$$

where $\pi_1 \pi_3$ gives the projection morphism onto the first factor: $x \oplus q_1 \oplus q_3 \to z$. By axiom (A) on $\mathcal{A}$, $\mathcal{C}$ is an additive category, hence we may apply Homology, Lemma 12.3.10 and conclude that

$$\text{Ker}(\pi_1 \pi_3) = q_1 \oplus q_3$$

in $\mathcal{C}$. Another application of Homology, Lemma 12.3.10 to $z = x \oplus q_2$ gives $\text{Ker}(\pi_1 \pi_3) = q_2$. Hence $q_2 \cong q_1 \oplus q_3$ in $\mathcal{C}$. It is clear that the dashed morphisms defined above give coprojection and projection.

Finally, we have to check that the morphism $\delta : q_3 \to q_1[1]$ induced by the admissible short exact sequence $q_1 \to q_2 \to q_3$ agrees with $p_1 \delta_3$. By the construction in Lemma 22.27.1, the morphism δ is given by

$$
\begin{aligned}
p_1 \pi_3 s_2 d(p_2 s_3) = & p_1 \pi_3 s_2 p_2 d(s_3) \\
= & p_1 \pi_3 (1 - \beta \alpha \pi_1 \pi_3) d(s_3) \\
= & p_1 \pi_3 d(s_3) \quad (\text{since } \pi_3 \beta = 0) \\
= & p_1 \delta_3
\end{aligned}
$$

as desired. The proof is complete. $\square$

Putting everything together we finally obtain the analogue of Proposition 22.10.3.

09QY Proposition 22.27.16. In Situation 22.27.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a triangulated category.

Proof. By Lemma 22.27.14 we know that $K(\mathcal{A})$ is pre-triangulated. Combining Lemmas 22.27.7 and 22.27.15 with Derived Categories, Lemma 13.4.15, we conclude that $K(\mathcal{A})$ is a triangulated category. □

0FQF Lemma 22.27.17. Let R be a ring. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between differential graded categories over R satisfying axioms (A), (B), and (C) such that $F(x[1]) = F(x)[1]$. Then F induces an exact functor $K(\mathcal{A}) \to K(\mathcal{B})$ of triangulated categories.

Proof. Namely, if $x \to y \to z$ is an admissible short exact sequence in $\text{Comp}(\mathcal{A})$, then $F(x) \to F(y) \to F(z)$ is an admissible short exact sequence in $\text{Comp}(\mathcal{B})$. Moreover, the "boundary" morphism $\delta = \pi\text{d}(s) : z \to x[1]$ constructed in Lemma 22.27.1 produces the morphism $F(\delta) : F(z) \to F(x[1]) = F(x)[1]$ which is equal to the boundary map $F(\pi)\text{d}(F(s))$ for the admissible short exact sequence $F(x) \to F(y) \to F(z)$. □

22.28. Bimodules

0FQG We continue the discussion started in Section 22.12.

0FQH Definition 22.28.1. Bimodules. Let R be a ring.

(1) Let A and B be R-algebras. An (A, B)-bimodule is an R-module M equippend with R-bilinear maps

$$A \times M \to M, (a, x) \mapsto ax \quad \text{and} \quad M \times B \to M, (x, b) \mapsto xb$$

such that the following hold
  (a) $a'(ax) = (a'a)x$ and $(xb)b' = x(bb')$,
  (b) $a(xb) = (ax)b$, and
  (c) $1x = x = x1$.

(2) Let A and B be $\mathbf{Z}$-graded R-algebras. A graded (A, B)-bimodule is an (A, B)-bimodule M which has a grading $M = \bigoplus M^n$ such that $A^n M^m \subset M^{n+m}$ and $M^n B^m \subset M^{n+m}$.

(3) Let A and B be differential graded R-algebras. A differential graded (A, B)-bimodule is a graded (A, B)-bimodule which comes equipped with a differential $\text{d} : M \to M$ homogeneous of degree 1 such that $\text{d}(ax) = \text{d}(a)x + (-1)^{\deg(a)}a\text{d}(x)$ and $\text{d}(xb) = \text{d}(x)b + (-1)^{\deg(x)}x\text{d}(b)$ for homogeneous elements $a \in A$, $x \in M$, $b \in B$.

Observe that a differential graded (A, B)-bimodule M is the same thing as a right differential graded B-module which is also a left differential graded A-module such that the grading and differentials agree and such that the A-module structure commutes with the B-module structure. Here is a precise statement.

0FQI Lemma 22.28.2. Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let M be a right differential graded B-module. There is a 1-to-1 correspondence between (A, B)-bimodule structures on M compatible with the given differential graded B-module structure and homomorphisms

$$A \longrightarrow \text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(M, M)$$

of differential graded R-algebras.

Proof. Let $\mu : A \times M \to M$ define a left differential graded A-module structure on the underlying complex of R-modules $M^\bullet$ of M. By Lemma 22.13.1 the structure μ corresponds to a map $\gamma : A \to \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$ of differential graded R-algebras. The assertion of the lemma is simply that μ commutes with the B-action, if and only if γ ends up inside

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(M, M) \subset \mathrm{Hom}^\bullet(M^\bullet, M^\bullet)$$

We omit the detailed calculation. □

Let M be a differential graded (A, B)-bimodule. Recall from Section 22.11 that the left differential graded A-module structure corresponds to a right differential graded A^{opp}-module structure. Since the A and B module structures commute this gives M the structure of a differential graded $A^{opp} \otimes_R B$-module:

$$x \cdot (a \otimes b) = (-1)^{\deg(a)\deg(x)} axb$$

Conversely, if we have a differential graded $A^{opp} \otimes_R B$-module M, then we can use the formula above to get a differential graded (A, B)-bimodule.

0FQJ Lemma 22.28.3. Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. The construction above defines an equivalence of categories

$$\begin{matrix} \text{differential graded} \\ (A,B)\text{-bimodules} \end{matrix} \longleftrightarrow \begin{matrix} \text{right differential graded} \\ A^{opp} \otimes_R B\text{-modules} \end{matrix}$$

Proof. Immediate from discussion the above. □

Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let P be a differential graded (A, B)-bimodule. We say P has property (P) if it there exists a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded (A, B)-bimodules such that

(1) $P = \bigcup F_pP$,
(2) the inclusions $F_iP \to F_{i+1}P$ are split as graded (A, B)-bimodule maps,
(3) the quotients $F_{i+1}P/F_iP$ are isomorphic as differential graded (A, B)-bimodules to a direct sum of $(A \otimes_R B)[k]$.

0FQK Lemma 22.28.4. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let M be a differential graded (A, B)-bimodule. There exists a homomorphism $P \to M$ of differential graded (A, B)-bimodules which is a quasi-isomorphism such that P has property (P) as defined above.

Proof. Immediate from Lemmas 22.28.3 and 22.20.4. □

0FQL Lemma 22.28.5. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let P be a differential graded (A, B)-bimodule having property (P) with corresponding filtration $F_\bullet$, then we obtain a short exact sequence

$$0 \to \bigoplus F_iP \to \bigoplus F_iP \to P \to 0$$

of differential graded (A, B)-bimodules which is split as a sequence of graded (A, B)-bimodules.

Proof. Immediate from Lemmas 22.28.3 and 22.20.1. □

22.29. Bimodules and tensor product

0FQM Let R be a ring. Let A and B be R-algebras. Let M be a right A-module. Let N be a (A, B)-bimodule. Then $M \otimes_A N$ is a right B-module.

If in the situation of the previous paragraph A and B are $\mathbf{Z}$-graded algebras, M is a graded A-module, and N is a graded (A, B)-bimodule, then $M \otimes_A N$ is a right graded B-module. The construction is functorial in M and defines a functor

$$- \otimes_A N : \text{Mod}^{gr}_A \longrightarrow \text{Mod}^{gr}_B$$

of graded categories as in Example 22.25.6. Namely, if M and M' are graded A-modules and $f : M \to M'$ is an A-module homomorphism homogeneous of degree n, then $f \otimes \text{id}_N : M \otimes_A N \to M' \otimes_A N$ is a B-module homomorphism homogeneous of degree n.

If in the situation of the previous paragraph (A, d) and (B, d) are differential graded algebras, M is a differential graded A-module, and N is a differential graded (A, B)-bimodule, then $M \otimes_A N$ is a right differential graded B-module.

09LM Lemma 22.29.1. Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Then $M \mapsto M \otimes_A N$ defines a functor

$$- \otimes_A N : \text{Mod}^{dg}_{(A,\text{d})} \longrightarrow \text{Mod}^{dg}_{(B,\text{d})}$$

of differential graded categories. This functor induces functors

$$\text{Mod}_{(A,\text{d})} \to \text{Mod}_{(B,\text{d})} \quad \text{and} \quad K(\text{Mod}_{(A,\text{d})}) \to K(\text{Mod}_{(B,\text{d})})$$

by an application of Lemma 22.26.5.

Proof. Above we have seen how the construction defines a functor of underlying graded categories. Thus it suffices to show that the construction is compatible with differentials. Let M and M' be differential graded A-modules and let $f : M \to M'$ be an A-module homomorphism which is homogeneous of degree n. Then we have

$$\text{d}(f) = \text{d}_{M'} \circ f - (-1)^n f \circ \text{d}_M$$

On the other hand, we have

$$\text{d}(f \otimes \text{id}_N) = \text{d}_{M' \otimes_A N} \circ (f \otimes \text{id}_N) - (-1)^n (f \otimes \text{id}_N) \circ \text{d}_{M \otimes_A N}$$

Applying this to an element $x \otimes y$ with $x \in M$ and $y \in N$ homogeneous we get

$$\begin{aligned} \text{d}(f \otimes \text{id}_N)(x \otimes y) = & \text{d}_{M'}(f(x)) \otimes y + (-1)^{n+\deg(x)} f(x) \otimes \text{d}_N(y) \\ & - (-1)^n f(\text{d}_M(x)) \otimes y - (-1)^{n+\deg(x)} f(x) \otimes \text{d}_N(y) \\ = & \text{d}(f)(x \otimes y) \end{aligned}$$

Thus we see that $\text{d}(f) \otimes \text{id}_N = \text{d}(f \otimes \text{id}_N)$ and the proof is complete. □

0FQN Remark 22.29.2. Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Let M be a right differential graded A-module. Then for every $k \in \mathbf{Z}$ there is an isomorphism

$$(M \otimes_A N)[k] \longrightarrow M[k] \otimes_A N$$

of right differential graded B-modules defined without the intervention of signs, see More on Algebra, Section 15.72.

If we have a ring R and R-algebras A, B, and C, a right A-module M, an (A,B)-bimodule N, and a (B,C)-bimodule N', then $N \otimes_B N'$ is a (A,C)-bimodule and we have

$$(M \otimes_A N) \otimes_B N' = M \otimes_A (N \otimes_B N')$$

This equality continuous to hold in the graded and in the differential graded case. See More on Algebra, Section 15.72 for sign rules.

22.30. Bimodules and internal hom

0FQP Let R be a ring. If A is an R-algebra (see our conventions in Section 22.2) and M, M' are right A-modules, then we define

$$\mathrm{Hom}_A(M, M') = \{f : M \to M' \mid f \text{ is } A\text{-linear}\}$$

as usual.

Let R-be a ring. Let A and B be R-algebras. Let N be an (A,B)-bimodule. Let N' be a right B-module. In this situation we will think of

$$\mathrm{Hom}_B(N, N')$$

as a right A-module using precomposition.

Let R-be a ring. Let A and B be $\mathbf{Z}$-graded R-algebras. Let N be a graded (A,B)-bimodule. Let N' be a right graded B-module. In this situation we will think of the graded R-module

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}_B}(N, N')$$

defined in Example 22.25.6 as a right graded A-module using precomposition. The construction is functorial in N' and defines a functor

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}_B}(N, -) : \mathrm{Mod}^{gr}_B \longrightarrow \mathrm{Mod}^{gr}_A$$

of graded categories as in Example 22.25.6. Namely, if N_1 and N_2 are graded B-modules and $f : N_1 \to N_2$ is a B-module homomorphism homogeneous of degree n, then the induced map $\mathrm{Hom}_{\mathrm{Mod}^{gr}_B}(N, N_1) \to \mathrm{Hom}_{\mathrm{Mod}^{gr}_B}(N, N_2)$ is an A-module homomorphism homogeneous of degree n.

Let R be a ring. Let A and B be differential $\mathbf{Z}$-graded R-algebras. Let N be a differential graded (A,B)-bimodule. Let N' be a right differential graded B-module. In this situation we will think of the differential graded R-module

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, N')$$

defined in Example 22.26.8 as a right differential graded A-module using precomposition as in the graded case. This is compatible with differentials because multiplication is the composition

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_B}(N, N') \otimes_R A \to \mathrm{Hom}_{\mathrm{Mod}^{dg}_B}(N, N') \otimes_R \mathrm{Hom}_{\mathrm{Mod}^{dg}_B}(N, N) \to \mathrm{Hom}_{\mathrm{Mod}^{dg}_B}(N, N')$$

The first arrow uses the map of Lemma 22.28.2 and the second arrow is the composition in the differential graded category $\mathrm{Mod}^{dg}_{(B,\mathrm{d})}$.

0FQQ **Lemma 22.30.1.** Let R be a ring. Let (A,d) and (B,d) be differential graded algebras over R. Let N be a differential graded (A,B)-bimodule. The construction above defines a functor

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, -) : \mathrm{Mod}^{dg}_{(B,\mathrm{d})} \longrightarrow \mathrm{Mod}^{dg}_{(A,\mathrm{d})}$$

of differential graded categories. This functor induces functors

$$\text{Mod}_{(B,\text{d})} \to \text{Mod}_{(A,\text{d})} \quad \text{and} \quad K(\text{Mod}_{(B,\text{d})}) \to K(\text{Mod}_{(A,\text{d})})$$

by an application of Lemma 22.26.5.

Proof. Above we have seen how the construction defines a functor of underlying graded categories. Thus it suffices to show that the construction is compatible with differentials. Let N_1 and N_2 be differential graded B-modules. Write

$$H_{12} = \text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N_1, N_2), \quad H_1 = \text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, N_1), \quad H_2 = \text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, N_2)$$

Consider the composition

$$c : H_{12} \otimes_R H_1 \longrightarrow H_2$$

in the differential graded category $\text{Mod}^{dg}_{(B,\text{d})}$. Let $f : N_1 \to N_2$ be a B-module homomorphism which is homogeneous of degree n, in other words, $f \in H^n_{12}$. The functor in the lemma sends f to $c_f : H_1 \to H_2$, $g \mapsto c(f, g)$. Simlarly for $\text{d}(f)$. On the other hand, the differential on

$$\text{Hom}_{\text{Mod}^{dg}_{(A,\text{d})}}(H_1, H_2)$$

sends c_f to $\text{d}_{H_2} \circ c_f - (-1)^n c_f \circ \text{d}_{H_1}$. As c is a morphism of complexes of R-modules we have $\text{d}c(f, g) = c(\text{d}f, g) + (-1)^n c(f, \text{d}g)$. Hence we see that

$$\begin{aligned}(\text{d}c_f)(g) &= \text{d}c(f, g) - (-1)^n c(f, \text{d}g) \\ &= c(\text{d}f, g) + (-1)^n c(f, \text{d}g) - (-1)^n c(f, \text{d}g) \\ &= c(\text{d}f, g) = c_{\text{d}f}(g)\end{aligned}$$

and the proof is complete. □

0FQR Remark 22.30.2. Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Let N' be a right differential graded B-module. Then for every $k \in \mathbf{Z}$ there is an isomorphism

$$\text{Hom}_{\text{Mod}^{gr}_B}(N, N')[k] \longrightarrow \text{Hom}_{\text{Mod}^{gr}_B}(N, N'[k])$$

of right differential graded A-modules defined without the intervention of signs, see More on Algebra, Section 15.72.

09LN Lemma 22.30.3. Let R be a ring. Let A and B be R-algebras. Let M be a right A-module, N an (A, B)-bimodule, and N' a right B-module. Then we have a canonical isomorphism

$$\text{Hom}_B(M \otimes_A N, N') = \text{Hom}_A(M, \text{Hom}_B(N, N'))$$

of R-modules. If A, B, M, N, N' are compatibly graded, then we have a canonical isomorphism

$$\text{Hom}_{\text{Mod}^{gr}_B}(M \otimes_A N, N') = \text{Hom}_{\text{Mod}^{gr}_A}(M, \text{Hom}_{\text{Mod}^{gr}_B}(N, N'))$$

of graded R-modules If A, B, M, N, N' are compatibly differential graded, then we have a canonical isomorphism

$$\text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(M \otimes_A N, N') = \text{Hom}_{\text{Mod}^{dg}_{(A,\text{d})}}(M, \text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, N'))$$

of complexes of R-modules.

Proof. Omitted. Hint: in the ungraded case interpret both sides as A-bilinear maps $\psi : M \times N \to N'$ which are B-linear on the right. In the (differential) graded case, use the isomorphism of More on Algebra, Lemma 15.71.1 and check it is compatible with the module structures. Alternatively, use the isomorphism of Lemma 22.13.2 and show that it is compatible with the B-module structures. $\square$

22.31. Derived Hom

09LF This section is analogous to More on Algebra, Section 15.73.

Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Consider the functor

09LG (22.31.0.1)
$$\text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, -) : \text{Mod}_{(B,\text{d})} \longrightarrow \text{Mod}_{(A,\text{d})}$$

of Section 22.30.

09LH Lemma 22.31.1. The functor (22.31.0.1) defines an exact functor $K(\text{Mod}_{(B,\text{d})}) \to K(\text{Mod}_{(A,\text{d})})$ of triangulated categories.

Proof. Via Lemma 22.30.1 and Remark 22.30.2 this follows from the general principle of Lemma 22.27.17. $\square$

Recall that we have an exact functor of triangulated categories

$$\text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, -) : K(\text{Mod}_{(B,\text{d})}) \to K(\text{Mod}_{(A,\text{d})})$$

see Lemma 22.31.1. Consider the diagram

$$\begin{tikzcd} K(\text{Mod}_{(B,\text{d})}) \arrow[r, "\text{see above}"'] \arrow[d] \arrow[dr, "F"] & K(\text{Mod}_{(A,\text{d})}) \arrow[d] \\ D(B, \text{d}) \arrow[r, dotted] & D(A, \text{d}) \end{tikzcd}$$

We would like to construct a dotted arrow as the right derived functor of the composition F. (Warning: in most interesting cases the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 13.14 we want to compute the right derived functor of F with respect to the multiplicative system of quasi-isomorphisms in $K(\text{Mod}_{(A,\text{d})})$.

09LI Lemma 22.31.2. In the situation above, the right derived functor of F exists. We denote it $R\,\text{Hom}(N, -) : D(B, \text{d}) \to D(A, \text{d})$.

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{I}$ of objects we will use the objects with property (I). Property (1) was shown in Lemma 22.21.4. Property (2) holds because if $s : I \to I'$ is a quasi-isomorphism of modules with property (I), then s is a homotopy equivalence by Lemma 22.22.3. $\square$

0BYV Lemma 22.31.3. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $f : N \to N'$ be a homomorphism of differential graded (A, B)-bimodules. Then f induces a morphism of functors

$$- \circ f : R\,\text{Hom}(N', -) \longrightarrow R\,\text{Hom}(N, -)$$

If f is a quasi-isomorphism, then $f \circ -$ is an isomorphism of functors.

Proof. Write $\mathcal{B} = \text{Mod}^{dg}_{(B,\text{d})}$ the differential graded category of differential graded B-modules, see Example 22.26.8. Let I be a differential graded B-module with property (I). Then $f \circ - : \text{Hom}_{\mathcal{B}}(N', I) \to \text{Hom}_{\mathcal{B}}(N, I)$ is a map of differential graded A-modules. Moreover, this is functorial with respect to I. Since the functors $R\text{Hom}(N', -)$ and $R\text{Hom}(N, -)$ are computed by applying $\text{Hom}_{\mathcal{B}}$ into objects with property (I) (Lemma 22.31.2) we obtain a transformation of functors as indicated.

Assume that f is a quasi-isomorphism. Let $F_\bullet$ be the given filtration on I. Since $I = \lim I/F_pI$ we see that $\text{Hom}_{\mathcal{B}}(N', I) = \lim \text{Hom}_{\mathcal{B}}(N', I/F_pI)$ and $\text{Hom}_{\mathcal{B}}(N, I) = \lim \text{Hom}_{\mathcal{B}}(N, I/F_pI)$. Since the transition maps in the system I/F_pI are split as graded modules, we see that the transition maps in the systems $\text{Hom}_{\mathcal{B}}(N', I/F_pI)$ and $\text{Hom}_{\mathcal{B}}(N, I/F_pI)$ are surjective. Hence $\text{Hom}_{\mathcal{B}}(N', I)$, resp. $\text{Hom}_{\mathcal{B}}(N, I)$ viewed as a complex of abelian groups computes $R\lim$ of the system of complexes $\text{Hom}_{\mathcal{B}}(N', I/F_pI)$, resp. $\text{Hom}_{\mathcal{B}}(N, I/F_pI)$. See More on Algebra, Lemma 15.86.1. Thus it suffices to prove each

$$\text{Hom}_{\mathcal{B}}(N', I/F_pI) \to \text{Hom}_{\mathcal{B}}(N, I/F_pI)$$

is a quasi-isomorphism. Since the surjections $I/F_{p+1}I \to I/F_pI$ are split as maps of graded B-modules we see that

$$0 \to \text{Hom}_{\mathcal{B}}(N', F_pI/F_{p+1}I) \to \text{Hom}_{\mathcal{B}}(N', I/F_{p+1}I) \to \text{Hom}_{\mathcal{B}}(N', I/F_pI) \to 0$$

is a short exact sequence of differential graded A-modules. There is a similar sequence for N and f induces a map of short exact sequences. Hence by induction on p (starting with $p = 0$ when $I/F_0I = 0$) we conclude that it suffices to show that the map $\text{Hom}_{\mathcal{B}}(N', F_pI/F_{p+1}I) \to \text{Hom}_{\mathcal{B}}(N, F_pI/F_{p+1}I)$ is a quasi-isomorphism. Since $F_pI/F_{p+1}I$ is a product of shifts of $A^\vee$ it suffice to prove $\text{Hom}_{\mathcal{B}}(N', B^\vee[k]) \to \text{Hom}_{\mathcal{B}}(N, B^\vee[k])$ is a quasi-isomorphism. By Lemma 22.19.3 it suffices to show $(N')^\vee \to N^\vee$ is a quasi-isomorphism. This is true because f is a quasi-isomorphism and $(\ )^\vee$ is an exact functor. $\square$

0CS5 Lemma 22.31.4. Let (A, d) and (B, d) be differential graded algebras over a ring R. Let N be a differential graded (A, B)-bimodule. Then for every $n \in \mathbf{Z}$ there are isomorphisms

$$H^n(R\text{Hom}(N, M)) = \text{Ext}^n_{D(B,\text{d})}(N, M)$$

of R-modules functorial in M. It is also functorial in N with respect to the operation described in Lemma 22.31.3.

Proof. In the proof of Lemma 22.31.2 we have seen

$$R\text{Hom}(N, M) = \text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, I)$$

as a differential graded A-module where $M \to I$ is a quasi-isomorphism of M into a differential graded B-module with property (I). Hence this complex has the correct cohomology modules by Lemma 22.22.3. We omit a discussion of the functorial nature of these identifications. $\square$

0BYW Lemma 22.31.5. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. If $\text{Hom}_{D(B,\text{d})}(N, N') = \text{Hom}_{K(\text{Mod}_{(B,\text{d})})}(N, N')$ for all $N' \in K(B, \text{d})$, for example if N has property (P) as a differential graded B-module, then

$$R\text{Hom}(N, M) = \text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, M)$$

functorially in M in $D(B, \text{d})$.

Proof. By construction (Lemma 22.31.2) to find $R\operatorname{Hom}(N, M)$ we choose a quasi-isomorphism $M \to I$ where I is a differential graded B-module with property (I) and we set $R\operatorname{Hom}(N, M) = \operatorname{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, I)$. By assumption the map

$$\operatorname{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, M) \longrightarrow \operatorname{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, I)$$

induced by $M \to I$ is a quasi-isomorphism, see discussion in Example 22.26.8. This proves the lemma. If N has property (P) as a B-module, then we see that the assumption is satisfied by Lemma 22.22.3. $\square$

22.32. Variant of derived Hom

09LJ Let $\mathcal{A}$ be an abelian category. Consider the differential graded category $\text{Comp}^{dg}(\mathcal{A})$ of complexes of $\mathcal{A}$, see Example 22.26.6. Let $K^\bullet$ be a complex of $\mathcal{A}$. Set

$$(E, \text{d}) = \operatorname{Hom}_{\text{Comp}^{dg}(\mathcal{A})}(K^\bullet, K^\bullet)$$

and consider the functor of differential graded categories

$$\text{Comp}^{dg}(\mathcal{A}) \longrightarrow \text{Mod}^{dg}_{(E,\text{d})}, \quad X^\bullet \longmapsto \operatorname{Hom}_{\text{Comp}^{dg}(\mathcal{A})}(K^\bullet, X^\bullet)$$

of Lemma 22.26.10.

09LK Lemma 22.32.1. In the situation above. If the right derived functor $R\operatorname{Hom}(K^\bullet, -)$ of $\operatorname{Hom}(K^\bullet, -) : K(\mathcal{A}) \to D(\text{Ab})$ is everywhere defined on $D(\mathcal{A})$, then we obtain a canonical exact functor

$$R\operatorname{Hom}(K^\bullet, -) : D(\mathcal{A}) \longrightarrow D(E, \text{d})$$

of triangulated categories which reduces to the usual one on taking associated complexes of abelian groups.

Proof. Note that we have an associated functor $K(\mathcal{A}) \to K(\text{Mod}_{(E,\text{d})})$ by Lemma 22.26.10. We claim this functor is an exact functor of triangulated categories. Namely, let $f : A^\bullet \to B^\bullet$ be a map of complexes of $\mathcal{A}$. Then a computation shows that

$$\operatorname{Hom}_{\text{Comp}^{dg}(\mathcal{A})}(K^\bullet, C(f)^\bullet) = C\left(\operatorname{Hom}_{\text{Comp}^{dg}(\mathcal{A})}(K^\bullet, A^\bullet) \to \operatorname{Hom}_{\text{Comp}^{dg}(\mathcal{A})}(K^\bullet, B^\bullet)\right)$$

where the right hand side is the cone in $\text{Mod}_{(E,\text{d})}$ defined earlier in this chapter. This shows that our functor is compatible with cones, hence with distinguished triangles. Let $X^\bullet$ be an object of $K(\mathcal{A})$. Consider the category of quasi-isomorphisms $s : X^\bullet \to Y^\bullet$. We are given that the functor $(s : X^\bullet \to Y^\bullet) \mapsto \operatorname{Hom}_{\mathcal{A}}(K^\bullet, Y^\bullet)$ is essentially constant when viewed in $D(\text{Ab})$. But since the forgetful functor $D(E, \text{d}) \to D(\text{Ab})$ is compatible with taking cohomology, the same thing is true in $D(E, \text{d})$. This proves the lemma. $\square$

Warning: Although the lemma holds as stated and may be useful as stated, the differential algebra E isn't the "correct" one unless $H^n(E) = \operatorname{Ext}^n_{D(\mathcal{A})}(K^\bullet, K^\bullet)$ for all $n \in \mathbf{Z}$.

22.33. Derived tensor product

09LP This section is analogous to More on Algebra, Section 15.60.

Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Consider the functor

09LQ (22.33.0.1)
$$\mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(B,\mathrm{d})}, \quad M \longmapsto M \otimes_A N$$

defined in Section 22.29.

09LR **Lemma 22.33.1.** *The functor (22.33.0.1) defines an exact functor of triangulated categories $K(\mathrm{Mod}_{(A,\mathrm{d})}) \to K(\mathrm{Mod}_{(B,\mathrm{d})})$.*

Proof. Via Lemma 22.29.1 and Remark 22.29.2 this follows from the general principle of Lemma 22.27.17. □

At this point we can consider the diagram

$$\begin{tikzcd} K(\mathrm{Mod}_{(A,\mathrm{d})}) \arrow[rr, "-\otimes_A N"'] \arrow[d] \arrow[drr, "F"] & & K(\mathrm{Mod}_{(B,\mathrm{d})}) \arrow[d] \\ D(A,\mathrm{d}) \arrow[rr, dotted] & & D(B,\mathrm{d}) \end{tikzcd}$$

The dotted arrow that we will construct below will be the left derived functor of the composition F. (Warning: the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 13.14 we want to compute the left derived functor of F with respect to the multiplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

09LS **Lemma 22.33.2.** *In the situation above, the left derived functor of F exists. We denote it $- \otimes_A^{\mathbf{L}} N : D(A, \mathrm{d}) \to D(B, \mathrm{d})$.*

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 22.20.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 22.22.3. □

09S3 **Lemma 22.33.3.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $f : N \to N'$ be a homomorphism of differential graded (A, B)-bimodules. Then f induces a morphism of functors*

$$1 \otimes f : - \otimes_A^{\mathbf{L}} N \longrightarrow - \otimes_A^{\mathbf{L}} N'$$

If f is a quasi-isomorphism, then $1 \otimes f$ is an isomorphism of functors.

Proof. Let M be a differential graded A-module with property (P). Then $1 \otimes f : M \otimes_A N \to M \otimes_A N'$ is a map of differential graded B-modules. Moreover, this is functorial with respect to M. Since the functors $- \otimes_A^{\mathbf{L}} N$ and $- \otimes_A^{\mathbf{L}} N'$ are computed by tensoring on objects with property (P) (Lemma 22.33.2) we obtain a transformation of functors as indicated.

Assume that f is a quasi-isomorphism. Let $F_\bullet$ be the given filtration on M. Observe that $M \otimes_A N = \operatorname{colim} F_i(M) \otimes_A N$ and $M \otimes_A N' = \operatorname{colim} F_i(M) \otimes_A N'$. Hence it suffices to show that $F_n(M) \otimes_A N \to F_n(M) \otimes_A N'$ is a quasi-isomorphism (filtered colimits are exact, see Algebra, Lemma 10.8.8). Since the inclusions $F_n(M) \to F_{n+1}(M)$ are split as maps of graded A-modules we see that

$$0 \to F_n(M) \otimes_A N \to F_{n+1}(M) \otimes_A N \to F_{n+1}(M)/F_n(M) \otimes_A N \to 0$$

is a short exact sequence of differential graded B-modules. There is a similar sequence for N' and f induces a map of short exact sequences. Hence by induction on n (starting with $n = -1$ when $F_{-1}(M) = 0$) we conclude that it suffices to show that the map $F_{n+1}(M)/F_n(M) \otimes_A N \to F_{n+1}(M)/F_n(M) \otimes_A N'$ is a quasi-isomorphism. This is true because $F_{n+1}(M)/F_n(M)$ is a direct sum of shifts of A and the result is true for $A[k]$ as $f : N \to N'$ is a quasi-isomorphism. □

0GZ2 Lemma 22.33.4. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule which has property (P) as a left differential graded A-module. Then $M \otimes_A^{\mathbf{L}} N$ is computed by $M \otimes_A N$ for all differential graded A-modules M.

Proof. Let $f : M \to M'$ be a homomorphism of differential graded A-modules which is a quasi-isomorphism. We claim that $f \otimes \text{id} : M \otimes_A N \to M' \otimes_A N$ is a quasi-isomorphism. If this is true, then by the construction of the derived tensor product in the proof of Lemma 22.33.2 we obtain the desired result. The construction of the map $f \otimes \text{id}$ only depends on the left differential graded A-module structure on N. Moreover, we have $M \otimes_A N = N \otimes_{A^{opp}} M = N \otimes_{A^{opp}}^{\mathbf{L}} M$ because N has property (P) as a differential graded A^{opp}-module. Hence the claim follows from Lemma 22.33.3. □

09LT Lemma 22.33.5. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Then the functor

$$- \otimes_A^{\mathbf{L}} N : D(A, \text{d}) \longrightarrow D(B, \text{d})$$

of Lemma 22.33.2 is a left adjoint to the functor

$$R\operatorname{Hom}(N, -) : D(B, \text{d}) \longrightarrow D(A, \text{d})$$

of Lemma 22.31.2.

Proof. This follows from Derived Categories, Lemma 13.30.1 and the fact that $- \otimes_A N$ and $\operatorname{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(N, -)$ are adjoint by Lemma 22.30.3. □

0BYX Example 22.33.6. Let R be a ring. Let $(A, \text{d}) \to (B, \text{d})$ be a homomorphism of differential graded R-algebras. Then we can view B as a differential graded (A, B)-bimodule and we get a functor

$$- \otimes_A B : D(A, \text{d}) \longrightarrow D(B, \text{d})$$

By Lemma 22.33.5 the left adjoint of this is the functor $R\operatorname{Hom}(B, -)$. For a differential graded B-module let us denote N_A the differential graded A-module obtained from N by restriction via $A \to B$. Then we clearly have a canonical isomorphism

$$\operatorname{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(B, N) \longrightarrow N_A, \quad f \longmapsto f(1)$$

functorial in the B-module N. Thus we see that $R\operatorname{Hom}(B, -)$ is the restriction functor and we obtain

$$\operatorname{Hom}_{D(A,\text{d})}(M, N_A) = \operatorname{Hom}_{D(B,\text{d})}(M \otimes_A^{\mathbf{L}} B, N)$$

bifunctorially in M and N exactly as in the case of commutative rings. Finally, observe that restriction is a tensor functor as well, since $N_A = N \otimes_B {}_BB_A = N \otimes_B^{\mathbf{L}} {}_BB_A$ where ${}_BB_A$ is B viewed as a differential graded (B, A)-bimodule.

09R9 Lemma 22.33.7. With notation and assumptions as in Lemma 22.33.5. Assume

(1) N defines a compact object of $D(B, \text{d})$, and
(2) the map $H^k(A) \to \text{Hom}_{D(B,\text{d})}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.

Then the functor $- \otimes_A^{\mathbf{L}} N$ is fully faithful.

Proof. Our functor has a left adjoint given by $R\,\text{Hom}(N, -)$ by Lemma 22.33.5. By Categories, Lemma 4.24.4 it suffices to show that for a differential graded A-module M the map

$$M \longrightarrow R\,\text{Hom}(N, M \otimes_A^{\mathbf{L}} N)$$

is an isomorphism in $D(A, \text{d})$. For this it suffices to show that

$$H^n(M) \longrightarrow \text{Ext}^n_{D(B,\text{d})}(N, M \otimes_A^{\mathbf{L}} N)$$

is an isomorphism, see Lemma 22.31.4. Since N is a compact object the right hand side commutes with direct sums. Thus by Remark 22.22.5 it suffices to prove this map is an isomorphism for $M = A[k]$. Since $(A[k] \otimes_A^{\mathbf{L}} N) = N[k]$ by Remark 22.29.2, assumption (2) on N is that the result holds for these. $\square$

0BYZ Lemma 22.33.8. Let $R \to R'$ be a ring map. Let (A, d) be a differential graded R-algebra. Let (A', d) be the base change, i.e., $A' = A \otimes_R R'$. If A is K-flat as a complex of R-modules, then

(1) $- \otimes_A^{\mathbf{L}} A' : D(A, \text{d}) \to D(A', \text{d})$ is equal to the right derived functor of
$$K(A, \text{d}) \longrightarrow K(A', \text{d}), \quad M \longmapsto M \otimes_R R'$$
(2) the diagram
$$\begin{tikzcd} D(A, \text{d}) \arrow[r, "- \otimes_A^{\mathbf{L}} A'"'] \arrow[d, "restriction"'] & D(A', \text{d}) \arrow[d, "restriction"] \\ D(R) \arrow[r, "- \otimes_R^{\mathbf{L}} R'"] & D(R') \end{tikzcd}$$
commutes, and
(3) if M is K-flat as a complex of R-modules, then the differential graded A'-module $M \otimes_R R'$ represents $M \otimes_A^{\mathbf{L}} A'$.

Proof. For any differential graded A-module M there is a canonical map

$$c_M : M \otimes_R R' \longrightarrow M \otimes_A A'$$

Let P be a differential graded A-module with property (P). We claim that c_P is an isomorphism and that P is K-flat as a complex of R-modules. This will prove all the results stated in the lemma by formal arguments using the definition of derived tensor product in Lemma 22.33.2 and More on Algebra, Section 15.59.

Let $F_\bullet$ be the filtration on P showing that P has property (P). Note that c_A is an isomorphism and A is K-flat as a complex of R-modules by assumption. Hence the same is true for direct sums of shifts of A (you can use More on Algebra, Lemma 15.59.8 to deal with direct sums if you like). Hence this holds for the complexes $F_{p+1}P/F_pP$. Since the short exact sequences

$$0 \to F_pP \to F_{p+1}P \to F_{p+1}P/F_pP \to 0$$

are split exact as sequences of graded modules, we can argue by induction that c_{F_pP} is an isomorphism for all p and that F_pP is K-flat as a complex of R-modules (use More on Algebra, Lemma 15.59.5). Finally, using that $P = \text{colim}\, F_pP$ we conclude that c_P is an isomorphism and that P is K-flat as a complex of R-modules (use More on Algebra, Lemma 15.59.8). $\square$

0BZ0 Lemma 22.33.9. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let T be a differential graded (A, B)-bimodule. Assume

(1) T defines a compact object of $D(B, \text{d})$, and
(2) $S = \text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(T, B)$ represents $R\text{Hom}(T, B)$ in $D(A, \text{d})$.

Then S has a structure of a differential graded (B, A)-bimodule and there is an isomorphism

$$N \otimes_B^{\mathbf{L}} S \longrightarrow R\text{Hom}(T, N)$$

functorial in N in $D(B, \text{d})$.

Proof. Write $\mathcal{B} = \text{Mod}^{dg}_{(B,\text{d})}$. The right A-module structure on S comes from the map $A \to \text{Hom}_{\mathcal{B}}(T, T)$ and the composition $\text{Hom}_{\mathcal{B}}(T, B) \otimes \text{Hom}_{\mathcal{B}}(T, T) \to \text{Hom}_{\mathcal{B}}(T, B)$ defined in Example 22.26.8. Using this multiplication a second time there is a map

$$c_N : N \otimes_B S = \text{Hom}_{\mathcal{B}}(B, N) \otimes_B \text{Hom}_{\mathcal{B}}(T, B) \longrightarrow \text{Hom}_{\mathcal{B}}(T, N)$$

functorial in N. Given N we can choose quasi-isomorphisms $P \to N \to I$ where P, resp. I is a differential graded B-module with property (P), resp. (I). Then using c_N we obtain a map $P \otimes_B S \to \text{Hom}_{\mathcal{B}}(T, I)$ between the objects representing $S \otimes_B^{\mathbf{L}} N$ and $R\text{Hom}(T, N)$. Clearly this defines a transformation of functors c as in the lemma.

To prove that c is an isomorphism of functors, we may assume N is a differential graded B-module which has property (P). Since T defines a compact object in $D(B, \text{d})$ and since both sides of the arrow define exact functors of triangulated categories, we reduce using Lemma 22.20.1 to the case where N has a finite filtration whose graded pieces are direct sums of $B[k]$. Using again that both sides of the arrow are exact functors of triangulated categories and compactness of T we reduce to the case $N = B[k]$. Assumption (2) is exactly the assumption that c is an isomorphism in this case. $\square$

22.34. Composition of derived tensor products

0BZ1 We encourage the reader to skip this section.

Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-module. We denote N_B the bimodule N viewed as a differential graded B-module (forgetting about the A-structure). There is a canonical map

0BZ2 (22.34.0.1)
$$N_B \otimes_B^{\mathbf{L}} N' \longrightarrow (N \otimes_B N')_C$$

in $D(C, \text{d})$. Here $(N \otimes_B N')_C$ denotes the (A, C)-bimodule $N \otimes_B N'$ viewed as a differential graded C-module. Namely, this map comes from the fact that the derived tensor product always maps to the plain tensor product (as it is a left derived functor).

0BZ3 Lemma 22.34.1. Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-module. Assume (22.34.0.1) is an isomorphism. Then the composition

$$D(A, \mathrm{d}) \xrightarrow{-\otimes_A^{\mathbf{L}} N} D(B, \mathrm{d}) \xrightarrow{-\otimes_B^{\mathbf{L}} N'} D(C, \mathrm{d})$$

is isomorphic to $- \otimes_A^{\mathbf{L}} N''$ with $N'' = N \otimes_B N'$ viewed as (A, C)-bimodule.

Proof. Let us define a transformation of functors

$$(- \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} N' \longrightarrow - \otimes_A^{\mathbf{L}} N''$$

To do this, let M be a differential graded A-module with property (P). According to the construction of the functor $- \otimes_A^{\mathbf{L}} N''$ of the proof of Lemma 22.33.2 the plain tensor product $M \otimes_A N''$ represents $M \otimes_A^{\mathbf{L}} N''$ in $D(C, \mathrm{d})$. Then we write

$$M \otimes_A N'' = M \otimes_A (N \otimes_B N') = (M \otimes_A N) \otimes_B N'$$

The module $M \otimes_A N$ represents $M \otimes_A^{\mathbf{L}} N$ in $D(B, \mathrm{d})$. Choose a quasi-isomorphism $Q \to M \otimes_A N$ where Q is a differential graded B-module with property (P). Then $Q \otimes_B N'$ represents $(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} N'$ in $D(C, \mathrm{d})$. Thus we can define our map via

$$(M \otimes_A^{\mathbf{L}} N) \otimes_B^{\mathbf{L}} N' = Q \otimes_B N' \to M \otimes_A N \otimes_B N' = M \otimes_A^{\mathbf{L}} N''$$

The construction of this map is functorial in M and compatible with distinguished triangles and direct sums; we omit the details. Consider the property T of objects M of $D(A, \mathrm{d})$ expressing that this map is an isomorphism. Then

(1) if T holds for M_i then T holds for $\bigoplus M_i$,
(2) if T holds for 2-out-of-3 in a distinguished triangle, then it holds for the third, and
(3) T holds for $A[k]$ because here we obtain a shift of the map (22.34.0.1) which we have assumed is an isomorphism.

Thus by Remark 22.22.5 property T always holds and the proof is complete. □

Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. We temporarily denote $(A \otimes_R B)_B$ the differential graded algebra $A \otimes_R B$ viewed as a (right) differential graded B-module, and ${}_B(B \otimes_R C)_C$ the differential graded algebra $B \otimes_R C$ viewed as a differential graded (B, C)-bimodule. Then there is a canonical map

0BZ4 (22.34.1.1) $$(A \otimes_R B)_B \otimes_B^{\mathbf{L}} {}_B(B \otimes_R C)_C \longrightarrow (A \otimes_R B \otimes_R C)_C$$

in $D(C, \mathrm{d})$ where $(A \otimes_R B \otimes_R C)_C$ denotes the differential graded R-algebra $A \otimes_R B \otimes_R C$ viewed as a (right) differential graded C-module. Namely, this map comes from the identification

$$(A \otimes_R B)_B \otimes_B {}_B(B \otimes_R C)_C = (A \otimes_R B \otimes_R C)_C$$

and the fact that the derived tensor product always maps to the plain tensor product (as it is a left derived functor).

0BZ5 Lemma 22.34.2. Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Assume that (22.34.1.1) is an isomorphism. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-bimodule. Then the composition

$$D(A, \mathrm{d}) \xrightarrow{-\otimes_A^{\mathbf{L}} N} D(B, \mathrm{d}) \xrightarrow{-\otimes_B^{\mathbf{L}} N'} D(C, \mathrm{d})$$

is isomorphic to $- \otimes_A^{\mathbf{L}} N''$ for a differential graded (A, C)-bimodule N'' described in the proof.

Proof. By Lemma 22.33.3 we may replace N and N' by quasi-isomorphic bimodules. Thus we may assume N, resp. N' has property (P) as differential graded (A, B)-bimodule, resp. (B, C)-bimodule, see Lemma 22.28.4. We claim the lemma holds with the (A, C)-bimodule $N'' = N \otimes_B N'$. To prove this, it suffices to show that

$$N_B \otimes_B^{\mathbf{L}} N' \longrightarrow (N \otimes_B N')_C$$

is an isomorphism in $D(C, \mathrm{d})$, see Lemma 22.34.1.

Let $F_\bullet$ be the filtration on N as in property (P) for bimodules. By Lemma 22.28.5 there is a short exact sequence

$$0 \to \bigoplus F_i N \to \bigoplus F_i N \to N \to 0$$

of differential graded (A, B)-bimodules which is split as a sequence of graded (A, B)-bimodules. A fortiori this is an admissible short exact sequence of differential graded B-modules and this produces a distinguished triangle

$$\bigoplus F_i N_B \to \bigoplus F_i N_B \to N_B \to \bigoplus F_i N_B[1]$$

in $D(B, \mathrm{d})$. Using that $- \otimes_B^{\mathbf{L}} N'$ is an exact functor of triangulated categories and commutes with direct sums and using that $- \otimes_B N'$ transforms admissible exact sequences into admissible exact sequences and commutes with direct sums we reduce to proving that

$$(F_p N)_B \otimes_B^{\mathbf{L}} N' \longrightarrow (F_p N)_B \otimes_B N'$$

is a quasi-isomorphism for all p. Repeating the argument with the short exact sequences of (A, B)-bimodules

$$0 \to F_p N \to F_{p+1} N \to F_{p+1} N / F_p N \to 0$$

which are split as graded (A, B)-bimodules we reduce to showing the same statement for $F_{p+1} N / F_p N$. Since these modules are direct sums of shifts of $(A \otimes_R B)_B$ we reduce to showing that

$$(A \otimes_R B)_B \otimes_B^{\mathbf{L}} N' \longrightarrow (A \otimes_R B)_B \otimes_B N'$$

is a quasi-isomorphism.

Choose a filtration $F_\bullet$ on N' as in property (P) for bimodules. Choose a quasi-isomorphism $P \to (A \otimes_R B)_B$ of differential graded B-modules where P has property (P). We have to show that $P \otimes_B N' \to (A \otimes_R B)_B \otimes_B N'$ is a quasi-isomorphism because $P \otimes_B N'$ represents $(A \otimes_R B)_B \otimes_B^{\mathbf{L}} N'$ in $D(C, \mathrm{d})$ by the construction in Lemma 22.33.2. As $N' = \operatorname{colim} F_p N'$ we find that it suffices to show that $P \otimes_B F_p N' \to (A \otimes_R B)_B \otimes_B F_p N'$ is a quasi-isomorphism. Using the short exact sequences $0 \to F_p N' \to F_{p+1} N' \to F_{p+1} N' / F_p N' \to 0$ which are split as graded (B, C)-bimodules we reduce to showing $P \otimes_B F_{p+1} N' / F_p N' \to (A \otimes_R B)_B \otimes_B F_{p+1} N' / F_p N'$ is a quasi-isomorphism for all p. Then finally using that $F_{p+1}N'/F_pN'$ is a direct sum of shifts of ${}_B(B \otimes_R C)_C$ we conclude that it suffices to show that

$$P \otimes_B {}_B(B \otimes_R C)_C \to (A \otimes_R B)_B \otimes_B {}_B(B \otimes_R C)_C$$

is a quasi-isomorphism. Since $P \to (A \otimes_R B)_B$ is a resolution by a module satisfying property (P) this map of differential graded C-modules represents the morphism (22.34.1.1) in $D(C, \text{d})$ and the proof is complete. $\square$

09S4 Lemma 22.34.3. Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. If C is K-flat as a complex of R-modules, then (22.34.1.1) is an isomorphism and the conclusion of Lemma 22.34.2 is valid.

Proof. Choose a quasi-isomorphism $P \to (A \otimes_R B)_B$ of differential graded B-modules, where P has property (P). Then we have to show that

$$P \otimes_B (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes_B (B \otimes_R C)$$

is a quasi-isomorphism. Equivalently we are looking at

$$P \otimes_R C \longrightarrow A \otimes_R B \otimes_R C$$

This is a quasi-isomorphism if C is K-flat as a complex of R-modules by More on Algebra, Lemma 15.59.2. $\square$

22.35. Variant of derived tensor product

09LU Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Then we have the functors

$$\text{Comp}(\mathcal{O}) \to K(\mathcal{O}) \to D(\mathcal{O})$$

and as we've seen above we have differential graded enhancement $\text{Comp}^{dg}(\mathcal{O})$. Namely, this is the differential graded category of Example 22.26.6 associated to the abelian category $\text{Mod}(\mathcal{O})$. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules in other words, an object of $\text{Comp}^{dg}(\mathcal{O})$. Set

$$(E, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$$

This is a differential graded $\mathbf{Z}$-algebra. We claim there is an analogue of the derived base change in this situation.

09LV Lemma 22.35.1. In the situation above there is a functor

$$- \otimes_E K^\bullet : \text{Mod}^{dg}_{(E,\text{d})} \longrightarrow \text{Comp}^{dg}(\mathcal{O})$$

of differential graded categories. This functor sends E to $K^\bullet$ and commutes with direct sums.

Proof. Let M be a differential graded E-module. For every object U of $\mathcal{C}$ the complex $K^\bullet(U)$ is a left differential graded E-module as well as a right $\mathcal{O}(U)$-module. The actions commute, so we have a bimodule. Thus, by the constructions in Sections 22.12 and 22.28 we can form the tensor product

$$M \otimes_E K^\bullet(U)$$

which is a differential graded $\mathcal{O}(U)$-module, i.e., a complex of $\mathcal{O}(U)$-modules. This construction is functorial with respect to U, hence we can sheafify to get a complex of $\mathcal{O}$-modules which we denote

$$M \otimes_E K^\bullet$$

Moreover, for each U the construction determines a functor $\text{Mod}^{dg}_{(E,\text{d})} \to \text{Comp}^{dg}(\mathcal{O}(U))$ of differential graded categories by Lemma 22.29.1. It is therefore clear that we obtain a functor as stated in the lemma. □

09LW Lemma 22.35.2. The functor of Lemma 22.35.1 defines an exact functor of triangulated categories $K(\text{Mod}_{(E,\text{d})}) \to K(\mathcal{O})$.

Proof. The functor induces a functor between homotopy categories by Lemma 22.26.5. We have to show that $- \otimes_E K^\bullet$ transforms distinguished triangles into distinguished triangles. Suppose that $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded E-modules. Let $s : M \to L$ be a graded E-module homomorphism which is left inverse to $L \to M$. Then s defines a map $M \otimes_E K^\bullet \to L \otimes_E K^\bullet$ of graded $\mathcal{O}$-modules (i.e., respecting $\mathcal{O}$-module structure and grading, but not differentials) which is left inverse to $L \otimes_E K^\bullet \to M \otimes_E K^\bullet$. Thus we see that

$$0 \to K \otimes_E K^\bullet \to L \otimes_E K^\bullet \to M \otimes_E K^\bullet \to 0$$

is a termwise split short exact sequences of complexes, i.e., a defines a distinguished triangle in $K(\mathcal{O})$. □

09LX Lemma 22.35.3. The functor $K(\text{Mod}_{(E,\text{d})}) \to K(\mathcal{O})$ of Lemma 22.35.2 has a left derived version defined on all of $D(E, \text{d})$. We denote it $- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \text{d}) \to D(\mathcal{O})$.

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 22.20.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 22.22.3. □

0CS6 Lemma 22.35.4. Let R be a ring. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of commutative R-algebras. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. The functor of Lemma 22.35.3 has the following property: For every M, N in $D(E, \text{d})$ there is a canonical map

$$R\operatorname{Hom}(M, N) \longrightarrow R\operatorname{Hom}_{\mathcal{O}}(M \otimes_E^{\mathbf{L}} K^\bullet, N \otimes_E^{\mathbf{L}} K^\bullet)$$

in $D(R)$ which on cohomology modules gives the maps

$$\text{Ext}^n_{D(E,\text{d})}(M, N) \to \text{Ext}^n_{D(\mathcal{O})}(M \otimes_E^{\mathbf{L}} K^\bullet, N \otimes_E^{\mathbf{L}} K^\bullet)$$

induced by the functor $- \otimes_E^{\mathbf{L}} K^\bullet$.

Proof. The right hand side of the arrow is the global derived hom introduced in Cohomology on Sites, Section 21.36 which has the correct cohomology modules. For the left hand side we think of M as a (R, A)-bimodule and we have the derived Hom introduced in Section 22.31 which also has the correct cohomology modules. To prove the lemma we may assume M and N are differential graded E-modules with property (P); this does not change the left hand side of the arrow by Lemma 22.31.3. By Lemma 22.31.5 this means that the left hand side of the arrow becomes $\text{Hom}_{\text{Mod}^{dg}_{(B,\text{d})}}(M, N)$. In Lemmas 22.35.1, 22.35.2, and 22.35.3 we have constructed a functor

$$- \otimes_E K^\bullet : \text{Mod}^{dg}_{(E,\text{d})} \longrightarrow \text{Comp}^{dg}(\mathcal{O})$$

of differential graded categories and we have shown that $- \otimes_E^{\mathbf{L}} K^\bullet$ is computed by evaluating this functor on differential graded E-modules with property (P). Hence we obtain a map of complexes of R-modules

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(M, N) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(M \otimes_E K^\bullet, N \otimes_E K^\bullet)$$

For any complexes of $\mathcal{O}$-modules $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ there is a canonical map

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \Gamma(\mathcal{C}, \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \longrightarrow R\,\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Combining these maps we obtain the desired map of the lemma. □

09LY Lemma 22.35.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Then the functor

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O})$$

of Lemma 22.35.3 is a left adjoint of the functor

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}) \longrightarrow D(E, \mathrm{d})$$

of Lemma 22.32.1.

Proof. The statement means that we have

$$\mathrm{Hom}_{D(E,\mathrm{d})}(M, R\,\mathrm{Hom}(K^\bullet, L^\bullet)) = \mathrm{Hom}_{D(\mathcal{O})}(M \otimes_E^{\mathbf{L}} K^\bullet, L^\bullet)$$

bifunctorially in M and $L^\bullet$. To see this we may replace M by a differential graded E-module P with property (P). We also may replace $L^\bullet$ by a K-injective complex of $\mathcal{O}$-modules $I^\bullet$. The computation of the derived functors given in the lemmas referenced in the statement combined with Lemma 22.22.3 translates the above into

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(E,\mathrm{d})})}(P, \mathrm{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

where $\mathcal{B} = \mathrm{Comp}^{dg}(\mathcal{O})$. There is an evaluation map from right to left functorial in P and $I^\bullet$ (details omitted). Choose a filtration $F_\bullet$ on P as in the definition of property (P). By Lemma 22.20.1 and the fact that both sides of the equation are homological functors in P on $K(\mathrm{Mod}_{(E,\mathrm{d})})$ we reduce to the case where P is replaced by the differential graded E-module $\bigoplus F_iP$. Since both sides turn direct sums in the variable P into direct products we reduce to the case where P is one of the differential graded E-modules F_iP. Since each F_iP has a finite filtration (given by admissible monomorphisms) whose graded pieces are graded projective E-modules we reduce to the case where P is a graded projective E-module. In this case we clearly have

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(E,\mathrm{d})}}(P, \mathrm{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

as graded $\mathbf{Z}$-modules (because this statement reduces to the case $P = E[k]$ where it is obvious). As the isomorphism is compatible with differentials we conclude. □

09LZ Lemma 22.35.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Assume

(1) $K^\bullet$ represents a compact object of $D(\mathcal{O})$, and
(2) $E = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$ computes the ext groups of $K^\bullet$ in $D(\mathcal{O})$.

Then the functor

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O})$$

of Lemma 22.35.3 is fully faithful.

Proof. Because our functor has a left adjoint given by $R\operatorname{Hom}(K^\bullet, -)$ by Lemma 22.35.5 it suffices to show for a differential graded E-module M that the map

$$H^0(M) \longrightarrow \operatorname{Hom}_{D(\mathcal{O})}(K^\bullet, M \otimes_E^{\mathbf{L}} K^\bullet)$$

is an isomorphism. We may assume that $M = P$ is a differential graded E-module which has property (P). Since $K^\bullet$ defines a compact object, we reduce using Lemma 22.20.1 to the case where P has a finite filtration whose graded pieces are direct sums of $E[k]$. Again using compactness we reduce to the case $P = E[k]$. The assumption on $K^\bullet$ is that the result holds for these. □

22.36. Characterizing compact objects

09QZ Compact objects of additive categories are defined in Derived Categories, Definition 13.37.1. In this section we characterize compact objects of the derived category of a differential graded algebra.

09R0 Remark 22.36.1. Let (A, d) be a differential graded algebra. Is there a characterization of those differential graded A-modules P for which we have

$$\operatorname{Hom}_{K(A,\text{d})}(P, M) = \operatorname{Hom}_{D(A,\text{d})}(P, M)$$

for all differential graded A-modules M? Let $\mathcal{D} \subset K(A, \text{d})$ be the full subcategory whose objects are the objects P satisfying the above. Then $\mathcal{D}$ is a strictly full saturated triangulated subcategory of $K(A, \text{d})$. If P is projective as a graded A-module, then to see where P is an object of $\mathcal{D}$ it is enough to check that $\operatorname{Hom}_{K(A,\text{d})}(P, M) = 0$ whenever M is acyclic. However, in general it is not enough to assume that P is projective as a graded A-module. Example: take $A = R = k[\epsilon]$ where k is a field and $k[\epsilon] = k[x]/(x^2)$ is the ring of dual numbers. Let P be the object with $P^n = R$ for all $n \in \mathbf{Z}$ and differential given by multiplication by ϵ. Then $\text{id}_P \in \operatorname{Hom}_{K(A,\text{d})}(P, P)$ is a nonzero element but P is acyclic.

09R1 Remark 22.36.2. Let (A, d) be a differential graded algebra. Let us say a differential graded A-module M is finite if M is generated, as a right A-module, by finitely many elements. If P is a differential graded A-module which is finite graded projective, then we can ask: Does P give a compact object of $D(A, \text{d})$? Presumably, this is not true in general, but we do not know a counter example. However, if P is also an object of the category $\mathcal{D}$ of Remark 22.36.1, then this is the case (this follows from the fact that direct sums in $D(A, \text{d})$ are given by direct sums of modules; details omitted).

09R2 Lemma 22.36.3. Let (A, d) be a differential graded algebra. Let E be a compact object of $D(A, \text{d})$. Let P be a differential graded A-module which has a finite filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset F_nP = P$$

by differential graded submodules such that

$$F_{i+1}P/F_iP \cong \bigoplus\nolimits_{j \in J_i} A[k_{i,j}]$$

as differential graded A-modules for some sets J_i and integers $k_{i,j}$. Let $E \to P$ be a morphism of $D(A, \text{d})$. Then there exists a differential graded submodule $P' \subset P$ such that $F_{i+1}P \cap P'/(F_iP \cap P')$ is equal to $\bigoplus_{j \in J'_i} A[k_{i,j}]$ for some finite subsets $J'_i \subset J_i$ and such that $E \to P$ factors through P'.

Proof. We will prove by induction on $-1 \leq m \leq n$ that there exists a differential graded submodule $P' \subset P$ such that

(1) $F_m P \subset P'$,
(2) for $i \geq m$ the quotient $F_{i+1}P \cap P'/(F_i P \cap P')$ is isomorphic to $\bigoplus_{j \in J'_i} A[k_{i,j}]$ for some finite subsets $J'_i \subset J_i$, and
(3) $E \to P$ factors through P'.

The base case is $m = n$ where we can take $P' = P$.

Induction step. Assume P' works for m. For $i \geq m$ and $j \in J'_i$ let $x_{i,j} \in F_{i+1}P \cap P'$ be a homogeneous element of degree $k_{i,j}$ whose image in $F_{i+1}P \cap P'/(F_i P \cap P')$ is the generator in the summand corresponding to $j \in J_i$. The $x_{i,j}$ generate $P'/F_m P$ as an A-module. Write

$$\mathrm{d}(x_{i,j}) = \sum x_{i',j'} a_{i,j}^{i',j'} + y_{i,j}$$

with $y_{i,j} \in F_m P$ and $a_{i,j}^{i',j'} \in A$. There exists a finite subset $J'_{m-1} \subset J_{m-1}$ such that each $y_{i,j}$ maps to an element of the submodule $\bigoplus_{j \in J'_{m-1}} A[k_{m-1,j}]$ of $F_m P/F_{m-1}P$. Let $P'' \subset F_m P$ be the inverse image of $\bigoplus_{j \in J'_{m-1}} A[k_{m-1,j}]$ under the map $F_m P \to F_m P/F_{m-1}P$. Then we see that the A-submodule

$$P'' + \sum x_{i,j} A$$

is a differential graded submodule of the type we are looking for. Moreover

$$P'/(P'' + \sum x_{i,j} A) = \bigoplus\nolimits_{j \in J_{m-1} \setminus J'_{m-1}} A[k_{m-1,j}]$$

Since E is compact, the composition of the given map $E \to P'$ with the quotient map, factors through a finite direct subsum of the module displayed above. Hence after enlarging J'_{m-1} we may assume $E \to P'$ factors through $P'' + \sum x_{i,j} A$ as desired. $\square$

It is not true that every compact object of $D(A, \mathrm{d})$ comes from a finite graded projective differential graded A-module, see Examples, Section 110.68.

09R3 **Proposition 22.36.4.** Let (A, d) be a differential graded algebra. Let E be an object of $D(A, \mathrm{d})$. Then the following are equivalent

(1) E is a compact object,
(2) E is a direct summand of an object of $D(A, \mathrm{d})$ which is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_i P/F_{i-1}P$ are finite direct sums of shifts of A.

Proof. Assume E is compact. By Lemma 22.20.4 we may assume that E is represented by a differential graded A-module P with property (P). Consider the distinguished triangle

$$\bigoplus F_i P \to \bigoplus F_i P \to P \xrightarrow{\delta} \bigoplus F_i P[1]$$

coming from the admissible short exact sequence of Lemma 22.20.1. Since E is compact we have $\delta = \sum_{i=1,\ldots,n} \delta_i$ for some $\delta_i : P \to F_i P[1]$. Since the composition of δ with the map $\bigoplus F_i P[1] \to \bigoplus F_i P[1]$ is zero (Derived Categories, Lemma 13.4.1) it follows that $\delta = 0$ (follows as $\bigoplus F_i P \to \bigoplus F_i P$ maps the summand $F_i P$ via the difference of id and the inclusion map into $F_{i-1}P$). Thus we see that the identity on E factors through $\bigoplus F_iP$ in $D(A, \text{d})$ (by Derived Categories, Lemma 13.4.11). Next, we use that P is compact again to see that the map $E \to \bigoplus F_iP$ factors through $\bigoplus_{i=1,\ldots,n} F_iP$ for some n. In other words, the identity on E factors through $\bigoplus_{i=1,\ldots,n} F_iP$. By Lemma 22.36.3 we see that the identity of E factors as $E \to P \to E$ where P is as in part (2) of the statement of the lemma. In other words, we have proven that (1) implies (2).

Assume (2). By Derived Categories, Lemma 13.37.2 it suffices to show that P gives a compact object. Observe that P has property (P), hence we have

$$\text{Hom}_{D(A,\text{d})}(P, M) = \text{Hom}_{K(A,\text{d})}(P, M)$$

for any differential graded module M by Lemma 22.22.3. As direct sums in $D(A, \text{d})$ are given by direct sums of graded modules (Lemma 22.22.4) we reduce to showing that $\text{Hom}_{K(A,\text{d})}(P, M)$ commutes with direct sums. Using that $K(A, \text{d})$ is a triangulated category, that Hom is a cohomological functor in the first variable, and the filtration on P, we reduce to the case that P is a finite direct sum of shifts of A. Thus we reduce to the case $P = A[k]$ which is clear. □

09RA Lemma 22.36.5. Let (A, d) be a differential graded algebra. For every compact object E of $D(A, \text{d})$ there exist integers $a \leq b$ such that $\text{Hom}_{D(A,\text{d})}(E, M) = 0$ if $H^i(M) = 0$ for $i \in [a, b]$.

Proof. Observe that the collection of objects of $D(A, \text{d})$ for which such a pair of integers exists is a saturated, strictly full triangulated subcategory of $D(A, \text{d})$. Thus by Proposition 22.36.4 it suffices to prove this when E is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A. Using the compatibility with triangles, we see that it suffices to prove it for $P = A$. In this case $\text{Hom}_{D(A,\text{d})}(A, M) = H^0(M)$ and the result holds with $a = b = 0$. □

If (A, d) is just an algebra placed in degree 0 with zero differential or more generally lives in only a finite number of degrees, then we do obtain the more precise description of compact objects.

09RB Lemma 22.36.6. Let (A, d) be a differential graded algebra. Assume that $A^n = 0$ for $|n| \gg 0$. Let E be an object of $D(A, \text{d})$. The following are equivalent

(1) E is a compact object, and
(2) E can be represented by a differential graded A-module P which is finite projective as a graded A-module and satisfies $\text{Hom}_{K(A,\text{d})}(P, M) = \text{Hom}_{D(A,\text{d})}(P, M)$ for every differential graded A-module M.

Proof. Let $\mathcal{D} \subset K(A, \text{d})$ be the triangulated subcategory discussed in Remark 22.36.1. Let P be an object of $\mathcal{D}$ which is finite projective as a graded A-module. Then P represents a compact object of $D(A, \text{d})$ by Remark 22.36.2.

To prove the converse, let E be a compact object of $D(A, \text{d})$. Fix $a \leq b$ as in Lemma 22.36.5. After decreasing a and increasing b if necessary, we may also assume that $H^i(E) = 0$ for $i \notin [a, b]$ (this follows from Proposition 22.36.4 and our assumption on A). Moreover, fix an integer $c > 0$ such that $A^n = 0$ if $|n| \geq c$.

By Proposition 22.36.4 we see that E is a direct summand, in $D(A, \text{d})$, of a differential graded A-module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A. In particular, P has property (P) and we have $\mathrm{Hom}_{D(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$ for any differential graded module M by Lemma 22.22.3. In other words, P is an object of the triangulated subcategory $\mathcal{D} \subset K(A, \mathrm{d})$ discussed in Remark 22.36.1. Note that P is finite free as a graded A-module.

Choose $n > 0$ such that $b + 4c - n < a$. Represent the projector onto E by an endomorphism $\varphi : P \to P$ of differential graded A-modules. Consider the distinguished triangle

$$P \xrightarrow{1-\varphi} P \to C \to P[1]$$

in $K(A, \mathrm{d})$ where C is the cone of the first arrow. Then C is an object of $\mathcal{D}$, we have $C \cong E \oplus E[1]$ in $D(A, \mathrm{d})$, and C is a finite graded free A-module. Next, consider a distinguished triangle

$$C[1] \to C \to C' \to C[2]$$

in $K(A, \mathrm{d})$ where C' is the cone on a morphism $C[1] \to C$ representing the composition

$$C[1] \cong E[1] \oplus E[2] \to E[1] \to E \oplus E[1] \cong C$$

in $D(A, \mathrm{d})$. Then we see that C' represents $E \oplus E[2]$. Continuing in this manner we see that we can find a differential graded A-module P which is an object of $\mathcal{D}$, is a finite free as a graded A-module, and represents $E \oplus E[n]$.

Choose a basis x_i, $i \in I$ of homogeneous elements for P as an A-module. Let $d_i = \deg(x_i)$. Let P_1 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \leq a - c - 1$. Let P_2 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \geq b - n + c$. We observe

(1) P_1 and P_2 are differential graded submodules of P,
(2) $P_1^t = 0$ for $t \geq a$,
(3) $P_1^t = P^t$ for $t \leq a - 2c$,
(4) $P_2^t = 0$ for $t \leq b - n$,
(5) $P_2^t = P^t$ for $t \geq b - n + 2c$.

As $b - n + 2c \geq a - 2c$ by our choice of n we obtain a short exact sequence of differential graded A-modules

$$0 \to P_1 \cap P_2 \to P_1 \oplus P_2 \xrightarrow{\pi} P \to 0$$

Since P is projective as a graded A-module this is an admissible short exact sequence (Lemma 22.16.1). Hence we obtain a boundary map $\delta : P \to (P_1 \cap P_2)[1]$ in $K(A, \mathrm{d})$, see Lemma 22.7.2. Since $P = E \oplus E[n]$ and since $P_1 \cap P_2$ lives in degrees $(b-n, a)$ we find that $\mathrm{Hom}_{D(A,\mathrm{d})}(E \oplus E[n], (P_1 \cap P_2)[1])$ is zero. Therefore $\delta = 0$ as a morphism in $K(A, \mathrm{d})$ as P is an object of $\mathcal{D}$. By Derived Categories, Lemma 13.4.11 we can find a map $s : P \to P_1 \oplus P_2$ such that $\pi \circ s = \mathrm{id}_P + \mathrm{d}h + h\mathrm{d}$ for some $h : P \to P$ of degree -1. Since $P_1 \oplus P_2 \to P$ is surjective and since P is projective as a graded A-module we can choose a homogeneous lift $\tilde{h} : P \to P_1 \oplus P_2$ of h. Then we change s into $s + \mathrm{d}\tilde{h} + \tilde{h}\mathrm{d}$ to get $\pi \circ s = \mathrm{id}_P$. This means we obtain a direct sum decomposition $P = s^{-1}(P_1) \oplus s^{-1}(P_2)$. Since $s^{-1}(P_2)$ is equal to P in degrees $\geq b - n + 2c$ we see that $s^{-1}(P_2) \to P \to E$ is a quasi-isomorphism, i.e., an isomorphism in $D(A, \mathrm{d})$. This finishes the proof. $\square$

22.37. Equivalences of derived categories

09S5 Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. A natural question that arises in nature is what it means that $D(A, \mathrm{d})$ is equivalent to $D(B, \mathrm{d})$ as an R-linear triangulated category. This is a rather subtle question and it will turn out it isn't always the correct question to ask. Nonetheless, in this section we collection some conditions that guarantee this is the case.

We strongly urge the reader to take a look at the groundbreaking paper [Ric89b] on this topic.

09S6 **Lemma 22.37.1.** *Let R be a ring. Let $(A, \mathrm{d}) \to (B, \mathrm{d})$ be a homomorphism of differential graded algebras over R, which induces an isomorphism on cohomology algebras. Then*

$$- \otimes_A^{\mathbf{L}} B : D(A, \mathrm{d}) \to D(B, \mathrm{d})$$

gives an R-linear equivalence of triangulated categories with quasi-inverse the restriction functor $N \mapsto N_A$.

Proof. By Lemma 22.33.7 the functor $M \longmapsto M \otimes_A^{\mathbf{L}} B$ is fully faithful. By Lemma 22.33.5 the functor $N \longmapsto R\operatorname{Hom}(B, N) = N_A$ is a right adjoint, see Example 22.33.6. It is clear that the kernel of $R\operatorname{Hom}(B, -)$ is zero. Hence the result follows from Derived Categories, Lemma 13.7.2. □

When we analyze the proof above we see that we obtain the following generalization for free.

09S7 **Lemma 22.37.2.** *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a differential graded (A, B)-bimodule. Assume that*

(1) *N defines a compact object of $D(B, \mathrm{d})$,*
(2) *if $N' \in D(B, \mathrm{d})$ and $\operatorname{Hom}_{D(B,\mathrm{d})}(N, N'[n]) = 0$ for $n \in \mathbf{Z}$, then $N' = 0$, and*
(3) *the map $H^k(A) \to \operatorname{Hom}_{D(B,\mathrm{d})}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.*

Then

$$- \otimes_A^{\mathbf{L}} N : D(A, \mathrm{d}) \to D(B, \mathrm{d})$$

gives an R-linear equivalence of triangulated categories.

Proof. By Lemma 22.33.7 the functor $M \longmapsto M \otimes_A^{\mathbf{L}} N$ is fully faithful. By Lemma 22.33.5 the functor $N' \longmapsto R\operatorname{Hom}(N, N')$ is a right adjoint. By assumption (3) the kernel of $R\operatorname{Hom}(N, -)$ is zero. Hence the result follows from Derived Categories, Lemma 13.7.2. □

09SS **Remark 22.37.3.** In Lemma 22.37.2 we can replace condition (2) by the condition that N is a classical generator for $D_{compact}(B, d)$, see Derived Categories, Proposition 13.37.6. Moreover, if we knew that $R\operatorname{Hom}(N, B)$ is a compact object of $D(A, \mathrm{d})$, then it suffices to check that N is a weak generator for $D_{compact}(B, \mathrm{d})$. We omit the proof; we will add it here if we ever need it in the Stacks project.

Sometimes the B-module P in the lemma below is called an "(A, B)-tilting complex".

09S8 **Lemma 22.37.4.** *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Assume that $A = H^0(A)$. The following are equivalent*

(1) $D(A, \mathrm{d})$ and $D(B, \mathrm{d})$ are equivalent as R-linear triangulated categories, and
(2) there exists an object P of $D(B, \mathrm{d})$ such that
   (a) P is a compact object of $D(B, \mathrm{d})$,
   (b) if $N \in D(B, \mathrm{d})$ with $\mathrm{Hom}_{D(B,\mathrm{d})}(P, N[i]) = 0$ for $i \in \mathbf{Z}$, then $N = 0$,
   (c) $\mathrm{Hom}_{D(B,\mathrm{d})}(P, P[i]) = 0$ for $i \neq 0$ and equal to A for $i = 0$.

The equivalence $D(A, \mathrm{d}) \to D(B, \mathrm{d})$ constructed in (2) sends A to P.

Proof. Let $F : D(A, \mathrm{d}) \to D(B, \mathrm{d})$ be an equivalence. Then F maps compact objects to compact objects. Hence $P = F(A)$ is compact, i.e., (2)(a) holds. Conditions (2)(b) and (2)(c) are immediate from the fact that F is an equivalence.

Let P be an object as in (2). Represent P by a differential graded module with property (P). Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(P, P)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 22.22.3 and assumption (2)(c). Viewing P as a (E, B)-bimodule and using Lemma 22.37.2 and assumption (2)(b) we obtain an equivalence

$$D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

sending E to P. Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \mathrm{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \to (E, \mathrm{d})$. Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \to D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

by Lemma 22.37.1. □

09S9 Remark 22.37.5. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Suppose given an R-linear equivalence

$$F : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

of triangulated categories. Set $N = F(A)$. Then N is a differential graded B-module. Since F is an equivalence and A is a compact object of $D(A, \mathrm{d})$, we conclude that N is a compact object of $D(B, \mathrm{d})$. Since A generates $D(A, \mathrm{d})$ and F is an equivalence, we see that N generates $D(B, \mathrm{d})$. Finally, $H^k(A) = \mathrm{Hom}_{D(A,\mathrm{d})}(A, A[k])$ and as F an equivalence we see that F induces an isomorphism $H^k(A) = \mathrm{Hom}_{D(B,\mathrm{d})}(N, N[k])$ for all k. In order to conclude that there is an equivalence $D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$ which arises from the construction in Lemma 22.37.2 all we need is a left A-module structure on N compatible with derivation and commuting with the given right B-module structure. In fact, it suffices to do this after replacing N by a quasi-isomorphic differential graded B-module. The module structure can be constructed in certain cases. For example, if we assume that F can be lifted to a differential graded functor

$$F^{dg} : \mathrm{Mod}^{dg}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$$

(for notation see Example 22.26.8) between the associated differential graded categories, then this holds. Another case is discussed in the proposition below.

09SA Proposition 22.37.6. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $F : D(A, \mathrm{d}) \to D(B, \mathrm{d})$ be an R-linear equivalence of triangulated categories. Assume that

(1) $A = H^0(A)$, and
(2) B is K-flat as a complex of R-modules.

Then there exists an (A, B)-bimodule N as in Lemma 22.37.2.

Proof. As in Remark 22.37.5 above, we set $N = F(A)$ in $D(B, \mathrm{d})$. We may assume that N is a differential graded B-module with property (P). Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, N)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 22.22.3. Moreover, by the discussion in Remark 22.37.5 and by Lemma 22.37.2 we see that N as a (E, B)-bimodule induces an equivalence $- \otimes_E^{\mathbf{L}} N : D(E, \mathrm{d}) \to D(B, \mathrm{d})$. Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \mathrm{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \to (E, \mathrm{d})$. Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \to D(E, \mathrm{d}) \to D(B, \mathrm{d})$$

by Lemma 22.37.1. Note that the quasi-inverse $D(A, \mathrm{d}) \to D(E', \mathrm{d})$ of the left vertical arrow is given by $M \mapsto M \otimes_A^{\mathbf{L}} A$ where A is viewed as a (A, E')-bimodule, see Example 22.33.6. On the other hand the functor $D(E', \mathrm{d}) \to D(B, \mathrm{d})$ is given by $M \mapsto M \otimes_{E'}^{\mathbf{L}} N$ where N is as above. We conclude by Lemma 22.34.3. $\square$

09SB Remark 22.37.7. Let A, B, F, N be as in Proposition 22.37.6. It is not clear that F and the functor $G(-) = - \otimes_A^{\mathbf{L}} N$ are isomorphic. By construction there is an isomorphism $N = G(A) \to F(A)$ in $D(B, \mathrm{d})$. It is straightforward to extend this to a functorial isomorphism $G(M) \to F(M)$ for M is a differential graded A-module which is graded projective (e.g., a sum of shifts of A). Then one can conclude that $G(M) \cong F(M)$ when M is a cone of a map between such modules. We don't know whether more is true in general.

09SC Lemma 22.37.8. Let R be a ring. Let A and B be R-algebras. The following are equivalent

(1) there is an R-linear equivalence $D(A) \to D(B)$ of triangulated categories,
(2) there exists an object P of $D(B)$ such that
    (a) P can be represented by a finite complex of finite projective B-modules,
    (b) if $K \in D(B)$ with $\mathrm{Ext}^i_B(P, K) = 0$ for $i \in \mathbf{Z}$, then $K = 0$, and
    (c) $\mathrm{Ext}^i_B(P, P) = 0$ for $i \neq 0$ and equal to A for $i = 0$.

Moreover, if B is flat as an R-module, then this is also equivalent to

(3) there exists an (A, B)-bimodule N such that $- \otimes_A^{\mathbf{L}} N : D(A) \to D(B)$ is an equivalence.

Proof. The equivalence of (1) and (2) is a special case of Lemma 22.37.4 combined with the result of Lemma 22.36.6 characterizing compact objects of $D(B)$ (small detail omitted). The equivalence with (3) if B is R-flat follows from Proposition 22.37.6. □

09SD Remark 22.37.9. Let R be a ring. Let A and B be R-algebras. If $D(A)$ and $D(B)$ are equivalent as R-linear triangulated categories, then the centers of A and B are isomorphic as R-algebras. In particular, if A and B are commutative, then $A \cong B$. The rather tricky proof can be found in [Ric89b, Proposition 9.2] or [KZ98, Proposition 6.3.2]. Another approach might be to use Hochschild cohomology (see remark below).

09ST Remark 22.37.10. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras which are derived equivalent, i.e., such that there exists an R-linear equivalence $D(A, \text{d}) \to D(B, \text{d})$ of triangulated categories. We would like to show that certain invariants of (A, d) and (B, d) coincide. In many situations one has more control of the situation. For example, it may happen that there is an equivalence of the form

$$- \otimes_A \Omega : D(A, \text{d}) \longrightarrow D(B, \text{d})$$

for some differential graded (A, B)-bimodule Ω (this happens in the situation of Proposition 22.37.6 and is often true if the equivalence comes from a geometric construction). If also the quasi-inverse of our functor is given as

$$- \otimes_B^{\mathbf{L}} \Omega' : D(B, \text{d}) \longrightarrow D(A, \text{d})$$

for a differential graded (B, A)-bimodule Ω' (and as before such a module Ω' often exists in practice). In this case we can consider the functor

$$D(A^{opp} \otimes_R A, \text{d}) \longrightarrow D(B^{opp} \otimes_R B, \text{d}), \quad M \longmapsto \Omega' \otimes_A^{\mathbf{L}} M \otimes_A^{\mathbf{L}} \Omega$$

on derived categories of bimodules (use Lemma 22.28.3 to turn bimodules into right modules). Observe that this functor sends the (A, A)-bimodule A to the (B, B)-bimodule B. Under suitable conditions (e.g., flatness of A, B, Ω over R, etc) this functor will be an equivalence as well. If this is the case, then it follows that we have isomorphisms of Hochschild cohomology groups

$$HH^i(A, \text{d}) = \text{Hom}_{D(A^{opp} \otimes_R A, \text{d})}(A, A[i]) \longrightarrow \text{Hom}_{D(B^{opp} \otimes_R B, \text{d})}(B, B[i]) = HH^i(B, \text{d}).$$

For example, if $A = H^0(A)$, then $HH^0(A, \text{d})$ is equal to the center of A, and this gives a conceptual proof of the result mentioned in Remark 22.37.9. If we ever need this remark we will provide a precise statement with a detailed proof here.

22.38. Resolutions of differential graded algebras

0BZ6 Let R be a ring. Under our assumptions the free R-algebra $R\langle S \rangle$ on a set S is the algebra with R-basis the expressions

$$s_1 s_2 \ldots s_n$$

where $n \geq 0$ and $s_1, \ldots, s_n \in S$ is a sequence of elements of S. Multiplication is given by concatenation

$$(s_1 s_2 \ldots s_n) \cdot (s'_1 s'_2 \ldots s'_m) = s_1 \ldots s_n s'_1 \ldots s'_m$$

This algebra is characterized by the property that the map

$$\text{Mor}_{R\text{-alg}}(R\langle S \rangle, A) \to \text{Map}(S, A), \quad \varphi \longmapsto (s \mapsto \varphi(s))$$

is a bijection for every R-algebra A.

In the category of graded R-algebras our set S should come with a grading, which we think of as a map $\deg : S \to \mathbf{Z}$. Then $R\langle S\rangle$ has a grading such that the monomials have degree

$$\deg(s_1 s_2 \ldots s_n) = \deg(s_1) + \ldots + \deg(s_n)$$

In this setting we have

$$\text{Mor}_{\text{graded } R\text{-alg}}(R\langle S\rangle, A) \to \text{Map}_{\text{graded sets}}(S, A), \quad \varphi \longmapsto (s \mapsto \varphi(s))$$

is a bijection for every graded R-algebra A.

If A is a graded R-algebra and S is a graded set, then we can similarly form $A\langle S\rangle$. Elements of $A\langle S\rangle$ are sums of elements of the form

$$a_0 s_1 a_1 s_2 \ldots a_{n-1} s_n a_n$$

with $a_i \in A$ modulo the relations that these expressions are R-multilinear in $(a_0, \ldots, a_n)$. Thus for every sequence $s_1, \ldots, s_n$ of elements of S there is an inclusion

$$A \otimes_R \ldots \otimes_R A \subset A\langle S\rangle$$

and the algebra is the direct sum of these. With this definition the reader shows that the map

$$\text{Mor}_{\text{graded } R\text{-alg}}(A\langle S\rangle, B) \to \text{Mor}_{\text{graded } R\text{-alg}}(A, B) \times \text{Map}_{\text{graded sets}}(S, B),$$

sending φ to $(\varphi|_A, (s \mapsto \varphi(s)))$ is a bijection for every graded R-algebra A. We observe that if A was a free graded R-algebra, then so is $A\langle S\rangle$.

Suppose that A is a differential graded R-algebra and that S is a graded set. Suppose moreover for every $s \in S$ we are given a homogeneous element $f_s \in A$ with $\deg(f_s) = \deg(s) + 1$ and $\mathrm{d}f_s = 0$. Then there exists a unique structure of differential graded algebra on $A\langle S\rangle$ with $\mathrm{d}(s) = f_s$. For example, given $a, b, c \in A$ and $s, t \in S$ we would define

$$\begin{aligned} \mathrm{d}(asbtc) = {} & \mathrm{d}(a)sbtc + (-1)^{\deg(a)} a f_s btc + (-1)^{\deg(a)+\deg(s)} as\mathrm{d}(b)tc \\ & + (-1)^{\deg(a)+\deg(s)+\deg(b)} asbf_t c + (-1)^{\deg(a)+\deg(s)+\deg(b)+\deg(t)} asbt\mathrm{d}(c) \end{aligned}$$

We omit the details.

0BZ7 **Lemma 22.38.1.** Let R be a ring. Let (B, d) be a differential graded R-algebra. There exists a quasi-isomorphism $(A, \mathrm{d}) \to (B, \mathrm{d})$ of differential graded R-algebras with the following properties

(1) A is K-flat as a complex of R-modules,
(2) A is a free graded R-algebra.

Proof. First we claim we can find $(A_0, \mathrm{d}) \to (B, \mathrm{d})$ having (1) and (2) inducing a surjection on cohomology. Namely, take a graded set S and for each $s \in S$ a homogeneous element $b_s \in \text{Ker}(d : B \to B)$ of degree $\deg(s)$ such that the classes $\overline{b}_s$ in $H^*(B)$ generate $H^*(B)$ as an R-module. Then we can set $A_0 = R\langle S\rangle$ with zero differential and $A_0 \to B$ given by mapping s to b_s.

Given $A_0 \to B$ inducing a surjection on cohomology we construct a sequence

$$A_0 \to A_1 \to A_2 \to \ldots B$$

by induction. Given $A_n \to B$ we set S_n be a graded set and for each $s \in S_n$ we let $a_s \in \text{Ker}(A_n \to A_n)$ be a homogeneous element of degree $\deg(s) + 1$ mapping to a class $\overline{a}_s$ in $H^*(A_n)$ which maps to zero in $H^*(B)$. We choose S_n large enough so that the elements $\overline{a}_s$ generate $\text{Ker}(H^*(A_n) \to H^*(B))$ as an R-module. Then we set

$$A_{n+1} = A_n\langle S_n \rangle$$

with differential given by $\text{d}(s) = a_s$ see discussion above. Then each (A_n, d) satisfies (1) and (2), we omit the details. The map $H^*(A_n) \to H^*(B)$ is surjective as this was true for $n = 0$.

It is clear that $A = \text{colim}\, A_n$ is a free graded R-algebra. It is K-flat by More on Algebra, Lemma 15.59.8. The map $H^*(A) \to H^*(B)$ is an isomorphism as it is surjective and injective: every element of $H^*(A)$ comes from an element of $H^*(A_n)$ for some n and if it dies in $H^*(B)$, then it dies in $H^*(A_{n+1})$ hence in $H^*(A)$. $\square$

As an application we prove the "correct" version of Lemma 22.34.2.

0BZ8 Lemma 22.38.2. Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded R-algebras. Assume $A \otimes_R C$ represents $A \otimes_R^{\mathbf{L}} C$ in $D(R)$. Let N be a differential graded (A, B)-bimodule. Let N' be a differential graded (B, C)-bimodule. Then the composition

$$D(A, \text{d}) \xrightarrow{-\otimes_A^{\mathbf{L}} N} D(B, \text{d}) \xrightarrow{-\otimes_B^{\mathbf{L}} N'} D(C, \text{d})$$

is isomorphic to $- \otimes_A^{\mathbf{L}} N''$ for some differential graded (A, C)-bimodule N''.

Proof. Using Lemma 22.38.1 we choose a quasi-isomorphism $(B', \text{d}) \to (B, \text{d})$ with B' K-flat as a complex of R-modules. By Lemma 22.37.1 the functor $- \otimes_{B'}^{\mathbf{L}} B : D(B', \text{d}) \to D(B, \text{d})$ is an equivalence with quasi-inverse given by restriction. Note that restriction is canonically isomorphic to the functor $- \otimes_B^{\mathbf{L}} B : D(B, \text{d}) \to D(B', \text{d})$ where B is viewed as a (B, B')-bimodule. Thus it suffices to prove the lemma for the compositions

$$D(A) \to D(B) \to D(B'), \quad D(B') \to D(B) \to D(C), \quad D(A) \to D(B') \to D(C).$$

The first one is Lemma 22.34.3 because B' is K-flat as a complex of R-modules. The second one is true because $B \otimes_B^{\mathbf{L}} N' = N' = B \otimes_B N'$ and hence Lemma 22.34.1 applies. Thus we reduce to the case where B is K-flat as a complex of R-modules.

Assume B is K-flat as a complex of R-modules. It suffices to show that (22.34.1.1) is an isomorphism, see Lemma 22.34.2. Choose a quasi-isomorphism $L \to A$ where L is a differential graded R-module which has property (P). Then it is clear that $P = L \otimes_R B$ has property (P) as a differential graded B-module. Hence we have to show that $P \to A \otimes_R B$ induces a quasi-isomorphism

$$P \otimes_B (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes_B (B \otimes_R C)$$

We can rewrite this as

$$P \otimes_R B \otimes_R C \longrightarrow A \otimes_R B \otimes_R C$$

Since B is K-flat as a complex of R-modules, it follows from More on Algebra, Lemma 15.59.2 that it is enough to show that

$$P \otimes_R C \to A \otimes_R C$$

is a quasi-isomorphism, which is exactly our assumption. $\square$

The following lemma does not really belong in this section, but there does not seem to be a good natural spot for it.

0CRM Lemma 22.38.3. Let (A, d) be a differential graded algebra with $H^i(A)$ countable for each i. Let M be an object of $D(A, \mathrm{d})$. Then the following are equivalent

(1) $M = \text{hocolim} E_n$ with E_n compact in $D(A, \mathrm{d})$, and
(2) $H^i(M)$ is countable for each i.

Proof. Assume (1) holds. Then we have $H^i(M) = \mathrm{colim}\, H^i(E_n)$ by Derived Categories, Lemma 13.33.8. Thus it suffices to prove that $H^i(E_n)$ is countable for each n. By Proposition 22.36.4 we see that E_n is isomorphic in $D(A, \mathrm{d})$ to a direct summand of a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_jP/F_{j-1}P$ are finite direct sums of shifts of A. By assumption the groups $H^i(F_jP/F_{j-1}P)$ are countable. Arguing by induction on the length of the filtration and using the long exact cohomology sequence we conclude that (2) is true. The interesting implication is the other one.

We claim there is a countable differential graded subalgebra $A' \subset A$ such that the inclusion map $A' \to A$ defines an isomorphism on cohomology. To construct A' we choose countable differential graded subalgebras

$$A_1 \subset A_2 \subset A_3 \subset \ldots$$

such that (a) $H^i(A_1) \to H^i(A)$ is surjective, and (b) for $n > 1$ the kernel of the map $H^i(A_{n-1}) \to H^i(A_n)$ is the same as the kernel of the map $H^i(A_{n-1}) \to H^i(A)$. To construct A_1 take any countable collection of cochains $S \subset A$ generating the cohomology of A (as a ring or as a graded abelian group) and let A_1 be the differential graded subalgebra of A generated by S. To construct A_n given A_{n-1} for each cochain $a \in A_{n-1}^i$ which maps to zero in $H^i(A)$ choose $s_a \in A^{i-1}$ with $\mathrm{d}(s_a) = a$ and let A_n be the differential graded subalgebra of A generated by A_{n-1} and the elements s_a. Finally, take $A' = \bigcup A_n$.

By Lemma 22.37.1 the restriction map $D(A, \mathrm{d}) \to D(A', \mathrm{d})$, $M \mapsto M_{A'}$ is an equivalence. Since the cohomology groups of M and $M_{A'}$ are the same, we see that it suffices to prove the implication (2) $\Rightarrow$ (1) for (A', d).

Assume A is countable. By the exact same type of argument as given above we see that for M in $D(A, \mathrm{d})$ the following are equivalent: $H^i(M)$ is countable for each i and M can be represented by a countable differential graded module. Hence in order to prove the implication (2) $\Rightarrow$ (1) we reduce to the situation described in the next paragraph.

Assume A is countable and that M is a countable differential graded module over A. We claim there exists a homomorphism $P \to M$ of differential graded A-modules such that

(1) $P \to M$ is a quasi-isomorphism,
(2) P has property (P), and
(3) P is countable.

Looking at the proof of the construction of P-resolutions in Lemma 22.20.4 we see that it suffices to show that we can prove Lemma 22.20.3 in the setting of countable differential graded modules. This is immediate from the proof.

Assume that A is countable and that M is a countable differential graded module with property (P). Choose a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded submodules such that we have

(1) $P = \bigcup F_pP$,
(2) $F_iP \to F_{i+1}P$ is an admissible monomorphism,
(3) isomorphisms of differential graded modules $F_iP/F_{i-1}P \to \bigoplus_{j \in J_i} A[k_j]$ for some sets J_i and integers k_j.

Of course J_i is countable for each i. For each i and $j \in J_i$ choose $x_{i,j} \in F_iP$ of degree k_j whose image in $F_iP/F_{i-1}P$ generates the summand corresponding to j.

Claim: Given n and finite subsets $S_i \subset J_i$, $i = 1, \ldots, n$ there exist finite subsets $S_i \subset T_i \subset J_i$, $i = 1, \ldots, n$ such that $P' = \bigoplus_{i \leq n} \bigoplus_{j \in T_i} Ax_{i,j}$ is a differential graded submodule of P. This was shown in the proof of Lemma 22.36.3 but it is also easily shown directly: the elements $x_{i,j}$ freely generate P as a right A-module. The structure of P shows that

$$\mathrm{d}(x_{i,j}) = \sum\nolimits_{i' < i} x_{i',j'} a_{i',j'}$$

where of course the sum is finite. Thus given $S_0, \ldots, S_n$ we can first choose $S_0 \subset S'_0, \ldots, S_{n-1} \subset S'_{n-1}$ with $\mathrm{d}(x_{n,j}) \in \bigoplus_{i' < n, j' \in S'_{i'}} x_{i',j'}A$ for all $j \in S_n$. Then by induction on n we can choose $S'_0 \subset T_0, \ldots, S'_{n-1} \subset T_{n-1}$ to make sure that $\bigoplus_{i' < n, j' \in T_{i'}} x_{i',j'}A$ is a differential graded A-submodule. Setting $T_n = S_n$ we find that $P' = \bigoplus_{i \leq n, j \in T_i} x_{i,j}A$ is as desired.

From the claim it is clear that $P = \bigcup P'_n$ is a countable rising union of P'_n as above. By construction each P'_n is a differential graded module with property (P) such that the filtration is finite and the succesive quotients are finite direct sums of shifts of A. Hence P'_n defines a compact object of $D(A, \mathrm{d})$, see for example Proposition 22.36.4. Since $P = \mathrm{hocolim} P'_n$ in $D(A, \mathrm{d})$ by Lemma 22.23.2 the proof of the implication (2) $\Rightarrow$ (1) is complete. $\square$

22.39. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Differential Graded Sheaves
(25) Hypercoverings

Schemes

(26) Schemes
(27) Constructions of Schemes
(28) Properties of Schemes

CHAPTER 23

Divided Power Algebra

09PD 23.1. Introduction

09PE In this chapter we talk about divided power algebras and what you can do with them. A reference is the book [Ber74].

23.2. Divided powers

07GK In this section we collect some results on divided power rings. We will use the convention $0! = 1$ (as empty products should give 1).

07GL Definition 23.2.1. Let A be a ring. Let I be an ideal of A. A collection of maps $\gamma_n : I \to I$, $n > 0$ is called a divided power structure on I if for all $n \geq 0$, $m > 0$, $x, y \in I$, and $a \in A$ we have

(1) $\gamma_1(x) = x$, we also set $\gamma_0(x) = 1$,
(2) $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$,
(3) $\gamma_n(ax) = a^n\gamma_n(x)$,
(4) $\gamma_n(x + y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$,
(5) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$.

Note that the rational numbers $\frac{(n+m)!}{n!m!}$ and $\frac{(nm)!}{n!(m!)^n}$ occurring in the definition are in fact integers; the first is the number of ways to choose n out of $n + m$ and the second counts the number of ways to divide a group of nm objects into n groups of m. We make some remarks about the definition which show that $\gamma_n(x)$ is a replacement for $x^n/n!$ in I.

07GM Lemma 23.2.2. Let A be a ring. Let I be an ideal of A.

(1) If γ is a divided power structure[1] on I, then $n!\gamma_n(x) = x^n$ for $n \geq 1$, $x \in I$.

Assume A is torsion free as a $\mathbf{Z}$-module.

(2) A divided power structure on I, if it exists, is unique.
(3) If $\gamma_n : I \to I$ are maps then
$$\gamma \text{ is a divided power structure} \Leftrightarrow n!\gamma_n(x) = x^n \ \forall x \in I, n \geq 1.$$
(4) The ideal I has a divided power structure if and only if there exists a set of generators x_i of I as an ideal such that for all $n \geq 1$ we have $x_i^n \in (n!)I$.

Proof. Proof of (1). If γ is a divided power structure, then condition (2) (applied to 1 and $n - 1$ instead of n and m) implies that $n\gamma_n(x) = \gamma_1(x)\gamma_{n-1}(x)$. Hence by induction and condition (1) we get $n!\gamma_n(x) = x^n$.

[1] Here and in the following, γ stands short for a sequence of maps $\gamma_1, \gamma_2, \gamma_3, \ldots$ from I to I.

Assume A is torsion free as a $\mathbf{Z}$-module. Proof of (2). This is clear from (1).

Proof of (3). Assume that $n!\gamma_n(x) = x^n$ for all $x \in I$ and $n \geq 1$. Since $A \subset A \otimes_{\mathbf{Z}} \mathbf{Q}$ it suffices to prove the axioms (1) – (5) of Definition 23.2.1 in case A is a $\mathbf{Q}$-algebra. In this case $\gamma_n(x) = x^n/n!$ and it is straightforward to verify (1) – (5); for example, (4) corresponds to the binomial formula

$$(x+y)^n = \sum_{i=0,\ldots,n} \frac{n!}{i!(n-i)!} x^i y^{n-i}$$

We encourage the reader to do the verifications to make sure that we have the coefficients correct.

Proof of (4). Assume we have generators x_i of I as an ideal such that $x_i^n \in (n!)I$ for all $n \geq 1$. We claim that for all $x \in I$ we have $x^n \in (n!)I$. If the claim holds then we can set $\gamma_n(x) = x^n/n!$ which is a divided power structure by (3). To prove the claim we note that it holds for $x = ax_i$. Hence we see that the claim holds for a set of generators of I as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove the claim for $x + y$ if it holds for x and y. This follows immediately from the binomial theorem. □

07GN Example 23.2.3. Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Then $I = pA$ has a canonical divided power structure. Namely, given $x = pa \in I$ we set

$$\gamma_n(x) = \frac{p^n}{n!} a^n$$

The reader verifies immediately that $p^n/n! \in p\mathbf{Z}_{(p)}$ for $n \geq 1$ (for instance, this can be derived from the fact that the exponent of p in the prime factorization of $n!$ is $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \ldots$), so that the definition makes sense and gives us a sequence of maps $\gamma_n : I \to I$. It is a straightforward exercise to verify that conditions (1) – (5) of Definition 23.2.1 are satisfied. Alternatively, it is clear that the definition works for $A_0 = \mathbf{Z}_{(p)}$ and then the result follows from Lemma 23.4.2.

We notice that $\gamma_n(0) = 0$ for any ideal I of A and any divided power structure γ on I. (This follows from axiom (3) in Definition 23.2.1, applied to $a = 0$.)

07GP Lemma 23.2.4. Let A be a ring. Let I be an ideal of A. Let $\gamma_n : I \to I$, $n \geq 1$ be a sequence of maps. Assume

(a) (1), (3), and (4) of Definition 23.2.1 hold for all $x, y \in I$, and
(b) properties (2) and (5) hold for x in some set of generators of I as an ideal.

Then γ is a divided power structure on I.

Proof. The numbers (1), (2), (3), (4), (5) in this proof refer to the conditions listed in Definition 23.2.1. Applying (3) we see that if (2) and (5) hold for x then (2) and (5) hold for ax for all $a \in A$. Hence we see (b) implies (2) and (5) hold for a set of generators of I as an abelian group. Hence, by induction of the length of an expression in terms of these it suffices to prove that, given $x, y \in I$ such that (2) and (5) hold for x and y, then (2) and (5) hold for $x + y$.

Proof of (2) for $x + y$. By (4) we have

$$\gamma_n(x+y)\gamma_m(x+y) = \sum\nolimits_{i+j=n,\ k+l=m} \gamma_i(x)\gamma_k(x)\gamma_j(y)\gamma_l(y)$$

Using (2) for x and y this equals

$$\sum \frac{(i+k)!}{i!k!}\frac{(j+l)!}{j!l!}\gamma_{i+k}(x)\gamma_{j+l}(y)$$

Comparing this with the expansion

$$\gamma_{n+m}(x+y) = \sum \gamma_a(x)\gamma_b(y)$$

we see that we have to prove that given $a+b=n+m$ we have

$$\sum\nolimits_{i+k=a,\ j+l=b,\ i+j=n,\ k+l=m} \frac{(i+k)!}{i!k!}\frac{(j+l)!}{j!l!} = \frac{(n+m)!}{n!m!}.$$

Instead of arguing this directly, we note that the result is true for the ideal $I=(x,y)$ in the polynomial ring $\mathbf{Q}[x,y]$ because $\gamma_n(f) = f^n/n!$, $f \in I$ defines a divided power structure on I. Hence the equality of rational numbers above is true.

Proof of (5) for $x+y$ given that (1) – (4) hold and that (5) holds for x and y. We will again reduce the proof to an equality of rational numbers. Namely, using (4) we can write $\gamma_n(\gamma_m(x+y)) = \gamma_n(\sum \gamma_i(x)\gamma_j(y))$. Using (4) we can write $\gamma_n(\gamma_m(x+y))$ as a sum of terms which are products of factors of the form $\gamma_k(\gamma_i(x)\gamma_j(y))$. If $i>0$ then

$$\begin{aligned}
\gamma_k(\gamma_i(x)\gamma_j(y)) &= \gamma_j(y)^k\gamma_k(\gamma_i(x)) \\
&= \frac{(ki)!}{k!(i!)^k}\gamma_j(y)^k\gamma_{ki}(x) \\
&= \frac{(ki)!}{k!(i!)^k}\frac{(kj)!}{(j!)^k}\gamma_{ki}(x)\gamma_{kj}(y)
\end{aligned}$$

using (3) in the first equality, (5) for x in the second, and (2) exactly k times in the third. Using (5) for y we see the same equality holds when $i=0$. Continuing like this using all axioms but (5) we see that we can write

$$\gamma_n(\gamma_m(x+y)) = \sum\nolimits_{i+j=nm} c_{ij}\gamma_i(x)\gamma_j(y)$$

for certain universal constants $c_{ij} \in \mathbf{Z}$. Again the fact that the equality is valid in the polynomial ring $\mathbf{Q}[x,y]$ implies that the coefficients c_{ij} are all equal to $(nm)!/n!(m!)^n$ as desired. □

07GQ Lemma 23.2.5. Let A be a ring with two ideals $I, J \subset A$. Let γ be a divided power structure on I and let δ be a divided power structure on J. Then

(1) γ and δ agree on IJ,
(2) if γ and δ agree on $I \cap J$ then they are the restriction of a unique divided power structure ϵ on $I+J$.

Proof. Let $x \in I$ and $y \in J$. Then

$$\gamma_n(xy) = y^n\gamma_n(x) = n!\delta_n(y)\gamma_n(x) = \delta_n(y)x^n = \delta_n(xy).$$

Hence γ and δ agree on a set of (additive) generators of IJ. By property (4) of Definition 23.2.1 it follows that they agree on all of IJ.

Assume γ and δ agree on $I \cap J$. Let $z \in I+J$. Write $z = x+y$ with $x \in I$ and $y \in J$. Then we set

$$\epsilon_n(z) = \sum \gamma_i(x)\delta_{n-i}(y)$$

for all $n \geq 1$. To see that this is well defined, suppose that $z = x' + y'$ is another representation with $x' \in I$ and $y' \in J$. Then $w = x - x' = y' - y \in I \cap J$. Hence

$$\begin{aligned}\sum\nolimits_{i+j=n} \gamma_i(x)\delta_j(y) &= \sum\nolimits_{i+j=n} \gamma_i(x'+w)\delta_j(y) \\ &= \sum\nolimits_{i'+l+j=n} \gamma_{i'}(x')\gamma_l(w)\delta_j(y) \\ &= \sum\nolimits_{i'+l+j=n} \gamma_{i'}(x')\delta_l(w)\delta_j(y) \\ &= \sum\nolimits_{i'+j'=n} \gamma_{i'}(x')\delta_{j'}(y+w) \\ &= \sum\nolimits_{i'+j'=n} \gamma_{i'}(x')\delta_{j'}(y')\end{aligned}$$

as desired. Hence, we have defined maps $\epsilon_n : I + J \to I + J$ for all $n \geq 1$; it is easy to see that $\epsilon_n \mid_I = \gamma_n$ and $\epsilon_n \mid_J = \delta_n$. Next, we prove conditions (1) – (5) of Definition 23.2.1 for the collection of maps ϵ_n. Properties (1) and (3) are clear. To see (4), suppose that $z = x + y$ and $z' = x' + y'$ with $x, x' \in I$ and $y, y' \in J$ and compute

$$\begin{aligned}\epsilon_n(z+z') &= \sum\nolimits_{a+b=n} \gamma_a(x+x')\delta_b(y+y') \\ &= \sum\nolimits_{i+i'+j+j'=n} \gamma_i(x)\gamma_{i'}(x')\delta_j(y)\delta_{j'}(y') \\ &= \sum\nolimits_{k=0,\ldots,n} \sum\nolimits_{i+j=k} \gamma_i(x)\delta_j(y) \sum\nolimits_{i'+j'=n-k} \gamma_{i'}(x')\delta_{j'}(y') \\ &= \sum\nolimits_{k=0,\ldots,n} \epsilon_k(z)\epsilon_{n-k}(z')\end{aligned}$$

as desired. Now we see that it suffices to prove (2) and (5) for elements of I or J, see Lemma 23.2.4. This is clear because γ and δ are divided power structures.

The existence of a divided power structure ϵ on $I + J$ whose restrictions to I and J are γ and δ is thus proven; its uniqueness is rather clear. □

07GR Lemma 23.2.6. Let p be a prime number. Let A be a ring, let $I \subset A$ be an ideal, and let γ be a divided power structure on I. Assume p is nilpotent in A/I. Then I is locally nilpotent if and only if p is nilpotent in A.

Proof. If $p^N = 0$ in A, then for $x \in I$ we have $x^{pN} = (pN)!\gamma_{pN}(x) = 0$ because $(pN)!$ is divisible by p^N. Conversely, assume I is locally nilpotent. We've also assumed that p is nilpotent in A/I, hence $p^r \in I$ for some r, hence p^r nilpotent, hence p nilpotent. □

23.3. Divided power rings

07GT There is a category of divided power rings. Here is the definition.

07GU Definition 23.3.1. A divided power ring is a triple (A, I, γ) where A is a ring, $I \subset A$ is an ideal, and $\gamma = (\gamma_n)_{n \geq 1}$ is a divided power structure on I. A homomorphism of divided power rings $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $\varphi : A \to B$ such that $\varphi(I) \subset J$ and such that $\delta_n(\varphi(x)) = \varphi(\gamma_n(x))$ for all $x \in I$ and $n \geq 1$.

We sometimes say "let (B, J, δ) be a divided power algebra over (A, I, γ)" to indicate that (B, J, δ) is a divided power ring which comes equipped with a homomorphism of divided power rings $(A, I, \gamma) \to (B, J, \delta)$.

07GV **Lemma 23.3.2.** The category of divided power rings has all limits and they agree with limits in the category of rings.

Proof. The empty limit is the zero ring (that's weird but we need it). The product of a collection of divided power rings (A_t, I_t, γ_t), $t \in T$ is given by $(\prod A_t, \prod I_t, \gamma)$ where $\gamma_n((x_t)) = (\gamma_{t,n}(x_t))$. The equalizer of $\alpha, \beta : (A, I, \gamma) \to (B, J, \delta)$ is just $C = \{a \in A \mid \alpha(a) = \beta(a)\}$ with ideal $C \cap I$ and induced divided powers. It follows that all limits exist, see Categories, Lemma 4.14.11. $\square$

The following lemma illustrates a very general category theoretic phenomenon in the case of divided power algebras.

07GW **Lemma 23.3.3.** Let $\mathcal{C}$ be the category of divided power rings. Let $F : \mathcal{C} \to \textit{Sets}$ be a functor. Assume that

(1) there exists a cardinal κ such that for every $f \in F(A, I, \gamma)$ there exists a morphism $(A', I', \gamma') \to (A, I, \gamma)$ of $\mathcal{C}$ such that f is the image of $f' \in F(A', I', \gamma')$ and $|A'| \leq \kappa$, and
(2) F commutes with limits.

Then F is representable, i.e., there exists an object (B, J, δ) of $\mathcal{C}$ such that

$$F(A, I, \gamma) = \mathrm{Hom}_{\mathcal{C}}((B, J, \delta), (A, I, \gamma))$$

functorially in (A, I, γ).

Proof. This is a special case of Categories, Lemma 4.25.1. $\square$

07GX **Lemma 23.3.4.** The category of divided power rings has all colimits.

Proof. The empty colimit is $\mathbf{Z}$ with divided power ideal (0). Let's discuss general colimits. Let $\mathcal{C}$ be a category and let $c \mapsto (A_c, I_c, \gamma_c)$ be a diagram. Consider the functor

$$F(B, J, \delta) = \lim_{c \in \mathcal{C}} Hom((A_c, I_c, \gamma_c), (B, J, \delta))$$

Note that any $f = (f_c)_{c \in C} \in F(B, J, \delta)$ has the property that all the images $f_c(A_c)$ generate a subring B' of B of bounded cardinality κ and that all the images $f_c(I_c)$ generate a divided power sub ideal J' of B'. And we get a factorization of f as a f' in $F(B')$ followed by the inclusion $B' \to B$. Also, F commutes with limits. Hence we may apply Lemma 23.3.3 to see that F is representable and we win. $\square$

07GY **Remark 23.3.5.** The forgetful functor $(A, I, \gamma) \mapsto A$ does not commute with colimits. For example, let

$$\begin{tikzcd} (B, J, \delta) \arrow[r] & (B'', J'', \delta'') \\ (A, I, \gamma) \arrow[u] \arrow[r] & (B', J', \delta') \arrow[u] \end{tikzcd}$$

be a pushout in the category of divided power rings. Then in general the map $B \otimes_A B' \to B''$ isn't an isomorphism. (It is always surjective.) An explicit example is given by $(A, I, \gamma) = (\mathbf{Z}, (0), \emptyset)$, $(B, J, \delta) = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta)$, and $(B', J', \delta') = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta')$ where $\delta_2(2) = 2$ and $\delta'_2(2) = 0$. More precisely, using Lemma 23.5.3 we let δ, resp. δ' be the unique divided power structure on J, resp. J' such that $\delta_2 : J \to J$, resp. $\delta'_2 : J' \to J'$ is the map $0 \mapsto 0, 2 \mapsto 2$, resp. $0 \mapsto 0, 2 \mapsto$

0. Then $(B'', J'', \delta'') = (\mathbf{F}_2, (0), \emptyset)$ which doesn't agree with the tensor product. However, note that it is always true that

$$B''/J'' = B/J \otimes_{A/I} B'/J'$$

as can be seen from the universal property of the pushout by considering maps into divided power algebras of the form $(C, (0), \emptyset)$.

23.4. Extending divided powers

07GZ Here is the definition.

07H0 Definition 23.4.1. Given a divided power ring (A, I, γ) and a ring map $A \to B$ we say γ extends to B if there exists a divided power structure $\bar{\gamma}$ on IB such that $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a homomorphism of divided power rings.

07H1 Lemma 23.4.2. Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. If γ extends to B then it extends uniquely. Assume (at least) one of the following conditions holds

(1) $IB = 0$,
(2) I is principal, or
(3) $A \to B$ is flat.

Then γ extends to B.

Proof. Any element of IB can be written as a finite sum $\sum_{i=1}^t b_i x_i$ with $b_i \in B$ and $x_i \in I$. If γ extends to $\bar{\gamma}$ on IB then $\bar{\gamma}_n(x_i) = \gamma_n(x_i)$. Thus, conditions (3) and (4) in Definition 23.2.1 imply that

$$\bar{\gamma}_n(\sum\nolimits_{i=1}^t b_i x_i) = \sum\nolimits_{n_1 + \ldots + n_t = n} \prod\nolimits_{i=1}^t b_i^{n_i} \gamma_{n_i}(x_i)$$

Thus we see that $\bar{\gamma}$ is unique if it exists.

If $IB = 0$ then setting $\bar{\gamma}_n(0) = 0$ works. If $I = (x)$ then we define $\bar{\gamma}_n(bx) = b^n \gamma_n(x)$. This is well defined: if $b'x = bx$, i.e., $(b - b')x = 0$ then

$$\begin{aligned} b^n \gamma_n(x) - (b')^n \gamma_n(x) &= (b^n - (b')^n)\gamma_n(x) \\ &= (b^{n-1} + \ldots + (b')^{n-1})(b - b')\gamma_n(x) = 0 \end{aligned}$$

because $\gamma_n(x)$ is divisible by x (since $\gamma_n(I) \subset I$) and hence annihilated by $b - b'$. Next, we prove conditions (1) – (5) of Definition 23.2.1. Parts (1), (2), (3), (5) are obvious from the construction. For (4) suppose that $y, z \in IB$, say $y = bx$ and $z = cx$. Then $y + z = (b + c)x$ hence

$$\begin{aligned} \bar{\gamma}_n(y + z) &= (b + c)^n \gamma_n(x) \\ &= \sum \frac{n!}{i!(n-i)!} b^i c^{n-i} \gamma_n(x) \\ &= \sum b^i c^{n-i} \gamma_i(x) \gamma_{n-i}(x) \\ &= \sum \bar{\gamma}_i(y) \bar{\gamma}_{n-i}(z) \end{aligned}$$

as desired.

Assume $A \to B$ is flat. Suppose that $b_1, \ldots, b_r \in B$ and $x_1, \ldots, x_r \in I$. Then

$$\bar{\gamma}_n(\sum b_i x_i) = \sum b_1^{e_1} \ldots b_r^{e_r} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r)$$

where the sum is over $e_1 + \ldots + e_r = n$ if $\bar\gamma_n$ exists. Next suppose that we have $c_1, \ldots, c_s \in B$ and $a_{ij} \in A$ such that $b_i = \sum a_{ij} c_j$. Setting $y_j = \sum a_{ij} x_i$ we claim that

$$\sum b_1^{e_1} \ldots b_r^{e_r} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r) = \sum c_1^{d_1} \ldots c_s^{d_s} \gamma_{d_1}(y_1) \ldots \gamma_{d_s}(y_s)$$

in B where on the right hand side we are summing over $d_1 + \ldots + d_s = n$. Namely, using the axioms of a divided power structure we can expand both sides into a sum with coefficients in $\mathbf{Z}[a_{ij}]$ of terms of the form $c_1^{d_1} \ldots c_s^{d_s} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r)$. To see that the coefficients agree we note that the result is true in $\mathbf{Q}[x_1, \ldots, x_r, c_1, \ldots, c_s, a_{ij}]$ with γ the unique divided power structure on $(x_1, \ldots, x_r)$. By Lazard's theorem (Algebra, Theorem 10.81.4) we can write B as a directed colimit of finite free A-modules. In particular, if $z \in IB$ is written as $z = \sum x_i b_i$ and $z = \sum x'_{i'} b'_{i'}$, then we can find $c_1, \ldots, c_s \in B$ and $a_{ij}, a'_{i'j} \in A$ such that $b_i = \sum a_{ij} c_j$ and $b'_{i'} = \sum a'_{i'j} c_j$ such that $y_j = \sum x_i a_{ij} = \sum x'_{i'} a'_{i'j}$ holds[2]. Hence the procedure above gives a well defined map $\bar\gamma_n$ on IB. By construction $\bar\gamma$ satisfies conditions (1), (3), and (4). Moreover, for $x \in I$ we have $\bar\gamma_n(x) = \gamma_n(x)$. Hence it follows from Lemma 23.2.4 that $\bar\gamma$ is a divided power structure on IB. $\square$

07H2 Lemma 23.4.3. Let (A, I, γ) be a divided power ring.

(1) If $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then $\mathrm{Ker}(\varphi) \cap I$ is preserved by γ_n for all $n \geq 1$.
(2) Let $\mathfrak{a} \subset A$ be an ideal and set $I' = I \cap \mathfrak{a}$. The following are equivalent
    (a) I' is preserved by γ_n for all $n > 0$,
    (b) γ extends to $A/\mathfrak{a}$, and
    (c) there exist a set of generators x_i of I' as an ideal such that $\gamma_n(x_i) \in I'$ for all $n > 0$.

Proof. Proof of (1). This is clear. Assume (2)(a). Define $\bar\gamma_n(x \bmod I') = \gamma_n(x) \bmod I'$ for $x \in I$. This is well defined since $\gamma_n(x + y) = \gamma_n(x) \bmod I'$ for $y \in I'$ by Definition 23.2.1 (4) and the fact that $\gamma_j(y) \in I'$ by assumption. It is clear that $\bar\gamma$ is a divided power structure as γ is one. Hence (2)(b) holds. Also, (2)(b) implies (2)(a) by part (1). It is clear that (2)(a) implies (2)(c). Assume (2)(c). Note that $\gamma_n(x) = a^n \gamma_n(x_i) \in I'$ for $x = a x_i$. Hence we see that $\gamma_n(x) \in I'$ for a set of generators of I' as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove $\forall n : \gamma_n(x + y) \in I'$ if $\forall n : \gamma_n(x), \gamma_n(y) \in I'$. This follows immediately from the fourth axiom of a divided power structure. $\square$

07H3 Lemma 23.4.4. Let (A, I, γ) be a divided power ring. Let $E \subset I$ be a subset. Then the smallest ideal $J \subset I$ preserved by γ and containing all $f \in E$ is the ideal J generated by $\gamma_n(f)$, $n \geq 1$, $f \in E$.

Proof. Follows immediately from Lemma 23.4.3. $\square$

07KD Lemma 23.4.5. Let (A, I, γ) be a divided power ring. Let p be a prime. If p is nilpotent in A/I, then

(1) the p-adic completion $A^\wedge = \lim_e A/p^e A$ surjects onto A/I,
(2) the kernel of this map is the p-adic completion $I^\wedge$ of I, and

[2] This can also be proven without recourse to Algebra, Theorem 10.81.4. Indeed, if $z = \sum x_i b_i$ and $z = \sum x'_{i'} b'_{i'}$, then $\sum x_i b_i - \sum x'_{i'} b'_{i'} = 0$ is a relation in the A-module B. Thus, Algebra, Lemma 10.39.11 (applied to the x_i and $x'_{i'}$ taking the place of the f_i, and the b_i and $b'_{i'}$ taking the role of the x_i) yields the existence of the $c_1, \ldots, c_s \in B$ and $a_{ij}, a'_{i'j} \in A$ as required.

(3) each γ_n is continuous for the p-adic topology and extends to $\gamma_n^\wedge : I^\wedge \to I^\wedge$ defining a divided power structure on $I^\wedge$.

If moreover A is a $\mathbf{Z}_{(p)}$-algebra, then

(4) for e large enough the ideal $p^eA \subset I$ is preserved by the divided power structure γ and
$$(A^\wedge, I^\wedge, \gamma^\wedge) = \lim_e (A/p^eA, I/p^eA, \bar{\gamma})$$
in the category of divided power rings.

Proof. Let $t \geq 1$ be an integer such that $p^tA/I = 0$, i.e., $p^tA \subset I$. The map $A^\wedge \to A/I$ is the composition $A^\wedge \to A/p^tA \to A/I$ which is surjective (for example by Algebra, Lemma 10.96.1). As $p^eI \subset p^eA \cap I \subset p^{e-t}I$ for $e \geq t$ we see that the kernel of the composition $A^\wedge \to A/I$ is the p-adic completion of I. The map γ_n is continuous because
$$\gamma_n(x + p^ey) = \sum\nolimits_{i+j=n} p^{je}\gamma_i(x)\gamma_j(y) = \gamma_n(x) \bmod p^eI$$
by the axioms of a divided power structure. It is clear that the axioms for divided power structures are inherited by the maps $\gamma_n^\wedge$ from the maps γ_n. Finally, to see the last statement say $e > t$. Then $p^eA \subset I$ and $\gamma_1(p^eA) \subset p^eA$ and for $n > 1$ we have
$$\gamma_n(p^ea) = p^n\gamma_n(p^{e-1}a) = \frac{p^n}{n!}p^{n(e-1)}a^n \in p^eA$$
as $p^n/n! \in \mathbf{Z}_{(p)}$ and as $n \geq 2$ and $e \geq 2$ so $n(e-1) \geq e$. This proves that γ extends to A/p^eA, see Lemma 23.4.3. The statement on limits is clear from the construction of limits in the proof of Lemma 23.3.2. □

23.5. Divided power polynomial algebras

07H4 A very useful example is the divided power polynomial algebra. Let A be a ring. Let $t \geq 1$. We will denote $A\langle x_1, \ldots, x_t\rangle$ the following A-algebra: As an A-module we set
$$A\langle x_1, \ldots, x_t\rangle = \bigoplus\nolimits_{n_1, \ldots, n_t \geq 0} Ax_1^{[n_1]} \ldots x_t^{[n_t]}$$
with multiplication given by
$$x_i^{[n]}x_i^{[m]} = \frac{(n+m)!}{n!m!}x_i^{[n+m]}.$$
We also set $x_i = x_i^{[1]}$. Note that $1 = x_1^{[0]} \ldots x_t^{[0]}$. There is a similar construction which gives the divided power polynomial algebra in infinitely many variables. There is an canonical A-algebra map $A\langle x_1, \ldots, x_t\rangle \to A$ sending $x_i^{[n]}$ to zero for $n > 0$. The kernel of this map is denoted $A\langle x_1, \ldots, x_t\rangle_+$.

07H5 Lemma 23.5.1. Let (A, I, γ) be a divided power ring. There exists a unique divided power structure δ on
$$J = IA\langle x_1, \ldots, x_t\rangle + A\langle x_1, \ldots, x_t\rangle_+$$
such that

(1) $\delta_n(x_i) = x_i^{[n]}$, and
(2) $(A, I, \gamma) \to (A\langle x_1, \ldots, x_t\rangle, J, \delta)$ is a homomorphism of divided power rings.

Moreover, $(A\langle x_1, \ldots, x_t\rangle, J, \delta)$ has the following universal property: A homomorphism of divided power rings $\varphi : (A\langle x_1, \ldots, x_t\rangle, J, \delta) \to (C, K, \epsilon)$ is the same thing as a homomorphism of divided power rings $A \to C$ and elements $k_1, \ldots, k_t \in K$.

Proof. We will prove the lemma in case of a divided power polynomial algebra in one variable. The result for the general case can be argued in exactly the same way, or by noting that $A\langle x_1, \ldots, x_t\rangle$ is isomorphic to the ring obtained by adjoining the divided power variables $x_1, \ldots, x_t$ one by one.

Let $A\langle x\rangle_+$ be the ideal generated by $x, x^{[2]}, x^{[3]}, \ldots$. Note that $J = IA\langle x\rangle + A\langle x\rangle_+$ and that

$$IA\langle x\rangle \cap A\langle x\rangle_+ = IA\langle x\rangle \cdot A\langle x\rangle_+$$

Hence by Lemma 23.2.5 it suffices to show that there exist divided power structures on the ideals $IA\langle x\rangle$ and $A\langle x\rangle_+$. The existence of the first follows from Lemma 23.4.2 as $A \to A\langle x\rangle$ is flat. For the second, note that if A is torsion free, then we can apply Lemma 23.2.2 (4) to see that δ exists. Namely, choosing as generators the elements $x^{[m]}$ we see that $(x^{[m]})^n = \frac{(nm)!}{(m!)^n} x^{[nm]}$ and $n!$ divides the integer $\frac{(nm)!}{(m!)^n}$. In general write $A = R/\mathfrak a$ for some torsion free ring R (e.g., a polynomial ring over $\mathbf{Z}$). The kernel of $R\langle x\rangle \to A\langle x\rangle$ is $\bigoplus \mathfrak a x^{[m]}$. Applying criterion (2)(c) of Lemma 23.4.3 we see that the divided power structure on $R\langle x\rangle_+$ extends to $A\langle x\rangle$ as desired.

Proof of the universal property. Given a homomorphism $\varphi : A \to C$ of divided power rings and $k_1, \ldots, k_t \in K$ we consider

$$A\langle x_1, \ldots, x_t\rangle \to C, \quad x_1^{[n_1]} \ldots x_t^{[n_t]} \longmapsto \epsilon_{n_1}(k_1) \ldots \epsilon_{n_t}(k_t)$$

using φ on coefficients. The only thing to check is that this is an A-algebra homomorphism (details omitted). The inverse construction is clear. $\square$

07H6 **Remark 23.5.2.** Let (A, I, γ) be a divided power ring. There is a variant of Lemma 23.5.1 for infinitely many variables. First note that if $s < t$ then there is a canonical map

$$A\langle x_1, \ldots, x_s\rangle \to A\langle x_1, \ldots, x_t\rangle$$

Hence if W is any set, then we set

$$A\langle x_w : w \in W\rangle = \operatorname{colim}_{E \subset W} A\langle x_e : e \in E\rangle$$

(colimit over E finite subset of W) with transition maps as above. By the definition of a colimit we see that the universal mapping property of $A\langle x_w : w \in W\rangle$ is completely analogous to the mapping property stated in Lemma 23.5.1.

The following lemma can be found in [BO83].

07GS **Lemma 23.5.3.** *Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Let $I \subset A$ be an ideal. Two divided power structures γ, γ' on I are equal if and only if $\gamma_p = \gamma'_p$. Moreover, given a map $\delta : I \to I$ such that*

1. *$p!\delta(x) = x^p$ for all $x \in I$,*
2. *$\delta(ax) = a^p\delta(x)$ for all $a \in A$, $x \in I$, and*
3. *$\delta(x + y) = \delta(x) + \sum_{i+j=p, i,j\geq 1} \frac{1}{i!j!} x^i y^j + \delta(y)$ for all $x, y \in I$,*

then there exists a unique divided power structure γ on I such that $\gamma_p = \delta$.

Proof. If n is not divisible by p, then $\gamma_n(x) = cx\gamma_{n-1}(x)$ where c is a unit in $\mathbf{Z}_{(p)}$. Moreover,

$$\gamma_{pm}(x) = c\gamma_m(\gamma_p(x))$$

where c is a unit in $\mathbf{Z}_{(p)}$. Thus the first assertion is clear. For the second assertion, we can, working backwards, use these equalities to define all γ_n. More precisely, if $n = a_0 + a_1 p + \ldots + a_e p^e$ with $a_i \in \{0, \ldots, p-1\}$ then we set

$$\gamma_n(x) = c_n x^{a_0} \delta(x)^{a_1} \ldots \delta^e(x)^{a_e}$$

for $c_n \in \mathbf{Z}_{(p)}$ defined by

$$c_n = (p!)^{a_1 + a_2(1+p) + \ldots + a_e(1+\ldots+p^{e-1})}/n!.$$

Now we have to show the axioms (1) – (5) of a divided power structure, see Definition 23.2.1. We observe that (1) and (3) are immediate. Verification of (2) and (5) is by a direct calculation which we omit. Let $x, y \in I$. We claim there is a ring map

$$\varphi : \mathbf{Z}_{(p)}\langle u, v\rangle \longrightarrow A$$

which maps $u^{[n]}$ to $\gamma_n(x)$ and $v^{[n]}$ to $\gamma_n(y)$. By construction of $\mathbf{Z}_{(p)}\langle u, v\rangle$ this means we have to check that

$$\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$$

in A and similarly for y. This is true because (2) holds for γ. Let ϵ denote the divided power structure on the ideal $\mathbf{Z}_{(p)}\langle u, v\rangle_+$ of $\mathbf{Z}_{(p)}\langle u, v\rangle$. Next, we claim that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ for $f \in \mathbf{Z}_{(p)}\langle u, v\rangle_+$ and all n. This is clear for $n = 0, 1, \ldots, p-1$. For $n = p$ it suffices to prove it for a set of generators of the ideal $\mathbf{Z}_{(p)}\langle u, v\rangle_+$ because both ϵ_p and $\gamma_p = \delta$ satisfy properties (1) and (3) of the lemma. Hence it suffices to prove that $\gamma_p(\gamma_n(x)) = \frac{(pn)!}{p!(n!)^p}\gamma_{pn}(x)$ and similarly for y, which follows as (5) holds for γ. Now, if $n = a_0 + a_1 p + \ldots + a_e p^e$ is an arbitrary integer written in p-adic expansion as above, then

$$\epsilon_n(f) = c_n f^{a_0} \gamma_p(f)^{a_1} \ldots \gamma_p^e(f)^{a_e}$$

because ϵ is a divided power structure. Hence we see that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ holds for all n. Applying this for $f = u + v$ we see that axiom (4) for γ follows from the fact that ϵ is a divided power structure. □

23.6. Tate resolutions

09PF In this section we briefly discuss the resolutions constructed in [Tat57] and [AH86] which combine divided power structures with differential graded algebras. In this section we will use homological notation for differential graded algebras. Our differential graded algebras will sit in nonnegative homological degrees. Thus our differential graded algebras (A, d) will be given as chain complexes

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to \ldots$$

endowed with a multiplication.

Let R be a ring (commutative, as usual). In this section we will often consider graded R-algebras $A = \bigoplus_{d \geq 0} A_d$ whose components are zero in negative degrees. We will set $A_+ = \bigoplus_{d > 0} A_d$. We will write $A_{even} = \bigoplus_{d \geq 0} A_{2d}$ and $A_{odd} = \bigoplus_{d \geq 0} A_{2d+1}$. Recall that A is graded commutative if $xy = (-1)^{\deg(x)\deg(y)} yx$

for homogeneous elements x, y. Recall that A is strictly graded commutative if in addition $x^2 = 0$ for homogeneous elements x of odd degree. Finally, to understand the following definition, keep in mind that $\gamma_n(x) = x^n/n!$ if A is a $\mathbf{Q}$-algebra.

09PG Definition 23.6.1. Let R be a ring. Let $A = \bigoplus_{d \geq 0} A_d$ be a graded R-algebra which is strictly graded commutative. A collection of maps $\gamma_n : A_{even,+} \to A_{even,+}$ defined for all $n > 0$ is called a divided power structure on A if we have

(1) $\gamma_n(x) \in A_{2nd}$ if $x \in A_{2d}$,
(2) $\gamma_1(x) = x$ for any x, we also set $\gamma_0(x) = 1$,
(3) $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$,
(4) $\gamma_n(xy) = x^n\gamma_n(y)$ for all $x \in A_{even}$ and $y \in A_{even,+}$,
(5) $\gamma_n(xy) = 0$ if $x, y \in A_{odd}$ homogeneous and $n > 1$
(6) if $x, y \in A_{even,+}$ then $\gamma_n(x + y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$,
(7) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$ for $x \in A_{even,+}$.

Observe that conditions (2), (3), (4), (6), and (7) imply that γ is a "usual" divided power structure on the ideal $A_{even,+}$ of the (commutative) ring A_{even}, see Sections 23.2, 23.3, 23.4, and 23.5. In particular, we have $n!\gamma_n(x) = x^n$ for all $x \in A_{even,+}$. Condition (1) states that γ is compatible with grading and condition (5) tells us γ_n for $n > 1$ vanishes on products of homogeneous elements of odd degree. But note that it may happen that

$$\gamma_2(z_1z_2 + z_3z_4) = z_1z_2z_3z_4$$

is nonzero if z_1, z_2, z_3, z_4 are homogeneous elements of odd degree.

09PH Example 23.6.2 (Adjoining odd variable). Let R be a ring. Let (A, γ) be a strictly graded commutative graded R-algebra endowed with a divided power structure as in the definition above. Let $d > 0$ be an odd integer. In this setting we can adjoin a variable T of degree d to A. Namely, set

$$A\langle T\rangle = A \oplus AT$$

with grading given by $A\langle T\rangle_m = A_m \oplus A_{m-d}T$. We claim there is a unique divided power structure on $A\langle T\rangle$ compatible with the given divided power structure on A. Namely, we set

$$\gamma_n(x + yT) = \gamma_n(x) + \gamma_{n-1}(x)yT$$

for $x \in A_{even,+}$ and $y \in A_{odd}$.

09PI Example 23.6.3 (Adjoining even variable). Let R be a ring. Let (A, γ) be a strictly graded commutative graded R-algebra endowed with a divided power structure as in the definition above. Let $d > 0$ be an even integer. In this setting we can adjoin a variable T of degree d to A. Namely, set

$$A\langle T\rangle = A \oplus AT \oplus AT^{(2)} \oplus AT^{(3)} \oplus \ldots$$

with multiplication given by

$$T^{(n)}T^{(m)} = \frac{(n+m)!}{n!m!}T^{(n+m)}$$

and with grading given by

$$A\langle T\rangle_m = A_m \oplus A_{m-d}T \oplus A_{m-2d}T^{(2)} \oplus \ldots$$

We claim there is a unique divided power structure on $A\langle T\rangle$ compatible with the given divided power structure on A such that $\gamma_n(T^{(i)}) = T^{(ni)}$. To define the divided power structure we first set

$$\gamma_n\left(\sum\nolimits_{i>0} x_i T^{(i)}\right) = \sum \prod\nolimits_{n=\sum e_i} x_i^{e_i} T^{(ie_i)}$$

if x_i is in A_{even}. If $x_0 \in A_{even,+}$ then we take

$$\gamma_n\left(\sum\nolimits_{i\geq 0} x_i T^{(i)}\right) = \sum\nolimits_{a+b=n} \gamma_a(x_0)\gamma_b\left(\sum\nolimits_{i>0} x_i T^{(i)}\right)$$

where γ_b is as defined above.

0F4I Remark 23.6.4. We can also adjoin a set (possibly infinite) of exterior or divided power generators in a given degree $d > 0$, rather than just one as in Examples 23.6.2 and 23.6.3. Namely, following Remark 23.5.2: for (A, γ) as above and a set J, let $A\langle T_j : j \in J\rangle$ be the directed colimit of the algebras $A\langle T_j : j \in S\rangle$ over all finite subsets S of J. It is immediate that this algebra has a unique divided power structure, compatible with the given structure on A and on each generator T_j.

At this point we tie in the definition of divided power structures with differentials. To understand the definition note that $\mathrm{d}(x^n/n!) = \mathrm{d}(x)x^{n-1}/(n-1)!$ if A is a $\mathbf{Q}$-algebra and $x \in A_{even,+}$.

09PJ Definition 23.6.5. Let R be a ring. Let $A = \bigoplus_{d\geq 0} A_d$ be a differential graded R-algebra which is strictly graded commutative. A divided power structure γ on A is compatible with the differential graded structure if $\mathrm{d}(\gamma_n(x)) = \mathrm{d}(x)\gamma_{n-1}(x)$ for all $x \in A_{even,+}$.

Warning: Let (A, d, γ) be as in Definition 23.6.5. It may not be true that $\gamma_n(x)$ is a boundary, if x is a boundary. Thus γ in general does not induce a divided power structure on the homology algebra $H(A)$. In some papers the authors put an additional compatibility condition in order to ensure that this is the case, but we elect not to do so.

09PK Lemma 23.6.6. Let (A, d, γ) and (B, d, γ) be as in Definition 23.6.5. Let $f : A \to B$ be a map of differential graded algebras compatible with divided power structures. Assume

(1) $H_k(A) = 0$ for $k > 0$, and
(2) f is surjective.

Then γ induces a divided power structure on the graded R-algebra $H(B)$.

Proof. Suppose that x and x' are homogeneous of the same degree $2d$ and define the same cohomology class in $H(B)$. Say $x' - x = \mathrm{d}(w)$. Choose a lift $y \in A_{2d}$ of x and a lift $z \in A_{2d+1}$ of w. Then $y' = y + \mathrm{d}(z)$ is a lift of x'. Hence

$$\gamma_n(y') = \sum \gamma_i(y)\gamma_{n-i}(\mathrm{d}(z)) = \gamma_n(y) + \sum\nolimits_{i<n} \gamma_i(y)\gamma_{n-i}(\mathrm{d}(z))$$

Since A is acyclic in positive degrees and since $\mathrm{d}(\gamma_j(\mathrm{d}(z))) = 0$ for all j we can write this as

$$\gamma_n(y') = \gamma_n(y) + \sum\nolimits_{i<n} \gamma_i(y)\mathrm{d}(z_i)$$

for some z_i in A. Moreover, for $0 < i < n$ we have

$$\mathrm{d}(\gamma_i(y)z_i) = \mathrm{d}(\gamma_i(y))z_i + \gamma_i(y)\mathrm{d}(z_i) = \mathrm{d}(y)\gamma_{i-1}(y)z_i + \gamma_i(y)\mathrm{d}(z_i)$$

and the first term maps to zero in B as $d(y)$ maps to zero in B. Hence $\gamma_n(x')$ and $\gamma_n(x)$ map to the same element of $H(B)$. Thus we obtain a well defined map $\gamma_n : H_{2d}(B) \to H_{2nd}(B)$ for all $d > 0$ and $n > 0$. We omit the verification that this defines a divided power structure on $H(B)$. $\square$

09PL Lemma 23.6.7. Let (A, d, γ) be as in Definition 23.6.5. Let $R \to R'$ be a ring map. Then d and γ induce similar structures on $A' = A \otimes_R R'$ such that (A', d, γ) is as in Definition 23.6.5.

Proof. Observe that $A'_{even} = A_{even} \otimes_R R'$ and $A'_{even,+} = A_{even,+} \otimes_R R'$. Hence we are trying to show that the divided powers γ extend to A'_{even} (terminology as in Definition 23.4.1). Once we have shown γ extends it follows easily that this extension has all the desired properties.

Choose a polynomial R-algebra P (on any set of generators) and a surjection of R-algebras $P \to R'$. The ring map $A_{even} \to A_{even} \otimes_R P$ is flat, hence the divided powers γ extend to $A_{even} \otimes_R P$ uniquely by Lemma 23.4.2. Let $J = \mathrm{Ker}(P \to R')$. To show that γ extends to $A \otimes_R R'$ it suffices to show that $I' = \mathrm{Ker}(A_{even,+} \otimes_R P \to A_{even,+} \otimes_R R')$ is generated by elements z such that $\gamma_n(z) \in I'$ for all $n > 0$. This is clear as I' is generated by elements of the form $x \otimes f$ with $x \in A_{even,+}$ and $f \in \mathrm{Ker}(P \to R')$. $\square$

09PM Lemma 23.6.8. Let (A, d, γ) be as in Definition 23.6.5. Let $d \geq 1$ be an integer. Let $A\langle T\rangle$ be the graded divided power polynomial algebra on T with $\deg(T) = d$ constructed in Example 23.6.2 or 23.6.3. Let $f \in A_{d-1}$ be an element with $\mathrm{d}(f) = 0$. There exists a unique differential d on $A\langle T\rangle$ such that $\mathrm{d}(T) = f$ and such that d is compatible with the divided power structure on $A\langle T\rangle$.

Proof. This is proved by a direct computation which is omitted. $\square$

In Lemma 23.12.3 we will compute the cohomology of $A\langle T\rangle$ in some special cases. Here is Tate's construction, as extended by Avramov and Halperin.

09PN Lemma 23.6.9. Let $R \to S$ be a homomorphism of commutative rings. There exists a factorization

$$R \to A \to S$$

with the following properties:

(1) (A, d, γ) is as in Definition 23.6.5,
(2) $A \to S$ is a quasi-isomorphism (if we endow S with the zero differential),
(3) $A_0 = R[x_j : j \in J] \to S$ is any surjection of a polynomial ring onto S, and
(4) A is a graded divided power polynomial algebra over R.

The last condition means that A is constructed out of A_0 by successively adjoining a set of variables T in each degree > 0 as in Example 23.6.2 or 23.6.3. Moreover, if R is Noetherian and $R \to S$ is of finite type, then A can be taken to have only finitely many generators in each degree.

Proof. We write out the construction for the case that R is Noetherian and $R \to S$ is of finite type. Without those assumptions, the proof is the same, except that we have to use some set (possibly infinite) of generators in each degree.

Start of the construction: Let $A(0) = R[x_1, \ldots, x_n]$ be a (usual) polynomial ring and let $A(0) \to S$ be a surjection. As grading we take $A(0)_0 = A(0)$ and $A(0)_d = 0$ for $d \neq 0$. Thus $\text{d} = 0$ and γ_n, $n > 0$, is zero as well.

Choose generators $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ for the kernel of the given map $A(0) = R[x_1, \ldots, x_n] \to S$. We apply Example 23.6.2 m times to get

$$A(1) = A(0)\langle T_1, \ldots, T_m\rangle$$

with $\deg(T_i) = 1$ as a graded divided power polynomial algebra. We set $\text{d}(T_i) = f_i$. Since $A(1)$ is a divided power polynomial algebra over $A(0)$ and since $\text{d}(f_i) = 0$ this extends uniquely to a differential on $A(1)$ by Lemma 23.6.8.

Induction hypothesis: Assume we are given factorizations

$$R \to A(0) \to A(1) \to \ldots \to A(m) \to S$$

where $A(0)$ and $A(1)$ are as above and each $R \to A(m') \to S$ for $2 \leq m' \leq m$ satisfies properties (1) and (4) of the statement of the lemma and (2) replaced by the condition that $H_i(A(m')) \to H_i(S)$ is an isomorphism for $m' > i \geq 0$. The base case is $m = 1$.

Induction step: Assume we have $R \to A(m) \to S$ as in the induction hypothesis. Consider the group $H_m(A(m))$. This is a module over $H_0(A(m)) = S$. In fact, it is a subquotient of $A(m)_m$ which is a finite type module over $A(m)_0 = R[x_1, \ldots, x_n]$. Thus we can pick finitely many elements

$$e_1, \ldots, e_t \in \text{Ker}(\text{d} : A(m)_m \to A(m)_{m-1})$$

which map to generators of this module. Applying Example 23.6.2 or 23.6.3 t times we get

$$A(m+1) = A(m)\langle T_1, \ldots, T_t\rangle$$

with $\deg(T_i) = m+1$ as a graded divided power algebra. We set $\text{d}(T_i) = e_i$. Since $A(m+1)$ is a divided power polynomial algebra over $A(m)$ and since $\text{d}(e_i) = 0$ this extends uniquely to a differential on $A(m+1)$ compatible with the divided power structure. Since we've added only material in degree $m+1$ and higher we see that $H_i(A(m+1)) = H_i(A(m))$ for $i < m$. Moreover, it is clear that $H_m(A(m+1)) = 0$ by construction.

To finish the proof we observe that we have shown there exists a sequence of maps

$$R \to A(0) \to A(1) \to \ldots \to A(m) \to A(m+1) \to \ldots \to S$$

and to finish the proof we set $A = \text{colim}\, A(m)$. □

0BZ9 Lemma 23.6.10. Let $R \to S$ be a pseudo-coherent ring map (More on Algebra, Definition 15.82.1). Then Lemma 23.6.9 holds, with the resolution A of S having finitely many generators in each degree.

Proof. This is proved in exactly the same way as Lemma 23.6.9. The only additional twist is that, given $A(m) \to S$ we have to show that $H_m = H_m(A(m))$ is a finite $R[x_1, \ldots, x_m]$-module (so that in the next step we need only add finitely many variables). Consider the complex

$$\ldots \to A(m)_{m-1} \to A(m)_m \to A(m)_{m-1} \to \ldots \to A(m)_0 \to S \to 0$$

Since S is a pseudo-coherent $R[x_1, \ldots, x_n]$-module and since $A(m)_i$ is a finite free $R[x_1, \ldots, x_n]$-module we conclude that this is a pseudo-coherent complex, see More on Algebra, Lemma 15.64.9. Since the complex is exact in (homological) degrees $> m$ we conclude that H_m is a finite R-module by More on Algebra, Lemma 15.64.3. □

09PP Lemma 23.6.11. Let R be a commutative ring. Suppose that (A, d, γ) and (B, d, γ) are as in Definition 23.6.5. Let $\overline{\varphi} : H_0(A) \to H_0(B)$ be an R-algebra map. Assume

(1) A is a graded divided power polynomial algebra over R.
(2) $H_k(B) = 0$ for $k > 0$.

Then there exists a map $\varphi : A \to B$ of differential graded R-algebras compatible with divided powers that lifts $\overline{\varphi}$.

Proof. The assumption means that A is obtained from R by successively adjoining some set of polynomial generators in degree zero, exterior generators in positive odd degrees, and divided power generators in positive even degrees. So we have a filtration $R \subset A(0) \subset A(1) \subset \ldots$ of A such that $A(m+1)$ is obtained from $A(m)$ by adjoining generators of the appropriate type (which we simply call "divided power generators") in degree $m+1$. In particular, $A(0) \to H_0(A)$ is a surjection from a (usual) polynomial algebra over R onto $H_0(A)$. Thus we can lift $\overline{\varphi}$ to an R-algebra map $\varphi(0) : A(0) \to B_0$.

Write $A(1) = A(0)\langle T_j : j \in J\rangle$ for some set J of divided power variables T_j of degree 1. Let $f_j \in B_0$ be $f_j = \varphi(0)(\mathrm{d}(T_j))$. Observe that f_j maps to zero in $H_0(B)$ as $\mathrm{d}T_j$ maps to zero in $H_0(A)$. Thus we can find $b_j \in B_1$ with $\mathrm{d}(b_j) = f_j$. By the universal property of divided power polynomial algebras from Lemma 23.5.1, we find a lift $\varphi(1) : A(1) \to B$ of $\varphi(0)$ mapping T_j to f_j.

Having constructed $\varphi(m)$ for some $m \geq 1$ we can construct $\varphi(m+1) : A(m+1) \to B$ in exactly the same manner. We omit the details. □

09PQ Lemma 23.6.12. Let R be a commutative ring. Let S and T be commutative R-algebras. Then there is a canonical structure of a strictly graded commutative R-algebra with divided powers on

$$\mathrm{Tor}_*^R(S, T).$$

Proof. Choose a factorization $R \to A \to S$ as above. Since $A \to S$ is a quasi-isomorphism and since A_d is a free R-module, we see that the differential graded algebra $B = A \otimes_R T$ computes the Tor groups displayed in the lemma. Choose a surjection $R[y_j : j \in J] \to T$. Then we see that B is a quotient of the differential graded algebra $A[y_j : j \in J]$ whose homology sits in degree 0 (it is equal to $S[y_j : j \in J]$). By Lemma 23.6.7 the differential graded algebras B and $A[y_j : j \in J]$ have divided power structures compatible with the differentials. Hence we obtain our divided power structure on $H(B)$ by Lemma 23.6.6.

The divided power algebra structure constructed in this way is independent of the choice of A. Namely, if A' is a second choice, then Lemma 23.6.11 implies there is a map $A \to A'$ preserving all structure and the augmentations towards S. Then the induced map $B = A \otimes_R T \to A' \otimes_R T' = B'$ also preserves all structure and is a quasi-isomorphism. The induced isomorphism of Tor algebras is therefore compatible with products and divided powers. □

23.7. Application to complete intersections

09PR Let R be a ring. Let (A, d, γ) be as in Definition 23.6.5. A derivation of degree 2 is an R-linear map $\theta : A \to A$ with the following properties

(1) $\theta(A_d) \subset A_{d-2}$,
(2) $\theta(xy) = \theta(x)y + x\theta(y)$,
(3) θ commutes with d,
(4) $\theta(\gamma_n(x)) = \theta(x)\gamma_{n-1}(x)$ for all $x \in A_{2d}$ all d.

In the following lemma we construct a derivation.

09PS **Lemma 23.7.1.** Let R be a ring. Let (A, d, γ) be as in Definition 23.6.5. Let $R' \to R$ be a surjection of rings whose kernel has square zero and is generated by one element f. If A is a graded divided power polynomial algebra over R with finitely many variables in each degree, then we obtain a derivation $\theta : A/IA \to A/IA$ where I is the annihilator of f in R.

Proof. Since A is a divided power polynomial algebra, we can find a divided power polynomial algebra A' over R' such that $A = A' \otimes_R R'$. Moreover, we can lift d to an R-linear operator d on A' such that

(1) $\mathrm{d}(xy) = \mathrm{d}(x)y + (-1)^{\deg(x)}x\mathrm{d}(y)$ for $x, y \in A'$ homogeneous, and
(2) $\mathrm{d}(\gamma_n(x)) = \mathrm{d}(x)\gamma_{n-1}(x)$ for $x \in A'_{even,+}$.

We omit the details (hint: proceed one variable at the time). However, it may not be the case that d^2 is zero on A'. It is clear that d^2 maps A' into $fA' \cong A/IA$. Hence d^2 annihilates fA' and factors as a map $A \to A/IA$. Since d^2 is R-linear we obtain our map $\theta : A/IA \to A/IA$. The verification of the properties of a derivation is immediate. $\square$

09PT **Lemma 23.7.2.** Assumption and notation as in Lemma 23.7.1. Suppose $S = H_0(A)$ is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some n, m, and $f_j \in R[x_1, \ldots, x_n]$. Moreover, suppose given a relation

$$\sum r_j f_j = 0$$

with $r_j \in R[x_1, \ldots, x_n]$. Choose $r'_j, f'_j \in R'[x_1, \ldots, x_n]$ lifting r_j, f_j. Write $\sum r'_j f'_j = gf$ for some $g \in R/I[x_1, \ldots, x_n]$. If $H_1(A) = 0$ and all the coefficients of each r_j are in I, then there exists an element $\xi \in H_2(A/IA)$ such that $\theta(\xi) = g$ in S/IS.

Proof. Let $A(0) \subset A(1) \subset A(2) \subset \ldots$ be the filtration of A such that $A(m)$ is gotten from $A(m-1)$ by adjoining divided power variables of degree m. Then $A(0)$ is a polynomial algebra over R equipped with an R-algebra surjection $A(0) \to S$. Thus we can choose a map

$$\varphi : R[x_1, \ldots, x_n] \to A(0)$$

lifting the augmentations to S. Next, $A(1) = A(0)\langle T_1, \ldots, T_t \rangle$ for some divided power variables T_i of degree 1. Since $H_0(A) = S$ we can pick $\xi_j \in \sum A(0)T_i$ with $\mathrm{d}(\xi_j) = \varphi(f_j)$. Then

$$\mathrm{d}\left(\sum \varphi(r_j)\xi_j\right) = \sum \varphi(r_j)\varphi(f_j) = \sum \varphi(r_j f_j) = 0$$

Since $H_1(A) = 0$ we can pick $\xi \in A_2$ with $\mathrm{d}(\xi) = \sum \varphi(r_j)\xi_j$. If the coefficients of r_j are in I, then the same is true for $\varphi(r_j)$. In this case $\mathrm{d}(\xi)$ dies in A_1/IA_1 and hence ξ defines a class in $H_2(A/IA)$.

The construction of θ in the proof of Lemma 23.7.1 proceeds by successively lifting $A(i)$ to $A'(i)$ and lifting the differential d. We lift φ to $\varphi' : R'[x_1, \ldots, x_n] \to A'(0)$. Next, we have $A'(1) = A'(0)\langle T_1, \ldots, T_t\rangle$. Moreover, we can lift ξ_j to $\xi'_j \in \sum A'(0)T_i$. Then $\mathrm{d}(\xi'_j) = \varphi'(f'_j) + fa_j$ for some $a_j \in A'(0)$. Consider a lift $\xi' \in A'_2$ of ξ. Then we know that

$$\mathrm{d}(\xi') = \sum \varphi'(r'_j)\xi'_j + \sum fb_iT_i$$

for some $b_i \in A(0)$. Applying d again we find

$$\theta(\xi) = \sum \varphi'(r'_j)\varphi'(f'_j) + \sum f\varphi'(r'_j)a_j + \sum fb_i\mathrm{d}(T_i)$$

The first term gives us what we want. The second term is zero because the coefficients of r_j are in I and hence are annihilated by f. The third term maps to zero in H_0 because $\mathrm{d}(T_i)$ maps to zero. □

The method of proof of the following lemma is apparently due to Gulliksen.

09PU Lemma 23.7.3. Let $R' \to R$ be a surjection of Noetherian rings whose kernel has square zero and is generated by one element f. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\sum r_jf_j = 0$ be a relation in $R[x_1, \ldots, x_n]$. Assume that

(1) each r_j has coefficients in the annihilator I of f in R,
(2) for some lifts $r'_j, f'_j \in R'[x_1, \ldots, x_n]$ we have $\sum r'_jf'_j = gf$ where g is not nilpotent in S/IS.

Then S does not have finite tor dimension over R (i.e., S is not a perfect R-algebra).

Proof. Choose a Tate resolution $R \to A \to S$ as in Lemma 23.6.9. Let $\xi \in H_2(A/IA)$ and $\theta : A/IA \to A/IA$ be the element and derivation found in Lemmas 23.7.1 and 23.7.2. Observe that

$$\theta^n(\gamma_n(\xi)) = g^n$$

in $H_0(A/IA) = S/IS$. Hence if g is not nilpotent in S/IS, then ξ^n is nonzero in $H_{2n}(A/IA)$ for all $n > 0$. Since $H_{2n}(A/IA) = \mathrm{Tor}^R_{2n}(S, R/I)$ we conclude. □

The following result can be found in [Rod88].

09PV Lemma 23.7.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset J \subset A$ be proper ideals. If A/J has finite tor dimension over A/I, then $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective.

Proof. Let $f \in I$ be an element mapping to a nonzero element of $I/\mathfrak{m}I$ which is mapped to zero in $J/\mathfrak{m}J$. We can choose an ideal I' with $\mathfrak{m}I \subset I' \subset I$ such that I/I' is generated by the image of f. Set $R = A/I$ and $R' = A/I'$. Let $J = (a_1, \ldots, a_m)$ for some $a_j \in A$. Then $f = \sum b_ja_j$ for some $b_j \in \mathfrak{m}$. Let $r_j, f_j \in R$ resp. $r'_j, f'_j \in R'$ be the image of b_j, a_j. Then we see we are in the situation of Lemma 23.7.3 (with the ideal I of that lemma equal to $\mathfrak{m}_R$) and the lemma is proved. □

09PW Lemma 23.7.5. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset J \subset A$ be proper ideals. Assume

(1) A/J has finite tor dimension over A/I, and
(2) J is generated by a regular sequence.

Then I is generated by a regular sequence and J/I is generated by a regular sequence.

Proof. By Lemma 23.7.4 we see that $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective. Thus we can find $s \leq r$ and a minimal system of generators $f_1, \ldots, f_r$ of J such that $f_1, \ldots, f_s$ are in I and form a minimal system of generators of I. The lemma follows as any minimal system of generators of J is a regular sequence by More on Algebra, Lemmas 15.30.15 and 15.30.7. □

09PX Lemma 23.7.6. Let $R \to S$ be a local ring map of Noetherian local rings. Let $I \subset R$ and $J \subset S$ be ideals with $IS \subset J$. If $R \to S$ is flat and $S/\mathfrak{m}_R S$ is regular, then the following are equivalent

(1) J is generated by a regular sequence and S/J has finite tor dimension as a module over R/I,
(2) J is generated by a regular sequence and $\text{Tor}_p^{R/I}(S/J, R/\mathfrak{m}_R)$ is nonzero for only finitely many p,
(3) I is generated by a regular sequence and J/IS is generated by a regular sequence in S/IS.

Proof. If (3) holds, then J is generated by a regular sequence, see for example More on Algebra, Lemmas 15.30.13 and 15.30.7. Moreover, if (3) holds, then $S/J = (S/I)/(J/I)$ has finite projective dimension over S/IS because the Koszul complex will be a finite free resolution of S/J over S/IS. Since $R/I \to S/IS$ is flat, it then follows that S/J has finite tor dimension over R/I by More on Algebra, Lemma 15.66.11. Thus (3) implies (1).

The implication (1) ⇒ (2) is trivial. Assume (2). By More on Algebra, Lemma 15.77.6 we find that S/J has finite tor dimension over S/IS. Thus we can apply Lemma 23.7.5 to conclude that IS and J/IS are generated by regular sequences. Let $f_1, \ldots, f_r \in I$ be a minimal system of generators of I. Since $R \to S$ is flat, we see that $f_1, \ldots, f_r$ form a minimal system of generators for IS in S. Thus $f_1, \ldots, f_r \in R$ is a sequence of elements whose images in S form a regular sequence by More on Algebra, Lemmas 15.30.15 and 15.30.7. Thus $f_1, \ldots, f_r$ is a regular sequence in R by Algebra, Lemma 10.68.5. □

23.8. Local complete intersection rings

09PY Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. By the Cohen structure theorem (see Algebra, Theorem 10.160.8) we can write A as the quotient of a regular Noetherian complete local ring R. Let us say that A is a complete intersection if there exists some surjection $R \to A$ with R a regular local ring such that the kernel is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the surjection.

09PZ Lemma 23.8.1. Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. The following are equivalent

(1) for every surjection of local rings $R \to A$ with R a regular local ring, the kernel of $R \to A$ is generated by a regular sequence, and
(2) for some surjection of local rings $R \to A$ with R a regular local ring, the kernel of $R \to A$ is generated by a regular sequence.

Proof. Let k be the residue field of A. If the characteristic of k is $p > 0$, then we denote Λ a Cohen ring (Algebra, Definition 10.160.5) with residue field k (Algebra, Lemma 10.160.6). If the characteristic of k is 0 we set $\Lambda = k$. Recall that $\Lambda[[x_1, \ldots, x_n]]$ for any n is formally smooth over $\mathbf{Z}$, resp. $\mathbf{Q}$ in the $\mathfrak{m}$-adic topology, see More on Algebra, Lemma 15.39.1. Fix a surjection $\Lambda[[x_1, \ldots, x_n]] \to A$ as in the Cohen structure theorem (Algebra, Theorem 10.160.8).

Let $R \to A$ be a surjection from a regular local ring R. Let $f_1, \ldots, f_r$ be a minimal sequence of generators of $\text{Ker}(R \to A)$. We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that $f_1, \ldots, f_r$ is a regular sequence in R if and only if $f_1, \ldots, f_r$ is a regular sequence in the completion $R^\wedge$ by Algebra, Lemmas 10.68.5 and 10.97.2. Moreover, we have

$$R^\wedge/(f_1, \ldots, f_r)R^\wedge = (R/(f_1, \ldots, f_n))^\wedge = A^\wedge = A$$

because A is $\mathfrak{m}_A$-adically complete (first equality by Algebra, Lemma 10.97.1). Finally, the ring $R^\wedge$ is regular since R is regular (More on Algebra, Lemma 15.43.4). Hence we may assume R is complete.

If R is complete we can choose a map $\Lambda[[x_1, \ldots, x_n]] \to R$ lifting the given map $\Lambda[[x_1, \ldots, x_n]] \to A$, see More on Algebra, Lemma 15.37.5. By adding some more variables $y_1, \ldots, y_m$ mapping to generators of the kernel of $R \to A$ we may assume that $\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \to R$ is surjective (some details omitted). Then we can consider the commutative diagram

$$\begin{tikzcd} \Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \arrow[r] \arrow[d] & R \arrow[d] \\ \Lambda[[x_1, \ldots, x_n]] \arrow[r] & A \end{tikzcd}$$

By Algebra, Lemma 10.135.6 we see that the condition for $R \to A$ is equivalent to the condition for the fixed chosen map $\Lambda[[x_1, \ldots, x_n]] \to A$. This finishes the proof of the lemma. $\square$

The following two lemmas are sanity checks on the definition given above.

09Q0 **Lemma 23.8.2.** Let R be a regular ring. Let $\mathfrak{p} \subset R$ be a prime. Let $f_1, \ldots, f_r \in \mathfrak{p}$ be a regular sequence. Then the completion of

$$A = (R/(f_1, \ldots, f_r))_\mathfrak{p} = R_\mathfrak{p}/(f_1, \ldots, f_r)R_\mathfrak{p}$$

is a complete intersection in the sense defined above.

Proof. The completion of A is equal to $A^\wedge = R_\mathfrak{p}^\wedge/(f_1, \ldots, f_r)R_\mathfrak{p}^\wedge$ because completion for finite modules over the Noetherian ring $R_\mathfrak{p}$ is exact (Algebra, Lemma 10.97.1). The image of the sequence $f_1, \ldots, f_r$ in $R_\mathfrak{p}$ is a regular sequence by Algebra, Lemmas 10.97.2 and 10.68.5. Moreover, $R_\mathfrak{p}^\wedge$ is a regular local ring by More on Algebra, Lemma 15.43.4. Hence the result holds by our definition of complete intersection for complete local rings. $\square$

The following lemma is the analogue of Algebra, Lemma 10.135.4.

09Q1 **Lemma 23.8.3.** Let R be a regular ring. Let $\mathfrak{p} \subset R$ be a prime. Let $I \subset \mathfrak{p}$ be an ideal. Set $A = (R/I)_\mathfrak{p} = R_\mathfrak{p}/I_\mathfrak{p}$. The following are equivalent

(1) the completion of A is a complete intersection in the sense above,
(2) $I_\mathfrak{p} \subset R_\mathfrak{p}$ is generated by a regular sequence,
(3) the module $(I/I^2)_\mathfrak{p}$ can be generated by $\dim(R_\mathfrak{p}) - \dim(A)$ elements,
(4) add more here.

Proof. We may and do replace R by its localization at $\mathfrak{p}$. Then $\mathfrak{p} = \mathfrak{m}$ is the maximal ideal of R and $A = R/I$. Let $f_1, \ldots, f_r \in I$ be a minimal sequence of generators. The completion of A is equal to $A^\wedge = R^\wedge/(f_1, \ldots, f_r)R^\wedge$ because completion for finite modules over the Noetherian ring $R_\mathfrak{p}$ is exact (Algebra, Lemma 10.97.1).

If (1) holds, then the image of the sequence $f_1, \ldots, f_r$ in $R^\wedge$ is a regular sequence by assumption. Hence it is a regular sequence in R by Algebra, Lemmas 10.97.2 and 10.68.5. Thus (1) implies (2).

Assume (3) holds. Set $c = \dim(R) - \dim(A)$ and let $f_1, \ldots, f_c \in I$ map to generators of I/I^2. by Nakayama's lemma (Algebra, Lemma 10.20.1) we see that $I = (f_1, \ldots, f_c)$. Since R is regular and hence Cohen-Macaulay (Algebra, Proposition 10.103.4) we see that $f_1, \ldots, f_c$ is a regular sequence by Algebra, Proposition 10.103.4. Thus (3) implies (2). Finally, (2) implies (1) by Lemma 23.8.2. □

The following result is due to Avramov, see [Avr75].

09Q2 Proposition 23.8.4. Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Then the following are equivalent

(1) $B^\wedge$ is a complete intersection,
(2) $A^\wedge$ and $(B/\mathfrak{m}_A B)^\wedge$ are complete intersections.

Proof. Consider the diagram

$$\begin{tikzcd} B \arrow[r] & B^\wedge \\ A \arrow[r] \arrow[u] & A^\wedge \arrow[u] \end{tikzcd}$$

Since the horizontal maps are faithfully flat (Algebra, Lemma 10.97.3) we conclude that the right vertical arrow is flat (for example by Algebra, Lemma 10.99.15). Moreover, we have $(B/\mathfrak{m}_A B)^\wedge = B^\wedge/\mathfrak{m}_{A^\wedge} B^\wedge$ by Algebra, Lemma 10.97.1. Thus we may assume A and B are complete local Noetherian rings.

Assume A and B are complete local Noetherian rings. Choose a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

as in More on Algebra, Lemma 15.39.3. Let $I = \mathrm{Ker}(R \to A)$ and $J = \mathrm{Ker}(S \to B)$. Note that since $R/I = A \to B = S/J$ is flat the map $J/IS \otimes_R R/\mathfrak{m}_R \to J/J \cap \mathfrak{m}_R S$ is an isomorphism. Hence a minimal system of generators of J/IS maps to a minimal system of generators of $\mathrm{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$. Finally, $S/\mathfrak{m}_R S$ is a regular local ring.

Assume (1) holds, i.e., J is generated by a regular sequence. Since $A = R/I \to B = S/J$ is flat we see Lemma 23.7.6 applies and we deduce that I and J/IS are generated by regular sequences. We have $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B)$ and $\dim(S/IS) = \dim(A) + \dim(S/\mathfrak{m}_R S)$ (Algebra, Lemma 10.112.7). Thus J/IS is generated by

$$\dim(S/IS) - \dim(S/J) = \dim(S/\mathfrak{m}_R S) - \dim(B/\mathfrak{m}_A B)$$

elements (Algebra, Lemma 10.60.13). It follows that $\mathrm{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$ is generated by the same number of elements (see above). Hence $\mathrm{Ker}(S/\mathfrak{m}_R S \to$

$B/\mathfrak{m}_A B$) is generated by a regular sequence, see for example Lemma 23.8.3. In this way we see that (2) holds.

If (2) holds, then I and $J/J \cap \mathfrak{m}_R S$ are generated by regular sequences. Lifting these generators (see above), using flatness of $R/I \to S/IS$, and using Grothendieck's lemma (Algebra, Lemma 10.99.3) we find that J/IS is generated by a regular sequence in S/IS. Thus Lemma 23.7.6 tells us that J is generated by a regular sequence, whence (1) holds. $\square$

09Q3 Definition 23.8.5. Let A be a Noetherian ring.

(1) If A is local, then we say A is a complete intersection if its completion is a complete intersection in the sense above.
(2) In general we say A is a local complete intersection if all of its local rings are complete intersections.

We will check below that this does not conflict with the terminology introduced in Algebra, Definitions 10.135.1 and 10.135.5. But first, we show this "makes sense" by showing that if A is a Noetherian local complete intersection, then A is a local complete intersection, i.e., all of its local rings are complete intersections.

09Q4 Lemma 23.8.6. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $\mathfrak{p} \subset A$ be a prime ideal. If A is a complete intersection, then $A_\mathfrak{p}$ is a complete intersection too.

Proof. Choose a prime $\mathfrak{q}$ of $A^\wedge$ lying over $\mathfrak{p}$ (this is possible as $A \to A^\wedge$ is faithfully flat by Algebra, Lemma 10.97.3). Then $A_\mathfrak{p} \to (A^\wedge)_\mathfrak{q}$ is a flat local ring homomorphism. Thus by Proposition 23.8.4 we see that $A_\mathfrak{p}$ is a complete intersection if and only if $(A^\wedge)_\mathfrak{q}$ is a complete intersection. Thus it suffices to prove the lemma in case A is complete (this is the key step of the proof).

Assume A is complete. By definition we may write $A = R/(f_1, \ldots, f_r)$ for some regular sequence $f_1, \ldots, f_r$ in a regular local ring R. Let $\mathfrak{q} \subset R$ be the prime corresponding to $\mathfrak{p}$. Observe that $f_1, \ldots, f_r \in \mathfrak{q}$ and that $A_\mathfrak{p} = R_\mathfrak{q}/(f_1, \ldots, f_r)R_\mathfrak{q}$. Hence $A_\mathfrak{p}$ is a complete intersection by Lemma 23.8.2. $\square$

09Q5 Lemma 23.8.7. Let A be a Noetherian ring. Then A is a local complete intersection if and only if $A_\mathfrak{m}$ is a complete intersection for every maximal ideal $\mathfrak{m}$ of A.

Proof. This follows immediately from Lemma 23.8.6 and the definitions. $\square$

09Q6 Lemma 23.8.8. Let S be a finite type algebra over a field k.

(1) for a prime $\mathfrak{q} \subset S$ the local ring $S_\mathfrak{q}$ is a complete intersection in the sense of Algebra, Definition 10.135.5 if and only if $S_\mathfrak{q}$ is a complete intersection in the sense of Definition 23.8.5, and
(2) S is a local complete intersection in the sense of Algebra, Definition 10.135.1 if and only if S is a local complete intersection in the sense of Definition 23.8.5.

Proof. Proof of (1). Let $k[x_1, \ldots, x_n] \to S$ be a surjection. Let $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak{q}$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of our surjection. Note that $k[x_1, \ldots, x_n]_\mathfrak{p} \to S_\mathfrak{q}$ is surjective with kernel $I_\mathfrak{p}$. Observe that $k[x_1, \ldots, x_n]$ is a regular ring by Algebra, Proposition 10.114.2. Hence the equivalence of the two notions in (1) follows by combining Lemma 23.8.3 with Algebra, Lemma 10.135.7.

Having proved (1) the equivalence in (2) follows from the definition and Algebra, Lemma 10.135.9. □

09Q7 **Lemma 23.8.9.** Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Then the following are equivalent

(1) B is a complete intersection,
(2) A and $B/\mathfrak{m}_A B$ are complete intersections.

Proof. Now that the definition makes sense this is a trivial reformulation of the (nontrivial) Proposition 23.8.4. □

23.9. Local complete intersection maps

09Q9 Let $A \to B$ be a local homomorphism of Noetherian complete local rings. A consequence of the Cohen structure theorem is that we can find a commutative diagram

$$\begin{tikzcd} S \arrow[r] & B \\ & A \arrow[ul] \arrow[u] \end{tikzcd}$$

of Noetherian complete local rings with $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ a regular local ring. This follows from More on Algebra, Lemma 15.39.3. Let us (temporarily) say $A \to S \to B$ is a good factorization of $A \to B$ if S is a Noetherian local ring, $A \to S \to B$ are local ring maps, $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ regular. Let us say that $A \to B$ is a complete intersection homomorphism if there exists some good factorization $A \to S \to B$ such that the kernel of $S \to B$ is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the diagram.

09QA **Lemma 23.9.1.** Let $A \to B$ be a local homomorphism of Noetherian complete local rings. The following are equivalent

(1) for some good factorization $A \to S \to B$ the kernel of $S \to B$ is generated by a regular sequence, and
(2) for every good factorization $A \to S \to B$ the kernel of $S \to B$ is generated by a regular sequence.

Proof. Let $A \to S \to B$ be a good factorization. As B is complete we obtain a factorization $A \to S^\wedge \to B$ where $S^\wedge$ is the completion of S. Note that this is also a good factorization: The ring map $S \to S^\wedge$ is flat (Algebra, Lemma 10.97.2), hence $A \to S^\wedge$ is flat. The ring $S^\wedge/\mathfrak{m}_A S^\wedge = (S/\mathfrak{m}_A S)^\wedge$ is regular since $S/\mathfrak{m}_A S$ is regular (More on Algebra, Lemma 15.43.4). Let $f_1, \ldots, f_r$ be a minimal sequence of generators of $\mathrm{Ker}(S \to B)$. We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that $f_1, \ldots, f_r$ is a regular sequence in S if and only if $f_1, \ldots, f_r$ is a regular sequence in the completion $S^\wedge$ by Algebra, Lemma 10.68.5. Moreover, we have

$$S^\wedge/(f_1, \ldots, f_r)R^\wedge = (S/(f_1, \ldots, f_n))^\wedge = B^\wedge = B$$

because B is $\mathfrak{m}_B$-adically complete (first equality by Algebra, Lemma 10.97.1). Thus the kernel of $S \to B$ is generated by a regular sequence if and only if the kernel of $S^\wedge \to B$ is generated by a regular sequence. Hence it suffices to consider good factorizations where S is complete.

Assume we have two factorizations $A \to S \to B$ and $A \to S' \to B$ with S and S' complete. By More on Algebra, Lemma 15.39.4 the ring $S \times_B S'$ is a Noetherian complete local ring. Hence, using More on Algebra, Lemma 15.39.3 we can choose a good factorization $A \to S'' \to S \times_B S'$ with S'' complete. Thus it suffices to show: If $A \to S' \to S \to B$ are comparable good factorizations, then $\text{Ker}(S \to B)$ is generated by a regular sequence if and only if $\text{Ker}(S' \to B)$ is generated by a regular sequence.

Let $A \to S' \to S \to B$ be comparable good factorizations. First, since $S'/\mathfrak{m}_R S' \to S/\mathfrak{m}_R S$ is a surjection of regular local rings, the kernel is generated by a regular sequence $\overline{x}_1, \ldots, \overline{x}_c \in \mathfrak{m}_{S'}/\mathfrak{m}_R S'$ which can be extended to a regular system of parameters for the regular local ring $S'/\mathfrak{m}_R S'$, see (Algebra, Lemma 10.106.4). Set $I = \text{Ker}(S' \to S)$. By flatness of S over R we have

$$I/\mathfrak{m}_R I = \text{Ker}(S'/\mathfrak{m}_R S' \to S/\mathfrak{m}_R S) = (\overline{x}_1, \ldots, \overline{x}_c).$$

Choose lifts $x_1, \ldots, x_c \in I$. These lifts form a regular sequence generating I as S' is flat over R, see Algebra, Lemma 10.99.3.

We conclude that if also $\text{Ker}(S \to B)$ is generated by a regular sequence, then so is $\text{Ker}(S' \to B)$, see More on Algebra, Lemmas 15.30.13 and 15.30.7.

Conversely, assume that $J = \text{Ker}(S' \to B)$ is generated by a regular sequence. Because the generators $x_1, \ldots, x_c$ of I map to linearly independent elements of $\mathfrak{m}_{S'}/\mathfrak{m}_{S'}^2$ we see that $I/\mathfrak{m}_{S'} I \to J/\mathfrak{m}_{S'} J$ is injective. Hence there exists a minimal system of generators $x_1, \ldots, x_c, y_1, \ldots, y_d$ for J. Then $x_1, \ldots, x_c, y_1, \ldots, y_d$ is a regular sequence and it follows that the images of $y_1, \ldots, y_d$ in S form a regular sequence generating $\text{Ker}(S \to B)$. This finishes the proof of the lemma. ☐

In the following proposition observe that the condition on vanishing of Tor's applies in particular if B has finite tor dimension over A and thus in particular if B is flat over A.

09QB **Proposition 23.9.2.** Let $A \to B$ be a local homomorphism of Noetherian local rings. Then the following are equivalent

(1) B is a complete intersection and $\text{Tor}_p^A(B, A/\mathfrak{m}_A)$ is nonzero for only finitely many p,
(2) A is a complete intersection and $A^\wedge \to B^\wedge$ is a complete intersection homomorphism in the sense defined above.

Proof. Let $F_\bullet \to A/\mathfrak{m}_A$ be a resolution by finite free A-modules. Observe that $\text{Tor}_p^A(B, A/\mathfrak{m}_A)$ is the pth homology of the complex $F_\bullet \otimes_A B$. Let $F_\bullet^\wedge = F_\bullet \otimes_A A^\wedge$ be the completion. Then $F_\bullet^\wedge$ is a resolution of $A^\wedge/\mathfrak{m}_{A^\wedge}$ by finite free $A^\wedge$-modules (as $A \to A^\wedge$ is flat and completion on finite modules is exact, see Algebra, Lemmas 10.97.1 and 10.97.2). It follows that

$$F_\bullet^\wedge \otimes_{A^\wedge} B^\wedge = F_\bullet \otimes_A B \otimes_B B^\wedge$$

By flatness of $B \to B^\wedge$ we conclude that

$$\text{Tor}_p^{A^\wedge}(B^\wedge, A^\wedge/\mathfrak{m}_{A^\wedge}) = \text{Tor}_p^A(B, A/\mathfrak{m}_A) \otimes_B B^\wedge$$

In this way we see that the condition in (1) on the local ring map $A \to B$ is equivalent to the same condition for the local ring map $A^\wedge \to B^\wedge$. Thus we may assume A and B are complete local Noetherian rings (since the other conditions are formulated in terms of the completions in any case).

Assume A and B are complete local Noetherian rings. Choose a diagram

$$\begin{tikzcd} S \arrow[r] & B \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in More on Algebra, Lemma 15.39.3. Let $I = \text{Ker}(R \to A)$ and $J = \text{Ker}(S \to B)$. The proposition now follows from Lemma 23.7.6. □

09QC Remark 23.9.3. It appears difficult to define an good notion of "local complete intersection homomorphisms" for maps between general Noetherian rings. The reason is that, for a local Noetherian ring A, the fibres of $A \to A^\wedge$ are not local complete intersection rings. Thus, if $A \to B$ is a local homomorphism of local Noetherian rings, and the map of completions $A^\wedge \to B^\wedge$ is a complete intersection homomorphism in the sense defined above, then $(A_\mathfrak{p})^\wedge \to (B_\mathfrak{q})^\wedge$ is in general not a complete intersection homomorphism in the sense defined above. A solution can be had by working exclusively with excellent Noetherian rings. More generally, one could work with those Noetherian rings whose formal fibres are complete intersections, see [Rod87]. We will develop this theory in Dualizing Complexes, Section 47.23.

To finish of this section we compare the notion defined above with the notion introduced in More on Algebra, Section 23.8.

09QD Lemma 23.9.4. Consider a commutative diagram

$$\begin{tikzcd} S \arrow[r] & B \\ & A \arrow[ul] \arrow[u] \end{tikzcd}$$

of Noetherian local rings with $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ a regular local ring. The following are equivalent

(1) $\text{Ker}(S \to B)$ is generated by a regular sequence, and
(2) $A^\wedge \to B^\wedge$ is a complete intersection homomorphism as defined above.

Proof. Omitted. Hint: the proof is identical to the argument given in the first paragraph of the proof of Lemma 23.9.1. □

09QE Lemma 23.9.5. Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. The following are equivalent

(1) $A \to B$ is a local complete intersection in the sense of More on Algebra, Definition 15.33.2,
(2) for every prime $\mathfrak{q} \subset B$ and with $\mathfrak{p} = A \cap \mathfrak{q}$ the ring map $(A_\mathfrak{p})^\wedge \to (B_\mathfrak{q})^\wedge$ is a complete intersection homomorphism in the sense defined above.

Proof. Choose a surjection $R = A[x_1, \ldots, x_n] \to B$. Observe that $A \to R$ is flat with regular fibres. Let I be the kernel of $R \to B$. Assume (2). Then we see that I is locally generated by a regular sequence by Lemma 23.9.4 and Algebra, Lemma 10.68.6. In other words, (1) holds. Conversely, assume (1). Then after localizing on R and B we can assume that I is generated by a Koszul regular sequence. By More on Algebra, Lemma 15.30.7 we find that I is locally generated by a regular sequence. Hence (2) hold by Lemma 23.9.4. Some details omitted. □

09QF Lemma 23.9.6. Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map such that the image of $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ contains all closed points of $\operatorname{Spec}(A)$. Then the following are equivalent

(1) B is a complete intersection and $A \to B$ has finite tor dimension,
(2) A is a complete intersection and $A \to B$ is a local complete intersection in the sense of More on Algebra, Definition 15.33.2.

Proof. This is a reformulation of Proposition 23.9.2 via Lemma 23.9.5. We omit the details. □

23.10. Smooth ring maps and diagonals

0FCV In this section we use the material above to characterize smooth ring maps as those whose diagonal is perfect.

0FCW Lemma 23.10.1. Let $A \to B$ be a local ring homomorphism of Noetherian local rings such that B is flat and essentially of finite type over A. If

$$B \otimes_A B \longrightarrow B$$

is a perfect ring map, i.e., if B has finite tor dimension over $B \otimes_A B$, then B is the localization of a smooth A-algebra.

Proof. As B is essentially of finite type over A, so is $B \otimes_A B$ and in particular $B \otimes_A B$ is Noetherian. Hence the quotient B of $B \otimes_A B$ is pseudo-coherent over $B \otimes_A B$ (More on Algebra, Lemma 15.64.17) which explains why perfectness of the ring map (More on Algebra, Definition 15.82.1) agrees with the condition of finite tor dimension.

We may write $B = R/K$ where R is the localization of $A[x_1, \ldots, x_n]$ at a prime ideal and $K \subset R$ is an ideal. Denote $\mathfrak{m} \subset R \otimes_A R$ the maximal ideal which is the inverse image of the maximal ideal of B via the surjection $R \otimes_A R \to B \otimes_A B \to B$. Then we have surjections

$$(R \otimes_A R)_\mathfrak{m} \to (B \otimes_A B)_\mathfrak{m} \to B$$

and hence ideals $I \subset J \subset (R \otimes_A R)_\mathfrak{m}$ as in Lemma 23.7.4. We conclude that $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective.

Let $K = (f_1, \ldots, f_r)$ with r minimal. We may and do assume that $f_i \in R$ is the image of an element of $A[x_1, \ldots, x_n]$ which we also denote f_i. Observe that I is generated by $f_1 \otimes 1, \ldots, f_r \otimes 1$ and $1 \otimes f_1, \ldots, 1 \otimes f_r$. We claim that this is a minimal set of generators of I. Namely, if κ is the common residue field of R, B, $(R \otimes_A R)_\mathfrak{m}$, and $(B \otimes_A B)_\mathfrak{m}$ then we have a map $R \otimes_A R \to R \otimes_A \kappa \oplus \kappa \otimes_A R$ which factors through $(R \otimes_A R)_\mathfrak{m}$. Since B is flat over A and since we have the short exact sequence $0 \to K \to R \to B \to 0$ we see that $K \otimes_A \kappa \subset R \otimes_A \kappa$, see Algebra, Lemma 10.39.12. Thus restricting the map $(R \otimes_A R)_{\mathfrak m} \to R \otimes_A \kappa \oplus \kappa \otimes_A R$ to I we obtain a map

$$I \to K \otimes_A \kappa \oplus \kappa \otimes_A K \to K \otimes_B \kappa \oplus \kappa \otimes_B K.$$

The elements $f_1 \otimes 1, \ldots, f_r \otimes 1, 1 \otimes f_1, \ldots, 1 \otimes f_r$ map to a basis of the target of this map, since by Nakayama's lemma (Algebra, Lemma 10.20.1) $f_1, \ldots, f_r$ map to a basis of $K \otimes_B \kappa$. This proves our claim.

The ideal J is generated by $f_1 \otimes 1, \ldots, f_r \otimes 1$ and the elements $x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n$ (for the proof it suffices to see that these elements are contained in the ideal J). Now we can write

$$f_i \otimes 1 - 1 \otimes f_i = \sum g_{ij}(x_j \otimes 1 - 1 \otimes x_j)$$

for some g_{ij} in $(R \otimes_A R)_{\mathfrak m}$. This is a general fact about elements of $A[x_1, \ldots, x_n]$ whose proof we omit. Denote $a_{ij} \in \kappa$ the image of g_{ij}. Another computation shows that a_{ij} is the image of $\partial f_i/\partial x_j$ in κ. The injectivity of $I/\mathfrak m I \to J/\mathfrak m J$ and the remarks made above forces the matrix (a_{ij}) to have maximal rank r. Set

$$C = A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

and consider the naive cotangent complex $NL_{C/A} \cong (C^{\oplus r} \to C^{\oplus n})$ where the map is given by the matrix of partial derivatives. Thus $NL_{C/A} \otimes_A B$ is isomorphic to a free B-module of rank $n - r$ placed in degree 0. Hence C_g is smooth over A for some $g \in C$ mapping to a unit in B, see Algebra, Lemma 10.137.12. This finishes the proof. ☐

0FCX Lemma 23.10.2. Let $A \to B$ be a flat finite type ring map of Noetherian rings. If

$$B \otimes_A B \longrightarrow B$$

is a perfect ring map, i.e., if B has finite tor dimension over $B \otimes_A B$, then B is a smooth A-algebra.

Proof. This follows from Lemma 23.10.1 and general facts about smooth ring maps, see Algebra, Lemmas 10.137.12 and 10.137.13. Alternatively, the reader can slightly modify the proof of Lemma 23.10.1 to prove this lemma. ☐

23.11. Freeness of the conormal module

0FJP Tate resolutions and derivations on them can be used to prove (stronger) versions of the results in this section, see [Iye01]. Two more elementary references are [Vas67] and [Fer67b].

0FJQ Lemma 23.11.1. Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of finite projective dimension over R. If $F \subset I/I^2$ is a direct summand isomorphic to R/I, then there exists a nonzerodivisor $x \in I$ such that the image of x in I/I^2 generates F. [Vas67]

Proof. By assumption we may choose a finite free resolution

$$0 \to R^{\oplus n_e} \to R^{\oplus n_{e-1}} \to \ldots \to R^{\oplus n_1} \to R \to R/I \to 0$$

Then $\varphi_1 : R^{\oplus n_1} \to R$ has rank 1 and we see that I contains a nonzerodivisor y by Algebra, Proposition 10.102.9. Let $\mathfrak p_1, \ldots, \mathfrak p_n$ be the associated primes of R, see Algebra, Lemma 10.63.5. Let $I^2 \subset J \subset I$ be an ideal such that $J/I^2 = F$. Then $J \not\subset \mathfrak p_i$ for all i as $y^2 \in J$ and $y^2 \not\in \mathfrak p_i$, see Algebra, Lemma 10.63.9. By

Nakayama's lemma (Algebra, Lemma 10.20.1) we have $J \not\subset \mathfrak{m}J + I^2$. By Algebra, Lemma 10.15.2 we can pick $x \in J$, $x \notin \mathfrak{m}J + I^2$ and $x \notin \mathfrak{p}_i$ for $i = 1, \ldots, n$. Then x is a nonzerodivisor and the image of x in I/I^2 generates (by Nakayama's lemma) the summand $J/I^2 \cong R/I$. □

0FJR **Lemma 23.11.2.** Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of finite projective dimension over R. If $F \subset I/I^2$ is a direct summand free of rank r, then there exists a regular sequence $x_1, \ldots, x_r \in I$ such that $x_1 \bmod I^2, \ldots, x_r \bmod I^2$ generate F.

Local version of [Vas67, Theorem 1.1]

Proof. If $r = 0$ there is nothing to prove. Assume $r > 0$. We may pick $x \in I$ such that x is a nonzerodivisor and $x \bmod I^2$ generates a summand of F isomorphic to R/I, see Lemma 23.11.1. Consider the ring $R' = R/(x)$ and the ideal $I' = I/(x)$. Of course $R'/I' = R/I$. The short exact sequence

$$0 \to R/I \xrightarrow{x} I/xI \to I' \to 0$$

splits because the map $I/xI \to I/I^2$ sends xR/xI to a direct summand. Now $I/xI = I \otimes_R^{\mathbf{L}} R'$ has finite projective dimension over R', see More on Algebra, Lemmas 15.74.3 and 15.74.9. Hence the summand I' has finite projective dimension over R'. On the other hand, we have the short exact sequence $0 \to xR/xI \to I/I^2 \to I'/(I')^2 \to 0$ and we conclude $I'/(I')^2$ has the free direct summand $F' = F/(R/I \cdot x)$ of rank $r - 1$. By induction on r we may we pick a regular sequence $x'_2, \ldots, x'_r \in I'$ such that there congruence classes freely generate F'. If $x_1 = x$ and $x_2, \ldots, x_r$ are any elements lifting $x'_1, \ldots, x'_r$ in R, then we see that the lemma holds. □

0FJS **Proposition 23.11.3.** Let R be a Noetherian ring. Let $I \subset R$ be an ideal which has finite projective dimension and such that I/I^2 is finite locally free over R/I. Then I is a regular ideal (More on Algebra, Definition 15.32.1).

Variant of [Vas67, Corollary 1]. See also [Iye01] and [Fer67b].

Proof. By Algebra, Lemma 10.68.6 it suffices to show that $I_\mathfrak{p} \subset R_\mathfrak{p}$ is generated by a regular sequence for every $\mathfrak{p} \supset I$. Thus we may assume R is local. If I/I^2 has rank r, then by Lemma 23.11.2 we find a regular sequence $x_1, \ldots, x_r \in I$ generating I/I^2. By Nakayama (Algebra, Lemma 10.20.1) we conclude that I is generated by $x_1, \ldots, x_r$. □

For any local complete intersection homomorphism $A \to B$ of rings, the naive cotangent complex $NL_{B/A}$ is perfect of tor-amplitude in $[-1, 0]$, see More on Algebra, Lemma 15.85.4. Using the above, we can show that this sometimes characterizes local complete intersection homomorphisms.

0FJT **Lemma 23.11.4.** Let $A \to B$ be a perfect (More on Algebra, Definition 15.82.1) ring homomorphism of Noetherian rings. Then the following are equivalent

(1) $NL_{B/A}$ has tor-amplitude in $[-1, 0]$,
(2) $NL_{B/A}$ is a perfect object of $D(B)$ with tor-amplitude in $[-1, 0]$, and
(3) $A \to B$ is a local complete intersection (More on Algebra, Definition 15.33.2).

Proof. Write $B = A[x_1, \ldots, x_n]/I$. Then $NL_{B/A}$ is represented by the complex

$$I/I^2 \longrightarrow \bigoplus B\mathrm{d}x_i$$

of B-modules with I/I^2 placed in degree -1. Since the term in degree 0 is finite free, this complex has tor-amplitude in $[-1, 0]$ if and only if I/I^2 is a flat B-module, see More on Algebra, Lemma 15.66.2. Since I/I^2 is a finite B-module and B is Noetherian, this is true if and only if I/I^2 is a finite locally free B-module (Algebra, Lemma 10.78.2). Thus the equivalence of (1) and (2) is clear. Moreover, the equivalence of (1) and (3) also follows if we apply Proposition 23.11.3 (and the observation that a regular ideal is a Koszul regular ideal as well as a quasi-regular ideal, see More on Algebra, Section 15.32). □

0FJV Lemma 23.11.5. Let $A \to B$ be a flat ring map of finite presentation. Then the following are equivalent

(1) $NL_{B/A}$ has tor-amplitude in $[-1, 0]$,
(2) $NL_{B/A}$ is a perfect object of $D(B)$ with tor-amplitude in $[-1, 0]$,
(3) $A \to B$ is syntomic (Algebra, Definition 10.136.1), and
(4) $A \to B$ is a local complete intersection (More on Algebra, Definition 15.33.2).

Proof. The equivalence of (3) and (4) is More on Algebra, Lemma 15.33.5.

If $A \to B$ is syntomic, then we can find a cocartesian diagram

$$\begin{tikzcd} B_0 \arrow[r] & B \\ A_0 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

such that $A_0 \to B_0$ is syntomic and A_0 is Noetherian, see Algebra, Lemmas 10.127.18 and 10.168.9. By Lemma 23.11.4 we see that NL_{B_0/A_0} is perfect of tor-amplitude in $[-1, 0]$. By More on Algebra, Lemma 15.85.3 we conclude the same thing is true for $NL_{B/A} = NL_{B_0/A_0} \otimes^{\mathbf{L}}_{B_0} B$ (see also More on Algebra, Lemmas 15.66.13 and 15.74.9). This proves that (3) implies (2).

Assume (1). By More on Algebra, Lemma 15.85.3 for every ring map $A \to k$ where k is a field, we see that $NL_{B\otimes_A k/k}$ has tor-amplitude in $[-1, 0]$ (see More on Algebra, Lemma 15.66.13). Hence by Lemma 23.11.4 we see that $k \to B \otimes_A k$ is a local complete intersection homomorphism. Thus $A \to B$ is syntomic by definition. This proves (1) implies (3) and finishes the proof. □

23.12. Koszul complexes and Tate resolutions

0GZ3 In this section we "lift" the result of More on Algebra, Lemma 15.94.1 to the category of differential graded algebras endowed with divided powers compatible with the differential graded structure (beware that in this section we represent Koszul complexes as chain complexes whereas in locus citatus we use cochain complexes).

Let R be a ring. Let $I \subset R$ be an ideal generated by $f_1, \ldots, f_r \in R$. For $n \geq 1$ we denote

$$K_n = K_{n,\bullet} = R\langle \xi_1, \ldots, \xi_r \rangle$$

the differential graded Koszul algebra with ξ_i in degree 1 and $\mathrm{d}(\xi_i) = f_i$. There exists a unique divided power structure on this (as in Definition 23.6.5), see Example 23.6.2. For $m > n$ the transition map $K_m \to K_n$ is the differential graded algebra map compatible with divided powers given by sending ξ_i to $f_i^{m-n}\xi_i$.

0GZ4 Lemma 23.12.1. In the situation above, if R is Noetherian, then for every n there exists an $N \geq n$ and maps

$$K_N \to A \to R/(f_1^N, \ldots, f_r^N) \quad \text{and} \quad A \to K_n$$

with the following properties

(1) (A, d, γ) is as in Definition 23.6.5,
(2) $A \to R/(f_1^N, \ldots, f_r^N)$ is a quasi-isomorphism,
(3) the composition $K_N \to A \to R/(f_1^N, \ldots, f_r^N)$ is the canonical map,
(4) the composition $K_N \to A \to K_n$ is the transition map,
(5) $A_0 = R \to R/(f_1^N, \ldots, f_r^N)$ is the canonical surjection,
(6) A is a graded divided power polynomial algebra over R with finitely many generators in each degree, and
(7) $A \to K_n$ is a homomorphism of differential graded R-algebras compatible with divided powers which induces the canonical map $R/(f_1^N, \ldots, f_r^N) \to R/(f_1^n, \ldots, f_r^n)$ on homology in degree 0.

Condition (4) means that A is constructed out of A_0 by successively adjoining a finite set of variables T in each degree > 0 as in Example 23.6.2 or 23.6.3.

Proof. Fix n. If $r = 0$, then we can just pick $A = R$. Assume $r > 0$. By More on Algebra, Lemma 15.94.1 (translated into the language of chain complexes) we can choose

$$n_r > n_{r-1} > \ldots > n_1 > n_0 = n$$

such that the transition maps $K_{n_{i+1}} \to K_{n_i}$ on Koszul algebras (see above) induce the zero map on homology in degrees > 0. We will prove the lemma with $N = n_r$.

We will construct A exactly as in the statement and proof of Lemma 23.6.9. Thus we will have

$$A = \operatorname{colim} A(m), \quad \text{and} \quad A(0) \to A(1) \to A(2) \to \ldots \to R/(f_1^N, \ldots, f_r^N)$$

This will immediately give us properties (1), (2), (5), and (6). To finish the proof we will construct the R-algebra maps $K_N \to A \to K_n$. To do this we will construct

(1) an isomorphism $A(1) \to K_N = K_{n_r}$,
(2) a map $A(2) \to K_{n_{r-1}}$,
(3) ...
(4) a map $A(r) \to K_{n_1}$,
(5) a map $A(r+1) \to K_{n_0} = K_n$, and
(6) a map $A \to K_n$.

In each of these steps the map constructed will be between differential graded algebras compatibly endowed with divided powers and each of the maps will be compatible with the previous one via the transition maps between the Koszul algebras and each of the maps will induce the obvious canonical map on homology in degree 0.

Recall that $A(0) = R$. For $m = 1$, the proof of Lemma 23.6.9 chooses $A(1) = R\langle T_1, \ldots, T_r \rangle$ with T_i of degree 1 and with $\mathrm{d}(T_i) = f_i^N$. Namely, the f_i^N are generators of the kernel of $A(0) \to R/(f_1^N, \ldots, f_r^N)$. Thus for $A(1) \to K_N = K_{n_r}$ we use the map

$$\varphi_1 : A(1) \longrightarrow K_{n_r}, \quad T_i \longmapsto \xi_i$$

which is an isomorphism.

For $m = 2$, the construction in the proof of Lemma 23.6.9 chooses generators $e_1, \ldots, e_t \in \mathrm{Ker}(\mathrm{d} : A(1)_1 \to A(1)_0)$. The construction proceeds by taking $A(2) = A(1)\langle T_1, \ldots, T_t\rangle$ as a divided power polynomial algebra with T_i of degree 2 and with $\mathrm{d}(T_i) = e_i$. Since $\varphi_1(e_i)$ is a cocycle in K_{n_r} we see that its image in $K_{n_{r-1}}$ is a coboundary by our choice of n_r and n_{r-1} above. Hence we can construct the following commutative diagram

$$\begin{tikzcd} A(1) \arrow[r, "\varphi_1"'] \arrow[d] & K_{n_r} \arrow[d] \\ A(2) \arrow[r, "\varphi_2"] & K_{n_{r-1}} \end{tikzcd}$$

by sending T_i to an element in degree 2 whose boundary is the image of $\varphi_1(e_i)$. The map φ_2 exists and is compatible with the differential and the divided powers by the universal of the divided power polynomial algebra.

The algebra $A(m)$ and the map $\varphi_m : A(m) \to K_{n_{r+1-m}}$ are constructed in exactly the same manner for $m = 2, \ldots, r$.

Given the map $A(r) \to K_{n_1}$ we see that the composition $H_r(A(r)) \to H_r(K_{n_1}) \to H_r(K_{n_0}) \subset (K_{n_0})_r$ is zero, hence we can extend this to $A(r+1) \to K_{n_0} = K_n$ by sending the new polynomial generators of $A(r+1)$ to zero.

Having constructed $A(r+1) \to K_{n_0} = K_n$ we can simply extend to $A(r+2), A(r+3), \ldots$ in the only possible way by sending the new polynomial generators to zero. This finishes the proof. $\square$

0GZ5 Remark 23.12.2. In the situation above, if R is Noetherian, we can inductively choose a sequence of integers $1 = n_0 < n_1 < n_2 < \ldots$ such that for $i = 1, 2, 3, \ldots$ we have maps $K_{n_i} \to A_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$ and $A_i \to K_{n_{i-1}}$ as in Lemma 23.12.1. Denote $A_{i+1} \to A_i$ the composition $A_{i+1} \to K_{n_i} \to A_i$. Then the diagram

$$\begin{tikzcd} K_{n_1} \arrow[d] & K_{n_2} \arrow[l] \arrow[d] & K_{n_3} \arrow[l] \arrow[d] & \ldots \arrow[l] \\ A_1 \arrow[d] & A_2 \arrow[l] \arrow[d] & A_3 \arrow[l] \arrow[d] & \ldots \arrow[l] \\ K_1 & K_{n_1} \arrow[l] & K_{n_2} \arrow[l] & \ldots \arrow[l] \end{tikzcd}$$

commutes. In this way we see that the inverse systems (K_n) and (A_n) are pro-isomorphic in the category of differential graded R-algebras with compatible divided powers.

0GZ6 Lemma 23.12.3. Let (A, d, γ), $d \geq 1$, $f \in A_{d-1}$, and $A\langle T\rangle$ be as in Lemma 23.6.8.

(1) If $d = 1$, then there is a long exact sequences
$$\ldots \to H_0(A) \xrightarrow{f} H_0(A) \to H_0(A\langle T\rangle) \to 0$$
(2) For $d = 2$ there is a bounded spectral sequence $(E_1)_{i,j} = H_{j-i}(A) \cdot T^{[i]}$ converging to $H_{i+j}(A\langle T\rangle)$. The differential $(d_1)_{i,j} : H_{j-i}(A) \cdot T^{[i]} \to H_{j-i+1}(A) \cdot T^{[i-1]}$ sends $\xi \cdot T^{[i]}$ to the class of $f\xi \cdot T^{[i-1]}$.
(3) Add more here for other degrees as needed.

Proof. For $d = 1$, we have a short exact sequence of complexes

$$0 \to A \to A\langle T\rangle \to A \cdot T \to 0$$

and the result (1) follows easily from this. For $d = 2$ we view $A\langle T\rangle$ as a filtered chain complex with subcomplexes

$$F^p A\langle T\rangle = \bigoplus\nolimits_{i \leq p} A \cdot T^{[i]}$$

Applying the spectral sequence of Homology, Section 12.24 (translated into chain complexes) we obtain (2). □

The following lemma will be needed later.

0GZ7 **Lemma 23.12.4.** In the situation above, for all $n \geq t \geq 1$ there exists an $N > n$ and a map

$$K_t \longrightarrow K_n \otimes_R K_t$$

in the derived category of left differential graded K_N-modules whose composition with the multiplication map is the transition map (in either direction).

Proof. We first prove this for $r = 1$. Set $f = f_1$. Write $K_t = R\langle x\rangle$, $K_n = R\langle y\rangle$, and $K_N = R\langle z\rangle$ with x, y, z of degree 1 and $\mathrm{d}(x) = f^t$, $\mathrm{d}(y) = f^n$, and $\mathrm{d}(z) = f^N$. For all $N > t$ we claim there is a quasi-isomorphism

$$B_{N,t} = R\langle x, z, u\rangle \longrightarrow K_t, \quad x \mapsto x, \quad z \mapsto f^{N-t}x, \quad u \mapsto 0$$

Here the left hand side denotes the divided power polynomial algebra in variables x and z of degree 1 and u of degree 2 with $\mathrm{d}(x) = f^t$, $\mathrm{d}(z) = f^N$, and $\mathrm{d}(u) = z - f^{N-t}x$. To prove the claim, we observe that the following three submodules of $H_*(R\langle x, z\rangle)$ are the same

1. the kernel of $H_*(R\langle x, z\rangle) \to H_*(K_t)$,
2. the image of $z - f^{N-t}x : H_*(R\langle x, z\rangle) \to H_*(R\langle x, z\rangle)$, and
3. the kernel of $z - f^{N-t}x : H_*(R\langle x, z\rangle) \to H_*(R\langle x, z\rangle)$.

This observation is proved by a direct computation[3] which we omit. Then we can apply Lemma 23.12.3 part (2) to see that the claim is true.

Via the homomorphism $K_N \to B_{N,t}$ of differential graded R-algebras sending z to z, we may view $B_{N,t} \to K_t$ as a quasi-isomorphism of left differential graded K_N-modules. To define the arrow in the statement of the lemma we use the homomorphism

$$B_{N,t} = R\langle x, z, u\rangle \to K_n \otimes_R K_t, \quad x \mapsto 1 \otimes x, \quad z \mapsto f^{N-n}y \otimes 1, \quad u \mapsto -f^{N-n-t}y \otimes x$$

This makes sense as long as we assume $N \geq n + t$. It is a pleasant computation to show that the (pre or post) composition with the multiplication map is the transition map.

For $r > 1$ we proceed by writing each of the Koszul algebras as a tensor product of Koszul algebras in 1 variable and we apply the previous construction. In other words, we write

$$K_t = R\langle x_1, \ldots, x_r\rangle = R\langle x_1\rangle \otimes_R \ldots \otimes_R R\langle x_r\rangle$$

[3] Hint: setting $z' = z - f^{N-t}x$ we see that $R\langle x, z\rangle = R\langle x, z'\rangle$ with $\mathrm{d}(z') = 0$ and moreover the map $R\langle x, z'\rangle \to K_t$ is the map killing z'.

where x_i is in degree 1 and $\mathrm{d}(x_i) = f_i^t$. In the case $r > 1$ we then use

$$B_{N,t} = R\langle x_1, z_1, u_1\rangle \otimes_R \ldots \otimes_R R\langle x_r, z_r, u_r\rangle$$

where x_i, z_i have degree 1 and u_i has degree 2 and we have $\mathrm{d}(x_i) = f_i^t$, $\mathrm{d}(z_i) = f_i^N$, and $\mathrm{d}(u_i) = z_i - f_i^{N-t}x_i$. The tensor product map $B_{N,t} \to K_t$ will be a quasi-isomorphism as it is a tensor product of quasi-isomorphisms between bounded above complexes of free R-modules. Finally, we define the map

$$B_{N,t} \to K_n \otimes_R K_t = R\langle y_1, \ldots, y_r\rangle \otimes_R R\langle x_1, \ldots, x_r\rangle$$

as the tensor product of the maps constructed in the case of $r = 1$ or simply by the rules $x_i \mapsto 1 \otimes x_i$, $z_i \mapsto f_i^{N-n}y_i \otimes 1$, and $u_i \mapsto -f_i^{N-n-t}y_i \otimes x_i$ which makes sense as long as $N \geq n + t$. We omit the details. $\square$

23.13. Other chapters

CHAPTER 24

Differential Graded Sheaves

0FQS 24.1. Introduction

0FQT This chapter is a continuation of the discussion started in Differential Graded Algebra, Section 22.1. A survey paper is [Kel06].

24.2. Conventions

0FQU In this chapter we hold on to the convention that ring means commutative ring with 1. If R is a ring, then an R-algebra A will be an R-module A endowed with an R-bilinear map $A \times A \to A$ (multiplication) such that multiplication is associative and has an identity. In other words, these are unital associative R-algebras such that the structure map $R \to A$ maps into the center of A.

24.3. Sheaves of graded algebras

0FQV Please skip this section.

0FQW Definition 24.3.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A sheaf of graded $\mathcal{O}$-algebras or a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$ is given by a family $\mathcal{A}^n$ indexed by $n \in \mathbf{Z}$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{A}^n \times \mathcal{A}^m \to \mathcal{A}^{n+m}, \quad (a, b) \longmapsto ab$$

called the multiplication maps with the following properties

(1) multiplication is associative, and
(2) there is a global section 1 of $\mathcal{A}^0$ which is a two-sided identity for multiplication.

We often denote such a structure $\mathcal{A}$. A homomorphism of graded $\mathcal{O}$-algebras $f : \mathcal{A} \to \mathcal{B}$ is a family of maps $f^n : \mathcal{A}^n \to \mathcal{B}^n$ of $\mathcal{O}$-modules compatible with the multiplication maps.

Given a graded $\mathcal{O}$-algebra $\mathcal{A}$ and an object $U \in \text{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{A}(U) = \Gamma(U, \mathcal{A}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{A}^n(U)$$

This is a graded $\mathcal{O}(U)$-algebra.

0FQX Remark 24.3.2. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. We have

(1) Let $\mathcal{A}$ be a graded $\mathcal{O}_\mathcal{C}$-algebra. The multiplication maps of $\mathcal{A}$ induce multiplication maps $f_*\mathcal{A}^n \times f_*\mathcal{A}^m \to f_*\mathcal{A}^{n+m}$ and via $f^\sharp$ we may view these as $\mathcal{O}_\mathcal{D}$-bilinear maps. We will denote $f_*\mathcal{A}$ the graded $\mathcal{O}_\mathcal{D}$-algebra we so obtain.

(2) Let $\mathcal{B}$ be a graded $\mathcal{O}_\mathcal{D}$-algebra. The multiplication maps of $\mathcal{B}$ induce multiplication maps $f^*\mathcal{B}^n \times f^*\mathcal{B}^m \to f^*\mathcal{B}^{n+m}$ and using $f^\sharp$ we may view these as $\mathcal{O}_\mathcal{C}$-bilinear maps. We will denote $f^*\mathcal{B}$ the graded $\mathcal{O}_\mathcal{C}$-algebra we so obtain.
(3) The set of homomorphisms $f^*\mathcal{B} \to \mathcal{A}$ of graded $\mathcal{O}_\mathcal{C}$-algebras is in 1-to-1 correspondence with the set of homomorphisms $\mathcal{B} \to f_*\mathcal{A}$ of graded $\mathcal{O}_\mathcal{C}$-algebras.

Part (3) follows immediately from the usual adjunction between f^* and f_* on sheaves of modules.

24.4. Sheaves of graded modules

0FQY Please skip this section.

0FQZ Definition 24.4.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. A (right) graded $\mathcal{A}$-module or (right) graded module over $\mathcal{A}$ is given by a family $\mathcal{M}^n$ indexed by $n \in \mathbf{Z}$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{M}^n \times \mathcal{A}^m \to \mathcal{M}^{n+m}, \quad (x, a) \longmapsto xa$$

called the multiplication maps with the following properties

(1) multiplication satisfies $(xa)a' = x(aa')$,
(2) the identity section 1 of $\mathcal{A}^0$ acts as the identity on $\mathcal{M}^n$ for all n.

We often say "let $\mathcal{M}$ be a graded $\mathcal{A}$-module" to indicate this situation. A homomorphism of graded $\mathcal{A}$-modules $f : \mathcal{M} \to \mathcal{N}$ is a family of maps $f^n : \mathcal{M}^n \to \mathcal{N}^n$ of $\mathcal{O}$-modules compatible with the multiplication maps. The category of (right) graded $\mathcal{A}$-modules is denoted $\text{Mod}(\mathcal{A})$.

We can define left graded modules in exactly the same manner but our default in the chapter will be right modules.

Given a graded $\mathcal{A}$-module $\mathcal{M}$ and an object $U \in \text{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a (right) graded $\mathcal{A}(U)$-module.

0FR0 Lemma 24.4.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a graded $\mathcal{O}$-algebra. The category $\text{Mod}(\mathcal{A})$ is an abelian category with the following properties

(1) $\text{Mod}(\mathcal{A})$ has arbitrary direct sums,
(2) $\text{Mod}(\mathcal{A})$ has arbitrary colimits,
(3) filtered colimit in $\text{Mod}(\mathcal{A})$ are exact,
(4) $\text{Mod}(\mathcal{A})$ has arbitrary products,
(5) $\text{Mod}(\mathcal{A})$ has arbitrary limits.

The functor

$$\text{Mod}(\mathcal{A}) \longrightarrow \text{Mod}(\mathcal{O}), \quad \mathcal{M} \longmapsto \mathcal{M}^n$$

sending a graded $\mathcal{A}$-module to its nth term commutes with all limits and colimits.

The lemma says that we may take limits and colimits termwise. It also says (or implies if you like) that the forgetful functor

$$\text{Mod}(\mathcal{A}) \longrightarrow \text{graded } \mathcal{O}\text{-modules}$$

commutes with all limits and colimits.

Proof. Let us denote $\mathrm{gr}^n : \mathrm{Mod}(\mathcal{A}) \to \mathrm{Mod}(\mathcal{O})$ the functor in the statement of the lemma. Consider a homomorphism $f : \mathcal{M} \to \mathcal{N}$ of graded $\mathcal{A}$-modules. The kernel and cokernel of f as maps of graded $\mathcal{O}$-modules are additionally endowed with multiplication maps as in Definition 24.4.1. Hence these are also the kernel and cokernel in $\mathrm{Mod}(\mathcal{A})$. Thus $\mathrm{Mod}(\mathcal{A})$ is an abelian category and taking kernels and cokernels commutes with gr^n.

To prove the existence of limits and colimits it is sufficient to prove the existence of products and direct sums, see Categories, Lemmas 4.14.11 and 4.14.12. The same lemmas show that proving the commutation of limits and colimits with gr^n follows if gr^n commutes with direct sums and products.

Let $\mathcal{M}_t$, $t \in T$ be a set of graded $\mathcal{A}$-modules. Then we can consider the graded $\mathcal{A}$-module whose degree n term is $\bigoplus_{t \in T} \mathcal{M}_t^n$ (with obvious multiplication maps). The reader easily verifies that this is a direct sum in $\mathrm{Mod}(\mathcal{A})$. Similarly for products.

Observe that gr^n is an exact functor for all n and that a complex $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ of $\mathrm{Mod}(\mathcal{A})$ is exact if and only if $\mathrm{gr}^n\mathcal{M}_1 \to \mathrm{gr}^n\mathcal{M}_2 \to \mathrm{gr}^n\mathcal{M}_3$ is exact in $\mathrm{Mod}(\mathcal{O})$ for all n. Hence we conclude that (3) holds as filtered colimits are exact in $\mathrm{Mod}(\mathcal{O})$; it is a Grothendieck abelian category, see Cohomology on Sites, Section 21.19. □

24.5. The graded category of sheaves of graded modules

0FR1 Please skip this section. This section is the analogue of Differential Graded Algebra, Example 22.25.6. For our conventions on graded categories, please see Differential Graded Algebra, Section 22.25.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. We will construct a graded category $\mathrm{Mod}^{gr}(\mathcal{A})$ over $R = \Gamma(\mathcal{C}, \mathcal{O})$ whose associated category $(\mathrm{Mod}^{gr}(\mathcal{A}))^0$ is the category of graded $\mathcal{A}$-modules. As objects of $\mathrm{Mod}^{gr}(\mathcal{A})$ we take right graded $\mathcal{A}$-modules (see Section 24.4). Given graded $\mathcal{A}$-modules $\mathcal{L}$ and $\mathcal{M}$ we set

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(\mathcal{L}, \mathcal{M})$$

where $\mathrm{Hom}^n(\mathcal{L}, \mathcal{M})$ is the set of right $\mathcal{A}$-module maps $f : \mathcal{L} \to \mathcal{M}$ which are homogeneous of degree n. More precisely, f is given by a family of maps $f : \mathcal{L}^i \to \mathcal{M}^{i+n}$ for $i \in \mathbf{Z}$ compatible with the multiplication maps. In terms of components, we have that

$$\mathrm{Hom}^n(\mathcal{L}, \mathcal{M}) \subset \prod\nolimits_{p+q=n} \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

(observe reversal of indices) is the subset consisting of those $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for local sections a of $\mathcal{A}^i$ and m of $\mathcal{L}^{-q-i}$. For graded $\mathcal{A}$-modules $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ we define composition in $\mathrm{Mod}^{gr}(\mathcal{A})$ via the maps

$$\mathrm{Hom}^m(\mathcal{L}, \mathcal{M}) \times \mathrm{Hom}^n(\mathcal{K}, \mathcal{L}) \longrightarrow \mathrm{Hom}^{n+m}(\mathcal{K}, \mathcal{M})$$

by simple composition of right $\mathcal{A}$-module maps: $(g, f) \mapsto g \circ f$.

24.6. Tensor product for sheaves of graded modules

0FR2 Please skip this section. This section is the analogue of part of Differential Graded Algebra, Section 22.12.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right graded $\mathcal{A}$-module and let $\mathcal{N}$ be a left graded $\mathcal{A}$-module. Then we define the tensor product $\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ to be the graded $\mathcal{O}$-module whose degree n term is

$$(\mathcal{M} \otimes_\mathcal{A} \mathcal{N})^n = \text{Coker}\left(\bigoplus\nolimits_{r+s+t=n} \mathcal{M}^r \otimes_\mathcal{O} \mathcal{A}^s \otimes_\mathcal{O} \mathcal{N}^t \longrightarrow \bigoplus\nolimits_{p+q=n} \mathcal{M}^p \otimes_\mathcal{O} \mathcal{N}^q\right)$$

where the map sends the local section $x \otimes a \otimes y$ of $\mathcal{M}^r \otimes_\mathcal{O} \mathcal{A}^s \otimes_\mathcal{O} \mathcal{N}^t$ to $xa \otimes y - x \otimes ay$. With this definition we have that $(\mathcal{M} \otimes_\mathcal{A} \mathcal{N})^n$ is the sheafification of the presheaf $U \mapsto (\mathcal{M}(U) \otimes_{\mathcal{A}(U)} \mathcal{N}(U))^n$ where the tensor product of graded modules is as defined in Differential Graded Algebra, Section 22.12.

If we fix the left graded $\mathcal{A}$-module $\mathcal{N}$ we obtain a functor

$$- \otimes_\mathcal{A} \mathcal{N} : \text{Mod}(\mathcal{A}) \longrightarrow \text{Gr}(\text{Mod}(\mathcal{O})) = \text{graded } \mathcal{O}\text{-modules}$$

For the notation $\text{Gr}(-)$ please see Homology, Definition 12.16.1. The graded category of graded $\mathcal{O}$-modules is denoted $\text{Gr}^{gr}(\text{Mod}(\mathcal{O}))$, see Differential Graded Algebra, Example 22.25.5. The functor above can be upgraded to a functor of graded categories

$$- \otimes_\mathcal{A} \mathcal{N} : \text{Mod}^{gr}(\mathcal{A}) \longrightarrow \text{Gr}^{gr}(\text{Mod}(\mathcal{O}))$$

by sending homomorphisms of degree n from $\mathcal{M} \to \mathcal{M}'$ to the induced map of degree n from $\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ to $\mathcal{M}' \otimes_\mathcal{A} \mathcal{N}$.

24.7. Internal hom for sheaves of graded modules

0FR3 We urge the reader to skip this section.

We are going to need the sheafified version of the construction in Section 24.5. Let $(\mathcal{C}, \mathcal{O})$, $\mathcal{A}$, $\mathcal{M}$, $\mathcal{L}$ be as in Section 24.5. Then we define

$$\mathcal{H}om^{gr}_\mathcal{A}(\mathcal{M}, \mathcal{L})$$

as the graded $\mathcal{O}$-module whose degree n term

$$\mathcal{H}om^n_\mathcal{A}(\mathcal{M}, \mathcal{L}) \subset \prod\nolimits_{p+q=n} \mathcal{H}om_\mathcal{O}(\mathcal{L}^{-q}, \mathcal{M}^p)$$

is the subsheaf consisting of those local sections $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for local sections a of $\mathcal{A}^i$ and m of $\mathcal{L}^{-q-i}$. As in Section 24.5 there is a composition map

$$\mathcal{H}om^{gr}_\mathcal{A}(\mathcal{L}, \mathcal{M}) \otimes_\mathcal{O} \mathcal{H}om^{gr}_\mathcal{A}(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om^{gr}_\mathcal{A}(\mathcal{K}, \mathcal{M})$$

where the left hand side is the tensor product of graded $\mathcal{O}$-modules defined in Section 24.6. This map is given by the composition map

$$\mathcal{H}om^m_\mathcal{A}(\mathcal{L}, \mathcal{M}) \otimes_\mathcal{O} \mathcal{H}om^n_\mathcal{A}(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om^{n+m}_\mathcal{A}(\mathcal{K}, \mathcal{M})$$

defined by simple composition (locally).

With these definitions we have

$$\text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{H}om^{gr}_\mathcal{A}(\mathcal{L}, \mathcal{M}))$$

as graded R-modules compatible with composition.

24.8. Sheaves of graded bimodules and tensor-hom adjunction

0FR4 Please skip this section.

0FR5 Definition 24.8.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. A graded $(\mathcal{A}, \mathcal{B})$-bimodule is given by a family $\mathcal{M}^n$ indexed by $n \in \mathbf{Z}$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{M}^n \times \mathcal{B}^m \to \mathcal{M}^{n+m}, \quad (x, b) \longmapsto xb$$

and

$$\mathcal{A}^n \times \mathcal{M}^m \to \mathcal{M}^{n+m}, \quad (a, x) \longmapsto ax$$

called the multiplication maps with the following properties

(1) multiplication satisfies $a(a'x) = (aa')x$ and $(xb)b' = x(bb')$,
(2) $(ax)b = a(xb)$,
(3) the identity section 1 of $\mathcal{A}^0$ acts as the identity by multiplication, and
(4) the identity section 1 of $\mathcal{B}^0$ acts as the identity by multiplication.

We often denote such a structure $\mathcal{M}$. A homomorphism of graded $(\mathcal{A}, \mathcal{B})$-bimodules $f : \mathcal{M} \to \mathcal{N}$ is a family of maps $f^n : \mathcal{M}^n \to \mathcal{N}^n$ of $\mathcal{O}$-modules compatible with the multiplication maps.

Given a graded $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a graded $(\mathcal{A}(U), \mathcal{B}(U))$-bimodule.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right graded $\mathcal{A}$-module and let $\mathcal{N}$ be a graded $(\mathcal{A}, \mathcal{B})$-bimodule. In this case the graded tensor product defined in Section 24.6

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$$

is a right graded $\mathcal{B}$-module with obvious multiplication maps. This construction defines a functor and a functor of graded categories

$$\otimes_{\mathcal{A}} \mathcal{N} : \mathrm{Mod}(\mathcal{A}) \longrightarrow \mathrm{Mod}(\mathcal{B}) \quad \text{and} \quad \otimes_{\mathcal{A}} \mathcal{N} : \mathrm{Mod}^{gr}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{gr}(\mathcal{B})$$

by sending homomorphisms of degree n from $\mathcal{M} \to \mathcal{M}'$ to the induced map of degree n from $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ to $\mathcal{M}' \otimes_{\mathcal{A}} \mathcal{N}$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{N}$ be a graded $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{L}$ be a right graded $\mathcal{B}$-module. In this case the graded internal hom defined in Section 24.7

$$\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$$

is a right graded $\mathcal{A}$-module with multiplication maps[1]

$$\mathcal{H}om^n_{\mathcal{B}}(\mathcal{N}, \mathcal{L}) \times \mathcal{A}^m \longrightarrow \mathcal{H}om^{n+m}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$$

sending a section $f = (f_{p,q})$ of $\mathcal{H}om^n_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$ over U and a section a of $\mathcal{A}^m$ over U to the section fa if $\mathcal{H}om^{n+m}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$ over U defined as the family of maps

$$\mathcal{N}^{-q-m}|_U \xrightarrow{a \cdot -} \mathcal{N}^{-q}|_U \xrightarrow{f_{p,q}} \mathcal{M}^p|_U$$

[1]Our conventions are here that this does not involve any signs.

We omit the verification that this is well defined. This construction defines a functor and a functor of graded categories

$$\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, -) : \text{Mod}(\mathcal{B}) \longrightarrow \text{Mod}(\mathcal{A}) \quad \text{and} \quad \mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, -) : \text{Mod}^{gr}(\mathcal{B}) \longrightarrow \text{Mod}^{gr}(\mathcal{A})$$

by sending homomorphisms of degree n from $\mathcal{L} \to \mathcal{L}'$ to the induced map of degree n from $\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$ to $\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}')$.

0FR6 Lemma 24.8.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right graded $\mathcal{A}$-module. Let $\mathcal{N}$ be a graded $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{L}$ be a right graded $\mathcal{B}$-module. With conventions as above we have

$$\text{Hom}_{\text{Mod}^{gr}(\mathcal{B})}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(\mathcal{M}, \mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}))$$

and

$$\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{M}, \mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}))$$

functorially in $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$.

Proof. Omitted. Hint: This follows by interpreting both sides as $\mathcal{A}$-bilinear graded maps $\psi : \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ which are $\mathcal{B}$-linear on the right. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of graded algebras on $(\mathcal{C}, \mathcal{O})$. As a special case of the above, suppose we are given a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of graded $\mathcal{O}$-algebras. Then we obtain a functor and a functor of graded categories

$$\otimes_{\mathcal{A},\varphi}\mathcal{B} : \text{Mod}(\mathcal{A}) \longrightarrow \text{Mod}(\mathcal{B}) \quad \text{and} \quad \otimes_{\mathcal{A},\varphi} \mathcal{B} : \text{Mod}^{gr}(\mathcal{A}) \longrightarrow \text{Mod}^{gr}(\mathcal{B})$$

On the other hand, we have the restriction functors

$$res_{\varphi} : \text{Mod}(\mathcal{B}) \longrightarrow \text{Mod}(\mathcal{A}) \quad \text{and} \quad res_{\varphi} : \text{Mod}^{gr}(\mathcal{B}) \longrightarrow \text{Mod}^{gr}(\mathcal{A})$$

We can use the lemma above to show these functors are adjoint to each other (as usual with restriction and base change). Namely, let us write ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$ for $\mathcal{B}$ viewed as a graded $(\mathcal{A}, \mathcal{B})$-bimodule. Then for any right graded $\mathcal{B}$-module $\mathcal{L}$ we have

$$\mathcal{H}om^{gr}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}, \mathcal{L}) = res_{\varphi}(\mathcal{L})$$

as right graded $\mathcal{A}$-modules. Thus Lemma 24.8.2 tells us that we have a functorial isomorphism

$$\text{Hom}_{\text{Mod}^{gr}(\mathcal{B})}(\mathcal{M} \otimes_{\mathcal{A},\varphi} \mathcal{B}, \mathcal{L}) = \text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(\mathcal{M}, res_{\varphi}(\mathcal{L}))$$

We usually drop the dependence on φ in this formula if it is clear from context. In the same manner we obtain the equality

$$\mathcal{H}om^{gr}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{L}) = \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{M}, \mathcal{L})$$

of graded $\mathcal{O}$-modules.

24.9. Pull and push for sheaves of graded modules

0FR7 We advise the reader to skip this section.

Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a graded $\mathcal{O}_{\mathcal{C}}$-algebra. Let $\mathcal{B}$ be a graded $\mathcal{O}_{\mathcal{D}}$-algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$$

of graded $f^{-1}\mathcal{O}_{\mathcal{D}}$-algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map $\varphi : f^*\mathcal{B} \to \mathcal{A}$ of graded $\mathcal{O}_{\mathcal{C}}$-algebras or equivalently φ can be viewed as a map

$$\varphi : \mathcal{B} \to f_*\mathcal{A}$$

of graded $\mathcal{O}_\mathcal{D}$-algebras. See Remark 24.3.2.

Let us define a functor

$$f_* : \text{Mod}(\mathcal{A}) \longrightarrow \text{Mod}(\mathcal{B})$$

Given a graded $\mathcal{A}$-module $\mathcal{M}$ we define $f_*\mathcal{M}$ to be the graded $\mathcal{B}$-module whose degree n term is $f_*\mathcal{M}^n$. As multiplication we use

$$f_*\mathcal{M}^n \times \mathcal{B}^m \xrightarrow{(\text{id},\varphi^m)} f_*\mathcal{M}^n \times f_*\mathcal{A}^m \xrightarrow{f_*\mu_{n,m}} f_*\mathcal{M}^{n+m}$$

where $\mu_{n,m} : \mathcal{M}^n \times \mathcal{A}^m \to \mathcal{M}^{n+m}$ is the multiplication map for $\mathcal{M}$ over $\mathcal{A}$. This uses that f_* commutes with products. The construction is clearly functorial in $\mathcal{M}$ and we obtain our functor.

Let us define a functor

$$f^* : \text{Mod}(\mathcal{B}) \longrightarrow \text{Mod}(\mathcal{A})$$

We will define this functor as a composite of functors

$$\text{Mod}(\mathcal{B}) \xrightarrow{f^{-1}} \text{Mod}(f^{-1}\mathcal{B}) \xrightarrow{-\otimes_{f^{-1}\mathcal{B}}\mathcal{A}} \text{Mod}(\mathcal{A})$$

First, given a graded $\mathcal{B}$-module $\mathcal{N}$ we define $f^{-1}\mathcal{N}$ to be the graded $f^{-1}\mathcal{B}$-module whose degree n term is $f^{-1}\mathcal{N}^n$. As multiplication we use

$$f^{-1}\nu_{n,m} : f^{-1}\mathcal{N}^n \times f^{-1}\mathcal{B}^m \longrightarrow f^{-1}\mathcal{N}^{n+m}$$

where $\nu_{n,m} : \mathcal{N}^n \times \mathcal{B}^m \to \mathcal{N}^{n+m}$ is the multiplication map for $\mathcal{N}$ over $\mathcal{B}$. This uses that f^{-1} commutes with products. The construction is clearly functorial in $\mathcal{N}$ and we obtain our functor f^{-1}. Having said this, we can use the tensor product discussion in Section 24.8 to define the functor

$$- \otimes_{f^{-1}\mathcal{B}} \mathcal{A} : \text{Mod}(f^{-1}\mathcal{B}) \longrightarrow \text{Mod}(\mathcal{A})$$

Finally, we set

$$f^*\mathcal{N} = f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{B},\varphi} \mathcal{A}$$

as already foretold above.

The functors f_* and f^* are readily enhanced to give functors of graded categories

$$f_* : \text{Mod}^{gr}(\mathcal{A}) \longrightarrow \text{Mod}^{gr}(\mathcal{B}) \quad \text{and} \quad f^* : \text{Mod}^{gr}(\mathcal{B}) \longrightarrow \text{Mod}^{gr}(\mathcal{A})$$

which do the same thing on underlying objects and are defined by functoriality of the constructions on homogenous morphisms of degree n.

0FR8 **Lemma 24.9.1.** In the situation above we have

$$\text{Hom}_{\text{Mod}^{gr}(\mathcal{B})}(\mathcal{N}, f_*\mathcal{M}) = \text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(f^*\mathcal{N}, \mathcal{M})$$

Proof. Omitted. Hints: First prove that f^{-1} and f_* are adjoint as functors between $\text{Mod}(\mathcal{B})$ and $\text{Mod}(f^{-1}\mathcal{B})$ using the adjunction between f^{-1} and f_* on sheaves of abelian groups. Next, use the adjunction between base change and restriction given in Section 24.8. $\square$

24.10. Localization and sheaves of graded modules

0FR9 We advise the reader to skip this section.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$ and denote

$$j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

the corresponding localization morphism (Modules on Sites, Section 18.19). Below we will use the following fact: for $\mathcal{O}_U$-modules $\mathcal{M}_i$, $i = 1, 2$ and a $\mathcal{O}$-module $\mathcal{A}$ there is a canonical map

$$j_! : \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U, \mathcal{M}_2) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}, j_!\mathcal{M}_2)$$

Namely, we have $j_!(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U) = j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}$ by Modules on Sites, Lemma 18.27.9.

Let $\mathcal{A}$ be a graded $\mathcal{O}$-algebra. We will denote $\mathcal{A}_U$ the restriction of $\mathcal{A}$ to $\mathcal{C}/U$, in other words, we have $\mathcal{A}_U = j^*\mathcal{A} = j^{-1}\mathcal{A}$. In Section 24.9 we have constructed adjoint functors

$$j_* : \mathrm{Mod}^{gr}(\mathcal{A}_U) \longrightarrow \mathrm{Mod}^{gr}(\mathcal{A}) \quad \text{and} \quad j^* : \mathrm{Mod}^{gr}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{gr}(\mathcal{A}_U)$$

with j^* left adjoint to j_*. We claim there is in addition an exact functor

$$j_! : \mathrm{Mod}^{gr}(\mathcal{A}_U) \longrightarrow \mathrm{Mod}^{gr}(\mathcal{A})$$

left adjoint to j^*. Namely, given a graded $\mathcal{A}_U$-module $\mathcal{M}$ we define $j_!\mathcal{M}$ to be the graded $\mathcal{A}$-module whose degree n term is $j_!\mathcal{M}^n$. As multiplication map we use

$$j_!\mu_{n,m} : j_!\mathcal{M}^n \times \mathcal{A}^m \to j_!\mathcal{M}^{n+m}$$

where $\mu_{m,n} : \mathcal{M}^n \times \mathcal{A}^m \to \mathcal{M}^{n+m}$ is the given multiplication map. Given a homogeneous map $f : \mathcal{M} \to \mathcal{M}'$ of degree n of graded $\mathcal{A}_U$-modules, we obtain a homogeneous map $j_!f : j_!\mathcal{M} \to j_!\mathcal{M}'$ of degree n. Thus we obtain our functor.

0FRA **Lemma 24.10.1.** *In the situation above we have*

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A})}(j_!\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A}_U)}(\mathcal{M}, j^*\mathcal{N})$$

Proof. By the discussion in Modules on Sites, Section 18.19 the functors $j_!$ and j^* on $\mathcal{O}$-modules are adjoint. Thus if we only look at the $\mathcal{O}$-module structures we know that

$$\mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathrm{Mod}(\mathcal{O}))}(j_!\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathrm{Mod}(\mathcal{O}_U))}(\mathcal{M}, j^*\mathcal{N})$$

(Recall that $\mathrm{Gr}^{gr}(\mathrm{Mod}(\mathcal{O}))$ denotes the graded category of graded $\mathcal{O}$-modules.) Then one has to check that these identifications map the $\mathcal{A}$-module maps on the left hand side to the $\mathcal{A}_U$-module maps on the right hand side. To check this, given $\mathcal{O}_U$-linear maps $f^n : \mathcal{M}^n \to j^*\mathcal{N}^{n+d}$ corresponding to $\mathcal{O}$-linear maps $g^n : j_!\mathcal{M}^n \to \mathcal{N}^{n+d}$ it suffices to show that

$$\begin{tikzcd}
\mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}_U^m \arrow[r, "f^n \otimes 1"'] \arrow[d] & j^*\mathcal{N}^{n+d} \otimes_{\mathcal{O}_U} \mathcal{A}_U^m \arrow[d] \\
\mathcal{M}^{n+m} \arrow[r, "f^{n+m}"] & j^*\mathcal{N}^{n+m+d}
\end{tikzcd}$$

commutes if and only if

$$\begin{tikzcd} j_!\mathcal{M}^n \otimes_\mathcal{O} \mathcal{A}^m \arrow[r, "g^n \otimes 1"'] \arrow[d] & \mathcal{N}^{n+d} \otimes_\mathcal{O} \mathcal{A}^m_U \arrow[d] \\ j_!\mathcal{M}^{n+m} \arrow[r, "g^{n+m}"] & \mathcal{N}^{n+m+d} \end{tikzcd}$$

commutes. However, we know that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}^m_U, j^*\mathcal{N}^{n+d+m}) &= \mathrm{Hom}_\mathcal{O}(j_!(\mathcal{M}^n \otimes_{\mathcal{O}_U} \mathcal{A}^m_U), \mathcal{N}^{n+d+m}) \\ &= \mathrm{Hom}_\mathcal{O}(j_!\mathcal{M}^n \otimes_\mathcal{O} \mathcal{A}^m, \mathcal{N}^{n+d+m}) \end{aligned}$$

by the already used Modules on Sites, Lemma 18.27.9. We omit the verification that shows that the obstruction to the commutativity of the first diagram in the first group maps to the obstruction to the commutativity of the second diagram in the last group. □

0FRB Lemma 24.10.2. In the situation above, let $\mathcal{M}$ be a right graded $\mathcal{A}_U$-module and let $\mathcal{N}$ be a left graded $\mathcal{A}$-module. Then

$$j_!\mathcal{M} \otimes_\mathcal{A} \mathcal{N} = j_!(\mathcal{M} \otimes_{\mathcal{A}_U} \mathcal{N}|_U)$$

as graded $\mathcal{O}$-modules functorially in $\mathcal{M}$ and $\mathcal{N}$.

Proof. Recall that the degree n component of $j_!\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ is the cokernel of the canonical map

$$\bigoplus\nolimits_{r+s+t=n} j_!\mathcal{M}^r \otimes_\mathcal{O} \mathcal{A}^s \otimes_\mathcal{O} \mathcal{N}^t \longrightarrow \bigoplus\nolimits_{p+q=n} j_!\mathcal{M}^p \otimes_\mathcal{O} \mathcal{N}^q$$

See Section 24.6. By Modules on Sites, Lemma 18.27.9 this is the same thing as the cokernel of

$$\bigoplus\nolimits_{r+s+t=n} j_!(\mathcal{M}^r \otimes_{\mathcal{O}_U} \mathcal{A}^s|_U \otimes_{\mathcal{O}_U} \mathcal{N}^t|_U) \longrightarrow \bigoplus\nolimits_{p+q=n} j_!(\mathcal{M}^p \otimes_{\mathcal{O}_U} \mathcal{N}^q|_U)$$

and we win. An alternative proof would be to redo the Yoneda argument given in the proof of the lemma cited above. □

24.11. Shift functors on sheaves of graded modules

0FRC We urge the reader to skip this section. It turns out that sheaves of graded modules over a graded algebra are an example of the phenomenon discussed in Differential Graded Algebra, Remark 22.25.7.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a graded $\mathcal{A}$-module. Let $k \in \mathbf{Z}$. We define the kth shift of $\mathcal{M}$, denoted $\mathcal{M}[k]$, to be the graded $\mathcal{A}$-module whose nth part is given by

$$(\mathcal{M}[k])^n = \mathcal{M}^{n+k}$$

is the $(n+k)$th part of $\mathcal{M}$. As multiplication maps

$$(\mathcal{M}[k])^n \times \mathcal{A}^m \longrightarrow (\mathcal{M}[k])^{n+m}$$

we simply use the multiplication maps

$$\mathcal{M}^{n+k} \times \mathcal{A}^m \longrightarrow \mathcal{M}^{n+m+k}$$

of $\mathcal{M}$. It is clear that we have defined a functor $[k]$, that we have $[k+l] = [k] \circ [l]$, and that we have

$$\mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}[k]) = \mathrm{Hom}_{\mathrm{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M})[k]$$

(without the intervention of signs) functorially in $\mathcal{M}$ and $\mathcal{L}$. Thus we see indeed that the graded category of graded $\mathcal{A}$-modules can be recovered from the ordinary category of graded $\mathcal{A}$-modules and the shift functors as discussed in Differential Graded Algebra, Remark 22.25.7.

0FRD Lemma 24.11.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a graded $\mathcal{O}$-algebra. The category $\mathrm{Mod}(\mathcal{A})$ is a Grothendieck abelian category.

Proof. By Lemma 24.4.2 and the definition of a Grothendieck abelian category (Injectives, Definition 19.10.1) it suffices to show that $\mathrm{Mod}(\mathcal{A})$ has a generator. We claim that

$$\mathcal{G} = \bigoplus\nolimits_{k,U} j_{U!}\mathcal{A}_U[k]$$

is a generator where the sum is over all objects U of $\mathcal{C}$ and $k \in \mathbf{Z}$. Indeed, given a graded $\mathcal{A}$-module $\mathcal{M}$ if there are no nonzero maps from $\mathcal{G}$ to $\mathcal{M}$, then we see that for all k and U we have

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A})}(j_{U!}\mathcal{A}_U[k], \mathcal{M}) = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{A}_U)}(\mathcal{A}_U[k], \mathcal{M}|_U) = \Gamma(U, \mathcal{M}^{-k})$$

is equal to zero. Hence $\mathcal{M}$ is zero. □

24.12. Sheaves of differential graded algebras

0FRE This section is the analogue of Differential Graded Algebra, Section 22.3.

0FRF Definition 24.12.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A sheaf of differential graded $\mathcal{O}$-algebras or a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$ is a cochain complex $\mathcal{A}^\bullet$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{A}^n \times \mathcal{A}^m \to \mathcal{A}^{n+m}, \quad (a, b) \longmapsto ab$$

called the multiplication maps with the following properties

(1) multiplication is associative,
(2) there is a global section 1 of $\mathcal{A}^0$ which is a two-sided identity for multiplication,
(3) for $U \in \mathrm{Ob}(\mathcal{C})$, $a \in \mathcal{A}^n(U)$, and $b \in \mathcal{A}^m(U)$ we have
$$\mathrm{d}^{n+m}(ab) = \mathrm{d}^n(a)b + (-1)^n a\mathrm{d}^m(b)$$

We often denote such a structure $(\mathcal{A}, \mathrm{d})$. A homomorphism of differential graded $\mathcal{O}$-algebras from $(\mathcal{A}, \mathrm{d})$ to $(\mathcal{B}, \mathrm{d})$ is a map $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ of complexes of $\mathcal{O}$-modules compatible with the multiplication maps.

Given a differential graded $\mathcal{O}$-algebra $(\mathcal{A}, \mathrm{d})$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{A}(U) = \Gamma(U, \mathcal{A}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{A}^n(U)$$

This is a differential graded $\mathcal{O}(U)$-algebra.

As much as possible, we will think of a differential graded $\mathcal{O}$-algebra $(\mathcal{A}, \mathrm{d})$ as a graded $\mathcal{O}$-algebra $\mathcal{A}$ endowed with the operator $\mathrm{d} : \mathcal{A} \to \mathcal{A}$ of degree 1 (where $\mathcal{A}$ is viewed as a graded $\mathcal{O}$-module) satisfying the Leibniz rule given in the definition.

0FRG Remark 24.12.2. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi.

(1) Let $(\mathcal{A}, \text{d})$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. The pushforward will be the differential graded $\mathcal{O}_\mathcal{D}$-algebra $(f_*\mathcal{A}, \text{d})$ where $f_*\mathcal{A}$ is as in Remark 24.3.2 and $\text{d} = f_*\text{d}$ as maps $f_*\mathcal{A}^n \to f_*\mathcal{A}^{n+1}$. We omit the verification that the Leibniz rule is satisfied.
(2) Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. The pullback will be the differential graded $\mathcal{O}_\mathcal{C}$-algebra $(f^*\mathcal{B}, \text{d})$ where $f^*\mathcal{B}$ is as in Remark 24.3.2 and $\text{d} = f^*\text{d}$ as maps $f^*\mathcal{B}^n \to f^*\mathcal{B}^{n+1}$. We omit the verification that the Leibniz rule is satisfied.
(3) The set of homomorphisms $f^*\mathcal{B} \to \mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras is in 1-to-1 correspondence with the set of homomorphisms $\mathcal{B} \to f_*\mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{D}$-algebras.

Part (3) follows immediately from the usual adjunction between f^* and f_* on sheaves of modules.

24.13. Sheaves of differential graded modules

0FRH This section is the analogue of Differential Graded Algebra, Section 22.4.

0FRI Definition 24.13.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. A (right) differential graded $\mathcal{A}$-module or (right) differential graded module over $\mathcal{A}$ is a cochain complex $\mathcal{M}^\bullet$ endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{M}^n \times \mathcal{A}^m \to \mathcal{M}^{n+m}, \quad (x, a) \longmapsto xa$$

called the multiplication maps with the following properties

(1) multiplication satisfies $(xa)a' = x(aa')$,
(2) the identity section 1 of $\mathcal{A}^0$ acts as the identity on $\mathcal{M}^n$ for all n,
(3) for $U \in \text{Ob}(\mathcal{C})$, $x \in \mathcal{M}^n(U)$, and $a \in \mathcal{A}^m(U)$ we have

$$\text{d}^{n+m}(xa) = \text{d}^n(x)a + (-1)^n x\text{d}^m(a)$$

We often say "let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module" to indicate this situation. A homomorphism of differential graded $\mathcal{A}$-modules from $\mathcal{M}$ to $\mathcal{N}$ is a map $f : \mathcal{M}^\bullet \to \mathcal{N}^\bullet$ of complexes of $\mathcal{O}$-modules compatible with the multiplication maps. The category of (right) differential graded $\mathcal{A}$-modules is denoted $\text{Mod}(\mathcal{A}, \text{d})$.

We can define left differential graded modules in exactly the same manner but our default in the chapter will be right modules.

Given a differential graded $\mathcal{A}$-module $\mathcal{M}$ and an object $U \in \text{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a (right) differential graded $\mathcal{A}(U)$-module.

0FRJ Lemma 24.13.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a differential graded $\mathcal{O}$-algebra. The category $\text{Mod}(\mathcal{A}, \text{d})$ is an abelian category with the following properties

(1) $\text{Mod}(\mathcal{A}, \text{d})$ has arbitrary direct sums,
(2) $\text{Mod}(\mathcal{A}, \text{d})$ has arbitrary colimits,
(3) filtered colimit in $\text{Mod}(\mathcal{A}, \text{d})$ are exact,

(4) $\text{Mod}(\mathcal{A}, \text{d})$ has arbitrary products,
(5) $\text{Mod}(\mathcal{A}, \text{d})$ has arbitrary limits.

The forgetful functor

$$\text{Mod}(\mathcal{A}, \text{d}) \longrightarrow \text{Mod}(\mathcal{A})$$

sending a differential graded $\mathcal{A}$-module to its underlying graded module commutes with all limits and colimits.

Proof. Let us denote $F : \text{Mod}(\mathcal{A}, \text{d}) \to \text{Mod}(\mathcal{A})$ the functor in the statement of the lemma. Observe that the category $\text{Mod}(\mathcal{A})$ has properties (1) – (5), see Lemma 24.4.2.

Consider a homomorphism $f : \mathcal{M} \to \mathcal{N}$ of graded $\mathcal{A}$-modules. The kernel and cokernel of f as maps of graded $\mathcal{A}$-modules are additionally endowed with differentials as in Definition 24.13.1. Hence these are also the kernel and cokernel in $\text{Mod}(\mathcal{A}, \text{d})$. Thus $\text{Mod}(\mathcal{A}, \text{d})$ is an abelian category and taking kernels and cokernels commutes with F.

To prove the existence of limits and colimits it is sufficient to prove the existence of products and direct sums, see Categories, Lemmas 4.14.11 and 4.14.12. The same lemmas show that proving the commutation of limits and colimits with F follows if F commutes with direct sums and products.

Let $\mathcal{M}_t$, $t \in T$ be a set of differential graded $\mathcal{A}$-modules. Then we can consider the direct sum $\bigoplus \mathcal{M}_t$ as a graded $\mathcal{A}$-module. Since the direct sum of graded modules is done termwise, it is clear that $\bigoplus \mathcal{M}_t$ comes endowed with a differential. The reader easily verifies that this is a direct sum in $\text{Mod}(\mathcal{A}, \text{d})$. Similarly for products.

Observe that F is an exact functor and that a complex $\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$ of $\text{Mod}(\mathcal{A}, \text{d})$ is exact if and only if $F(\mathcal{M}_1) \to F(\mathcal{M}_2) \to F(\mathcal{M}_3)$ is exact in $\text{Mod}(\mathcal{A})$. Hence we conclude that (3) holds as filtered colimits are exact in $\text{Mod}(\mathcal{A})$. □

Combining Lemmas 24.13.2 and 24.4.2 we find that there is an exact and faithful functor

$$\text{Mod}(\mathcal{A}, \text{d}) \longrightarrow \text{Comp}(\mathcal{O})$$

of abelian categories. For a differential graded $\mathcal{A}$-module $\mathcal{M}$ the cohomology $\mathcal{O}$-modules, denoted $H^i(\mathcal{M})$, are defined as the cohomology of the complex of $\mathcal{O}$-modules corresponding to $\mathcal{M}$. Therefore, a short exact sequence $0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{M} \to 0$ of differential graded $\mathcal{A}$-modules gives rise to a long exact sequence

0FRK (24.13.2.1) $$H^n(\mathcal{K}) \to H^n(\mathcal{L}) \to H^n(\mathcal{M}) \to H^{n+1}(\mathcal{K})$$

of cohomology modules, see Homology, Lemma 12.13.12.

Moreover, from now on we borrow all the terminology used for complexes of modules. For example, we say that a differential graded $\mathcal{A}$-module $\mathcal{M}$ is acyclic if $H^k(\mathcal{M}) = 0$ for all $k \in \mathbf{Z}$. We say that a homomorphism $\mathcal{M} \to \mathcal{N}$ of differential graded $\mathcal{A}$-modules is a quasi-isomorphism if it induces isomorphisms $H^k(\mathcal{M}) \to H^k(\mathcal{N})$ for all $k \in \mathbf{Z}$. And so on and so forth.

24.14. The differential graded category of modules

0FRL This section is the analogue of Differential Graded Algebra, Example 22.26.8. For our conventions on differential graded categories, please see Differential Graded Algebra, Section 22.26.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. We will construct a differential graded category

$$\text{Mod}^{dg}(\mathcal{A}, \text{d})$$

over $R = \Gamma(\mathcal{C}, \mathcal{O})$ whose associated category of complexes is the category of differential graded $\mathcal{A}$-modules:

$$\text{Mod}(\mathcal{A}, \text{d}) = \text{Comp}(\text{Mod}^{dg}(\mathcal{A}, \text{d}))$$

As objects of $\text{Mod}^{dg}(\mathcal{A}, \text{d})$ we take right differential graded $\mathcal{A}$-modules, see Section 24.13. Given differential graded $\mathcal{A}$-modules $\mathcal{L}$ and $\mathcal{M}$ we set

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, \text{d})}(\mathcal{L}, \mathcal{M}) = \text{Hom}_{\text{Mod}^{gr}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \text{Hom}^n(\mathcal{L}, \mathcal{M})$$

as a graded R-module, see Section 24.5. In other words, the nth graded piece $\text{Hom}^n(\mathcal{L}, \mathcal{M})$ is the R-module of right $\mathcal{A}$-module maps homogeneous of degree n. For an element $f \in \text{Hom}^n(\mathcal{L}, \mathcal{M})$ we set

$$\text{d}(f) = \text{d}_\mathcal{M} \circ f - (-1)^n f \circ \text{d}_\mathcal{L}$$

To make sense of this we think of $\text{d}_\mathcal{M}$ and $\text{d}_\mathcal{L}$ as graded $\mathcal{O}$-module maps and we use composition of graded $\mathcal{O}$-module maps. It is clear that $\text{d}(f)$ is homogeneous of degree $n+1$ as a graded $\mathcal{O}$-module map, and it is $\mathcal{A}$-linear because for homogeneous local sections x and a of $\mathcal{M}$ and $\mathcal{A}$ we have

$$\begin{aligned} \text{d}(f)(xa) &= \text{d}_\mathcal{M}(f(x)a) - (-1)^n f(\text{d}_\mathcal{L}(xa)) \\ &= \text{d}_\mathcal{M}(f(x))a + (-1)^{\deg(x)+n} f(x)\text{d}(a) - (-1)^n f(\text{d}_\mathcal{L}(x))a - (-1)^{n+\deg(x)} f(x)\text{d}(a) \\ &= \text{d}(f)(x)a \end{aligned}$$

as desired (observe that this calculation would not work without the sign in the definition of our differential on Hom).

For differential graded $\mathcal{A}$-modules $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ we have already defined the composition

$$\text{Hom}^m(\mathcal{L}, \mathcal{M}) \times \text{Hom}^n(\mathcal{K}, \mathcal{L}) \longrightarrow \text{Hom}^{n+m}(\mathcal{K}, \mathcal{M})$$

in Section 24.5 by the usual composition of maps of sheaves. This defines a map of differential graded modules

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, \text{d})}(\mathcal{L}, \mathcal{M}) \otimes_R \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, \text{d})}(\mathcal{K}, \mathcal{L}) \longrightarrow \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, \text{d})}(\mathcal{K}, \mathcal{M})$$

as required in Differential Graded Algebra, Definition 22.26.1 because

$$\begin{aligned} \text{d}(g \circ f) &= \text{d}_\mathcal{M} \circ g \circ f - (-1)^{n+m} g \circ f \circ \text{d}_\mathcal{K} \\ &= (\text{d}_\mathcal{M} \circ g - (-1)^m g \circ \text{d}_L) \circ f + (-1)^m g \circ (\text{d}_\mathcal{L} \circ f - (-1)^n f \circ \text{d}_\mathcal{K}) \\ &= \text{d}(g) \circ f + (-1)^m g \circ \text{d}(f) \end{aligned}$$

if f has degree n and g has degree m as desired.

24.15. Tensor product for sheaves of differential graded modules

0FRM This section is the analogue of part of Differential Graded Algebra, Section 22.12.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right differential graded $\mathcal{A}$-module and let $\mathcal{N}$ be a left differential graded $\mathcal{A}$-module. In this situation we define the tensor product $\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ as follows. As a graded $\mathcal{O}$-module it is given by the construction in Section 24.6. It comes endowed with a differential

$$\mathrm{d}_{\mathcal{M}\otimes_\mathcal{A}\mathcal{N}} : (\mathcal{M} \otimes_\mathcal{A} \mathcal{N})^n \longrightarrow (\mathcal{M} \otimes_\mathcal{A} \mathcal{N})^{n+1}$$

defined by the rule that

$$\mathrm{d}_{\mathcal{M}\otimes_\mathcal{A}\mathcal{N}}(x \otimes y) = \mathrm{d}_\mathcal{M}(x) \otimes y + (-1)^{\deg(x)} x \otimes \mathrm{d}_\mathcal{N}(y)$$

for homogeneous local sections x and y of $\mathcal{M}$ and $\mathcal{N}$. To see that this is well defined we have to show that $\mathrm{d}_{\mathcal{M}\otimes_\mathcal{A}\mathcal{N}}$ annihilates elements of the form $xa \otimes y - x \otimes ay$ for homogeneous local sections x, a, y of $\mathcal{M}$, $\mathcal{A}$, $\mathcal{N}$. We compute

$$\begin{aligned}
&\mathrm{d}_{\mathcal{M}\otimes_\mathcal{A}\mathcal{N}}(xa \otimes y - x \otimes ay)\\
&= \mathrm{d}_\mathcal{M}(xa) \otimes y + (-1)^{\deg(x)+\deg(a)} xa \otimes \mathrm{d}_\mathcal{N}(y) - \mathrm{d}_\mathcal{M}(x) \otimes ay - (-1)^{\deg(x)} x \otimes \mathrm{d}_\mathcal{N}(ay)\\
&= \mathrm{d}_\mathcal{M}(x)a \otimes y + (-1)^{\deg(x)} x\mathrm{d}(a) \otimes y + (-1)^{\deg(x)+\deg(a)} xa \otimes \mathrm{d}_\mathcal{N}(y)\\
&- \mathrm{d}_\mathcal{M}(x) \otimes ay - (-1)^{\deg(x)} x \otimes \mathrm{d}(a)y - (-1)^{\deg(x)+\deg(a)} x \otimes a\mathrm{d}_\mathcal{N}(y)
\end{aligned}$$

then we observe that the elements

$$\mathrm{d}_\mathcal{M}(x)a \otimes y - \mathrm{d}_\mathcal{M}(x) \otimes ay, \quad x\mathrm{d}(a) \otimes y - x \otimes \mathrm{d}(a)y, \quad \text{and} \quad xa \otimes \mathrm{d}_\mathcal{N}(y) - x \otimes a\mathrm{d}_\mathcal{N}(y)$$

map to zero in $\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ and we conclude. We omit the verification that $\mathrm{d}_{\mathcal{M}\otimes_\mathcal{A}\mathcal{N}} \circ \mathrm{d}_{\mathcal{M}\otimes_\mathcal{A}\mathcal{N}} = 0$.

If we fix the left differential graded $\mathcal{A}$-module $\mathcal{N}$ we obtain a functor

$$- \otimes_\mathcal{A} \mathcal{N} : \mathrm{Mod}(\mathcal{A}, \mathrm{d}) \longrightarrow \mathrm{Comp}(\mathcal{O})$$

where on the right hand side we have the category of complexes of $\mathcal{O}$-modules. This can be upgraded to a functor of differential graded categories

$$- \otimes_\mathcal{A} \mathcal{N} : \mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow \mathrm{Comp}^{dg}(\mathcal{O})$$

On underlying graded objects, we send a homomorphism $f : \mathcal{M} \to \mathcal{M}'$ of degree n to the degree n map $f \otimes \mathrm{id}_\mathcal{N} : \mathcal{M} \otimes_\mathcal{A} \mathcal{N} \to \mathcal{M}' \otimes_\mathcal{A} \mathcal{N}$, because this is what we did in Section 24.6. To show that this works, we have to verify that the map

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{M}, \mathcal{M}') \longrightarrow \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(\mathcal{M} \otimes_\mathcal{A} \mathcal{N}, \mathcal{M}' \otimes_\mathcal{A} \mathcal{N})$$

is compatible with differentials. To see this for f as above we have to show that

$$(\mathrm{d}_{\mathcal{M}'} \circ f - (-1)^n f \circ \mathrm{d}_\mathcal{M}) \otimes \mathrm{id}_\mathcal{N}$$

is equal to

$$\mathrm{d}_{\mathcal{M}'\otimes_\mathcal{A}\mathcal{N}} \circ (f \otimes \mathrm{id}_\mathcal{N}) - (-1)^n (f \otimes \mathrm{id}_\mathcal{N}) \circ \mathrm{d}_{\mathcal{M}\otimes_\mathcal{A}\mathcal{N}}$$

Let us compute the effect of these operators on a local section of the form $x \otimes y$ with x and y homogeneous local sections of $\mathcal{M}$ and $\mathcal{N}$. For the first we obtain

$$(\mathrm{d}_{\mathcal{M}'}(f(x)) - (-1)^n f(\mathrm{d}_\mathcal{M}(x))) \otimes y$$

and for the second we obtain

$$\begin{aligned} &\mathrm{d}_{\mathcal{M}'\otimes_{\mathcal{A}}\mathcal{N}}(f(x)\otimes y) - (-1)^n(f\otimes \mathrm{id}_{\mathcal{N}})(\mathrm{d}_{\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}}(x\otimes y) \\ &= \mathrm{d}_{\mathcal{M}'}(f(x))\otimes y + (-1)^{\deg(x)+n} f(x)\otimes \mathrm{d}_{\mathcal{N}}(y) \\ &- (-1)^n f(\mathrm{d}_{\mathcal{M}}(x))\otimes y - (-1)^n(-1)^{\deg(x)} f(x)\otimes \mathrm{d}_{\mathcal{N}}(y) \end{aligned}$$

which is indeeed the same local section.

24.16. Internal hom for sheaves of differential graded modules

0FRN We are going to need the sheafified version of the construction in Section 24.14. Let $(\mathcal{C}, \mathcal{O})$, $\mathcal{A}$, $\mathcal{M}$, $\mathcal{L}$ be as in Section 24.14. Then we define

$$\mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{M}, \mathcal{L}) = \mathcal{H}om^{gr}_{\mathcal{A}}(\mathcal{M}, \mathcal{L}) = \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{H}om^n_{\mathcal{A}}(\mathcal{M}, \mathcal{L})$$

as a graded $\mathcal{O}$-module, see Section 24.7. In other words, a section f of the nth graded piece $\mathcal{H}om^n_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ over U is a map of right $\mathcal{A}_U$-module map $\mathcal{L}|_U \to \mathcal{M}|_U$ homogeneous of degree n. For such f we set

$$\mathrm{d}(f) = \mathrm{d}_{\mathcal{M}}|_U \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}}|_U$$

To make sense of this we think of $\mathrm{d}_{\mathcal{M}}|_U$ and $\mathrm{d}_{\mathcal{L}}|_U$ as graded $\mathcal{O}_U$-module maps and we use composition of graded $\mathcal{O}_U$-module maps. It is clear that $\mathrm{d}(f)$ is homogeneous of degree $n+1$ as a graded $\mathcal{O}_U$-module map. Using the exact same computation as in Section 24.14 we see that $\mathrm{d}(f)$ is $\mathcal{A}_U$-linear.

As in Section 24.14 there is a composition map

$$\mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{K}, \mathcal{M})$$

where the left hand side is the tensor product of differential graded $\mathcal{O}$-modules defined in Section 24.15. This map is given by the composition map

$$\mathcal{H}om^m(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}} \mathcal{H}om^n(\mathcal{K}, \mathcal{L}) \longrightarrow \mathcal{H}om^{n+m}(\mathcal{K}, \mathcal{M})$$

defined by simple composition (locally). Using the exact same computation as in Section 24.14 on local sections we see that the composition map is a morphism of differential graded $\mathcal{O}$-modules.

With these definitions we have

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A})}(\mathcal{L}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{L}, \mathcal{M}))$$

as graded R-modules compatible with composition.

24.17. Sheaves of differential graded bimodules and tensor-hom adjunction

0FRP This section is the analogue of part of Differential Graded Algebra, Section 22.12.

0FRQ Definition 24.17.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. A differential graded $(\mathcal{A}, \mathcal{B})$-bimodule is given by a complex $\mathcal{M}^\bullet$ of $\mathcal{O}$-modules endowed with $\mathcal{O}$-bilinear maps

$$\mathcal{M}^n \times \mathcal{B}^m \to \mathcal{M}^{n+m}, \quad (x, b) \longmapsto xb$$

and

$$\mathcal{A}^n \times \mathcal{M}^m \to \mathcal{M}^{n+m}, \quad (a, x) \longmapsto ax$$

called the multiplication maps with the following properties

(1) multiplication satisfies $a(a'x) = (aa')x$ and $(xb)b' = x(bb')$,

(2) $(ax)b = a(xb)$,
(3) $\mathrm{d}(ax) = \mathrm{d}(a)x + (-1)^{\deg(a)}a\mathrm{d}(x)$ and $\mathrm{d}(xb) = \mathrm{d}(x)b + (-1)^{\deg(x)}x\mathrm{d}(b)$,
(4) the identity section 1 of $\mathcal{A}^0$ acts as the identity by multiplication, and
(5) the identity section 1 of $\mathcal{B}^0$ acts as the identity by multiplication.

We often denote such a structure $\mathcal{M}$ and sometimes we write ${}_\mathcal{A}\mathcal{M}_\mathcal{B}$. A homomorphism of differential graded $(\mathcal{A}, \mathcal{B})$-bimodules $f : \mathcal{M} \to \mathcal{N}$ is a map of complexes $f : \mathcal{M}^\bullet \to \mathcal{N}^\bullet$ of $\mathcal{O}$-modules compatible with the multiplication maps.

Given a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ and an object $U \in \mathrm{Ob}(\mathcal{C})$ we use the notation

$$\mathcal{M}(U) = \Gamma(U, \mathcal{M}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{M}^n(U)$$

This is a differential graded $(\mathcal{A}(U), \mathcal{B}(U))$-bimodule.

Observe that a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ is the same thing as a right differential graded $\mathcal{B}$-module which is also a left differential graded $\mathcal{A}$-module such that the grading and differentials agree and such that the $\mathcal{A}$-module structure commutes with the $\mathcal{B}$-module structure. Here is a precise statement.

0FRR Lemma 24.17.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{N}$ be a right differential graded $\mathcal{B}$-module. There is a 1-to-1 correspondence between $(\mathcal{A}, \mathcal{B})$-bimodule structures on $\mathcal{N}$ compatible with the given differential graded $\mathcal{B}$-module structure and homomorphisms

$$\mathcal{A} \longrightarrow \mathcal{H}om^{dg}_\mathcal{B}(\mathcal{N}, \mathcal{N})$$

of differential graded $\mathcal{O}$-algebras.

Proof. Omitted. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right differential graded $\mathcal{A}$-module and let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. In this case the differential graded tensor product defined in Section 24.15

$$\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$$

is a right differential graded $\mathcal{B}$-module with multiplication maps as in Section 24.8. This construction defines a functor and a functor of graded categories

$$\otimes_\mathcal{A}\mathcal{N} : \mathrm{Mod}(\mathcal{A}, \mathrm{d}) \longrightarrow \mathrm{Mod}(\mathcal{B}, \mathrm{d}) \quad \text{and} \quad \otimes_\mathcal{A} \mathcal{N} : \mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow \mathrm{Mod}^{dg}(\mathcal{B}, \mathrm{d})$$

by sending homomorphisms of degree n from $\mathcal{M} \to \mathcal{M}'$ to the induced map of degree n from $\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ to $\mathcal{M}' \otimes_\mathcal{A} \mathcal{N}$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{L}$ be a right differential graded $\mathcal{B}$-module. In this case the differential graded internal hom defined in Section 24.16

$$\mathcal{H}om^{dg}_\mathcal{B}(\mathcal{N}, \mathcal{L})$$

is a right differential graded $\mathcal{A}$-module where the right graded $\mathcal{A}$-module structure is the one defined in Section 24.8. Another way to define the multiplication is the use the composition

$$\mathcal{H}om^{dg}_\mathcal{B}(\mathcal{N}, \mathcal{L}) \otimes_\mathcal{O} \mathcal{A} \to \mathcal{H}om^{dg}_\mathcal{B}(\mathcal{N}, \mathcal{L}) \otimes_\mathcal{O} \mathcal{H}om^{dg}_\mathcal{B}(\mathcal{N}, \mathcal{N}) \to \mathcal{H}om^{dg}_\mathcal{B}(\mathcal{N}, \mathcal{L})$$

where the first arrow comes from Lemma 24.17.2 and the second arrow is the composition of Section 24.16. Since these arrows are both compatible with differentials, we conclude that we indeed obtain a differential graded $\mathcal{A}$-module. This construction defines a functor and a functor of differential graded categories

$$\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, -) : \text{Mod}(\mathcal{B}, \text{d}) \longrightarrow \text{Mod}(\mathcal{A}) \quad \text{and} \quad \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, -) : \text{Mod}^{dg}(\mathcal{B}, \text{d}) \longrightarrow \text{Mod}^{dg}(\mathcal{A}, \text{d})$$

by sending homomorphisms of degree n from $\mathcal{L} \to \mathcal{L}'$ to the induced map of degree n from $\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L})$ to $\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}')$.

0FRS Lemma 24.17.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a right differential graded $\mathcal{A}$-module. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{L}$ be a right differential graded $\mathcal{B}$-module. With conventions as above we have

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{B}, \text{d})}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, \text{d})}(\mathcal{M}, \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}))$$

and

$$\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) = \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{M}, \mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, \mathcal{L}))$$

functorially in $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$.

Proof. Omitted. Hint: On the graded level we have seen this is true in Lemma 24.8.2. Thus it suffices to check the isomorphisms are compatible with differentials which can be done by a computation on the level of local sections. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ and $\mathcal{B}$ be a sheaves of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. As a special case of the above, suppose we are given a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of differential graded $\mathcal{O}$-algebras. Then we obtain a functor and a functor of differential graded categories

$$\otimes_{\mathcal{A},\varphi}\mathcal{B} : \text{Mod}(\mathcal{A}, \text{d}) \longrightarrow \text{Mod}(\mathcal{B}, \text{d}) \quad \text{and} \quad \otimes_{\mathcal{A},\varphi} \mathcal{B} : \text{Mod}^{dg}(\mathcal{A}, \text{d}) \longrightarrow \text{Mod}^{dg}(\mathcal{B}, \text{d})$$

On the other hand, we have the restriction functors

$$res_{\varphi} : \text{Mod}(\mathcal{B}, \text{d}) \longrightarrow \text{Mod}(\mathcal{A}, \text{d}) \quad \text{and} \quad res_{\varphi} : \text{Mod}^{dg}(\mathcal{B}, \text{d}) \longrightarrow \text{Mod}^{dg}(\mathcal{A}, \text{d})$$

We can use the lemma above to show these functors are adjoint to each other (as usual with restriction and base change). Namely, let us write ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$ for $\mathcal{B}$ viewed as a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. Then for any right differential graded $\mathcal{B}$-module $\mathcal{L}$ we have

$$\mathcal{H}om^{dg}_{\mathcal{B}}({}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}, \mathcal{L}) = res_{\varphi}(\mathcal{L})$$

as right differential graded $\mathcal{A}$-modules. Thus Lemma 24.8.2 tells us that we have a functorial isomorphism

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{B}, \text{d})}(\mathcal{M} \otimes_{\mathcal{A},\varphi} \mathcal{B}, \mathcal{L}) = \text{Hom}_{\text{Mod}^{dg}(\mathcal{A}, \text{d})}(\mathcal{M}, res_{\varphi}(\mathcal{L}))$$

We usually drop the dependence on φ in this formula if it is clear from context. In the same manner we obtain the equality

$$\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{L}) = \mathcal{H}om^{dg}_{\mathcal{A}}(\mathcal{M}, \mathcal{L})$$

of graded $\mathcal{O}$-modules.

24.18. Pull and push for sheaves of differential graded modules

0FRT Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$$

of differential graded $f^{-1}\mathcal{O}_\mathcal{D}$-algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map $\varphi : f^*\mathcal{B} \to \mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras or equivalently φ can be viewed as a map

$$\varphi : \mathcal{B} \to f_*\mathcal{A}$$

of differential graded $\mathcal{O}_\mathcal{D}$-algebras. See Remark 24.12.2.

Let us define a functor

$$f_* : \text{Mod}(\mathcal{A}, \text{d}) \longrightarrow \text{Mod}(\mathcal{B}, \text{d})$$

Given a differential graded $\mathcal{A}$-module $\mathcal{M}$ we define $f_*\mathcal{M}$ to be the graded $\mathcal{B}$-module constructed in Section 24.9 with differential given by the maps $f_*d : f_*\mathcal{M}^n \to f_*\mathcal{M}^{n+1}$. The construction is clearly functorial in $\mathcal{M}$ and we obtain our functor.

Let us define a functor

$$f^* : \text{Mod}(\mathcal{B}, \text{d}) \longrightarrow \text{Mod}(\mathcal{A}, \text{d})$$

Given a differential graded $\mathcal{B}$-module $\mathcal{N}$ we define $f^*\mathcal{N}$ to be the graded $\mathcal{A}$-module constructed in Section 24.9. Recall that

$$f^*\mathcal{N} = f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{B}} \mathcal{A}$$

Since $f^{-1}\mathcal{N}$ comes with the differentials $f^{-1}\text{d} : f^{-1}\mathcal{N}^n \to f^{-1}\mathcal{N}^{n+1}$ we can view this tensor product as an example of the tensor product discussed in Section 24.17 which provides us with a differential. The construction is clearly functorial in $\mathcal{N}$ and we obtain our functor f^*.

The functors f_* and f^* are readily enhanced to give functors of differential graded categories

$$f_* : \text{Mod}^{dg}(\mathcal{A}, \text{d}) \longrightarrow \text{Mod}^{dg}(\mathcal{B}, \text{d}) \quad \text{and} \quad f^* : \text{Mod}^{dg}(\mathcal{B}, \text{d}) \longrightarrow \text{Mod}^{dg}(\mathcal{A}, \text{d})$$

which do the same thing on underlying objects and are defined by functoriality of the constructions on homogenous morphisms of degree n.

0FRU Lemma 24.18.1. In the situation above we have

$$\text{Hom}_{\text{Mod}^{dg}(\mathcal{B},\text{d})}(\mathcal{N}, f_*\mathcal{M}) = \text{Hom}_{\text{Mod}^{dg}(\mathcal{A},\text{d})}(f^*\mathcal{N}, \mathcal{M})$$

Proof. Omitted. Hints: This is true for the underlying graded categories by Lemma 24.9.1. A calculation shows that these isomorphisms are compatible with differentials. $\square$

24.19. Localization and sheaves of differential graded modules

0FRV Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$ and denote

$$j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

the corresponding localization morphism (Modules on Sites, Section 18.19). Below we will use the following fact: for $\mathcal{O}_U$-modules $\mathcal{M}_i$, $i = 1, 2$ and a $\mathcal{O}$-module $\mathcal{A}$ there is a canonical map

$$j_! : \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U, \mathcal{M}_2) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}, j_!\mathcal{M}_2)$$

Namely, we have $j_!(\mathcal{M}_1 \otimes_{\mathcal{O}_U} \mathcal{A}|_U) = j_!\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{A}$ by Modules on Sites, Lemma 18.27.9.

Let $\mathcal{A}$ be a differential graded $\mathcal{O}$-algebra. We will denote $\mathcal{A}_U$ the restriction of $\mathcal{A}$ to $\mathcal{C}/U$, in other words, we have $\mathcal{A}_U = j^*\mathcal{A} = j^{-1}\mathcal{A}$. In Section 24.18 we have constructed adjoint functors

$$j_* : \mathrm{Mod}^{dg}(\mathcal{A}_U, \mathrm{d}) \longrightarrow \mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d}) \quad \text{and} \quad j^* : \mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow \mathrm{Mod}^{dg}(\mathcal{A}_U, \mathrm{d})$$

with j^* left adjoint to j_*. We claim there is in addition an exact functor

$$j_! : \mathrm{Mod}^{dg}(\mathcal{A}_U, \mathrm{d}) \longrightarrow \mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d})$$

right adjoint to j_*. Namely, given a differential graded $\mathcal{A}_U$-module $\mathcal{M}$ we define $j_!\mathcal{M}$ to be the graded $\mathcal{A}$-module constructed in Section 24.10 with differentials $j_!\mathrm{d} : j_!\mathcal{M}^n \to j_!\mathcal{M}^{n+1}$. Given a homogeneous map $f : \mathcal{M} \to \mathcal{M}'$ of degree n of differential graded $\mathcal{A}_U$-modules, we obtain a homogeneous map $j_!f : j_!\mathcal{M} \to j_!\mathcal{M}'$ of degree n of differential graded $\mathcal{A}$-modules. We omit the straightforward verification that this construction is compatible with differentials. Thus we obtain our functor.

0FRW **Lemma 24.19.1.** In the situation above we have

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},\mathrm{d})}(j_!\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A}_U,\mathrm{d})}(\mathcal{M}, j^*\mathcal{N})$$

Proof. Omitted. Hint: We have seen in Lemma 24.10.1 that the lemma is true on graded level. Thus all that needs to be checked is that the resulting isomorphism is compatible with differentials. □

0FRX **Lemma 24.19.2.** In the situation above, let $\mathcal{M}$ be a right differential graded $\mathcal{A}_U$-module and let $\mathcal{N}$ be a left differential graded $\mathcal{A}$-module. Then

$$j_!\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} = j_!(\mathcal{M} \otimes_{\mathcal{A}_U} \mathcal{N}|_U)$$

as complexes of $\mathcal{O}$-modules functorially in $\mathcal{M}$ and $\mathcal{N}$.

Proof. As graded modules, this follows from Lemma 24.10.2. We omit the verification that this isomorphism is compatible with differentials. □

24.20. Shift functors on sheaves of differential graded modules

0FRY Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module. Let $k \in \mathbf{Z}$. We define the kth shift of $\mathcal{M}$, denoted $\mathcal{M}[k]$, as follows

(1) as a graded $\mathcal{A}$-module we let $\mathcal{M}[k]$ be as defined in Section 24.11,
(2) the differential $d_{\mathcal{M}[k]} : (\mathcal{M}[k])^n \to (\mathcal{M}[k])^{n+1}$ is defined to be $(-1)^k\mathrm{d}_{\mathcal{M}} : \mathcal{M}^{n+k} \to \mathcal{M}^{n+k+1}$.

For a homomorphism $f : \mathcal{L} \to \mathcal{M}$ of $\mathcal{A}$-modules homogeneous of degree n, we let $f[k] : \mathcal{L}[k] \to \mathcal{M}[k]$ be given by the same component maps as f. Then $f[k]$ is a homogeneous $\mathcal{A}$-module map of degree n. This gives a map

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}, \mathcal{M}) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}[k], \mathcal{M}[k])$$

compatible with differentials (it follows from the fact that the signs of the differentials of $\mathcal{L}$ and $\mathcal{M}$ are changed by the same amount). These choices are compatible with the choice in Differential Graded Algebra, Definition 22.4.3. It is clear that we have defined a functor

$$[k] : \mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d}) \longrightarrow \mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d})$$

of differential graded categories and that we have $[k + l] = [k] \circ [l]$.

We claim that the isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}, \mathcal{M}[k]) = \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L}, \mathcal{M})[k]$$

defined in Section 24.11 on underlying graded modules is compatible with the differentials. To see this, suppose we have a right $\mathcal{A}$-module map $f : \mathcal{L} \to \mathcal{M}[k]$ homogeneous of degree n; this is an element of degree n of the LHS. Denote $f' : \mathcal{L} \to \mathcal{M}$ the homogeneous $\mathcal{A}$-module map of degree $n+k$ with the same component maps as f. By our conventions, this is the corresponding element of degree n of the RHS. By definition of the differential of LHS we obtain

$$\mathrm{d}_{LHS}(f) = \mathrm{d}_{\mathcal{M}[k]} \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}} = (-1)^k \mathrm{d}_{\mathcal{M}} \circ f - (-1)^n f \circ \mathrm{d}_{\mathcal{L}}$$

and for the differential on the RHS we obtain

$$\mathrm{d}_{RHS}(f') = (-1)^k \left(\mathrm{d}_{\mathcal{M}} \circ f' - (-1)^{n+k} f' \circ \mathrm{d}_{\mathcal{L}}\right) = (-1)^k \mathrm{d}_{\mathcal{M}} \circ f' - (-1)^n f' \circ \mathrm{d}_{\mathcal{L}}$$

These maps have the same component maps and the proof is complete.

24.21. The homotopy category

0FRZ This section is the analogue of Differential Graded Algebra, Section 22.5.

0FS0 Definition 24.21.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $f, g : \mathcal{M} \to \mathcal{N}$ be homomorphisms of differential graded $\mathcal{A}$-modules. A homotopy between f and g is a graded $\mathcal{A}$-module map $h : \mathcal{M} \to \mathcal{N}$ homogeneous of degree -1 such that

$$f - g = \mathrm{d}_{\mathcal{N}} \circ h + h \circ \mathrm{d}_{\mathcal{M}}$$

If a homotopy exists, then we say f and g are homotopic.

In the situation of the definition, if we have maps $a : \mathcal{K} \to \mathcal{M}$ and $c : \mathcal{N} \to \mathcal{L}$ then we see that

$$\begin{matrix} h \text{ is a homotopy} \\ \text{between } f \text{ and } g \end{matrix} \Rightarrow \begin{matrix} c \circ h \circ a \text{ is a homotopy} \\ \text{between } c \circ f \circ a \text{ and } c \circ g \circ a \end{matrix}$$

Thus we can define composition of homotopy classes of morphisms in $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$.

0FS1 Definition 24.21.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The homotopy category, denoted $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$, is the category whose objects are the objects of $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ and whose morphisms are homotopy classes of homomorphisms of differential graded $\mathcal{A}$-modules.

The notation $K(\text{Mod}(\mathcal{A}, \text{d}))$ is not standard but at least is consistent with the use of $K(-)$ in other places of the Stacks project.

In Differential Graded Algebra, Definition 22.26.3 we have defined what we mean by the category of complexes $\text{Comp}(\mathcal{S})$ and the homotopy category $K(\mathcal{S})$ of a differential graded category $\mathcal{S}$. Applying this to the differential graded category $\text{Mod}^{dg}(\mathcal{A}, \text{d})$ we obtain

$$\text{Mod}(\mathcal{A}, \text{d}) = \text{Comp}(\text{Mod}^{dg}(\mathcal{A}, \text{d}))$$

(see discussion in Section 24.14) and we obtain

$$K(\text{Mod}(\mathcal{A}, \text{d})) = K(\text{Mod}^{dg}(\mathcal{A}, \text{d}))$$

To see that this last equality is true, note that we have the equality

$$\text{d}_{\text{Hom}_{\text{Mod}^{dg}(\mathcal{A},\text{d})}(\mathcal{M},\mathcal{N})}(h) = \text{d}_\mathcal{N} \circ h + h \circ \text{d}_\mathcal{M}$$

when h is as in Definition 24.21.1. We omit the details.

0FS2 Lemma 24.21.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The homotopy category $K(\text{Mod}(\mathcal{A}, \text{d}))$ has direct sums and products.

Proof. Omitted. Hint: Just use the direct sums and products as in Lemma 24.13.2. This works because we saw that these functors commute with the forgetful functor to the category of graded $\mathcal{A}$-modules and because $\prod$ and $\bigoplus$ are exact functors on the category of families of abelian groups. $\square$

24.22. Cones and triangles

0FS3 In this section we use the material from Differential Graded Algebra, Section 22.27 to conclude that the homotopy category of the category of differential graded $\mathcal{A}$-modules is a triangulated category.

0FS4 Lemma 24.22.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The differential graded category $\text{Mod}^{dg}(\mathcal{A}, \text{d})$ satisfies axioms (A) and (B) of Differential Graded Algebra, Section 22.27.

Proof. Suppose given differential graded $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$. Consider the differential graded $\mathcal{A}$-module $\mathcal{M} \oplus \mathcal{N}$ defined in the obvious manner. Then the coprojections $i : \mathcal{M} \to \mathcal{M} \oplus \mathcal{N}$ and $j : \mathcal{N} \to \mathcal{M} \oplus \mathcal{N}$ and the projections $p : \mathcal{M} \oplus \mathcal{N} \to \mathcal{N}$ and $q : \mathcal{M} \oplus \mathcal{N} \to \mathcal{M}$ are morphisms of differential graded $\mathcal{A}$-modules. Hence i, j, p, q are homogeneous of degree 0 and closed, i.e., $\text{d}(i) = 0$, etc. Thus this direct sum is a differential graded sum in the sense of Differential Graded Algebra, Definition 22.26.4. This proves axiom (A).

Axiom (B) was shown in Section 24.20. $\square$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Recall that a sequence

$$0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{N} \to 0$$

in $\text{Mod}(\mathcal{A}, \text{d})$ is called an admissible short exact sequence (in Differential Graded Algebra, Section 22.27) if it is split in $\text{Mod}(\mathcal{A})$. In other words, if it is split as a sequence of graded $\mathcal{A}$-modules. Denote $s : \mathcal{N} \to \mathcal{L}$ and $\pi : \mathcal{L} \to \mathcal{K}$ graded

$\mathcal{A}$-module splittings. Combining Lemma 24.22.1 and Differential Graded Algebra, Lemma 22.27.1 we obtain a triangle

$$\mathcal{K} \to \mathcal{L} \to \mathcal{N} \to \mathcal{K}[1]$$

where the arrow $\mathcal{N} \to \mathcal{K}[1]$ in the proof of Differential Graded Algebra, Lemma 22.27.1 is constructed as

$$\delta = \pi \circ \mathrm{d}_{\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A},\mathrm{d})}(\mathcal{L},\mathcal{M})}(s) = \pi \circ \mathrm{d}_{\mathcal{L}} \circ s - \pi \circ s \circ \mathrm{d}_{\mathcal{N}} = \pi \circ \mathrm{d}_{\mathcal{L}} \circ s$$

with apologies for the horrendous notation. In any case, we see that in our setting the boundary map δ as constructed in Differential Graded Algebra, Lemma 22.27.1 agrees on underlying complexes of $\mathcal{O}$-modules with the usual boundary map used throughout the Stacks project for termwise split short exact sequences of complexes, see Derived Categories, Definition 13.9.9.

0FS5 Definition 24.22.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $f : \mathcal{K} \to \mathcal{L}$ be a homomorphism of differential graded $\mathcal{A}$-modules. The cone of f is the differential graded $\mathcal{A}$-module $C(f)$ defined as follows:

(1) the underlying complex of $\mathcal{O}$-modules is the cone of the corresponding map $f : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ of complexes of $\mathcal{A}$-modules, i.e., we have $C(f)^n = \mathcal{L}^n \oplus \mathcal{K}^{n+1}$ and differential
$$d_{C(f)} = \begin{pmatrix} \mathrm{d}_{\mathcal{L}} & f \\ 0 & -\mathrm{d}_{\mathcal{K}} \end{pmatrix}$$
(2) the multiplication map
$$C(f)^n \times \mathcal{A}^m \to C(f)^{n+m}$$
is the direct sum of the multiplication map $\mathcal{L}^n \times \mathcal{A}^m \to \mathcal{L}^{n+m}$ and the multiplication map $\mathcal{K}^{n+1} \times \mathcal{A}^m \to \mathcal{K}^{n+1+m}$.

It comes equipped with canonical hommorphisms of differential graded $\mathcal{A}$-modules $i : \mathcal{L} \to C(f)$ and $p : C(f) \to \mathcal{K}[1]$ induced by the obvious maps.

Observe that in the situation of the definition the sequence

$$0 \to \mathcal{L} \to C(f) \to \mathcal{K}[1] \to 0$$

is an addmissible short exact sequence.

0FS6 Lemma 24.22.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The differential graded category $\mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d})$ satisfies axiom (C) formulated in Differential Graded Algebra, Situation 22.27.2.

Proof. Let $f : \mathcal{K} \to \mathcal{L}$ be a homomorphism of differential graded $\mathcal{A}$-modules. By the above we have an admissible short exact sequence

$$0 \to \mathcal{L} \to C(f) \to \mathcal{K}[1] \to 0$$

To finish the proof we have to show that the boundary map

$$\delta : \mathcal{K}[1] \to \mathcal{L}[1]$$

associated to this (see discussion above) is equal to $f[1]$. For the section $s : \mathcal{K}[1] \to C(f)$ we use in degree n the embeddding $\mathcal{K}^{n+1} \to C(f)^n$. Then in degree n the map π is given by the projections $C(f)^n \to \mathcal{L}^n$. Then finally we have to compute

$$\delta = \pi \circ \mathrm{d}_{C(f)} \circ s$$

(see discussion above). In matrix notation this is equal to

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}_\mathcal{L} & f \\ 0 & -\mathrm{d}_\mathcal{K} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f$$

as desired. $\square$

At this point we know that all lemmas proved in Differential Graded Algebra, Section 22.27 are valid for the differential graded category $\mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d})$. In particular, we have the following.

0FS7 Proposition 24.22.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The homotopy category $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$ is a triangulated category where

(1) the shift functors are those constructed in Section 24.20,
(2) the distinghuished triangles are those triangles in $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$ which are isomorphic as a triangle to a triangle

$$\mathcal{K} \to \mathcal{L} \to \mathcal{N} \xrightarrow{\delta} \mathcal{K}[1], \qquad \delta = \pi \circ \mathrm{d}_\mathcal{L} \circ s$$

constructed from an admissible short exact sequence $0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{N} \to 0$ in $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ above.

Proof. Recall that $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d})) = K(\mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d}))$, see Section 24.21. Having said this, the proposition follows from Lemmas 24.22.1 and 24.22.3 and Differential Graded Algebra, Proposition 22.27.16. $\square$

0FS8 Remark 24.22.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $C = C(\mathrm{id}_\mathcal{A})$ be the cone on the identity map $\mathcal{A} \to \mathcal{A}$ viewed as a map of differential graded $\mathcal{A}$-modules. Then

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A}, \mathrm{d})}(C, \mathcal{M}) = \{(x, y) \in \Gamma(\mathcal{C}, \mathcal{M}^0) \times \Gamma(\mathcal{C}, \mathcal{M}^{-1}) \mid x = \mathrm{d}(y)\}$$

where the map from left to right sends f to the pair (x, y) where x is the image of the global section $(0, 1)$ of $C^{-1} = \mathcal{A}^{-1} \oplus \mathcal{A}^0$ and where y is the image of the global section $(1, 0)$ of $C^0 = \mathcal{A}^0 \oplus \mathcal{A}^1$.

0FS9 Lemma 24.22.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a differential graded $\mathcal{O}$-algebra. The category $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ is a Grothendieck abelian category.

Proof. By Lemma 24.13.2 and the definition of a Grothendieck abelian category (Injectives, Definition 19.10.1) it suffices to show that $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ has a generator. For every object U of $\mathcal{C}$ we denote C_U the cone on the identity map $\mathcal{A}_U \to \mathcal{A}_U$ as in Remark 24.22.5. We claim that

$$\mathcal{G} = \bigoplus\nolimits_{k, U} j_{U!} C_U[k]$$

is a generator where the sum is over all objects U of $\mathcal{C}$ and $k \in \mathbf{Z}$. Indeed, given a differential graded $\mathcal{A}$-module $\mathcal{M}$ if there are no nonzero maps from $\mathcal{G}$ to $\mathcal{M}$, then we see that for all k and U we have

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A})}(j_{U!} C_U[k], \mathcal{M}) \\ &\quad = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{A}_U)}(C_U[k], \mathcal{M}|_U) \\ &\quad = \{(x, y) \in \mathcal{M}^{-k}(U) \times \mathcal{M}^{-k-1}(U) \mid x = \mathrm{d}(y)\} \end{aligned}$$

is equal to zero. Hence $\mathcal{M}$ is zero. $\square$

24.23. Flat resolutions

0FSA This section is the analogue of Differential Graded Algebra, Section 22.20.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let us call a right differential graded $\mathcal{A}$-module $\mathcal{P}$ good if

(1) the functor $\mathcal{N} \mapsto \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$ is exact on the category of graded left $\mathcal{A}$-modules,
(2) if $\mathcal{N}$ is an acyclic differential graded left $\mathcal{A}$-module, then $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$ is acyclic,
(3) for any morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi and any differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$ and any map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}_{\mathcal{D}}$-algebras we have properties (1) and (2) for the pullback $f^*\mathcal{P}$ (Section 24.18) viewed as a differential graded $\mathcal{A}'$-module.

The first condition means that $\mathcal{P}$ is flat as a right graded $\mathcal{A}$-module, the second condition means that $\mathcal{P}$ is K-flat in the sense of Spaltenstein (see Cohomology on Sites, Section 21.17), and the third condition is that this holds after arbitrary base change.

Perhaps surprisingly, there are many good modules.

0FSB **Lemma 24.23.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then $j_!\mathcal{A}_U$ is a good differential graded $\mathcal{A}$-module.

Proof. Let $\mathcal{N}$ be a left graded $\mathcal{A}$-module. By Lemma 24.10.2 we have

$$j_!\mathcal{A}_U \otimes_{\mathcal{A}} \mathcal{N} = j_!(\mathcal{A}_U \otimes_{\mathcal{A}_U} \mathcal{N}|_U) = j_!(\mathcal{N}_U)$$

as graded modules. Since both restriction to U and $j_!$ are exact this proves condition (1). The same argument works for (2) using Lemma 24.19.2.

Consider a morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi, a differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$, and a map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}$-algebras. We have to show that

$$f^*j_!\mathcal{A}_U = f^{-1}j_!\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}'$$

satisfies (1) and (2) for the ringed topos $(Sh(\mathcal{C}'), \mathcal{O}')$ endowed with the sheaf of differential graded $\mathcal{O}'$-algebras $\mathcal{A}'$. To prove this we may replace $(Sh(\mathcal{C}), \mathcal{O})$ and $(Sh(\mathcal{C}'), \mathcal{O}')$ by equivalent ringed topoi. Thus by Modules on Sites, Lemma 18.7.2 we may assume that f comes from a morphism of sites $f : \mathcal{C} \to \mathcal{C}'$ given by the continuous functor $u : \mathcal{C} \to \mathcal{C}'$. In this case, set $U' = u(U)$ and denote $j' : Sh(\mathcal{C}'/U') \to Sh(\mathcal{C}')$ the corresponding localization morphism. We obtain a commutative square of morphisms of ringed topoi

$$\begin{tikzcd}
(Sh(\mathcal{C}'/U'), \mathcal{O}'_{U'}) \arrow[r, "{(j', (j')^\sharp)}"'] \arrow[d, "{(f', (f')^\sharp)}"'] & (Sh(\mathcal{C}'), \mathcal{O}') \arrow[d, "{(f, f^\sharp)}"] \\
(Sh(\mathcal{C}/U), \mathcal{O}_U) \arrow[r, "{(j, j^\sharp)}"] & (Sh(\mathcal{C}), \mathcal{O}).
\end{tikzcd}$$

and we have $f'_*(j')^{-1} = j^{-1}f_*$. See Modules on Sites, Lemma 18.20.1. By uniqueness of adjoints we obtain $f^{-1}j_! = j'_!(f')^{-1}$. Thus we obtain

$$\begin{aligned} f^*j_!\mathcal{A}_U &= f^{-1}j_!\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}' \\ &= j'_!(f')^{-1}\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}} \mathcal{A}' \\ &= j'_!\left((f')^{-1}\mathcal{A}_U \otimes_{f^{-1}\mathcal{A}|_{U'}} \mathcal{A}'|_{U'}\right) \\ &= j'_!\mathcal{A}'_{U'} \end{aligned}$$

The first equation is the definition of the pullback of $j_!\mathcal{A}_U$ to a differential graded module over $\mathcal{A}'$. The second equation because $f^{-1}j_! = j'_!(f')^{-1}$. The third equation by Lemma 24.19.2 applied to the ringed site $(\mathcal{C}', f^{-1}\mathcal{O})$ with sheaf of differential graded algebras $f^{-1}\mathcal{A}$ and with differential graded modules $(f')^{-1}\mathcal{A}_U$ on $\mathcal{C}'/U'$ and $\mathcal{A}'$ on $\mathcal{C}'$. The fourth equation holds because of course we have $(f')^{-1}\mathcal{A}_U = f^{-1}\mathcal{A}|_{U'}$. Hence we see that the pullback is another module of the same kind and we've proven conditions (1) and (2) for it above. □

0FSC Lemma 24.23.2. et $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $0 \to \mathcal{P} \to \mathcal{P}' \to \mathcal{P}'' \to 0$ be an admissible short exact sequence of differential graded $\mathcal{A}$-modules. If two-out-of-three of these modules are good, so is the third.

Proof. For condition (1) this is immediate as the sequence is a direct sum at the graded level. For condition (2) note that for any left differential graded $\mathcal{A}$-module, the sequence

$$0 \to \mathcal{P} \otimes_\mathcal{A} \mathcal{N} \to \mathcal{P}' \otimes_\mathcal{A} \mathcal{N} \to \mathcal{P}'' \otimes_\mathcal{A} \mathcal{N} \to 0$$

is an admissible short exact sequence of differential graded $\mathcal{O}$-modules (since forgetting the differential the tensor product is just taken in the category of graded modules). Hence if two out of three are exact as complexes of $\mathcal{O}$-modules, so is the third. Finally, the same argument shows that given a morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi, a differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$, and a map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}$-algebras we have that

$$0 \to f^*\mathcal{P} \to f^*\mathcal{P}' \to f^*\mathcal{P}'' \to 0$$

is an admissible short exact sequence of differential graded $\mathcal{A}'$-modules and the same argument as above applies here. □

0FSD Lemma 24.23.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. An arbitrary direct sum of good differential graded $\mathcal{A}$-modules is good. A filtered colimit of good differential graded $\mathcal{A}$-modules is good.

Proof. Omitted. Hint: direct sums and filtered colimits commute with tensor products and with pullbacks. □

0FSE Lemma 24.23.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module. There exists a homomorphism $\mathcal{P} \to \mathcal{M}$ of differential graded $\mathcal{A}$-modules with the following properties

1. $\mathcal{P} \to \mathcal{M}$ is surjective,
2. $\mathrm{Ker}(d_\mathcal{P}) \to \mathrm{Ker}(d_\mathcal{M})$ is surjective, and
3. $\mathcal{P}$ is good.

Proof. Consider triples (U, k, x) where U is an object of $\mathcal{C}$, $k \in \mathbf{Z}$, and x is a section of $\mathcal{M}^k$ over U with $d_{\mathcal{M}}(x) = 0$. Then we obtain a unique morphism of differential graded $\mathcal{A}_U$-modules $\varphi_x : \mathcal{A}_U[-k] \to \mathcal{M}|_U$ mapping 1 to x. This is adjoint to a morphism $\psi_x : j_{U!}\mathcal{A}_U[-k] \to \mathcal{M}$. Observe that $1 \in \mathcal{A}_U(U)$ corresponds to a section $1 \in j_{U!}\mathcal{A}_U[-k](U)$ of degree k whose differential is zero and which is mapped to x by ψ_x. Thus if we consider the map

$$\bigoplus\nolimits_{(U,k,x)} j_{U!}\mathcal{A}_U[-k] \longrightarrow \mathcal{M}$$

then we will have conditions (2) and (3). Namely, the objects $j_{U!}\mathcal{A}_U[-k]$ are good (Lemma 24.23.1) and any direct sum of good objects is good (Lemma 24.23.3).

Next, consider triples (U, k, x) where U is an object of $\mathcal{C}$, $k \in \mathbf{Z}$, and x is a section of $\mathcal{M}^k$ (not necessarily annihilated by the differential). Then we can consider the cone C_U on the identity map $\mathcal{A}_U \to \mathcal{A}_U$ as in Remark 24.22.5. The element x will determine a map $\varphi_x : C_U[-k-1] \to \mathcal{A}_U$, see Remark 24.22.5. Now, since we have an admissible short exact sequence

$$0 \to \mathcal{A}_U \to C_U \to \mathcal{A}_U[1] \to 0$$

we conclude that $j_{U!}C_U$ is a good module by Lemma 24.23.2 and the already used Lemma 24.23.1. As above we conclude that the direct sum of the maps ψ_x : $j_{U!}C_U \to \mathcal{M}$ adjoint to the φ_x

$$\bigoplus\nolimits_{(U,k,x)} j_{U!}C_U \longrightarrow \mathcal{M}$$

is surjective. Taking the direct sum with the map produced in the first paragraph we conclude. □

0FSF Remark 24.23.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A sheaf of graded sets on $\mathcal{C}$ is a sheaf of sets $\mathcal{S}$ endowed with a map $\deg : \mathcal{S} \to \underline{\mathbf{Z}}$ of sheaves of sets. Let us denote $\mathcal{O}[\mathcal{S}]$ the graded $\mathcal{O}$-module which is the free $\mathcal{O}$-module on the graded sheaf of sets $\mathcal{S}$. More precisely, the nth graded part of $\mathcal{O}[\mathcal{S}]$ is the sheafification of the rule

$$U \longmapsto \bigoplus\nolimits_{s \in \mathcal{S}(U),\ \deg(s)=n} s \cdot \mathcal{O}(U)$$

With zero differential we also may consider this as a differential graded $\mathcal{O}$-module. Let $\mathcal{A}$ be a sheaf of graded $\mathcal{O}$-algebras Then we similarly define $\mathcal{A}[\mathcal{S}]$ to be the graded $\mathcal{A}$-module whose nth graded part is the sheafification of the rule

$$U \longmapsto \bigoplus\nolimits_{s \in \mathcal{S}(U)} s \cdot \mathcal{A}^{n-\deg(s)}(U)$$

If $\mathcal{A}$ is a differential graded $\mathcal{O}$-algebra, the we turn this into a differential graded $\mathcal{O}$-module by setting $d(s) = 0$ for all $s \in \mathcal{S}(U)$ and sheafifying.

0FSG Lemma 24.23.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a differential graded $\mathcal{A}$-algebra. Let $\mathcal{S}$ be a sheaf of graded sets on $\mathcal{C}$. Then the free graded module $\mathcal{A}[\mathcal{S}]$ on $\mathcal{S}$ endowed with differential as in Remark 24.23.5 is a good differential graded $\mathcal{A}$-module.

Proof. Let $\mathcal{N}$ be a left graded $\mathcal{A}$-module. Then we have

$$\mathcal{A}[\mathcal{S}] \otimes_{\mathcal{A}} \mathcal{N} = \mathcal{O}[\mathcal{S}] \otimes_{\mathcal{O}} \mathcal{N} = \mathcal{N}[\mathcal{S}]$$

where $\mathcal{N}[\mathcal{S}$ is the graded $\mathcal{O}$-module whose degree n part is the sheaf associated to the presheaf

$$U \longmapsto \bigoplus\nolimits_{s \in \mathcal{S}(U)} s \cdot \mathcal{N}^{n-\deg(s)}(U)$$

It is clear that $\mathcal{N} \to \mathcal{N}[\mathcal{S}]$ is an exact functor, hence $\mathcal{A}[\mathcal{S}$ is flat as a graded $\mathcal{A}$-module. Next, suppose that $\mathcal{N}$ is a differential graded left $\mathcal{A}$-module. Then we have

$$H^*(\mathcal{A}[\mathcal{S}] \otimes_\mathcal{A} \mathcal{N}) = H^*(\mathcal{O}[\mathcal{S}] \otimes_\mathcal{O} \mathcal{N})$$

as graded sheaves of $\mathcal{O}$-modules, which by the flatness (over $\mathcal{O}$) is equal to

$$H^*(\mathcal{N})[\mathcal{S}]$$

as a graded $\mathcal{O}$-module. Hence if $\mathcal{N}$ is acyclic, then $\mathcal{A}[\mathcal{S}] \otimes_\mathcal{A} \mathcal{N}$ is acyclic.

Finally, consider a morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi, a differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$, and a map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}$-algebras. Then it is straightforward to see that

$$f^*\mathcal{A}[\mathcal{S}] = \mathcal{A}'[f^{-1}\mathcal{S}]$$

which finishes the proof that our module is good. □

0FSH Lemma 24.23.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module. There exists a homomorphism $\mathcal{P} \to \mathcal{M}$ of differential graded $\mathcal{A}$-modules with the following properties

(1) $\mathcal{P} \to \mathcal{M}$ is a quasi-isomorphism, and
(2) $\mathcal{P}$ is good.

First proof. Let $\mathcal{S}_0$ be the sheaf of graded sets (Remark 24.23.5) whose degree n part is $\text{Ker}(\text{d}^n_\mathcal{M})$. Consider the homomorphism of differential graded modules

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{S}_0] \longrightarrow \mathcal{M}$$

where the left hand side is as in Remark 24.23.5 and the map sends a local section s of $\mathcal{S}_0$ to the corresponding local section of $\mathcal{M}^{\deg(s)}$ (which is in the kernel of the differential, so our map is a map of differential graded modules indeed). By construction the induced maps on cohomology sheaves $H^n(\mathcal{P}_0) \to H^n(\mathcal{M})$ are surjective. We are going to inductively construct maps

$$\mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \ldots \to \mathcal{M}$$

Observe that of course $H^*(\mathcal{P}_i) \to H^*(\mathcal{M})$ will be surjective for all i. Given $\mathcal{P}_i \to \mathcal{M}$ denote $\mathcal{S}_{i+1}$ the sheaf of graded sets whose degree n part is

$$\text{Ker}(\text{d}^{n+1}_{\mathcal{P}_i}) \times_{\mathcal{M}^{n+1}, \text{d}} \mathcal{M}^n$$

Then we set

$$\mathcal{P}_{i+1} = \mathcal{P}_i \oplus \mathcal{A}[\mathcal{S}_{i+1}]$$

as graded $\mathcal{A}$-module with differential and map to $\mathcal{M}$ defined as follows

(1) for local sections of $\mathcal{P}_i$ use the differential on $\mathcal{P}_i$ and the given map to $\mathcal{M}$,
(2) for a local section $s = (p, m)$ of $\mathcal{S}_{i+1}$ we set $\text{d}(s)$ equal to p viewed as a section of $\mathcal{P}_i$ of degree $\deg(s) + 1$ and we map s to m in $\mathcal{M}$, and
(3) extend the differential uniquely so that the Leibniz rule holds.

This makes sense because $\mathrm{d}(m)$ is the image of p and $\mathrm{d}(p) = 0$. Finally, we set $\mathcal{P} = \operatorname{colim} \mathcal{P}_i$ with the induced map to $\mathcal{M}$.

The map $\mathcal{P} \to \mathcal{M}$ is a quasi-isomorphism: we have $H^n(\mathcal{P}) = \operatorname{colim} H^n(\mathcal{P}_i)$ and for each i the map $H^n(\mathcal{P}_i) \to H^n(\mathcal{M})$ is surjective with kernel annihilated by the map $H^n(\mathcal{P}_i) \to H^n(\mathcal{P}_{i+1})$ by construction. Each $\mathcal{P}_i$ is good because $\mathcal{P}_0$ is good by Lemma 24.23.6 and each $\mathcal{P}_{i+1}$ is in the middle of the admissible short exact sequence $0 \to \mathcal{P}_i \to \mathcal{P}_{i+1} \to \mathcal{A}[\mathcal{S}_{i+1}] \to 0$ whose outer terms are good by induction. Hence $\mathcal{P}_{i+1}$ is good by Lemma 24.23.2. Finally, we conclude that $\mathcal{P}$ is good by Lemma 24.23.3. $\square$

Second proof. We urge the reader to read the proof of Differential Graded Algebra, Lemma 22.20.4 before reading this proof. Set $\mathcal{M} = \mathcal{M}_0$. We inductively choose short exact sequences

$$0 \to \mathcal{M}_{i+1} \to \mathcal{P}_i \to \mathcal{M}_i \to 0$$

where the maps $\mathcal{P}_i \to \mathcal{M}_i$ are chosen as in Lemma 24.23.4. This gives a "resolution"

$$\ldots \to \mathcal{P}_2 \xrightarrow{f_2} \mathcal{P}_1 \xrightarrow{f_1} \mathcal{P}_0 \to \mathcal{M} \to 0$$

Then we let $\mathcal{P}$ be the differential graded $\mathcal{A}$-module defined as follows

(1) as a graded $\mathcal{A}$-module we set $\mathcal{P} = \bigoplus_{a \leq 0} \mathcal{P}_{-a}[-a]$, i.e., the degree n part is given by $\mathcal{P}^n = \bigoplus_{a+b=n} \mathcal{P}_{-a}^b$,
(2) the differential on $\mathcal{P}$ is as in the construction of the total complex associated to a double complex given by
$$\mathrm{d}_{\mathcal{P}}(x) = f_{-a}(x) + (-1)^a \mathrm{d}_{\mathcal{P}_{-a}}(x)$$
for x a local section of $\mathcal{P}_{-a}^b$.

With these conventions $\mathcal{P}$ is indeed a differential graded $\mathcal{A}$-module; we omit the details. There is a map $\mathcal{P} \to \mathcal{M}$ of differential graded $\mathcal{A}$-modules which is zero on the summands $\mathcal{P}_{-a}[-a]$ for $a < 0$ and the given map $\mathcal{P}_0 \to \mathcal{M}$ for $a = 0$. Observe that we have

$$\mathcal{P} = \operatorname{colim}_i F_i\mathcal{P}$$

where $F_i\mathcal{P} \subset \mathcal{P}$ is the differential graded $\mathcal{A}$-submodule whose underlying graded $\mathcal{A}$-module is

$$F_i\mathcal{P} = \bigoplus\nolimits_{i \geq -a \geq 0} \mathcal{P}_{-a}[-a]$$

It is immediate that the maps

$$0 \to F_1\mathcal{P} \to F_2\mathcal{P} \to F_3\mathcal{P} \to \ldots \to \mathcal{P}$$

are all admissible monomorphisms and we have admissible short exact sequences

$$0 \to F_i\mathcal{P} \to F_{i+1}\mathcal{P} \to \mathcal{P}_{i+1}[i+1] \to 0$$

By induction and Lemma 24.23.2 we find that $F_i\mathcal{P}$ is a good differential graded $\mathcal{A}$-module. Since $\mathcal{P} = \operatorname{colim} F_i\mathcal{P}$ we find that $\mathcal{P}$ is good by Lemma 24.23.3.

Finally, we have to show that $\mathcal{P} \to \mathcal{M}$ is a quasi-isomorphism. If $\mathcal{C}$ has enough points, then this follows from the elementary Homology, Lemma 12.26.2 by checking on stalks. In general, we can argue as follows (this proof is far too long — there is an alternative argument by working with local sections as in the elementary proof but it is also rather long). Since filtered colimits are exact on the category of abelian sheaves, we have

$$H^d(\mathcal{P}) = \operatorname{colim} H^d(F_i\mathcal{P})$$

We claim that for each $i \geq 0$ and $d \in \mathbf{Z}$ we have (a) a short exact sequence

$$0 \to H^d(\mathcal{M}_{i+1}[i]) \to H^d(F_i\mathcal{P}) \to H^d(\mathcal{M}) \to 0$$

where the second arrow comes from $F_i\mathcal{P} \to \mathcal{P} \to \mathcal{M}$ and (b) the composition

$$H^d(\mathcal{M}_{i+1}[i]) \to H^d(F_i\mathcal{P}) \to H^d(F_{i+1}\mathcal{P})$$

is zero. It is clear that the claim suffices to finish the proof.

Proof of the claim. For any $i \geq 0$ there is a map $\mathcal{M}_{i+1}[i] \to F_i\mathcal{P}$ coming from the inclusion of $\mathcal{M}_{i+1}$ into $\mathcal{P}_i$ as the kernel of f_i. Consider the short exact sequence

$$0 \to \mathcal{M}_{i+1}[i] \to F_i\mathcal{P} \to C_i \to 0$$

of complexes of $\mathcal{O}$-modules defining C_i. Observe that $C_0 = \mathcal{M}_0 = \mathcal{M}$. Also, observe that C_i is the total complex associated to the double complex $C_i^{\bullet,\bullet}$ with columns

$$\mathcal{M}_i = \mathcal{P}_i/\mathcal{M}_{i+1}, \mathcal{P}_{i-1}, \ldots, \mathcal{P}_0$$

in degree $-i, -i+1, \ldots, 0$. There is a map of double complexes $C_i^{\bullet,\bullet} \to C_{i-1}^{\bullet,\bullet}$ which is 0 on the column in degree $-i$, is the surjection $\mathcal{P}_{i-1} \to \mathcal{M}_{i-1}$ in degree $-i+1$, and is the identity on the other columns. Hence there are maps of complexes

$$C_i \longrightarrow C_{i-1}$$

These maps are surjective quasi-isomorphisms because the kernel is the total complex on the double complex with columns $\mathcal{M}_i, \mathcal{M}_i$ in degrees $-i, -i+1$ and the identity map between these two columns. Using the resulting identifications $H^d(C_i) = H^d(C_{i-1} = \ldots = H^d(\mathcal{M})$ this already shows we get a long exact sequence

$$H^d(\mathcal{M}_{i+1}[i]) \to H^d(F_i\mathcal{P}) \to H^d(\mathcal{M}) \to H^{d+1}(\mathcal{M}_{i+1}[i])$$

from the short exact sequence of complexes above. However, we also have the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{i+2}[i+1] \arrow[r, "a"'] & T_{i+1} \arrow[r] & F_{i+1}\mathcal{P} \arrow[r] & C_{i+1} \arrow[d] \\
& \mathcal{M}_{i+1}[i] \arrow[r] \arrow[u, "b"] & F_i\mathcal{P} \arrow[r] \arrow[u] & C_i
\end{tikzcd}$$

where T_{i+1} is the total complex on the double complex with columns $\mathcal{P}_{i+1}, \mathcal{M}_{i+1}$ placed in degrees $-i-1$ and $-i$. In other words, T_{i+1} is a shift of the cone on the map $\mathcal{P}_{i+1} \to \mathcal{M}_{i+1}$ and we find that a is a quasi-isomorphism and the map $a^{-1} \circ b$ is a shift of the third map of the distinguished triangle in $D(\mathcal{O})$ associated to the short exact sequence

$$0 \to \mathcal{M}_{i+2} \to \mathcal{P}_{i+1} \to \mathcal{M}_{i+1} \to 0$$

The map $H^d(\mathcal{P}_{i+1}) \to H^d(\mathcal{M}_{i+1})$ is surjective because we chose our maps such that $\text{Ker}(\text{d}_{\mathcal{P}_{i+1}}) \to \text{Ker}(\text{d}_{\mathcal{M}_{i+1}})$ is surjective. Thus we see that $a^{-1} \circ b$ is zero on cohomology sheaves. This proves part (b) of the claim. Since T_{i+1} is the kernel of the surjective map of complexes $F_{i+1}\mathcal{P} \to C_i$ we find a map of long exact cohomology sequences

$$\begin{tikzcd} H^d(T_{i+1}) \arrow[r] & H^d(F_{i+1}\mathcal{P}) \arrow[r] & H^d(\mathcal{M}) \arrow[r] & H^{d+1}(T_{i+1}) \\ H^d(\mathcal{M}_{i+1}[i]) \arrow[r] \arrow[u] & H^d(F_i\mathcal{P}) \arrow[r] \arrow[u] & H^d(\mathcal{M}) \arrow[r] \arrow[u] & H^{d+1}(\mathcal{M}_{i+1}[i]) \arrow[u] \end{tikzcd}$$

Here we know, by the discussion above, that the vertical maps on the outside are zero. Hence the maps $H^d(F_{i+1}\mathcal{P}) \to H^d(\mathcal{M})$ are surjective and part (a) of the claim follows. More precisely, the claim follows for $i > 0$ and we leave the claim for $i = 0$ to the reader (actually it suffices to prove the claim for all $i \gg 0$ in order to get the lemma). □

0FSI Lemma 24.23.8. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{P}$ be a good acyclic right differential graded $\mathcal{A}$-module.

(1) for any differential graded left $\mathcal{A}$-module $\mathcal{N}$ the tensor product $\mathcal{P} \otimes_\mathcal{A} \mathcal{N}$ is acyclic,
(2) for any morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi and any differential graded $\mathcal{O}'$-algebra $\mathcal{A}'$ and any map $\varphi : f^{-1}\mathcal{A} \to \mathcal{A}'$ of differential graded $f^{-1}\mathcal{O}$-algebras the pullback $f^*\mathcal{P}$ is acyclic and good.

Proof. Proof of (1). By Lemma 24.23.7 we can choose a good left differential graded $\mathcal{Q}$ and a quasi-isomorphism $\mathcal{Q} \to \mathcal{N}$. Then $\mathcal{P} \otimes_\mathcal{A} \mathcal{Q}$ is acyclic because $\mathcal{Q}$ is good. Let $\mathcal{N}'$ be the cone on the map $\mathcal{Q} \to \mathcal{N}$. Then $\mathcal{P} \otimes_\mathcal{A} \mathcal{N}'$ is acyclic because $\mathcal{P}$ is good and because $\mathcal{N}'$ is acyclic (as the cone on a quasi-isomorphism). We have a distinguished triangle

$$\mathcal{Q} \to \mathcal{N} \to \mathcal{N}' \to \mathcal{Q}[1]$$

in $K(\text{Mod}(\mathcal{A}, \text{d}))$ by our construction of the triangulated structure. Since $\mathcal{P} \otimes_\mathcal{A} -$ sends distinguished triangles to distinguished triangles, we obtain a distinguished triangle

$$\mathcal{P} \otimes_\mathcal{A} \mathcal{Q} \to \mathcal{P} \otimes_\mathcal{A} \mathcal{N} \to \mathcal{P} \otimes_\mathcal{A} \mathcal{N}' \to \mathcal{P} \otimes_\mathcal{A} \mathcal{Q}[1]$$

in $K(\text{Mod}(\mathcal{O}))$. Thus we conclude.

Proof of (2). Observe that $f^*\mathcal{P}$ is good by our definition of good modules. Recall that $f^*\mathcal{P} = f^{-1}\mathcal{P} \otimes_{f^{-1}\mathcal{A}} \mathcal{A}'$. Then $f^{-1}\mathcal{P}$ is a good acyclic (because f^{-1} is exact) differential graded $f^{-1}\mathcal{A}$-module. Hence we see that $f^*\mathcal{P}$ is acyclic by part (1). □

24.24. The differential graded hull of a graded module

0FSJ The differential graded hull of a graded module $\mathcal{N}$ is the result of applying the functor G in the following lemma.

0FSK Lemma 24.24.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The forgetful functor $F : \text{Mod}(\mathcal{A}, \text{d}) \to \text{Mod}(\mathcal{A})$ has a left adjoint $G : \text{Mod}(\mathcal{A}) \to \text{Mod}(\mathcal{A}, \text{d})$.

Proof. To prove the existence of G we can use the adjoint functor theorem, see Categories, Theorem 4.25.3 (observe that we have switched the roles of F and G). The exactness conditions on F are satisfied by Lemma 24.13.2. The set theoretic condition can be seen as follows: suppose given a graded $\mathcal{A}$-module $\mathcal{N}$. Then for any map

$$\varphi : \mathcal{N} \longrightarrow F(\mathcal{M})$$

we can consider the smallest differential graded $\mathcal{A}$-submodule $\mathcal{M}' \subset \mathcal{M}$ with $\mathrm{Im}(\varphi) \subset F(\mathcal{M}')$. It is clear that $\mathcal{M}'$ is the image of the map of graded $\mathcal{A}$-modules

$$\mathcal{N} \oplus \mathcal{N}[-1] \otimes_{\mathcal{O}} \mathcal{A} \longrightarrow \mathcal{M}$$

defined by

$$(n, \sum n_i \otimes a_i) \longmapsto \varphi(n) + \sum \mathrm{d}(\varphi(n_i))a_i$$

because the image of this map is easily seen to be a differential graded submodule of $\mathcal{M}$. Thus the number of possible isomorphism classes of these $\mathcal{M}'$ is bounded and we conclude. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module and suppose we have a short exact sequence

$$0 \to \mathcal{N} \to F(\mathcal{M}) \to \mathcal{N}' \to 0$$

in $\mathrm{Mod}(\mathcal{A})$. Then we obtain a canonical graded $\mathcal{A}$-module homomorphism

$$\overline{\mathrm{d}} : \mathcal{N} \to \mathcal{N}'[1]$$

as follows: given a local section x of $\mathcal{N}$ denote $\overline{\mathrm{d}}(x)$ the image in $\mathcal{N}'$ of $\mathrm{d}_{\mathcal{M}}(x)$ when x is viewed as a local section of $\mathcal{M}$.

0FSL **Lemma 24.24.2.** The functors F, G of Lemma 24.24.1 have the following properties. Given a graded $\mathcal{A}$-module $\mathcal{N}$ we have

(1) the counit $\mathcal{N} \to F(G(\mathcal{N}))$ is injective,
(2) the map $\overline{\mathrm{d}} : \mathcal{N} \to \mathrm{Coker}(\mathcal{N} \to F(G(\mathcal{N})))[1]$ is an isomorphism, and
(3) $G(\mathcal{N})$ is an acyclic differential graded $\mathcal{A}$-module.

Proof. We observe that property (3) is a consequence of properties (1) and (2). Namely, if s is a nonzero local section of $F(G(\mathcal{N}))$ with $\mathrm{d}(s) = 0$, then s cannot be in the image of $\mathcal{N} \to F(G(\mathcal{N}))$. Hence we can write the image $\overline{s}$ of s in the cokernel as $\overline{\mathrm{d}}(s')$ for some local section s' of $\mathcal{N}$. Then we see that $s = \mathrm{d}(s')$ because the difference $s - \mathrm{d}(s')$ is still in the kernel of d and is contained in the image of the counit.

Let us write temporarily $\mathcal{A}_{gr}$, respectively $\mathcal{A}_{dg}$ the sheaf $\mathcal{A}$ viewed as a (right) graded module over itself, respectively as a (right) differential graded module over itself. The most important case of the lemma is to understand what is $G(\mathcal{A}_{gr})$. Of course $G(\mathcal{A}_{gr})$ is the object of $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ representing the functor

$$\mathcal{M} \longmapsto \mathrm{Hom}_{\mathrm{Mod}(\mathcal{A})}(\mathcal{A}_{gr}, F(\mathcal{M})) = \Gamma(\mathcal{C}, \mathcal{M})$$

By Remark 24.22.5 we see that this functor represented by $C[-1]$ where C is the cone on the identity of $\mathcal{A}_{dg}$. We have a short exact sequence

$$0 \to \mathcal{A}_{dg}[-1] \to C[-1] \to \mathcal{A}_{dg} \to 0$$

in $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ which is split by the counit $\mathcal{A}_{gr} \to F(C[-1])$ in $\mathrm{Mod}(\mathcal{A})$. Thus $G(\mathcal{A}_{gr})$ satisfies properties (1) and (2).

Let U be an object of $\mathcal{C}$. Denote $j_U : \mathcal{C}/U \to \mathcal{C}$ the localization morphism. Denote $\mathcal{A}_U$ the restriction of $\mathcal{A}$ to U. We will use the notation $\mathcal{A}_{U,gr}$ to denote $\mathcal{A}_U$ viewed as a graded $\mathcal{A}_U$-module. Denote $F_U : \text{Mod}(\mathcal{A}_U, \text{d}) \to \text{Mod}(\mathcal{A}_U)$ the forgetful functor and denote G_U its adjoint. Then we have the commutative diagrams

$$\begin{tikzcd} \text{Mod}(\mathcal{A}, \text{d}) \arrow[r, "F"'] \arrow[d, "j_U^*"'] & \text{Mod}(\mathcal{A}) \arrow[d, "j_U^*"] \\ \text{Mod}(\mathcal{A}_U, \text{d}) \arrow[r, "F_U"] & \text{Mod}(\mathcal{A}_U) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \text{Mod}(\mathcal{A}_U, \text{d}) \arrow[r, "F_U"'] \arrow[d, "j_{U!}"'] & \text{Mod}(\mathcal{A}_U) \arrow[d, "j_{U!}"] \\ \text{Mod}(\mathcal{A}, \text{d}) \arrow[r, "F"] & \text{Mod}(\mathcal{A}) \end{tikzcd}$$

by the construction of j_U^* and $j_{U!}$ in Sections 24.9, 24.18, 24.10, and 24.19. By uniqueness of adjoints we obtain $j_{U!} \circ G_U = G \circ j_{U!}$. Since $j_{U!}$ is an exact functor, we see that the properties (1) and (2) for the counit $\mathcal{A}_{U,gr} \to F_U(G_U(\mathcal{A}_{U,gr}))$ which we've seen in the previous part of the proof imply properties (1) and (2) for the counit $j_{U!}\mathcal{A}_{U,gr} \to F(G(j_{U!}\mathcal{A}_{U,gr})) = j_{U!}F_U(G_U(\mathcal{A}_{U,gr}))$.

In the proof of Lemma 24.11.1 we have seen that any object of $\text{Mod}(\mathcal{A})$ is a quotient of a direct sum of copies of $j_{U!}\mathcal{A}_{U,gr}$. Since G is a left adjoint, we see that G commutes with direct sums. Thus properties (1) and (2) hold for direct sums of objects for which they hold. Thus we see that every object $\mathcal{N}$ of $\text{Mod}(\mathcal{A})$ fits into an exact sequence

$$\mathcal{N}_1 \to \mathcal{N}_0 \to \mathcal{N} \to 0$$

such that (1) and (2) hold for $\mathcal{N}_1$ and $\mathcal{N}_0$. We leave it to the reader to deduce (1) and (2) for $\mathcal{N}$ using that G is right exact. $\square$

24.25. K-injective differential graded modules

0FSM This section is the analogue of Injectives, Section 19.12 in the setting of sheaves of differential graded modules over a sheaf of differential graded algebras.

0FSN **Lemma 24.25.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a sheaf of graded algebras on $(\mathcal{C}, \mathcal{O})$. There exists a set T and for each $t \in T$ an injective map $\mathcal{N}_t \to \mathcal{N}'_t$ of graded $\mathcal{A}$-modules such that an object $\mathcal{I}$ of $\text{Mod}(\mathcal{A})$ is injective if and only if for every solid diagram

$$\begin{tikzcd} \mathcal{N}_t \arrow[r] \arrow[d] & \mathcal{I} \\ \mathcal{N}'_t \arrow[ur, dotted] & \end{tikzcd}$$

a dotted arrow exists in $\text{Mod}(\mathcal{A})$ making the diagram commute.

Proof. This is true in any Grothendieck abelian category, see Injectives, Lemma 19.11.6. By Lemma 24.11.1 the category $\text{Mod}(\mathcal{A})$ is a Grothendieck abelian category. $\square$

0FSP **Definition 24.25.2.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. A diffential graded $\mathcal{A}$-module $\mathcal{I}$ is said to be *graded injective*[2] if $\mathcal{M}$ viewed as a graded $\mathcal{A}$-module is an injective object of the category $\text{Mod}(\mathcal{A})$ of graded $\mathcal{A}$-modules.

[2] This may be nonstandard terminology.

0FSQ Remark 24.25.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{I}$ be a graded injective diffential graded $\mathcal{A}$-module. Let

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

be a short exact sequence of differential graded $\mathcal{A}$-modules. Since $\mathcal{I}$ is graded injective we obtain a short exact sequence of complexes

$$0 \to \text{Hom}_{\text{Mod}^{dg}(\mathcal{A},\text{d})}(\mathcal{M}_3, \mathcal{I}) \to \text{Hom}_{\text{Mod}^{dg}(\mathcal{A},\text{d})}(\mathcal{M}_2, \mathcal{I}) \to \text{Hom}_{\text{Mod}^{dg}(\mathcal{A},\text{d})}(\mathcal{M}_1, \mathcal{I}) \to 0$$

of $\Gamma(\mathcal{C}, \mathcal{O})$-modules. Taking cohomology we obtain a long exact sequence

$$\begin{tikzcd}
\text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}_3, \mathcal{I}) \arrow[d] & \text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}_3, \mathcal{I})[1] \arrow[d] \\
\text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}_2, \mathcal{I}) \arrow[d] & \text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}_2, \mathcal{I})[1] \arrow[d] \\
\text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}_1, \mathcal{I}) \arrow[ruu] & \text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}_1, \mathcal{I})[1]
\end{tikzcd}$$

of groups of homomorphisms in the homotopy category. The point is that we get this even though we didn't assume that our short exact sequence is admissible (so the short exact sequence in general does not define a distinguished triangle in the homotopy category).

0FSR Lemma 24.25.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let T be a set and for each $t \in T$ let $\mathcal{I}_t$ be a graded injective diffential graded $\mathcal{A}$-module. Then $\prod \mathcal{I}_t$ is a graded injective differential graded $\mathcal{A}$-module.

Proof. This is true because products of injectives are injectives, see Homology, Lemma 12.27.3, and because products in $\text{Mod}(\mathcal{A}, \text{d})$ are compatible with products in $\text{Mod}(\mathcal{A})$ via the forgetful functor. $\square$

0FSS Lemma 24.25.5. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. There exists a set T and for each $t \in T$ an injective map $\mathcal{M}_t \to \mathcal{M}'_t$ of acyclic differential graded $\mathcal{A}$-modules such that for an object $\mathcal{I}$ of $\text{Mod}(\mathcal{A}, \text{d})$ the following are equivalent

(1) $\mathcal{I}$ is graded injective, and
(2) for every solid diagram

$$\begin{tikzcd}
\mathcal{M}_t \arrow[r] \arrow[d] & \mathcal{I} \\
\mathcal{M}'_t \arrow[ru, dotted] &
\end{tikzcd}$$

a dotted arrow exists in $\text{Mod}(\mathcal{A}, \text{d})$ making the diagram commute.

Proof. Let T and $\mathcal{N}_t \to \mathcal{N}'_t$ be as in Lemma 24.25.1. Denote $F : \text{Mod}(\mathcal{A}, \text{d}) \to \text{Mod}(\mathcal{A})$ the forgetful functor. Let G be the left adjoint functor to F as in Lemma 24.24.1. Set

$$\mathcal{M}_t = G(\mathcal{N}_t) \to G(\mathcal{N}'_t) = \mathcal{M}'_t$$

This is an injective map of acyclic differential graded $\mathcal{A}$-modules by Lemma 24.24.2. Since G is the left adjoint to F we see that there exists a dotted arrow in the diagram

$$\begin{tikzcd} \mathcal{M}_t \arrow[r] \arrow[d] & \mathcal{I} \\ \mathcal{M}'_t \arrow[ru, dotted] & \end{tikzcd}$$

if and only if there exists a dotted arrow in the diagram

$$\begin{tikzcd} \mathcal{N}_t \arrow[r] \arrow[d] & F(\mathcal{I}) \\ \mathcal{N}'_t \arrow[ru, dotted] & \end{tikzcd}$$

Hence the result follows from the choice of our collection of arrows $\mathcal{N}_t \to \mathcal{N}'_t$. □

0FST Lemma 24.25.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. There exists a set S and for each s an acyclic differential graded $\mathcal{A}$-module $\mathcal{M}_s$ such that for every nonzero acyclic differential graded $\mathcal{A}$-module $\mathcal{M}$ there is an $s \in S$ and an injective map $\mathcal{M}_s \to \mathcal{M}$ in $\text{Mod}(\mathcal{A}, \text{d})$.

Proof. Before we start recall that our conventions guarantee the site $\mathcal{C}$ has a set of objects and morphisms and a set $\text{Cov}(\mathcal{C})$ of coverings. If $\mathcal{F}$ is a differential graded $\mathcal{A}$-module, let us define $|\mathcal{F}|$ to be the sum of the cardinality of

$$\coprod\nolimits_{(U,n)} \mathcal{F}^n(U)$$

as U ranges over the objects of $\mathcal{C}$ and $n \in \mathbf{Z}$. Choose an infinite cardinal κ bigger than the cardinals $|\text{Ob}(\mathcal{C})|$, $|\text{Arrows}(\mathcal{C})|$, $|\text{Cov}(\mathcal{C})|$, $\sup |I|$ for $\{U_i \to U\}_{i \in I} \in \text{Cov}(\mathcal{C})$, and $|\mathcal{A}|$.

Let $\mathcal{F} \subset \mathcal{M}$ be an inclusion of differential graded $\mathcal{A}$-modules. Suppose given a set K and for each $k \in K$ a triple (U_k, n_k, x_k) consisting of an object U_k of $\mathcal{C}$, integer n_k, and a section $x_k \in \mathcal{M}^{n_k}(U_k)$. Then we can consider the smallest differential graded $\mathcal{A}$-submodule $\mathcal{F}' \subset \mathcal{M}$ containing $\mathcal{F}$ and the sections x_k for $k \in K$. We can describe

$$(\mathcal{F}')^n(U) \subset \mathcal{M}^n(U)$$

as the set of elements $x \in \mathcal{M}^n(U)$ such that there exists $\{f_i : U_i \to U\}_{i \in I} \in \text{Cov}(\mathcal{C})$ such that for each $i \in I$ there is a finite set T_i and morphisms $g_t : U_i \to U_{k_t}$

$$f_i^* x = y_i + \sum\nolimits_{t \in T_i} a_{it} g_t^* x_{k_t} + b_{it} g_t^* \text{d}(x_{k_t})$$

for some section $y_i \in \mathcal{F}^n(U)$ and sections $a_{it} \in \mathcal{A}^{n-n_{k_t}}(U_i)$ and $b_{it} \in \mathcal{A}^{n-n_{k_t}-1}(U_i)$. (Details omitted; hints: these sections are certainly in $\mathcal{F}'$ and you show conversely that this rule defines a differential graded $\mathcal{A}$-submodule.) It follows from this description that $|\mathcal{F}'| \leq \max(|\mathcal{F}|, |K|, \kappa)$.

Let $\mathcal{M}$ be a nonzero acyclic differential graded $\mathcal{A}$-module. Then we can find an integer n and a nonzero section x of $\mathcal{M}^n$ over some object U of $\mathcal{C}$. Let

$$\mathcal{F}_0 \subset \mathcal{M}$$

be the smallest differential graded $\mathcal{A}$-submodule containing x. By the previous paragraph we have $|\mathcal{F}_0| \leq \kappa$. By induction, given $\mathcal{F}_0, \ldots, \mathcal{F}_n$ define $\mathcal{F}_{n+1}$ as follows. Consider the set

$$L = \{(U, n, x)\}\{U_i \to U\}_{i \in I}, (x_i)_{i \in I})\}$$

of triples where U is an object of $\mathcal{C}$, $n \in \mathbf{Z}$, and $x \in \mathcal{F}_n(U)$ with $\mathrm{d}(x) = 0$. Since $\mathcal{M}$ is acyclic for each triple $l = (U_l, n_l, x_l) \in L$ we can choose $\{(U_{l,i} \to U_l\}_{i \in I_l} \in \mathrm{Cov}(\mathcal{C})$ and $x_{l,i} \in \mathcal{M}^{n_l - 1}(U_{l,i})$ such that $\mathrm{d}(x_{l,i}) = x|_{U_{l,i}}$. Then we set

$$K = \{(U_{l,i}, n_l - 1, x_{l,i}) \mid l \in L, i \in I_l\}$$

and we let $\mathcal{F}_{n+1}$ be the smallest differential graded $\mathcal{A}$-submodule of $\mathcal{M}$ containing $\mathcal{F}_n$ and the sections $x_{l,i}$. Since $|K| \leq \max(\kappa, |\mathcal{F}_n|)$ we conclude that $|\mathcal{F}_{n+1}| \leq \kappa$ by induction.

By construction the inclusion $\mathcal{F}_n \to \mathcal{F}_{n+1}$ induces the zero map on cohomology sheaves. Hence we see that $\mathcal{F} = \bigcup \mathcal{F}_n$ is a nonzero acyclic submodule with $|\mathcal{F}| \leq \kappa$. Since there is only a set of isomorphism classes of differential graded $\mathcal{A}$-modules $\mathcal{F}$ with $|\mathcal{F}|$ bounded, we conclude. $\square$

0FSU Definition 24.25.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. A diffential graded $\mathcal{A}$-module $\mathcal{I}$ is K-injective if for every acyclic differential graded $\mathcal{M}$ we have

$$\mathrm{Hom}_{K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))}(\mathcal{M}, \mathcal{I}) = 0$$

Please note the similarity with Derived Categories, Definition 13.31.1.

0FSV Lemma 24.25.8. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let T be a set and for each $t \in T$ let $\mathcal{I}_t$ be a K-injective diffential graded $\mathcal{A}$-module. Then $\prod \mathcal{I}_t$ is a K-injective differential graded $\mathcal{A}$-module.

Proof. Let $\mathcal{K}$ be an acyclic differential graded $\mathcal{A}$-module. Then we have

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d})}(\mathcal{K}, \prod\nolimits_{t \in T} \mathcal{I}_t) = \prod\nolimits_{t \in T} \mathrm{Hom}_{\mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d})}(\mathcal{K}, \mathcal{I}_t)$$

because taking products in $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ commutes with the forgetful functor to graded $\mathcal{A}$-modules. Since taking products is an exact functor on the category of abelian groups we conclude. $\square$

0FSW Lemma 24.25.9. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{I}$ be a K-injective and graded injective object of $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$. For every solid diagram in $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$

$$\begin{tikzcd} \mathcal{M} \arrow[r, "a"] \arrow[d, "b"'] & \mathcal{I} \\ \mathcal{M}' \arrow[ru, dotted] & \end{tikzcd}$$

where b is injective and $\mathcal{M}$ is acyclic a dotted arrow exists making the diagram commute.

Proof. Since $\mathcal{M}$ is acyclic and $\mathcal{I}$ is K-injective, there exists a graded $\mathcal{A}$-module map $h : \mathcal{M} \to \mathcal{I}$ of degree -1 such that $a = \mathrm{d}(h)$. Since $\mathcal{I}$ is graded injective and b is injective, there exists a graded $\mathcal{A}$-module map $h' : \mathcal{M}' \to \mathcal{I}$ of degree -1 such that $h = h' \circ b$. Then we can take $a' = \mathrm{d}(h')$ as the dotted arrow. $\square$

0FSX Lemma 24.25.10. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{I}$ be a K-injective and graded injective object of $\text{Mod}(\mathcal{A}, \text{d})$. For every solid diagram in $\text{Mod}(\mathcal{A}, \text{d})$

$$\begin{tikzcd} \mathcal{M} \arrow[r, "a"'] \arrow[d, "b"'] & \mathcal{I} \\ \mathcal{M}' \arrow[ur, dotted] & \end{tikzcd}$$

where b is a quasi-isomorphism a dotted arrow exists making the diagram commute up to homotopy.

Proof. After replacing $\mathcal{M}'$ by the direct sum of $\mathcal{M}'$ and the cone on the identity on $\mathcal{M}$ (which is acyclic) we may assume b is also injective. Then the cokernel $\mathcal{Q}$ of b is acyclic. Thus we see that

$$\text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{Q}, \mathcal{I}) = \text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{Q}, \mathcal{I})[1] = 0$$

as $\mathcal{I}$ is K-injective. As $\mathcal{I}$ is graded injective by Remark 24.25.3 we see that

$$\text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}', \mathcal{I}) \longrightarrow \text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}, \mathcal{I})$$

is bijective and the proof is complete. $\square$

0FSY Lemma 24.25.11. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. There exists a set R and for each $r \in R$ an injective map $\mathcal{M}_r \to \mathcal{M}'_r$ of acyclic differential graded $\mathcal{A}$-modules such that for an object $\mathcal{I}$ of $\text{Mod}(\mathcal{A}, \text{d})$ the following are equivalent

(1) $\mathcal{I}$ is K-injective and graded injective, and
(2) for every solid diagram

$$\begin{tikzcd} \mathcal{M}_r \arrow[r] \arrow[d] & \mathcal{I} \\ \mathcal{M}'_r \arrow[ur, dotted] & \end{tikzcd}$$

a dotted arrow exists in $\text{Mod}(\mathcal{A}, \text{d})$ making the diagram commute.

Proof. Let T and $\mathcal{M}_t \to \mathcal{M}'_t$ be as in Lemma 24.25.5. Let S and $\mathcal{M}_s$ be as in Lemma 24.25.6. Choose an injective map $\mathcal{M}_s \to \mathcal{M}'_s$ of acyclic differential graded $\mathcal{A}$-modules which is homotopic to zero. This is possible because we may take $\mathcal{M}'_s$ to be the cone on the identity; in that case it is even true that the identity on $\mathcal{M}'_s$ is homotopic to zero, see Differential Graded Algebra, Lemma 22.27.4 which applies by the discussion in Section 24.22. We claim that $R = T \coprod S$ with the given maps works.

The implication (1) $\Rightarrow$ (2) holds by Lemma 24.25.9.

Assume (2). First, by Lemma 24.25.5 we see that $\mathcal{I}$ is graded injective. Next, let $\mathcal{M}$ be an acyclic differential graded $\mathcal{A}$-module. We have to show that

$$\text{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{M}, \mathcal{I}) = 0$$

The proof will be exactly the same as the proof of Injectives, Lemma 19.12.3.

We are going to construct by induction on the ordinal α an acyclic differential graded submodule $\mathcal{K}_\alpha \subset \mathcal{M}$ as follows. For $\alpha = 0$ we set $\mathcal{K}_0 = 0$. For $\alpha > 0$ we proceed as follows:

(1) If $\alpha = \beta + 1$ and $\mathcal{K}_\beta = \mathcal{M}$ then we choose $\mathcal{K}_\alpha = \mathcal{K}_\beta$.
(2) If $\alpha = \beta + 1$ and $\mathcal{K}_\beta \neq \mathcal{M}$ then $\mathcal{M}/\mathcal{K}_\beta$ is a nonzero acyclic differential graded $\mathcal{A}$-module. We choose a differential graded $\mathcal{A}$ submodule $\mathcal{N}_\alpha \subset \mathcal{M}/\mathcal{K}_\beta$ isomorphic to $\mathcal{M}_s$ for some $s \in S$, see Lemma 24.25.6. Finally, we let $\mathcal{K}_\alpha \subset \mathcal{M}$ be the inverse image of $\mathcal{N}_\alpha$.
(3) If α is a limit ordinal we set $\mathcal{K}_\beta = \operatorname{colim} \mathcal{K}_\alpha$.

It is clear that $\mathcal{M} = \mathcal{K}_\alpha$ for a suitably large ordinal α. We will prove that

$$\operatorname{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{K}_\alpha, \mathcal{I})$$

is zero by transfinite induction on α. It holds for $\alpha = 0$ since $\mathcal{K}_0$ is zero. Suppose it holds for β and $\alpha = \beta + 1$. In case (1) of the list above the result is clear. In case (2) there is a short exact sequence

$$0 \to \mathcal{K}_\beta \to \mathcal{K}_\alpha \to \mathcal{N}_\alpha \to 0$$

By Remark 24.25.3 and since we've seen that $\mathcal{I}$ is graded injective, we obtain an exact sequence

$$\operatorname{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{K}_\beta, \mathcal{I}) \to \operatorname{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{K}_\alpha, \mathcal{I}) \to \operatorname{Hom}_{K(\text{Mod}(\mathcal{A},\text{d}))}(\mathcal{N}_\alpha, \mathcal{I})$$

By induction the term on the left is zero. By assumption (2) the term on the right is zero: any map $\mathcal{M}_s \to \mathcal{I}$ factors through $\mathcal{M}'_s$ and hence is homotopic to zero. Thus the middle group is zero too. Finally, suppose that α is a limit ordinal. Because we also have $\mathcal{K}_\alpha = \operatorname{colim} \mathcal{K}_\alpha$ as graded $\mathcal{A}$-modules we see that

$$\operatorname{Hom}_{\text{Mod}^{dg}(\mathcal{A},\text{d})}(\mathcal{K}_\alpha, \mathcal{I}) = \lim_{\beta < \alpha} \operatorname{Hom}_{\text{Mod}^{dg}(\mathcal{A},\text{d})}(\mathcal{K}_\beta, \mathcal{I})$$

as complexes of abelian groups. The cohomology groups of these complexes compute morphisms in $K(\text{Mod}(\mathcal{A}, \text{d}))$ between shifts. The transition maps in the system of complexes are surjective by Remark 24.25.3 because $\mathcal{I}$ is graded injective. Moreover, for a limit ordinal $\beta \leq \alpha$ we have equality of limit and value. Thus we may apply Homology, Lemma 12.31.8 to conclude. $\square$

0FSZ Lemma 24.25.12. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let R be a set and for each $r \in R$ let an injective map $\mathcal{M}_r \to \mathcal{M}'_r$ of acyclic differential graded $\mathcal{A}$-modules be given. There exists a functor $M : \text{Mod}(\mathcal{A}, \text{d}) \to \text{Mod}(\mathcal{A}, \text{d})$ and a natural transformation $j : \text{id} \to M$ such that

(1) $j_\mathcal{M} : \mathcal{M} \to M(\mathcal{M})$ is injective and a quasi-isomorphism,
(2) for every solid diagram

$$\begin{tikzcd} \mathcal{M}_r \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "j_\mathcal{M}"] \\ \mathcal{M}'_r \arrow[r, dotted] & M(\mathcal{M}) \end{tikzcd}$$

a dotted arrow exists in $\text{Mod}(\mathcal{A}, \text{d})$ making the diagram commute.

Proof. We define $M(\mathcal{M})$ as the pushout in the following diagram

$$\begin{tikzcd} \bigoplus_{(r,\varphi)} \mathcal{M}_r \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \bigoplus_{(r,\varphi)} \mathcal{M}'_r \arrow[r] & M(\mathcal{M}) \end{tikzcd}$$

where the direct sum is over all pairs (r, φ) with $r \in R$ and $\varphi \in \mathrm{Hom}_{\mathrm{Mod}(\mathcal{A},\mathrm{d})}(\mathcal{M}_r, \mathcal{M})$. Since the pushout of an injective map is injective, we see that $\mathcal{M} \to M(\mathcal{M})$ is injective. Since the cokernel of the left vertical arrow is acyclic, we see that the (isomorphic) cokernel of $\mathcal{M} \to M(\mathcal{M})$ is acyclic, hence $\mathcal{M} \to M(\mathcal{M})$ is a quasi-isomorphism. Property (2) holds by construction. We omit the verification that this procedure can be turned into a functor. □

0FT0 Theorem 24.25.13. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. For every differential graded $\mathcal{A}$-module $\mathcal{M}$ there exists a quasi-isomorphism $\mathcal{M} \to \mathcal{I}$ where $\mathcal{I}$ is a graded injective and K-injective differential graded $\mathcal{A}$-module. Moreover, the construction is functorial in $\mathcal{M}$.

Proof. Let R and $\mathcal{M}_r \to \mathcal{M}'_r$ be a set of morphisms of $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ found in Lemma 24.25.11. Let M with transformation $\mathrm{id} \to M$ be as constructed in Lemma 24.25.12 using R and $\mathcal{M}_r \to \mathcal{M}'_r$. Using transfinite recursion we define a sequence of functors M_α and natural transformations $M_\beta \to M_\alpha$ for $\alpha < \beta$ by setting

(1) $M_0 = \mathrm{id}$,
(2) $M_{\alpha+1} = M \circ M_\alpha$ with natural transformation $M_\beta \to M_{\alpha+1}$ for $\beta < \alpha + 1$ coming from the already constructed $M_\beta \to M_\alpha$ and the maps $M_\alpha \to M \circ M_\alpha$ coming from $\mathrm{id} \to M$, and
(3) $M_\alpha = \mathrm{colim}_{\beta<\alpha} M_\beta$ if α is a limit ordinal with the coprojections as transformations $M_\beta \to M_\alpha$ for $\alpha < \beta$.

Observe that for every differential graded $\mathcal{A}$-module the maps $\mathcal{M} \to M_\beta(\mathcal{M}) \to M_\alpha(\mathcal{M})$ are injective quasi-isomorphisms (as filtered colimits are exact).

Recall that $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ is a Grothendieck abelian category. Thus by Injectives, Proposition 19.11.5 (applied to the direct sum of $\mathcal{M}_r$ for all $r \in R$) there is a limit ordinal α such that $\mathcal{M}_r$ is α-small with respect to injections for every $r \in R$. We claim that $\mathcal{M} \to M_\alpha(\mathcal{M})$ is the desired functorial embedding of $\mathcal{M}$ into a graded injective K-injective module.

Namely, any map $\mathcal{M}_r \to M_\alpha(\mathcal{M})$ factors through $M_\beta(\mathcal{M})$ for some $\beta < \alpha$. However, by the construction of M we see that this means that $\mathcal{M}_r \to M_{\beta+1}(\mathcal{M}) = M(M_\beta(\mathcal{M}))$ factors through $\mathcal{M}'_r$. Since $M_\beta(\mathcal{M}) \subset M_{\beta+1}(\mathcal{M}) \subset M_\alpha(\mathcal{M})$ we get the desired factorizaton into $M_\alpha(\mathcal{M})$. We conclude by our choice of R and $\mathcal{M}_r \to \mathcal{M}'_r$ in Lemma 24.25.11. □

24.26. The derived category

0FT1 This section is the analogue of Differential Graded Algebra, Section 22.22.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. We will construct the derived category $D(\mathcal{A}, \mathrm{d})$ by inverting the quasi-isomorphisms in $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$.

0FT2 Lemma 24.26.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The functor $H^0 : \mathrm{Mod}(\mathcal{A}, \mathrm{d}) \to \mathrm{Mod}(\mathcal{O})$ of Section 24.13 factors through a functor

$$H^0 : K(\mathrm{Mod}(\mathcal{A}, \mathrm{d})) \to \mathrm{Mod}(\mathcal{O})$$

which is homological in the sense of Derived Categories, Definition 13.3.5.

Proof. It follows immediately from the definitions that there is a commutative diagram

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{A}, \mathrm{d}) \arrow[r] \arrow[d] & K(\mathrm{Mod}(\mathcal{A}, \mathrm{d})) \arrow[d] \\ \mathrm{Comp}(\mathcal{O}) \arrow[r] & K(\mathrm{Mod}(\mathcal{O})) \end{tikzcd}$$

Since $H^0(\mathcal{M})$ is defined as the zeroth cohomology sheaf of the underlying complex of $\mathcal{O}$-modules of $\mathcal{M}$ the lemma follows from the case of complexes of $\mathcal{O}$-modules which is a special case of Derived Categories, Lemma 13.11.1. □

0FT3 Lemma 24.26.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The full subcategory Ac of the homotopy category $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$ consisting of acyclic modules is a strictly full saturated triangulated subcategory of $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$.

Proof. Of course an object $\mathcal{M}$ of $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$ is in Ac if and only if $H^i(\mathcal{M}) = H^0(\mathcal{M}[i])$ is zero for all i. The lemma follows from this, Lemma 24.26.1, and Derived Categories, Lemma 13.6.3. See also Derived Categories, Definitions 13.6.1 and 13.3.4 and Lemma 13.4.16. □

0FT4 Lemma 24.26.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Consider the subclass $\mathrm{Qis} \subset \mathrm{Arrows}(K(\mathrm{Mod}(\mathcal{A}, \mathrm{d})))$ consisting of quasi-isomorphisms. This is a saturated multiplicative system compatible with the triangulated structure on $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$.

Proof. Observe that if $f, g : \mathcal{M} \to \mathcal{N}$ are morphisms of $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ which are homotopic, then f is a quasi-isomorphism if and only if g is a quasi-isomorphism. Namely, the maps $H^i(f) = H^0(f[i])$ and $H^i(g) = H^0(g[i])$ are the same by Lemma 24.26.1. Thus it is unambiguous to say that a morphism of the homotopy category $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$ is a quasi-isomorphism. For definitions of "multiplicative system", "saturated", and "compatible with the triangulated structure" see Derived Categories, Definition 13.5.1 and Categories, Definitions 4.27.1 and 4.27.20.

To actually prove the lemma consider the composition of exact functors of triangulated categories

$$K(\mathrm{Mod}(\mathcal{A}, \mathrm{d})) \longrightarrow K(\mathrm{Mod}(\mathcal{O})) \longrightarrow D(\mathcal{O})$$

and observe that a morphism $f : \mathcal{M} \to \mathcal{N}$ of $K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$ is in Qis if and only if it maps to an isomorphism in $D(\mathcal{O})$. Thus the lemma follows from Derived Categories, Lemma 13.5.4. □

In the situation of Lemma 24.26.3 we can apply Derived Categories, Proposition 13.5.6 to obtain an exact functor of triangulated categories

$$Q : K(\mathrm{Mod}(\mathcal{A}, \mathrm{d})) \longrightarrow \mathrm{Qis}^{-1} K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$$

However, as $\mathrm{Mod}(\mathcal{A}, \mathrm{d})$ is a "big" category, i.e., its objects form a proper class, it isn't immediately clear that given $\mathcal{M}$ and $\mathcal{N}$ the construction of $\mathrm{Qis}^{-1} K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))$ produces a set

$$\mathrm{Mor}_{\mathrm{Qis}^{-1} K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))}(\mathcal{M}, \mathcal{N})$$

of morphisms. However, this is true thanks to our construction of K-injective complexes. Namely, by Theorem 24.25.13 we can choose a quasi-isomorphism $s_0 : \mathcal{N} \to \mathcal{I}$ where $\mathcal{I}$ is a graded injective and K-injective differential graded $\mathcal{A}$-module.

Next, recall that elements of the displayed set are equivalence classes of pairs $(f : \mathcal{M} \to \mathcal{N}', s : \mathcal{N} \to \mathcal{N}')$ where f is an arbitrary morphism of $K(\text{Mod}(\mathcal{A}, \text{d}))$ and s is a quasi-isomorphsm, see the description of the left calculus of fractions in Categories, Section 4.27. By Lemma 24.25.10 we can choose the dotted arrow

$$\begin{tikzcd} \mathcal{M} \arrow[dr, "f"] & & \mathcal{N} \arrow[dl, "s"'] \arrow[dr, "s_0"] & \\ & \mathcal{N}' \arrow[rr, dotted, "s'"] & & \mathcal{I} \end{tikzcd}$$

making the diagram commute (in the homotopy category). Thus the pair (f, s) is equivalent to the pair $(s' \circ f, s_0)$ and we find that the collection of equivalence classes forms a set.

0FT5 Definition 24.26.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let Qis be as in Lemma 24.26.3. The derived category of $(\mathcal{A}, \text{d})$ is the triangulated category

$$D(\mathcal{A}, \text{d}) = \text{Qis}^{-1} K(\text{Mod}(\mathcal{A}, \text{d}))$$

discussed in more detail above.

We prove some facts about this construction.

0FT6 Lemma 24.26.5. In Definition 24.26.4 the kernel of the localization functor $Q : K(\text{Mod}(\mathcal{A}, \text{d})) \to D(\mathcal{A}, \text{d})$ is the category Ac of Lemma 24.26.2.

Proof. This is immediate from Derived Categories, Lemma 13.5.9 and the fact that $0 \to \mathcal{M}$ is a quasi-isomorphism if and only if $\mathcal{M}$ is acyclic. $\square$

0FT7 Lemma 24.26.6. In Definition 24.26.4 the functor $H^0 : K(\text{Mod}(\mathcal{A}, \text{d})) \to \text{Mod}(\mathcal{O})$ factors through a homological functor $H^0 : D(\mathcal{A}, \text{d}) \to \text{Mod}(\mathcal{O})$.

Proof. Follows immediately from Derived Categories, Lemma 13.5.7. $\square$

Here is the promised lemma computing morphism sets in the derived category.

0FT8 Lemma 24.26.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}$ and $\mathcal{N}$ be differential graded $\mathcal{A}$-modules. Let $\mathcal{N} \to \mathcal{I}$ be a quasi-isomorphism with $\mathcal{I}$ a graded injective and K-injective differential graded $\mathcal{A}$-module. Then

$$\text{Hom}_{D(\mathcal{A}, \text{d})}(\mathcal{M}, \mathcal{N}) = \text{Hom}_{K(\text{Mod}(\mathcal{A}, \text{d}))}(\mathcal{M}, \mathcal{I})$$

Proof. Since $\mathcal{N} \to \mathcal{I}$ is a quasi-isomorphism we see that

$$\text{Hom}_{D(\mathcal{A}, \text{d})}(\mathcal{M}, \mathcal{N}) = \text{Hom}_{D(\mathcal{A}, \text{d})}(\mathcal{M}, \mathcal{I})$$

In the discussion preceding Definition 24.26.4 we found, using Lemma 24.25.10, that any morphism $\mathcal{M} \to \mathcal{I}$ in $D(\mathcal{A}, \text{d})$ can be represented by a morphism $f : \mathcal{M} \to \mathcal{I}$ in $K(\text{Mod}(\mathcal{A}, \text{d}))$. Now, if $f, f' : \mathcal{M} \to \mathcal{I}$ are two morphism in $K(\text{Mod}(\mathcal{A}, \text{d}))$, then they define the same morphism in $D(\mathcal{A}, \text{d})$ if and only if there exists a quasi-isomorphism $g : \mathcal{I} \to \mathcal{K}$ in $K(\text{Mod}(\mathcal{A}, \text{d}))$ such that $g \circ f = g \circ f'$, see Categories, Lemma 4.27.6. However, by Lemma 24.25.10 there exists a map $h : \mathcal{K} \to \mathcal{I}$ such that $h \circ g = \text{id}_{\mathcal{I}}$ in in $K(\text{Mod}(\mathcal{A}, \text{d}))$. Thus $g \circ f = g \circ f'$ implies $f = f'$ and the proof is complete. $\square$

0FT9 **Lemma 24.26.8.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Then

(1) $D(\mathcal{A}, \text{d})$ has both direct sums and products,
(2) direct sums are obtained by taking direct sums of differential graded $\mathcal{A}$-modules,
(3) products are obtained by taking products of K-injective differential graded modules.

Proof. We will use that $\text{Mod}(\mathcal{A}, \text{d})$ is an abelian category with arbitrary direct sums and products, and that these give rise to direct sums and products in $K(\text{Mod}(\mathcal{A}, \text{d}))$. See Lemmas 24.13.2 and 24.21.3.

Let $\mathcal{M}_j$ be a family of differential graded $\mathcal{A}$-modules. Consider the direct sum $\mathcal{M} = \bigoplus \mathcal{M}_j$ as a differential graded $\mathcal{A}$-module. For a differential graded $\mathcal{A}$-module $\mathcal{N}$ choose a quasi-isomorphism $\mathcal{N} \to \mathcal{I}$ where $\mathcal{I}$ is graded injective and K-injective as a differential graded $\mathcal{A}$-module. See Theorem 24.25.13. Using Lemma 24.26.7 we have

$$\begin{aligned} \text{Hom}_{D(\mathcal{A},\text{d})}(\mathcal{M}, \mathcal{N}) &= \text{Hom}_{K(\mathcal{A},\text{d})}(\mathcal{M}, \mathcal{I}) \\ &= \prod \text{Hom}_{K(\mathcal{A},\text{d})}(\mathcal{M}_j, \mathcal{I}) \\ &= \prod \text{Hom}_{D(\mathcal{A},\text{d})}(\mathcal{M}_j, \mathcal{I}) \end{aligned}$$

whence the existence of direct sums in $D(A, \text{d})$ as given in part (2) of the lemma.

Let $\mathcal{M}_j$ be a family of differential graded $\mathcal{A}$-modules. For each j choose a quasi-isomorphism $\mathcal{M} \to \mathcal{I}_j$ where $\mathcal{I}_j$ is graded injective and K-injective as a differential graded $\mathcal{A}$-module. Consider the product $\mathcal{I} = \prod \mathcal{I}_j$ of differential graded $\mathcal{A}$-modules. By Lemmas 24.25.8 and 24.25.4 we see that $\mathcal{I}$ is graded injective and K-injective as a differential graded $\mathcal{A}$-module. For a differential graded $\mathcal{A}$-module $\mathcal{N}$ using Lemma 24.26.7 we have

$$\begin{aligned} \text{Hom}_{D(\mathcal{A},\text{d})}(\mathcal{N}, \mathcal{I}) &= \text{Hom}_{K(\mathcal{A},\text{d})}(\mathcal{N}, \mathcal{I}) \\ &= \prod \text{Hom}_{K(\mathcal{A},\text{d})}(\mathcal{N}, \mathcal{I}_j) \\ &= \prod \text{Hom}_{D(\mathcal{A},\text{d})}(\mathcal{N}, \mathcal{M}_j) \end{aligned}$$

whence the existence of products in $D(\mathcal{A}, \text{d})$ as given in part (3) of the lemma. $\square$

24.27. The canonical delta-functor

0FTA Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Consider the functor $\text{Mod}(\mathcal{A}, \text{d}) \to K(\text{Mod}(\mathcal{A}, \text{d}))$. This functor is not a δ-functor in general. However, it turns out that the functor $\text{Mod}(\mathcal{A}, \text{d}) \to D(A, \text{d})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to \mathcal{K} \xrightarrow{a} \mathcal{L} \xrightarrow{b} \mathcal{M} \to 0$$

in the abelian category $\text{Mod}(\mathcal{A}, \text{d})$. Consider the cone $C(a)$ of the morphism a together with its canonical morphisms $i : \mathcal{L} \to C(a)$ and $p : C(a) \to \mathcal{K}[1]$, see Definition 24.22.2. There is a homomorphism of differential graded $\mathcal{A}$-modules

$$q : C(a) \longrightarrow \mathcal{M}$$

by Differential Graded Algebra, Lemma 22.27.3 (which we may use by the discussion in Section 24.22) applied to the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "a"] \arrow[d] & \mathcal{L} \arrow[d, "b"] \\ 0 \arrow[r] & \mathcal{M} \end{tikzcd}$$

The map q is a quasi-isomorphism for example because this is true in the category of morphisms of complexes of $\mathcal{O}$-modules, see discussion in Derived Categories, Section 13.12. According to Differential Graded Algebra, Lemma 22.27.13 (which we may use by the discussion in Section 24.22) the triangle

$$(\mathcal{K}, \mathcal{L}, C(a), a, i, -p)$$

is a distinguished triangle in $K(\text{Mod}(\mathcal{A}, \text{d}))$. As the localization functor $K(\text{Mod}(\mathcal{A}, \text{d})) \to D(\mathcal{A}, \text{d})$ is exact we see that $(\mathcal{K}, \mathcal{L}, C(a), a, i, -p)$ is a distinguished triangle in $D(\mathcal{A}, \text{d})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(\mathcal{A}, \text{d})$. Hence we deduce that

$$(\mathcal{K}, \mathcal{L}, \mathcal{M}, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(\mathcal{A}, \text{d})$. This suggests the following lemma.

0FTB Lemma 24.27.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. The localization functor $\text{Mod}(\mathcal{A}, \text{d}) \to D(\mathcal{A}, \text{d})$ has the natural structure of a δ-functor, with

$$\delta_{\mathcal{K} \to \mathcal{L} \to \mathcal{M}} = -p \circ q^{-1}$$

with p and q as explained above.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show functoriality of this construction, see Derived Categories, Definition 13.3.6. This follows from Differential Graded Algebra, Lemma 22.27.3 (which we may use by the discussion in Section 24.22) with a bit of work. Compare with Derived Categories, Lemma 13.12.1. □

0FTC Lemma 24.27.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Let $\mathcal{M}_n$ be a system of differential graded $\mathcal{A}$-modules. Then the derived colimit $\text{hocolim} \mathcal{M}_n$ in $D(\mathcal{A}, \text{d})$ is represented by the differential graded module $\text{colim}\, \mathcal{M}_n$.

Proof. Set $\mathcal{M} = \text{colim}\, \mathcal{M}_n$. We have an exact sequence of differential graded $\mathcal{A}$-modules

$$0 \to \bigoplus \mathcal{M}_n \to \bigoplus \mathcal{M}_n \to \mathcal{M} \to 0$$

by Derived Categories, Lemma 13.33.6 (applied the underlying complexes of $\mathcal{O}$-modules). The direct sums are direct sums in $D(\mathcal{A}, \text{d})$ by Lemma 24.26.8. Thus the result follows from the definition of derived colimits in Derived Categories, Definition 13.33.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Lemma 24.27.1). □

24.28. Derived pullback

0FTD Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Suppose we are given a map

$$\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$$

of differential graded $f^{-1}\mathcal{O}_\mathcal{D}$-algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map $\varphi : f^*\mathcal{B} \to \mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras or equivalently φ can be viewed as a map

$$\varphi : \mathcal{B} \to f_*\mathcal{A}$$

of differential graded $\mathcal{O}_\mathcal{D}$-algebras. See Remark 24.12.2.

In addition to the above, let $\mathcal{A}'$ be a second differential graded $\mathcal{O}_\mathcal{C}$-algebra and let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{A}')$-bimodule. In this setting we can consider the functor

$$\text{Mod}(\mathcal{B}, \text{d}) \longrightarrow \text{Mod}(\mathcal{A}', \text{d}), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$$

Observe that this extends to a functor

$$\text{Mod}^{dg}(\mathcal{B}, \text{d}) \longrightarrow \text{Mod}^{dg}(\mathcal{A}', \text{d}), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$$

of differential graded categories by the discussion in Sections 24.18 and 24.17. It follows formally that we also obtain an exact functor

0FTE (24.28.0.1)
$$K(\text{Mod}(\mathcal{B}, \text{d})) \longrightarrow K(\text{Mod}(\mathcal{A}', \text{d})), \quad \mathcal{M} \longmapsto f^*\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$$

of triangulated categories.

0FTF **Lemma 24.28.1.** *In the situation above, the functor (24.28.0.1) composed with the localization functor $K(\text{Mod}(\mathcal{A}', \text{d})) \to D(\mathcal{A}', \text{d})$ has a left derived extension $D(\mathcal{B}, \text{d}) \to D(\mathcal{A}', \text{d})$ whose value on a good right differential graded $\mathcal{B}$-module $\mathcal{P}$ is $f^*\mathcal{P} \otimes_\mathcal{A} \mathcal{N}$.*

Proof. Recall that for any (right) differential graded $\mathcal{B}$-module $\mathcal{M}$ there exists a quasi-isomorphism $\mathcal{P} \to \mathcal{M}$ with $\mathcal{P}$ a good differential graded $\mathcal{B}$-module. See Lemma 24.23.7. Hence by Derived Categories, Lemma 13.14.15 it suffices to show that given a quasi-isomorphism $\mathcal{P} \to \mathcal{P}'$ of good differential graded $\mathcal{B}$-modules the induced map

$$f^*\mathcal{P} \otimes_\mathcal{A} \mathcal{N} \longrightarrow f^*\mathcal{P}' \otimes_\mathcal{A} \mathcal{N}$$

is a quasi-isomorphism. The cone $\mathcal{P}''$ on $\mathcal{P} \to \mathcal{P}'$ is a good differential graded $\mathcal{A}$-module by Lemma 24.23.2. Since we have a distinguished triangle

$$\mathcal{P} \to \mathcal{P}' \to \mathcal{P}'' \to \mathcal{P}[1]$$

in $K(\text{Mod}(\mathcal{B}, \text{d}))$ we obtain a distinguished triangle

$$f^*\mathcal{P} \otimes_\mathcal{A} \mathcal{N} \to f^*\mathcal{P}' \otimes_\mathcal{A} \mathcal{N} \to f^*\mathcal{P}'' \otimes_\mathcal{A} \mathcal{N} \to f^*\mathcal{P}[1] \otimes_\mathcal{A} \mathcal{N}$$

in $K(\text{Mod}(\mathcal{A}', \text{d}))$. By Lemma 24.23.8 the differential graded module $f^*\mathcal{P}'' \otimes_\mathcal{A} \mathcal{N}$ is acyclic and the proof is complete. $\square$

0FTG **Definition 24.28.2.** Derived tensor product and derived pullback.

(1) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. The functor $D(\mathcal{A}, \text{d}) \to D(\mathcal{B}, \text{d})$ constructed in Lemma 24.28.1 is called the *derived tensor product* and denoted $- \otimes_\mathcal{A}^\mathbf{L} \mathcal{N}$.

(2) Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Let $\varphi : \mathcal{B} \to f_*\mathcal{A}$ be a homomorphism of differential graded $\mathcal{O}_\mathcal{D}$-algebras. The functor $D(\mathcal{B}, \text{d}) \to D(\mathcal{A}, \text{d})$ constructed in Lemma 24.28.1 is called derived pullback and denote Lf^*.

With this language in place we can express some obvious compatibilities.

0FTH Lemma 24.28.3. In Lemma 24.28.1 the functor $D(\mathcal{B}, \text{d}) \to D(\mathcal{A}', \text{d})$ is equal to $\mathcal{M} \mapsto Lf^*\mathcal{M} \otimes_\mathcal{A}^\mathbf{L} \mathcal{N}$.

Proof. Immediate from the fact that we can compute these functors by representing objects by good differential graded modules and because $f^*\mathcal{P}$ is a good differential graded $\mathcal{A}$-module if $\mathcal{P}$ is a good differential graded $\mathcal{B}$-module. ☐

0FTI Lemma 24.28.4. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and $(g, g^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}''), \mathcal{O}'')$ be morphisms of ringed topoi. Let $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ be a differential graded $\mathcal{O}$-algebra, $\mathcal{O}'$-algebra, and $\mathcal{O}''$-algebra. Let $\varphi : \mathcal{A}' \to f_*\mathcal{A}$ and $\varphi' : \mathcal{A}'' \to g_*\mathcal{A}'$ be a homomorphism of differential graded $\mathcal{O}'$-algebras and $\mathcal{O}''$-algebras. Then we have $L(g \circ f)^* = Lf^* \circ Lg^* : D(\mathcal{A}'', \text{d}) \to D(\mathcal{A}, \text{d})$.

Proof. Immediate from the fact that we can compute these functors by representing objects by good differential graded modules and because $f^*\mathcal{P}$ is a good differential graded $\mathcal{A}'$-module of $\mathcal{P}$ is a good differential graded $\mathcal{A}$-module. ☐

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N} \to \mathcal{N}'$ be a homomorphism of differential graded $(\mathcal{A}, \mathcal{B})$-bimodules. Then we obtain canonical maps

$$t : \mathcal{M} \otimes_\mathcal{A}^\mathbf{L} \mathcal{N} \longrightarrow \mathcal{M} \otimes_\mathcal{A}^\mathbf{L} \mathcal{N}'$$

functorial in $\mathcal{M}$ in $D(\mathcal{A}, \text{d})$ which define a natural transformation between exact functors $D(\mathcal{A}, \text{d}) \to D(\mathcal{B}, \text{d})$ of triangulated categories. The value of t on a good differential graded $\mathcal{A}$-module $\mathcal{P}$ is the obvious map

$$\mathcal{P} \otimes_\mathcal{A}^\mathbf{L} \mathcal{N} = \mathcal{P} \otimes_\mathcal{A} \mathcal{N} \longrightarrow \mathcal{P} \otimes_\mathcal{A} \mathcal{N}' = \mathcal{P} \otimes_\mathcal{A}^\mathbf{L} \mathcal{N}'$$

0FTJ Lemma 24.28.5. In the situation above, if $\mathcal{N} \to \mathcal{N}'$ is an isomorphism on cohomology sheaves, then t is an isomorphism of functors $(- \otimes_\mathcal{A}^\mathbf{L} \mathcal{N}) \to (- \otimes_\mathcal{A}^\mathbf{L} \mathcal{N}')$.

Proof. It is enough to show that $\mathcal{P} \otimes_\mathcal{A} \mathcal{N} \to \mathcal{P} \otimes_\mathcal{A} \mathcal{N}'$ is an isomorphism on cohomology sheaves for any good differential graded $\mathcal{A}$-module $\mathcal{P}$. To do this, let $\mathcal{N}''$ be the cone on the map $\mathcal{N} \to \mathcal{N}'$ as a left differential graded $\mathcal{A}$-module, see Definition 24.22.2. (To be sure, $\mathcal{N}''$ is a bimodule too but we don't need this.) By functoriality of the tensor construction (it is a functor of differential graded categories) we see that $\mathcal{P} \otimes_\mathcal{A} \mathcal{N}''$ is the cone (as a complex of $\mathcal{O}$-modules) on the map $\mathcal{P} \otimes_\mathcal{A} \mathcal{N} \to \mathcal{P} \otimes_\mathcal{A} \mathcal{N}'$. Hence it suffices to show that $\mathcal{P} \otimes_\mathcal{A} \mathcal{N}''$ is acyclic. This follows from the fact that $\mathcal{P}$ is good and the fact that $\mathcal{N}''$ is acyclic as a cone on a quasi-isomorphism. ☐

0FTK Lemma 24.28.6. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. If $\mathcal{N}$ is good as a left differential graded $\mathcal{A}$-module, then we have $\mathcal{M} \otimes_\mathcal{A}^\mathbf{L} \mathcal{N} = \mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ for all differential graded $\mathcal{A}$-modules $\mathcal{M}$.

Proof. Let $\mathcal{P} \to \mathcal{M}$ be a quasi-isomorphism where $\mathcal{P}$ is a good (right) differential graded $\mathcal{A}$-module. To prove the lemma we have to show that $\mathcal{P} \otimes_\mathcal{A} \mathcal{N} \to \mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ is a quasi-isomorphism. The cone C on the map $\mathcal{P} \to \mathcal{M}$ is an acyclic right differential graded $\mathcal{A}$-module. Hence $C \otimes_\mathcal{A} \mathcal{N}$ is acyclic as $\mathcal{N}$ is assumed good as a left differential graded $\mathcal{A}$-module. Since $C \otimes_\mathcal{A} \mathcal{N}$ is the cone on the maps $\mathcal{P} \otimes_\mathcal{A} \mathcal{N} \to \mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ as a complex of $\mathcal{O}$-modules we conclude. $\square$

0FTL Lemma 24.28.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ and $\mathcal{N}'$ be a differential graded $(\mathcal{A}, \mathcal{A}')$-bimodule and $(\mathcal{A}', \mathcal{A}'')$-bimodule. Assume that the canonical map

$$\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' \longrightarrow \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

in $D(\mathcal{A}'', \text{d})$ is a quasi-isomorphism. Then we have

$$(\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}) \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' = \mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}')$$

as functors $D(\mathcal{A}, \text{d}) \to D(\mathcal{A}'', \text{d})$.

Proof. Choose a good differential graded $\mathcal{A}$-module $\mathcal{P}$ and a quasi-isomorphism $\mathcal{P} \to \mathcal{M}$, see Lemma 24.23.7. Then

$$\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}') = \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

and we have

$$(\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N}) \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' = (\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}) \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}'$$

Thus we have to show the canonical map

$$(\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}) \otimes^{\mathbf{L}}_{\mathcal{A}'} \mathcal{N}' \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

is a quasi-isomorphism. Choose a quasi-isomorphism $\mathcal{Q} \to \mathcal{N}'$ where $\mathcal{Q}$ is a good left differential graded $\mathcal{A}'$-module (Lemma 24.23.7). By Lemma 24.28.6 the map above as a map in the derived category of $\mathcal{O}$-modules is the map

$$\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q} \longrightarrow \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

Since $\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q} \to \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$ is a quasi-isomorphism by assumption and $\mathcal{P}$ is a good differential graded $\mathcal{A}$-module this map is an quasi-isomorphism by Lemma 24.28.5 (the left and right hand side compute $\mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{Q})$ and $\mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}')$ or you can just repeat the argument in the proof of the lemma). $\square$

24.29. Derived pushforward

0FTM The existence of enough K-injective guarantees that we can take the right derived functor of any exact functor on the homotopy category.

0FTN Lemma 24.29.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded algebras on $(\mathcal{C}, \mathcal{O})$. Then any exact functor

$$T : K(\text{Mod}(\mathcal{A}, \text{d})) \longrightarrow \mathcal{D}$$

of triangulated categories has a right derived extension $RT : D(\mathcal{A}, \text{d}) \to \mathcal{D}$ whose value on a graded injective and K-injective differential graded $\mathcal{A}$-module $\mathcal{I}$ is $T(\mathcal{I})$.

Proof. By Theorem 24.25.13 for any (right) differential graded $\mathcal{A}$-module $\mathcal{M}$ there exists a quasi-isomorphism $\mathcal{M} \to \mathcal{I}$ where $\mathcal{I}$ is a graded injective and K-injective differential graded $\mathcal{A}$-module. Hence by Derived Categories, Lemma 13.14.15 it suffices to show that given a quasi-isomorphism $\mathcal{I} \to \mathcal{I}'$ of differential graded $\mathcal{A}$-modules which are both graded injective and K-injective then $T(\mathcal{I}) \to T(\mathcal{I}')$ is an isomorphism. This is true because the map $\mathcal{I} \to \mathcal{I}'$ is an isomorphism in $K(\text{Mod}(\mathcal{A}, \text{d}))$ as follows for example from Lemma 24.26.7 (or one can deduce it from Lemma 24.25.10). $\square$

There are a number of functors we have already seen to which this applies. Here are two examples.

0FTP Definition 24.29.2. Derived internal hom and derived pushforward.

(1) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. The right derived extension
$$R\mathcal{H}om_{\mathcal{B}}(\mathcal{N}, -) : D(\mathcal{B}, \text{d}) \longrightarrow D(\mathcal{A}, \text{d})$$
of the internal hom functor $\mathcal{H}om^{dg}_{\mathcal{B}}(\mathcal{N}, -)$ is called derived internal hom.

(2) Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Let $\varphi : \mathcal{B} \to f_*\mathcal{A}$ be a homomorphism of differential graded $\mathcal{O}_\mathcal{D}$-algebras. The right derived extension
$$Rf_* : D(\mathcal{A}, \text{d}) \longrightarrow D(\mathcal{B}, \text{d})$$
of the pushforward f_* is called derived pushforward.

It turns out that $Rf_* : D(\mathcal{A}, \text{d}) \to D(\mathcal{B}, \text{d})$ agrees with derived pusforward on underlying complexes of $\mathcal{O}$-modules, see Lemma 24.29.8.

These functors are the adjoints of derived pullback and derived tensor product.

0FTQ Lemma 24.29.3. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{B}$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{B})$-bimodule. Then
$$R\mathcal{H}om_{\mathcal{B}}(\mathcal{N}, -) : D(\mathcal{B}, \text{d}) \longrightarrow D(\mathcal{A}, \text{d})$$
is right adjoint to
$$- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{N} : D(\mathcal{A}, \text{d}) \longrightarrow D(\mathcal{B}, \text{d})$$

Proof. This follows from Derived Categories, Lemma 13.30.1 and Lemma 24.17.3. $\square$

0FTR Lemma 24.29.4. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Let $\varphi : \mathcal{B} \to f_*\mathcal{A}$ be a homomorphism of differential graded $\mathcal{O}_\mathcal{D}$-algebras. Then
$$Rf_* : D(\mathcal{A}, \text{d}) \longrightarrow D(\mathcal{B}, \text{d})$$
is right adjoint to
$$Lf^* : D(\mathcal{B}, \text{d}) \longrightarrow D(\mathcal{A}, \text{d})$$

Proof. This follows from Derived Categories, Lemma 13.30.1 and Lemma 24.18.1. $\square$

Next, we discuss what happens in the situation considered in Section 24.28.

Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Suppose we are given a map
$$\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$$

of differential graded $f^{-1}\mathcal{O}_\mathcal{D}$-algebras. By the adjunction of restriction and extension of scalars, this is the same thing as a map $\varphi : f^*\mathcal{B} \to \mathcal{A}$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras or equivalently φ can be viewed as a map

$$\varphi : \mathcal{B} \to f_*\mathcal{A}$$

of differential graded $\mathcal{O}_\mathcal{D}$-algebras. See Remark 24.12.2.

In addition to the above, let $\mathcal{A}'$ be a second differential graded $\mathcal{O}_\mathcal{C}$-algebra and let $\mathcal{N}$ be a differential graded $(\mathcal{A}, \mathcal{A}')$-bimodule. In this setting we can consider the functor

$$\text{Mod}(\mathcal{A}', \text{d}) \longrightarrow \text{Mod}(\mathcal{B}, \text{d}), \quad \mathcal{M} \longmapsto f_* \mathcal{H}om^{dg}_{\mathcal{A}'}(\mathcal{N}, \mathcal{M})$$

Observe that this extends to a functor

$$\text{Mod}^{dg}(\mathcal{A}', \text{d}) \longrightarrow \text{Mod}^{dg}(\mathcal{B}, \text{d}), \quad \mathcal{M} \longmapsto f_* \mathcal{H}om^{dg}_{\mathcal{A}'}(\mathcal{N}, \mathcal{M})$$

of differential graded categories by the discussion in Sections 24.18 and 24.17. It follows formally that we also obtain an exact functor

0FTS (24.29.4.1) $$K(\text{Mod}(\mathcal{A}', \text{d})) \longrightarrow K(\text{Mod}(\mathcal{B}, \text{d})), \quad \mathcal{M} \longmapsto f_* \mathcal{H}om^{dg}_{\mathcal{A}'}(\mathcal{N}, \mathcal{M})$$

of triangulated categories.

0FTT Lemma 24.29.5. In the situation above, denote $RT : D(\mathcal{A}', \text{d}) \to D(\mathcal{B}, \text{d})$ the right derived extension of (24.29.4.1). Then we have

$$RT(\mathcal{M}) = Rf_*R\mathcal{H}om(\mathcal{N}, \mathcal{M})$$

functorially in $\mathcal{M}$.

Proof. By Lemmas 24.17.3 and 24.18.1 the functor (24.29.4.1) is right adjoint to the functor (24.28.0.1). By Derived Categories, Lemma 13.30.1 the functor RT is right adjoint to the functor of Lemma 24.28.1 which is equal to $Lf^*(-) \otimes^\mathbf{L}_\mathcal{A} \mathcal{N}$ by Lemma 24.28.3. By Lemmas 24.29.3 and 24.29.4 the functor $Lf^*(-) \otimes^\mathbf{L}_\mathcal{A} \mathcal{N}$ is left adjoint to $Rf_*R\mathcal{H}om(\mathcal{N}, -)$ Thus we conclude by uniqueness of adjoints. $\square$

0FTU Lemma 24.29.6. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and $(g, g^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}''), \mathcal{O}'')$ be morphisms of ringed topoi. Let $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ be a differential graded $\mathcal{O}$-algebra, $\mathcal{O}'$-algebra, and $\mathcal{O}''$-algebra. Let $\varphi : \mathcal{A}' \to f_*\mathcal{A}$ and $\varphi' : \mathcal{A}'' \to g_*\mathcal{A}'$ be a homomorphism of differential graded $\mathcal{O}'$-algebras and $\mathcal{O}''$-algebras. Then we have $R(g \circ f)_* = Rg_* \circ Rf_* : D(\mathcal{A}, \text{d}) \to D(\mathcal{A}'', \text{d})$.

Proof. Follows from Lemmas 24.28.4 and 24.29.4 and uniqueness of adjoints. $\square$

0FTV Lemma 24.29.7. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ be differential graded $\mathcal{O}$-algebras. Let $\mathcal{N}$ and $\mathcal{N}'$ be a differential graded $(\mathcal{A}, \mathcal{A}')$-bimodule and $(\mathcal{A}', \mathcal{A}'')$-bimodule. Assume that the canonical map

$$\mathcal{N} \otimes^\mathbf{L}_{\mathcal{A}'} \mathcal{N}' \longrightarrow \mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}'$$

in $D(\mathcal{A}'', \text{d})$ is a quasi-isomorphism. Then we have

$$R\mathcal{H}om_{\mathcal{A}''}(\mathcal{N} \otimes_{\mathcal{A}'} \mathcal{N}', -) = R\mathcal{H}om_{\mathcal{A}'}(\mathcal{N}, R\mathcal{H}om_{\mathcal{A}''}(\mathcal{N}', -))$$

as functors $D(\mathcal{A}'', \text{d}) \to D(\mathcal{A}, \text{d})$.

Proof. Follows from Lemmas 24.28.7 and 24.29.3 and uniqueness of adjoints. $\square$

0FTW Lemma 24.29.8. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a differential graded $\mathcal{O}_\mathcal{C}$-algebra. Let $\mathcal{B}$ be a differential graded $\mathcal{O}_\mathcal{D}$-algebra. Let $\varphi : \mathcal{B} \to f_*\mathcal{A}$ be a homomorphism of differential graded $\mathcal{O}_\mathcal{D}$-algebras. The diagram

$$\begin{tikzcd} D(\mathcal{A}, \mathrm{d}) \arrow[r, "forget"'] \arrow[d, "Rf_*"'] & D(\mathcal{O}_\mathcal{C}) \arrow[d, "Rf_*"] \\ D(\mathcal{B}, \mathrm{d}) \arrow[r, "forget"] & D(\mathcal{O}_\mathcal{D}) \end{tikzcd}$$

commutes.

Proof. Besides identifying some categories, this lemma follows immediately from Lemma 24.29.6.

We may view $(\mathcal{O}_\mathcal{C}, 0)$ as a differential graded $\mathcal{O}_\mathcal{C}$-algebra by placing $\mathcal{O}_\mathcal{C}$ in degree 0 and endowing it with the zero differential. It is clear that we have

$$\mathrm{Mod}(\mathcal{O}_\mathcal{C}, 0) = \mathrm{Comp}(\mathcal{O}_\mathcal{C}) \quad \text{and} \quad D(\mathcal{O}_\mathcal{C}, 0) = D(\mathcal{O}_\mathcal{C})$$

Via this identification the forgetful functor $\mathrm{Mod}(\mathcal{A}, \mathrm{d}) \to \mathrm{Comp}(\mathcal{O}_\mathcal{C})$ is the "pushforward" $\mathrm{id}_{\mathcal{C},*}$ defined in Section 24.18 corresponding to the identity morphism $\mathrm{id}_\mathcal{C} : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{C}, \mathcal{O}_\mathcal{C})$ of ringed topoi and the map $(\mathcal{O}_\mathcal{C}, 0) \to (\mathcal{A}, \mathrm{d})$ of differential graded $\mathcal{O}_\mathcal{C}$-algebras. Since $\mathrm{id}_{\mathcal{C},*}$ is exact, we immediately see that

$$R\mathrm{id}_{\mathcal{C},*} = forget : D(\mathcal{A}, \mathrm{d}) \longrightarrow D(\mathcal{O}_\mathcal{C}, 0) = D(\mathcal{O}_\mathcal{C})$$

The exact same reasoning shows that

$$R\mathrm{id}_{\mathcal{D},*} = forget : D(\mathcal{B}, \mathrm{d}) \longrightarrow D(\mathcal{O}_\mathcal{D}, 0) = D(\mathcal{O}_\mathcal{D})$$

Moreover, the construction of $Rf_* : D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_\mathcal{D})$ of Cohomology on Sites, Section 21.19 agrees with the construction of $Rf_* : D(\mathcal{O}_\mathcal{C}, 0) \to D(\mathcal{O}_\mathcal{D}, 0)$ in Definition 24.29.2 as both functors are defined as the right derived extension of pushforward on underlying complexes of modules. By Lemma 24.29.6 we see that both $Rf_* \circ R\mathrm{id}_{\mathcal{C},*}$ and $R\mathrm{id}_{\mathcal{D},*} \circ Rf_*$ are the derived functors of $f_* \circ forget = forget \circ f_*$ and hence equal by uniqueness of adjoints. $\square$

0FTX Lemma 24.29.9. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{A}$ be a differential graded $\mathcal{O}$-algebra. Let $\mathcal{M}$ be a differential graded $\mathcal{A}$-module. Let $n \in \mathbf{Z}$. We have

$$H^n(\mathcal{C}, \mathcal{M}) = \mathrm{Hom}_{D(\mathcal{A}, \mathrm{d})}(\mathcal{A}, \mathcal{M}[n])$$

where on the left hand side we have the cohomology of $\mathcal{M}$ viewed as a complex of $\mathcal{O}$-modules.

Proof. To prove the formula, observe that

$$R\Gamma(\mathcal{C}, \mathcal{M}) = \Gamma(\mathcal{C}, \mathcal{I})$$

where $\mathcal{M} \to \mathcal{I}$ is a quasi-isomorphism to a graded injective and K-injective differential graded $\mathcal{A}$-module $\mathcal{I}$ (combine Lemmas 24.29.1 and 24.29.8). By Lemma 24.26.7 we have

$$\mathrm{Hom}_{D(\mathcal{A}, \mathrm{d})}(\mathcal{A}, \mathcal{M}[n]) = \mathrm{Hom}_{K(\mathrm{Mod}(\mathcal{A}, \mathrm{d}))}(\mathcal{M}, \mathcal{I}[n]) = H^0(\Gamma(\mathcal{C}, \mathcal{I}[n])) = H^n(\Gamma(\mathcal{C}, \mathcal{I}))$$

Combining these two results we obtain our equality. $\square$

24.30. Equivalences of derived categories

0FTY This section is the analogue of Differential Graded Algebra, Section 22.37.

0FTZ **Lemma 24.30.1.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of differential graded $\mathcal{O}$-algebras which induces an isomorphism on cohomology sheaves, then

$$D(\mathcal{A}, \mathrm{d}) \longrightarrow D(\mathcal{B}, \mathrm{d}), \quad \mathcal{M} \longmapsto \mathcal{M} \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{B}$$

is an equivalence of categories.

Proof. Recall that the restriction functor

$$\mathrm{Mod}^{dg}(\mathcal{B}, \mathrm{d}) \to \mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d}), \quad \mathcal{N} \mapsto res_\varphi \mathcal{N}$$

is a right adjoint to

$$\mathrm{Mod}^{dg}(\mathcal{A}, \mathrm{d}) \to \mathrm{Mod}^{dg}(\mathcal{B}, \mathrm{d}), \quad \mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$$

See Section 24.17. Since restriction sends quasi-isomorphisms to quasi-isomorphisms, we see that it trivially has a left derived extension (given by restriction). This functor will be right adjoint to $- \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{B}$ by Derived Categories, Lemma 13.30.1. The adjunction map

$$\mathcal{M} \to res_\varphi(\mathcal{M} \otimes_{\mathcal{A}}^{\mathbf{L}} \mathcal{B})$$

is an isomorphism in $D(\mathcal{A}, \mathrm{d})$ by our assumption that $\mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of (left) differential graded $\mathcal{A}$-modules. In particular, the functor of the lemma is fully faithful, see Categories, Lemma 4.24.4. It is clear that the kernel of the restriction functor $D(\mathcal{B}, \mathrm{d}) \to D(\mathcal{A}, \mathrm{d})$ is zero. Thus we conclude by Derived Categories, Lemma 13.7.2. □

24.31. Resolutions of differential graded algebras

0FU0 This section is the analogue of Differential Graded Algebra, Section 22.38.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. As in Remark 24.23.5 consider a sheaf of graded sets $\mathcal{S}$ on $\mathcal{C}$. Let us think of the r-fold self product $\mathcal{S} \times \ldots \times \mathcal{S}$ as a sheaf of graded sets with the rule $\deg(s_1 \cdot \ldots \cdot s_r) = \sum \deg(s_i)$. Here given local sections $s_i \in \mathcal{S}(U)$, $i = 1, \ldots, r$ we use $s_1 \cdot \ldots \cdot s_r$ to denote the corresponding section of $\mathcal{S} \times \ldots \times \mathcal{S}$ over U. Let us denote $\mathcal{O}\langle\mathcal{S}\rangle$ the free graded $\mathcal{O}$-algebra on $\mathcal{S}$. More precisely, we set

$$\mathcal{O}\langle\mathcal{S}\rangle = \mathcal{O} \oplus \bigoplus\nolimits_{r \geq 1} \mathcal{O}[\mathcal{S} \times \ldots \times \mathcal{S}]$$

with notation as in Remark 24.23.5. This becomes a sheaf of graded $\mathcal{O}$-algebras by concatenation

$$(s_1 \cdot \ldots \cdot s_r)(s'_1 \cdot \ldots \cdot s'_{r'}) = s_1 \cdot \ldots s_r \cdot s'_1 \cdot \ldots \cdot s'_{r'}$$

We may endow $\mathcal{O}\langle\mathcal{S}\rangle$ with a differential by setting $\mathrm{d}(s) = 0$ for all local sections s of $\mathcal{S}$ and extending uniquely using the Leibniz rule although it is important to also consider other differentials.

Indeed, suppose that we are given a system of the following kind

(1) for $i = 0, 1, 2, \ldots$ sheaves of graded sets $\mathcal{S}_i$,
(2) for $i = 0, 1, 2, \ldots$ maps

$$\delta_{i+1} : \mathcal{S}_{i+1} \longrightarrow \mathcal{A}_i = \mathcal{O}\langle\mathcal{S}_0 \amalg \ldots \amalg \mathcal{S}_i\rangle$$

of sheaves of graded sets of degree 1 whose image is contained in the kernel of the inductively defined differential on the target.

More precisely, we first set $\mathcal{A}_0 = \mathcal{O}\langle \mathcal{S}_0 \rangle$ and we endow it with the unique differential satisfying the Leibniz rule where $\text{d}(s) = 0$ for any local section s of $\mathcal{S}$. By induction, assume given a differential d on $\mathcal{A}_i$. Then we extend it to the unique differential on $\mathcal{A}_{i+1}$ satisfying the Leibniz rule and with

$$\text{d}(s) = \delta(s)$$

where $\delta(s) = \delta_j(s)$ if s is in the summand $\mathcal{S}_j$ of $\mathcal{S}_0 \amalg \ldots \amalg \mathcal{S}_{i+1}$. This makes sense exactly because $\delta(s)$ is in the kernel of the inductively defined differential.

0FU1 Lemma 24.31.1. In the situation above the differential graded $\mathcal{O}$-algebra

$$\mathcal{A} = \text{colim}\, \mathcal{A}_i$$

has the following property: for any morphism $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ of ringed topoi, the pullback $f^*\mathcal{A}$ is flat as a graded $\mathcal{O}'$-module and is K-flat as a complex of $\mathcal{O}'$-modules.

Proof. Observe that $f^*\mathcal{A} = \text{colim}\, f^*\mathcal{A}_i$ and that

$$f^*\mathcal{A}_i = \mathcal{O}'\langle f^{-1}\mathcal{S}_0 \amalg \ldots \amalg f^{-1}\mathcal{S}_i \rangle$$

with differential given by the inductive procedure above using $f^{-1}\delta_{i+1}$. Thus it suffices to prove that $\mathcal{A}$ is flat as a graded $\mathcal{O}$-module and is K-flat as a complex of $\mathcal{O}$-modules. For this it suffices to prove that each $\mathcal{A}_i$ is flat as a graded $\mathcal{O}$-module and is K-flat as a complex of $\mathcal{O}$-modules, compare with Lemma 24.23.3.

For $i \geq 1$ write $\mathcal{S} = \mathcal{S}_0 \amalg \ldots \amalg \mathcal{S}_i$ so that we have $\mathcal{A}_i = \mathcal{O}\langle \mathcal{S} \rangle$ as a graded $\mathcal{O}$-algebra. We are going to construct a filtration of this algebra by differential graded $\mathcal{O}$-submodules.

Set $W = \mathbf{Z}_{\geq 0}^{i+1}$ considered with lexicographical ordering. Namely, given $w = (w_0, \ldots w_i)$ and $w' = (w'_0, \ldots, w'_i)$ in W we say

$$w > w' \Leftrightarrow \exists j,\ 0 \leq j \leq i : w_i = w'_i,\ w_{i-1} = w'_{i-1},\ \ldots,\ w_{j+1} = w'_{j+1},\ w_j > w'_j$$

and so on. Suppose given a section $s = s_1 \cdot \ldots \cdot s_r$ of $\mathcal{S} \times \ldots \times \mathcal{S}$ over U. We say that the weight of s is defined if we have $s_a \in \mathcal{S}_{j_a}(U)$ for a unique $0 \leq j_a \leq i$. In this case we define the weight

$$w(s) = (w_0(s), \ldots, w_i(s)) \in W, \quad w_j(s) = |\{a \mid j_a = j\}|$$

The weight of any section of $\mathcal{S} \times \ldots \times \mathcal{S}$ is defined locally. The reader checks easily that we obtain a disjoint union decompostion

$$\mathcal{S} \times \ldots \times \mathcal{S} = \coprod\nolimits_{w \in W} (\mathcal{S} \times \ldots \times \mathcal{S})_w$$

into the subsheaves of sections of a given weight. Of course only $w \in W$ with $\sum_{0 \leq j \leq i} w_j = r$ show up for a given r. We correspondingly obtain a decomposition

$$\mathcal{A}_i = \mathcal{O} \oplus \bigoplus\nolimits_{r \geq 1} \bigoplus\nolimits_{w \in W} \mathcal{O}[(\mathcal{S} \times \ldots \times \mathcal{S})_w]$$

The rest of the proof relies on the following trivial observation: given r, w and local section $s = s_1 \cdot \ldots \cdot s_r$ of $(\mathcal{S} \times \ldots \times \mathcal{S})_w$ we have

$$\text{d}(s) \text{ is a local section of } \mathcal{O} \oplus \bigoplus\nolimits_{r' \geq 1} \bigoplus\nolimits_{w' \in W,\ w' < w} \mathcal{O}[(\mathcal{S} \times \ldots \times \mathcal{S})_{w'}]$$

The reason is that in each of the expressions

$$(-1)^{\deg(s_1) + \ldots + \deg(s_{a-1})} s_1 \cdot \ldots s_{a-1} \cdot \delta(s_a) \cdot s_{a+1} \cdot \ldots \cdot s_r$$

whose sum give the element $\mathrm{d}(s)$ the element $\delta(s_a)$ is locally a $\mathcal{O}$-linear combination of elements $s'_1 \cdot \ldots \cdot s'_{r'}$ with $s'_{a'}$ in $\mathcal{S}_{j'_a}$ for some $0 \leq j'_{a'} < j_a$ where j_a is such that s_a is section of $\mathcal{S}_{j_a}$.

What this means is the following. Suppose for $w \in W$ we set

$$F_w\mathcal{A}_i = \mathcal{O} \oplus \bigoplus\nolimits_{r \geq 1} \bigoplus\nolimits_{w' \in W,\ w' \leq w} \mathcal{O}[(\mathcal{S} \times \ldots \times \mathcal{S})_{w'}]$$

By the observation above this is a differential graded $\mathcal{O}$-submodule. We get admissible short exact sequences

$$0 \to \mathrm{colim}_{w' < w} F_{w'}\mathcal{A}_i \to F_w\mathcal{A}_i \to \bigoplus\nolimits_{r \geq 1} \mathcal{O}[(\mathcal{S} \times \ldots \times \mathcal{S})_w] \to 0$$

of differential graded $\mathcal{A}$-modules where the differential on the right hand side is zero.

Now we finish the proof by transfinite induction over the ordered set W. The differential graded complex $F_0\mathcal{A}_0$ is the summand $\mathcal{O}$ and this is K-flat and graded flat. For $w \in W$ if the result is true for $F_{w'}\mathcal{A}_i$ for $w' < w$, then by Lemmas 24.23.3, 24.23.2, and 24.23.6 we obtain the result for w. Finally, we have $\mathcal{A}_i = \mathrm{colim}_{w \in W} F_w\mathcal{A}_i$ and we conclude. □

0FU2 Lemma 24.31.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{B}, \mathrm{d})$ be a differential graded $\mathcal{O}$-algebra. There exists a quasi-isomorphism of differential graded $\mathcal{O}$-algebras $(\mathcal{A}, \mathrm{d}) \to (\mathcal{B}, \mathrm{d})$ such that $\mathcal{A}$ is graded flat and K-flat as a complex of $\mathcal{O}$-modules and such that the same is true after pullback by any morphism of ringed topoi.

Proof. The proof is exactly the same as the first proof of Lemma 24.23.7 but now working with free graded algebras instead of free graded modules.

We will construct $\mathcal{A} = \mathrm{colim}\,\mathcal{A}_i$ as in Lemma 24.31.1 by constructing

$$\mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \ldots \to \mathcal{B}$$

Let $\mathcal{S}_0$ be the sheaf of graded sets (Remark 24.23.5) whose degree n part is $\mathrm{Ker}(\mathrm{d}^n_{\mathcal{B}})$. Consider the homomorphism of differential graded modules

$$\mathcal{A}_0 = \mathcal{O}\langle \mathcal{S}_0 \rangle \longrightarrow \mathcal{B}$$

where map sends a local section s of $\mathcal{S}_0$ to the corresponding local section of $\mathcal{A}^{\deg(s)}$ (which is in the kernel of the differential, so our map is a map of differential graded algebras indeed). By construction the induced maps on cohomology sheaves $H^n(\mathcal{A}_0) \to H^n(\mathcal{B})$ are surjective and hence the same will remain true for all i.

Induction step of the construction. Given $\mathcal{A}_i \to \mathcal{B}$ denote $\mathcal{S}_{i+1}$ the sheaf of graded sets whose degree n part is

$$\mathrm{Ker}(\mathrm{d}^{n+1}_{\mathcal{A}_i}) \times_{\mathcal{B}^{n+1}, \mathrm{d}} \mathcal{B}^n$$

This comes equipped with a canonical map

$$\delta_{i+1} : \mathcal{S}_{i+1} \longrightarrow \mathcal{A}_i$$

whose image is contained in the kernel of $\mathrm{d}_{\mathcal{A}_i}$ by construction. Hence $\mathcal{A}_{i+1} = \mathcal{O}\langle \mathcal{S}_0 \amalg \ldots \mathcal{S}_{i+1} \rangle$ has a differential exteding the differential on $\mathcal{A}_i$, see discussion at the start of this section. The map from $\mathcal{A}_{i+1}$ to $\mathcal{B}$ is the unique map of graded algebras which restricts to the given map on $\mathcal{A}_i$ and sends a local section $s = (a, b)$

of $\mathcal{S}_{i+1}$ to b in $\mathcal{B}$. This is compatible with differentials exactly because $\text{d}(b)$ is the image of a in $\mathcal{B}$.

The map $\mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism: we have $H^n(\mathcal{A}) = \text{colim}\, H^n(\mathcal{A}_i)$ and for each i the map $H^n(\mathcal{A}_i) \to H^n(\mathcal{B})$ is surjective with kernel annihilated by the map $H^n(\mathcal{A}_i) \to H^n(\mathcal{A}_{i+1})$ by construction. Finally, the flatness condition for $\mathcal{A}$ where shown in Lemma 24.31.1. $\square$

24.32. Miscellany

0FU3 Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a sheaf of differential graded $\mathcal{O}$-algebras. Using the composition[3]

$$\mathcal{A} \otimes_{\mathcal{O}}^{\mathbf{L}} \mathcal{A} \longrightarrow \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A} \longrightarrow \mathcal{A}$$

and the relative cup product (see Cohomology on Sites, Remark 21.19.7 and Section 21.33) we obtain a multiplication[4]

$$\mu : Rf_*\mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \longrightarrow Rf_*\mathcal{A}$$

in $D(\mathcal{O}')$. This multiplication is associative in the sense that the diagram

$$\begin{tikzcd} Rf_*\mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \arrow[r, "\mu\otimes 1"'] \arrow[d, "1\otimes\mu"'] & Rf_*\mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \arrow[d, "\mu"] \\ Rf_*\mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \arrow[r, "\mu"] & Rf_*\mathcal{A} \end{tikzcd}$$

commutes in $D(\mathcal{O}')$; this follows from Cohomology on Sites, Lemma 21.33.2. In exactly the same way, given a right differential graded $\mathcal{A}$-module $\mathcal{M}$ we obtain a multiplication

$$\mu_\mathcal{M} : Rf_*\mathcal{M} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \longrightarrow Rf_*\mathcal{M}$$

in $D(\mathcal{O}')$. This multiplication is compatible with μ above in the sense that the diagram

$$\begin{tikzcd} Rf_*\mathcal{M} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \arrow[r, "\mu_\mathcal{M}\otimes 1"'] \arrow[d, "1\otimes\mu"'] & Rf_*\mathcal{M} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \arrow[d, "\mu_\mathcal{M}"] \\ Rf_*\mathcal{M} \otimes_{\mathcal{O}'}^{\mathbf{L}} Rf_*\mathcal{A} \arrow[r, "\mu_\mathcal{M}"] & Rf_*\mathcal{M} \end{tikzcd}$$

commutes in $D(\mathcal{O}')$; again this follows from Cohomology on Sites, Lemma 21.33.2.

A particular example of the above is when one takes f to be the morphism to the punctual topos $Sh(pt)$. In that case μ is just the cup product map

$$R\Gamma(\mathcal{C}, \mathcal{A}) \otimes_{\Gamma(\mathcal{C},\mathcal{O})}^{\mathbf{L}} R\Gamma(\mathcal{C}, \mathcal{A}) \longrightarrow R\Gamma(\mathcal{C}, \mathcal{A}), \quad \eta \otimes \theta \mapsto \eta \cup \theta$$

and similarly $\mu_\mathcal{M}$ is the cup product map

$$R\Gamma(\mathcal{C}, \mathcal{M}) \otimes_{\Gamma(\mathcal{C},\mathcal{O})}^{\mathbf{L}} R\Gamma(\mathcal{C}, \mathcal{A}) \longrightarrow R\Gamma(\mathcal{C}, \mathcal{M}), \quad \eta \otimes \theta \mapsto \eta \cup \theta$$

[3]It would be more precise to write $F(\mathcal{A}) \otimes_{\mathcal{O}}^{\mathbf{L}} F(\mathcal{A}) \to F(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) \to F(\mathcal{A})$ were F denotes the forgetful functor to complexes of $\mathcal{O}$-modules. Also, note that $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}$ indicates the tensor product of Section 24.15 so that $F(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) = \text{Tot}(F(\mathcal{A}) \otimes_{\mathcal{O}} F(\mathcal{A}))$. The first arrow of the sequence is the canonical map from the derived tensor product of two complexes of $\mathcal{O}$-modules to the usual tensor product of complexes of $\mathcal{O}$-modules.

[4]Here and below $Rf_* : D(\mathcal{O}) \to D(\mathcal{O}')$ is the derived functor studied in Cohomology on Sites, Section 21.19 ff.

In general, via the identifications

$$R\Gamma(\mathcal{C}, \mathcal{A}) = R\Gamma(\mathcal{C}', Rf_*\mathcal{A}) \quad \text{and} \quad R\Gamma(\mathcal{C}, \mathcal{M}) = R\Gamma(\mathcal{C}', Rf_*\mathcal{M})$$

of Cohomology on Sites, Remark 21.14.4 the map $\mu_\mathcal{M}$ induces the cup product on cohomology. To see this use Cohomology on Sites, Lemma 21.33.4 where the second morphism of topoi is the morphism from $Sh(\mathcal{C}')$ to the punctual topos as above.

If $\mathcal{M}_1 \to \mathcal{M}_2$ is a homomorphism of right differential graded $\mathcal{A}$-modules, then the diagram

$$\begin{tikzcd}
Rf_*\mathcal{M}_1 \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \arrow[r, "\mu_{\mathcal{M}_1}"'] \arrow[d] & Rf_*\mathcal{M}_1 \arrow[d] \\
Rf_*\mathcal{M}_2 \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \arrow[r, "\mu_{\mathcal{M}_2}"] & Rf_*\mathcal{M}_2
\end{tikzcd}$$

commutes in $D(\mathcal{O}')$; this follows from the fact that the relative cup product is functorial. Suppose we have a short exact sequence

$$0 \to \mathcal{M}_1 \xrightarrow{a} \mathcal{M}_2 \to \mathcal{M}_3 \to 0$$

of right differential graded $\mathcal{A}$-modules. Then we claim that the diagram

$$\begin{tikzcd}
Rf_*\mathcal{M}_3 \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \arrow[r, "\mu_{\mathcal{M}_3}"'] \arrow[d, "Rf_*\delta\otimes\mathrm{id}"'] & Rf_*\mathcal{M}_3 \arrow[d, "Rf_*\delta"] \\
Rf_*\mathcal{M}_1[1] \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \arrow[r, "\mu_{\mathcal{M}_1[1]}"] & Rf_*\mathcal{M}_1[1]
\end{tikzcd}$$

commutes in $D(\mathcal{O}')$ where $\delta : \mathcal{M}_3 \to \mathcal{M}_1[1]$ is the morphism of $D(\mathcal{O})$ coming from the given short exact sequence (see Derived Categories, Section 13.12). This is clear if our sequence is split as a sequence of graded right $\mathcal{A}$-modules, because in this case δ can be represented by a map of right $\mathcal{A}$-modules and the discussion above applies. In general we argue using the cone on a and the diagram

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, "a"'] \arrow[d] & \mathcal{M}_2 \arrow[r, "i"'] \arrow[d] & C(a) \arrow[r, "-p"'] \arrow[d, "q"] & \mathcal{M}_1[1] \arrow[d] \\
\mathcal{M}_1 \arrow[r] & \mathcal{M}_2 \arrow[r] & \mathcal{M}_3 \arrow[r, "\delta"] & \mathcal{M}_1[1]
\end{tikzcd}$$

where the right square is commutative in $D(\mathcal{O})$ by the definition of δ in Derived Categories, Lemma 13.12.1. Now the cone $C(a)$ has the structure of a right differential graded $\mathcal{A}$-module such that i, p, q are homomorphisms of right differential graded $\mathcal{A}$-modules, see Definition 24.22.2. Hence by the above we know that the corresponding diagrams commute for the morphisms q and $-p$. Since q is an isomorphism in $D(\mathcal{O})$ we conclude the same is true for δ as desired.

In the situation above given a right differential graded $\mathcal{A}$-module $\mathcal{M}$ let

$$\xi \in H^n(\mathcal{C}, \mathcal{M})$$

In other words, ξ is a degree n cohomology class in the cohomology of $\mathcal{M}$ viewed as a complex of $\mathcal{O}$-modules. By Lemma 24.29.9 we can construct maps

$$x : \mathcal{A} \to \mathcal{M}'[n] \quad \text{and} \quad s : \mathcal{M} \to \mathcal{M}'$$

of right differential graded $\mathcal{A}$-modules where s is a quasi-isomorphism and such that ξ is the image of $1 \in H^0(\mathcal{C}, \mathcal{A})$ via the morphism $s[n]^{-1} \circ x$ in the derived category $D(\mathcal{A}, \text{d})$ and a fortiori in the derived category $D(\mathcal{O})$. It follows that the corresponding map

$$\xi' = (s[n])^{-1} \circ x : \mathcal{A} \longrightarrow \mathcal{M}[n]$$

in $D(\mathcal{O})$ is uniquely characterized by the following two properties

(1) ξ' can be lifted to a morphism in $D(\mathcal{A}, \text{d})$, and
(2) $\xi = \xi'(1)$ in $H^0(\mathcal{C}, \mathcal{M}[n]) = H^n(\mathcal{C}, \mathcal{M})$.

Using the compatibilities of x and s with the relative cup product discussed above it follows that for every[5] morphism of ringed topoi $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ the derived pushforward

$$Rf_*\xi' : Rf_*\mathcal{A} \longrightarrow Rf_*\mathcal{M}[n]$$

of ξ' is compatible with the maps μ and $\mu_{\mathcal{M}[n]}$ constructed above in the sense that the diagram

$$\begin{tikzcd} Rf_*\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \arrow[r, "\mu"'] \arrow[d, "Rf_*\xi' \otimes \text{id}"'] & Rf_*\mathcal{A} \arrow[d, "Rf_*\xi'"] \\ Rf_*\mathcal{M}[n] \otimes^{\mathbf{L}}_{\mathcal{O}'} Rf_*\mathcal{A} \arrow[r, "\mu_{\mathcal{M}[n]}"] & Rf_*\mathcal{M}[n] \end{tikzcd}$$

commutes in $D(\mathcal{O}')$. Using this compatibility for the map to the punctual topos, we see in particular that

$$\begin{tikzcd} R\Gamma(\mathcal{C}, \mathcal{A}) \otimes^{\mathbf{L}}_{\Gamma(\mathcal{C}, \mathcal{O})} R\Gamma(\mathcal{C}, \mathcal{A}) \arrow[r] \arrow[d, "\xi' \otimes \text{id}"'] & R\Gamma(\mathcal{C}, \mathcal{A}) \arrow[d, "\xi'"] \\ R\Gamma(\mathcal{C}, \mathcal{M}[n]) \otimes^{\mathbf{L}}_{\Gamma(\mathcal{C}, \mathcal{O})} R\Gamma(\mathcal{C}, \mathcal{A}) \arrow[r] & R\Gamma(\mathcal{C}, \mathcal{M}[n]) \end{tikzcd}$$

commutes. Combined with $\xi'(1) = \xi$ this implies that the induced map on cohomology

$$\xi' : R\Gamma(\mathcal{C}, \mathcal{A}) \to R\Gamma(\mathcal{C}, \mathcal{M}[n]), \quad \eta \mapsto \xi \cup \eta$$

is given by left cup product by ξ as indicated.

24.33. Differential graded modules on a category

0GZ8 This section is the continuation of Cohomology on Sites, Section 21.43.

Let $\mathcal{C}$ be a category. We think of $\mathcal{C}$ as a site with the chaotic topology. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $(\mathcal{A}, \text{d})$ be a sheaf of differential graded $\mathcal{O}$-algebras. In other words, $\mathcal{O}$ is a presheaf of rings on the category $\mathcal{C}$ and $(\mathcal{A}, \text{d})$ is a presheaf of differential graded $\mathcal{O}$-algebras on $\mathcal{C}$, see Categories, Definition 4.3.3.

0GZ9 **Definition 24.33.1.** In the situation above, we denote $QC(\mathcal{A}, \text{d})$ the full subcategory of $D(\mathcal{A}, \text{d})$ consisting of objects M such that for all $U \to V$ in $\mathcal{C}$ the canonical map

$$R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}(U) \longrightarrow R\Gamma(U, M)$$

is an isomorphism in $D(\mathcal{A}(U), \text{d})$.

0GZA **Lemma 24.33.2.** *In the situation above, the subcategory $QC(\mathcal{A}, \text{d})$ is a strictly full, saturated, triangulated subcategory of $D(\mathcal{A}, \text{d})$ preserved by arbitrary direct sums.*

[5]For example the identity morphism.

Proof. Let U be an object of $\mathcal{C}$. Since the topology on $\mathcal{C}$ is chaotic, the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is exact and commutes with direct sums. Hence the exact functor $M \mapsto R\Gamma(U, M)$ is computed by representing K by any differential graded $\mathcal{A}$-module $\mathcal{M}$ and taking $\mathcal{M}(U)$. Thus $R\Gamma(U, -)$ commutes with direct sums, see Lemma 24.26.8. Similarly, given a morphism $U \to V$ of $\mathcal{C}$ the derived tensor product functor $- \otimes^{\mathbf{L}}_{\mathcal{O}(A)} \mathcal{A}(U) : D(\mathcal{A}(V)) \to D(\mathcal{A}(U))$ is exact and commutes with direct sums. The lemma follows from these observations in a straightforward manner; details omitted. $\square$

0GZB Remark 24.33.3. As above, let $\mathcal{C}$ be a category viewed as a site with the chaotic topology, let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$, and let $(\mathcal{A}, \mathrm{d})$ be a sheaf of differential graded $\mathcal{O}$-algebras. Then the analogue of Cohomology on Sites, Proposition 21.43.9 holds for $QC(\mathcal{A}, \mathrm{d})$ with almost exactly the same proof:

(1) any contravariant cohomological functor $H : QC(\mathcal{A}, \mathrm{d}) \to \mathrm{Ab}$ which transforms direct sums into products is representable,
(2) any exact functor $F : QC(\mathcal{A}, \mathrm{d}) \to \mathcal{D}$ of triangulated categories which transforms direct sums into direct sums has an exact right adjoint, and
(3) the inclusion functor $QC(\mathcal{A}, \mathrm{d}) \to D(\mathcal{A}, \mathrm{d})$ has an exact right adjoint.

If we ever need this we will precisely formulate and prove this here.

Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor between categories. If we view $\mathcal{C}$ and $\mathcal{C}'$ as sites with the chaotic topology, then u is a continuous and cocontinuous functor. Hence we obtain a morphism $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ of topoi, see Sites, Lemma 7.21.1. Additionally, suppose given sheaves of rings $\mathcal{O}$ on $\mathcal{C}$ and $\mathcal{O}'$ on $\mathcal{C}'$ and a map $g^\sharp : g^{-1}\mathcal{O} \to \mathcal{O}'$. We denote the corresponding morphism of ringed topoi simply $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$, see Modules on Sites, Section 18.7. Finally, suppose that $(\mathcal{A}, \mathrm{d})$ is a sheaf of differential graded $\mathcal{O}$-algebras and that $(\mathcal{A}', \mathrm{d})$ is a sheaf of differential graded $\mathcal{O}'$-algebras and moreover that we are given a map $\varphi : g^*\mathcal{A} \to \mathcal{A}'$ of differential graded $\mathcal{O}'$-algebras (see Section 24.18).

0GZC Lemma 24.33.4. Let $g : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ and $\varphi : g^*\mathcal{A} \to \mathcal{A}'$ be as above. Then the functor $Lg^* : D(\mathcal{A}, \mathrm{d}) \to D(\mathcal{A}', \mathrm{d})$ maps $QC(\mathcal{A}, \mathrm{d})$ into $QC(\mathcal{A}', \mathrm{d})$.

Proof. Let $U' \in \mathrm{Ob}(\mathcal{C}')$ with image $U = u(U')$ in $\mathcal{C}$. Let pt denote the category with a single object and a single morphism. Denote $(Sh(pt), \mathcal{O}'(U'))$ and $(Sh(pt), \mathcal{O}(U))$ the ringed topoi as indicated endowed with the differential graded algebras $\mathcal{A}'(U)$ and $\mathcal{A}(U)$. Of course we identify the derived category of differential graded modules on these with $D(\mathcal{A}'(U'), \mathrm{d})$ and $D(\mathcal{A}(U), \mathrm{d})$. Then we have a commutative diagram of ringed topoi

$$\begin{tikzcd}
(Sh(pt), \mathcal{O}'(U')) \arrow[r, "U'"'] \arrow[d] & (Sh(\mathcal{C}'), \mathcal{O}') \arrow[d, "g"] \\
(Sh(pt), \mathcal{O}(U)) \arrow[r, "U"] & (Sh(\mathcal{C}), \mathcal{O})
\end{tikzcd}$$

each endowed with corresponding differential graded algebras. Pullback along the lower horizontal morphism sends M in $D(\mathcal{A}, \mathrm{d})$ to $R\Gamma(U, K)$ viewed as an object in $D(\mathcal{A}(U), \mathrm{d})$. Pullback by the left vertical arrow sends M to $M \otimes^{\mathbf{L}}_{\mathcal{A}(U)} \mathcal{A}'(U')$. Going around the diagram either direction produces the same result (Lemma 24.28.4) and hence we conclude

$$R\Gamma(U', Lg^*K) = R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{A}(U)} \mathcal{A}'(U')$$

Finally, let $f' : U' \to V'$ be a morphism in $\mathcal{C}'$ and denote $f = u(f') : U = u(U') \to V = u(V')$ the image in $\mathcal{C}$. If K is in $QC(\mathcal{A}, \text{d})$ then we have

$$\begin{aligned} R\Gamma(V', Lg^*K) \otimes^{\mathbf{L}}_{\mathcal{A}'(V')} \mathcal{A}'(U') &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}'(V') \otimes^{\mathbf{L}}_{\mathcal{A}'(V')} \mathcal{A}'(U') \\ &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}'(U') \\ &= R\Gamma(V, K) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}(U) \otimes^{\mathbf{L}}_{\mathcal{A}(U)} \mathcal{A}'(U') \\ &= R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{A}(U)} \mathcal{A}'(U') \\ &= R\Gamma(U', Lg^*K) \end{aligned}$$

as desired. Here we have used the observation above both for U' and V'. □

24.34. Differential graded modules on a category, bis

0GZD We develop a few more results on the notion of quasi-coherent modules introduced in Section 24.33.

0GZE Lemma 24.34.1. Let $\mathcal{C}, \mathcal{O}, \mathcal{A}$ be as in Section 24.33. Let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory with the following property: for every $U \in \text{Ob}(\mathcal{C})$ the category $U/\mathcal{C}'$ of arrows $U \to U'$ is cofiltered. Denote $\mathcal{O}', \mathcal{A}'$ the restrictions of $\mathcal{O}, \mathcal{A}$ to $\mathcal{C}'$. Then restrictions induces an equivalence $QC(\mathcal{A}, \text{d}) \to QC(\mathcal{A}', \text{d})$.

Proof. We will construct a quasi-inverse of the functor. Namely, let M' be an object of $QC(\mathcal{A}', \text{d})$. We may represent M' by a good differential graded module $\mathcal{M}'$, see Lemma 24.23.7. Then for every $U' \in \text{Ob}(\mathcal{C}')$ the differential graded $\mathcal{A}'(U')$-module $\mathcal{M}'(U)$ is K-flat and graded flat and for every morphism $U'_1 \to U'_2$ of $\mathcal{C}'$ the map

$$\mathcal{M}'(U'_2) \otimes_{\mathcal{A}'(U'_2)} \mathcal{A}'(U'_1) \longrightarrow \mathcal{M}'(U'_1)$$

is a quasi-isomorphism (as the source represents the derived tensor product). Consider the differential graded $\mathcal{A}$-module $\mathcal{M}$ defined by the rule

$$\mathcal{M}(U) = \text{colim}_{U \to U' \in U/\mathcal{C}'} \mathcal{M}'(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$$

This is a filtered colimit of complexes by our assumption in the lemma. Since M' is in $QC(\mathcal{A}', \text{d})$ all the transition maps in the system are quasi-isomorphisms. Since filtered colimits are exact, we see that $\mathcal{M}(U)$ in $D(\mathcal{A}(U), \text{d})$ is isomorphic to $\mathcal{M}'(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$ for any morphism $U \to U'$ with $U' \in \text{Ob}(\mathcal{C}')$.

We claim that $\mathcal{M}$ is in $QC(\mathcal{A}, \text{d})$: namely, given $U \to V$ in $\mathcal{C}$ we choose a map $V \to V'$ with $V' \in \text{Ob}(\mathcal{C}')$. By the above we see that the map $\mathcal{M}(V) \to \mathcal{M}(U)$ is identified with the map

$$\mathcal{M}'(V') \otimes_{\mathcal{A}'(V')} \mathcal{A}(V) \longrightarrow \mathcal{M}'(V') \otimes_{\mathcal{A}'(V')} \mathcal{A}(U)$$

Since $\mathcal{M}'(V')$ is K-flat as differential gradede $\mathcal{A}'(V')$-module, we conclude the claim is true.

The natural map $\mathcal{M}|_{\mathcal{C}'} \to \mathcal{M}'$ is an isomorphism in $D(\mathcal{A}', d)$ as follows immediately from the above.

Conversely, if we have an object E of $QC(\mathcal{A}, \text{d})$, then we represent it by a good differential graded module $\mathcal{E}$. Setting $\mathcal{M}' = \mathcal{E}|_{\mathcal{C}'}$ (this is another good differential graded module) we see that there is a map

$$\mathcal{E} \to \mathcal{M}$$

wich over U in $\mathcal{C}$ is given by the map

$$\mathcal{E}(U) \longrightarrow \mathrm{colim}_{U \to U' \in U/\mathcal{C}'} \mathcal{E}(U') \otimes_{\mathcal{A}'(U')} \mathcal{A}(U)$$

which is a quasi-isomorphism by the same reason. Thus restriction and the construction above are quasi-inverse functors as desired. $\square$

0GZF Lemma 24.34.2. Let $\mathcal{C}, \mathcal{O}$ be as in Section 24.33. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of differential graded $\mathcal{O}$-algebras which induces an isomorphism on cohomology sheaves, then the equivalence $D(\mathcal{A}, \mathrm{d}) \to D(\mathcal{B}, \mathrm{d})$ of Lemma 24.30.1 induces an equivalence $QC(\mathcal{A}, \mathrm{d}) \to QC(\mathcal{B}, \mathrm{d})$.

Proof. It suffices to show the following: given a morphism $U \to V$ of $\mathcal{C}$ and M in $D(\mathcal{A}, \mathrm{d})$ the following are equivalent

(1) $R\Gamma(V, M) \otimes^{\mathbf{L}}_{\mathcal{A}(V)} \mathcal{A}(U) \to \Gamma(U, M)$ is an isomorphism in $D(\mathcal{A}(U), \mathrm{d})$, and
(2) $R\Gamma(V, M \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{B}) \otimes^{\mathbf{L}}_{\mathcal{B}(V)} \mathcal{B}(U) \to \Gamma(U, M \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{B})$ is an isomorphism in $D(\mathcal{B}(U), \mathrm{d})$.

Since the topology on $\mathcal{C}$ is chaotic, this simply boils down to fact that $\mathcal{A}(U) \to \mathcal{B}(U)$ and $\mathcal{A}(V) \to \mathcal{B}(V)$ are quasi-isomorphisms. Details omitted. $\square$

24.35. Inverse systems of differential graded algebras

0GZG In this section we consider the following special case of the situation discussed in Section 24.33:

(1) $\mathcal{C}$ is the category $\mathbf{N}$ with a unique morphism $i \to j$ if and only if $i \leq j$,
(2) $\mathcal{O}$ is the constant (pre)sheaf of rings with value a given ring R.

In this setting a sheaf $\mathcal{A}$ of differential graded $\mathcal{O}$-algebras is the same thing as an inverse system (A_n) of differential graded R-algebras. A sheaf $\mathcal{M}$ of differential graded $\mathcal{A}$-modules is the same thing as an inverse system (M_n) where M_n is a differential graded A_n-module and the transition maps $M_{n+1} \to M_n$ are A_{n+1}-module maps.

Suppose that $\mathcal{B} = (B_n)$ is a second inverse system of differential graded R-algebras. Given a morphism $\varphi : (A_n) \to (B_n)$ of pro-objects we will construct an exact functor from $QC(\mathcal{A}, \mathrm{d})$ to $QC(\mathcal{B}, \mathrm{d})$. Namely, according to Categories, Example 4.22.6 the morphism φ is given by a sequence $\ldots \geq m(3) \geq m(2) \geq m(1)$ of integers and a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & A_{m(3)} \arrow[r] \arrow[d, "\varphi_3"] & A_{m(2)} \arrow[r] \arrow[d, "\varphi_2"] & A_{m(1)} \arrow[d, "\varphi_1"] \\
\ldots \arrow[r] & B_3 \arrow[r] & B_2 \arrow[r] & B_1
\end{tikzcd}$$

of differential graded R-algebras. Then given a good sheaf of differential graded $\mathcal{A}$-modules $\mathcal{M} = (M_n)$ representing an object of $QC(\mathcal{A}, \mathrm{d})$ we can set

$$N_n = M_{m(n)} \otimes_{A_{m(n)}} B_n$$

This inverse system determines an object of $QC(\mathcal{B}, \mathrm{d})$ because the $A_{m(n)}$-modules $M_{m(n)}$ are K-flat; details omitted. We also leave it to the reader to show that the resulting functor is independent of the choices made in its construction.

0GZH Lemma 24.35.1. In the situation above, suppose that $\mathcal{A} = (A_n)$ and $\mathcal{B} = (B_n)$ are inverse systems of differential graded R-algebras. If $\varphi : (A_n) \to (B_n)$ is an isomorphism of pro-objects, then the functor $QC(\mathcal{A}, \text{d}) \to QC(\mathcal{B}, \text{d})$ constructed above is an equivalence.

Proof. Let $\psi : (B_n) \to (A_n)$ be a morphism of pro-objects which is inverse to φ. According to the discussion in Categories, Example 4.22.6 we may assume that φ is given by a system of maps as above and ψ is given $n(1) < n(2) < \ldots$ and a commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & B_{n(3)} \arrow[r] \arrow[d, "\psi_3"] & B_{n(2)} \arrow[r] \arrow[d, "\psi_2"] & B_{n(1)} \arrow[d, "\psi_1"] \\ \ldots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1 \end{tikzcd}$$

of differential graded R-algebras. Since $\varphi \circ \psi = \text{id}$ we may, after possibly increasing the values of the functions $n(\cdot)$ and $m(\cdot)$ assume that $B_{n(m(i))} \to A_{m(i)} \to B_i$ is the identity. It follows that the composition of the functors

$$QC(\mathcal{B}, \text{d}) \to QC(\mathcal{A}, \text{d}) \to QC(\mathcal{B}, \text{d})$$

sends a good sheaf of differential graded $\mathcal{B}$-modules $\mathcal{N} = (N_n)$ to the inverse system $\mathcal{N}' = (N'_i)$ with values

$$N'_i = N_{n(m(i))} \otimes_{B_{n(m(i))}} B_i$$

which is canonically quasi-isomorphic to $\mathcal{N}$ exactly because $\mathcal{N}$ is an object of $QC(\mathcal{B}, \text{d})$ and because N_j is a K-flat differential graded module for all j. Since the same is true for the composition the other way around we conclude. $\square$

Let $\mathcal{C} = \mathbf{N}$ and $\mathcal{O}$ the constant sheaf with value a ring R and let $\mathcal{A}$ be given by an inverse system (A_n) of differential graded R-algebras. Suppose given two left differential graded $\mathcal{A}$-modules $\mathcal{N}$ and $\mathcal{N}'$ given by inverse systems (N_n) and (N'_n). Thus each N_n and N'_n is a left differential graded A_n-module. Let us temporarily say that (N_n) and (N'_n) are pro-isomorphic in the derived category if there exist a sequence of integers

$$1 = n_0 < n_1 < n_2 < n_3 < \ldots$$

and maps

$$N_{n_{2i}} \to N'_{n_{2i-1}} \quad \text{in} \quad D(A_{n_{2i}}^{opp}, \text{d})$$

and

$$N'_{n_{2i+1}} \to N'_{n_{2i}} \quad \text{in} \quad D(A_{n_{2i+1}}^{opp}, \text{d})$$

such that the compositions $N_{n_{2i}} \to N_{n_{2i-2}}$ and $N'_{n_{2i+1}} \to N'_{2i-1}$ are given by the transition maps of the respective systems.

0GZI Lemma 24.35.2. If (N_n) and (N'_n) are pro-isomorphic in the derived category as defined above, then for every object (M_n) of $D(\mathbf{N}, \mathcal{A})$ we have

$$R\lim(M_n \otimes_{A_n}^{\mathbf{L}} N_n) = R\lim(M_n \otimes_{A_n}^{\mathbf{L}} N'_n)$$

in $D(R)$.

Proof. The assumption implies that the inverse system $(M_n \otimes_{A_n}^{\mathbf{L}} N_n)$ of $D(R)$ is pro-isomorphic (in the usual sense) to the inverse system $(M_n \otimes_{A_n}^{\mathbf{L}} N'_n)$ of $D(R)$. Hence the result follows from the fact that taking $R\lim$ is well defined for inverse systems in the derived category, see discussion in More on Algebra, Section 15.87. $\square$

0GZJ Lemma 24.35.3. Let R be a ring. Let $f_1, \ldots, f_r \in R$. Let K_n be the Koszul complex on $f_1^n, \ldots, f_r^n$ viewed as a differential graded R-algebra. Let (M_n) be an object of $D(\mathbf{N}, (K_n))$. Then for any $t \geq 1$ we have This is a variant of [BS13, Lemma 3.5.4]

$$R\lim(M_n \otimes_R^{\mathbf{L}} K_t) = R\lim(M_n \otimes_{K_n}^{\mathbf{L}} K_t)$$

in $D(R)$.

Proof. We fix $t \geq 1$. For $n \geq t$ let us denote ${}_nK_t$ the differential graded R-algebra K_t viewed as a left differential graded K_n-module. Observe that

$$M_n \otimes_R^{\mathbf{L}} K_t = M_n \otimes_{K_n}^{\mathbf{L}} (K_n \otimes_R^{\mathbf{L}} K_t) = M_n \otimes_{K_n}^{\mathbf{L}} (K_n \otimes_R K_t)$$

Hence by Lemma 24.35.2 it suffices to show that $({}_nK_t)$ and $(K_n \otimes_R K_t)$ are pro-isomorphic in the derived category. The multiplication maps

$$K_n \otimes_R K_t \longrightarrow {}_nK_t$$

are maps of left differential graded K_n-modules. Thus to finish the proof it suffices to show that for all $n \geq 1$ there exists an $N > n$ and a map

$${}_NK_t \longrightarrow {}_NK_n \otimes_R K_t$$

in $D(K_N^{opp}, \text{d})$ whose composition with the multiplication map is the transition map (in either direction). This is done in Divided Power Algebra, Lemma 23.12.4 by an explicit construction. $\square$

0GZK Proposition 24.35.4. Let R be a Noetherian ring. Let $I \subset R$ be an ideal. The following three categories are canonically equivalent:

(1) Let $\mathcal{A}$ be the sheaf of R-algebras on $\mathbf{N}$ corresponding to the inverse system of R-algebras $A_n = R/I^n$. The category $QC(\mathcal{A})$.
(2) Choose generators $f_1, \ldots, f_r$ of I. Let $\mathcal{B}$ be the sheaf of differential graded R-algebras on $\mathbf{N}$ corresponding to the inverse system of Koszul algebras on $f_1^n, \ldots, f_r^n$. The category $QC(\mathcal{B})$.
(3) The full subcategory $D_{comp}(R, I) \subset D(R)$ of derived complete objects, see More on Algebra, Definition 15.91.4 and text following.

Proof. Consider the obvious morphism $f : (Sh(\mathbf{N}), \mathcal{A}) \to (Sh(pt), R)$ of ringed topoi and let us consider the adjoint functors Lf^* and Rf_*. The first restricts to a functor

$$F : D_{comp}(R, I) \longrightarrow QC(\mathcal{A})$$

which sends an object K of $D_{comp}(R, I)$ represented by a K-flat complex $K^\bullet$ to the object $(K^\bullet \otimes_R R/I^n)$ of $QC(\mathcal{A})$. The second restricts to a functor

$$G : QC(\mathcal{A}) \longrightarrow D_{comp}(R, I)$$

which sends an object $(M_n^\bullet)$ of $QC(\mathcal{A})$ to $R\lim M_n^\bullet$. The output is derived complete for example by More on Algebra, Lemma 15.91.14. Also, it follows from More on Algebra, Proposition 15.94.2 that $G \circ F = \text{id}$. Thus to see that F and G are quasi-inverse equivalences it suffices to see that the kernel of G is zero (see Derived Categories, Lemma 13.7.2). However, it does not appear easy to show this directly!

In this paragraph we will show that $QC(\mathcal{A})$ and $QC(\mathcal{B})$ are equivalent. Write $\mathcal{B} = (B_n)$ where B_n is the Koszul complex viewed as a cochain complex in degrees $-r, -r+1, \ldots, 0$. By Divided Power Algebra, Remark 23.12.2 (but with chain complexes turned into cochain complexes) we can find $1 < n_1 < n_2 < \ldots$ and maps of differential graded R-algebras $B_{n_i} \to E_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$ and $E_i \to B_{n_{i-1}}$ such that

$$\begin{tikzcd} B_{n_1} \arrow[d] & B_{n_2} \arrow[l] \arrow[d] & B_{n_3} \arrow[l] \arrow[d] & \ldots \arrow[l] \\ E_1 \arrow[d] & E_2 \arrow[l] \arrow[d] & E_3 \arrow[l] \arrow[d] & \ldots \arrow[l] \\ B_1 & B_{n_1} \arrow[l] & B_{n_2} \arrow[l] & \ldots \arrow[l] \end{tikzcd}$$

is a commutative diagram of differential graded R-algebras and such that $E_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$ is a quasi-isomorphism. We conclude

(1) there is an equivalence between $QC(\mathcal{B})$ and $QC((E_i))$,
(2) there is an equivalence between $QC((E_i))$ and $QC((R/(f_1^{n_i}, \ldots, f_r^{n_i})))$,
(3) there is an equivalence between $QC((R/(f_1^{n_i}, \ldots, f_r^{n_i})))$ and $QC(\mathcal{A})$.

Namely, for (1) we can apply Lemma 24.35.1 to the diagram above which shows that (E_i) and (B_n) are pro-isomorphic. For (2) we can apply Lemma 24.34.2 to the inverse system of quasi-isomorphisms $E_i \to R/(f_1^{n_i}, \ldots, f_r^{n_i})$. For (3) we can apply Lemma 24.35.1 and the elementary fact that the inverse systems (R/I^n) and $(R/(f_1^{n_i}, \ldots, f_r^{n_i})$ are pro-isomorphic.

Exactly as in the first paragraph of the proof we can define adjoint functors[6]

$$F' : D_{comp}(R, I) \longrightarrow QC(\mathcal{B}) \quad \text{and} \quad G' : QC(\mathcal{B}) \longrightarrow D_{comp}(R, I).$$

The first sends an object K of $D_{comp}(R, I)$ represented by a K-flat complex $K^\bullet$ to the object $(K^\bullet \otimes_R B_n)$ of $QC(\mathcal{B})$. The second sends an object (M_n) of $QC(\mathcal{B})$ to $R\lim M_n$. Arguing as above it suffices to show that the kernel of G' is zero. So let $\mathcal{M} = (M_n)$ be a good sheaf of differential graded modules over $\mathcal{B}$ which represents an object of $QC(\mathcal{B})$ in the kernel of G'. Then

$$0 = R\lim M_n \Rightarrow 0 = (R\lim M_n) \otimes_R^{\mathbf{L}} B_t = R\lim(M_n \otimes_R^{\mathbf{L}} B_t)$$

By Lemma 24.35.3 we have $R\lim(M_n \otimes_R^{\mathbf{L}} B_t) = R\lim(M_n \otimes_{B_n}^{\mathbf{L}} B_t)$. Since (M_n) is an object of $QC(\mathcal{B})$ we see that the inverse system $M_n \otimes_{B_n}^{\mathbf{L}} B_t$ is eventually constant with value M_t. Hence $M_t = 0$ as desired. $\square$

0H1E Remark 24.35.5. Let R be a ring and let $f_1, \ldots, f_r \in R$ be a sequence of elements generating an ideal I. Let K_n be the Koszul complex on $f_1^n, \ldots, f_r^n$ viewed as a differential graded R-algebra. We say $f_1, \ldots, f_r$ is a weakly proregular sequence if for all n there is an $m > n$ such that $K_m \to K_n$ induces the zero map on cohomology except in degree 0. If so, then the arguments in the proof of Proposition 24.35.4 continue to work even when R is not Noetherian. In particular we see that $QC(\{R/I^n\})$ is equivalent as an R-linear triangulated category to the category $D_{comp}(R, I)$ of derived complete objects, provided I can be generated by a weakly proregular sequence. If the need arises, we will precisely state and prove this here.

[6] It can be shown that these functors are, via the equivalences above, compatible with F and G defined before.

24.36. Other chapters

CHAPTER 25

Hypercoverings

01FX 25.1. Introduction

01FY Let $\mathcal{C}$ be a site, see Sites, Definition 7.6.2. Let X be an object of $\mathcal{C}$. Given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we would like to compute its cohomology groups

$$H^i(X, \mathcal{F}).$$

According to our general definitions (Cohomology on Sites, Section 21.2) this cohomology group is computed by choosing an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \ldots$ and setting

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \ldots)$$

The goal of this chapter is to show that we may also compute these cohomology groups without choosing an injective resolution (in the case that $\mathcal{C}$ has fibre products). To do this we will use hypercoverings.

A hypercovering in a site is a generalization of a covering, see [AGV71, Exposé V, Sec. 7]. Given a hypercovering K of an object X, there is a Čech to cohomology spectral sequence expressing the cohomology of an abelian sheaf $\mathcal{F}$ over X in terms of the cohomology of the sheaf over the components K_n of K. It turns out that there are always enough hypercoverings, so that taking the colimit over all hypercoverings, the spectral sequence degenerates and the cohomology of $\mathcal{F}$ over X is computed by the colimit of the Čech cohomology groups.

A more general gadget one can consider is a simplicial augmentation where one has cohomological descent, see [AGV71, Exposé Vbis]. A nice manuscript on cohomological descent is the text by Brian Conrad, see `https://math.stanford.edu/~conrad/papers/hypercover.pdf`. We will come back to these issue in the chapter on simplicial spaces where we will show, for example, that proper hypercoverings of "locally compact" topological spaces are of cohomological descent (Simplicial Spaces, Section 85.25). Our method of attack will be to reduce this statement to the Čech to cohomology spectral sequence constructed in this chapter.

25.2. Semi-representable objects

0DBB In order to start we make the following definition. The letters "SR" stand for Semi-Representable.

01G0 Definition 25.2.1. Let $\mathcal{C}$ be a category. We denote $\text{SR}(\mathcal{C})$ the category of semi-representable objects defined as follows

(1) objects are families of objects $\{U_i\}_{i \in I}$, and
(2) morphisms $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ of $\mathcal{C}$.

Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. The category of semi-representable objects over X is the category $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$.

This definition is essentially equivalent to [AGV71, Exposé V, Subsection 7.3.0]. Note that this is a "big" category. We will later "bound" the size of the index sets I that we need for hypercoverings of X. We can then redefine $\mathrm{SR}(\mathcal{C}, X)$ to become a category. Let's spell out the objects and morphisms $\mathrm{SR}(\mathcal{C}, X)$:

(1) objects are families of morphisms $\{U_i \to X\}_{i \in I}$, and
(2) morphisms $\{U_i \to X\}_{i \in I} \to \{V_j \to X\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ over X.

There is a forgetful functor $\mathrm{SR}(\mathcal{C}, X) \to \mathrm{SR}(\mathcal{C})$.

01G1 Definition 25.2.2. Let $\mathcal{C}$ be a category. We denote F the functor which associates a presheaf to a semi-representable object. In a formula

$$\begin{aligned} F : \mathrm{SR}(\mathcal{C}) & \longrightarrow \mathrm{PSh}(\mathcal{C}) \\ \{U_i\}_{i \in I} & \longmapsto \amalg_{i \in I} h_{U_i} \end{aligned}$$

where h_U denotes the representable presheaf associated to the object U.

Given a morphism $U \to X$ we obtain a morphism $h_U \to h_X$ of representable presheaves. Thus we often think of F on $\mathrm{SR}(\mathcal{C}, X)$ as a functor into the category of presheaves of sets over h_X, namely $\mathrm{PSh}(\mathcal{C})/h_X$. Here is a picture:

$$\begin{tikzcd} \mathrm{SR}(\mathcal{C}, X) \arrow[r, "F"'] \arrow[d] & \mathrm{PSh}(\mathcal{C})/h_X \arrow[d] \\ \mathrm{SR}(\mathcal{C}) \arrow[r, "F"] & \mathrm{PSh}(\mathcal{C}) \end{tikzcd}$$

Next we discuss the existence of limits in the category of semi-representable objects.

01G2 Lemma 25.2.3. Let $\mathcal{C}$ be a category.

(1) the category $\mathrm{SR}(\mathcal{C})$ has coproducts and F commutes with them,
(2) the functor $F : \mathrm{SR}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ commutes with limits,
(3) if $\mathcal{C}$ has fibre products, then $\mathrm{SR}(\mathcal{C})$ has fibre products,
(4) if $\mathcal{C}$ has products of pairs, then $\mathrm{SR}(\mathcal{C})$ has products of pairs,
(5) if $\mathcal{C}$ has equalizers, so does $\mathrm{SR}(\mathcal{C})$, and
(6) if $\mathcal{C}$ has a final object, so does $\mathrm{SR}(\mathcal{C})$.

Let $X \in \mathrm{Ob}(\mathcal{C})$.

(1) the category $\mathrm{SR}(\mathcal{C}, X)$ has coproducts and F commutes with them,
(2) if $\mathcal{C}$ has fibre products, then $\mathrm{SR}(\mathcal{C}, X)$ has finite limits and $F : \mathrm{SR}(\mathcal{C}, X) \to \mathrm{PSh}(\mathcal{C})/h_X$ commutes with them.

Proof. Proof of the results on $\mathrm{SR}(\mathcal{C})$. Proof of (1). The coproduct of $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ is $\{U_i\}_{i \in I} \amalg \{V_j\}_{j \in J}$, in other words, the family of objects whose index set is $I \amalg J$ and for an element $k \in I \amalg J$ gives U_i if $k = i \in I$ and gives V_j if $k = j \in J$. Similarly for coproducts of families of objects. It is clear that F commutes with these.

Proof of (2). For U in $\mathrm{Ob}(\mathcal{C})$ consider the object $\{U\}$ of $\mathrm{SR}(\mathcal{C})$. It is clear that $\mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\}, K)) = F(K)(U)$ for $K \in \mathrm{Ob}(\mathrm{SR}(\mathcal{C}))$. Since limits of presheaves are computed at the level of sections (Sites, Section 7.4) we conclude that F commutes with limits.

Proof of (3). Suppose given a morphism $(\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ and a morphism $(\beta, g_k) : \{W_k\}_{k \in K} \to \{V_j\}_{j \in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k\}_{(i,j,k) \in I \times J \times K \text{ such that } j=\alpha(i)=\beta(k)}$$

The fibre products exist if $\mathcal{C}$ has fibre products.

Proof of (4). The product of $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ is $\{U_i \times V_j\}_{i \in I, j \in J}$. The products exist if $\mathcal{C}$ has products.

Proof of (5). The equalizer of two maps $(\alpha, f_i), (\alpha', f_i') : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ is

$$\{\mathrm{Eq}(f_i, f_i' : U_i \to V_{\alpha(i)})\}_{i \in I,\ \alpha(i)=\alpha'(i)}$$

The equalizers exist if $\mathcal{C}$ has equalizers.

Proof of (6). If X is a final object of $\mathcal{C}$, then $\{X\}$ is a final object of $\mathrm{SR}(\mathcal{C})$.

Proof of the statements about $\mathrm{SR}(\mathcal{C}, X)$. These follow from the results above applied to the category $\mathcal{C}/X$ using that $\mathrm{SR}(\mathcal{C}/X) = \mathrm{SR}(\mathcal{C}, X)$ and that $\mathrm{PSh}(\mathcal{C}/X) = \mathrm{PSh}(\mathcal{C})/h_X$ (Sites, Lemma 7.25.4 applied to $\mathcal{C}$ endowed with the chaotic topology). However we also argue directly as follows. It is clear that the coproduct of $\{U_i \to X\}_{i \in I}$ and $\{V_j \to X\}_{j \in J}$ is $\{U_i \to X\}_{i \in I} \amalg \{V_j \to X\}_{j \in J}$ and similarly for coproducts of families of families of morphisms with target X. The object $\{X \to X\}$ is a final object of $\mathrm{SR}(\mathcal{C}, X)$. Suppose given a morphism $(\alpha, f_i) : \{U_i \to X\}_{i \in I} \to \{V_j \to X\}_{j \in J}$ and a morphism $(\beta, g_k) : \{W_k \to X\}_{k \in K} \to \{V_j \to X\}_{j \in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k \to X\}_{(i,j,k) \in I \times J \times K \text{ such that } j=\alpha(i)=\beta(k)}$$

The fibre products exist by the assumption that $\mathcal{C}$ has fibre products. Thus $\mathrm{SR}(\mathcal{C}, X)$ has finite limits, see Categories, Lemma 4.18.4. We omit verifying the statements on the functor F in this case. $\square$

25.3. Hypercoverings

01FZ If we assume our category is a site, then we can make the following definition.

01G3 Definition 25.3.1. Let $\mathcal{C}$ be a site. Let $f = (\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ be a morphism in the category $\mathrm{SR}(\mathcal{C})$. We say that f is a covering if for every $j \in J$ the family of morphisms $\{U_i \to V_j\}_{i \in I, \alpha(i)=j}$ is a covering for the site $\mathcal{C}$. Let X be an object of $\mathcal{C}$. A morphism $K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$ is a covering if its image in $\mathrm{SR}(\mathcal{C})$ is a covering.

01G4 Lemma 25.3.2. Let $\mathcal{C}$ be a site.

(1) A composition of coverings in $\mathrm{SR}(\mathcal{C})$ is a covering.
(2) If $K \to L$ is a covering in $\mathrm{SR}(\mathcal{C})$ and $L' \to L$ is a morphism, then $L' \times_L K$ exists and $L' \times_L K \to L'$ is a covering.
(3) If $\mathcal{C}$ has products of pairs, and $A \to B$ and $K \to L$ are coverings in $\mathrm{SR}(\mathcal{C})$, then $A \times K \to B \times L$ is a covering.

Let $X \in \mathrm{Ob}(\mathcal{C})$. Then (1) and (2) holds for $\mathrm{SR}(\mathcal{C}, X)$ and (3) holds if $\mathcal{C}$ has fibre products.

Proof. Part (1) is immediate from the axioms of a site. Part (2) follows by the construction of fibre products in $\mathrm{SR}(\mathcal{C})$ in the proof of Lemma 25.2.3 and the requirement that the morphisms in a covering of $\mathcal{C}$ are representable. Part (3) follows by thinking of $A \times K \to B \times L$ as the composition $A \times K \to B \times K \to B \times L$ and hence a composition of basechanges of coverings. The final statement follows because $\text{SR}(\mathcal{C}, X) = \text{SR}(\mathcal{C}/X)$. $\square$

By Lemma 25.2.3 and Simplicial, Lemma 14.19.2 the coskeleton of a truncated simplicial object of $\text{SR}(\mathcal{C}, X)$ exists if $\mathcal{C}$ has fibre products. Hence the following definition makes sense.

01G5 Definition 25.3.3. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has fibre products. Let $X \in \text{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. A hypercovering of X is a simplicial object K of $\text{SR}(\mathcal{C}, X)$ such that

(1) The object K_0 is a covering of X for the site $\mathcal{C}$.
(2) For every $n \geq 0$ the canonical morphism

$$K_{n+1} \longrightarrow (\text{cosk}_n \text{sk}_n K)_{n+1}$$

is a covering in the sense defined above.

Condition (1) makes sense since each object of $\text{SR}(\mathcal{C}, X)$ is after all a family of morphisms with target X. It could also be formulated as saying that the morphism of K_0 to the final object of $\text{SR}(\mathcal{C}, X)$ is a covering.

01G6 Example 25.3.4 (Čech hypercoverings). Let $\mathcal{C}$ be a site with fibre products. Let $\{U_i \to X\}_{i \in I}$ be a covering of $\mathcal{C}$. Set $K_0 = \{U_i \to X\}_{i \in I}$. Then K_0 is a 0-truncated simplicial object of $\text{SR}(\mathcal{C}, X)$. Hence we may form

$$K = \text{cosk}_0 K_0.$$

Clearly K passes condition (1) of Definition 25.3.3. Since all the morphisms $K_{n+1} \to (\text{cosk}_n \text{sk}_n K)_{n+1}$ are isomorphisms by Simplicial, Lemma 14.19.10 it also passes condition (2). Note that the terms K_n are the usual

$$K_n = \{U_{i_0} \times_X U_{i_1} \times_X \ldots \times_X U_{i_n} \to X\}_{(i_0, i_1, \ldots, i_n) \in I^{n+1}}$$

A hypercovering of X of this form is called a Čech hypercovering of X.

0GM9 Example 25.3.5 (Hypercovering by a simplicial object of the site). Let $\mathcal{C}$ be a site with fibre products. Let $X \in \text{Ob}(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. As usual we denote $U_n = U([n])$. Finally, assume given an augmentation

$$a : U \to X$$

In this situation we can consider the simplicial object K of $\text{SR}(\mathcal{C}, X)$ with terms $K_n = \{U_n \to X\}$. Then K is a hypercovering of X in the sense of Definition 25.3.3 if and only if the following three conditions[1] hold:

(1) $\{U_0 \to X\}$ is a covering of $\mathcal{C}$,
(2) $\{U_1 \to U_0 \times_X U_0\}$ is a covering of $\mathcal{C}$,
(3) $\{U_{n+1} \to (\text{cosk}_n \text{sk}_n U)_{n+1}\}$ is a covering of $\mathcal{C}$ for $n \geq 1$.

We omit the straightforward verification.

[1]As $\mathcal{C}$ has fibre products, the category $\mathcal{C}/X$ has all finite limits. Hence the required coskeleta exist by Simplicial, Lemma 14.19.2.

0GMA Example 25.3.6 (Čech hypercovering associated to a cover). Let $\mathcal{C}$ be a site with fibre products. Let $U \to X$ be a morphism of $\mathcal{C}$ such that $\{U \to X\}$ is a covering of $\mathcal{C}$[2]. Consider the simplical object K of $\mathrm{SR}(\mathcal{C}, X)$ with terms

$$K_n = \{U \times_X U \times_X \ldots \times_X U \to X\} \quad (n+1 \text{ factors})$$

Then K is a hypercovering of X. This example is a special case of both Example 25.3.4 and of Example 25.3.5.

01G7 Lemma 25.3.7. Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. The collection of all hypercoverings of X forms a set.

Proof. Since $\mathcal{C}$ is a site, the set of all coverings of X forms a set. Thus we see that the collection of possible K_0 forms a set. Suppose we have shown that the collection of all possible $K_0, \ldots, K_n$ form a set. Then it is enough to show that given $K_0, \ldots, K_n$ the collection of all possible K_{n+1} forms a set. And this is clearly true since we have to choose K_{n+1} among all possible coverings of $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$. □

01G8 Remark 25.3.8. The lemma does not just say that there is a cofinal system of choices of hypercoverings that is a set, but that really the hypercoverings form a set.

The category of presheaves on $\mathcal{C}$ has finite (co)limits. Hence the functors cosk_n exists for presheaves of sets.

01G9 Lemma 25.3.9. Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. Let K be a hypercovering of X. Consider the simplicial object $F(K)$ of $\mathrm{PSh}(\mathcal{C})$, endowed with its augmentation to the constant simplicial presheaf h_X.

(1) The morphism of presheaves $F(K)_0 \to h_X$ becomes a surjection after sheafification.
(2) The morphism
$$(d^1_0, d^1_1) : F(K)_1 \longrightarrow F(K)_0 \times_{h_X} F(K)_0$$
becomes a surjection after sheafification.
(3) For every $n \geq 1$ the morphism
$$F(K)_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n F(K))_{n+1}$$
turns into a surjection after sheafification.

Proof. We will use the fact that if $\{U_i \to U\}_{i \in I}$ is a covering of the site $\mathcal{C}$, then the morphism

$$\amalg_{i \in I} h_{U_i} \to h_U$$

becomes surjective after sheafification, see Sites, Lemma 7.12.4. Thus the first assertion follows immediately.

For the second assertion, note that according to Simplicial, Example 14.19.1 the simplicial object $\mathrm{cosk}_0 \mathrm{sk}_0 K$ has terms $K_0 \times \ldots \times K_0$. Thus according to the definition of a hypercovering we see that $(d^1_0, d^1_1) : K_1 \to K_0 \times K_0$ is a covering. Hence (2) follows from the claim above and the fact that F transforms products into fibred products over h_X.

For the third, we claim that $\mathrm{cosk}_n \mathrm{sk}_n F(K) = F(\mathrm{cosk}_n \mathrm{sk}_n K)$ for $n \geq 1$. To prove this, denote temporarily F' the functor $\mathrm{SR}(\mathcal{C}, X) \to \mathrm{PSh}(\mathcal{C})/h_X$. By Lemma 25.2.3

[2]A morphism of $\mathcal{C}$ with this property is sometimes called a “cover”.

the functor F' commutes with finite limits. By our description of the cosk_n functor in Simplicial, Section 14.12 we see that $\text{cosk}_n\text{sk}_n F'(K) = F'(\text{cosk}_n\text{sk}_n K)$. Recall that the category used in the description of $(\text{cosk}_n U)_m$ in Simplicial, Lemma 14.19.2 is the category $(\Delta/[m])^{opp}_{\leq n}$. It is an amusing exercise to show that $(\Delta/[m])_{\leq n}$ is a connected category (see Categories, Definition 4.16.1) as soon as $n \geq 1$. Hence, Categories, Lemma 4.16.2 shows that $\text{cosk}_n\text{sk}_n F'(K) = \text{cosk}_n\text{sk}_n F(K)$. Whence the claim. Property (2) follows from this, because now we see that the morphism in (2) is the result of applying the functor F to a covering as in Definition 25.3.1, and the result follows from the first fact mentioned in this proof. □

25.4. Acyclicity

01GA Let $\mathcal{C}$ be a site. For a presheaf of sets $\mathcal{F}$ we denote $\mathbf{Z}_\mathcal{F}$ the presheaf of abelian groups defined by the rule

$$\mathbf{Z}_\mathcal{F}(U) = \text{free abelian group on } \mathcal{F}(U).$$

We will sometimes call this the free abelian presheaf on $\mathcal{F}$. Of course the construction $\mathcal{F} \mapsto \mathbf{Z}_\mathcal{F}$ is a functor and it is left adjoint to the forgetful functor $\text{PAb}(\mathcal{C}) \to \text{PSh}(\mathcal{C})$. Of course the sheafification $\mathbf{Z}_\mathcal{F}^\#$ is a sheaf of abelian groups, and the functor $\mathcal{F} \mapsto \mathbf{Z}_\mathcal{F}^\#$ is a left adjoint as well. We sometimes call $\mathbf{Z}_\mathcal{F}^\#$ the free abelian sheaf on $\mathcal{F}$.

For an object X of the site $\mathcal{C}$ we denote $\mathbf{Z}_X$ the free abelian presheaf on h_X, and we denote $\mathbf{Z}_X^\#$ its sheafification.

01GB Definition 25.4.1. Let $\mathcal{C}$ be a site. Let K be a simplicial object of $\text{PSh}(\mathcal{C})$. By the above we get a simplicial object $\mathbf{Z}_K^\#$ of $\text{Ab}(\mathcal{C})$. We can take its associated complex of abelian presheaves $s(\mathbf{Z}_K^\#)$, see Simplicial, Section 14.23. The homology of K is the homology of the complex of abelian sheaves $s(\mathbf{Z}_K^\#)$.

In other words, the ith homology $H_i(K)$ of K is the sheaf of abelian groups $H_i(K) = H_i(s(\mathbf{Z}_K^\#))$. In this section we worry about the homology in case K is a hypercovering of an object X of $\mathcal{C}$.

01GC Lemma 25.4.2. Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of sets. Denote K the simplicial object of $\text{PSh}(\mathcal{C})$ whose nth term is the $(n+1)$st fibre product of $\mathcal{F}$ over $\mathcal{G}$, see Simplicial, Example 14.3.5. Then, if $\mathcal{F} \to \mathcal{G}$ is surjective after sheafification, we have

$$H_i(K) = \begin{cases} 0 & \text{if} \quad i > 0 \\ \mathbf{Z}_\mathcal{G}^\# & \text{if} \quad i = 0 \end{cases}$$

The isomorphism in degree 0 is given by the morphism $H_0(K) \to \mathbf{Z}_\mathcal{G}^\#$ coming from the map $(\mathbf{Z}_K^\#)_0 = \mathbf{Z}_\mathcal{F}^\# \to \mathbf{Z}_\mathcal{G}^\#$.

Proof. Let $\mathcal{G}' \subset \mathcal{G}$ be the image of the morphism $\mathcal{F} \to \mathcal{G}$. Let $U \in \text{Ob}(\mathcal{C})$. Set $A = \mathcal{F}(U)$ and $B = \mathcal{G}'(U)$. Then the simplicial set $K(U)$ is equal to the simplicial set with n-simplices given by

$$A \times_B A \times_B \ldots \times_B A \ (n+1 \text{ factors}).$$

By Simplicial, Lemma 14.32.3 the morphism $K(U) \to B$ is a trivial Kan fibration. Thus it is a homotopy equivalence (Simplicial, Lemma 14.30.8). Hence applying the functor "free abelian group on" to this we deduce that

$$\mathbf{Z}_K(U) \longrightarrow \mathbf{Z}_B$$

is a homotopy equivalence. Note that $s(\mathbf{Z}_B)$ is the complex

$$\ldots \to \bigoplus\nolimits_{b\in B} \mathbf{Z} \xrightarrow{0} \bigoplus\nolimits_{b\in B} \mathbf{Z} \xrightarrow{1} \bigoplus\nolimits_{b\in B} \mathbf{Z} \xrightarrow{0} \bigoplus\nolimits_{b\in B} \mathbf{Z} \to 0$$

see Simplicial, Lemma 14.23.3. Thus we see that $H_i(s(\mathbf{Z}_K(U))) = 0$ for $i > 0$, and $H_0(s(\mathbf{Z}_K(U))) = \bigoplus_{b\in B} \mathbf{Z} = \bigoplus_{s\in \mathcal{G}'(U)} \mathbf{Z}$. These identifications are compatible with restriction maps.

We conclude that $H_i(s(\mathbf{Z}_K)) = 0$ for $i > 0$ and $H_0(s(\mathbf{Z}_K)) = \mathbf{Z}_{\mathcal{G}'}$, where here we compute homology groups in $\text{PAb}(\mathcal{C})$. Since sheafification is an exact functor we deduce the result of the lemma. Namely, the exactness implies that $H_0(s(\mathbf{Z}_K))^\# = H_0(s(\mathbf{Z}_K^\#))$, and similarly for other indices. $\square$

01GD Lemma 25.4.3. Let $\mathcal{C}$ be a site. Let $f : L \to K$ be a morphism of simplicial objects of $\text{PSh}(\mathcal{C})$. Let $n \geq 0$ be an integer. Assume that

(1) For $i < n$ the morphism $L_i \to K_i$ is an isomorphism.
(2) The morphism $L_n \to K_n$ is surjective after sheafification.
(3) The canonical map $L \to \text{cosk}_n\text{sk}_n L$ is an isomorphism.
(4) The canonical map $K \to \text{cosk}_n\text{sk}_n K$ is an isomorphism.

Then $H_i(f) : H_i(L) \to H_i(K)$ is an isomorphism.

Proof. This proof is exactly the same as the proof of Lemma 25.4.2 above. Namely, we first let $K'_n \subset K_n$ be the sub presheaf which is the image of the map $L_n \to K_n$. Assumption (2) means that the sheafification of K'_n is equal to the sheafification of K_n. Moreover, since $L_i = K_i$ for all $i < n$ we see that get an n-truncated simplicial presheaf U by taking $U_0 = L_0 = K_0, \ldots, U_{n-1} = L_{n-1} = K_{n-1}, U_n = K'_n$. Denote $K' = \text{cosk}_n U$, a simplicial presheaf. Because we can construct K'_m as a finite limit, and since sheafification is exact, we see that $(K'_m)^\# = K_m$. In other words, $(K')^\# = K^\#$. We conclude, by exactness of sheafification once more, that $H_i(K) = H_i(K')$. Thus it suffices to prove the lemma for the morphism $L \to K'$, in other words, we may assume that $L_n \to K_n$ is a surjective morphism of presheaves!

In this case, for any object U of $\mathcal{C}$ we see that the morphism of simplicial sets

$$L(U) \longrightarrow K(U)$$

satisfies all the assumptions of Simplicial, Lemma 14.32.1. Hence it is a trivial Kan fibration. In particular it is a homotopy equivalence (Simplicial, Lemma 14.30.8). Thus

$$\mathbf{Z}_L(U) \longrightarrow \mathbf{Z}_K(U)$$

is a homotopy equivalence too. This for all U. The result follows. $\square$

01GE Lemma 25.4.4. Let $\mathcal{C}$ be a site. Let K be a simplicial presheaf. Let $\mathcal{G}$ be a presheaf. Let $K \to \mathcal{G}$ be an augmentation of K towards $\mathcal{G}$. Assume that

(1) The morphism of presheaves $K_0 \to \mathcal{G}$ becomes a surjection after sheafification.
(2) The morphism

$$(d^1_0, d^1_1) : K_1 \longrightarrow K_0 \times_{\mathcal{G}} K_0$$

becomes a surjection after sheafification.

(3) For every $n \geq 1$ the morphism
$$K_{n+1} \longrightarrow (\text{cosk}_n \text{sk}_n K)_{n+1}$$
turns into a surjection after sheafification.

Then $H_i(K) = 0$ for $i > 0$ and $H_0(K) = \mathbf{Z}_\mathcal{G}^\#$.

Proof. Denote $K^n = \text{cosk}_n \text{sk}_n K$ for $n \geq 1$. Define K^0 as the simplicial object with terms $(K^0)_n$ equal to the $(n+1)$-fold fibred product $K_0 \times_\mathcal{G} \ldots \times_\mathcal{G} K_0$, see Simplicial, Example 14.3.5. We have morphisms
$$K \longrightarrow \ldots \to K^n \to K^{n-1} \to \ldots \to K^1 \to K^0.$$
The morphisms $K \to K^i$, $K^j \to K^i$ for $j \geq i \geq 1$ come from the universal properties of the cosk_n functors. The morphism $K^1 \to K^0$ is the canonical morphism from Simplicial, Remark 14.20.4. We also recall that $K^0 \to \text{cosk}_1 \text{sk}_1 K^0$ is an isomorphism, see Simplicial, Lemma 14.20.3.

By Lemma 25.4.2 we see that $H_i(K^0) = 0$ for $i > 0$ and $H_0(K^0) = \mathbf{Z}_\mathcal{G}^\#$.

Pick $n \geq 1$. Consider the morphism $K^n \to K^{n-1}$. It is an isomorphism on terms of degree $< n$. Note that $K^n \to \text{cosk}_n \text{sk}_n K^n$ and $K^{n-1} \to \text{cosk}_n \text{sk}_n K^{n-1}$ are isomorphisms. Note that $(K^n)_n = K_n$ and that $(K^{n-1})_n = (\text{cosk}_{n-1} \text{sk}_{n-1} K)_n$. Hence by assumption, we have that $(K^n)_n \to (K^{n-1})_n$ is a morphism of presheaves which becomes surjective after sheafification. By Lemma 25.4.3 we conclude that $H_i(K^n) = H_i(K^{n-1})$. Combined with the above this proves the lemma. □

01GF Lemma 25.4.5. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. The homology of the simplicial presheaf $F(K)$ is 0 in degrees > 0 and equal to $\mathbf{Z}_X^\#$ in degree 0.

Proof. Combine Lemmas 25.4.4 and 25.3.9. □

25.5. Čech cohomology and hypercoverings

01GU Let $\mathcal{C}$ be a site. Consider a presheaf of abelian groups $\mathcal{F}$ on the site $\mathcal{C}$. It defines a functor
$$\begin{array}{rcl} \mathcal{F} : \text{SR}(\mathcal{C})^{opp} & \longrightarrow & \text{Ab} \\ \{U_i\}_{i \in I} & \longmapsto & \prod_{i \in I} \mathcal{F}(U_i) \end{array}$$
Thus a simplicial object K of $\text{SR}(\mathcal{C})$ is turned into a cosimplicial object $\mathcal{F}(K)$ of Ab. The cochain complex $s(\mathcal{F}(K))$ associated to $\mathcal{F}(K)$ (Simplicial, Section 14.25) is called the Čech complex of $\mathcal{F}$ with respect to the simplicial object K. We set
$$\check{H}^i(K, \mathcal{F}) = H^i(s(\mathcal{F}(K))).$$
and we call it the ith Čech cohomology group of $\mathcal{F}$ with respect to K. In this section we prove analogues of some of the results for Čech cohomology of open coverings proved in Cohomology, Sections 20.9, 20.10 and 20.11.

01GV Lemma 25.5.1. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$.

Proof. We have
$$\check{H}^0(K, \mathcal{F}) = \mathrm{Ker}(\mathcal{F}(K_0) \longrightarrow \mathcal{F}(K_1))$$
Write $K_0 = \{U_i \to X\}$. It is a covering in the site $\mathcal{C}$. As well, we have that $K_1 \to K_0 \times K_0$ is a covering in $\mathrm{SR}(\mathcal{C}, X)$. Hence we may write $K_1 = \amalg_{i_0, i_1 \in I}\{V_{i_0 i_1 j} \to X\}$ so that the morphism $K_1 \to K_0 \times K_0$ is given by coverings $\{V_{i_0 i_1 j} \to U_{i_0} \times_X U_{i_1}\}$ of the site $\mathcal{C}$. Thus we can further identify
$$\check{H}^0(K, \mathcal{F}) = \mathrm{Ker}(\prod\nolimits_i \mathcal{F}(U_i) \longrightarrow \prod\nolimits_{i_0 i_1 j} \mathcal{F}(V_{i_0 i_1 j}))$$
with obvious map. The sheaf property of $\mathcal{F}$ implies that $\check{H}^0(K, \mathcal{F}) = H^0(X, \mathcal{F})$. □

In fact this property characterizes the abelian sheaves among all abelian presheaves on $\mathcal{C}$ of course. The analogue of Cohomology, Lemma 25.5.2 in this case is the following.

01GW **Lemma 25.5.2.** Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then
$$\check{H}^p(K, \mathcal{I}) = \begin{cases} \mathcal{I}(X) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. Observe that for any object $Z = \{U_i \to X\}$ of $\mathrm{SR}(\mathcal{C}, X)$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have
$$\begin{aligned} \mathcal{F}(Z) &= \prod \mathcal{F}(U_i) \\ &= \prod \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(h_{U_i}, \mathcal{F}) \\ &= \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(F(Z), \mathcal{F}) \\ &= \mathrm{Mor}_{\mathrm{PAb}(\mathcal{C})}(\mathbf{Z}_{F(Z)}, \mathcal{F}) \\ &= \mathrm{Mor}_{\mathrm{Ab}(\mathcal{C})}(\mathbf{Z}_{F(Z)}^\#, \mathcal{F}) \end{aligned}$$
Thus we see, for any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ that we have

01GX (25.5.2.1)
$$s(\mathcal{F}(K)) = \mathrm{Hom}_{\mathrm{Ab}(\mathcal{C})}(s(\mathbf{Z}_{F(K)}^\#), \mathcal{F})$$
see Definition 25.4.1 for notation. The complex of sheaves $s(\mathbf{Z}_{F(K)}^\#)$ is quasi-isomorphic to $\mathbf{Z}_X^\#$ if K is a hypercovering, see Lemma 25.4.5. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, and K a hypercovering, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have
$$\check{H}^i(K, \mathcal{I}) = 0$$
for $i > 0$. Combined with Lemma 25.5.1 the lemma is proved. □

Next we come to the analogue of Cohomology on Sites, Lemma 21.10.6. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Recall that $\underline{H}^i(\mathcal{F})$ indicates the presheaf of abelian groups on $\mathcal{C}$ which is defined by the rule $\underline{H}^i(\mathcal{F}) : U \longmapsto H^i(U, \mathcal{F})$. We extend this to $\mathrm{SR}(\mathcal{C})$ as in the introduction to this section.

01GY **Lemma 25.5.3.** Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map
$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D^+(\text{Ab})$ functorial in $\mathcal{F}$, which induces natural transformations

$$\check{H}^i(K, -) \longrightarrow H^i(X, -)$$

as functors $\text{Ab}(\mathcal{C}) \to \text{Ab}$. Moreover, there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. We could prove this by the same method as employed in the corresponding lemma in the chapter on cohomology. Instead let us prove this by a double complex argument.

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential on the complex $s(\mathcal{I}^q(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential $\mathcal{I}^q \to \mathcal{I}^{q+1}$. Denote $\text{Tot}(A^{\bullet,\bullet})$ the total complex associated to the double complex $A^{\bullet,\bullet}$, see Homology, Section 12.18. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.25.

By Lemma 25.5.2 the complexes $s(\mathcal{I}^q(K))$ are acyclic in positive degrees and have H^0 equal to $\mathcal{I}^q(X)$. Hence by Homology, Lemma 12.25.4 the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow \text{Tot}(A^{\bullet,\bullet})$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(\text{Tot}(A^{\bullet,\bullet})) = H^n(X, \mathcal{F})$.

The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$ of the lemma is the composition of the map $s(\mathcal{F}(K)) \to \text{Tot}(A^{\bullet,\bullet})$ followed by the inverse of the displayed quasi-isomorphism above. This works because $\mathcal{I}^\bullet(X)$ is a representative of $R\Gamma(X, \mathcal{F})$.

Consider the spectral sequence $({}'E_r, {}'d_r)_{r \geq 0}$. By Homology, Lemma 12.25.1 we see that

$${}'E_2^{p,q} = H_I^p(H_{II}^q(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to d_2 which gives the groups ${}'E_1^{p,q} = \underline{H}^q(\mathcal{F})(K_p)$. Hence it is indeed the case (by the description of the differential ${}'d_1$) that ${}'E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. By the above and Homology, Lemma 12.25.3 we see that this converges to $H^n(X, \mathcal{F})$ as desired.

We omit the proof of the statements regarding the functoriality of the above constructions in the abelian sheaf $\mathcal{F}$ and the hypercovering K. □

25.6. Hypercoverings a la Verdier

09VT The astute reader will have noticed that all we need in order to get the Čech to cohomology spectral sequence for a hypercovering of an object X, is the conclusion of Lemma 25.3.9. Therefore the following definition makes sense.

09VU Definition 25.6.1. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has equalizers and fibre products. Let $\mathcal{G}$ be a presheaf of sets. A hypercovering of $\mathcal{G}$ is a simplicial object K of $\text{SR}(\mathcal{C})$ endowed with an augmentation $F(K) \to \mathcal{G}$ such that

(1) $F(K_0) \to \mathcal{G}$ becomes surjective after sheafification,
(2) $F(K_1) \to F(K_0) \times_{\mathcal{G}} F(K_0)$ becomes surjective after sheafification, and
(3) $F(K_{n+1}) \longrightarrow F((\text{cosk}_n \text{sk}_n K)_{n+1})$ for $n \geq 1$ becomes surjective after sheafification.

We say that a simplicial object K of $\text{SR}(\mathcal{C})$ is a hypercovering if K is a hypercovering of the final object $*$ of $\text{PSh}(\mathcal{C})$.

The assumption that $\mathcal{C}$ has fibre products and equalizers guarantees that $\text{SR}(\mathcal{C})$ has fibre products and equalizers and F commutes with these (Lemma 25.2.3) which suffices to define the coskeleton functors used (see Simplicial, Remark 14.19.11 and Categories, Lemma 4.18.2). If $\mathcal{C}$ is general, we can replace the condition (3) by the condition that $F(K_{n+1}) \longrightarrow ((\text{cosk}_n \text{sk}_n F(K))_{n+1})$ for $n \geq 1$ becomes surjective after sheafification and the results of this section remain valid.

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. In the previous section, we defined the Čech complex of $\mathcal{F}$ with respect to a simplicial object K of $\text{SR}(\mathcal{C})$. Next, given a presheaf $\mathcal{G}$ we set

$$H^0(\mathcal{G}, \mathcal{F}) = \text{Mor}_{\text{PSh}(\mathcal{C})}(\mathcal{G}, \mathcal{F}) = \text{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^\#, \mathcal{F}) = H^0(\mathcal{G}^\#, \mathcal{F})$$

with notation as in Cohomology on Sites, Section 21.13. This is a left exact functor and its higher derived functors (briefly studied in Cohomology on Sites, Section 21.13) are denoted $H^i(\mathcal{G}, \mathcal{F})$. We will show that given a hypercovering K of $\mathcal{G}$, there is a Čech to cohomology spectral sequence converging to the cohomology $H^i(\mathcal{G}, \mathcal{F})$. Note that if $\mathcal{G} = *$, then $H^i(*, \mathcal{F}) = H^i(\mathcal{C}, \mathcal{F})$ recovers the cohomology of $\mathcal{F}$ on the site $\mathcal{C}$.

09VV Lemma 25.6.2. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = H^0(\mathcal{G}, \mathcal{F})$.

Proof. This follows from the definition of $H^0(\mathcal{G}, \mathcal{F})$ and the fact that

$$F(K_1) \rightrightarrows F(K_0) \longrightarrow \mathcal{G}$$

becomes an coequalizer diagram after sheafification. □

09VW Lemma 25.6.3. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} H^0(\mathcal{G}, \mathcal{I}) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. By (25.5.2.1) we have

$$s(\mathcal{F}(K)) = \text{Hom}_{\text{Ab}(\mathcal{C})}(s(\mathbf{Z}^\#_{F(K)}), \mathcal{F})$$

The complex $s(\mathbf{Z}^\#_{F(K)})$ is quasi-isomorphic to $\mathbf{Z}^\#_{\mathcal{G}}$, see Lemma 25.4.4. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have $\check{H}^i(K, \mathcal{I}) = 0$ for $i > 0$. Combined with Lemma 25.6.2 the lemma is proved. □

09VX Lemma 25.6.4. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$$

in $D^+(\text{Ab})$ functorial in $\mathcal{F}$, which induces a natural transformation

$$\check{H}^i(K, -) \longrightarrow H^i(\mathcal{G}, -)$$

of functors $\text{Ab}(\mathcal{C}) \to \text{Ab}$. Moreover, there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(\mathcal{G}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $\mathcal{I}^p \to \mathcal{I}^{p+1}$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential on the complex $s(\mathcal{I}^p(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ as explained above. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.25.

By Lemma 25.6.3 the complexes $s(\mathcal{I}^p(K))$ are acyclic in positive degrees and have H^0 equal to $H^0(\mathcal{G}, \mathcal{I}^p)$. Hence by Homology, Lemma 12.25.4 and its proof the spectral sequence $({}'E_r, {}'d_r)$ degenerates, and the natural map

$$H^0(\mathcal{G}, \mathcal{I}^\bullet) \longrightarrow \text{Tot}(A^{\bullet,\bullet})$$

is a quasi-isomorphism of complexes of abelian groups. The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$ of the lemma is the composition of the natural map $s(\mathcal{F}(K)) \to \text{Tot}(A^{\bullet,\bullet})$ followed by the inverse of the displayed quasi-isomorphism above. This works because $H^0(\mathcal{G}, \mathcal{I}^\bullet)$ is a representative of $R\Gamma(\mathcal{G}, \mathcal{F})$.

Consider the spectral sequence $({}''E_r, {}''d_r)_{r \geq 0}$. By Homology, Lemma 12.25.1 we see that

$${}''E_2^{p,q} = H^p_{II}(H^q_I(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to d_1 which gives the groups ${}''E_1^{p,q} = \underline{H}^p(\mathcal{F})(K_q)$. Hence it is indeed the case (by the description of the differential ${}''d_1$) that ${}''E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. Since this spectral sequence converges to the cohomology of $\text{Tot}(A^{\bullet,\bullet})$ the proof is finished. □

09VY Lemma 25.6.5. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Let $\mathcal{F}$ be an abelian sheaf. There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to the global cohomology groups $H^{p+q}(\mathcal{F})$.

Proof. This is a special case of Lemma 25.6.4. □

25.7. Covering hypercoverings

01GG Here are some ways to construct hypercoverings. We note that since the category $\text{SR}(\mathcal{C}, X)$ has fibre products the category of simplicial objects of $\text{SR}(\mathcal{C}, X)$ has fibre products as well, see Simplicial, Lemma 14.7.2.

01GH **Lemma 25.7.1.** Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L, M be simplicial objects of $\text{SR}(\mathcal{C}, X)$. Let $a : K \to L$, $b : M \to L$ be morphisms. Assume

(1) K is a hypercovering of X,
(2) the morphism $M_0 \to L_0$ is a covering, and
(3) for all $n \geq 0$ in the diagram

$$\begin{tikzcd}
M_{n+1} \arrow[rr] \arrow[dr, "\gamma"] \arrow[dd] & & (\text{cosk}_n \text{sk}_n M)_{n+1} \arrow[dd] \\
& L_{n+1} \times_{(\text{cosk}_n \text{sk}_n L)_{n+1}} (\text{cosk}_n \text{sk}_n M)_{n+1} \arrow[ur] \arrow[dl] & \\
L_{n+1} \arrow[rr] & & (\text{cosk}_n \text{sk}_n L)_{n+1}
\end{tikzcd}$$

the arrow γ is a covering.

Then the fibre product $K \times_L M$ is a hypercovering of X.

Proof. The morphism $(K \times_L M)_0 = K_0 \times_{L_0} M_0 \to K_0$ is a base change of a covering by (2), hence a covering, see Lemma 25.3.2. And $K_0 \to \{X \to X\}$ is a covering by (1). Thus $(K \times_L M)_0 \to \{X \to X\}$ is a covering by Lemma 25.3.2. Hence $K \times_L M$ satisfies the first condition of Definition 25.3.3.

We still have to check that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K \times_L M)_{n+1} \longrightarrow (\text{cosk}_n \text{sk}_n (K \times_L M))_{n+1}$$

is a covering for all $n \geq 0$. We abbreviate as follows: $A = (\text{cosk}_n \text{sk}_n K)_{n+1}$, $B = (\text{cosk}_n \text{sk}_n L)_{n+1}$, and $C = (\text{cosk}_n \text{sk}_n M)_{n+1}$. The functor $\text{cosk}_n \text{sk}_n$ commutes with fibre products, see Simplicial, Lemma 14.19.13. Thus the right hand side above is equal to $A \times_B C$. Consider the following commutative diagram

$$\begin{tikzcd}
K_{n+1} \times_{L_{n+1}} M_{n+1} \arrow[r] \arrow[d] & M_{n+1} \arrow[d] \arrow[dr, "\gamma"] \arrow[drr] & & \\
K_{n+1} \arrow[r] \arrow[dr] & L_{n+1} \arrow[drr] & L_{n+1} \times_B C \arrow[l] \arrow[r] & C \arrow[d] \\
& A \arrow[rr] & & B
\end{tikzcd}$$

This diagram shows that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1}$$

Now, $K_{n+1} \times_B C \to A \times_B C$ is a base change of the covering $K_{n+1} \to A$ via the morphism $A \times_B C \to A$, hence is a covering. By assumption (3) the morphism γ is a covering. Hence the morphism

$$(K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1} \longrightarrow K_{n+1} \times_B C$$

is a covering as a base change of a covering. The lemma follows as a composition of coverings is a covering. □

01GI Lemma 25.7.2. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. If K, L are hypercoverings of X, then $K \times L$ is a hypercovering of X.

Proof. You can either verify this directly, or use Lemma 25.7.1 above and check that $L \to \{X \to X\}$ has property (3). □

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Since the category $\text{SR}(\mathcal{C}, X)$ has coproducts and finite limits, it is permissible to speak about the objects $U \times K$ and $\text{Hom}(U, K)$ for certain simplicial sets U (for example those with finitely many nondegenerate simplices) and any simplicial object K of $\text{SR}(\mathcal{C}, X)$. See Simplicial, Sections 14.13 and 14.17.

01GJ Lemma 25.7.3. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $k \geq 0$ be an integer. Let $u : Z \to K_k$ be a covering in $\text{SR}(\mathcal{C}, X)$. Then there exists a morphism of hypercoverings $f : L \to K$ such that $L_k \to K_k$ factors through u.

Proof. Denote $Y = K_k$. Let $C[k]$ be the cosimplicial set defined in Simplicial, Example 14.5.6. We will use the description of $\text{Hom}(C[k], Y)$ and $\text{Hom}(C[k], Z)$ given in Simplicial, Lemma 14.15.2. There is a canonical morphism $K \to \text{Hom}(C[k], Y)$ corresponding to $\text{id} : K_k = Y \to Y$. Consider the morphism $\text{Hom}(C[k], Z) \to \text{Hom}(C[k], Y)$ which on degree n terms is the morphism

$$\prod\nolimits_{\alpha:[k]\to[n]} Z \longrightarrow \prod\nolimits_{\alpha:[k]\to[n]} Y$$

using the given morphism $Z \to Y$ on each factor. Set

$$L = K \times_{\text{Hom}(C[k], Y)} \text{Hom}(C[k], Z).$$

The morphism $L_k \to K_k$ sits in to a commutative diagram

$$\begin{tikzcd}
L_k \arrow[r] \arrow[d] & \prod_{\alpha:[k]\to[k]} Z \arrow[r, "\text{pr}_{\text{id}_{[k]}}"] \arrow[d] & Z \arrow[d] \\
K_k \arrow[r] & \prod_{\alpha:[k]\to[k]} Y \arrow[r, "\text{pr}_{\text{id}_{[k]}}"] & Y
\end{tikzcd}$$

Since the composition of the two bottom arrows is the identity we conclude that we have the desired factorization.

We still have to show that L is a hypercovering of X. To see this we will use Lemma 25.7.1. Condition (1) is satisfied by assumption. For (2), the morphism

$$\text{Hom}(C[k], Z)_0 \to \text{Hom}(C[k], Y)_0$$

is a covering because it is isomorphic to $Z \to Y$ as there is only one morphism $[k] \to [0]$.

Let us consider condition (3) for $n = 0$. Then, since $(\text{cosk}_0 T)_1 = T \times T$ (Simplicial, Example 14.19.1) and since $\text{Hom}(C[k], Z)_1 = \prod_{\alpha:[k]\to[1]} Z$ we obtain the diagram

$$\begin{tikzcd} \prod_{\alpha:[k]\to[1]} Z \arrow[r] \arrow[d] & Z \times Z \arrow[d] \\ \prod_{\alpha:[k]\to[1]} Y \arrow[r] & Y \times Y \end{tikzcd}$$

with horizontal arrows corresponding to the projection onto the factors corresponding to the two nonsurjective α. Thus the arrow γ is the morphism

$$\prod\nolimits_{\alpha:[k]\to[1]} Z \longrightarrow \prod\nolimits_{\alpha:[k]\to[1] \text{ not onto}} Z \times \prod\nolimits_{\alpha:[k]\to[1] \text{ onto}} Y$$

which is a product of coverings and hence a covering by Lemma 25.3.2.

Let us consider condition (3) for $n > 0$. We claim there is an injective map $\tau : S' \to S$ of finite sets, such that for any object T of $\text{SR}(\mathcal{C}, X)$ the morphism

0B16 (25.7.3.1)
$$\text{Hom}(C[k], T)_{n+1} \to (\text{cosk}_n \text{sk}_n \text{Hom}(C[k], T))_{n+1}$$

is isomorphic to the projection $\prod_{s\in S} T \to \prod_{s'\in S'} T$ functorially in T. If this is true, then we see, arguing as in the previous paragraph, that the arrow γ is the morphism

$$\prod\nolimits_{s\in S} Z \longrightarrow \prod\nolimits_{s\in S'} Z \times \prod\nolimits_{s\not\in\tau(S')} Y$$

which is a product of coverings and hence a covering by Lemma 25.3.2. By construction, we have $\text{Hom}(C[k], T)_{n+1} = \prod_{\alpha:[k]\to[n+1]} T$ (see Simplicial, Lemma 14.15.2). Correspondingly we take $S = \text{Map}([k], [n+1])$. On the other hand, Simplicial, Lemma 14.19.5, provides a description of points of $(\text{cosk}_n \text{sk}_n \text{Hom}(C[k], T))_{n+1}$ as sequences $(f_0, \ldots, f_{n+1})$ of points of $\text{Hom}(C[k], T)_n$ satisfying $d^n_{j-1} f_i = d^n_i f_j$ for $0 \leq i < j \leq n+1$. We can write $f_i = (f_{i,\alpha})$ with $f_{i,\alpha}$ a point of T and $\alpha \in \text{Map}([k], [n])$. The conditions translate into

$$f_{i,\delta^n_{j-1}\circ\beta} = f_{j,\delta^n_i\circ\beta}$$

for any $0 \leq i < j \leq n+1$ and $\beta : [k] \to [n-1]$. Thus we see that

$$S' = \{0, \ldots, n+1\} \times \text{Map}([k], [n])/\sim$$

where the equivalence relation is generated by the equivalences

$$(i, \delta^n_{j-1} \circ \beta) \sim (j, \delta^n_i \circ \beta)$$

for $0 \leq i < j \leq n+1$ and $\beta : [k] \to [n-1]$. A computation (omitted) shows that the morphism (25.7.3.1) corresponds to the map $S' \to S$ which sends (i, α) to $\delta^{n+1}_i \circ \alpha \in S$. (It may be a comfort to the reader to see that this map is well defined by part (1) of Simplicial, Lemma 14.2.3.) To finish the proof it suffices to show that if $\alpha, \alpha' : [k] \to [n]$ and $0 \leq i < j \leq n+1$ are such that

$$\delta^{n+1}_i \circ \alpha = \delta^{n+1}_j \circ \alpha'$$

then we have $\alpha = \delta^n_{j-1} \circ \beta$ and $\alpha' = \delta^n_i \circ \beta$ for some $\beta : [k] \to [n-1]$. This is easy to see and omitted. $\square$

01GK Lemma 25.7.4. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $n \geq 0$ be an integer. Let $u : \mathcal{F} \to F(K_n)$ be a morphism of presheaves which becomes surjective on sheafification. Then there exists a morphism of hypercoverings $f : L \to K$ such that $F(f_n) : F(L_n) \to F(K_n)$ factors through u.

Proof. Write $K_n = \{U_i \to X\}_{i \in I}$. Thus the map u is a morphism of presheaves of sets $u : \mathcal{F} \to \amalg h_{u_i}$. The assumption on u means that for every $i \in I$ there exists a covering $\{U_{ij} \to U_i\}_{j \in I_i}$ of the site $\mathcal{C}$ and a morphism of presheaves $t_{ij} : h_{U_{ij}} \to \mathcal{F}$ such that $u \circ t_{ij}$ is the map $h_{U_{ij}} \to h_{U_i}$ coming from the morphism $U_{ij} \to U_i$. Set $J = \amalg_{i \in I} I_i$, and let $\alpha : J \to I$ be the obvious map. For $j \in J$ denote $V_j = U_{\alpha(j)j}$. Set $Z = \{V_j \to X\}_{j \in J}$. Finally, consider the morphism $u' : Z \to K_n$ given by $\alpha : J \to I$ and the morphisms $V_j = U_{\alpha(j)j} \to U_{\alpha(j)}$ above. Clearly, this is a covering in the category $\text{SR}(\mathcal{C}, X)$, and by construction $F(u') : F(Z) \to F(K_n)$ factors through u. Thus the result follows from Lemma 25.7.3 above. □

25.8. Adding simplices

01GL In this section we prove some technical lemmas which we will need later. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. As we pointed out in Section 25.7 above, the objects $U \times K$ and $\text{Hom}(U, K)$ for certain simplicial sets U and any simplicial object K of $\text{SR}(\mathcal{C}, X)$ are defined. See Simplicial, Sections 14.13 and 14.17.

01GM Lemma 25.8.1. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U has finitely many nondegenerate simplices. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that

1. $V_i = U_i$ for $i < n$,
2. $V_n = U_n \cup \{x\}$,
3. any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.

Then the morphism

$$\text{Hom}(V, K)_0 \longrightarrow \text{Hom}(U, K)_0$$

of $\text{SR}(\mathcal{C}, X)$ is a covering.

Proof. If $n = 0$, then it follows easily that $V = U \amalg \Delta[0]$ (see below). In this case $\text{Hom}(V, K)_0 = \text{Hom}(U, K)_0 \times K_0$. The result, in this case, then follows from Lemma 25.3.2.

Let $a : \Delta[n] \to V$ be the morphism associated to x as in Simplicial, Lemma 14.11.3. Let us write $\partial\Delta[n] = i_{(n-1)!}\text{sk}_{n-1}\Delta[n]$ for the $(n-1)$-skeleton of $\Delta[n]$. Let $b : \partial\Delta[n] \to U$ be the restriction of a to the $(n-1)$ skeleton of $\Delta[n]$. By Simplicial, Lemma 14.21.7 we have $V = U \amalg_{\partial\Delta[n]} \Delta[n]$. By Simplicial, Lemma 14.17.5 we get that

$$\begin{tikzcd} \text{Hom}(V, K)_0 \arrow[r] \arrow[d] & \text{Hom}(U, K)_0 \arrow[d] \\ \text{Hom}(\Delta[n], K)_0 \arrow[r] & \text{Hom}(\partial\Delta[n], K)_0 \end{tikzcd}$$

is a fibre product square. Thus it suffices to show that the bottom horizontal arrow is a covering. By Simplicial, Lemma 14.21.11 this arrow is identified with

$$K_n \to (\mathrm{cosk}_{n-1}\mathrm{sk}_{n-1}K)_n$$

and hence is a covering by definition of a hypercovering. □

01GN Lemma 25.8.2. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then the morphism

$$\mathrm{Hom}(V, K)_0 \longrightarrow \mathrm{Hom}(U, K)_0$$

of $\mathrm{SR}(\mathcal{C}, X)$ is a covering.

Proof. By Lemma 25.8.1 above, it suffices to prove a simple lemma about inclusions of simplicial sets $U \subset V$ as in the lemma. And this is exactly the result of Simplicial, Lemma 14.21.8. □

0DEQ Lemma 25.8.3. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Then

(1) K_n is a covering of X for each $n \geq 0$,
(2) $d^n_i : K_n \to K_{n-1}$ is a covering for all $n \geq 1$ and $0 \leq i \leq n$.

Proof. Recall that K_0 is a covering of X by Definition 25.3.3 and that this is equivalent to saying that $K_0 \to \{X \to X\}$ is a covering in the sense of Definition 25.3.1. Hence (1) follows from (2) because it will prove that the composition $K_n \to K_{n-1} \to \ldots \to K_0 \to \{X \to X\}$ is a covering by Lemma 25.3.2.

Proof of (2). Observe that $\mathrm{Mor}(\Delta[n], K)_0 = K_n$ by Simplicial, Lemma 14.17.4. Therefore (2) follows from Lemma 25.8.2 applied to the $n + 1$ different inclusions $\Delta[n - 1] \to \Delta[n]$. □

0DER Remark 25.8.4. A useful special case of Lemmas 25.8.2 and 25.8.3 is the following. Suppose we have a category $\mathcal{C}$ having fibre products. Let $P \subset \mathrm{Arrows}(\mathcal{C})$ be a subset stable under base change, stable under composition, and containing all isomorphisms. Then one says a P-hypercovering is an augmentation $a : U \to X$ from a simplicial object of $\mathcal{C}$ such that

(1) $U_0 \to X$ is in P,
(2) $U_1 \to U_0 \times_X U_0$ is in P,
(3) $U_{n+1} \to (\mathrm{cosk}_n\mathrm{sk}_nU)_{n+1}$ is in P for $n \geq 1$.

The category $\mathcal{C}/X$ has all finite limits, hence the coskeleta used in the formulation above exist (see Categories, Lemma 4.18.4). Then we claim that the morphisms $U_n \to X$ and $d^n_i : U_n \to U_{n-1}$ are in P. This follows from the aforementioned lemmas by turning $\mathcal{C}$ into a site whose coverings are $\{f : V \to U\}$ with $f \in P$ and taking K given by $K_n = \{U_n \to X\}$.

25.9. Homotopies

01GO Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $\mathrm{SR}(\mathcal{C}, X)$. According to Simplicial, Lemma 14.17.4 there exists an object $\mathrm{Hom}(\Delta[1], L)$ in the category $\mathrm{Simp}(\mathrm{SR}(\mathcal{C}, X))$ which represents the functor

$$T \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathrm{SR}(\mathcal{C},X))}(\Delta[1] \times T, L)$$

There is a canonical morphism

$$\mathrm{Hom}(\Delta[1], L) \to L \times L$$

coming from $e_i : \Delta[0] \to \Delta[1]$ and the identification $\mathrm{Hom}(\Delta[0], L) = L$.

01GP **Lemma 25.9.1.** Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $\mathrm{SR}(\mathcal{C}, X)$. Let $n \geq 0$. Consider the commutative diagram

01GQ (25.9.1.1)
$$\begin{tikzcd} \mathrm{Hom}(\Delta[1], L)_{n+1} \arrow[r] \arrow[d] & (\mathrm{cosk}_n \mathrm{sk}_n \mathrm{Hom}(\Delta[1], L))_{n+1} \arrow[d] \\ (L \times L)_{n+1} \arrow[r] & (\mathrm{cosk}_n \mathrm{sk}_n (L \times L))_{n+1} \end{tikzcd}$$

coming from the morphism defined above. We can identify the terms in this diagram as follows, where $\partial\Delta[n+1] = i_{n!}\mathrm{sk}_n\Delta[n+1]$ is the n-skeleton of the $(n+1)$-simplex:

$$\begin{matrix} \mathrm{Hom}(\Delta[1], L)_{n+1} & = & \mathrm{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \\ (\mathrm{cosk}_n \mathrm{sk}_n \mathrm{Hom}(\Delta[1], L))_{n+1} & = & \mathrm{Hom}(\Delta[1] \times \partial\Delta[n+1], L)_0 \\ (L \times L)_{n+1} & = & \mathrm{Hom}((\Delta[n+1] \amalg \Delta[n+1], L)_0 \\ (\mathrm{cosk}_n \mathrm{sk}_n (L \times L))_{n+1} & = & \mathrm{Hom}(\partial\Delta[n+1] \amalg \partial\Delta[n+1], L)_0 \end{matrix}$$

and the morphism between these objects of $\mathrm{SR}(\mathcal{C}, X)$ come from the commutative diagram of simplicial sets

01GR (25.9.1.2)
$$\begin{tikzcd} \Delta[1] \times \Delta[n+1] & \Delta[1] \times \partial\Delta[n+1] \arrow[l] \\ \Delta[n+1] \amalg \Delta[n+1] \arrow[u] & \partial\Delta[n+1] \amalg \partial\Delta[n+1] \arrow[l] \arrow[u] \end{tikzcd}$$

Moreover the fibre product of the bottom arrow and the right arrow in (25.9.1.1) is equal to

$$\mathrm{Hom}(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$ is the smallest simplicial subset such that both $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ map into it.

Proof. The first and third equalities are Simplicial, Lemma 14.17.4. The second and fourth follow from the cited lemma combined with Simplicial, Lemma 14.21.11. The last assertion follows from the fact that U is the push-out of the bottom and right arrow of the diagram (25.9.1.2), via Simplicial, Lemma 14.17.5. To see that U is equal to this push-out it suffices to see that the intersection of $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ in $\Delta[1] \times \Delta[n+1]$ is equal to $\partial\Delta[n+1] \amalg \partial\Delta[n+1]$. This we leave to the reader. $\square$

01GS **Lemma 25.9.2.** Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L be hypercoverings of X. Let $a, b : K \to L$ be morphisms of hypercoverings. There exists a morphism of hypercoverings $c : K' \to K$ such that $a \circ c$ is homotopic to $b \circ c$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} K' \arrow[r, equal, "def"] \arrow[drr, "c"'] & K \times_{(L\times L)} \mathrm{Hom}(\Delta[1], L) \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta[1], L) \arrow[d] \\ & K \arrow[r, "(a,b)"] & L \times L \end{tikzcd}$$

By the functorial property of $\mathrm{Hom}(\Delta[1], L)$ the composition of the horizontal morphisms corresponds to a morphism $K' \times \Delta[1] \to L$ which defines a homotopy between $c \circ a$ and $c \circ b$. Thus if we can show that K' is a hypercovering of X, then we obtain the lemma. To see this we will apply Lemma 25.7.1 to the pair of morphisms $K \to L \times L$ and $\mathrm{Hom}(\Delta[1], L) \to L \times L$. Condition (1) of Lemma 25.7.1 is satisfied. Condition (2) of Lemma 25.7.1 is true because $\mathrm{Hom}(\Delta[1], L)_0 = L_1$, and the morphism $(d^1_0, d^1_1) : L_1 \to L_0 \times L_0$ is a covering of $\mathrm{SR}(\mathcal{C}, X)$ by our assumption that L is a hypercovering. To prove condition (3) of Lemma 25.7.1 we use Lemma 25.9.1 above. According to this lemma the morphism γ of condition (3) of Lemma 25.7.1 is the morphism

$$\mathrm{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \longrightarrow \mathrm{Hom}(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$. According to Lemma 25.8.2 this is a covering and hence the claim has been proven. $\square$

01GT Remark 25.9.3. Note that the crux of the proof is to use Lemma 25.8.2. This lemma is completely general and does not care about the exact shape of the simplicial sets (as long as they have only finitely many nondegenerate simplices). It seems altogether reasonable to expect a result of the following kind: Given any morphism $a : K \times \partial\Delta[k] \to L$, with K and L hypercoverings, there exists a morphism of hypercoverings $c : K' \to K$ and a morphism $g : K' \times \Delta[k] \to L$ such that $g|_{K' \times \partial\Delta[k]} = a \circ (c \times \mathrm{id}_{\partial\Delta[k]})$. In other words, the category of hypercoverings is in a suitable sense contractible.

25.10. Cohomology and hypercoverings

01GZ Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Let K, L be hypercoverings of X. If $a, b : K \to L$ are homotopic maps, then $\mathcal{F}(a), \mathcal{F}(b) : \mathcal{F}(K) \to \mathcal{F}(L)$ are homotopic maps, see Simplicial, Lemma 14.28.4. Hence have the same effect on cohomology groups of the associated cochain complexes, see Simplicial, Lemma 14.28.6. We are going to use this to define the colimit over all hypercoverings.

Let us temporarily denote $\mathrm{HC}(\mathcal{C}, X)$ the category whose objects are hypercoverings of X and whose morphisms are maps between hypercoverings of X up to homotopy. We have seen that this is a category and not a "big" category, see Lemma 25.3.7. The opposite to $\mathrm{HC}(\mathcal{C}, X)$ will be the index category for our diagram, see Categories, Section 4.14 for terminology. Consider the diagram

$$\check{H}^i(-, \mathcal{F}) : \mathrm{HC}(\mathcal{C}, X)^{opp} \longrightarrow \mathrm{Ab}.$$

By Lemmas 25.7.2 and 25.9.2 and the remark on homotopies above, this diagram is directed, see Categories, Definition 4.19.1. Thus the colimit

$$\check{H}^i_{\mathrm{HC}}(X, \mathcal{F}) = \mathrm{colim}_{K \in \mathrm{HC}(\mathcal{C}, X)} \check{H}^i(K, \mathcal{F})$$

has a particularly simple description (see location cited).

01H0 **Theorem 25.10.1.** Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $i \geq 0$. The functors

$$\begin{array}{ccc} \text{Ab}(\mathcal{C}) & \longrightarrow & \text{Ab} \\ \mathcal{F} & \longmapsto & H^i(X, \mathcal{F}) \\ \mathcal{F} & \longmapsto & \check{H}^i_{\text{HC}}(X, \mathcal{F}) \end{array}$$

are canonically isomorphic.

Proof using spectral sequences. Suppose that $\xi \in H^p(X, \mathcal{F})$ for some $p \geq 0$. Let us show that ξ is in the image of the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 25.5.3 for some hypercovering K of X.

This is true if $p = 0$ by Lemma 25.5.1. If $p = 1$, choose a Čech hypercovering K of X as in Example 25.3.4 starting with a covering $K_0 = \{U_i \to X\}$ in the site $\mathcal{C}$ such that $\xi|_{U_i} = 0$, see Cohomology on Sites, Lemma 21.7.3. It follows immediately from the spectral sequence in Lemma 25.5.3 that ξ comes from an element of $\check{H}^1(K, \mathcal{F})$ in this case. In general, choose any hypercovering K of X such that ξ maps to zero in $\underline{H}^p(\mathcal{F})(K_0)$ (using Example 25.3.4 and Cohomology on Sites, Lemma 21.7.3 again). By the spectral sequence of Lemma 25.5.3 the obstruction for ξ to come from an element of $\check{H}^p(K, \mathcal{F})$ is a sequence of elements $\xi_1, \ldots, \xi_{p-1}$ with $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$ (more precisely the images of the ξ_q in certain subquotients of these groups).

We can inductively replace the hypercovering K by refinements such that the obstructions $\xi_1, \ldots, \xi_{p-1}$ restrict to zero (and not just the images in the subquotients – so no subtlety here). Indeed, suppose we have already managed to reach the situation where $\xi_{q+1}, \ldots, \xi_{p-1}$ are zero. Note that $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$ is the class of some element

$$\tilde{\xi}_q \in \underline{H}^q(\mathcal{F})(K_{p-q}) = \prod H^q(U_i, \mathcal{F})$$

if $K_{p-q} = \{U_i \to X\}_{i \in I}$. Let $\xi_{q,i}$ be the component of $\tilde{\xi}_q$ in $H^q(U_i, \mathcal{F})$. As $q \geq 1$ we can use Cohomology on Sites, Lemma 21.7.3 yet again to choose coverings $\{U_{i,j} \to U_i\}$ of the site such that each restriction $\xi_{q,i}|_{U_{i,j}} = 0$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\text{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_{p-q}$. It is clear that u is a covering, see Definition 25.3.1. By Lemma 25.7.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_{p-q} \to K_{p-q}$ factors through u. Then clearly the image of ξ_q in $\underline{H}^q(\mathcal{F})(L_{p-q})$. is zero. Since the spectral sequence of Lemma 25.5.3 is functorial this means that after replacing K by L we reach the situation where $\xi_q, \ldots, \xi_{p-1}$ are all zero. Continuing like this we end up with a hypercovering where they are all zero and hence ξ is in the image of the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$.

Suppose that K is a hypercovering of X, that $\xi \in \check{H}^p(K, \mathcal{F})$ and that the image of ξ under the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 25.5.3 is zero. To finish the proof of the theorem we have to show that there exists a morphism of hypercoverings $L \to K$ such that ξ restricts to zero in $\check{H}^p(L, \mathcal{F})$. By the spectral sequence of Lemma 25.5.3 the vanishing of the image of ξ in $H^p(X, \mathcal{F})$ means that there exist elements $\xi_1, \ldots, \xi_{p-2}$ with $\xi_q \in \check{H}^{p-1-q}(K, \underline{H}^q(\mathcal{F}))$ (more precisely the images of these in certain subquotients) such that the images $d_{q+1}^{p-1-q,q}\xi_q$ (in the spectral sequence) add up to ξ. Hence by exactly the same mechanism as above we can find a morphism of hypercoverings $L \to K$ such that the restrictions of the elements ξ_q, $q = 1, \ldots, p-2$ in $\check{H}^{p-1-q}(L, \underline{H}^q(\mathcal{F}))$ are zero. Then it follows that ξ is zero since the morphism $L \to K$ induces a morphism of spectral sequences according to Lemma 25.5.3. □

Proof without using spectral sequences. We have seen the result for $i = 0$, see Lemma 25.5.1. We know that the functors $H^i(X, -)$ form a universal δ-functor, see Derived Categories, Lemma 13.20.4. In order to prove the theorem it suffices to show that the sequence of functors $\check{H}^i_{HC}(X, -)$ forms a δ-functor. Namely we know that Čech cohomology is zero on injective sheaves (Lemma 25.5.2) and then we can apply Homology, Lemma 12.12.4.

Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. Choose a hypercovering K of X and an element $\sigma \in \mathcal{H}(K_p)$ representing ξ in cohomology. There is a corresponding exact sequence of complexes

$$0 \to s(\mathcal{F}(K)) \to s(\mathcal{G}(K)) \to s(\mathcal{H}(K))$$

but we are not assured that there is a zero on the right also and this is the only thing that prevents us from defining $\delta(\xi)$ by a simple application of the snake lemma. Recall that

$$\mathcal{H}(K_p) = \prod \mathcal{H}(U_i)$$

if $K_p = \{U_i \to X\}$. Let $\sigma = \prod \sigma_i$ with $\sigma_i \in \mathcal{H}(U_i)$. Since $\mathcal{G} \to \mathcal{H}$ is a surjection of sheaves we see that there exist coverings $\{U_{i,j} \to U_i\}$ such that $\sigma_i|_{U_{i,j}}$ is the image of some element $\tau_{i,j} \in \mathcal{G}(U_{i,j})$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\text{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_p$. It is clear that u is a covering, see Definition 25.3.1. By Lemma 25.7.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_p \to K_p$ factors through u. After replacing K by L we may therefore assume that σ is the image of an element $\tau \in \mathcal{G}(K_p)$. Note that $d(\sigma) = 0$, but not necessarily $d(\tau) = 0$. Thus $d(\tau) \in \mathcal{F}(K_{p+1})$ is a cocycle. In this situation we define $\delta(\xi)$ as the class of the cocycle $d(\tau)$ in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$.

At this point there are several things to verify: (a) $\delta(\xi)$ does not depend on the choice of τ, (b) $\delta(\xi)$ does not depend on the choice of the hypercovering $L \to K$ such that σ lifts, and (c) $\delta(\xi)$ does not depend on the initial hypercovering and σ chosen to represent ξ. We omit the verification of (a), (b), and (c); the independence of the choices of the hypercoverings really comes down to Lemmas 25.7.2 and 25.9.2. We also omit the verification that δ is functorial with respect to morphisms of short exact sequences of abelian sheaves on $\mathcal{C}$.

Finally, we have to verify that with this definition of δ our short exact sequence of abelian sheaves above leads to a long exact sequence of Čech cohomology groups. First we show that if $\delta(\xi) = 0$ (with ξ as above) then ξ is the image of some element $\xi' \in \check{H}^p_{HC}(X, \mathcal{G})$. Namely, if $\delta(\xi) = 0$, then, with notation as above, we see that the class of $d(\tau)$ is zero in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$. Hence there exists a morphism of hypercoverings $L \to K$ such that the restriction of $d(\tau)$ to an element of $\mathcal{F}(L_{p+1})$ is equal to $d(\upsilon)$ for some $\upsilon \in \mathcal{F}(L_p)$. This implies that $\tau|_{L_p} + \upsilon$ form a cocycle, and determine a class $\xi' \in \check{H}^p(L, \mathcal{G})$ which maps to ξ as desired.

We omit the proof that if $\xi' \in \check{H}^{p+1}_{HC}(X, \mathcal{F})$ maps to zero in $\check{H}^{p+1}_{HC}(X, \mathcal{G})$, then it is equal to $\delta(\xi)$ for some $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. $\square$

Next, we deduce Verdier's case of Theorem 25.10.1 by a sleight of hand.

09VZ Proposition 25.10.2. Let $\mathcal{C}$ be a site with fibre products and products of pairs. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $i \geq 0$. Then

(1) for every $\xi \in H^i(\mathcal{F})$ there exists a hypercovering K such that ξ is in the image of the canonical map $\check{H}^i(K, \mathcal{F}) \to H^i(\mathcal{F})$, and
(2) if K, L are hypercoverings and $\xi_K \in \check{H}^i(K, \mathcal{F})$, $\xi_L \in \check{H}^i(L, \mathcal{F})$ are elements mapping to the same element of $H^i(\mathcal{F})$, then there exists a hypercovering M and morphisms $M \to K$ and $M \to L$ such that ξ_K and ξ_L map to the same element of $\check{H}^i(M, \mathcal{F})$.

In other words, modulo set theoretical issues, the cohomology groups of $\mathcal{F}$ on $\mathcal{C}$ are the colimit of the Čech cohomology groups of $\mathcal{F}$ over all hypercoverings.

Proof. This result is a trivial consequence of Theorem 25.10.1. Namely, we can artificially replace $\mathcal{C}$ with a slightly bigger site $\mathcal{C}'$ such that (I) $\mathcal{C}'$ has a final object X and (II) hypercoverings in $\mathcal{C}$ are more or less the same thing as hypercoverings of X in $\mathcal{C}'$. But due to the nature of things, there is quite a bit of bookkeeping to do.

Let us call a family of morphisms $\{U_i \to U\}$ in $\mathcal{C}$ with fixed target a weak covering if the sheafification of the map $\coprod_{i \in I} h_{U_i} \to h_U$ becomes surjective. We construct a new site $\mathcal{C}'$ as follows

(1) as a category set $\text{Ob}(\mathcal{C}') = \text{Ob}(\mathcal{C}) \amalg \{X\}$ and add a unique morphism to X from every object of $\mathcal{C}'$,
(2) $\mathcal{C}'$ has fibre products as fibre products and products of pairs exist in $\mathcal{C}$,
(3) coverings of $\mathcal{C}'$ are weak coverings of $\mathcal{C}$ together with those $\{U_i \to X\}_{i \in I}$ such that either $U_i = X$ for some i, or $U_i \neq X$ for all i and the map $\coprod h_{U_i} \to *$ of presheaves on $\mathcal{C}$ becomes surjective after sheafification on $\mathcal{C}$,
(4) we apply Sets, Lemma 3.11.1 to restrict the coverings to obtain our site $\mathcal{C}'$.

Then $Sh(\mathcal{C}') = Sh(\mathcal{C})$ because the inclusion functor $\mathcal{C} \to \mathcal{C}'$ is a special cocontinuous functor (see Sites, Definition 7.29.2). We omit the straightforward verifications.

Choose a covering $\{U_i \to X\}$ of $\mathcal{C}'$ such that U_i is an object of $\mathcal{C}$ for all i (possible because $\mathcal{C} \to \mathcal{C}'$ is special cocontinuous). Then $K_0 = \{U_i \to X\}$ is a covering in the site $\mathcal{C}'$ constructed above. We view K_0 as an object of $\text{SR}(\mathcal{C}', X)$ and we set $K_{init} = \text{cosk}_0(K_0)$. Then K_{init} is a hypercovering of X, see Example 25.3.4. Note that every $K_{init,n}$ has the shape $\{W_j \to X\}$ with $W_j \in \text{Ob}(\mathcal{C})$.

Proof of (1). Choose $\xi \in H^i(\mathcal{F}) = H^i(X, \mathcal{F}')$ where $\mathcal{F}'$ is the abelian sheaf on $\mathcal{C}'$ corresponding to $\mathcal{F}$ on $\mathcal{C}$. By Theorem 25.10.1 there exists a morphism of hypercoverings $K' \to K_{init}$ of X in $\mathcal{C}'$ such that ξ comes from an element of $\check{H}^i(K', \mathcal{F})$. Write $K'_n = \{U_{n,j} \to X\}$. Now since K'_n maps to $K_{init,n}$ we see that $U_{n,j}$ is an object of $\mathcal{C}$. Hence we can define a simplicial object K of $\text{SR}(\mathcal{C})$ by setting $K_n = \{U_{n,j}\}$. Since coverings in $\mathcal{C}'$ consisting of families of morphisms of $\mathcal{C}$ are weak coverings, we see that K is a hypercovering in the sense of Definition 25.6.1. Finally, since $\mathcal{F}'$ is the unique sheaf on $\mathcal{C}'$ whose restriction to $\mathcal{C}$ is equal to $\mathcal{F}$ we see that the Čech complexes $s(\mathcal{F}(K))$ and $s(\mathcal{F}'(K'))$ are identical and (1) follows. (Compatibility with map into cohomology groups omitted.)

Proof of (2). Let K and L be hypercoverings in $\mathcal{C}$. Let K' and L' be the simplicial objects of $\text{SR}(\mathcal{C}', X)$ gotten from K and L by the functor $\text{SR}(\mathcal{C}) \to \text{SR}(\mathcal{C}', X)$, $\{U_i\} \mapsto \{U_i \to X\}$. As before we have equality of Čech complexes and hence we obtain $\xi_{K'}$ and $\xi_{L'}$ mapping to the same cohomology class of $\mathcal{F}'$ over $\mathcal{C}'$. After possibly enlarging our choice of coverings in $\mathcal{C}'$ (due to a set theoretical issue) we may assume that K' and L' are hypercoverings of X in $\mathcal{C}'$; this is true by our definition of hypercoverings in Definition 25.6.1 and the fact that weak coverings in $\mathcal{C}$ give coverings in $\mathcal{C}'$. By Theorem 25.10.1 there exists a hypercovering M' of X in $\mathcal{C}'$ and morphisms $M' \to K'$, $M' \to L'$, and $M' \to K_{init}$ such that $\xi_{K'}$ and $\xi_{L'}$ restrict to the same element of $\check{H}^i(M', \mathcal{F})$. Unwinding this statement as above we find that (2) is true. □

25.11. Hypercoverings of spaces

01H1 The theory above is mildly interesting even in the case of topological spaces. In this case we can work out what a hypercovering is and see what the result actually says.

Let X be a topological space. Consider the site X_{Zar} of Sites, Example 7.6.4. Recall that an object of X_{Zar} is simply an open of X and that morphisms of X_{Zar} correspond simply to inclusions. So what is a hypercovering of X for the site X_{Zar}?

Let us first unwind Definition 25.2.1. An object of $\text{SR}(X_{Zar}, X)$ is simply given by a set I and for each $i \in I$ an open $U_i \subset X$. Let us denote this by $\{U_i\}_{i\in I}$ since there can be no confusion about the morphism $U_i \to X$. A morphism $\{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ between two such objects is given by a map of sets $\alpha : I \to J$ such that $U_i \subset V_{\alpha(i)}$ for all $i \in I$. When is such a morphism a covering? This is the case if and only if for every $j \in J$ we have $V_j = \bigcup_{i\in I,\ \alpha(i)=j} U_i$ (and is a covering in the site X_{Zar}).

Using the above we get the following description of a hypercovering in the site X_{Zar}. A hypercovering of X in X_{Zar} is given by the following data

(1) a simplicial set I (see Simplicial, Section 14.11), and
(2) for each $n \geq 0$ and every $i \in I_n$ an open set $U_i \subset X$.

We will denote such a collection of data by the notation $(I, \{U_i\})$. In order for this to be a hypercovering of X we require the following properties

- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i \subset U_{d^n_a(i)}$,
- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i = U_{s^n_a(i)}$,
- we have

01H2 (25.11.0.1)
$$X = \bigcup\nolimits_{i\in I_0} U_i,$$

- for every $i_0, i_1 \in I_0$, we have

01H3 (25.11.0.2)
$$U_{i_0} \cap U_{i_1} = \bigcup\nolimits_{i\in I_1,\ d^1_0(i)=i_0,\ d^1_1(i)=i_1} U_i,$$

- for every $n \geq 1$ and every $(i_0, \ldots, i_{n+1}) \in (I_n)^{n+2}$ such that $d^n_{b-1}(i_a) = d^n_a(i_b)$ for all $0 \leq a < b \leq n+1$ we have

01H4 (25.11.0.3)
$$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup\nolimits_{i\in I_{n+1},\ d^{n+1}_a(i)=i_a,\ a=0,\ldots,n+1} U_i,$$

- each of the open coverings (25.11.0.1), (25.11.0.2), and (25.11.0.3) is an element of $\text{Cov}(X_{Zar})$ (this is a set theoretic condition, bounding the size of the index sets of the coverings).

Conditions (25.11.0.1) and (25.11.0.2) should be familiar from the chapter on sheaves on spaces for example, and condition (25.11.0.3) is the natural generalization.

01H5 Remark 25.11.1. One feature of this description is that if one of the multiple intersections $U_{i_0} \cap \ldots \cap U_{i_{n+1}}$ is empty then the covering on the right hand side may be the empty covering. Thus it is not automatically the case that the maps $I_{n+1} \to (\text{cosk}_n \text{sk}_n I)_{n+1}$ are surjective. This means that the geometric realization of I may be an interesting (non-contractible) space.

In fact, let $I'_n \subset I_n$ be the subset consisting of those simplices $i \in I_n$ such that $U_i \neq \emptyset$. It is easy to see that $I' \subset I$ is a subsimplicial set, and that $(I', \{U_i\})$ is a hypercovering. Hence we can always refine a hypercovering to a hypercovering where none of the opens U_i is empty.

02N9 Remark 25.11.2. Let us repackage this information in yet another way. Namely, suppose that $(I, \{U_i\})$ is a hypercovering of the topological space X. Given this data we can construct a simplicial topological space $U_\bullet$ by setting

$$U_n = \coprod\nolimits_{i \in I_n} U_i,$$

and where for given $\varphi : [n] \to [m]$ we let morphisms $U(\varphi) : U_n \to U_m$ be the morphism coming from the inclusions $U_i \subset U_{\varphi(i)}$ for $i \in I_n$. This simplicial topological space comes with an augmentation $\epsilon : U_\bullet \to X$. With this morphism the simplicial space $U_\bullet$ becomes a hypercovering of X along which one has cohomological descent in the sense of [AGV71, Exposé Vbis]. In other words, $H^n(U_\bullet, \epsilon^*\mathcal{F}) = H^n(X, \mathcal{F})$. (Insert future reference here to cohomology over simplicial spaces and cohomological descent formulated in those terms.) Suppose that $\mathcal{F}$ is an abelian sheaf on X. In this case the spectral sequence of Lemma 25.5.3 becomes the spectral sequence with E_1-term

$$E_1^{p,q} = H^q(U_p, \epsilon_q^*\mathcal{F}) \Rightarrow H^{p+q}(U_\bullet, \epsilon^*\mathcal{F}) = H^{p+q}(X, \mathcal{F})$$

comparing the total cohomology of $\epsilon^*\mathcal{F}$ to the cohomology groups of $\mathcal{F}$ over the pieces of $U_\bullet$. (Insert future reference to this spectral sequence here.)

In topology we often want to find hypercoverings of X which have the property that all the U_i come from a given basis for the topology of X and that all the coverings (25.11.0.2) and (25.11.0.3) are from a given cofinal collection of coverings. Here are two example lemmas.

01H6 Lemma 25.11.3. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. There exists a hypercovering $(I, \{U_i\})$ of X such that each U_i is an element of $\mathcal{B}$.

Proof. Let $n \geq 0$. Let us say that an n-truncated hypercovering of X is given by an n-truncated simplicial set I and for each $i \in I_a$, $0 \leq a \leq n$ an open U_i of X such that the conditions defining a hypercovering hold whenever they make sense. In other words we require the inclusion relations and covering conditions only when all simplices that occur in them are a-simplices with $a \leq n$. The lemma follows if we can prove that given a n-truncated hypercovering $(I, \{U_i\})$ with all $U_i \in \mathcal{B}$ we can extend it to an $(n+1)$-truncated hypercovering without adding any a-simplices for $a \leq n$. This we do as follows. First we consider the $(n+1)$-truncated simplicial set I' defined by $I' = \text{sk}_{n+1}(\text{cosk}_n I)$. Recall that

$$I'_{n+1} = \left\{ \begin{matrix} (i_0, \ldots, i_{n+1}) \in (I_n)^{n+2} \text{ such that} \\ d^n_{b-1}(i_a) = d^n_a(i_b) \text{ for all } 0 \leq a < b \leq n+1 \end{matrix} \right\}$$

If $i' \in I'_{n+1}$ is degenerate, say $i' = s^n_a(i)$ then we set $U_{i'} = U_i$ (this is forced on us anyway by the second condition). We also set $J_{i'} = \{i'\}$ in this case. If $i' \in I'_{n+1}$ is nondegenerate, say $i' = (i_0, \ldots, i_{n+1})$, then we choose a set $J_{i'}$ and an open covering

071K (25.11.3.1) $$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup\nolimits_{i \in J_{i'}} U_i,$$

with $U_i \in \mathcal{B}$ for $i \in J_{i'}$. Set

$$I_{n+1} = \coprod\nolimits_{i' \in I'_{n+1}} J_{i'}$$

There is a canonical map $\pi : I_{n+1} \to I'_{n+1}$ which is a bijection over the set of degenerate simplices in I'_{n+1} by construction. For $i \in I_{n+1}$ we define $d^{n+1}_a(i) = d^{n+1}_a(\pi(i))$. For $i \in I_n$ we define $s^n_a(i) \in I_{n+1}$ as the unique simplex lying over the degenerate simplex $s^n_a(i) \in I'_{n+1}$. We omit the verification that this defines an $(n+1)$-truncated hypercovering of X. □

01H7 Lemma 25.11.4. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. Assume that

(1) X is quasi-compact,
(2) each $U \in \mathcal{B}$ is quasi-compact open, and
(3) the intersection of any two quasi-compact opens in X is quasi-compact.

Then there exists a hypercovering $(I, \{U_i\})$ of X with the following properties

(1) each U_i is an element of the basis $\mathcal{B}$,
(2) each of the I_n is a finite set, and in particular
(3) each of the coverings (25.11.0.1), (25.11.0.2), and (25.11.0.3) is finite.

Proof. This follows directly from the construction in the proof of Lemma 25.11.3 if we choose finite coverings by elements of $\mathcal{B}$ in (25.11.3.1). Details omitted. □

25.12. Constructing hypercoverings

094J Let $\mathcal{C}$ be a site. In this section we will think of a simplicial object of $\text{SR}(\mathcal{C})$ as follows. As usual, we set $K_n = K([n])$ and we denote $K(\varphi) : K_n \to K_m$ the morphism associated to $\varphi : [m] \to [n]$. We may write $K_n = \{U_{n,i}\}_{i \in I_n}$. For $\varphi : [m] \to [n]$ the morphism $K(\varphi) : K_n \to K_m$ is given by a map $\alpha(\varphi) : I_n \to I_m$ and morphisms $f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ for $i \in I_n$. The fact that K is a simplicial object of $\text{SR}(\mathcal{C})$ implies that $(I_n, \alpha(\varphi))$ is a simplicial set and that $f_{\psi,\alpha(\varphi)(i)} \circ f_{\varphi,i} = f_{\varphi \circ \psi,i}$ when $\psi : [l] \to [m]$.

0DAU Lemma 25.12.1. Let $\mathcal{C}$ be a site. Let K be an r-truncated simplicial object of $\text{SR}(\mathcal{C})$. The following are equivalent

(1) K is split (Simplicial, Definition 14.18.1),
(2) $f_{\varphi,i} : U_{n,i} \to U_{m,\alpha(\varphi)(i)}$ is an isomorphism for $r \geq n \geq 0$, $\varphi : [m] \to [n]$ surjective, $i \in I_n$, and
(3) $f_{\sigma^n_j,i} : U_{n,i} \to U_{n+1,\alpha(\sigma^n_j)(i)}$ is an isomorphism for $0 \leq j \leq n < r$, $i \in I_n$.

The same holds for simplicial objects if in (2) and (3) we set $r = \infty$.

Proof. The splitting of a simplicial set is unique and is given by the nondegenerate indices $N(I_n)$ in each degree n, see Simplicial, Lemma 14.18.2. The coproduct of two objects $\{U_i\}_{i \in I}$ and $\{U_j\}_{j \in J}$ of $\text{SR}(\mathcal{C})$ is given by $\{U_l\}_{l \in I \amalg J}$ with obvious notation. Hence a splitting of K must be given by $N(K_n) = \{U_i\}_{i \in N(I_n)}$. The equivalence of (1) and (2) now follows by unwinding the definitions. The equivalence of (2) and (3) follows from the fact that any surjection $\varphi : [m] \to [n]$ is a composition of morphisms σ^k_j with $k = n, n+1, \ldots, m-1$. $\square$

0DAV Lemma 25.12.2. Let $\mathcal{C}$ be a site with fibre products. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Assume

1. any object U of $\mathcal{C}$ has a covering $\{U_j \to U\}_{j \in J}$ with $U_j \in \mathcal{B}$, and
2. if $\{U_j \to U\}_{j \in J}$ is a covering with $U_j \in \mathcal{B}$ and $\{U' \to U\}$ is a morphism with $U' \in \mathcal{B}$, then $\{U_j \to U\}_{j \in J} \amalg \{U' \to U\}$ is a covering.

Then for any X in $\mathcal{C}$ there is a hypercovering K of X such that $K_n = \{U_{n,i}\}_{i \in I_n}$ with $U_{n,i} \in \mathcal{B}$ for all $i \in I_n$.

Proof. A warmup for this proof is the proof of Lemma 25.11.3 and we encourage the reader to read that proof first.

First we replace $\mathcal{C}$ by the site $\mathcal{C}/X$. After doing so we may assume that X is the final object of $\mathcal{C}$ and that $\mathcal{C}$ has all finite limits (Categories, Lemma 4.18.4).

Let $n \geq 0$. Let us say that an n-truncated $\mathcal{B}$-hypercovering of X is given by an n-truncated simplicial object K of $\text{SR}(\mathcal{C})$ such that for $i \in I_a$, $0 \leq a \leq n$ we have $U_{a,i} \in \mathcal{B}$ and such that K_0 is a covering of X and $K_{a+1} \to (\text{cosk}_a \text{sk}_a K)_{a+1}$ for $a = 0, \ldots, n-1$ is a covering as in Definition 25.3.1.

Since X has a covering $\{U_{0,i} \to X\}_{i \in I_0}$ with $U_i \in \mathcal{B}$ by assumption, we get a 0-truncated $\mathcal{B}$-hypercovering of X. Observe that any 0-truncated $\mathcal{B}$-hypercovering of X is split, see Lemma 25.12.1.

The lemma follows if we can prove for $n \geq 0$ that given a split n-truncated $\mathcal{B}$-hypercovering K of X we can extend it to a split $(n+1)$-truncated $\mathcal{B}$-hypercovering of X.

Construction of the extension. Consider the $(n+1)$-truncated simplicial object $K' = \text{sk}_{n+1}(\text{cosk}_n K)$ of $\text{SR}(\mathcal{C})$. Write

$$K'_{n+1} = \{U'_{n+1,i}\}_{i \in I'_{n+1}}$$

Since $K = \text{sk}_n K'$ we have $K_a = K'_a$ for $0 \leq a \leq n$. For every $i' \in I'_{n+1}$ we choose a covering

0DAW (25.12.2.1)
$$\{g_{n+1,j} : U_{n+1,j} \to U'_{n+1,i'}\}_{j \in J_{i'}}$$

with $U_{n+1,j} \in \mathcal{B}$ for $j \in J_{i'}$. This is possible by our assumption on $\mathcal{B}$ in the lemma. For $0 \leq m \leq n$ denote $N_m \subset I_m$ the subset of nondegenerate indices. We set

$$I_{n+1} = \coprod\nolimits_{\varphi : [n+1] \to [m] \text{ surjective}, \ 0 \leq m \leq n} N_m \amalg \coprod\nolimits_{i' \in I'_{n+1}} J_{i'}$$

For $j \in I_{n+1}$ we set

$$U_{n+1,j} = \begin{cases} U_{m,i} & \text{if} \quad j = (\varphi, i) \quad \text{where} \quad \varphi : [n+1] \to [m], i \in N_m \\ U_{n+1,j} & \text{if} \quad j \in J_{i'} \quad \text{where} \quad i' \in I'_{n+1} \end{cases}$$

with obvious notation. We set $K_{n+1} = \{U_{n+1,j}\}_{j \in I_{n+1}}$. By construction $U_{n+1,j}$ is an element of $\mathcal{B}$ for all $j \in I_{n+1}$. Let us define compatible maps

$$I_{n+1} \to I'_{n+1} \quad \text{and} \quad K_{n+1} \to K'_{n+1}$$

Namely, the first map is given by $(\varphi, i) \mapsto \alpha'(\varphi)(i)$ and $(j \in J_{i'}) \mapsto i'$. For the second map we use the morphisms

$$f'_{\varphi,i} : U_{m,i} \to U'_{n+1,\alpha'(\varphi)(i)} \quad \text{and} \quad g_{n+1,j} : U_{n+1,j} \to U'_{n+1,i'}$$

We claim the morphism

$$K_{n+1} \to K'_{n+1} = (\text{cosk}_n \text{sk}_n K')_{n+1} = (\text{cosk}_n K)_{n+1}$$

is a covering as in Definition 25.3.1. Namely, if $i' \in I'_{n+1}$, then either i' is nondegenerate and the inverse image of i' in I_{n+1} is equal to $J_{i'}$ and we get a covering of $U'_{n+1,i'}$ by our choice (25.12.2.1), or i' is degenerate and the inverse image of i' in I_{n+1} is $J_{i'} \amalg \{(\varphi, i)\}$ for a unique pair (φ, i) and we get a covering by our choice (25.12.2.1) and assumption (2) of the lemma.

To finish the proof we have to define the morphisms $K(\varphi) : K_{n+1} \to K_m$ corresponding to morphisms $\varphi : [m] \to [n+1]$, $0 \leq m \leq n$ and the morphisms $K(\varphi) : K_m \to K_{n+1}$ corresponding to morphisms $\varphi : [n+1] \to [m]$, $0 \leq m \leq n$ satisfying suitable composition relations. For the first kind we use the composition

$$K_{n+1} \to K'_{n+1} \xrightarrow{K'(\varphi)} K'_m = K_m$$

to define $K(\varphi) : K_{n+1} \to K_m$. For the second kind, suppose given $\varphi : [n+1] \to [m]$, $0 \leq m \leq n$. We define the corresponding morphism $K(\varphi) : K_m \to K_{n+1}$ as follows:

(1) for $i \in I_m$ there is a unique surjective map $\psi : [m] \to [m_0]$ and a unique $i_0 \in I_{m_0}$ nondegenerate such that $\alpha(\psi)(i_0) = i$[3],
(2) we set $\varphi_0 = \psi_0 \circ \varphi : [n+1] \to [m_0]$ and we map $i \in I_m$ to $(\varphi_0, i_0) \in I_{n+1}$, in other words, $\alpha(\varphi)(i) = (\varphi_0, i_0)$, and
(3) the morphism $f_{\varphi,i} : U_{m,i} \to U_{n+1,\alpha(\varphi)(i)} = U_{m_0,i_0}$ is the inverse of the isomorphism $f_{\psi,i_0} : U_{m_0,i_0} \to U_{m,i}$ (see Lemma 25.12.1).

We omit the straightforward but cumbersome verification that this defines a split $(n+1)$-truncated $\mathcal{B}$-hypercovering of X extending the given n-truncated one. In fact, everything is clear from the above, except for the verification that the morphisms $K(\varphi)$ compose correctly for all $\varphi : [a] \to [b]$ with $0 \leq a, b \leq n+1$. □

0DAX Lemma 25.12.3. Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Assume that any object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$. Then there is a hypercovering K such that $K_n = \{U_i\}_{i \in I_n}$ with $U_i \in \mathcal{B}$ for all $i \in I_n$.

Proof. This proof is almost the same as the proof of Lemma 25.12.2. We will only explain the differences.

Let $n \geq 1$. Let us say that an n-truncated $\mathcal{B}$-hypercovering is given by an n-truncated simplicial object K of $\text{SR}(\mathcal{C})$ such that for $i \in I_a$, $0 \leq a \leq n$ we have $U_{a,i} \in \mathcal{B}$ and such that

(1) $F(K_0)^\# \to *$ is surjective,
(2) $F(K_1)^\# \to F(K_0)^\# \times F(K_0)^\#$ is surjective,

[3]For example, if i is nondegenerate, then $m = m_0$ and $\psi = \text{id}_{[m]}$.

(3) $F(K_{a+1})^\# \to F((\text{cosk}_a\text{sk}_a K)_{a+1})^\#$ for $a = 1, \ldots, n-1$ is surjective.

We first explicitly construct a split 1-truncated $\mathcal{B}$-hypercovering.

Take $I_0 = \mathcal{B}$ and $K_0 = \{U\}_{U \in \mathcal{B}}$. Then (1) holds by our assumption on $\mathcal{B}$. Set

$$\Omega = \{(U, V, W, a, b) \mid U, V, W \in \mathcal{B}, a : U \to V, b : U \to W\}$$

Then we set $I_1 = I_0 \amalg \Omega$. For $i \in I_1$ we set $U_{1,i} = U_{0,i}$ if $i \in I_0$ and $U_{1,i} = U$ if $i = (U, V, W, a, b) \in \Omega$. The map $K(\sigma^0_0) : K_0 \to K_1$ corresponds to the inclusion $\alpha(\sigma^0_0) : I_0 \to I_1$ and the identity $f_{\sigma^0_0, i} : U_{0,i} \to U_{1,i}$ on objects. The maps $K(\delta^1_0), K(\delta^1_1) : K_1 \to K_0$ correspond to the two maps $I_1 \to I_0$ which are the identity on $I_0 \subset I_1$ and map $(U, V, W, a, b) \in \Omega \subset I_1$ to V, resp. W. The corresponding morphisms $f_{\delta^1_0, i}, f_{\delta^1_1, i} : U_{1,i} \to U_{0,i}$ are the identity if $i \in I_0$ and a, b in case $i = (U, V, W, a, b) \in \Omega$. The reason that (2) holds is that any section of $F(K_0)^\# \times F(K_0)^\#$ over an object U of $\mathcal{C}$ comes, after replacing U by the members of a covering, from a map $U \to F(K_0) \times F(K_0)$. This in turn means we have $V, W \in \mathcal{B}$ and two morphisms $U \to V$ and $U \to W$. Further replacing U by the members of a covering we may assume $U \in \mathcal{B}$ as desired.

The lemma follows if we can prove that given a split n-truncated $\mathcal{B}$-hypercovering K for $n \geq 1$ we can extend it to a split $(n+1)$-truncated $\mathcal{B}$-hypercovering. Here the argument proceeds exactly as in the proof of Lemma 25.12.2. We omit the precise details, except for the following comments. First, we do not need assumption (2) in the proof of the current lemma as we do not need the morphism $K_{n+1} \to (\text{cosk}_n K)_{n+1}$ to be covering; we only need it to induce a surjection on associated sheaves of sets which follows from Sites, Lemma 7.12.4. Second, the assumption that $\mathcal{C}$ has fibre products and equalizers guarantees that $\text{SR}(\mathcal{C})$ has fibre products and equalizers and F commutes with these (Lemma 25.2.3). This suffices assure us the coskeleton functors used exist (see Simplicial, Remark 14.19.11 and Categories, Lemma 4.18.2). □

0DAY Lemma 25.12.4. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the functor $u : \mathcal{D} \to \mathcal{C}$. Assume $\mathcal{D}$ and $\mathcal{C}$ have equalizers and fibre products and u commutes with them. If a simplicial object K of $\text{SR}(\mathcal{D})$ is a hypercovering, then $u(K)$ is a hypercovering.

Proof. If we write $K_n = \{U_{n,i}\}_{i \in I_n}$ as in the introduction to this section, then $u(K)$ is the object of $\text{SR}(\mathcal{C})$ given by $u(K_n) = \{u(U_i)\}_{i \in I_n}$. By Sites, Lemma 7.13.5 we have $f^{-1}h_U^\# = h_{u(U)}^\#$ for $U \in \text{Ob}(\mathcal{D})$. This means that $f^{-1}F(K_n)^\# = F(u(K_n))^\#$ for all n. Let us check the conditions (1), (2), (3) for $u(K)$ to be a hypercovering from Definition 25.6.1. Since f^{-1} is an exact functor, we find that

$$F(u(K_0))^\# = f^{-1}F(K_0)^\# \to f^{-1}* = *$$

is surjective as a pullback of a surjective map and we get (1). Similarly,

$$F(u(K_1))^\# = f^{-1}F(K_1)^\# \to f^{-1}(F(K_0) \times F(K_0))^\# = F(u(K_0))^\# \times F(u(K_0))^\#$$

is surjective as a pullback and we get (2). For condition (3), in order to conclude by the same method it suffices if

$$F((\text{cosk}_n\text{sk}_n u(K))_{n+1})^\# = f^{-1}F((\text{cosk}_n\text{sk}_n K)_{n+1})^\#$$

The above shows that $f^{-1}F(-) = F(u(-))$. Thus it suffices to show that u commutes with the limits used in defining $(\text{cosk}_n\text{sk}_n K)_{n+1}$ for $n \geq 1$. By Simplicial, Remark 14.19.11 these limits are finite connected limits and u commutes with these by assumption. $\square$

0DAZ Lemma 25.12.5. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a continuous functor. Assume $\mathcal{D}$ and $\mathcal{C}$ have fibre products and u commutes with them. Let $Y \in \mathcal{D}$ and $K \in \text{SR}(\mathcal{D}, Y)$ a hypercovering of Y. Then $u(K)$ is a hypercovering of $u(Y)$.

Proof. This is easier than the proof of Lemma 25.12.4 because the notion of being a hypercovering of an object is stronger, see Definitions 25.3.3 and 25.3.1. Namely, u sends coverings to coverings by the definition of a morphism of sites. It suffices to check u commutes with the limits used in defining $(\text{cosk}_n \text{sk}_n K)_{n+1}$ for $n \geq 1$. This is clear because the induced functor $\mathcal{D}/Y \to \mathcal{C}/X$ commutes with all finite limits (and source and target have all finite limits by Categories, Lemma 4.18.4). $\square$

094K Lemma 25.12.6. Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \text{Ob}(\mathcal{C})$ be a subset. Assume

(1) $\mathcal{C}$ has fibre products,
(2) for all $X \in \text{Ob}(\mathcal{C})$ there exists a finite covering $\{U_i \to X\}_{i \in I}$ with $U_i \in \mathcal{B}$,
(3) if $\{U_i \to X\}_{i \in I}$ is a finite covering with $U_i \in \mathcal{B}$ and $U \to X$ is a morphism with $U \in \mathcal{B}$, then $\{U_i \to X\}_{i \in I} \amalg \{U \to X\}$ is a covering.

Then for every X there exists a hypercovering K of X such that each $K_n = \{U_{n,i} \to X\}_{i \in I_n}$ with I_n finite and $U_{n,i} \in \mathcal{B}$.

Proof. This lemma is the analogue of Lemma 25.11.4 for sites. To prove the lemma we follow exactly the proof of Lemma 25.12.2 paying attention to the following two points

(a) We choose our initial covering $\{U_{0,i} \to X\}_{i \in I_0}$ with $U_{0,i} \in \mathcal{B}$ such that the index set I_0 is finite, and
(b) in choosing the coverings (25.12.2.1) we choose $J_{i'}$ finite.

The reader sees easily that with these modifications we end up with finite index sets I_n for all n. $\square$

0DB0 Remark 25.12.7. Let $\mathcal{C}$ be a site. Let K and L be objects of $\text{SR}(\mathcal{C})$. Write $K = \{U_i\}_{i \in I}$ and $L = \{V_j\}_{j \in J}$. Assume $U = \coprod_{i \in I} U_i$ and $V = \coprod_{j \in J} V_j$ exist. Then we get

$$\text{Mor}_{\text{SR}(\mathcal{C})}(K, L) \longrightarrow \text{Mor}_\mathcal{C}(U, V)$$

as follows. Given $f : K \to L$ given by $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ we obtain a transformation of functors

$$\text{Mor}_\mathcal{C}(V, -) = \prod\nolimits_{j \in J} \text{Mor}_\mathcal{C}(V_j, -) \to \prod\nolimits_{i \in I} \text{Mor}_\mathcal{C}(U_i, -) = \text{Mor}_\mathcal{C}(U, -)$$

sending $(g_j)_{j \in J}$ to $(g_{\alpha(i)} \circ f_i)_{i \in I}$. Hence the Yoneda lemma produces the corresponding map $U \to V$. Of course, $U \to V$ maps the summand U_i into the summand $V_{\alpha(i)}$ via the morphism f_i.

0DB1 Remark 25.12.8. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has fibre products and equalizers and let K be a hypercovering. Write $K_n = \{U_{n,i}\}_{i \in I_n}$. Suppose that

(a) $U_n = \coprod_{i \in I_n} U_{n,i}$ exists, and
(b) $\coprod_{i \in I_n} h_{U_{n,i}} \to h_{U_n}$ induces an isomorphism on sheafifications.

Then we get another simplicial object L of $\text{SR}(\mathcal{C})$ with $L_n = \{U_n\}$, see Remark 25.12.7. Now we claim that L is a hypercovering. To see this we check conditions

(1), (2), (3) of Definition 25.6.1. Condition (1) follows from (b) and (1) for K. Condition (2) follows in exactly the same way. Condition (3) follows because

$$\begin{aligned} F((\text{cosk}_n\text{sk}_n L)_{n+1})^\# &= ((\text{cosk}_n\text{sk}_n F(L)^\#)_{n+1}) \\ &= ((\text{cosk}_n\text{sk}_n F(K)^\#)_{n+1}) \\ &= F((\text{cosk}_n\text{sk}_n K)_{n+1})^\# \end{aligned}$$

for $n \geq 1$ and hence the condition for K implies the condition for L exactly as in (1) and (2). Note that F commutes with connected limits and sheafification is exact proving the first and last equality; the middle equality follows as $F(K)^\# = F(L)^\#$ by (b).

0DB2 Remark 25.12.9. Let $\mathcal{C}$ be a site. Let $X \in \text{Ob}(\mathcal{C})$. Assume $\mathcal{C}$ has fibre products and let K be a hypercovering of X. Write $K_n = \{U_{n,i}\}_{i \in I_n}$. Suppose that

(a) $U_n = \coprod_{i \in I_n} U_{n,i}$ exists,
(b) given morphisms $(\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ and $(\beta, g_k) : \{W_k\}_{k \in K} \to \{V_j\}_{j \in J}$ in $\text{SR}(\mathcal{C})$ such that $U = \coprod U_i$, $V = \coprod V_j$, and $W = \coprod W_j$ exist, then $U \times_V W = \coprod_{(i,j,k), \alpha(i) = j = \beta(k)} U_i \times_{V_j} W_k$,
(c) if $(\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ is a covering in the sense of Definition 25.3.1 and $U = \coprod U_i$ and $V = \coprod V_j$ exist, then the corresponding morphism $U \to V$ of Remark 25.12.7 is a covering of $\mathcal{C}$.

Then we get another simplicial object L of $\text{SR}(\mathcal{C})$ with $L_n = \{U_n\}$, see Remark 25.12.7. Now we claim that L is a hypercovering of X. To see this we check conditions (1), (2) of Definition 25.3.3. Condition (1) follows from (c) and (1) for K because (1) for K says $K_0 = \{U_{0,i}\}_{i \in I_0}$ is a covering of $\{X\}$ in the sense of Definition 25.3.1. Condition (2) follows because $\mathcal{C}/X$ has all finite limits hence $\text{SR}(\mathcal{C}/X)$ has all finite limits, and condition (b) says the construction of "taking disjoint unions" commutes with these fimite limits. Thus the morphism

$$L_{n+1} \longrightarrow (\text{cosk}_n\text{sk}_n L)_{n+1}$$

is a covering as it is the consequence of applying our "taking disjoint unions" functor to the morphism

$$K_{n+1} \longrightarrow (\text{cosk}_n\text{sk}_n K)_{n+1}$$

which is assumed to be a covering in the sense of Definition 25.3.1 by (2) for K. This makes sense because property (b) in particular assures us that if we start with a finite diagram of semi-representable objects over X for which we can take disjoint unions, then the limit of the diagram in $\text{SR}(\mathcal{C}/X)$ still is a semi-representable object over X for which we can take disjoint unions.

25.13. Other chapters

Preliminaries

Part 2

Schemes

CHAPTER 26

Schemes

01H8 26.1. Introduction

01H9 In this document we define schemes. A basic reference is [DG67].

26.2. Locally ringed spaces

01HA Recall that we defined ringed spaces in Sheaves, Section 6.25. Briefly, a ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$. You can think of $f^\sharp$ as a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$, see Sheaves, Definition 6.21.7 and Lemma 6.21.8.

A good geometric example of this to keep in mind is $\mathcal{C}^\infty$-manifolds and morphisms of $\mathcal{C}^\infty$-manifolds. Namely, if M is a $\mathcal{C}^\infty$-manifold, then the sheaf $\mathcal{C}^\infty_M$ of smooth functions is a sheaf of rings on M. And any map $f : M \to N$ of manifolds is smooth if and only if for every local section h of $\mathcal{C}^\infty_N$ the composition $h \circ f$ is a local section of $\mathcal{C}^\infty_M$. Thus a smooth map f gives rise in a natural way to a morphism of ringed spaces

$$f : (M, \mathcal{C}^\infty_M) \longrightarrow (N, \mathcal{C}^\infty_N)$$

see Sheaves, Example 6.25.2. It is instructive to consider what happens to stalks. Namely, let $m \in M$ with image $f(m) = n \in N$. Recall that the stalk $\mathcal{C}^\infty_{M,m}$ is the ring of germs of smooth functions at m, see Sheaves, Example 6.11.4. The algebra of germs of functions on (M, m) is a local ring with maximal ideal the functions which vanish at m. Similarly for $\mathcal{C}^\infty_{N,n}$. The map on stalks $f^\sharp : \mathcal{C}^\infty_{N,n} \to \mathcal{C}^\infty_{M,m}$ maps the maximal ideal into the maximal ideal, simply because $f(m) = n$.

In algebraic geometry we study schemes. On a scheme the sheaf of rings is not determined by an intrinsic property of the space. The spectrum of a ring R (see Algebra, Section 10.17) endowed with a sheaf of rings constructed out of R (see below), will be our basic building block. It will turn out that the stalks of $\mathcal{O}$ on $\operatorname{Spec}(R)$ are the local rings of R at its primes. There are two reasons to introduce locally ringed spaces in this setting: (1) There is in general no mechanism that assigns to a continuous map of spectra a map of the corresponding rings. This is why we add as an extra datum the map $f^\sharp$. (2) If we consider morphisms of these spectra in the category of ringed spaces, then the maps on stalks may not be local homomorphisms. Since our geometric intuition says it should we introduce locally ringed spaces as follows.

01HB Definition 26.2.1. Locally ringed spaces.

(1) A locally ringed space $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.

(2) Given a locally ringed space $(X, \mathcal{O}_X)$ we say that $\mathcal{O}_{X,x}$ is the local ring of X at x. We denote $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the residue field of X at x is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
(3) A morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We will usually suppress the sheaf of rings $\mathcal{O}_X$ in the notation when discussing locally ringed spaces. We will simply refer to "the locally ringed space X". We will by abuse of notation think of X also as the underlying topological space. Finally we will denote the corresponding sheaf of rings $\mathcal{O}_X$ as the structure sheaf of X. In addition, it is customary to denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$ by $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$. We will say "let $f : X \to Y$ be a morphism of locally ringed spaces" thereby suppressing the structure sheaves even further. In this case, we will by abuse of notation think of $f : X \to Y$ also as the underlying continuous map of topological spaces. The f-map corresponding to f will customarily be denoted $f^\sharp$. The condition that f is a morphism of locally ringed spaces can then be expressed by saying that for every $x \in X$ the map on stalks

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

maps the maximal ideal $\mathfrak{m}_{Y,f(x)}$ into $\mathfrak{m}_{X,x}$.

Let us use these notational conventions to show that the collection of locally ringed spaces and morphisms of locally ringed spaces forms a category. In order to see this we have to show that the composition of morphisms of locally ringed spaces is a morphism of locally ringed spaces. OK, so let $f : X \to Y$ and $g : Y \to Z$ be morphism of locally ringed spaces. The composition of f and g is defined in Sheaves, Definition 6.25.3. Let $x \in X$. By Sheaves, Lemma 6.21.10 the composition

$$\mathcal{O}_{Z,g(f(x))} \xrightarrow{g^\sharp} \mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp} \mathcal{O}_{X,x}$$

is the associated map on stalks for the morphism $g \circ f$. The result follows since a composition of local ring homomorphisms is a local ring homomorphism.

A pleasing feature of the definition is the fact that the functor

$$\text{Locally ringed spaces} \longrightarrow \text{Ringed spaces}$$

reflects isomorphisms (plus more). Here is a less abstract statement.

01HC Lemma 26.2.2. Let X, Y be locally ringed spaces. If $f : X \to Y$ is an isomorphism of ringed spaces, then f is an isomorphism of locally ringed spaces.

Proof. This follows trivially from the corresponding fact in algebra: Suppose A, B are local rings. Any isomorphism of rings $A \to B$ is a local ring homomorphism. $\square$

26.3. Open immersions of locally ringed spaces

01HD

01HE Definition 26.3.1. Let $f : X \to Y$ be a morphism of locally ringed spaces. We say that f is an open immersion if f is a homeomorphism of X onto an open subset of Y, and the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

The following construction is parallel to Sheaves, Definition 6.31.2 (3).

01HF Example 26.3.2. Let X be a locally ringed space. Let $U \subset X$ be an open subset. Let $\mathcal{O}_U = \mathcal{O}_X|_U$ be the restriction of $\mathcal{O}_X$ to U. For $u \in U$ the stalk $\mathcal{O}_{U,u}$ is equal to the stalk $\mathcal{O}_{X,u}$, and hence is a local ring. Thus $(U, \mathcal{O}_U)$ is a locally ringed space and the morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is an open immersion.

01HG Definition 26.3.3. Let X be a locally ringed space. Let $U \subset X$ be an open subset. The locally ringed space $(U, \mathcal{O}_U)$ of Example 26.3.2 above is the open subspace of X associated to U.

01HH Lemma 26.3.4. Let $f : X \to Y$ be an open immersion of locally ringed spaces. Let $j : V = f(X) \to Y$ be the open subspace of Y associated to the image of f. There is a unique isomorphism $f' : X \cong V$ of locally ringed spaces such that $f = j \circ f'$.

Proof. Let f' be the homeomorphism between X and V induced by f. Then $f = j \circ f'$ as maps of topological spaces. Since there is an isomorphism of sheaves $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, there is an isomorphism of rings $f^\sharp : \Gamma(U, f^{-1}(\mathcal{O}_Y)) \to \Gamma(U, \mathcal{O}_X)$ for each open subset $U \subset X$. Since $\mathcal{O}_V = j^{-1}\mathcal{O}_Y$ and $f^{-1} = f'^{-1}j^{-1}$ (Sheaves, Lemma 6.21.6) we see that $f^{-1}\mathcal{O}_Y = f'^{-1}\mathcal{O}_V$, hence $\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, f^{-1}(\mathcal{O}_Y))$ is an isomorphism for every $U \subset X$ open. By composing these we get an isomorphism of rings

$$\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, \mathcal{O}_X)$$

for each open subset $U \subset X$, and therefore an isomorphism of sheaves $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$. In other words, we have an isomorphism $f'^\sharp : f'^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$ and therefore an isomorphism of locally ringed spaces $(f', f'^\sharp) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ (use Lemma 26.2.2). Note that $f = j \circ f'$ as morphisms of locally ringed spaces by construction.

Suppose we have another morphism $f'' : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ such that $f = j \circ f''$. At any point $x \in X$, we have $j(f'(x)) = j(f''(x))$ from which it follows that $f'(x) = f''(x)$ since j is the inclusion map; therefore f' and f'' are the same as morphisms of topological spaces. On structure sheaves, for each open subset $U \subset X$ we have a commutative diagram

$$\begin{tikzcd}
\Gamma(U, f^{-1}(\mathcal{O}_Y)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Gamma(U, \mathcal{O}_X) \\
\Gamma(U, f'^{-1}(\mathcal{O}_V)) \arrow[ur, bend left, "f'^\sharp"] \arrow[ur, bend right, "f''^\sharp"']
\end{tikzcd}$$

from which we see that $f'^\sharp$ and $f''^\sharp$ define the same morphism of sheaves. $\square$

From now on we do not distinguish between open subsets and their associated subspaces.

01HI Lemma 26.3.5. Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $U \subset X$, and $V \subset Y$ be open subsets. Suppose that $f(U) \subset V$. There exists a unique morphism of locally ringed spaces $f|_U : U \to V$ such that the following diagram is a commutative square of locally ringed spaces

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r] & Y \end{tikzcd}$$

Proof. Omitted. □

In the following we will use without further mention the following fact which follows from the lemma above. Given any morphism $f : Y \to X$ of locally ringed spaces, and any open subset $U \subset X$ such that $f(Y) \subset U$, then there exists a unique morphism of locally ringed spaces $Y \to U$ such that the composition $Y \to U \to X$ is equal to f. In fact, we will even by abuse of notation write $f : Y \to U$ since this rarely gives rise to confusion.

26.4. Closed immersions of locally ringed spaces

01HJ We follow our conventions introduced in Modules, Definition 17.13.1.

01HK **Definition 26.4.1.** Let $i : Z \to X$ be a morphism of locally ringed spaces. We say that i is a *closed immersion* if:

(1) The map i is a homeomorphism of Z onto a closed subset of X.
(2) The map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective; let $\mathcal{I}$ denote the kernel.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

01HL **Lemma 26.4.2.** Let $f : Z \to X$ be a morphism of locally ringed spaces. In order for f to be a closed immersion it suffices that there exists an open covering $X = \bigcup U_i$ such that each $f : f^{-1}U_i \to U_i$ is a closed immersion.

Proof. Omitted. □

01HM **Example 26.4.3.** Let X be a locally ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals which is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Let Z be the support of the sheaf of rings $\mathcal{O}_X/\mathcal{I}$. This is a closed subset of X, by Modules, Lemma 17.5.3. Denote $i : Z \to X$ the inclusion map. By Modules, Lemma 17.6.1 there is a unique sheaf of rings $\mathcal{O}_Z$ on Z with $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. For any $z \in Z$ the stalk $\mathcal{O}_{Z,z}$ is equal to a quotient $\mathcal{O}_{X,i(z)}/\mathcal{I}_{i(z)}$ of a local ring and nonzero, hence a local ring. Thus $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a closed immersion of locally ringed spaces.

01HN **Definition 26.4.4.** Let X be a locally ringed space. Let $\mathcal{I}$ be a sheaf of ideals on X which is locally generated by sections. The locally ringed space $(Z, \mathcal{O}_Z)$ of Example 26.4.3 above is the *closed subspace of X associated to the sheaf of ideals $\mathcal{I}$*.

01HO **Lemma 26.4.5.** Let $f : X \to Y$ be a closed immersion of locally ringed spaces. Let $\mathcal{I}$ be the kernel of the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $i : Z \to Y$ be the closed subspace of Y associated to $\mathcal{I}$. There is a unique isomorphism $f' : X \cong Z$ of locally ringed spaces such that $f = i \circ f'$.

Proof. Omitted. □

01HP **Lemma 26.4.6.** Let X, Y be locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals locally generated by sections. Let $i : Z \to X$ be the associated closed subspace. A morphism $f : Y \to X$ factors through Z if and only if the map $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$ is zero. If this is the case the morphism $g : Y \to Z$ such that $f = i \circ g$ is unique.

Proof. Clearly if f factors as $Y \to Z \to X$ then the map $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Conversely suppose that $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Pick any $y \in Y$, and consider the ring map $f^\sharp_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Since the composition $\mathcal{I}_{f(y)} \to \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is zero by assumption and since $f^\sharp_y(1) = 1$ we see that $1 \notin \mathcal{I}_{f(y)}$, i.e., $\mathcal{I}_{f(y)} \neq \mathcal{O}_{X,f(y)}$. We conclude that $f(Y) \subset Z = \text{Supp}(\mathcal{O}_X/\mathcal{I})$. Hence $f = i \circ g$ where $g : Y \to Z$ is continuous. Consider the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. The assumption $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero implies that the composition $\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$ is zero by adjointness of f_* and f^*. In other words, we obtain a morphism of sheaves of rings $\overline{f^\sharp} : \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y$. Note that $f_*\mathcal{O}_Y = i_*g_*\mathcal{O}_Y$ and that $\mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$. By Sheaves, Lemma 6.32.4 we obtain a unique morphism of sheaves of rings $g^\sharp : \mathcal{O}_Z \to g_*\mathcal{O}_Y$ whose pushforward under i is $\overline{f^\sharp}$. We omit the verification that $(g, g^\sharp)$ defines a morphism of locally ringed spaces and that $f = i \circ g$ as a morphism of locally ringed spaces. The uniqueness of $(g, g^\sharp)$ was pointed out above. □

01HQ Lemma 26.4.7. Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals which is locally generated by sections. Let $i : Z \to Y$ be the closed subspace associated to the sheaf of ideals $\mathcal{I}$. Let $\mathcal{J}$ be the image of the map $f^*\mathcal{I} \to f^*\mathcal{O}_Y = \mathcal{O}_X$. Then this ideal is locally generated by sections. Moreover, let $i' : Z' \to X$ be the associated closed subspace of X. There exists a unique morphism of locally ringed spaces $f' : Z' \to Z$ such that the following diagram is a commutative square of locally ringed spaces

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

Moreover, this diagram is a fibre square in the category of locally ringed spaces.

Proof. The ideal $\mathcal{J}$ is locally generated by sections by Modules, Lemma 17.8.2. The rest of the lemma follows from the characterization, in Lemma 26.4.6 above, of what it means for a morphism to factor through a closed subspace. □

26.5. Affine schemes

01HR Let R be a ring. Consider the topological space $\text{Spec}(R)$ associated to R, see Algebra, Section 10.17. We will endow this space with a sheaf of rings $\mathcal{O}_{\text{Spec}(R)}$ and the resulting pair $(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ will be an affine scheme.

Recall that $\text{Spec}(R)$ has a basis of open sets $D(f)$, $f \in R$ which we call standard opens, see Algebra, Definition 10.17.3. In addition, the intersection of two standard opens is another: $D(f) \cap D(g) = D(fg)$, $f, g \in R$.

01HS Lemma 26.5.1. Let R be a ring. Let $f \in R$.

(1) If $g \in R$ and $D(g) \subset D(f)$, then
  (a) f is invertible in R_g,
  (b) $g^e = af$ for some $e \geq 1$ and $a \in R$,
  (c) there is a canonical ring map $R_f \to R_g$, and
  (d) there is a canonical R_f-module map $M_f \to M_g$ for any R-module M.
(2) Any open covering of $D(f)$ can be refined to a finite open covering of the form $D(f) = \bigcup_{i=1}^n D(g_i)$.

(3) If $g_1, \ldots, g_n \in R$, then $D(f) \subset \bigcup D(g_i)$ if and only if $g_1, \ldots, g_n$ generate the unit ideal in R_f.

Proof. Recall that $D(g) = \text{Spec}(R_g)$ (see Algebra, Lemma 10.17.6). Thus (a) holds because f maps to an element of R_g which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 10.17.2. Write the inverse of f in R_g as a/g^d. This means $g^d - af$ is annihilated by a power of g, whence (b). For (c), the map $R_f \to R_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. The equality $M_f = M \otimes_R R_f$ can be used to obtain the map on modules, or we can define $M_f \to M_g$ by mapping x/f^n to $a^n x/g^{ne}$.

Recall that $D(f)$ is quasi-compact, see Algebra, Lemma 10.29.1. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 10.17.2. □

In Sheaves, Section 6.30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 6.30.6 and 6.30.9. Moreover, we showed in Sheaves, Lemma 6.30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

01HT Definition 26.5.2. Let R be a ring.

(1) A standard open covering of $\text{Spec}(R)$ is a covering $\text{Spec}(R) = \bigcup_{i=1}^n D(f_i)$, where $f_1, \ldots, f_n \in R$.
(2) Suppose that $D(f) \subset \text{Spec}(R)$ is a standard open. A standard open covering of $D(f)$ is a covering $D(f) = \bigcup_{i=1}^n D(g_i)$, where $g_1, \ldots, g_n \in R$.

Let R be a ring. Let M be an R-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \text{Spec}(R)$ is a standard open. If $f, g \in R$ are such that $D(f) = D(g)$, then by Lemma 26.5.1 above there are canonical maps $M_f \to M_g$ and $M_g \to M_f$ which are mutually inverse. Hence we may choose any f such that $U = D(f)$ and define

$$\widetilde{M}(U) = M_f.$$

Note that if $D(g) \subset D(f)$, then by Lemma 26.5.1 above we have a canonical map

$$\widetilde{M}(D(f)) = M_f \longrightarrow M_g = \widetilde{M}(D(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = R$, then $\widetilde{R}$ is a presheaf of rings on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \text{Spec}(R)$. Suppose that x corresponds to the prime $\mathfrak{p} \subset R$. By definition of the stalk we see that

$$\widetilde{M}_x = \text{colim}_{f \in R, f \notin \mathfrak{p}} M_f$$

Here the set $\{f \in R, f \notin \mathfrak{p}\}$ is preordered by the rule $f \geq f' \Leftrightarrow D(f) \subset D(f')$. If $f_1, f_2 \in R \setminus \mathfrak{p}$, then we have $f_1 f_2 \geq f_1$ in this ordering. Hence by Algebra, Lemma 10.9.9 we conclude that

$$\widetilde{M}_x = M_{\mathfrak{p}}.$$

Next, we check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_f \to \bigoplus M_{g_i} \to \bigoplus M_{g_i g_j}.$$

Note that $D(g_i) = D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_f \to \bigoplus M_{fg_i} \to \bigoplus M_{fg_i g_j}.$$

In addition, by Lemma 26.5.1 above we see that $g_1, \ldots, g_n$ generate the unit ideal in R_f. Thus we may apply Algebra, Lemma 10.24.1 to the module M_f over R_f and the elements $g_1, \ldots, g_n$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 6.30 that there exists a unique sheaf of rings $\mathcal{O}_{\text{Spec}(R)}$ which agrees with $\widetilde{R}$ on the standard opens. Note that by our computation of stalks above, the stalks of this sheaf of rings are all local rings.

Similarly, for any R-module M there exists a unique sheaf of $\mathcal{O}_{\text{Spec}(R)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 6.30.12.

01HU **Definition 26.5.3.** Let R be a ring.

(1) The structure sheaf $\mathcal{O}_{\text{Spec}(R)}$ of the spectrum of R is the unique sheaf of rings $\mathcal{O}_{\text{Spec}(R)}$ which agrees with $\widetilde{R}$ on the basis of standard opens.
(2) The locally ringed space $(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ is called the spectrum of R and denoted $\text{Spec}(R)$.
(3) The sheaf of $\mathcal{O}_{\text{Spec}(R)}$-modules extending $\widetilde{M}$ to all opens of $\text{Spec}(R)$ is called the sheaf of $\mathcal{O}_{\text{Spec}(R)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

01HV **Lemma 26.5.4.** Let R be a ring. Let M be an R-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{\text{Spec}(R)}$-modules associated to M.

(1) We have $\Gamma(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}) = R$.
(2) We have $\Gamma(\text{Spec}(R), \widetilde{M}) = M$ as an R-module.
(3) For every $f \in R$ we have $\Gamma(D(f), \mathcal{O}_{\text{Spec}(R)}) = R_f$.
(4) For every $f \in R$ we have $\Gamma(D(f), \widetilde{M}) = M_f$ as an R_f-module.
(5) Whenever $D(g) \subset D(f)$ the restriction mappings on $\mathcal{O}_{\text{Spec}(R)}$ and $\widetilde{M}$ are the maps $R_f \to R_g$ and $M_f \to M_g$ from Lemma 26.5.1.
(6) Let $\mathfrak{p}$ be a prime of R, and let $x \in \text{Spec}(R)$ be the corresponding point. We have $\mathcal{O}_{\text{Spec}(R),x} = R_{\mathfrak{p}}$.
(7) Let $\mathfrak{p}$ be a prime of R, and let $x \in \text{Spec}(R)$ be the corresponding point. We have $\widetilde{M}_x = M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module.

Moreover, all these identifications are functorial in the R module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of R-modules to the category of $\mathcal{O}_{\text{Spec}(R)}$-modules.

Proof. Assertions (1) - (7) are clear from the discussion above. The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{\mathfrak p}$ is exact and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 17.3.1. $\square$

01HW Definition 26.5.5. An affine scheme is a locally ringed space isomorphic as a locally ringed space to $\text{Spec}(R)$ for some ring R. A morphism of affine schemes is a morphism in the category of locally ringed spaces.

It turns out that affine schemes play a special role among all locally ringed spaces, which is what the next section is about.

26.6. The category of affine schemes

01HX Note that if Y is an affine scheme, then its points are in canonical $1-1$ bijection with prime ideals in $\Gamma(Y, \mathcal{O}_Y)$.

01HY Lemma 26.6.1. Let X be a locally ringed space. Let Y be an affine scheme. Let $f \in \text{Mor}(X, Y)$ be a morphism of locally ringed spaces. Given a point $x \in X$ consider the ring maps

$$\Gamma(Y, \mathcal{O}_Y) \xrightarrow{f^\sharp} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$$

Let $\mathfrak p \subset \Gamma(Y, \mathcal{O}_Y)$ denote the inverse image of $\mathfrak m_x$. Let $y \in Y$ be the corresponding point. Then $f(x) = y$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r] & \mathcal{O}_{X,x} \\ \Gamma(Y, \mathcal{O}_Y) \arrow[r] \arrow[u] & \mathcal{O}_{Y,f(x)} \arrow[u] \end{tikzcd}$$

(see the discussion of f-maps below Sheaves, Definition 6.21.7). Since the right vertical arrow is local we see that $\mathfrak m_{f(x)}$ is the inverse image of $\mathfrak m_x$. The result follows. $\square$

01HZ Lemma 26.6.2. Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The set

$$D(f) = \{x \in X \mid \text{image } f \not\in \mathfrak m_x\}$$

is open. Moreover $f|_{D(f)}$ has an inverse.

Proof. This is a special case of Modules, Lemma 17.25.10, but we also give a direct proof. Suppose that $U \subset X$ and $V \subset X$ are two open subsets such that $f|_U$ has an inverse g and $f|_V$ has an inverse h. Then clearly $g|_{U \cap V} = h|_{U \cap V}$. Thus it suffices to show that f is invertible in an open neighbourhood of any $x \in D(f)$. This is clear because $f \not\in \mathfrak m_x$ implies that $f \in \mathcal{O}_{X,x}$ has an inverse $g \in \mathcal{O}_{X,x}$ which means there is some open neighbourhood $x \in U \subset X$ so that $g \in \mathcal{O}_X(U)$ and $g \cdot f|_U = 1$. $\square$

01I0 Lemma 26.6.3. In Lemma 26.6.2 above, if X is an affine scheme, then the open $D(f)$ agrees with the standard open $D(f)$ defined previously (in Algebra, Definition 10.17.1).

Proof. Omitted. $\square$

01I1 Lemma 26.6.4. Let X be a locally ringed space. Let Y be an affine scheme. The map

$$\mathrm{Mor}(X, Y) \longrightarrow \mathrm{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

A reference for this fact is [DG67, II, Err 1, Prop. 1.8.1] where it is attributed to J. Tate.

Proof. Since Y is affine we have $(Y, \mathcal{O}_Y) \cong (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some ring R. During the proof we will use facts about Y and its structure sheaf which are direct consequences of things we know about the spectrum of a ring, see e.g. Lemma 26.5.4.

Motivated by the lemmas above we construct the inverse map. Let $\psi_Y : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. First, we define the corresponding map of spaces

$$\Psi : X \longrightarrow Y$$

by the rule of Lemma 26.6.1. In other words, given $x \in X$ we define $\Psi(x)$ to be the point of Y corresponding to the prime in $\Gamma(Y, \mathcal{O}_Y)$ which is the inverse image of $\mathfrak{m}_x$ under the composition $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\psi_Y} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$.

We claim that the map $\Psi : X \to Y$ is continuous. The standard opens $D(g)$, for $g \in \Gamma(Y, \mathcal{O}_Y)$ are a basis for the topology of Y. Thus it suffices to prove that $\Psi^{-1}(D(g))$ is open. By construction of Ψ the inverse image $\Psi^{-1}(D(g))$ is exactly the set $D(\psi_Y(g)) \subset X$ which is open by Lemma 26.6.2. Hence Ψ is continuous.

Next we construct a Ψ-map of sheaves from $\mathcal{O}_Y$ to $\mathcal{O}_X$. By Sheaves, Lemma 6.30.14 it suffices to define ring maps $\psi_{D(g)} : \Gamma(D(g), \mathcal{O}_Y) \to \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X)$ compatible with restriction maps. We have a canonical isomorphism $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, because Y is an affine scheme. Because $\psi_Y(g)$ is invertible on $D(\psi_Y(g))$ we see that there is a canonical map

$$\Gamma(Y, \mathcal{O}_Y)_g \longrightarrow \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X) = \Gamma(D(\psi_Y(g)), \mathcal{O}_X)$$

extending the map ψ_Y by the universal property of localization. Note that there is no choice but to take the canonical map here! And we take this, combined with the canonical identification $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, to be $\psi_{D(g)}$. This is compatible with localization since the restriction mapping on the affine schemes are defined in terms of the universal properties of localization also, see Lemmas 26.5.4 and 26.5.1.

Thus we have defined a morphism of ringed spaces $(\Psi, \psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ recovering ψ_Y on global sections. To see that it is a morphism of locally ringed spaces we have to show that the induced maps on local rings

$$\psi_x : \mathcal{O}_{Y,\Psi(x)} \longrightarrow \mathcal{O}_{X,x}$$

are local. This follows immediately from the commutative diagram of the proof of Lemma 26.6.1 and the definition of Ψ.

Finally, we have to show that the constructions $(\Psi, \psi) \mapsto \psi_Y$ and the construction $\psi_Y \mapsto (\Psi, \psi)$ are inverse to each other. Clearly, $\psi_Y \mapsto (\Psi, \psi) \mapsto \psi_Y$. Hence the only thing to prove is that given ψ_Y there is at most one pair (Ψ, ψ) giving rise to it. The uniqueness of Ψ was shown in Lemma 26.6.1 and given the uniqueness of Ψ the uniqueness of the map ψ was pointed out during the course of the proof above. $\square$

01I2 **Lemma 26.6.5.** The category of affine schemes is equivalent to the opposite of the category of rings. The equivalence is given by the functor that associates to an affine scheme the global sections of its structure sheaf.

Proof. This is now clear from Definition 26.5.5 and Lemma 26.6.4. $\square$

01I3 **Lemma 26.6.6.** Let Y be an affine scheme. Let $f \in \Gamma(Y, \mathcal{O}_Y)$. The open subspace $D(f)$ is an affine scheme.

Proof. We may assume that $Y = \mathrm{Spec}(R)$ and $f \in R$. Consider the morphism of affine schemes $\phi : U = \mathrm{Spec}(R_f) \to \mathrm{Spec}(R) = Y$ induced by the ring map $R \to R_f$. By Algebra, Lemma 10.17.6 we know that it is a homeomorphism onto $D(f)$. On the other hand, the map $\phi^{-1}\mathcal{O}_Y \to \mathcal{O}_U$ is an isomorphism on stalks, hence an isomorphism. Thus we see that ϕ is an open immersion. We conclude that $D(f)$ is isomorphic to U by Lemma 26.3.4. $\square$

01I4 **Lemma 26.6.7.** The category of affine schemes has finite products, and fibre products. In other words, it has finite limits. Moreover, the products and fibre products in the category of affine schemes are the same as in the category of locally ringed spaces. In a formula, we have (in the category of locally ringed spaces)

$$\mathrm{Spec}(R) \times \mathrm{Spec}(S) = \mathrm{Spec}(R \otimes_{\mathbf{Z}} S)$$

and given ring maps $R \to A$, $R \to B$ we have

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_R B).$$

Proof. This is just an application of Lemma 26.6.4. First of all, by that lemma, the affine scheme $\mathrm{Spec}(\mathbf{Z})$ is the final object in the category of locally ringed spaces. Thus the first displayed formula follows from the second. To prove the second note that for any locally ringed space X we have

$$\begin{aligned} \mathrm{Mor}(X, \mathrm{Spec}(A \otimes_R B)) &= \mathrm{Hom}(A \otimes_R B, \mathcal{O}_X(X)) \\ &= \mathrm{Hom}(A, \mathcal{O}_X(X)) \times_{\mathrm{Hom}(R, \mathcal{O}_X(X))} \mathrm{Hom}(B, \mathcal{O}_X(X)) \\ &= \mathrm{Mor}(X, \mathrm{Spec}(A)) \times_{\mathrm{Mor}(X, \mathrm{Spec}(R))} \mathrm{Mor}(X, \mathrm{Spec}(B)) \end{aligned}$$

which proves the formula. See Categories, Section 4.6 for the relevant definitions. $\square$

01I5 **Lemma 26.6.8.** Let X be a locally ringed space. Assume $X = U \amalg V$ with U and V open and such that U, V are affine schemes. Then X is an affine scheme.

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ by the sheaf property. By Lemma 26.6.4 there is a canonical morphism of locally ringed spaces $X \to \mathrm{Spec}(R)$. By Algebra, Lemma 10.21.2 we see that as a topological space $\mathrm{Spec}(\mathcal{O}_X(U)) \amalg \mathrm{Spec}(\mathcal{O}_X(V)) = \mathrm{Spec}(R)$ with the maps coming from the ring homomorphisms $R \to \mathcal{O}_X(U)$ and $R \to \mathcal{O}_X(V)$. This of course means that $\mathrm{Spec}(R)$ is the coproduct in the category of locally ringed spaces as well. By assumption the morphism $X \to \mathrm{Spec}(R)$ induces an isomorphism of $\mathrm{Spec}(\mathcal{O}_X(U))$ with U and similarly for V. Hence $X \to \mathrm{Spec}(R)$ is an isomorphism. $\square$

26.7. Quasi-coherent sheaves on affines

01I6 Recall that we have defined the abstract notion of a quasi-coherent sheaf in Modules, Definition 17.10.1. In this section we show that any quasi-coherent sheaf on an affine scheme $\mathrm{Spec}(R)$ corresponds to the sheaf $\widetilde{M}$ associated to an R-module M.

01I7 **Lemma 26.7.1.** Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let M be an R-module. There exists a canonical isomorphism between the sheaf $\widetilde{M}$ associated to the R-module M (Definition 26.5.3) and the sheaf $\mathcal{F}_M$ associated to the R-module M (Modules, Definition 17.10.6). This isomorphism is functorial in M. In particular, the sheaves $\widetilde{M}$ are quasi-coherent. Moreover, they are characterized by the following mapping property

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{F}))$$

for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Here a map $\alpha : \widetilde{M} \to \mathcal{F}$ corresponds to its effect on global sections.

Proof. By Modules, Lemma 17.10.5 we have a morphism $\mathcal{F}_M \to \widetilde{M}$ corresponding to the map $M \to \Gamma(X, \widetilde{M}) = M$. Let $x \in X$ correspond to the prime $\mathfrak{p} \subset R$. The induced map on stalks are the maps $\mathcal{O}_{X,x} \otimes_R M \to M_{\mathfrak{p}}$ which are isomorphisms because $R_{\mathfrak{p}} \otimes_R M = M_{\mathfrak{p}}$. Hence the map $\mathcal{F}_M \to \widetilde{M}$ is an isomorphism. The mapping property follows from the mapping property of the sheaves $\mathcal{F}_M$. □

01I8 **Lemma 26.7.2.** Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. There are canonical isomorphisms

(1) $\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, see Modules, Section 17.16.
(2) $\widetilde{\mathrm{T}^n(M)} \cong \mathrm{T}^n(\widetilde{M})$, $\widetilde{\mathrm{Sym}^n(M)} \cong \mathrm{Sym}^n(\widetilde{M})$, and $\widetilde{\wedge^n(M)} \cong \wedge^n(\widetilde{M})$, see Modules, Section 17.21.
(3) if M is a finitely presented R-module, then $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_R(M, N)}$, see Modules, Section 17.22.

First proof. Using Lemma 26.7.1 and Modules, Lemma 17.10.5 we see that the functor $M \mapsto \widetilde{M}$ can be viewed as π^* for a morphism π of ringed spaces. And pulling back modules commutes with tensor constructions by Modules, Lemmas 17.16.4 and 17.21.3. The morphism $\pi : (X, \mathcal{O}_X) \to (\{*\}, R)$ is flat for example because the stalks of $\mathcal{O}_X$ are localizations of R (Lemma 26.5.4) and hence flat over R. Thus pullback by π commutes with internal hom if the first module is finitely presented by Modules, Lemma 17.22.5. □

Second proof. Proof of (1). By Lemma 26.7.1 to give a map $\widetilde{M \otimes_R N}$ into $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ we have to give a map on global sections $M \otimes_R N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})$ which exists by definition of the tensor product of sheaves of modules. To see that this map is an isomorphism it suffices to check that it is an isomorphism on stalks. And this follows from the description of the stalks of $\widetilde{M}$ (either in Lemma 26.5.4 or in Modules, Lemma 17.10.5), the fact that tensor product commutes with localization (Algebra, Lemma 10.12.16) and Modules, Lemma 17.16.1.

Proof of (2). This is similar to the proof of (1), using Algebra, Lemma 10.13.6 and Modules, Lemma 17.21.2.

Proof of (3). Since the construction $M \mapsto \widetilde{M}$ is functorial there is an R-linear map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. The target of this map is the global sections of $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. Hence by Lemma 26.7.1 we obtain a map of $\mathcal{O}_X$-modules $\widetilde{\mathrm{Hom}_R(M, N)} \to \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. We check that this is an isomorphism by comparing stalks. If M is finitely presented as an R-module then $\widetilde{M}$ has a global finite presentation as an $\mathcal{O}_X$-module. Hence we conclude using Algebra, Lemma 10.10.2 and Modules, Lemma 17.22.4. □

Third proof of part (1). For any $\mathcal{O}_X$-module $\mathcal{F}$ we have the following isomorphisms functorial in M, N, and $\mathcal{F}$

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}, \mathcal{F}) &= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, \Gamma(X, \mathcal{F}))) \\
&= \mathrm{Hom}_R(M \otimes_R N, \Gamma(X, \mathcal{F})) \\
&= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M \otimes_R N}, \mathcal{F})
\end{aligned}$$

The first equality is Modules, Lemma 17.22.1. The second equality is the universal property of $\widetilde{M}$, see Lemma 26.7.1. The third equality holds by definition of $\mathcal{H}om$. The fourth equality is the universal property of $\widetilde{N}$. Then fifth equality is Algebra, Lemma 10.12.8. The final equality is the universal property of $\widetilde{M \otimes_R N}$. By the Yoneda lemma (Categories, Lemma 4.3.5) we obtain (1). □

01I9 Lemma 26.7.3. Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)})$, $(Y, \mathcal{O}_Y) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be affine schemes. Let $\psi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of affine schemes, corresponding to the ring map $\psi^\sharp : R \to S$ (see Lemma 26.6.5).

(1) We have $\psi^*\widetilde{M} = \widetilde{S \otimes_R M}$ functorially in the R-module M.
(2) We have $\psi_*\widetilde{N} = \widetilde{N_R}$ functorially in the S-module N.

Proof. The first assertion follows from the identification in Lemma 26.7.1 and the result of Modules, Lemma 17.10.7. The second assertion follows from the fact that $\psi^{-1}(D(f)) = D(\psi^\sharp(f))$ and hence

$$\psi_*\widetilde{N}(D(f)) = \widetilde{N}(D(\psi^\sharp(f))) = N_{\psi^\sharp(f)} = (N_R)_f = \widetilde{N_R}(D(f))$$

as desired. □

Lemma 26.7.3 above says in particular that if you restrict the sheaf $\widetilde{M}$ to a standard affine open subspace $D(f)$, then you get $\widetilde{M_f}$. We will use this from now on without further mention.

01IA Lemma 26.7.4. Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is isomorphic to the sheaf associated to the R-module $\Gamma(X, \mathcal{F})$.

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Since every standard open $D(f)$ is quasi-compact we see that X is a locally quasi-compact, i.e., every point has a fundamental system of quasi-compact neighbourhoods, see Topology, Definition 5.13.1. Hence by Modules, Lemma 17.10.8 for every prime $\mathfrak{p} \subset R$ corresponding to

$x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the quasi-coherent sheaf associated to some $\mathcal{O}_X(U)$-module M. In other words, we get an open covering by U's with this property. By Lemma 26.5.1 for example we can refine this covering to a standard open covering. Thus we get a covering $\text{Spec}(R) = \bigcup D(f_i)$ and R_{f_i}-modules M_i and isomorphisms $\varphi_i : \mathcal{F}|_{D(f_i)} \to \mathcal{F}_{M_i}$ for some R_{f_i}-module M_i. On the overlaps we get isomorphisms

$$\mathcal{F}_{M_i}|_{D(f_if_j)} \xrightarrow{\varphi_i^{-1}|_{D(f_if_j)}} \mathcal{F}|_{D(f_if_j)} \xrightarrow{\varphi_j|_{D(f_if_j)}} \mathcal{F}_{M_j}|_{D(f_if_j)}.$$

Let us denote these ψ_{ij}. It is clear that we have the cocycle condition

$$\psi_{jk}|_{D(f_if_jf_k)} \circ \psi_{ij}|_{D(f_if_jf_k)} = \psi_{ik}|_{D(f_if_jf_k)}$$

on triple overlaps.

Recall that each of the open subspaces $D(f_i)$, $D(f_if_j)$, $D(f_if_jf_k)$ is an affine scheme. Hence the sheaves $\mathcal{F}_{M_i}$ are isomorphic to the sheaves $\widetilde{M_i}$ by Lemma 26.7.1 above. In particular we see that $\mathcal{F}_{M_i}(D(f_if_j)) = (M_i)_{f_j}$, etc. Also by Lemma 26.7.1 above we see that ψ_{ij} corresponds to a unique $R_{f_if_j}$-module isomorphism

$$\psi_{ij} : (M_i)_{f_j} \longrightarrow (M_j)_{f_i}$$

namely, the effect of ψ_{ij} on sections over $D(f_if_j)$. Moreover these then satisfy the cocycle condition that

$$\begin{tikzcd}
(M_i)_{f_jf_k} \arrow[rr, "\psi_{ik}"] \arrow[rd, "\psi_{ij}"'] & & (M_k)_{f_if_j} \\
& (M_j)_{f_if_k} \arrow[ru, "\psi_{jk}"'] &
\end{tikzcd}$$

commutes (for any triple i, j, k).

Now Algebra, Lemma 10.24.5 shows that there exist an R-module M such that $M_i = M_{f_i}$ compatible with the morphisms ψ_{ij}. Consider $\mathcal{F}_M = \widetilde{M}$. At this point it is a formality to show that $\widetilde{M}$ is isomorphic to the quasi-coherent sheaf $\mathcal{F}$ we started out with. Namely, the sheaves $\mathcal{F}$ and $\widetilde{M}$ give rise to isomorphic sets of glueing data of sheaves of $\mathcal{O}_X$-modules with respect to the covering $X = \bigcup D(f_i)$, see Sheaves, Section 6.33 and in particular Lemma 6.33.4. Explicitly, in the current situation, this boils down to the following argument: Let us construct an R-module map

$$M \longrightarrow \Gamma(X, \mathcal{F}).$$

Namely, given $m \in M$ we get $m_i = m/1 \in M_{f_i} = M_i$ by construction of M. By construction of M_i this corresponds to a section $s_i \in \mathcal{F}(U_i)$. (Namely, $\varphi_i^{-1}(m_i)$.) We claim that $s_i|_{D(f_if_j)} = s_j|_{D(f_if_j)}$. This is true because, by construction of M, we have $\psi_{ij}(m_i) = m_j$, and by the construction of the ψ_{ij}. By the sheaf condition of $\mathcal{F}$ this collection of sections gives rise to a unique section s of $\mathcal{F}$ over X. We leave it to the reader to show that $m \mapsto s$ is a R-module map. By Lemma 26.7.1 we obtain an associated $\mathcal{O}_X$-module map

$$\widetilde{M} \longrightarrow \mathcal{F}.$$

By construction this map reduces to the isomorphisms φ_i^{-1} on each $D(f_i)$ and hence is an isomorphism. $\square$

01IB **Lemma 26.7.5.** Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. The functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ define quasi-inverse equivalences of categories

$$\begin{tikzcd} QCoh(\mathcal{O}_X) \arrow[r, shift left] & \mathrm{Mod}_R \arrow[l, shift left] \end{tikzcd}$$

between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of R-modules.

Proof. See Lemmas 26.7.1 and 26.7.4 above. □

From now on we will not distinguish between quasi-coherent sheaves on affine schemes and sheaves of the form $\widetilde{M}$.

01IC **Lemma 26.7.6.** Let $X = \mathrm{Spec}(R)$ be an affine scheme. Kernels and cokernels of maps of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.

Proof. This follows from the exactness of the functor $\widetilde{\ }$ since by Lemma 26.7.1 we know that any map $\psi : \widetilde{M} \to \widetilde{N}$ comes from an R-module map $\varphi : M \to N$. (So we have $\mathrm{Ker}(\psi) = \widetilde{\mathrm{Ker}(\varphi)}$ and $\mathrm{Coker}(\psi) = \widetilde{\mathrm{Coker}(\varphi)}$.) □

01ID **Lemma 26.7.7.** Let $X = \mathrm{Spec}(R)$ be an affine scheme. The direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. The same holds for colimits.

Proof. Suppose $\mathcal{F}_i$, $i \in I$ is a collection of quasi-coherent sheaves on X. By Lemma 26.7.5 above we can write $\mathcal{F}_i = \widetilde{M_i}$ for some R-module M_i. Set $M = \bigoplus M_i$. Consider the sheaf $\widetilde{M}$. For each standard open $D(f)$ we have

$$\widetilde{M}(D(f)) = M_f = \left(\bigoplus M_i\right)_f = \bigoplus M_{i,f}.$$

Hence we see that the quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ is the direct sum of the sheaves $\mathcal{F}_i$. A similar argument works for general colimits. □

01IE **Lemma 26.7.8.** Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. This is clear in case both $\mathcal{F}_1$ and $\mathcal{F}_2$ are quasi-coherent because the functor $M \mapsto \widetilde{M}$ is exact, see Lemma 26.5.4. Similarly in case both $\mathcal{F}_2$ and $\mathcal{F}_3$ are quasi-coherent. Now, suppose that $\mathcal{F}_1 = \widetilde{M_1}$ and $\mathcal{F}_3 = \widetilde{M_3}$ are quasi-coherent. Set $M_2 = \Gamma(X, \mathcal{F}_2)$. We claim it suffices to show that the sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact. Namely, if this is the case, then (by using the mapping property of Lemma 26.7.1) we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{M_1} \arrow[r] \arrow[d] & \widetilde{M_2} \arrow[r] \arrow[d] & \widetilde{M_3} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0 \end{tikzcd}$$

and we win by the snake lemma.

The "correct" argument here would be to show first that $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$. This is actually not all that hard, but it is perhaps better to postpone this till later. Instead we use a small trick.

Pick $m \in M_3 = \Gamma(X, \mathcal{F}_3)$. Consider the following set

$$I = \{f \in R \mid \text{the element } fm \text{ comes from } M_2\}.$$

Clearly this is an ideal. It suffices to show $1 \in I$. Hence it suffices to show that for any prime $\mathfrak{p}$ there exists an $f \in I$, $f \notin \mathfrak{p}$. Let $x \in X$ be the point corresponding to $\mathfrak{p}$. Because surjectivity can be checked on stalks there exists an open neighbourhood U of x such that $m|_U$ comes from a local section $s \in \mathcal{F}_2(U)$. In fact we may assume that $U = D(f)$ is a standard open, i.e., $f \in R$, $f \notin \mathfrak{p}$. We will show that for some $N \gg 0$ we have $f^N \in I$, which will finish the proof.

Take any point $z \in V(f)$, say corresponding to the prime $\mathfrak{q} \subset R$. We can also find a $g \in R$, $g \notin \mathfrak{q}$ such that $m|_{D(g)}$ lifts to some $s' \in \mathcal{F}_2(D(g))$. Consider the difference $s|_{D(fg)} - s'|_{D(fg)}$. This is an element m' of $\mathcal{F}_1(D(fg)) = (M_1)_{fg}$. For some integer $n = n(z)$ the element $f^n m'$ comes from some $m'_1 \in (M_1)_g$. We see that $f^n s$ extends to a section σ of $\mathcal{F}_2$ on $D(f) \cup D(g)$ because it agrees with the restriction of $f^n s' + m'_1$ on $D(f) \cap D(g) = D(fg)$. Moreover, σ maps to the restriction of $f^n m$ to $D(f) \cup D(g)$.

Since $V(f)$ is quasi-compact, there exists a finite list of elements $g_1, \ldots, g_m \in R$ such that $V(f) \subset \bigcup D(g_j)$, an integer $n > 0$ and sections $\sigma_j \in \mathcal{F}_2(D(f) \cup D(g_j))$ such that $\sigma_j|_{D(f)} = f^n s$ and σ_j maps to the section $f^n m|_{D(f)\cup D(g_j)}$ of $\mathcal{F}_3$. Consider the differences

$$\sigma_j|_{D(f)\cup D(g_j g_k)} - \sigma_k|_{D(f)\cup D(g_j g_k)}.$$

These correspond to sections of $\mathcal{F}_1$ over $D(f) \cup D(g_j g_k)$ which are zero on $D(f)$. In particular their images in $\mathcal{F}_1(D(g_j g_k)) = (M_1)_{g_j g_k}$ are zero in $(M_1)_{g_j g_k f}$. Thus some high power of f kills each and every one of these. In other words, the elements $f^N \sigma_j$, for some $N \gg 0$ satisfy the glueing condition of the sheaf property and give rise to a section σ of $\mathcal{F}_2$ over $\bigcup (D(f) \cup D(g_j)) = X$ as desired. □

26.8. Closed subspaces of affine schemes

01IF

01IG Example 26.8.1. Let R be a ring. Let $I \subset R$ be an ideal. Consider the morphism of affine schemes $i : Z = \text{Spec}(R/I) \to \text{Spec}(R) = X$. By Algebra, Lemma 10.17.7 this is a homeomorphism of Z onto a closed subset of X. Moreover, if $I \subset \mathfrak{p} \subset R$ is a prime corresponding to a point $x = i(z)$, $x \in X$, $z \in Z$, then on stalks we get the map

$$\mathcal{O}_{X,x} = R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/IR_{\mathfrak{p}} = \mathcal{O}_{Z,z}$$

Thus we see that i is a closed immersion of locally ringed spaces, see Definition 26.4.1. Clearly, this is (isomorphic) to the closed subspace associated to the quasi-coherent sheaf of ideals $\widetilde{I}$, as in Example 26.4.3.

01IH Lemma 26.8.2. Let $(X, \mathcal{O}_X) = (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ be an affine scheme. Let $i : Z \to X$ be any closed immersion of locally ringed spaces. Then there exists a unique ideal $I \subset R$ such that the morphism $i : Z \to X$ can be identified with the closed immersion $\text{Spec}(R/I) \to \text{Spec}(R)$ constructed in Example 26.8.1 above.

Proof. This is kind of silly! Namely, by Lemma 26.4.5 we can identify $Z \to X$ with the closed subspace associated to a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ as in Definition 26.4.4 and Example 26.4.3. By our conventions this sheaf of ideals is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Hence the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ is locally on X the cokernel of a map $\bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{O}_U$. Thus by definition, $\mathcal{O}_X/\mathcal{I}$ is quasi-coherent. By our results in Section 26.7 it is of the form $\widetilde{S}$ for some R-module S. Moreover, since $\mathcal{O}_X = \widetilde{R} \to \widetilde{S}$ is surjective we see by Lemma 26.7.8 that also $\mathcal{I}$ is quasi-coherent, say $\mathcal{I} = \widetilde{I}$. Of course $I \subset R$ and $S = R/I$ and everything is clear. □

26.9. Schemes

01II

01IJ Definition 26.9.1. A scheme is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes will be denoted *Sch*.

Let X be a scheme. We will use the following (very slight) abuse of language. We will say $U \subset X$ is an affine open, or an open affine if the open subspace U is an affine scheme. We will often write $U = \mathrm{Spec}(R)$ to indicate that U is isomorphic to $\mathrm{Spec}(R)$ and moreover that we will identify (temporarily) U and $\mathrm{Spec}(R)$.

01IK Lemma 26.9.2. Let X be a scheme. Let $j : U \to X$ be an open immersion of locally ringed spaces. Then U is a scheme. In particular, any open subspace of X is a scheme.

Proof. Let $U \subset X$. Let $u \in U$. Pick an affine open neighbourhood $u \in V \subset X$. Because standard opens of V form a basis of the topology on V we see that there exists a $f \in \mathcal{O}_V(V)$ such that $u \in D(f) \subset U$. And $D(f)$ is an affine scheme by Lemma 26.6.6. This proves that every point of U has an open neighbourhood which is affine. □

Clearly the lemma (or its proof) shows that any scheme X has a basis (see Topology, Section 5.5) for the topology consisting of affine opens.

01IL Example 26.9.3. Let k be a field. An example of a scheme which is not affine is given by the open subspace $U = \mathrm{Spec}(k[x,y]) \setminus \{(x,y)\}$ of the affine scheme $X = \mathrm{Spec}(k[x,y])$. It is covered by two affines, namely $D(x) = \mathrm{Spec}(k[x,y,1/x])$ and $D(y) = \mathrm{Spec}(k[x,y,1/y])$ whose intersection is $D(xy) = \mathrm{Spec}(k[x,y,1/xy])$. By the sheaf property for $\mathcal{O}_U$ there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_U) \to k[x,y,1/x] \times k[x,y,1/y] \to k[x,y,1/xy]$$

We conclude that the map $k[x,y] \to \Gamma(U, \mathcal{O}_U)$ (coming from the morphism $U \to X$) is an isomorphism. Therefore U cannot be affine since if it was then by Lemma 26.6.5 we would have $U \cong X$.

26.10. Immersions of schemes

01IM In Lemma 26.9.2 we saw that any open subspace of a scheme is a scheme. Below we will prove that the same holds for a closed subspace of a scheme.

Note that the notion of a quasi-coherent sheaf of $\mathcal{O}_X$-modules is defined for any ringed space X in particular when X is a scheme. By our efforts in Section 26.7 we know that such a sheaf is on any affine open $U \subset X$ of the form $\widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

01IN Lemma 26.10.1. Let X be a scheme. Let $i : Z \to X$ be a closed immersion of locally ringed spaces.

(1) The locally ringed space Z is a scheme,
(2) the kernel $\mathcal{I}$ of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of ideals,
(3) for any affine open $U = \mathrm{Spec}(R)$ of X the morphism $i^{-1}(U) \to U$ can be identified with $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ for some ideal $I \subset R$, and
(4) we have $\mathcal{I}|_U = \widetilde{I}$.

In particular, any sheaf of ideals locally generated by sections is a quasi-coherent sheaf of ideals (and vice versa), and any closed subspace of X is a scheme.

Proof. Let $i : Z \to X$ be a closed immersion. Let $z \in Z$ be a point. Choose any affine open neighbourhood $i(z) \in U \subset X$. Say $U = \mathrm{Spec}(R)$. By Lemma 26.8.2 we know that $i^{-1}(U) \to U$ can be identified with the morphism of affine schemes $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. First of all this implies that $z \in i^{-1}(U) \subset Z$ is an affine neighbourhood of z. Thus Z is a scheme. Second this implies that $\mathcal{I}|_U$ is $\widetilde{I}$. In other words for every point $x \in i(Z)$ there exists an open neighbourhood such that $\mathcal{I}$ is quasi-coherent in that neighbourhood. Note that $\mathcal{I}|_{X \setminus i(Z)} \cong \mathcal{O}_{X \setminus i(Z)}$. Thus the restriction of the sheaf of ideals is quasi-coherent on $X \setminus i(Z)$ also. We conclude that $\mathcal{I}$ is quasi-coherent. $\square$

01IO Definition 26.10.2. Let X be a scheme.

(1) A morphism of schemes is called an open immersion if it is an open immersion of locally ringed spaces (see Definition 26.3.1).
(2) An open subscheme of X is an open subspace of X in the sense of Definition 26.3.3; an open subscheme of X is a scheme by Lemma 26.9.2.
(3) A morphism of schemes is called a closed immersion if it is a closed immersion of locally ringed spaces (see Definition 26.4.1).
(4) A closed subscheme of X is a closed subspace of X in the sense of Definition 26.4.4; a closed subscheme is a scheme by Lemma 26.10.1.
(5) A morphism of schemes $f : X \to Y$ is called an immersion, or a locally closed immersion if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.

It follows from the lemmas in Sections 26.3 and 26.4 that any open (resp. closed) immersion of schemes is isomorphic to the inclusion of an open (resp. closed) subscheme of the target.

Our definition of a closed immersion is halfway between Hartshorne and EGA. Hartshorne defines a closed immersion as a morphism $f : X \to Y$ of schemes which induces a homeomorphism of X onto a closed subset of Y such that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, see [Har77, Page 85]. We will show this is equivalent to our notion in Lemma 26.24.2. In [DG67], Grothendieck and Dieudonné first define closed subschemes via the construction of Example 26.4.3 using quasi-coherent sheaves of ideals and then define a closed immersion as a morphism $f : X \to Y$ which induces an isomorphism with a closed subscheme. It follows from Lemma 26.10.1 that this agrees with our notion.

Pedagogically speaking the definition above is a disaster/nightmare. In teaching this material to students, we have found it often convenient to define a closed immersion as an affine morphism $f : X \to Y$ of schemes such that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. Namely, it turns out that the notion of an affine morphism (Morphisms, Section 29.11) is quite natural and easy to understand.

For more information on closed immersions we suggest the reader visit Morphisms, Sections 29.2 and 29.4.

We will discuss locally closed subschemes and immersions at the end of this section.

01IP Remark 26.10.3. If $f : X \to Y$ is an immersion of schemes, then it is in general not possible to factor f as an open immersion followed by a closed immersion. See Morphisms, Example 29.3.4.

01IQ Lemma 26.10.4. Let $f : Y \to X$ be an immersion of schemes. Then f is a closed immersion if and only if $f(Y) \subset X$ is a closed subset.

Proof. If f is a closed immersion then $f(Y)$ is closed by definition. Conversely, suppose that $f(Y)$ is closed. By definition there exists an open subscheme $U \subset X$ such that f is the composition of a closed immersion $i : Y \to U$ and the open immersion $j : U \to X$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals associated to the closed immersion i. Note that $\mathcal{I}|_{U \setminus i(Y)} = \mathcal{O}_{U \setminus i(Y)} = \mathcal{O}_{X \setminus i(Y)}|_{U \setminus i(Y)}$. Thus we may glue (see Sheaves, Section 6.33) $\mathcal{I}$ and $\mathcal{O}_{X \setminus i(Y)}$ to a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$. Since every point of X has a neighbourhood where $\mathcal{J}$ is quasi-coherent, we see that $\mathcal{J}$ is quasi-coherent (in particular locally generated by sections). By construction $\mathcal{O}_X/\mathcal{J}$ is supported on U and equal to $\mathcal{O}_U/\mathcal{I}$. Thus we see that the closed subspaces associated to $\mathcal{I}$ and $\mathcal{J}$ are canonically isomorphic, see Example 26.4.3. In particular the closed subspace of U associated to $\mathcal{I}$ is isomorphic to a closed subspace of X. Since $Y \to U$ is identified with the closed subspace associated to $\mathcal{I}$, see Lemma 26.4.5, we conclude that $Y \to U \to X$ is a closed immersion. $\square$

Let $f : Y \to X$ be an immersion. Let $Z = \overline{f(Y)} \setminus f(Y)$ which is a closed subset of X. Let $U = X \setminus Z$. The lemma implies that U is the biggest open subspace of X such that $f : Y \to X$ factors through a closed immersion into U. We define a locally closed subscheme of X as a pair (Z, U) consisting of a closed subscheme Z of an open subscheme U of X such that in addition $\overline{Z} \cup U = X$. We usually just say "let Z be a locally closed subscheme of X" since we may recover U from the morphism $Z \to X$. The above then shows that any immersion $f : Y \to X$ factors uniquely as $Y \to Z \to X$ where Z is a locally closed subspace of X and $Y \to Z$ is an isomorphism.

The interest of this is that the collection of locally closed subschemes of X forms a set. We may define a partial ordering on this set, which we call inclusion for obvious reasons. To be explicit, if $Z \to X$ and $Z' \to X$ are two locally closed subschemes of X, then we say that Z is contained in Z' simply if the morphism $Z \to X$ factors through Z'. If it does, then of course Z is identified with a unique locally closed subscheme of Z', and so on.

For more information on immersions, we refer the reader to Morphisms, Section 29.3.

26.11. Zariski topology of schemes

01IR See Topology, Section 5.1 for some basic material in topology adapted to the Zariski topology of schemes.

01IS Lemma 26.11.1. Let X be a scheme. Any irreducible closed subset of X has a unique generic point. In other words, X is a sober topological space, see Topology, Definition 5.8.6.

Proof. Let $Z \subset X$ be an irreducible closed subset. For every affine open $U \subset X$, $U = \mathrm{Spec}(R)$ we know that $Z \cap U = V(I)$ for a unique radical ideal $I \subset R$. Note that $Z \cap U$ is either empty or irreducible. In the second case (which occurs for at least one U) we see that $I = \mathfrak{p}$ is a prime ideal, which is a generic point ξ of $Z \cap U$. It follows that $Z = \overline{\{\xi\}}$, in other words ξ is a generic point of Z. If ξ' was a second generic point, then $\xi' \in Z \cap U$ and it follows immediately that $\xi' = \xi$. $\square$

01IT Lemma 26.11.2. Let X be a scheme. The collection of affine opens of X forms a basis for the topology on X.

Proof. This follows from the discussion on open subschemes in Section 26.9. $\square$

01IU Remark 26.11.3. In general the intersection of two affine opens in X is not affine open. See Example 26.14.3.

01IV Lemma 26.11.4. The underlying topological space of any scheme is locally quasi-compact, see Topology, Definition 5.13.1.

Proof. This follows from Lemma 26.11.2 above and the fact that the spectrum of ring is quasi-compact, see Algebra, Lemma 10.17.10. $\square$

01IW Lemma 26.11.5. Let X be a scheme. Let U, V be affine opens of X, and let $x \in U \cap V$. There exists an affine open neighbourhood W of x such that W is a standard open of both U and V.

Proof. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$. Say x corresponds to the prime $\mathfrak{p} \subset A$ and the prime $\mathfrak{q} \subset B$. We may choose an $f \in A$, $f \not\in \mathfrak{p}$ such that $D(f) \subset U \cap V$. Note that any standard open of $D(f)$ is a standard open of $\mathrm{Spec}(A) = U$. Hence we may assume that $U \subset V$. In other words, now we may think of U as an affine open of V. Next we choose a $g \in B$, $g \not\in \mathfrak{q}$ such that $D(g) \subset U$. In this case we see that $D(g) = D(g_A)$ where $g_A \in A$ denotes the image of g by the map $B \to A$. Thus the lemma is proved. $\square$

01IX Lemma 26.11.6. Let X be a scheme. Let $X = \bigcup_i U_i$ be an affine open covering. Let $V \subset X$ be an affine open. There exists a standard open covering $V = \bigcup_{j=1,\ldots,m} V_j$ (see Definition 26.5.2) such that each V_j is a standard open in one of the U_i.

Proof. Pick $v \in V$. Then $v \in U_i$ for some i. By Lemma 26.11.5 above there exists an open $v \in W_v \subset V \cap U_i$ such that W_v is a standard open in both V and U_i. Since V is quasi-compact the lemma follows. $\square$

0F1A Lemma 26.11.7. Let X be a scheme. Let $\mathcal{B}$ be the set of affine opens of X. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$, see Sheaves, Definition 6.30.1. The following are equivalent

(1) $\mathcal{F}$ is the restriction of a sheaf on X to $\mathcal{B}$,
(2) $\mathcal{F}$ is a sheaf on $\mathcal{B}$, and

(3) $\mathcal{F}(\emptyset)$ is a singleton and whenever $U = V \cup W$ with $U, V, W \in \mathcal{B}$ and $V, W \subset U$ standard open (Algebra, Definition 10.17.3) the map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \times \mathcal{F}(W)$$

is injective with image the set of pairs (s, t) such that $s|_{V \cap W} = t|_{V \cap W}$.

Proof. The equivalence of (1) and (2) is Sheaves, Lemma 6.30.7. It is clear that (2) implies (3). Hence it suffices to prove that (3) implies (2). By Sheaves, Lemma 6.30.4 and Lemma 26.5.1 it suffices to prove the sheaf condition holds for standard open coverings (Definition 26.5.2) of elements of $\mathcal{B}$. Let $U = U_1 \cup \ldots \cup U_n$ be a standard open covering with $U \subset X$ affine open. We will prove the sheaf condition for this covering by induction on n. If $n = 0$, then U is empty and we get the sheaf condition by assumption. If $n = 1$, then there is nothing to prove. If $n = 2$, then this is assumption (3). If $n > 2$, then we write $U_i = D(f_i)$ for $f_i \in A = \mathcal{O}_X(U)$. Suppose that $s_i \in \mathcal{F}(U_i)$ are sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $1 \leq i < j \leq n$. Since $U = U_1 \cup \ldots \cup U_n$ we have $1 = \sum_{i=1,\ldots,n} a_i f_i$ in A for some $a_i \in A$, see Algebra, Lemma 10.17.2. Set $g = \sum_{i=1,\ldots,n-1} a_i f_i$. Then $U = D(g) \cup D(f_n)$. Observe that $D(g) = D(gf_1) \cup \ldots \cup D(gf_{n-1})$ is a standard open covering. By induction there is a unique section $s' \in \mathcal{F}(D(g))$ which agrees with $s_i|_{D(gfi)}$ for $i = 1, \ldots, n-1$. We claim that s' and s_n have the same restriction to $D(gf_n)$. This is true by induction and the covering $D(gf_n) = D(gf_n f_1) \cup \ldots \cup D(gf_n f_{n-1})$. Thus there is a unique section $s \in \mathcal{F}(U)$ whose restriction to $D(g)$ is s' and whose restriction to $D(f_n)$ is s_n. We omit the verification that s restricts to s_i on $D(f_i)$ for $i = 1, \ldots, n-1$ and we omit the verification that s is unique. $\square$

02O0 Lemma 26.11.8. Let X be a scheme whose underlying topological space is a finite discrete set. Then X is affine.

Proof. Say $X = \{x_1, \ldots, x_n\}$. Then $U_i = \{x_i\}$ is an open neighbourhood of x_i. By Lemma 26.11.2 it is affine. Hence X is a finite disjoint union of affine schemes, and hence is affine by Lemma 26.6.8. $\square$

01IY Example 26.11.9. There exists a scheme without closed points. Namely, let R be a local domain whose spectrum looks like $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{m}$. Then the open subscheme $\text{Spec}(R) \setminus \{\mathfrak{m}\}$ does not have a closed point. To see that such a ring R exists, we use that given any totally ordered group $(\Gamma, \geq)$ there exists a valuation ring A with valuation group $(\Gamma, \geq)$, see [Kru32]. See Algebra, Section 10.50 for notation. We take $\Gamma = \mathbf{Z}x_1 \oplus \mathbf{Z}x_2 \oplus \mathbf{Z}x_3 \oplus \ldots$ and we define $\sum_i a_i x_i \geq 0$ if and only if the first nonzero a_i is > 0, or all $a_i = 0$. So $x_1 \geq x_2 \geq x_3 \geq \ldots \geq 0$. The subsets $x_i + \Gamma_{\geq 0}$ are prime ideals of $(\Gamma, \geq)$, see Algebra, notation above Lemma 10.50.17. These together with $\emptyset$ and $\Gamma_{\geq 0}$ are the only prime ideals. Hence A is an example of a ring with the given structure of its spectrum, by Algebra, Lemma 10.50.17.

26.12. Reduced schemes

01IZ

01J0 Definition 26.12.1. Let X be a scheme. We say X is reduced if every local ring $\mathcal{O}_{X,x}$ is reduced.

01J1 Lemma 26.12.2. A scheme X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for all $U \subset X$ open.

Proof. Assume that X is reduced. Let $f \in \mathcal{O}_X(U)$ be a section such that $f^n = 0$. Then the image of f in $\mathcal{O}_{U,u}$ is zero for all $u \in U$. Hence f is zero, see Sheaves, Lemma 6.11.1. Conversely, assume that $\mathcal{O}_X(U)$ is reduced for all opens U. Pick any nonzero element $f \in \mathcal{O}_{X,x}$. Any representative $(U, f \in \mathcal{O}(U))$ of f is nonzero and hence not nilpotent. Hence f is not nilpotent in $\mathcal{O}_{X,x}$. $\square$

01J2 Lemma 26.12.3. An affine scheme $\mathrm{Spec}(R)$ is reduced if and only if R is reduced.

Proof. The direct implication follows immediately from Lemma 26.12.2 above. In the other direction it follows since any localization of a reduced ring is reduced, and in particular the local rings of a reduced ring are reduced. $\square$

01J3 Lemma 26.12.4. Let X be a scheme. Let $T \subset X$ be a closed subset. There exists a unique closed subscheme $Z \subset X$ with the following properties: (a) the underlying topological space of Z is equal to T, and (b) Z is reduced.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sub presheaf defined by the rule

$$\mathcal{I}(U) = \{f \in \mathcal{O}_X(U) \mid f(t) = 0 \text{ for all } t \in T \cap U\}$$

Here we use $f(t)$ to indicate the image of f in the residue field $\kappa(t)$ of X at t. Because of the local nature of the condition it is clear that $\mathcal{I}$ is a sheaf of ideals. Moreover, let $U = \mathrm{Spec}(R)$ be an affine open. We may write $T \cap U = V(I)$ for a unique radical ideal $I \subset R$. Given a prime $\mathfrak{p} \in V(I)$ corresponding to $t \in T \cap U$ and an element $f \in R$ we have $f(t) = 0 \Leftrightarrow f \in \mathfrak{p}$. Hence $\mathcal{I}(U) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = I$ by Algebra, Lemma 10.17.2. Moreover, for any standard open $D(g) \subset \mathrm{Spec}(R) = U$ we have $\mathcal{I}(D(g)) = I_g$ by the same reasoning. Thus $\widetilde{I}$ and $\mathcal{I}|_U$ agree (as ideals) on a basis of opens and hence are equal. Therefore $\mathcal{I}$ is a quasi-coherent sheaf of ideals.

At this point we may define Z as the closed subspace associated to the sheaf of ideals $\mathcal{I}$. For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z \cap U = \mathrm{Spec}(R/I)$ where I is a radical ideal and hence Z is reduced (by Lemma 26.12.3 above). By construction the underlying closed subset of Z is T. Hence we have found a closed subscheme with properties (a) and (b).

Let $Z' \subset X$ be a second closed subscheme with properties (a) and (b). For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z' \cap U = \mathrm{Spec}(R/I')$ for some ideal $I' \subset R$. By Lemma 26.12.3 the ring R/I' is reduced and hence I' is radical. Since $V(I') = T \cap U = V(I)$ we deduced that $I = I'$ by Algebra, Lemma 10.17.2. Hence Z' and Z are defined by the same sheaf of ideals and hence are equal. $\square$

01J4 Definition 26.12.5. Let X be a scheme. Let $Z \subset X$ be a closed subset. A scheme structure on Z is given by a closed subscheme Z' of X whose underlying set is equal to Z. We often say "let $(Z, \mathcal{O}_Z)$ be a scheme structure on Z" to indicate this. The reduced induced scheme structure on Z is the one constructed in Lemma 26.12.4. The reduction X_{red} of X is the reduced induced scheme structure on X itself.

Often when we say "let $Z \subset X$ be an irreducible component of X" we think of Z as a reduced closed subscheme of X using the reduced induced scheme structure.

0F2L Remark 26.12.6. Let X be a scheme. Let $T \subset X$ be a locally closed subset. In this situation we sometimes also use the phrase "reduced induced scheme structure on T". It refers to the reduced induced scheme structure from Definition 26.12.5 when we view T as a closed subset of the open subscheme $X \setminus \partial T$ of X. Here $\partial T = \overline{T} \setminus T$ is the "boundary" of T in the topological space of X.

0356 Lemma 26.12.7. Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let Y be a reduced scheme. A morphism $f : Y \to X$ factors through Z if and only if $f(Y) \subset Z$ (set theoretically). In particular, any morphism $Y \to X$ factors as $Y \to X_{red} \to X$.

Proof. Assume $f(Y) \subset Z$ (set theoretically). Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. For any affine opens $U \subset X$, $\text{Spec}(B) = V \subset Y$ with $f(V) \subset U$ and any $g \in \mathcal{I}(U)$ the pullback $b = f^\sharp(g) \in \Gamma(V, \mathcal{O}_Y) = B$ maps to zero in the residue field of any $y \in V$. In other words $b \in \bigcap_{\mathfrak{p} \subset B} \mathfrak{p}$. This implies $b = 0$ as B is reduced (Lemma 26.12.2, and Algebra, Lemma 10.17.2). Hence f factors through Z by Lemma 26.4.6. $\square$

26.13. Points of schemes

01J5 Given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow \text{Sets}, \quad T \longmapsto \text{Mor}(T, X).$$

See Categories, Example 4.3.4. This is called the functor of points of X. A fun part of scheme theory is to find descriptions of the internal geometry of X in terms of this functor h_X. In this section we find a simple way to describe points of X.

Let X be a scheme. Let R be a local ring with maximal ideal $\mathfrak{m} \subset R$. Suppose that $f : \text{Spec}(R) \to X$ is a morphism of schemes. Let $x \in X$ be the image of the closed point $\mathfrak{m} \in \text{Spec}(R)$. Then we obtain a local homomorphism of local rings

$$f^\sharp : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{\text{Spec}(R),\mathfrak{m}} = R.$$

01J6 Lemma 26.13.1. Let X be a scheme. Let R be a local ring. The construction above gives a bijective correspondence between morphisms $\text{Spec}(R) \to X$ and pairs (x, φ) consisting of a point $x \in X$ and a local homomorphism of local rings $\varphi : \mathcal{O}_{X,x} \to R$.

Proof. Let A be a ring. For any ring homomorphism $\psi : A \to R$ there exists a unique prime ideal $\mathfrak{p} \subset A$ and a factorization $A \to A_\mathfrak{p} \to R$ where the last map is a local homomorphism of local rings. Namely, $\mathfrak{p} = \psi^{-1}(\mathfrak{m})$. Via Lemma 26.6.4 this proves that the lemma holds if X is an affine scheme.

Let X be a general scheme. Any $x \in X$ is contained in an open affine $U \subset X$. By the affine case we conclude that every pair (x, φ) occurs as the end product of the construction above the lemma.

To finish the proof it suffices to show that any morphism $f : \text{Spec}(R) \to X$ has image contained in any affine open containing the image x of the closed point of $\text{Spec}(R)$. In fact, let $x \in V \subset X$ be any open neighbourhood containing x. Then $f^{-1}(V) \subset \text{Spec}(R)$ is an open containing the unique closed point and hence equal to $\text{Spec}(R)$. $\square$

As a special case of the lemma above we obtain for every point x of a scheme X a canonical morphism

02NA (26.13.1.1) $$\text{Spec}(\mathcal{O}_{X,x}) \longrightarrow X$$

corresponding to the identity map on the local ring of X at x. We may reformulate the lemma above as saying that for any morphism $f : \text{Spec}(R) \to X$ there exists a unique point $x \in X$ such that f factors as $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ where the first map comes from a local homomorphism $\mathcal{O}_{X,x} \to R$.

In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{O}_{X,x}) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{S,s}) \arrow[r] & S \end{tikzcd}$$

where the left vertical map corresponds to the local ring map $f^\sharp_x : \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$.

01J7 Lemma 26.13.2. Let X be a scheme. Let $x, x' \in X$ be points of X. Then $x' \in X$ is a generalization of x if and only if x' is in the image of the canonical morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$.

Proof. A continuous map preserves the relation of specialization/generalization. Since every point of $\mathrm{Spec}(\mathcal{O}_{X,x})$ is a generalization of the closed point we see every point in the image of $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ is a generalization of x. Conversely, suppose that x' is a generalization of x. Choose an affine open neighbourhood $U = \mathrm{Spec}(R)$ of x. Then $x' \in U$. Say $\mathfrak{p} \subset R$ and $\mathfrak{p}' \subset R$ are the primes corresponding to x and x'. Since x' is a generalization of x we see that $\mathfrak{p}' \subset \mathfrak{p}$. This means that $\mathfrak{p}'$ is in the image of the morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) = \mathrm{Spec}(R_\mathfrak{p}) \to \mathrm{Spec}(R) = U \subset X$ as desired. $\square$

Now, let us discuss morphisms from spectra of fields. Let $(R, \mathfrak{m}, \kappa)$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let K be a field. A local homomorphism $R \to K$ by definition factors as $R \to \kappa \to K$, i.e., is the same thing as a morphism $\kappa \to K$. Thus we see that morphisms

$$\mathrm{Spec}(K) \longrightarrow X$$

correspond to pairs $(x, \kappa(x) \to K)$. We may define a preorder on morphisms of spectra of fields to X by saying that $\mathrm{Spec}(K) \to X$ dominates $\mathrm{Spec}(L) \to X$ if $\mathrm{Spec}(K) \to X$ factors through $\mathrm{Spec}(L) \to X$. This suggests the following notion: Let us temporarily say that two morphisms $p : \mathrm{Spec}(K) \to X$ and $q : \mathrm{Spec}(L) \to X$ are equivalent if there exists a third field Ω and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\Omega) \arrow[r] \arrow[d] & \mathrm{Spec}(L) \arrow[d, "q"] \\ \mathrm{Spec}(K) \arrow[r, "p"] & X \end{tikzcd}$$

Of course this immediately implies that the unique points of all three of the schemes $\mathrm{Spec}(K)$, $\mathrm{Spec}(L)$, and $\mathrm{Spec}(\Omega)$ map to the same $x \in X$. Thus a diagram (by the remarks above) corresponds to a point $x \in X$ and a commutative diagram

$$\begin{tikzcd} \Omega & L \arrow[l] \\ K \arrow[u] & \kappa(x) \arrow[l] \arrow[u] \end{tikzcd}$$

of fields. This defines an equivalence relation, because given any set of field extensions K_i/κ there exists some field extension Ω/κ such that all the field extensions K_i are contained in the extension Ω.

01J9 Lemma 26.13.3. Let X be a scheme. Points of X correspond bijectively to equivalence classes of morphisms from spectra of fields into X. Moreover, each equivalence class contains a (unique up to unique isomorphism) smallest element $\mathrm{Spec}(\kappa(x)) \to X$.

Proof. Follows from the discussion above. □

Of course the morphisms $\mathrm{Spec}(\kappa(x)) \to X$ factor through the canonical morphisms $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$. And the content of Lemma 26.13.2 is in this setting that the morphism $\mathrm{Spec}(\kappa(x')) \to X$ factors as $\mathrm{Spec}(\kappa(x')) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ whenever x' is a generalization of x. In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\kappa(x)) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}_{X,x}) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\kappa(s)) \arrow[r] & \mathrm{Spec}(\mathcal{O}_{S,s}) \arrow[r] & S. \end{tikzcd}$$

26.14. Glueing schemes

01JA Let I be a set. For each $i \in I$ let $(X_i, \mathcal{O}_i)$ be a locally ringed space. (Actually the construction that follows works equally well for ringed spaces.) For each pair $i, j \in I$ let $U_{ij} \subset X_i$ be an open subspace. For each pair $i, j \in I$, let

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

be an isomorphism of locally ringed spaces. For convenience we assume that $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$. For each triple $i, j, k \in I$ assume that

(1) we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$, and
(2) the diagram

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & U_{ki} \cap U_{kj} \\ & U_{ji} \cap U_{jk} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

is commutative.

Let us call a collection $(I, (X_i)_{i \in I}, (U_{ij})_{i,j \in I}, (\varphi_{ij})_{i,j \in I})$ satisfying the conditions above a glueing data.

01JB Lemma 26.14.1. Given any glueing data of locally ringed spaces there exists a locally ringed space X and open subspaces $U_i \subset X$ together with isomorphisms $\varphi_i : X_i \to U_i$ of locally ringed spaces such that

(1) $X = \bigcup_{i \in I} U_i$,
(2) $\varphi_i(U_{ij}) = U_i \cap U_j$, and
(3) $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

The locally ringed space X is characterized by the following mapping properties: Given a locally ringed space Y we have

$$\begin{array}{rcl} \mathrm{Mor}(X, Y) & = & \{(f_i)_{i\in I} \mid f_i : X_i \to Y,\ f_j \circ \varphi_{ij} = f_i|_{U_{ij}}\} \\ f & \mapsto & (f|_{U_i} \circ \varphi_i)_{i\in I} \\ \mathrm{Mor}(Y, X) & = & \left\{\begin{array}{c}\text{open covering } Y = \bigcup_{i\in I} V_i \text{ and } (g_i : V_i \to X_i)_{i\in I} \text{ such that} \\ g_i^{-1}(U_{ij}) = V_i \cap V_j \text{ and } g_j|_{V_i\cap V_j} = \varphi_{ij} \circ g_i|_{V_i\cap V_j}\end{array}\right\} \\ g & \mapsto & V_i = g^{-1}(U_i),\ g_i = \varphi_i^{-1} \circ g|_{V_i} \end{array}$$

Proof. We construct X in stages. As a set we take

$$X = (\coprod X_i)/\sim .$$

Here given $x \in X_i$ and $x' \in X_j$ we say $x \sim x'$ if and only if $x \in U_{ij}$, $x' \in U_{ji}$ and $\varphi_{ij}(x) = x'$. This is an equivalence relation since if $x \in X_i$, $x' \in X_j$, $x'' \in X_k$, and $x \sim x'$ and $x' \sim x''$, then $x' \in U_{ji} \cap U_{jk}$, hence by condition (1) of a glueing data also $x \in U_{ij} \cap U_{ik}$ and $x'' \in U_{ki} \cap U_{kj}$ and by condition (2) we see that $\varphi_{ik}(x) = x''$. (Reflexivity and symmetry follows from our assumptions that $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$.) Denote $\varphi_i : X_i \to X$ the natural maps. Denote $U_i = \varphi_i(X_i) \subset X$. Note that $\varphi_i : X_i \to U_i$ is a bijection.

The topology on X is defined by the rule that $U \subset X$ is open if and only if $\varphi_i^{-1}(U)$ is open for all i. We leave it to the reader to verify that this does indeed define a topology. Note that in particular U_i is open since $\varphi_j^{-1}(U_i) = U_{ji}$ which is open in X_j for all j. Moreover, for any open set $W \subset X_i$ the image $\varphi_i(W) \subset U_i$ is open because $\varphi_j^{-1}(\varphi_i(W)) = \varphi_{ji}^{-1}(W \cap U_{ij})$. Therefore $\varphi_i : X_i \to U_i$ is a homeomorphism.

To obtain a locally ringed space we have to construct the sheaf of rings $\mathcal{O}_X$. We do this by glueing the sheaves of rings $\mathcal{O}_{U_i} := \varphi_{i,*}\mathcal{O}_i$. Namely, in the commutative diagram

$$\begin{tikzcd} U_{ij} \arrow[rr, "\varphi_{ij}"'] \arrow[dr, "\varphi_i|_{U_{ij}}"'] & & U_{ji} \arrow[dl, "\varphi_j|_{U_{ji}}"] \\ & U_i \cap U_j & \end{tikzcd}$$

the arrow on top is an isomorphism of ringed spaces, and hence we get unique isomorphisms of sheaves of rings

$$\mathcal{O}_{U_i}|_{U_i\cap U_j} \longrightarrow \mathcal{O}_{U_j}|_{U_i\cap U_j}.$$

These satisfy a cocycle condition as in Sheaves, Section 6.33. By the results of that section we obtain a sheaf of rings $\mathcal{O}_X$ on X such that $\mathcal{O}_X|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}$ compatibly with the glueing maps displayed above. In particular $(X, \mathcal{O}_X)$ is a locally ringed space since the stalks of $\mathcal{O}_X$ are equal to the stalks of $\mathcal{O}_i$ at corresponding points.

The proof of the mapping properties is omitted. □

01JC Lemma 26.14.2. In Lemma 26.14.1 above, assume that all X_i are schemes. Then the resulting locally ringed space X is a scheme.

Proof. This is clear since each of the U_i is a scheme and hence every $x \in X$ has an affine neighbourhood. □

It is customary to think of X_i as an open subspace of X via the isomorphisms φ_i. We will do this in the next two examples.

01JD Example 26.14.3 (Affine space with zero doubled). Let k be a field. Let $n \geq 1$. Let $X_1 = \mathrm{Spec}(k[x_1, \ldots, x_n])$, let $X_2 = \mathrm{Spec}(k[y_1, \ldots, y_n])$. Let $0_1 \in X_1$ be the point corresponding to the maximal ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$. Let $0_2 \in X_2$ be the point corresponding to the maximal ideal $(y_1, \ldots, y_n) \subset k[y_1, \ldots, y_n]$. Let $U_{12} = X_1 \setminus \{0_1\}$ and let $U_{21} = X_2 \setminus \{0_2\}$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ mapping y_i to x_i (which induces $X_1 \cong X_2$ mapping 0_1 to 0_2). Let X be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_1, X_2 \subset X$ as open subschemes. There is a morphism $f : X \to \mathrm{Spec}(k[t_1, \ldots, t_n])$ which on X_1 (resp. X_2) corresponds to k algebra map $k[t_1, \ldots, t_n] \to k[x_1, \ldots, x_n]$ (resp. $k[t_1, \ldots, t_n] \to k[y_1, \ldots, y_n]$) mapping t_i to x_i (resp. t_i to y_i). It is easy to see that this morphism identifies $k[t_1, \ldots, t_n]$ with $\Gamma(X, \mathcal{O}_X)$. Since $f(0_1) = f(0_2)$ we see that X is not affine.

Note that X_1 and X_2 are affine opens of X. But, if $n = 2$, then $X_1 \cap X_2$ is the scheme described in Example 26.9.3 and hence not affine. Thus in general the intersection of affine opens of a scheme is not affine. (This fact holds more generally for any $n > 1$.)

Another curious feature of this example is the following. If $n > 1$ there are many irreducible closed subsets $T \subset X$ (take the closure of any non closed point in X_1 for example). But unless $T = \{0_1\}$, or $T = \{0_2\}$ we have $0_1 \in T \Leftrightarrow 0_2 \in T$. Proof omitted.

01JE Example 26.14.4 (Projective line). Let k be a field. Let $X_1 = \mathrm{Spec}(k[x])$, let $X_2 = \mathrm{Spec}(k[y])$. Let $0 \in X_1$ be the point corresponding to the maximal ideal $(x) \subset k[x]$. Let $\infty \in X_2$ be the point corresponding to the maximal ideal $(y) \subset k[y]$. Let $U_{12} = X_1 \setminus \{0\} = D(x) = \mathrm{Spec}(k[x, 1/x])$ and let $U_{21} = X_2 \setminus \{\infty\} = D(y) = \mathrm{Spec}(k[y, 1/y])$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y, 1/y] \to k[x, 1/x]$ mapping y to $1/x$. Let $\mathbf{P}^1_k$ be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_i \subset \mathbf{P}^1_k$ as open subschemes. In this case we see that $\Gamma(\mathbf{P}^1_k, \mathcal{O}) = k$ because the only polynomials $g(x)$ in x such that $g(1/y)$ is also a polynomial in y are constant polynomials. Since $\mathbf{P}^1_k$ is infinite we see that $\mathbf{P}^1_k$ is not affine.

We claim that there exists an affine open $U \subset \mathbf{P}^1_k$ which contains both 0 and ∞. Namely, let $U = \mathbf{P}^1_k \setminus \{1\}$, where 1 is the point of X_1 corresponding to the maximal ideal $(x - 1)$ and also the point of X_2 corresponding to the maximal ideal $(y - 1)$. Then it is easy to see that $s = 1/(x - 1) = y/(1 - y) \in \Gamma(U, \mathcal{O}_U)$. In fact you can show that $\Gamma(U, \mathcal{O}_U)$ is equal to the polynomial ring $k[s]$ and that the corresponding morphism $U \to \mathrm{Spec}(k[s])$ is an isomorphism of schemes. Details omitted.

26.15. A representability criterion

01JF In this section we reformulate the glueing lemma of Section 26.14 in terms of functors. We recall some of the material from Categories, Section 4.3. Recall that given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow \mathrm{Sets}, \quad T \longmapsto \mathrm{Mor}(T, X).$$

This is called the functor of points of X.

Let F be a contravariant functor from the category of schemes to the category of sets. In a formula

$$F : Sch^{opp} \longrightarrow \text{Sets}.$$

We will use the same terminology as in Sites, Section 7.2. Namely, given a scheme T, an element $\xi \in F(T)$, and a morphism $f : T' \to T$ we will denote $f^*\xi$ the element $F(f)(\xi)$, and sometimes we will even use the notation $\xi|_{T'}$

01JG Definition 26.15.1. (See Categories, Definition 4.3.6.) Let F be a contravariant functor from the category of schemes to the category of sets (as above). We say that F is representable by a scheme or representable if there exists a scheme X such that $h_X \cong F$.

Suppose that F is representable by the scheme X and that $s : h_X \to F$ is an isomorphism. By Categories, Yoneda Lemma 4.3.5 the pair $(X, s : h_X \to F)$ is unique up to unique isomorphism if it exists. Moreover, the Yoneda lemma says that given any contravariant functor F as above and any scheme Y, we have a bijection

$$\text{Mor}_{\text{Fun}(Sch^{opp},\text{Sets})}(h_Y, F) \longrightarrow F(Y), \quad s \longmapsto s(\text{id}_Y).$$

Here is the reverse construction. Given any $\xi \in F(Y)$ the transformation of functors $s_\xi : h_Y \to F$ associates to any morphism $f : T \to Y$ the element $f^*\xi \in F(T)$.

In particular, in the case that F is representable, there exists a scheme X and an element $\xi \in F(X)$ such that the corresponding morphism $h_X \to F$ is an isomorphism. In this case we also say the pair (X, ξ) represents F. The element $\xi \in F(X)$ is often called the "universal family" for reasons that will become more clear when we talk about algebraic stacks (insert future reference here). For the moment we simply observe that the fact that if the pair (X, ξ) represents F, then every element $\xi' \in F(T)$ for any T is of the form $\xi' = f^*\xi$ for a unique morphism $f : T \to X$.

01JH Example 26.15.2. Consider the rule which associates to every scheme T the set $F(T) = \Gamma(T, \mathcal{O}_T)$. We can turn this into a contravariant functor by using for a morphism $f : T' \to T$ the pullback map $f^\sharp : \Gamma(T, \mathcal{O}_T) \to \Gamma(T', \mathcal{O}_{T'})$. Given a ring R and an element $t \in R$ there exists a unique ring homomorphism $\mathbf{Z}[x] \to R$ which maps x to t. Thus, using Lemma 26.6.4, we see that

$$\text{Mor}(T, \text{Spec}(\mathbf{Z}[x])) = \text{Hom}(\mathbf{Z}[x], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T).$$

This does indeed give an isomorphism $h_{\text{Spec}(\mathbf{Z}[x])} \to F$. What is the "universal family" ξ? To get it we have to apply the identifications above to $\text{id}_{\text{Spec}(\mathbf{Z}[x])}$. Clearly under the identifications above this gives that $\xi = x \in \Gamma(\text{Spec}(\mathbf{Z}[x]), \mathcal{O}_{\text{Spec}(\mathbf{Z}[x])}) = \mathbf{Z}[x]$ as expected.

01JI Definition 26.15.3. Let F be a contravariant functor on the category of schemes with values in sets.

(1) We say that F satisfies the sheaf property for the Zariski topology if for every scheme T and every open covering $T = \bigcup_{i \in I} U_i$, and for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
(2) A subfunctor $H \subset F$ is a rule that associates to every scheme T a subset $H(T) \subset F(T)$ such that the maps $F(f) : F(T) \to F(T')$ maps $H(T)$ into $H(T')$ for all morphisms of schemes $f : T' \to T$.

(3) Let $H \subset F$ be a subfunctor. We say that $H \subset F$ is representable by open immersions if for all pairs (T, ξ), where T is a scheme and $\xi \in F(T)$ there exists an open subscheme $U_\xi \subset T$ with the following property:
   (*) A morphism $f : T' \to T$ factors through U_ξ if and only if $f^*\xi \in H(T')$.
(4) Let I be a set. For each $i \in I$ let $H_i \subset F$ be a subfunctor. We say that the collection $(H_i)_{i\in I}$ covers F if and only if for every $\xi \in F(T)$ there exists an open covering $T = \bigcup U_i$ such that $\xi|_{U_i} \in H_i(U_i)$.

In condition (4), if $H_i \subset F$ is representable by open immersions for all i, then to check $(H_i)_{i\in I}$ covers F, it suffices to check $F(T) = \bigcup H_i(T)$ whenever T is the spectrum of a field.

01JJ Lemma 26.15.4. Let F be a contravariant functor on the category of schemes with values in the category of sets. Suppose that

(1) F satisfies the sheaf property for the Zariski topology,
(2) there exists a set I and a collection of subfunctors $F_i \subset F$ such that
   (a) each F_i is representable,
   (b) each $F_i \subset F$ is representable by open immersions, and
   (c) the collection $(F_i)_{i\in I}$ covers F.

Then F is representable.

Proof. Let X_i be a scheme representing F_i and let $\xi_i \in F_i(X_i) \subset F(X_i)$ be the "universal family". Because $F_j \subset F$ is representable by open immersions, there exists an open $U_{ij} \subset X_i$ such that $T \to X_i$ factors through U_{ij} if and only if $\xi_i|_T \in F_j(T)$. In particular $\xi_i|_{U_{ij}} \in F_j(U_{ij})$ and therefore we obtain a canonical morphism $\varphi_{ij} : U_{ij} \to X_j$ such that $\varphi_{ij}^*\xi_j = \xi_i|_{U_{ij}}$. By definition of U_{ji} this implies that φ_{ij} factors through U_{ji}. Since $(\varphi_{ij} \circ \varphi_{ji})^*\xi_j = \varphi_{ji}^*(\varphi_{ij}^*\xi_j) = \varphi_{ji}^*\xi_i = \xi_j$ we conclude that $\varphi_{ij} \circ \varphi_{ji} = \mathrm{id}_{U_{ji}}$ because the pair (X_j, ξ_j) represents F_j. In particular the maps $\varphi_{ij} : U_{ij} \to U_{ji}$ are isomorphisms of schemes. Next we have to show that $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$. This is true because (a) $U_{ji} \cap U_{jk}$ is the largest open of U_{ji} such that ξ_j restricts to an element of F_k, (b) $U_{ij} \cap U_{ik}$ is the largest open of U_{ij} such that ξ_i restricts to an element of F_k, and (c) $\varphi_{ij}^*\xi_j = \xi_i$. Moreover, the cocycle condition in Section 26.14 follows because both $\varphi_{jk}|_{U_{ji}\cap U_{jk}} \circ \varphi_{ij}|_{U_{ij}\cap U_{ik}}$ and $\varphi_{ik}|_{U_{ij}\cap U_{ik}}$ pullback ξ_k to the element ξ_i. Thus we may apply Lemma 26.14.2 to obtain a scheme X with an open covering $X = \bigcup U_i$ and isomorphisms $\varphi_i : X_i \to U_i$ with properties as in Lemma 26.14.1. Let $\xi'_i = (\varphi_i^{-1})^*\xi_i$. The conditions of Lemma 26.14.1 imply that $\xi'_i|_{U_i\cap U_j} = \xi'_j|_{U_i\cap U_j}$. Therefore, by the condition that F satisfies the sheaf condition in the Zariski topology we see that there exists an element $\xi' \in F(X)$ such that $\xi_i = \varphi_i^*\xi'|_{U_i}$ for all i. Since φ_i is an isomorphism we also get that $(U_i, \xi'|_{U_i})$ represents the functor F_i.

We claim that the pair (X, ξ') represents the functor F. To show this, let T be a scheme and let $\xi \in F(T)$. We will construct a unique morphism $g : T \to X$ such that $g^*\xi' = \xi$. Namely, by the condition that the subfunctors F_i cover F there exists an open covering $T = \bigcup V_i$ such that for each i the restriction $\xi|_{V_i} \in F_i(V_i)$. Moreover, since each of the inclusions $F_i \subset F$ are representable by open immersions we may assume that each $V_i \subset T$ is maximal open with this property. Because, (U_i, ξ'_{U_i}) represents the functor F_i we get a unique morphism $g_i : V_i \to U_i$ such that $g_i^*\xi'|_{U_i} = \xi|_{V_i}$. On the overlaps $V_i \cap V_j$ the morphisms g_i and g_j agree, for example because they both pull back $\xi'|_{U_i \cap U_j} \in F_i(U_i \cap U_j)$ to the same element. Thus the morphisms g_i glue to a unique morphism from $T \to X$ as desired. □

01JK Remark 26.15.5. Suppose the functor F is defined on all locally ringed spaces, and if conditions of Lemma 26.15.4 are replaced by the following:

(1) F satisfies the sheaf property on the category of locally ringed spaces,
(2) there exists a set I and a collection of subfunctors $F_i \subset F$ such that
    (a) each F_i is representable by a scheme,
    (b) each $F_i \subset F$ is representable by open immersions on the category of locally ringed spaces, and
    (c) the collection $(F_i)_{i \in I}$ covers F as a functor on the category of locally ringed spaces.

We leave it to the reader to spell this out further. Then the end result is that the functor F is representable in the category of locally ringed spaces and that the representing object is a scheme.

26.16. Existence of fibre products of schemes

01JL A very basic question is whether or not products and fibre products exist on the category of schemes. We first prove abstractly that products and fibre products exist, and in the next section we show how we may think in a reasonable way about fibre products of schemes.

01JM Lemma 26.16.1. The category of schemes has a final object, products and fibre products. In other words, the category of schemes has finite limits, see Categories, Lemma 4.18.4.

Proof. Please skip this proof. It is more important to learn how to work with the fibre product which is explained in the next section.

By Lemma 26.6.4 the scheme $\text{Spec}(\mathbf{Z})$ is a final object in the category of locally ringed spaces. Thus it suffices to prove that fibred products exist.

Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. We have to show that the functor

$$\begin{array}{rcl} F : Sch^{opp} & \longrightarrow & \text{Sets} \\ T & \longmapsto & \text{Mor}(T, X) \times_{\text{Mor}(T,S)} \text{Mor}(T, Y) \end{array}$$

is representable. We claim that Lemma 26.15.4 applies to the functor F. If we prove this then the lemma is proved.

First we show that F satisfies the sheaf property in the Zariski topology. Namely, suppose that T is a scheme, $T = \bigcup_{i \in I} U_i$ is an open covering, and $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ for all pairs i, j. By definition ξ_i corresponds to a pair (a_i, b_i) where $a_i : U_i \to X$ and $b_i : U_i \to Y$ are morphisms of schemes such that $f \circ a_i = g \circ b_i$. The glueing condition says that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ and $b_i|_{U_i \cap U_j} = b_j|_{U_i \cap U_j}$. Thus by glueing the morphisms a_i we obtain a morphism of locally ringed spaces (i.e., a morphism of schemes) $a : T \to X$ and similarly $b : T \to Y$ (see for example the mapping property of Lemma 26.14.1). Moreover, on the members of an open covering the compositions $f \circ a$ and $g \circ b$ agree. Therefore $f \circ a = g \circ b$ and the pair (a, b) defines an element of $F(T)$ which restricts to the pairs (a_i, b_i) on each U_i. The sheaf condition is verified.

Next, we construct the family of subfunctors. Choose an open covering by open affines $S = \bigcup_{i \in I} U_i$. For every $i \in I$ choose open coverings by open affines $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ and $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$. Note that $X = \bigcup_{i \in I} \bigcup_{j \in J_i} V_j$ is an open covering and similarly for Y. For any $i \in I$ and each pair $(j, k) \in J_i \times K_i$ we have a commutative diagram

$$\begin{tikzcd} & W_k \arrow[d] \arrow[dr] & \\ V_j \arrow[r] \arrow[dr] & U_i \arrow[dr] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

where all the skew arrows are open immersions. For such a triple we get a functor

$$\begin{aligned} F_{i,j,k} : Sch^{opp} & \longrightarrow \text{Sets} \\ T & \longmapsto \text{Mor}(T, V_j) \times_{\text{Mor}(T, U_i)} \text{Mor}(T, W_k). \end{aligned}$$

There is an obvious transformation of functors $F_{i,j,k} \to F$ (coming from the huge commutative diagram above) which is injective, so we may think of $F_{i,j,k}$ as a subfunctor of F.

We check condition (2)(a) of Lemma 26.15.4. This follows directly from Lemma 26.6.7. (Note that we use here that the fibre products in the category of affine schemes are also fibre products in the whole category of locally ringed spaces.)

We check condition (2)(b) of Lemma 26.15.4. Let T be a scheme and let $\xi \in F(T)$. In other words, $\xi = (a, b)$ where $a : T \to X$ and $b : T \to Y$ are morphisms of schemes such that $f \circ a = g \circ b$. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$. For any further morphism $h : T' \to T$ we have $h^*\xi = (a \circ h, b \circ h)$. Hence we see that $h^*\xi \in F_{i,j,k}(T')$ if and only if $a(h(T')) \subset V_j$ and $b(h(T')) \subset W_k$. In other words, if and only if $h(T') \subset V_{i,j,k}$. This proves condition (2)(b).

We check condition (2)(c) of Lemma 26.15.4. Let T be a scheme and let $\xi = (a, b) \in F(T)$ as above. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$ as above. Condition (2)(c) just means that $T = \bigcup V_{i,j,k}$ which is evident. Thus the lemma is proved and fibre products exist. $\square$

01JN Remark 26.16.2. Using Remark 26.15.5 you can show that the fibre product of morphisms of schemes exists in the category of locally ringed spaces and is a scheme.

26.17. Fibre products of schemes

01JO Here is a review of the general definition, even though we have already shown that fibre products of schemes exist.

01JP Definition 26.17.1. Given morphisms of schemes $f : X \to S$ and $g : Y \to S$ the fibre product is a scheme $X \times_S Y$ together with projection morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ sitting into the following commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"'] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

which is universal among all diagrams of this sort, see Categories, Definition 4.6.1.

In other words, given any solid commutative diagram of morphisms of schemes

$$\begin{tikzcd} T \arrow[rrd, bend left] \arrow[rdd, bend right] \arrow[rd, dashed] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute. We will prove some lemmas which will tell us how to think about fibre products.

01JQ Lemma 26.17.2. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. If X, Y, S are all affine then $X \times_S Y$ is affine.

Proof. Suppose that $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. By Lemma 26.6.7 the affine scheme $\mathrm{Spec}(A \otimes_R B)$ is the fibre product $X \times_S Y$ in the category of locally ringed spaces. Hence it is a fortiori the fibre product in the category of schemes. □

01JR Lemma 26.17.3. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $X \times_S Y$, p, q be the fibre product. Suppose that $U \subset S$, $V \subset X$, $W \subset Y$ are open subschemes such that $f(V) \subset U$ and $g(W) \subset U$. Then the canonical morphism $V \times_U W \to X \times_S Y$ is an open immersion which identifies $V \times_U W$ with $p^{-1}(V) \cap q^{-1}(W)$.

Proof. Let T be a scheme. Suppose $a : T \to V$ and $b : T \to W$ are morphisms such that $f \circ a = g \circ b$ as morphisms into U. Then they agree as morphisms into S. By the universal property of the fibre product we get a unique morphism $T \to X \times_S Y$. Of course this morphism has image contained in the open $p^{-1}(V) \cap q^{-1}(W)$. Thus $p^{-1}(V) \cap q^{-1}(W)$ is a fibre product of V and W over U. The result follows from the uniqueness of fibre products, see Categories, Section 4.6. □

In particular this shows that $V \times_U W = V \times_S W$ in the situation of the lemma. Moreover, if U, V, W are all affine, then we know that $V \times_U W$ is affine. And of course we may cover $X \times_S Y$ by such affine opens $V \times_U W$. We formulate this as a lemma.

01JS Lemma 26.17.4. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $S = \bigcup U_i$ be any affine open covering of S. For each $i \in I$, let $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ be an affine open covering of $f^{-1}(U_i)$ and let $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$ be an affine open covering of $g^{-1}(U_i)$. Then

$$X \times_S Y = \bigcup_{i \in I} \bigcup_{j \in J_i,\ k \in K_i} V_j \times_{U_i} W_k$$

is an affine open covering of $X \times_S Y$.

Proof. See discussion above the lemma. □

In other words, we might have used the previous lemma to construct the fibre product directly by glueing the affine schemes. (Which is of course exactly what we did in the proof of Lemma 26.16.1 anyway.) Here is a way to describe the set of points of a fibre product of schemes.

01JT Lemma 26.17.5. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Points z of $X \times_S Y$ are in bijective correspondence to quadruples

$$(x, y, s, \mathfrak{p})$$

where $x \in X$, $y \in Y$, $s \in S$ are points with $f(x) = s$, $g(y) = s$ and $\mathfrak{p}$ is a prime ideal of the ring $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. The residue field of z corresponds to the residue field of the prime $\mathfrak{p}$.

Proof. Let z be a point of $X \times_S Y$ and let us construct a quadruple as above. Recall that we may think of z as a morphism $\mathrm{Spec}(\kappa(z)) \to X \times_S Y$, see Lemma 26.13.3. This morphism corresponds to morphisms $a : \mathrm{Spec}(\kappa(z)) \to X$ and $b : \mathrm{Spec}(\kappa(z)) \to Y$ such that $f \circ a = g \circ b$. By the same lemma again we get points $x \in X$, $y \in Y$ lying over the same point $s \in S$ as well as field maps $\kappa(x) \to \kappa(z)$, $\kappa(y) \to \kappa(z)$ such that the compositions $\kappa(s) \to \kappa(x) \to \kappa(z)$ and $\kappa(s) \to \kappa(y) \to \kappa(z)$ are the same. In other words we get a ring map $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)$. We let $\mathfrak{p}$ be the kernel of this map.

Conversely, given a quadruple $(x, y, s, \mathfrak{p})$ we get a commutative solid diagram

$$\begin{tikzcd}
X \times_S Y \arrow[rrrd, bend left] \arrow[rddd, bend right] & & & \\
& \mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)/\mathfrak{p}) \arrow[r] \arrow[d] \arrow[ul, dashed] & \mathrm{Spec}(\kappa(y)) \arrow[r] \arrow[d] & Y \arrow[dd] \\
& \mathrm{Spec}(\kappa(x)) \arrow[r] \arrow[d] & \mathrm{Spec}(\kappa(s)) \arrow[rd] & \\
& X \arrow[rr] & & S
\end{tikzcd}$$

see the discussion in Section 26.13. Thus we get the dotted arrow. The corresponding point z of $X \times_S Y$ is the image of the generic point of $\mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)/\mathfrak{p})$. We omit the verification that the two constructions are inverse to each other. $\square$

01JU Lemma 26.17.6. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target.

(1) If $f : X \to S$ is a closed immersion, then $X \times_S Y \to Y$ is a closed immersion. Moreover, if $X \to S$ corresponds to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$, then $X \times_S Y \to Y$ corresponds to the sheaf of ideals $\mathrm{Im}(g^*\mathcal{I} \to \mathcal{O}_Y)$.
(2) If $f : X \to S$ is an open immersion, then $X \times_S Y \to Y$ is an open immersion.
(3) If $f : X \to S$ is an immersion, then $X \times_S Y \to Y$ is an immersion.

Proof. Assume that $X \to S$ is a closed immersion corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$. By Lemma 26.4.7 the closed subspace $Z \subset Y$ defined by the sheaf of ideals $\mathrm{Im}(g^*\mathcal{I} \to \mathcal{O}_Y)$ is the fibre product in the category of locally ringed spaces. By Lemma 26.10.1 Z is a scheme. Hence $Z = X \times_S Y$ and the first statement follows. The second follows from Lemma 26.17.3 for example. The third is a combination of the first two. □

01JV Definition 26.17.7. Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be a closed subscheme of Y. The inverse image $f^{-1}(Z)$ of the closed subscheme Z is the closed subscheme $Z \times_Y X$ of X. See Lemma 26.17.6 above.

We may occasionally also use this terminology with locally closed and open subschemes.

26.18. Base change in algebraic geometry

01JW One motivation for the introduction of the language of schemes is that it gives a very precise notion of what it means to define a variety over a particular field. For example a variety X over $\mathbf{Q}$ is synonymous (Varieties, Definition 33.3.1) with $X \to \text{Spec}(\mathbf{Q})$ which is of finite type, separated, irreducible and reduced[1]. In any case, the idea is more generally to work with schemes over a given base scheme, often denoted S. We use the language: "let X be a scheme over S" to mean simply that X comes equipped with a morphism $X \to S$. In diagrams we will try to picture the structure morphism $X \to S$ as a downward arrow from X to S. We are often more interested in the properties of X relative to S rather than the internal geometry of X. For example, we would like to know things about the fibres of $X \to S$, what happens to X after base change, and so on.

We introduce some of the language that is customarily used. Of course this language is just a special case of thinking about the category of objects over a given object in a category, see Categories, Example 4.2.13.

01JX Definition 26.18.1. Let S be a scheme.

(1) We say X is a scheme over S to mean that X comes equipped with a morphism of schemes $X \to S$. The morphism $X \to S$ is sometimes called the structure morphism.
(2) If R is a ring we say X is a scheme over R instead of X is a scheme over $\text{Spec}(R)$.
(3) A morphism $f : X \to Y$ of schemes over S is a morphism of schemes such that the composition $X \to Y \to S$ of f with the structure morphism of Y is equal to the structure morphism of X.
(4) We denote $\text{Mor}_S(X, Y)$ the set of all morphisms from X to Y over S.
(5) Let X be a scheme over S. Let $S' \to S$ be a morphism of schemes. The base change of X is the scheme $X_{S'} = S' \times_S X$ over S'.
(6) Let $f : X \to Y$ be a morphism of schemes over S. Let $S' \to S$ be a morphism of schemes. The base change of f is the induced morphism $f' : X_{S'} \to Y_{S'}$ (namely the morphism $\text{id}_{S'} \times_{\text{id}_S} f$).
(7) Let R be a ring. Let X be a scheme over R. Let $R \to R'$ be a ring map. The base change $X_{R'}$ is the scheme $\text{Spec}(R') \times_{\text{Spec}(R)} X$ over R'.

Here is a typical result.

[1] Of course algebraic geometers still quibble over whether one should require X to be geometrically irreducible over $\mathbf{Q}$.

01JY **Lemma 26.18.2.** Let S be a scheme. Let $f : X \to Y$ be an immersion (resp. closed immersion, resp. open immersion) of schemes over S. Then any base change of f is an immersion (resp. closed immersion, resp. open immersion).

Proof. We can think of the base change of f via the morphism $S' \to S$ as the top left vertical arrow in the following commutative diagram:

$$\begin{tikzcd} X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \arrow[dd, bend left] \\ Y_{S'} \arrow[r] \arrow[d] & Y \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

The diagram implies $X_{S'} \cong Y_{S'} \times_Y X$, and the lemma follows from Lemma 26.17.6. □

In fact this type of result is so typical that there is a piece of language to express it. Here it is.

01JZ **Definition 26.18.3.** Properties and base change.

(1) Let $\mathcal{P}$ be a property of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that $\mathcal{P}$ is *preserved under base change* if whenever X/S has $\mathcal{P}$, any base change $X_{S'}/S'$ has $\mathcal{P}$.
(2) Let $\mathcal{P}$ be a property of morphisms of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that *preserved under base change* if whenever $f : X \to Y$ over S has $\mathcal{P}$, any base change $f' : X_{S'} \to Y_{S'}$ over S' has $\mathcal{P}$.

At this point we can say that "being a closed immersion" is preserved under arbitrary base change.

01K0 **Definition 26.18.4.** Let $f : X \to S$ be a morphism of schemes. Let $s \in S$ be a point. The *scheme theoretic fibre X_s of f over s*, or simply the *fibre of f over s*, is the scheme fitting in the following fibre product diagram

$$\begin{tikzcd} X_s = \mathrm{Spec}(\kappa(s)) \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd}$$

We think of the fibre X_s always as a scheme over $\kappa(s)$.

01K1 **Lemma 26.18.5.** Let $f : X \to S$ be a morphism of schemes. Consider the diagrams

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}(\mathcal{O}_{S,s}) \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{S,s}) \arrow[r] & S \end{tikzcd}$$

In both cases the top horizontal arrow is a homeomorphism onto its image.

Proof. Choose an open affine $U \subset S$ that contains s. The bottom horizontal morphisms factor through U, see Lemma 26.13.1 for example. Thus we may assume that S is affine. If X is also affine, then the result follows from Algebra, Remark 10.17.8. In the general case the result follows by covering X by open affines. □

26.19. Quasi-compact morphisms

01K2 A scheme is quasi-compact if its underlying topological space is quasi-compact. There is a relative notion which is defined as follows.

01K3 Definition 26.19.1. A morphism of schemes is called quasi-compact if the underlying map of topological spaces is quasi-compact, see Topology, Definition 5.12.1.

01K4 Lemma 26.19.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) $f : X \to S$ is quasi-compact,
(2) the inverse image of every affine open is quasi-compact, and
(3) there exists some affine open covering $S = \bigcup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is quasi-compact for all i.

Proof. Suppose we are given a covering $S = \bigcup_{i \in I} U_i$ as in (3). First, let $U \subset S$ be any affine open. For any $u \in U$ we can find an index $i(u) \in I$ such that $u \in U_{i(u)}$. As standard opens form a basis for the topology on $U_{i(u)}$ we can find $W_u \subset U \cap U_{i(u)}$ which is standard open in $U_{i(u)}$. By compactness we can find finitely many points $u_1, \ldots, u_n \in U$ such that $U = \bigcup_{j=1}^{n} W_{u_j}$. For each j write $f^{-1}U_{i(u_j)} = \bigcup_{k \in K_j} V_{jk}$ as a finite union of affine opens. Since $W_{u_j} \subset U_{i(u_j)}$ is a standard open we see that $f^{-1}(W_{u_j}) \cap V_{jk}$ is a standard open of V_{jk}, see Algebra, Lemma 10.17.4. Hence $f^{-1}(W_{u_j}) \cap V_{jk}$ is affine, and so $f^{-1}(W_{u_j})$ is a finite union of affines. This proves that the inverse image of any affine open is a finite union of affine opens.

Next, assume that the inverse image of every affine open is a finite union of affine opens. Let $K \subset S$ be any quasi-compact open. Since S has a basis of the topology consisting of affine opens we see that K is a finite union of affine opens. Hence the inverse image of K is a finite union of affine opens. Hence f is quasi-compact.

Finally, assume that f is quasi-compact. In this case the argument of the previous paragraph shows that the inverse image of any affine is a finite union of affine opens. □

01K5 Lemma 26.19.3. Being quasi-compact is a property of morphisms of schemes over a base which is preserved under arbitrary base change.

Proof. Omitted. □

01K6 Lemma 26.19.4. The composition of quasi-compact morphisms is quasi-compact.

Proof. This follows from the definitions and Topology, Lemma 5.12.2. □

01K7 Lemma 26.19.5. A closed immersion is quasi-compact.

Proof. Follows from the definitions and Topology, Lemma 5.12.3. □

01K8 Example 26.19.6. An open immersion is in general not quasi-compact. The standard example of this is the open subspace $U \subset X$, where $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$, where U is $X \setminus \{0\}$, and where 0 is the point of X corresponding to the maximal ideal $(x_1, x_2, x_3, \ldots)$.

05JL Lemma 26.19.7. Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent

(1) $f(X) \subset S$ is closed, and
(2) $f(X) \subset S$ is stable under specialization.

Proof. We have (1) $\Rightarrow$ (2) by Topology, Lemma 5.19.2. Assume (2). Let $U \subset S$ be an affine open. It suffices to prove that $f(X) \cap U$ is closed. Since $U \cap f(X)$ is stable under specializations in U, we have reduced to the case where S is affine. Because f is quasi-compact we deduce that $X = f^{-1}(S)$ is quasi-compact as S is affine. Thus we may write $X = \bigcup_{i=1}^{n} U_i$ with $U_i \subset X$ open affine. Say $S = \text{Spec}(R)$ and $U_i = \text{Spec}(A_i)$ for some R-algebra A_i. Then $f(X) = \text{Im}(\text{Spec}(A_1 \times \ldots \times A_n) \to \text{Spec}(R))$. Thus the lemma follows from Algebra, Lemma 10.41.5. □

01K9 Lemma 26.19.8. Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is closed if and only if specializations lift along f, see Topology, Definition 5.19.4.

Proof. According to Topology, Lemma 5.19.7 if f is closed then specializations lift along f. Conversely, suppose that specializations lift along f. Let $Z \subset X$ be a closed subset. We may think of Z as a scheme with the reduced induced scheme structure, see Definition 26.12.5. Since $Z \subset X$ is closed the restriction of f to Z is still quasi-compact. Moreover specializations lift along $Z \to S$ as well, see Topology, Lemma 5.19.5. Hence it suffices to prove $f(X)$ is closed if specializations lift along f. In particular $f(X)$ is stable under specializations, see Topology, Lemma 5.19.6. Thus $f(X)$ is closed by Lemma 26.19.7. □

26.20. Valuative criterion for universal closedness

01KA In Topology, Section 5.17 there is a discussion of proper maps as closed maps of topological spaces all of whose fibres are quasi-compact, or as maps such that all base changes are closed maps. Here is the corresponding notion in algebraic geometry.

01KB Definition 26.20.1. A morphism of schemes $f : X \to S$ is said to be universally closed if every base change $f' : X_{S'} \to S'$ is closed.

In fact the adjective "universally" is often used in this way. In other words, given a property $\mathcal{P}$ of morphisms then we say that "$X \to S$ is universally $\mathcal{P}$" if and only if every base change $X_{S'} \to S'$ has $\mathcal{P}$.

Please take a look at Morphisms, Section 29.41 for a more detailed discussion of the properties of universally closed morphisms. In this section we restrict the discussion to the relationship between universal closed morphisms and morphisms satisfying the existence part of the valuative criterion.

01KC Lemma 26.20.2. Let $f : X \to S$ be a morphism of schemes.

(1) If f is universally closed then specializations lift along any base change of f, see Topology, Definition 5.19.4.
(2) If f is quasi-compact and specializations lift along any base change of f, then f is universally closed.

Proof. Part (1) is a direct consequence of Topology, Lemma 5.19.7. Part (2) follows from Lemmas 26.19.8 and 26.19.3. □

01KD Definition 26.20.3. Let $f : X \to S$ be a morphism of schemes. We say f satisfies the existence part of the valuative criterion if given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & S \end{tikzcd}$$

where A is a valuation ring with field of fractions K, the dotted arrow exists. We say f satisfies the uniqueness part of the valuative criterion if there is at most one dotted arrow given any diagram as above (without requiring existence of course).

A valuation ring is a local domain maximal among the relation of domination in its fraction field, see Algebra, Definition 10.50.1. Hence the spectrum of a valuation ring has a unique generic point η and a unique closed point 0, and of course we have the specialization $\eta \leadsto 0$. The significance of valuation rings is that any specialization of points in any scheme is the image of $\eta \leadsto 0$ under some morphism from the spectrum of some valuation ring. Here is the precise result.

01J8 Lemma 26.20.4. Let S be a scheme. Let $s' \leadsto s$ be a specialization of points of S. Then

(1) there exists a valuation ring A and a morphism $f : \mathrm{Spec}(A) \to S$ such that the generic point η of $\mathrm{Spec}(A)$ maps to s' and the special point maps to s, and
(2) given a field extension $K/\kappa(s')$ we may arrange it so that the extension $\kappa(\eta)/\kappa(s')$ induced by f is isomorphic to the given extension.

Proof. Let $s' \leadsto s$ be a specialization in S, and let $K/\kappa(s')$ be an extension of fields. By Lemma 26.13.2 and the discussion following Lemma 26.13.3 this leads to ring maps $\mathcal{O}_{S,s} \to \kappa(s') \to K$. Let $A \subset K$ be any valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{S,s} \to K$, see Algebra, Lemma 10.50.2. The ring map $\mathcal{O}_{S,s} \to A$ induces the morphism $f : \mathrm{Spec}(A) \to S$, see Lemma 26.13.1. This morphism has all the desired properties by construction. $\square$

01KE Lemma 26.20.5. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) Specializations lift along any base change of f
(2) The morphism f satisfies the existence part of the valuative criterion.

Proof. Assume (1) holds. Let a solid diagram as in Definition 26.20.3 be given. In order to find the dotted arrow we may replace $X \to S$ by $X_{\mathrm{Spec}(A)} \to \mathrm{Spec}(A)$ since after all the assumption is stable under base change. Thus we may assume $S = \mathrm{Spec}(A)$. Let $x' \in X$ be the image of $\mathrm{Spec}(K) \to X$, so that we have $\kappa(x') \subset K$, see Lemma 26.13.3. By assumption there exists a specialization $x' \leadsto x$ in X such that x maps to the closed point of $S = \mathrm{Spec}(A)$. We get a local ring map $A \to \mathcal{O}_{X,x}$ and a ring map $\mathcal{O}_{X,x} \to \kappa(x')$, see Lemma 26.13.2 and the discussion following Lemma 26.13.3. The composition $A \to \mathcal{O}_{X,x} \to \kappa(x') \to K$ is the given injection $A \to K$. Since $A \to \mathcal{O}_{X,x}$ is local, the image of $\mathcal{O}_{X,x} \to K$ dominates A and hence is equal to A, by Algebra, Definition 10.50.1. Thus we obtain a ring map $\mathcal{O}_{X,x} \to A$ and hence a morphism $\mathrm{Spec}(A) \to X$ (see Lemma 26.13.1 and discussion following it). This proves (2).

Conversely, assume (2) holds. It is immediate that the existence part of the valuative criterion holds for any base change $X_{S'} \to S'$ of f by considering the following commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] \arrow[rru, dashed] & S' \arrow[r] & S \end{tikzcd}$$

Namely, the more horizontal dotted arrow will lead to the other one by definition of the fibre product. OK, so it clearly suffices to show that specializations lift along f. Let $s' \rightsquigarrow s$ be a specialization in S, and let $x' \in X$ be a point lying over s'. Apply Lemma 26.20.4 to $s' \rightsquigarrow s$ and the extension of fields $K = \kappa(x')/\kappa(s')$. We get a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[rr] \arrow[d] & & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[rru, dashed] & \mathrm{Spec}(\mathcal{O}_{S,s}) \arrow[r] & S \end{tikzcd}$$

and by condition (2) we get the dotted arrow. The image x of the closed point of $\mathrm{Spec}(A)$ in X will be a solution to our problem, i.e., x is a specialization of x' and maps to s. □

01KF Proposition 26.20.6 (Valuative criterion of universal closedness). Let f be a quasi-compact morphism of schemes. Then f is universally closed if and only if f satisfies the existence part of the valuative criterion.

Proof. This is a formal consequence of Lemmas 26.20.2 and 26.20.5 above. □

01KG Example 26.20.7. Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to \mathrm{Spec}(k)$ of the projective line over k, see Example 26.14.4. Let us use the valuative criterion above to prove that p is universally closed. By construction $\mathbf{P}^1_k$ is covered by two affine opens and hence p is quasi-compact. Let a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "\xi"'] \arrow[d] & \mathbf{P}^1_k \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "\varphi"] & \mathrm{Spec}(k) \end{tikzcd}$$

be given, where A is a valuation ring and K is its field of fractions. Recall that $\mathbf{P}^1_k$ is gotten by glueing $\mathrm{Spec}(k[x])$ to $\mathrm{Spec}(k[y])$ by glueing $D(x)$ to $D(y)$ via $x = y^{-1}$ (or more symmetrically $xy = 1$). To show there is a morphism $\mathrm{Spec}(A) \to \mathbf{P}^1_k$ fitting diagonally into the diagram above we may assume that ξ maps into the open $\mathrm{Spec}(k[x])$ (by symmetry). This gives the following commutative diagram of rings

$$\begin{tikzcd} K & k[x] \arrow[l, "\xi^\sharp"'] \\ A \arrow[u] & k \arrow[l, "\varphi^\sharp"'] \arrow[u] \end{tikzcd}$$

By Algebra, Lemma 10.50.4 we see that either $\xi^\sharp(x) \in A$ or $\xi^\sharp(x)^{-1} \in A$. In the first case we get a ring map

$$k[x] \to A,\ \lambda \mapsto \varphi^\sharp(\lambda),\ x \mapsto \xi^\sharp(x)$$

fitting into the diagram of rings above, and we win. In the second case we see that we get a ring map

$$k[y] \to A,\ \lambda \mapsto \varphi^\sharp(\lambda),\ y \mapsto \xi^\sharp(x)^{-1}.$$

This gives a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(k[y]) \to \mathbf{P}^1_k$ which fits diagonally into the initial commutative diagram of this example (check omitted).

26.21. Separation axioms

01KH A topological space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is a closed subset. The analogue in algebraic geometry is, given a scheme X over a base scheme S, to consider the diagonal morphism

$$\Delta_{X/S} : X \longrightarrow X \times_S X.$$

This is the unique morphism of schemes such that $\mathrm{pr}_1 \circ \Delta_{X/S} = \mathrm{id}_X$ and $\mathrm{pr}_2 \circ \Delta_{X/S} = \mathrm{id}_X$ (it exists in any category with fibre products).

01KI **Lemma 26.21.1.** *The diagonal morphism of a morphism between affines is closed.*

Proof. The diagonal morphism associated to the morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the morphism on spectra corresponding to the ring map $S \otimes_R S \to S$, $a \otimes b \mapsto ab$. This map is clearly surjective, so $S \cong S \otimes_R S/J$ for some ideal $J \subset S \otimes_R S$. Hence Δ is a closed immersion according to Example 26.8.1 $\square$

01KJ **Lemma 26.21.2.** *Let X be a scheme over S. The diagonal morphism $\Delta_{X/S}$ is an immersion.*

Proof. Recall that if $V \subset X$ is affine open and maps into $U \subset S$ affine open, then $V \times_U V$ is affine open in $X \times_S X$, see Lemmas 26.17.2 and 26.17.3. Consider the open subscheme W of $X \times_S X$ which is the union of these affine opens $V \times_U V$. By Lemma 26.4.2 it is enough to show that each morphism $\Delta_{X/S}^{-1}(V \times_U V) \to V \times_U V$ is a closed immersion. Since $V = \Delta_{X/S}^{-1}(V \times_U V)$ we are just checking that $\Delta_{V/U}$ is a closed immersion, which is Lemma 26.21.1. $\square$

01KK **Definition 26.21.3.** Let $f : X \to S$ be a morphism of schemes.

1. We say f is *separated* if the diagonal morphism $\Delta_{X/S}$ is a closed immersion.
2. We say f is *quasi-separated* if the diagonal morphism $\Delta_{X/S}$ is a quasi-compact morphism.
3. We say a scheme Y is *separated* if the morphism $Y \to \mathrm{Spec}(\mathbf{Z})$ is separated.
4. We say a scheme Y is *quasi-separated* if the morphism $Y \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated.

By Lemmas 26.21.2 and 26.10.4 we see that $\Delta_{X/S}$ is a closed immersion if an only if $\Delta_{X/S}(X) \subset X \times_S X$ is a closed subset. Moreover, by Lemma 26.19.5 we see that a separated morphism is quasi-separated. The reason for introducing quasi-separated morphisms is that nonseparated morphisms come up naturally in studying algebraic varieties (especially when doing moduli, algebraic stacks, etc). But most often they are still quasi-separated.

01KL Example 26.21.4. Here is an example of a non-quasi-separated morphism. Suppose $X = X_1 \cup X_2 \to S = \mathrm{Spec}(k)$ with $X_1 = X_2 = \mathrm{Spec}(k[t_1, t_2, t_3, \ldots])$ glued along the complement of $\{0\} = \{(t_1, t_2, t_3, \ldots)\}$ (glued as in Example 26.14.3). In this case the inverse image of the affine scheme $X_1 \times_S X_2$ under $\Delta_{X/S}$ is the scheme $\mathrm{Spec}(k[t_1, t_2, t_3, \ldots]) \setminus \{0\}$ which is not quasi-compact.

01KM Lemma 26.21.5. Let X, Y be schemes over S. Let $a, b : X \to Y$ be morphisms of schemes over S. There exists a largest locally closed subscheme $Z \subset X$ such that $a|_Z = b|_Z$. In fact Z is the equalizer of (a, b). Moreover, if Y is separated over S, then Z is a closed subscheme.

Proof. The equalizer of (a, b) is for categorical reasons the fibre product Z in the following diagram

$$\begin{tikzcd} Z = Y \times_{(Y \times_S Y)} X \arrow[r] \arrow[d] & X \arrow[d, "{(a,b)}"] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

Thus the lemma follows from Lemmas 26.18.2, 26.21.2 and Definition 26.21.3. □

01KO Lemma 26.21.6. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is quasi-separated.
(2) For every pair of affine opens $U, V \subset X$ which map into a common affine open of S the intersection $U \cap V$ is a finite union of affine opens of X.
(3) There exists an affine open covering $S = \bigcup_{i \in I} U_i$ and for each i an affine open covering $f^{-1}U_i = \bigcup_{j \in I_i} V_j$ such that for each i and each pair $j, j' \in I_i$ the intersection $V_j \cap V_{j'}$ is a finite union of affine opens of X.

Proof. Let us prove that (3) implies (1). By Lemma 26.17.4 the covering $X \times_S X = \bigcup_i \bigcup_{j,j'} V_j \times_{U_i} V_{j'}$ is an affine open covering of $X \times_S X$. Moreover, $\Delta_{X/S}^{-1}(V_j \times_{U_i} V_{j'}) = V_j \cap V_{j'}$. Hence the implication follows from Lemma 26.19.2.

The implication (1) $\Rightarrow$ (2) follows from the fact that under the hypotheses of (2) the fibre product $U \times_S V$ is an affine open of $X \times_S X$. The implication (2) $\Rightarrow$ (3) is trivial. □

01KP Lemma 26.21.7. Let $f : X \to S$ be a morphism of schemes.

(1) If f is separated then for every pair of affine opens (U, V) of X which map into a common affine open of S we have
  (a) the intersection $U \cap V$ is affine.
  (b) the ring map $\mathcal{O}_X(U) \otimes_{\mathbf{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.
(2) If any pair of points $x_1, x_2 \in X$ lying over a common point $s \in S$ are contained in affine opens $x_1 \in U$, $x_2 \in V$ which map into a common affine open of S such that (a), (b) hold, then f is separated.

Proof. Assume f separated. Suppose (U, V) is a pair as in (1). Let $W = \mathrm{Spec}(R)$ be an affine open of S containing both $f(U)$ and $f(V)$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ for R-algebras A and B. By Lemma 26.17.3 we see that $U \times_S V = U \times_W V = \mathrm{Spec}(A \otimes_R B)$ is an affine open of $X \times_S X$. Hence, by Lemma 26.10.1 we see that $\Delta^{-1}(U \times_S V) \to U \times_S V$ can be identified with $\mathrm{Spec}((A \otimes_R B)/J)$ for some ideal $J \subset A \otimes_R B$. Thus $U \cap V = \Delta^{-1}(U \times_S V)$ is affine. Assertion (1)(b) holds because $A \otimes_{\mathbf{Z}} B \to (A \otimes_R B)/J$ is surjective.

Assume the hypothesis formulated in (2) holds. Clearly the collection of affine opens $U \times_S V$ for pairs (U, V) as in (2) form an affine open covering of $X \times_S X$ (see e.g. Lemma 26.17.4). Hence it suffices to show that each morphism $U \cap V = \Delta_{X/S}^{-1}(U \times_S V) \to U \times_S V$ is a closed immersion, see Lemma 26.4.2. By assumption (a) we have $U \cap V = \mathrm{Spec}(C)$ for some ring C. After choosing an affine open $W = \mathrm{Spec}(R)$ of S into which both U and V map and writing $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(B)$ we see that the assumption (b) means that the composition

$$A \otimes_{\mathbf{Z}} B \to A \otimes_R B \to C$$

is surjective. Hence $A \otimes_R B \to C$ is surjective and we conclude that $\mathrm{Spec}(C) \to \mathrm{Spec}(A \otimes_R B)$ is a closed immersion. $\square$

01KQ Example 26.21.8. Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to \mathrm{Spec}(k)$ of the projective line over k, see Example 26.14.4. Let us use the lemma above to prove that p is separated. By construction $\mathbf{P}^1_k$ is covered by two affine opens $U = \mathrm{Spec}(k[x])$ and $V = \mathrm{Spec}(k[y])$ with intersection $U \cap V = \mathrm{Spec}(k[x, y]/(xy - 1))$ (using obvious notation). Thus it suffices to check that conditions (2)(a) and (2)(b) of Lemma 26.21.7 hold for the pairs of affine opens (U, U), (U, V), (V, U) and (V, V). For the pairs (U, U) and (V, V) this is trivial. For the pair (U, V) this amounts to proving that $U \cap V$ is affine, which is true, and that the ring map

$$k[x] \otimes_{\mathbf{Z}} k[y] \longrightarrow k[x, y]/(xy - 1)$$

is surjective. This is clear because any element in the right hand side can be written as a sum of a polynomial in x and a polynomial in y.

01KR Lemma 26.21.9. Let $f : X \to T$ and $g : Y \to T$ be morphisms of schemes with the same target. Let $h : T \to S$ be a morphism of schemes. Then the induced morphism $i : X \times_T Y \to X \times_S Y$ is an immersion. If $T \to S$ is separated, then i is a closed immersion. If $T \to S$ is quasi-separated, then i is a quasi-compact morphism.

Proof. By general category theory the following diagram

$$\begin{tikzcd} X \times_T Y \arrow[r] \arrow[d] & X \times_S Y \arrow[d] \\ T \arrow[r, "\Delta_{T/S}"] & T \times_S T \end{tikzcd}$$

is a fibre product diagram. The lemma follows from Lemmas 26.21.2, 26.17.6 and 26.19.3. $\square$

01KS Lemma 26.21.10. Let $g : X \to Y$ be a morphism of schemes over S. The morphism $i : X \to X \times_S Y$ is an immersion. If Y is separated over S it is a closed immersion. If Y is quasi-separated over S it is quasi-compact.

Proof. This is a special case of Lemma 26.21.9 applied to the morphism $X = X \times_Y Y \to X \times_S Y$. $\square$

01KT Lemma 26.21.11. Let $f : X \to S$ be a morphism of schemes. Let $s : S \to X$ be a section of f (in a formula $f \circ s = \mathrm{id}_S$). Then s is an immersion. If f is separated then s is a closed immersion. If f is quasi-separated, then s is quasi-compact.

Proof. This is a special case of Lemma 26.21.10 applied to $g = s$ so the morphism $i = s : S \to S \times_S X$. □

01KU Lemma 26.21.12. Permanence properties.

(1) A composition of separated morphisms is separated.
(2) A composition of quasi-separated morphisms is quasi-separated.
(3) The base change of a separated morphism is separated.
(4) The base change of a quasi-separated morphism is quasi-separated.
(5) A (fibre) product of separated morphisms is separated.
(6) A (fibre) product of quasi-separated morphisms is quasi-separated.

Proof. Let $X \to Y \to Z$ be morphisms. Assume that $X \to Y$ and $Y \to Z$ are separated. The composition

$$X \to X \times_Y X \to X \times_Z X$$

is closed because the first one is by assumption and the second one by Lemma 26.21.9. The same argument works for "quasi-separated" (with the same references).

Let $f : X \to Y$ be a morphism of schemes over a base S. Let $S' \to S$ be a morphism of schemes. Let $f' : X_{S'} \to Y_{S'}$ be the base change of f. Then the diagonal morphism of f' is a morphism

$$\Delta_{f'} : X_{S'} = S' \times_S X \longrightarrow X_{S'} \times_{Y_{S'}} X_{S'} = S' \times_S (X \times_Y X)$$

which is easily seen to be the base change of Δ_f. Thus (3) and (4) follow from the fact that closed immersions and quasi-compact morphisms are preserved under arbitrary base change (Lemmas 26.17.6 and 26.19.3).

If $f : X \to Y$ and $g : U \to V$ are morphisms of schemes over a base S, then $f \times g$ is the composition of $X \times_S U \to X \times_S V$ (a base change of g) and $X \times_S V \to Y \times_S V$ (a base change of f). Hence (5) and (6) follow from (1) – (4). □

01KV Lemma 26.21.13. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated then so is f. If $g \circ f$ is quasi-separated then so is f.

Proof. Assume that $g \circ f$ is separated. Consider the factorization $X \to X \times_Y X \to X \times_Z X$ of the diagonal morphism of $g \circ f$. By Lemma 26.21.9 the last morphism is an immersion. By assumption the image of X in $X \times_Z X$ is closed. Hence it is also closed in $X \times_Y X$. Thus we see that $X \to X \times_Y X$ is a closed immersion by Lemma 26.10.4.

Assume that $g \circ f$ is quasi-separated. Let $V \subset Y$ be an affine open which maps into an affine open of Z. Let $U_1, U_2 \subset X$ be affine opens which map into V. Then $U_1 \cap U_2$ is a finite union of affine opens because U_1, U_2 map into a common affine open of Z. Since we may cover Y by affine opens like V we deduce the lemma from Lemma 26.21.6. □

03GI Lemma 26.21.14. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 26.21.11 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 26.21.12). The second map is quasi-compact as it is the base change of $g \circ f$, see Lemma

26.19.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 26.19.4. □

01KN Lemma 26.21.15. An affine scheme is separated. A morphism from an affine scheme to another scheme is separated.

Proof. Let $U = \text{Spec}(A)$ be an affine scheme. Then $U \to \text{Spec}(\mathbf{Z})$ has closed diagonal by Lemma 26.21.1. Thus U is separated by Definition 26.21.3. If $U \to X$ is a morphism of schemes, then we can apply Lemma 26.21.13 to the morphisms $U \to X \to \text{Spec}(\mathbf{Z})$ to conclude that $U \to X$ is separated. □

You may have been wondering whether the condition of only considering pairs of affine opens whose image is contained in an affine open is really necessary to be able to conclude that their intersection is affine. Often it isn't!

01KW Lemma 26.21.16. Let $f : X \to S$ be a morphism. Assume f is separated and S is a separated scheme. Suppose $U \subset X$ and $V \subset X$ are affine. Then $U \cap V$ is affine (and a closed subscheme of $U \times V$).

Proof. In this case X is separated by Lemma 26.21.12. Hence $U \cap V$ is affine by applying Lemma 26.21.7 to the morphism $X \to \text{Spec}(\mathbf{Z})$. □

On the other hand, the following example shows that we cannot expect the image of an affine to be contained in an affine.

01KX Example 26.21.17. Consider the nonaffine scheme $U = \text{Spec}(k[x, y]) \setminus \{(x, y)\}$ of Example 26.9.3. On the other hand, consider the scheme

$$\mathbf{GL}_{2,k} = \text{Spec}(k[a, b, c, d, 1/ad - bc]).$$

There is a morphism $\mathbf{GL}_{2,k} \to U$ corresponding to the ring map $x \mapsto a$, $y \mapsto b$. It is easy to see that this is a surjective morphism, and hence the image is not contained in any affine open of U. In fact, the affine scheme $\mathbf{GL}_{2,k}$ also surjects onto $\mathbf{P}^1_k$, and $\mathbf{P}^1_k$ does not even have an immersion into any affine scheme.

0816 Remark 26.21.18. The category of quasi-compact and quasi-separated schemes $\mathcal{C}$ has the following properties. If $X, Y \in \text{Ob}(\mathcal{C})$, then any morphism of schemes $f : X \to Y$ is quasi-compact and quasi-separated by Lemmas 26.21.14 and 26.21.13 with $Z = \text{Spec}(\mathbf{Z})$. Moreover, if $X \to Y$ and $Z \to Y$ are morphisms $\mathcal{C}$, then $X \times_Y Z$ is an object of $\mathcal{C}$ too. Namely, the projection $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of the morphism $Z \to Y$, see Lemmas 26.21.12 and 26.19.3. Hence the composition $X \times_Y Z \to Z \to \text{Spec}(\mathbf{Z})$ is quasi-compact and quasi-separated, see Lemmas 26.21.12 and 26.19.4.

26.22. Valuative criterion of separatedness

01KY

01KZ Lemma 26.22.1. Let $f : X \to S$ be a morphism of schemes. If f is separated, then f satisfies the uniqueness part of the valuative criterion.

Proof. Let a diagram as in Definition 26.20.3 be given. Suppose there are two morphisms $a, b : \text{Spec}(A) \to X$ fitting into the diagram. Let $Z \subset \text{Spec}(A)$ be the equalizer of a and b. By Lemma 26.21.5 this is a closed subscheme of $\text{Spec}(A)$. By assumption it contains the generic point of $\text{Spec}(A)$. Since A is a domain this implies $Z = \text{Spec}(A)$. Hence $a = b$ as desired. □

01L0 **Lemma 26.22.2** (Valuative criterion separatedness). Let $f : X \to S$ be a morphism. Assume [DG67, II Proposition 7.2.3]

(1) the morphism f is quasi-separated, and
(2) the morphism f satisfies the uniqueness part of the valuative criterion.

Then f is separated.

Proof. By assumption (1), Proposition 26.20.6, and Lemmas 26.21.2 and 26.10.4 we see that it suffices to prove the morphism $\Delta_{X/S} : X \to X \times_S X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & X \times_S X \end{tikzcd}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ over S. By (2) we see that $a = b$. Hence using a as the dotted arrow works. $\square$

26.23. Monomorphisms

01L1

01L2 **Definition 26.23.1.** A morphism of schemes is called a *monomorphism* if it is a monomorphism in the category of schemes, see Categories, Definition 4.13.1.

01L3 **Lemma 26.23.2.** Let $j : X \to Y$ be a morphism of schemes. Then j is a monomorphism if and only if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.

Proof. This is true in any category with fibre products. $\square$

01L4 **Lemma 26.23.3.** A monomorphism of schemes is separated.

Proof. This is true because an isomorphism is a closed immersion, and Lemma 26.23.2 above. $\square$

01L5 **Lemma 26.23.4.** A composition of monomorphisms is a monomorphism.

Proof. True in any category. $\square$

02YC **Lemma 26.23.5.** The base change of a monomorphism is a monomorphism.

Proof. True in any category with fibre products. $\square$

0DVA **Lemma 26.23.6.** Let $j : X \to Y$ be a morphism of schemes. If j is injective on points, then j is separated.

Proof. Let z be a point of $X \times_Y X$. Then $x = \mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ are the same because j maps these points to the same point y of Y. Then we can choose an affine open neighbourhood $V \subset Y$ of y and an affine open neighbourhood $U \subset X$ of x with $j(U) \subset V$. Then $z \in U \times_V U \subset X \times_Y X$. Hence $X \times_Y X$ is the union of the affine opens $U \times_V U$. Since $\Delta_{X/Y}^{-1}(U \times_V U) = U$ and since $U \to U \times_V U$ is a closed immersion, we conclude that $\Delta_{X/Y}$ is a closed immersion (see argument in the proof of Lemma 26.21.2). $\square$

01L6 **Lemma 26.23.7.** Let $j : X \to Y$ be a morphism of schemes. If

(1) j is injective on points, and

(2) for any $x \in X$ the ring map $j_x^\sharp : \mathcal{O}_{Y,j(x)} \to \mathcal{O}_{X,x}$ is surjective,

then j is a monomorphism.

Proof. Let $a, b : Z \to X$ be two morphisms of schemes such that $j \circ a = j \circ b$. Then (1) implies $a = b$ as underlying maps of topological spaces. For any $z \in Z$ we have $a_z^\sharp \circ j_{a(z)}^\sharp = b_z^\sharp \circ j_{b(z)}^\sharp$ as maps $\mathcal{O}_{Y,j(a(z))} \to \mathcal{O}_{Z,z}$. The surjectivity of the maps $j_x^\sharp$ forces $a_z^\sharp = b_z^\sharp$, $\forall z \in Z$. This implies that $a^\sharp = b^\sharp$. Hence we conclude $a = b$ as morphisms of schemes as desired. □

01L7 Lemma 26.23.8. An immersion of schemes is a monomorphism. In particular, any immersion is separated.

Proof. We can see this by checking that the criterion of Lemma 26.23.7 applies. More elegantly perhaps, we can use that Lemmas 26.3.5 and 26.4.6 imply that open and closed immersions are monomorphisms and hence any immersion (which is a composition of such) is a monomorphism. □

01L8 Lemma 26.23.9. Let $f : X \to S$ be a separated morphism. Any locally closed subscheme $Z \subset X$ is separated over S.

Proof. Follows from Lemma 26.23.8 and the fact that a composition of separated morphisms is separated (Lemma 26.21.12). □

01L9 Example 26.23.10. The morphism $\text{Spec}(\mathbf{Q}) \to \text{Spec}(\mathbf{Z})$ is a monomorphism. This is true because $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}$. More generally, for any scheme S and any point $s \in S$ the canonical morphism

$$\text{Spec}(\mathcal{O}_{S,s}) \longrightarrow S$$

is a monomorphism.

03DP Lemma 26.23.11. Let $k_1, \ldots, k_n$ be fields. For any monomorphism of schemes $X \to \text{Spec}(k_1 \times \ldots \times k_n)$ there exists a subset $I \subset \{1, \ldots, n\}$ such that $X \cong \text{Spec}(\prod_{i \in I} k_i)$ as schemes over $\text{Spec}(k_1 \times \ldots \times k_n)$. More generally, if $X = \coprod_{i \in I} \text{Spec}(k_i)$ is a disjoint union of spectra of fields and $Y \to X$ is a monomorphism, then there exists a subset $J \subset I$ such that $Y = \coprod_{i \in J} \text{Spec}(k_i)$.

Proof. First reduce to the case $n = 1$ (or $\#I = 1$) by taking the inverse images of the open and closed subschemes $\text{Spec}(k_i)$. In this case X has only one point hence is affine. The corresponding algebra problem is this: If $k \to R$ is an algebra map with $R \otimes_k R \cong R$, then $R \cong k$ or $R = 0$. This holds for dimension reasons. See also Algebra, Lemma 10.107.8 □

26.24. Functoriality for quasi-coherent modules

01LA Let X be a scheme. We denote $QCoh(\mathcal{O}_X)$ the category of quasi-coherent $\mathcal{O}_X$-modules as defined in Modules, Definition 17.10.1. We have seen in Section 26.7 that the category $QCoh(\mathcal{O}_X)$ has a lot of good properties when X is affine. Since the property of being quasi-coherent is local on X, these properties are inherited by the category of quasi-coherent sheaves on any scheme X. We enumerate them here.

(1) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each affine open $U = \text{Spec}(R)$ is of the form $\widetilde{M}$ for some R-module M.

(2) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each of the members of an affine open covering is quasi-coherent.
(3) Any direct sum of quasi-coherent sheaves is quasi-coherent.
(4) Any colimit of quasi-coherent sheaves is quasi-coherent.
01LB (5) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.
(6) Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.
(7) Given a morphism of schemes $f : Y \to X$ the pullback of a quasi-coherent $\mathcal{O}_X$-module is a quasi-coherent $\mathcal{O}_Y$-module. See Modules, Lemma 17.10.4.
(8) Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent, see Modules, Lemma 17.16.6.
(9) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the tensor, symmetric and exterior algebras on $\mathcal{F}$ are quasi-coherent, see Modules, Lemma 17.21.6.
(10) Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F}$ is of finite presentation, then the internal hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, see Modules, Lemma 17.22.6 and (5) above.

On the other hand, it is in general not the case that the pushforward of a quasi-coherent module is quasi-coherent. Here is a case where this does hold.

01LC Lemma 26.24.1. Let $f : X \to S$ be a morphism of schemes. If f is quasi-compact and quasi-separated then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_S$-modules.

Proof. The question is local on S and hence we may assume that S is affine. Because X is quasi-compact we may write $X = \bigcup_{i=1}^n U_i$ with each U_i open affine. Because f is quasi-separated we may write $U_i \cap U_j = \bigcup_{k=1}^{n_{ij}} U_{ijk}$ for some affine open U_{ijk}, see Lemma 26.21.6. Denote $f_i : U_i \to S$ and $f_{ijk} : U_{ijk} \to S$ the restrictions of f. For any open V of S and any sheaf $\mathcal{F}$ on X we have

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}V) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i \mathcal{F}(f^{-1}V \cap U_i) \to \bigoplus\nolimits_{i,j,k} \mathcal{F}(f^{-1}V \cap U_{ijk})\right) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i f_{i,*}(\mathcal{F}|_{U_i})(V) \to \bigoplus\nolimits_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})(V)\right) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i f_{i,*}(\mathcal{F}|_{U_i}) \to \bigoplus\nolimits_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})\right)(V) \end{aligned}$$

In other words there is an exact sequence of sheaves

$$0 \to f_*\mathcal{F} \to \bigoplus f_{i,*}\mathcal{F}_i \to \bigoplus f_{ijk,*}\mathcal{F}_{ijk}$$

where $\mathcal{F}_i, \mathcal{F}_{ijk}$ denotes the restriction of $\mathcal{F}$ to the corresponding open. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module then $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_{U_i}$-module and $\mathcal{F}_{ijk}$ is a quasi-coherent $\mathcal{O}_{U_{ijk}}$-module. Hence by Lemma 26.7.3 we see that the second and third term of the exact sequence are quasi-coherent $\mathcal{O}_S$-modules. Thus we conclude that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module. $\square$

Using this we can characterize (closed) immersions of schemes as follows.

01LD Lemma 26.24.2. Let $f : X \to Y$ be a morphism of schemes. Suppose that
(1) f induces a homeomorphism of X with a closed subset of Y, and
(2) $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

Then f is a closed immersion of schemes.

Proof. Assume (1) and (2). By (1) the morphism f is quasi-compact (see Topology, Lemma 5.12.3). Conditions (1) and (2) imply conditions (1) and (2) of Lemma 26.23.7. Hence $f : X \to Y$ is a monomorphism. In particular, f is separated, see Lemma 26.23.3. Hence Lemma 26.24.1 above applies and we conclude that $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Therefore the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is quasi-coherent by Lemma 26.7.8. Since a quasi-coherent sheaf is locally generated by sections (see Modules, Definition 17.10.1) this implies that f is a closed immersion, see Definition 26.4.1. □

We can use this lemma to prove the following lemma.

02V0 Lemma 26.24.3. A composition of immersions of schemes is an immersion, a composition of closed immersions of schemes is a closed immersion, and a composition of open immersions of schemes is an open immersion.

Proof. This is clear for the case of open immersions since an open subspace of an open subspace is also an open subspace.

Suppose $a : Z \to Y$ and $b : Y \to X$ are closed immersions of schemes. We will verify that $c = b \circ a$ is also a closed immersion. The assumption implies that a and b are homeomorphisms onto closed subsets, and hence also $c = b \circ a$ is a homeomorphism onto a closed subset. Moreover, the map $\mathcal{O}_X \to c_*\mathcal{O}_Z$ is surjective since it factors as the composition of the surjective maps $\mathcal{O}_X \to b_*\mathcal{O}_Y$ and $b_*\mathcal{O}_Y \to b_*a_*\mathcal{O}_Z$ (surjective as b_* is exact, see Modules, Lemma 17.6.1). Hence by Lemma 26.24.2 above c is a closed immersion.

Finally, we come to the case of immersions. Suppose $a : Z \to Y$ and $b : Y \to X$ are immersions of schemes. This means there exist open subschemes $V \subset Y$ and $U \subset X$ such that $a(Z) \subset V$, $b(Y) \subset U$ and $a : Z \to V$ and $b : Y \to U$ are closed immersions. Since the topology on Y is induced from the topology on U we can find an open $U' \subset U$ such that $V = b^{-1}(U')$. Then we see that $Z \to V = b^{-1}(U') \to U'$ is a composition of closed immersions and hence a closed immersion. This proves that $Z \to X$ is an immersion and we win. □

26.25. Other chapters

CHAPTER 27

Constructions of Schemes

01LE 27.1. Introduction

01LF In this chapter we introduce ways of constructing schemes out of others. A basic reference is [DG67].

27.2. Relative glueing

01LG The following lemma is relevant in case we are trying to construct a scheme X over S, and we already know how to construct the restriction of X to the affine opens of S. The actual result is completely general and works in the setting of (locally) ringed spaces, although our proof is written in the language of schemes.

01LH Lemma 27.2.1. Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:

(1) For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.
(2) For $U, V \in \mathcal{B}$ with $V \subset U$ a morphism $\rho^U_V : X_V \to X_U$ over U.

Assume that

(a) each ρ^U_V induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,
(b) whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho^U_W = \rho^U_V \circ \rho^V_W$.

Then there exists a morphism $f : X \to S$ of schemes and isomorphisms $i_U : f^{-1}(U) \to X_U$ over $U \in \mathcal{B}$ such that for $V, U \in \mathcal{B}$ with $V \subset U$ the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{inclusion} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ^U_V. Moreover X is unique up to unique isomorphism over S.

Proof. To prove this we will use Schemes, Lemma 26.15.4. First we define a contravariant functor F from the category of schemes to the category of sets. Namely, for a scheme T we set

$$F(T) = \left\{ \begin{matrix} (g, \{h_U\}_{U\in\mathcal{B}}),\ g : T \to S,\ h_U : g^{-1}(U) \to X_U, \\ f_U \circ h_U = g|_{g^{-1}(U)},\ h_U|_{g^{-1}(V)} = \rho^U_V \circ h_V\ \forall\, V, U \in \mathcal{B}, V \subset U \end{matrix} \right\}.$$

The restriction mapping $F(T) \to F(T')$ given a morphism $T' \to T$ is just gotten by composition. For any $W \in \mathcal{B}$ we consider the subfunctor $F_W \subset F$ consisting of those systems $(g, \{h_U\})$ such that $g(T) \subset W$.

First we show F satisfies the sheaf property for the Zariski topology. Suppose that T is a scheme, $T = \bigcup V_i$ is an open covering, and $\xi_i \in F(V_i)$ is an element such that $\xi_i|_{V_i\cap V_j} = \xi_j|_{V_i\cap V_j}$. Say $\xi_i = (g_i, \{h_{i,U}\})$. Then we immediately see that the morphisms g_i glue to a unique global morphism $g : T \to S$. Moreover, it is clear that $g^{-1}(U) = \bigcup g_i^{-1}(U)$. Hence the morphisms $h_{i,U} : g_i^{-1}(U) \to X_U$ glue to a unique morphism $h_U : g^{-1}(U) \to X_U$. It is easy to verify that the system $(g, \{h_U\})$ is an element of $F(T)$. Hence F satisfies the sheaf property for the Zariski topology.

Next we verify that each F_W, $W \in \mathcal{B}$ is representable. Namely, we claim that the transformation of functors

$$F_W \longrightarrow \mathrm{Mor}(-, X_W), \ (g, \{h_U\}) \longmapsto h_W$$

is an isomorphism. To see this suppose that T is a scheme and $\alpha : T \to X_W$ is a morphism. Set $g = f_W \circ \alpha$. For any $U \in \mathcal{B}$ such that $U \subset W$ we can define $h_U : g^{-1}(U) \to X_U$ be the composition $(\rho^W_U)^{-1} \circ \alpha|_{g^{-1}(U)}$. This works because the image $\alpha(g^{-1}(U))$ is contained in $f_W^{-1}(U)$ and condition (a) of the lemma. It is clear that $f_U \circ h_U = g|_{g^{-1}(U)}$ for such a U. Moreover, if also $V \in \mathcal{B}$ and $V \subset U \subset W$, then $\rho^U_V \circ h_V = h_U|_{g^{-1}(V)}$ by property (b) of the lemma. We still have to define h_U for an arbitrary element $U \in \mathcal{B}$. Since $\mathcal{B}$ is a basis for the topology on S we can find an open covering $U \cap W = \bigcup U_i$ with $U_i \in \mathcal{B}$. Since g maps into W we have $g^{-1}(U) = g^{-1}(U \cap W) = \bigcup g^{-1}(U_i)$. Consider the morphisms $h_i = \rho^U_{U_i} \circ h_{U_i} : g^{-1}(U_i) \to X_U$. It is a simple matter to use condition (b) of the lemma to prove that $h_i|_{g^{-1}(U_i) \cap g^{-1}(U_j)} = h_j|_{g^{-1}(U_i) \cap g^{-1}(U_j)}$. Hence these morphisms glue to give the desired morphism $h_U : g^{-1}(U) \to X_U$. We omit the (easy) verification that the system $(g, \{h_U\})$ is an element of $F_W(T)$ which maps to α under the displayed arrow above.

Next, we verify each $F_W \subset F$ is representable by open immersions. This is clear from the definitions.

Finally we have to verify the collection $(F_W)_{W \in \mathcal{B}}$ covers F. This is clear by construction and the fact that $\mathcal{B}$ is a basis for the topology of S.

Let X be a scheme representing the functor F. Let $(f, \{i_U\}) \in F(X)$ be a "universal family". Since each F_W is representable by X_W (via the morphism of functors displayed above) we see that $i_W : f^{-1}(W) \to X_W$ is an isomorphism as desired. The lemma is proved. □

01LI Lemma 27.2.2. Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:

(1) For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.
(2) For every $U \in \mathcal{B}$ a quasi-coherent sheaf $\mathcal{F}_U$ over X_U.
(3) For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\rho^U_V : X_V \to X_U$.
(4) For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\theta^U_V : (\rho^U_V)^*\mathcal{F}_U \to \mathcal{F}_V$.

Assume that

(a) each ρ^U_V induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,
(b) each θ^U_V is an isomorphism,
(c) whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho^U_W = \rho^U_V \circ \rho^V_W$,
(d) whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\theta^U_W = \theta^V_W \circ (\rho^V_W)^*\theta^U_V$.

Then there exists a morphism of schemes $f : X \to S$ together with a quasi-coherent sheaf $\mathcal{F}$ on X and isomorphisms $i_U : f^{-1}(U) \to X_U$ and $\theta_U : i_U^*\mathcal{F}_U \to \mathcal{F}|_{f^{-1}(U)}$ over $U \in \mathcal{B}$ such that for $V, U \in \mathcal{B}$ with $V \subset U$ the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{inclusion} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ_V^U, and the composition

01LJ (27.2.2.1) $$(\rho_V^U)^*\mathcal{F}_U = (i_V^{-1})^*((i_U^*\mathcal{F}_U)|_{f^{-1}(V)}) \xrightarrow{\theta_U|_{f^{-1}(V)}} (i_V^{-1})^*(\mathcal{F}|_{f^{-1}(V)}) \xrightarrow{\theta_V^{-1}} \mathcal{F}_V$$

is equal to θ_V^U. Moreover $(X, \mathcal{F})$ is unique up to unique isomorphism over S.

Proof. By Lemma 27.2.1 we get the scheme X over S and the isomorphisms i_U. Set $\mathcal{F}'_U = i_U^*\mathcal{F}_U$ for $U \in \mathcal{B}$. This is a quasi-coherent $\mathcal{O}_{f^{-1}(U)}$-module. The maps

$$\mathcal{F}'_U|_{f^{-1}(V)} = i_U^*\mathcal{F}_U|_{f^{-1}(V)} = i_V^*(\rho_V^U)^*\mathcal{F}_U \xrightarrow{i_V^*\theta_V^U} i_V^*\mathcal{F}_V = \mathcal{F}'_V$$

define isomorphisms $(\theta')_V^U : \mathcal{F}'_U|_{f^{-1}(V)} \to \mathcal{F}'_V$ whenever $V \subset U$ are elements of $\mathcal{B}$. Condition (d) says exactly that this is compatible in case we have a triple of elements $W \subset V \subset U$ of $\mathcal{B}$. This allows us to get well defined isomorphisms

$$\varphi_{12} : \mathcal{F}'_{U_1}|_{f^{-1}(U_1 \cap U_2)} \longrightarrow \mathcal{F}'_{U_2}|_{f^{-1}(U_1 \cap U_2)}$$

whenever $U_1, U_2 \in \mathcal{B}$ by covering the intersection $U_1 \cap U_2 = \bigcup V_j$ by elements V_j of $\mathcal{B}$ and taking

$$\varphi_{12}|_{V_j} = \left((\theta')_{V_j}^{U_2}\right)^{-1} \circ (\theta')_{V_j}^{U_1}.$$

We omit the verification that these maps do indeed glue to a φ_{12} and we omit the verification of the cocycle condition of a glueing datum for sheaves (as in Sheaves, Section 6.33). By Sheaves, Lemma 6.33.2 we get our $\mathcal{F}$ on X. We omit the verification of (27.2.2.1). $\square$

01LK Remark 27.2.3. There is a functoriality property for the constructions explained in Lemmas 27.2.1 and 27.2.2. Namely, suppose given two collections of data $(f_U : X_U \to U, \rho_V^U)$ and $(g_U : Y_U \to U, \sigma_V^U)$ as in Lemma 27.2.1. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U. Functoriality means that this gives rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, where $f : X \to S$ is obtained from the datum $(f_U : X_U \to U, \rho_V^U)$ and $g : Y \to S$ is obtained from the datum $(g_U : Y_U \to U, \sigma_V^U)$.

Similarly, suppose given two collections of data $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$ as in Lemma 27.2.2. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U, and a morphism $\tau_U : h_U^*\mathcal{G}_U \to \mathcal{F}_U$ compatible with the maps θ_V^U and η_V^U. Functoriality means that these give rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, and a morphism $h^*\mathcal{G} \to \mathcal{F}$ restricting back to the maps h_U where $(f : X \to S, \mathcal{F})$ is obtained from the datum $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and where $(g : Y \to S, \mathcal{G})$ is obtained from the datum $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$.

We omit the verifications and we omit a suitable formulation of "equivalence of categories" between relative glueing data and relative objects.

27.3. Relative spectrum via glueing

01LL

01LM Situation 27.3.1. Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra. This means that $\mathcal{A}$ is a sheaf of $\mathcal{O}_S$-algebras which is quasi-coherent as an $\mathcal{O}_S$-module.

In this section we outline how to construct a morphism of schemes

$$\underline{\mathrm{Spec}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the spectra $\mathrm{Spec}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 27.2.1.

01LN Lemma 27.3.2. In Situation 27.3.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of rings $A' \to A$ induces a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and the diagram

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Spec}(A') \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

is cartesian.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 26.7.3 for example). The result follows from the description of the fibre product of affine schemes in Schemes, Lemma 26.6.7. $\square$

In particular the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ of the lemma is an open immersion.

01LO Lemma 27.3.3. In Situation 27.3.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and $\mathrm{Spec}(A') \to \mathrm{Spec}(A'')$ of Lemma 27.3.2 gives the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A'')$ of Lemma 27.3.2.

Proof. This follows as the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$ (because $\mathcal{A}$ is a sheaf). $\square$

01LP Lemma 27.3.4. In Situation 27.3.1. There exists a morphism of schemes

$$\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

(1) for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to \mathrm{Spec}(\mathcal{A}(U))$, and
(2) for $U \subset U' \subset S$ affine open the composition

$$\mathrm{Spec}(\mathcal{A}(U)) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} \mathrm{Spec}(\mathcal{A}(U'))$$

is the open immersion of Lemma 27.3.2 above.

Proof. Follows immediately from Lemmas 27.2.1, 27.3.2, and 27.3.3. $\square$

27.4. Relative spectrum as a functor

01LQ We place ourselves in Situation 27.3.1, i.e., S is a scheme and $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras.

For any $f : T \to S$ the pullback $f^*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_T$-algebras. We are going to consider pairs $(f : T \to S, \varphi)$ where f is a morphism of schemes and $\varphi : f^*\mathcal{A} \to \mathcal{O}_T$ is a morphism of $\mathcal{O}_T$-algebras. Note that this is the same as giving a $f^{-1}\mathcal{O}_S$-algebra homomorphism $\varphi : f^{-1}\mathcal{A} \to \mathcal{O}_T$, see Sheaves, Lemma 6.20.2. This is also the same as giving an $\mathcal{O}_S$-algebra map $\varphi : \mathcal{A} \to f_*\mathcal{O}_T$, see Sheaves, Lemma 6.24.7. We will use all three ways of thinking about φ, without further mention.

Given such a pair $(f : T \to S, \varphi)$ and a morphism $a : T' \to T$ we get a second pair $(f' = f \circ a, \varphi' = a^*\varphi)$ which we call the pullback of (f, φ). One way to describe $\varphi' = a^*\varphi$ is as the composition $\mathcal{A} \to f_*\mathcal{O}_T \to f'_*\mathcal{O}_{T'}$ where the second map is $f_*a^\sharp$ with $a^\sharp : \mathcal{O}_T \to a_*\mathcal{O}_{T'}$. In this way we have defined a functor

01LR (27.4.0.1)
$$\begin{matrix} F : Sch^{opp} & \longrightarrow & Sets \\ T & \longmapsto & F(T) = \{\text{pairs } (f, \varphi) \text{ as above}\} \end{matrix}$$

01LS **Lemma 27.4.1.** In Situation 27.3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F' be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism

$$F' \cong h_{S'} \times_{h_S} F$$

of functors.

Proof. A pair $(f' : T \to S', \varphi' : (f')^*\mathcal{A}' \to \mathcal{O}_T)$ is the same as a pair $(f, \varphi : f^*\mathcal{A} \to \mathcal{O}_T)$ together with a factorization of f as $f = g \circ f'$. Namely with this notation we have $(f')^*\mathcal{A}' = (f')^*g^*\mathcal{A} = f^*\mathcal{A}$. Hence the lemma. □

01LT **Lemma 27.4.2.** In Situation 27.3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. If S is affine, then F is representable by the affine scheme $\mathrm{Spec}(\Gamma(S, \mathcal{A}))$.

Proof. Write $S = \mathrm{Spec}(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is an R-algebra and $\mathcal{A} = \widetilde{A}$. The ring map $R \to A$ gives rise to a canonical map

$$f_{univ} : \mathrm{Spec}(A) \longrightarrow S = \mathrm{Spec}(R).$$

We have $f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A}$ by Schemes, Lemma 26.7.3. Hence there is a canonical map

$$\varphi_{univ} : f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A} \longrightarrow \widetilde{A} = \mathcal{O}_{\mathrm{Spec}(A)}$$

coming from the A-module map $A \otimes_R A \to A$, $a \otimes a' \mapsto aa'$. We claim that the pair $(f_{univ}, \varphi_{univ})$ represents F in this case. In other words we claim that for any scheme T the map

$$\mathrm{Mor}(T, \mathrm{Spec}(A)) \longrightarrow \{\text{pairs } (f, \varphi)\}, \quad a \longmapsto (f_{univ} \circ a, a^*\varphi_{univ})$$

is bijective.

Let us construct the inverse map. For any pair $(f : T \to S, \varphi)$ we get the induced ring map

$$A = \Gamma(S, \mathcal{A}) \xrightarrow{f^*} \Gamma(T, f^*\mathcal{A}) \xrightarrow{\varphi} \Gamma(T, \mathcal{O}_T)$$

This induces a morphism of schemes $T \to \mathrm{Spec}(A)$ by Schemes, Lemma 26.6.4.

The verification that this map is inverse to the map displayed above is omitted. □

01LU Lemma 27.4.3. In Situation 27.3.1. The functor F is representable by a scheme.

Proof. We are going to use Schemes, Lemma 26.15.4.

First we check that F satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i \in I} U_i$ is an open covering, and that $(f_i, \varphi_i) \in F(U_i)$ such that $(f_i, \varphi_i)|_{U_i \cap U_j} = (f_j, \varphi_j)|_{U_i \cap U_j}$. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see Schemes, Section 26.14. Thus $f_i^*\mathcal{A} = f^*\mathcal{A}|_{U_i}$ and by assumption the morphisms φ_i agree on $U_i \cap U_j$. Hence by Sheaves, Section 6.33 these glue to a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. This proves that F satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i \in I} U_i$ be an affine open covering. Let $F_i \subset F$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each F_i is representable. This is the case because F_i is identified with the functor associated to U_i equipped with the quasi-coherent $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$, by Lemma 27.4.1. Thus the result follows from Lemma 27.4.2.

Next we show that $F_i \subset F$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of F_i that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_i(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_i)_{i \in I}$ covers F. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i \in I} U_i$ is an open covering of S we see that $T = \bigcup_{i \in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_i(V_i)$. This finishes the proof of the lemma. $\square$

01LV Lemma 27.4.4. In Situation 27.3.1. The scheme $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ constructed in Lemma 27.3.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.

Proof. Let $X \to S$ be the scheme representing the functor F. Consider the sheaf of $\mathcal{O}_S$-algebras $\mathcal{R} = \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$. By construction of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ we have isomorphisms $\mathcal{A}(U) \to \mathcal{R}(U)$ for every affine open $U \subset S$; this follows from Lemma 27.3.4 part (1). For $U \subset U' \subset S$ open these isomorphisms are compatible with the restriction mappings; this follows from Lemma 27.3.4 part (2). Hence by Sheaves, Lemma 6.30.13 these isomorphisms result from an isomorphism of $\mathcal{O}_S$-algebras $\varphi : \mathcal{A} \to \mathcal{R}$. Hence this gives an element $(\underline{\mathrm{Spec}}_S(\mathcal{A}), \varphi) \in F(\underline{\mathrm{Spec}}_S(\mathcal{A}))$. Since X represents the functor F we get a corresponding morphism of schemes $can : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to X$ over S.

Let $U \subset S$ be any affine open. Let $F_U \subset F$ be the subfunctor of F corresponding to pairs (f, φ) over schemes T with $f(T) \subset U$. Clearly the base change X_U represents F_U. Moreover, F_U is represented by $\mathrm{Spec}(\mathcal{A}(U)) = \pi^{-1}(U)$ according to Lemma 27.4.2. In other words $X_U \cong \pi^{-1}(U)$. We omit the verification that this identification is brought about by the base change of the morphism can to U. $\square$

01LW Definition 27.4.5. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. The relative spectrum of $\mathcal{A}$ over S, or simply the spectrum of $\mathcal{A}$ over S is the scheme constructed in Lemma 27.3.4 which represents the functor F (27.4.0.1), see Lemma 27.4.4. We denote it $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$. The "universal family" is a morphism of $\mathcal{O}_S$-algebras

$$\mathcal{A} \longrightarrow \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$$

The following lemma says among other things that forming the relative spectrum commutes with base change.

01LX Lemma 27.4.6. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Let $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ be the relative spectrum of $\mathcal{A}$ over S.

(1) For every affine open $U \subset S$ the inverse image $\pi^{-1}(U)$ is affine.
(2) For every morphism $g : S' \to S$ we have $S' \times_S \underline{\mathrm{Spec}}_S(\mathcal{A}) = \underline{\mathrm{Spec}}_{S'}(g^*\mathcal{A})$.
(3) The universal map

$$\mathcal{A} \longrightarrow \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$$

is an isomorphism of $\mathcal{O}_S$-algebras.

Proof. Part (1) comes from the description of the relative spectrum by glueing, see Lemma 27.3.4. Part (2) follows immediately from Lemma 27.4.1. Part (3) follows because it is local on S and it is clear in case S is affine by Lemma 27.4.2 for example. $\square$

01LY Lemma 27.4.7. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. By Schemes, Lemma 26.24.1 the sheaf $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. There is a canonical morphism

$$can : X \longrightarrow \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$$

of schemes over S. For any affine open $U \subset S$ the restriction $can|_{f^{-1}(U)}$ is identified with the canonical morphism

$$f^{-1}(U) \longrightarrow \mathrm{Spec}(\Gamma(f^{-1}(U), \mathcal{O}_X))$$

coming from Schemes, Lemma 26.6.4.

Proof. The morphism comes, via the definition of $\underline{\mathrm{Spec}}$ as the scheme representing the functor F, from the canonical map $\varphi : f^*f_*\mathcal{O}_X \to \mathcal{O}_X$ (which by adjointness of push and pull corresponds to $\mathrm{id} : f_*\mathcal{O}_X \to f_*\mathcal{O}_X$). The statement on the restriction to $f^{-1}(U)$ follows from the description of the relative spectrum over affines, see Lemma 27.4.2. $\square$

27.5. Affine n-space

01LZ As an application of the relative spectrum we define affine n-space over a base scheme S as follows. For any integer $n \geq 0$ we can consider the quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n]$. It is quasi-coherent because as a sheaf of $\mathcal{O}_S$-modules it is just the direct sum of copies of $\mathcal{O}_S$ indexed by multi-indices.

01M0 Definition 27.5.1. Let S be a scheme and $n \geq 0$. The scheme

$$\mathbf{A}^n_S = \underline{\mathrm{Spec}}_S(\mathcal{O}_S[T_1, \ldots, T_n])$$

over S is called affine n-space over S. If $S = \mathrm{Spec}(R)$ is affine then we also call this affine n-space over R and we denote it $\mathbf{A}^n_R$.

Note that $\mathbf{A}^n_R = \mathrm{Spec}(R[T_1, \ldots, T_n])$. For any morphism $g : S' \to S$ of schemes we have $g^*\mathcal{O}_S[T_1, \ldots, T_n] = \mathcal{O}_{S'}[T_1, \ldots, T_n]$ and hence $\mathbf{A}^n_{S'} = S' \times_S \mathbf{A}^n_S$ is the base change. Therefore an alternative definition of affine n-space is the formula

$$\mathbf{A}^n_S = S \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{A}^n_{\mathbf{Z}}.$$

Also, a morphism from an S-scheme $f : X \to S$ to $\mathbf{A}^n_S$ is given by a homomorphism of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n] \to f_*\mathcal{O}_X$. This is clearly the same thing as giving the images of the T_i. In other words, a morphism from X to $\mathbf{A}^n_S$ over S is the same as giving n elements $h_1, \ldots, h_n \in \Gamma(X, \mathcal{O}_X)$.

27.6. Vector bundles

01M1 Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 17.21.6 the symmetric algebra $\mathrm{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Hence it makes sense to apply the construction of the previous section to it.

01M2 Definition 27.6.1. Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[1]. The vector bundle associated to $\mathcal{E}$ is

$$\mathbf{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_S(\mathrm{Sym}(\mathcal{E})).$$

The vector bundle associated to $\mathcal{E}$ comes with a bit of extra structure. Namely, we have a grading

$$\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})} = \bigoplus\nolimits_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}).$$

which turns $\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})}$ into a graded $\mathcal{O}_S$-algebra. Conversely, we can recover $\mathcal{E}$ from the degree 1 part of this. Thus we define an abstract vector bundle as follows.

062M Definition 27.6.2. Let S be a scheme. A vector bundle $\pi : V \to S$ over S is an affine morphism of schemes such that $\pi_*\mathcal{O}_V$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_V = \bigoplus_{n \geq 0} \mathcal{E}_n$ such that $\mathcal{E}_0 = \mathcal{O}_S$ and such that the maps

$$\mathrm{Sym}^n(\mathcal{E}_1) \longrightarrow \mathcal{E}_n$$

are isomorphisms for all $n \geq 0$. A morphism of vector bundles over S is a morphism $f : V \to V'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{V'} \longrightarrow \pi_*\mathcal{O}_V$$

is compatible with the given gradings.

An example of a vector bundle over S is affine n-space $\mathbf{A}^n_S$ over S, see Definition 27.5.1. This is true because $\mathcal{O}_S[T_1, \ldots, T_n] = \mathrm{Sym}(\mathcal{O}_S^{\oplus n})$.

062N Lemma 27.6.3. The category of vector bundles over a scheme S is anti-equivalent to the category of quasi-coherent $\mathcal{O}_S$-modules.

Proof. Omitted. Hint: In one direction one uses the functor $\underline{\mathrm{Spec}}_S(\mathrm{Sym}^*_{\mathcal{O}_S}(-))$ and in the other the functor $(\pi : V \to S) \rightsquigarrow (\pi_*\mathcal{O}_V)_1$ where the subscript indicates we take the degree 1 part. $\square$

[1] The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [DG67, II, Definition 1.7.8].

27.7. Cones

062P In algebraic geometry cones correspond to graded algebras. By our conventions a graded ring or algebra A comes with a grading $A = \bigoplus_{d \geq 0} A_d$ by the nonnegative integers, see Algebra, Section 10.56.

062Q Definition 27.7.1. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Assume that $\mathcal{O}_S \to \mathcal{A}_0$ is an isomorphism[2]. The cone associated to $\mathcal{A}$ or the affine cone associated to $\mathcal{A}$ is

$$C(\mathcal{A}) = \underline{\text{Spec}}_S(\mathcal{A}).$$

The cone associated to a graded sheaf of $\mathcal{O}_S$-algebras comes with a bit of extra structure. Namely, we obtain a grading

$$\pi_*\mathcal{O}_{C(\mathcal{A})} = \bigoplus\nolimits_{n \geq 0} \mathcal{A}_n$$

Thus we can define an abstract cone as follows.

062R Definition 27.7.2. Let S be a scheme. A cone $\pi : C \to S$ over S is an affine morphism of schemes such that $\pi_*\mathcal{O}_C$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_C = \bigoplus_{n \geq 0} \mathcal{A}_n$ such that $\mathcal{A}_0 = \mathcal{O}_S$. A morphism of cones from $\pi : C \to S$ to $\pi' : C' \to S$ is a morphism $f : C \to C'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{C'} \longrightarrow \pi_*\mathcal{O}_C$$

is compatible with the given gradings.

Any vector bundle is an example of a cone. In fact the category of vector bundles over S is a full subcategory of the category of cones over S.

27.8. Proj of a graded ring

01M3 In this section we construct Proj of a graded ring following [DG67, II, Section 2].

Let S be a graded ring. Consider the topological space $\text{Proj}(S)$ associated to S, see Algebra, Section 10.57. We will endow this space with a sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ such that the resulting pair $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ will be a scheme.

Recall that $\text{Proj}(S)$ has a basis of open sets $D_+(f)$, $f \in S_d$, $d \geq 1$ which we call standard opens, see Algebra, Section 10.57. This terminology will always imply that f is homogeneous of positive degree even if we forget to mention it. In addition, the intersection of two standard opens is another: $D_+(f) \cap D_+(g) = D_+(fg)$, for $f, g \in S$ homogeneous of positive degree.

01M4 Lemma 27.8.1. Let S be a graded ring. Let $f \in S$ homogeneous of positive degree.

(1) If $g \in S$ homogeneous of positive degree and $D_+(g) \subset D_+(f)$, then
   (a) f is invertible in S_g, and $f^{\deg(g)}/g^{\deg(f)}$ is invertible in $S_{(g)}$,
   (b) $g^e = af$ for some $e \geq 1$ and $a \in S$ homogeneous,
   (c) there is a canonical S-algebra map $S_f \to S_g$,
   (d) there is a canonical S_0-algebra map $S_{(f)} \to S_{(g)}$ compatible with the map $S_f \to S_g$,
   (e) the map $S_{(f)} \to S_{(g)}$ induces an isomorphism

$$(S_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong S_{(g)},$$

[2]Often one imposes the assumption that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{O}_S$. We do not assume this in order to be consistent with [DG67, II, (8.3.1)].

(f) these maps induce a commutative diagram of topological spaces

$$\begin{tikzcd} D_+(g) \arrow[d] & \{\mathbf{Z}\text{-graded primes of } S_g\} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Spec}(S_{(g)}) \arrow[d] \\ D_+(f) & \{\mathbf{Z}\text{-graded primes of } S_f\} \arrow[l] \arrow[r] & \mathrm{Spec}(S_{(f)}) \end{tikzcd}$$

where the horizontal maps are homeomorphisms and the vertical maps are open immersions,

(g) there are compatible canonical S_f and $S_{(f)}$-module maps $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ for any graded S-module M, and

(h) the map $M_{(f)} \to M_{(g)}$ induces an isomorphism

$$(M_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong M_{(g)}.$$

(2) Any open covering of $D_+(f)$ can be refined to a finite open covering of the form $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$.

(3) Let $g_1, \ldots, g_n \in S$ be homogeneous of positive degree. Then $D_+(f) \subset \bigcup D_+(g_i)$ if and only if $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$.

Proof. Recall that $D_+(g) = \mathrm{Spec}(S_{(g)})$ with identification given by the ring maps $S \to S_g \leftarrow S_{(g)}$, see Algebra, Lemma 10.57.3. Thus $f^{\deg(g)}/g^{\deg(f)}$ is an element of $S_{(g)}$ which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 10.17.2. We conclude that (a) holds. Write the inverse of f in S_g as a/g^d. We may replace a by its homogeneous part of degree $d\deg(g) - \deg(f)$. This means $g^d - af$ is annihilated by a power of g, whence $g^e = af$ for some $a \in S$ homogeneous of degree $e\deg(g) - \deg(f)$. This proves (b). For (c), the map $S_f \to S_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. This clearly induces a map of the subrings $S_{(f)} \to S_{(g)}$ of degree zero elements as well. We can similarly define $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ by mapping x/f^n to $a^n x/g^{ne}$. The statements writing $S_{(g)}$ resp. $M_{(g)}$ as principal localizations of $S_{(f)}$ resp. $M_{(f)}$ are clear from the formulas above. The maps in the commutative diagram of topological spaces correspond to the ring maps given above. The horizontal arrows are homeomorphisms by Algebra, Lemma 10.57.3. The vertical arrows are open immersions since the left one is the inclusion of an open subset.

The open $D_+(f)$ is quasi-compact because it is homeomorphic to $\mathrm{Spec}(S_{(f)})$, see Algebra, Lemma 10.17.10. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 10.17.2. $\square$

In Sheaves, Section 6.30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 6.30.6 and 6.30.9. Moreover, we showed in Sheaves, Lemma 6.30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

01M5 Definition 27.8.2. Let S be a graded ring. Suppose that $D_+(f) \subset \text{Proj}(S)$ is a standard open. A standard open covering of $D_+(f)$ is a covering $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, where $g_1, \ldots, g_n \in S$ are homogeneous of positive degree.

Let S be a graded ring. Let M be a graded S-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \text{Proj}(S)$ is a standard open. If $f, g \in S$ are homogeneous of positive degree such that $D_+(f) = D_+(g)$, then by Lemma 27.8.1 above there are canonical maps $M_{(f)} \to M_{(g)}$ and $M_{(g)} \to M_{(f)}$ which are mutually inverse. Hence we may choose any f such that $U = D_+(f)$ and define

$$\widetilde{M}(U) = M_{(f)}.$$

Note that if $D_+(g) \subset D_+(f)$, then by Lemma 27.8.1 above we have a canonical map

$$\widetilde{M}(D_+(f)) = M_{(f)} \longrightarrow M_{(g)} = \widetilde{M}(D_+(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = S$, then $\widetilde{S}$ is a presheaf of rings on the basis of standard opens. And for general M we see that $\widetilde{M}$ is a presheaf of $\widetilde{S}$-modules on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \text{Proj}(S)$. Suppose that x corresponds to the homogeneous prime ideal $\mathfrak{p} \subset S$. By definition of the stalk we see that

$$\widetilde{M}_x = \text{colim}_{f \in S_d, d > 0, f \not\in \mathfrak{p}} M_{(f)}$$

Here the set $\{f \in S_d, d > 0, f \not\in \mathfrak{p}\}$ is preordered by the rule $f \geq f' \Leftrightarrow D_+(f) \subset D_+(f')$. If $f_1, f_2 \in S \setminus \mathfrak{p}$ are homogeneous of positive degree, then we have $f_1 f_2 \geq f_1$ in this ordering. In Algebra, Section 10.57 we defined $M_{(\mathfrak{p})}$ as the module whose elements are fractions x/f with x, f homogeneous, $\deg(x) = \deg(f)$, $f \not\in \mathfrak{p}$. Since $\mathfrak{p} \in \text{Proj}(S)$ there exists at least one $f_0 \in S$ homogeneous of positive degree with $f_0 \not\in \mathfrak{p}$. Hence $x/f = f_0 x / f f_0$ and we see that we may always assume the denominator of an element in $M_{(\mathfrak{p})}$ has positive degree. From these remarks it follows easily that

$$\widetilde{M}_x = M_{(\mathfrak{p})}.$$

Next, we check the sheaf condition for the standard open coverings. If $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(g_i)} \to \bigoplus M_{(g_i g_j)}.$$

Note that $D_+(g_i) = D_+(f g_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(f g_i)} \to \bigoplus M_{(f g_i g_j)}.$$

By Lemma 27.8.1 we see that $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$, and that the modules $M_{(f g_i)}$, $M_{(f g_i g_j)}$ are the principal localizations of the $S_{(f)}$-module $M_{(f)}$ at these elements and their products. Thus we may apply Algebra, Lemma 10.24.1 to the module $M_{(f)}$ over $S_{(f)}$ and the elements $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 6.30 that there exists a unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the standard opens. Note that by our computation of stalks above and Algebra, Lemma 10.57.5 the stalks of this sheaf of rings are all local rings.

Similarly, for any graded S-module M there exists a unique sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 6.30.12.

01M6 **Definition 27.8.3.** Let S be a graded ring.

(1) The structure sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$ of the homogeneous spectrum of S is the unique sheaf of rings $\mathcal{O}_{\mathrm{Proj}(S)}$ which agrees with $\widetilde{S}$ on the basis of standard opens.
(2) The locally ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is called the homogeneous spectrum of S and denoted $\mathrm{Proj}(S)$.
(3) The sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules extending $\widetilde{M}$ to all opens of $\mathrm{Proj}(S)$ is called the sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

01M7 **Lemma 27.8.4.** Let S be a graded ring. Let M be a graded S-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules associated to M.

(1) For every $f \in S$ homogeneous of positive degree we have
$$\Gamma(D_+(f), \mathcal{O}_{\mathrm{Proj}(S)}) = S_{(f)}.$$
(2) For every $f \in S$ homogeneous of positive degree we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ as an $S_{(f)}$-module.
(3) Whenever $D_+(g) \subset D_+(f)$ the restriction mappings on $\mathcal{O}_{\mathrm{Proj}(S)}$ and $\widetilde{M}$ are the maps $S_{(f)} \to S_{(g)}$ and $M_{(f)} \to M_{(g)}$ from Lemma 27.8.1.
(4) Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in \mathrm{Proj}(S)$ be the corresponding point. We have $\mathcal{O}_{\mathrm{Proj}(S),x} = S_{(\mathfrak{p})}$.
(5) Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in \mathrm{Proj}(S)$ be the corresponding point. We have $\mathcal{F}_x = M_{(\mathfrak{p})}$ as an $S_{(\mathfrak{p})}$-module.

01M8 (6) There is a canonical ring map $S_0 \longrightarrow \Gamma(\mathrm{Proj}(S), \widetilde{S})$ and a canonical S_0-module map $M_0 \longrightarrow \Gamma(\mathrm{Proj}(S), \widetilde{M})$ compatible with the descriptions of sections over standard opens and stalks above.

Moreover, all these identifications are functorial in the graded S-module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of graded S-modules to the category of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules.

Proof. Assertions (1) - (5) are clear from the discussion above. We see (6) since there are canonical maps $M_0 \to M_{(f)}$, $x \mapsto x/1$ compatible with the restriction maps described in (3). The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{(\mathfrak{p})}$ is exact (see Algebra, Lemma 10.57.5) and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 17.3.1. $\square$

01M9 **Remark 27.8.5.** The map from M_0 to the global sections of $\widetilde{M}$ is generally far from being an isomorphism. A trivial example is to take $S = k[x, y, z]$ with $1 = \deg(x) = \deg(y) = \deg(z)$ (or any number of variables) and to take $M = S/(x^{100}, y^{100}, z^{100})$. It is easy to see that $\widetilde{M} = 0$, but $M_0 = k$.

01MA Lemma 27.8.6. Let S be a graded ring. Let $f \in S$ be homogeneous of positive degree. Suppose that $D(g) \subset \text{Spec}(S_{(f)})$ is a standard open. Then there exists an $h \in S$ homogeneous of positive degree such that $D(g)$ corresponds to $D_+(h) \subset D_+(f)$ via the homeomorphism of Algebra, Lemma 10.57.3. In fact we can take h such that $g = h/f^n$ for some n.

Proof. Write $g = h/f^n$ for some h homogeneous of positive degree and some $n \geq 1$. If $D_+(h)$ is not contained in $D_+(f)$ then we replace h by hf and n by $n+1$. Then h has the required shape and $D_+(h) \subset D_+(f)$ corresponds to $D(g) \subset \text{Spec}(S_{(f)})$. $\square$

01MB Lemma 27.8.7. Let S be a graded ring. The locally ringed space $\text{Proj}(S)$ is a scheme. The standard opens $D_+(f)$ are affine opens. For any graded S-module M the sheaf $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules.

Proof. Consider a standard open $D_+(f) \subset \text{Proj}(S)$. By Lemmas 27.8.1 and 27.8.4 we have $\Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}) = S_{(f)}$, and we have a homeomorphism $\varphi : D_+(f) \to \text{Spec}(S_{(f)})$. For any standard open $D(g) \subset \text{Spec}(S_{(f)})$ we may pick an $h \in S_+$ as in Lemma 27.8.6. Then $\varphi^{-1}(D(g)) = D_+(h)$, and by Lemmas 27.8.4 and 27.8.1 we see

$$\Gamma(D_+(h), \mathcal{O}_{\text{Proj}(S)}) = S_{(h)} = (S_{(f)})_{h^{\deg(f)}/f^{\deg(h)}} = (S_{(f)})_g = \Gamma(D(g), \mathcal{O}_{\text{Spec}(S_{(f)})}).$$

Thus the restriction of $\mathcal{O}_{\text{Proj}(S)}$ to $D_+(f)$ corresponds via the homeomorphism φ exactly to the sheaf $\mathcal{O}_{\text{Spec}(S_{(f)})}$ as defined in Schemes, Section 26.5. We conclude that $D_+(f)$ is an affine scheme isomorphic to $\text{Spec}(S_{(f)})$ via φ and hence that $\text{Proj}(S)$ is a scheme.

In exactly the same way we show that $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules. Namely, the argument above will show that

$$\widetilde{M}|_{D_+(f)} \cong \varphi^* \left(\widetilde{M_{(f)}}\right)$$

which shows that $\widetilde{M}$ is quasi-coherent. $\square$

01MC Lemma 27.8.8. Let S be a graded ring. The scheme $\text{Proj}(S)$ is separated.

Proof. We have to show that the canonical morphism $\text{Proj}(S) \to \text{Spec}(\mathbf{Z})$ is separated. We will use Schemes, Lemma 26.21.7. Thus it suffices to show given any pair of standard opens $D_+(f)$ and $D_+(g)$ that $D_+(f) \cap D_+(g) = D_+(fg)$ is affine (clear) and that the ring map

$$S_{(f)} \otimes_{\mathbf{Z}} S_{(g)} \longrightarrow S_{(fg)}$$

is surjective. Any element s in $S_{(fg)}$ is of the form $s = h/(f^n g^m)$ with $h \in S$ homogeneous of degree $n \deg(f) + m \deg(g)$. We may multiply h by a suitable monomial $f^i g^j$ and assume that $n = n' \deg(g)$, and $m = m' \deg(f)$. Then we can rewrite s as $s = h/f^{(n'+m')\deg(g)} \cdot f^{m' \deg(g)}/g^{m' \deg(f)}$. So s is indeed in the image of the displayed arrow. $\square$

01MD Lemma 27.8.9. Let S be a graded ring. The scheme $\text{Proj}(S)$ is quasi-compact if and only if there exist finitely many homogeneous elements $f_1, \ldots, f_n \in S_+$ such that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$. In this case $\text{Proj}(S) = D_+(f_1) \cup \ldots \cup D_+(f_n)$.

Proof. Given such a collection of elements the standard affine opens $D_+(f_i)$ cover $\text{Proj}(S)$ by Algebra, Lemma 10.57.3. Conversely, if $\text{Proj}(S)$ is quasi-compact, then we may cover it by finitely many standard opens $D_+(f_i)$, $i = 1, \ldots, n$ and we see that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$ by the lemma referenced above. $\square$

01ME Lemma 27.8.10. Let S be a graded ring. The scheme $\text{Proj}(S)$ has a canonical morphism towards the affine scheme $\text{Spec}(S_0)$, agreeing with the map on topological spaces coming from Algebra, Definition 10.57.1.

Proof. We saw above that our construction of $\widetilde{S}$, resp. $\widetilde{M}$ gives a sheaf of S_0-algebras, resp. S_0-modules. Hence we get a morphism by Schemes, Lemma 26.6.4. This morphism, when restricted to $D_+(f)$ comes from the canonical ring map $S_0 \to S_{(f)}$. The maps $S \to S_f$, $S_{(f)} \to S_f$ are S_0-algebra maps, see Lemma 27.8.1. Hence if the homogeneous prime $\mathfrak{p} \subset S$ corresponds to the $\mathbf{Z}$-graded prime $\mathfrak{p}' \subset S_f$ and the (usual) prime $\mathfrak{p}'' \subset S_{(f)}$, then each of these has the same inverse image in S_0. $\square$

01MF Lemma 27.8.11. Let S be a graded ring. If S is finitely generated as an algebra over S_0, then the morphism $\text{Proj}(S) \to \text{Spec}(S_0)$ satisfies the existence and uniqueness parts of the valuative criterion, see Schemes, Definition 26.20.3.

Proof. The uniqueness part follows from the fact that $\text{Proj}(S)$ is separated (Lemma 27.8.8 and Schemes, Lemma 26.22.1). Choose $x_i \in S_+$ homogeneous, $i = 1, \ldots, n$ which generate S over S_0. Let $d_i = \deg(x_i)$ and set $d = \text{lcm}\{d_i\}$. Suppose we are given a diagram

$$\begin{tikzcd} \text{Spec}(K) \arrow[r] \arrow[d] & \text{Proj}(S) \arrow[d] \\ \text{Spec}(A) \arrow[r] & \text{Spec}(S_0) \end{tikzcd}$$

as in Schemes, Definition 26.20.3. Denote $v : K^* \to \Gamma$ the valuation of A, see Algebra, Definition 10.50.13. We may choose an $f \in S_+$ homogeneous such that $\text{Spec}(K)$ maps into $D_+(f)$. Then we get a commutative diagram of ring maps

$$\begin{tikzcd} K & S_{(f)} \arrow[l, "\varphi"'] \\ A \arrow[u] & S_0 \arrow[l] \arrow[u] \end{tikzcd}$$

After renumbering we may assume that $\varphi(x_i^{\deg(f)}/f^{d_i})$ is nonzero for $i = 1, \ldots, r$ and zero for $i = r+1, \ldots, n$. Since the open sets $D_+(x_i)$ cover $\text{Proj}(S)$ we see that $r \geq 1$. Let $i_0 \in \{1, \ldots, r\}$ be an index minimizing $\gamma_i = (d/d_i)v(\varphi(x_i^{\deg(f)}/f^{d_i}))$ in Γ. For convenience set $x_0 = x_{i_0}$ and $d_0 = d_{i_0}$. The ring map φ factors though a map $\varphi' : S_{(fx_0)} \to K$ which gives a ring map $S_{(x_0)} \to S_{(fx_0)} \to K$. The algebra $S_{(x_0)}$ is generated over S_0 by the elements $x_1^{e_1} \ldots x_n^{e_n}/x_0^{e_0}$, where $\sum e_i d_i = e_0 d_0$. If $e_i > 0$ for some $i > r$, then $\varphi'(x_1^{e_1} \ldots x_n^{e_n}/x_0^{e_0}) = 0$. If $e_i = 0$ for $i > r$, then we have

$$\begin{aligned}\deg(f)v(\varphi'(x_1^{e_1}\ldots x_r^{e_r}/x_0^{e_0})) &= v(\varphi'(x_1^{e_1\deg(f)}\ldots x_r^{e_r\deg(f)}/x_0^{e_0\deg(f)}))\\ &= \sum e_i v(\varphi'(x_i^{\deg(f)}/f^{d_i})) - e_0 v(\varphi'(x_0^{\deg(f)}/f^{d_0}))\\ &= \sum e_i d_i \gamma_i - e_0 d_0 \gamma_0\\ &\geq \sum e_i d_i \gamma_0 - e_0 d_0 \gamma_0 = 0\end{aligned}$$

because γ_0 is minimal among the γ_i. This implies that $S_{(x_0)}$ maps into A via φ'. The corresponding morphism of schemes $\text{Spec}(A) \to \text{Spec}(S_{(x_0)}) = D_+(x_0) \subset \text{Proj}(S)$ provides the morphism fitting into the first commutative diagram of this proof. □

We saw in the proof of Lemma 27.8.11 that, under the hypotheses of that lemma, the morphism $\text{Proj}(S) \to \text{Spec}(S_0)$ is quasi-compact as well. Hence (by Schemes, Proposition 26.20.6) we see that $\text{Proj}(S) \to \text{Spec}(S_0)$ is universally closed in the situation of the lemma. We give several examples showing these results do not hold without some assumption on the graded ring S.

01MG Example 27.8.12. Let $\mathbf{C}[X_1, X_2, X_3, \ldots]$ be the graded $\mathbf{C}$-algebra with each X_i in degree 1. Consider the ring map

$$\mathbf{C}[X_1, X_2, X_3, \ldots] \longrightarrow \mathbf{C}[t^\alpha; \alpha \in \mathbf{Q}_{\geq 0}]$$

which maps X_i to $t^{1/i}$. The right hand side becomes a valuation ring A upon localization at the ideal $\mathfrak{m} = (t^\alpha; \alpha > 0)$. Let K be the fraction field of A. The above gives a morphism $\text{Spec}(K) \to \text{Proj}(\mathbf{C}[X_1, X_2, X_3, \ldots])$ which does not extend to a morphism defined on all of $\text{Spec}(A)$. The reason is that the image of $\text{Spec}(A)$ would be contained in one of the $D_+(X_i)$ but then X_{i+1}/X_i would map to an element of A which it doesn't since it maps to $t^{1/(i+1)-1/i}$.

01MH Example 27.8.13. Let $R = \mathbf{C}[t]$ and

$$S = R[X_1, X_2, X_3, \ldots]/(X_i^2 - tX_{i+1}).$$

The grading is such that $R = S_0$ and $\deg(X_i) = 2^{i-1}$. Note that if $\mathfrak{p} \in \text{Proj}(S)$ then $t \notin \mathfrak{p}$ (otherwise $\mathfrak{p}$ has to contain all of the X_i which is not allowed for an element of the homogeneous spectrum). Thus we see that $D_+(X_i) = D_+(X_{i+1})$ for all i. Hence $\text{Proj}(S)$ is quasi-compact; in fact it is affine since it is equal to $D_+(X_1)$. It is easy to see that the image of $\text{Proj}(S) \to \text{Spec}(R)$ is $D(t)$. Hence the morphism $\text{Proj}(S) \to \text{Spec}(R)$ is not closed. Thus the valuative criterion cannot apply because it would imply that the morphism is closed (see Schemes, Proposition 26.20.6).

01MI Example 27.8.14. Let A be a ring. Let $S = A[T]$ as a graded A algebra with T in degree 1. Then the canonical morphism $\text{Proj}(S) \to \text{Spec}(A)$ (see Lemma 27.8.10) is an isomorphism.

0G5W Example 27.8.15. Let $X = \text{Spec}(A)$ be an affine scheme, and let $U \subset X$ be an open subscheme. Grade $A[T]$ by setting $\deg T = 1$. Define S to be the subring of $A[T]$ generated by A and all fT^i, where $i \geq 0$ and where $f \in A$ is such that $D(f) \subset U$. We claim that S is a graded ring with $S_0 = A$ such that $\text{Proj}(S) \cong U$, and this isomorphism identifies the canonical morphism $\text{Proj}(S) \to \text{Spec}(A)$ of Lemma 27.8.10 with the inclusion $U \subset X$.

Suppose $\mathfrak{p} \in \text{Proj}(S)$ is such that every $fT \in S_1$ is in $\mathfrak{p}$. Then every generator fT^i with $i \geq 1$ is in $\mathfrak{p}$ because $(fT^i)^2 = (fT)(fT^{2i-1}) \in \mathfrak{p}$ and $\mathfrak{p}$ is radical. But then $\mathfrak{p} \supset S_+$, which is impossible. Consequently $\text{Proj}(S)$ is covered by the standard open affine subsets $\{D_+(fT)\}_{fT \in S_1}$.

Observe that, if $fT \in S_1$, then the inclusion $S \subset A[T]$ induces a graded isomorphism of $S[(fT)^{-1}]$ with $A[T, T^{-1}, f^{-1}]$. Hence the standard open subset $D_+(fT) \cong \text{Spec}(S_{(fT)})$ is isomorphic to $\text{Spec}(A[T, T^{-1}, f^{-1}]_0) = \text{Spec}(A[f^{-1}])$. It is clear that this isomorphism is a restriction of the canonical morphism $\text{Proj}(S) \to \text{Spec}(A)$. If in addition $gT \in S_1$, then $S[(fT)^{-1}, (gT)^{-1}] \cong A[T, T^{-1}, f^{-1}, g^{-1}]$ as graded rings, so $D_+(fT) \cap D_+(gT) \cong \text{Spec}(A[f^{-1}, g^{-1}])$. Therefore $\text{Proj}(S)$ is the union of open subschemes $D_+(fT)$ which are isomorphic to the open subschemes $D(f) \subset X$ under the canonical morphism, and these open subschemes intersect in $\text{Proj}(S)$ in the same way they do in X. We conclude that the canonical morphism is an isomorphism of $\text{Proj}(S)$ with the union of all $D(f) \subset U$, which is U.

27.9. Quasi-coherent sheaves on Proj

01MJ Let S be a graded ring. Let M be a graded S-module. We saw in Lemma 27.8.4 how to construct a quasi-coherent sheaf of modules $\widetilde{M}$ on $\text{Proj}(S)$ and a map

0AG1 (27.9.0.1)
$$M_0 \longrightarrow \Gamma(\text{Proj}(S), \widetilde{M})$$

of the degree 0 part of M to the global sections of $\widetilde{M}$. The degree 0 part of the nth twist $M(n)$ of the graded module M (see Algebra, Section 10.56) is equal to M_n. Hence we can get maps

0AG2 (27.9.0.2)
$$M_n \longrightarrow \Gamma(\text{Proj}(S), \widetilde{M(n)}).$$

We would like to be able to perform this operation for any quasi-coherent sheaf $\mathcal{F}$ on $\text{Proj}(S)$. We will do this by tensoring with the nth twist of the structure sheaf, see Definition 27.10.1. In order to relate the two notions we will use the following lemma.

01MK **Lemma 27.9.1.** Let S be a graded ring. Let $(X, \mathcal{O}_X) = (\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ be the scheme of Lemma 27.8.7. Let $f \in S_+$ be homogeneous. Let $x \in X$ be a point corresponding to the homogeneous prime $\mathfrak{p} \subset S$. Let M, N be graded S-modules. There is a canonical map of $\mathcal{O}_{\text{Proj}(S)}$-modules

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

which induces the canonical map $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$ on sections over $D_+(f)$ and the canonical map $M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \to (M \otimes_S N)_{(\mathfrak{p})}$ on stalks at x. Moreover, the following diagram

$$\begin{tikzcd}
M_0 \otimes_{S_0} N_0 \arrow[r] \arrow[d] & (M \otimes_S N)_0 \arrow[d] \\
\Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \arrow[r] & \Gamma(X, \widetilde{M \otimes_S N})
\end{tikzcd}$$

is commutative where the vertical maps are given by (27.9.0.1).

Proof. To construct a morphism as displayed is the same as constructing a $\mathcal{O}_X$-bilinear map

$$\widetilde{M} \times \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

see Modules, Section 17.16. It suffices to define this on sections over the opens $D_+(f)$ compatible with restriction mappings. On $D_+(f)$ we use the $S_{(f)}$-bilinear map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$, $(x/f^n, y/f^m) \mapsto (x \otimes y)/f^{n+m}$. Details omitted. $\square$

01ML Remark 27.9.2. In general the map constructed in Lemma 27.9.1 above is not an isomorphism. Here is an example. Let k be a field. Let $S = k[x, y, z]$ with k in degree 0 and $\deg(x) = 1$, $\deg(y) = 2$, $\deg(z) = 3$. Let $M = S(1)$ and $N = S(2)$, see Algebra, Section 10.56 for notation. Then $M \otimes_S N = S(3)$. Note that

$$\begin{array}{rcl} S_z & = & k[x, y, z, 1/z] \\ S_{(z)} & = & k[x^3/z, xy/z, y^3/z^2] \cong k[u, v, w]/(uw - v^3) \\ M_{(z)} & = & S_{(z)} \cdot x + S_{(z)} \cdot y^2/z \subset S_z \\ N_{(z)} & = & S_{(z)} \cdot y + S_{(z)} \cdot x^2 \subset S_z \\ S(3)_{(z)} & = & S_{(z)} \cdot z \subset S_z \end{array}$$

Consider the maximal ideal $\mathfrak{m} = (u, v, w) \subset S_{(z)}$. It is not hard to see that both $M_{(z)}/\mathfrak{m}M_{(z)}$ and $N_{(z)}/\mathfrak{m}N_{(z)}$ have dimension 2 over $\kappa(\mathfrak{m})$. But $S(3)_{(z)}/\mathfrak{m}S(3)_{(z)}$ has dimension 1. Thus the map $M_{(z)} \otimes N_{(z)} \to S(3)_{(z)}$ is not an isomorphism.

27.10. Invertible sheaves on Proj

01MM Recall from Algebra, Section 10.56 the construction of the twisted module $M(n)$ associated to a graded module over a graded ring.

01MN Definition 27.10.1. Let S be a graded ring. Let $X = \text{Proj}(S)$.

(1) We define $\mathcal{O}_X(n) = \widetilde{S(n)}$. This is called the nth twist of the structure sheaf of $\text{Proj}(S)$.
(2) For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

We are going to use Lemma 27.9.1 to construct some canonical maps. Since $S(n) \otimes_S S(m) = S(n + m)$ we see that there are canonical maps

01MO (27.10.1.1)
$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n + m).$$

These maps are not isomorphisms in general, see the example in Remark 27.9.2. The same example shows that $\mathcal{O}_X(n)$ is not an invertible sheaf on X in general. Tensoring with an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ we get maps

03GJ (27.10.1.2)
$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n + m).$$

The maps (27.10.1.1) on global sections give a map of graded rings

01MP (27.10.1.3)
$$S \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)).$$

And for an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ the maps (27.10.1.2) give a graded module structure

03GK (27.10.1.4)
$$\bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)) \times \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \longrightarrow \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m))$$

and via (27.10.1.3) also a S-module structure. More generally, given any graded S-module M we have $M(n) = M \otimes_S S(n)$. Hence we get maps

01MQ (27.10.1.5) $$\widetilde{M}(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \longrightarrow \widetilde{M(n)}.$$

On global sections (27.9.0.2) defines a map of graded S-modules

01MR (27.10.1.6) $$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M(n)}).$$

Here is an important fact which follows basically immediately from the definitions.

01MS Lemma 27.10.2. Let S be a graded ring. Set $X = \text{Proj}(S)$. Let $f \in S$ be homogeneous of degree $d > 0$. The sheaves $\mathcal{O}_X(nd)|_{D_+(f)}$ are invertible, and in fact trivial for all $n \in \mathbf{Z}$ (see Modules, Definition 17.25.1). The maps (27.10.1.1) restricted to $D_+(f)$

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(nd + m)|_{D_+(f)},$$

the maps (27.10.1.2) restricted to $D_+(f)$

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{F}(m)|_{D_+(f)} \longrightarrow \mathcal{F}(nd + m)|_{D_+(f)},$$

and the maps (27.10.1.5) restricted to $D_+(f)$

$$\widetilde{M}(nd)|_{D_+(f)} = \widetilde{M}|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(nd)|_{D_+(f)} \longrightarrow \widetilde{M(nd)}|_{D_+(f)}$$

are isomorphisms for all $n, m \in \mathbf{Z}$.

Proof. The (not graded) S-module maps $S \to S(nd)$, and $M \to M(nd)$, given by $x \mapsto f^n x$ become isomorphisms after inverting f. The first shows that $S_{(f)} \cong S(nd)_{(f)}$ which gives an isomorphism $\mathcal{O}_{D_+(f)} \cong \mathcal{O}_X(nd)|_{D_+(f)}$. The second shows that the map $S(nd)_{(f)} \otimes_{S_{(f)}} M_{(f)} \to M(nd)_{(f)}$ is an isomorphism. The case of the map (27.10.1.2) is a consequence of the case of the map (27.10.1.1). □

01MT Lemma 27.10.3. Let S be a graded ring. Let M be a graded S-module. Set $X = \text{Proj}(S)$. Assume X is covered by the standard opens $D_+(f)$ with $f \in S_1$, e.g., if S is generated by S_1 over S_0. Then the sheaves $\mathcal{O}_X(n)$ are invertible and the maps (27.10.1.1), (27.10.1.2), and (27.10.1.5) are isomorphisms. In particular, these maps induce isomorphisms

$$\mathcal{O}_X(1)^{\otimes n} \cong \mathcal{O}_X(n) \quad \text{and} \quad \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{M}(n) \cong \widetilde{M(n)}$$

Thus (27.9.0.2) becomes a map

0AG3 (27.10.3.1) $$M_n \longrightarrow \Gamma(X, \widetilde{M}(n))$$

and (27.10.1.6) becomes a map

0AG4 (27.10.3.2) $$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M}(n)).$$

Proof. Under the assumptions of the lemma X is covered by the open subsets $D_+(f)$ with $f \in S_1$ and the lemma is a consequence of Lemma 27.10.2 above. □

01MU Lemma 27.10.4. Let S be a graded ring. Set $X = \text{Proj}(S)$. Fix $d \geq 1$ an integer. The following open subsets of X are equal:

(1) The largest open subset $W = W_d \subset X$ such that each $\mathcal{O}_X(dn)|_W$ is invertible and all the multiplication maps $\mathcal{O}_X(nd)|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(md)|_W \to \mathcal{O}_X(nd + md)|_W$ (see 27.10.1.1) are isomorphisms.

(2) The union of the open subsets $D_+(fg)$ with $f, g \in S$ homogeneous and $\deg(f) = \deg(g) + d$.

Moreover, all the maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 27.10.1.5) are isomorphisms.

Proof. If $x \in D_+(fg)$ with $\deg(f) = \deg(g)+d$ then on $D_+(fg)$ the sheaves $\mathcal{O}_X(dn)$ are generated by the element $(f/g)^n = f^{2n}/(fg)^n$. This implies x is in the open subset W defined in (1) by arguing as in the proof of Lemma 27.10.2.

Conversely, suppose that $\mathcal{O}_X(d)$ is free of rank 1 in an open neighbourhood V of $x \in X$ and all the multiplication maps $\mathcal{O}_X(nd)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_X(md)|_V \to \mathcal{O}_X(nd+md)|_V$ are isomorphisms. We may choose $h \in S_+$ homogeneous such that $x \in D_+(h) \subset V$. By the definition of the twists of the structure sheaf we conclude there exists an element s of $(S_h)_d$ such that s^n is a basis of $(S_h)_{nd}$ as a module over $S_{(h)}$ for all $n \in \mathbf{Z}$. We may write $s = f/h^m$ for some $m \geq 1$ and $f \in S_{d+m\deg(h)}$. Set $g = h^m$ so $s = f/g$. Note that $x \in D_+(g)$ by construction. Note that $g^d \in (S_h)_{-d\deg(g)}$. By assumption we can write this as a multiple of $s^{\deg(g)} = f^{\deg(g)}/g^{\deg(g)}$, say $g^d = a/g^e \cdot f^{\deg(g)}/g^{\deg(g)}$. Then we conclude that $g^{d+e+\deg(g)} = af^{\deg(g)}$ and hence also $x \in D_+(f)$. So x is an element of the set defined in (2).

The existence of the generating section $s = f/g$ over the affine open $D_+(fg)$ whose powers freely generate the sheaves of modules $\mathcal{O}_X(nd)$ easily implies that the multiplication maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 27.10.1.5) are isomorphisms. Compare with the proof of Lemma 27.10.2. □

Recall from Modules, Lemma 17.25.10 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x\mathcal{L}_x\}$ is open.

01MV Lemma 27.10.5. Let S be a graded ring. Set $X = \text{Proj}(S)$. Fix $d \geq 1$ an integer. Let $W = W_d \subset X$ be the open subscheme defined in Lemma 27.10.4. Let $n \geq 1$ and $f \in S_{nd}$. Denote $s \in \Gamma(W, \mathcal{O}_W(nd))$ the section which is the image of f via (27.10.1.3) restricted to W. Then

$$W_s = D_+(f) \cap W.$$

Proof. Let $D_+(ab) \subset W$ be a standard affine open with $a, b \in S$ homogeneous and $\deg(a) = \deg(b) + d$. Note that $D_+(ab) \cap D_+(f) = D_+(abf)$. On the other hand the restriction of s to $D_+(ab)$ corresponds to the element $f/1 = b^n f/a^n (a/b)^n \in (S_{ab})_{nd}$. We have seen in the proof of Lemma 27.10.4 that $(a/b)^n$ is a generator for $\mathcal{O}_W(nd)$ over $D_+(ab)$. We conclude that $W_s \cap D_+(ab)$ is the principal open associated to $b^n f/a^n \in \mathcal{O}_X(D_+(ab))$. Thus the result of the lemma is clear. □

The following lemma states the properties that we will later use to characterize schemes with an ample invertible sheaf.

01MW Lemma 27.10.6. Let S be a graded ring. Let $X = \text{Proj}(S)$. Let $Y \subset X$ be a quasi-compact open subscheme. Denote $\mathcal{O}_Y(n)$ the restriction of $\mathcal{O}_X(n)$ to Y. There exists an integer $d \geq 1$ such that

(1) the subscheme Y is contained in the open W_d defined in Lemma 27.10.4,
(2) the sheaf $\mathcal{O}_Y(dn)$ is invertible for all $n \in \mathbf{Z}$,
(3) all the maps $\mathcal{O}_Y(nd) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \longrightarrow \mathcal{O}_Y(nd+m)$ of Equation (27.10.1.1) are isomorphisms,

(4) all the maps $\widetilde{M}(nd)|_Y = \widetilde{M}|_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X(nd)|_Y \to \widetilde{M(nd)}|_Y$ (see 27.10.1.5) are isomorphisms,
(5) given $f \in S_{nd}$ denote $s \in \Gamma(Y, \mathcal{O}_Y(nd))$ the image of f via (27.10.1.3) restricted to Y, then $D_+(f) \cap Y = Y_s$,
(6) a basis for the topology on Y is given by the collection of opens Y_s, where $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$, and
(7) a basis for the topology of Y is given by those opens $Y_s \subset Y$, for $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$ which are affine.

Proof. Since Y is quasi-compact there exist finitely many homogeneous $f_i \in S_+$, $i = 1, \ldots, n$ such that the standard opens $D_+(f_i)$ give an open covering of Y. Let $d_i = \deg(f_i)$ and set $d = d_1 \ldots d_n$. Note that $D_+(f_i) = D_+(f_i^{d/d_i})$ and hence we see immediately that $Y \subset W_d$, by characterization (2) in Lemma 27.10.4 or by (1) using Lemma 27.10.2. Note that (1) implies (2), (3) and (4) by Lemma 27.10.4. (Note that (3) is a special case of (4).) Assertion (5) follows from Lemma 27.10.5. Assertions (6) and (7) follow because the open subsets $D_+(f)$ form a basis for the topology of X and are affine. □

0B5I Lemma 27.10.7. Let S be a graded ring. Set $X = \text{Proj}(S)$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ as a graded S-module, using (27.10.1.4) and (27.10.1.3). Then there is a canonical $\mathcal{O}_X$-module map

$$\widetilde{M} \longrightarrow \mathcal{F}$$

functorial in $\mathcal{F}$ such that the induced map $M_0 \to \Gamma(X, \mathcal{F})$ is the identity.

Proof. Let $f \in S$ be homogeneous of degree $d > 0$. Recall that $\widetilde{M}|_{D_+(f)}$ corresponds to the $S_{(f)}$-module $M_{(f)}$ by Lemma 27.8.4. Thus we can define a canonical map

$$M_{(f)} \longrightarrow \Gamma(D_+(f), \mathcal{F}), \quad m/f^n \longmapsto m|_{D_+(f)} \otimes f|_{D_+(f)}^{-n}$$

which makes sense because $f|_{D_+(f)}$ is a trivializing section of the invertible sheaf $\mathcal{O}_X(d)|_{D_+(f)}$, see Lemma 27.10.2 and its proof. Since $\widetilde{M}$ is quasi-coherent, this leads to a canonical map

$$\widetilde{M}|_{D_+(f)} \longrightarrow \mathcal{F}|_{D_+(f)}$$

via Schemes, Lemma 26.7.1. We obtain a global map if we prove that the displayed maps glue on overlaps. Proof of this is omitted. We also omit the proof of the final statement. □

27.11. Functoriality of Proj

01MX A graded ring map $\psi : A \to B$ does not always give rise to a morphism of associated projective homogeneous spectra. The reason is that the inverse image $\psi^{-1}(\mathfrak{q})$ of a homogeneous prime $\mathfrak{q} \subset B$ may contain the irrelevant prime A_+ even if $\mathfrak{q}$ does not contain B_+. The correct result is stated as follows.

01MY Lemma 27.11.1. Let A, B be two graded rings. Set $X = \text{Proj}(A)$ and $Y = \text{Proj}(B)$. Let $\psi : A \to B$ be a graded ring map. Set

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \text{ homogeneous}} D_+(\psi(f)) \subset Y.$$

Then there is a canonical morphism of schemes

$$r_\psi : U(\psi) \longrightarrow X$$

and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^*\left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the following properties:

(1) For every $d \geq 0$ the diagram

$$\begin{tikzcd} A_d \arrow[rr, "\psi"'] \arrow[d] & & B_d \arrow[d] \\ \Gamma(X, \mathcal{O}_X(d)) \arrow[r, "\theta"] & \Gamma(U(\psi), \mathcal{O}_Y(d)) & \Gamma(Y, \mathcal{O}_Y(d)) \arrow[l] \end{tikzcd}$$

is commutative.

(2) For any $f \in A_+$ homogeneous we have $r_\psi^{-1}(D_+(f)) = D_+(\psi(f))$ and the restriction of r_ψ to $D_+(\psi(f))$ corresponds to the ring map $A_{(f)} \to B_{(\psi(f))}$ induced by ψ.

Proof. Clearly condition (2) uniquely determines the morphism of schemes and the open subset $U(\psi)$. Pick $f \in A_d$ with $d \geq 1$. Note that $\mathcal{O}_X(n)|_{D_+(f)}$ corresponds to the $A_{(f)}$-module $(A_f)_n$ and that $\mathcal{O}_Y(n)|_{D_+(\psi(f))}$ corresponds to the $B_{(\psi(f))}$-module $(B_{\psi(f)})_n$. In other words θ when restricted to $D_+(\psi(f))$ corresponds to a map of $\mathbf{Z}$-graded $B_{(\psi(f))}$-algebras

$$A_f \otimes_{A_{(f)}} B_{(\psi(f))} \longrightarrow B_{\psi(f)}$$

Condition (1) determines the images of all elements of A. Since f is an invertible element which is mapped to $\psi(f)$ we see that $1/f^m$ is mapped to $1/\psi(f)^m$. It easily follows from this that θ is uniquely determined, namely it is given by the rule

$$a/f^m \otimes b/\psi(f)^e \longmapsto \psi(a)b/\psi(f)^{m+e}.$$

To show existence we remark that the proof of uniqueness above gave a well defined prescription for the morphism r and the map θ when restricted to every standard open of the form $D_+(\psi(f)) \subset U(\psi)$ into $D_+(f)$. Call these r_f and θ_f. Hence we only need to verify that if $D_+(f) \subset D_+(g)$ for some $f, g \in A_+$ homogeneous, then the restriction of r_g to $D_+(\psi(f))$ matches r_f. This is clear from the formulas given for r and θ above. $\square$

01MZ **Lemma 27.11.2.** Let A, B, and C be graded rings. Set $X = \text{Proj}(A)$, $Y = \text{Proj}(B)$ and $Z = \text{Proj}(C)$. Let $\varphi : A \to B$, $\psi : B \to C$ be graded ring maps. Then we have

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi \circ \varphi} = r_\varphi \circ r_\psi|_{U(\psi \circ \varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi \circ \varphi}$$

with obvious notation.

Proof. Omitted. $\square$

01N0 **Lemma 27.11.3.** With hypotheses and notation as in Lemma 27.11.1 above. Assume $A_d \to B_d$ is surjective for all $d \gg 0$. Then

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ is a closed immersion, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are surjective but not isomorphisms in general (even if $A \to B$ is surjective).

Proof. Part (1) follows from the definition of $U(\psi)$ and the fact that $D_+(f) = D_+(f^n)$ for any $n > 0$. For $f \in A_+$ homogeneous we see that $A_{(f)} \to B_{(\psi(f))}$ is surjective because any element of $B_{(\psi(f))}$ can be represented by a fraction $b/\psi(f)^n$ with n arbitrarily large (which forces the degree of $b \in B$ to be large). This proves (2). The same argument shows the map

$$A_f \to B_{\psi(f)}$$

is surjective which proves the surjectivity of θ. For an example where this map is not an isomorphism consider the graded ring $A = k[x, y]$ where k is a field and $\deg(x) = 1$, $\deg(y) = 2$. Set $I = (x)$, so that $B = k[y]$. Note that $\mathcal{O}_Y(1) = 0$ in this case. But it is easy to see that $r_\psi^*\mathcal{O}_X(1)$ is not zero. (There are less silly examples.) $\square$

07ZE **Lemma 27.11.4.** With hypotheses and notation as in Lemma 27.11.1 above. Assume $A_d \to B_d$ is an isomorphism for all $d \gg 0$. Then

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ is an isomorphism, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.

Proof. We have (1) by Lemma 27.11.3. Let $f \in A_+$ be homogeneous. The assumption on ψ implies that $A_f \to B_f$ is an isomorphism (details omitted). Thus it is clear that r_ψ and θ restrict to isomorphisms over $D_+(f)$. The lemma follows. $\square$

01N1 **Lemma 27.11.5.** With hypotheses and notation as in Lemma 27.11.1 above. Assume $A_d \to B_d$ is surjective for $d \gg 0$ and that A is generated by A_1 over A_0. Then

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ is a closed immersion, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.

Proof. By Lemmas 27.11.4 and 27.11.2 we may replace B by the image of $A \to B$ without changing X or the sheaves $\mathcal{O}_X(n)$. Thus we may assume that $A \to B$ is surjective. By Lemma 27.11.3 we get (1) and (2) and surjectivity in (3). By Lemma 27.10.3 we see that both $\mathcal{O}_X(n)$ and $\mathcal{O}_Y(n)$ are invertible. Hence θ is an isomorphism. $\square$

01N2 **Lemma 27.11.6.** With hypotheses and notation as in Lemma 27.11.1 above. Assume there exists a ring map $R \to A_0$ and a ring map $R \to R'$ such that $B = R' \otimes_R A$. Then

(1) $U(\psi) = Y$,
(2) the diagram

$$\begin{tikzcd} Y = \mathrm{Proj}(B) \arrow[r, "r_\psi"'] \arrow[d] & \mathrm{Proj}(A) = X \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

is a fibre product square, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.

Proof. This follows immediately by looking at what happens over the standard opens $D_+(f)$ for $f \in A_+$. $\square$

01N3 **Lemma 27.11.7.** With hypotheses and notation as in Lemma 27.11.1 above. Assume there exists a $g \in A_0$ such that ψ induces an isomorphism $A_g \to B$. Then $U(\psi) = Y$, $r_\psi : Y \to X$ is an open immersion which induces an isomorphism of Y with the inverse image of $D(g) \subset \mathrm{Spec}(A_0)$. Moreover the map θ is an isomorphism.

Proof. This is a special case of Lemma 27.11.6 above. $\square$

0B5J **Lemma 27.11.8.** Let S be a graded ring. Let $d \geq 1$. Set $S' = S^{(d)}$ with notation as in Algebra, Section 10.56. Set $X = \mathrm{Proj}(S)$ and $X' = \mathrm{Proj}(S')$. There is a canonical isomorphism $i : X \to X'$ of schemes such that

(1) for any graded S-module M setting $M' = M^{(d)}$, we have a canonical isomorphism $\widetilde{M} \to i^*\widetilde{M'}$,
(2) we have canonical isomorphisms $\mathcal{O}_X(nd) \to i^*\mathcal{O}_{X'}(n)$

and these isomorphisms are compatible with the multiplication maps of Lemma 27.9.1 and hence with the maps (27.10.1.1), (27.10.1.2), (27.10.1.3), (27.10.1.4), (27.10.1.5), and (27.10.1.6) (see proof for precise statements.

Proof. The injective ring map $S' \to S$ (which is not a homomorphism of graded rings due to our conventions), induces a map $j : \mathrm{Spec}(S) \to \mathrm{Spec}(S')$. Given a graded prime ideal $\mathfrak{p} \subset S$ we see that $\mathfrak{p}' = j(\mathfrak{p}) = S' \cap \mathfrak{p}$ is a graded prime ideal of S'. Moreover, if $f \in S_+$ is homogeneous and $f \notin \mathfrak{p}$, then $f^d \in S'_+$ and $f^d \notin \mathfrak{p}'$. Conversely, if $\mathfrak{p}' \subset S'$ is a graded prime ideal not containing some homogeneous element $f \in S'_+$, then $\mathfrak{p} = \{g \in S \mid g^d \in \mathfrak{p}'\}$ is a graded prime ideal of S not containing f whose image under j is $\mathfrak{p}'$. To see that $\mathfrak{p}$ is an ideal, note that if $g, h \in \mathfrak{p}$, then $(g+h)^{2d} \in \mathfrak{p}'$ by the binomial formula and hence $g + h \in \mathfrak{p}'$ as $\mathfrak{p}'$ is a prime. In this way we see that j induces a homeomorphism $i : X \to X'$. Moreover, given $f \in S_+$ homogeneous, then we have $S_{(f)} \cong S'_{(f^d)}$. Since these isomorphisms are compatible with the restrictions mappings of Lemma 27.8.1, we see that there exists an isomorphism $i^\sharp : i^{-1}\mathcal{O}_{X'} \to \mathcal{O}_X$ of structure sheaves on X and X', hence i is an isomorphism of schemes.

Let M be a graded S-module. Given $f \in S_+$ homogeneous, we have $M_{(f)} \cong M'_{(f^d)}$, hence in exactly the same manner as above we obtain the isomorphism in (1). The isomorphisms in (2) are a special case of (1) for $M = S(nd)$ which gives $M' = S'(n)$. Let M and N be graded S-modules. Then we have

$$M' \otimes_{S'} N' = (M \otimes_S N)^{(d)} = (M \otimes_S N)'$$

as can be verified directly from the definitions. Having said this the compatibility with the multiplication maps of Lemma 27.9.1 is the commutativity of the diagram

$$\begin{tikzcd}
\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \arrow[r] \arrow[d, "(1)\otimes(1)"'] & \widetilde{M \otimes_S N} \arrow[d, "(1)"] \\
i^*(\widetilde{M'} \otimes_{\mathcal{O}_{X'}} \widetilde{N'}) \arrow[r] & i^*(\widetilde{M' \otimes_{S'} N'})
\end{tikzcd}$$

This can be seen by looking at the construction of the maps over the open $D_+(f) = D_+(f^d)$ where the top horizontal arrow is given by the map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$ and the lower horizontal arrow by the map $M'_{(f^d)} \times N'_{(f^d)} \to (M' \otimes_{S'} N')_{(f^d)}$. Since these maps agree via the identifications $M_{(f)} = M'_{(f^d)}$, etc, we get the desired compatibility. We omit the proof of the other compatibilities. $\square$

27.12. Morphisms into Proj

01N4 Let S be a graded ring. Let $X = \text{Proj}(S)$ be the homogeneous spectrum of S. Let $d \geq 1$ be an integer. Consider the open subscheme

01N5 (27.12.0.1)
$$U_d = \bigcup\nolimits_{f \in S_d} D_+(f) \quad \subset \quad X = \text{Proj}(S)$$

Note that $d | d' \Rightarrow U_d \subset U_{d'}$ and $X = \bigcup_d U_d$. Neither X nor U_d need be quasi-compact, see Algebra, Lemma 10.57.3. Let us write $\mathcal{O}_{U_d}(n) = \mathcal{O}_X(n)|_{U_d}$. By Lemma 27.10.2 we know that $\mathcal{O}_{U_d}(nd)$, $n \in \mathbf{Z}$ is an invertible $\mathcal{O}_{U_d}$-module and that all the multiplication maps $\mathcal{O}_{U_d}(nd) \otimes_{\mathcal{O}_{U_d}} \mathcal{O}_{U_d}(m) \to \mathcal{O}_{U_d}(nd+m)$ of (27.10.1.1) are isomorphisms. In particular we have $\mathcal{O}_{U_d}(nd) \cong \mathcal{O}_{U_d}(d)^{\otimes n}$. The graded ring map (27.10.1.3) on global sections combined with restriction to U_d give a homomorphism of graded rings

01N6 (27.12.0.2)
$$\psi^d : S^{(d)} \longrightarrow \Gamma_*(U_d, \mathcal{O}_{U_d}(d)).$$

For the notation $S^{(d)}$, see Algebra, Section 10.56. For the notation Γ_* see Modules, Definition 17.25.7. Moreover, since U_d is covered by the opens $D_+(f)$, $f \in S_d$ we see that $\mathcal{O}_{U_d}(d)$ is globally generated by the sections in the image of $\psi^d_1 : S^{(d)}_1 = S_d \to \Gamma(U_d, \mathcal{O}_{U_d}(d))$, see Modules, Definition 17.4.1.

Let Y be a scheme, and let $\varphi : Y \to X$ be a morphism of schemes. Assume the image $\varphi(Y)$ is contained in the open subscheme U_d of X. By the discussion following Modules, Definition 17.25.7 we obtain a homomorphism of graded rings

$$\Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d)).$$

The composition of this and ψ^d gives a graded ring homomorphism

01N7 (27.12.0.3)
$$\psi^d_\varphi : S^{(d)} \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d))$$

which has the property that the invertible sheaf $\varphi^*\mathcal{O}_X(d)$ is globally generated by the sections in the image of $(S^{(d)})_1 = S_d \to \Gamma(Y, \varphi^*\mathcal{O}_X(d))$.

01N8 **Lemma 27.12.1.** *Let S be a graded ring, and $X = \text{Proj}(S)$. Let $d \geq 1$ and $U_d \subset X$ as above. Let Y be a scheme. Let $\mathcal{L}$ be an invertible sheaf on Y. Let $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ be a graded ring homomorphism such that $\mathcal{L}$ is generated by the sections in the image of $\psi|_{S_d} : S_d \to \Gamma(Y, \mathcal{L})$. Then there exist a morphism $\varphi : Y \to X$ such that $\varphi(Y) \subset U_d$ and an isomorphism $\alpha : \varphi^*\mathcal{O}_{U_d}(d) \to \mathcal{L}$ such that ψ^d_φ agrees with ψ via α:*

$$\begin{tikzcd}
\Gamma_*(Y, \mathcal{L}) & \Gamma_*(Y, \varphi^*\mathcal{O}_{U_d}(d)) \arrow[l, "\alpha"'] & \Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \arrow[l, "\varphi^*"'] \\
S^{(d)} \arrow[u, "\psi"] & & S^{(d)} \arrow[ll, "\mathrm{id}"] \arrow[u, "\psi^d"'] \arrow[ul, "\psi^d_\varphi"]
\end{tikzcd}$$

commutes. Moreover, the pair (φ, α) is unique.

Proof. Pick $f \in S_d$. Denote $s = \psi(f) \in \Gamma(Y, \mathcal{L})$. On the open set Y_s where s does not vanish multiplication by s induces an isomorphism $\mathcal{O}_{Y_s} \to \mathcal{L}|_{Y_s}$, see Modules, Lemma 17.25.10. We will denote the inverse of this map $x \mapsto x/s$, and similarly for powers of $\mathcal{L}$. Using this we define a ring map $\psi_{(f)} : S_{(f)} \to \Gamma(Y_s, \mathcal{O})$ by mapping the fraction a/f^n to $\psi(a)/s^n$. By Schemes, Lemma 26.6.4 this corresponds to a morphism $\varphi_f : Y_s \to \text{Spec}(S_{(f)}) = D_+(f)$. We also introduce the isomorphism $\alpha_f : \varphi_f^*\mathcal{O}_{D_+(f)}(d) \to \mathcal{L}|_{Y_s}$ which maps the pullback of the trivializing section f over

$D_+(f)$ to the trivializing section s over Y_s. With this choice the commutativity of the diagram in the lemma holds with Y replaced by Y_s, φ replaced by φ_f, and α replaced by α_f; verification omitted.

Suppose that $f' \in S_d$ is a second element, and denote $s' = \psi(f') \in \Gamma(Y, \mathcal{L})$. Then $Y_s \cap Y_{s'} = Y_{ss'}$ and similarly $D_+(f) \cap D_+(f') = D_+(ff')$. In Lemma 27.10.6 we saw that $D_+(f') \cap D_+(f)$ is the same as the set of points of $D_+(f)$ where the section of $\mathcal{O}_X(d)$ defined by f' does not vanish. Hence $\varphi_f^{-1}(D_+(f') \cap D_+(f)) = Y_s \cap Y_{s'} = \varphi_{f'}^{-1}(D_+(f') \cap D_+(f))$. On $D_+(f) \cap D_+(f')$ the fraction f/f' is an invertible section of the structure sheaf with inverse f'/f. Note that $\psi_{(f')}(f/f') = \psi(f)/s' = s/s'$ and $\psi_{(f)}(f'/f) = \psi(f')/s = s'/s$. We claim there is a unique ring map $S_{(ff')} \to \Gamma(Y_{ss'}, \mathcal{O})$ making the following diagram commute

$$\begin{tikzcd}
\Gamma(Y_s, \mathcal{O}) \arrow[r] & \Gamma(Y_{ss'}, \mathcal{O}) & \Gamma(Y_{s'}, \mathcal{O}) \arrow[l] \\
S_{(f)} \arrow[u, "\psi_{(f)}"] \arrow[r] & S_{(ff')} \arrow[u] & S_{(f')} \arrow[l] \arrow[u, "\psi_{(f')}"']
\end{tikzcd}$$

It exists because we may use the rule $x/(ff')^n \mapsto \psi(x)/(ss')^n$, which "works" by the formulas above. Uniqueness follows as $\text{Proj}(S)$ is separated, see Lemma 27.8.8 and its proof. This shows that the morphisms φ_f and $\varphi_{f'}$ agree over $Y_s \cap Y_{s'}$. The restrictions of α_f and $\alpha_{f'}$ agree over $Y_s \cap Y_{s'}$ because the regular functions s/s' and $\psi_{(f')}(f)$ agree. This proves that the morphisms ψ_f glue to a global morphism from Y into $U_d \subset X$, and that the maps α_f glue to an isomorphism satisfying the conditions of the lemma.

We still have to show the pair (φ, α) is unique. Suppose (φ', α') is a second such pair. Let $f \in S_d$. By the commutativity of the diagrams in the lemma we have that the inverse images of $D_+(f)$ under both φ and φ' are equal to $Y_{\psi(f)}$. Since the opens $D_+(f)$ are a basis for the topology on X, and since X is a sober topological space (see Schemes, Lemma 26.11.1) this means the maps φ and φ' are the same on underlying topological spaces. Let us use $s = \psi(f)$ to trivialize the invertible sheaf $\mathcal{L}$ over $Y_{\psi(f)}$. By the commutativity of the diagrams we have that $\alpha^{\otimes n}(\psi^d_\varphi(x)) = \psi(x) = (\alpha')^{\otimes n}(\psi^d_{\varphi'}(x))$ for all $x \in S_{nd}$. By construction of ψ^d_φ and $\psi^d_{\varphi'}$ we have $\psi^d_\varphi(x) = \varphi^\sharp(x/f^n)\psi^d_\varphi(f^n)$ over $Y_{\psi(f)}$, and similarly for $\psi^d_{\varphi'}$. By the commutativity of the diagrams of the lemma we deduce that $\varphi^\sharp(x/f^n) = (\varphi')^\sharp(x/f^n)$. This proves that φ and φ' induce the same morphism from $Y_{\psi(f)}$ into the affine scheme $D_+(f) = \text{Spec}(S_{(f)})$. Hence φ and φ' are the same as morphisms. Finally, it remains to show that the commutativity of the diagram of the lemma singles out, given φ, a unique α. We omit the verification. □

We continue the discussion from above the lemma. Let S be a graded ring. Let Y be a scheme. We will consider triples $(d, \mathcal{L}, \psi)$ where

(1) $d \geq 1$ is an integer,
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and
(3) $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_d$.

Given a morphism $h : Y' \to Y$ and a triple $(d, \mathcal{L}, \psi)$ over Y we can pull it back to the triple $(d, h^*\mathcal{L}, h^* \circ \psi)$. Given two triples $(d, \mathcal{L}, \psi)$ and $(d, \mathcal{L}', \psi')$ with the same integer d we say they are strictly equivalent if there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded ring maps $S^{(d)} \to \Gamma_*(Y, \mathcal{L}')$.

For each integer $d \geq 1$ we define

$$\begin{array}{rcl} F_d : Sch^{opp} & \longrightarrow & \text{Sets}, \\ Y & \longmapsto & \{\text{strict equivalence classes of triples } (d, \mathcal{L}, \psi) \text{ as above}\} \end{array}$$

with pullbacks as defined above.

01N9 Lemma 27.12.2. Let S be a graded ring. Let $X = \text{Proj}(S)$. The open subscheme $U_d \subset X$ (27.12.0.1) represents the functor F_d and the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$ defined above is the universal family (see Schemes, Section 26.15).

Proof. This is a reformulation of Lemma 27.12.1 $\square$

01NA Lemma 27.12.3. Let S be a graded ring generated as an S_0-algebra by the elements of S_1. In this case the scheme $X = \text{Proj}(S)$ represents the functor which associates to a scheme Y the set of pairs $(\mathcal{L}, \psi)$, where

(1) $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and
(2) $\psi : S \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_1$

up to strict equivalence as above.

Proof. Under the assumptions of the lemma we have $X = U_1$ and the lemma is a reformulation of Lemma 27.12.2 above. $\square$

We end this section with a discussion of a functor corresponding to $\text{Proj}(S)$ for a general graded ring S. We advise the reader to skip the rest of this section.

Fix an arbitrary graded ring S. Let T be a scheme. We will say two triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ over T with possibly different integers d, d' are equivalent if there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ of invertible sheaves over T such that $\beta \circ \psi|_{S^{(dd')}}$ and $\psi'|_{S^{(dd')}}$ agree as graded ring maps $S^{(dd')} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes dd'})$.

01NB Lemma 27.12.4. Let S be a graded ring. Set $X = \text{Proj}(S)$. Let T be a scheme. Let $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ be two triples over T. The following are equivalent:

(1) Let $n = \text{lcm}(d, d')$. Write $n = ad = a'd'$. There exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.
(2) The triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equivalent.
(3) For some positive integer $n = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.
(4) The morphisms $\varphi : T \to X$ and $\varphi' : T \to X$ associated to $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equal.

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Also (3) implies (4) by the uniqueness statement in Lemma 27.12.1. Thus we have to prove that (4) implies (1). Assume (4), in other words $\varphi = \varphi'$. Note that this implies that we may write $\mathcal{L} = \varphi^*\mathcal{O}_X(d)$

and $\mathcal{L}' = \varphi^*\mathcal{O}_X(d')$. Moreover, via these identifications we have that the graded ring maps ψ and ψ' correspond to the restriction of the canonical graded ring map

$$S \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

to $S^{(d)}$ and $S^{(d')}$ composed with pullback by φ (by Lemma 27.12.1 again). Hence taking β to be the isomorphism

$$(\varphi^*\mathcal{O}_X(d))^{\otimes a} = \varphi^*\mathcal{O}_X(n) = (\varphi^*\mathcal{O}_X(d'))^{\otimes a'}$$

works. □

Let S be a graded ring. Let $X = \text{Proj}(S)$. Over the open subscheme scheme $U_d \subset X = \text{Proj}(S)$ (27.12.0.1) we have the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$. Clearly, if $d|d'$ the triples $(d, \mathcal{O}_{U_d}(d), \psi^d)$ and $(d', \mathcal{O}_{U_{d'}}(d'), \psi^{d'})$ are equivalent when restricted to the open U_d (which is a subset of $U_{d'}$). This, combined with Lemma 27.12.1 shows that morphisms $Y \to X$ correspond roughly to equivalence classes of triples over Y. This is not quite true since if Y is not quasi-compact, then there may not be a single triple which works. Thus we have to be slightly careful in defining the corresponding functor.

Here is one possible way to do this. Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the triple $(d, \mathcal{L}, \psi)$ over T the triple $(d', \mathcal{L}^{\otimes a}, \psi|_{S^{(d')}})$. One of the implications of Lemma 27.12.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d \in \mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = \lim\nolimits_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

$$F : Sch^{opp} \longrightarrow \text{Sets}$$

01NC Lemma 27.12.5. Let S be a graded ring. Let $X = \text{Proj}(S)$. The functor F defined above is representable by the scheme X.

Proof. We have seen above that the functor F_d corresponds to the open subscheme $U_d \subset X$. Moreover the transformation of functors $F_d \to F_{d'}$ (if $d|d'$) defined above corresponds to the inclusion morphism $U_d \to U_{d'}$ (see discussion above). Hence to show that F is represented by X it suffices to show that $T \to X$ for a quasi-compact scheme T ends up in some U_d, and that for a general scheme T we have

$$\text{Mor}(T, X) = \lim\nolimits_{V \subset T \text{ quasi-compact open}} \text{Mor}(V, X).$$

These verifications are omitted. □

27.13. Projective space

01ND Projective space is one of the fundamental objects studied in algebraic geometry. In this section we just give its construction as Proj of a polynomial ring. Later we will discover many of its beautiful properties.

01NE **Lemma 27.13.1.** Let $S = \mathbf{Z}[T_0, \ldots, T_n]$ with $\deg(T_i) = 1$. The scheme

$$\mathbf{P}^n_{\mathbf{Z}} = \text{Proj}(S)$$

represents the functor which associates to a scheme Y the pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ where

(1) $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and
(2) $s_0, \ldots, s_n$ are global sections of $\mathcal{L}$ which generate $\mathcal{L}$

up to the following equivalence: $(\mathcal{L}, (s_0, \ldots, s_n)) \sim (\mathcal{N}, (t_0, \ldots, t_n)) \Leftrightarrow$ there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{N}$ with $\beta(s_i) = t_i$ for $i = 0, \ldots, n$.

Proof. This is a special case of Lemma 27.12.3 above. Namely, for any graded ring A we have

$$\begin{array}{rcl} \text{Mor}_{graded rings}(\mathbf{Z}[T_0, \ldots, T_n], A) & = & A_1 \times \ldots \times A_1 \\ \psi & \mapsto & (\psi(T_0), \ldots, \psi(T_n)) \end{array}$$

and the degree 1 part of $\Gamma_*(Y, \mathcal{L})$ is just $\Gamma(Y, \mathcal{L})$. $\square$

01NF **Definition 27.13.2.** The scheme $\mathbf{P}^n_{\mathbf{Z}} = \text{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ is called *projective n-space over $\mathbf{Z}$*. Its base change $\mathbf{P}^n_S$ to a scheme S is called *projective n-space over S*. If R is a ring the base change to $\text{Spec}(R)$ is denoted $\mathbf{P}^n_R$ and called *projective n-space over R*.

Given a scheme Y over S and a pair $(\mathcal{L}, (s_0, \ldots, s_n))$ as in Lemma 27.13.1 the induced morphism to $\mathbf{P}^n_S$ is denoted

$$\varphi_{(\mathcal{L}, (s_0, \ldots, s_n))} : Y \longrightarrow \mathbf{P}^n_S$$

This makes sense since the pair defines a morphism into $\mathbf{P}^n_{\mathbf{Z}}$ and we already have the structure morphism into S so combined we get a morphism into $\mathbf{P}^n_S = \mathbf{P}^n_{\mathbf{Z}} \times S$. Note that this is the S-morphism characterized by

$$\mathcal{L} = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} \mathcal{O}_{\mathbf{P}^n_R}(1) \quad \text{and} \quad s_i = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} T_i$$

where we think of T_i as a global section of $\mathcal{O}_{\mathbf{P}^n_S}(1)$ via (27.10.1.3).

01NG **Lemma 27.13.3.** Projective n-space over $\mathbf{Z}$ is covered by $n + 1$ standard opens

$$\mathbf{P}^n_{\mathbf{Z}} = \bigcup\nolimits_{i = 0, \ldots, n} D_+(T_i)$$

where each $D_+(T_i)$ is isomorphic to $\mathbf{A}^n_{\mathbf{Z}}$ affine n-space over $\mathbf{Z}$.

Proof. This is true because $\mathbf{Z}[T_0, \ldots, T_n]_+ = (T_0, \ldots, T_n)$ and since

$$\text{Spec}\left(\mathbf{Z}\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\right) \cong \mathbf{A}^n_{\mathbf{Z}}$$

in an obvious way. $\square$

01NH **Lemma 27.13.4.** Let S be a scheme. The structure morphism $\mathbf{P}^n_S \to S$ is

(1) separated,
(2) quasi-compact,

(3) satisfies the existence and uniqueness parts of the valuative criterion, and
(4) universally closed.

Proof. All these properties are stable under base change (this is clear for the last two and for the other two see Schemes, Lemmas 26.21.12 and 26.19.3). Hence it suffices to prove them for the morphism $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$. Separatedness is Lemma 27.8.8. Quasi-compactness follows from Lemma 27.13.3. Existence and uniqueness of the valuative criterion follow from Lemma 27.8.11. Universally closed follows from the above and Schemes, Proposition 26.20.6. □

01NI Remark 27.13.5. What's missing in the list of properties above? Well to be sure the property of being of finite type. The reason we do not list this here is that we have not yet defined the notion of finite type at this point. (Another property which is missing is "smoothness". And I'm sure there are many more you can think of.)

01WD Lemma 27.13.6 (Segre embedding). Let S be a scheme. There exists a closed immersion

$$\mathbf{P}^n_S \times_S \mathbf{P}^m_S \longrightarrow \mathbf{P}^{nm+n+m}_S$$

called the Segre embedding.

Proof. It suffices to prove this when $S = \mathrm{Spec}(\mathbf{Z})$. Hence we will drop the index S and work in the absolute setting. Write $\mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[X_0, \ldots, X_n])$, $\mathbf{P}^m = \mathrm{Proj}(\mathbf{Z}[Y_0, \ldots, Y_m])$, and $\mathbf{P}^{nm+n+m} = \mathrm{Proj}(\mathbf{Z}[Z_0, \ldots, Z_{nm+n+m}])$. In order to map into $\mathbf{P}^{nm+n+m}$ we have to write down an invertible sheaf $\mathcal{L}$ on the left hand side and $(n+1)(m+1)$ sections s_i which generate it. See Lemma 27.13.1. The invertible sheaf we take is

$$\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}^m}(1)$$

The sections we take are

$$s_0 = X_0Y_0,\ s_1 = X_1Y_0, \ldots,\ s_n = X_nY_0,\ s_{n+1} = X_0Y_1, \ldots,\ s_{nm+n+m} = X_nY_m.$$

These generate $\mathcal{L}$ since the sections X_i generate $\mathcal{O}_{\mathbf{P}^n}(1)$ and the sections Y_j generate $\mathcal{O}_{\mathbf{P}^m}(1)$. The induced morphism φ has the property that

$$\varphi^{-1}(D_+(Z_{i+(n+1)j})) = D_+(X_i) \times D_+(Y_j).$$

Hence it is an affine morphism. The corresponding ring map in case $(i, j) = (0, 0)$ is the map

$$\mathbf{Z}[Z_1/Z_0, \ldots, Z_{nm+n+m}/Z_0] \longrightarrow \mathbf{Z}[X_1/X_0, \ldots, X_n/X_0, Y_1/Y_0, \ldots, Y_n/Y_0]$$

which maps Z_i/Z_0 to the element X_i/X_0 for $i \leq n$ and the element $Z_{(n+1)j}/Z_0$ to the element Y_j/Y_0. Hence it is surjective. A similar argument works for the other affine open subsets. Hence the morphism φ is a closed immersion (see Schemes, Lemma 26.4.2 and Example 26.8.1.) □

The following two lemmas are special cases of more general results later, but perhaps it makes sense to prove these directly here now.

03GL Lemma 27.13.7. Let R be a ring. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme. Let

$$I_d = \mathrm{Ker}\left(R[T_0, \ldots, T_n]_d \longrightarrow \Gamma(Z, \mathcal{O}_{\mathbf{P}^n_R}(d)|_Z)\right)$$

Then $I = \bigoplus I_d \subset R[T_0, \ldots, T_n]$ is a graded ideal and $Z = \mathrm{Proj}(R[T_0, \ldots, T_n]/I)$.

Proof. It is clear that I is a graded ideal. Set $Z' = \text{Proj}(R[T_0, \ldots, T_n]/I)$. By Lemma 27.11.5 we see that Z' is a closed subscheme of $\mathbf{P}^n_R$. To see the equality $Z = Z'$ it suffices to check on an standard affine open $D_+(T_i)$. By renumbering the homogeneous coordinates we may assume $i = 0$. Say $Z \cap D_+(T_0)$, resp. $Z' \cap D_+(T_0)$ is cut out by the ideal J, resp. J' of $R[T_1/T_0, \ldots, T_n/T_0]$. Then J' is the ideal generated by the elements $F/T_0^{\deg(F)}$ where $F \in I$ is homogeneous. Suppose the degree of $F \in I$ is d. Since F vanishes as a section of $\mathcal{O}_{\mathbf{P}^n_R}(d)$ restricted to Z we see that F/T_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $f \in J$. If f has total degree d in $T_1/T_0, \ldots, T_n/T_0$, then we can write $f = F/T_0^d$ for some $F \in R[T_0, \ldots, T_n]_d$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(T_i)$ is cut out by some ideal $J_i \subset R[T_0/T_i, \ldots, T_n/T_i]$. Moreover,

$$J \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] = J_i \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]$$

The left hand side is the localization of J with respect to the element T_i/T_0 and the right hand side is the localization of J_i with respect to the element T_0/T_i. It follows that $T_0^{d_i} F/T_i^{d+d_i}$ is an element of J_i for some d_i sufficiently large. This proves that $T_0^{\max(d_i)} F$ is an element of I, because its restriction to each standard affine open $D_+(T_i)$ vanishes on the closed subscheme $Z \cap D_+(T_i)$. Hence $f \in J'$ and we conclude $J \subset J'$ as desired. $\square$

The following lemma is a special case of the more general Properties, Lemmas 28.28.3 or 28.28.5.

03GM Lemma 27.13.8. Let R be a ring. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbf{P}^n_R$. For $d \geq 0$ set

$$M_d = \Gamma(\mathbf{P}^n_R, \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(d)) = \Gamma(\mathbf{P}^n_R, \mathcal{F}(d))$$

Then $M = \bigoplus_{d \geq 0} M_d$ is a graded $R[T_0, \ldots, R_n]$-module and there is a canonical isomorphism $\mathcal{F} = \widetilde{M}$.

Proof. The multiplication maps

$$R[T_0, \ldots, R_n]_e \times M_d \longrightarrow M_{d+e}$$

come from the natural isomorphisms

$$\mathcal{O}_{\mathbf{P}^n_R}(e) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{F}(d) \longrightarrow \mathcal{F}(e + d)$$

see Equation (27.10.1.4). Let us construct the map $c : \widetilde{M} \to \mathcal{F}$. On each of the standard affines $U_i = D_+(T_i)$ we see that $\Gamma(U_i, \widetilde{M}) = (M[1/T_i])_0$ where the subscript $_0$ means degree 0 part. An element of this can be written as m/T_i^d with $m \in M_d$. Since T_i is a generator of $\mathcal{O}(1)$ over U_i we can always write $m|_{U_i} = m_i \otimes T_i^d$ where $m_i \in \Gamma(U_i, \mathcal{F})$ is a unique section. Thus a natural guess is $c(m/T_i^d) = m_i$. A small argument, which is omitted here, shows that this gives a well defined map $c : \widetilde{M} \to \mathcal{F}$ if we can show that

$$(T_i/T_j)^d m_i|_{U_i \cap U_j} = m_j|_{U_i \cap U_j}$$

in $M[1/T_iT_j]$. But this is clear since on the overlap the generators T_i and T_j of $\mathcal{O}(1)$ differ by the invertible function T_i/T_j.

Injectivity of c. We may check for injectivity over the affine opens U_i. Let $i \in \{0, \ldots, n\}$ and let s be an element $s = m/T_i^d \in \Gamma(U_i, \widetilde{M})$ such that $c(m/T_i^d) = 0$.

By the description of c above this means that $m_i = 0$, hence $m|_{U_i} = 0$. Hence $T_i^e m = 0$ in M for some e. Hence $s = m/T_i^d = T_i^e/T_i^{e+d} = 0$ as desired.

Surjectivity of c. We may check for surjectivity over the affine opens U_i. By renumbering it suffices to check it over U_0. Let $s \in \mathcal{F}(U_0)$. Let us write $\mathcal{F}|_{U_i} = \widetilde{N_i}$ for some $R[T_0/T_i, \ldots, T_0/T_i]$-module N_i, which is possible because $\mathcal{F}$ is quasi-coherent. So s corresponds to an element $x \in N_0$. Then we have that

$$(N_i)_{T_j/T_i} \cong (N_j)_{T_i/T_j}$$

(where the subscripts mean "principal localization at") as modules over the ring

$$R\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}, \frac{T_0}{T_j}, \ldots, \frac{T_n}{T_j}\right].$$

This means that for some large integer d there exist elements $s_i \in N_i$, $i = 1, \ldots, n$ such that

$$s = (T_i/T_0)^d s_i$$

on $U_0 \cap U_i$. Next, we look at the difference

$$t_{ij} = s_i - (T_j/T_i)^d s_j$$

on $U_i \cap U_j$, $0 < i < j$. By our choice of s_i we know that $t_{ij}|_{U_0 \cap U_i \cap U_j} = 0$. Hence there exists a large integer e such that $(T_0/T_i)^e t_{ij} = 0$. Set $s_i' = (T_0/T_i)^e s_i$, and $s_0' = s$. Then we will have

$$s_a' = (T_b/T_a)^{e+d} s_b'$$

on $U_a \cap U_b$ for all a, b. This is exactly the condition that the elements s_a' glue to a global section $m \in \Gamma(\mathbf{P}^n_R, \mathcal{F}(e+d))$. And moreover $c(m/T_0^{e+d}) = s$ by construction. Hence c is surjective and we win. □

0B3B Lemma 27.13.9. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf and let $s_0, \ldots, s_n$ be global sections of $\mathcal{L}$ which generate it. Let $\mathcal{F}$ be the kernel of the induced map $\mathcal{O}_X^{\oplus n+1} \to \mathcal{L}$. Then $\mathcal{F} \otimes \mathcal{L}$ is globally generated.

Proof. In fact the result is true if X is any locally ringed space. The sheaf $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module of rank n. The elements

$$s_{ij} = (0, \ldots, 0, s_j, 0, \ldots, 0, -s_i, 0, \ldots, 0) \in \Gamma(X, \mathcal{L}^{\oplus n+1})$$

with s_j in the ith spot and $-s_i$ in the jth spot map to zero in $\mathcal{L}^{\otimes 2}$. Hence $s_{ij} \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L})$. A local computation shows that these sections generate $\mathcal{F} \otimes \mathcal{L}$.

Alternative proof. Consider the morphism $\varphi : X \to \mathbf{P}^n_{\mathbf{Z}}$ associated to the pair $(\mathcal{L}, (s_0, \ldots, s_n))$. Since the pullback of $\mathcal{O}(1)$ is $\mathcal{L}$ and since the pullback of T_i is s_i, it suffices to prove the lemma in the case of $\mathbf{P}^n_{\mathbf{Z}}$. In this case the sheaf $\mathcal{F}$ corresponds to the graded $S = \mathbf{Z}[T_0, \ldots, T_n]$ module M which fits into the short exact sequence

$$0 \to M \to S^{\oplus n+1} \to S(1) \to 0$$

where the second map is given by $T_0, \ldots, T_n$. In this case the statement above translates into the statement that the elements

$$T_{ij} = (0, \ldots, 0, T_j, 0, \ldots, 0, -T_i, 0, \ldots, 0) \in M(1)_0$$

generate the graded module $M(1)$ over S. We omit the details. □

27.14. Invertible sheaves and morphisms into Proj

01NJ Let T be a scheme and let $\mathcal{L}$ be an invertible sheaf on T. For a section $s \in \Gamma(T, \mathcal{L})$ we denote T_s the open subset of points where s does not vanish. See Modules, Lemma 17.25.10. We can view the following lemma as a slight generalization of Lemma 27.12.3. It also is a generalization of Lemma 27.11.1.

01NK **Lemma 27.14.1.** Let A be a graded ring. Set $X = \text{Proj}(A)$. Let T be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_T$-module. Let $\psi : A \to \Gamma_*(T, \mathcal{L})$ be a homomorphism of graded rings. Set

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \text{ homogeneous}} T_{\psi(f)}$$

The morphism ψ induces a canonical morphism of schemes

$$r_{\mathcal{L}, \psi} : U(\psi) \longrightarrow X$$

together with a map of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras

$$\theta : r^*_{\mathcal{L}, \psi}\left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{L}^{\otimes d}|_{U(\psi)}.$$

The triple $(U(\psi), r_{\mathcal{L}, \psi}, \theta)$ is characterized by the following properties:

(1) For $f \in A_+$ homogeneous we have $r_{\mathcal{L}, \psi}^{-1}(D_+(f)) = T_{\psi(f)}$.
(2) For every $d \geq 0$ the diagram

$$\begin{tikzcd} A_d \arrow[r, "\psi"'] \arrow[d, "(27.10.1.3)"'] & \Gamma(T, \mathcal{L}^{\otimes d}) \arrow[d, "restrict"] \\ \Gamma(X, \mathcal{O}_X(d)) \arrow[r, "\theta"] & \Gamma(U(\psi), \mathcal{L}^{\otimes d}) \end{tikzcd}$$

is commutative.

Moreover, for any $d \geq 1$ and any open subscheme $V \subset T$ such that the sections in $\psi(A_d)$ generate $\mathcal{L}^{\otimes d}|_V$ the morphism $r_{\mathcal{L}, \psi}|_V$ agrees with the morphism $\varphi : V \to \text{Proj}(A)$ and the map $\theta|_V$ agrees with the map $\alpha : \varphi^*\mathcal{O}_X(d) \to \mathcal{L}^{\otimes d}|_V$ where (φ, α) is the pair of Lemma 27.12.1 associated to $\psi|_{A^{(d)}} : A^{(d)} \to \Gamma_*(V, \mathcal{L}^{\otimes d})$.

Proof. Suppose that we have two triples $(U, r : U \to X, \theta)$ and $(U', r' : U' \to X, \theta')$ satisfying (1) and (2). Property (1) implies that $U = U' = U(\psi)$ and that $r = r'$ as maps of underlying topological spaces, since the opens $D_+(f)$ form a basis for the topology on X, and since X is a sober topological space (see Algebra, Section 10.57 and Schemes, Lemma 26.11.1). Let $f \in A_+$ be homogeneous. Note that $\Gamma(D_+(f), \bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(n)) = A_f$ as a $\mathbf{Z}$-graded algebra. Consider the two $\mathbf{Z}$-graded ring maps

$$\theta, \theta' : A_f \longrightarrow \Gamma(T_{\psi(f)}, \bigoplus \mathcal{L}^{\otimes n}).$$

We know that multiplication by f (resp. $\psi(f)$) is an isomorphism on the left (resp. right) hand side. We also know that $\theta(x/1) = \theta'(x/1) = \psi(x)|_{T_{\psi(f)}}$ by (2) for all $x \in A$. Hence we deduce easily that $\theta = \theta'$ as desired. Considering the degree 0 parts we deduce that $r^\sharp = (r')^\sharp$, i.e., that $r = r'$ as morphisms of schemes. This proves the uniqueness.

Now we come to existence. By the uniqueness just proved, it is enough to construct the pair (r, θ) locally on T. Hence we may assume that $T = \text{Spec}(R)$ is affine, that $\mathcal{L} = \mathcal{O}_T$ and that for some $f \in A_+$ homogeneous we have $\psi(f)$ generates

$\mathcal{O}_T = \mathcal{O}_T^{\otimes \deg(f)}$. In other words, $\psi(f) = u \in R^*$ is a unit. In this case the map ψ is a graded ring map

$$A \longrightarrow R[x] = \Gamma_*(T, \mathcal{O}_T)$$

which maps f to $ux^{\deg(f)}$. Clearly this extends (uniquely) to a **Z**-graded ring map $\theta : A_f \to R[x, x^{-1}]$ by mapping $1/f$ to $u^{-1}x^{-\deg(f)}$. This map in degree zero gives the ring map $A_{(f)} \to R$ which gives the morphism $r : T = \mathrm{Spec}(R) \to \mathrm{Spec}(A_{(f)}) = D_+(f) \subset X$. Hence we have constructed (r, θ) in this special case.

Let us show the last statement of the lemma. According to Lemma 27.12.1 the morphism constructed there is the unique one such that the displayed diagram in its statement commutes. The commutativity of the diagram in the lemma implies the commutativity when restricted to V and $A^{(d)}$. Whence the result. □

01NL Remark 27.14.2. Assumptions as in Lemma 27.14.1 above. The image of the morphism $r_{\mathcal{L},\psi}$ need not be contained in the locus where the sheaf $\mathcal{O}_X(1)$ is invertible. Here is an example. Let k be a field. Let $S = k[A, B, C]$ graded by $\deg(A) = 1$, $\deg(B) = 2$, $\deg(C) = 3$. Set $X = \mathrm{Proj}(S)$. Let $T = \mathbf{P}^2_k = \mathrm{Proj}(k[X_0, X_1, X_2])$. Recall that $\mathcal{L} = \mathcal{O}_T(1)$ is invertible and that $\mathcal{O}_T(n) = \mathcal{L}^{\otimes n}$. Consider the composition ψ of the maps

$$S \to k[X_0, X_1, X_2] \to \Gamma_*(T, \mathcal{L}).$$

Here the first map is $A \mapsto X_0$, $B \mapsto X_1^2$, $C \mapsto X_2^3$ and the second map is (27.10.1.3). By the lemma this corresponds to a morphism $r_{\mathcal{L},\psi} : T \to X = \mathrm{Proj}(S)$ which is easily seen to be surjective. On the other hand, in Remark 27.9.2 we showed that the sheaf $\mathcal{O}_X(1)$ is not invertible at all points of X.

27.15. Relative Proj via glueing

01NM

01NN Situation 27.15.1. Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra.

In this section we outline how to construct a morphism of schemes

$$\underline{\mathrm{Proj}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the homogeneous spectra $\mathrm{Proj}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the homogeneous spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 27.2.1.

01NO Lemma 27.15.2. In Situation 27.15.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of graded rings $A' \to A$ induces a morphism $r : \mathrm{Proj}(A) \to \mathrm{Proj}(A')$, and the diagram

$$\begin{tikzcd} \mathrm{Proj}(A) \arrow[r] \arrow[d] & \mathrm{Proj}(A') \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

is cartesian. Moreover there are canonical isomorphisms $\theta : r^*\mathcal{O}_{\mathrm{Proj}(A')}(n) \to \mathcal{O}_{\mathrm{Proj}(A)}(n)$ compatible with multiplication maps.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 26.7.3 for example). Hence the lemma follows from Lemma 27.11.6. □

In particular the morphism $\text{Proj}(A) \to \text{Proj}(A')$ of the lemma is an open immersion.

01NP Lemma 27.15.3. In Situation 27.15.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $r : \text{Proj}(A) \to \text{Proj}(A')$, and $r' : \text{Proj}(A') \to \text{Proj}(A'')$ of Lemma 27.15.2 gives the morphism $r'' : \text{Proj}(A) \to \text{Proj}(A'')$ of Lemma 27.15.2. A similar statement holds for the isomorphisms θ.

Proof. This follows from Lemma 27.11.2 since the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$. $\square$

01NQ Lemma 27.15.4. In Situation 27.15.1. There exists a morphism of schemes

$$\pi : \underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

(1) for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to \text{Proj}(A)$ with $A = \mathcal{A}(U)$, and
(2) for $U \subset U' \subset S$ affine open the composition

$$\text{Proj}(A) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} \text{Proj}(A')$$

with $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ is the open immersion of Lemma 27.15.2 above.

Proof. Follows immediately from Lemmas 27.2.1, 27.15.2, and 27.15.3. $\square$

01NR Lemma 27.15.5. In Situation 27.15.1. The morphism $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \to S$ of Lemma 27.15.4 comes with the following additional structure. There exists a quasi-coherent $\mathbf{Z}$-graded sheaf of $\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$, and a morphism of graded $\mathcal{O}_S$-algebras

$$\psi : \mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \pi_*\left(\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)\right)$$

uniquely determined by the following property: For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ there is an isomorphism

$$\theta_U : i_U^*\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\text{Proj}(A)}(n)\right) \longrightarrow \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)\right)|_{\pi^{-1}(U)}$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\pi^{-1}(U)}$-algebras such that

$$\begin{tikzcd} A_n \arrow[rr, "\psi"] \arrow[dr, "(27.10.1.3)"'] & & \Gamma(\pi^{-1}(U), \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)) \\ & \Gamma(\text{Proj}(A), \mathcal{O}_{\text{Proj}(A)}(n)) \arrow[ur, "\theta_U"'] & \end{tikzcd}$$

is commutative.

Proof. We are going to use Lemma 27.2.2 to glue the sheaves of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\text{Proj}(A)}(n)$ for $A = \mathcal{A}(U)$, $U \subset S$ affine open over the scheme $\underline{\text{Proj}}_S(\mathcal{A})$. We have constructed the data necessary for this in Lemma 27.15.2 and we have checked condition (d) of Lemma 27.2.2 in Lemma 27.15.3. Hence we get the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$ together with the isomorphisms θ_U for all $U \subset S$ affine open and all $n \in \mathbf{Z}$. For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ we have a map $A \to \Gamma(\text{Proj}(A), \bigoplus_{n\geq 0} \mathcal{O}_{\text{Proj}(A)}(n))$. Hence the map

ψ exists by functoriality of relative glueing, see Remark 27.2.3. The diagram of the lemma commutes by construction. This characterizes the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$ because the proof of Lemma 27.11.1 shows that having these diagrams commute uniquely determines the maps θ_U. Some details omitted. $\square$

27.16. Relative Proj as a functor

01NS We place ourselves in Situation 27.15.1. So S is a scheme and $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. In this section we relativize the construction of Proj by constructing a functor which the relative homogeneous spectrum will represent. As a result we will construct a morphism of schemes

$$\underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow S$$

which above affine opens of S will look like the homogeneous spectrum of a graded ring. The discussion will be modeled after our discussion of the relative spectrum in Section 27.4. The easier method using glueing schemes of the form $\text{Proj}(A)$, $A = \Gamma(U, \mathcal{A})$, $U \subset S$ affine open, is explained in Section 27.15, and the result in this section will be shown to be isomorphic to that one.

Fix for the moment an integer $d \geq 1$. We denote $\mathcal{A}^{(d)} = \bigoplus_{n \geq 0} \mathcal{A}_{nd}$ similarly to the notation in Algebra, Section 10.56. Let T be a scheme. Let us consider quadruples $(d, f : T \to S, \mathcal{L}, \psi)$ over T where

1. d is the integer we fixed above,
2. $f : T \to S$ is a morphism of schemes,
3. $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
4. $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

Given a morphism $h : T' \to T$ and a quadruple $(d, f, \mathcal{L}, \psi)$ over T we can pull it back to the quadruple $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$ over T'. Given two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d, f', \mathcal{L}', \psi')$ over T with the same integer d we say they are strictly equivalent if $f = f'$ and there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded $\mathcal{O}_T$-algebra maps $f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes n}$.

For each integer $d \geq 1$ we define

$$\begin{array}{rcl} F_d : Sch^{opp} & \longrightarrow & Sets, \\ T & \longmapsto & \{\text{strict equivalence classes of } (d, f : T \to S, \mathcal{L}, \psi) \text{ as above}\} \end{array}$$

with pullbacks as defined above.

01NT **Lemma 27.16.1.** In Situation 27.15.1. Let $d \geq 1$. Let F_d be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F'_d be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism

$$F'_d \cong h_{S'} \times_{h_S} F_d$$

of functors.

Proof. A quadruple $(d, f' : T \to S', \mathcal{L}', \psi' : (f')^*(\mathcal{A}')^{(d)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes n})$ is the same as a quadruple $(d, f, \mathcal{L}, \psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n})$ together with a factorization of f as $f = g \circ f'$. Namely, the correspondence is $f = g \circ f'$, $\mathcal{L} = \mathcal{L}'$ and $\psi = \psi'$ via the identifications $(f')^*(\mathcal{A}')^{(d)} = (f')^*g^*(\mathcal{A}^{(d)}) = f^*\mathcal{A}^{(d)}$. Hence the lemma. $\square$

01NU Lemma 27.16.2. In Situation 27.15.1. Let F_d be the functor associated to $(d, S, \mathcal{A})$ above. If S is affine, then F_d is representable by the open subscheme U_d (27.12.0.1) of the scheme $\text{Proj}(\Gamma(S, \mathcal{A}))$.

Proof. Write $S = \text{Spec}(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is a graded R-algebra and $\mathcal{A} = \widetilde{A}$. To prove the lemma we have to identify the functor F_d with the functor $F_d^{triples}$ of triples defined in Section 27.12.

Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple. We may think of ψ as a $\mathcal{O}_S$-module map $\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}$. Since $\mathcal{A}^{(d)}$ is quasi-coherent this is the same thing as an R-linear homomorphism of graded rings $A^{(d)} \to \Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n})$. Clearly, $\Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}) = \Gamma_*(T, \mathcal{L})$. Thus we may associate to the quadruple the triple $(d, \mathcal{L}, \psi)$.

Conversely, let $(d, \mathcal{L}, \psi)$ be a triple. The composition $R \to A_0 \to \Gamma(T, \mathcal{O}_T)$ determines a morphism $f : T \to S = \text{Spec}(R)$, see Schemes, Lemma 26.6.4. With this choice of f the map $A^{(d)} \to \Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n})$ is R-linear, and hence corresponds to a ψ which we can use for a quadruple $(d, f : T \to S, \mathcal{L}, \psi)$. We omit the verification that this establishes an isomorphism of functors $F_d = F_d^{triples}$. $\square$

01NV Lemma 27.16.3. In Situation 27.15.1. The functor F_d is representable by a scheme.

Proof. We are going to use Schemes, Lemma 26.15.4.

First we check that F_d satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i \in I} U_i$ is an open covering, and that $(d, f_i, \mathcal{L}_i, \psi_i) \in F_d(U_i)$ such that $(d, f_i, \mathcal{L}_i, \psi_i)|_{U_i \cap U_j}$ and $(d, f_j, \mathcal{L}_j, \psi_j)|_{U_i \cap U_j}$ are strictly equivalent. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see Schemes, Section 26.14. Thus $f_i^*\mathcal{A}^{(d)} = f^*\mathcal{A}^{(d)}|_{U_i}$. It also implies there exist isomorphisms $\beta_{ij} : \mathcal{L}_i|_{U_i \cap U_j} \to \mathcal{L}_j|_{U_i \cap U_j}$ such that $\beta_{ij} \circ \psi_i = \psi_j$ on $U_i \cap U_j$. Note that the isomorphisms β_{ij} are uniquely determined by this requirement because the maps $f_i^*\mathcal{A}_d \to \mathcal{L}_i$ are surjective. In particular we see that $\beta_{jk} \circ \beta_{ij} = \beta_{ik}$ on $U_i \cap U_j \cap U_k$. Hence by Sheaves, Section 6.33 the invertible sheaves $\mathcal{L}_i$ glue to an invertible $\mathcal{O}_T$-module $\mathcal{L}$ and the morphisms ψ_i glue to morphism of $\mathcal{O}_T$-algebras $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$. This proves that F_d satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i \in I} U_i$ be an affine open covering. Let $F_{d,i} \subset F_d$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each $F_{d,i}$ is representable. This is the case because $F_{d,i}$ is identified with the functor associated to U_i equipped with the quasi-coherent graded $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$ by Lemma 27.16.1. Thus the result follows from Lemma 27.16.2.

Next we show that $F_{d,i} \subset F_d$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of $F_{d,i}$ that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_{d,i}(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_{d,i})_{i \in I}$ covers F_d. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i \in I} U_i$ is an open covering of S we see that $T = \bigcup_{i \in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_{d,i}(V_i)$. This finishes the proof of the lemma. $\square$

At this point we can redo the material at the end of Section 27.12 in the current relative setting and define a functor which is representable by $\underline{\text{Proj}}_S(\mathcal{A})$. To do this we introduce the notion of equivalence between two quadruples $(d, f : T \to S, \mathcal{L}, \psi)$ and $(d', f' : T \to S, \mathcal{L}', \psi')$ with possibly different values of the integers d, d'. Namely, we say these are equivalent if $f = f'$, and there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ such that $\beta \circ \psi|_{f^*\mathcal{A}^{(dd')}} = \psi'|_{f^*\mathcal{A}^{(dd')}}$. The following lemma implies that this defines an equivalence relation. (This is not a complete triviality.)

01NW Lemma 27.16.4. In Situation 27.15.1. Let T be a scheme. Let $(d, f, \mathcal{L}, \psi)$, $(d', f', \mathcal{L}', \psi')$ be two quadruples over T. The following are equivalent:

(1) Let $m = \text{lcm}(d, d')$. Write $m = ad = a'd'$. We have $f = f'$ and there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n \geq 0}(\mathcal{L}')^{\otimes mn}$.
(2) The quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are equivalent.
(3) We have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n \geq 0}(\mathcal{L}')^{\otimes mn}$.

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Assume (3) for some integer $m = ad = a'd'$. Let $m_0 = \text{lcm}(d, d')$ and write it as $m_0 = a_0 d = a_0' d'$. We are given an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property described in (3). We want to find an isomorphism $\beta_0 : \mathcal{L}^{\otimes a_0} \to (\mathcal{L}')^{\otimes a_0'}$ having that property as well. Since by assumption the maps $\psi : f^*\mathcal{A}_d \to \mathcal{L}$ and $\psi' : (f')^*\mathcal{A}_{d'} \to \mathcal{L}'$ are surjective the same is true for the maps $\psi : f^*\mathcal{A}_{m_0} \to \mathcal{L}^{\otimes a_0}$ and $\psi' : (f')^*\mathcal{A}_{m_0} \to (\mathcal{L}')^{\otimes a_0}$. Hence if β_0 exists it is uniquely determined by the condition that $\beta_0 \circ \psi = \psi'$. This means that we may work locally on T. Hence we may assume that $f = f' : T \to S$ maps into an affine open, in other words we may assume that S is affine. In this case the result follows from the corresponding result for triples (see Lemma 27.12.4) and the fact that triples and quadruples correspond in the affine base case (see proof of Lemma 27.16.2). $\square$

Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the quadruple $(d, f, \mathcal{L}, \psi)$ over T the quadruple $(d', f, \mathcal{L}^{\otimes a}, \psi|_{f^*\mathcal{A}^{(d')}})$. One of the implications of Lemma 27.16.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d \in \mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = \lim\nolimits_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

01NX (27.16.4.1)
$$F : Sch^{opp} \longrightarrow \text{Sets}$$

01NY Lemma 27.16.5. In Situation 27.15.1. The functor F above is representable by a scheme.

Proof. Let $U_d \to S$ be the scheme representing the functor F_d defined above. Let $\mathcal{L}_d$, $\psi^d : \pi_d^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}_d^{\otimes n}$ be the universal object. If $d|d'$, then we may consider the quadruple $(d', \pi_d, \mathcal{L}_d^{\otimes d'/d}, \psi^d|_{\mathcal{A}^{(d')}})$ which determines a canonical morphism $U_d \to U_{d'}$ over S. By construction this morphism corresponds to the transformation of functors $F_d \to F_{d'}$ defined above.

For every affine open $\text{Spec}(R) = V \subset S$ setting $A = \Gamma(V, \mathcal{A})$ we have a canonical identification of the base change $U_{d,V}$ with the corresponding open subscheme of $\text{Proj}(A)$, see Lemma 27.16.2. Moreover, the morphisms $U_{d,V} \to U_{d',V}$ constructed above correspond to the inclusions of opens in $\text{Proj}(A)$. Thus we conclude that $U_d \to U_{d'}$ is an open immersion.

This allows us to construct X by glueing the schemes U_d along the open immersions $U_d \to U_{d'}$. Technically, it is convenient to choose a sequence $d_1|d_2|d_3|\ldots$ such that every positive integer divides one of the d_i and to simply take $X = \bigcup U_{d_i}$ using the open immersions above. It is then a simple matter to prove that X represents the functor F. □

01NZ Lemma 27.16.6. In Situation 27.15.1. The scheme $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \to S$ constructed in Lemma 27.15.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.

Proof. Let X be the scheme representing the functor F. Note that X is a scheme over S since the functor F comes equipped with a natural transformation $F \to h_S$. Write $Y = \underline{\text{Proj}}_S(\mathcal{A})$. We have to show that $X \cong Y$ as S-schemes. We give two arguments.

The first argument uses the construction of X as the union of the schemes U_d representing F_d in the proof of Lemma 27.16.5. Over each affine open of S we can identify X with the homogeneous spectrum of the sections of $\mathcal{A}$ over that open, since this was true for the opens U_d. Moreover, these identifications are compatible with further restrictions to smaller affine opens. On the other hand, Y was constructed by glueing these homogeneous spectra. Hence we can glue these isomorphisms to an isomorphism between X and $\underline{\text{Proj}}_S(\mathcal{A})$ as desired. Details omitted.

Here is the second argument. Lemma 27.15.5 shows that there exists a morphism of graded algebras

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus_{n \geq 0} \mathcal{O}_Y(n)$$

over Y which on sections over affine opens of S agrees with (27.10.1.3). Hence for every $y \in Y$ there exists an open neighbourhood $V \subset Y$ of y and an integer $d \geq 1$ such that for $d|n$ the sheaf $\mathcal{O}_Y(n)|_V$ is invertible and the multiplication maps $\mathcal{O}_Y(n)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y(m)|_V \to \mathcal{O}_Y(n+m)|_V$ are isomorphisms. Thus ψ restricted to the sheaf $\pi^*\mathcal{A}^{(d)}|_V$ gives an element of $F_d(V)$. Since the opens V cover Y we see "ψ" gives rise to an element of $F(Y)$. Hence a canonical morphism $Y \to X$ over S. Because this construction is completely canonical to see that it is an isomorphism we may work locally on S. Hence we reduce to the case S affine where the result is clear. □

01O0 Definition 27.16.7. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The relative homogeneous spectrum of $\mathcal{A}$ over S, or the homogeneous spectrum of $\mathcal{A}$ over S, or the relative Proj of $\mathcal{A}$ over S is the scheme constructed in Lemma 27.15.4 which represents the functor F (27.16.4.1), see Lemma 27.16.6. We denote it $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \to S$.

The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$ (the twists of the structure sheaf) and a "universal" homomorphism of graded algebras

$$\psi_{univ} : \mathcal{A} \longrightarrow \pi_*\left(\bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)\right)$$

see Lemma 27.15.5. We may also think of this as a homomorphism

$$\psi_{univ} : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n)$$

if we like. The following lemma is a formulation of the universality of this object.

01O1 Lemma 27.16.8. In Situation 27.15.1. Let $(f : T \to S, d, \mathcal{L}, \psi)$ be a quadruple. Let $r_{d,\mathcal{L},\psi} : T \to \underline{\text{Proj}}_S(\mathcal{A})$ be the associated S-morphism. There exists an isomorphism of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras

$$\theta : r_{d,\mathcal{L},\psi}^*\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(nd)\right) \longrightarrow \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}^{(d)} \arrow[rr, "\psi"] \arrow[dr, "\psi_{univ}"'] & & f_*\left(\bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}\right) \\
& \pi_*\left(\bigoplus_{n\geq 0} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(nd)\right) \arrow[ur, "\theta"'] &
\end{tikzcd}$$

The commutativity of this diagram uniquely determines θ.

Proof. Note that the quadruple $(f : T \to S, d, \mathcal{L}, \psi)$ defines an element of $F_d(T)$. Let $U_d \subset \underline{\text{Proj}}_S(\mathcal{A})$ be the locus where the sheaf $\mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)$ is invertible and generated by the image of $\psi_{univ} : \pi^*\mathcal{A}_d \to \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)$. Recall that U_d represents the functor F_d, see the proof of Lemma 27.16.5. Hence the result will follow if we can show the quadruple $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal family, i.e., the representing object in $F_d(U_d)$. We may do this after restricting to an affine open of S because (a) the formation of the functors F_d commutes with base change (see Lemma 27.16.1), and (b) the pair $(\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(n), \psi_{univ})$ is constructed by glueing over affine opens in S (see Lemma 27.15.5). Hence we may assume that S is affine. In this case the functor of quadruples F_d and the functor of triples F_d agree (see proof of Lemma 27.16.2) and moreover Lemma 27.12.2 shows that $(d, \mathcal{O}_{U_d}(d), \psi^d)$ is the universal triple over U_d. Going backwards through the identifications in the proof of Lemma 27.16.2 shows that $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal quadruple as desired. □

01O2 Lemma 27.16.9. Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The morphism $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \to S$ is separated.

Proof. To prove a morphism is separated we may work locally on the base, see Schemes, Section 26.21. By construction $\underline{\text{Proj}}_S(\mathcal{A})$ is over any affine $U \subset S$ isomorphic to $\text{Proj}(A)$ with $A = \mathcal{A}(U)$. By Lemma 27.8.8 we see that $\text{Proj}(A)$ is separated. Hence $\text{Proj}(A) \to U$ is separated (see Schemes, Lemma 26.21.13) as desired. □

01O3 Lemma 27.16.10. Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. Let $g : S' \to S$ be any morphism of schemes. Then there is a canonical isomorphism

$$r : \underline{\text{Proj}}_{S'}(g^*\mathcal{A}) \longrightarrow S' \times_S \underline{\text{Proj}}_S(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^*\text{pr}_2^* \left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{\text{Proj}}_{S'}(g^*\mathcal{A})}(d)$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\text{Proj}}_{S'}(g^*\mathcal{A})}$-algebras.

Proof. This follows from Lemma 27.16.1 and the construction of $\underline{\text{Proj}}_S(\mathcal{A})$ in Lemma 27.16.5 as the union of the schemes U_d representing the functors F_d. In terms of the construction of relative Proj via glueing this isomorphism is given by the isomorphisms constructed in Lemma 27.11.6 which provides us with the isomorphism θ. Some details omitted. □

01O4 Lemma 27.16.11. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. In this case the scheme $X = \underline{\text{Proj}}_S(\mathcal{A})$ represents the functor F_1 which associates to a scheme $f : T \to S$ over S the set of pairs $(\mathcal{L}, \psi)$, where

(1) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(2) $\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a graded $\mathcal{O}_T$-algebra homomorphism such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective

up to strict equivalence as above. Moreover, in this case all the quasi-coherent sheaves $\mathcal{O}_{\underline{\text{Proj}}(\mathcal{A})}(n)$ are invertible $\mathcal{O}_{\underline{\text{Proj}}(\mathcal{A})}$-modules and the multiplication maps induce isomorphisms $\mathcal{O}_{\underline{\text{Proj}}(\mathcal{A})}(n) \otimes_{\mathcal{O}_{\underline{\text{Proj}}(\mathcal{A})}} \mathcal{O}_{\underline{\text{Proj}}(\mathcal{A})}(m) = \mathcal{O}_{\underline{\text{Proj}}(\mathcal{A})}(n+m)$.

Proof. Under the assumptions of the lemma the sheaves $\mathcal{O}_{\underline{\text{Proj}}(\mathcal{A})}(n)$ are invertible and the multiplication maps isomorphisms by Lemma 27.16.5 and Lemma 27.12.3 over affine opens of S. Thus X actually represents the functor F_1, see proof of Lemma 27.16.5. □

27.17. Quasi-coherent sheaves on relative Proj

01O5 We briefly discuss how to deal with graded modules in the relative setting.

We place ourselves in Situation 27.15.1. So S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{M} = \bigoplus_{n\in\mathbf{Z}} \mathcal{M}_n$ be a graded $\mathcal{A}$-module, quasi-coherent as an $\mathcal{O}_S$-module. We are going to describe the associated quasi-coherent sheaf of modules on $\underline{\text{Proj}}_S(\mathcal{A})$. We first describe the value of this sheaf on schemes T mapping into the relative Proj.

Let T be a scheme. Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple over T, as in Section 27.16. We define a quasi-coherent sheaf $\widetilde{\mathcal{M}}_T$ of $\mathcal{O}_T$-modules as follows

01O6 (27.17.0.1)
$$\widetilde{\mathcal{M}}_T = \left(f^*\mathcal{M}^{(d)} \otimes_{f^*\mathcal{A}^{(d)}} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}\right)\right)_0$$

So $\widetilde{\mathcal{M}}_T$ is the degree 0 part of the tensor product of the graded $f^*\mathcal{A}^{(d)}$-modules $\mathcal{M}^{(d)}$ and $\bigoplus_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n}$. Note that the sheaf $\widetilde{\mathcal{M}}_T$ depends on the quadruple even though we suppressed this in the notation. This construction has the pleasing property that given any morphism $g : T' \to T$ we have $\widetilde{\mathcal{M}}_{T'} = g^*\widetilde{\mathcal{M}}_T$ where $\widetilde{\mathcal{M}}_{T'}$ denotes the quasi-coherent sheaf associated to the pullback quadruple $(d, f \circ g, g^*\mathcal{L}, g^*\psi)$.

Since all sheaves in (27.17.0.1) are quasi-coherent we can spell out the construction over an affine open $\mathrm{Spec}(C) = V \subset T$ which maps into an affine open $\mathrm{Spec}(R) = U \subset S$. Namely, suppose that $\mathcal{A}|_U$ corresponds to the graded R-algebra A, that $\mathcal{M}|_U$ corresponds to the graded A-module M, and that $\mathcal{L}|_V$ corresponds to the invertible C-module L. The map ψ gives rise to a graded R-algebra map $\gamma : A^{(d)} \to \bigoplus_{n \geq 0} L^{\otimes n}$. (Tensor powers of L over C.) Then $(\widetilde{\mathcal{M}}_T)|_V$ is the quasi-coherent sheaf associated to the C-module

$$N_{R,C,A,M,\gamma} = \left(M^{(d)} \otimes_{A^{(d)}, \gamma} \left(\bigoplus\nolimits_{n \in \mathbf{Z}} L^{\otimes n}\right)\right)_0$$

By assumption we may even cover T by affine opens V such that there exists some $a \in A_d$ such that $\gamma(a) \in L$ is a C-basis for the module L. In that case any element of $N_{R,C,A,M,\gamma}$ is a sum of pure tensors $\sum m_i \otimes \gamma(a)^{-n_i}$ with $m \in M_{n_i d}$. In fact we may multiply each m_i with a suitable positive power of a and collect terms to see that each element of $N_{R,C,A,M,\gamma}$ can be written as $m \otimes \gamma(a)^{-n}$ with $m \in M_{nd}$ and $n \gg 0$. In other words we see that in this case

$$N_{R,C,A,M,\gamma} = M_{(a)} \otimes_{A_{(a)}} C$$

where the map $A_{(a)} \to C$ is the map $x/a^n \mapsto \gamma(x)/\gamma(a)^n$. In other words, this is the value of $\widetilde{M}$ on $D_+(a) \subset \mathrm{Proj}(A)$ pulled back to $\mathrm{Spec}(C)$ via the morphism $\mathrm{Spec}(C) \to D_+(a)$ coming from γ.

01O7 Lemma 27.17.1. In Situation 27.15.1. For any quasi-coherent sheaf of graded $\mathcal{A}$-modules $\mathcal{M}$ on S, there exists a canonical associated sheaf of $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}$-modules $\widetilde{\mathcal{M}}$ with the following properties:

(1) Given a scheme T and a quadruple $(T \to S, d, \mathcal{L}, \psi)$ over T corresponding to a morphism $h : T \to \underline{\mathrm{Proj}}_S(\mathcal{A})$ there is a canonical isomorphism $\widetilde{\mathcal{M}}_T = h^*\widetilde{\mathcal{M}}$ where $\widetilde{\mathcal{M}}_T$ is defined by (27.17.0.1).
(2) The isomorphisms of (1) are compatible with pullbacks.
(3) There is a canonical map
$$\pi^*\mathcal{M}_0 \longrightarrow \widetilde{\mathcal{M}}.$$
(4) The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is functorial in $\mathcal{M}$.
(5) The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is exact.
(6) There are canonical maps
$$\widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}} \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}$$
as in Lemma 27.9.1.
(7) There exist canonical maps
$$\pi^*\mathcal{M} \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \widetilde{\mathcal{M}(n)}$$
generalizing (27.10.1.6).
(8) The formation of $\widetilde{\mathcal{M}}$ commutes with base change.

Proof. Omitted. We should split this lemma into parts and prove the parts separately. □

27.18. Functoriality of relative Proj

07ZF This section is the analogue of Section 27.11 for the relative Proj. Let S be a scheme. A graded $\mathcal{O}_S$-algebra map $\psi : \mathcal{A} \to \mathcal{B}$ does not always give rise to a morphism of associated relative Proj. The correct result is stated as follows.

07ZG Lemma 27.18.1. Let S be a scheme. Let $\mathcal{A}$, $\mathcal{B}$ be two graded quasi-coherent $\mathcal{O}_S$-algebras. Set $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ and $q : Y = \underline{\text{Proj}}_S(\mathcal{B}) \to S$. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of graded $\mathcal{O}_S$-algebras. There is a canonical open $U(\psi) \subset Y$ and a canonical morphism of schemes

$$r_\psi : U(\psi) \longrightarrow X$$

over S and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^*\left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any affine open $W \subset S$ the triple

$$(U(\psi) \cap p^{-1}W, \quad r_\psi|_{U(\psi) \cap p^{-1}W} : U(\psi) \cap p^{-1}W \to q^{-1}W, \quad \theta|_{U(\psi) \cap p^{-1}W})$$

is equal to the triple associated to $\psi : \mathcal{A}(W) \to \mathcal{B}(W)$ in Lemma 27.11.1 via the identifications $p^{-1}W = \text{Proj}(\mathcal{A}(W))$ and $q^{-1}W = \text{Proj}(\mathcal{B}(W))$ of Section 27.15.

Proof. This lemma proves itself by glueing the local triples. □

07ZH Lemma 27.18.2. Let S be a scheme. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be quasi-coherent graded $\mathcal{O}_S$-algebras. Set $X = \underline{\text{Proj}}_S(\mathcal{A})$, $Y = \underline{\text{Proj}}_S(\mathcal{B})$ and $Z = \underline{\text{Proj}}_S(\mathcal{C})$. Let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ be graded $\mathcal{O}_S$-algebra maps. Then we have

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi \circ \varphi} = r_\varphi \circ r_\psi|_{U(\psi \circ \varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi \circ \varphi}$$

with obvious notation.

Proof. Omitted. □

07ZI Lemma 27.18.3. With hypotheses and notation as in Lemma 27.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$. Then

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ is a closed immersion, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \to \mathcal{B}$ is surjective).

Proof. Follows on combining Lemma 27.18.1 with Lemma 27.11.3. □

07ZJ Lemma 27.18.4. With hypotheses and notation as in Lemma 27.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ is an isomorphism, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.

Proof. Follows on combining Lemma 27.18.1 with Lemma 27.11.4. □

07ZK Lemma 27.18.5. With hypotheses and notation as in Lemma 27.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$ and that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Then

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ is a closed immersion, and
(3) the maps $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.

Proof. Follows on combining Lemma 27.18.1 with Lemma 27.11.5. □

27.19. Invertible sheaves and morphisms into relative Proj

01O8 It seems that we may need the following lemma somewhere. The situation is the following:

(1) Let S be a scheme.
(2) Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra.
(3) Denote $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \to S$ the relative homogeneous spectrum over S.
(4) Let $f : X \to S$ be a morphism of schemes.
(5) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.
(6) Let $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ be a homomorphism of graded $\mathcal{O}_X$-algebras.

Given this data set

$$U(\psi) = \bigcup\nolimits_{(U,V,a)} U_{\psi(a)}$$

where (U, V, a) satisfies:

(1) $V \subset S$ affine open,
(2) $U = f^{-1}(V)$, and
(3) $a \in \mathcal{A}(V)_+$ is homogeneous.

Namely, then $\psi(a) \in \Gamma(U, \mathcal{L}^{\otimes \deg(a)})$ and $U_{\psi(a)}$ is the corresponding open (see Modules, Lemma 17.25.10).

01O9 Lemma 27.19.1. With assumptions and notation as above. The morphism ψ induces a canonical morphism of schemes over S

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow \underline{\text{Proj}}_S(\mathcal{A})$$

together with a map of graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta : r_{\mathcal{L},\psi}^*\left(\bigoplus\nolimits_{d \geq 0} \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{L}^{\otimes d}|_{U(\psi)}$$

characterized by the following properties:

(1) For every open $V \subset S$ and every $d \geq 0$ the diagram

$$\begin{tikzcd} \mathcal{A}_d(V) \arrow[r, "\psi"'] \arrow[d, "\psi"'] & \Gamma(f^{-1}(V), \mathcal{L}^{\otimes d}) \arrow[d, "restrict"] \\ \Gamma(\pi^{-1}(V), \mathcal{O}_{\underline{\text{Proj}}_S(\mathcal{A})}(d)) \arrow[r, "\theta"] & \Gamma(f^{-1}(V) \cap U(\psi), \mathcal{L}^{\otimes d}) \end{tikzcd}$$

is commutative.

(2) For any $d \geq 1$ and any open subscheme $W \subset X$ such that $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ is surjective the restriction of the morphism $r_{\mathcal{L},\psi}$ agrees with the morphism $W \to \underline{\text{Proj}}_S(\mathcal{A})$ which exists by the construction of the relative homogeneous spectrum, see Definition 27.16.7.

(3) For any affine open $V \subset S$, the restriction

$$(U(\psi) \cap f^{-1}(V), r_{\mathcal{L},\psi}|_{U(\psi)\cap f^{-1}(V)}, \theta|_{U(\psi)\cap f^{-1}(V)})$$

agrees via i_V (see Lemma 27.15.4) with the triple $(U(\psi'), r_{\mathcal{L},\psi'}, \theta')$ of Lemma 27.14.1 associated to the map $\psi' : A = \mathcal{A}(V) \to \Gamma_*(f^{-1}(V), \mathcal{L}|_{f^{-1}(V)})$ induced by ψ.

Proof. Use characterization (3) to construct the morphism $r_{\mathcal{L},\psi}$ and θ locally over S. Use the uniqueness of Lemma 27.14.1 to show that the construction glues. Details omitted. $\square$

27.20. Twisting by invertible sheaves and relative Proj

02NB Let S be a scheme. Let $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{L}$ be an invertible sheaf on S. In this situation we obtain another quasi-coherent graded $\mathcal{O}_S$-algebra, namely

$$\mathcal{B} = \bigoplus\nolimits_{d \geq 0} \mathcal{A}_d \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes d}$$

It turns out that $\mathcal{A}$ and $\mathcal{B}$ have isomorphic relative homogeneous spectra.

02NC **Lemma 27.20.1.** With notation S, $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{B}$ as above. There is a canonical isomorphism

$$\begin{tikzcd} P = \underline{\text{Proj}}_S(\mathcal{A}) \arrow[rr, "g"] \arrow[dr, "\pi"'] & & \underline{\text{Proj}}_S(\mathcal{B}) = P' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

with the following properties

(1) There are isomorphisms $\theta_n : g^*\mathcal{O}_{P'}(n) \to \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$ which fit together to give an isomorphism of $\mathbf{Z}$-graded algebras

$$\theta : g^*\left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{O}_{P'}(n)\right) \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$$

(2) For every open $V \subset S$ the diagrams

$$\begin{tikzcd} \mathcal{A}_n(V) \otimes \mathcal{L}^{\otimes n}(V) \arrow[r, "multiply"'] \arrow[d, "\psi \otimes \pi^*"] & \mathcal{B}_n(V) \arrow[dd, "\psi"] \\ \Gamma(\pi^{-1}V, \mathcal{O}_P(n)) \otimes \Gamma(\pi^{-1}V, \pi^*\mathcal{L}^{\otimes n}) \arrow[d, "multiply"] & \\ \Gamma(\pi^{-1}V, \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}) & \Gamma(\pi'^{-1}V, \mathcal{O}_{P'}(n)) \arrow[l, "\theta_n"'] \end{tikzcd}$$

are commutative.

(3) Add more here as necessary.

Proof. This is the identity map when $\mathcal{L} \cong \mathcal{O}_S$. In general choose an open covering of S such that $\mathcal{L}$ is trivialized over the pieces and glue the corresponding maps. Details omitted. $\square$

27.21. Projective bundles

01OA Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 17.21.6 the symmetric algebra $\text{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Note that it is generated in degree 1 over $\mathcal{O}_S$. Hence it makes sense to apply the construction of the previous section to it, specifically Lemmas 27.16.5 and 27.16.11.

01OB Definition 27.21.1. Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[3]. We denote

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E})) \longrightarrow S$$

and we call it the projective bundle associated to $\mathcal{E}$. The symbol $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)$ indicates the invertible $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$-module of Lemma 27.16.11 and is called the nth twist of the structure sheaf.

According to Lemma 27.15.5 there are canonical $\mathcal{O}_S$-module homomorphisms

$$\text{Sym}^n(\mathcal{E}) \longrightarrow \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n) \quad \text{equivalently} \quad \pi^*\text{Sym}^n(\mathcal{E}) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)$$

for all $n \geq 0$. In particular, for $n = 1$ we have

$$\mathcal{E} \longrightarrow \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \quad \text{equivalently} \quad \pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

and the map $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is a surjection by Lemma 27.16.11. This is a good way to remember how we have normalized our construction of $\mathbf{P}(\mathcal{E})$.

Warning: In some references the scheme $\mathbf{P}(\mathcal{E})$ is only defined for $\mathcal{E}$ finite locally free on S. Moreover sometimes $\mathbf{P}(\mathcal{E})$ is actually defined as our $\mathbf{P}(\mathcal{E}^\vee)$ where $\mathcal{E}^\vee$ is the dual of $\mathcal{E}$ (and this is done only when $\mathcal{E}$ is finite locally free).

Let S, $\mathcal{E}$, $\mathbf{P}(\mathcal{E}) \to S$ be as in Definition 27.21.1. Let $f : T \to S$ be a scheme over S. Let $\psi : f^*\mathcal{E} \to \mathcal{L}$ be a surjection where $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module. The induced graded $\mathcal{O}_T$-algebra map

$$f^*\text{Sym}(\mathcal{E}) = \text{Sym}(f^*\mathcal{E}) \to \text{Sym}(\mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \mathcal{L}^{\otimes n}$$

corresponds to a morphism

$$\varphi_{\mathcal{L},\psi} : T \longrightarrow \mathbf{P}(\mathcal{E})$$

over S by our construction of the relative Proj as the scheme representing the functor F in Section 27.16. On the other hand, given a morphism $\varphi : T \to \mathbf{P}(\mathcal{E})$ over S we can set $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ and $\psi : f^*\mathcal{E} \to \mathcal{L}$ equal to the pullback by φ of the canonical surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By Lemma 27.16.11 these constructions are inverse bijections between the set of isomorphism classes of pairs $(\mathcal{L}, \psi)$ and the set of morphisms $\varphi : T \to \mathbf{P}(\mathcal{E})$ over S. Thus we see that $\mathbf{P}(\mathcal{E})$ represents the functor which associates to $f : T \to S$ the set of $\mathcal{O}_T$-module quotients of $f^*\mathcal{E}$ which are locally free of rank 1.

0FCY Example 27.21.2 (Projective space of a vector space). Let k be a field. Let V be a k-vector space. The corresponding projective space is the k-scheme

$$\mathbf{P}(V) = \text{Proj}(\text{Sym}(V))$$

where $\text{Sym}(V)$ is the symmetric algebra on V over k. Of course we have $\mathbf{P}(V) \cong \mathbf{P}^n_k$ if $\dim(V) = n + 1$ because then the symmetric algebra on V is isomorphic to a

[3]The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [DG67, II, Definition 4.1.1].

polynomial ring in $n+1$ variables. If we think of V as a quasi-coherent module on $\mathrm{Spec}(k)$, then $\mathbf{P}(V)$ is the corresponding projective space bundle over $\mathrm{Spec}(k)$. By the discussion above a k-valued point p of $\mathbf{P}(V)$ corresponds to a surjection of k-vector spaces $V \to L_p$ with $\dim(L_p) = 1$. More generally, let X be a scheme over k, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and let $\psi : V \to \Gamma(X, \mathcal{L})$ be a k-linear map such that $\mathcal{L}$ is generated as an $\mathcal{O}_X$-module by the sections in the image of ψ. Then the discussion above gives a canonical morphism

$$\varphi_{\mathcal{L},\psi} : X \longrightarrow \mathbf{P}(V)$$

of schemes over k such that there is an isomorphism $\theta : \varphi_{\mathcal{L},\psi}^*\mathcal{O}_{\mathbf{P}(V)}(1) \to \mathcal{L}$ and such that ψ agrees with the composition

$$V \to \Gamma(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)) \to \Gamma(X, \varphi_{\mathcal{L},\psi}^*\mathcal{O}_{\mathbf{P}(V)}(1)) \to \Gamma(X, \mathcal{L})$$

See Lemma 27.14.1. If $V \subset \Gamma(X, \mathcal{L})$ is a subspace, then we will denote the morphism constructed above simply as $\varphi_{\mathcal{L},V}$. If $\dim(V) = n+1$ and we choose a basis $v_0, \ldots, v_n$ of V then the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi_{\mathcal{L},\psi}"'] \arrow[d, equal] & \mathbf{P}(V) \arrow[d, "\cong"] \\ X \arrow[r, "\varphi_{(\mathcal{L},(s_0,\ldots,s_n))}"] & \mathbf{P}^n_k \end{tikzcd}$$

is commutative, where $s_i = \psi(v_i) \in \Gamma(X, \mathcal{L})$, where $\varphi_{(\mathcal{L},(s_0,\ldots,s_n))}$ is as in Section 27.13, and where the right vertical arrow corresponds to the isomorphism $k[T_0, \ldots, T_n] \to \mathrm{Sym}(V)$ sending T_i to v_i.

01OC Example 27.21.3. The map $\mathrm{Sym}^n(\mathcal{E}) \to \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n))$ is an isomorphism if $\mathcal{E}$ is locally free, but in general need not be an isomorphism. In fact we will give an example where this map is not injective for $n = 1$. Set $S = \mathrm{Spec}(A)$ with

$$A = k[u, v, s_1, s_2, t_1, t_2]/I$$

where k is a field and

$$I = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1).$$

Denote $\overline{u}$ the class of u in A and similarly for the other variables. Let $M = (Ax \oplus Ay)/A(\overline{u}x + \overline{v}y)$ so that

$$\mathrm{Sym}(M) = A[x, y]/(\overline{u}x + \overline{v}y) = k[x, y, u, v, s_1, s_2, t_1, t_2]/J$$

where

$$J = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1, ux + vy).$$

In this case the projective bundle associated to the quasi-coherent sheaf $\mathcal{E} = \widetilde{M}$ on $S = \mathrm{Spec}(A)$ is the scheme

$$P = \mathrm{Proj}(\mathrm{Sym}(M)).$$

Note that this scheme as an affine open covering $P = D_+(x) \cup D_+(y)$. Consider the element $m \in M$ which is the image of the element $us_1x + vt_2y$. Note that

$$x(us_1x + vt_2y) = (s_1x + s_2y)(ux + vy) \bmod I$$

and

$$y(us_1x + vt_2y) = (t_1x + t_2y)(ux + vy) \bmod I.$$

The first equation implies that m maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(x)$ and the second that it maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(y)$. This shows that m maps to zero in $\Gamma(P, \mathcal{O}_P(1))$. On the other hand we claim that $m \neq 0$, so that m gives an example of a nonzero global section of $\mathcal{E}$ mapping to zero in $\Gamma(P, \mathcal{O}_P(1))$. Assume $m = 0$ to get a contradiction. In this case there exists an element $f \in k[u, v, s_1, s_2, t_1, t_2]$ such that

$$us_1x + vt_2y = f(ux + vy) \bmod I$$

Since I is generated by homogeneous polynomials of degree 2 we may decompose f into its homogeneous components and take the degree 1 component. In other words we may assume that

$$f = au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2$$

for some $a, b, \alpha_1, \alpha_2, \beta_1, \beta_2 \in k$. The resulting conditions are that

$$\begin{aligned} us_1 - u(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I \\ vt_2 - v(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I \end{aligned}$$

There are no terms u^2, uv, v^2 in the generators of I and hence we see $a = b = 0$. Thus we get the relations

$$\begin{aligned} us_1 - u(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I \\ vt_2 - v(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I \end{aligned}$$

We may use the first generator of I to replace any occurrence of us_1 by $vt_1 + ut_2$, the second generator of I to replace any occurrence of vs_1 by $-us_2 + vt_2$, the third generator to remove occurrences of vs_2 and the third to remove occurrences of ut_1. Then we get the relations

$$\begin{aligned} (1 - \alpha_1)vt_1 + (1 - \alpha_1)ut_2 - \alpha_2 us_2 - \beta_2 ut_2 = 0 \\ (1 - \alpha_1)vt_2 + \alpha_1 us_2 - \beta_1 vt_1 - \beta_2 vt_2 = 0 \end{aligned}$$

This implies that α_1 should be both 0 and 1 which is a contradiction as desired.

01OD Lemma 27.21.4. Let S be a scheme. The structure morphism $\mathbf{P}(\mathcal{E}) \to S$ of a projective bundle over S is separated.

Proof. Immediate from Lemma 27.16.9. □

01OE Lemma 27.21.5. Let S be a scheme. Let $n \geq 0$. Then $\mathbf{P}^n_S$ is a projective bundle over S.

Proof. Note that

$$\mathbf{P}^n_{\mathbf{Z}} = \text{Proj}(\mathbf{Z}[T_0, \ldots, T_n]) = \underline{\text{Proj}}_{\text{Spec}(\mathbf{Z})}\left(\mathbf{Z}[\widetilde{T_0, \ldots}, T_n]\right)$$

where the grading on the ring $\mathbf{Z}[T_0, \ldots, T_n]$ is given by $\deg(T_i) = 1$ and the elements of $\mathbf{Z}$ are in degree 0. Recall that $\mathbf{P}^n_S$ is defined as $\mathbf{P}^n_{\mathbf{Z}} \times_{\text{Spec}(\mathbf{Z})} S$. Moreover, forming the relative homogeneous spectrum commutes with base change, see Lemma 27.16.10. For any scheme $g : S \to \text{Spec}(\mathbf{Z})$ we have $g^*\mathcal{O}_{\text{Spec}(\mathbf{Z})}[T_0, \ldots, T_n] = \mathcal{O}_S[T_0, \ldots, T_n]$. Combining the above we see that

$$\mathbf{P}^n_S = \underline{\text{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$

Finally, note that $\mathcal{O}_S[T_0, \ldots, T_n] = \text{Sym}(\mathcal{O}_S^{\oplus n+1})$. Hence we see that $\mathbf{P}^n_S$ is a projective bundle over S. □

27.22. Grassmannians

089R In this section we introduce the standard Grassmannian functors and we show that they are represented by schemes. Pick integers k, n with $0 < k < n$. We will construct a functor

089S (27.22.0.1)
$$G(k, n) : \mathit{Sch} \longrightarrow \text{Sets}$$

which will loosely speaking parametrize k-dimensional subspaces of n-space. However, for technical reasons it is more convenient to parametrize $(n-k)$-dimensional quotients and this is what we will do.

More precisely, $G(k, n)$ associates to a scheme S the set $G(k, n)(S)$ of isomorphism classes of surjections

$$q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{Q}$$

where $\mathcal{Q}$ is a finite locally free $\mathcal{O}_S$-module of rank $n-k$. Note that this is indeed a set, for example by Modules, Lemma 17.9.8 or by the observation that the isomorphism class of the surjection q is determined by the kernel of q (and given a sheaf there is a set of subsheaves). Given a morphism of schemes $f : T \to S$ we let $G(k, n)(f) : G(k, n)(S) \to G(k, n)(T)$ which sends the isomorphism class of $q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{Q}$ to the isomorphism class of $f^*q : \mathcal{O}_T^{\oplus n} \longrightarrow f^*\mathcal{Q}$. This makes sense since (1) $f^*\mathcal{O}_S = \mathcal{O}_T$, (2) f^* is additive, (3) f^* preserves locally free modules (Modules, Lemma 17.14.3), and (4) f^* is right exact (Modules, Lemma 17.3.3).

089T **Lemma 27.22.1.** Let $0 < k < n$. The functor $G(k, n)$ of (27.22.0.1) is representable by a scheme.

Proof. Set $F = G(k, n)$. To prove the lemma we will use the criterion of Schemes, Lemma 26.15.4. The reason F satisfies the sheaf property for the Zariski topology is that we can glue sheaves, see Sheaves, Section 6.33 (some details omitted).

The family of subfunctors F_i. Let I be the set of subsets of $\{1, \ldots, n\}$ of cardinality $n-k$. Given a scheme S and $j \in \{1, \ldots, n\}$ we denote e_j the global section

$$e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \quad (1 \text{ in } j\text{th spot})$$

of $\mathcal{O}_S^{\oplus n}$. Of course these sections freely generate $\mathcal{O}_S^{\oplus n}$. Similarly, for $j \in \{1, \ldots, n-k\}$ we denote f_j the global section of $\mathcal{O}_S^{\oplus n-k}$ which is zero in all summands except the jth where we put a 1. For $i \in I$ we let

$$s_i : \mathcal{O}_S^{\oplus n-k} \longrightarrow \mathcal{O}_S^{\oplus n}$$

which is the direct sum of the coprojections $\mathcal{O}_S \to \mathcal{O}_S^{\oplus n}$ corresponding to elements of i. More precisely, if $i = \{i_1, \ldots, i_{n-k}\}$ with $i_1 < i_2 < \ldots < i_{n-k}$ then s_i maps f_j to e_{i_j} for $j \in \{1, \ldots, n-k\}$. With this notation we can set

$$F_i(S) = \{q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q} \in F(S) \mid q \circ s_i \text{ is surjective}\} \subset F(S)$$

Given a morphism $f : T \to S$ of schemes the pullback f^*s_i is the corresponding map over T. Since f^* is right exact (Modules, Lemma 17.3.3) we conclude that F_i is a subfunctor of F.

Representability of F_i. To prove this we may assume (after renumbering) that $i = \{1, \ldots, n-k\}$. This means s_i is the inclusion of the first $n-k$ summands. Observe that if $q \circ s_i$ is surjective, then $q \circ s_i$ is an isomorphism as a surjective map between finite locally free modules of the same rank (Modules, Lemma 17.14.5).

Thus if $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ is an element of $F_i(S)$, then we can use $q \circ s_i$ to identify $\mathcal{Q}$ with $\mathcal{O}_S^{\oplus n-k}$. After doing so we obtain

$$q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{O}_S^{\oplus n-k}$$

mapping e_j to f_j (notation as above) for $j = 1, \ldots, n-k$. To determine q completely we have to fix the images $q(e_{n-k+1}), \ldots, q(e_n)$ in $\Gamma(S, \mathcal{O}_S^{\oplus n-k})$. It follows that F_i is isomorphic to the functor

$$S \longmapsto \prod\nolimits_{j=n-k+1,\ldots,n} \Gamma(S, \mathcal{O}_S^{\oplus n-k})$$

This functor is isomorphic to the $k(n-k)$-fold self product of the functor $S \mapsto \Gamma(S, \mathcal{O}_S)$. By Schemes, Example 26.15.2 the latter is representable by $\mathbf{A}^1_{\mathbf{Z}}$. It follows F_i is representable by $\mathbf{A}^{k(n-k)}_{\mathbf{Z}}$ since fibred product over $\mathrm{Spec}(\mathbf{Z})$ is the product in the category of schemes.

The inclusion $F_i \subset F$ is representable by open immersions. Let S be a scheme and let $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ be an element of $F(S)$. By Modules, Lemma 17.9.4. the set $U_i = \{s \in S \mid (q \circ s_i)_s \text{ surjective}\}$ is open in S. Since $\mathcal{O}_{S,s}$ is a local ring and $\mathcal{Q}_s$ a finite $\mathcal{O}_{S,s}$-module by Nakayama's lemma (Algebra, Lemma 10.20.1) we have

$$s \in U_i \Leftrightarrow \big(\text{the map } \kappa(s)^{\oplus n-k} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s \text{ induced by } (q \circ s_i)_s \text{ is surjective}\big)$$

Let $f : T \to S$ be a morphism of schemes and let $t \in T$ be a point mapping to $s \in S$. We have $(f^*\mathcal{Q})_t = \mathcal{Q}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{T,t}$ (Sheaves, Lemma 6.26.4) and so on. Thus the map

$$\kappa(t)^{\oplus n-k} \to (f^*\mathcal{Q})_t/\mathfrak{m}_t(f^*\mathcal{Q})_t$$

induced by $(f^*q \circ f^*s_i)_t$ is the base change of the map $\kappa(s)^{\oplus n-k} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$ above by the field extension $\kappa(t)/\kappa(s)$. It follows that $s \in U_i$ if and only if t is in the corresponding open for f^*q. In particular $T \to S$ factors through U_i if and only if $f^*q \in F_i(T)$ as desired.

The collection F_i, $i \in I$ covers F. Let $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ be an element of $F(S)$. We have to show that for every point s of S there exists an $i \in I$ such that s_i is surjective in a neighbourhood of s. Thus we have to show that one of the compositions

$$\kappa(s)^{\oplus n-k} \xrightarrow{s_i} \kappa(s)^{\oplus n} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$$

is surjective (see previous paragraph). As $\mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$ is a vector space of dimension $n-k$ this follows from the theory of vector spaces. □

089U Definition 27.22.2. Let $0 < k < n$. The scheme $\mathbf{G}(k,n)$ representing the functor $G(k,n)$ is called Grassmannian over $\mathbf{Z}$. Its base change $\mathbf{G}(k,n)_S$ to a scheme S is called Grassmannian over S. If R is a ring the base change to $\mathrm{Spec}(R)$ is denoted $\mathbf{G}(k,n)_R$ and called Grassmannian over R.

The definition makes sense as we've shown in Lemma 27.22.1 that these functors are indeed representable.

089V Lemma 27.22.3. Let $n \geq 1$. There is a canonical isomorphism $\mathbf{G}(n, n+1) = \mathbf{P}^n_{\mathbf{Z}}$.

Proof. According to Lemma 27.13.1 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ represents the functor which assigns to a scheme S the set of isomorphisms classes of pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ consisting of an invertible module $\mathcal{L}$ and an $(n+1)$-tuple of global sections generating $\mathcal{L}$. Given such a pair we obtain a quotient

$$\mathcal{O}_S^{\oplus n+1} \longrightarrow \mathcal{L}, \quad (h_0, \ldots, h_n) \longmapsto \sum h_i s_i.$$

Conversely, given an element $q : \mathcal{O}_S^{\oplus n+1} \to \mathcal{Q}$ of $G(n, n+1)(S)$ we obtain such a pair, namely $(\mathcal{Q}, (q(e_1), \ldots, q(e_{n+1})))$. Here e_i, $i = 1, \ldots, n+1$ are the standard generating sections of the free module $\mathcal{O}_S^{\oplus n+1}$. We omit the verification that these constructions define mutually inverse transformations of functors. $\square$

27.23. Other chapters

CHAPTER 28

Properties of Schemes

01OH 28.1. Introduction

01OI In this chapter we introduce some absolute properties of schemes. A foundational reference is [DG67].

28.2. Constructible sets

054B Constructible and locally constructible sets are introduced in Topology, Section 5.15. We may characterize locally constructible subsets of schemes as follows.

054C Lemma 28.2.1. Let X be a scheme. A subset E of X is locally constructible in X if and only if $E \cap U$ is constructible in U for every affine open U of X.

Proof. Assume E is locally constructible. Then there exists an open covering $X = \bigcup U_i$ such that $E \cap U_i$ is constructible in U_i for each i. Let $V \subset X$ be any affine open. We can find a finite open affine covering $V = V_1 \cup \ldots \cup V_m$ such that for each j we have $V_j \subset U_i$ for some $i = i(j)$. By Topology, Lemma 5.15.4 we see that each $E \cap V_j$ is constructible in V_j. Since the inclusions $V_j \to V$ are quasi-compact (see Schemes, Lemma 26.19.2) we conclude that $E \cap V$ is constructible in V by Topology, Lemma 5.15.6. The converse implication is immediate. $\square$

0AAW Lemma 28.2.2. Let X be a scheme and let $E \subset X$ be a locally constructible subset. Let $\xi \in X$ be a generic point of an irreducible component of X.

(1) If $\xi \in E$, then an open neighbourhood of ξ is contained in E.
(2) If $\xi \notin E$, then an open neighbourhood of ξ is disjoint from E.

Proof. As the complement of a locally constructible subset is locally constructible it suffices to show (2). We may assume X is affine and hence E constructible (Lemma 28.2.1). In this case X is a spectral space (Algebra, Lemma 10.26.2). Then $\xi \notin E$ implies $\xi \notin \overline{E}$ by Topology, Lemma 5.23.6 and the fact that there are no points of X different from ξ which specialize to ξ. $\square$

054D Lemma 28.2.3. Let X be a quasi-separated scheme. The intersection of any two quasi-compact opens of X is a quasi-compact open of X. Every quasi-compact open of X is retrocompact in X.

Proof. If U and V are quasi-compact open then $U \cap V = \Delta^{-1}(U \times V)$, where $\Delta : X \to X \times X$ is the diagonal. As X is quasi-separated we see that Δ is quasi-compact. Hence we see that $U \cap V$ is quasi-compact as $U \times V$ is quasi-compact (details omitted; use Schemes, Lemma 26.17.4 to see $U \times V$ is a finite union of affines). The other assertions follow from the first and Topology, Lemma 5.27.1. $\square$

094L Lemma 28.2.4. Let X be a quasi-compact and quasi-separated scheme. Then the underlying topological space of X is a spectral space.

Proof. By Topology, Definition 5.23.1 we have to check that X is sober, quasi-compact, has a basis of quasi-compact opens, and the intersection of any two quasi-compact opens is quasi-compact. This follows from Schemes, Lemma 26.11.1 and 26.11.2 and Lemma 28.2.3 above. $\square$

054E Lemma 28.2.5. Let X be a quasi-compact and quasi-separated scheme. Any locally constructible subset of X is constructible.

Proof. As X is quasi-compact we can choose a finite affine open covering $X = V_1 \cup \ldots \cup V_m$. As X is quasi-separated each V_i is retrocompact in X by Lemma 28.2.3. Hence by Topology, Lemma 5.15.6 we see that $E \subset X$ is constructible in X if and only if $E \cap V_j$ is constructible in V_j. Thus we win by Lemma 28.2.1. $\square$

07ZL Lemma 28.2.6. Let X be a scheme. A subset E of X is retrocompact in X if and only if $E \cap U$ is quasi-compact for every affine open U of X.

Proof. Immediate from the fact that every quasi-compact open of X is a finite union of affine opens. $\square$

0F2M Lemma 28.2.7. A partition $X = \coprod_{i \in I} X_i$ of a scheme X with retrocompact parts is locally finite if and only if the parts are locally constructible.

Proof. See Topology, Definitions 5.12.1, 5.28.1, and 5.28.4 for the definitions of retrocompact, partition, and locally finite.

If the partition is locally finite and $U \subset X$ is an affine open, then we see that $U = \coprod_{i \in I} U \cap X_i$ is a finite partition (more precisely, all but a finite number of its parts are empty). Hence $U \cap X_i$ is quasi-compact and its complement is retrocompact in U as a finite union of retrocompact parts. Thus $U \cap X_i$ is constructible by Topology, Lemma 5.15.13. It follows that X_i is locally constructible by Lemma 28.2.1.

Assume the parts are locally constructible. Then for any affine open $U \subset X$ we obtain a covering $U = \coprod X_i \cap U$ by constructible subsets. Since the constructible topology is quasi-compact, see Topology, Lemma 5.23.2, this covering has a finite refinement, i.e., the partition is locally finite. $\square$

28.3. Integral, irreducible, and reduced schemes

01OJ

01OK Definition 28.3.1. Let X be a scheme. We say X is integral if it is nonempty and for every nonempty affine open $\mathrm{Spec}(R) = U \subset X$ the ring R is an integral domain.

01OL Lemma 28.3.2. Let X be a scheme. The following are equivalent.

(1) The scheme X is reduced, see Schemes, Definition 26.12.1.
(2) There exists an affine open covering $X = \bigcup U_i$ such that each $\Gamma(U_i, \mathcal{O}_X)$ is reduced.
(3) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.
(4) For every open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.

Proof. See Schemes, Lemmas 26.12.2 and 26.12.3. $\square$

01OM Lemma 28.3.3. Let X be a scheme. The following are equivalent.

(1) The scheme X is irreducible.
(2) There exists an affine open covering $X = \bigcup_{i \in I} U_i$ such that I is not empty, U_i is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$.

(3) The scheme X is nonempty and every nonempty affine open $U \subset X$ is irreducible.

Proof. Assume (1). By Schemes, Lemma 26.11.1 we see that X has a unique generic point η. Then $X = \overline{\{\eta\}}$. Hence η is an element of every nonempty affine open $U \subset X$. This implies that $\eta \in U$ is dense hence U is irreducible. It also implies any two nonempty affines meet. Thus (1) implies both (2) and (3).

Assume (2). Suppose $X = Z_1 \cup Z_2$ is a union of two closed subsets. For every i we see that either $U_i \subset Z_1$ or $U_i \subset Z_2$. Pick some $i \in I$ and assume $U_i \subset Z_1$ (possibly after renumbering Z_1, Z_2). For any $j \in I$ the open subset $U_i \cap U_j$ is dense in U_j and contained in the closed subset $Z_1 \cap U_j$. We conclude that also $U_j \subset Z_1$. Thus $X = Z_1$ as desired.

Assume (3). Choose an affine open covering $X = \bigcup_{i \in I} U_i$. We may assume that each U_i is nonempty. Since X is nonempty we see that I is not empty. By assumption each U_i is irreducible. Suppose $U_i \cap U_j = \emptyset$ for some pair $i, j \in I$. Then the open $U_i \amalg U_j = U_i \cup U_j$ is affine, see Schemes, Lemma 26.6.8. Hence it is irreducible by assumption which is absurd. We conclude that (3) implies (2). The lemma is proved. □

01ON Lemma 28.3.4. A scheme X is integral if and only if it is reduced and irreducible.

Proof. If X is irreducible, then every affine open $\text{Spec}(R) = U \subset X$ is irreducible. If X is reduced, then R is reduced, by Lemma 28.3.2 above. Hence R is reduced and (0) is a prime ideal, i.e., R is an integral domain.

If X is integral, then for every nonempty affine open $\text{Spec}(R) = U \subset X$ the ring R is reduced and hence X is reduced by Lemma 28.3.2. Moreover, every nonempty affine open is irreducible. Hence X is irreducible, see Lemma 28.3.3. □

In Examples, Section 110.6 we construct a connected affine scheme all of whose local rings are domains, but which is not integral.

28.4. Types of schemes defined by properties of rings

01OO In this section we study what properties of rings allow one to define local properties of schemes.

01OP Definition 28.4.1. Let P be a property of rings. We say that P is local if the following hold:

(1) For any ring R, and any $f \in R$ we have $P(R) \Rightarrow P(R_f)$.
(2) For any ring R, and $f_i \in R$ such that $(f_1, \ldots, f_n) = R$ then $\forall i, P(R_{f_i}) \Rightarrow P(R)$.

01OQ Definition 28.4.2. Let P be a property of rings. Let X be a scheme. We say X is locally P if for any $x \in X$ there exists an affine open neighbourhood U of x in X such that $\mathcal{O}_X(U)$ has property P.

This is only a good notion if the property is local. Even if P is a local property we will not automatically use this definition to say that a scheme is "locally P" unless we also explicitly state the definition elsewhere.

01OR Lemma 28.4.3. Let X be a scheme. Let P be a local property of rings. The following are equivalent:

(1) The scheme X is locally P.
(2) For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally P.

Moreover, if X is locally P then every open subscheme is locally P.

Proof. Of course (1) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (1). If (3) $\Rightarrow$ (2), then the final statement of the lemma holds and it follows easily that (4) is also equivalent to (1). Thus we show (3) $\Rightarrow$ (2).

Let $X = \bigcup U_i$ be an affine open covering, say $U_i = \text{Spec}(R_i)$. Assume $P(R_i)$. Let $\text{Spec}(R) = U \subset X$ be an arbitrary affine open. By Schemes, Lemma 26.11.6 there exists a standard covering of $U = \text{Spec}(R)$ by standard opens $D(f_j)$ such that each ring R_{f_j} is a principal localization of one of the rings R_i. By Definition 28.4.1 (1) we get $P(R_{f_j})$. Whereupon $P(R)$ by Definition 28.4.1 (2). $\square$

Here is a sample application.

01OS Lemma 28.4.4. Let X be a scheme. Then X is reduced if and only if X is "locally reduced" in the sense of Definition 28.4.2.

Proof. This is clear from Lemma 28.3.2. $\square$

01OT Lemma 28.4.5. The following properties of a ring R are local.

(1) (Cohen-Macaulay.) The ring R is Noetherian and CM, see Algebra, Definition 10.104.6.
(2) (Regular.) The ring R is Noetherian and regular, see Algebra, Definition 10.110.7.
(3) (Absolutely Noetherian.) The ring R is of finite type over Z.
(4) Add more here as needed.[1]

Proof. Omitted. $\square$

28.5. Noetherian schemes

01OU Recall that a ring R is Noetherian if it satisfies the ascending chain condition of ideals. Equivalently every ideal of R is finitely generated.

01OV Definition 28.5.1. Let X be a scheme.

(1) We say X is locally Noetherian if every $x \in X$ has an affine open neighbourhood $\text{Spec}(R) = U \subset X$ such that the ring R is Noetherian.
(2) We say X is Noetherian if X is locally Noetherian and quasi-compact.

Here is the standard result characterizing locally Noetherian schemes.

01OW Lemma 28.5.2. Let X be a scheme. The following are equivalent:

(1) The scheme X is locally Noetherian.
(2) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian.

[1]But we only list those properties here which we have not already dealt with separately somewhere else.

(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally Noetherian.

Moreover, if X is locally Noetherian then every open subscheme is locally Noetherian.

Proof. To show this it suffices to show that being Noetherian is a local property of rings, see Lemma 28.4.3. Any localization of a Noetherian ring is Noetherian, see Algebra, Lemma 10.31.1. By Algebra, Lemma 10.23.2 we see the second property to Definition 28.4.1. □

01OX Lemma 28.5.3. Any immersion $Z \to X$ with X locally Noetherian is quasi-compact.

Proof. A closed immersion is clearly quasi-compact. A composition of quasi-compact morphisms is quasi-compact, see Topology, Lemma 5.12.2. Hence it suffices to show that an open immersion into a locally Noetherian scheme is quasi-compact. Using Schemes, Lemma 26.19.2 we reduce to the case where X is affine. Any open subset of the spectrum of a Noetherian ring is quasi-compact (for example combine Algebra, Lemma 10.31.5 and Topology, Lemmas 5.9.2 and 5.12.13). □

01OY Lemma 28.5.4. A locally Noetherian scheme is quasi-separated.

Proof. By Schemes, Lemma 26.21.6 we have to show that the intersection $U \cap V$ of two affine opens of X is quasi-compact. This follows from Lemma 28.5.3 above on considering the open immersion $U \cap V \to U$ for example. (But really it is just because any open of the spectrum of a Noetherian ring is quasi-compact.) □

01OZ Lemma 28.5.5. A (locally) Noetherian scheme has a (locally) Noetherian underlying topological space, see Topology, Definition 5.9.1.

Proof. This is because a Noetherian scheme is a finite union of spectra of Noetherian rings and Algebra, Lemma 10.31.5 and Topology, Lemma 5.9.4. □

02IK Lemma 28.5.6. Any locally closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian.

Proof. Omitted. Hint: Any quotient, and any localization of a Noetherian ring is Noetherian. For the Noetherian case use again that any subset of a Noetherian space is a Noetherian space (with induced topology). □

0BA8 Lemma 28.5.7. A Noetherian scheme has a finite number of irreducible components.

Proof. The underlying topological space of a Noetherian scheme is Noetherian (Lemma 28.5.5) and we conclude because a Noetherian topological space has only finitely many irreducible components (Topology, Lemma 5.9.2). □

01P0 Lemma 28.5.8. Any morphism of schemes $f : X \to Y$ with X Noetherian is quasi-compact.

Proof. Use Lemma 28.5.5 and use that any subset of a Noetherian topological space is quasi-compact (see Topology, Lemmas 5.9.2 and 5.12.13). □

Here is a fun lemma. It says that every locally Noetherian scheme has plenty of closed points (at least one in every closed subset).

02IL Lemma 28.5.9. Any nonempty locally Noetherian scheme has a closed point. Any nonempty closed subset of a locally Noetherian scheme has a closed point. Equivalently, any point of a locally Noetherian scheme specializes to a closed point.

Proof. The second assertion follows from the first (using Schemes, Lemma 26.12.4 and Lemma 28.5.6). Consider any nonempty affine open $U \subset X$. Let $x \in U$ be a closed point. If x is a closed point of X then we are done. If not, let $X_0 \subset X$ be the reduced induced closed subscheme structure on $\overline{\{x\}}$. Then $U_0 = U \cap X_0$ is an affine open of X_0 by Schemes, Lemma 26.10.1 and $U_0 = \{x\}$. Let $y \in X_0$, $y \neq x$ be a specialization of x. Consider the local ring $R = \mathcal{O}_{X_0,y}$. This is a Noetherian local ring as X_0 is Noetherian by Lemma 28.5.6. Denote $V \subset \text{Spec}(R)$ the inverse image of U_0 in $\text{Spec}(R)$ by the canonical morphism $\text{Spec}(R) \to X_0$ (see Schemes, Section 26.13.) By construction V is a singleton with unique point corresponding to x (use Schemes, Lemma 26.13.2). By Algebra, Lemma 10.61.1 we see that $\dim(R) = 1$. In other words, we see that y is an immediate specialization of x (see Topology, Definition 5.20.1). In other words, any point $y \neq x$ such that $x \rightsquigarrow y$ is an immediate specialization of x. Clearly each of these points is a closed point as desired. $\square$

054F Lemma 28.5.10. Let X be a locally Noetherian scheme. Let $x' \rightsquigarrow x$ be a specialization of points of X. Then

(1) there exists a discrete valuation ring R and a morphism $f : \text{Spec}(R) \to X$ such that the generic point η of $\text{Spec}(R)$ maps to x' and the special point maps to x, and
(2) given a finitely generated field extension $K/\kappa(x')$ we may arrange it so that the extension $\kappa(\eta)/\kappa(x')$ induced by f is isomorphic to the given one.

Proof. Let $x' \rightsquigarrow x$ be a specialization in X, and let $K/\kappa(x')$ be a finitely generated extension of fields. By Schemes, Lemma 26.13.2 and the discussion following Schemes, Lemma 26.13.3 this leads to ring maps $\mathcal{O}_{X,x} \to \kappa(x') \to K$. Let $R \subset K$ be any discrete valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{X,x} \to K$, see Algebra, Lemma 10.119.13. The ring map $\mathcal{O}_{X,x} \to R$ induces the morphism $f : \text{Spec}(R) \to X$, see Schemes, Lemma 26.13.1. This morphism has all the desired properties by construction. $\square$

0CXG Lemma 28.5.11. Let S be a Noetherian scheme. Let $T \subset S$ be an infinite subset. Then there exists an infinite subset $T' \subset T$ such that there are no nontrivial specializations among the points T'.

Proof. Let $T_0 \subset T$ be the set of $t \in T$ which do not specialize to another point of T. If T_0 is infinite, then $T' = T_0$ works. Hence we may and do assume T_0 is finite. Inductively, for $i > 0$, consider the set $T_i \subset T$ of $t \in T$ such that

(1) $t \not\in T_{i-1} \cup T_{i-2} \cup \ldots \cup T_0$,
(2) there exist a nontrivial specialization $t \rightsquigarrow t'$ with $t' \in T_{i-1}$, and
(3) for any nontrivial specialization $t \rightsquigarrow t'$ with $t' \in T$ we have $t' \in T_{i-1} \cup T_{i-2} \cup \ldots \cup T_0$.

Again, if T_i is infinite, then $T' = T_i$ works. Let d be the maximum of the dimensions of the local rings $\mathcal{O}_{S,t}$ for $t \in T_0$; then d is an integer because T_0 is finite and the dimensions of the local rings are finite by Algebra, Proposition 10.60.9. Then $T_i = \emptyset$ for $i > d$. Namely, if $t \in T_i$ then we can find a sequence of nontrivial specializations $t = t_i \rightsquigarrow t_{i-1} \rightsquigarrow \ldots \rightsquigarrow t_0$ with $t_0 \in T_0$. As the points $t = t_i, t_{i-1}, \ldots, t_0$ are in

$\mathrm{Spec}(\mathcal{O}_{S,t_0})$ (Schemes, Lemma 26.13.2), we see that $i \leq d$. Thus $\bigcup T_i = T_d \cup \ldots \cup T_0$ is a finite subset of T.

Suppose $t \in T$ is not in $\bigcup T_i$. Then there must be a specialization $t \rightsquigarrow t'$ with $t' \in T$ and $t' \notin \bigcup T_i$. (Namely, if every specialization of t is in the finite set $T_d \cup \ldots \cup T_0$, then there is a maximum i such that there is some specialization $t \rightsquigarrow t'$ with $t' \in T_i$ and then $t \in T_{i+1}$ by construction.) Hence we get an infinite sequence

$$t \rightsquigarrow t' \rightsquigarrow t'' \rightsquigarrow \ldots$$

of nontrivial specializations between points of $T \setminus \bigcup T_i$. This is impossible because the underlying topological space of S is Noetherian by Lemma 28.5.4. □

0G2R Lemma 28.5.12. Let S be a Noetherian scheme. Let $T \subset S$ be a subset. Let $T_0 \subset T$ be the set of $t \in T$ such that there is no nontrivial specialization $t' \rightsquigarrow t$ with $t' \in T'$. Then (a) there are no specializations among the points of T_0, (b) every point of T is a specialization of a point of T_0, and (c) the closures of T and T_0 are the same.

Proof. Recall that $\dim(\mathcal{O}_{S,s}) < \infty$ for any $s \in S$, see Algebra, Proposition 10.60.9. Let $t \in T$. If $t' \rightsquigarrow t$, then by dimension theory $\dim(\mathcal{O}_{S,t'}) \leq \dim(\mathcal{O}_{S,t})$ with equality if and only if $t' = t$. Thus if we pick $t' \rightsquigarrow t$ with $\dim(\mathcal{O}_{T,t'})$ minimal, then $t' \in T_0$. In other words, every $t \in T$ is the specialization of an element of T_0. □

0G2F Lemma 28.5.13. Let S be a Noetherian scheme. Let $T \subset S$ be an infinite dense subset. Then there exist a countable subset $E \subset T$ which is dense in S.

Proof. Let T' be the set of points $s \in S$ such that $\overline{\{s\}} \cap T$ contains a countable subset whose closure is $\overline{\{s\}}$. Since a finite set is countable we have $T \subset T'$. For $s \in T'$ choose such a countable subset $E_s \subset \overline{\{s\}} \cap T$. Let $E' = \{s_1, s_2, s_3, \ldots\} \subset T'$ be a countable subset. Then the closure of E' in S is the closure of the countable subset $\bigcup_n E_{s_n}$ of T. It follows that if Z is an irreducible component of the closure of E', then the generic point of Z is in T'.

Denote $T'_0 \subset T'$ the subset of $t \in T'$ such that there is no nontrivial specialization $t' \rightsquigarrow t$ with $t' \in T'$ as in Lemma 28.5.12 whose results we will use without further mention. If T'_0 is infinite, then we choose a countable subset $E' \subset T'_0$. By the argument in the first paragraph, the generic points of the irreducible components of the closure of E' are in T'. However, since one of these points specializes to infinitely many distinct elements of $E' \subset T'_0$ this is a contradiction. Thus T'_0 is finite, say $T'_0 = \{s_1, \ldots, s_m\}$. Then it follows that S, which is the closure of T, is contained in the closure of $\{s_1, \ldots, s_m\}$, which in turn is contained in the closure of the countable subset $E_{s_1} \cup \ldots \cup E_{s_m} \subset T$ as desired. □

28.6. Jacobson schemes

01P1 Recall that a space is said to be Jacobson if the closed points are dense in every closed subset, see Topology, Section 5.18.

01P2 Definition 28.6.1. A scheme S is said to be Jacobson if its underlying topological space is Jacobson.

Recall that a ring R is Jacobson if every radical ideal of R is the intersection of maximal ideals, see Algebra, Definition 10.35.1.

01P3 **Lemma 28.6.2.** An affine scheme $\mathrm{Spec}(R)$ is Jacobson if and only if the ring R is Jacobson.

Proof. This is Algebra, Lemma 10.35.4. $\square$

Here is the standard result characterizing Jacobson schemes. Intuitively it claims that Jacobson $\Leftrightarrow$ locally Jacobson.

01P4 **Lemma 28.6.3.** Let X be a scheme. The following are equivalent:

(1) The scheme X is Jacobson.
(2) The scheme X is "locally Jacobson" in the sense of Definition 28.4.2.
(3) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Jacobson.
(4) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Jacobson.
(5) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Jacobson.

Moreover, if X is Jacobson then every open subscheme is Jacobson.

Proof. The final assertion of the lemma holds by Topology, Lemma 5.18.5. The equivalence of (5) and (1) is Topology, Lemma 5.18.4. Hence, using Lemma 28.6.2, we see that (1) $\Leftrightarrow$ (2). To finish proving the lemma it suffices to show that "Jacobson" is a local property of rings, see Lemma 28.4.3. Any localization of a Jacobson ring at an element is Jacobson, see Algebra, Lemma 10.35.14. Suppose R is a ring, $f_1, \ldots, f_n \in R$ generate the unit ideal and each R_{f_i} is Jacobson. Then we see that $\mathrm{Spec}(R) = \bigcup D(f_i)$ is a union of open subsets which are all Jacobson, and hence $\mathrm{Spec}(R)$ is Jacobson by Topology, Lemma 5.18.4 again. This proves the second property of Definition 28.4.1. $\square$

Many schemes used commonly in algebraic geometry are Jacobson, see Morphisms, Lemma 29.16.10. We mention here the following interesting case.

02IM **Lemma 28.6.4.** Examples of Noetherian Jacobson schemes.

(1) If $(R, \mathfrak{m})$ is a Noetherian local ring, then the punctured spectrum $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ is a Jacobson scheme.
(2) If R is a Noetherian ring with Jacobson radical $\mathrm{rad}(R)$ then $\mathrm{Spec}(R) \setminus V(\mathrm{rad}(R))$ is a Jacobson scheme.
(3) If (R, I) is a Zariski pair (More on Algebra, Definition 15.10.1) with R Noetherian, then $\mathrm{Spec}(R) \setminus V(I)$ is a Jacobson scheme.

Proof. Proof of (3). Observe that $\mathrm{Spec}(R) - V(I)$ has a covering by the affine opens $\mathrm{Spec}(R_f)$ for $f \in I$. The rings R_f are Jacobson by More on Algebra, Lemma 15.10.5. Hence $\mathrm{Spec}(R) \setminus V(I)$ is Jacobson by Lemma 28.6.3. Parts (1) and (2) are special cases of (3).

Direct proof of case (1). Since $\mathrm{Spec}(R)$ is a Noetherian scheme, S is a Noetherian scheme (Lemma 28.5.6). Hence S is a sober, Noetherian topological space (use Schemes, Lemma 26.11.1). Assume S is not Jacobson to get a contradiction. By Topology, Lemma 5.18.3 there exists some non-closed point $\xi \in S$ such that $\{\xi\}$ is locally closed. This corresponds to a prime $\mathfrak{p} \subset R$ such that (1) there exists a prime $\mathfrak{q}$, $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$ with both inclusions strict, and (2) $\{\mathfrak{p}\}$ is open in $\mathrm{Spec}(R/\mathfrak{p})$. This is impossible by Algebra, Lemma 10.61.1. $\square$

28.7. Normal schemes

033H Recall that a ring R is said to be normal if all its local rings are normal domains, see Algebra, Definition 10.37.11. A normal domain is a domain which is integrally closed in its field of fractions, see Algebra, Definition 10.37.1. Thus it makes sense to define a normal scheme as follows.

033I **Definition 28.7.1.** A scheme X is *normal* if and only if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a normal domain.

This seems to be the definition used in EGA, see [DG67, 0, 4.1.4]. Suppose $X = \text{Spec}(A)$, and A is reduced. Then saying that X is normal is not equivalent to saying that A is integrally closed in its total ring of fractions. However, if A is Noetherian then this is the case (see Algebra, Lemma 10.37.16).

033J **Lemma 28.7.2.** Let X be a scheme. The following are equivalent:

(1) The scheme X is normal.
(2) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is normal.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is normal.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is normal.

Moreover, if X is normal then every open subscheme is normal.

Proof. This is clear from the definitions. $\square$

033K **Lemma 28.7.3.** A normal scheme is reduced.

Proof. Immediate from the definitions. $\square$

033L **Lemma 28.7.4.** Let X be an integral scheme. Then X is normal if and only if for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is a normal domain.

Proof. This follows from Algebra, Lemma 10.37.10. $\square$

0357 **Lemma 28.7.5.** Let X be a scheme such that any quasi-compact open has a finite number of irreducible components. The following are equivalent:

(1) X is normal, and
(2) X is a disjoint union of normal integral schemes.

Proof. It is immediate from the definitions that (2) implies (1). Let X be a normal scheme such that every quasi-compact open has a finite number of irreducible components. If X is affine then X satisfies (2) by Algebra, Lemma 10.37.16. For a general X, let $X = \bigcup X_i$ be an affine open covering. Note that also each X_i has but a finite number of irreducible components, and the lemma holds for each X_i. Let $T \subset X$ be an irreducible component. By the affine case each intersection $T \cap X_i$ is open in X_i and an integral normal scheme. Hence $T \subset X$ is open, and an integral normal scheme. This proves that X is the disjoint union of its irreducible components, which are integral normal schemes. $\square$

033M **Lemma 28.7.6.** Let X be a Noetherian scheme. The following are equivalent:

(1) X is normal, and
(2) X is a finite disjoint union of normal integral schemes.

Proof. This is a special case of Lemma 28.7.5 because a Noetherian scheme has a Noetherian underlying topological space (Lemma 28.5.5 and Topology, Lemma 5.9.2). ☐

033N **Lemma 28.7.7.** Let X be a locally Noetherian scheme. The following are equivalent:

1. X is normal, and
2. X is a disjoint union of integral normal schemes.

Proof. Omitted. Hint: This is purely topological from Lemma 28.7.6. ☐

033O **Remark 28.7.8.** Let X be a normal scheme. If X is locally Noetherian then we see that X is integral if and only if X is connected, see Lemma 28.7.7. But there exists a connected affine scheme X such that $\mathcal{O}_{X,x}$ is a domain for all $x \in X$, but X is not irreducible, see Examples, Section 110.6. This example is even a normal scheme (proof omitted), so beware!

0358 **Lemma 28.7.9.** Let X be an integral normal scheme. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. It is clear that R is a domain. Suppose $f = a/b$ is an element of its fraction field which is integral over R. Say we have $f^d + \sum_{i=0,\ldots,d-1} a_i f^i = 0$ with $a_i \in R$. Let $U \subset X$ be a nonempty affine open. Since $b \in R$ is not zero and since X is integral we see that also $b|_U \in \mathcal{O}_X(U)$ is not zero. Hence a/b is an element of the fraction field of $\mathcal{O}_X(U)$ which is integral over $\mathcal{O}_X(U)$ (because we can use the same polynomial $f^d + \sum_{i=0,\ldots,d-1} a_i|_U f^i = 0$ on U). Since $\mathcal{O}_X(U)$ is a normal domain (Lemma 28.7.2), we see that $f_U = (a|_U)/(b|_U) \in \mathcal{O}_X(U)$. It is clear that $f_U|_V = f_V$ whenever $V \subset U \subset X$ are nonempty affine open. Hence the local sections f_U glue to an element $g \in R = \Gamma(X, \mathcal{O}_X)$. Then bg and a restrict to the same element of $\mathcal{O}_X(U)$ for all U as above, hence $bg = a$, in other words, g maps to f in the fraction field of R. ☐

28.8. Cohen-Macaulay schemes

02IN Recall, see Algebra, Definition 10.104.1, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be *Cohen-Macaulay* if $\text{depth}_{\mathfrak{m}}(R) = \dim(R)$. Recall that a Noetherian ring R is said to be *Cohen-Macaulay* if every local ring $R_{\mathfrak{p}}$ of R is Cohen-Macaulay, see Algebra, Definition 10.104.6.

02IO **Definition 28.8.1.** Let X be a scheme. We say X is *Cohen-Macaulay* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.

02IP **Lemma 28.8.2.** Let X be a scheme. The following are equivalent:

1. X is Cohen-Macaulay,
2. X is locally Noetherian and all of its local rings are Cohen-Macaulay, and
3. X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay.

Proof. Algebra, Lemma 10.104.5 says that the localization of a Cohen-Macaulay local ring is Cohen-Macaulay. The lemma follows by combining this with Lemma 28.5.2, with the existence of closed points on locally Noetherian schemes (Lemma 28.5.9), and the definitions. ☐

02IQ **Lemma 28.8.3.** Let X be a scheme. The following are equivalent:

(1) The scheme X is Cohen-Macaulay.
(2) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and Cohen-Macaulay.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Cohen-Macaulay.

Moreover, if X is Cohen-Macaulay then every open subscheme is Cohen-Macaulay.

Proof. Combine Lemmas 28.5.2 and 28.8.2. $\square$

More information on Cohen-Macaulay schemes and depth can be found in Cohomology of Schemes, Section 30.11.

28.9. Regular schemes

02IR Recall, see Algebra, Definition 10.60.10, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be regular if $\mathfrak{m}$ can be generated by $\dim(R)$ elements. Recall that a Noetherian ring R is said to be regular if every local ring $R_\mathfrak{p}$ of R is regular, see Algebra, Definition 10.110.7.

02IS Definition 28.9.1. Let X be a scheme. We say X is regular, or nonsingular if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and regular.

02IT Lemma 28.9.2. Let X be a scheme. The following are equivalent:

(1) X is regular,
(2) X is locally Noetherian and all of its local rings are regular, and
(3) X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular.

Proof. By the discussion in Algebra preceding Algebra, Definition 10.110.7 we know that the localization of a regular local ring is regular. The lemma follows by combining this with Lemma 28.5.2, with the existence of closed points on locally Noetherian schemes (Lemma 28.5.9), and the definitions. $\square$

02IU Lemma 28.9.3. Let X be a scheme. The following are equivalent:

(1) The scheme X is regular.
(2) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and regular.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and regular.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is regular.

Moreover, if X is regular then every open subscheme is regular.

Proof. Combine Lemmas 28.5.2 and 28.9.2. $\square$

0569 Lemma 28.9.4. A regular scheme is normal.

Proof. See Algebra, Lemma 10.157.5. $\square$

28.10. Dimension

04MS The dimension of a scheme is just the dimension of its underlying topological space.

04MT Definition 28.10.1. Let X be a scheme.

(1) The dimension of X is just the dimension of X as a topological spaces, see Topology, Definition 5.10.1.
(2) For $x \in X$ we denote $\dim_x(X)$ the dimension of the underlying topological space of X at x as in Topology, Definition 5.10.1. We say $\dim_x(X)$ is the dimension of X at x.

As a scheme has a sober underlying topological space (Schemes, Lemma 26.11.1) we may compute the dimension of X as the supremum of the lengths n of chains

$$T_0 \subset T_1 \subset \ldots \subset T_n$$

of irreducible closed subsets of X, or as the supremum of the lengths n of chains of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of X.

04MU Lemma 28.10.2. Let X be a scheme. The following are equal

(1) The dimension of X.
(2) The supremum of the dimensions of the local rings of X.
(3) The supremum of $\dim_x(X)$ for $x \in X$.

Proof. Note that given a chain of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of X all of the points ξ_i correspond to prime ideals of the local ring of X at ξ_0 by Schemes, Lemma 26.13.2. Hence we see that the dimension of X is the supremum of the dimensions of its local rings. In particular $\dim_x(X) \geq \dim(\mathcal{O}_{X,x})$ as $\dim_x(X)$ is the minimum of the dimensions of open neighbourhoods of x. Thus $\sup_{x \in X} \dim_x(X) \geq \dim(X)$. On the other hand, it is clear that $\sup_{x \in X} \dim_x(X) \leq \dim(X)$ as $\dim(U) \leq \dim(X)$ for any open subset of X. $\square$

02IZ Lemma 28.10.3. Let X be a scheme. Let $Y \subset X$ be an irreducible closed subset. Let $\xi \in Y$ be the generic point. Then

$$\operatorname{codim}(Y, X) = \dim(\mathcal{O}_{X,\xi})$$

where the codimension is as defined in Topology, Definition 5.11.1.

Proof. By Topology, Lemma 5.11.2 we may replace X by an affine open neighbourhood of ξ. In this case the result follows easily from Algebra, Lemma 10.26.3. $\square$

0BA9 Lemma 28.10.4. Let X be a scheme. Let $x \in X$. Then x is a generic point of an irreducible component of X if and only if $\dim(\mathcal{O}_{X,x}) = 0$.

Proof. This follows from Lemma 28.10.3 for example. $\square$

0AAX Lemma 28.10.5. A locally Noetherian scheme of dimension 0 is a disjoint union of spectra of Artinian local rings.

Proof. A Noetherian ring of dimension 0 is a finite product of Artinian local rings, see Algebra, Proposition 10.60.7. Hence an affine open of a locally Noetherian scheme X of dimension 0 has discrete underlying topological space. This implies that the topology on X is discrete. The lemma follows easily from these remarks. $\square$

0CKV **Lemma 28.10.6.** Let X be a scheme of dimension zero. The following are equivalent Email from Ofer Gabber dated June 4, 2016

1. X is quasi-separated,
2. X is separated,
3. X is Hausdorff,
4. every affine open is closed.

In this case the connected components of X are points and every quasi-compact open of X is affine. In particular, if X is quasi-compact, then X is affine.

Proof. As the dimension of X is zero, we see that for any affine open $U \subset X$ the space U is profinite and satisfies a bunch of other properties which we will use freely below, see Algebra, Lemma 10.26.5. We choose an affine open covering $X = \bigcup U_i$.

If (4) holds, then $U_i \cap U_j$ is a closed subset of U_i, hence quasi-compact, hence X is quasi-separated, by Schemes, Lemma 26.21.6, hence (1) holds.

If (1) holds, then $U_i \cap U_j$ is a quasi-compact open of U_i hence closed in U_i. Then $U_i \cap U_j \to U_i$ is an open immersion whose image is closed, hence it is a closed immersion. In particular $U_i \cap U_j$ is affine and $\mathcal{O}(U_i) \to \mathcal{O}_X(U_i \cap U_j)$ is surjective. Thus X is separated by Schemes, Lemma 26.21.7, hence (2) holds.

Assume (2) and let $x, y \in X$. Say $x \in U_i$. If $y \in U_i$ too, then we can find disjoint open neighbourhoods of x and y because U_i is Hausdorff. Say $y \notin U_i$ and $y \in U_j$. Then $y \notin U_i \cap U_j$ which is an affine open of U_j and hence closed in U_j. Thus we can find an open neighbourhood of y not meeting U_i and we conclude that X is Hausdorff, hence (3) holds.

Assume (3). Let $U \subset X$ be affine open. Then U is closed in X by Topology, Lemma 5.12.4. This proves (4) holds.

Assume X satisfies the equivalent conditions (1) – (4). We prove the final statements of the lemma. Say $x, y \in X$ with $x \neq y$. Since y does not specialize to x we can choose $U \subset X$ affine open with $x \in U$ and $y \notin U$. Then we see that $X = U \amalg (X \setminus U)$ is a decomposistion into open and closed subsets which shows that x and y do not belong to the same connected component of X. Next, assume $U \subset X$ is a quasi-compact open. Write $U = U_1 \cup \ldots \cup U_n$ as a union of affine opens. We will prove by induction on n that U is affine. This immediately reduces us to the case $n = 2$. In this case we have $U = (U_1 \setminus U_2) \amalg (U_1 \cap U_2) \amalg (U_2 \setminus U_1)$ and the arguments above show that each of the pieces is affine. $\square$

28.11. Catenary schemes

02IV Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 5.11.4.

02IW Definition 28.11.1. Let S be a scheme. We say S is catenary if the underlying topological space of S is catenary.

Recall that a ring A is called catenary if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 10.105.1.

02IX Lemma 28.11.2. Let S be a scheme. The following are equivalent

(1) S is catenary,
(2) there exists an open covering of S all of whose members are catenary schemes,
(3) for every affine open $\mathrm{Spec}(R) = U \subset S$ the ring R is catenary, and
(4) there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a catenary ring.

Moreover, in this case any locally closed subscheme of S is catenary as well.

Proof. Combine Topology, Lemma 5.11.5, and Algebra, Lemma 10.105.2. $\square$

02IY Lemma 28.11.3. Let S be a locally Noetherian scheme. The following are equivalent:

(1) S is catenary, and
(2) locally in the Zariski topology there exists a dimension function on S (see Topology, Definition 5.20.1).

Proof. This follows from Topology, Lemmas 5.11.5, 5.20.2, and 5.20.4, Schemes, Lemma 26.11.1 and finally Lemma 28.5.5. $\square$

It turns out that a scheme is catenary if and only if its local rings are catenary.

02J0 Lemma 28.11.4. Let X be a scheme. The following are equivalent

(1) X is catenary, and
(2) for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is catenary.

Proof. Assume X is catenary. Let $x \in X$. By Lemma 28.11.2 we may replace X by an affine open neighbourhood of x, and then $\Gamma(X, \mathcal{O}_X)$ is a catenary ring. By Algebra, Lemma 10.105.4 any localization of a catenary ring is catenary. Whence $\mathcal{O}_{X,x}$ is catenary.

Conversely assume all local rings of X are catenary. Let $Y \subset Y'$ be an inclusion of irreducible closed subsets of X. Let $\xi \in Y$ be the generic point. Let $\mathfrak{p} \subset \mathcal{O}_{X,\xi}$ be the prime corresponding to the generic point of Y', see Schemes, Lemma 26.13.2. By that same lemma the irreducible closed subsets of X in between Y and Y' correspond to primes $\mathfrak{q} \subset \mathcal{O}_{X,\xi}$ with $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}_\xi$. Hence we see all maximal chains of these are finite and have the same length as $\mathcal{O}_{X,\xi}$ is a catenary ring. $\square$

28.12. Serre's conditions

033P Here are two technical notions that are often useful. See also Cohomology of Schemes, Section 30.11.

033Q Definition 28.12.1. Let X be a locally Noetherian scheme. Let $k \geq 0$.

(1) We say X is regular in codimension k, or we say X has property (R_k) if for every $x \in X$ we have
$$\dim(\mathcal{O}_{X,x}) \leq k \Rightarrow \mathcal{O}_{X,x} \text{ is regular}$$
(2) We say X has property (S_k) if for every $x \in X$ we have $\text{depth}(\mathcal{O}_{X,x}) \geq \min(k, \dim(\mathcal{O}_{X,x}))$.

The phrase "regular in codimension k" makes sense since we have seen in Section 28.11 that if $Y \subset X$ is irreducible closed with generic point x, then $\dim(\mathcal{O}_{X,x}) = \text{codim}(Y, X)$. For example condition (R_0) means that for every generic point $\eta \in X$ of an irreducible component of X the local ring $\mathcal{O}_{X,\eta}$ is a field. But for general Noetherian schemes it can happen that the regular locus of X is badly behaved, so care has to be taken.

0B3C Lemma 28.12.2. Let X be a locally Noetherian scheme. Then X is regular if and only if X has (R_k) for all $k \geq 0$.

Proof. Follows from Lemma 28.9.2 and the definitions. □

0342 Lemma 28.12.3. Let X be a locally Noetherian scheme. Then X is Cohen-Macaulay if and only if X has (S_k) for all $k \geq 0$.

Proof. By Lemma 28.8.2 we reduce to looking at local rings. Hence the lemma is true because a Noetherian local ring is Cohen-Macaulay if and only if it has depth equal to its dimension. □

0344 Lemma 28.12.4. Let X be a locally Noetherian scheme. Then X is reduced if and only if X has properties (S_1) and (R_0).

Proof. This is Algebra, Lemma 10.157.3. □

0345 Lemma 28.12.5. Let X be a locally Noetherian scheme. Then X is normal if and only if X has properties (S_2) and (R_1).

Proof. This is Algebra, Lemma 10.157.4. □

0BX2 Lemma 28.12.6. Let X be a locally Noetherian scheme which is normal and has dimension ≤ 1. Then X is regular.

Proof. This follows from Lemma 28.12.5 and the definitions. □

0B3D Lemma 28.12.7. Let X be a locally Noetherian scheme which is normal and has dimension ≤ 2. Then X is Cohen-Macaulay.

Proof. This follows from Lemma 28.12.5 and the definitions. □

28.13. Japanese and Nagata schemes

033R The notions considered in this section are not prominently defined in EGA. A "universally Japanese scheme" is mentioned and defined in [DG67, IV Corollary 5.11.4]. A "Japanese scheme" is mentioned in [DG67, IV Remark 10.4.14 (ii)] but no definition is given. A Nagata scheme (as given below) occurs in a few places in the literature (see for example [Liu02, Definition 8.2.30] and [Gre76, Page 142]).

We briefly recall that a domain R is called Japanese if the integral closure of R in any finite extension of its fraction field is finite over R. A ring R is called universally Japanese if for any finite type ring map $R \to S$ with S a domain S is Japanese. A ring R is called Nagata if it is Noetherian and $R/\mathfrak{p}$ is Japanese for every prime $\mathfrak{p}$ of R.

033S Definition 28.13.1. Let X be a scheme.

(1) Assume X integral. We say X is Japanese if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Japanese (see Algebra, Definition 10.161.1).
(2) We say X is universally Japanese if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is universally Japanese (see Algebra, Definition 10.162.1).
(3) We say X is Nagata if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Nagata (see Algebra, Definition 10.162.1).

Being Nagata is the same thing as being locally Noetherian and universally Japanese, see Lemma 28.13.8.

033T Remark 28.13.2. In [Hoo72] a (locally Noetherian) scheme X is called Japanese if for every $x \in X$ and every associated prime $\mathfrak{p}$ of $\mathcal{O}_{X,x}$ the ring $\mathcal{O}_{X,x}/\mathfrak{p}$ is Japanese. We do not use this definition since there exists a one dimensional Noetherian domain with excellent (in particular Japanese) local rings whose normalization is not finite. See [Hoc73, Example 1] or [HL07] or [ILO14, Exposé XIX]. On the other hand, we could circumvent this problem by calling a scheme X Japanese if for every affine open $\text{Spec}(A) \subset X$ the ring $A/\mathfrak{p}$ is Japanese for every associated prime $\mathfrak{p}$ of A.

033U Lemma 28.13.3. A Nagata scheme is locally Noetherian.

Proof. This is true because a Nagata ring is Noetherian by definition. □

033V Lemma 28.13.4. Let X be an integral scheme. The following are equivalent:

(1) The scheme X is Japanese.
(2) For every affine open $U \subset X$ the domain $\mathcal{O}_X(U)$ is Japanese.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Japanese.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Japanese.

Moreover, if X is Japanese then every open subscheme is Japanese.

Proof. This follows from Lemma 28.4.3 and Algebra, Lemmas 10.161.3 and 10.161.4. □

033W Lemma 28.13.5. Let X be a scheme. The following are equivalent:

(1) The scheme X is universally Japanese.
(2) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is universally Japanese.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is universally Japanese.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is universally Japanese.

Moreover, if X is universally Japanese then every open subscheme is universally Japanese.

Proof. This follows from Lemma 28.4.3 and Algebra, Lemmas 10.162.4 and 10.162.7. □

033X Lemma 28.13.6. Let X be a scheme. The following are equivalent:

(1) The scheme X is Nagata.
(2) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Nagata.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Nagata.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Nagata.

Moreover, if X is Nagata then every open subscheme is Nagata.

Proof. This follows from Lemma 28.4.3 and Algebra, Lemmas 10.162.6 and 10.162.7. □

033Y Lemma 28.13.7. Let X be a locally Noetherian scheme. Then X is Nagata if and only if every integral closed subscheme $Z \subset X$ is Japanese.

Proof. Assume X is Nagata. Let $Z \subset X$ be an integral closed subscheme. Let $z \in Z$. Let $\text{Spec}(A) = U \subset X$ be an affine open containing z such that A is Nagata. Then $Z \cap U \cong \text{Spec}(A/\mathfrak{p})$ for some prime $\mathfrak{p}$, see Schemes, Lemma 26.10.1 (and Definition 28.3.1). By Algebra, Definition 10.162.1 we see that $A/\mathfrak{p}$ is Japanese. Hence Z is Japanese by definition.

Assume every integral closed subscheme of X is Japanese. Let $\text{Spec}(A) = U \subset X$ be any affine open. As X is locally Noetherian we see that A is Noetherian (Lemma 28.5.2). Let $\mathfrak{p} \subset A$ be a prime ideal. We have to show that $A/\mathfrak{p}$ is Japanese. Let $T \subset U$ be the closed subset $V(\mathfrak{p}) \subset \text{Spec}(A)$. Let $\overline{T} \subset X$ be the closure. Then $\overline{T}$ is irreducible as the closure of an irreducible subset. Hence the reduced closed subscheme defined by $\overline{T}$ is an integral closed subscheme (called $\overline{T}$ again), see Schemes, Lemma 26.12.4. In other words, $\text{Spec}(A/\mathfrak{p})$ is an affine open of an integral closed subscheme of X. This subscheme is Japanese by assumption and by Lemma 28.13.4 we see that $A/\mathfrak{p}$ is Japanese. □

033Z Lemma 28.13.8. Let X be a scheme. The following are equivalent:

(1) X is Nagata, and
(2) X is locally Noetherian and universally Japanese.

Proof. This is Algebra, Proposition 10.162.15. □

This discussion will be continued in Morphisms, Section 29.18.

28.14. The singular locus

07R0 Here is the definition.

07R1 Definition 28.14.1. Let X be a locally Noetherian scheme. The regular locus $\text{Reg}(X)$ of X is the set of $x \in X$ such that $\mathcal{O}_{X,x}$ is a regular local ring. The singular locus $\text{Sing}(X)$ is the complement $X \setminus \text{Reg}(X)$, i.e., the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not a regular local ring.

The regular locus of a locally Noetherian scheme is stable under generalizations, see the discussion preceding Algebra, Definition 10.110.7. However, for general locally Noetherian schemes the regular locus need not be open. In More on Algebra, Section 15.47 the reader can find some criteria for when this is the case. We will discuss this further in Morphisms, Section 29.19.

28.15. Local irreducibility

0BQ1 Recall that in More on Algebra, Section 15.106 we introduced the notion of a (geometrically) unibranch local ring.

0BQ2 **Definition 28.15.1.** Let X be a scheme. Let $x \in X$. We say X is *unibranch at x* if the local ring $\mathcal{O}_{X,x}$ is unibranch. We say X is *geometrically unibranch at x* if the local ring $\mathcal{O}_{X,x}$ is geometrically unibranch. We say X is *unibranch* if X is unibranch at all of its points. We say X is *geometrically unibranch* if X is geometrically unibranch at all of its points. [GD67, Chapter IV (6.15.1)]

To be sure, it can happen that a local ring A is geometrically unibranch (in the sense of More on Algebra, Definition 15.106.1) but the scheme $\mathrm{Spec}(A)$ is not geometrically unibranch in the sense of Definition 28.15.1. For example this happens if A is the local ring at the vertex of the cone over an irreducible plane curve which has ordinary double point singularity (a node).

0BQ3 **Lemma 28.15.2.** *A normal scheme is geometrically unibranch.*

Proof. This follows from the definitions. Namely, a scheme is normal if the local rings are normal domains. It is immediate from the More on Algebra, Definition 15.106.1 that a local normal domain is geometrically unibranch. □

0BQ4 **Lemma 28.15.3.** *Let X be a Noetherian scheme. The following are equivalent* Compare with [Art66, Proposition 2.3]

1. *X is geometrically unibranch (Definition 28.15.1),*
2. *for every point $x \in X$ which is not the generic point of an irreducible component of X, the punctured spectrum of the strict henselization $\mathcal{O}_{X,x}^{sh}$ is connected.*

Proof. More on Algebra, Lemma 15.106.5 shows that (1) implies that the punctured spectra in (2) are irreducible and in particular connected.

Assume (2). Let $x \in X$. We have to show that $\mathcal{O}_{X,x}$ is geometrically unibranch. By induction on $\dim(\mathcal{O}_{X,x})$ we may assume that the result holds for every nontrivial generalization of x. We may replace X by $\mathrm{Spec}(\mathcal{O}_{X,x})$. In other words, we may assume that $X = \mathrm{Spec}(A)$ with A local and that $A_\mathfrak{p}$ is geometrically unibranch for each nonmaximal prime $\mathfrak{p} \subset A$.

Let A^{sh} be the strict henselization of A. If $\mathfrak{q} \subset A^{sh}$ is a prime lying over $\mathfrak{p} \subset A$, then $A_\mathfrak{p} \to A^{sh}_\mathfrak{q}$ is a filtered colimit of étale algebras. Hence the strict henselizations of $A_\mathfrak{p}$ and $A^{sh}_\mathfrak{q}$ are isomorphic. Thus by More on Algebra, Lemma 15.106.5 we conclude that $A^{sh}_\mathfrak{q}$ has a unique minimal prime ideal for every nonmaximal prime $\mathfrak{q}$ of A^{sh}.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of A^{sh}. We have to show that $r = 1$. By the above we see that $V(\mathfrak{q}_1) \cap V(\mathfrak{q}_j) = \{\mathfrak{m}^{sh}\}$ for $j = 2, \ldots, r$. Hence $V(\mathfrak{q}_1) \setminus \{\mathfrak{m}^{sh}\}$ is an open and closed subset of the punctured spectrum of A^{sh} which is a contradiction with the assumption that this punctured spectrum is connected unless $r = 1$. □

0C38 **Definition 28.15.4.** Let X be a scheme. Let $x \in X$. The *number of branches of X at x* is the number of branches of the local ring $\mathcal{O}_{X,x}$ as defined in More on Algebra, Definition 15.106.6. The *number of geometric branches of X at x* is the number of geometric branches of the local ring $\mathcal{O}_{X,x}$ as defined in More on Algebra, Definition 15.106.6.

Often we want to compare this with the branches of the complete local ring, but the comparison is not straightforward in general; some information on this topic can be found in More on Algebra, Section 15.108.

0E20 Lemma 28.15.5. Let X be a scheme and $x \in X$. Let X_i, $i \in I$ be the irreducible components of X passing through x. Then the number of (geometric) branches of X at x is the sum over $i \in I$ of the number of (geometric) branches of X_i at x.

Proof. We view the X_i as integral closed subschemes of X, see Schemes, Definition 26.12.5 and Lemma 28.3.4. Observe that the number of (geometric) branches of X_i at x is at least 1 for all i (essentially by definition). Recall that the X_i correspond 1-to-1 with the minimal prime ideals $\mathfrak{p}_i \subset \mathcal{O}_{X,x}$, see Algebra, Lemma 10.26.3. Thus, if I is infinite, then $\mathcal{O}_{X,x}$ has infinitely many minimal primes, whence both $\mathcal{O}^h_{X,x}$ and $\mathcal{O}^{sh}_{X,x}$ have infinitely many minimal primes (combine Algebra, Lemmas 10.30.5 and 10.30.7 and the injectivity of the maps $\mathcal{O}_{X,x} \to \mathcal{O}^h_{X,x} \to \mathcal{O}^{sh}_{X,x}$). In this case the number of (geometric) branches of X at x is defined to be ∞ which is also true for the sum. Thus we may assume I is finite. Let A' be the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions Q of $(\mathcal{O}_{X,x})_{red}$. Let A'_i be the integral closure of $\mathcal{O}_{X,x}/\mathfrak{p}_i$ in the total ring of fractions Q_i of $\mathcal{O}_{X,x}/\mathfrak{p}_i$. By Algebra, Lemma 10.25.4 we have $Q = \prod_{i \in I} Q_i$. Thus $A' = \prod A'_i$. Then the equality of the lemma follows from More on Algebra, Lemma 15.106.7 which expresses the number of (geometric) branches in terms of the maximal ideals of A'. $\square$

0C39 Lemma 28.15.6. Let X be a scheme. Let $x \in X$.

(1) The number of branches of X at x is 1 if and only if X is unibranch at x.
(2) The number of geometric branches of X at x is 1 if and only if X is geometrically unibranch at x.

Proof. This lemma follows immediately from the definitions and the corresponding result for rings, see More on Algebra, Lemma 15.106.7. $\square$

28.16. Characterizing modules of finite type and finite presentation

01PA Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following lemma implies that $\mathcal{F}$ is of finite type (see Modules, Definition 17.9.1) if and only if $\mathcal{F}$ is on each open affine $\text{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finite type A-module M. Similarly, $\mathcal{F}$ is of finite presentation (see Modules, Definition 17.11.1) if and only if $\mathcal{F}$ is on each open affine $\text{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finitely presented A-module M.

01PB Lemma 28.16.1. Let $X = \text{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module if and only if M is a finite R-module.

Proof. Assume $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module. This means there exists an open covering of X such that $\widetilde{M}$ restricted to the members of this covering is globally generated by finitely many sections. Thus there also exists a standard open covering $X = \bigcup_{i=1,\ldots,n} D(f_i)$ such that $\widetilde{M}|_{D(f_i)}$ is generated by finitely many sections. Thus M_{f_i} is finitely generated for each i. Hence we conclude by Algebra, Lemma 10.23.2. $\square$

01PC Lemma 28.16.2. Let $X = \mathrm{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation if and only if M is an R-module of finite presentation.

Proof. Assume $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation. By Lemma 28.16.1 we see that M is a finite R-module. Choose a surjection $R^n \to M$ with kernel K. By Schemes, Lemma 26.5.4 there is a short exact sequence

$$0 \to \widetilde{K} \to \bigoplus \mathcal{O}_X^{\oplus n} \to \widetilde{M} \to 0$$

By Modules, Lemma 17.11.3 we see that $\widetilde{K}$ is a finite type $\mathcal{O}_X$-module. Hence by Lemma 28.16.1 again we see that K is a finite R-module. Hence M is an R-module of finite presentation. $\square$

28.17. Sections over principal opens

0B5K Here is a typical result of this kind. We will use a more naive but more direct method of proof in later lemmas.

01P7 Lemma 28.17.1. Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Denote $X_f \subset X$ the open where f is invertible, see Schemes, Lemma 26.6.2. If X is quasi-compact and quasi-separated, the canonical map

$$\Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma(X_f, \mathcal{O}_X)$$

is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules the map

$$\Gamma(X, \mathcal{F})_f \longrightarrow \Gamma(X_f, \mathcal{F})$$

is an isomorphism.

Proof. Write $R = \Gamma(X, \mathcal{O}_X)$. Consider the canonical morphism

$$\varphi : X \longrightarrow \mathrm{Spec}(R)$$

of schemes, see Schemes, Lemma 26.6.4. Then the inverse image of the standard open $D(f)$ on the right hand side is X_f on the left hand side. Moreover, since X is assumed quasi-compact and quasi-separated the morphism φ is quasi-compact and quasi-separated, see Schemes, Lemma 26.19.2 and 26.21.13. Hence by Schemes, Lemma 26.24.1 we see that $\varphi_*\mathcal{F}$ is quasi-coherent. Hence we see that $\varphi_*\mathcal{F} = \widetilde{M}$ with $M = \Gamma(X, \mathcal{F})$ as an R-module. Thus we see that

$$\Gamma(X_f, \mathcal{F}) = \Gamma(D(f), \varphi_*\mathcal{F}) = \Gamma(D(f), \widetilde{M}) = M_f$$

which is exactly the content of the lemma. The first displayed isomorphism of the lemma follows by taking $\mathcal{F} = \mathcal{O}_X$. $\square$

Recall that given a scheme X, an invertible sheaf $\mathcal{L}$ on X, and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we get a graded ring $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ and a graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ see Modules, Definition 17.25.7. If we have moreover a section $s \in \Gamma(X, \mathcal{L})$, then we obtain a map

0B5L (28.17.1.1)
$$\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{F}|_{X_s})$$

which sends t/s^n where $t \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ to $t|_{X_s} \otimes s|_{X_s}^{-n}$. This makes sense because $X_s \subset X$ is by definition the open over which s has an inverse, see Modules, Lemma 17.25.10.

01PW Lemma 28.17.2. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) If X is quasi-compact, then (28.17.1.1) is injective, and
(2) if X is quasi-compact and quasi-separated, then (28.17.1.1) is an isomorphism.

In particular, the canonical map

$$\Gamma_*(X, \mathcal{L})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{O}_X), \quad a/s^n \longmapsto a \otimes s^{-n}$$

is an isomorphism if X is quasi-compact and quasi-separated.

Proof. Assume X is quasi-compact. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ with U_j affine and $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. Via this isomorphism, the image $s|_{U_j}$ corresponds to some $f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$. Then $X_s \cap U_j = D(f_j)$.

Proof of (1). Let t/s^n be an element in the kernel of (28.17.1.1). Then $t|_{X_s} = 0$. Hence $(t|_{U_j})|_{D(f_j)} = 0$. By Lemma 28.17.1 we conclude that $f_j^{e_j} t|_{U_j} = 0$ for some $e_j \geq 0$. Let $e = \max(e_j)$. Then we see that $t \otimes s^e$ restricts to zero on U_j for all j, hence is zero. Since t/s^n is equal to $t \otimes s^e/s^{n+e}$ in $\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)}$ we conclude that $t/s^n = 0$ as desired.

Proof of (2). Assume X is quasi-compact and quasi-separated. Then $U_j \cap U_{j'}$ is quasi-compact for all pairs j, j', see Schemes, Lemma 26.21.6. By part (1) we know (28.17.1.1) is injective. Let $t' \in \Gamma(X_s, \mathcal{F}|_{X_s})$. For every j, there exist an integer $e_j \geq 0$ and $t'_j \in \Gamma(U_j, \mathcal{F}|_{U_j})$ such that $t'|_{D(f_j)}$ corresponds to $t'_j/f_j^{e_j}$ via the isomorphism of Lemma 28.17.1. Set $e = \max(e_j)$ and

$$t_j = f_j^{e-e_j} t'_j \otimes q_j^e \in \Gamma(U_j, (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e})|_{U_j})$$

where $q_j \in \Gamma(U_j, \mathcal{L}|_{U_j})$ is the trivializing section coming from the isomorphism $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. In particular we have $s|_{U_j} = f_j q_j$. Using this a calculation shows that $t_j|_{U_j \cap U_{j'}}$ and $t_{j'}|_{U_j \cap U_{j'}}$ map to the same section of $\mathcal{F}$ over $U_j \cap U_{j'} \cap X_s$. By quasi-compactness of $U_j \cap U_{j'}$ and part (1) there exists an integer $e' \geq 0$ such that

$$t_j|_{U_j \cap U_{j'}} \otimes s^{e'}|_{U_j \cap U_{j'}} = t_{j'}|_{U_j \cap U_{j'}} \otimes s^{e'}|_{U_j \cap U_{j'}}$$

as sections of $\mathcal{F} \otimes \mathcal{L}^{\otimes e+e'}$ over $U_j \cap U_{j'}$. We may choose the same e' to work for all pairs j, j'. Then the sheaf conditions implies there is a section $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes e+e'})$ whose restriction to U_j is $t_j \otimes s^{e'}|_{U_j}$. A simple computation shows that $t/s^{e+e'}$ maps to t' as desired. □

Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Consider the graded $\Gamma_*(X, \mathcal{L})$-module

$$M = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

Next, let $s \in \Gamma(X, \mathcal{L})$ be a section. Then there is a canonical map

0B5M (28.17.2.1) $$M_{(s)} \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_s}}(\mathcal{F}|_{X_s}, \mathcal{G}|_{X_s})$$

which sends α/s^n to the map $\alpha|_{X_s} \otimes s|_{X_s}^{-n}$. The following lemma, combined with Lemma 28.22.4, says roughly that, if X is quasi-compact and quasi-separated, the category of finitely presented $\mathcal{O}_{X_s}$-modules is the category of finitely presented $\mathcal{O}_X$-modules with the multiplicative system of maps $s^n : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ inverted.

01XQ **Lemma 28.17.3.** Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules.

(1) If X is quasi-compact and $\mathcal{F}$ is of finite type, then (28.17.2.1) is injective, and
(2) if X is quasi-compact and quasi-separated and $\mathcal{F}$ is of finite presentation, then (28.17.2.1) is bijective.

Proof. We first prove the lemma in case $X = \mathrm{Spec}(A)$ is affine and $\mathcal{L} = \mathcal{O}_X$. In this case s corresponds to an element $f \in A$. Say $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$ for some A-modules M and N. Then the lemma translates (via Lemmas 28.16.1 and 28.16.2) into the following algebra statements

(1) If M is a finite A-module and $\varphi : M \to N$ is an A-module map such that the induced map $M_f \to N_f$ is zero, then $f^n \varphi = 0$ for some n.
(2) If M is a finitely presented A-module, then $\mathrm{Hom}_A(M, N)_f = \mathrm{Hom}_{A_f}(M_f, N_f)$.

The second statement is Algebra, Lemma 10.10.2 and we omit the proof of the first statement.

Next, we prove (1) for general X. Assume X is quasi-compact and hoose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ with U_j affine and $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. Via this isomorphism, the image $s|_{U_j}$ corresponds to some $f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$. Then $X_s \cap U_j = D(f_j)$. Let α/s^n be an element in the kernel of (28.17.2.1). Then $\alpha|_{X_s} = 0$. Hence $(\alpha|_{U_j})|_{D(f_j)} = 0$. By the affine case treated above we conclude that $f_j^{e_j} \alpha|_{U_j} = 0$ for some $e_j \geq 0$. Let $e = \max(e_j)$. Then we see that $\alpha \otimes s^e$ restricts to zero on U_j for all j, hence is zero. Since α/s^n is equal to $\alpha \otimes s^e/s^{n+e}$ in $M_{(s)}$ we conclude that $\alpha/s^n = 0$ as desired.

Proof of (2). Since $\mathcal{F}$ is of finite presentation, the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, see Schemes, Section 26.24. Moreover, it is clear that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

for all n. Hence in this case the statement follows from Lemma 28.17.2 applied to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. □

28.18. Quasi-affine schemes

01P5

01P6 **Definition 28.18.1.** A scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme.

0EHM **Lemma 28.18.2.** Let A be a ring and let $U \subset \mathrm{Spec}(A)$ be a quasi-compact open subscheme. For $\mathcal{F}$ quasi-coherent on U the canonical map

$$\widetilde{H^0(U, \mathcal{F})}|_U \to \mathcal{F}$$

is an isomorphism.

Proof. Denote $j : U \to \mathrm{Spec}(A)$ the inclusion morphism. Then $H^0(U, \mathcal{F}) = H^0(\mathrm{Spec}(A), j_*\mathcal{F})$ and $j_*\mathcal{F}$ is quasi-coherent by Schemes, Lemma 26.24.1. Hence $j_*\mathcal{F} = \widetilde{H^0(U, \mathcal{F})}$ by Schemes, Lemma 26.7.5. Restricting back to U we get the lemma. □

01P8 Lemma 28.18.3. Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Assume X is quasi-compact and quasi-separated and assume that X_f is affine. Then the canonical morphism

$$j : X \longrightarrow \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 26.6.4 induces an isomorphism of $X_f = j^{-1}(D(f))$ onto the standard affine open $D(f) \subset \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$.

Proof. This is clear as j induces an isomorphism of rings $\Gamma(X, \mathcal{O}_X)_f \to \mathcal{O}_X(X_f)$ by Lemma 28.17.1 above. $\square$

01P9 Lemma 28.18.4. Let X be a scheme. Then X is quasi-affine if and only if the canonical morphism

$$X \longrightarrow \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 26.6.4 is a quasi-compact open immersion.

Proof. If the displayed morphism is a quasi-compact open immersion then X is isomorphic to a quasi-compact open subscheme of $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ and clearly X is quasi-affine.

Assume X is quasi-affine, say $X \subset \mathrm{Spec}(R)$ is quasi-compact open. This in particular implies that X is separated, see Schemes, Lemma 26.23.9. Let $A = \Gamma(X, \mathcal{O}_X)$. Consider the ring map $R \to A$ coming from $R = \Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ and the restriction mapping of the sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$. By Schemes, Lemma 26.6.4 we obtain a factorization:

$$X \longrightarrow \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$$

of the inclusion morphism. Let $x \in X$. Choose $r \in R$ such that $x \in D(r)$ and $D(r) \subset X$. Denote $f \in A$ the image of r in A. The open X_f of Lemma 28.17.1 above is equal to $D(r) \subset X$ and hence $A_f \cong R_r$ by the conclusion of that lemma. Hence $D(r) \to \mathrm{Spec}(A)$ is an isomorphism onto the standard affine open $D(f)$ of $\mathrm{Spec}(A)$. Since X can be covered by such affine opens $D(f)$ we win. $\square$

0ARY Lemma 28.18.5. Let $U \to V$ be an open immersion of quasi-affine schemes. Then

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[d] & & \mathrm{Spec}(\Gamma(U, \mathcal{O}_U)) \arrow[d] \\ U \arrow[r] & V \arrow[r, "j'"] & \mathrm{Spec}(\Gamma(V, \mathcal{O}_V)) \end{tikzcd}$$

is cartesian.

Proof. The diagram is commutative by Schemes, Lemma 26.6.4. Write $A = \Gamma(U, \mathcal{O}_U)$ and $B = \Gamma(V, \mathcal{O}_V)$. Let $g \in B$ be such that V_g is affine and contained in U. This means that if f is the image of g in A, then $U_f = V_g$. By Lemma 28.18.3 we see that j' induces an isomorphism of V_g with the standard open $D(g)$ of $\mathrm{Spec}(B)$. Thus $V_g \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is an isomorphism onto $D(f) \subset \mathrm{Spec}(A)$. By Lemma 28.18.3 again j maps U_f isomorphically to $D(f)$. Thus we see that $U_f = U_f \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$. Since by Lemma 28.18.4 we can cover U by $V_g = U_f$ as above, we see that $U \to U \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$ is an isomorphism. $\square$

0F82 Lemma 28.18.6. Let X be a quasi-affine scheme. There exists an integer $n \geq 0$, an affine scheme T, and a morphism $T \to X$ such that for every morphism $X' \to X$ with X' affine the fibre product $X' \times_X T$ is isomorphic to $\mathbf{A}^n_{X'}$ over X'.

Proof. By definition, there exists a ring A such that X is isomorphic to a quasi-compact open subscheme $U \subset \mathrm{Spec}(A)$. Recall that the standard opens $D(f) \subset \mathrm{Spec}(A)$ form a basis for the topology, see Algebra, Section 10.17. Since U is quasi-compact we can choose $f_1, \ldots, f_n \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_n)$. Thus we may assume $X = \mathrm{Spec}(A) \setminus V(I)$ where $I = (f_1, \ldots, f_n)$. We set

$$T = \mathrm{Spec}(A[t, x_1, \ldots, x_n]/(f_1x_1 + \ldots + f_nx_n - 1))$$

The structure morphism $T \to \mathrm{Spec}(A)$ factors through the open X to give the morphism $T \to X$. If $X' = \mathrm{Spec}(A')$ and the morphism $X' \to X$ corresponds to the ring map $A \to A'$, then the images $f'_1, \ldots, f'_n \in A'$ of $f_1, \ldots, f_n$ generate the unit ideal in A'. Say $1 = f'_1a'_1 + \ldots + f'_na'_n$. The base change $X' \times_X T$ is the spectrum of $A'[t, x_1, \ldots, x_n]/(f'_1x_1 + \ldots + f'_nx_n - 1)$. We claim the A'-algebra homomorphism

$$\varphi : A'[y_1, \ldots, y_n] \longrightarrow A'[t, x_1, \ldots, x_n, x_{n+1}]/(f'_1x_1 + \ldots + f'_nx_n - 1)$$

sending y_i to $a'_it + x_i$ is an isomorphism. The claim finishes the proof of the lemma. The inverse of φ is given by the A'-algebra homomorphism

$$\psi : A'[t, x_1, \ldots, x_n, x_{n+1}]/(f'_1x_1 + \ldots + f'_nx_n - 1) \longrightarrow A'[y_1, \ldots, y_n]$$

sending t to $-1 + f'_1y_1 + \ldots + f'_ny_n$ and x_i to $y_i + a'_i - a'_i(f'_1y_1 + \ldots + f'_ny_n)$ for $i = 1, \ldots, n$. This makes sense because $\sum f'_ix_i$ is mapped to

$$\sum f'_i(y_i + a'_i - a'_i(\sum f'_jy_j)) = (\sum f'_iy_i) + 1 - (\sum f'_jy_j) = 1$$

To see the maps are mutually inverse one computes as follows:

$$\varphi(\psi(t) = \varphi(-1 + \sum f'_iy_i) = -1 + \sum f'_i(a'_it + x_i) = t$$
$$\varphi(\psi(x_i)) = \varphi(y_i + a'_i - a'_i(\sum f'_jy_j)) = a'_it + x_i + a'_i - a'_i(\sum f'_ja'_jt + f'_jx_j) = x_i$$
$$\psi(\varphi(y_i)) = \psi(a'_it + x_i) = a'_i(-1 + \sum f'_jy_j) + y_i + a'_i - a'_i(\sum f'_jy_j) = y_i$$

This finishes the proof. ☐

28.19. Flat modules

05NZ On any ringed space $(X, \mathcal{O}_X)$ we know what it means for an $\mathcal{O}_X$-module to be flat (at a point), see Modules, Definition 17.17.1 (Definition 17.17.3). For quasi-coherent sheaves on an affine scheme this matches the notion defined in the algebra chapter.

05P0 Lemma 28.19.1. Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if M is a flat R-module.

Proof. Flatness of $\mathcal{F}$ may be checked on the stalks, see Modules, Lemma 17.17.2. The same is true in the case of modules over a ring, see Algebra, Lemma 10.39.18. And since $\mathcal{F}_x = M_{\mathfrak{p}}$ if x corresponds to $\mathfrak{p}$ the lemma is true. ☐

28.20. Locally free modules

05P1 On any ringed space we know what it means for an $\mathcal{O}_X$-module to be (finite) locally free. On an affine scheme this matches the notion defined in the algebra chapter.

05JM Lemma 28.20.1. Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a (finite) locally free $\mathcal{O}_X$-module of if and only if M is a (finite) locally free R-module.

Proof. Follows from the definitions, see Modules, Definition 17.14.1 and Algebra, Definition 10.78.1. $\square$

We can characterize finite locally free modules in many different ways.

05P2 Lemma 28.20.2. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:

(1) $\mathcal{F}$ is a flat $\mathcal{O}_X$-module of finite presentation,
(2) $\mathcal{F}$ is $\mathcal{O}_X$-module of finite presentation and for all $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module,
(3) $\mathcal{F}$ is a locally free, finite type $\mathcal{O}_X$-module,
(4) $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module, and
(5) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, for every $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, and the function
$$\rho_{\mathcal{F}} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is locally constant in the Zariski topology on X.

Proof. This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 10.78.2. The translation uses Lemmas 28.16.1, 28.16.2, 28.19.1, and 28.20.1. $\square$

0FWH Lemma 28.20.3. Let X be a reduced scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the equivalent conditions of Lemma 28.20.2 are also equivalent to

(6) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type and the function
$$\rho_{\mathcal{F}} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is locally constant in the Zariski topology on X.

Proof. This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 10.78.3. $\square$

28.21. Locally projective modules

05JN A consequence of the work done in the algebra chapter is that it makes sense to define a locally projective module as follows.

05JP Definition 28.21.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is locally projective if for every affine open $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is projective.

05JQ Lemma 28.21.2. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is locally projective, and
(2) there exists an affine open covering $X = \bigcup U_i$ such that the $\mathcal{O}_X(U_i)$-module $\mathcal{F}(U_i)$ is projective for every i.

In particular, if $X = \text{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$ then $\mathcal{F}$ is locally projective if and only if M is a projective A-module.

Proof. First, note that if M is a projective A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a projective B-module, see Algebra, Lemma 10.94.1. Hence if U is an affine open such that $\mathcal{F}(U)$ is a projective $\mathcal{O}_X(U)$-module, then the standard open $D(f)$ is an affine open such that $\mathcal{F}(D(f))$ is a projective $\mathcal{O}_X(D(f))$-module for all $f \in \mathcal{O}_X(U)$. Assume (2) holds. Let $U \subset X$ be an arbitrary affine open. We can find an open covering $U = \bigcup_{j=1,\ldots,m} D(f_j)$ by finitely many standard opens $D(f_j)$ such that for each j the open $D(f_j)$ is a standard open of some U_i, see Schemes, Lemma 26.11.5. Hence, if we set $A = \mathcal{O}_X(U)$ and if M is an A-module such that $\mathcal{F}|_U$ corresponds to M, then we see that M_{f_j} is a projective A_{f_j}-module. It follows that $A \to B = \prod A_{f_j}$ is a faithfully flat ring map such that $M \otimes_A B$ is a projective B-module. Hence M is projective by Algebra, Theorem 10.95.6. $\square$

060M Lemma 28.21.3. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.

Proof. See Algebra, Lemma 10.94.1. $\square$

28.22. Extending quasi-coherent sheaves

01PD It is sometimes useful to be able to show that a given quasi-coherent sheaf on an open subscheme extends to the whole scheme.

01PE Lemma 28.22.1. Let $j : U \to X$ be a quasi-compact open immersion of schemes.

1. Any quasi-coherent sheaf on U extends to a quasi-coherent sheaf on X.
2. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent subsheaf. There exists a quasi-coherent subsheaf $\mathcal{H}$ of $\mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ as subsheaves of $\mathcal{F}|_U$.
3. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on U. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. There exists a quasi-coherent sheaf $\mathcal{H}$ of $\mathcal{O}_X$-modules and a map $\psi : \mathcal{H} \to \mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ and that $\psi|_U = \varphi$.

Proof. An immersion is separated (see Schemes, Lemma 26.23.8) and j is quasi-compact by assumption. Hence for any quasi-coherent sheaf $\mathcal{G}$ on U the sheaf $j_*\mathcal{G}$ is an extension to X. See Schemes, Lemma 26.24.1 and Sheaves, Section 6.31.

Assume $\mathcal{F}$, $\mathcal{G}$ are as in (2). Then $j_*\mathcal{G}$ is a quasi-coherent sheaf on X (see above). It is a subsheaf of $j_*j^*\mathcal{F}$. Hence the kernel

$$\mathcal{H} = \mathrm{Ker}(\mathcal{F} \oplus j_*\mathcal{G} \longrightarrow j_*j^*\mathcal{F})$$

is quasi-coherent as well, see Schemes, Section 26.24. It is formal to check that $\mathcal{H} \subset \mathcal{F}$ and that $\mathcal{H}|_U = \mathcal{G}$ (using the material in Sheaves, Section 6.31 again).

Part (3) is proved in the same manner as (2). Just take $\mathcal{H} = \mathrm{Ker}(\mathcal{F} \oplus j_*\mathcal{G} \to j_*j^*\mathcal{F})$ with its obvious map to $\mathcal{F}$ and its obvious identification with $\mathcal{G}$ over U. $\square$

01PF Lemma 28.22.2. Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.

Proof. Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend $\mathcal{G}$ to a $\mathcal{G}_1$ over $U \cup U_1$ to a $\mathcal{G}_2$ over $U \cup U_1 \cup U_2$ to a $\mathcal{G}_3$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U, V are quasi-compact open, we see that $U \cap V$ is a quasi-compact open. It suffices to prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U \cap V})$ since we can glue the resulting sheaf $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = \text{Spec}(R)$. Write $\mathcal{F} = \widetilde{M}$ for some R-module M. By Lemma 28.22.1 above we may find a quasi-coherent subsheaf $\mathcal{H} \subset \mathcal{F}$ which restricts to $\mathcal{G}$ over U. Write $\mathcal{H} = \widetilde{N}$ for some R-module N. For every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that N_f is finitely generated, see Lemma 28.16.1. Since U is quasi-compact we can cover it by finitely many $D(f_i)$ such that N_{f_i} is generated by finitely many elements, say $x_{i,1}/f_i^N, \ldots, x_{i,r_i}/f_i^N$. Let $N' \subset N$ be the submodule generated by the elements $x_{i,j}$. Then the subsheaf $\mathcal{G}' = \widetilde{N'} \subset \mathcal{H} \subset \mathcal{F}$ works. □

01PG Lemma 28.22.3. Let X be a quasi-compact and quasi-separated scheme. Any quasi-coherent sheaf of $\mathcal{O}_X$-modules is the directed colimit of its quasi-coherent $\mathcal{O}_X$-submodules which are of finite type.

Proof. The colimit is directed because if $\mathcal{G}_1$, $\mathcal{G}_2$ are quasi-coherent subsheaves of finite type, then the image of $\mathcal{G}_1 \oplus \mathcal{G}_2 \to \mathcal{F}$ is a quasi-coherent submodule of finite type. Let $U \subset X$ be any affine open, and let $s \in \Gamma(U, \mathcal{F})$ be any section. Let $\mathcal{G} \subset \mathcal{F}|_U$ be the subsheaf generated by s. Then clearly $\mathcal{G}$ is quasi-coherent and has finite type as an $\mathcal{O}_U$-module. By Lemma 28.22.2 we see that $\mathcal{G}$ is the restriction of a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{F}$ which has finite type. Since X has a basis for the topology consisting of affine opens we conclude that every local section of $\mathcal{F}$ is locally contained in a quasi-coherent submodule of finite type. Thus we win. □

01PI Lemma 28.22.4. Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module which is of finite presentation. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. Then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation, and a morphism of $\mathcal{O}_X$-modules $\varphi' : \mathcal{G}' \to \mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$ and such that $\varphi'|_U = \varphi$.

Proof. The beginning of the proof is a repeat of the beginning of the proof of Lemma 28.22.2. We write it out carefuly anyway.

Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend the pair $(\mathcal{G}, \varphi)$ to a pair $(\mathcal{G}_1, \varphi_1)$ over $U \cup U_1$ to a pair $(\mathcal{G}_2, \varphi_2)$ over $U \cup U_1 \cup U_2$ to a pair $(\mathcal{G}_3, \varphi_3)$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U quasi-compact, we see that $U \cap V \subset V$ is quasi-compact. Suppose we prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U \cap V}, \varphi|_{U \cap V})$ thereby producing $(\mathcal{G}', \varphi')$ over V. Then we can glue $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$, and similarly we can glue the map φ' to the map φ along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = \text{Spec}(R)$. By Lemma 28.22.1 above we may find a quasi-coherent sheaf $\mathcal{H}$ with a map $\psi : \mathcal{H} \to \mathcal{F}$ over X which restricts to $\mathcal{G}$ and φ over U. By Lemma 28.22.2 we can find a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}' \subset \mathcal{H}$

such that $\mathcal{H}'|_U = \mathcal{G}$. Thus after replacing $\mathcal{H}$ by $\mathcal{H}'$ and ψ by the restriction of ψ to $\mathcal{H}'$ we may assume that $\mathcal{H}$ is of finite type. By Lemma 28.16.2 we conclude that $\mathcal{H} = \widetilde{N}$ with N a finitely generated R-module. Hence there exists a surjection as in the following short exact sequence of quasi-coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus n} \to \mathcal{H} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Since $\mathcal{G}$ is of finite presentation and $\mathcal{H}|_U = \mathcal{G}$ by Modules, Lemma 17.11.3 the restriction $\mathcal{K}|_U$ is an $\mathcal{O}_U$-module of finite type. Hence by Lemma 28.22.2 again we see that there exists a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{K}' \subset \mathcal{K}$ such that $\mathcal{K}'|_U = \mathcal{K}|_U$. The solution to the problem posed in the lemma is to set

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}'$$

which is clearly of finite presentation and restricts to give $\mathcal{G}$ on U with φ' equal to the composition

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}' \to \mathcal{O}_X^{\oplus n}/\mathcal{K} = \mathcal{H} \xrightarrow{\psi} \mathcal{F}.$$

This finishes the proof of the lemma. □

0G41 Lemma 28.22.5. Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module.

(1) If $\mathcal{G}$ is quasi-coherent and of finite type, then there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}'$ of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.
(2) If $\mathcal{G}$ is of finite presentation, then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation such that $\mathcal{G}'|_U = \mathcal{G}$.

Proof. Part (2) is the special case of Lemma 28.22.4 where $\mathcal{F} = 0$. For part (1) we first write $\mathcal{G} = \mathcal{F}|_U$ for some quasi-coherent $\mathcal{O}_X$-module by Lemma 28.22.1 and then we apply Lemma 28.22.2 with $\mathcal{G} = \mathcal{F}|_U$. □

The following lemma says that every quasi-coherent sheaf on a quasi-compact and quasi-separated scheme is a filtered colimit of $\mathcal{O}$-modules of finite presentation. Actually, we reformulate this in (perhaps more familiar) terms of directed colimits over directed sets in the next lemma.

01PJ Lemma 28.22.6. Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist

(1) a filtered index category $\mathcal{I}$ (see Categories, Definition 4.19.1),
(2) a diagram $\mathcal{I} \to \text{Mod}(\mathcal{O}_X)$ (see Categories, Section 4.14), $i \mapsto \mathcal{F}_i$,
(3) morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\text{colim}_i \mathcal{F}_i = \mathcal{F}.$$

Proof. Choose a set I and for each $i \in I$ an $\mathcal{O}_X$-module of finite presentation and a homomorphism of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$ with the following property: For any $\psi : \mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ of finite presentation there is an $i \in I$ such that there exists an isomorphism $\alpha : \mathcal{F}_i \to \mathcal{G}$ with $\varphi_i = \psi \circ \alpha$. It is clear from Modules, Lemma 17.9.8 that such a set exists (see also its proof). We denote $\mathcal{I}$ the category with $\text{Ob}(\mathcal{I}) = I$ and given $i, i' \in I$ we set

$$\text{Mor}_{\mathcal{I}}(i, i') = \{\alpha : \mathcal{F}_i \to \mathcal{F}_{i'} \mid \alpha \circ \varphi_{i'} = \varphi_i\}.$$

We claim that $\mathcal{I}$ is a filtered category and that $\mathcal{F} = \text{colim}_i \mathcal{F}_i$.

Let $i, i' \in I$. Then we can consider the morphism

$$\mathcal{F}_i \oplus \mathcal{F}_{i'} \longrightarrow \mathcal{F}$$

which is the direct sum of φ_i and $\varphi_{i'}$. Since a direct sum of finitely presented $\mathcal{O}_X$-modules is finitely presented we see that there exists some $i'' \in I$ such that $\varphi_{i''} : \mathcal{F}_{i''} \to \mathcal{F}$ is isomorphic to the displayed arrow towards $\mathcal{F}$ above. Since there are commutative diagrams

$$\begin{tikzcd} \mathcal{F}_i \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, equal] \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} \arrow[r] & \mathcal{F} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}_{i'} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, equal] \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} \arrow[r] & \mathcal{F} \end{tikzcd}$$

we see that there are morphisms $i \to i''$ and $i' \to i''$ in $\mathcal{I}$. Next, suppose that we have $i, i' \in I$ and morphisms $\alpha, \beta : i \to i'$ (corresponding to $\mathcal{O}_X$-module maps $\alpha, \beta : \mathcal{F}_i \to \mathcal{F}_{i'}$). In this case consider the coequalizer

$$\mathcal{G} = \text{Coker}(\mathcal{F}_i \xrightarrow{\alpha - \beta} \mathcal{F}_{i'})$$

Note that $\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation. Since by definition of morphisms in the category $\mathcal{I}$ we have $\varphi_{i'} \circ \alpha = \varphi_{i'} \circ \beta$ we see that we get an induced map $\psi : \mathcal{G} \to \mathcal{F}$. Hence again the pair $(\mathcal{G}, \psi)$ is isomorphic to the pair $(\mathcal{F}_{i''}, \varphi_{i''})$ for some i''. Hence we see that there exists a morphism $i' \to i''$ in $\mathcal{I}$ which equalizes α and β. Thus we have shown that the category $\mathcal{I}$ is filtered.

We still have to show that the colimit of the diagram is $\mathcal{F}$. By definition of the colimit, and by our definition of the category $\mathcal{I}$ there is a canonical map

$$\varphi : \text{colim}_i \mathcal{F}_i \longrightarrow \mathcal{F}.$$

Pick $x \in X$. Let us show that φ_x is an isomorphism. Recall that

$$(\text{colim}_i \mathcal{F}_i)_x = \text{colim}_i \mathcal{F}_{i,x},$$

see Sheaves, Section 6.29. First we show that the map φ_x is injective. Suppose that $s \in \mathcal{F}_{i,x}$ is an element such that s maps to zero in $\mathcal{F}_x$. Then there exists a quasi-compact open U such that s comes from $s \in \mathcal{F}_i(U)$ and such that $\varphi_i(s) = 0$ in $\mathcal{F}(U)$. By Lemma 28.22.2 we can find a finite type quasi-coherent subsheaf $\mathcal{K} \subset \text{Ker}(\varphi_i)$ which restricts to the quasi-coherent $\mathcal{O}_U$-submodule of $\mathcal{F}_i$ generated by s: $\mathcal{K}|_U = \mathcal{O}_U \cdot s \subset \mathcal{F}_i|_U$. Clearly, $\mathcal{F}_i/\mathcal{K}$ is of finite presentation and the map φ_i factors through the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Hence we can find an $i' \in I$ and a morphism $\alpha : \mathcal{F}_i \to \mathcal{F}_{i'}$ in $\mathcal{I}$ which can be identified with the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Then it follows that the section s maps to zero in $\mathcal{F}_{i'}(U)$ and in particular in $(\text{colim}_i \mathcal{F}_i)_x = \text{colim}_i \mathcal{F}_{i,x}$. The injectivity follows. Finally, we show that the map φ_x is surjective. Pick $s \in \mathcal{F}_x$. Choose a quasi-compact open neighbourhood $U \subset X$ of x such that s corresponds to a section $s \in \mathcal{F}(U)$. Consider the map $s : \mathcal{O}_U \to \mathcal{F}$ (multiplication by s). By Lemma 28.22.4 there exists an $\mathcal{O}_X$-module $\mathcal{G}$ of finite presentation and an $\mathcal{O}_X$-module map $\mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}|_U \to \mathcal{F}|_U$ is identified with $s : \mathcal{O}_U \to \mathcal{F}$. Again by definition of $\mathcal{I}$ there exists an $i \in I$ such that $\mathcal{G} \to \mathcal{F}$ is isomorphic to $\varphi_i : \mathcal{F}_i \to \mathcal{F}$. Clearly there exists a section $s' \in \mathcal{F}_i(U)$ mapping to $s \in \mathcal{F}(U)$. This proves surjectivity and the proof of the lemma is complete. □

01PK Lemma 28.22.7. Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist

(1) a directed set I (see Categories, Definition 4.21.1),
(2) a system $(\mathcal{F}_i, \varphi_{ii'})$ over I in $\text{Mod}(\mathcal{O}_X)$ (see Categories, Definition 4.21.2)
(3) morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\operatorname{colim}_i \mathcal{F}_i = \mathcal{F}.$$

Proof. This is a direct consequence of Lemma 28.22.6 and Categories, Lemma 4.21.5 (combined with the fact that colimits exist in the category of sheaves of $\mathcal{O}_X$-modules, see Sheaves, Section 6.29). $\square$

086M Lemma 28.22.8. Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ with $\mathcal{F}_i$ of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.

Proof. Write $\mathcal{F} = \operatorname{colim} \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 28.22.7). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some i. Namely, choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$. Choose sections $s_{jl} \in \mathcal{F}(U_j)$ generating $\mathcal{F}|_{U_j}$, see Lemma 28.16.1. By Sheaves, Lemma 6.29.1 we see that s_{jl} is in the image of $\mathcal{G}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \operatorname{colim} \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 28.22.3). Then $\mathcal{F} = \operatorname{colim} \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. $\square$

080V Lemma 28.22.9. Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open such that $\mathcal{F}|_U$ is of finite presentation. Then there exists a map of $\mathcal{O}_X$-modules $\varphi : \mathcal{G} \to \mathcal{F}$ with (a) $\mathcal{G}$ of finite presentation, (b) φ is surjective, and (c) $\varphi|_U$ is an isomorphism.

Proof. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a directed colimit with each $\mathcal{F}_i$ of finite presentation, see Lemma 28.22.7. Choose a finite affine open covering $X = \bigcup V_j$ and choose finitely many sections $s_{jl} \in \mathcal{F}(V_j)$ generating $\mathcal{F}|_{V_j}$, see Lemma 28.16.1. By Sheaves, Lemma 6.29.1 we see that s_{jl} is in the image of $\mathcal{F}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{F}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{F}_i$ be the kernel of the map $\mathcal{F}_i \to \mathcal{F}$. Since $\mathcal{F}_U$ is of finite presentation, we see that $\mathcal{K}|_U$ is of finite type, see Modules, Lemma 17.11.3. Hence we can find a finite type quasi-coherent submodule $\mathcal{K}' \subset \mathcal{K}$ with $\mathcal{K}'|_U = \mathcal{K}|_U$, see Lemma 28.22.2. Then $\mathcal{G} = \mathcal{F}_i/\mathcal{K}'$ with the given map $\mathcal{G} \to \mathcal{F}$ is a solution. $\square$

Let X be a scheme. In the following lemma we use the notion of a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite presentation. This means that for every affine open $\text{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

05JS Lemma 28.22.10. Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. There exist

(1) a directed set I (see Categories, Definition 4.21.1),
(2) a system $(\mathcal{A}_i, \varphi_{ii'})$ over I in the category of $\mathcal{O}_X$-algebras,
(3) morphisms of $\mathcal{O}_X$-algebras $\varphi_i : \mathcal{A}_i \to \mathcal{A}$

such that each $\mathcal{A}_i$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\operatorname{colim}_i \mathcal{A}_i = \mathcal{A}.$$

Proof. First we write $\mathcal{A} = \operatorname{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 28.22.7. For each i let $\mathcal{B}_i = \operatorname{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \operatorname{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \operatorname{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 28.22.3. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \operatorname{colim}_{i,j} \mathcal{A}_{i,j}$. □

Let X be a scheme. In the following lemma we use the notion of a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type. This means that for every affine open $\operatorname{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

05JT Lemma 28.22.11. Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.

Proof. If $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are quasi-coherent $\mathcal{O}_X$-subalgebras of finite type, then the image of $\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2 \to \mathcal{A}$ is also a quasi-coherent $\mathcal{O}_X$-subalgebra of finite type (some details omitted) which contains both $\mathcal{A}_1$ and $\mathcal{A}_2$. In this way we see that the system is directed. To show that $\mathcal{A}$ is the colimit of this system, write $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$ as a directed colimit of finitely presented quasi-coherent $\mathcal{O}_X$-algebras as in Lemma 28.22.10. Then the images $\mathcal{A}'_i = \operatorname{Im}(\mathcal{A}_i \to \mathcal{A})$ are quasi-coherent subalgebras of $\mathcal{A}$ of finite type. Since $\mathcal{A}$ is the colimit of these the result follows. □

Let X be a scheme. In the following lemma we use the notion of a finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$. This means that for every affine open $\operatorname{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is finite (resp. integral) as an R-algebra.

086N Lemma 28.22.12. Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras such that all transition maps $\mathcal{A}_{i'} \to \mathcal{A}_i$ are surjective.

Proof. By Lemma 28.22.8 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \operatorname{colim} \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 28.22.3). Set

$$\mathcal{A}_i = \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjections, and $\mathcal{A} = \text{colim}\,\mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for i sufficiently large. To do this we choose an affine open covering $X = U_1 \cup \ldots \cup U_m$. Take generators $f_{j,1}, \ldots, f_{j,N_j} \in \Gamma(U_i, \mathcal{F})$. As $\mathcal{A}(U_j)$ is a finite $\mathcal{O}_X(U_j)$-algebra we see that for each k there exists a monic polynomial $P_{j,k} \in \mathcal{O}(U_j)[T]$ such that $P_{j,k}(f_{j,k})$ is zero in $\mathcal{A}(U_j)$. Since $\mathcal{A} = \text{colim}\,\mathcal{A}_i$ by construction, we have $P_{j,k}(f_{j,k}) = 0$ in $\mathcal{A}_i(U_j)$ for all sufficiently large i. For such i the algebras $\mathcal{A}_i$ are finite. □

0817 Lemma 28.22.13. Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then

(1) $\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and
(2) $\mathcal{A}$ is a direct colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras.

Proof. By Lemma 28.22.11 we have $\mathcal{A} = \text{colim}\,\mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (apply Algebra, Lemma 10.36.5 to $\mathcal{A}_i(U) \subset \mathcal{A}(U)$ for affine opens U in X). This proves (1).

To prove (2), write $\mathcal{A} = \text{colim}\,\mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 28.22.7. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \text{colim}\,\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above). Write $\mathcal{J}_i = \text{colim}\,\mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{tikzcd} \mathcal{J}_i \arrow[r] & \mathcal{J}_{i'} \\ \mathcal{E}_{ik} \arrow[u] \arrow[r] & \mathcal{E}_{i'k'} \arrow[u] \end{tikzcd}$$

commute. This follows from Modules, Lemma 17.22.8. This induces a map

$$\mathcal{A}_{ik} = \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \text{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 28.22.12). Finally, we have

$$\text{colim}\,\mathcal{A}_{ik} = \text{colim}\,\mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 28.22.12 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. □

28.23. Gabber's result

077K In this section we prove a result of Gabber which guarantees that on every scheme there exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the union of its quasi-coherent κ-generated subsheaves. It follows that the category of quasi-coherent sheaves on a scheme is a Grothendieck abelian category having limits and enough injectives[2].

077L Definition 28.23.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be an infinite cardinal. We say a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is κ-generated if there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is generated by a subset $R_i \subset \mathcal{F}(U_i)$ whose cardinality is at most κ.

Note that a direct sum of at most κ κ-generated modules is again κ-generated because $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6. In particular this holds for the direct sum of two κ-generated modules. Moreover, a quotient of a κ-generated sheaf is κ-generated. (But the same needn't be true for submodules.)

077M Lemma 28.23.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t)_{t \in T}$ of κ-generated $\mathcal{O}_X$-modules such that every κ-generated $\mathcal{O}_X$-module is isomorphic to one of the $\mathcal{F}_t$.

Proof. There is a set of coverings of X (provided we disallow repeats). Suppose $X = \bigcup U_i$ is a covering and suppose $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module. Then there is a set of isomorphism classes of $\mathcal{O}_X$-modules $\mathcal{F}$ with the property that $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$ since there is a set of glueing maps. This reduces us to proving there is a set of (isomorphism classes of) quotients $\oplus_{k \in \kappa} \mathcal{O}_X \to \mathcal{F}$ for any ringed space X. This is clear. $\square$

Here is the result the title of this section refers to.

077N Lemma 28.23.3. Let X be a scheme. There exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the directed colimit of its quasi-coherent κ-generated submodules.

Proof. Choose an affine open covering $X = \bigcup_{i \in I} U_i$. For each pair i, j choose an affine open covering $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$. Write $U_i = \mathrm{Spec}(A_i)$ and $U_{ijk} = \mathrm{Spec}(A_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}.

Let $\mathcal{F}$ be a quasi-coherent sheaf. Set $M_i = \mathcal{F}(U_i)$ and $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}.$$

see Schemes, Lemma 26.7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ as above. Fix such a map.

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S'_i)$ where

$$S'_j = \bigcup\nolimits_{(i,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

[2] Nicely explained in a blog post by Akhil Mathew.

Note that $S_i \subset S'_i$. Note that S'_i has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk}.$$

as submodules of M_{ijk}. Thus there exists a quasi-coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$. Moreover, by construction the sheaf $\mathcal{G}$ is κ-generated.

Let $\{\mathcal{G}_t\}_{t \in T}$ be the set of κ-generated quasi-coherent subsheaves. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent subsheaf as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $\text{colim}_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. $\square$

077P Proposition 28.23.4. Let X be a scheme.

(1) The category $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_X)$ has enough injectives and all limits.
(2) The inclusion functor $QCoh(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ has a right adjoint[3]

$$Q : \text{Mod}(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

Proof. Part (1) means $QCoh(\mathcal{O}_X)$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Schemes, Section 26.24 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $\text{Mod}(\mathcal{O}_X)$. By Modules, Lemma 17.3.2 filtered colimits are exact. Hence (a) and (b) hold. To construct a generator U, pick a cardinal κ as in Lemma 28.23.3. Pick a collection $(\mathcal{F}_t)_{t \in T}$ of κ-generated quasi-coherent sheaves as in Lemma 28.23.2. Set $U = \bigoplus_{t \in T} \mathcal{F}_t$. Since every object of $QCoh(\mathcal{O}_X)$ is a filtered colimit of κ-generated quasi-coherent modules, i.e., of objects isomorphic to $\mathcal{F}_t$, it is clear that U is a generator. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.7 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $\text{Mod}(\mathcal{O}_X)$ we consider the functor

$$QCoh(\mathcal{O}_X)^{opp} \longrightarrow \text{Sets}, \quad \mathcal{G} \longmapsto \text{Hom}_X(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\text{Hom}_X(\mathcal{G}, \mathcal{F}) = \text{Hom}_X(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_X)$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that

[3]This functor is sometimes called the coherator.

$Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ is fully faithful. $\square$

28.24. Sections with support in a closed subset

07ZM Given any topological space X, a closed subset $Z \subset X$, and an abelian sheaf $\mathcal{F}$ you can take the subsheaf of sections whose support is contained in Z. If X is a scheme, Z a closed subscheme, and $\mathcal{F}$ a quasi-coherent module there is a variant where you take sections which are scheme theoretically supported on Z. However, in the scheme setting you have to be careful because the resulting $\mathcal{O}_X$-module may not be quasi-coherent.

01PH **Lemma 28.24.1.** Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be an open subscheme. The following are equivalent:

1. U is retrocompact in X,
2. U is quasi-compact,
3. U is a finite union of affine opens, and
4. there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $X \setminus U = V(\mathcal{I})$ (set theoretically).

Proof. The equivalence of (1), (2), and (3) follows from Lemma 28.2.3. Assume (1), (2), (3). Let $T = X \setminus U$. By Schemes, Lemma 26.12.4 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subscheme structure on T. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. By Lemma 28.22.2 there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $X \setminus U = V(\mathcal{I})$ and we obtain (4). Conversely, if $\mathcal{I}$ is as in (4) and $W = \text{Spec}(R) \subset X$ is an affine open, then $\mathcal{I}|_W = \widetilde{I}$ for some finitely generated ideal $I \subset R$, see Lemma 28.16.1. It follows that $U \cap W = \text{Spec}(R) \setminus V(I)$ is quasi-compact, see Algebra, Lemma 10.29.1. Hence $U \subset X$ is retrocompact by Lemma 28.2.6. $\square$

01PO **Lemma 28.24.2.** Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

1. $\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,
2. on any affine open $U \subset X$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and
3. $\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$ (the sheaf condition is easy to verify). Hence we may work locally on X to verify the other statements. In other words we may assume that $X = \text{Spec}(A)$, $\mathcal{F} = \widetilde{M}$ and $\mathcal{I} = \widetilde{I}$. It is clear that in this case $\mathcal{F}'(U) = \{x \in M \mid Ix = 0\} =: M'$ because $\widetilde{I}$ is generated by its global sections I which proves (2). To show $\mathcal{F}'$ is quasi-coherent it suffices to show that for every $f \in A$ we have $\{x \in M_f \mid I_f x = 0\} = (M')_f$. Write $I = (g_1, \ldots, g_t)$, which is possible because $\mathcal{I}$ is of finite type, see Lemma 28.16.1. If $x = y/f^n$ and $I_f x = 0$, then that means that for every i there exists an $m \geq 0$ such that $f^m g_i x = 0$. We may choose one m which works for all i (and this is where we use that I is finitely generated). Then we see that $f^m x \in M'$ and $x/f^n = f^m x/f^{n+m}$ in $(M')_f$ as desired. The proof of (3) is similar and omitted. $\square$

01PP Definition 28.24.3. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 28.24.2 above is called the subsheaf of sections annihilated by $\mathcal{I}$.

07ZN Lemma 28.24.4. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.

Proof. Omitted. (Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent so that Lemma 28.24.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$.) $\square$

For an abelian sheaf on a topological space we have discussed the subsheaf of sections with support in a closed subset in Modules, Remark 17.6.2. For quasi-coherent modules this submodule isn't always a quasi-coherent module, but if the closed subset has a retrocompact complement, then it is.

07ZP Lemma 28.24.5. Let X be a scheme. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \text{the support of } s \text{ is contained in } Z \cap U\}$$

If $X \setminus Z$ is a retrocompact open of X, then

(1) for an affine open $U \subset X$ there exist a finitely generated ideal $I \subset \mathcal{O}_X(U)$ such that $Z \cap U = V(I)$,
(2) for U and I as in (1) we have $\mathcal{F}'(U) = \{x \in \mathcal{F}(U) \mid I^n x = 0 \text{ for some } n\}$,
(3) $\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

Proof. Part (1) is Algebra, Lemma 10.29.1. Let $U = \mathrm{Spec}(A)$ and I be as in (1). Then $\mathcal{F}|_U$ is the quasi-coherent sheaf associated to some A-module M. We have

$$\mathcal{F}'(U) = \{x \in M \mid x = 0 \text{ in } M_\mathfrak{p} \text{ for all } \mathfrak{p} \notin Z\}.$$

by Modules, Definition 17.5.1. Thus $x \in \mathcal{F}'(U)$ if and only if $V(\mathrm{Ann}(x)) \subset V(I)$, see Algebra, Lemma 10.40.7. Since I is finitely generated this is equivalent to $I^n x = 0$ for some n. This proves (2).

Proof of (3). Observe that given $U \subset X$ open there is an exact sequence

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}(U \setminus Z)$$

If we denote $j : X \setminus Z \to X$ the inclusion morphism, then we observe that $\mathcal{F}(U \setminus Z)$ is the sections of the module $j_*(\mathcal{F}|_{X\setminus Z})$ over U. Thus we have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to j_*(\mathcal{F}|_{X\setminus Z})$$

The restriction $\mathcal{F}|_{X\setminus Z}$ is quasi-coherent. Hence $j_*(\mathcal{F}|_{X\setminus Z})$ is quasi-coherent by Schemes, Lemma 26.24.1 and our assumption that j is quasi-compact (any open immersion is separated). Hence $\mathcal{F}'$ is quasi-coherent as a kernel of a map of quasi-coherent modules, see Schemes, Section 26.24. $\square$

084L Definition 28.24.6. Let X be a scheme. Let $T \subset X$ be a closed subset whose complement is retrocompact in X. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 28.24.5 is called the subsheaf of sections supported on T.

07ZQ Lemma 28.24.7. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Let $Z \subset Y$ be a closed subset such that $Y \setminus Z$ is retrocompact in Y. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported in $f^{-1}Z$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported in Z.

Proof. Omitted. (Hint: First show that $X \setminus f^{-1}Z$ is retrocompact in X as $Y \setminus Z$ is retrocompact in Y. Hence Lemma 28.24.5 applies to $f^{-1}Z$ and $\mathcal{F}$. As f is quasi-compact and quasi-separated we see that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 28.24.5 applies to Z and $f_*\mathcal{F}$. Finally, match the sheaves directly.) $\square$

28.25. Sections of quasi-coherent sheaves

01PL Here is a computation of sections of a quasi-coherent sheaf on a quasi-compact open of an affine spectrum.

01PM Lemma 28.25.1. Let A be a ring. Let $I \subset A$ be a finitely generated ideal. Let M be an A-module. Then there is a canonical map

$$\mathrm{colim}_n \mathrm{Hom}_A(I^n, M) \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}).$$

This map is always injective. If for all $x \in M$ we have $Ix = 0 \Rightarrow x = 0$ then this map is an isomorphism. In general, set $M_n = \{x \in M \mid I^n x = 0\}$, then there is an isomorphism

$$\mathrm{colim}_n \mathrm{Hom}_A(I^n, M/M_n) \longrightarrow \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}).$$

Proof. Since $I^{n+1} \subset I^n$ and $M_n \subset M_{n+1}$ we can use composition via these maps to get canonical maps of A-modules

$$\mathrm{Hom}_A(I^n, M) \longrightarrow \mathrm{Hom}_A(I^{n+1}, M)$$

and

$$\mathrm{Hom}_A(I^n, M/M_n) \longrightarrow \mathrm{Hom}_A(I^{n+1}, M/M_{n+1})$$

which we will use as the transition maps in the systems. Given an A-module map $\varphi : I^n \to M$, then we get a map of sheaves $\widetilde{\varphi} : \widetilde{I^n} \to \widetilde{M}$ which we can restrict to the open $\mathrm{Spec}(A) \setminus V(I)$. Since $\widetilde{I^n}$ restricted to this open gives the structure sheaf we get an element of $\Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this is compatible with the transition maps in the system $\mathrm{Hom}_A(I^n, M)$. This gives the first arrow. To get the second arrow we note that $\widetilde{M}$ and $\widetilde{M/M_n}$ agree over the open $\mathrm{Spec}(A) \setminus V(I)$ since the sheaf $\widetilde{M_n}$ is clearly supported on $V(I)$. Hence we can use the same mechanism as before.

Next, we work out how to define this arrow in terms of algebra. Say $I = (f_1, \ldots, f_t)$. Then $\mathrm{Spec}(A) \setminus V(I) = \bigcup_{i=1,\ldots,t} D(f_i)$. Hence

$$0 \to \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M}) \to \bigoplus\nolimits_i M_{f_i} \to \bigoplus\nolimits_{i,j} M_{f_i f_j}$$

is exact. Suppose that $\varphi : I^n \to M$ is an A-module map. Consider the vector of elements $\varphi(f_i^n)/f_i^n \in M_{f_i}$. It is easy to see that this vector maps to zero in the second direct sum of the exact sequence above. Whence an element of

$\Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this description agrees with the one given above.

Let us show that the first arrow is injective using this description. Namely, if φ maps to zero, then for each i the element $\varphi(f_i^n)/f_i^n$ is zero in M_{f_i}. In other words we see that for each i we have $f_i^m\varphi(f_i^n) = 0$ for some $m \geq 0$. We may choose a single m which works for all i. Then we see that $\varphi(f_i^{n+m}) = 0$ for all i. It is easy to see that this means that $\varphi|_{I^{t(n+m-1)+1}} = 0$ in other words that φ maps to zero in the $t(n+m-1)+1$st term of the colimit. Hence injectivity follows.

Note that each $M_n = 0$ in case we have $Ix = 0 \Rightarrow x = 0$ for $x \in M$. Thus to finish the proof of the lemma it suffices to show that the second arrow is an isomorphism.

Let us attempt to construct an inverse of the second map of the lemma. Let $s \in \Gamma(\mathrm{Spec}(A) \setminus V(I), \widetilde{M})$. This corresponds to a vector x_i/f_i^n with $x_i \in M$ of the first direct sum of the exact sequence above. Hence for each i, j there exists $m \geq 0$ such that $f_i^m f_j^m (f_j^n x_i - f_i^n x_j) = 0$ in M. We may choose a single m which works for all pairs i, j. After replacing x_i by $f_i^m x_i$ and n by $n + m$ we see that we get $f_j^n x_i = f_i^n x_j$ in M for all i, j. Let us introduce

$$K_n = \{x \in M \mid f_1^n x = \ldots = f_t^n x = 0\}$$

We claim there is an A-module map

$$\varphi : I^{t(n-1)+1} \longrightarrow M/K_n$$

which maps the monomial $f_1^{e_1} \ldots f_t^{e_t}$ with $\sum e_i = t(n-1)+1$ to the class modulo K_n of the expression $f_1^{e_1} \ldots f_i^{e_i - n} \ldots f_t^{e_t} x_i$ where i is chosen such that $e_i \geq n$ (note that there is at least one such i). To see that this is indeed the case suppose that

$$\sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} = 0$$

is a relation between the monomials with coefficients a_E in A. Then we would map this to

$$z = \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_{i(E)}^{e_{i(E)}-n} \ldots f_t^{e_t} x_{i(E)}$$

where for each multiindex E we have chosen a particular $i(E)$ such that $e_{i(E)} \geq n$. Note that if we multiply this by f_j^n for any j, then we get zero, since by the relations $f_j^n x_i = f_i^n x_j$ above we get

$$\begin{aligned} f_j^n z &= \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_j^{e_j+n} \ldots f_{i(E)}^{e_{i(E)}-n} \ldots f_t^{e_t} x_{i(E)} \\ &= \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} x_j = 0. \end{aligned}$$

Hence $z \in K_n$ and we see that every relation gets mapped to zero in M/K_n. This proves the claim.

Note that $K_n \subset M_{t(n-1)+1}$. Hence the map φ in particular gives rise to an A-module map $I^{t(n-1)+1} \to M/M_{t(n-1)+1}$. This proves the second arrow of the lemma is surjective. We omit the proof of injectivity. $\square$

01PN Example 28.25.2. We will give two examples showing that the first displayed map of Lemma 28.25.1 is not an isomorphism.

Let k be a field. Consider the ring

$$A = k[x, y, z_1, z_2, \ldots]/(x^n z_n).$$

Set $I = (x)$ and let $M = A$. Then the element y/x defines a section of the structure sheaf of $\mathrm{Spec}(A)$ over $D(x) = \mathrm{Spec}(A) \setminus V(I)$. We claim that y/x is not in the image of the canonical map $\mathrm{colim}\, \mathrm{Hom}_A(I^n, A) \to A_x = \mathcal{O}(D(x))$. Namely, if so it would come from a homomorphism $\varphi : I^n \to A$ for some n. Set $a = \varphi(x^n)$. Then we would have $x^m(xa - x^n y) = 0$ for some $m > 0$. This would mean that $x^{m+1}a = x^{m+n}y$. This would mean that $\varphi(x^{n+m+1}) = x^{m+n}y$. This leads to a contradiction because it would imply that

$$0 = \varphi(0) = \varphi(z_{n+m+1}x^{n+m+1}) = x^{m+n}yz_{n+m+1}$$

which is not true in the ring A.

Let k be a field. Consider the ring

$$A = k[f, g, x, y, \{a_n, b_n\}_{n\geq 1}]/(fy - gx, \{a_n f^n + b_n g^n\}_{n\geq 1}).$$

Set $I = (f, g)$ and let $M = A$. Then $x/f \in A_f$ and $y/g \in A_g$ map to the same element of A_{fg}. Hence these define a section s of the structure sheaf of $\mathrm{Spec}(A)$ over $D(f) \cup D(g) = \mathrm{Spec}(A) \setminus V(I)$. However, there is no $n \geq 0$ such that s comes from an A-module map $\varphi : I^n \to A$ as in the source of the first displayed arrow of Lemma 28.25.1. Namely, given such a module map set $x_n = \varphi(f^n)$ and $y_n = \varphi(g^n)$. Then $f^m x_n = f^{n+m-1}x$ and $g^m y_n = g^{n+m-1}y$ for some $m \geq 0$ (see proof of the lemma). But then we would have $0 = \varphi(0) = \varphi(a_{n+m}f^{n+m} + b_{n+m}g^{n+m}) = a_{n+m}f^{n+m-1}x + b_{n+m}g^{n+m-1}y$ which is not the case in the ring A.

We will improve on the following lemma in the Noetherian case, see Cohomology of Schemes, Lemma 30.10.5.

01PQ Lemma 28.25.3. Let X be a quasi-compact scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $Z \subset X$ be the closed subscheme defined by $\mathcal{I}$ and set $U = X \setminus Z$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The canonical map

$$\mathrm{colim}_n\, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})$$

is injective. Assume further that X is quasi-separated. Let $\mathcal{F}_n \subset \mathcal{F}$ be subsheaf of sections annihilated by $\mathcal{I}^n$. The canonical map

$$\mathrm{colim}_n\, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}/\mathcal{F}_n) \longrightarrow \Gamma(U, \mathcal{F})$$

is an isomorphism.

Proof. Let $\mathrm{Spec}(A) = W \subset X$ be an affine open. Write $\mathcal{F}|_W = \widetilde{M}$ for some A-module M and $\mathcal{I}|_W = \widetilde{I}$ for some finite type ideal $I \subset A$. Restricting the first displayed map of the lemma to W we obtain the first displayed map of Lemma 28.25.1. Since we can cover X by a finite number of affine opens this proves the first displayed map of the lemma is injective.

We have $\mathcal{F}_n|_W = \widetilde{M_n}$ where $M_n \subset M$ is defined as in Lemma 28.25.1 (details omitted). The lemma guarantees that we have a bijection

$$\mathrm{colim}_n\, \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{I}^n|_W, (\mathcal{F}/\mathcal{F}_n)|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

for any such affine open W.

To see the second displayed arrow of the lemma is bijective, we choose a finite affine open covering $X = \bigcup_{j=1,\ldots,m} W_j$. The injectivity follows immediately from the above and the finiteness of the covering. If X is quasi-separated, then for each pair j, j' we choose a finite affine open covering

$$W_j \cap W_{j'} = \bigcup\nolimits_{k=1,\ldots,m_{jj'}} W_{jj'k}.$$

Let $s \in \Gamma(U, \mathcal{F})$. As seen above for each j there exists an n_j and a map $\varphi_j : \mathcal{I}^{n_j}|_{W_j} \to (\mathcal{F}/\mathcal{F}_{n_j})|_{W_j}$ which corresponds to $s|_{U \cap W_j}$. By the same token for each triple (j, j', k) there exists an integer $n_{jj'k}$ such that the restriction of φ_j and $\varphi_{j'}$ as maps $\mathcal{I}^{n_{jj'k}} \to \mathcal{F}/\mathcal{F}_{n_{jj'k}}$ agree over $W_{jj'k}$. Let $n = \max\{n_j, n_{jj'k}\}$ and we see that the φ_j glue as maps $\mathcal{I}^n \to \mathcal{F}/\mathcal{F}_n$ over X. This proves surjectivity of the map. □

28.26. Ample invertible sheaves

01PR Recall from Modules, Lemma 17.25.10 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where ss' denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

01PS Definition 28.26.1. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is ample if [DG67, II Definition 4.5.3]

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

01PT Lemma 28.26.2. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample. [DG67, II Proposition 4.5.6(i)]

Proof. This follows from the fact that $X_{s^n} = X_s$. □

01PU Lemma 28.26.3. Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme $Z \subset X$ the restriction of $\mathcal{L}$ to Z is ample.

Proof. This is clear since a closed subset of a quasi-compact space is quasi-compact and a closed subscheme of an affine scheme is affine (see Schemes, Lemma 26.8.2). □

01PV Lemma 28.26.4. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. For any affine $U \subset X$ the intersection $U \cap X_s$ is affine.

Proof. This translates into the following algebra problem. Let R be a ring. Let N be an invertible R-module (i.e., locally free of rank 1). Let $s \in N$ be an element. Then $U = \{\mathfrak{p} \mid s \notin \mathfrak{p}N\}$ is an affine open subset of $\mathrm{Spec}(R)$.

Let $A = \bigoplus_{n \geq 0} A_n$ be the symmetric algebra of N (which is commutative) and view s as an element of A_1. Set $B = A/(s-1)A$. This is an R-algebra whose construction commutes with any base change $R \to R'$. Thus $B' = B \otimes_R R'$ is the zero ring if s maps to zero in $N' = N \otimes_R R'$. It follows that if $x \in \mathrm{Spec}(R) \setminus U$, then $B \otimes_R \kappa(x) = 0$. We conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ factors through U as the fibres over $x \notin U$ are empty. On the other hand, if $\mathrm{Spec}(R') \subset U$ is an affine open, then s maps to a basis element of N' and we see that $B' = R'[s]/(s-1) \cong R'$. It follows that $\mathrm{Spec}(B) \to U$ is an isomorphism and U is indeed affine. □

0890 Lemma 28.26.5. Let X be a scheme. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible $\mathcal{O}_X$-modules. If [DG67, II Proposition 4.5.6(ii)]

(1) $\mathcal{L}$ is ample, and

(2) the open sets X_t where $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ for $m > 0$ cover X,

then $\mathcal{L} \otimes \mathcal{M}$ is ample.

Proof. We check the conditions of Definition 28.26.1. As $\mathcal{L}$ is ample we see that X is quasi-compact. Let $x \in X$. Choose $n \geq 1$, $m \geq 1$, $s \in \Gamma(X, \mathcal{L}^{\otimes n})$, and $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ such that $x \in X_s$, $x \in X_t$ and X_s is affine. Then $s^m t^n \in \Gamma(X, (\mathcal{L} \otimes \mathcal{M})^{\otimes nm})$, $x \in X_{s^m t^n}$, and $X_{s^m t^n}$ is affine by Lemma 28.26.4. $\square$

01PX Lemma 28.26.6. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume the open sets X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, form a basis for the topology on X. Then among those opens, the open sets X_s which are affine form a basis for the topology on X.

Proof. Let $x \in X$. Choose an affine open neighbourhood $\text{Spec}(R) = U \subset X$ of x. By assumption, there exists a $n \geq 1$ and a $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s \subset U$. By Lemma 28.26.4 above the intersection $X_s = U \cap X_s$ is affine. Since U can be chosen arbitrarily small we win. $\square$

01PY Lemma 28.26.7. Let X be a scheme and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume for every point x of X there exists $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine. Then X is separated.

Proof. We show first that X is quasi-separated. By assumption we can find a covering of X by affine opens of the form X_s. By Lemma 28.26.4, the intersection of any two such sets is affine, so Schemes, Lemma 26.21.6 implies that X is quasi-separated.

To show that X is separated, we can use the valuative criterion, Schemes, Lemma 26.22.2. Thus, let A be a valuation ring with fraction field K and consider two morphisms $f, g : \text{Spec}(A) \to X$ such that the two compositions $\text{Spec}(K) \to \text{Spec}(A) \to X$ agree. As A is local, there exists $p, q \geq 1$, $s \in \Gamma(X, \mathcal{L}^{\otimes p})$, and $t \in \Gamma(X, \mathcal{L}^{\otimes q})$ such that X_s and X_t are affine, $f(\text{Spec}\, A) \subseteq X_s$, and $g(\text{Spec}\, A) \subseteq X_t$. We now replace s by s^q, t by t^p, and $\mathcal{L}$ by $\mathcal{L}^{\otimes pq}$. This is harmless as $X_s = X_{s^q}$ and $X_t = X_{t^p}$, and now s and t are both sections of the same sheaf $\mathcal{L}$.

The quasi-coherent module $f^*\mathcal{L}$ corresponds to an A-module M and $g^*\mathcal{L}$ corresponds to an A-module N by our classification of quasi-coherent modules over affine schemes (Schemes, Lemma 26.7.4). The A-modules M and N are locally free of rank 1 (Lemma 28.20.1) and as A is local they are free (Algebra, Lemma 10.55.8). Therefore we may identify M and N with A-submodules of $M \otimes_A K$ and $N \otimes_A K$. The equality $f|_{\text{Spec}(K)} = g|_{\text{Spec}(K)}$ determines an isomorphism $\phi\colon M \otimes_A K \to N \otimes_A K$.

Let $x \in M$ and $y \in N$ be the elements corresponding to the pullback of s along f and g, respectively. These satisfy $\phi(x \otimes 1) = y \otimes 1$. The image of f is contained in X_s, so $x \notin \mathfrak{m}_A M$, that is, x generates M. Hence ϕ determines an isomorphism of M with the submodule of N generated by y. Arguing symmetrically using t, ϕ^{-1} determines an isomorphism of N with a submodule of M. Consequently ϕ restricts to an isomorphism of M and N. Since x generates M, its image y generates N, implying $y \notin \mathfrak{m}_A N$. Therefore $g(\text{Spec}(A)) \subseteq X_s$. Because X_s is affine, it is separated by Schemes, Lemma 26.21.15, and we conclude $f = g$. $\square$

09MP Lemma 28.26.8. Let X be a scheme. If there exists an ample invertible sheaf on X then X is separated.

Proof. Follows immediately from Lemma 28.26.7 and Definition 28.26.1. $\square$

01PZ Lemma 28.26.9. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes

$$f : X \longrightarrow Y = \text{Proj}(S),$$

to the homogeneous spectrum of S (see Constructions, Section 27.8). This morphism has the following properties

(1) $f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,
(2) there are $\mathcal{O}_X$-module maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (27.10.1.1),
(3) the composition $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ is the identity map, and
(4) for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of x such that $f^*\mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.

Proof. Denote $\psi : S \to \Gamma_*(X, \mathcal{L})$ the identity map. We are going to use the triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ of Constructions, Lemma 27.14.1. By assumption the open subscheme $U(\psi)$ of equals X. Hence $r_{\mathcal{L},\psi} : U(\psi) \to Y$ is defined on all of X. We set $f = r_{\mathcal{L},\psi}$. The maps in part (2) are the components of θ. Part (3) follows from condition (2) in the lemma cited above. Part (1) follows from (3) combined with condition (1) in the lemma cited above. Part (4) follows from the last statement in Constructions, Lemma 27.14.1 since the map α mentioned there is an isomorphism. $\square$

01Q0 Lemma 28.26.10. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume (a) every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, and (b) X is quasi-compact. Then the canonical morphism of schemes $f : X \longrightarrow \text{Proj}(S)$ of Lemma 28.26.9 above is quasi-compact with dense image.

Proof. To prove f is quasi-compact it suffices to show that $f^{-1}(D_+(s))$ is quasi-compact for any $s \in S_+$ homogeneous. Write $X = \bigcup_{i=1,\ldots,n} X_i$ as a finite union of affine opens. By Lemma 28.26.4 each intersection $X_s \cap X_i$ is affine. Hence $X_s = \bigcup_{i=1,\ldots,n} X_s \cap X_i$ is quasi-compact. Assume that the image of f is not dense to get a contradiction. Then, since the opens $D_+(s)$ with $s \in S_+$ homogeneous form a basis for the topology on $\text{Proj}(S)$, we can find such an s with $D_+(s) \neq \emptyset$ and $f(X) \cap D_+(s) = \emptyset$. By Lemma 28.26.9 this means $X_s = \emptyset$. By Lemma 28.17.2 this means that a power s^n is the zero section of $\mathcal{L}^{\otimes n \deg(s)}$. This in turn means that $D_+(s) = \emptyset$ which is the desired contradiction. $\square$

01Q1 Lemma 28.26.11. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume $\mathcal{L}$ is ample. Then the canonical morphism of schemes $f : X \longrightarrow \text{Proj}(S)$ of Lemma 28.26.9 is an open immersion with dense image.

Proof. By Lemma 28.26.7 we see that X is quasi-separated. Choose finitely many $s_1, \ldots, s_n \in S_+$ homogeneous such that X_{s_i} are affine, and $X = \bigcup X_{s_i}$. Say s_i has degree d_i. The inverse image of $D_+(s_i)$ under f is X_{s_i}, see Lemma 28.26.9. By Lemma 28.17.2 the ring map

$$(S^{(d_i)})_{(s_i)} = \Gamma(D_+(s_i), \mathcal{O}_{\text{Proj}(S)}) \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_X)$$

is an isomorphism. Hence f induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus f is an isomorphism of X onto the open subscheme $\bigcup_{i=1,\ldots,n} D_+(s_i)$ of $\text{Proj}(S)$. The image is dense by Lemma 28.26.10. □

01Q2 Lemma 28.26.12. Let X be a scheme. Let S be a graded ring. Assume X is quasi-compact, and assume there exists an open immersion

$$j : X \longrightarrow Y = \text{Proj}(S).$$

Then $j^*\mathcal{O}_Y(d)$ is an invertible ample sheaf for some $d > 0$.

Proof. This is Constructions, Lemma 27.10.6. □

01Q3 Proposition 28.26.13. Let X be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$. The following are equivalent:

01Q4 (1) $\mathcal{L}$ is ample,
01Q5 (2) the open sets X_s, with $s \in S_+$ homogeneous, cover X and the associated morphism $X \to \text{Proj}(S)$ is an open immersion,
01Q6 (3) the open sets X_s, with $s \in S_+$ homogeneous, form a basis for the topology of X,
01Q7 (4) the open sets X_s, with $s \in S_+$ homogeneous, which are affine form a basis for the topology of X,
01Q8 (5) for every quasi-coherent sheaf $\mathcal{F}$ on X the sum of the images of the canonical maps
$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$
with $n \geq 1$ equals $\mathcal{F}$,
01Q9 (6) same property as (5) with $\mathcal{F}$ ranging over all quasi-coherent sheaves of ideals,
01QA (7) X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exists an integer n_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$,
01QB (8) X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exist integers $n > 0$, $k \geq 0$ such that $\mathcal{F}$ is a quotient of a direct sum of k copies of $\mathcal{L}^{\otimes -n}$, and
01QC (9) same as in (8) with $\mathcal{F}$ ranging over all sheaves of ideals of finite type on X.

Proof. Lemma 28.26.11 is (1) ⇒ (2). Lemmas 28.26.2 and 28.26.12 provide the implication (1) ⇐ (2). The implications (2) ⇒ (4) ⇒ (3) are clear from Constructions, Section 27.8. Lemma 28.26.6 is (3) ⇒ (1). Thus we see that the first 4 conditions are all equivalent.

Assume the equivalent conditions (1) – (4). Note that in particular X is separated (as an open subscheme of the separated scheme $\text{Proj}(S)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Choose $s \in S_+$ homogeneous such that X_s is affine. We claim that any section $m \in \Gamma(X_s, \mathcal{F})$ is in the image of one of the maps displayed in (5) above. This will imply (5) since these affines X_s cover X. Namely, by Lemma 28.17.2 we may write m as the image of $m' \otimes s^{-n}$ for some $n \geq 1$, some $m' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This proves the claim.

Clearly (5) ⇒ (6). Let us assume (6) and prove $\mathcal{L}$ is ample. Pick $x \in X$. Let $U \subset X$ be an affine open which contains x. Set $Z = X \setminus U$. We may think of Z as a reduced closed subscheme, see Schemes, Section 26.12. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to the closed subscheme Z. By assumption (6), there exists an $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ such that s does not vanish at x (more precisely such that $s \not\in \mathfrak{m}_x\mathcal{I}_x \otimes \mathcal{L}_x^{\otimes n}$). We may think of s as a section of $\mathcal{L}^{\otimes n}$. Since it clearly vanishes along Z we see that $X_s \subset U$. Hence X_s is affine, see Lemma 28.26.4. This proves that $\mathcal{L}$ is ample. At this point we have proved that (1) – (6) are equivalent.

Assume the equivalent conditions (1) – (6). In the following we will use the fact that the tensor product of two sheaves of modules which are globally generated is globally generated without further mention (see Modules, Lemma 17.4.3). By (1) we can find elements $s_i \in S_{d_i}$ with $d_i \geq 1$ such that $X = \bigcup_{i=1,\ldots,n} X_{s_i}$. Set $d = d_1 \ldots d_n$. It follows that $\mathcal{L}^{\otimes d}$ is globally generated by

$$s_1^{d/d_1}, \ldots, s_n^{d/d_n}.$$

This means that if $\mathcal{L}^{\otimes j}$ is globally generated then so is $\mathcal{L}^{\otimes j+dn}$ for all $n \geq 0$. Fix a $j \in \{0, \ldots, d-1\}$. For any point $x \in X$ there exists an $n \geq 1$ and a global section s of $\mathcal{L}^{j+dn}$ which does not vanish at x, as follows from (5) applied to $\mathcal{F} = \mathcal{L}^{\otimes j}$ and ample invertible sheaf $\mathcal{L}^{\otimes d}$. Since X is quasi-compact there we may find a finite list of integers n_i and global sections s_i of $\mathcal{L}^{\otimes j+dn_i}$ which do not vanish at any point of X. Since $\mathcal{L}^{\otimes d}$ is globally generated this means that $\mathcal{L}^{\otimes j+dn}$ is globally generated where $n = \max\{n_i\}$. Since we proved this for every congruence class mod d we conclude that there exists an $n_0 = n_0(\mathcal{L})$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. At this point we see that if $\mathcal{F}$ is globally generated then so is $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0$.

We continue to assume the equivalent conditions (1) – (6). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. Denote $\mathcal{F}_n \subset \mathcal{F}$ the image of the canonical map of (5). By construction $\mathcal{F}_n \otimes \mathcal{L}^{\otimes n}$ is globally generated. By (5) we see $\mathcal{F}$ is the sum of the subsheaves $\mathcal{F}_n$, $n \geq 1$. By Modules, Lemma 17.9.7 we see that $\mathcal{F} = \sum_{n=1,\ldots,N} \mathcal{F}_n$ for some $N \geq 1$. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated whenever $n \geq N + n_0(\mathcal{L})$ with $n_0(\mathcal{L})$ as above. We conclude that (1) – (6) implies (7).

Assume (7). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. By (7) there exists an integer $n \geq 1$ such that the canonical map

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

is surjective. Let I be the set of finite subsets of $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ partially ordered by inclusion. Then I is a directed partially ordered set. For $i = \{s_1, \ldots, s_{r(i)}\}$ let $\mathcal{F}_i \subset \mathcal{F}$ be the image of the map

$$\bigoplus\nolimits_{j=1,\ldots,r(i)} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

which is multiplication by s_j on the jth factor. The surjectivity above implies that $\mathcal{F} = \mathrm{colim}_{i \in I} \mathcal{F}_i$. Hence Modules, Lemma 17.9.7 applies and we conclude that $\mathcal{F} = \mathcal{F}_i$ for some i. Hence we have proved (8). In other words, (7) $\Rightarrow$ (8).

The implication (8) $\Rightarrow$ (9) is trivial.

Finally, assume (9). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. By Lemma 28.22.3 (this is where we use the condition that X be quasi-separated) we see that $\mathcal{I} = \mathrm{colim}_\alpha I_\alpha$ with each I_α quasi-coherent of finite type. Since by assumption each of the I_α is a quotient of negative tensor powers of $\mathcal{L}$ we conclude the same for

$\mathcal{I}$ (but of course without the finiteness or boundedness of the powers). Hence we conclude that (9) implies (6). This ends the proof of the proposition. $\square$

0B3E Lemma 28.26.14. Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $i : X' \to X$ be a morphism of schemes. Assume at least one of the following conditions holds

(1) i is a quasi-compact immersion,
(2) X' is quasi-compact and i is an immersion,
(3) i is quasi-compact and induces a homeomorphism between X' and $i(X')$,
(4) X' is quasi-compact and i induces a homeomorphism between X' and $i(X')$.

Then $i^*\mathcal{L}$ is ample on X'.

Proof. Observe that in cases (1) and (3) the scheme X' is quasi-compact as X is quasi-compact by Definition 28.26.1. Thus it suffices to prove (2) and (4). Since (2) is a special case of (4) it suffices to prove (4).

Assume condition (4) holds. For $s \in \Gamma(X, \mathcal{L}^{\otimes d})$ denote $s' = i^*s$ the pullback of s to X'. Note that s' is a section of $(i^*\mathcal{L})^{\otimes d}$. By Proposition 28.26.13 the opens X_s, for $s \in \Gamma(X, \mathcal{L}^{\otimes d})$, form a basis for the topology on X. Since $X'_{s'} = i^{-1}(X_s)$ and since $X' \to i(X')$ is a homeomorphism, we conclude the opens $X'_{s'}$ form a basis for the topology of X'. Hence $i^*\mathcal{L}$ is ample by Proposition 28.26.13. $\square$

0DNK Lemma 28.26.15. Let S be a quasi-separated scheme. Let X, Y be schemes over S. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module and let $\mathcal{N}$ be an ample invertible $\mathcal{O}_Y$-module. Then $\mathcal{M} = \text{pr}_1^*\mathcal{L} \otimes_{\mathcal{O}_{X \times_S Y}} \text{pr}_2^*\mathcal{N}$ is an ample invertible sheaf on $X \times_S Y$.

Proof. The morphism $i : X \times_S Y \to X \times Y$ is a quasi-compact immersion, see Schemes, Lemma 26.21.9. On the other hand, $\mathcal{M}$ is the pullback by i of the corresponding invertible module on $X \times Y$. By Lemma 28.26.14 it suffices to prove the lemma for $X \times Y$. We check (1) and (2) of Definition 28.26.1 for $\mathcal{M}$ on $X \times Y$.

Since X and Y are quasi-compact, so is $X \times Y$. Let $z \in X \times Y$ be a point. Let $x \in X$ and $y \in Y$ be the projections. Choose $n > 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is an affine open neighbourhood of x. Choose $m > 0$ and $t \in \Gamma(Y, \mathcal{N}^{\otimes m})$ such that Y_t is an affine open neighbourhood of y. Then $r = \text{pr}_1^*s \otimes \text{pr}_2^*t$ is a section of $\mathcal{M}$ with $(X \times Y)_r = X_s \times Y_t$. This is an affine open neighbourhood of z and the proof is complete. $\square$

28.27. Affine and quasi-affine schemes

01QD

01QE Lemma 28.27.1. Let X be a scheme. Then X is quasi-affine if and only if $\mathcal{O}_X$ is ample.

Proof. Suppose that X is quasi-affine. Set $A = \Gamma(X, \mathcal{O}_X)$. Consider the open immersion

$$j : X \longrightarrow \text{Spec}(A)$$

from Lemma 28.18.4. Note that $\text{Spec}(A) = \text{Proj}(A[T])$, see Constructions, Example 27.8.14. Hence we can apply Lemma 28.26.12 to deduce that $\mathcal{O}_X$ is ample.

Suppose that $\mathcal{O}_X$ is ample. Note that $\Gamma_*(X, \mathcal{O}_X) \cong A[T]$ as graded rings. Hence the result follows from Lemmas 28.26.11 and 28.18.4 taking into account that $\text{Spec}(A) = \text{Proj}(A[T])$ for any ring A as seen above. $\square$

0BCK Lemma 28.27.2. Let X be a quasi-affine scheme. For any quasi-compact immersion $i : X' \to X$ the scheme X' is quasi-affine.

Proof. This can be proved directly without making use of the material on ample invertible sheaves; we urge the reader to do this on a napkin. Since X is quasi-affine, we have that $\mathcal{O}_X$ is ample by Lemma 28.27.1. Then $\mathcal{O}_{X'}$ is ample by Lemma 28.26.14. Then X' is quasi-affine by Lemma 28.27.1. $\square$

01QF Lemma 28.27.3. Let X be a scheme. Suppose that there exist finitely many elements $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that

(1) each X_{f_i} is an affine open of X, and
(2) the ideal generated by $f_1, \ldots, f_n$ in $\Gamma(X, \mathcal{O}_X)$ is equal to the unit ideal.

Then X is affine.

Proof. Assume we have $f_1, \ldots, f_n$ as in the lemma. We may write $1 = \sum g_i f_i$ for some $g_j \in \Gamma(X, \mathcal{O}_X)$ and hence it is clear that $X = \bigcup X_{f_i}$. (The f_i's cannot all vanish at a point.) Since each X_{f_i} is quasi-compact (being affine) it follows that X is quasi-compact. Hence we see that X is quasi-affine by Lemma 28.27.1 above. Consider the open immersion

$$j : X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)),$$

see Lemma 28.18.4. The inverse image of the standard open $D(f_i)$ on the right hand side is equal to X_{f_i} on the left hand side and the morphism j induces an isomorphism $X_{f_i} \cong D(f_i)$, see Lemma 28.18.3. Since the f_i generate the unit ideal we see that $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) = \bigcup_{i=1,\ldots,n} D(f_i)$. Thus j is an isomorphism. $\square$

28.28. Quasi-coherent sheaves and ample invertible sheaves

01QG Theme of this section: in the presence of an ample invertible sheaf every quasi-coherent sheaf comes from a graded module.

01QH Situation 28.28.1. Let X be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. Set $Y = \mathrm{Proj}(S)$. Let $f : X \to Y$ be the canonical morphism of Lemma 28.26.9. It comes equipped with a $\mathbf{Z}$-graded $\mathcal{O}_X$-algebra map $\bigoplus f^*\mathcal{O}_Y(n) \to \bigoplus \mathcal{L}^{\otimes n}$.

The following lemma is really a special case of the next lemma but it seems like a good idea to point out its validity first.

01QI Lemma 28.28.2. In Situation 28.28.1. The canonical morphism $f : X \to Y$ maps X into the open subscheme $W = W_1 \subset Y$ where $\mathcal{O}_Y(1)$ is invertible and where all multiplication maps $\mathcal{O}_Y(n) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \to \mathcal{O}_Y(n+m)$ are isomorphisms (see Constructions, Lemma 27.10.4). Moreover, the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are all isomorphisms.

Proof. By Proposition 28.26.13 there exists an integer n_0 such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. Let $x \in X$ be a point. By the above we can find $a \in S_{n_0}$ and $b \in S_{n_0+1}$ such that a and b do not vanish at x. Hence $f(x) \in D_+(a) \cap D_+(b) = D_+(ab)$. By Constructions, Lemma 27.10.4 we see that $f(x) \in W_1$ as desired. By Constructions, Lemma 27.14.1 which was used in the construction of the map f the maps $f^*\mathcal{O}_Y(n_0) \to \mathcal{L}^{\otimes n_0}$ and $f^*\mathcal{O}_Y(n_0+1) \to \mathcal{L}^{\otimes n_0+1}$ are isomorphisms in a neighbourhood of x. By compatibility with the algebra structure and the fact that f maps into W we conclude all the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are isomorphisms in a neighbourhood of x. Hence we win. $\square$

Recall from Modules, Definition 17.25.7 that given a locally ringed space X, an invertible sheaf $\mathcal{L}$, and a $\mathcal{O}_X$-module $\mathcal{F}$ we have the graded $\Gamma_*(X, \mathcal{L})$-module

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

The following lemma says that, in Situation 28.28.1, we can recover a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ from this graded module. Take a look also at Constructions, Lemma 27.13.8 where we prove this lemma in the special case $X = \mathbf{P}^n_R$.

01QJ Lemma 28.28.3. In Situation 28.28.1. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Set $M = \Gamma_*(X, \mathcal{L}, \mathcal{F})$ as a graded S-module. There are isomorphisms

$$f^*\widetilde{M} \longrightarrow \mathcal{F}$$

functorial in $\mathcal{F}$ such that $M_0 \to \Gamma(\text{Proj}(S), \widetilde{M}) \to \Gamma(X, \mathcal{F})$ is the identity map.

Proof. Let $s \in S_+$ be homogeneous such that X_s is affine open in X. Recall that $\widetilde{M}|_{D_+(s)}$ corresponds to the $S_{(s)}$-module $M_{(s)}$, see Constructions, Lemma 27.8.4. Recall that $f^{-1}(D_+(s)) = X_s$. As X carries an ample invertible sheaf it is quasi-compact and quasi-separated, see Section 28.26. By Lemma 28.17.2 there is a canonical isomorphism $M_{(s)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \to \Gamma(X_s, \mathcal{F})$. Since $\mathcal{F}$ is quasi-coherent this leads to a canonical isomorphism

$$f^*\widetilde{M}|_{X_s} \to \mathcal{F}|_{X_s}$$

Since $\mathcal{L}$ is ample on X we know that X is covered by the affine opens of the form X_s. Hence it suffices to prove that the displayed maps glue on overlaps. Proof of this is omitted. $\square$

01QK Remark 28.28.4. With assumptions and notation of Lemma 28.28.3. Denote the displayed map of the lemma by $\theta_{\mathcal{F}}$. Note that the isomorphism $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ of Lemma 28.28.2 is just $\theta_{\mathcal{L}^{\otimes n}}$. Consider the multiplication maps

$$\widetilde{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n) \longrightarrow \widetilde{M(n)}$$

see Constructions, Equation (27.10.1.5). Pull this back to X and consider

$$\begin{tikzcd} f^*\widetilde{M} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(n) \arrow[r] \arrow[d, "\theta_{\mathcal{F}} \otimes \theta_{\mathcal{L}^{\otimes n}}"'] & f^*\widetilde{M(n)} \arrow[d, "\theta_{\mathcal{F} \otimes \mathcal{L}^{\otimes n}}"] \\ \mathcal{F} \otimes \mathcal{L}^{\otimes n} \arrow[r, "\text{id}"] & \mathcal{F} \otimes \mathcal{L}^{\otimes n} \end{tikzcd}$$

Here we have used the obvious identification $M(n) = \Gamma_*(X, \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This diagram commutes. Proof omitted.

It should be possible to deduce the following lemma from Lemma 28.28.3 (or conversely) but it seems simpler to just repeat the proof.

0AG5 Lemma 28.28.5. Let S be a graded ring such that $X = \text{Proj}(S)$ is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ as a graded S-module, see Constructions, Section 27.10. The map

$$\widetilde{M} \longrightarrow \mathcal{F}$$

of Constructions, Lemma 27.10.7 is an isomorphism. If X is covered by standard opens $D_+(f)$ where f has degree 1, then the induced maps $M_n \to \Gamma(X, \mathcal{F}(n))$ are the identity maps.

Proof. Since X is quasi-compact we can find homogeneous elements $f_1, \ldots, f_n \in S$ of positive degrees such that $X = D_+(f_1) \cup \ldots \cup D_+(f_n)$. Let d be the least common multiple of the degrees of $f_1, \ldots, f_n$. After replacing f_i by a power we may assume that each f_i has degree d. Then we see that $\mathcal{L} = \mathcal{O}_X(d)$ is invertible, the multiplication maps $\mathcal{O}_X(ad) \otimes \mathcal{O}_X(bd) \to \mathcal{O}_X((a+b)d)$ are isomorphisms, and each f_i determines a global section s_i of $\mathcal{L}$ such that $X_{s_i} = D_+(f_i)$, see Constructions, Lemmas 27.10.4 and 27.10.5. Thus $\Gamma(X, \mathcal{F}(ad)) = \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$. Recall that $\widetilde{M}|_{D_+(f_i)}$ corresponds to the $S_{(f_i)}$-module $M_{(f_i)}$, see Constructions, Lemma 27.8.4. Since the degree of f_i is d, the isomorphism class of $M_{(f_i)}$ depends only on the homogeneous summands of M of degree divisible by d. More precisely, the isomorphism class of $M_{(f_i)}$ depends only on the graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{L}, \mathcal{F})$ and the image s_i of f_i in $\Gamma_*(X, \mathcal{L})$. The scheme X is quasi-compact by assumption and separated by Constructions, Lemma 27.8.8. By Lemma 28.17.2 there is a canonical isomorphism

$$M_{(f_i)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s_i)} \to \Gamma(X_{s_i}, \mathcal{F}).$$

The construction of the map in Constructions, Lemma 27.10.7 then shows that it is an isomorphism over $D_+(f_i)$ hence an isomorphism as X is covered by these opens. We omit the proof of the final statement. $\square$

28.29. Finding suitable affine opens

01ZU In this section we collect some results on the existence of affine opens in more and less general situations.

01ZV Lemma 28.29.1. Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X, see Topology, Section 5.8. Let $\eta_i \in Z_i$ be their generic points, see Schemes, Lemma 26.11.1. There exist affine open neighbourhoods $\eta_i \in U_i$ such that $U_i \cap U_j = \emptyset$ for all $i \neq j$. In particular, $U = U_1 \cup \ldots \cup U_n$ is an affine open containing all of the points $\eta_1, \ldots, \eta_n$.

Proof. Let V_i be any affine open containing η_i and disjoint from the closed set $Z_1 \cup \ldots \hat{Z}_i \ldots \cup Z_n$. Since X is quasi-separated for each i the union $W_i = \bigcup_{j, j \neq i} V_i \cap V_j$ is a quasi-compact open of V_i not containing η_i. We can find open neighbourhoods $U_i \subset V_i$ containing η_i and disjoint from W_i by Algebra, Lemma 10.26.4. Finally, U is affine since it is the spectrum of the ring $R_1 \times \ldots \times R_n$ where $R_i = \mathcal{O}_X(U_i)$, see Schemes, Lemma 26.6.8. $\square$

01ZW Remark 28.29.2. Lemma 28.29.1 above is false if X is not quasi-separated. Here is an example. Take $R = \mathbf{Q}[x, y_1, y_2, \ldots]/((x - i)y_i)$. Consider the minimal prime ideal $\mathfrak{p} = (y_1, y_2, \ldots)$ of R. Glue two copies of $\text{Spec}(R)$ along the (not quasi-compact) open $\text{Spec}(R) \setminus V(\mathfrak{p})$ to get a scheme X (glueing as in Schemes, Example 26.14.3). Then the two maximal points of X corresponding to $\mathfrak{p}$ are not contained in a common affine open. The reason is that any open of $\text{Spec}(R)$ containing $\mathfrak{p}$ contains infinitely many of the "lines" $x = i$, $y_j = 0$, $j \neq i$ with parameter y_i. Details omitted.

Notwithstanding the example above, for "most" finite sets of irreducible closed subsets one can apply Lemma 28.29.1 above, at least if X is quasi-compact. This is true because X contains a dense open which is separated.

03J1 Lemma 28.29.3. Let X be a quasi-compact scheme. There exists a dense open $V \subset X$ which is separated.

Proof. Say $X = \bigcup_{i=1,\ldots,n} U_i$ is a union of n affine open subschemes. We will prove the lemma by induction on n. It is trivial for $n = 1$. Let $V' \subset \bigcup_{i=1,\ldots,n-1} U_i$ be a separated dense open subscheme, which exists by induction hypothesis. Consider

$$V = V' \amalg (U_n \setminus \overline{V'}).$$

It is clear that V is separated and a dense open subscheme of X. □

It turns out that, even if X is quasi-separated as well as quasi-compact, there does not exist a separated, quasi-compact dense open, see Examples, Lemma 110.26.2. Here is a slight refinement of Lemma 28.29.1 above.

01ZX Lemma 28.29.4. Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X. Let $\eta_i \in Z_i$ be their generic points. Let $x \in X$ be arbitrary. There exists an affine open $U \subset X$ containing x and all the η_i.

Proof. Suppose that $x \in Z_1 \cap \ldots \cap Z_r$ and $x \notin Z_{r+1}, \ldots, Z_n$. Then we may choose an affine open $W \subset X$ such that $x \in W$ and $W \cap Z_i = \emptyset$ for $i = r+1, \ldots, n$. Note that clearly $\eta_i \in W$ for $i = 1, \ldots, r$. By Lemma 28.29.1 we may choose affine opens $U_i \subset X$ which are pairwise disjoint such that $\eta_i \in U_i$ for $i = r+1, \ldots, n$. Since X is quasi-separated the opens $W \cap U_i$ are quasi-compact and do not contain η_i for $i = r+1, \ldots, n$. Hence by Algebra, Lemma 10.26.4 we may shrink U_i such that $W \cap U_i = \emptyset$ for $i = r+1, \ldots, n$. Then the union $U = W \cup \bigcup_{i=r+1,\ldots,n} U_i$ is disjoint and hence (by Schemes, Lemma 26.6.8) a suitable affine open. □

01ZY Lemma 28.29.5. Let X be a scheme. Assume either

(1) The scheme X is quasi-affine.
(2) The scheme X is isomorphic to a locally closed subscheme of an affine scheme.
(3) There exists an ample invertible sheaf on X.
(4) The scheme X is isomorphic to a locally closed subscheme of $\text{Proj}(S)$ for some graded ring S.

Then for any finite subset $E \subset X$ there exists an affine open $U \subset X$ with $E \subset U$.

Proof. By Properties, Definition 28.18.1 a quasi-affine scheme is a quasi-compact open subscheme of an affine scheme. Any affine scheme $\text{Spec}(R)$ is isomorphic to $\text{Proj}(R[X])$ where $R[X]$ is graded by setting $\deg(X) = 1$. By Proposition 28.26.13 if X has an ample invertible sheaf then X is isomorphic to an open subscheme of $\text{Proj}(S)$ for some graded ring S. Hence, it suffices to prove the lemma in case (4). (We urge the reader to prove case (2) directly for themselves.)

Thus assume $X \subset \text{Proj}(S)$ is a locally closed subscheme where S is some graded ring. Let $T = \overline{X} \setminus X$. Recall that the standard opens $D_+(f)$ form a basis of the topology on $\text{Proj}(S)$. Since E is finite we may choose finitely many homogeneous elements $f_i \in S_+$ such that

$$E \subset D_+(f_1) \cup \ldots \cup D_+(f_n) \subset \text{Proj}(S) \setminus T$$

Suppose that $E = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ as a subset of $\text{Proj}(S)$. Consider the ideal $I = (f_1, \ldots, f_n) \subset S$. Since $I \not\subset \mathfrak{p}_j$ for all $j = 1, \ldots, m$ we see from Algebra, Lemma 10.57.6 that there exists a homogeneous element $f \in I$, $f \notin \mathfrak{p}_j$ for all $j = 1, \ldots, m$. Then $E \subset D_+(f) \subset D_+(f_1) \cup \ldots \cup D_+(f_n)$. Since $D_+(f)$ does not meet T we see that $X \cap D_+(f)$ is a closed subscheme of the affine scheme $D_+(f)$, hence is an affine open of X as desired. □

09NV Lemma 28.29.6. Let X be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let

$$E \subset W \subset X$$

with E finite and W open in X. Then there exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and $E \subset X_s \subset W$.

Proof. The reader can modify the proof of Lemma 28.29.5 to prove this lemma; we will instead deduce the lemma from it. By Lemma 28.29.5 we can choose an affine open $U \subset W$ such that $E \subset U$. Consider the graded ring $S = \Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. For each $x \in E$ let $\mathfrak{p}_x \subset S$ be the graded ideal of sections vanishing at x. It is clear that $\mathfrak{p}_x$ is a prime ideal and since some power of $\mathcal{L}$ is globally generated, it is clear that $S_+ \not\subset \mathfrak{p}_x$. Let $I \subset S$ be the graded ideal of sections vanishing on all points of $X \setminus U$. Since the sets X_s form a basis for the topology we see that $I \not\subset \mathfrak{p}_x$ for all $x \in E$. By (graded) prime avoidance (Algebra, Lemma 10.57.6) we can find $s \in I$ homogeneous with $s \notin \mathfrak{p}_x$ for all $x \in E$. Then $E \subset X_s \subset U$ and X_s is affine by Lemma 28.26.4. $\square$

0F20 Lemma 28.29.7. Let X be a quasi-affine scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $E \subset W \subset X$ with E finite and W open. Then there exists an $s \in \Gamma(X, \mathcal{L})$ such that X_s is affine and $E \subset X_s \subset W$.

Proof. The proof of this lemma has a lot in common with the proof of Algebra, Lemma 10.15.2. Say $E = \{x_1, \ldots, x_n\}$. If $E = W = \emptyset$, then $s = 0$ works. If $W \neq \emptyset$, then we may assume $E \neq \emptyset$ by adding a point if necessary. Thus we may assume $n \geq 1$. We will prove the lemma by induction on n.

Base case: $n = 1$. After replacing W by an affine open neighbourhood of x_1 in W, we may assume W is affine. Combining Lemmas 28.27.1 and Proposition 28.26.13 we see that every quasi-coherent $\mathcal{O}_X$-module is globally generated. Hence there exists a global section s of $\mathcal{L}$ which does not vanish at x_1. On the other hand, let $Z \subset X$ be the reduced induced closed subscheme on $X \setminus W$. Applying global generation to the quasi-coherent ideal sheaf $\mathcal{I}$ of Z we find a global section f of $\mathcal{I}$ which does not vanish at x_1. Then $s' = fs$ is a global section of $\mathcal{L}$ which does not vanish at x_1 such that $X_{s'} \subset W$. Then $X_{s'}$ is affine by Lemma 28.26.4.

Induction step for $n > 1$. If there is a specialization $x_i \rightsquigarrow x_j$ for $i \neq j$, then it suffices to prove the lemma for $\{x_1, \ldots, x_n\} \setminus \{x_i\}$ and we are done by induction. Thus we may assume there are no specializations among the x_i. By either Lemma 28.29.5 or Lemma 28.29.6 we may assume W is affine. By induction we can find a global section s of $\mathcal{L}$ such that $X_s \subset W$ is affine and contains $x_1, \ldots, x_{n-1}$. If $x_n \in X_s$ then we are done. Assume s is zero at x_n. By the case $n = 1$ we can find a global section s' of $\mathcal{L}$ with $\{x_n\} \subset X_{s'} \subset W \setminus \overline{\{x_1, \ldots, x_{n-1}\}}$. Here we use that x_n is not a specialization of $x_1, \ldots, x_{n-1}$. Then $s + s'$ is a global section of $\mathcal{L}$ which is nonvanishing at $x_1, \ldots, x_n$ with $X_{s+s'} \subset W$ and we conclude as before. $\square$

0BX3 Lemma 28.29.8. Let X be a scheme and $x \in X$ a point. There exists an affine open neighbourhood $U \subset X$ of x such that the canonical map $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is injective in each of the following cases:

(1) X is integral,
(2) X is locally Noetherian,
(3) X is reduced and has a finite number of irreducible components.

Proof. After translation into algebra, this follows from Algebra, Lemma 10.31.9. □

28.30. Other chapters

CHAPTER 29

Morphisms of Schemes

01QL 29.1. Introduction

01QM In this chapter we introduce some types of morphisms of schemes. A basic reference is [DG67].

29.2. Closed immersions

01QN In this section we elucidate some of the results obtained previously on closed immersions of schemes. Recall that a morphism of schemes $i : Z \to X$ is defined to be a closed immersion if (a) i induces a homeomorphism onto a closed subset of X, (b) $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and (c) the kernel of $i^\sharp$ is locally generated by sections, see Schemes, Definitions 26.10.2 and 26.4.1. It turns out that, given that Z and X are schemes, there are many different ways of characterizing a closed immersion.

01QO **Lemma 29.2.1.** *Let $i : Z \to X$ be a morphism of schemes. The following are equivalent:*

1. *The morphism i is a closed immersion.*
2. *For every affine open $\mathrm{Spec}(R) = U \subset X$, there exists an ideal $I \subset R$ such that $i^{-1}(U) = \mathrm{Spec}(R/I)$ as schemes over $U = \mathrm{Spec}(R)$.*
3. *There exists an affine open covering $X = \bigcup_{j \in J} U_j$, $U_j = \mathrm{Spec}(R_j)$ and for every $j \in J$ there exists an ideal $I_j \subset R_j$ such that $i^{-1}(U_j) = \mathrm{Spec}(R_j/I_j)$ as schemes over $U_j = \mathrm{Spec}(R_j)$.*
4. *The morphism i induces a homeomorphism of Z with a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.*
5. *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\mathrm{Ker}(i^\sharp) \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals.*
6. *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $\mathrm{Ker}(i^\sharp) \subset \mathcal{O}_X$ is a sheaf of ideals which is locally generated by sections.*

Proof. Condition (6) is our definition of a closed immersion, see Schemes, Definitions 26.4.1 and 26.10.2. So (6) $\Leftrightarrow$ (1). We have (1) $\Rightarrow$ (2) by Schemes, Lemma 26.10.1. Trivially (2) $\Rightarrow$ (3).

Assume (3). Each of the morphisms $\mathrm{Spec}(R_j/I_j) \to \mathrm{Spec}(R_j)$ is a closed immersion, see Schemes, Example 26.8.1. Hence $i^{-1}(U_j) \to U_j$ is a homeomorphism onto its image and $i^\sharp|_{U_j}$ is surjective. Hence i is a homeomorphism onto its image and $i^\sharp$ is surjective since this may be checked locally. We conclude that (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (1) is Schemes, Lemma 26.24.2. The implication (5) $\Rightarrow$ (6) is trivial. And the implication (6) $\Rightarrow$ (5) follows from Schemes, Lemma 26.10.1. $\square$

01QP Lemma 29.2.2. Let X be a scheme. Let $i : Z \to X$ and $i' : Z' \to X$ be closed immersions and consider the ideal sheaves $\mathcal{I} = \text{Ker}(i^\sharp)$ and $\mathcal{I}' = \text{Ker}((i')^\sharp)$ of $\mathcal{O}_X$.

1. (1) The morphism $i : Z \to X$ factors as $Z \to Z' \to X$ for some $a : Z \to Z'$ if and only if $\mathcal{I}' \subset \mathcal{I}$. If this happens, then a is a closed immersion.
2. (2) We have $Z \cong Z'$ over X if and only if $\mathcal{I} = \mathcal{I}'$.

Proof. This follows from our discussion of closed subspaces in Schemes, Section 26.4 especially Schemes, Lemmas 26.4.5 and 26.4.6. It also follows in a straightforward way from characterization (3) in Lemma 29.2.1 above. $\square$

01QQ Lemma 29.2.3. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. The following are equivalent:

1. (1) $\mathcal{I}$ is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules,
2. (2) $\mathcal{I}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules, and
3. (3) there exists a closed immersion $i : Z \to X$ of schemes whose corresponding sheaf of ideals $\text{Ker}(i^\sharp)$ is equal to $\mathcal{I}$.

Proof. The equivalence of (1) and (2) is immediate from Schemes, Lemma 26.10.1. If (1) holds, then there is a closed subspace $i : Z \to X$ with $\mathcal{I} = \text{Ker}(i^\sharp)$ by Schemes, Definition 26.4.4 and Example 26.4.3. By Schemes, Lemma 26.10.1 this is a closed immersion of schemes and (3) holds. Conversely, if (3) holds, then (2) holds by Schemes, Lemma 26.10.1 (which applies because a closed immersion of schemes is a fortiori a closed immersion of locally ringed spaces). $\square$

01QR Lemma 29.2.4. The base change of a closed immersion is a closed immersion.

Proof. See Schemes, Lemma 26.18.2. $\square$

01QS Lemma 29.2.5. A composition of closed immersions is a closed immersion.

Proof. We have seen this in Schemes, Lemma 26.24.3, but here is another proof. Namely, it follows from the characterization (3) of closed immersions in Lemma 29.2.1. Since if $I \subset R$ is an ideal, and $\overline{J} \subset R/I$ is an ideal, then $\overline{J} = J/I$ for some ideal $J \subset R$ which contains I and $(R/I)/\overline{J} = R/J$. $\square$

01QT Lemma 29.2.6. A closed immersion is quasi-compact.

Proof. This lemma is a duplicate of Schemes, Lemma 26.19.5. $\square$

01QU Lemma 29.2.7. A closed immersion is separated.

Proof. This lemma is a special case of Schemes, Lemma 26.23.8. $\square$

29.3. Immersions

07RJ In this section we collect some facts on immersions.

07RK Lemma 29.3.1. Let $Z \to Y \to X$ be morphisms of schemes.

1. (1) If $Z \to X$ is an immersion, then $Z \to Y$ is an immersion.
2. (2) If $Z \to X$ is a quasi-compact immersion and $Y \to X$ is quasi-separated, then $Z \to Y$ is a quasi-compact immersion.

(3) If $Z \to X$ is a closed immersion and $Y \to X$ is separated, then $Z \to Y$ is a closed immersion.

Proof. In each case the proof is to contemplate the commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ & Y \arrow[r] & X \end{tikzcd}$$

where the composition of the top horizontal arrows is the identity. Let us prove (1). The first horizontal arrow is a section of $Y \times_X Z \to Z$, whence an immersion by Schemes, Lemma 26.21.11. The arrow $Y \times_X Z \to Y$ is a base change of $Z \to X$ hence an immersion (Schemes, Lemma 26.18.2). Finally, a composition of immersions is an immersion (Schemes, Lemma 26.24.3). This proves (1). The other two results are proved in exactly the same manner. $\square$

01QV Lemma 29.3.2. Let $h : Z \to X$ be an immersion. If h is quasi-compact, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.

Proof. Note that h is quasi-compact and quasi-separated (see Schemes, Lemma 26.23.8). Hence $h_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules by Schemes, Lemma 26.24.1. This implies that $\mathcal{I} = \text{Ker}(\mathcal{O}_X \to h_*\mathcal{O}_Z)$ is a quasi-coherent sheaf of ideals, see Schemes, Section 26.24. Let $\overline{Z} \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Lemma 29.2.3. By Schemes, Lemma 26.4.6 the morphism h factors as $h = i \circ j$ where $i : \overline{Z} \to X$ is the inclusion morphism. To see that j is an open immersion, choose an open subscheme $U \subset X$ such that h induces a closed immersion of Z into U. Then it is clear that $\mathcal{I}|_U$ is the sheaf of ideals corresponding to the closed immersion $Z \to U$. Hence we see that $Z = \overline{Z} \cap U$. $\square$

03DQ Lemma 29.3.3. Let $h : Z \to X$ be an immersion. If Z is reduced, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.

Proof. Let $\overline{Z} \subset X$ be the closure of $h(Z)$ with the reduced induced closed subscheme structure, see Schemes, Definition 26.12.5. By Schemes, Lemma 26.12.7 the morphism h factors as $h = i \circ j$ with $i : \overline{Z} \to X$ the inclusion morphism and $j : Z \to \overline{Z}$. From the definition of an immersion we see there exists an open subscheme $U \subset X$ such that h factors through a closed immersion into U. Hence $\overline{Z} \cap U$ and $h(Z)$ are reduced closed subschemes of U with the same underlying closed set. Hence by the uniqueness in Schemes, Lemma 26.12.4 we see that $h(Z) \cong \overline{Z} \cap U$. So j induces an isomorphism of Z with $\overline{Z} \cap U$. In other words j is an open immersion. $\square$

01QW Example 29.3.4. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = \text{Spec}(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\widetilde{I_n}$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\widetilde{I_n}|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if

$n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

0FCZ Lemma 29.3.5. Let $f : Y \to X$ be a morphism of schemes. If for all $y \in Y$ there is an open subscheme $f(y) \in U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is an immersion, then f is an immersion.

Proof. This statement follows readily from the discussion of closed subschemes at the end of Schemes, Section 26.10 but we will also give a detailed proof. Let $Z \subset X$ be the closure of $f(Y)$. Since taking closures commutes with restricting to opens, we see from the assumption that $f(Y) \subset Z$ is open. Hence $Z' = Z \setminus f(Y)$ is closed. Hence $X' = X \setminus Z'$ is an open subscheme of X and f factors as $f : Y \to X'$ followed by the inclusion. If $y \in Y$ and $U \subset X$ is as in the statement of the lemma, then $U' = X' \cap U$ is an open neighbourhood of $f'(y)$ such that $(f')^{-1}(U') \to U'$ is an immersion (Lemma 29.3.1) with closed image. Hence it is a closed immersion, see Schemes, Lemma 26.10.4. Since being a closed immersion is local on the target (for example by Lemma 29.2.1) we conclude that f' is a closed immersion as desired. $\square$

29.4. Closed immersions and quasi-coherent sheaves

01QX The following lemma finally does for quasi-coherent sheaves on schemes what Modules, Lemma 17.6.1 does for abelian sheaves. See also the discussion in Modules, Section 17.13.

01QY Lemma 29.4.1. Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. A closed immersion is quasi-compact and separated, see Lemmas 29.2.6 and 29.2.7. Hence Schemes, Lemma 26.24.1 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X.

By Modules, Lemma 17.13.4 the functor i_* is fully faithful.

Now we turn to the description of the essential image of the functor i_*. We have $\mathcal{I}(i_*\mathcal{F}) = 0$ for any quasi-coherent $\mathcal{O}_Z$-module, for example by Modules, Lemma 17.13.4. Next, suppose that $\mathcal{G}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{G} = 0$. It suffices to show that the canonical map

$$\mathcal{G} \longrightarrow i_*i^*\mathcal{G}$$

is an isomorphism[1]. In the case of schemes and quasi-coherent modules, working affine locally on X and using Lemma 29.2.1 and Schemes, Lemma 26.7.3 it suffices

[1]This was proved in a more general situation in the proof of Modules, Lemma 17.13.4.

to prove the following algebraic statement: Given a ring R, an ideal I and an R-module N such that $IN = 0$ the canonical map

$$N \longrightarrow N \otimes_R R/I, \quad n \longmapsto n \otimes 1$$

is an isomorphism of R-modules. Proof of this easy algebra fact is omitted. $\square$

Let $i : Z \to X$ be a closed immersion. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

01QZ Lemma 29.4.2. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 26.24. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. $\square$

01R0 Lemma 29.4.3. Let $i : Z \to X$ be a closed immersion of schemes. There is a functor[2] $i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 17.6.3.)

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 29.4.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. $\square$

Using the 1-to-1 corresponding between quasi-coherent sheaves of ideals and closed subschemes (see Lemma 29.2.3) we can define scheme theoretic intersections and unions of closed subschemes.

0C4H Definition 29.4.4. Let X be a scheme. Let $Z, Y \subset X$ be closed subschemes corresponding to quasi-coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$. The scheme theoretic intersection of Z and Y is the closed subscheme of X cut out by $\mathcal{I} + \mathcal{J}$. The scheme theoretic union of Z and Y is the closed subscheme of X cut out by $\mathcal{I} \cap \mathcal{J}$.

[2]This is likely nonstandard notation.

0C4I Lemma 29.4.5. Let X be a scheme. Let $Z, Y \subset X$ be closed subschemes. Let $Z \cap Y$ be the scheme theoretic intersection of Z and Y. Then $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions and

$$\begin{tikzcd} Z \cap Y \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of schemes, i.e., $Z \cap Y = Z \times_X Y$.

Proof. The morphisms $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions by Lemma 29.2.2. Let $U = \mathrm{Spec}(A)$ be an affine open of X and let $Z \cap U$ and $Y \cap U$ correspond to the ideals $I \subset A$ and $J \subset A$. Then $Z \cap Y \cap U$ corresponds to $I + J \subset A$. Since $A/I \otimes_A A/J = A/(I + J)$ we see that the diagram is cartesian by our description of fibre products of schemes in Schemes, Section 26.17. □

0C4J Lemma 29.4.6. Let S be a scheme. Let $X, Y \subset S$ be closed subschemes. Let $X \cup Y$ be the scheme theoretic union of X and Y. Let $X \cap Y$ be the scheme theoretic intersection of X and Y. Then $X \to X \cup Y$ and $Y \to X \cup Y$ are closed immersions, there is a short exact sequence

$$0 \to \mathcal{O}_{X \cup Y} \to \mathcal{O}_X \times \mathcal{O}_Y \to \mathcal{O}_{X \cap Y} \to 0$$

of $\mathcal{O}_S$-modules, and the diagram

$$\begin{tikzcd} X \cap Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \cup Y \end{tikzcd}$$

is cocartesian in the category of schemes, i.e., $X \cup Y = X \amalg_{X \cap Y} Y$.

Proof. The morphisms $X \to X \cup Y$ and $Y \to X \cup Y$ are closed immersions by Lemma 29.2.2. In the short exact sequence we use the equivalence of Lemma 29.4.1 to think of quasi-coherent modules on closed subschemes of S as quasi-coherent modules on S. For the first map in the sequence we use the canonical maps $\mathcal{O}_{X \cup Y} \to \mathcal{O}_X$ and $\mathcal{O}_{X \cup Y} \to \mathcal{O}_Y$ and for the second map we use the canonical map $\mathcal{O}_X \to \mathcal{O}_{X \cap Y}$ and the negative of the canonical map $\mathcal{O}_Y \to \mathcal{O}_{X \cap Y}$. Then to check exactness we may work affine locally. Let $U = \mathrm{Spec}(A)$ be an affine open of S and let $X \cap U$ and $Y \cap U$ correspond to the ideals $I \subset A$ and $J \subset A$. Then $(X \cup Y) \cap U$ corresponds to $I \cap J \subset A$ and $X \cap Y \cap U$ corresponds to $I + J \subset A$. Thus exactness follows from the exactness of

$$0 \to A/I \cap J \to A/I \times A/J \to A/(I + J) \to 0$$

To show the diagram is cocartesian, suppose we are given a scheme T and morphisms of schemes $f : X \to T$, $g : Y \to T$ agreeing as morphisms $X \cap Y \to T$. Goal: Show there exists a unique morphism $h : X \cup Y \to T$ agreeing with f and g. To construct h we may work affine locally on $X \cup Y$, see Schemes, Section 26.14. If $s \in X$, $s \notin Y$, then $X \to X \cup Y$ is an isomorphism in a neighbourhood of s and it is clear how to construct h. Similarly for $s \in Y$, $s \notin X$. For $s \in X \cap Y$ we can pick an affine open $V = \mathrm{Spec}(B) \subset T$ containing $f(s) = g(s)$. Then we can choose an affine open $U = \mathrm{Spec}(A) \subset S$ containing s such that $f(X \cap U)$ and $g(Y \cap U)$ are contained in V. The morphisms $f|_{X \cap U}$ and $g|_{Y \cap V}$ into V correspond to ring maps

$$B \to A/I \quad \text{and} \quad B \to A/J$$

which agree as maps into $A/(I+J)$. By the short exact sequence displayed above there is a unique lift of these ring homomorphism to a ring map $B \to A/I \cap J$ as desired. □

29.5. Supports of modules

056H In this section we collect some elementary results on supports of quasi-coherent modules on schemes. Recall that the support of a sheaf of modules has been defined in Modules, Section 17.5. On the other hand, the support of a module was defined in Algebra, Section 10.62. These match.

056I Lemma 29.5.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent

(1) $\mathfrak{p}$ is in the support of M, and
(2) x is in the support of $\mathcal{F}$.

Proof. This follows from the equality $\mathcal{F}_x = M_\mathfrak{p}$, see Schemes, Lemma 26.5.4 and the definitions. □

05AC Lemma 29.5.2. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. The support of $\mathcal{F}$ is closed under specialization.

Proof. If $x' \rightsquigarrow x$ is a specialization and $\mathcal{F}_x = 0$ then $\mathcal{F}_{x'}$ is zero, as $\mathcal{F}_{x'}$ is a localization of the module $\mathcal{F}_x$. Hence the complement of $\mathrm{Supp}(\mathcal{F})$ is closed under generalization. □

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

056J Lemma 29.5.3. Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. Then

(1) The support of $\mathcal{F}$ is closed.
(2) For $x \in X$ we have
$$x \in \mathrm{Supp}(\mathcal{F}) \Leftrightarrow \mathcal{F}_x \neq 0 \Leftrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0.$$
(3) For any morphism of schemes $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $\mathrm{Supp}(f^*\mathcal{F}) = f^{-1}(\mathrm{Supp}(\mathcal{F}))$.

Proof. Part (1) is a reformulation of Modules, Lemma 17.9.6. You can also combine Lemma 29.5.1, Properties, Lemma 28.16.1, and Algebra, Lemma 10.40.5 to see this. The first equivalence in (2) is the definition of support, and the second equivalence follows from Nakayama's lemma, see Algebra, Lemma 10.20.1. Let $f : Y \to X$ be a morphism of schemes. Note that $f^*\mathcal{F}$ is of finite type by Modules, Lemma 17.9.2. For the final assertion, let $y \in Y$ with image $x \in X$. Recall that
$$(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y},$$
see Sheaves, Lemma 6.26.4. Hence $(f^*\mathcal{F})_y \otimes \kappa(y)$ is nonzero if and only if $\mathcal{F}_x \otimes \kappa(x)$ is nonzero. By (2) this implies $x \in \mathrm{Supp}(\mathcal{F})$ if and only if $y \in \mathrm{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). □

05JU Lemma 29.5.4. Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. There exists a smallest closed subscheme $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:

(1) If $\mathrm{Spec}(A) \subset X$ is any affine open, and $\mathcal{F}|_{\mathrm{Spec}(A)} = \widetilde{M}$ then $Z \cap \mathrm{Spec}(A) = \mathrm{Spec}(A/I)$ where $I = \mathrm{Ann}_A(M)$.
(2) The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.
(3) The quasi-coherent sheaf $\mathcal{G}$ is of finite type.
(4) The support of $\mathcal{G}$ and of $\mathcal{F}$ is Z.

Proof. Suppose that $i' : Z' \to X$ is a closed subscheme which satisfies the description on open affines from the lemma. Then by Lemma 29.4.1 we see that $\mathcal{F} \cong i'_*\mathcal{G}'$ for some unique quasi-coherent sheaf $\mathcal{G}'$ on Z'. Furthermore, it is clear that Z' is the smallest closed subscheme with this property (by the same lemma). Finally, using Properties, Lemma 28.16.1 and Algebra, Lemma 10.5.5 it follows that $\mathcal{G}'$ is of finite type. We have $\mathrm{Supp}(\mathcal{G}') = Z$ by Algebra, Lemma 10.40.5. Hence, in order to prove the lemma it suffices to show that the characterization in (1) actually does define a closed subscheme. And, in order to do this it suffices to prove that the given rule produces a quasi-coherent sheaf of ideals, see Lemma 29.2.3. This comes down to the following algebra fact: If A is a ring, $f \in A$, and M is a finite A-module, then $\mathrm{Ann}_A(M)_f = \mathrm{Ann}_{A_f}(M_f)$. We omit the proof. □

05JV Definition 29.5.5. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. The scheme theoretic support of $\mathcal{F}$ is the closed subscheme $Z \subset X$ constructed in Lemma 29.5.4.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 29.4.1).

29.6. Scheme theoretic image

01R5 Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

01R6 Lemma 29.6.1. Let $f : X \to Y$ be a morphism of schemes. There exists a closed subscheme $Z \subset Y$ such that f factors through Z and such that for any other closed subscheme $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.

Proof. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 29.2.3. This works by Schemes, Lemma 26.4.6. In general the same lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 29.4.2. □

01R7 Definition 29.6.2. Let $f : X \to Y$ be a morphism of schemes. The scheme theoretic image of f is the smallest closed subscheme $Z \subset Y$ through which f factors, see Lemma 29.6.1 above.

For a morphism $f : X \to Y$ of schemes with scheme theoretic image Z we often denote $f : X \to Z$ the factorization of f through its scheme theoretic image. If the morphism f is not quasi-compact, then (in general)

(1) the set theoretic inclusion $\overline{f(X)} \subset Z$ is not an equality, i.e., $f(X) \subset Z$ is not a dense subset, and
(2) the construction of the scheme theoretic image does not commute with restriction to open subschemes to Y.

In Examples, Section 110.23 the reader finds an example for both phenomena. These phenomena can arise even for immersions, see Examples, Section 110.25. However, the next lemma shows that both disasters are avoided when the morphism is quasi-compact.

01R8 Lemma 29.6.3. Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then

1. the sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,
2. the scheme theoretic image Z is the closed subscheme determined by $\mathcal{I}$,
3. for any open $U \subset Y$ the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is equal to $Z \cap U$, and
4. the image $f(X) \subset Z$ is a dense subset of Z, in other words the morphism $X \to Z$ is dominant (see Definition 29.8.1).

Proof. Part (4) follows from part (3). To show (3) it suffices to prove (1) since the formation of $\mathcal{I}$ commutes with restriction to open subschemes of Y. And if (1) holds then in the proof of Lemma 29.6.1 we showed (2). Thus it suffices to prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is local we may assume Y is affine. As f is quasi-compact, we can find a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. Denote f' the composition

$$X' = \coprod U_i \longrightarrow X \longrightarrow Y.$$

Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$, and hence $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f'_*\mathcal{O}_{X'})$. By Schemes, Lemma 26.24.1 the sheaf $f'_*\mathcal{O}_{X'}$ is quasi-coherent on Y. Hence we win. $\square$

056A Example 29.6.4. If $A \to B$ is a ring map with kernel I, then the scheme theoretic image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the closed subscheme $\mathrm{Spec}(A/I)$ of $\mathrm{Spec}(A)$. This follows from Lemma 29.6.3.

If the morphism is quasi-compact, then the scheme theoretic image only adds points which are specializations of points in the image.

02JQ Lemma 29.6.5. Let $f : X \to Y$ be a quasi-compact morphism. Let Z be the scheme theoretic image of f. Let $z \in Z$[3]. There exists a valuation ring A with fraction field K and a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[rr] \arrow[d] & & X \arrow[dl] \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & Z \arrow[r] & Y
\end{tikzcd}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z. In particular any point of Z is the specialization of a point of $f(X)$.

Proof. Let $z \in \mathrm{Spec}(R) = V \subset Y$ be an affine open neighbourhood of z. By Lemma 29.6.3 the intersection $Z \cap V$ is the scheme theoretic image of $f^{-1}(V) \to V$. Hence we may replace Y by V and assume $Y = \mathrm{Spec}(R)$ is affine. In this case X is quasi-compact as f is quasi-compact. Say $X = U_1 \cup \ldots \cup U_n$ is a finite affine open covering. Write $U_i = \mathrm{Spec}(A_i)$. Let $I = \mathrm{Ker}(R \to A_1 \times \ldots \times A_n)$. By Lemma 29.6.3 again we see that Z corresponds to the closed subscheme $\mathrm{Spec}(R/I)$ of Y.

[3]By Lemma 29.6.3 set-theoretically Z agrees with the closure of $f(X)$ in Y.

If $\mathfrak{p} \subset R$ is the prime corresponding to z, then we see that $I_\mathfrak{p} \subset R_\mathfrak{p}$ is not an equality. Hence (as localization is exact, see Algebra, Proposition 10.9.12) we see that $R_\mathfrak{p} \to (A_1)_\mathfrak{p} \times \ldots \times (A_n)_\mathfrak{p}$ is not zero. Hence one of the rings $(A_i)_\mathfrak{p}$ is not zero. Hence there exists an i and a prime $\mathfrak{q}_i \subset A_i$ lying over a prime $\mathfrak{p}_i \subset \mathfrak{p}$. By Algebra, Lemma 10.50.2 we can choose a valuation ring $A \subset K = \kappa(\mathfrak{q}_i)$ dominating the local ring $R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p} \subset \kappa(\mathfrak{q}_i)$. This gives the desired diagram. Some details omitted. $\square$

01R9 Lemma 29.6.6. Let

$$\begin{tikzcd} X_1 \arrow[r, "f_1"'] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

be a commutative diagram of schemes. Let $Z_i \subset Y_i$, $i = 1, 2$ be the scheme theoretic image of f_i. Then the morphism $Y_1 \to Y_2$ induces a morphism $Z_1 \to Z_2$ and a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r] & Z_2 \arrow[r] & Y_2 \end{tikzcd}$$

Proof. The scheme theoretic inverse image of Z_2 in Y_1 is a closed subscheme of Y_1 through which f_1 factors. Hence Z_1 is contained in this. This proves the lemma. $\square$

056B Lemma 29.6.7. Let $f : X \to Y$ be a morphism of schemes. If X is reduced, then the scheme theoretic image of f is the reduced induced scheme structure on $\overline{f(X)}$.

Proof. This is true because the reduced induced scheme structure on $\overline{f(X)}$ is clearly the smallest closed subscheme of Y through which f factors, see Schemes, Lemma 26.12.7. $\square$

0CNG Lemma 29.6.8. Let $f : X \to Y$ be a separated morphism of schemes. Let $V \subset Y$ be a retrocompact open. Let $s : V \to X$ be a morphism such that $f \circ s = \mathrm{id}_V$. Let Y' be the scheme theoretic image of s. Then $Y' \to Y$ is an isomorphism over V.

Proof. The assumption that V is retrocompact in Y (Topology, Definition 5.12.1) means that $V \to Y$ is a quasi-compact morphism. By Schemes, Lemma 26.21.14 the morphism $s : V \to X$ is quasi-compact. Hence the construction of the scheme theoretic image Y' of s commutes with restriction to opens by Lemma 29.6.3. In particular, we see that $Y' \cap f^{-1}(V)$ is the scheme theoretic image of a section of the separated morphism $f^{-1}(V) \to V$. Since a section of a separated morphism is a closed immersion (Schemes, Lemma 26.21.11), we conclude that $Y' \cap f^{-1}(V) \to V$ is an isomorphism as desired. $\square$

29.7. Scheme theoretic closure and density

01RA We take the following definition from [DG67, IV, Definition 11.10.2].

01RB Definition 29.7.1. Let X be a scheme. Let $U \subset X$ be an open subscheme.

(1) The scheme theoretic image of the morphism $U \to X$ is called the scheme theoretic closure of U in X.

(2) We say U is scheme theoretically dense in X if for every open $V \subset X$ the scheme theoretic closure of $U \cap V$ in V is equal to V.

With this definition it is not the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X, see Example 29.7.2. This is somewhat inelegant; but see Lemmas 29.7.3 and 29.7.8 below. On the other hand, with this definition U is scheme theoretically dense in X if and only if for every $V \subset X$ open the ring map $\mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is injective, see Lemma 29.7.5 below. In particular we see that scheme theoretically dense implies dense which is pleasing.

01RC Example 29.7.2. Here is an example where scheme theoretic closure being X does not imply dense for the underlying topological spaces. Let k be a field. Set $A = k[x, z_1, z_2, \ldots]/(x^n z_n)$ Set $I = (z_1, z_2, \ldots) \subset A$. Consider the affine scheme $X = \text{Spec}(A)$ and the open subscheme $U = X \setminus V(I)$. Since $A \to \prod_n A_{z_n}$ is injective we see that the scheme theoretic closure of U is X. Consider the morphism $X \to \text{Spec}(k[x])$. This morphism is surjective (set all $z_n = 0$ to see this). But the restriction of this morphism to U is not surjective because it maps to the point $x = 0$. Hence U cannot be topologically dense in X.

01RD Lemma 29.7.3. Let X be a scheme. Let $U \subset X$ be an open subscheme. If the inclusion morphism $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.

Proof. Follows from Lemma 29.6.3 part (3). $\square$

056C Example 29.7.4. Let A be a ring and $X = \text{Spec}(A)$. Let $f_1, \ldots, f_n \in A$ and let $U = D(f_1) \cup \ldots \cup D(f_n)$. Let $I = \text{Ker}(A \to \prod A_{f_i})$. Then the scheme theoretic closure of U in X is the closed subscheme $\text{Spec}(A/I)$ of X. Note that $U \to X$ is quasi-compact. Hence by Lemma 29.7.3 we see U is scheme theoretically dense in X if and only if $I = 0$.

01RE Lemma 29.7.5. Let $j : U \to X$ be an open immersion of schemes. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any open V of X. Hence the scheme theoretic closure of $U \cap V$ in V is equal to V, see proof of Lemma 29.6.1. Conversely, suppose that the scheme theoretic closure of $U \cap V$ is equal to V for all opens V. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine open, say $\text{Spec}(A) = V \subset X$ and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \cap U, \mathcal{O}_X)$. In this case the scheme theoretic closure of $V \cap U$ in V is clearly contained in $\text{Spec}(A/(f))$ a contradiction. $\square$

01RF Lemma 29.7.6. Let X be a scheme. If U, V are scheme theoretically dense open subschemes of X, then so is $U \cap V$.

Proof. Let $W \subset X$ be any open. Consider the map $\mathcal{O}_X(W) \to \mathcal{O}_X(W \cap V) \to \mathcal{O}_X(W \cap V \cap U)$. By Lemma 29.7.5 both maps are injective. Hence the composite is injective. Hence by Lemma 29.7.5 $U \cap V$ is scheme theoretically dense in X. $\square$

01RG Lemma 29.7.7. Let $h : Z \to X$ be an immersion. Assume either h is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subscheme of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.

Proof. By Lemma 29.3.2 or Lemma 29.3.3 we can factor $Z \to X$ as $Z \to \overline{Z}_1 \to X$ with $Z \to \overline{Z}_1$ open and $\overline{Z}_1 \to X$ closed. On the other hand, let $Z \to \overline{Z} \subset X$ be the scheme theoretic closure of $Z \to X$. We conclude that $\overline{Z} \subset \overline{Z}_1$. Since Z is an open subscheme of $\overline{Z}_1$ it follows that Z is an open subscheme of $\overline{Z}$ as well. In the case that Z is reduced we know that $Z \subset \overline{Z}_1$ is topologically dense by the construction of $\overline{Z}_1$ in the proof of Lemma 29.3.3. Hence $\overline{Z}_1$ and $\overline{Z}$ have the same underlying topological spaces. Thus $\overline{Z} \subset \overline{Z}_1$ is a closed immersion into a reduced scheme which induces a bijection on underlying topological spaces, and hence it is an isomorphism. In the case that $Z \to X$ is quasi-compact we argue as follows: The assertion that Z is scheme theoretically dense in $\overline{Z}$ follows from Lemma 29.6.3 part (3). The last assertion follows from Lemma 29.6.3 part (4). □

056D Lemma 29.7.8. Let X be a reduced scheme and let $U \subset X$ be an open subscheme. Then the following are equivalent

(1) U is topologically dense in X,
(2) the scheme theoretic closure of U in X is X, and
(3) U is scheme theoretically dense in X.

Proof. This follows from Lemma 29.7.7 and the fact that a closed subscheme Z of X whose underlying topological space equals X must be equal to X as a scheme. □

056E Lemma 29.7.9. Let X be a scheme and let $U \subset X$ be a reduced open subscheme. Then the following are equivalent

(1) the scheme theoretic closure of U in X is X, and
(2) U is scheme theoretically dense in X.

If this holds then X is a reduced scheme.

Proof. This follows from Lemma 29.7.7 and the fact that the scheme theoretic closure of U in X is reduced by Lemma 29.6.7. □

01RH Lemma 29.7.10. Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms of schemes over S. Let $U \subset X$ be an open subscheme such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to S$ is separated, then $f = g$.

Proof. Follows from the definitions and Schemes, Lemma 26.21.5. □

29.8. Dominant morphisms

01RI The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

01RJ Definition 29.8.1. A morphism $f : X \to S$ of schemes is called dominant if the image of f is a dense subset of S.

So for example, if k is an infinite field and $\lambda_1, \lambda_2, \ldots$ is a countable collection of distinct elements of k, then the morphism

$$\coprod\nolimits_{i=1,2,\ldots} \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k[x])$$

with ith factor mapping to the point $x = \lambda_i$ is dominant.

01RK Lemma 29.8.2. Let $f : X \to S$ be a morphism of schemes. If every generic point of every irreducible component of S is in the image of f, then f is dominant.

Proof. This is a topological fact which follows directly from the fact that the topological space underlying a scheme is sober, see Schemes, Lemma 26.11.1, and that every point of S is contained in an irreducible component of S, see Topology, Lemma 5.8.3. $\square$

The expectation that morphisms are dominant only if generic points of the target are in the image does hold if the morphism is quasi-compact.

01RL Lemma 29.8.3. Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is dominant if and only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f.

Proof. Let $V \subset S$ be an affine open. Because f is quasi-compact we may choose finitely many affine opens $U_i \subset f^{-1}(V)$, $i = 1, \ldots, n$ covering $f^{-1}(V)$. Consider the morphism of affines

$$f' : \coprod\nolimits_{i=1,\ldots,n} U_i \longrightarrow V.$$

A disjoint union of affines is affine, see Schemes, Lemma 26.6.8. Generic points of irreducible components of V are exactly the generic points of the irreducible components of S that meet V. Also, f is dominant if and only if f' is dominant no matter what choices of V, n, U_i we make above. Thus we have reduced the lemma to the case of a morphism of affine schemes. The affine case is Algebra, Lemma 10.30.6. $\square$

0H3F Lemma 29.8.4. Let $f : X \to S$ be a quasi-compact dominant morphism of schemes. Let $g : S' \to S$ be a morphism of schemes and denote $f' : X' \to S'$ the base change of f by g. If generalizations lift along g, then f' is dominant.

Proof. Observe that f' is quasi-compact by Schemes, Lemma 26.19.3. Let $\eta' \in S'$ be the generic point of an irreducible component of S'. If generalizations lift along g, then $\eta = g(\eta')$ is the generic point of an irreducible component of S. By Lemma 29.8.3 we see that η is in the image of f. Hence η' is in the image of f' by Schemes, Lemma 26.17.5. It follows that f' is dominant by Lemma 29.8.3. $\square$

02NE Lemma 29.8.5. Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S. If $\eta \notin f(X)$ then there exists an open neighbourhood $V \subset S$ of η such that $f^{-1}(V) = \emptyset$.

Proof. Let $Z \subset S$ be the scheme theoretic image of f. We have to show that $\eta \notin Z$. This follows from Lemma 29.6.5 but can also be seen as follows. By Lemma 29.6.3 the morphism $X \to Z$ is dominant, which by Lemma 29.8.3 means all the generic points of all irreducible components of Z are in the image of $X \to Z$. By assumption we see that $\eta \notin Z$ since η would be the generic point of some irreducible component of Z if it were in Z. $\square$

There is another case where dominant is the same as having all generic points of irreducible components in the image.

01RM Lemma 29.8.6. Let $f : X \to S$ be a morphism of schemes. Suppose that X has finitely many irreducible components. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f. If so, then S has finitely many irreducible components as well.

Proof. Assume f is dominant. Say $X = Z_1 \cup Z_2 \cup \ldots \cup Z_n$ is the decomposition of X into irreducible components. Let $\xi_i \in Z_i$ be its generic point, so $Z_i = \overline{\{\xi_i\}}$. Note that $f(Z_i)$ is an irreducible subset of S. Hence

$$S = \overline{f(X)} = \bigcup \overline{f(Z_i)} = \bigcup \overline{\{f(\xi_i)\}}$$

is a finite union of irreducible subsets whose generic points are in the image of f. The lemma follows. $\square$

0CC1 Lemma 29.8.7. Let $f : X \to Y$ be a morphism of integral schemes. The following are equivalent

(1) f is dominant,
(2) f maps the generic point of X to the generic point of Y,
(3) for some nonempty affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective,
(4) for all nonempty affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective,
(5) for some $x \in X$ with image $y = f(x) \in Y$ the local ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective, and
(6) for all $x \in X$ with image $y = f(x) \in Y$ the local ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective.

Proof. The equivalence of (1) and (2) follows from Lemma 29.8.6. Let $U \subset X$ and $V \subset Y$ be nonempty affine opens with $f(U) \subset V$. Recall that the rings $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(V)$ are integral domains. The morphism $f|_U : U \to V$ corresponds to a ring map $\varphi : B \to A$. The generic points of X and Y correspond to the prime ideals $(0) \subset A$ and $(0) \subset B$. Thus (2) is equivalent to the condition $(0) = \varphi^{-1}((0))$, i.e., to the condition that φ is injective. In this way we see that (2), (3), and (4) are equivalent. Similarly, given x and y as in (5) the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are domains and the prime ideals $(0) \subset \mathcal{O}_{X,x}$ and $(0) \subset \mathcal{O}_{Y,y}$ correspond to the generic points of X and Y (via the identification of the spectrum of the local ring at x with the set of points specializing to x, see Schemes, Lemma 26.13.2). Thus we can argue in the exact same manner as above to see that (2), (5), and (6) are equivalent. $\square$

29.9. Surjective morphisms

01RY

01RZ Definition 29.9.1. A morphism of schemes is said to be surjective if it is surjective on underlying topological spaces.

01S0 Lemma 29.9.2. The composition of surjective morphisms is surjective.

Proof. Omitted. $\square$

0495 Lemma 29.9.3. Let X and Y be schemes over a base scheme S. Given points $x \in X$ and $y \in Y$, there is a point of $X \times_S Y$ mapping to x and y under the projections if and only if x and y lie above the same point of S.

Proof. The condition is obviously necessary, and the converse follows from the proof of Schemes, Lemma 26.17.5. $\square$

01S1 Lemma 29.9.4. The base change of a surjective morphism is surjective.

Proof. Let $f : X \to Y$ be a morphism of schemes over a base scheme S. If $S' \to S$ is a morphism of schemes, let $p : X_{S'} \to X$ and $q : Y_{S'} \to Y$ be the canonical projections. The commutative square

$$\begin{tikzcd} X_{S'} \arrow[r, "p"] \arrow[d, "f_{S'}"] & X \arrow[d, "f"] \\ Y_{S'} \arrow[r, "q"] & Y. \end{tikzcd}$$

identifies $X_{S'}$ as a fibre product of $X \to Y$ and $Y_{S'} \to Y$. Let Z be a subset of the underlying topological space of X. Then $q^{-1}(f(Z)) = f_{S'}(p^{-1}(Z))$, because $y' \in q^{-1}(f(Z))$ if and only if $q(y') = f(x)$ for some $x \in Z$, if and only if, by Lemma 29.9.3, there exists $x' \in X_{S'}$ such that $f_{S'}(x') = y'$ and $p(x') = x$. In particular taking $Z = X$ we see that if f is surjective so is the base change $f_{S'} : X_{S'} \to Y_{S'}$. □

0496 Example 29.9.5. Bijectivity is not stable under base change, and so neither is injectivity. For example consider the bijection $\text{Spec}(\mathbf{C}) \to \text{Spec}(\mathbf{R})$. The base change $\text{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}) \to \text{Spec}(\mathbf{C})$ is not injective, since there is an isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \times \mathbf{C}$ (the decomposition comes from the idempotent $\frac{1\otimes 1 + i \otimes i}{2}$) and hence $\text{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C})$ has two points.

04ZD Lemma 29.9.6. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $W \subset Z$ be a quasi-compact open. By assumption $p^{-1}(W)$ is quasi-compact. Hence by Topology, Lemma 5.12.7 the inverse image $q^{-1}(W) = f(p^{-1}(W))$ is quasi-compact too. This proves the lemma. □

29.10. Radicial and universally injective morphisms

01S2 In this section we define what it means for a morphism of schemes to be radicial and what it means for a morphism of schemes to be universally injective. We then show that these notions agree. The reason for introducing both is that in the case of algebraic spaces there are corresponding notions which may not always agree.

01S3 Definition 29.10.1. Let $f : X \to S$ be a morphism.

(1) We say that f is universally injective if and only if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is injective (on underlying topological spaces).
(2) We say f is radicial if f is injective as a map of topological spaces, and for every $x \in X$ the field extension $\kappa(x)/\kappa(f(x))$ is purely inseparable.

01S4 Lemma 29.10.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) For every field K the induced map $\text{Mor}(\text{Spec}(K), X) \to \text{Mor}(\text{Spec}(K), S)$ is injective.
(2) The morphism f is universally injective.

(3) The morphism f is radicial.
(4) The diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$ is surjective.

Proof. Let K be a field, and let $s : \mathrm{Spec}(K) \to S$ be a morphism. Giving a morphism $x : \mathrm{Spec}(K) \to X$ such that $f \circ x = s$ is the same as giving a section of the projection $X_K = \mathrm{Spec}(K) \times_S X \to \mathrm{Spec}(K)$, which in turn is the same as giving a point $x \in X_K$ whose residue field is K. Hence we see that (2) implies (1).

Conversely, suppose that (1) holds. Assume that $x, x' \in X_{S'}$ map to the same point $s' \in S'$. Choose a commutative diagram

$$\begin{tikzcd} K & \kappa(x) \arrow[l] \\ \kappa(x') \arrow[u] & \kappa(s') \arrow[l] \arrow[u] \end{tikzcd}$$

of fields. By Schemes, Lemma 26.13.3 we get two morphisms $a, a' : \mathrm{Spec}(K) \to X_{S'}$. One corresponding to the point x and the embedding $\kappa(x) \subset K$ and the other corresponding to the point x' and the embedding $\kappa(x') \subset K$. Also we have $f' \circ a = f' \circ a'$. Condition (1) now implies that the compositions of a and a' with $X_{S'} \to X$ are equal. Since $X_{S'}$ is the fibre product of S' and X over S we see that $a = a'$. Hence $x = x'$. Thus (1) implies (2).

If there are two different points $x, x' \in X$ mapping to the same point of s then (2) is violated. If for some $s = f(x)$, $x \in X$ the field extension $\kappa(x)/\kappa(s)$ is not purely inseparable, then we may find a field extension $K/\kappa(s)$ such that $\kappa(x)$ has two $\kappa(s)$-homomorphisms into K. By Schemes, Lemma 26.13.3 this implies that the map $\mathrm{Mor}(\mathrm{Spec}(K), X) \to \mathrm{Mor}(\mathrm{Spec}(K), S)$ is not injective, and hence (1) is violated. Thus we see that the equivalent conditions (1) and (2) imply f is radicial, i.e., they imply (3).

Assume (3). By Schemes, Lemma 26.13.3 a morphism $\mathrm{Spec}(K) \to X$ is given by a pair $(x, \kappa(x) \to K)$. Property (3) says exactly that associating to the pair $(x, \kappa(x) \to K)$ the pair $(s, \kappa(s) \to \kappa(x) \to K)$ is injective. In other words (1) holds. At this point we know that (1), (2) and (3) are all equivalent.

Finally, we prove the equivalence of (4) with (1), (2) and (3). A point of $X \times_S X$ is given by a quadruple $(x_1, x_2, s, \mathfrak{p})$, where $x_1, x_2 \in X$, $f(x_1) = f(x_2) = s$ and $\mathfrak{p} \subset \kappa(x_1) \otimes_{\kappa(s)} \kappa(x_2)$ is a prime ideal, see Schemes, Lemma 26.17.5. If f is universally injective, then by taking $S' = X$ in the definition of universally injective, $\Delta_{X/S}$ must be surjective since it is a section of the injective morphism $X \times_S X \longrightarrow X$. Conversely, if $\Delta_{X/S}$ is surjective, then always $x_1 = x_2 = x$ and there is exactly one such prime ideal $\mathfrak{p}$, which means that $\kappa(s) \subset \kappa(x)$ is purely inseparable. Hence f is radicial. Alternatively, if $\Delta_{X/S}$ is surjective, then for any $S' \to S$ the base change $\Delta_{X_{S'}/S'}$ is surjective which implies that f is universally injective. This finishes the proof of the lemma. $\square$

05VE Lemma 29.10.3. A universally injective morphism is separated.

Proof. Combine Lemma 29.10.2 with the remark that $X \to S$ is separated if and only if the image of $\Delta_{X/S}$ is closed in $X \times_S X$, see Schemes, Definition 26.21.3 and the discussion following it. $\square$

0472 Lemma 29.10.4. A base change of a universally injective morphism is universally injective.

Proof. This is formal. □

02V1 Lemma 29.10.5. A composition of radicial morphisms is radicial, and so the same holds for the equivalent condition of being universally injective.

Proof. Omitted. □

29.11. Affine morphisms

01S5

01S6 Definition 29.11.1. A morphism of schemes $f : X \to S$ is called affine if the inverse image of every affine open of S is an affine open of X.

01S7 Lemma 29.11.2. An affine morphism is separated and quasi-compact.

Proof. Let $f : X \to S$ be affine. Quasi-compactness is immediate from Schemes, Lemma 26.19.2. We will show f is separated using Schemes, Lemma 26.21.7. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s. By assumption $f^{-1}(W)$ is affine. Apply the lemma cited with $U = V = f^{-1}(W)$. □

01S8 Lemma 29.11.3. Let $f : X \to S$ be a morphism of schemes. The following are equivalent [DG67, II, Corollary 1.3.2]

(1) The morphism f is affine.
(2) There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is affine.
(3) There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and an isomorphism $X \cong \underline{\mathrm{Spec}}_S(\mathcal{A})$ of schemes over S. See Constructions, Section 27.4 for notation.

Moreover, in this case $X = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$.

Proof. It is obvious that (1) implies (2).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. By Schemes, Lemma 26.19.2 we see that f is quasi-compact. By Schemes, Lemma 26.21.6 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 26.24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Thus we have the scheme $g : Y = \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 27.4.7. By assumption and the lemma just cited the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is an isomorphism. Thus can is an isomorphism. We have shown that (2) implies (3).

Assume (3). By Constructions, Lemma 27.4.6 we see that the inverse image of every affine open is affine, and hence the morphism is affine by definition. □

01S9 Remark 29.11.4. We can also argue directly that (2) implies (1) in Lemma 29.11.3 above as follows. Assume $S = \bigcup W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. First argue that $\mathcal{A} = f_*\mathcal{O}_X$ is quasi-coherent as in the proof above. Let $\mathrm{Spec}(R) = V \subset S$ be affine open. We have to show that $f^{-1}(V)$ is affine. Set $A = \mathcal{A}(V) = f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$. By Schemes, Lemma 26.6.4

there is a canonical morphism $\psi : f^{-1}(V) \to \text{Spec}(A)$ over $\text{Spec}(R) = V$. By Schemes, Lemma 26.11.6 there exists an integer $n \geq 0$, a standard open covering $V = \bigcup_{i=1,\ldots,n} D(h_i)$, $h_i \in R$, and a map $a : \{1, \ldots, n\} \to J$ such that each $D(h_i)$ is also a standard open of the affine scheme $W_{a(i)}$. The inverse image of a standard open under a morphism of affine schemes is standard open, see Algebra, Lemma 10.17.4. Hence we see that $f^{-1}(D(h_i))$ is a standard open of $f^{-1}(W_{a(i)})$, in particular that $f^{-1}(D(h_i))$ is affine. Because $\mathcal{A}$ is quasi-coherent we have $A_{h_i} = \mathcal{A}(D(h_i)) = \mathcal{O}_X(f^{-1}(D(h_i)))$, so $f^{-1}(D(h_i))$ is the spectrum of A_{h_i}. It follows that the morphism ψ induces an isomorphism of the open $f^{-1}(D(h_i))$ with the open $\text{Spec}(A_{h_i})$ of $\text{Spec}(A)$. Since $f^{-1}(V) = \bigcup f^{-1}(D(h_i))$ and $\text{Spec}(A) = \bigcup \text{Spec}(A_{h_i})$ we win.

01SA Lemma 29.11.5. Let S be a scheme. There is an anti-equivalence of categories

$$\begin{matrix} \text{Schemes affine} \\ \text{over } S \end{matrix} \longleftrightarrow \begin{matrix} \text{quasi-coherent sheaves} \\ \text{of } \mathcal{O}_S\text{-algebras} \end{matrix}$$

which associates to $f : X \to S$ the sheaf $f_*\mathcal{O}_X$. Moreover, this equivalence is compatible with arbitrary base change.

Proof. The functor from right to left is given by $\underline{\text{Spec}}_S$. The two functors are mutually inverse by Lemma 29.11.3 and Constructions, Lemma 27.4.6 part (3). The final statement is Constructions, Lemma 27.4.6 part (2). $\square$

01SB Lemma 29.11.6. Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{A} = f_*\mathcal{O}_X$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories

$$\left\{ \begin{matrix} \text{category of quasi-coherent} \\ \mathcal{O}_X\text{-modules} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{category of quasi-coherent} \\ \mathcal{A}\text{-modules} \end{matrix} \right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_S$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. Omitted. $\square$

01SC Lemma 29.11.7. The composition of affine morphisms is affine.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is affine by assumption on g. Whereupon $f^{-1}(g^{-1}(U))$ is affine by assumption on f. Hence $(g \circ f)^{-1}(U)$ is affine. $\square$

01SD Lemma 29.11.8. The base change of an affine morphism is affine.

Proof. Let $f : X \to S$ be an affine morphism. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V \subset S'$ which maps into some open affine $U \subset S$. By assumption $f^{-1}(U)$ is affine. By the material in Schemes, Section 26.17 we see that $f^{-1}(U)_V = V \times_U f^{-1}(U)$ is affine and equal to $(f')^{-1}(V)$. This proves that S' has an open covering by affines whose inverse image under f' is affine. We conclude by Lemma 29.11.3 above. $\square$

01SE Lemma 29.11.9. A closed immersion is affine.

Proof. The first indication of this is Schemes, Lemma 26.8.2. See Schemes, Lemma 26.10.1 for a complete statement. $\square$

01SF Lemma 29.11.10. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. The inclusion morphism $j : X_s \to X$ is affine.

Proof. This follows from Properties, Lemma 28.26.4 and the definition. □

01SG Lemma 29.11.11. Suppose $g : X \to Y$ is a morphism of schemes over S.

(1) If X is affine over S and $\Delta : Y \to Y \times_S Y$ is affine, then g is affine.
(2) If X is affine over S and Y is separated over S, then g is affine.
(3) A morphism from an affine scheme to a scheme with affine diagonal is affine.
(4) A morphism from an affine scheme to a separated scheme is affine.

Proof. Proof of (1). The base change $X \times_S Y \to Y$ is affine by Lemma 29.11.8. The morphism $(1, g) : X \to X \times_S Y$ is the base change of $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$. Hence it is affine by Lemma 29.11.8. The composition of affine morphisms is affine (see Lemma 29.11.7) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 29.11.9) and Y/S separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). □

01SH Lemma 29.11.12. A morphism between affine schemes is affine.

Proof. Immediate from Lemma 29.11.11 with $S = \mathrm{Spec}(\mathbf{Z})$. It also follows directly from the equivalence of (1) and (2) in Lemma 29.11.3. □

01SI Lemma 29.11.13. Let S be a scheme. Let A be an Artinian ring. Any morphism $\mathrm{Spec}(A) \to S$ is affine.

Proof. Omitted. □

0C3A Lemma 29.11.14. Let $j : Y \to X$ be an immersion of schemes. Assume there exists an open $U \subset X$ with complement $Z = X \setminus U$ such that

(1) $U \to X$ is affine,
(2) $j^{-1}(U) \to U$ is affine, and
(3) $j(Y) \cap Z$ is closed.

Then j is affine. In particular, if X is affine, so is Y.

Proof. By Schemes, Definition 26.10.2 there exists an open subscheme $W \subset X$ such that j factors as a closed immersion $i : Y \to W$ followed by the inclusion morphism $W \to X$. Since a closed immersion is affine (Lemma 29.11.9), we see that for every $x \in W$ there is an affine open neighbourhood of x in X whose inverse image under j is affine. If $x \in U$, then the same thing is true by assumption (2). Finally, assume $x \in Z$ and $x \notin W$. Then $x \notin j(Y) \cap Z$. By assumption (3) we can find an affine open neighbourhood $V \subset X$ of x which does not meet $j(Y) \cap Z$. Then $j^{-1}(V) = j^{-1}(V \cap U)$ which is affine by assumptions (1) and (2). It follows that j is affine by Lemma 29.11.3. □

29.12. Families of ample invertible modules

0FXQ A short section on the notion of a family of ample invertible modules.

0FXR Definition 29.12.1. Let X be a scheme. Let $\{\mathcal{L}_i\}_{i \in I}$ be a family of invertible $\mathcal{O}_X$-modules. We say $\{\mathcal{L}_i\}_{i \in I}$ is an ample family of invertible modules on X if [BGI71, II Definition 2.2.4]

(1) X is quasi-compact, and

(2) for every $x \in X$ there exists an $i \in I$, an $n \geq 1$, and $s \in \Gamma(X, \mathcal{L}_i^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

If $\{\mathcal{L}_i\}_{i \in I}$ is an ample family of invertible modules on a scheme X, then there exists a finite subset $I' \subset I$ such that $\{\mathcal{L}_i\}_{i \in I'}$ is an ample family of invertible modules on X (follows immediately from quasi-compactness). A scheme having an ample family of invertible modules has an affine diagonal by the next lemma and hence is a fortiori quasi-separated.

0FXS Lemma 29.12.2. Let X be a scheme such that for every point $x \in X$ there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a global section $s \in \Gamma(X, \mathcal{L})$ such that $x \in X_s$ and X_s is affine. Then the diagonal of X is an affine morphism.

Proof. Given invertible $\mathcal{O}_X$-modules $\mathcal{L}$, $\mathcal{M}$ and global sections $s \in \Gamma(X, \mathcal{L})$, $t \in \Gamma(X, \mathcal{M})$ such that X_s and X_t are affine we have to prove $X_s \cap X_t$ is affine. Namely, then Lemma 29.11.3 applied to $\Delta : X \to X \times X$ and the fact that $\Delta^{-1}(X_s \times X_t) = X_s \cap X_t$ shows that Δ is affine. The fact that $X_s \cap X_t$ is affine follows from Properties, Lemma 28.26.4. □

0FXT Remark 29.12.3. In Properties, Lemma 28.26.7 we see that a scheme which has an ample invertible module is separated. This is wrong for schemes having an ample family of invertible modules. Namely, let X be as in Schemes, Example 26.14.3 with $n = 1$, i.e., the affine line with zero doubled. We use the notation of that example except that we write x for x_1 and y for y_1. There is, for every integer n, an invertible sheaf $\mathcal{L}_n$ on X which is trivial on X_1 and X_2 and whose transition function $U_{12} \to U_{21}$ is $f(x) \mapsto y^n f(y)$. The global sections of $\mathcal{L}_n$ are pairs $(f(x), g(y)) \in k[x] \oplus k[y]$ such that $y^n f(y) = g(y)$. The sections $s = (1, y)$ of $\mathcal{L}_1$ and $t = (x, 1)$ of $\mathcal{L}_{-1}$ determine an open affine cover because $X_s = X_1$ and $X_t = X_2$. Therefore X has an ample family of invertible modules but it is not separated.

29.13. Quasi-affine morphisms

01SJ Recall that a scheme X is called quasi-affine if it is quasi-compact and isomorphic to an open subscheme of an affine scheme, see Properties, Definition 28.18.1.

01SK Definition 29.13.1. A morphism of schemes $f : X \to S$ is called quasi-affine if the inverse image of every affine open of S is a quasi-affine scheme.

01SL Lemma 29.13.2. A quasi-affine morphism is separated and quasi-compact.

Proof. Let $f : X \to S$ be quasi-affine. Quasi-compactness is immediate from Schemes, Lemma 26.19.2. Let $U \subset S$ be an affine open. If we can show that $f^{-1}(U)$ is a separated scheme, then f is separated (Schemes, Lemma 26.21.7 shows that being separated is local on the base). By assumption $f^{-1}(U)$ is isomorphic to an open subscheme of an affine scheme. An affine scheme is separated and hence every open subscheme of an affine scheme is separated as desired. □

01SM Lemma 29.13.3. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is quasi-affine.
(2) There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is quasi-affine.

(3) There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion

$$\begin{tikzcd} X \arrow[rr] \arrow[rd] & & \underline{\mathrm{Spec}}_S(\mathcal{A}) \arrow[ld] \\ & S & \end{tikzcd}$$

over S.

(4) Same as in (3) but with $\mathcal{A} = f_*\mathcal{O}_X$ and the horizontal arrow the canonical morphism of Constructions, Lemma 27.4.7.

Proof. It is obvious that (1) implies (2) and that (4) implies (3).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is quasi-affine. By Schemes, Lemma 26.19.2 we see that f is quasi-compact. By Schemes, Lemma 26.21.6 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 26.24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Thus we have the scheme $g : Y = \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 27.4.7. By assumption, the lemma just cited, and Properties, Lemma 28.18.4 the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is a quasi-compact open immersion. Thus can is a quasi-compact open immersion. We have shown that (2) implies (4).

Assume (3). Choose any affine open $U \subset S$. By Constructions, Lemma 27.4.6 we see that the inverse image of U in the relative spectrum is affine. Hence we conclude that $f^{-1}(U)$ is quasi-affine (note that quasi-compactness is encoded in (3) as well). Thus (3) implies (1). □

01SN Lemma 29.13.4. The composition of quasi-affine morphisms is quasi-affine.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasi-affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is quasi-affine by assumption on g. Let $j : g^{-1}(U) \to V$ be a quasi-compact open immersion into an affine scheme V. By Lemma 29.13.3 above we see that $f^{-1}(g^{-1}(U))$ is a quasi-compact open subscheme of the relative spectrum $\underline{\mathrm{Spec}}_{g^{-1}(U)}(\mathcal{A})$ for some quasi-coherent sheaf of $\mathcal{O}_{g^{-1}(U)}$-algebras $\mathcal{A}$. By Schemes, Lemma 26.24.1 the sheaf $\mathcal{A}' = j_*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_V$-algebras with the property that $j^*\mathcal{A}' = \mathcal{A}$. Hence we get a commutative diagram

$$\begin{tikzcd} f^{-1}(g^{-1}(U)) \arrow[r] & \underline{\mathrm{Spec}}_{g^{-1}(U)}(\mathcal{A}) \arrow[r] \arrow[d] & \underline{\mathrm{Spec}}_V(\mathcal{A}') \arrow[d] \\ & g^{-1}(U) \arrow[r, "j"] & V \end{tikzcd}$$

with the square being a fibre square, see Constructions, Lemma 27.4.6. Note that the upper right corner is an affine scheme. Hence $(g \circ f)^{-1}(U)$ is quasi-affine. □

01SO Lemma 29.13.5. The base change of a quasi-affine morphism is quasi-affine.

Proof. Let $f : X \to S$ be a quasi-affine morphism. By Lemma 29.13.3 above we can find a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion $X \to \underline{\mathrm{Spec}}_S(\mathcal{A})$ over S. Let $g : S' \to S$ be any morphism. Denote

$f' : X_{S'} = S' \times_S X \to S'$ the base change of f. Since the base change of a quasi-compact open immersion is a quasi-compact open immersion we see that $X_{S'} \to \underline{\mathrm{Spec}}_{S'}(g^*\mathcal{A})$ is a quasi-compact open immersion (we have used Schemes, Lemmas 26.19.3 and 26.18.2 and Constructions, Lemma 27.4.6). By Lemma 29.13.3 again we conclude that $X_{S'} \to S'$ is quasi-affine. $\square$

02JR Lemma 29.13.6. A quasi-compact immersion is quasi-affine.

Proof. Let $X \to S$ be a quasi-compact immersion. We have to show the inverse image of every affine open is quasi-affine. Hence, assuming S is an affine scheme, we have to show X is quasi-affine. By Lemma 29.7.7 the morphism $X \to S$ factors as $X \to Z \to S$ where Z is a closed subscheme of S and $X \subset Z$ is a quasi-compact open. Since S is affine Lemma 29.2.1 implies Z is affine. Hence we win. $\square$

01SP Lemma 29.13.7. Let S be a scheme. Let X be an affine scheme. A morphism $f : X \to S$ is quasi-affine if and only if it is quasi-compact. In particular any morphism from an affine scheme to a quasi-separated scheme is quasi-affine.

Proof. Let $V \subset S$ be an affine open. Then $f^{-1}(V)$ is an open subscheme of the affine scheme X, hence quasi-affine if and only if it is quasi-compact. This proves the first assertion. The quasi-compactness of any $f : X \to S$ where X is affine and S quasi-separated follows from Schemes, Lemma 26.21.14 applied to $X \to S \to \mathrm{Spec}(\mathbf{Z})$. $\square$

054G Lemma 29.13.8. Suppose $g : X \to Y$ is a morphism of schemes over S. If X is quasi-affine over S and Y is quasi-separated over S, then g is quasi-affine. In particular, any morphism from a quasi-affine scheme to a quasi-separated scheme is quasi-affine.

Proof. The base change $X \times_S Y \to Y$ is quasi-affine by Lemma 29.13.5. The morphism $X \to X \times_S Y$ is a quasi-compact immersion as $Y \to S$ is quasi-separated, see Schemes, Lemma 26.21.11. A quasi-compact immersion is quasi-affine by Lemma 29.13.6 and the composition of quasi-affine morphisms is quasi-affine (see Lemma 29.13.4). Thus we win. $\square$

29.14. Types of morphisms defined by properties of ring maps

01SQ In this section we study what properties of ring maps allow one to define local properties of morphisms of schemes.

01SR Definition 29.14.1. Let P be a property of ring maps.

(1) We say that P is local if the following hold:
 (a) For any ring map $R \to A$, and any $f \in R$ we have $P(R \to A) \Rightarrow P(R_f \to A_f)$.
 (b) For any rings R, A, any $f \in R$, $a \in A$, and any ring map $R_f \to A$ we have $P(R_f \to A) \Rightarrow P(R \to A_a)$.
 (c) For any ring map $R \to A$, and $a_i \in A$ such that $(a_1, \ldots, a_n) = A$ then $\forall i, P(R \to A_{a_i}) \Rightarrow P(R \to A)$.
(2) We say that P is stable under base change if for any ring maps $R \to A$, $R \to R'$ we have $P(R \to A) \Rightarrow P(R' \to R' \otimes_R A)$.
(3) We say that P is stable under composition if for any ring maps $A \to B$, $B \to C$ we have $P(A \to B) \wedge P(B \to C) \Rightarrow P(A \to C)$.

01SS Definition 29.14.2. Let P be a property of ring maps. Let $f : X \to S$ be a morphism of schemes. We say f is locally of type P if for any $x \in X$ there exists an affine open neighbourhood U of x in X which maps into an affine open $V \subset S$ such that the induced ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ has property P.

This is not a "good" definition unless the property P is a local property. Even if P is a local property we will not automatically use this definition to say that a morphism is "locally of type P" unless we also explicitly state the definition elsewhere.

01ST Lemma 29.14.3. Let $f : X \to S$ be a morphism of schemes. Let P be a property of ring maps. Let U be an affine open of X, and V an affine open of S such that $f(U) \subset V$. If f is locally of type P and P is local, then $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds.

Proof. As f is locally of type P for every $u \in U$ there exists an affine open $U_u \subset X$ mapping into an affine open $V_u \subset S$ such that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U_u))$ holds. Choose an open neighbourhood $U'_u \subset U \cap U_u$ of u which is standard affine open in both U and U_u, see Schemes, Lemma 26.11.5. By Definition 29.14.1 (1)(b) we see that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U'_u))$ holds. Hence we may assume that $U_u \subset U$ is a standard affine open. Choose an open neighbourhood $V'_u \subset V \cap V_u$ of $f(u)$ which is standard affine open in both V and V_u, see Schemes, Lemma 26.11.5. Then $U'_u = f^{-1}(V'_u) \cap U_u$ is a standard affine open of U_u (hence of U) and we have $P(\mathcal{O}_S(V'_u) \to \mathcal{O}_X(U'_u))$ by Definition 29.14.1 (1)(a). Hence we may assume both $U_u \subset U$ and $V_u \subset V$ are standard affine open. Applying Definition 29.14.1 (1)(b) one more time we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U_u))$ holds. Because U is quasi-compact we may choose a finite number of points $u_1, \ldots, u_n \in U$ such that

$$U = U_{u_1} \cup \ldots \cup U_{u_n}.$$

By Definition 29.14.1 (1)(c) we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds. □

01SU Lemma 29.14.4. Let P be a local property of ring maps. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is locally of type P.
(2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ we have $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$.
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of type P.
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $P(\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i))$ holds, for all $j \in J, i \in I_j$.

Moreover, if f is locally of type P then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of type P.

Proof. This follows from Lemma 29.14.3 above. □

01SV Lemma 29.14.5. Let P be a property of ring maps. Assume P is local and stable under composition. The composition of morphisms locally of type P is locally of type P.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms locally of type P. Let $x \in X$. Choose an affine open neighbourhood $W \subset Z$ of $g(f(x))$. Choose an affine open neighbourhood $V \subset g^{-1}(W)$ of $f(x)$. Choose an affine open neighbourhood

$U \subset f^{-1}(V)$ of x. By Lemma 29.14.4 the ring maps $\mathcal{O}_Z(W) \to \mathcal{O}_Y(V)$ and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ satisfy P. Hence $\mathcal{O}_Z(W) \to \mathcal{O}_X(U)$ satisfies P as P is assumed stable under composition. □

01SW Lemma 29.14.6. Let P be a property of ring maps. Assume P is local and stable under base change. The base change of a morphism locally of type P is locally of type P.

Proof. Let $f : X \to S$ be a morphism locally of type P. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V' \subset S'$ which maps into some open affine $V \subset S$. By Lemma 29.14.4 the open $f^{-1}(V)$ is a union of affines U_i such that the ring maps $\mathcal{O}_S(V) \to \mathcal{O}_X(U_i)$ all satisfy P. By the material in Schemes, Section 26.17 we see that $f^{-1}(U)_{V'} = V' \times_V f^{-1}(V)$ is the union of the affine opens $V' \times_V U_i$. Since $\mathcal{O}_{X_{S'}}(V' \times_V U_i) = \mathcal{O}_{S'}(V') \otimes_{\mathcal{O}_S(V)} \mathcal{O}_X(U_i)$ we see that the ring maps $\mathcal{O}_{S'}(V') \to \mathcal{O}_{X_{S'}}(V' \times_V U_i)$ satisfy P as P is assumed stable under base change. □

01SX Lemma 29.14.7. The following properties of a ring map $R \to A$ are local.

(1) (Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.
(2) (Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.
(3) (Reduced fibres.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ is reduced.
(4) (Fibres of dimension at most n.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ has Krull dimension at most n.
(5) (Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.
(6) Add more here as needed[4].

Proof. Omitted. □

01SY Lemma 29.14.8. The following properties of ring maps are stable under base change.

(1) (Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.
(2) (Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.
(3) Add more here as needed[5].

Proof. Omitted. □

01SZ Lemma 29.14.9. The following properties of ring maps are stable under composition.

(1) (Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.
(2) (Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.

[4]But only those properties that are not already dealt with separately elsewhere.
[5]But only those properties that are not already dealt with separately elsewhere.

(3) (Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.
(4) Add more here as needed[6].

Proof. Omitted. □

29.15. Morphisms of finite type

01T0 Recall that a ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, see Algebra, Definition 10.6.1.

01T1 Definition 29.15.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is of finite type at $x \in X$ if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
(2) We say that f is locally of finite type if it is of finite type at every point of X.
(3) We say that f is of finite type if it is locally of finite type and quasi-compact.

01T2 Lemma 29.15.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is locally of finite type.
(2) For all affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type.
(3) There exist an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite type.
(4) There exist an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite type, for all $j \in J, i \in I_j$.

Moreover, if f is locally of finite type then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite type.

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is of finite type" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. By Algebra, Lemma 10.14.2 being of finite type is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.6.2 being of finite type is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite type. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.23.3. □

01T3 Lemma 29.15.3. The composition of two morphisms which are locally of finite type is locally of finite type. The same is true for morphisms of finite type.

Proof. In the proof of Lemma 29.15.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being of finite type is a property of ring maps that is stable under composition, see Algebra, Lemma 10.6.2. By the above and the fact that compositions of quasi-compact morphisms are quasi-compact, see

[6]But only those properties that are not already dealt with separately elsewhere.

Schemes, Lemma 26.19.4 we see that the composition of morphisms of finite type is of finite type. ☐

01T4 Lemma 29.15.4. The base change of a morphism which is locally of finite type is locally of finite type. The same is true for morphisms of finite type.

Proof. In the proof of Lemma 29.15.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.6 combined with the fact that being of finite type is a property of ring maps that is stable under base change, see Algebra, Lemma 10.14.2. By the above and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 26.19.3 we see that the base change of a morphism of finite type is a morphism of finite type. ☐

01T5 Lemma 29.15.5. A closed immersion is of finite type. An immersion is locally of finite type.

Proof. This is true because an open immersion is a local isomorphism, and a closed immersion is obviously of finite type. ☐

01T6 Lemma 29.15.6. Let $f : X \to S$ be a morphism. If S is (locally) Noetherian and f (locally) of finite type then X is (locally) Noetherian.

Proof. This follows immediately from the fact that a ring of finite type over a Noetherian ring is Noetherian, see Algebra, Lemma 10.31.1. (Also: use the fact that the source of a quasi-compact morphism with quasi-compact target is quasi-compact.) ☐

01T7 Lemma 29.15.7. Let $f : X \to S$ be locally of finite type with S locally Noetherian. Then f is quasi-separated.

Proof. In fact, it is true that X is quasi-separated, see Properties, Lemma 28.5.4 and Lemma 29.15.6 above. Then apply Schemes, Lemma 26.21.13 to conclude that f is quasi-separated. ☐

01T8 Lemma 29.15.8. Let $X \to Y$ be a morphism of schemes over a base scheme S. If X is locally of finite type over S, then $X \to Y$ is locally of finite type.

Proof. Via Lemma 29.15.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite type, then $B \to C$ is of finite type. (See Algebra, Lemma 10.6.2). ☐

29.16. Points of finite type and Jacobson schemes

01T9 Let S be a scheme. A finite type point s of S is a point such that the morphism $\text{Spec}(\kappa(s)) \to S$ is of finite type. The reason for studying this is that finite type points can replace closed points in a certain sense and in certain situations. There are always enough of them for example. Moreover, a scheme is Jacobson if and only if all finite type points are closed points.

01TA Lemma 29.16.1. Let S be a scheme. Let k be a field. Let $f : \text{Spec}(k) \to S$ be a morphism. The following are equivalent:

(1) The morphism f is of finite type.
(2) The morphism f is locally of finite type.

(3) There exists an affine open $U = \mathrm{Spec}(R)$ of S such that f corresponds to a finite ring map $R \to k$.
(4) There exists an affine open $U = \mathrm{Spec}(R)$ of S such that the image of f consists of a closed point u in U and the field extension $k/\kappa(u)$ is finite.

Proof. The equivalence of (1) and (2) is obvious as $\mathrm{Spec}(k)$ is a singleton and hence any morphism from it is quasi-compact.

Suppose f is locally of finite type. Choose any affine open $\mathrm{Spec}(R) = U \subset S$ such that the image of f is contained in U, and the ring map $R \to k$ is of finite type. Let $\mathfrak p \subset R$ be the kernel. Then $R/\mathfrak p \subset k$ is of finite type. By Algebra, Lemma 10.34.2 there exist a $\overline{f} \in R/\mathfrak p$ such that $(R/\mathfrak p)_{\overline{f}}$ is a field and $(R/\mathfrak p)_{\overline{f}} \to k$ is a finite field extension. If $f \in R$ is a lift of $\overline{f}$, then we see that k is a finite R_f-module. Thus (2) $\Rightarrow$ (3).

Suppose that $\mathrm{Spec}(R) = U \subset S$ is an affine open such that f corresponds to a finite ring map $R \to k$. Then f is locally of finite type by Lemma 29.15.2. Thus (3) $\Rightarrow$ (2).

Suppose $R \to k$ is finite. The image of $R \to k$ is a field over which k is finite by Algebra, Lemma 10.36.18. Hence the kernel of $R \to k$ is a maximal ideal. Thus (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (3) is immediate. $\square$

02HV Lemma 29.16.2. Let S be a scheme. Let A be an Artinian local ring with residue field κ. Let $f : \mathrm{Spec}(A) \to S$ be a morphism of schemes. Then f is of finite type if and only if the composition $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A) \to S$ is of finite type.

Proof. Since the morphism $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A)$ is of finite type it is clear that if f is of finite type so is the composition $\mathrm{Spec}(\kappa) \to S$ (see Lemma 29.15.3). For the converse, note that $\mathrm{Spec}(A) \to S$ maps into some affine open $U = \mathrm{Spec}(B)$ of S as $\mathrm{Spec}(A)$ has only one point. To finish apply Algebra, Lemma 10.54.4 to $B \to A$. $\square$

Recall that given a point s of a scheme S there is a canonical morphism $\mathrm{Spec}(\kappa(s)) \to S$, see Schemes, Section 26.13.

02J1 Definition 29.16.3. Let S be a scheme. Let us say that a point s of S is a finite type point if the canonical morphism $\mathrm{Spec}(\kappa(s)) \to S$ is of finite type. We denote $S_{\text{ft-pts}}$ the set of finite type points of S.

We can describe the set of finite type points as follows.

02J2 Lemma 29.16.4. Let S be a scheme. We have

$$S_{\text{ft-pts}} = \bigcup\nolimits_{U \subset S \text{ open}} U_0$$

where U_0 is the set of closed points of U. Here we may let U range over all opens or over all affine opens of S.

Proof. Immediate from Lemma 29.16.1. $\square$

02J3 Lemma 29.16.5. Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type, then $f(T_{\text{ft-pts}}) \subset S_{\text{ft-pts}}$.

Proof. If T is the spectrum of a field this is Lemma 29.16.1. In general it follows since the composition of morphisms locally of finite type is locally of finite type (Lemma 29.15.3). $\square$

06EB Lemma 29.16.6. Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type and surjective, then $f(T_{\text{ft-pts}}) = S_{\text{ft-pts}}$.

Proof. We have $f(T_{\text{ft-pts}}) \subset S_{\text{ft-pts}}$ by Lemma 29.16.5. Let $s \in S$ be a finite type point. As f is surjective the scheme $T_s = \text{Spec}(\kappa(s)) \times_S T$ is nonempty, therefore has a finite type point $t \in T_s$ by Lemma 29.16.4. Now $T_s \to T$ is a morphism of finite type as a base change of $s \to S$ (Lemma 29.15.4). Hence the image of t in T is a finite type point by Lemma 29.16.5 which maps to s by construction. $\square$

02J4 Lemma 29.16.7. Let S be a scheme. For any locally closed subset $T \subset S$ we have

$$T \neq \emptyset \Rightarrow T \cap S_{\text{ft-pts}} \neq \emptyset.$$

In particular, for any closed subset $T \subset S$ we see that $T \cap S_{\text{ft-pts}}$ is dense in T.

Proof. Note that T carries a scheme structure (see Schemes, Lemma 26.12.4) such that $T \to S$ is a locally closed immersion. Any locally closed immersion is locally of finite type, see Lemma 29.15.5. Hence by Lemma 29.16.5 we see $T_{\text{ft-pts}} \subset S_{\text{ft-pts}}$. Finally, any nonempty affine open of T has at least one closed point which is a finite type point of T by Lemma 29.16.4. $\square$

It follows that most of the material from Topology, Section 5.18 goes through with the set of closed points replaced by the set of points of finite type. In fact, if S is Jacobson then we recover the closed points as the finite type points.

01TB Lemma 29.16.8. Let S be a scheme. The following are equivalent:

(1) the scheme S is Jacobson,
(2) $S_{\text{ft-pts}}$ is the set of closed points of S,
(3) for all $T \to S$ locally of finite type closed points map to closed points, and
(4) for all $T \to S$ locally of finite type closed points $t \in T$ map to closed points $s \in S$ with $\kappa(s) \subset \kappa(t)$ finite.

Proof. We have trivially (4) $\Rightarrow$ (3) $\Rightarrow$ (2). Lemma 29.16.7 shows that (2) implies (1). Hence it suffices to show that (1) implies (4). Suppose that $T \to S$ is locally of finite type. Choose $t \in T$ closed and let $s \in S$ be the image. Choose affine open neighbourhoods $\text{Spec}(R) = U \subset S$ of s and $\text{Spec}(A) = V \subset T$ of t with V mapping into U. The induced ring map $R \to A$ is of finite type (see Lemma 29.15.2) and R is Jacobson by Properties, Lemma 28.6.3. Thus the result follows from Algebra, Proposition 10.35.19. $\square$

02J5 Lemma 29.16.9. Let S be a Jacobson scheme. Any scheme locally of finite type over S is Jacobson.

Proof. This is clear from Algebra, Proposition 10.35.19 (and Properties, Lemma 28.6.3 and Lemma 29.15.2). $\square$

02J6 Lemma 29.16.10. The following types of schemes are Jacobson.

(1) Any scheme locally of finite type over a field.
(2) Any scheme locally of finite type over $\mathbf{Z}$.
(3) Any scheme locally of finite type over a 1-dimensional Noetherian domain with infinitely many primes.

(4) A scheme of the form $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ where $(R, \mathfrak{m})$ is a Noetherian local ring. Also any scheme locally of finite type over it.

Proof. We will use Lemma 29.16.9 without mention. The spectrum of a field is clearly Jacobson. The spectrum of $\mathbf{Z}$ is Jacobson, see Algebra, Lemma 10.35.6. For (3) see Algebra, Lemma 10.61.4. For (4) see Properties, Lemma 28.6.4. □

29.17. Universally catenary schemes

02J7 Recall that a topological space X is called catenary if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 5.11.4. Recall that a scheme is catenary if its underlying topological space is catenary. See Properties, Definition 28.11.1.

02J8 Definition 29.17.1. Let S be a scheme. Assume S is locally Noetherian. We say S is universally catenary if for every morphism $X \to S$ locally of finite type the scheme X is catenary.

This is a "better" notion than catenary as there exist Noetherian schemes which are catenary but not universally catenary. See Examples, Section 110.18. Many schemes are universally catenary, see Lemma 29.17.5 below.

Recall that a ring A is called catenary if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 10.105.1. We have seen the relationship between catenary schemes and catenary rings in Properties, Section 28.11. Recall that a ring A is called universally catenary if A is Noetherian and for every finite type ring map $A \to B$ the ring B is catenary. See Algebra, Definition 10.105.3. Many interesting rings which come up in algebraic geometry satisfy this property.

02J9 Lemma 29.17.2. Let S be a locally Noetherian scheme. The following are equivalent

(1) S is universally catenary,
(2) there exists an open covering of S all of whose members are universally catenary schemes,
(3) for every affine open $\mathrm{Spec}(R) = U \subset S$ the ring R is universally catenary, and
(4) there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a universally catenary ring.

Moreover, in this case any scheme locally of finite type over S is universally catenary as well.

Proof. By Lemma 29.15.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 29.15.3). Thus it is clear that if S is universally catenary then any open and any scheme locally of finite type over S is universally catenary as well. This proves the final statement of the lemma and that (1) implies (2).

If $\text{Spec}(R)$ is a universally catenary scheme, then every scheme $\text{Spec}(A)$ with A a finite type R-algebra is catenary. Hence all these rings A are catenary by Algebra, Lemma 10.105.2. Thus R is universally catenary. Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (3) implies (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $X \to S$ be a morphism locally of finite type. We can find an affine open covering $X = \bigcup V_j$ such that each $V_j \to S$ maps into one of the U_i. By Lemma 29.15.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence $\mathcal{O}(V_j)$ is catenary. Hence X is catenary by Properties, Lemma 28.11.2. □

02JA Lemma 29.17.3. Let S be a locally Noetherian scheme. The following are equivalent:

(1) S is universally catenary, and
(2) all local rings $\mathcal{O}_{S,s}$ of S are universally catenary.

Proof. Assume that all local rings of S are universally catenary. Let $f : X \to S$ be locally of finite type. We know that X is catenary if and only if $\mathcal{O}_{X,x}$ is catenary for all $x \in X$. If $f(x) = s$, then $\mathcal{O}_{X,x}$ is essentially of finite type over $\mathcal{O}_{S,s}$. Hence $\mathcal{O}_{X,x}$ is catenary by the assumption that $\mathcal{O}_{S,s}$ is universally catenary.

Conversely, assume that S is universally catenary. Let $s \in S$. We may replace S by an affine open neighbourhood of s by Lemma 29.17.2. Say $S = \text{Spec}(R)$ and s corresponds to the prime ideal $\mathfrak{p}$. Any finite type $R_\mathfrak{p}$-algebra A' is of the form $A_\mathfrak{p}$ for some finite type R-algebra A. By assumption (and Lemma 29.17.2 if you like) the ring A is catenary, and hence A' (a localization of A) is catenary. Thus $R_\mathfrak{p}$ is universally catenary. □

0G42 Lemma 29.17.4. Let S be a locally Noetherian scheme. Then S is universally catenary if and only if the irreducible components of S are universally catenary.

Proof. Omitted. For the affine case, please see Algebra, Lemma 10.105.8. □

02JB Lemma 29.17.5. The following types of schemes are universally catenary.

(1) Any scheme locally of finite type over a field.
(2) Any scheme locally of finite type over a Cohen-Macaulay scheme.
(3) Any scheme locally of finite type over $\mathbf{Z}$.
(4) Any scheme locally of finite type over a 1-dimensional Noetherian domain.
(5) And so on.

Proof. All of these follow from the fact that a Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 10.105.9. Also, use the last assertion of Lemma 29.17.2. Some details omitted. □

29.18. Nagata schemes, reprise

0359 See Properties, Section 28.13 for the definitions and basic properties of Nagata and universally Japanese schemes.

035A Lemma 29.18.1. Let $f : X \to S$ be a morphism. If S is Nagata and f locally of finite type then X is Nagata. If S is universally Japanese and f locally of finite type then X is universally Japanese.

Proof. For "universally Japanese" this follows from Algebra, Lemma 10.162.4. For "Nagata" this follows from Algebra, Proposition 10.162.15. ☐

035B Lemma 29.18.2. The following types of schemes are Nagata.

(1) Any scheme locally of finite type over a field.
(2) Any scheme locally of finite type over a Noetherian complete local ring.
(3) Any scheme locally of finite type over $\mathbf{Z}$.
(4) Any scheme locally of finite type over a Dedekind ring of characteristic zero.
(5) And so on.

Proof. By Lemma 29.18.1 we only need to show that the rings mentioned above are Nagata rings. For this see Algebra, Proposition 10.162.16. ☐

29.19. The singular locus, reprise

07R2 We look for a criterion that implies openness of the regular locus for any scheme locally of finite type over the base. Here is the definition.

07R3 Definition 29.19.1. Let X be a locally Noetherian scheme. We say X is J-2 if for every morphism $Y \to X$ which is locally of finite type the regular locus $\text{Reg}(Y)$ is open in Y.

This is the analogue of the corresponding notion for Noetherian rings, see More on Algebra, Definition 15.47.1.

07R4 Lemma 29.19.2. Let X be a locally Noetherian scheme. The following are equivalent

(1) X is J-2,
(2) there exists an open covering of X all of whose members are J-2 schemes,
(3) for every affine open $\text{Spec}(R) = U \subset X$ the ring R is J-2, and
(4) there exists an affine open covering $S = \bigcup U_i$ such that each $\mathcal{O}(U_i)$ is J-2 for all i.

Moreover, in this case any scheme locally of finite type over X is J-2 as well.

Proof. By Lemma 29.15.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 29.15.3). Thus it is clear that if X is J-2 then any open and any scheme locally of finite type over X is J-2 as well. This proves the final statement of the lemma.

If $\text{Spec}(R)$ is J-2, then for every finite type R-algebra A the regular locus of the scheme $\text{Spec}(A)$ is open. Hence R is J-2, by definition (see More on Algebra, Definition 15.47.1). Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $Y \to X$ be a morphism locally of finite type. We can find an affine open covering $Y = \bigcup V_j$ such that each $V_j \to X$ maps into one of the U_i. By Lemma 29.15.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence the regular locus of $V_j = \text{Spec}(\mathcal{O}(V_j))$ is open. Since $\text{Reg}(Y) \cap V_j = \text{Reg}(V_j)$ we conclude that $\text{Reg}(Y)$ is open as desired. ☐

07R5 Lemma 29.19.3. The following types of schemes are J-2.

(1) Any scheme locally of finite type over a field.

(2) Any scheme locally of finite type over a Noetherian complete local ring.
(3) Any scheme locally of finite type over $\mathbf{Z}$.
(4) Any scheme locally of finite type over a Noetherian local ring of dimension 1.
(5) Any scheme locally of finite type over a Nagata ring of dimension 1.
(6) Any scheme locally of finite type over a Dedekind ring of characteristic zero.
(7) And so on.

Proof. By Lemma 29.19.2 we only need to show that the rings mentioned above are J-2. For this see More on Algebra, Proposition 15.48.7. □

29.20. Quasi-finite morphisms

01TC A solid treatment of quasi-finite morphisms is the basis of many developments further down the road. It will lead to various versions of Zariski's Main Theorem, behaviour of dimensions of fibres, descent for étale morphisms, etc, etc. Before reading this section it may be a good idea to take a look at the algebra results in Algebra, Section 10.122.

Recall that a finite type ring map $R \to A$ is quasi-finite at a prime $\mathfrak{q}$ if $\mathfrak{q}$ defines an isolated point of its fibre, see Algebra, Definition 10.122.3.

01TD Definition 29.20.1. Let $f : X \to S$ be a morphism of schemes. [DG67, II Definition 6.2.3]

(1) We say that f is quasi-finite at a point $x \in X$ if there exist an affine neighbourhood $\text{Spec}(A) = U \subset X$ of x and an affine open $\text{Spec}(R) = V \subset S$ such that $f(U) \subset V$, the ring map $R \to A$ is of finite type, and $R \to A$ is quasi-finite at the prime of A corresponding to x (see above).
(2) We say f is locally quasi-finite if f is quasi-finite at every point x of X.
(3) We say that f is quasi-finite if f is of finite type and every point x is an isolated point of its fibre.

Trivially, a locally quasi-finite morphism is locally of finite type. We will see below that a morphism f which is locally of finite type is quasi-finite at x if and only if x is isolated in its fibre. Moreover, the set of points at which a morphism is quasi-finite is open; we will see this in Section 29.56 on Zariski's Main Theorem.

01TE Lemma 29.20.2. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If $\kappa(x)/\kappa(s)$ is an algebraic field extension, then

(1) x is a closed point of its fibre, and
(2) if in addition s is a closed point of S, then x is a closed point of X.

Proof. The second statement follows from the first by elementary topology. According to Schemes, Lemma 26.18.5 to prove the first statement we may replace X by X_s and S by $\text{Spec}(\kappa(s))$. Thus we may assume that $S = \text{Spec}(k)$ is the spectrum of a field. In this case, let $\text{Spec}(A) = U \subset X$ be any affine open containing x. The point x corresponds to a prime ideal $\mathfrak{q} \subset A$ such that $\kappa(\mathfrak{q})/k$ is an algebraic field extension. By Algebra, Lemma 10.35.9 we see that $\mathfrak{q}$ is a maximal ideal, i.e., $x \in U$ is a closed point. Since the affine opens form a basis of the topology of X we conclude that $\{x\}$ is closed. □

The following lemma is a version of the Hilbert Nullstellensatz.

01TF Lemma 29.20.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Assume f is locally of finite type. Then x is a closed point of its fibre if and only if $\kappa(x)/\kappa(s)$ is a finite field extension.

Proof. If the extension is finite, then x is a closed point of the fibre by Lemma 29.20.2 above. For the converse, assume that x is a closed point of its fibre. Choose affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 29.15.2 the ring map $R \to A$ is of finite type. Let $\mathfrak q \subset A$, resp. $\mathfrak p \subset R$ be the prime ideal corresponding to x, resp. s. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak p)$. Let $\overline{\mathfrak q}$ be the prime of $\overline{A}$ corresponding to $\mathfrak q$. The assumption that x is a closed point of its fibre implies that $\overline{\mathfrak q}$ is a maximal ideal of $\overline{A}$. Since $\overline{A}$ is an algebra of finite type over the field $\kappa(\mathfrak p)$ we see by the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1, that $\kappa(\overline{\mathfrak q})$ is a finite extension of $\kappa(\mathfrak p)$. Since $\kappa(s) = \kappa(\mathfrak p)$ and $\kappa(x) = \kappa(\mathfrak q) = \kappa(\overline{\mathfrak q})$ we win. □

053M Lemma 29.20.4. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $g : S' \to S$ be any morphism. Denote $f' : X' \to S'$ the base change. If $x' \in X'$ maps to a point $x \in X$ which is closed in $X_{f(x)}$ then x' is closed in $X'_{f'(x')}$.

Proof. The residue field $\kappa(x')$ is a quotient of $\kappa(f'(x')) \otimes_{\kappa(f(x))} \kappa(x)$, see Schemes, Lemma 26.17.5. Hence it is a finite extension of $\kappa(f'(x'))$ as $\kappa(x)$ is a finite extension of $\kappa(f(x))$ by Lemma 29.20.3. Thus we see that x' is closed in its fibre by applying that lemma one more time. □

01TG Lemma 29.20.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If f is quasi-finite at x, then the residue field extension $\kappa(x)/\kappa(s)$ is finite.

Proof. This is clear from Algebra, Definition 10.122.3. □

01TH Lemma 29.20.6. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Let X_s be the fibre of f at s. Assume f is locally of finite type. The following are equivalent:

(1) The morphism f is quasi-finite at x.
(2) The point x is isolated in X_s.
(3) The point x is closed in X_s and there is no point $x' \in X_s$, $x' \neq x$ which specializes to x.
(4) For any pair of affine opens $\text{Spec}(A) = U \subset X$, $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$ corresponding to $\mathfrak q \subset A$ the ring map $R \to A$ is quasi-finite at $\mathfrak q$.

Proof. Assume f is quasi-finite at x. By assumption there exist opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$, $x \in U$ and x an isolated point of U_s. Hence $\{x\} \subset U_s$ is an open subset. Since $U_s = U \cap X_s \subset X_s$ is also open we conclude that $\{x\} \subset X_s$ is an open subset also. Thus we conclude that x is an isolated point of X_s.

Note that X_s is a Jacobson scheme by Lemma 29.16.10 (and Lemma 29.15.4). If x is isolated in X_s, i.e., $\{x\} \subset X_s$ is open, then $\{x\}$ contains a closed point (by the Jacobson property), hence x is closed in X_s. It is clear that there is no point $x' \in X_s$, distinct from x, specializing to x.

Assume that x is closed in X_s and that there is no point $x' \in X_s$, distinct from x, specializing to x. Consider a pair of affine opens $\text{Spec}(A) = U \subset X$, $\text{Spec}(R) =$

$V \subset S$ with $f(U) \subset V$ and $x \in U$. Let $\mathfrak{q} \subset A$ correspond to x and $\mathfrak{p} \subset R$ correspond to s. By Lemma 29.15.2 the ring map $R \to A$ is of finite type. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. Since $\text{Spec}(\overline{A})$ is an open subscheme of the fibre X_s we see that $\overline{q}$ is a maximal ideal of $\overline{A}$ and that there is no point of $\text{Spec}(\overline{A})$ specializing to $\overline{\mathfrak{q}}$. This implies that $\dim(\overline{A}_{\overline{q}}) = 0$. Hence by Algebra, Definition 10.122.3 we see that $R \to A$ is quasi-finite at $\mathfrak{q}$, i.e., $X \to S$ is quasi-finite at x by definition.

At this point we have shown conditions (1) – (3) are all equivalent. It is clear that (4) implies (1). And it is also clear that (2) implies (4) since if x is an isolated point of X_s then it is also an isolated point of U_s for any open U which contains it. $\square$

02NG Lemma 29.20.7. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that

(1) f is locally of finite type, and
(2) $f^{-1}(\{s\})$ is a finite set.

Then X_s is a finite discrete topological space, and f is quasi-finite at each point of X lying over s.

Proof. Suppose T is a scheme which (a) is locally of finite type over a field k, and (b) has finitely many points. Then Lemma 29.16.10 shows T is a Jacobson scheme. A finite Jacobson space is discrete, see Topology, Lemma 5.18.6. Apply this remark to the fibre X_s which is locally of finite type over $\text{Spec}(\kappa(s))$ to see the first statement. Finally, apply Lemma 29.20.6 to see the second. $\square$

06RT Lemma 29.20.8. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Then the following are equivalent

(1) f is locally quasi-finite,
(2) for every $s \in S$ the fibre X_s is a discrete topological space, and
(3) for every morphism $\text{Spec}(k) \to S$ where k is a field the base change X_k has an underlying discrete topological space.

Proof. It is immediate that (3) implies (2). Lemma 29.20.6 shows that (2) is equivalent to (1). Assume (2) and let $\text{Spec}(k) \to S$ be as in (3). Denote $s \in S$ the image of $\text{Spec}(k) \to S$. Then X_k is the base change of X_s via $\text{Spec}(k) \to \text{Spec}(\kappa(s))$. Hence every point of X_k is closed by Lemma 29.20.4. As $X_k \to \text{Spec}(k)$ is locally of finite type (by Lemma 29.15.4), we may apply Lemma 29.20.6 to conclude that every point of X_k is isolated, i.e., X_k has a discrete underlying topological space. $\square$

01TJ Lemma 29.20.9. Let $f : X \to S$ be a morphism of schemes. Then f is quasi-finite if and only if f is locally quasi-finite and quasi-compact.

Proof. Assume f is quasi-finite. It is quasi-compact by Definition 29.15.1. Let $x \in X$. We see that f is quasi-finite at x by Lemma 29.20.6. Hence f is quasi-compact and locally quasi-finite.

Assume f is quasi-compact and locally quasi-finite. Then f is of finite type. Let $x \in X$ be a point. By Lemma 29.20.6 we see that x is an isolated point of its fibre. The lemma is proved. $\square$

02NH Lemma 29.20.10. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) f is quasi-finite, and

(2) f is locally of finite type, quasi-compact, and has finite fibres.

Proof. Assume f is quasi-finite. In particular f is locally of finite type and quasi-compact (since it is of finite type). Let $s \in S$. Since every $x \in X_s$ is isolated in X_s we see that $X_s = \bigcup_{x \in X_s} \{x\}$ is an open covering. As f is quasi-compact, the fibre X_s is quasi-compact. Hence we see that X_s is finite.

Conversely, assume f is locally of finite type, quasi-compact and has finite fibres. Then it is locally quasi-finite by Lemma 29.20.7. Hence it is quasi-finite by Lemma 29.20.9. □

Recall that a ring map $R \to A$ is quasi-finite if it is of finite type and quasi-finite at all primes of A, see Algebra, Definition 10.122.3.

01TK Lemma 29.20.11. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is locally quasi-finite.
(2) For every pair of affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is quasi-finite.
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally quasi-finite.
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is quasi-finite, for all $j \in J, i \in I_j$.

Moreover, if f is locally quasi-finite then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally quasi-finite.

Proof. For a ring map $R \to A$ let us define $P(R \to A)$ to mean "$R \to A$ is quasi-finite" (see remark above lemma). We claim that P is a local property of ring maps. We check conditions (a), (b) and (c) of Definition 29.14.1. In the proof of Lemma 29.15.2 we have seen that (a), (b) and (c) hold for the property of being "of finite type". Note that, for a finite type ring map $R \to A$, the property $R \to A$ is quasi-finite at $\mathfrak{q}$ depends only on the local ring $A_\mathfrak{q}$ as an algebra over $R_\mathfrak{p}$ where $\mathfrak{p} = R \cap \mathfrak{q}$ (usual abuse of notation). Using these remarks (a), (b) and (c) of Definition 29.14.1 follow immediately. For example, suppose $R \to A$ is a ring map such that all of the ring maps $R \to A_{a_i}$ are quasi-finite for $a_1, \ldots, a_n \in A$ generating the unit ideal. We conclude that $R \to A$ is of finite type. Also, for any prime $\mathfrak{q} \subset A$ the local ring $A_\mathfrak{q}$ is isomorphic as an R-algebra to the local ring $(A_{a_i})_{\mathfrak{q}_i}$ for some i and some $\mathfrak{q}_i \subset A_{a_i}$. Hence we conclude that $R \to A$ is quasi-finite at $\mathfrak{q}$.

We conclude that Lemma 29.14.3 applies with P as in the previous paragraph. Hence it suffices to prove that f is locally quasi-finite is equivalent to f is locally of type P. Since $P(R \to A)$ is "$R \to A$ is quasi-finite" which means $R \to A$ is quasi-finite at every prime of A, this follows from Lemma 29.20.6. □

01TL Lemma 29.20.12. The composition of two morphisms which are locally quasi-finite is locally quasi-finite. The same is true for quasi-finite morphisms.

Proof. In the proof of Lemma 29.20.11 we saw that $P =$"quasi-finite" is a local property of ring maps, and that a morphism of schemes is locally quasi-finite if and only if it is locally of type P as in Definition 29.14.2. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being quasi-finite is a property of ring maps that is stable under composition, see Algebra, Lemma 10.122.7. By the above, Lemma 29.20.9 and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 26.19.4 we see that the composition of quasi-finite morphisms is quasi-finite. □

We will see later (Lemma 29.56.2) that the set U of the following lemma is open.

01TM Lemma 29.20.13. Let $f : X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Denote $f' : X' \to S'$ the base change of f by g and denote $g' : X' \to X$ the projection. Assume X is locally of finite type over S.

(1) Let $U \subset X$ (resp. $U' \subset X'$) be the set of points where f (resp. f') is quasi-finite. Then $U' = U \times_S S' = (g')^{-1}(U)$.
(2) The base change of a locally quasi-finite morphism is locally quasi-finite.
(3) The base change of a quasi-finite morphism is quasi-finite.

Proof. The first and second assertion follow from the corresponding algebra result, see Algebra, Lemma 10.122.8 (combined with the fact that f' is also locally of finite type by Lemma 29.15.4). By the above, Lemma 29.20.9 and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 26.19.3 we see that the base change of a quasi-finite morphism is quasi-finite. □

0AAY Lemma 29.20.14. Let $f : X \to S$ be a morphism of schemes of finite type. Let $s \in S$. There are at most finitely many points of X lying over s at which f is quasi-finite.

Proof. The fibre X_s is a scheme of finite type over a field, hence Noetherian (Lemma 29.15.6). Hence the topology on X_s is Noetherian (Properties, Lemma 28.5.5) and can have at most a finite number of isolated points (by elementary topology). Thus our lemma follows from Lemma 29.20.6. □

0CT8 Lemma 29.20.15. Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite type and a monomorphism, then f is separated and locally quasi-finite.

Proof. A monomorphism is separated by Schemes, Lemma 26.23.3. A monomorphism is injective, hence we get f is quasi-finite at every $x \in X$ for example by Lemma 29.20.6. □

01TN Lemma 29.20.16. Any immersion is locally quasi-finite.

Proof. This is true because an open immersion is a local isomorphism and a closed immersion is clearly quasi-finite. □

03WR Lemma 29.20.17. Let $X \to Y$ be a morphism of schemes over a base scheme S. Let $x \in X$. If $X \to S$ is quasi-finite at x, then $X \to Y$ is quasi-finite at x. If X is locally quasi-finite over S, then $X \to Y$ is locally quasi-finite.

Proof. Via Lemma 29.20.11 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is quasi-finite, then $B \to C$ is quasi-finite. This follows from Algebra, Lemma 10.122.6 with $R = A$, $S = S' = C$ and $R' = B$. □

0GWS Lemma 29.20.18. Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If f is surjective, $g \circ f$ locally quasi-finite, and g locally of finite type, then $g : Y \to S$ is locally quasi-finite.

Proof. Let $x \in X$ with images $y \in Y$ and $s \in S$. Since $g \circ f$ is locally quasi-finite by Lemma 29.20.5 the extension $\kappa(x)/\kappa(s)$ is finite. Hence $\kappa(y)/\kappa(s)$ is finite. Hence y is a closed point of Y_s by Lemma 29.20.2. Since f is surjective, we see that every point of Y is closed in its fibre over S. Thus by Lemma 29.20.6 we conclude that g is quasi-finite at every point. $\square$

29.21. Morphisms of finite presentation

01TO Recall that a ring map $R \to A$ is of finite presentation if A is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ as an R-algebra for some n, m and some polynomials f_j, see Algebra, Definition 10.6.1.

01TP Definition 29.21.1. Let $f : X \to S$ be a morphism of schemes.

1. We say that f is of finite presentation at $x \in X$ if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite presentation.
2. We say that f is locally of finite presentation if it is of finite presentation at every point of X.
3. We say that f is of finite presentation if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is not just a quasi-compact morphism which is locally of finite presentation. Later we will characterize morphisms which are locally of finite presentation as those morphisms such that

$$\mathrm{colim}\, \mathrm{Mor}_S(T_i, X) = \mathrm{Mor}_S(\lim T_i, X)$$

for any directed system of affine schemes T_i over S. See Limits, Proposition 32.6.1. In Limits, Section 32.10 we show that, if $S = \lim_i S_i$ is a limit of affine schemes, any scheme X of finite presentation over S descends to a scheme X_i over S_i for some i.

01TQ Lemma 29.21.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

1. The morphism f is locally of finite presentation.
2. For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation.
3. There exist an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite presentation.
4. There exist an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite presentation, for all $j \in J, i \in I_j$.

Moreover, if f is locally of finite presentation then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite presentation.

Proof. This follows from Lemma 29.14.4 if we show that the property "$R \to A$ is of finite presentation" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. By Algebra, Lemma 10.14.2 being of finite presentation is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.6.2 being of finite presentation is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite presentation. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.23.3. □

01TR Lemma 29.21.3. The composition of two morphisms which are locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.

Proof. In the proof of Lemma 29.21.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under composition, see Algebra, Lemma 10.6.2. By the above and the fact that compositions of quasi-compact, quasi-separated morphisms are quasi-compact and quasi-separated, see Schemes, Lemmas 26.19.4 and 26.21.12 we see that the composition of morphisms of finite presentation is of finite presentation. □

01TS Lemma 29.21.4. The base change of a morphism which is locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.

Proof. In the proof of Lemma 29.21.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under base change, see Algebra, Lemma 10.14.2. By the above and the fact that a base change of a quasi-compact, quasi-separated morphism is quasi-compact and quasi-separated, see Schemes, Lemmas 26.19.3 and 26.21.12 we see that the base change of a morphism of finite presentation is a morphism of finite presentation. □

01TT Lemma 29.21.5. Any open immersion is locally of finite presentation.

Proof. This is true because an open immersion is a local isomorphism. □

01TU Lemma 29.21.6. Any open immersion is of finite presentation if and only if it is quasi-compact.

Proof. We have seen (Lemma 29.21.5) that an open immersion is locally of finite presentation. We have seen (Schemes, Lemma 26.23.8) that an immersion is separated and hence quasi-separated. From this and Definition 29.21.1 the lemma follows. □

01TV Lemma 29.21.7. A closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \text{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).

Proof. On any affine open $\text{Spec}(R) \subset X$ we have $i^{-1}(\text{Spec}(R)) = \text{Spec}(R/I)$ and $\mathcal{I} = \widetilde{I}$. Moreover, $\mathcal{I}$ is of finite type if and only if I is a finite R-module for every such affine open (see Properties, Lemma 28.16.1). And R/I is of finite presentation over R if and only if I is a finite R-module. Hence we win. □

01TW Lemma 29.21.8. A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.

Proof. Omitted. □

01TX Lemma 29.21.9. Let $f : X \to S$ be a morphism.

(1) If S is locally Noetherian and f locally of finite type then f is locally of finite presentation.
(2) If S is locally Noetherian and f of finite type then f is of finite presentation.

Proof. The first statement follows from the fact that a ring of finite type over a Noetherian ring is of finite presentation, see Algebra, Lemma 10.31.4. Suppose that f is of finite type and S is locally Noetherian. Then f is quasi-compact and locally of finite presentation by (1). Hence it suffices to prove that f is quasi-separated. This follows from Lemma 29.15.7 (and Lemma 29.21.8). $\square$

01TY Lemma 29.21.10. Let S be a scheme which is quasi-compact and quasi-separated. If X is of finite presentation over S, then X is quasi-compact and quasi-separated.

Proof. Omitted. $\square$

02FV Lemma 29.21.11. Let $f : X \to Y$ be a morphism of schemes over S.

(1) If X is locally of finite presentation over S and Y is locally of finite type over S, then f is locally of finite presentation.
(2) If X is of finite presentation over S and Y is quasi-separated and locally of finite type over S, then f is of finite presentation.

Proof. Proof of (1). Via Lemma 29.21.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite presentation and $A \to B$ is of finite type, then $B \to C$ is of finite presentation. See Algebra, Lemma 10.6.2.

Part (2) follows from (1) and Schemes, Lemmas 26.21.13 and 26.21.14. $\square$

0818 Lemma 29.21.12. Let $f : X \to Y$ be a morphism of schemes with diagonal $\Delta : X \to X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.

Proof. Note that Δ is a morphism of schemes over X (via the second projection $X \times_Y X \to X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is locally of finite type over X (by Lemma 29.15.4). Thus the first statement holds by Lemma 29.21.11. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is a monomorphism, hence separated (Schemes, Lemma 26.23.3). $\square$

29.22. Constructible sets

054H Constructible and locally constructible sets of schemes have been discussed in Properties, Section 28.2. In this section we prove some results concerning images and inverse images of (locally) constructible sets. The main result is Chevalley's theorem which states that the image of a locally constructible set under a morphism of finite presentation is locally constructible.

054I Lemma 29.22.1. Let $f : X \to Y$ be a morphism of schemes. Let $E \subset Y$ be a subset. If E is (locally) constructible in Y, then $f^{-1}(E)$ is (locally) constructible in X.

Proof. To show that the inverse image of every constructible subset is constructible it suffices to show that the inverse image of every retrocompact open V of Y is retrocompact in X, see Topology, Lemma 5.15.3. The significance of V being retrocompact in Y is just that the open immersion $V \to Y$ is quasi-compact. Hence the base change $f^{-1}(V) = X \times_Y V \to X$ is quasi-compact too, see Schemes, Lemma 26.19.3. Hence we see $f^{-1}(V)$ is retrocompact in X. Suppose E is locally constructible in Y. Choose $x \in X$. Choose an affine neighbourhood V of $f(x)$ and an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. Thus we think of $f|_U : U \to V$ as a morphism into V. By Properties, Lemma 28.2.1 we see that $E \cap V$ is constructible in V. By the constructible case we see that $(f|_U)^{-1}(E \cap V)$ is constructible in U. Since $(f|_U)^{-1}(E \cap V) = f^{-1}(E) \cap U$ we win. ☐

054J Lemma 29.22.2. Let $f : X \to Y$ be a morphism of schemes. Assume

(1) f is quasi-compact and locally of finite presentation, and
(2) Y is quasi-compact and quasi-separated.

Then the image of every constructible subset of X is constructible in Y.

Proof. By Properties, Lemma 28.2.5 it suffices to prove this lemma in case Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is constructible, then each $E \cap U_i$ is constructible too, see Topology, Lemma 5.15.4. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 10.29.10. ☐

054K Theorem 29.22.3 (Chevalley's Theorem). Let $f : X \to Y$ be a morphism of schemes. Assume f is quasi-compact and locally of finite presentation. Then the image of every locally constructible subset is locally constructible. [DG67, IV, Theorem 1.8.4]

Proof. Let $E \subset X$ be locally constructible. We have to show that $f(E)$ is locally constructible too. We will show that $f(E) \cap V$ is constructible for any affine open $V \subset Y$. Thus we reduce to the case where Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is locally constructible, then each $E \cap U_i$ is constructible, see Properties, Lemma 28.2.1. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 10.29.10. ☐

05LW Lemma 29.22.4. Let X be a scheme. Let $x \in X$. Let $E \subset X$ be a locally constructible subset. If $\{x' \mid x' \rightsquigarrow x\} \subset E$, then E contains an open neighbourhood of x.

Proof. Assume $\{x' \mid x' \rightsquigarrow x\} \subset E$. We may assume X is affine. In this case E is constructible, see Properties, Lemma 28.2.1. In particular, also the complement E^c is constructible. By Algebra, Lemma 10.29.4 we can find a morphism of affine schemes $f : Y \to X$ such that $E^c = f(Y)$. Let $Z \subset X$ be the scheme theoretic image of f. By Lemma 29.6.5 and the assumption $\{x' \mid x' \rightsquigarrow x\} \subset E$ we see that $x \notin Z$. Hence $X \setminus Z \subset E$ is an open neighbourhood of x contained in E. ☐

29.23. Open morphisms

01TZ

01U0 Definition 29.23.1. Let $f : X \to S$ be a morphism.

(1) We say f is open if the map on underlying topological spaces is open.
(2) We say f is universally open if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is open.

According to Topology, Lemma 5.19.7 generalizations lift along certain types of open maps of topological spaces. In fact generalizations lift along any open morphism of schemes (see Lemma 29.23.5). Also, we will see that generalizations lift along flat morphisms of schemes (Lemma 29.25.9). This sometimes in turn implies that the morphism is open.

01U1 Lemma 29.23.2. Let $f : X \to S$ be a morphism.

(1) If f is locally of finite presentation and generalizations lift along f, then f is open.
(2) If f is locally of finite presentation and generalizations lift along every base change of f, then f is universally open.

Proof. It suffices to prove the first assertion. This reduces to the case where both X and S are affine. In this case the result follows from Algebra, Lemma 10.41.3 and Proposition 10.41.8. $\square$

See also Lemma 29.25.10 for the case of a morphism flat of finite presentation.

02V2 Lemma 29.23.3. A composition of (universally) open morphisms is (universally) open.

Proof. Omitted. $\square$

0383 Lemma 29.23.4. Let k be a field. Let X be a scheme over k. The structure morphism $X \to \mathrm{Spec}(k)$ is universally open.

Proof. Let $S \to \mathrm{Spec}(k)$ be a morphism. We have to show that the base change $X_S \to S$ is open. The question is local on S and X, hence we may assume that S and X are affine. In this case the result is Algebra, Lemma 10.41.10. $\square$

040F Lemma 29.23.5. Let $\varphi : X \to Y$ be a morphism of schemes. If φ is open, then φ is generizing (i.e., generalizations lift along φ). If φ is universally open, then φ is universally generizing.

Follows from the implication (a) $\Rightarrow$ (b) in [DG67, IV, Corollary 1.10.4]

Proof. Assume φ is open. Let $y' \rightsquigarrow y$ be a specialization of points of Y. Let $x \in X$ with $\varphi(x) = y$. Choose affine opens $U \subset X$ and $V \subset Y$ such that $\varphi(U) \subset V$ and $x \in U$. Then also $y' \in V$. Hence we may replace X by U and Y by V and assume X, Y affine. The affine case is Algebra, Lemma 10.41.2 (combined with Algebra, Lemma 10.41.3). $\square$

04ZE Lemma 29.23.6. Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be open and surjective such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.

Proof. Let $V \subset Y$ be a quasi-compact open. As g is open and surjective we can find a quasi-compact open $W' \subset W$ such that $g(W') = V$. By assumption $(f')^{-1}(W')$ is quasi-compact. The image of $(f')^{-1}(W')$ in X is equal to $f^{-1}(V)$, see Lemma 29.9.3. Hence $f^{-1}(V)$ is quasi-compact as the image of a quasi-compact space, see Topology, Lemma 5.12.7. Thus f is quasi-compact. $\square$

29.24. Submersive morphisms

040G

040H Definition 29.24.1. Let $f : X \to Y$ be a morphism of schemes.

(1) We say f is submersive[7] if the continuous map of underlying topological spaces is submersive, see Topology, Definition 5.6.3.
(2) We say f is universally submersive if for every morphism of schemes $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

0CES Lemma 29.24.2. The base change of a universally submersive morphism of schemes by any morphism of schemes is universally submersive.

Proof. This is immediate from the definition. □

0CET Lemma 29.24.3. The composition of a pair of (universally) submersive morphisms of schemes is (universally) submersive.

Proof. Omitted. □

29.25. Flat morphisms

01U2 Flatness is one of the most important technical tools in algebraic geometry. In this section we introduce this notion. We intentionally limit the discussion to straightforward observations, apart from Lemma 29.25.10. A very important class of results, namely criteria for flatness, are discussed in Algebra, Sections 10.99, 10.101, 10.128, and More on Morphisms, Section 37.16. There is a chapter dedicated to advanced material on flat morphisms of schemes, namely More on Flatness, Section 38.1.

Recall that a module M over a ring R is flat if the functor $-\otimes_R M : \text{Mod}_R \to \text{Mod}_R$ is exact. A ring map $R \to A$ is said to be flat if A is flat as an R-module. See Algebra, Definition 10.39.1.

01U3 Definition 29.25.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

(1) We say f is flat at a point $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.
(2) We say that $\mathcal{F}$ is flat over S at a point $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module.
(3) We say f is flat if f is flat at every point of X.
(4) We say that $\mathcal{F}$ is flat over S if $\mathcal{F}$ is flat over S at every point x of X.

Thus we see that f is flat if and only if the structure sheaf $\mathcal{O}_X$ is flat over S.

01U4 Lemma 29.25.2. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The following are equivalent

(1) The sheaf $\mathcal{F}$ is flat over S.
(2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_S(V)$-module $\mathcal{F}(U)$ is flat.

[7] This is very different from the notion of a submersion of differential manifolds.

(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the modules $\mathcal{F}|_{U_i}$ is flat over V_j, for all $j \in J, i \in I_j$.
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{F}(U_i)$ is a flat $\mathcal{O}_S(V_j)$-module, for all $j \in J, i \in I_j$.

Moreover, if $\mathcal{F}$ is flat over S then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is flat over V.

Proof. Let $R \to A$ be a ring map. Let M be an A-module. If M is R-flat, then for all primes $\mathfrak{q}$ the module $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$. Conversely, if $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ for all primes $\mathfrak{q}$ of A, then M is flat over R. See Algebra, Lemma 10.39.18. This equivalence easily implies the statements of the lemma. □

01U5 Lemma 29.25.3. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is flat.
(2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is flat.
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is flat.
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is flat, for all $j \in J, i \in I_j$.

Moreover, if f is flat then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is flat.

Proof. This is a special case of Lemma 29.25.2 above. □

0FLM Lemma 29.25.4. Let $f : X \to Y$ be an affine morphism of schemes over a base scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat over S if and only if $f_*\mathcal{F}$ is flat over S.

Proof. By Lemma 29.25.2 and the fact that f is an affine morphism, this reduces us to the affine case. Say $X \to Y \to S$ corresponds to the ring maps $C \leftarrow B \leftarrow A$. Let N be the C-module corresponding to $\mathcal{F}$. Recall that $f_*\mathcal{F}$ corresponds to N viewed as a B-module, see Schemes, Lemma 26.7.3. Thus the result is clear. □

01U6 Lemma 29.25.5. Let $X \to Y \to Z$ be morphisms of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image y in Y. If $\mathcal{F}$ is flat over Y at x, and Y is flat over Z at y, then $\mathcal{F}$ is flat over Z at x.

Proof. See Algebra, Lemma 10.39.4. □

01U7 Lemma 29.25.6. The composition of flat morphisms is flat.

Proof. This is a special case of Lemma 29.25.5. □

01U8 Lemma 29.25.7. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Let $g : S' \to S$ be a morphism of schemes. Denote $g' : X' = X_{S'} \to X$ the projection. Let $x' \in X'$ be a point with image $x = g'(x') \in$

X. If $\mathcal{F}$ is flat over S at x, then $(g')^*\mathcal{F}$ is flat over S' at x'. In particular, if $\mathcal{F}$ is flat over S, then $(g')^*\mathcal{F}$ is flat over S'.

Proof. See Algebra, Lemma 10.39.7. □

01U9 Lemma 29.25.8. The base change of a flat morphism is flat.

Proof. This is a special case of Lemma 29.25.7. □

03HV Lemma 29.25.9. Let $f : X \to S$ be a flat morphism of schemes. Then generalizations lift along f, see Topology, Definition 5.19.4.

Proof. See Algebra, Section 10.41. □

01UA Lemma 29.25.10. A flat morphism locally of finite presentation is universally open.

Proof. This follows from Lemmas 29.25.9 and Lemma 29.23.2 above. We can also argue directly as follows.

Let $f : X \to S$ be flat and locally of finite presentation. By Lemmas 29.25.8 and 29.21.4 any base change of f is flat and locally of finite presentation. Hence it suffices to show f is open. To show f is open it suffices to show that we may cover X by open affines $X = \bigcup U_i$ such that $U_i \to S$ is open. We may cover X by affine opens $U_i \subset X$ such that each U_i maps into an affine open $V_i \subset S$ and such that the induced ring map $\mathcal{O}_S(V_i) \to \mathcal{O}_X(U_i)$ is flat and of finite presentation (Lemmas 29.25.3 and 29.21.2). Then $U_i \to V_i$ is open by Algebra, Proposition 10.41.8 and the proof is complete. □

0CVT Lemma 29.25.11. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f locally finite presentation, $\mathcal{F}$ of finite type, $X = \text{Supp}(\mathcal{F})$, and $\mathcal{F}$ flat over Y. Then f is universally open.

Proof. By Lemmas 29.25.7, 29.21.4, and 29.5.3 the assumptions are preserved under base change. By Lemma 29.23.2 it suffices to show that generalizations lift along f. This follows from Algebra, Lemma 10.41.12. □

02JY Lemma 29.25.12. Let $f : X \to Y$ be a quasi-compact, surjective, flat morphism. A subset $T \subset Y$ is open (resp. closed) if and only $f^{-1}(T)$ is open (resp. closed). In other words, f is a submersive morphism. [Gro71, Expose VIII, Corollaire 4.3] and [DG67, IV, Corollaire 2.3.12]

Proof. The question is local on Y, hence we may assume that Y is affine. In this case X is quasi-compact as f is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ as a finite union of affine opens. Then $f' : X' = X_1 \amalg \ldots \amalg X_n \to Y$ is a surjective flat morphism of affine schemes. Note that for $T \subset Y$ we have $(f')^{-1}(T) = f^{-1}(T) \cap X_1 \amalg \ldots \amalg f^{-1}(T) \cap X_n$. Hence, $f^{-1}(T)$ is open if and only if $(f')^{-1}(T)$ is open. Thus we may assume both X and Y are affine.

Let $f : \text{Spec}(B) \to \text{Spec}(A)$ be a surjective morphism of affine schemes corresponding to a flat ring map $A \to B$. Suppose that $f^{-1}(T)$ is closed, say $f^{-1}(T) = V(J)$ for $J \subset B$ an ideal. Then $T = f(f^{-1}(T)) = f(V(J))$ is the image of $\text{Spec}(B/J) \to \text{Spec}(A)$ (here we use that f is surjective). On the other hand, generalizations lift along f (Lemma 29.25.9). Hence by Topology, Lemma 5.19.6 we see that $Y \setminus T = f(X \setminus f^{-1}(T))$ is stable under generalization. Hence T is stable under specialization (Topology, Lemma 5.19.2). Thus T is closed by Algebra, Lemma 10.41.5. □

02JZ **Lemma 29.25.13.** Let $h : X \to Y$ be a morphism of schemes over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in X$ with $y = h(x) \in Y$. If h is flat at x, then

$$\mathcal{G} \text{ flat over } S \text{ at } y \Leftrightarrow h^*\mathcal{G} \text{ flat over } S \text{ at } x.$$

In particular: If h is surjective and flat, then $\mathcal{G}$ is flat over S, if and only if $h^*\mathcal{G}$ is flat over S. If h is surjective and flat, and X is flat over S, then Y is flat over S.

Proof. You can prove this by applying Algebra, Lemma 10.39.9. Here is a direct proof. Let $s \in S$ be the image of y. Consider the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. By assumption the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is faithfully flat, see Algebra, Lemma 10.39.17. Let $N = \mathcal{G}_y$. Note that $h^*\mathcal{G}_x = N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$, see Sheaves, Lemma 6.26.4. Let $M' \to M$ be an injection of $\mathcal{O}_{S,s}$-modules. By the faithful flatness mentioned above we have

$$\begin{aligned} & \mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \to M \otimes_{\mathcal{O}_{S,s}} N) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \\ = & \mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to M \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}) \end{aligned}$$

Hence the equivalence of the lemma follows from the second characterization of flatness in Algebra, Lemma 10.39.5. $\square$

07T9 **Lemma 29.25.14.** Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.

Proof. Using the characterization of scheme theoretic support on affines as given in Lemma 29.5.4 we reduce to Algebra, Lemma 10.40.4. $\square$

081H **Lemma 29.25.15.** Let $f : X \to Y$ be a flat morphism of schemes. Let $V \subset Y$ be a retrocompact open which is scheme theoretically dense. Then $f^{-1}V$ is scheme theoretically dense in X.

Proof. We will use the characterization of Lemma 29.7.5. We have to show that for any open $U \subset X$ the map $\mathcal{O}_X(U) \to \mathcal{O}_X(U \cap f^{-1}V)$ is injective. It suffices to prove this when U is an affine open which maps into an affine open $W \subset Y$. Say $W = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(B)$. Then $V \cap W = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 10.29.1. Thus we have to show that $B \to B_{f_1} \times \ldots \times B_{f_n}$ is injective. We are given that $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective and that $A \to B$ is flat. Since $B_{f_i} = A_{f_i} \otimes_A B$ we win. $\square$

081I **Lemma 29.25.16.** Let $f : X \to Y$ be a flat morphism of schemes. Let $g : V \to Y$ be a quasi-compact morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \to X$. Then $Z' = f^{-1}Z$.

Proof. Recall that Z is cut out by $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to g_*\mathcal{O}_V)$ and Z' is cut out by $\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_X \to (V \times_Y X \to X)_*\mathcal{O}_{V \times_Y X})$, see Lemma 29.6.3. Hence the question is local on X and Y and we may assume X and Y affine. Note that we may replace V by $\coprod V_i$ where $V = V_1 \cup \ldots \cup V_n$ is a finite affine open covering. Hence we may assume g is affine. In this case $(V \times_Y X \to X)_*\mathcal{O}_{V \times_Y X}$ is the pullback of $g_*\mathcal{O}_V$ by f. Since f is flat we conclude that $f^*\mathcal{I} = \mathcal{I}'$ and the lemma holds. $\square$

29.26. Flat closed immersions

04PV Connected components of schemes are not always open. But they do always have a canonical scheme structure. We explain this in this section.

04PW Lemma 29.26.1. Let X be a scheme. The rule which associates to a closed subscheme of X its underlying closed subset defines a bijection

$$\left\{\begin{matrix}\text{closed subschemes } Z \subset X\\ \text{such that } Z \to X \text{ is flat}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{closed subsets } Z \subset X\\ \text{closed under generalizations}\end{matrix}\right\}$$

If $Z \subset X$ is such a closed subscheme, every morphism of schemes $g : Y \to X$ with $g(Y) \subset Z$ set theoretically factors (scheme theoretically) through Z.

Proof. The affine case of the bijection is Algebra, Lemma 10.108.4. For general schemes X the bijection follows by covering X by affines and glueing. Details omitted. For the final assertion, observe that the projection $Z \times_{X,g} Y \to Y$ is a flat (Lemma 29.25.8) closed immersion which is bijective on underlying topological spaces and hence must be an isomorphism by the bijection esthablished in the first part of the proof. $\square$

0819 Lemma 29.26.2. A flat closed immersion of finite presentation is the open immersion of an open and closed subscheme.

Proof. The affine case is Algebra, Lemma 10.108.5. In general the lemma follows by covering X by affines. Details omitted. $\square$

Note that a connected component T of a scheme X is a closed subset stable under generalization. Hence the following definition makes sense.

04PX Definition 29.26.3. Let X be a scheme. Let $T \subset X$ be a connected component. The canonical scheme structure on T is the unique scheme structure on T such that the closed immersion $T \to X$ is flat, see Lemma 29.26.1.

It turns out that we can determine when every finite flat $\mathcal{O}_X$-module is finite locally free using the previous lemma.

053N Lemma 29.26.4. Let X be a scheme. The following are equivalent

(1) every finite flat quasi-coherent $\mathcal{O}_X$-module is finite locally free, and
(2) every closed subset $Z \subset X$ which is closed under generalizations is open.

Proof. In the affine case this is Algebra, Lemma 10.108.6. The scheme case does not follow directly from the affine case, so we simply repeat the arguments.

Assume (1). Consider a closed immersion $i : Z \to X$ such that i is flat. Then $i_*\mathcal{O}_Z$ is quasi-coherent and flat, hence finite locally free by (1). Thus $Z = \text{Supp}(i_*\mathcal{O}_Z)$ is also open and we see that (2) holds. Hence the implication (1) $\Rightarrow$ (2) follows from the characterization of flat closed immersions in Lemma 29.26.1.

For the converse assume that X satisfies (2). Let $\mathcal{F}$ be a finite flat quasi-coherent $\mathcal{O}_X$-module. The support $Z = \text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is closed, see Modules, Lemma 17.9.6. On the other hand, if $x \rightsquigarrow x'$ is a specialization, then by Algebra, Lemma 10.78.5 the module $\mathcal{F}_{x'}$ is free over $\mathcal{O}_{X,x'}$, and

$$\mathcal{F}_x = \mathcal{F}_{x'} \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{X,x}.$$

Hence $x' \in \text{Supp}(\mathcal{F}) \Rightarrow x \in \text{Supp}(\mathcal{F})$, in other words, the support is closed under generalization. As X satisfies (2) we see that the support of $\mathcal{F}$ is open and closed.

The modules $\wedge^i(\mathcal{F})$, $i = 1, 2, 3, \ldots$ are finite flat quasi-coherent $\mathcal{O}_X$-modules also, see Modules, Section 17.21. Note that $\text{Supp}(\wedge^{i+1}(\mathcal{F})) \subset \text{Supp}(\wedge^i(\mathcal{F}))$. Thus we see that there exists a decomposition

$$X = U_0 \amalg U_1 \amalg U_2 \amalg \ldots$$

by open and closed subsets such that the support of $\wedge^i(\mathcal{F})$ is $U_i \cup U_{i+1} \cup \ldots$ for all i. Let x be a point of X, and say $x \in U_r$. Note that $\wedge^i(\mathcal{F})_x \otimes \kappa(x) = \wedge^i(\mathcal{F}_x \otimes \kappa(x))$. Hence, $x \in U_r$ implies that $\mathcal{F}_x \otimes \kappa(x)$ is a vector space of dimension r. By Nakayama's lemma, see Algebra, Lemma 10.20.1 we can choose an affine open neighbourhood $U \subset U_r \subset X$ of x and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ such that the induced map

$$\mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U, \quad (f_1, \ldots, f_r) \longmapsto \sum f_i s_i$$

is surjective. This means that $\wedge^r(\mathcal{F}|_U)$ is a finite flat quasi-coherent $\mathcal{O}_U$-module whose support is all of U. By the above it is generated by a single element, namely $s_1 \wedge \ldots \wedge s_r$. Hence $\wedge^r(\mathcal{F}|_U) \cong \mathcal{O}_U/\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I}$ such that $\mathcal{O}_U/\mathcal{I}$ is flat over $\mathcal{O}_U$ and such that $V(\mathcal{I}) = U$. It follows that $\mathcal{I} = 0$ by applying Lemma 29.26.1. Thus $s_1 \wedge \ldots \wedge s_r$ is a basis for $\wedge^r(\mathcal{F}|_U)$ and it follows that the displayed map is injective as well as surjective. This proves that $\mathcal{F}$ is finite locally free as desired. □

29.27. Generic flatness

0529 A scheme of finite type over an integral base is flat over a dense open of the base. In Algebra, Section 10.118 we proved a Noetherian version, a version for morphisms of finite presentation, and a general version. We only state and prove the general version here. However, it turns out that this will be superseded by Proposition 29.27.2 which shows the result holds if we only assume the base is reduced.

052A **Proposition 29.27.1** (Generic flatness). Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume

(1) S is integral,
(2) f is of finite type, and
(3) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. As S is integral it is irreducible (see Properties, Lemma 28.3.4) and any nonempty open is dense. Hence we may replace S by an affine open of S and assume that $S = \text{Spec}(A)$ is affine. As S is integral we see that A is a domain. As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = \text{Spec}(B_i)$. Then B_i is a finite type A-algebra, see Lemma 29.15.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 28.16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \text{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \text{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 10.118.3 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain nonzero $f_{ij}, f_i \in A$ such that (a)

$A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (b) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. We claim that taking $U = D(f)$ works.

To prove our claim we may replace A by A_f, i.e., perform the base change by $U = \mathrm{Spec}(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 29.21.2 and Properties, Lemma 28.16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \mathrm{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 10.40.8. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 26.21.6. This proves the proposition. □

It actually turns out that there is also a version of generic flatness over an arbitrary reduced base. Here it is.

052B Proposition 29.27.2 (Generic flatness, reduced case). Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume

(1) S is reduced,
(2) f is of finite type, and
(3) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. For the impatient reader: This proof is a repeat of the proof of Proposition 29.27.1 using Algebra, Lemma 10.118.7 instead of Algebra, Lemma 10.118.3.

Since being flat and being of finite presentation is local on the base, see Lemmas 29.25.2 and 29.21.2, we may work affine locally on S. Thus we may assume that $S = \mathrm{Spec}(A)$, where A is a reduced ring (see Properties, Lemma 28.3.2). As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = \mathrm{Spec}(B_i)$. Then B_i is a finite type A-algebra, see Lemma 29.15.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 28.16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \mathrm{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \mathrm{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 10.118.7 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain dense opens $U(A \to B_i, M_{ij}) \subset S$ and dense opens $U(A \to B_i, M_i) \subset S$ with notation as in Algebra, Equation (10.118.3.2). Since a finite intersection of dense opens is dense open, we see that

$$U = \bigcap\nolimits_{i,j} U(A \to B_i, M_{ij}) \quad \cap \quad \bigcap\nolimits_{i} U(A \to B_i, M_i)$$

is open and dense in S. We claim that U is the desired open.

Pick $u \in U$. By definition of the loci $U(A \to B_i, M_{ij})$ and $U(A \to B, M_i)$ there exist $f_{ij}, f_i \in A$ such that (a) $u \in D(f_i)$ and $u \in D(f_{ij})$, (b) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (c) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. Now it suffices to prove that $X \to S$ is flat and of finite presentation over $D(f)$ and that $\mathcal{F}$ restricted to $X_{D(f)}$ is flat over $D(f)$ and of finite presentation over the structure sheaf of $X_{D(f)}$.

Hence we may replace A by A_f, i.e., perform the base change by $\text{Spec}(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 29.21.2 and Properties, Lemma 28.16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \text{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 10.40.8. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 26.21.6. This proves the proposition. □

052C Remark 29.27.3. The results above are a first step towards more refined flattening techniques for morphisms of schemes. The article [GR71] by Raynaud and Gruson contains many wonderful results in this direction.

29.28. Morphisms and dimensions of fibres

02FW Let X be a topological space, and $x \in X$. Recall that we have defined $\dim_x(X)$ as the minimum of the dimensions of the open neighbourhoods of x in X. See Topology, Definition 5.10.1.

02FX Lemma 29.28.1. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ and set $s = f(x)$. Assume f is locally of finite type. Then

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)).$$

Proof. This immediately reduces to the case $S = s$, and X affine. In this case the result follows from Algebra, Lemma 10.116.3. □

02JS Lemma 29.28.2. Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. Let $x \in X$ and set $y = f(x)$, $s = g(y)$. Assume f and g locally of finite type. Then

$$\dim_x(X_s) \leq \dim_x(X_y) + \dim_y(Y_s).$$

Moreover, equality holds if $\mathcal{O}_{X_s,x}$ is flat over $\mathcal{O}_{Y_s,y}$, which holds for example if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$.

Proof. Note that $\text{trdeg}_{\kappa(s)}(\kappa(x)) = \text{trdeg}_{\kappa(y)}(\kappa(x)) + \text{trdeg}_{\kappa(s)}(\kappa(y))$. Thus by Lemma 29.28.1 the statement is equivalent to

$$\dim(\mathcal{O}_{X_s,x}) \leq \dim(\mathcal{O}_{X_y,x}) + \dim(\mathcal{O}_{Y_s,y}).$$

For this see Algebra, Lemma 10.112.6. For the flat case see Algebra, Lemma 10.112.7. □

02FY Lemma 29.28.3. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a fibre product diagram of schemes. Assume f locally of finite type. Suppose that $x' \in X'$, $x = g'(x')$, $s' = f'(x')$ and $s = g(s') = f(x)$. Then

(1) $\dim_x(X_s) = \dim_{x'}(X'_{s'})$,
(2) if F is the fibre of the morphism $X'_{s'} \to X_s$ over x, then

$$\dim(\mathcal{O}_{F,x'}) = \dim(\mathcal{O}_{X'_{s'},x'}) - \dim(\mathcal{O}_{X_s,x}) = \text{trdeg}_{\kappa(s)}(\kappa(x)) - \text{trdeg}_{\kappa(s')}(\kappa(x'))$$

In particular $\dim(\mathcal{O}_{X'_{s'},x'}) \geq \dim(\mathcal{O}_{X_s,x})$ and $\text{trdeg}_{\kappa(s)}(\kappa(x)) \geq \text{trdeg}_{\kappa(s')}(\kappa(x'))$.
(3) given s', s, x there exists a choice of x' such that $\dim(\mathcal{O}_{X'_{s'},x'}) = \dim(\mathcal{O}_{X_s,x})$ and $\text{trdeg}_{\kappa(s)}(\kappa(x)) = \text{trdeg}_{\kappa(s')}(\kappa(x'))$.

Proof. Part (1) follows immediately from Algebra, Lemma 10.116.6. Parts (2) and (3) from Algebra, Lemma 10.116.7. □

The following lemma follows from a nontrivial algebraic result. Namely, the algebraic version of Zariski's main theorem.

02FZ Lemma 29.28.4. Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite type. The set [DG67, IV Theorem 13.1.3]

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

is open in X.

Proof. This is immediate from Algebra, Lemma 10.125.6 □

0A3V Lemma 29.28.5. Let $f : X \to Y$ be a morphism of finite type with Y quasi-compact. Then the dimension of the fibres of f is bounded.

Proof. By Lemma 29.28.4 the set $U_n \subset X$ of points where the dimension of the fibre is $\leq n$ is open. Since f is of finite type, every point is contained in some U_n (because the dimension of a finite type algebra over a field is finite). Since Y is quasi-compact and f is of finite type, we see that X is quasi-compact. Hence $X = U_n$ for some n. □

02G0 Lemma 29.28.6. Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite presentation. The open

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

of Lemma 29.28.4 is retrocompact in X. (See Topology, Definition 5.12.1.)

Proof. The topological space X has a basis for its topology consisting of affine opens $U \subset X$ such that the induced morphism $f|_U : U \to S$ factors through an affine open $V \subset S$. Hence it is enough to show that $U \cap U_n$ is quasi-compact for such a U. Note that $U_n \cap U$ is the same as the open $\{x \in U \mid \dim_x U_{f(x)} \leq n\}$. This reduces us to the case where X and S are affine. In this case the lemma follows from Algebra, Lemma 10.125.8 (and Lemma 29.21.2). □

06RU Lemma 29.28.7. Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a nontrivial specialization of points in X lying over the same point $s \in S$. Assume f is locally of finite type. Then

(1) $\dim_x(X_s) \leq \dim_{x'}(X_s)$,
(2) $\dim(\mathcal{O}_{X_s,x}) < \dim(\mathcal{O}_{X_s,x'})$, and
(3) $\text{trdeg}_{\kappa(s)}(\kappa(x)) > \text{trdeg}_{\kappa(s)}(\kappa(x'))$.

Proof. Part (1) follows from the fact that any open of X_s containing x' also contains x. Part (2) follows since $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{X_s,x'}$ at a prime ideal, hence any chain of prime ideals in $\mathcal{O}_{X_s,x}$ is part of a strictly longer chain of primes in $\mathcal{O}_{X_s,x'}$. The last inequality follows from Algebra, Lemma 10.116.2. □

29.29. Morphisms of given relative dimension

02NI In order to be able to speak comfortably about morphisms of a given relative dimension we introduce the following notion.

02NJ Definition 29.29.1. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type.

(1) We say f is of relative dimension $\leq d$ at x if $\dim_x(X_{f(x)}) \leq d$.
(2) We say f is of relative dimension $\leq d$ if $\dim_x(X_{f(x)}) \leq d$ for all $x \in X$.
(3) We say f is of relative dimension d if all nonempty fibres X_s are equidimensional of dimension d.

This is not a particularly well behaved notion, but it works well in a number of situations.

02NK Lemma 29.29.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If f has relative dimension d, then so does any base change of f. Same for relative dimension $\leq d$.

Proof. This is immediate from Lemma 29.28.3. □

02NL Lemma 29.29.3. Let $f : X \to Y$, $g : Y \to Z$ be locally of finite type. If f has relative dimension $\leq d$ and g has relative dimension $\leq e$ then $g \circ f$ has relative dimension $\leq d + e$. If

(1) f has relative dimension d,
(2) g has relative dimension e, and
(3) f is flat,

then $g \circ f$ has relative dimension $d + e$.

Proof. This is immediate from Lemma 29.28.2. □

In general it is not possible to decompose a morphism into its pieces where the relative dimension is a given one. However, it is possible if the morphism has Cohen-Macaulay fibres and is flat of finite presentation.

02NM Lemma 29.29.4. Let $f : X \to S$ be a morphism of schemes. Assume that

(1) f is flat,
(2) f is locally of finite presentation, and
(3) for all $s \in S$ the fibre X_s is Cohen-Macaulay (Properties, Definition 28.8.1)

Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \to S$ has relative dimension d.

Proof. This is immediate from Algebra, Lemma 10.130.8. □

0397 Lemma 29.29.5. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Let $x \in X$ with $s = f(x)$. Then f is quasi-finite at x if and only if $\dim_x(X_s) = 0$. In particular, f is locally quasi-finite if and only if f has relative dimension 0.

Proof. If f is quasi-finite at x then $\kappa(x)$ is a finite extension of $\kappa(s)$ (by Lemma 29.20.5) and x is isolated in X_s (by Lemma 29.20.6), hence $\dim_x(X_s) = 0$ by Lemma 29.28.1. Conversely, if $\dim_x(X_s) = 0$ then by Lemma 29.28.1 we see $\kappa(s) \subset \kappa(x)$ is algebraic and there are no other points of X_s specializing to x. Hence x is closed in its fibre by Lemma 29.20.2 and by Lemma 29.20.6 (3) we conclude that f is quasi-finite at x. □

0AFE **Lemma 29.29.6.** Let $f : X \to Y$ be a morphism of locally Noetherian schemes which is flat, locally of finite type and of relative dimension d. For every point x in X with image y in Y we have $\dim_x(X) = \dim_y(Y) + d$.

Proof. After shrinking X and Y to open neighborhoods of x and y, we can assume that $\dim(X) = \dim_x(X)$ and $\dim(Y) = \dim_y(Y)$, by definition of the dimension of a scheme at a point (Properties, Definition 28.10.1). The morphism f is open by Lemmas 29.21.9 and 29.25.10. Hence we can shrink Y to arrange that f is surjective. It remains to show that $\dim(X) = \dim(Y) + d$.

Let a be a point in X with image b in Y. By Algebra, Lemma 10.112.7,

$$\dim(\mathcal{O}_{X,a}) = \dim(\mathcal{O}_{Y,b}) + \dim(\mathcal{O}_{X_b,a}).$$

Taking the supremum over all points a in X, it follows that $\dim(X) = \dim(Y) + d$, as we want, see Properties, Lemma 28.10.2. □

29.30. Syntomic morphisms

01UB An algebra A over a field k is called a *global complete intersection over k* if $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $\dim(A) = n - c$. An algebra A over a field k is called a *local complete intersection* if $\text{Spec}(A)$ can be covered by standard opens each of which are global complete intersections over k. See Algebra, Section 10.135. Recall that a ring map $R \to A$ is syntomic if it is of finite presentation, flat with local complete intersection rings as fibres, see Algebra, Definition 10.136.1.

01UC **Definition 29.30.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *syntomic at $x \in X$* if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is syntomic.
(2) We say that f is *syntomic* if it is syntomic at every point of X.
(3) If $S = \text{Spec}(k)$ and f is syntomic, then we say that X is a *local complete intersection over k*.
(4) A morphism of affine schemes $f : X \to S$ is called *standard syntomic* if there exists a global relative complete intersection $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 10.136.5) such that $X \to S$ is isomorphic to

$$\text{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \text{Spec}(R).$$

In the literature a syntomic morphism is sometimes referred to as a *flat local complete intersection morphism*. It turns out this is a convenient class of morphisms. For example one can define a syntomic topology using these, which is finer than the smooth and étale topologies, but has many of the same formal properties.

A global relative complete intersection (which we used to define standard syntomic ring maps) is in particular flat. In More on Morphisms, Section 37.62 we will consider morphisms $X \to S$ which locally are of the form

$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \mathrm{Spec}(R).$$

for some Koszul-regular sequence $f_1, \ldots, f_r$ in $R[x_1, \ldots, x_n]$. Such a morphism will be called a local complete intersection morphism. Once we have this definition in place it will be the case that a morphism is syntomic if and only if it is a flat, local complete intersection morphism.

Note that there is no separation or quasi-compactness hypotheses in the definition of a syntomic morphism. Hence the question of being syntomic is local in nature on the source. Here is the precise result.

01UD Lemma 29.30.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is syntomic.
(2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is syntomic.
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is syntomic.
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is syntomic, for all $j \in J, i \in I_j$.

Moreover, if f is syntomic then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is syntomic.

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is syntomic" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. By Algebra, Lemma 10.136.3 being syntomic is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.136.17 being syntomic is stable under composition and trivially for any ring R the ring map $R \to R_f$ is syntomic. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.136.4. $\square$

01UH Lemma 29.30.3. The composition of two morphisms which are syntomic is syntomic.

Proof. In the proof of Lemma 29.30.2 we saw that being syntomic is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being syntomic is a property of ring maps that is stable under composition, see Algebra, Lemma 10.136.17. $\square$

01UI Lemma 29.30.4. The base change of a morphism which is syntomic is syntomic.

Proof. In the proof of Lemma 29.30.2 we saw that being syntomic is a local property of ring maps. Hence the lemma follows from Lemma 29.14.5 combined with the fact that being syntomic is a property of ring maps that is stable under base change, see Algebra, Lemma 10.136.3. $\square$

01UJ Lemma 29.30.5. Any open immersion is syntomic.

Proof. This is true because an open immersion is a local isomorphism. $\square$

01UK Lemma 29.30.6. A syntomic morphism is locally of finite presentation.

Proof. True because a syntomic ring map is of finite presentation by definition. □

01UL Lemma 29.30.7. A syntomic morphism is flat.

Proof. True because a syntomic ring map is flat by definition. □

056F Lemma 29.30.8. A syntomic morphism is universally open.

Proof. Combine Lemmas 29.30.6, 29.30.7, and 29.25.10. □

Let k be a field. Let A be a local k-algebra essentially of finite type over k. Recall that A is called a complete intersection over k if we can write $A \cong R/(f_1, \ldots, f_c)$ where R is a regular local ring essentially of finite type over k, and $f_1, \ldots, f_c$ is a regular sequence in R, see Algebra, Definition 10.135.5.

01UG Lemma 29.30.9. Let k be a field. Let X be a scheme locally of finite type over k. The following are equivalent:

(1) X is a local complete intersection over k,
(2) for every $x \in X$ there exists an affine open $U = \mathrm{Spec}(R) \subset X$ neighbourhood of x such that $R \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a global complete intersection over k, and
(3) for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a complete intersection over k.

Proof. The corresponding algebra results can be found in Algebra, Lemmas 10.135.8 and 10.135.9. □

The following lemma says locally any syntomic morphism is standard syntomic. Hence we can use standard syntomic morphisms as a local model for a syntomic morphism. Moreover, it says that a flat morphism of finite presentation is syntomic if and only if the fibres are local complete intersection schemes.

01UE Lemma 29.30.10. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s = f(x)$. Let $V \subset S$ be an affine open neighbourhood of s. The following are equivalent

(1) The morphism f is syntomic at x.
(2) There exist an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that $f|_U : U \to V$ is standard syntomic.
(3) The morphism f is of finite presentation at x, the local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a complete intersection over $\kappa(s)$ (see Algebra, Definition 10.135.5).

Proof. Follows from the definitions and Algebra, Lemma 10.136.15. □

01UF Lemma 29.30.11. Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are local complete intersections, then f is syntomic.

Proof. Clear from Lemmas 29.30.9 and 29.30.10 and the isomorphisms of local rings $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x} \cong \mathcal{O}_{X_s,x}$. □

02V3 Lemma 29.30.12. Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set

$$T = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is a complete intersection over } \kappa(f(x))\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed flat, and locally of finite presentation then the same holds for the open set of points where f is syntomic.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 10.135.10. The second part follows from the first because in that case T is the set of points where f is syntomic according to Lemma 29.30.10. □

02K0 Lemma 29.30.13. Let R be a ring. Let $R \to A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Set $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. Consider the morphism $f : X \to S$ associated to the ring map $R \to A$. The function $x \mapsto \dim_x(X_{f(x)})$ is constant with value $n - c$.

Proof. By Algebra, Definition 10.136.5 $R \to A$ being a relative global complete intersection means all nonzero fibre rings have dimension $n - c$. Thus for a prime $\mathfrak{p}$ of R the fibre ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ is either zero or a global complete intersection ring of dimension $n - c$. By the discussion following Algebra, Definition 10.135.1 this implies it is equidimensional of dimension $n - c$. Whence the lemma. □

02K1 Lemma 29.30.14. Let $f : X \to S$ be a syntomic morphism. The function $x \mapsto \dim_x(X_{f(x)})$ is locally constant on X.

Proof. By Lemma 29.30.10 the morphism f locally looks like a standard syntomic morphism of affines. Hence the result follows from Lemma 29.30.13. □

Lemma 29.30.14 says that the following definition makes sense.

02K2 Definition 29.30.15. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is syntomic of relative dimension d if f is syntomic and the function $\dim_x(X_{f(x)}) = d$ for all $x \in X$.

In other words, f is syntomic and the nonempty fibres are equidimensional of dimension d.

02K3 Lemma 29.30.16. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) f is surjective and syntomic,
(2) p is syntomic, and
(3) q is locally of finite presentation[8].

[8]In fact, if f is surjective, flat, and locally of finite presentation and p is syntomic, then both q and f are syntomic, see Descent, Lemma 35.14.7.

Then q is syntomic.

Proof. By Lemma 29.25.13 we see that q is flat. Hence it suffices to show that the fibres of $Y \to S$ are local complete intersections, see Lemma 29.30.11. Let $s \in S$. Consider the morphism $X_s \to Y_s$. This is a base change of the morphism $X \to Y$ and hence surjective, and syntomic (Lemma 29.30.4). For the same reason X_s is syntomic over $\kappa(s)$. Moreover, Y_s is locally of finite type over $\kappa(s)$ (Lemma 29.15.4). In this way we reduce to the case where S is the spectrum of a field.

Assume $S = \mathrm{Spec}(k)$. Let $y \in Y$. Choose an affine open $\mathrm{Spec}(A) \subset Y$ neighbourhood of y. Let $\mathrm{Spec}(B) \subset X$ be an affine open such that $f(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$, containing a point $x \in X$ such that $f(x) = y$. Choose a surjection $k[x_1, \ldots, x_n] \to A$ with kernel I. Choose a surjection $A[y_1, \ldots, y_m] \to B$, which gives rise in turn to a surjection $k[x_i, y_j] \to B$ with kernel J. Let $\mathfrak q \subset k[x_i, y_j]$ be the prime corresponding to $y \in \mathrm{Spec}(B)$ and let $\mathfrak p \subset k[x_i]$ the prime corresponding to $x \in \mathrm{Spec}(A)$. Since x maps to y we have $\mathfrak p = \mathfrak q \cap k[x_i]$. Consider the following commutative diagram of local rings:

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, equal] & B_{\mathfrak q} & k[x_1, \ldots, x_n, y_1, \ldots, y_m]_{\mathfrak q} \arrow[l] \\
\mathcal{O}_{Y,y} \arrow[u] \arrow[r, equal] & A_{\mathfrak p} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak p} \arrow[l] \arrow[u]
\end{tikzcd}$$

We claim that the hypotheses of Algebra, Lemma 10.135.12 are satisfied. Conditions (1) and (2) are trivial. Condition (4) follows as $X \to Y$ is flat. Condition (3) follows as the rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak m_y \mathcal{O}_{X,x}$ are complete intersection rings by our assumptions that f and p are syntomic, see Lemma 29.30.10. The output of Algebra, Lemma 10.135.12 is exactly that $\mathcal{O}_{Y,y}$ is a complete intersection ring! Hence by Lemma 29.30.10 again we see that Y is syntomic over k at y as desired. $\square$

29.31. Conormal sheaf of an immersion

01R1 Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Lemma 29.4.1. This quasi-coherent $\mathcal{O}_Z$-module is called the conormal sheaf of Z in X and is often simply denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Section 29.4.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

01R2 Definition 29.31.1. Let $i : Z \to X$ be an immersion. The conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X or the conormal sheaf of i is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [DG67, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the normal sheaf of the immersion. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

01R3 **Lemma 29.31.2.** Let $i : Z \to X$ be an immersion. The conormal sheaf of i has the following properties:

(1) Let $U \subset X$ be any open subscheme such that i factors as $Z \xrightarrow{i'} U \to X$ where i' is a closed immersion. Let $\mathcal{I} = \text{Ker}((i')^\sharp) \subset \mathcal{O}_U$. Then

$$\mathcal{C}_{Z/X} = (i')^*\mathcal{I} \quad \text{and} \quad i'_*\mathcal{C}_{Z/X} = \mathcal{I}/\mathcal{I}^2$$

(2) For any affine open $\text{Spec}(R) = U \subset X$ such that $Z \cap U = \text{Spec}(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X}) = I/I^2$.

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I/I^2 = I \otimes_R R/I$. Details omitted. $\square$

01R4 **Lemma 29.31.3.** Let

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

characterized by the following property: For every pair of affine opens $(\text{Spec}(R) = U \subset X, \text{Spec}(R') = U' \subset X')$ with $f(U) \subset U'$ such that $Z \cap U = \text{Spec}(R/I)$ and $Z' \cap U' = \text{Spec}(R'/I')$ the induced map

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X'}) = I'/I'^2 \longrightarrow I/I^2 = \Gamma(Z \cap U, \mathcal{C}_{Z/X})$$

is the one induced by the ring map $f^\sharp : R' \to R$ which has the property $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \big(g^{-1}(\partial Z') \cup \partial Z\big)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 26.4.6 and 26.4.7. Hence we get an induced map of quasi-coherent sheaves $g^*(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back by i gives $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) \to i^*(\mathcal{I}/\mathcal{I}^2)$. Note that $i^*(\mathcal{I}/\mathcal{I}^2) = \mathcal{C}_{Z/X}$. On the other hand, $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*(i')^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*\mathcal{C}_{Z'/X'}$. This gives the desired map.

Checking that the map is locally described as the given map $I'/(I')^2 \to I/I^2$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). ☐

0473 Lemma 29.31.4. Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 29.31.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $I'/(I')^2 \otimes_{R'} R \to I/I^2$ is surjective. If $R' \to R$ is flat, then $I = I' \otimes_{R'} R$ and $I^2 = (I')^2 \otimes_{R'} R$ and we see the map is an isomorphism. ☐

062S Lemma 29.31.5. Let $Z \to Y \to X$ be immersions of schemes. Then there is a canonical exact sequence

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 29.31.3 and $i : Z \to Y$ is the first morphism.

Proof. Via Lemma 29.31.3 this translates into the following algebra fact. Suppose that $C \to B \to A$ are surjective ring maps. Let $I = \text{Ker}(B \to A)$, $J = \text{Ker}(C \to A)$ and $K = \text{Ker}(C \to B)$. Then there is an exact sequence

$$K/K^2 \otimes_B A \to J/J^2 \to I/I^2 \to 0.$$

This follows immediately from the observation that $I = J/K$. ☐

29.32. Sheaf of differentials of a morphism

01UM We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.131) and the corresponding section in the chapter on sheaves of modules (Modules, Section 17.28).

01UQ Definition 29.32.1. Let $f : X \to S$ be a morphism of schemes. The sheaf of differentials $\Omega_{X/S}$ of X over S is the sheaf of differentials of f viewed as a morphism of ringed spaces (Modules, Definition 17.28.10) equipped with its universal S-derivation

$$\text{d}_{X/S} : \mathcal{O}_X \longrightarrow \Omega_{X/S}.$$

It turns out that $\Omega_{X/S}$ is a quasi-coherent $\mathcal{O}_X$-module for example as it is isomorphic to the conormal sheaf of the diagonal morphism $\Delta : X \to X \times_S X$ (Lemma 29.32.7). We have defined the module of differentials of X over S using a universal property, namely as the receptacle of the universal derivation. If you have any other construction of the sheaf of relative differentials which satisfies this universal property then, by the Yoneda lemma, it will be canonically isomorphic to the one defined above. For convenience we restate the universal property here.

01UR Lemma 29.32.2. Let $f : X \to S$ be a morphism of schemes. The map

$$\text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \longrightarrow \text{Der}_S(\mathcal{O}_X, \mathcal{F}), \quad \alpha \longmapsto \alpha \circ \text{d}_{X/S}$$

is an isomorphism of functors $\text{Mod}(\mathcal{O}_X) \to \text{Sets}$.

Proof. This is just a restatement of the definition. □

01US Lemma 29.32.3. Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$, $V \subset S$ be open subschemes such that $f(U) \subset V$. Then there is a unique isomorphism $\Omega_{X/S}|_U = \Omega_{U/V}$ of $\mathcal{O}_U$-modules such that $\text{d}_{X/S}|_U = \text{d}_{U/V}$.

Proof. This is a special case of Modules, Lemma 17.28.5 if we use the canonical identification $f^{-1}\mathcal{O}_S|_U = (f|_U)^{-1}\mathcal{O}_V$. □

From now on we will use these canonical identifications and simply write $\Omega_{U/S}$ or $\Omega_{U/V}$ for the restriction of $\Omega_{X/S}$ to U.

01UO Lemma 29.32.4. Let $R \to A$ be a ring map. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X = \text{Spec}(A)$. Set $S = \text{Spec}(R)$. The rule which associates to an S-derivation on $\mathcal{F}$ its action on global sections defines a bijection between the set of S-derivations of $\mathcal{F}$ and the set of R-derivations on $M = \Gamma(X, \mathcal{F})$.

Proof. Let $D : A \to M$ be an R-derivation. We have to show there exists a unique S-derivation on $\mathcal{F}$ which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Any element of $\Gamma(U, \mathcal{O}_X)$ is of the form a/f^n for some $a \in A$ and $n \geq 0$. By the Leibniz rule we have

$$D(a)|_U = a/f^n D(f^n)|_U + f^n D(a/f^n)$$

in $\Gamma(U, \mathcal{F})$. Since f acts invertibly on $\Gamma(U, \mathcal{F})$ this completely determines the value of $D(a/f^n) \in \Gamma(U, \mathcal{F})$. This proves uniqueness. Existence follows by simply defining

$$D(a/f^n) := (1/f^n)D(a)|_U - a/f^{2n} D(f^n)|_U$$

and proving this has all the desired properties (on the basis of standard opens of X). Details omitted. □

01UT Lemma 29.32.5. Let $f : X \to S$ be a morphism of schemes. For any pair of affine opens $\text{Spec}(A) = U \subset X$, $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ there is a unique isomorphism

$$\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}.$$

compatible with $\text{d}_{X/S}$ and $\text{d} : A \to \Omega_{A/R}$.

Proof. By Lemma 29.32.3 we may replace X and S by U and V. Thus we may assume $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$ and we have to show the lemma with $U = X$ and $V = S$. Consider the A-module $M = \Gamma(X, \Omega_{X/S})$ together with the R-derivation $\text{d}_{X/S} : A \to M$. Let N be another A-module and denote $\widetilde{N}$ the quasi-coherent $\mathcal{O}_X$-module associated to N, see Schemes, Section 26.7. Precomposing by $\text{d}_{X/S} : A \to M$ we get an arrow

$$\alpha : \text{Hom}_A(M, N) \longrightarrow \text{Der}_R(A, N)$$

Using Lemmas 29.32.2 and 29.32.4 we get identifications

$$\text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \widetilde{N}) = \text{Der}_S(\mathcal{O}_X, \widetilde{N}) = \text{Der}_R(A, N)$$

Taking global sections determines an arrow $\text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \widetilde{N}) \to \text{Hom}_R(M, N)$. Combining this arrow and the identifications above we get an arrow

$$\beta : \text{Der}_R(A, N) \longrightarrow \text{Hom}_R(M, N)$$

Checking what happens on global sections, we find that α and β are each others inverse. Hence we see that $\text{d}_{X/S} : A \to M$ satisfies the same universal property as

$\mathrm{d} : A \to \Omega_{A/R}$, see Algebra, Lemma 10.131.3. Thus the Yoneda lemma (Categories, Lemma 4.3.5) implies there is a unique isomorphism of A-modules $M \cong \Omega_{A/R}$ compatible with derivations. □

01UU Remark 29.32.6. The lemma above gives a second way of constructing the module of differentials. Namely, let $f : X \to S$ be a morphism of schemes. Consider the collection of all affine opens $U \subset X$ which map into an affine open of S. These form a basis for the topology on X. Thus it suffices to define $\Gamma(U, \Omega_{X/S})$ for such U. We simply set $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$ if A, R are as in Lemma 29.32.5 above. This works, but it takes somewhat more algebraic preliminaries to construct the restriction mappings and to verify the sheaf condition with this ansatz.

The following lemma gives yet another way to define the sheaf of differentials and it in particular shows that $\Omega_{X/S}$ is quasi-coherent if X and S are schemes.

08S2 Lemma 29.32.7. Let $f : X \to S$ be a morphism of schemes. There is a canonical isomorphism between $\Omega_{X/S}$ and the conormal sheaf of the diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$.

Proof. We first establish the existence of a couple of "global" sheaves and global maps of sheaves, and further down we describe the constructions over some affine opens.

Recall that $\Delta = \Delta_{X/S} : X \to X \times_S X$ is an immersion, see Schemes, Lemma 26.21.2. Let $\mathcal{J}$ be the ideal sheaf of the immersion which lives over some open subscheme W of $X \times_S X$ such that $\Delta(X) \subset W$ is closed. Let us take the one that was found in the proof of Schemes, Lemma 26.21.2. Note that the sheaf of rings $\mathcal{O}_W/\mathcal{J}^2$ is supported on $\Delta(X)$. Moreover it sits in a short exact sequence of sheaves

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_W/\mathcal{J}^2 \to \Delta_*\mathcal{O}_X \to 0.$$

Using Δ^{-1} we can think of this as a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras with kernel the conormal sheaf of Δ (see Definition 29.31.1 and Lemma 29.31.2).

$$0 \to \mathcal{C}_{X/X\times_S X} \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \to \mathcal{O}_X \to 0$$

This places us in the situation of Modules, Lemma 17.28.11. The projection morphisms $p_i : X \times_S X \to X$, $i = 1, 2$ induce maps of sheaves of rings $(p_i)^\sharp : (p_i)^{-1}\mathcal{O}_X \to \mathcal{O}_{X\times_S X}$. We may restrict to W and quotient by $\mathcal{J}^2$ to get $(p_i)^{-1}\mathcal{O}_X \to \mathcal{O}_W/\mathcal{J}^2$. Since $\Delta^{-1}p_i^{-1}\mathcal{O}_X = \mathcal{O}_X$ we get maps

$$s_i : \mathcal{O}_X \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2).$$

Both s_1 and s_2 are sections to the map $\Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \to \mathcal{O}_X$, as in Modules, Lemma 17.28.11. Thus we get an S-derivation $\mathrm{d} = s_2 - s_1 : \mathcal{O}_X \to \mathcal{C}_{X/X\times_S X}$. By the universal property of the module of differentials we find a unique $\mathcal{O}_X$-linear map

$$\Omega_{X/S} \longrightarrow \mathcal{C}_{X/X\times_S X}, \quad f\mathrm{d}g \longmapsto fs_2(g) - fs_1(g)$$

To see the map is an isomorphism, let us work this out over suitable affine opens. We can cover X by affine opens $\mathrm{Spec}(A) = U \subset X$ whose image is contained in an affine open $\mathrm{Spec}(R) = V \subset S$. According to the proof of Schemes, Lemma 26.21.2 $U \times_V U \subset X \times_S X$ is an affine open contained in the open W mentioned above. Also $U \times_V U = \mathrm{Spec}(A \otimes_R A)$. The sheaf $\mathcal{J}$ corresponds to the ideal

$J = \mathrm{Ker}(A \otimes_R A \to A)$. The short exact sequence to the short exact sequence of $A \otimes_R A$-modules

$$0 \to J/J^2 \to (A \otimes_R A)/J^2 \to A \to 0$$

The sections s_i correspond to the ring maps

$$A \longrightarrow (A \otimes_R A)/J^2, \quad s_1 : a \mapsto a \otimes 1, \quad s_2 : a \mapsto 1 \otimes a.$$

By Lemma 29.31.2 we have $\Gamma(U, \mathcal{C}_{X/X \times_S X}) = J/J^2$ and by Lemma 29.32.5 we have $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$. The map above is the map $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$ which is shown to be an isomorphism in Algebra, Lemma 10.131.13. □

01UV Lemma 29.32.8. Let

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

be a commutative diagram of schemes. The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*\mathrm{d}_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}.$$

It is uniquely characterized by the property that $f^*\mathrm{d}_{X/S}(h)$ maps to $\mathrm{d}_{X'/S'}(f^*h)$ for any local section h of $\mathcal{O}_X$.

Proof. This is a special case of Modules, Lemma 17.28.12. In the case of schemes we can also use the functoriality of the conormal sheaves (see Lemma 29.31.3) and Lemma 29.32.7 to define c_f. Or we can use the characterization in the last line of the lemma to glue maps defined on affine patches (see Algebra, Equation (10.131.4.1)). □

01UX Lemma 29.32.9. Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Then there is a canonical exact sequence

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 29.32.8.

Proof. This is the sheafified version of Algebra, Lemma 10.131.7. □

01V0 Lemma 29.32.10. Let $X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Let $X' = X_{S'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map

$$(g')^*\Omega_{X/S} \to \Omega_{X'/S'}$$

of Lemma 29.32.8 is an isomorphism.

Proof. This is the sheafified version of Algebra, Lemma 10.131.12. □

01V1 Lemma 29.32.11. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projection morphisms. The maps from Lemma 29.32.8

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X \times_S Y/S}$$

give an isomorphism.

Proof. By Lemma 29.32.10 the composition $p^*\Omega_{X/S} \to \Omega_{X \times_S Y/S} \to \Omega_{X \times_S Y/Y}$ is an isomorphism, and similarly for q. Moreover, the cokernel of $p^*\Omega_{X/S} \to \Omega_{X \times_S Y/S}$ is $\Omega_{X \times_S Y/X}$ by Lemma 29.32.9. The result follows. $\square$

01V2 Lemma 29.32.12. Let $f : X \to S$ be a morphism of schemes. If f is locally of finite type, then $\Omega_{X/S}$ is a finite type $\mathcal{O}_X$-module.

Proof. Immediate from Algebra, Lemma 10.131.16, Lemma 29.32.5, Lemma 29.15.2, and Properties, Lemma 28.16.1. $\square$

01V3 Lemma 29.32.13. Let $f : X \to S$ be a morphism of schemes. If f is locally of finite presentation, then $\Omega_{X/S}$ is an $\mathcal{O}_X$-module of finite presentation.

Proof. Immediate from Algebra, Lemma 10.131.15, Lemma 29.32.5, Lemma 29.21.2, and Properties, Lemma 28.16.2. $\square$

01UY Lemma 29.32.14. If $X \to S$ is an immersion, or more generally a monomorphism, then $\Omega_{X/S}$ is zero.

Proof. This is true because $\Delta_{X/S}$ is an isomorphism in this case and hence has trivial conormal sheaf. Hence $\Omega_{X/S} = 0$ by Lemma 29.32.7. The algebraic version is Algebra, Lemma 10.131.4. $\square$

01UZ Lemma 29.32.15. Let $i : Z \to X$ be an immersion of schemes over S. There is a canonical exact sequence

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

where the first arrow is induced by $\mathrm{d}_{X/S}$ and the second arrow comes from Lemma 29.32.8.

Proof. This is the sheafified version of Algebra, Lemma 10.131.9. However we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/S}$" here. Namely, we may assume that i is a closed immersion by shrinking X. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Lemma 29.4.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. $\square$

0474 Lemma 29.32.16. Let $i : Z \to X$ be an immersion of schemes over S, and assume i (locally) has a left inverse. Then the canonical sequence

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 29.32.15 is (locally) split exact. In particular, if $s : S \to X$ is a section of the structure morphism $X \to S$ then the map $\mathcal{C}_{S/X} \to s^*\Omega_{X/S}$ induced by $\mathrm{d}_{X/S}$ is an isomorphism.

Proof. Follows from Algebra, Lemma 10.131.10. Clarification: if $g : X \to Z$ is a left inverse of i, then i^*c_g is a right inverse of the map $i^*\Omega_{X/S} \to \Omega_{Z/S}$. Also, if s is a section, then it is an immersion $s : Z = S \to X$ over S (see Schemes, Lemma 26.21.11) and in that case $\Omega_{Z/S} = 0$. $\square$

060N Remark 29.32.17. Let $X \to S$ be a morphism of schemes. According to Lemma 29.32.11 we have

$$\Omega_{X \times_S X/S} = \mathrm{pr}_1^*\Omega_{X/S} \oplus \mathrm{pr}_2^*\Omega_{X/S}$$

On the other hand, the diagonal morphism $\Delta : X \to X \times_S X$ is an immersion, which locally has a left inverse. Hence by Lemma 29.32.16 we obtain a canonical short exact sequence

$$0 \to \mathcal{C}_{X/X \times_S X} \to \Omega_{X/S} \oplus \Omega_{X/S} \to \Omega_{X/S} \to 0$$

Note that the right arrow is $(1, 1)$ which is indeed a split surjection. On the other hand, by Lemma 29.32.7 we have an identification $\Omega_{X/S} = \mathcal{C}_{X/X \times_S X}$. Because we chose $\mathrm{d}_{X/S}(f) = s_2(f) - s_1(f)$ in this identification it turns out that the left arrow is the map $(-1, 1)$[9].

067L Lemma 29.32.18. Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[rd, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are immersions. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 29.31.3 and the second from Lemma 29.32.15.

Proof. The algebraic version of this is Algebra, Lemma 10.134.7. □

29.33. Finite order differential operators

0G43 We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.133) and the corresponding section in the chapter on sheaves of modules (Modules, Section 17.29).

0G44 Lemma 29.33.1. Let $R \to A$ be a ring map. Denote $f : X \to S$ the corresponding morphism of affine schemes. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is quasi-coherent then the map

$$\mathrm{Diff}^k_{X/S}(\mathcal{F}, \mathcal{G}) \to \mathrm{Diff}^k_{A/R}(\Gamma(X, \mathcal{F}), \Gamma(X, \mathcal{G}))$$

sending a differential operator to its action on global sections is bijective.

Proof. Write $\mathcal{F} = \widetilde{M}$ for some A-module M. Set $N = \Gamma(X, \mathcal{G})$. Let $D : M \to N$ be a differential operator of order k. We have to show there exists a unique differential operator $\mathcal{F} \to \mathcal{G}$ of order k which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Then $\mathcal{F}(U) = M_f$ is the localization. By Algebra, Lemma 10.133.10 the differential operator D extends to a unique differential operator

$$D_f : \mathcal{F}(U) = \widetilde{M}(U) = M_f \to N_f = \widetilde{N}(U)$$

The uniqueness shows that these maps D_f glue to give a map of sheaves $\widetilde{M} \to \widetilde{N}$ on the basis of all standard opens of X. Hence we get a unique map of sheaves $\widetilde{D} : \widetilde{M} \to \widetilde{N}$ agreeing with these maps by the material in Sheaves, Section 6.30. Since $\widetilde{D}$ is given by differential operators of order k on the standard opens, we find that $\widetilde{D}$ is a differential operator of order k (small detail omitted). Finally, we can

[9]Namely, the local section $\mathrm{d}_{X/S}(f) = 1 \otimes f - f \otimes 1$ of the ideal sheaf of Δ maps via $\mathrm{d}_{X \times_S X/X}$ to the local section $1 \otimes 1 \otimes 1 \otimes f - 1 \otimes f \otimes 1 \otimes 1 - 1 \otimes 1 \otimes f \otimes 1 + f \otimes 1 \otimes 1 \otimes 1 = \mathrm{pr}_2^*\mathrm{d}_{X/S}(f) - \mathrm{pr}_1^*\mathrm{d}_{X/S}(f)$.

post-compose with the canonical $\mathcal{O}_X$-module map $c : \widetilde{N} \to \mathcal{G}$ (Schemes, Lemma 26.7.1) to get $c \circ \widetilde{D} : \mathcal{F} \to \mathcal{G}$ which is a differential operator of order k by Modules, Lemma 17.29.2. This proves existence. We omit the proof of uniqueness. □

0G45 Lemma 29.33.2. Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Let $\mathcal{F}$ and $\mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. Let $D : \mathcal{F} \to \mathcal{F}'$ be a differential operator of order k on X/S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Then there is a unique differential operator

$$D' : \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} \mathrm{pr}_2^*\mathcal{G} \longrightarrow \mathrm{pr}_1^*\mathcal{F}' \otimes_{\mathcal{O}_{X \times_S Y}} \mathrm{pr}_2^*\mathcal{G}$$

of order k on $X \times_S Y/Y$ such that $D'(s \otimes t) = D(s) \otimes t$ for local sections s of $\mathcal{F}$ and t of $\mathcal{G}$.

Proof. In case X, Y, and S are affine, this follows, via Lemma 29.33.1, from the corresponding algebra result, see Algebra, Lemma 10.133.11. In general, one uses coverings by affines (for example as in Schemes, Lemma 26.17.4) to construct D' globally. Details omitted. □

0G46 Remark 29.33.3. Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Denote $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ the projections. In this remark, given an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$ let us set

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}$$

Denote $\mathcal{A}_{X/S}$ the additive category whose objects are quasi-coherent $\mathcal{O}_X$-modules and whose morphisms are differential operators of finite order on X/S. Similarly for $\mathcal{A}_{Y/S}$ and $\mathcal{A}_{X \times_S Y/S}$. The construction of Lemma 29.33.2 determines a functor

$$\boxtimes : \mathcal{A}_{X/S} \times \mathcal{A}_{Y/S} \longrightarrow \mathcal{A}_{X \times_S Y/S}, \quad (\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \boxtimes \mathcal{G}$$

which is bilinear on morphisms. If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(R)$, then via the identification of quasi-coherent sheaves with modules this functor is given by $(M, N) \mapsto M \otimes_R N$ on objects and sends the morphism $(D, D') : (M, N) \to (M', N')$ to $D \otimes D' : M \otimes_R N \to M' \otimes_R N'$.

29.34. Smooth morphisms

01V4 Let $f : X \to Y$ be a continuous map of topological spaces. Consider the following condition: For every $x \in X$ there exist open neighbourhoods $x \in U \subset X$ and $f(x) \in V \subset Y$, and an integer d such that $f(U) \subset V$ and such that we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"'] \arrow[d] & V \times \mathbf{R}^d \arrow[dl] \\ Y & V \arrow[l] & \end{tikzcd}$$

where π is a homeomorphism onto an open subset. Smooth morphisms of schemes are the analogue of these maps in the category of schemes. See Lemma 29.34.11 and Lemma 29.36.20.

Contrary to expectations (perhaps) the notion of a smooth ring map is not defined solely in terms of the module of differentials. Namely, recall that $R \to A$ is a smooth ring map if A is of finite presentation over R and if the naive cotangent complex of A over R is quasi-isomorphic to a projective module placed in degree 0, see Algebra, Definition 10.137.1.

01V5 **Definition 29.34.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *smooth at $x \in X$* if there exist an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is smooth.
(2) We say that f is *smooth* if it is smooth at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called *standard smooth* if there exists a standard smooth ring map $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 10.137.6) such that $X \to S$ is isomorphic to

$$\text{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \text{Spec}(R).$$

A pleasing feature of this definition is that the set of points where a morphism is smooth is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being smooth is local in nature on the source. Here is the precise result.

01V6 **Lemma 29.34.2.** Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is smooth.
(2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is smooth.
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is smooth.
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is smooth, for all $j \in J, i \in I_j$.

Moreover, if f is smooth then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is smooth.

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is smooth" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. By Algebra, Lemma 10.137.4 being smooth is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.137.14 being smooth is stable under composition and for any ring R the ring map $R \to R_f$ is (standard) smooth. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.137.13. $\square$

The following lemma characterizes a smooth morphism as a flat, finitely presented morphism with smooth fibres. Note that schemes smooth over a field are discussed in more detail in Varieties, Section 33.25.

01V8 **Lemma 29.34.3.** Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are smooth, then f is smooth.

Proof. Follows from Algebra, Lemma 10.137.17. $\square$

01VA **Lemma 29.34.4.** The composition of two morphisms which are smooth is smooth.

Proof. In the proof of Lemma 29.34.2 we saw that being smooth is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being smooth is a property of ring maps that is stable under composition, see Algebra, Lemma 10.137.14. $\square$

01VB **Lemma 29.34.5.** *The base change of a morphism which is smooth is smooth.*

Proof. In the proof of Lemma 29.34.2 we saw that being smooth is a local property of ring maps. Hence the lemma follows from Lemma 29.14.5 combined with the fact that being smooth is a property of ring maps that is stable under base change, see Algebra, Lemma 10.137.4. □

01VC **Lemma 29.34.6.** *Any open immersion is smooth.*

Proof. This is true because an open immersion is a local isomorphism. □

01VD **Lemma 29.34.7.** *A smooth morphism is syntomic.*

Proof. See Algebra, Lemma 10.137.10. □

01VE **Lemma 29.34.8.** *A smooth morphism is locally of finite presentation.*

Proof. True because a smooth ring map is of finite presentation by definition. □

01VF **Lemma 29.34.9.** *A smooth morphism is flat.*

Proof. Combine Lemmas 29.30.7 and 29.34.7. □

056G **Lemma 29.34.10.** *A smooth morphism is universally open.*

Proof. Combine Lemmas 29.34.9, 29.34.8, and 29.25.10. Or alternatively, combine Lemmas 29.34.7, 29.30.8. □

The following lemma says locally any smooth morphism is standard smooth. Hence we can use standard smooth morphisms as a local model for a smooth morphism.

01V7 **Lemma 29.34.11.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Let $V \subset S$ be an affine open neighbourhood of $f(x)$. The following are equivalent*

(1) *The morphism f is smooth at x.*
(2) *There exists an affine open $U \subset X$, with $x \in U$ and $f(U) \subset V$ such that the induced morphism $f|_U : U \to V$ is standard smooth.*

Proof. Follows from the definitions and Algebra, Lemmas 10.137.7 and 10.137.10. □

02G1 **Lemma 29.34.12.** *Let $f : X \to S$ be a morphism of schemes. Assume f is smooth. Then the module of differentials $\Omega_{X/S}$ of X over S is finite locally free and*

$$\text{rank}_x(\Omega_{X/S}) = \dim_x(X_{f(x)})$$

for every $x \in X$.

Proof. The statement is local on X and S. By Lemma 29.34.11 above we may assume that f is a standard smooth morphism of affines. In this case the result follows from Algebra, Lemma 10.137.7 (and the definition of a relative global complete intersection, see Algebra, Definition 10.136.5). □

Lemma 29.34.12 says that the following definition makes sense.

02G2 **Definition 29.34.13.** Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *smooth of relative dimension d* if f is smooth and $\Omega_{X/S}$ is finite locally free of constant rank d.

In other words, f is smooth and the nonempty fibres are equidimensional of dimension d. By Lemma 29.34.14 below this is also the same as requiring: (a) f is locally of finite presentation, (b) f is flat, (c) all nonempty fibres equidimensional of dimension d, and (d) $\Omega_{X/S}$ finite locally free of rank d. It is not enough to simply assume that f is flat, of finite presentation, and $\Omega_{X/S}$ is finite locally free of rank d. A counter example is given by $\text{Spec}(\mathbf{F}_p[t]) \to \text{Spec}(\mathbf{F}_p[t^p])$.

Here is a differential criterion of smoothness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

01V9 Lemma 29.34.14. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:

(1) The morphism f is smooth at x.
(2) The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \text{Spec}(\kappa(s))$ is smooth at x.
(3) The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ can be generated by at most $\dim_x(X_{f(x)})$ elements.
(4) The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
can be generated by at most $\dim_x(X_{f(x)})$ elements.
(5) There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth.
(6) There exist affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation
$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$
with
$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$
mapping to an element of A not in $\mathfrak{q}$.

Proof. Note that if f is smooth at x, then we see from Lemma 29.34.11 that (5) holds, and (6) is a slightly weakened version of (5). Moreover, f smooth implies that the ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat (see Lemma 29.34.9) and that $\Omega_{X/S}$ is finite locally free of rank equal to $\dim_x(X_s)$ (see Lemma 29.34.12). Thus (1) implies (3) and (4). By Lemma 29.34.5 we also see that (1) implies (2).

By Lemma 29.32.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. Hence the displayed equality in part (4) of the lemma. By Lemma 29.32.12 these modules are of finite type. Hence the minimal number of generators of the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ is the same and equal to the dimension of this $\kappa(x)$-vector space by Nakayama's Lemma (Algebra, Lemma 10.20.1). This in particular shows that (3) and (4) are equivalent.

Algebra, Lemma 10.137.17 shows that (2) implies (1). Algebra, Lemma 10.140.3 shows that (3) and (4) imply (2). Finally, (6) implies (5) see for example Algebra, Example 10.137.8 and (5) implies (1) by Algebra, Lemma 10.137.7. □

02V4 Lemma 29.34.15. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Let $W \subset X$, resp. $W' \subset X'$ be the open subscheme of points where f, resp. f' is smooth. Then $W' = (g')^{-1}(W)$ if

(1) f is flat and locally of finite presentation, or
(2) f is locally of finite presentation and g is flat.

Proof. Assume first that f locally of finite type. Consider the set

$$T = \{x \in X \mid X_{f(x)} \text{ is smooth over } \kappa(f(x)) \text{ at } x\}$$

and the corresponding set $T' \subset X'$ for f'. Then we claim $T' = (g')^{-1}(T)$. Namely, let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \text{Spec}(\kappa(s')) \times_{\text{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the claim is equivalent to Algebra, Lemma 10.137.19.

Thus case (1) follows because in case (1) T is the (open) set of points where f is smooth by Lemma 29.34.14.

In case (2) let $x' \in W'$. Then g' is flat at x' (Lemma 29.25.7) and $g \circ f$ is flat at x' (Lemma 29.25.5). It follows that f is flat at $x = g'(x')$ by Lemma 29.25.13. On the other hand, since $x' \in T'$ (Lemma 29.34.5) we see that $x \in T$. Hence f is smooth at x by Lemma 29.34.14. □

Here is a lemma that actually uses the vanishing of H^{-1} of the naive cotangent complex for a smooth ring map.

02K4 Lemma 29.34.16. Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is smooth. Then

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Lemma 29.32.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 10.131.7 is exact. This is Algebra, Lemma 10.139.1. □

06AA Lemma 29.34.17. Let $i : Z \to X$ be an immersion of schemes over S. Assume that Z is smooth over S. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 29.32.15 is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ smooth and $B \to C$ surjective with kernel J, then the sequence

$$0 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$$

of Algebra, Lemma 10.131.9 is exact. This is Algebra, Lemma 10.139.2. □

06AB Lemma 29.34.18. Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are immersions and $X \to Y$ is smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 29.32.18 is exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ surjective and $A \to B$ smooth, then the sequence

$$0 \to I/I^2 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to 0$$

of Algebra, Lemma 10.134.7 is exact. This is Algebra, Lemma 10.139.3. □

02K5 Lemma 29.34.19. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) f is surjective, and smooth,
(2) p is smooth, and
(3) q is locally of finite presentation[10].

Then q is smooth.

Proof. By Lemma 29.25.13 we see that q is flat. Pick a point $y \in Y$. Pick a point $x \in X$ mapping to y. Suppose f has relative dimension a at x and p has relative dimension b at x. By Lemma 29.34.12 this means that $\Omega_{X/S,x}$ is free of rank b and $\Omega_{X/Y,x}$ is free of rank a. By the short exact sequence of Lemma 29.34.16 this means that $(f^*\Omega_{Y/S})_x$ is free of rank $b - a$. By Nakayama's Lemma this implies that $\Omega_{Y/S,y}$ can be generated by $b - a$ elements. Also, by Lemma 29.28.2 we see that $\dim_y(Y_s) = b - a$. Hence we conclude that $Y \to S$ is smooth at y by Lemma 29.34.14 part (2). □

In the situation of the following lemma the image of σ is locally on X cut out by a regular sequence, see Divisors, Lemma 31.22.8.

[10]In fact this is implied by (1) and (2), see Descent, Lemma 35.14.3. Moreover, it suffices to assume f is surjective, flat and locally of finite presentation, see Descent, Lemma 35.14.5.

05D9 Lemma 29.34.20. Let $f : X \to S$ be a morphism of schemes. Let $\sigma : S \to X$ be a section of f. Let $s \in S$ be a point such that f is smooth at $x = \sigma(s)$. Then there exist affine open neighbourhoods $\mathrm{Spec}(A) = U \subset S$ of s and $\mathrm{Spec}(B) = V \subset X$ of x such that

(1) $f(V) \subset U$ and $\sigma(U) \subset V$,
(2) with $I = \mathrm{Ker}(\sigma^\# : B \to A)$ the module I/I^2 is a free A-module, and
(3) $B^\wedge \cong A[[x_1, \ldots, x_d]]$ as A-algebras where $B^\wedge$ denotes the completion of B with respect to I.

Proof. Pick an affine open $U \subset S$ containing s Pick an affine open $V \subset f^{-1}(U)$ containing x. Pick an affine open $U' \subset \sigma^{-1}(V)$ containing s. Note that $V' = f^{-1}(U') \cap V$ is affine as it is equal to the fibre product $V' = U' \times_U V$. Then U' and V' satisfy (1). Write $U' = \mathrm{Spec}(A')$ and $V' = \mathrm{Spec}(B')$. By Algebra, Lemma 10.139.4 the module $I'/(I')^2$ is finite locally free as a A'-module. Hence after replacing U' by a smaller affine open $U'' \subset U'$ and V' by $V'' = V' \cap f^{-1}(U'')$ we obtain the situation where $I''/(I'')^2$ is free, i.e., (2) holds. In this case (3) holds also by Algebra, Lemma 10.139.4. □

The dimension of a scheme X at a point x (Properties, Definition 28.10.1) is just the dimension of X at x as a topological space, see Topology, Definition 5.10.1. This is not the dimension of the local ring $\mathcal{O}_{X,x}$, in general.

0AFF Lemma 29.34.21. Let $f : X \to Y$ be a smooth morphism of locally Noetherian schemes. For every point x in X with image y in Y,

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where X_y denotes the fiber over y.

Proof. After replacing X by an open neighborhood of x, there is a natural number d such that all fibers of $X \to Y$ have dimension d at every point, see Lemma 29.34.12. Then f is flat (Lemma 29.34.9), locally of finite type (Lemma 29.34.8), and of relative dimension d. Hence the result follows from Lemma 29.29.6. □

29.35. Unramified morphisms

02G3 We briefly discuss unramified morphisms before the (perhaps) more interesting class of étale morphisms. Recall that a ring map $R \to A$ is unramified if it is of finite type and $\Omega_{A/R} = 0$ (this is the definition of [Ray70]). A ring map $R \to A$ is called G-unramified if it is of finite presentation and $\Omega_{A/R} = 0$ (this is the definition of [DG67]). See Algebra, Definition 10.151.1.

02G4 Definition 29.35.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is unramified at $x \in X$ if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is unramified.
(2) We say that f is G-unramified at $x \in X$ if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is G-unramified.
(3) We say that f is unramified if it is unramified at every point of X.
(4) We say that f is G-unramified if it is G-unramified at every point of X.

Note that a G-unramified morphism is unramified. Hence any result for unramified morphisms implies the corresponding result for G-unramified morphisms. Moreover, if S is locally Noetherian then there is no difference between G-unramified and unramified morphisms, see Lemma 29.35.6. A pleasing feature of this definition is that the set of points where a morphism is unramified (resp. G-unramified) is automatically open.

02G5 Lemma 29.35.2. Let $f : X \to S$ be a morphism of schemes. Then

(1) f is unramified if and only if f is locally of finite type and $\Omega_{X/S} = 0$, and
(2) f is G-unramified if and only if f is locally of finite presentation and $\Omega_{X/S} = 0$.

Proof. By definition a ring map $R \to A$ is unramified (resp. G-unramified) if and only if it is of finite type (resp. finite presentation) and $\Omega_{A/R} = 0$. Hence the lemma follows directly from the definitions and Lemma 29.32.5. $\square$

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being unramified is local in nature on the source. Here is the precise result.

02G6 Lemma 29.35.3. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is unramified (resp. G-unramified).
(2) For every affine open $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is unramified (resp. G-unramified).
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is unramified (resp. G-unramified).
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is unramified (resp. G-unramified), for all $j \in J, i \in I_j$.

Moreover, if f is unramified (resp. G-unramified) then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is unramified (resp. G-unramified).

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is unramified" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. These properties are proved in Algebra, Lemma 10.151.3. $\square$

02G9 Lemma 29.35.4. The composition of two morphisms which are unramified is unramified. The same holds for G-unramified morphisms.

Proof. The proof of Lemma 29.35.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under composition, see Algebra, Lemma 10.151.3. $\square$

02GA Lemma 29.35.5. The base change of a morphism which is unramified is unramified. The same holds for G-unramified morphisms.

Proof. The proof of Lemma 29.35.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the lemma follows from Lemma 29.14.6 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under base change, see Algebra, Lemma 10.151.3. □

04EV Lemma 29.35.6. Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. Then f is unramified if and only if f is G-unramified.

Proof. Follows from the definitions and Lemma 29.21.9. □

02GB Lemma 29.35.7. Any open immersion is G-unramified.

Proof. This is true because an open immersion is a local isomorphism. □

02GC Lemma 29.35.8. A closed immersion $i : Z \to X$ is unramified. It is G-unramified if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).

Proof. Follows from Lemma 29.21.7 and Algebra, Lemma 10.151.3. □

02GD Lemma 29.35.9. An unramified morphism is locally of finite type. A G-unramified morphism is locally of finite presentation.

Proof. An unramified ring map is of finite type by definition. A G-unramified ring map is of finite presentation by definition. □

02V5 Lemma 29.35.10. Let $f : X \to S$ be a morphism of schemes. If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.

Proof. See Algebra, Lemma 10.151.6. □

02G7 Lemma 29.35.11. Fibres of unramified morphisms.

(1) Let X be a scheme over a field k. The structure morphism $X \to \mathrm{Spec}(k)$ is unramified if and only if X is a disjoint union of spectra of finite separable field extensions of k.
(2) If $f : X \to S$ is an unramified morphism then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.

Proof. Part (2) follows from part (1) and Lemma 29.35.5. Let us prove part (1). We first use Algebra, Lemma 10.151.7. This lemma implies that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to \mathrm{Spec}(k)$ is unramified. Conversely, suppose that $X \to \mathrm{Spec}(k)$ is unramified. By Algebra, Lemma 10.151.5 for every $x \in X$ the residue field extension $\kappa(x)/k$ is finite separable. Since $X \to \mathrm{Spec}(k)$ is locally quasi-finite (Lemma 29.35.10) we see that all points of X are isolated closed points, see Lemma 29.20.6. Thus X is a discrete space, in particular the disjoint union of the spectra of its local rings. By Algebra, Lemma 10.151.5 again these local rings are fields, and we win. □

The following lemma characterizes an unramified morphisms as morphisms locally of finite type with unramified fibres.

02G8 Lemma 29.35.12. Let $f : X \to S$ be a morphism of schemes.

(1) If f is unramified then for any $x \in X$ the field extension $\kappa(x)/\kappa(f(x))$ is finite separable.

(2) If f is locally of finite type, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is unramified.
(3) If f is locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is G-unramified.

Proof. Follows from Algebra, Lemmas 10.151.5 and 10.151.7. □

Here is a characterization of unramified morphisms in terms of the diagonal morphism.

02GE Lemma 29.35.13. Let $f : X \to S$ be a morphism.

(1) If f is unramified, then the diagonal morphism $\Delta : X \to X \times_S X$ is an open immersion.
(2) If f is locally of finite type and Δ is an open immersion, then f is unramified.
(3) If f is locally of finite presentation and Δ is an open immersion, then f is G-unramified.

Proof. The first statement follows from Algebra, Lemma 10.151.4. The second statement from the fact that $\Omega_{X/S}$ is the conormal sheaf of the diagonal morphism (Lemma 29.32.7) and hence clearly zero if Δ is an open immersion. □

02GF Lemma 29.35.14. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite type (resp. locally of finite presentation). The following are equivalent:

(1) The morphism f is unramified (resp. G-unramified) at x.
(2) The fibre X_s is unramified over $\kappa(s)$ at x.
(3) The $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.
(4) The $\mathcal{O}_{X_s,x}$-module $\Omega_{X_s/s,x}$ is zero.
(5) The $\kappa(x)$-vector space
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
(6) We have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(x)/\kappa(s)$ is finite separable.

Proof. Note that if f is unramified at x, then we see that $\Omega_{X/S} = 0$ in a neighbourhood of x by the definitions and the results on modules of differentials in Section 29.32. Hence (1) implies (3) and the vanishing of the right hand vector space in (5). It also implies (2) because by Lemma 29.32.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. This fact on modules of differentials also implies the displayed equality of vector spaces in part (4). By Lemma 29.32.12 the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are of finite type. Hence the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are zero if and only if the corresponding $\kappa(x)$-vector space in (4) is zero by Nakayama's Lemma (Algebra, Lemma 10.20.1). This in particular shows that (3), (4) and (5) are equivalent. The support of $\Omega_{X/S}$ is closed in X, see Modules, Lemma 17.9.6. Assumption (3) implies that x is not in the support. Hence $\Omega_{X/S}$ is zero in a neighbourhood of x, which implies (1). The equivalence of (1) and (3) applied to $X_s \to s$ implies the equivalence of (2) and (4). At this point we have seen that (1) – (5) are equivalent.

Alternatively you can use Algebra, Lemma 10.151.3 to see the equivalence of (1) – (5) more directly.

The equivalence of (1) and (6) follows from Lemma 29.35.12. It also follows more directly from Algebra, Lemmas 10.151.5 and 10.151.7. ☐

0475 Lemma 29.35.15. Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the open set

$$\begin{aligned} T &= \{x \in X \mid X_{f(x)} \text{ is unramified over } \kappa(f(x)) \text{ at } x\} \\ &= \{x \in X \mid X \text{ is unramified over } S \text{ at } x\} \end{aligned}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. If f is assumed locally of finite presentation then the same holds for the open set of points where f is G-unramified.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. In particular

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x') = \Omega_{X'_{s'}/s',x'} \otimes_{\mathcal{O}_{X'_{s'},x'}} \kappa(x')$$

see Lemma 29.32.10. Whence $x' \in T'$ if and only if $x \in T$ by Lemma 29.35.14. The second part follows from the first because in that case T is the (open) set of points where f is G-unramified according to Lemma 29.35.14. ☐

02GG Lemma 29.35.16. Let $f : X \to Y$ be a morphism of schemes over S.

(1) If X is unramified over S, then f is unramified.
(2) If X is G-unramified over S and Y is locally of finite type over S, then f is G-unramified.

Proof. Assume that X is unramified over S. By Lemma 29.15.8 we see that f is locally of finite type. By assumption we have $\Omega_{X/S} = 0$. Hence $\Omega_{X/Y} = 0$ by Lemma 29.32.9. Thus f is unramified. If X is G-unramified over S and Y is locally of finite type over S, then by Lemma 29.21.11 we see that f is locally of finite presentation and we conclude that f is G-unramified. ☐

04HB Lemma 29.35.17. Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms over S. Let $x \in X$. Assume that

(1) the structure morphism $Y \to S$ is unramified,
(2) $f(x) = g(x)$ in Y, say $y = f(x) = g(x)$, and
(3) the induced maps $f^\sharp, g^\sharp : \kappa(y) \to \kappa(x)$ are equal.

Then there exists an open neighbourhood of x in X on which f and g are equal.

Proof. Consider the morphism $(f, g) : X \to Y \times_S Y$. By assumption (1) and Lemma 29.35.13 the inverse image of $\Delta_{Y/S}(Y)$ is open in X. And assumptions (2) and (3) imply that x is in this open subset. ☐

29.36. Étale morphisms

02GH The Zariski topology of a scheme is a very coarse topology. This is particularly clear when looking at varieties over $\mathbf{C}$. It turns out that declaring an étale morphism to be the analogue of a local isomorphism in topology introduces a much finer topology. On varieties over $\mathbf{C}$ this topology gives rise to the "correct" Betti numbers when computing cohomology with finite coefficients. Another observable is that if $f : X \to Y$ is an étale morphism of varieties over $\mathbf{C}$, and if x is a closed point of X, then f induces an isomorphism $\mathcal{O}^\wedge_{Y,f(x)} \to \mathcal{O}^\wedge_{X,x}$ of complete local rings.

In this section we start our study of these matters. In fact we deliberately restrict our discussion to a minimum since we will discuss more interesting results elsewhere. Recall that a ring map $R \to A$ is said to be étale if it is smooth and $\Omega_{A/R} = 0$, see Algebra, Definition 10.143.1.

02GI Definition 29.36.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is étale at $x \in X$ if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is étale.
(2) We say that f is étale if it is étale at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called standard étale if $X \to S$ is isomorphic to
$$\text{Spec}(R[x]_h/(g)) \to \text{Spec}(R)$$
where $R \to R[x]_h/(g)$ is a standard étale ring map, see Algebra, Definition 10.144.1, i.e., g is monic and g' invertible in $R[x]_h/(g)$.

A morphism is étale if and only if it is smooth of relative dimension 0 (see Definition 29.34.13). A pleasing feature of the definition is that the set of points where a morphism is étale is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being étale is local in nature on the source. Here is the precise result.

02GJ Lemma 29.36.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is étale.
(2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is étale.
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is étale.
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is étale, for all $j \in J, i \in I_j$.

Moreover, if f is étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is étale.

Proof. This follows from Lemma 29.14.3 if we show that the property "$R \to A$ is étale" is local. We check conditions (a), (b) and (c) of Definition 29.14.1. These all follow from Algebra, Lemma 10.143.3. □

02GN Lemma 29.36.3. The composition of two morphisms which are étale is étale.

Proof. In the proof of Lemma 29.36.2 we saw that being étale is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 29.14.5 combined with the fact that being étale is a property of ring maps that is stable under composition, see Algebra, Lemma 10.143.3. □

02GO Lemma 29.36.4. The base change of a morphism which is étale is étale.

Proof. In the proof of Lemma 29.36.2 we saw that being étale is a local property of ring maps. Hence the lemma follows from Lemma 29.14.5 combined with the fact that being étale is a property of ring maps that is stable under base change, see Algebra, Lemma 10.143.3. □

02GK Lemma 29.36.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Then f is étale at x if and only if f is smooth and unramified at x.

Proof. This follows immediately from the definitions. □

03WS Lemma 29.36.6. An étale morphism is locally quasi-finite.

Proof. By Lemma 29.36.5 an étale morphism is unramified. By Lemma 29.35.10 an unramified morphism is locally quasi-finite. □

02GL Lemma 29.36.7. Fibres of étale morphisms.

(1) Let X be a scheme over a field k. The structure morphism $X \to \mathrm{Spec}(k)$ is étale if and only if X is a disjoint union of spectra of finite separable field extensions of k.
(2) If $f : X \to S$ is an étale morphism, then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.

Proof. You can deduce this from Lemma 29.35.11 via Lemma 29.36.5 above. Here is a direct proof.

We will use Algebra, Lemma 10.143.4. Hence it is clear that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to \mathrm{Spec}(k)$ is étale. Conversely, suppose that $X \to \mathrm{Spec}(k)$ is étale. Then for any affine open $U \subset X$ we see that U is a finite disjoint union of spectra of finite separable field extensions of k. Hence all points of X are closed points (see Lemma 29.20.2 for example). Thus X is a discrete space and we win. □

The following lemma characterizes an étale morphism as a flat, finitely presented morphism with "étale fibres".

02GM Lemma 29.36.8. Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$, then f is étale.

Proof. You can deduce this from Algebra, Lemma 10.143.7. Here is another proof.

By Lemma 29.36.7 a fibre X_s is étale and hence smooth over s. By Lemma 29.34.3 we see that $X \to S$ is smooth. By Lemma 29.35.12 we see that f is unramified. We conclude by Lemma 29.36.5. □

02GP Lemma 29.36.9. Any open immersion is étale.

Proof. This is true because an open immersion is a local isomorphism. □

02GQ **Lemma 29.36.10.** An étale morphism is syntomic.

Proof. See Algebra, Lemma 10.137.10 and use that an étale morphism is the same as a smooth morphism of relative dimension 0. □

02GR **Lemma 29.36.11.** An étale morphism is locally of finite presentation.

Proof. True because an étale ring map is of finite presentation by definition. □

02GS **Lemma 29.36.12.** An étale morphism is flat.

Proof. Combine Lemmas 29.30.7 and 29.36.10. □

03WT **Lemma 29.36.13.** An étale morphism is open.

Proof. Combine Lemmas 29.36.12, 29.36.11, and 29.25.10. □

The following lemma says locally any étale morphism is standard étale. This is actually kind of a tricky result to prove in complete generality. The tricky parts are hidden in the chapter on commutative algebra. Hence a standard étale morphism is a local model for a general étale morphism.

02GT **Lemma 29.36.14.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Let $V \subset S$ be an affine open neighbourhood of $f(x)$. The following are equivalent

(1) The morphism f is étale at x.
(2) There exist an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that the induced morphism $f|_U : U \to V$ is standard étale (see Definition 29.36.1).

Proof. Follows from the definitions and Algebra, Proposition 10.144.4. □

Here is a differential criterion of étaleness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

02GU **Lemma 29.36.15.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:

(1) The morphism f is étale at x.
(2) The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \mathrm{Spec}(\kappa(s))$ is étale at x.
(3) The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and $X_s \to \mathrm{Spec}(\kappa(s))$ is unramified at x.
(4) The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.
(5) The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
(6) The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat, we have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(x)/\kappa(s)$ is finite separable.
(7) There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth of relative dimension 0.

(8) There exist affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation
$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$
with
$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$
mapping to an element of A not in $\mathfrak{q}$.

(9) There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard étale.

(10) There exist affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation
$$A = R[x]_Q/(P) = R[x, 1/Q]/(P)$$
with $P, Q \in R[x]$, P monic and $P' = \mathrm{d}P/\mathrm{d}x$ mapping to an element of A not in $\mathfrak{q}$.

Proof. Use Lemma 29.36.14 and the definitions to see that (1) implies all of the other conditions. For each of the conditions (2) – (10) combine Lemmas 29.34.14 and 29.35.14 to see that (1) holds by showing f is both smooth and unramified at x and applying Lemma 29.36.5. Some details omitted. $\square$

02GV Lemma 29.36.16. A morphism is étale at a point if and only if it is flat and G-unramified at that point. A morphism is étale if and only if it is flat and G-unramified.

Proof. This is clear from Lemmas 29.36.15 and 29.35.14. $\square$

0476 Lemma 29.36.17. Let
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$
be a cartesian diagram of schemes. Let $W \subset X$, resp. $W' \subset X'$ be the open subscheme of points where f, resp. f' is étale. Then $W' = (g')^{-1}(W)$ if

(1) f is flat and locally of finite presentation, or
(2) f is locally of finite presentation and g is flat.

Proof. Assume first that f locally of finite type. Consider the set
$$T = \{x \in X \mid f \text{ is unramified at } x\}$$
and the corresponding set $T' \subset X'$ for f'. Then $T' = (g')^{-1}(T)$ by Lemma 29.35.15.

Thus case (1) follows because in case (1) T is the (open) set of points where f is étale by Lemma 29.36.16.

In case (2) let $x' \in W'$. Then g' is flat at x' (Lemma 29.25.7) and $g \circ f'$ is flat at x' (Lemma 29.25.5). It follows that f is flat at $x = g'(x')$ by Lemma 29.25.13. On the other hand, since $x' \in T'$ (Lemma 29.34.5) we see that $x \in T$. Hence f is étale at x by Lemma 29.36.15. □

Our proof of the following lemma is somewhat complicated. It uses the "Critère de platitude par fibres" to see that a morphism $X \to Y$ over S between schemes étale over S is automatically flat. The details are in the chapter on commutative algebra.

02GW Lemma 29.36.18. Let $f : X \to Y$ be a morphism of schemes over S. If X and Y are étale over S, then f is étale.

Proof. See Algebra, Lemma 10.143.8. □

02K6 Lemma 29.36.19. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) f is surjective, and étale,
(2) p is étale, and
(3) q is locally of finite presentation[11].

Then q is étale.

Proof. By Lemma 29.34.19 we see that q is smooth. Thus we only need to see that q has relative dimension 0. This follows from Lemma 29.28.2 and the fact that f and p have relative dimension 0. □

A final characterization of smooth morphisms is that a smooth morphism $f : X \to S$ is locally the composition of an étale morphism by a projection $\mathbf{A}^d_S \to S$.

054L Lemma 29.36.20. Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. Let $V \subset Y$ be an affine open neighbourhood of $\varphi(x)$. If φ is smooth at x, then there exists an integer $d \geq 0$ and an affine open $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"] \arrow[d] & \mathbf{A}^d_V \arrow[dl] \\ Y & V \arrow[l] & \end{tikzcd}$$

where π is étale.

Proof. By Lemma 29.34.11 we can find an affine open U as in the lemma such that $\varphi|_U : U \to V$ is standard smooth. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$ so that we can write

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

[11]In fact this is implied by (1) and (2), see Descent, Lemma 35.14.3. Moreover, it suffices to assume that f is surjective, flat and locally of finite presentation, see Descent, Lemma 35.14.5.

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

mapping to an invertible element of A. Then it is clear that $R[x_{c+1}, \ldots, x_n] \to A$ is standard smooth of relative dimension 0. Hence it is smooth of relative dimension 0. In other words the ring map $R[x_{c+1}, \ldots, x_n] \to A$ is étale. As $\mathbf{A}^{n-c}_V = \mathrm{Spec}(R[x_{c+1}, \ldots, x_n])$ the lemma with $d = n - c$. □

29.37. Relatively ample sheaves

01VG Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample on X if X is quasi-compact and every point of X is contained in an affine open of the form X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, see Properties, Definition 28.26.1. We turn this into a relative notion as follows.

01VH **Definition 29.37.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively ample*, or *f-relatively ample*, or *ample on X/S*, or *f-ample* if $f : X \to S$ is quasi-compact, and if for every affine open $V \subset S$ the restriction of $\mathcal{L}$ to the open subscheme $f^{-1}(V)$ of X is ample. [DG67, II Definition 4.6.1]

We note that the existence of a relatively ample sheaf on X does not force the morphism $X \to S$ to be of finite type.

02NN **Lemma 29.37.2.** Let $X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is f-ample if and only if $\mathcal{L}^{\otimes n}$ is f-ample.

Proof. This follows from Properties, Lemma 28.26.2. □

01VI **Lemma 29.37.3.** Let $f : X \to S$ be a morphism of schemes. If there exists an f-ample invertible sheaf, then f is separated.

Proof. Being separated is local on the base (see Schemes, Lemma 26.21.7 for example; it also follows easily from the definition). Hence we may assume S is affine and X has an ample invertible sheaf. In this case the result follows from Properties, Lemma 28.26.8. □

There are many ways to characterize relatively ample invertible sheaves, analogous to the equivalent conditions in Properties, Proposition 28.26.13. We will add these here as needed.

01VJ **Lemma 29.37.4.** Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. The following are equivalent: [DG67, II, Proposition 4.6.3]

(1) The invertible sheaf $\mathcal{L}$ is f-ample.
(2) There exists an open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample relative to $f^{-1}(V_i) \to V_i$.
(3) There exists an affine open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample.
(4) There exists a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ such that $U(\psi) = X$ and
$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{\mathrm{Proj}}_S(\mathcal{A})$$
is an open immersion (see Constructions, Lemma 27.19.1 for notation).

(5) The morphism f is quasi-separated and part (4) above holds with $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ and ψ the adjunction mapping.
(6) Same as (4) but just requiring $r_{\mathcal{L},\psi}$ to be an immersion.

Proof. It is immediate from the definition that (1) implies (2) and (2) implies (3). It is clear that (5) implies (4).

Assume (3) holds for the affine open covering $S = \bigcup V_i$. We are going to show (5) holds. Since each $f^{-1}(V_i)$ has an ample invertible sheaf we see that $f^{-1}(V_i)$ is separated (Properties, Lemma 28.26.8). Hence f is separated. By Schemes, Lemma 26.24.1 we see that $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Denote $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ the adjunction mapping. The description of the open $U(\psi)$ in Constructions, Section 27.19 and the definition of ampleness of $\mathcal{L}|_{f^{-1}(V_i)}$ show that $U(\psi) = X$. Moreover, Constructions, Lemma 27.19.1 part (3) shows that the restriction of $r_{\mathcal{L},\psi}$ to $f^{-1}(V_i)$ is the same as the morphism from Properties, Lemma 28.26.9 which is an open immersion according to Properties, Lemma 28.26.11. Hence (5) holds.

Let us show that (4) implies (1). Assume (4). Denote $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \to S$ the structure morphism. Choose $V \subset S$ affine open. By Constructions, Definition 27.16.7 we see that $\pi^{-1}(V) \subset \underline{\text{Proj}}_S(\mathcal{A})$ is equal to $\text{Proj}(A)$ where $A = \mathcal{A}(V)$ as a graded ring. Hence $r_{\mathcal{L},\psi}$ maps $f^{-1}(V)$ isomorphically onto a quasi-compact open of $\text{Proj}(A)$. Moreover, $\mathcal{L}^{\otimes d}$ is isomorphic to the pullback of $\mathcal{O}_{\text{Proj}(A)}(d)$ for some $d \geq 1$. (See part (3) of Constructions, Lemma 27.19.1 and the final statement of Constructions, Lemma 27.14.1.) This implies that $\mathcal{L}|_{f^{-1}(V)}$ is ample by Properties, Lemmas 28.26.12 and 28.26.2.

Assume (6). By the equivalence of (1) - (5) above we see that the property of being relatively ample on X/S is local on S. Hence we may assume that S is affine, and we have to show that $\mathcal{L}$ is ample on X. In this case the morphism $r_{\mathcal{L},\psi}$ is identified with the morphism, also denoted $r_{\mathcal{L},\psi} : X \to \text{Proj}(A)$ associated to the map $\psi : A = \mathcal{A}(V) \to \Gamma_*(X, \mathcal{L})$. (See references above.) As above we also see that $\mathcal{L}^{\otimes d}$ is the pullback of the sheaf $\mathcal{O}_{\text{Proj}(A)}(d)$ for some $d \geq 1$. Moreover, since X is quasi-compact we see that X gets identified with a closed subscheme of a quasi-compact open subscheme $Y \subset \text{Proj}(A)$. By Constructions, Lemma 27.10.6 (see also Properties, Lemma 28.26.12) we see that $\mathcal{O}_Y(d')$ is an ample invertible sheaf on Y for some $d' \geq 1$. Since the restriction of an ample sheaf to a closed subscheme is ample, see Properties, Lemma 28.26.3 we conclude that the pullback of $\mathcal{O}_Y(d')$ is ample. Combining these results with Properties, Lemma 28.26.2 we conclude that $\mathcal{L}$ is ample as desired. $\square$

01VK Lemma 29.37.5. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine. Then $\mathcal{L}$ is f-relatively ample if and only if $\mathcal{L}$ is ample on X. [DG67, II Corollary 4.6.6]

Proof. Immediate from Lemma 29.37.4 and the definitions. $\square$

0891 Lemma 29.37.6. Let $f : X \to S$ be a morphism of schemes. Then f is quasi-affine if and only if $\mathcal{O}_X$ is f-relatively ample. [DG67, II Proposition 5.1.6]

Proof. Follows from Properties, Lemma 28.27.1 and the definitions. $\square$

0892 Lemma 29.37.7. Let $f : X \to Y$ be a morphism of schemes, $\mathcal{M}$ an invertible $\mathcal{O}_Y$-module, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.

(1) If $\mathcal{L}$ is f-ample and $\mathcal{M}$ is ample, then $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample for $a \gg 0$.
(2) If $\mathcal{M}$ is ample and f quasi-affine, then $f^*\mathcal{M}$ is ample.

Proof. Assume $\mathcal{L}$ is f-ample and $\mathcal{M}$ ample. By assumption Y and f are quasi-compact (see Definition 29.37.1 and Properties, Definition 28.26.1). Hence X is quasi-compact. By Properties, Lemma 28.26.8 the scheme Y is separated and by Lemma 29.37.3 the morphism f is separated. Hence X is separated by Schemes, Lemma 26.21.12. Pick $x \in X$. We can choose $m \geq 1$ and $t \in \Gamma(Y, \mathcal{M}^{\otimes m})$ such that Y_t is affine and $f(x) \in Y_t$. Since $\mathcal{L}$ restricts to an ample invertible sheaf on $f^{-1}(Y_t) = X_{f^*t}$ we can choose $n \geq 1$ and $s \in \Gamma(X_{f^*t}, \mathcal{L}^{\otimes n})$ with $x \in (X_{f^*t})_s$ with $(X_{f^*t})_s$ affine. By Properties, Lemma 28.17.2 part (2) whose assumptions are satisfied by the above, there exists an integer $e \geq 1$ and a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes em})$ which restricts to $s(f^*t)^e$ on X_{f^*t}. For any $b > 0$ consider the section $s'' = s'(f^*t)^b$ of $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes (e+b)m}$. Then $X_{s''} = (X_{f^*t})_s$ is an affine open of X containing x. Picking b such that n divides $e + b$ we see $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes (e+b)m}$ is the nth power of $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ for some a and we can get any a divisible by m and big enough. Since X is quasi-compact a finite number of these affine opens cover X. We conclude that for some a sufficiently divisible and large enough the invertible sheaf $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample on X. On the other hand, we know that $\mathcal{M}^{\otimes c}$ (and hence its pullback to X) is globally generated for all $c \gg 0$ by Properties, Proposition 28.26.13. Thus $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a+c}$ is ample (Properties, Lemma 28.26.5) for $c \gg 0$ and (1) is proved.

Part (2) follows from Lemma 29.37.6, Properties, Lemma 28.26.2, and part (1). $\square$

0C4K Lemma 29.37.8. Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. Let $\mathcal{M}$ be an invertible $\mathcal{O}_Y$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If S is quasi-compact, $\mathcal{M}$ is g-ample, and $\mathcal{L}$ is f-ample, then $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is $g \circ f$-ample for $a \gg 0$.

Proof. Let $S = \bigcup_{i=1,\ldots,n} V_i$ be a finite affine open covering. By Lemma 29.37.4 it suffices to prove that $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample on $(g \circ f)^{-1}(V_i)$ for $i = 1, \ldots, n$. Thus the lemma follows from Lemma 29.37.7. $\square$

0893 Lemma 29.37.9. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes. Let $f' : X' \to S'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-ample, then $\mathcal{L}'$ is f'-ample.

Proof. By Lemma 29.37.4 it suffices to find an affine open covering $S' = \bigcup U'_i$ such that $\mathcal{L}'$ restricts to an ample invertible sheaf on $(f')^{-1}(U'_i)$ for all i. We may choose U'_i mapping into an affine open $U_i \subset S$. In this case the morphism $(f')^{-1}(U'_i) \to f^{-1}(U_i)$ is affine as a base change of the affine morphism $U'_i \to U_i$ (Lemma 29.11.8). Thus $\mathcal{L}'|_{(f')^{-1}(U'_i)}$ is ample by Lemma 29.37.7. $\square$

0C4L Lemma 29.37.10. Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\mathcal{L}$ is $g \circ f$-ample and f is quasi-compact[12] then $\mathcal{L}$ is f-ample.

[12]This follows if g is quasi-separated by Schemes, Lemma 26.21.14.

Proof. Assume f is quasi-compact and $\mathcal{L}$ is $g \circ f$-ample. Let $U \subset S$ be an affine open and let $V \subset Y$ be an affine open with $g(V) \subset U$. Then $\mathcal{L}|_{(g\circ f)^{-1}(U)}$ is ample on $(g \circ f)^{-1}(U)$ by assumption. Since $f^{-1}(V) \subset (g \circ f)^{-1}(U)$ we see that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)$ by Properties, Lemma 28.26.14. Namely, $f^{-1}(V) \to (g \circ f)^{-1}(U)$ is a quasi-compact open immersion by Schemes, Lemma 26.21.14 as $(g \circ f)^{-1}(U)$ is separated (Properties, Lemma 28.26.8) and $f^{-1}(V)$ is quasi-compact (as f is quasi-compact). Thus we conclude that $\mathcal{L}$ is f-ample by Lemma 29.37.4. □

29.38. Very ample sheaves

01VL Recall that given a quasi-coherent sheaf $\mathcal{E}$ on a scheme S the projective bundle associated to $\mathcal{E}$ is the morphism $\mathbf{P}(\mathcal{E}) \to S$, where $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E}))$, see Constructions, Definition 27.21.1.

01VM **Definition 29.38.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively very ample* or more precisely *f-relatively very ample*, or *very ample on X/S*, or *f-very ample* if there exist a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Since there is no assumption of quasi-compactness in this definition it is not true in general that a relatively very ample invertible sheaf is a relatively ample invertible sheaf.

01VN **Lemma 29.38.2.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If f is quasi-compact and $\mathcal{L}$ is a relatively very ample invertible sheaf, then $\mathcal{L}$ is a relatively ample invertible sheaf. [DG67, II, Proposition 4.6.2]

Proof. By definition there exists quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Set $\mathcal{A} = \text{Sym}(\mathcal{E})$, so $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\mathcal{A})$ by definition. The graded $\mathcal{O}_S$-algebra $\mathcal{A}$ comes equipped with a map

$$\psi : \mathcal{A} \to \bigoplus\nolimits_{n\geq 0} \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n) \to \bigoplus\nolimits_{n\geq 0} f_*\mathcal{L}^{\otimes n}$$

where the second arrow uses the identification $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By adjointness of f_* and f^* we get a morphism $\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$. We omit the verification that the morphism $r_{\mathcal{L},\psi}$ associated to this map is exactly the immersion i. Hence the result follows from part (6) of Lemma 29.37.4. □

To arrive at the correct converse of this lemma we ask whether given a relatively ample invertible sheaf $\mathcal{L}$ there exists an integer $n \geq 1$ such that $\mathcal{L}^{\otimes n}$ is relatively very ample? In general this is false. There are several things that prevent this from being true:

(1) Even if S is affine, it can happen that no finite integer n works because $X \to S$ is not of finite type, see Example 29.38.4.
(2) The base not being quasi-compact means the result can be prevented from being true even with f finite type. Namely, given a field k there exists a scheme X_d of finite type over k with an ample invertible sheaf $\mathcal{O}_{X_d}(1)$ so that the smallest tensor power of $\mathcal{O}_{X_d}(1)$ which is very ample is the dth power. See Example 29.38.5. Taking f to be the disjoint union of the schemes X_d mapping to the disjoint union of copies of $\text{Spec}(k)$ gives an example.

To see our version of the converse take a look at Lemma 29.39.5 below. We will do some preliminary work before proving it.

07ZR Example 29.38.3. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Set $X = \underline{\text{Proj}}_S(\mathcal{A})$. In this case $\mathcal{O}_X(1)$ is a very ample invertible sheaf on X. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \to \mathbf{P}(\mathcal{A}_1)$ which pulls back $\mathcal{O}_{\mathbf{P}(\mathcal{A}_1)}(1)$ to $\mathcal{O}_X(1)$, see Constructions, Lemma 27.18.5.

01VO Example 29.38.4. Let k be a field. Consider the graded k-algebra

$$A = k[U, V, Z_1, Z_2, Z_3, \ldots]/I \quad \text{with} \quad I = (U^2 - Z_1^2, U^4 - Z_2^2, U^6 - Z_3^2, \ldots)$$

with grading given by $\deg(U) = \deg(V) = \deg(Z_1) = 1$ and $\deg(Z_d) = d$. Note that $X = \text{Proj}(A)$ is covered by $D_+(U)$ and $D_+(V)$. Hence the sheaves $\mathcal{O}_X(n)$ are all invertible and isomorphic to $\mathcal{O}_X(1)^{\otimes n}$. In particular $\mathcal{O}_X(1)$ is ample and f-ample for the morphism $f : X \to \text{Spec}(k)$. We claim that no power of $\mathcal{O}_X(1)$ is f-relatively very ample. Namely, it is easy to see that $\Gamma(X, \mathcal{O}_X(n))$ is the degree n summand of the algebra A. Hence if $\mathcal{O}_X(n)$ were very ample, then X would be a closed subscheme of a projective space over k and hence of finite type over k. On the other hand $D_+(V)$ is the spectrum of $k[t, t_1, t_2, \ldots]/(t^2 - t_1^2, t^4 - t_2^2, t^6 - t_3^2, \ldots)$ which is not of finite type over k.

01VP Example 29.38.5. Let k be an infinite field. Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be pairwise distinct elements of k^*. (This is not strictly necessary, and in fact the example works perfectly well even if all λ_i are equal to 1.) Consider the graded k-algebra

$$A_d = k[U, V, Z]/I_d \quad \text{with} \quad I_d = (Z^2 - \prod\nolimits_{i=1}^{2d} (U - \lambda_i V)).$$

with grading given by $\deg(U) = \deg(V) = 1$ and $\deg(Z) = d$. Then $X_d = \text{Proj}(A_d)$ has ample invertible sheaf $\mathcal{O}_{X_d}(1)$. We claim that if $\mathcal{O}_{X_d}(n)$ is very ample, then $n \geq d$. The reason for this is that Z has degree d, and hence $\Gamma(X_d, \mathcal{O}_{X_d}(n)) = k[U, V]_n$ for $n < d$. Details omitted.

01VQ Lemma 29.38.6. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. If $\mathcal{L}$ is relatively very ample on X/S then f is separated.

Proof. Being separated is local on the base (see Schemes, Section 26.21). An immersion is separated (see Schemes, Lemma 26.23.8). Hence the lemma follows since locally X has an immersion into the homogeneous spectrum of a graded ring which is separated, see Constructions, Lemma 27.8.8. □

01VR Lemma 29.38.7. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is quasi-compact. The following are equivalent

(1) $\mathcal{L}$ is relatively very ample on X/S,
(2) there exists an open covering $S = \bigcup V_j$ such that $\mathcal{L}|_{f^{-1}(V_j)}$ is relatively very ample on $f^{-1}(V_j)/V_j$ for all j,
(3) there exists a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras $\mathcal{A}$ generated in degree 1 over $\mathcal{O}_S$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective and the associated morphism $r_{\mathcal{L},\psi} : X \to \underline{\text{Proj}}_S(\mathcal{A})$ is an immersion, and

(4) f is quasi-separated, the canonical map $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective, and the associated map $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ is an immersion.

Proof. It is clear that (1) implies (2). It is also clear that (4) implies (1); the hypothesis of quasi-separation in (4) is used to guarantee that $f_*\mathcal{L}$ is quasi-coherent via Schemes, Lemma 26.24.1.

Assume (2). We will prove (4). Let $S = \bigcup V_j$ be an open covering as in (2). Set $X_j = f^{-1}(V_j)$ and $f_j : X_j \to V_j$ the restriction of f. We see that f is separated by Lemma 29.38.6 (as being separated is local on the base). By assumption there exists a quasi-coherent $\mathcal{O}_{V_j}$-module $\mathcal{E}_j$ and an immersion $i_j : X_j \to \mathbf{P}(\mathcal{E}_j)$ with $\mathcal{L}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_j)}(1)$. The morphism i_j corresponds to a surjection $f_j^*\mathcal{E}_j \to \mathcal{L}|_{X_j}$, see Constructions, Section 27.21. This map is adjoint to a map $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ such that the composition

$$f_j^*\mathcal{E}_j \to (f^*f_*\mathcal{L})|_{X_j} \to \mathcal{L}|_{X_j}$$

is surjective. We conclude that $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective. Let $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ be the associated morphism. We still have to show that $r_{\mathcal{L},\psi}$ is an immersion; we urge the reader to prove this for themselves. The $\mathcal{O}_{V_j}$-module map $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ determines a homomorphism on symmetric algebras, which in turn defines a morphism

$$\mathbf{P}(f_*\mathcal{L}|_{V_j}) \supset U_j \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

where U_j is the open subscheme of Constructions, Lemma 27.18.1. The compatibility of ψ with $\mathcal{E}_j \to f_*\mathcal{L}|_{V_j}$ shows that $r_{\mathcal{L},\psi}(X_j) \subset U_j$ and that there is a factorization

$$X_j \xrightarrow{r_{\mathcal{L},\psi}} U_j \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

We omit the verification. This shows that $r_{\mathcal{L},\psi}$ is an immersion.

At this point we see that (1), (2) and (4) are equivalent. Clearly (4) implies (3). Assume (3). We will prove (1). Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras generated in degree 1 over $\mathcal{O}_S$. Consider the map of graded $\mathcal{O}_S$-algebras $\text{Sym}(\mathcal{A}_1) \to \mathcal{A}$. This is surjective by hypothesis and hence induces a closed immersion

$$\underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow \mathbf{P}(\mathcal{A}_1)$$

which pulls back $\mathcal{O}(1)$ to $\mathcal{O}(1)$, see Constructions, Lemma 27.18.5. Hence it is clear that (3) implies (1). $\square$

0B3F Lemma 29.38.8. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes. Let $f' : X' \to S'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-very ample, then $\mathcal{L}'$ is f'-very ample.

Proof. By Definition 29.38.1 there exists there exist a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. The base change of $\mathbf{P}(\mathcal{E})$ to S' is the projective bundle associated to the pullback $\mathcal{E}'$ of $\mathcal{E}$ and the pullback of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is $\mathcal{O}_{\mathbf{P}(\mathcal{E}')}(1)$, see Constructions, Lemma 27.16.10. Finally, the base change of an immersion is an immersion (Schemes, Lemma 26.18.2). $\square$

29.39. Ample and very ample sheaves relative to finite type morphisms

02NO In fact most of the material in this section is about the notion of a (quasi-)projective morphism which we have not defined yet.

02NP **Lemma 29.39.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that

(1) the invertible sheaf $\mathcal{L}$ is very ample on X/S,
(2) the morphism $X \to S$ is of finite type, and
(3) S is affine.

Then there exist an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Assume (1), (2) and (3). Condition (3) means $S = \text{Spec}(R)$ for some ring R. Condition (1) means by definition there exists a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $\alpha : X \to \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} = \alpha^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Write $\mathcal{E} = \widetilde{M}$ for some R-module M. Thus we have

$$\mathbf{P}(\mathcal{E}) = \text{Proj}(\text{Sym}_R(M)).$$

Since α is an immersion, and since the topology of $\text{Proj}(\text{Sym}_R(M))$ is generated by the standard opens $D_+(f)$, $f \in \text{Sym}^d_R(M)$, $d \geq 1$, we can find for each $x \in X$ an $f \in \text{Sym}^d_R(M)$, $d \geq 1$, with $\alpha(x) \in D_+(f)$ such that

$$\alpha|_{\alpha^{-1}(D_+(f))} : \alpha^{-1}(D_+(f)) \to D_+(f)$$

is a closed immersion. Condition (2) implies X is quasi-compact. Hence we can find a finite collection of elements $f_j \in \text{Sym}^{d_j}_R(M)$, $d_j \geq 1$ such that for each $f = f_j$ the displayed map above is a closed immersion and such that $\alpha(X) \subset \bigcup D_+(f_j)$. Write $U_j = \alpha^{-1}(D_+(f_j))$. Note that U_j is affine as a closed subscheme of the affine scheme $D_+(f_j)$. Write $U_j = \text{Spec}(A_j)$. Condition (2) also implies that A_j is of finite type over R, see Lemma 29.15.2. Choose finitely many $x_{j,k} \in A_j$ which generate A_j as a R-algebra. Since $\alpha|_{U_j}$ is a closed immersion we see that $x_{j,k}$ is the image of an element

$$f_{j,k}/f_j^{e_{j,k}} \in \text{Sym}_R(M)_{(f_j)} = \Gamma(D_+(f_j), \mathcal{O}_{\text{Proj}(\text{Sym}_R(M))}).$$

Finally, choose $n \geq 1$ and elements $y_0, \ldots, y_n \in M$ such that each of the polynomials $f_j, f_{j,k} \in \text{Sym}_R(M)$ is a polynomial in the elements y_t with coefficients in R. Consider the graded ring map

$$\psi : R[Y_0, \ldots, Y_n] \longrightarrow \text{Sym}_R(M), \quad Y_i \longmapsto y_i.$$

Denote F_j, $F_{j,k}$ the elements of $R[Y_0, \ldots, Y_n]$ such that $\psi(F_j) = f_j$ and $\psi(F_{j,k}) = f_{j,k}$. By Constructions, Lemma 27.11.1 we obtain an open subscheme

$$U(\psi) \subset \text{Proj}(\text{Sym}_R(M))$$

and a morphism $r_\psi : U(\psi) \to \mathbf{P}^n_R$. This morphism satisfies $r_\psi^{-1}(D_+(F_j)) = D_+(f_j)$, and hence we see that $\alpha(X) \subset U(\psi)$. Moreover, it is clear that

$$i = r_\psi \circ \alpha : X \longrightarrow \mathbf{P}^n_R$$

is still an immersion since $i^\sharp(F_{j,k}/F_j^{e_{j,k}}) = x_{j,k} \in A_j = \Gamma(U_j, \mathcal{O}_X)$ by construction. Moreover, the morphism r_ψ comes equipped with a map $\theta : r_\psi^*\mathcal{O}_{\mathbf{P}^n_R}(1) \to \mathcal{O}_{\text{Proj}(\text{Sym}_R(M))}(1)|_{U(\psi)}$ which is an isomorphism in this case (for construction θ see lemma cited above; some details omitted). Since the original map α was assumed to have the property that $\mathcal{L} = \alpha^*\mathcal{O}_{\text{Proj}(\text{Sym}_R(M))}(1)$ we win. □

04II Lemma 29.39.2. Let $\pi : X \to S$ be a morphism of schemes. Assume that X is quasi-affine and that π is locally of finite type. Then there exist $n \geq 0$ and an immersion $i : X \to \mathbf{A}^n_S$ over S.

Proof. Let $A = \Gamma(X, \mathcal{O}_X)$. By assumption X is quasi-compact and is identified with an open subscheme of $\text{Spec}(A)$, see Properties, Lemma 28.18.4. Moreover, the set of opens X_f, for those $f \in A$ such that X_f is affine, forms a basis for the topology of X, see the proof of Properties, Lemma 28.18.4. Hence we can find a finite number of $f_j \in A$, $j = 1, \ldots, m$ such that $X = \bigcup X_{f_j}$, and such that $\pi(X_{f_j}) \subset V_j$ for some affine open $V_j \subset S$. By Lemma 29.15.2 the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(X_{f_j}) = A_{f_j}$ are of finite type. Thus we may choose $a_1, \ldots, a_N \in A$ such that the elements $a_1, \ldots, a_N, 1/f_j$ generate A_{f_j} over $\mathcal{O}(V_j)$ for each j. Take $n = m + N$ and let

$$i : X \longrightarrow \mathbf{A}^n_S$$

be the morphism given by the global sections $f_1, \ldots, f_m, a_1, \ldots, a_N$ of the structure sheaf of X. Let $D(x_j) \subset \mathbf{A}^n_S$ be the open subscheme where the jth coordinate function is nonzero. Then for $1 \leq j \leq m$ we have $i^{-1}(D(x_j)) = X_{f_j}$ and the induced morphism $X_{f_j} \to D(x_j)$ factors through the affine open $\text{Spec}(\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j])$ of $D(x_j)$. Since the ring map $\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j] \to A_{f_j}$ is surjective by construction we conclude that $i^{-1}(D(x_j)) \to D(x_j)$ is an immersion as desired. □

01VS Lemma 29.39.3. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that

(1) the invertible sheaf $\mathcal{L}$ is ample on X, and
(2) the morphism $X \to S$ is locally of finite type.

Then there exists a $d_0 \geq 1$ such that for every $d \geq d_0$ there exist an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Let $A = \Gamma_*(X, \mathcal{L}) = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$. By Properties, Proposition 28.26.13 the set of affine opens X_a with $a \in A_+$ homogeneous forms a basis for the topology of X. Hence we can find finitely many such elements $a_0, \ldots, a_n \in A_+$ such that

(1) we have $X = \bigcup_{i=0,\ldots,n} X_{a_i}$,
(2) each X_{a_i} is affine, and
(3) each X_{a_i} maps into an affine open $V_i \subset S$.

By Lemma 29.15.2 we see that the ring maps $\mathcal{O}_S(V_i) \to \mathcal{O}_X(X_{a_i})$ are of finite type. Hence we can find finitely many elements $f_{ij} \in \mathcal{O}_X(X_{a_i})$, $j = 1, \ldots, n_i$ which generate $\mathcal{O}_X(X_{a_i})$ as an $\mathcal{O}_S(V_i)$-algebra. By Properties, Lemma 28.17.2 we may write each f_{ij} as $a_{ij}/a_i^{e_{ij}}$ for some $a_{ij} \in A_+$ homogeneous. Let N be a positive integer which is a common multiple of all the degrees of the elements a_i, a_{ij}. Consider the elements

$$a_i^{N/\deg(a_i)},\ a_{ij}a_i^{(N/\deg(a_i))-e_{ij}} \in A_N.$$

By construction these generate the invertible sheaf $\mathcal{L}^{\otimes N}$ over X. Hence they give rise to a morphism

$$j : X \longrightarrow \mathbf{P}^m_S \quad \text{with } m = n + \sum n_i$$

over S, see Constructions, Lemma 27.13.1 and Definition 27.13.2. Moreover, $j^*\mathcal{O}_{\mathbf{P}_S}(1) = \mathcal{L}^{\otimes N}$. We name the homogeneous coordinates $T_0, \ldots, T_n, T_{ij}$ instead of $T_0, \ldots, T_m$. For $i = 0, \ldots, n$ we have $i^{-1}(D_+(T_i)) = X_{a_i}$. Moreover, pulling back the element T_{ij}/T_i via $j^\sharp$ we get the element $f_{ij} \in \mathcal{O}_X(X_{a_i})$. Hence the morphism j restricted to X_{a_i} gives a closed immersion of X_{a_i} into the affine open $D_+(T_i) \cap \mathbf{P}^m_{V_i}$ of $\mathbf{P}^N_S$. Hence we conclude that the morphism j is an immersion. This implies the lemma holds for some d and n which is enough in virtually all applications.

This proves that for one $d_2 \geq 1$ (namely $d_2 = N$ above), some $m \geq 0$ there exists some immersion $j : X \to \mathbf{P}^m_S$ given by global sections $s'_0, \ldots, s'_m \in \Gamma(X, \mathcal{L}^{\otimes d_2})$. By Properties, Proposition 28.26.13 we know there exists an integer d_1 such that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq d_1$. Set $d_0 = d_1 + d_2$. We claim that the lemma holds with this value of d_0. Namely, given an integer $d \geq d_0$ we may choose $s''_1, \ldots, s''_t \in \Gamma(X, \mathcal{L}^{\otimes d - d_2})$ which generate $\mathcal{L}^{\otimes d - d_2}$ over X. Set $k = (m+1)t$ and denote $s_0, \ldots, s_k$ the collection of sections $s'_\alpha s''_\beta$, $\alpha = 0, \ldots, m$, $\beta = 1, \ldots, t$. These generate $\mathcal{L}^{\otimes d}$ over X and therefore define a morphism

$$i : X \longrightarrow \mathbf{P}^{k-1}_S$$

such that $i^*\mathcal{O}_{\mathbf{P}^n_S}(1) \cong \mathcal{L}^{\otimes d}$. To see that i is an immersion, observe that i is the composition

$$X \longrightarrow \mathbf{P}^m_S \times_S \mathbf{P}^{t-1}_S \longrightarrow \mathbf{P}^{k-1}_S$$

where the first morphism is (j, j') with j' given by $s''_1, \ldots, s''_t$ and the second morphism is the Segre embedding (Constructions, Lemma 27.13.6). Since j is an immersion, so is (j, j') (apply Lemma 29.3.1 to $X \to \mathbf{P}^m_S \times_S \mathbf{P}^{t-1}_S \to \mathbf{P}^m_S$). Thus i is a composition of immersions and hence an immersion (Schemes, Lemma 26.24.3). □

01VT Lemma 29.39.4. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine and f of finite type. The following are equivalent

(1) $\mathcal{L}$ is ample on X,
(2) $\mathcal{L}$ is f-ample,
(3) $\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,
(4) $\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$,
(5) for some $d \geq 1$ there exist $n \geq 1$ and an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$, and
(6) for all $d \gg 1$ there exist $n \geq 1$ and an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. The equivalence of (1) and (2) is Lemma 29.37.5. The implication (2) ⇒ (6) is Lemma 29.39.3. Trivially (6) implies (5). As $\mathbf{P}^n_S$ is a projective bundle over S (see Constructions, Lemma 27.21.5) we see that (5) implies (3) and (6) implies (4) from the definition of a relatively very ample sheaf. Trivially (4) implies (3). To finish we have to show that (3) implies (2) which follows from Lemma 29.38.2 and Lemma 29.37.2. □

01VU Lemma 29.39.5. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S quasi-compact and f of finite type. The following are equivalent

(1) $\mathcal{L}$ is f-ample,
(2) $\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,
(3) $\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$.

Proof. Trivially (3) implies (2). Lemma 29.38.2 guarantees that (2) implies (1) since a morphism of finite type is quasi-compact by definition. Assume that $\mathcal{L}$ is f-ample. Choose a finite affine open covering $S = V_1 \cup \ldots \cup V_m$. Write $X_i = f^{-1}(V_i)$. By Lemma 29.39.4 above we see there exists a d_0 such that $\mathcal{L}^{\otimes d}$ is relatively very ample on X_i/V_i for all $d \geq d_0$. Hence we conclude (1) implies (3) by Lemma 29.38.7. □

The following two lemmas provide the most used and most useful characterizations of relatively very ample and relatively ample invertible sheaves when the morphism is of finite type.

02NQ Lemma 29.39.6. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is of finite type. The following are equivalent:

(1) $\mathcal{L}$ is f-relatively very ample, and
(2) there exist an open covering $S = \bigcup V_j$, for each j an integer n_j, and immersions
$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$
over V_j such that $\mathcal{L}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 29.39.1 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 29.38.7. □

02NR Lemma 29.39.7. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is of finite type. The following are equivalent:

(1) $\mathcal{L}$ is f-relatively ample, and
(2) there exist an open covering $S = \bigcup V_j$, for each j an integers $d_j \geq 1$, $n_j \geq 0$, and immersions
$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$
over V_j such that $\mathcal{L}^{\otimes d_j}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 29.39.4 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 29.37.4. □

0FVC Lemma 29.39.8. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{N}$, $\mathcal{L}$ be invertible $\mathcal{O}_X$-modules. Assume S is quasi-compact, f is of finite type, and $\mathcal{L}$ is f-ample. Then $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$.

Proof. By Lemma 29.39.6 we reduce to the case S is affine. Combining Lemma 29.39.4 and Properties, Proposition 28.26.13 we can find an integer d_0 such that $\mathcal{N} \otimes \mathcal{L}^{\otimes d_0}$ is globally generated. Choose global sections $s_0, \ldots, s_n$ of $\mathcal{N} \otimes \mathcal{L}^{\otimes d_0}$ which generate it. This determines a morphism $j : X \to \mathbf{P}^n_S$ over S. By Lemma 29.39.4 we can also pick an integer d_1 such that for all $d \geq d_1$ there exist sections $t_{d,0}, \ldots, t_{d,n(d)}$ of $\mathcal{L}^{\otimes d}$ which generate it and define an immersion
$$j_d = \varphi_{\mathcal{L}^{\otimes d}, t_{d,0}, \ldots, t_{d,n(d)}} : X \longrightarrow \mathbf{P}^{n(d)}_S$$
over S. Then for $d \geq d_0 + d_1$ we can consider the morphism
$$\varphi_{\mathcal{N} \otimes \mathcal{L}^{\otimes d}, s_j \otimes t_{d-d_0,i}} : X \longrightarrow \mathbf{P}^{(n+1)(n(d-d_0)+1)-1}_S$$

This morphism is an immersion as it is the composition

$$X \to \mathbf{P}^n_S \times_S \mathbf{P}^{n(d-d_0)}_S \to \mathbf{P}^{(n+1)(n(d-d_0)+1)-1}_S$$

where the first morphism is (j, j_{d-d_0}) and the second is the Segre embedding (Constructions, Lemma 27.13.6). Since j is an immersion, so is (j, j_{d-d_0}) (apply Lemma 29.3.1). We have a composition of immersions and hence an immersion (Schemes, Lemma 26.24.3). $\square$

29.40. Quasi-projective morphisms

01VV The discussion in the previous section suggests the following definitions. We take our definition of quasi-projective from [DG67]. The version with the letter "H" is the definition in [Har77].

01VW Definition 29.40.1. Let $f : X \to S$ be a morphism of schemes. [DG67, II, Definition 5.3.1] and [Har77, page 103]

(1) We say f is quasi-projective if f is of finite type and there exists an f-relatively ample invertible $\mathcal{O}_X$-module.
(2) We say f is H-quasi-projective if there exists a quasi-compact immersion $X \to \mathbf{P}^n_S$ over S for some n.[13]
(3) We say f is locally quasi-projective if there exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is quasi-projective.

As this definition suggests the property of being quasi-projective is not local on S. At a later stage we will be able to say more about the category of quasi-projective schemes, see More on Morphisms, Section 37.49.

0B3G Lemma 29.40.2. A base change of a quasi-projective morphism is quasi-projective.

Proof. This follows from Lemmas 29.15.4 and 29.37.9. $\square$

0C4M Lemma 29.40.3. Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If S is quasi-compact and f and g are quasi-projective, then $g \circ f$ is quasi-projective.

Proof. This follows from Lemmas 29.15.3 and 29.37.8. $\square$

01VX Lemma 29.40.4. Let $f : X \to S$ be a morphism of schemes. If f is quasi-projective, or H-quasi-projective or locally quasi-projective, then f is separated of finite type.

Proof. Omitted. $\square$

01VY Lemma 29.40.5. A H-quasi-projective morphism is quasi-projective.

Proof. Omitted. $\square$

01VZ Lemma 29.40.6. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is locally quasi-projective.
(2) There exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is H-quasi-projective.

[13] This is not exactly the same as the definition in Hartshorne. Namely, the definition in Hartshorne (8th corrected printing, 1997) is that f should be the composition of an open immersion followed by a H-projective morphism (see Definition 29.43.1), which does not imply f is quasi-compact. See Lemma 29.43.11 for the implication in the other direction.

Proof. By Lemma 29.40.5 we see that (2) implies (1). Assume (1). The question is local on S and hence we may assume S is affine, X of finite type over S and $\mathcal{L}$ is a relatively ample invertible sheaf on X/S. By Lemma 29.39.4 we may assume $\mathcal{L}$ is ample on X. By Lemma 29.39.3 we see that there exists an immersion of X into a projective space over S, i.e., X is H-quasi-projective over S as desired. □

0B3H Lemma 29.40.7. A quasi-affine morphism of finite type is quasi-projective. [DG67, II, Proposition 5.3.4 (i)]

Proof. This follows from Lemma 29.37.6. □

0C4N Lemma 29.40.8. Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. If $g \circ f$ is quasi-projective and f is quasi-compact[14], then f is quasi-projective.

Proof. Observe that f is of finite type by Lemma 29.15.8. Thus the lemma follows from Lemma 29.37.10 and the definitions. □

29.41. Proper morphisms

01W0 The notion of a proper morphism plays an important role in algebraic geometry. An important example of a proper morphism will be the structure morphism $\mathbf{P}^n_S \to S$ of projective n-space, and this is in fact the motivating example leading to the definition.

01W1 Definition 29.41.1. Let $f : X \to S$ be a morphism of schemes. We say f is proper if f is separated, finite type, and universally closed.

The morphism from the affine line with zero doubled to the affine line is of finite type and universally closed, so the separation condition is necessary in the definition above. In the rest of this section we prove some of the basic properties of proper morphisms and of universally closed morphisms.

02K7 Lemma 29.41.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is universally closed.
(2) There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is universally closed for all indices j.

Proof. This is clear from the definition. □

01W2 Lemma 29.41.3. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is proper.
(2) There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is proper for all indices j.

Proof. Omitted. □

01W3 Lemma 29.41.4. The composition of proper morphisms is proper. The same is true for universally closed morphisms.

[14] This follows if g is quasi-separated by Schemes, Lemma 26.21.14.

Proof. A composition of closed morphisms is closed. If $X \to Y \to Z$ are universally closed morphisms and $Z' \to Z$ is any morphism, then we see that $Z' \times_Z X = (Z' \times_Z Y) \times_Y X \to Z' \times_Z Y$ is closed and $Z' \times_Z Y \to Z'$ is closed. Hence the result for universally closed morphisms. We have seen that "separated" and "finite type" are preserved under compositions (Schemes, Lemma 26.21.12 and Lemma 29.15.3). Hence the result for proper morphisms. $\square$

01W4 Lemma 29.41.5. The base change of a proper morphism is proper. The same is true for universally closed morphisms.

Proof. This is true by definition for universally closed morphisms. It is true for separated morphisms (Schemes, Lemma 26.21.12). It is true for morphisms of finite type (Lemma 29.15.4). Hence it is true for proper morphisms. $\square$

01W5 Lemma 29.41.6. A closed immersion is proper, hence a fortiori universally closed.

Proof. The base change of a closed immersion is a closed immersion (Schemes, Lemma 26.18.2). Hence it is universally closed. A closed immersion is separated (Schemes, Lemma 26.23.8). A closed immersion is of finite type (Lemma 29.15.5). Hence a closed immersion is proper. $\square$

01W6 Lemma 29.41.7. Suppose given a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[rr] \arrow[rd] & & Y \arrow[ld] \\ & S & \end{tikzcd}$$

with Y separated over S.

(1) If $X \to S$ is universally closed, then the morphism $X \to Y$ is universally closed.
(2) If X is proper over S, then the morphism $X \to Y$ is proper.

In particular, in both cases the image of X in Y is closed.

Proof. Assume that $X \to S$ is universally closed (resp. proper). We factor the morphism as $X \to X \times_S Y \to Y$. The first morphism is a closed immersion, see Schemes, Lemma 26.21.10. Hence the first morphism is proper (Lemma 29.41.6). The projection $X \times_S Y \to Y$ is the base change of a universally closed (resp. proper) morphism and hence universally closed (resp. proper), see Lemma 29.41.5. Thus $X \to Y$ is universally closed (resp. proper) as the composition of universally closed (resp. proper) morphisms (Lemma 29.41.4). $\square$

The proof of the following lemma is due to Bjorn Poonen, see this location.

04XU Lemma 29.41.8. A universally closed morphism of schemes is quasi-compact. Due to Bjorn Poonen.

Proof. Let $f : X \to S$ be a morphism. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Schemes, Lemma 26.19.2 there exists an affine open $V \subset S$ such that $f^{-1}(V)$ is not quasi-compact. To achieve our goal it suffices to show that $f^{-1}(V) \to V$ is not universally closed, hence we may assume that $S = \mathrm{Spec}(A)$ for some ring A.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are affine open subschemes of X. Let $T = \mathrm{Spec}(A[y_i; i \in I])$. Let $T_i = D(y_i) \subset T$. Let Z be the closed set $(X \times_S T) -$

$\bigcup_{i \in I}(X_i \times_S T_i)$. It suffices to prove that the image $f_T(Z)$ of Z under $f_T : X \times_S T \to T$ is not closed.

There exists a point $s \in S$ such that there is no neighborhood U of s in S such that X_U is quasi-compact. Otherwise we could cover S with finitely many such U and Schemes, Lemma 26.19.2 would imply f quasi-compact. Fix such an $s \in S$.

First we check that $f_T(Z_s) \neq T_s$. Let $t \in T$ be the point lying over s with $\kappa(t) = \kappa(s)$ such that $y_i = 1$ in $\kappa(t)$ for all i. Then $t \in T_i$ for all i, and the fiber of $Z_s \to T_s$ above t is isomorphic to $(X - \bigcup_{i \in I} X_i)_s$, which is empty. Thus $t \in T_s - f_T(Z_s)$.

Assume $f_T(Z)$ is closed in T. Then there exists an element $g \in A[y_i; i \in I]$ with $f_T(Z) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(s)$. Hence this coefficient is invertible on some neighborhood U of s. Let J be the finite set of $j \in I$ such that y_j appears in g. Since X_U is not quasi-compact, we may choose a point $x \in X - \bigcup_{j \in J} X_j$ lying above some $u \in U$. Since g has a coefficient that is invertible on U, we can find a point $t' \in T$ lying above u such that $t' \notin V(g)$ and $t' \in V(y_i)$ for all $i \notin J$. This is true because $V(y_i; i \in I, i \notin J) = \mathrm{Spec}(A[t_j; j \in J])$ and the set of points of this scheme lying over u is bijective with $\mathrm{Spec}(\kappa(u)[t_j; j \in J])$. In other words $t' \notin T_i$ for each $i \notin J$. By Schemes, Lemma 26.17.5 we can find a point z of $X \times_S T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin X_j$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in Z$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(Z)$ closed" is wrong and we conclude indeed that f_T is not closed, as desired. □

The following lemma says that the image of a proper scheme (in a separated scheme of finite type over the base) is proper.

03GN Lemma 29.41.9. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X is universally closed over S and f is surjective then Y is universally closed over S. In particular, if also Y is separated and locally of finite type over S, then Y is proper over S.

Proof. Assume X is universally closed and f surjective. Denote $p : X \to S$, $q : Y \to S$ the structure morphisms. Let $S' \to S$ be a morphism of schemes. The base change $f' : X_{S'} \to Y_{S'}$ is surjective (Lemma 29.9.4), and the base change $p' : X_{S'} \to S'$ is closed. If $T \subset Y_{S'}$ is closed, then $(f')^{-1}(T) \subset X_{S'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. This proves the first statement. Thus $Y \to S$ is quasi-compact by Lemma 29.41.8 and hence $Y \to S$ is proper by definition if in addition $Y \to S$ is locally of finite type and separated. □

0AH6 Lemma 29.41.10. Suppose given a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Assume

(1) $X \to S$ is a universally closed (for example proper) morphism, and
(2) $Y \to S$ is separated and locally of finite type.

Then the scheme theoretic image $Z \subset Y$ of h is proper over S and $X \to Z$ is surjective.

Proof. The scheme theoretic image of h is constructed in Section 29.6. Since f is quasi-compact (Lemma 29.41.8) we find that h is quasi-compact (Schemes, Lemma 26.21.14). Hence $h(X) \subset Z$ is dense (Lemma 29.6.3). On the other hand $h(X)$ is closed in Y (Lemma 29.41.7) hence $X \to Z$ is surjective. Thus $Z \to S$ is a proper (Lemma 29.41.9). ☐

The target of a separated scheme under a surjective universally closed morphism is separated.

09MQ Lemma 29.41.11. Let S be a scheme. Let $f : X \to Y$ be a surjective universally closed morphism of schemes over S.

(1) If X is quasi-separated, then Y is quasi-separated.
(2) If X is separated, then Y is separated.
(3) If X is quasi-separated over S, then Y is quasi-separated over S.
(4) If X is separated over S, then Y is separated over S.

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = \mathrm{Spec}(\mathbf{Z})$ (see Schemes, Definition 26.21.3). Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/S}"'] \arrow[d] & X \times_S X \arrow[d] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_S X \to X \times_S Y \to Y \times_S Y$. Hence it is also quasi-compact, see Lemma 29.41.8.

Assume X is quasi-separated over S, i.e., $\Delta_{X/S}$ is quasi-compact. If $V \subset Y \times_S Y$ is a quasi-compact open, then $V \times_{Y \times_S Y} X \to \Delta_{Y/S}^{-1}(V)$ is surjective and $V \times_{Y \times_S Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/S}$ is quasi-compact, i.e., Y is quasi-separated over S.

Assume X is separated over S, i.e., $\Delta_{X/S}$ is a closed immersion. Then $X \to Y \times_S Y$ is closed as a composition of closed morphisms. Since $X \to Y$ is surjective, it follows that $\Delta_{Y/S}(Y)$ is closed in $Y \times_S Y$. Hence Y is separated over S by the discussion following Schemes, Definition 26.21.3. ☐

29.42. Valuative criteria

0BX4 We have already discussed the valuative criterion for universal closedness and for separatedness in Schemes, Sections 26.20 and 26.22. In this section we will discuss some consequences and variants. In Limits, Section 32.15 we will show that it suffices to consider discrete valuation rings when working with locally Noetherian schemes and morphisms of finite type.

0BX5 Lemma 29.42.1 (Valuative criterion for properness). Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume f is of finite type and quasi-separated. Then the following are equivalent [DG67, II Theorem 7.3.8]

(1) f is proper,

(2) f satisfies the valuative criterion (Schemes, Definition 26.20.3),
(3) given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute.

Proof. Part (3) is a reformulation of (2). Thus the lemma is a formal consequence of Schemes, Proposition 26.20.6 and Lemma 26.22.2 and the definitions. □

One usually does not have to consider all possible diagrams when testing the valuative criterion. We will call a valuative criterion as in the next lemma a "refined valuative criterion".

0894 Lemma 29.42.2. Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that f and h are quasi-compact and that $h(U)$ is dense in X. If given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dashed] & & S \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is universally closed. If moreover f is quasi-separated, then f is separated.

Proof. To prove f is universally closed we will verify the existence part of the valuative criterion for f which suffices by Schemes, Proposition 26.20.6. To do this, consider a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & S \end{tikzcd}$$

where A is a valuation ring and K is the fraction field of A. Note that since valuation rings and fields are reduced, we may replace U, X, and S by their respective reductions by Schemes, Lemma 26.12.7. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \to X$ is X, see Lemma 29.6.7. We may also replace S by an affine open through which the morphism $\mathrm{Spec}(A) \to S$ factors. Thus we may assume that $S = \mathrm{Spec}(R)$.

Let $\mathrm{Spec}(B) \subset X$ be an affine open through which the morphism $\mathrm{Spec}(K) \to X$ factors. Choose a polynomial algebra P over B and a B-algebra surjection $P \to K$. Then $\mathrm{Spec}(P) \to X$ is flat. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \to \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma 29.25.16. By Lemma 29.6.5 we can find a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U \times_X \mathrm{Spec}(P) \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] & \mathrm{Spec}(P) \end{tikzcd}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B-algebra map $\varphi : K \to A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A'/\mathfrak{m}_{A'}$ (Algebra, Lemma 10.50.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 10.50.10. As C is an R-algebra with fraction field K', we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & U \arrow[r] & X \arrow[d] \\ \mathrm{Spec}(C) \arrow[rr] \arrow[rru, dashed] & & S \end{tikzcd}$$

as in the statement of the lemma. Thus a dotted arrow fitting into the diagram as indicated. By the uniqueness assumption of the lemma the composition $\mathrm{Spec}(A') \to \mathrm{Spec}(C) \to X$ agrees with the given morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(P) \to \mathrm{Spec}(B) \subset X$. Hence the restriction of the morphism to the spectrum of $C/\mathfrak{m}_{A'} = A''$ induces the given morphism $\mathrm{Spec}(K'') = \mathrm{Spec}(A'/\mathfrak{m}_{A'}) \to \mathrm{Spec}(K) \to X$. Let $x \in X$ be the image of the closed point of $\mathrm{Spec}(A'') \to X$. The image of the induced ring map $\mathcal{O}_{X,x} \to A''$ is a local subring which is contained in $K \subset K''$. Since A is maximal for the relation of domination in K and since $A \subset A''$, we have $A = K \cap A''$. We conclude that $\mathcal{O}_{X,x} \to A''$ factors through $A \subset A''$. In this way we obtain our desired arrow $\mathrm{Spec}(A) \to X$.

Finally, assume f is quasi-separated. Then $\Delta : X \to X \times_S X$ is quasi-compact. Given a solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dashed] & & X \times_S X \end{tikzcd}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute. Namely, the lower horizontal arrow is the same thing as a pair of morphisms $\mathrm{Spec}(A) \to X$ which can serve as the dotted arrow in the diagram of the lemma. Thus the required uniqueness shows that the lower horizontal arrow factors through Δ. Hence we can apply the result we just proved to $\Delta : X \to X \times_S X$ and $h : U \to X$ and conclude that Δ is universally closed. Clearly this means that f is separated. $\square$

0895 Remark 29.42.3. The assumption on uniqueness of the dotted arrows in Lemma 29.42.2 is necessary (details omitted). Of course, uniqueness is guaranteed if f is separated (Schemes, Lemma 26.22.1).

0BX6 **Lemma 29.42.4.** Let S be a scheme. Let X, Y be schemes over S. Let $s \in S$ and $x \in X$, $y \in Y$ points over s.

(1) Let $f, g : X \to Y$ be morphisms over S such that $f(x) = g(x) = y$ and $f_x^\sharp = g_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Then there is an open neighbourhood $U \subset X$ with $f|_U = g|_U$ in the following cases
   (a) Y is locally of finite type over S,
   (b) X is integral,
   (c) X is locally Noetherian, or
   (d) X is reduced with finitely many irreducible components.
(2) Let $\varphi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a local $\mathcal{O}_{S,s}$-algebra map. Then there exists an open neighbourhood $U \subset X$ of x and a morphism $f : U \to Y$ mapping x to y with $f_x^\sharp = \varphi$ in the following cases
   (a) Y is locally of finite presentation over S,
   (b) Y is locally of finite type and X is integral,
   (c) Y is locally of finite type and X is locally Noetherian, or
   (d) Y is locally of finite type and X is reduced with finitely many irreducible components.

Proof. Proof of (1). We may replace X, Y, S by suitable affine open neighbourhoods of x, y, s and reduce to the following algebra problem: given a ring R, two R-algebra maps $\varphi, \psi : B \to A$ such that

(1) $R \to B$ is of finite type, or A is a domain, or A is Noetherian, or A is reduced and has finitely many minimal primes,
(2) the two maps $B \to A_\mathfrak{p}$ are the same for some prime $\mathfrak{p} \subset A$,

show that φ, ψ define the same map $B \to A_g$ for a suitable $g \in A$, $g \notin \mathfrak{p}$. If $R \to B$ is of finite type, let $t_1, \ldots, t_m \in B$ be generators of B as an R-algebra. For each j we can find $g_j \in A$, $g_j \notin \mathfrak{p}$ such that $\varphi(t_j)$ and $\psi(t_j)$ have the same image in A_{g_j}. Then we set $g = \prod g_j$. In the other cases (if A is a domain, Noetherian, or reduced with finitely many minimal primes), we can find a $g \in A$, $g \notin \mathfrak{p}$ such that $A_g \subset A_\mathfrak{p}$. See Algebra, Lemma 10.31.9. Thus the maps $B \to A_g$ are equal as desired.

Proof of (2). To do this we may replace X, Y, and S by suitable affine opens. Say $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(R)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. Let $\mathfrak{q} \subset B$ be the prime corresponding to y. Then φ is a local R-algebra map $\varphi : B_\mathfrak{q} \to A_\mathfrak{p}$. If $R \to B$ is a ring map of finite presentation, then there exists a $g \in A \setminus \mathfrak{p}$ and an R-algebra map $B \to A_g$ such that

$$\begin{tikzcd} B_\mathfrak{q} \arrow[r, "\varphi"'] & A_\mathfrak{p} \\ B \arrow[u] \arrow[r] & A_g \arrow[u] \end{tikzcd}$$

commutes, see Algebra, Lemmas 10.127.3 and 10.9.9. The induced morphism $\mathrm{Spec}(A_g) \to \mathrm{Spec}(B)$ works. If B is of finite type over R, let $t_1, \ldots, t_m \in B$ be generators of B as an R-algebra. Then we can choose $g_j \in A$, $g_j \notin \mathfrak{p}$ such that $\varphi(t_j) \in \mathrm{Im}(A_{g_j} \to A_\mathfrak{p})$. Thus after replacing A by $A[1/\prod g_j]$ we may assume that B maps into the image of $A \to A_\mathfrak{p}$. If we can find a $g \in A$, $g \notin \mathfrak{p}$ such that $A_g \to A_\mathfrak{p}$ is injective, then we'll get the desired R-algebra map $B \to A_g$. Thus the proof is finished by another application of See Algebra, Lemma 10.31.9. $\square$

0BX7 **Lemma 29.42.5.** Let S be a scheme. Let X, Y be schemes over S. Let $x \in X$. Let $U \subset X$ be an open and let $f : U \to Y$ be a morphism over S. Assume

(1) x is in the closure of U,
(2) X is reduced with finitely many irreducible components or X is Noetherian,
(3) $\mathcal{O}_{X,x}$ is a valuation ring,
(4) $Y \to S$ is proper

Then there exists an open $U \subset U' \subset X$ containing x and an S-morphism $f' : U' \to Y$ extending f.

Proof. It is harmless to replace X by an open neighbourhood of x in X (small detail omitted). By Properties, Lemma 28.29.8 we may assume X is affine with $\Gamma(X, \mathcal{O}_X) \subset \mathcal{O}_{X,x}$. In particular X is integral with a unique generic point ξ whose residue field is the fraction field K of the valuation ring $\mathcal{O}_{X,x}$. Since x is in the closure of U we see that U is not empty, hence U contains ξ. Thus by the valuative criterion of properness (Lemma 29.42.1) there is a morphism $t : \mathrm{Spec}(\mathcal{O}_{X,x}) \to Y$ fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d, "\xi"'] & \mathrm{Spec}(\mathcal{O}_{X,x}) \arrow[d, "t"] \\ U \arrow[r, "f"] & Y \end{tikzcd}$$

of morphisms of schemes over S. Applying Lemma 29.42.4 with $y = t(x)$ and $\varphi = t_x^\sharp$ we obtain an open neighbourhood $V \subset X$ of x and a morphism $g : V \to Y$ over S which sends x to y and such that $g_x^\sharp = t_x^\sharp$. As $Y \to S$ is separated, the equalizer E of $f|_{U \cap V}$ and $g|_{U \cap V}$ is a closed subscheme of $U \cap V$, see Schemes, Lemma 26.21.5. Since f and g determine the same morphism $\mathrm{Spec}(K) \to Y$ by construction we see that E contains the generic point of the integral scheme $U \cap V$. Hence $E = U \cap V$ and we conclude that f and g glue to a morphism $U' = U \cup V \to Y$ as desired. $\square$

29.43. Projective morphisms

01W7 We will use the definition of a projective morphism from [DG67]. The version of the definition with the "H" is the one from [Har77]. The resulting definitions are different. Both are useful.

01W8 **Definition 29.43.1.** Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *projective* if X is isomorphic as an S-scheme to a closed subscheme of a projective bundle $\mathbf{P}(\mathcal{E})$ for some quasi-coherent, finite type $\mathcal{O}_S$-module $\mathcal{E}$.
(2) We say f is *H-projective* if there exists an integer n and a closed immersion $X \to \mathbf{P}^n_S$ over S.
(3) We say f is *locally projective* if there exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is projective.

As expected, a projective morphism is quasi-projective, see Lemma 29.43.10. Conversely, quasi-projective morphisms are often compositions of open immersions and projective morphisms, see Lemma 29.43.12. For an overview of properties of projective morphisms over a quasi-projective base, see More on Morphisms, Section 37.50.

07ZS Example 29.43.2. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Assume furthermore that $\mathcal{A}_1$ is of finite type over $\mathcal{O}_S$. Set $X = \underline{\text{Proj}}_S(\mathcal{A})$. In this case $X \to S$ is projective. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion, see Constructions, Lemma 27.18.5.

01W9 Lemma 29.43.3. An H-projective morphism is H-quasi-projective. An H-projective morphism is projective.

Proof. The first statement is immediate from the definitions. The second holds as $\mathbf{P}^n_S$ is a projective bundle over S, see Constructions, Lemma 27.21.5. □

01WB Lemma 29.43.4. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is locally projective.
(2) There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is H-projective.

Proof. By Lemma 29.43.3 we see that (2) implies (1). Assume (1). For every point $s \in S$ we can find $\text{Spec}(R) = U \subset S$ an affine open neighbourhood of s such that X_U is isomorphic to a closed subscheme of $\mathbf{P}(\mathcal{E})$ for some finite type, quasi-coherent sheaf of $\mathcal{O}_U$-modules $\mathcal{E}$. Write $\mathcal{E} = \widetilde{M}$ for some finite type R-module M (see Properties, Lemma 28.16.1). Choose generators $x_0, \ldots, x_n \in M$ of M as an R-module. Consider the surjective graded R-algebra map

$$R[X_0, \ldots, X_n] \longrightarrow \text{Sym}_R(M).$$

According to Constructions, Lemma 27.11.3 the corresponding morphism

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^n_R$$

is a closed immersion. Hence we conclude that $f^{-1}(U)$ is isomorphic to a closed subscheme of $\mathbf{P}^n_U$ (as a scheme over U). In other words: (2) holds. □

01WC Lemma 29.43.5. A locally projective morphism is proper.

Proof. Let $f : X \to S$ be locally projective. In order to show that f is proper we may work locally on the base, see Lemma 29.41.3. Hence, by Lemma 29.43.4 above we may assume there exists a closed immersion $X \to \mathbf{P}^n_S$. By Lemmas 29.41.4 and 29.41.6 it suffices to prove that $\mathbf{P}^n_S \to S$ is proper. Since $\mathbf{P}^n_S \to S$ is the base change of $\mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z})$ it suffices to show that $\mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z})$ is proper, see Lemma 29.41.5. By Constructions, Lemma 27.8.8 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is separated. By Constructions, Lemma 27.8.9 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is quasi-compact. It is clear that $\mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z})$ is locally of finite type since $\mathbf{P}^n_{\mathbf{Z}}$ is covered by the affine opens $D_+(X_i)$ each of which is the spectrum of the finite type $\mathbf{Z}$-algebra

$$\mathbf{Z}[X_0/X_i, \ldots, X_n/X_i].$$

Finally, we have to show that $\mathbf{P}^n_{\mathbf{Z}} \to \text{Spec}(\mathbf{Z})$ is universally closed. This follows from Constructions, Lemma 27.8.11 and the valuative criterion (see Schemes, Proposition 26.20.6). □

0B5N Lemma 29.43.6. Let $f : X \to S$ be a proper morphism of schemes. If there exists an f-ample invertible sheaf on X, then f is locally projective.

Proof. If there exists an f-ample invertible sheaf, then we can locally on S find an immersion $i : X \to \mathbf{P}^n_S$, see Lemma 29.39.4. Since $X \to S$ is proper the morphism i is a closed immersion, see Lemma 29.41.7. $\square$

01WE **Lemma 29.43.7.** A composition of H-projective morphisms is H-projective.

Proof. Suppose $X \to Y$ and $Y \to Z$ are H-projective. Then there exist closed immersions $X \to \mathbf{P}^n_Y$ over Y, and $Y \to \mathbf{P}^m_Z$ over Z. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbf{P}^n_Y \arrow[r] \arrow[dl] & \mathbf{P}^n_{\mathbf{P}^m_Z} \arrow[r, equal] \arrow[dl] & \mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z \arrow[r] & \mathbf{P}^{nm+n+m}_Z \arrow[ddllll] \\
Y \arrow[r] \arrow[d] & \mathbf{P}^m_Z \arrow[dl] & & & \\
Z & & & &
\end{tikzcd}$$

Here the rightmost top horizontal arrow is the Segre embedding, see Constructions, Lemma 27.13.6. The diagram identifies X as a closed subscheme of $\mathbf{P}^{nm+n+m}_Z$ as desired. $\square$

01WF **Lemma 29.43.8.** A base change of a H-projective morphism is H-projective.

Proof. This is true because the base change of projective space over a scheme is projective space, and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 26.18.2. $\square$

02V6 **Lemma 29.43.9.** A base change of a (locally) projective morphism is (locally) projective.

Proof. This is true because the base change of a projective bundle over a scheme is a projective bundle, the pullback of a finite type $\mathcal{O}$-module is of finite type (Modules, Lemma 17.9.2) and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 26.18.2. Some details omitted. $\square$

07RL **Lemma 29.43.10.** A projective morphism is quasi-projective.

Proof. Let $f : X \to S$ be a projective morphism. Choose a closed immersion $i : X \to \mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is a quasi-coherent, finite type $\mathcal{O}_S$-module. Then $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is f-very ample. Since f is proper (Lemma 29.43.5) it is quasi-compact. Hence Lemma 29.38.2 implies that $\mathcal{L}$ is f-ample. Since f is proper it is of finite type. Thus we've checked all the defining properties of quasi-projective holds and we win. $\square$

01WA **Lemma 29.43.11.** Let $f : X \to S$ be a H-quasi-projective morphism. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is H-projective.

Proof. By definition we can factor f as a quasi-compact immersion $i : X \to \mathbf{P}^n_S$ followed by the projection $\mathbf{P}^n_S \to S$. By Lemma 29.7.7 there exists a closed subscheme $X' \subset \mathbf{P}^n_S$ such that i factors through an open immersion $X \to X'$. The lemma follows. $\square$

07RM Lemma 29.43.12. Let $f : X \to S$ be a quasi-projective morphism with S quasi-compact and quasi-separated. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is projective.

Proof. Let $\mathcal{L}$ be f-ample. Since f is of finite type and S is quasi-compact $\mathcal{L}^{\otimes n}$ is f-very ample for some $n > 0$, see Lemma 29.39.5. Replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$. Write $\mathcal{F} = f_*\mathcal{L}$. This is a quasi-coherent $\mathcal{O}_S$-module by Schemes, Lemma 26.24.1 (quasi-projective morphisms are quasi-compact and separated, see Lemma 29.40.4). By Properties, Lemma 28.22.7 we can find a directed set I and a system of finite type quasi-coherent $\mathcal{O}_S$-modules $\mathcal{E}_i$ over I such that $\mathcal{F} = \operatorname{colim} \mathcal{E}_i$. Consider the compositions $\psi_i : f^*\mathcal{E}_i \to f^*\mathcal{F} \to \mathcal{L}$. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,m} V_j$. For each j we can choose sections

$$s_{j,0}, \ldots, s_{j,n_j} \in \Gamma(f^{-1}(V_j), \mathcal{L}) = f_*\mathcal{L}(V_j) = \mathcal{F}(V_j)$$

which generate $\mathcal{L}$ over $f^{-1}V_j$ and define an immersion

$$f^{-1}V_j \longrightarrow \mathbf{P}^{n_j}_{V_j},$$

see Lemma 29.39.1. Choose i such that there exist sections $e_{j,t} \in \mathcal{E}_i(V_j)$ mapping to $s_{j,t}$ in $\mathcal{F}$ for all $j = 1, \ldots, m$ and $t = 1, \ldots, n_j$. Then the map ψ_i is surjective as the sections $f^*e_{j,t}$ have the same image as the sections $s_{j,t}$ which generate $\mathcal{L}|_{f^{-1}V_j}$. Whence we obtain a morphism

$$r_{\mathcal{L},\psi_i} : X \longrightarrow \mathbf{P}(\mathcal{E}_i)$$

over S such that over V_j we have a factorization

$$f^{-1}V_j \to \mathbf{P}(\mathcal{E}_i)|_{V_j} \to \mathbf{P}^{n_j}_{V_j}$$

of the immersion given above. It follows that $r_{\mathcal{L},\psi_i}|_{V_j}$ is an immersion, see Lemma 29.3.1. Since $S = \bigcup V_j$ we conclude that $r_{\mathcal{L},\psi_i}$ is an immersion. Note that $r_{\mathcal{L},\psi_i}$ is quasi-compact as $X \to S$ is quasi-compact and $\mathbf{P}(\mathcal{E}_i) \to S$ is separated (see Schemes, Lemma 26.21.14). By Lemma 29.7.7 there exists a closed subscheme $X' \subset \mathbf{P}(\mathcal{E}_i)$ such that i factors through an open immersion $X \to X'$. Then $X' \to S$ is projective by definition and we win. $\square$

0BCL Lemma 29.43.13. Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of schemes. Then

(1) f is projective if and only if f is quasi-projective and proper, and
(2) f is H-projective if and only if f is H-quasi-projective and proper.

Proof. If f is projective, then f is quasi-projective by Lemma 29.43.10 and proper by Lemma 29.43.5. Conversely, if $X \to S$ is quasi-projective and proper, then we can choose an open immersion $X \to X'$ with $X' \to S$ projective by Lemma 29.43.12. Since $X \to S$ is proper, we see that X is closed in X' (Lemma 29.41.7), i.e., $X \to X'$ is a (open and) closed immersion. Since X' is isomorphic to a closed subscheme of a projective bundle over S (Definition 29.43.1) we see that the same thing is true for X, i.e., $X \to S$ is a projective morphism. This proves (1). The proof of (2) is the same, except it uses Lemmas 29.43.3 and 29.43.11. $\square$

0C4P Lemma 29.43.14. Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If S is quasi-compact and quasi-separated and f and g are projective, then $g \circ f$ is projective.

Proof. By Lemmas 29.43.10 and 29.43.5 we see that f and g are quasi-projective and proper. By Lemmas 29.41.4 and 29.40.3 we see that $g \circ f$ is proper and quasi-projective. Thus $g \circ f$ is projective by Lemma 29.43.13. $\square$

0C4Q Lemma 29.43.15. Let $g : Y \to S$ and $f : X \to Y$ be morphisms of schemes. If $g \circ f$ is projective and g is separated, then f is projective.

Proof. Choose a closed immersion $X \to \mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is a quasi-coherent, finite type $\mathcal{O}_S$-module. Then we get a morphism $X \to \mathbf{P}(\mathcal{E}) \times_S Y$. This morphism is a closed immersion because it is the composition

$$X \to X \times_S Y \to \mathbf{P}(\mathcal{E}) \times_S Y$$

where the first morphism is a closed immersion by Schemes, Lemma 26.21.10 (and the fact that g is separated) and the second as the base change of a closed immersion. Finally, the fibre product $\mathbf{P}(\mathcal{E}) \times_S Y$ is isomorphic to $\mathbf{P}(g^*\mathcal{E})$ and pullback preserves quasi-coherent, finite type modules. $\square$

087S Lemma 29.43.16. Let S be a scheme which admits an ample invertible sheaf. Then

(1) any projective morphism $X \to S$ is H-projective, and
(2) any quasi-projective morphism $X \to S$ is H-quasi-projective.

Proof. The assumptions on S imply that S is quasi-compact and separated, see Properties, Definition 28.26.1 and Lemma 28.26.11 and Constructions, Lemma 27.8.8. Hence Lemma 29.43.12 applies and we see that (1) implies (2). Let $\mathcal{E}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. By our definition of projective morphisms it suffices to show that $\mathbf{P}(\mathcal{E}) \to S$ is H-projective. If $\mathcal{E}$ is generated by finitely many global sections, then the corresponding surjection $\mathcal{O}_S^{\oplus n} \to \mathcal{E}$ induces a closed immersion

$$\mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{P}(\mathcal{O}_S^{\oplus n}) = \mathbf{P}^n_S$$

as desired. In general, let $\mathcal{L}$ be an invertible sheaf on S. By Properties, Proposition 28.26.13 there exists an integer n such that $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n}$ is globally generated by finitely many sections. Since $\mathbf{P}(\mathcal{E}) = \mathbf{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n})$ by Constructions, Lemma 27.20.1 this finishes the proof. $\square$

0C6J Lemma 29.43.17. Let $f : X \to S$ be a universally closed morphism. Let $\mathcal{L}$ be an f-ample invertible $\mathcal{O}_X$-module. Then the canonical morphism

$$r : X \longrightarrow \underline{\text{Proj}}_S \left(\bigoplus\nolimits_{d \geq 0} f_*\mathcal{L}^{\otimes d} \right)$$

of Lemma 29.37.4 is an isomorphism.

Proof. Observe that f is quasi-compact because the existence of an f-ample invertible module forces f to be quasi-compact. By the lemma cited the morphism r is an open immersion. On the other hand, the image of r is closed by Lemma 29.41.7 (the target of r is separated over S by Constructions, Lemma 27.16.9). Finally, the image of r is dense by Properties, Lemma 28.26.11 (here we also use that it was shown in the proof of Lemma 29.37.4 that the morphism r over affine opens of S is given by the canonical morphism of Properties, Lemma 28.26.9). Thus we conclude that r is a surjective open immersion, i.e., an isomorphism. $\square$

0EKE Lemma 29.43.18. Let $f : X \to S$ be a universally closed morphism. Let $\mathcal{L}$ be an f-ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Then $X_s \to S$ is an affine morphism.

Proof. The question is local on S (Lemma 29.11.3) hence we may assume S is affine. By Lemma 29.43.17 we can write $X = \text{Proj}(A)$ where A is a graded ring and s corresponds to $f \in A_1$ and $X_s = D_+(f)$ (Properties, Lemma 28.26.9) which proves the lemma by construction of $\text{Proj}(A)$, see Constructions, Section 27.8. □

29.44. Integral and finite morphisms

01WG Recall that a ring map $R \to A$ is said to be integral if every element of A satisfies a monic equation with coefficients in R. Recall that a ring map $R \to A$ is said to be finite if A is finite as an R-module. See Algebra, Definition 10.36.1.

01WH Definition 29.44.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is integral if f is affine and if for every affine open $\text{Spec}(R) = V \subset S$ with inverse image $\text{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is integral.
(2) We say that f is finite if f is affine and if for every affine open $\text{Spec}(R) = V \subset S$ with inverse image $\text{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is finite.

It is clear that integral/finite morphisms are separated and quasi-compact. It is also clear that a finite morphism is a morphism of finite type. Most of the lemmas in this section are completely standard. But note the fun Lemma 29.44.7 at the end of the section.

02K8 Lemma 29.44.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is integral.
(2) There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is integral.
(3) There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is integral.

Moreover, if f is integral then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is integral.

Proof. See Algebra, Lemma 10.36.14. Some details omitted. □

01WI Lemma 29.44.3. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is finite.
(2) There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is finite.
(3) There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is finite.

Moreover, if f is finite then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is finite.

Proof. See Algebra, Lemma 10.36.14. Some details omitted. □

01WJ Lemma 29.44.4. A finite morphism is integral. An integral morphism which is locally of finite type is finite.

Proof. See Algebra, Lemma 10.36.3 and Lemma 10.36.5. □

01WK Lemma 29.44.5. A composition of finite morphisms is finite. Same is true for integral morphisms.

Proof. See Algebra, Lemmas 10.7.3 and 10.36.6. □

01WL Lemma 29.44.6. A base change of a finite morphism is finite. Same is true for integral morphisms.

Proof. See Algebra, Lemma 10.36.13. □

01WM Lemma 29.44.7. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) f is integral, and
(2) f is affine and universally closed.

Proof. Assume (1). An integral morphism is affine by definition. A base change of an integral morphism is integral so in order to prove (2) it suffices to show that an integral morphism is closed. This follows from Algebra, Lemmas 10.36.22 and 10.41.6.

Assume (2). We may assume f is the morphism $f : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ coming from a ring map $R \to A$. Let a be an element of A. We have to show that a is integral over R, i.e. that in the kernel I of the map $R[x] \to A$ sending x to a there is a monic polynomial. Consider the ring $B = A[x]/(ax-1)$ and let J be the kernel of the composition $R[x] \to A[x] \to B$. If $f \in J$ there exists $q \in A[x]$ such that $f = (ax-1)q$ in $A[x]$ so if $f = \sum_i f_i x^i$ and $q = \sum_i q_i x^i$, for all $i \geq 0$ we have $f_i = aq_{i-1} - q_i$. For $n \geq \deg q + 1$ the polynomial

$$\sum\nolimits_{i\geq 0} f_i x^{n-i} = \sum\nolimits_{i\geq 0}(aq_{i-1} - q_i)x^{n-i} = (a-x)\sum\nolimits_{i\geq 0} q_i x^{n-i-1}$$

is clearly in I; if $f_0 = 1$ this polynomial is also monic, so we are reduced to prove that J contains a polynomial with constant term 1. We do it by proving $\mathrm{Spec}(R[x]/(J + (x))$ is empty.

Since f is universally closed the base change $\mathrm{Spec}(A[x]) \to \mathrm{Spec}(R[x])$ is closed. Hence the image of the closed subset $\mathrm{Spec}(B) \subset \mathrm{Spec}(A[x])$ is the closed subset $\mathrm{Spec}(R[x]/J) \subset \mathrm{Spec}(R[x])$, see Example 29.6.4 and Lemma 29.6.3. In particular $\mathrm{Spec}(B) \to \mathrm{Spec}(R[x]/J)$ is surjective. Consider the following diagram where every square is a pullback:

$$\begin{tikzcd}
\mathrm{Spec}(B) \arrow[r, two heads, "g"] & \mathrm{Spec}(R[x]/J) \arrow[r] & \mathrm{Spec}(R[x]) \\
\emptyset \arrow[u] \arrow[r] & \mathrm{Spec}(R[x]/(J+(x))) \arrow[u] \arrow[r] & \mathrm{Spec}(R) \arrow[u, "0"]
\end{tikzcd}$$

The bottom left corner is empty because it is the spectrum of $R \otimes_{R[x]} B$ where the map $R[x] \to B$ sends x to an invertible element and $R[x] \to R$ sends x to 0. Since g is surjective this implies $\mathrm{Spec}(R[x]/(J + (x)))$ is empty, as we wanted to show. □

02NT Lemma 29.44.8. Let $f : X \to S$ be an integral morphism. Then every point of X is closed in its fibre.

Proof. See Algebra, Lemma 10.36.20. □

0ECG **Lemma 29.44.9.** Let $f : X \to Y$ be an integral morphism. Then $\dim(X) \leq \dim(Y)$. If f is surjective then $\dim(X) = \dim(Y)$.

Proof. Since the dimension of X and Y is the supremum of the dimensions of the members of an affine open covering, we may assume Y and X are affine. The inequality follows from Algebra, Lemma 10.112.3. The equality then follows from Algebra, Lemmas 10.112.1 and 10.36.22. $\square$

02NU **Lemma 29.44.10.** A finite morphism is quasi-finite.

Proof. This is implied by Algebra, Lemma 10.122.4 and Lemma 29.20.9. Alternatively, all points in fibres are closed points by Lemma 29.44.8 (and the fact that a finite morphism is integral) and use Lemma 29.20.6 (3) to see that f is quasi-finite at x for all $x \in X$. $\square$

01WN **Lemma 29.44.11.** Let $f : X \to S$ be a morphism of schemes. The following are equivalent

1. f is finite, and
2. f is affine and proper.

Proof. This follows formally from Lemma 29.44.7, the fact that a finite morphism is integral and separated, the fact that a proper morphism is the same thing as a finite type, separated, universally closed morphism, and the fact that an integral morphism of finite type is finite (Lemma 29.44.4). $\square$

035C **Lemma 29.44.12.** A closed immersion is finite (and a fortiori integral).

Proof. True because a closed immersion is affine (Lemma 29.11.9) and a surjective ring map is finite and integral. $\square$

0CYI **Lemma 29.44.13.** Let $X_i \to Y$, $i = 1, \ldots, n$ be finite morphisms of schemes. Then $X_1 \amalg \ldots \amalg X_n \to Y$ is finite too.

Proof. Follows from the algebra fact that if $R \to A_i$, $i = 1, \ldots, n$ are finite ring maps, then $R \to A_1 \times \ldots \times A_n$ is finite too. $\square$

035D **Lemma 29.44.14.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms.

1. If $g \circ f$ is finite and g separated then f is finite.
2. If $g \circ f$ is integral and g separated then f is integral.

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 29.44.6. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Schemes, Lemma 26.21.11. A closed immersion is finite (resp. integral), see Lemma 29.44.12. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 29.44.5. Thus we win. $\square$

03BB **Lemma 29.44.15.** Let $f : X \to Y$ be a morphism of schemes. If f is finite and a monomorphism, then f is a closed immersion.

Proof. This reduces to Algebra, Lemma 10.107.6. $\square$

0B3I **Lemma 29.44.16.** A finite morphism is projective.

Proof. Let $f : X \to S$ be a finite morphism. Then $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_S$-module (Lemma 29.11.5) of finite type (by our definition of finite morphisms and Properties, Lemma 28.16.1). We claim there is a closed immersion

$$\sigma : X \longrightarrow \mathbf{P}(f_*\mathcal{O}_X) = \underline{\text{Proj}}_S(\text{Sym}^*_{\mathcal{O}_S}(f_*\mathcal{O}_X))$$

over S, which finishes the proof. Namely, we let σ be the morphism which corresponds (via Constructions, Lemma 27.16.11) to the surjection

$$f^*f_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$$

coming from the adjunction map $f^*f_* \to \text{id}$. Then σ is a closed immersion by Schemes, Lemma 26.21.11 and Constructions, Lemma 27.21.4. $\square$

29.45. Universal homeomorphisms

04DC The following definition is really superfluous since a universal homeomorphism is really just an integral, universally injective and surjective morphism, see Lemma 29.45.5.

04DD **Definition 29.45.1.** A morphism $f : X \to Y$ of schemes is called a *universal homeomorphism* if the base change $f' : Y' \times_Y X \to Y'$ is a homeomorphism for every morphism $Y' \to Y$.

First we state the obligatory lemmas.

0CEU **Lemma 29.45.2.** *The base change of a universal homeomorphism of schemes by any morphism of schemes is a universal homeomorphism.*

Proof. This is immediate from the definition. $\square$

0CEV **Lemma 29.45.3.** *The composition of a pair of universal homeomorphisms of schemes is a universal homeomorphism.*

Proof. Omitted. $\square$

The following simple lemma is the key to characterizing universal homeomorphisms.

04DE **Lemma 29.45.4.** *Let $f : X \to Y$ be a morphism of schemes. If f is a homeomorphism onto a closed subset of Y then f is affine.*

Proof. Let $y \in Y$ be a point. If $y \notin f(X)$, then there exists an affine neighbourhood of y which is disjoint from $f(X)$. If $y \in f(X)$, let $x \in X$ be the unique point of X mapping to y. Let $y \in V$ be an affine open neighbourhood. Let $U \subset X$ be an affine open neighbourhood of x which maps into V. Since $f(U) \subset V \cap f(X)$ is open in the induced topology by our assumption on f we may choose a $h \in \Gamma(V, \mathcal{O}_Y)$ such that $y \in D(h)$ and $D(h) \cap f(X) \subset f(U)$. Denote $h' \in \Gamma(U, \mathcal{O}_X)$ the restriction of $f^\sharp(h)$ to U. Then we see that $D(h') \subset U$ is equal to $f^{-1}(D(h))$. In other words, every point of Y has an open neighbourhood whose inverse image is affine. Thus f is affine, see Lemma 29.11.3. $\square$

04DF **Lemma 29.45.5.** *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f is a universal homeomorphism, and*
(2) *f is integral, universally injective and surjective.*

Proof. Assume f is a universal homeomorphism. By Lemma 29.45.4 we see that f is affine. Since f is clearly universally closed we see that f is integral by Lemma 29.44.7. It is also clear that f is universally injective and surjective.

Assume f is integral, universally injective and surjective. By Lemma 29.44.7 f is universally closed. Since it is also universally bijective (see Lemma 29.9.4) we see that it is a universal homeomorphism. □

054M Lemma 29.45.6. Let X be a scheme. The canonical closed immersion $X_{red} \to X$ (see Schemes, Definition 26.12.5) is a universal homeomorphism.

Proof. Omitted. □

0896 Lemma 29.45.7. Let $f : X \to S$ and $S' \to S$ be morphisms of schemes. Assume

(1) $S' \to S$ is a closed immersion,
(2) $S' \to S$ is bijective on points,
(3) $X \times_S S' \to S'$ is a closed immersion, and
(4) $X \to S$ is of finite type or $S' \to S$ is of finite presentation.

Then $f : X \to S$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $S' \to S$ is a universal homeomorphism (for example because $S_{red} = S'_{red}$ and using Lemma 29.45.6). Hence (3) implies that $X \to S$ is homeomorphism onto a closed subset of S. Then $X \to S$ is affine by Lemma 29.45.4. Let $U \subset S$ be an affine open, say $U = \mathrm{Spec}(A)$. Then $S' = \mathrm{Spec}(A/I)$ by (1) for a locally nilpotent ideal I by (2). As f is affine we see that $f^{-1}(U) = \mathrm{Spec}(B)$. Assumption (4) tells us B is a finite type A-algebra (Lemma 29.15.2) or that I is finitely generated (Lemma 29.21.7). Assumption (3) is that $A/I \to B/IB$ is surjective. From Algebra, Lemma 10.126.9 if $A \to B$ is of finite type or Algebra, Lemma 10.20.1 if I is finitely generated and hence nilpotent we deduce that $A \to B$ is surjective. This means that f is a closed immersion, see Lemma 29.2.1. □

0H2M Lemma 29.45.8. Let $f : X \to Z$ be the composition of two morphisms $g : X \to Y$ and $h : Y \to Z$. If two of the morphisms $\{f, g, h\}$ are universal homeomorphisms, so is the third morphism.

Proof. If both of g and h are universal homeomorphisms, so is f by Lemma 29.45.3.

Suppose both of f and g are universal homeomorphisms. We want to show that h is also. Now base change the diagram along an arbitrary morphism $\alpha : Z' \to Z$ of schemes, we get the following diagram with all squares Cartesian:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & Y' \arrow[r, "h'"] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, "g"] & Y \arrow[r, "h"] & Z. \end{tikzcd}$$

Our assumption implies that the composition $f' = h' \circ g' : X' \to Z'$ and $g' : X' \to Y'$ are homeomorphisms, therefore so is h'. This finishes the proof of h being a universal homeomorphism.

Finally, assume f and h are universal homeomorphisms. We want to show that g is a universal homeomorphism. Let $\beta : Y' \to Y$ be an arbitrary morphism of schemes.

We get the following diagram with all squares Cartesian:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & Y' \arrow[d, "\gamma"] & \\ X'' \arrow[r, "g''"] \arrow[d] & Y'' \arrow[r, "h''"] \arrow[d] & Y' \arrow[d, "h\circ\beta"] \\ X \arrow[r, "g"] & Y \arrow[r, "h"] & Z. \end{tikzcd}$$

Here the morphism $\gamma : Y' \to Y''$ is defined by the universal property of fiber products and the two morphisms $id_{Y'} : Y' \to Y'$ and $\beta : Y' \to Y$. We shall prove that g' is a homeomorphism. Since the property of being a homeomorphism has 2-out-of-3 property, we see that g'' is a homeomorphism. Staring at the top square, it suffices to prove that γ is a universal homeomorphism. Since h'' is a homeomorphism, we see that it is an affine morphism by Lemma 29.45.4 and a fortiori separated (Lemma 29.11.2). Since $h'' \circ \gamma$ is the identity, we see that γ is a closed immersion by Schemes, Lemma 26.21.11. Since h'' is bijective, it follows that γ is a bijective closed immersion and hence a universal homeomorphism (for example by the characterization in Lemma 29.45.5) as desired. □

29.46. Universal homeomorphisms of affine schemes

0CN6 In this section we characterize universal homeomorphisms of affine schemes.

0CN7 Lemma 29.46.1. Let $A \to B$ be a ring map such that the induced morphism of schemes $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism, resp. a universal homeomorphism inducing isomorphisms on residue fields, resp. universally closed, resp. universally closed and universally injective. Then for any A-subalgebra $B' \subset B$ the same thing is true for $f' : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$.

Proof. If f is universally closed, then B is integral over A by Lemma 29.44.7. Hence B' is integral over A and f' is universally closed (by the same lemma). This proves the case where f is universally closed.

Continuing, we see that B is integral over B' (Algebra, Lemma 10.36.15) which implies $\mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is surjective (Algebra, Lemma 10.36.17). Thus if $A \to B$ induces purely inseparable extensions of residue fields, then the same is true for $A \to B'$. This proves the case where f is universally closed and universally injective, see Lemma 29.10.2.

The case where f is a universal homeomorphism follows from the remarks above, Lemma 29.45.5, and the obvious observation that if f is surjective, then so is f'.

If $A \to B$ induces isomorphisms on residue fields, then so does $A \to B'$ (see argument in second paragraph). In this way we see that the lemma holds in the remaining case. □

0CN8 Lemma 29.46.2. Let A be a ring. Let $B = \mathrm{colim}\, B_\lambda$ be a filtered colimit of A-algebras. If each $f_\lambda : \mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ is a universal homeomorphism, resp. a universal homeomorphism inducing isomorphisms on residue fields, resp. universally closed, resp. universally closed and universally injective, then the same thing is true for $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

Proof. If f_λ is universally closed, then B_λ is integral over A by Lemma 29.44.7. Hence B is integral over A and f is universally closed (by the same lemma). This proves the case where each f_λ is universally closed.

For a prime $\mathfrak q \subset B$ lying over $\mathfrak p \subset A$ denote $\mathfrak q_\lambda \subset B_\lambda$ the inverse image. Then $\kappa(\mathfrak q) = \operatorname{colim} \kappa(\mathfrak q_\lambda)$. Thus if $A \to B_\lambda$ induces purely inseparable extensions of residue fields, then the same is true for $A \to B$. This proves the case where f_λ is universally closed and universally injective, see Lemma 29.10.2.

The case where f is a universal homeomorphism follows from the remarks above and Lemma 29.45.5 combined with the fact that prime ideals in B are the same thing as compatible sequences of prime ideals in all of the B_λ.

If $A \to B_\lambda$ induces isomorphisms on residue fields, then so does $A \to B$ (see argument in second paragraph). In this way we see that the lemma holds in the remaining case. $\square$

0CN9 Lemma 29.46.3. Let $A \subset B$ be a ring extension. Let $S \subset A$ be a multiplicative subset. Let $n \geq 1$ and $b_i \in B$ for $1 \leq i \leq n$. Any $x \in S^{-1}B$ such that

$$x \notin S^{-1}A \text{ and } b_i x^i \in S^{-1}A \text{ for } i = 1, \ldots, n$$

is equal to $s^{-1}y$ with $s \in S$ and $y \in B$ such that

$$y \notin A \text{ and } b_i y^i \in A \text{ for } i = 1, \ldots, n$$

Proof. Omitted. Hint: clear denominators. $\square$

0CNA Lemma 29.46.4. Let $A \subset B$ be a ring extension. If there exists $b \in B$, $b \notin A$ and an integer $n \geq 2$ with $b^n \in A$ and $b^{n+1} \in A$, then there exists a $b' \in B$, $b' \notin A$ with $(b')^2 \in A$ and $(b')^3 \in A$.

Proof. Let b and n be as in the lemma. Then all sufficiently large powers of b are in A. Namely, $(b^n)^k(b^{n+1})^i = b^{(k+i)n+i}$ which implies any power b^m with $m \geq n^2$ is in A. Hence if $i \geq 1$ is the largest integer such that $b^i \notin A$, then $(b^i)^2 \in A$ and $(b^i)^3 \in A$. $\square$

0CNB Lemma 29.46.5. Let $A \subset B$ be a ring extension such that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields. If $A \neq B$, then there exists a $b \in B$, $b \notin A$ with $b^2 \in A$ and $b^3 \in A$.

Proof. Recall that $A \subset B$ is integral (Lemma 29.44.7). By Lemma 29.46.1 we may assume that B is generated by a single element over A. Hence B is finite over A (Algebra, Lemma 10.36.5). Hence the support of B/A as an A-module is closed and not empty (Algebra, Lemmas 10.40.5 and 10.40.2). Let $\mathfrak p \subset A$ be a minimal prime of the support. After replacing $A \subset B$ by $A_\mathfrak p \subset B_\mathfrak p$ (permissible by Lemma 29.46.3) we may assume that $(A, \mathfrak m)$ is a local ring, that B is finite over A, and that B/A has support $\{\mathfrak m\}$ as an A-module. Since B/A is a finite module, we see that $I = \operatorname{Ann}_A(B/A)$ satisfies $\mathfrak m = \sqrt{I}$ (Algebra, Lemma 10.40.5). Let $\mathfrak m' \subset B$ be the unique prime ideal lying over $\mathfrak m$. Because $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a homeomorphism, we find that $\mathfrak m' = \sqrt{IB}$. For $f \in \mathfrak m'$ pick $n \geq 1$ such that $f^n \in IB$. Then also $f^{n+1} \in IB$. Since $IB \subset A$ by our choice of I we conclude that $f^n, f^{n+1} \in A$. Using Lemma 29.46.4 we conclude our lemma is true if $\mathfrak m' \not\subset A$. However, if $\mathfrak m' \subset A$, then $\mathfrak m' = \mathfrak m$ and we conclude that $A = B$ as the residue fields are isomorphic as well by assumption. This contradiction finishes the proof. $\square$

0CNC Lemma 29.46.6. Let $A \subset B$ be a ring extension such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism. If $A \neq B$, then either there exists a $b \in B$, $b \notin A$ with $b^2 \in A$ and $b^3 \in A$ or there exists a prime number p and a $b \in B$, $b \notin A$ with $pb \in A$ and $b^p \in A$.

Proof. The argument is almost exactly the same as in the proof of Lemma 29.46.5 but we write everything out to make sure it works.

Recall that $A \subset B$ is integral (Lemma 29.44.7). By Lemma 29.46.1 we may assume that B is generated by a single element over A. Hence B is finite over A (Algebra, Lemma 10.36.5). Hence the support of B/A as an A-module is closed and not empty (Algebra, Lemmas 10.40.5 and 10.40.2). Let $\mathfrak p \subset A$ be a minimal prime of the support. After replacing $A \subset B$ by $A_{\mathfrak p} \subset B_{\mathfrak p}$ (permissible by Lemma 29.46.3) we may assume that $(A, \mathfrak m)$ is a local ring, that B is finite over A, and that B/A has support $\{\mathfrak m\}$ as an A-module. Since B/A is a finite module, we see that $I = \mathrm{Ann}_A(B/A)$ satisfies $\mathfrak m = \sqrt{I}$ (Algebra, Lemma 10.40.5). Let $\mathfrak m' \subset B$ be the unique prime ideal lying over $\mathfrak m$. Because $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a homeomorphism, we find that $\mathfrak m' = \sqrt{IB}$. For $f \in \mathfrak m'$ pick $n \geq 1$ such that $f^n \in IB$. Then also $f^{n+1} \in IB$. Since $IB \subset A$ by our choice of I we conclude that $f^n, f^{n+1} \in A$. Using Lemma 29.46.4 we conclude our lemma is true if $\mathfrak m' \not\subset A$. If $\mathfrak m' \subset A$, then $\mathfrak m' = \mathfrak m$. Since $A \neq B$ we conclude the map $\kappa = A/\mathfrak m \to B/\mathfrak m' = \kappa'$ of residue fields cannot be an isomorphism. By Lemma 29.10.2 we conclude that the characteristic of κ is a prime number p and that the extension κ'/κ is purely inseparable. Pick $b \in B$ whose image in κ' is an element not contained in κ but whose pth power is in κ. Then $b \notin A$, $b^p \in A$, and $pb \in A$ (because $pb \in \mathfrak m' = \mathfrak m \subset A$) as desired. $\square$

0CND Proposition 29.46.7. Let $A \subset B$ be a ring extension. The following are equivalent

(1) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields, and
(2) every finite subset $E \subset B$ is contained in an extension
$$A[b_1, \ldots, b_n] \subset B$$
such that $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$ for $i = 1, \ldots, n$.

Proof. Assume (1). Using transfinite recursion we construct for each ordinal α an A-subalgebra $B_\alpha \subset B$ as follows. Set $B_0 = A$. If α is a limit ordinal, then we set $B_\alpha = \mathrm{colim}_{\beta<\alpha} B_\beta$. If $\alpha = \beta + 1$, then either $B_\beta = B$ in which case we set $B_\alpha = B_\beta$ or $B_\beta \neq B$, in which case we apply Lemma 29.46.5 to choose a $b_\alpha \in B$, $b_\alpha \notin B_\beta$ with $b_\alpha^2, b_\alpha^3 \in B_\beta$ and we set $B_\alpha = B_\beta[b_\alpha] \subset B$. Clearly, $B = \mathrm{colim}\, B_\alpha$ (in fact $B = B_\alpha$ for some ordinal α as one sees by looking at cardinalities). We will prove, by transfinite induction, that (2) holds for $A \to B_\alpha$ for every ordinal α. It is clear for $\alpha = 0$. Assume the statement holds for every $\beta < \alpha$ and let $E \subset B_\alpha$ be a finite subset. If α is a limit ordinal, then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ and we see that $E \subset B_\beta$ for some $\beta < \alpha$ which proves the result in this case. If $\alpha = \beta + 1$, then $B_\alpha = B_\beta[b_\alpha]$. Thus any $e \in E$ can be written as a polynomial $e = \sum d_{e,i} b_\alpha^i$ with $d_{e,i} \in B_\beta$. Let $D \subset B_\beta$ be the set $D = \{d_{e,i}\} \cup \{b_\alpha^2, b_\alpha^3\}$. By induction assumption there exists an A-subalgebra $A[b_1, \ldots, b_n] \subset B_\beta$ as in the statement of the lemma containing D. Then $A[b_1, \ldots, b_n, b_\alpha] \subset B_\alpha$ is an A-subalgebra of B_α as in the statement of the lemma containing E.

Assume (2). Write $B = \text{colim}\, B_\lambda$ as the colimit of its finite A-subalgebras. By Lemma 29.46.2 it suffices to show that $\text{Spec}(B_\lambda) \to \text{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields. Compositions of universally closed morphisms are universally closed and the same thing for morphisms which induce isomorphisms on residue fields. Thus it suffices to show that if $A \subset B$ and B is generated by a single element b with $b^2, b^3 \in A$, then (1) holds. Such an extension is integral and hence $\text{Spec}(B) \to \text{Spec}(A)$ is universally closed and surjective (Lemma 29.44.7 and Algebra, Lemma 10.36.17). Note that $(b^2)^3 = (b^3)^2$ in A. For any ring map $\varphi : A \to K$ to a field K we see that there exists a $\lambda \in K$ with $\varphi(b^2) = \lambda^2$ and $\varphi(b^3) = \lambda^3$. Namely, $\lambda = 0$ if $\varphi(b^2) = 0$ and $\lambda = \varphi(b^3)/\varphi(b^2)$ if not. Thus $B \otimes_A K$ is a quotient of $K[x]/(x^2 - \lambda^2, x^3 - \lambda^3)$. This ring has exactly one prime with residue field K. This implies that $\text{Spec}(B) \to \text{Spec}(A)$ is bijective and induces isomorphisms on residue fields. Combined with universal closedness this shows (1) is true, see Lemmas 29.45.5 and 29.10.2. $\square$

0CNE Proposition 29.46.8. Let $A \subset B$ be a ring extension. The following are equivalent

(1) $\text{Spec}(B) \to \text{Spec}(A)$ is a universal homeomorphism, and
(2) every finite subset $E \subset B$ is contained in an extension

$$A[b_1, \ldots, b_n] \subset B$$

such that for $i = 1, \ldots, n$ we have
(a) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, or
(b) there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.

Proof. The proof is exactly the same as the proof of Proposition 29.46.7 except for the following changes:

(1) Use Lemma 29.46.6 instead of Lemma 29.46.5 which means that for each successor ordinal $\alpha = \beta + 1$ we either have $b_\alpha^2, b_\alpha^3 \in B_\beta$ or we have a prime p and $pb_\alpha, b_\alpha^p \in B_\beta$.
(2) If α is a successor ordinal, then take $D = \{d_{e,i}\} \cup \{b_\alpha^2, b_\alpha^3\}$ or take $D = \{d_{e,i}\} \cup \{pb_\alpha, b_\alpha^p\}$ depending on which case α falls into.
(3) In the proof of (2) $\Rightarrow$ (1) we also need to consider the case where B is generated over A by a single element b with $pb, b^p \in B$ for some prime number p. Here $A \subset B$ induces a universal homeomorphism on spectra for example by Algebra, Lemma 10.46.7.

This finishes the proof. $\square$

0CNF Lemma 29.46.9. Let p be a prime number. Let $A \to B$ be a ring map which induces an isomorphism $A[1/p] \to B[1/p]$ (for example if p is nilpotent in A). The following are equivalent

(1) $\text{Spec}(B) \to \text{Spec}(A)$ is a universal homeomorphism, and
(2) the kernel of $A \to B$ is a locally nilpotent ideal and for every $b \in B$ there exists a p-power q with qb and b^q in the image of $A \to B$.

Proof. If (2) holds, then (1) holds by Algebra, Lemma 10.46.7. Assume (1). Then the kernel of $A \to B$ consists of nilpotent elements by Algebra, Lemma 10.30.6. Thus we may replace A by the image of $A \to B$ and assume that $A \subset B$. By Algebra, Lemma 10.46.5 the set

$$B' = \{b \in B \mid p^n b, b^{p^n} \in A \text{ for some } n \geq 0\}$$

is an A-subalgebra of B (being closed under products is trivial). We have to show $B' = B$. If not, then according to Lemma 29.46.6 there exists a $b \in B$, $b \notin B'$ with either $b^2, b^3 \in B'$ or there exists a prime number ℓ with $\ell b, b^\ell \in B'$. We will show both cases lead to a contradiction, thereby proving the lemma.

Since $A[1/p] = B[1/p]$ we can choose a p-power q such that $qb \in A$.

If $b^2, b^3 \in B'$ then also $b^q \in B'$. By definition of B' we find that $(b^q)^{q'} \in A$ for some p-power q'. Then $qq'b, b^{qq'} \in A$ whence $b \in B'$ which is a contradiction.

Assume now there exists a prime number ℓ with $\ell b, b^\ell \in B'$. If $\ell \neq p$ then $\ell b \in B'$ and $qb \in A \subset B'$ imply $b \in B'$ a contradiction. Thus $\ell = p$ and $b^p \in B'$ and we get a contradiction exactly as before. □

0EUI Lemma 29.46.10. Let A be a ring. Let $x, y \in A$.

1. If $x^3 = y^2$ in A, then $A \to B = A[t]/(t^2 - x, t^3 - y)$ induces bijections on residue fields and a universal homeomorphism on spectra.
2. If there is a prime number p such that $p^p x = y^p$ in A, then $A \to B = A[t]/(t^p - x, pt - y)$ induces a universal homeomorphism on spectra.

Proof. We will use the criterion of Lemma 29.45.5 to check this. In both cases the ring map is integral. Thus it suffices to show that given a field k and a ring map $\varphi : A \to k$ the k-algebra $B \otimes_A k$ has a unique prime ideal whose residue field is equal to k in case (1) and purely inseparable over k in case (2). See Lemma 29.10.2.

In case (1) set $\lambda = 0$ if $\varphi(x) = 0$ and set $\lambda = \varphi(y)/\varphi(x)$ if not. Then $B = k[t]/(t^2 - \lambda^2, t^3 - \lambda^2)$. Thus the result is clear.

In case (2) if the characteristic of k is p, then we obtain $\varphi(y) = 0$ and $B = k[t]/(t^p - \varphi(x))$ which is a local Artinian k-algebra whose residue field is either k or a degree p purely inseparable extension of k. If the characteristic of k is not p, then setting $\lambda = \varphi(y)/p$ we see $B = k[t]/(t - \lambda) = k$ and we conclude as well. □

0EUJ Lemma 29.46.11. Let $A \to B$ be a ring map.

1. If $A \to B$ induces a universal homeomorphism on spectra, then $B = \operatorname{colim} B_i$ is a filtered colimit of finitely presented A-algebras B_i such that $A \to B_i$ induces a universal homeomorphism on spectra.
2. If $A \to B$ induces isomorphisms on residue fields and a universal homeomorphism on spectra, then $B = \operatorname{colim} B_i$ is a filtered colimit of finitely presented A-algebras B_i such that $A \to B_i$ induces isomorphisms on residue fields and a universal homeomorphism on spectra.

Proof. Proof of (1). We will use the criterion of Algebra, Lemma 10.127.4. Let $A \to C$ be of finite presentation and let $\varphi : C \to B$ be an A-algebra map. Let $B' = \varphi(C) \subset B$ be the image. Then $A \to B'$ induces a universal homeomorphism on spectra by Lemma 29.46.1. By Algebra, Lemma 10.127.2 we can write $B' = \operatorname{colim}_{i \in I} B_i$ with $A \to B_i$ of finite presentation and surjective transition maps. By Algebra, Lemma 10.127.3 we can choose an index $0 \in I$ and a factorization $C \to B_0 \to B'$ of the map $C \to B'$. We claim that $\text{Spec}(B_i) \to \text{Spec}(A)$ is a universal homeomorphism for i sufficiently large. The claim finishes the proof of (1).

Proof of the claim. By Lemma 29.45.6 the ring map $A_{red} \to B'_{red}$ induces a universal homeomorphism on spectra. Thus $A_{red} \subset B'_{red}$ by Algebra, Lemma

10.30.6. Setting $A' = \mathrm{Im}(A \to B')$ we have surjections $A \to A' \to A_{red}$ inducing bijections $\mathrm{Spec}(A_{red}) = \mathrm{Spec}(A') = \mathrm{Spec}(A)$. Thus $A' \subset B'$ induces a universal homeomorphism on spectra. By Proposition 29.46.8 and the fact that B' is finite type over A' we can find n and $b'_1, \ldots, b'_n \in B'$ such that $B' = A'[b'_1, \ldots, b'_n]$ and such that for $j = 1, \ldots, n$ we have

(1) $(b'_j)^2, (b'_j)^3 \in A'[b'_1, \ldots, b'_{j-1}]$, or
(2) there exists a prime number p with $pb'_j, (b'_j)^p \in A'[b'_1, \ldots, b'_{j-1}]$.

Choose $b_1, \ldots, b_n \in B_0$ lifting $b'_1, \ldots, b'_n$. For $i \geq 0$ denote $b_{j,i}$ the image of b_j in B_i. For large enough i we will have for $j = 1, \ldots, n$

(1) $b_{j,i}^2, b_{j,i}^3 \in A_i[b_{1,i}, \ldots, b_{j-1,i}]$, or
(2) there exists a prime number p with $pb_{j,i}, b_{j,i}^p \in A_i[b_{1,i}, \ldots, b_{j-1,i}]$.

Here $A_i \subset B_i$ is the image of $A \to B_i$. Observe that $A \to A_i$ is a surjective ring map whose kernel is a locally nilpotent ideal. After increasing i more if necessary, we may assume B_i is generated by $b_1, \ldots, b_n$ over A_i, in other words $B_i = A_i[b_1, \ldots, b_n]$. By Algebra, Lemmas 10.46.7 and 10.46.4 we conclude that $A \to A_i \to A_i[b_1] \to \ldots \to A_i[b_1, \ldots, b_n] = B_i$ induce universal homeomorphisms on spectra. This finishes the proof of the claim.

The proof of (2) is exactly the same. □

29.47. Absolute weak normalization and seminormalization

0EUK Motivated by the results proved in the previous section we give the following definition.

0EUL **Definition 29.47.1.** Let A be a ring.

(1) We say A is *seminormal* if for all $x, y \in A$ with $x^3 = y^2$ there is a unique $a \in A$ with $x = a^2$ and $y = a^3$.
(2) We say A is *absolutely weakly normal* if (a) A is seminormal and (b) for any prime number p and $x, y \in A$ with $p^p x = y^p$ there is a unique $a \in A$ with $x = a^p$ and $y = pa$.

An amusing observation, see [Cos82], is that in the definition of seminormal rings it suffices[15] to assume the existence of a. Absolutely weakly normal schemes were defined in [Ryd07b, Appendix B].

0EUM **Lemma 29.47.2.** *Being seminormal or being absolutely weakly normal is a local property of rings, see Properties, Definition 28.4.1.*

Proof. Suppose that A is seminormal and $f \in A$. Let $x', y' \in A_f$ with $(x')^3 = (y')^2$. Write $x' = x/f^{2n}$ and $y' = y/f^{3n}$ for some $n \geq 0$ and $x, y \in A$. After replacing x, y by $f^{2m}x, f^{3m}y$ and n by $n + m$, we see that $x^3 = y^2$ in A. Then we find a unique $a \in A$ with $x = a^2$ and $y = a^3$. Setting $a' = a/f^n$ we get $x' = (a')^2$ and $y' = (a')^3$ as desired. Uniqueness of a' follows from uniqueness of a. In exactly the same manner the reader shows that if A is absolutely weakly normal, then A_f is absolutely weakly normal.

[15]Let A be a ring such that for all $x, y \in A$ with $x^3 = y^2$ there is an $a \in A$ with $x = a^2$ and $y = a^3$. Then A is reduced: if $x^2 = 0$, then $x^2 = x^3$ and hence there exists an a such that $x = a^3$ and $x = a^2$. Then $x = a^3 = ax = a^4 = x^2 = 0$. Finally, if $a_1^2 = a_2^2$ and $a_1^3 = a_2^3$ for a_1, a_2 in a reduced ring, then $(a_1 - a_2)^3 = a_1^3 - 3a_1^2a_2 + 3a_1a_2^2 - a_2^3 = (1 - 3 + 3 - 1)a_1^3 = 0$ and hence $a_1 = a_2$.

Assume A is a ring and $f_1, \ldots, f_n \in A$ generate the unit ideal. Assume A_{f_i} is seminormal for each i. Let $x, y \in A$ with $x^3 = y^2$. For each i we find a unique $a_i \in A_{f_i}$ with $x = a_i^2$ and $y = a_i^3$ in A_{f_i}. By the uniqueness and the result of the first paragraph (which tells us that $A_{f_i f_j}$ is seminormal) we see that a_i and a_j map to the same element of $A_{f_i f_j}$. By Algebra, Lemma 10.24.2 we find a unique $a \in A$ mapping to a_i in A_{f_i} for all i. Then $x = a^2$ and $y = a^3$ by the same token. Clearly this a is unique. Thus A is seminormal. If we assume A_{f_i} is absolutely weakly normal, then the exact same argument shows that A is absolutely weakly normal. □

Next we define seminormal schemes and absolutely weakly normal schemes.

0EUN Definition 29.47.3. Let X be a scheme.

(1) We say X is seminormal if every $x \in X$ has an affine open neighbourhood $\text{Spec}(R) = U \subset X$ such that the ring R is seminormal.
(2) We say X is absolutely weakly normal if every $x \in X$ has an affine open neighbourhood $\text{Spec}(R) = U \subset X$ such that the ring R is absolutely weakly normal.

Here is the obligatory lemma.

0EUP Lemma 29.47.4. Let X be a scheme. The following are equivalent:

(1) The scheme X is seminormal.
(2) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is seminormal.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is seminormal.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is seminormal.

Moreover, if X is seminormal then every open subscheme is seminormal. The same statements are true with “seminormal” replaced by “absolutely weakly normal”.

Proof. Combine Properties, Lemma 28.4.3 and Lemma 29.47.2. □

0EUQ Lemma 29.47.5. A seminormal scheme or ring is reduced. A fortiori the same is true for absolutely weakly normal schemes or rings.

Proof. Let A be a ring. If $a \in A$ is nonzero but $a^2 = 0$, then $a^2 = 0^2$ and $a^3 = 0^3$ and hence A is not seminormal. □

0EUR Lemma 29.47.6. Let A be a ring.

(1) The category of ring maps $A \to B$ inducing a universal homeomorphism on spectra has a final object $A \to A^{awn}$.
(2) Given $A \to B$ in the category of (1) the resulting map $B \to A^{awn}$ is an isomorphism if and only if B is absolutely weakly normal.
(3) The category of ring maps $A \to B$ inducing isomorphisms on residue fields and a universal homeomorphism on spectra has a final object $A \to A^{sn}$.
(4) Given $A \to B$ in the category of (3) the resulting map $B \to A^{sn}$ is an isomorphism if and only if B is seminormal.

For any ring map $\varphi : A \to A'$ there are unique maps $\varphi^{awn} : A^{awn} \to (A')^{awn}$ and $\varphi^{sn} : A^{sn} \to (A')^{sn}$ compatible with φ.

Proof. We prove (1) and (2) and we omit the proof of (3) and (4) and the final statement. Consider the category of A-algebras of the form

$$B = A[x_1, \ldots, x_n]/J$$

where J is a finitely generated ideal such that $A \to B$ defines a universal homeomorphism on spectra. We claim this category is directed (Categories, Definition 4.19.1). Namely, given

$$B = A[x_1, \ldots, x_n]/J \quad \text{and} \quad B' = A[x_1, \ldots, x_{n'}]/J'$$

then we can consider

$$B'' = A[x_1, \ldots, x_{n+n'}]/J''$$

where J'' is generated by the elements of J and the elements $f(x_{n+1}, \ldots, x_{n+n'})$ where $f \in J'$. Then we have A-algebra homomorphisms $B \to B''$ and $B' \to B''$ which induce an isomorphism $B \otimes_A B' \to B''$. It follows from Lemmas 29.45.2 and 29.45.3 that $\mathrm{Spec}(B'') \to \mathrm{Spec}(A)$ is a universal homeomorphism and hence $A \to B''$ is in our category. Finally, given $\varphi, \varphi' : B \to B'$ in our category with B as displayed above, then we consider the quotient B'' of B' by the ideal generated by $\varphi(x_i) - \varphi'(x_i)$, $i = 1, \ldots, n$. Since $\mathrm{Spec}(B') = \mathrm{Spec}(B)$ we see that $\mathrm{Spec}(B'') \to \mathrm{Spec}(B')$ is a bijective closed immersion hence a universal homeomorphism. Thus B'' is in our category and φ, φ' are equalized by $B' \to B''$. This completes the proof of our claim. We set

$$A^{awn} = \mathrm{colim}\, B$$

where the colimit is over the category just described. Observe that $A \to A^{awn}$ induces a universal homeomorphism on spectra by Lemma 29.46.2 (this is where we use the category is directed).

Given a ring map $A \to B$ of finite presentation inducing a universal homeomorphism on spectra, we get a canonical map $B \to A^{awn}$ by the very construction of A^{awn}. Since every $A \to B$ as in (1) is a filtered colimit of $A \to B$ as in (1) of finite presentation (Lemma 29.46.11), we see that $A \to A^{awn}$ is final in the category (1).

Let $x, y \in A^{awn}$ be elements such that $x^3 = y^2$. Then $A^{awn} \to A^{awn}[t]/(t^2 - x, t^3 - y)$ induces a universal homeomorphism on spectra by Lemma 29.46.10. Thus $A \to A^{awn}[t]/(t^2 - x, t^3 - y)$ is in the category (1) and we obtain a unique A-algebra map $A^{awn}[t]/(t^2 - x, t^3 - y) \to A^{awn}$. The image $a \in A^{awn}$ of t is therefore the unique element such that $a^2 = x$ and $a^3 = y$ in A^{awn}. In exactly the same manner, given a prime p and $x, y \in A^{awn}$ with $p^p x = y^p$ we find a unique $a \in A^{awn}$ with $a^p = x$ and $pq = y$. Thus A^{awn} is absolutely weakly normal by definition.

Finally, let $A \to B$ be in the category (1) with B absolutely weakly normal. Since $A^{awn} \to B^{awn}$ induces a universal homeomorphism on spectra and since A^{awn} is reduced (Lemma 29.47.5) we find $A^{awn} \subset B^{awn}$ (see Algebra, Lemma 10.30.6). If this inclusion is not an equality, then Lemma 29.46.6 implies there is an element $b \in B^{awn}$, $b \notin A^{awn}$ such that either $b^2, b^3 \in A^{awn}$ or $pb, b^p \in A^{awn}$ for some prime number p. However, by the existence and uniqueness in Definition 29.47.1 this forces $b \in A^{awn}$ and hence we obtain the contradiction that finishes the proof. $\square$

0EUS Lemma 29.47.7. Let X be a scheme.

(1) The category of universal homeomorphisms $Y \to X$ has an initial object $X^{awn} \to X$.

(2) Given $Y \to X$ in the category of (1) the resulting morphism $X^{awn} \to Y$ is an isomorphism if and only if Y is absolutely weakly normal.
(3) The category of universal homeomorphisms $Y \to X$ which induce ismomorphisms on residue fields has an initial object $X^{sn} \to X$.
(4) Given $Y \to X$ in the category of (3) the resulting morphism $X^{sn} \to Y$ is an isomorphism if and only if Y is seminormal.

For any morphism $h : X' \to X$ of schemes there are unique morphisms $h^{awn} : (X')^{awn} \to X^{awn}$ and $h^{sn} : (X')^{sn} \to X^{sn}$ compatible with h.

Proof. We will prove (1) and (2) and omit the proof of (3) and (4). Let $h : X' \to X$ be a morphism of schemes. If (1) holds for X and X', then $X' \times_X X^{awn} \to X'$ is a universal homeomorphism and hence we get a unique morphism $(X')^{awn} \to X' \times_X X^{awn}$ over X' by the universal property of $(X')^{awn} \to X'$. Composed with the projection $X' \times_X X^{awn} \to X^{awn}$ we obtain h^{awn}. If in addition (2) holds for X and X' and h is an open immersion, then $X' \times_X X^{awn}$ is absolutely weakly normal (Lemma 29.47.4) and we deduce that $(X')^{awn} \to X' \times_X X^{awn}$ is an isomorphism.

Recall that any universal homeomorphism is affine, see Lemma 29.45.4. Thus if X is affine then (1) and (2) follow immediately from Lemma 29.47.6. Let X be a scheme and let $\mathcal{B}$ be the set of affine opens of X. For each $U \in \mathcal{B}$ we obtain $U^{awn} \to U$ and for $V \subset U$, $V, U \in \mathcal{B}$ we obtain a canonical isomorphism $\rho_{V,U} : V^{awn} \to V \times_U U^{awn}$ by the discussion in the previous paragraph. Thus by relative glueing (Constructions, Lemma 27.2.1) we obtain a morphism $X^{awn} \to X$ which restricts to U^{awn} over U compatibly with the $\rho_{V,U}$. Next, let $Y \to X$ be a universal homeomorphism. Then $U \times_X Y \to U$ is a universal homeomorphism for $U \in \mathcal{B}$ and we obtain a unique morphism $g_U : U^{awn} \to U \times_X Y$ over U. These g_U are compatible with the morphisms $\rho_{V,U}$; details omitted. Hence there is a unique morphism $g : X^{awn} \to Y$ over X agreeing with g_U over U, see Constructions, Remark 27.2.3. This proves (1) for X. Part (2) follows because it holds affine locally. $\square$

0EUT Definition 29.47.8. Let X be a scheme.

(1) The morphism $X^{sn} \to X$ constructed in Lemma 29.47.7 is the seminormalization of X.
(2) The morphism $X^{awn} \to X$ constructed in Lemma 29.47.7 is the absolute weak normalization of X.

To be sure, the seminormalization X^{sn} of X is a seminormal scheme and the absolute weak normalization X^{awn} is an absolutely weakly normal scheme. Moreover, for any morphism $h : Y \to X$ of schemes we obtain a canonical commutative diagram

$$\begin{tikzcd} Y^{awn} \arrow[r] \arrow[d, "h^{awn}"] & Y^{sn} \arrow[r] \arrow[d, "h^{sn}"] & Y \arrow[d, "h"] \\ X^{awn} \arrow[r] & X^{sn} \arrow[r] & X \end{tikzcd}$$

of schemes; the arrows h^{sn} and h^{awn} are the unique ones compatible with h.

0H3G Lemma 29.47.9. Let X be a scheme. The following are equivalent

(1) X is seminormal,
(2) X is equal to its own seminormalization, i.e., the morphism $X^{sn} \to X$ is an isomorphism,

(3) if $\pi : Y \to X$ is a universal homomorphism inducing isomorphisms on residue fields with Y reduced, then π is an isomorphism.

Proof. The equivalence of (1) and (2) is clear from Lemma 29.47.7. If (3) holds, then $X^{sn} \to X$ is an isomorphism and we see that (2) holds.

Assume (2) holds and let $\pi : Y \to X$ be a universal homomorphism inducing isomorphisms on residue fields with Y reduced. Then there exists a factorization $X \to Y \to X$ of id_X by Lemma 29.47.7. Then $X \to Y$ is a closed immersion (by Schemes, Lemma 26.21.11 and the fact that π is separated for example by Lemma 29.10.3). Since $X \to Y$ is also a bijection on points, the reducedness of Y shows that it has to be an isomorphism. This finishes the proof. □

0H3H Lemma 29.47.10. Let X be a scheme. The following are equivalent

(1) X is absolutely weakly normal,
(2) X is equal to its own absolute weak normalization, i.e., the morphism $X^{awn} \to X$ is an isomorphism,
(3) if $\pi : Y \to X$ is a universal homomorphism with Y reduced, then π is an isomorphism.

Proof. This is proved in exactly the same manner as Lemma 29.47.9. □

29.48. Finite locally free morphisms

02K9 In many papers the authors use finite flat morphisms when they really mean finite locally free morphisms. The reason is that if the base is locally Noetherian then this is the same thing. But in general it is not, see Exercises, Exercise 111.5.3.

02KA Definition 29.48.1. Let $f : X \to S$ be a morphism of schemes. We say f is finite locally free if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module. In this case we say f is has rank or degree d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of degree d.

Note that if $f : X \to S$ is finite locally free then S is the disjoint union of open and closed subschemes S_d such that $f^{-1}(S_d) \to S_d$ is finite locally free of degree d.

02KB Lemma 29.48.2. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) f is finite locally free,
(2) f is finite, flat, and locally of finite presentation.

If S is locally Noetherian these are also equivalent to

(3) f is finite and flat.

Proof. Let $V \subset S$ be affine open. In all three cases the morphism is affine hence $f^{-1}(V)$ is affine. Thus we may write $V = \text{Spec}(R)$ and $f^{-1}(V) = \text{Spec}(A)$ for some R-algebra A. Assume (1). This means we can cover S by affine opens $V = \text{Spec}(R)$ such that A is finite free as an R-module. Then $R \to A$ is of finite presentation by Algebra, Lemma 10.7.4. Thus (2) holds. Conversely, assume (2). For every affine open $V = \text{Spec}(R)$ of S the ring map $R \to A$ is finite and of finite presentation and A is flat as an R-module. By Algebra, Lemma 10.36.23 we see that A is finitely presented as an R-module. Thus Algebra, Lemma 10.78.2 implies A is finite locally free. Thus (1) holds. The Noetherian case follows as a finite module over a Noetherian ring is a finitely presented module, see Algebra, Lemma 10.31.4. □

02KC **Lemma 29.48.3.** A composition of finite locally free morphisms is finite locally free.

Proof. Omitted. □

02KD **Lemma 29.48.4.** A base change of a finite locally free morphism is finite locally free.

Proof. Omitted. □

04MH **Lemma 29.48.5.** Let $f : X \to S$ be a finite locally free morphism of schemes. There exists a disjoint union decomposition $S = \coprod_{d \geq 0} S_d$ by open and closed subschemes such that setting $X_d = f^{-1}(S_d)$ the restrictions $f|_{X_d}$ are finite locally free morphisms $X_d \to S_d$ of degree d.

Proof. This is true because a finite locally free sheaf locally has a well defined rank. Details omitted. □

03HW **Lemma 29.48.6.** Let $f : Y \to X$ be a finite morphism with X affine. There exists a diagram

$$\begin{tikzcd} Z' \arrow[rd] & Y' \arrow[l, "i"'] \arrow[r] \arrow[d] & Y \arrow[d] \\ & X' \arrow[r] & X \end{tikzcd}$$

where

(1) $Y' \to Y$ and $X' \to X$ are surjective finite locally free,
(2) $Y' = X' \times_X Y$,
(3) $i : Y' \to Z'$ is a closed immersion,
(4) $Z' \to X'$ is finite locally free, and
(5) $Z' = \bigcup_{j=1,\ldots,m} Z'_j$ is a (set theoretic) finite union of closed subschemes, each of which maps isomorphically to X'.

Proof. Write $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. See also More on Algebra, Section 15.21. Let $x_1, \ldots, x_n \in B$ be generators of B over A. For each i we can choose a monic polynomial $P_i(T) \in A[T]$ such that $P(x_i) = 0$ in B. By Algebra, Lemma 10.136.14 (applied n times) there exists a finite locally free ring extension $A \subset A'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{k=1,\ldots,d_i} (T - \alpha_{ik})$$

for certain $\alpha_{ik} \in A'$. Set

$$C = A'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

and $B' = A' \otimes_A B$. The map $C \to B'$, $T_i \mapsto 1 \otimes x_i$ is an A'-algebra surjection. Setting $X' = \text{Spec}(A')$, $Y' = \text{Spec}(B')$ and $Z' = \text{Spec}(C)$ we see that (1) – (4) hold. Part (5) holds because set theoretically $\text{Spec}(C)$ is the union of the closed subschemes cut out by the ideals

$$(T_1 - \alpha_{1k_1}, T_2 - \alpha_{2k_2}, \ldots, T_n - \alpha_{nk_n})$$

for any $1 \leq k_i \leq d_i$. □

The following lemma is stated in the correct generality in Lemma 29.56.4 below.

03HX **Lemma 29.48.7.** Let $f : Y \to X$ be a finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a closed nowhere dense subset of X.

Proof. By Lemma 29.44.11 we know that $f(T) \subset X$ is closed. Let $X = \bigcup X_i$ be an affine covering. Since T is nowhere dense in Y, we see that also $T \cap f^{-1}(X_i)$ is nowhere dense in $f^{-1}(X_i)$. Hence if we can prove the theorem in the affine case, then we see that $f(T) \cap X_i$ is nowhere dense. This then implies that T is nowhere dense in X by Topology, Lemma 5.21.4.

Assume X is affine. Choose a diagram

$$\begin{tikzcd} Z' \arrow[rd] & Y' \arrow[l, "i"'] \arrow[r, "a"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ & X' \arrow[r, "b"] & X \end{tikzcd}$$

as in Lemma 29.48.6. The morphisms a, b are open since they are finite locally free (Lemmas 29.48.2 and 29.25.10). Hence $T' = a^{-1}(T)$ is nowhere dense, see Topology, Lemma 5.21.6. The morphism b is surjective and open. Hence, if we can prove $f'(T') = b^{-1}(f(T))$ is nowhere dense, then $f(T)$ is nowhere dense, see Topology, Lemma 5.21.6. As i is a closed immersion, by Topology, Lemma 5.21.5 we see that $i(T') \subset Z'$ is closed and nowhere dense. Thus we have reduced the problem to the case discussed in the following paragraph.

Assume that $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of closed subsets, each mapping isomorphically to X. Consider $T_i = Y_i \cap T$. If each of the T_i is nowhere dense in Y_i, then each $f(T_i)$ is nowhere dense in X as $Y_i \to X$ is an isomorphism. Hence $f(T) = f(T_i)$ is a finite union of nowhere dense closed subsets of X and we win, see Topology, Lemma 5.21.2. Suppose not, say T_1 contains a nonempty open $V \subset Y_1$. We are going to show this leads to a contradiction. Consider $Y_2 \cap V \subset V$. This is either a proper closed subset, or equal to V. In the first case we replace V by $V \setminus V \cap Y_2$, so $V \subset T_1$ is open in Y_1 and does not meet Y_2. In the second case we have $V \subset Y_1 \cap Y_2$ is open in both Y_1 and Y_2. Repeat sequentially with $i = 3, \ldots, n$. The result is a disjoint union decomposition

$$\{1, \ldots, n\} = I_1 \amalg I_2, \quad 1 \in I_1$$

and an open V of Y_1 contained in T_1 such that $V \subset Y_i$ for $i \in I_1$ and $V \cap Y_i = \emptyset$ for $i \in I_2$. Set $U = f(V)$. This is an open of X since $f|_{Y_1} : Y_1 \to X$ is an isomorphism. Then

$$f^{-1}(U) = V \ \amalg \ \bigcup\nolimits_{i \in I_2} (Y_i \cap f^{-1}(U))$$

As $\bigcup_{i \in I_2} Y_i$ is closed, this implies that $V \subset f^{-1}(U)$ is open, hence $V \subset Y$ is open. This contradicts the assumption that T is nowhere dense in Y, as desired. $\square$

29.49. Rational maps

01RR Let X be a scheme. Note that if U, V are dense open in X, then so is $U \cap V$.

01RS **Definition 29.49.1.** Let X, Y be schemes.

1. Let $f : U \to Y$, $g : V \to Y$ be morphisms of schemes defined on dense open subsets U, V of X. We say that f is equivalent to g if $f|_W = g|_W$ for some $W \subset U \cap V$ dense open in X.

(2) A rational map from X to Y is an equivalence class for the equivalence relation defined in (1).
(3) If X, Y are schemes over a base scheme S we say that a rational map from X to Y is an S-rational map from X to Y if there exists a representative $f : U \to Y$ of the equivalence class which is an S-morphism.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent. In some cases rational maps are determined by maps on local rings at generic points.

0BX8 Lemma 29.49.2. Let S be a scheme. Let X and Y be schemes over S. Assume X has finitely many irreducible components with generic points $x_1, \ldots, x_n$. Let $s_i \in S$ be the image of x_i. Consider the map

$$\left\{ \begin{matrix} S\text{-rational maps} \\ \text{from } X \text{ to } Y \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} (y_1, \varphi_1, \ldots, y_n, \varphi_n) \text{ where } y_i \in Y \text{ lies over } s_i \text{ and} \\ \varphi_i : \mathcal{O}_{Y,y_i} \to \mathcal{O}_{X,x_i} \text{ is a local } \mathcal{O}_{S,s_i}\text{-algebra map} \end{matrix} \right\}$$

which sends $f : U \to Y$ to the $2n$-tuple with $y_i = f(x_i)$ and $\varphi_i = f^\sharp_{x_i}$. Then

(1) If $Y \to S$ is locally of finite type, then the map is injective.
(2) If $Y \to S$ is locally of finite presentation, then the map is bijective.
(3) If $Y \to S$ is locally of finite type and X reduced, then the map is bijective.

Proof. Observe that any dense open of X contains the points x_i so the construction makes sense. To prove (1) or (2) we may replace X by any dense open. Thus if $Z_1, \ldots, Z_n$ are the irreducible components of X, then we may replace X by $X \setminus \bigcup_{i \neq j} Z_i \cap Z_j$. After doing this X is the disjoint union of its irreducible components (viewed as open and closed subschemes). Then both the right hand side and the left hand side of the arrow are products over the irreducible components and we reduce to the case where X is irreducible.

Assume X is irreducible with generic point x lying over $s \in S$. Part (1) follows from part (1) of Lemma 29.42.4. Parts (2) and (3) follow from part (2) of the same lemma. $\square$

01RT Definition 29.49.3. Let X be a scheme. A rational function on X is a rational map from X to $\mathbf{A}^1_{\mathbf{Z}}$.

See Constructions, Definition 27.5.1 for the definition of the affine line $\mathbf{A}^1$. Let X be a scheme over S. For any open $U \subset X$ a morphism $U \to \mathbf{A}^1_{\mathbf{Z}}$ is the same as a morphism $U \to \mathbf{A}^1_S$ over S. Hence a rational function is also the same as a S-rational map from X into $\mathbf{A}^1_S$.

Recall that we have the canonical identification $\mathrm{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$ for any scheme T, see Schemes, Example 26.15.2. Hence $\mathbf{A}^1_{\mathbf{Z}}$ is a ring-object in the category of schemes. More precisely, the morphisms

$$\begin{matrix} + : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} & \longrightarrow & \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) & \longmapsto & f + g \\ * : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} & \longrightarrow & \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) & \longmapsto & fg \end{matrix}$$

satisfy all the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}^1_{\mathbf{Z}}$ has a natural ring structure.

01RU Definition 29.49.4. Let X be a scheme. The ring of rational functions on X is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

For schemes with finitely many irreducible components we can compute this.

01RV Lemma 29.49.5. Let X be a scheme with finitely many irreducible components $X_1, \ldots, X_n$. If $\eta_i \in X_i$ is the generic point, then

$$R(X) = \mathcal{O}_{X,\eta_1} \times \ldots \times \mathcal{O}_{X,\eta_n}$$

If X is reduced this is equal to $\prod \kappa(\eta_i)$. If X is integral then $R(X) = \mathcal{O}_{X,\eta} = \kappa(\eta)$ is a field.

Proof. Let $U \subset X$ be an open dense subset. Then $U_i = (U \cap X_i) \setminus (\bigcup_{j \neq i} X_j)$ is nonempty open as it contained η_i, contained in X_i, and $\bigcup U_i \subset U \subset X$ is dense. Thus the identification in the lemma comes from the string of equalities

$$\begin{aligned} R(X) &= \mathrm{colim}_{U \subset X \text{ open dense}} \mathrm{Mor}(U, \mathbf{A}^1_{\mathbf{Z}}) \\ &= \mathrm{colim}_{U \subset X \text{ open dense}} \mathcal{O}_X(U) \\ &= \mathrm{colim}_{\eta_i \in U_i \subset X \text{ open}} \prod \mathcal{O}_X(U_i) \\ &= \prod \mathrm{colim}_{\eta_i \in U_i \subset X \text{ open}} \mathcal{O}_X(U_i) \\ &= \prod \mathcal{O}_{X,\eta_i} \end{aligned}$$

where the second equality is Schemes, Example 26.15.2. The final statement follows from Algebra, Lemma 10.25.1. □

01RW Definition 29.49.6. Let X be an integral scheme. The function field, or the field of rational functions of X is the field $R(X)$.

We may occasionally indicate this field $k(X)$ instead of $R(X)$. We can use the notion of the function field to elucidate the separation condition on an integral scheme. Note that by Lemma 29.49.5 on an integral scheme every local ring $\mathcal{O}_{X,x}$ may be viewed as a local subring of $R(X)$.

02NF Lemma 29.49.7. Let X be an integral separated scheme. Let Z_1, Z_2 be distinct irreducible closed subsets of X. Let η_i be the generic point of Z_i. If $Z_1 \not\subset Z_2$, then $\mathcal{O}_{X,\eta_1} \not\subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. In particular, if $Z_1 = \{x\}$ consists of one closed point x, there exists a function regular in a neighborhood of x which is not in $\mathcal{O}_{X,\eta_2}$.

Proof. First observe that under the assumption of X being separated, there is a unique map of schemes $\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \to X$ over X such that the composition

$$\mathrm{Spec}(R(X)) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow X$$

is the canonical map $\mathrm{Spec}(R(X)) \to X$. Namely, there is the canonical map can : $\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \to X$, see Schemes, Equation (26.13.1.1). Given a second morphism a to X, we have that a agrees with can on the generic point of $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$ by assumption. Now X being separated guarantees that the subset in $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$ where these two maps agree is closed, see Schemes, Lemma 26.21.5. Hence $a = can$ on all of $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$.

Assume $Z_1 \not\subset Z_2$ and assume on the contrary that $\mathcal{O}_{X,\eta_1} \subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. Then we would obtain a second morphism

$$\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,\eta_1}) \longrightarrow X.$$

By the above this composition would have to be equal to *can*. This implies that η_2 specializes to η_1 (see Schemes, Lemma 26.13.2). But this contradicts our assumption $Z_1 \not\subset Z_2$. □

0A1X Definition 29.49.8. Let φ be a rational map between two schemes X and Y. We say φ is defined in a point $x \in X$ if there exists a representative (U, f) of φ with $x \in U$. The domain of definition of φ is the set of all points where φ is defined.

With this definition it isn't true in general that φ has a representative which is defined on all of the domain of definition.

0A1Y Lemma 29.49.9. Let X and Y be schemes. Assume X reduced and Y separated. Let φ be a rational map from X to Y with domain of definition $U \subset X$. Then there exists a unique morphism $f : U \to Y$ representing φ. If X and Y are schemes over a separated scheme S and if φ is an S-rational map, then f is a morphism over S.

Proof. Let (V, g) and (V', g') be representatives of φ. Then g, g' agree on a dense open subscheme $W \subset V \cap V'$. On the other hand, the equalizer E of $g|_{V \cap V'}$ and $g'|_{V \cap V'}$ is a closed subscheme of $V \cap V'$ (Schemes, Lemma 26.21.5). Now $W \subset E$ implies that $E = V \cap V'$ set theoretically. As $V \cap V'$ is reduced we conclude $E = V \cap V'$ scheme theoretically, i.e., $g|_{V \cap V'} = g'|_{V \cap V'}$. It follows that we can glue the representatives $g : V \to Y$ of φ to a morphism $f : U \to Y$, see Schemes, Lemma 26.14.1. We omit the proof of the final statement. □

In general it does not make sense to compose rational maps. The reason is that the image of a representative of the first rational map may have empty intersection with the domain of definition of the second. However, if we assume that our schemes are irreducible and we look at dominant rational maps, then we can compose rational maps.

0A1Z Definition 29.49.10. Let X and Y be irreducible schemes. A rational map from X to Y is called dominant if any representative $f : U \to Y$ is a dominant morphism of schemes.

By Lemma 29.8.6 it is equivalent to require that the generic point $\eta \in X$ maps to the generic point ξ of Y, i.e., $f(\eta) = \xi$ for any representative $f : U \to Y$. We can compose a dominant rational map φ between irreducible schemes X and Y with an arbitrary rational map ψ from Y to Z. Namely, choose representatives $f : U \to Y$ with $U \subset X$ open dense and $g : V \to Z$ with $V \subset Y$ open dense. Then $W = f^{-1}(V) \subset X$ is open nonempty (because it contains the generic point of X) and we let $\psi \circ \varphi$ be the equivalence class of $g \circ f|_W : W \to Z$. We omit the verification that this is well defined.

In this way we obtain a category whose objects are irreducible schemes and whose morphisms are dominant rational maps. Given a base scheme S we can similarly define a category whose objects are irreducible schemes over S and whose morphisms are dominant S-rational maps.

0A20 Definition 29.49.11. Let X and Y be irreducible schemes.

(1) We say X and Y are birational if X and Y are isomorphic in the category of irreducible schemes and dominant rational maps.
(2) Assume X and Y are schemes over a base scheme S. We say X and Y are S-birational if X and Y are isomorphic in the category of irreducible schemes over S and dominant S-rational maps.

If X and Y are birational irreducible schemes, then the set of rational maps from X to Z is bijective with the set of rational map from Y to Z for all schemes Z (functorially in Z). For "general" irreducible schemes this is just one possible definition. Another would be to require X and Y have isomorphic rings of rational functions. For varieties these conditions are equivalent, see Lemma 29.50.6.

0BAA Lemma 29.49.12. Let X and Y be irreducible schemes.

(1) The schemes X and Y are birational if and only if they have isomorphic nonempty opens.
(2) Assume X and Y are schemes over a base scheme S. Then X and Y are S-birational if and only if there are nonempty opens $U \subset X$ and $V \subset Y$ which are S-isomorphic.

Proof. Assume X and Y are birational. Let $f : U \to Y$ and $g : V \to X$ define inverse dominant rational maps from X to Y and from Y to X. We may assume V affine. We may replace U by an affine open of $f^{-1}(V)$. As $g \circ f$ is the identity as a dominant rational map, we see that the composition $U \to V \to X$ is the identity on a dense open of U. Thus after replacing U by a smaller affine open we may assume that $U \to V \to X$ is the inclusion of U into X. It follows that $U \to V$ is an immersion (apply Schemes, Lemma 26.21.11 to $U \to g^{-1}(U) \to U$). However, switching the roles of U and V and redoing the argument above, we see that there exists a nonempty affine open $V' \subset V$ such that the inclusion factors as $V' \to U \to V$. Then $V' \to U$ is necessarily an open immersion. Namely, $V' \to f^{-1}(V') \to V'$ are monomorphisms (Schemes, Lemma 26.23.8) composing to the identity, hence isomorphisms. Thus V' is isomorphic to an open of both X and Y. In the S-rational maps case, the exact same argument works. $\square$

0BX9 Remark 29.49.13. Here is a generalization of the category of irreducible schemes and dominant rational maps. For a scheme X denote X^0 the set of points $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 0$, in other words, X^0 is the set of generic points of irreducible components of X. Then we can consider the category with

(1) objects are schemes X such that every quasi-compact open has finitely many irreducible components, and
(2) morphisms from X to Y are rational maps $f : U \to Y$ from X to Y such that $f(U^0) = Y^0$.

If $U \subset X$ is a dense open of a scheme, then $U^0 \subset X^0$ need not be an equality, but if X is an object of our category, then this is the case. Thus given two morphisms in our category, the composition is well defined and a morphism in our category.

01RX Remark 29.49.14. There is a variant of Definition 29.49.1 where we consider only those morphism $U \to Y$ defined on scheme theoretically dense open subschemes $U \subset X$. We use Lemma 29.7.6 to see that we obtain an equivalence relation. An equivalence class of these is called a pseudo-morphism from X to Y. If X is reduced the two notions coincide.

29.50. Birational morphisms

01RN You may be used to the notion of a birational map of varieties having the property that it is an isomorphism over an open subset of the target. However, in general a birational morphism may not be an isomorphism over any nonempty open, see Example 29.50.4. Here is the formal definition.

01RO **Definition 29.50.1.** Let X, Y be schemes. Assume X and Y have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if [GD60, (2.2.9)]

1. f induces a bijection between the set of generic points of irreducible components of X and the set of generic points of the irreducible components of Y, and
2. for every generic point $\eta \in X$ of an irreducible component of X the local ring map $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

We will see below that the fibres of a birational morphism over generic points are singletons. Moreover, we will see that in most cases one encounters in practice the existence a birational morphism between irreducible schemes X and Y implies X and Y are birational schemes.

01RP **Lemma 29.50.2.** Let $f : X \to Y$ be a morphism of schemes having finitely many irreducible components. If f is birational then f is dominant.

Proof. Follows from Lemma 29.8.2 and the definition. □

0BAB **Lemma 29.50.3.** Let $f : X \to Y$ be a birational morphism of schemes having finitely many irreducible components. If $y \in Y$ is the generic point of an irreducible component, then the base change $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism.

Proof. We may assume $Y = \mathrm{Spec}(B)$ is affine and irreducible. Then X is irreducible too. If we prove the result for any nonempty affine open $U \subset X$, then the result holds for X (small argument omitted). Hence we may assume X is affine too, say $X = \mathrm{Spec}(A)$. Let $y \in Y$ correspond to the minimal prime $\mathfrak{q} \subset B$. By assumption A has a unique minimal prime $\mathfrak{p}$ lying over $\mathfrak{q}$ and $B_\mathfrak{q} \to A_\mathfrak{p}$ is an isomorphism. It follows that $A_\mathfrak{q} \to \kappa(\mathfrak{p})$ is surjective, hence $\mathfrak{p}A_\mathfrak{q}$ is a maximal ideal. On the other hand $\mathfrak{p}A_\mathfrak{q}$ is the unique minimal prime of $A_\mathfrak{q}$. We conclude that $\mathfrak{p}A_\mathfrak{q}$ is the unique prime of $A_\mathfrak{q}$ and that $A_\mathfrak{q} = A_\mathfrak{p}$. Since $A_\mathfrak{q} = A \otimes_B B_\mathfrak{q}$ the lemma follows. □

01RQ **Example 29.50.4.** Here are two examples of birational morphisms which are not isomorphisms over any open of the target.

First example. Let k be an infinite field. Let $A = k[x]$. Let $B = k[x, \{y_\alpha\}_{\alpha \in k}]/((x-\alpha)y_\alpha, y_\alpha y_\beta)$. There is an inclusion $A \subset B$ and a retraction $B \to A$ setting all y_α equal to zero. Both the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are birational but not an isomorphism over any open.

Second example. Let A be a domain. Let $S \subset A$ be a multiplicative subset not containing 0. With $B = S^{-1}A$ the morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is birational. If there exists an open U of $\mathrm{Spec}(A)$ such that $f^{-1}(U) \to U$ is an isomorphism, then there exists an $a \in A$ such that each every element of S becomes invertible in the principal localization A_a. Taking $A = \mathbf{Z}$ and S the set of odd integers give a counter example.

0BAC Lemma 29.50.5. Let $f : X \to Y$ be a birational morphism of schemes having finitely many irreducible components over a base scheme S. Assume one of the following conditions is satisfied

(1) f is locally of finite type and Y reduced,
(2) f is locally of finite presentation.

Then there exist dense opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is an isomorphism. In particular if X and Y are irreducible, then X and Y are S-birational.

Proof. There is an immediate reduction to the case where X and Y are irreducible which we omit. Moreover, after shrinking further and we may assume X and Y are affine, say $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. By assumption A, resp. B has a unique minimal prime $\mathfrak p$, resp. $\mathfrak q$, the prime $\mathfrak p$ lies over $\mathfrak q$, and $B_\mathfrak q = A_\mathfrak p$. By Lemma 29.50.3 we have $B_\mathfrak q = A_\mathfrak q = A_\mathfrak p$.

Suppose $B \to A$ is of finite type, say $A = B[x_1, \ldots, x_n]$. There exist a $b_i \in B$ and $g_i \in B \setminus \mathfrak q$ such that b_i/g_i maps to the image of x_i in $A_\mathfrak q$. Hence $b_i - g_i x_i$ maps to zero in $A_{g'_i}$ for some $g'_i \in B \setminus \mathfrak q$. Setting $g = \prod g_i g'_i$ we see that $B_g \to A_g$ is surjective. If moreover Y is reduced, then the map $B_g \to B_\mathfrak q$ is injective and hence $B_g \to A_g$ is injective as well. This proves case (1).

Proof of (2). By the argument given in the previous paragraph we may assume that $B \to A$ is surjective. As f is locally of finite presentation the kernel $J \subset B$ is a finitely generated ideal. Say $J = (b_1, \ldots, b_r)$. Since $B_\mathfrak q = A_\mathfrak q$ there exist $g_i \in B \setminus \mathfrak q$ such that $g_i b_i = 0$. Setting $g = \prod g_i$ we see that $B_g \to A_g$ is an isomorphism. $\square$

0BAD Lemma 29.50.6. Let S be a scheme. Let X and Y be irreducible schemes locally of finite presentation over S. Let $x \in X$ and $y \in Y$ be the generic points. The following are equivalent

(1) X and Y are S-birational,
(2) there exist nonempty opens of X and Y which are S-isomorphic, and
(3) x and y map to the same point s of S and $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as $\mathcal{O}_{S,s}$-algebras.

Proof. We have seen the equivalence of (1) and (2) in Lemma 29.49.12. It is immediate that (2) implies (3). To finish we assume (3) holds and we prove (1). By Lemma 29.49.2 there is a rational map $f : U \to Y$ which sends $x \in U$ to y and induces the given isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. Thus f is a birational morphism and hence induces an isomorphism on nonempty opens by Lemma 29.50.5. This finishes the proof. $\square$

0552 Lemma 29.50.7. Let S be a scheme. Let X and Y be integral schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be the generic points. The following are equivalent

(1) X and Y are S-birational,
(2) there exist nonempty opens of X and Y which are S-isomorphic, and
(3) x and y map to the same point $s \in S$ and $\kappa(x) \cong \kappa(y)$ as $\kappa(s)$-extensions.

Proof. We have seen the equivalence of (1) and (2) in Lemma 29.49.12. It is immediate that (2) implies (3). To finish we assume (3) holds and we prove (1). Observe that $\mathcal{O}_{X,x} = \kappa(x)$ and $\mathcal{O}_{Y,y} = \kappa(y)$ by Algebra, Lemma 10.25.1. By Lemma 29.49.2 there is a rational map $f : U \to Y$ which sends $x \in U$ to y and induces the given isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. Thus f is a birational morphism and hence induces an isomorphism on nonempty opens by Lemma 29.50.5. This finishes the proof. $\square$

29.51. Generically finite morphisms

02NV In this section we characterize maps between schemes which are locally of finite type and which are "generically finite" in some sense.

02NW Lemma 29.51.1. Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let $\eta \in Y$ be a generic point of an irreducible component of Y. The following are equivalent:

(1) the set $f^{-1}(\{\eta\})$ is finite,
(2) there exist affine opens $U_i \subset X$, $i = 1, \ldots, n$ and $V \subset Y$ with $f(U_i) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset \bigcup U_i$ such that each $f|_{U_i} : U_i \to V$ is finite.

If f is quasi-separated, then these are also equivalent to

(3) there exist affine opens $V \subset Y$, and $U \subset X$ with $f(U) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset U$ such that $f|_U : U \to V$ is finite.

If f is quasi-compact and quasi-separated, then these are also equivalent to

(4) there exists an affine open $V \subset Y$, $\eta \in V$ such that $f^{-1}(V) \to V$ is finite.

Proof. The question is local on the base. Hence we may replace Y by an affine neighbourhood of η, and we may and do assume throughout the proof below that Y is affine, say $Y = \text{Spec}(R)$.

It is clear that (2) implies (1). Assume that $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$ is finite. Choose affine opens $U_i \subset X$ with $\xi_i \in U_i$. By Algebra, Lemma 10.122.10 we see that after replacing Y by a standard open in Y each of the morphisms $U_i \to Y$ is finite. In other words (2) holds.

It is clear that (3) implies (1). Assume f is quasi-separated and (1). Write $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$. There are no specializations among the ξ_i by Lemma 29.20.7. Since each ξ_i maps to the generic point η of an irreducible component of Y, there cannot be a nontrivial specialization $\xi \leadsto \xi_i$ in X (since ξ would map to η as well). We conclude each ξ_i is a generic point of an irreducible component of X. Since Y is affine and f quasi-separated we see X is quasi-separated. By Properties, Lemma 28.29.1 we can find an affine open $U \subset X$ containing each ξ_i. By Algebra, Lemma 10.122.10 we see that after replacing Y by a standard open in Y the morphisms $U \to Y$ is finite. In other words (3) holds.

It is clear that (4) implies all of (1) – (3) with no further assumptions on f. Suppose that f is quasi-compact and quasi-separated. We have to show that the equivalent conditions (1) – (3) imply (4). Let U, V be as in (3). Replace Y by V. Since f is quasi-compact and Y is quasi-compact (being affine) we see that X is quasi-compact. Hence $Z = X \setminus U$ is quasi-compact, hence the morphism $f|_Z : Z \to Y$ is quasi-compact. By construction of Z we see that $\eta \notin f(Z)$. Hence by Lemma 29.8.5 we see that there exists an affine open neighbourhood V' of η in Y such that $f^{-1}(V') \cap Z = \emptyset$. Then we have $f^{-1}(V') \subset U$ and this means that $f^{-1}(V') \to V'$ is finite. $\square$

03HY Example 29.51.2. Let $A = \prod_{n \in \mathbf{N}} \mathbf{F}_2$. Every element of A is an idempotent. Hence every prime ideal is maximal with residue field $\mathbf{F}_2$. Thus the topology on $X = \text{Spec}(A)$ is totally disconnected and quasi-compact. The projection maps

$A \to \mathbf{F}_2$ define open points of $\text{Spec}(A)$. It cannot be the case that all the points of X are open since X is quasi-compact. Let $x \in X$ be a closed point which is not open. Then we can form a scheme Y which is two copies of X glued along $X \setminus \{x\}$. In other words, this is X with x doubled, compare Schemes, Example 26.14.3. The morphism $f : Y \to X$ is quasi-compact, finite type and has finite fibres but is not quasi-separated. The point $x \in X$ is a generic point of an irreducible component of X (since X is totally disconnected). But properties (3) and (4) of Lemma 29.51.1 do not hold. The reason is that for any open neighbourhood $x \in U \subset X$ the inverse image $f^{-1}(U)$ is not affine because functions on $f^{-1}(U)$ cannot separate the two points lying over x (proof omitted; this is a nice exercise). Hence the condition that f is quasi-separated is necessary in parts (3) and (4) of the lemma.

03HZ Remark 29.51.3. An alternative to Lemma 29.51.1 is the statement that a quasi-finite morphism is finite over a dense open of the target. This will be shown in More on Morphisms, Lemma 37.45.2.

0BAH Lemma 29.51.4. Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let X^0, resp. Y^0 denote the set of generic points of irreducible components of X, resp. Y. Let $\eta \in Y^0$. The following are equivalent

(1) $f^{-1}(\{\eta\}) \subset X^0$,
(2) f is quasi-finite at all points lying over η,
(3) f is quasi-finite at all $\xi \in X^0$ lying over η.

Proof. Condition (1) implies there are no specializations among the points of the fibre X_η. Hence (2) holds by Lemma 29.20.6. The implication (2) $\Rightarrow$ (3) is immediate. Since η is a generic point of Y, the generic points of X_η are generic points of X. Hence (3) and Lemma 29.20.6 imply the generic points of X_η are also closed. Thus all points of X_η are generic and we see that (1) holds. □

0BAI Lemma 29.51.5. Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let X^0, resp. Y^0 denote the set of generic points of irreducible components of X, resp. Y. Assume

(1) X^0 and Y^0 are finite and $f^{-1}(Y^0) = X^0$,
(2) either f is quasi-compact or f is separated.

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.

Proof. Since Y has finitely many irreducible components, we can find a dense open which is a disjoint union of its irreducible components. Thus we may assume Y is irreducible affine with generic point η. Then the fibre over η is finite as X^0 is finite.

Assume f is separated and Y irreducible affine. Choose $V \subset Y$ and $U \subset X$ as in Lemma 29.51.1 part (3). Since $f|_U : U \to V$ is finite, we see that $U \subset f^{-1}(V)$ is closed as well as open (Lemmas 29.41.7 and 29.44.11). Thus $f^{-1}(V) = U \amalg W$ for some open subscheme W of X. However, since U contains all the generic points of X we conclude that $W = \emptyset$ as desired.

Assume f is quasi-compact and Y irreducible affine. Then X is quasi-compact, hence there exists a dense open subscheme $U \subset X$ which is separated (Properties, Lemma 28.29.3). Since the set of generic points X^0 is finite, we see that $X^0 \subset U$. Thus $\eta \notin f(X \setminus U)$. Since $X \setminus U \to Y$ is quasi-compact, we conclude that there is a nonempty open $V \subset Y$ such that $f^{-1}(V) \subset U$, see Lemma 29.8.3. After replacing

X by $f^{-1}(V)$ and Y by V we reduce to the separated case which we dealt with in the preceding paragraph. □

0BAJ Lemma 29.51.6. Let X, Y be schemes. Let $f : X \to Y$ be a birational morphism between schemes which have finitely many irreducible components. Assume

(1) either f is quasi-compact or f is separated, and
(2) either f is locally of finite type and Y is reduced or f is locally of finite presentation.

Then there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism.

Proof. By Lemma 29.51.5 we may assume that f is finite. Since Y has finitely many irreducible components, we can find a dense open which is a disjoint union of its irreducible components. Thus we may assume Y is irreducible. By Lemma 29.50.5 we find a nonempty open $U \subset X$ such that $f|_U : U \to Y$ is an open immersion. After removing the closed (as f finite) subset $f(X \setminus U)$ from Y we see that f is an isomorphism. □

02NX Lemma 29.51.7. Let X, Y be integral schemes. Let $f : X \to Y$ be locally of finite type. Assume f is dominant. The following are equivalent:

(1) the extension $R(Y) \subset R(X)$ has transcendence degree 0,
(2) the extension $R(Y) \subset R(X)$ is finite,
(3) there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite, and
(4) the generic point of X is the only point of X mapping to the generic point of Y.

If f is separated or if f is quasi-compact, then these are also equivalent to

(5) there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.

Proof. Choose any affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset Y$ such that $f(U) \subset V$. Then R and A are domains by definition. The ring map $R \to A$ is of finite type (Lemma 29.15.2). By Lemma 29.8.6 the generic point of X maps to the generic point of Y hence $R \to A$ is injective. Let $K = R(Y)$ be the fraction field of R and $L = R(X)$ the fraction field of A. Then L/K is a finitely generated field extension. Hence we see that (1) is equivalent to (2).

Suppose (2) holds. Let $x_1, \ldots, x_n \in A$ be generators of A over R. By assumption there exist nonzero polynomials $P_i(X) \in R[X]$ such that $P_i(x_i) = 0$. Let $f_i \in R$ be the leading coefficient of P_i. Then we conclude that $R_{f_1 \ldots f_n} \to A_{f_1 \ldots f_n}$ is finite, i.e., (3) holds. Note that (3) implies (2). So now we see that (1), (2) and (3) are all equivalent.

Let η be the generic point of X, and let $\eta' \in Y$ be the generic point of Y. Assume (4). Then $\dim_\eta(X_{\eta'}) = 0$ and we see that $R(X) = \kappa(\eta)$ has transcendence degree 0 over $R(Y) = \kappa(\eta')$ by Lemma 29.28.1. In other words (1) holds. Assume the equivalent conditions (1), (2) and (3). Suppose that $x \in X$ is a point mapping to η'. As x is a specialization of η, this gives inclusions $R(Y) \subset \mathcal{O}_{X,x} \subset R(X)$, which implies $\mathcal{O}_{X,x}$ is a field, see Algebra, Lemma 10.36.19. Hence $x = \eta$. Thus we see that (1) – (4) are all equivalent.

It is clear that (5) implies (3) with no additional assumptions on f. What remains is to prove that if f is either separated or quasi-compact, then the equivalent conditions (1) – (4) imply (5). This follows from Lemma 29.51.5. □

02NY Definition 29.51.8. Let X and Y be integral schemes. Let $f : X \to Y$ be locally of finite type and dominant. Assume $[R(X) : R(Y)] < \infty$, or any other of the equivalent conditions (1) – (4) of Lemma 29.51.7. Then the positive integer

$$\deg(X/Y) = [R(X) : R(Y)]$$

is called the degree of X over Y.

It is possible to extend this notion to a morphism $f : X \to Y$ if (a) Y is integral with generic point η, (b) f is locally of finite type, and (c) $f^{-1}(\{\eta\})$ is finite. In this case we can define

$$\deg(X/Y) = \sum\nolimits_{\xi \in X,\ f(\xi)=\eta} \dim_{R(Y)}(\mathcal{O}_{X,\xi}).$$

Namely, given that $R(Y) = \kappa(\eta) = \mathcal{O}_{Y,\eta}$ (Lemma 29.49.5) the dimensions above are finite by Lemma 29.51.1 above. However, for most applications the definition given above is the right one.

02NZ Lemma 29.51.9. Let X, Y, Z be integral schemes. Let $f : X \to Y$ and $g : Y \to Z$ be dominant morphisms locally of finite type. Assume that $[R(X) : R(Y)] < \infty$ and $[R(Y) : R(Z)] < \infty$. Then

$$\deg(X/Z) = \deg(X/Y) \deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields, see Fields, Lemma 9.7.7. □

073A Remark 29.51.10. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. There are (at least) two properties that we could use to define generically finite morphisms. These correspond to whether you want the property to be local on the source or local on the target:

(1) (Local on the target; suggested by Ravi Vakil.) Assume every quasi-compact open of Y has finitely many irreducible components (for example if Y is locally Noetherian). The requirement is that the inverse image of each generic point is finite, see Lemma 29.51.1.
(2) (Local on the source.) The requirement is that there exists a dense open $U \subset X$ such that $U \to Y$ is locally quasi-finite.

In case (1) the requirement can be formulated without the auxiliary condition on Y, but probably doesn't give the right notion for general schemes. Property (2) as formulated doesn't imply that the fibres over generic points are finite; however, if f is quasi-compact and Y is as in (1) then it does.

0AAZ Definition 29.51.11. Let X be an integral scheme. A modification of X is a birational proper morphism $f : X' \to X$ with X' integral.

Let $f : X' \to X$ be a modification as in the definition. By Lemma 29.51.7 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By generic flatness (Proposition 29.27.1) we may assume $f^{-1}(U) \to U$ is flat and of finite presentation. So $f^{-1}(U) \to U$ is finite locally free (Lemma 29.48.2). Since f is birational, the degree of X' over X is 1. Hence $f^{-1}(U) \to U$ is finite locally free of degree 1, in other words it is an isomorphism. Thus we can redefine a modification to be a proper morphism $f : X' \to X$ of integral schemes such that $f^{-1}(U) \to U$ is an isomorphism for some nonempty open $U \subset X$.

0AB0 **Definition 29.51.12.** Let X be an integral scheme. An *alteration* of X is a proper dominant morphism $f : Y \to X$ with Y integral such that $f^{-1}(U) \to U$ is finite for some nonempty open $U \subset X$. [dJ96, Definition 2.20]

This is the definition as given in [dJ96], except that here we do not require X and Y to be Noetherian. Arguing as above we see that an alteration is a proper dominant morphism $f : Y \to X$ of integral schemes which induces a finite extension of function fields, i.e., such that the equivalent conditions of Lemma 29.51.7 hold.

29.52. The dimension formula

02JT For morphisms between Noetherian schemes we can say a little more about dimensions of local rings. Here is an important (and not so hard to prove) result. Recall that $R(X)$ denotes the function field of an integral scheme X.

02JU **Lemma 29.52.1.** Let S be a scheme. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume

(1) S is locally Noetherian,
(2) f is locally of finite type,
(3) X and S integral, and
(4) f dominant.

We have

02JV (29.52.1.1) $$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + \text{trdeg}_{R(S)} R(X) - \text{trdeg}_{\kappa(s)} \kappa(x).$$

Moreover, equality holds if S is universally catenary.

Proof. The corresponding algebra statement is Algebra, Lemma 10.113.1. □

0BAE **Lemma 29.52.2.** Let S be a scheme. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume S is locally Noetherian and f is locally of finite type, We have

0BAF (29.52.2.1) $$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + E - \text{trdeg}_{\kappa(s)} \kappa(x).$$

where E is the maximum of $\text{trdeg}_{\kappa(f(\xi))}(\kappa(\xi))$ where ξ runs over the generic points of irreducible components of X containing x.

Proof. Let $X_1, \ldots, X_n$ be the irreducible components of X containing x endowed with their reduced induced scheme structure. These correspond to the minimal primes $\mathfrak{q}_i$ of $\mathcal{O}_{X,x}$ and hence there are finitely many of them (Schemes, Lemma 26.13.2 and Algebra, Lemma 10.31.6). Then $\dim(\mathcal{O}_{X,x}) = \max \dim(\mathcal{O}_{X,x}/\mathfrak{q}_i) = \max \dim(\mathcal{O}_{X_i,x})$. The ξ's occurring in the definition of E are exactly the generic points $\xi_i \in X_i$. Let $Z_i = \overline{\{f(\xi_i)\}} \subset S$ endowed with the reduced induced scheme structure. The composition $X_i \to X \to S$ factors through Z_i (Schemes, Lemma 26.12.7). Thus we may apply the dimension formula (Lemma 29.52.1) to see that $\dim(\mathcal{O}_{X_i,x}) \leq \dim(\mathcal{O}_{Z_i,x}) + \text{trdeg}_{\kappa(f(\xi))}(\kappa(\xi)) - \text{trdeg}_{\kappa(s)} \kappa(x)$. Putting everything together we obtain the lemma. □

An application is the construction of a dimension function on any scheme of finite type over a universally catenary scheme endowed with a dimension function. For the definition of dimension functions, see Topology, Definition 5.20.1.

02JW Lemma 29.52.3. Let S be a locally Noetherian and universally catenary scheme. Let $\delta : S \to \mathbf{Z}$ be a dimension function. Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Then the map

$$\begin{aligned} \delta = \delta_{X/S} : X &\longrightarrow \mathbf{Z} \\ x &\longmapsto \delta(f(x)) + \text{trdeg}_{\kappa(f(x))}\kappa(x) \end{aligned}$$

is a dimension function on X.

Proof. Let $f : X \to S$ be locally of finite type. Let $x \rightsquigarrow y$, $x \neq y$ be a specialization in X. We have to show that $\delta_{X/S}(x) > \delta_{X/S}(y)$ and that $\delta_{X/S}(x) = \delta_{X/S}(y) + 1$ if y is an immediate specialization of x.

Choose an affine open $V \subset S$ containing the image of y and choose an affine open $U \subset X$ mapping into V and containing y. We may clearly replace X by U and S by V. Thus we may assume that $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$ and that f is given by a ring map $R \to A$. The ring R is universally catenary (Lemma 29.17.2) and the map $R \to A$ is of finite type (Lemma 29.15.2).

Let $\mathfrak{q} \subset A$ be the prime ideal corresponding to the point x and let $\mathfrak{p} \subset R$ be the prime ideal corresponding to $f(x)$. The restriction δ' of δ to $S' = \text{Spec}(R/\mathfrak{p}) \subset S$ is a dimension function. The ring $R/\mathfrak{p}$ is universally catenary. The restriction of $\delta_{X/S}$ to $X' = \text{Spec}(A/\mathfrak{q})$ is clearly equal to the function $\delta_{X'/S'}$ constructed using the dimension function δ'. Hence we may assume in addition to the above that $R \subset A$ are domains, in other words that X and S are integral schemes, and that x is the generic point of X and $f(x)$ is the generic point of S.

Note that $\mathcal{O}_{X,x} = R(X)$ and that since $x \rightsquigarrow y$, $x \neq y$, the spectrum of $\mathcal{O}_{X,y}$ has at least two points (Schemes, Lemma 26.13.2) hence $\dim(\mathcal{O}_{X,y}) > 0$. If y is an immediate specialization of x, then $\text{Spec}(\mathcal{O}_{X,y}) = \{x, y\}$ and $\dim(\mathcal{O}_{X,y}) = 1$.

Write $s = f(x)$ and $t = f(y)$. We compute

$$\begin{aligned} \delta_{X/S}(x) - \delta_{X/S}(y) &= \delta(s) + \text{trdeg}_{\kappa(s)}\kappa(x) - \delta(t) - \text{trdeg}_{\kappa(t)}\kappa(y) \\ &= \delta(s) - \delta(t) + \text{trdeg}_{R(S)}R(X) - \text{trdeg}_{\kappa(t)}\kappa(y) \\ &= \delta(s) - \delta(t) + \dim(\mathcal{O}_{X,y}) - \dim(\mathcal{O}_{S,t}) \end{aligned}$$

where we use equality in (29.52.1.1) in the last step. Since δ is a dimension function on the scheme S and $s \in S$ is the generic point, the difference $\delta(s) - \delta(t)$ is equal to $\text{codim}(\overline{\{t\}}, S)$ by Topology, Lemma 5.20.2. This is equal to $\dim(\mathcal{O}_{S,t})$ by Properties, Lemma 28.10.3. Hence we conclude that

$$\delta_{X/S}(x) - \delta_{X/S}(y) = \dim(\mathcal{O}_{X,y})$$

and the lemma follows from what we said above about $\dim(\mathcal{O}_{X,y})$. $\square$

Another application of the dimension formula is that the dimension does not change under "alterations" (to be defined later).

02JX Lemma 29.52.4. Let $f : X \to Y$ be a morphism of schemes. Assume that

(1) Y is locally Noetherian,
(2) X and Y are integral schemes,
(3) f is dominant, and
(4) f is locally of finite type.

Then we have

$$\dim(X) \leq \dim(Y) + \text{trdeg}_{R(Y)} R(X).$$

If f is closed[16] then equality holds.

Proof. Let $f : X \to Y$ be as in the lemma. Let $\xi_0 \leadsto \xi_1 \leadsto \ldots \leadsto \xi_e$ be a sequence of specializations in X. Set $x = \xi_e$ and $y = f(x)$. Observe that $e \leq \dim(\mathcal{O}_{X,x})$ as the given specializations occur in the spectrum of $\mathcal{O}_{X,x}$, see Schemes, Lemma 26.13.2. By the dimension formula, Lemma 29.52.1, we see that

$$\begin{aligned} e &\leq \dim(\mathcal{O}_{X,x}) \\ &\leq \dim(\mathcal{O}_{Y,y}) + \text{trdeg}_{R(Y)} R(X) - \text{trdeg}_{\kappa(y)} \kappa(x) \\ &\leq \dim(\mathcal{O}_{Y,y}) + \text{trdeg}_{R(Y)} R(X) \end{aligned}$$

Hence we conclude that $e \leq \dim(Y) + \text{trdeg}_{R(Y)} R(X)$ as desired.

Next, assume f is also closed. Say $\overline{\xi}_0 \leadsto \overline{\xi}_1 \leadsto \ldots \leadsto \overline{\xi}_d$ is a sequence of specializations in Y. We want to show that $\dim(X) \geq d + r$. We may assume that $\overline{\xi}_0 = \eta$ is the generic point of Y. The generic fibre X_η is a scheme locally of finite type over $\kappa(\eta) = R(Y)$. It is nonempty as f is dominant. Hence by Lemma 29.16.10 it is a Jacobson scheme. Thus by Lemma 29.16.8 we can find a closed point $\xi_0 \in X_\eta$ and the extension $\kappa(\eta) \subset \kappa(\xi_0)$ is a finite extension. Note that $\mathcal{O}_{X,\xi_0} = \mathcal{O}_{X_\eta,\xi_0}$ because η is the generic point of Y. Hence we see that $\dim(\mathcal{O}_{X,\xi_0}) = r$ by Lemma 29.52.1 applied to the scheme X_η over the universally catenary scheme $\text{Spec}(\kappa(\eta))$ (see Lemma 29.17.5) and the point ξ_0. This means that we can find $\xi_{-r} \leadsto \ldots \leadsto \xi_{-1} \leadsto \xi_0$ in X. On the other hand, as f is closed specializations lift along f, see Topology, Lemma 5.19.7. Thus, as ξ_0 lies over $\eta = \overline{\xi}_0$ we can find specializations $\xi_0 \leadsto \xi_1 \leadsto \ldots \leadsto \xi_d$ lying over $\overline{\xi}_0 \leadsto \overline{\xi}_1 \leadsto \ldots \leadsto \overline{\xi}_d$. In other words we have

$$\xi_{-r} \leadsto \ldots \leadsto \xi_{-1} \leadsto \xi_0 \leadsto \xi_1 \leadsto \ldots \leadsto \xi_d$$

which means that $\dim(X) \geq d + r$ as desired. □

0BAG Lemma 29.52.5. Let $f : X \to Y$ be a morphism of schemes. Assume that Y is locally Noetherian and f is locally of finite type. Then

$$\dim(X) \leq \dim(Y) + E$$

where E is the supremum of $\text{trdeg}_{\kappa(f(\xi))}(\kappa(\xi))$ where ξ runs through the generic points of the irreducible components of X.

Proof. Immediate consequence of Lemma 29.52.2 and Properties, Lemma 28.10.2. □

29.53. Relative normalization

0BAK In this section we construct the normalization of one scheme in another.

035F Lemma 29.53.1. Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The subsheaf $\mathcal{A}' \subset \mathcal{A}$ defined by the rule

$$U \longmapsto \{f \in \mathcal{A}(U) \mid f_x \in \mathcal{A}_x \text{ integral over } \mathcal{O}_{X,x} \text{ for all } x \in U\}$$

is a quasi-coherent $\mathcal{O}_X$-algebra, the stalk $\mathcal{A}'_x$ is the integral closure of $\mathcal{O}_{X,x}$ in $\mathcal{A}_x$, and for any affine open $U \subset X$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.

[16]For example if f is proper, see Definition 29.41.1.

Proof. This is a subsheaf by the local nature of the conditions. It is an $\mathcal{O}_X$-algebra by Algebra, Lemma 10.36.7. Let $U \subset X$ be an affine open. Say $U = \mathrm{Spec}(R)$ and say $\mathcal{A}$ is the quasi-coherent sheaf associated to the R-algebra A. Then according to Algebra, Lemma 10.36.12 the value of $\mathcal{A}'$ over U is given by the integral closure A' of R in A. This proves the last assertion of the lemma. To prove that $\mathcal{A}'$ is quasi-coherent, it suffices to show that $\mathcal{A}'(D(f)) = A'_f$. This follows from the fact that integral closure and localization commute, see Algebra, Lemma 10.36.11. The same fact shows that the stalks are as advertised. □

035G Definition 29.53.2. Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_X$ in $\mathcal{A}$ is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 29.53.1 above.

In the setting of the definition above we can consider the morphism of relative spectra

$$\begin{tikzcd} Y = \underline{\mathrm{Spec}}_X(\mathcal{A}) \arrow[rr] \arrow[dr] & & X' = \underline{\mathrm{Spec}}_X(\mathcal{A}') \arrow[dl] \\ & X & \end{tikzcd}$$

see Lemma 29.11.5. The scheme $X' \to X$ will be the normalization of X in the scheme Y. Here is a slightly more general setting. Suppose we have a quasi-compact and quasi-separated morphism $f : Y \to X$ of schemes. In this case the sheaf of $\mathcal{O}_X$-algebras $f_*\mathcal{O}_Y$ is quasi-coherent, see Schemes, Lemma 26.24.1. Taking the integral closure $\mathcal{O}' \subset f_*\mathcal{O}_Y$ we obtain a quasi-coherent sheaf of $\mathcal{O}_X$-algebras whose relative spectrum is the normalization of X in Y. Here is the formal definition.

035H Definition 29.53.3. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The normalization of X in Y is the scheme[17]

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over X. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

The factorization is the composition of the canonical morphism $Y \to \underline{\mathrm{Spec}}(f_*\mathcal{O}_Y)$ (see Constructions, Lemma 27.4.7) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \to f_*\mathcal{O}_Y$. We can characterize the normalization as follows.

035I Lemma 29.53.4. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:

(1) the morphism ν is integral, and

[17] The scheme X' need not be normal, for example if $Y = X$ and $f = \mathrm{id}_X$, then $X' = X$.

(2) for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f'"'] & Z \arrow[d, "\pi"] \\ X' \arrow[r, "\nu"] \arrow[ur, "h"] & X \end{tikzcd}$$

for some unique morphism $h : X' \to Z$.

Moreover, the morphism $f' : Y \to X'$ is dominant and in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 29.53.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 29.44.1 π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : Y \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine open $U \subset X$ (by Definition 29.44.1) we see from Lemma 29.53.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum Lemma 29.11.5 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category.

From the universal property in (2) we see that f' does not factor through a proper closed subscheme of X'. Hence the scheme theoretic image of f' is X'. Since f' is quasi-compact (by Schemes, Lemma 26.21.14 and the fact that ν is separated as an affine morphism) we see that $f'(Y)$ is dense in X'. Hence f' is dominant.

The morphism h in (2) is integral by Lemma 29.44.14. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the last statement of the lemma. □

035J Lemma 29.53.5. Let

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "f_2"'] & Y_1 \arrow[d, "f_1"] \\ X_2 \arrow[r] & X_1 \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume f_1, f_2 quasi-compact and quasi-separated. Let $f_i = \nu_i \circ f_i'$, $i = 1, 2$ be the canonical factorizations, where $\nu_i : X_i' \to X_i$ is the normalization of X_i in Y_i. Then there exists a unique arrow

$X_2' \to X_1'$ fitting into a commutative diagram

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "f_2'"'] & Y_1 \arrow[d, "f_1'"] \\ X_2' \arrow[r] \arrow[d, "\nu_2"'] & X_1' \arrow[d, "\nu_1"] \\ X_2 \arrow[r] & X_1 \end{tikzcd}$$

Proof. By Lemmas 29.53.4 (1) and 29.44.6 the base change $X_2 \times_{X_1} X_1' \to X_2$ is integral. Note that f_2 factors through this morphism. Hence we get a unique morphism $X_2' \to X_2 \times_{X_1} X_1'$ from Lemma 29.53.4 (2). This gives the arrow $X_2' \to X_1'$ fitting into the commutative diagram and uniqueness follows as well. □

035K Lemma 29.53.6. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $U \subset X$ be an open subscheme and set $V = f^{-1}(U)$. Then the normalization of U in V is the inverse image of U in the normalization of X in Y.

Proof. Clear from the construction. □

0BXA Lemma 29.53.7. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let X' be the normalization of X in Y. Then the normalization of X' in Y is X'.

Proof. If $Y \to X'' \to X'$ is the normalization of X' in Y, then we can apply Lemma 29.53.4 to the composition $X'' \to X$ to get a canonical morphism $h : X' \to X''$ over X. We omit the verification that the morphisms h and $X'' \to X'$ are mutually inverse (using uniqueness of the factorization in the lemma). □

0AXN Lemma 29.53.8. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. If Y is reduced, so is X'.

Proof. This follows from the fact that a subring of a reduced ring is reduced. Some details omitted. □

0AXP Lemma 29.53.9. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $X' \to X$ be the normalization of X in Y. Every generic point of an irreducible component of X' is the image of a generic point of an irreducible component of Y.

Proof. By Lemma 29.53.6 we may assume $X = \text{Spec}(A)$ is affine. Choose a finite affine open covering $Y = \bigcup \text{Spec}(B_i)$. Then $X' = \text{Spec}(A')$ and the morphisms $\text{Spec}(B_i) \to Y \to X'$ jointly define an injective A-algebra map $A' \to \prod B_i$. Thus the lemma follows from Algebra, Lemma 10.30.5. □

03GO Lemma 29.53.10. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Suppose that $Y = Y_1 \amalg Y_2$ is a disjoint union of two schemes. Write $f_i = f|_{Y_i}$. Let X_i' be the normalization of X in Y_i. Then $X_1' \amalg X_2'$ is the normalization of X in Y.

Proof. In terms of integral closures this corresponds to the following fact: Let $A \to B$ be a ring map. Suppose that $B = B_1 \times B_2$. Let A_i' be the integral closure of A in B_i. Then $A_1' \times A_2'$ is the integral closure of A in B. The reason this works is that the elements $(1, 0)$ and $(0, 1)$ of B are idempotents and hence integral over A. Thus the integral closure A' of A in B is a product and it is not hard to see that the factors are the integral closures A'_i as described above (some details omitted). $\square$

03GQ **Lemma 29.53.11.** Let $f : X \to S$ be a quasi-compact, quasi-separated and universally closed morphisms of schemes. Then $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_S$. In other words, the normalization of S in X is equal to the factorization

$$X \longrightarrow \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X) \longrightarrow S$$

of Constructions, Lemma 27.4.7.

Proof. The question is local on S, hence we may assume $S = \mathrm{Spec}(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R. Think of h as a morphism as in the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "h"'] \arrow[dr, "f"'] & & \mathbf{A}^1_S \arrow[dl] \\ & S & \end{tikzcd}$$

Let $Z \subset \mathbf{A}^1_S$ be the scheme theoretic image of h, see Definition 29.6.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}^1_S \to S$ is separated, see Schemes, Lemma 26.21.14. By Lemma 29.6.3 the morphism $X \to Z$ is dominant. By Lemma 29.41.7 the morphism $X \to Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 29.41.9 to conclude that $Z \to S$ is universally closed (and even proper). Since $Z \subset \mathbf{A}^1_S$, we see that $Z \to S$ is affine and proper, hence integral by Lemma 29.44.7. Writing $\mathbf{A}^1_S = \mathrm{Spec}(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. $\square$

03GP **Lemma 29.53.12.** Let $f : Y \to X$ be an integral morphism. Then the normalization of X in Y is equal to Y.

Proof. By Lemma 29.44.7 this is a special case of Lemma 29.53.11. $\square$

035L **Lemma 29.53.13.** Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let X' be the normalization of X in Y. Assume

(1) Y is a normal scheme,
(2) quasi-compact opens of Y have finitely many irreducible components.

Then X' is a disjoint union of integral normal schemes. Moreover, the morphism $Y \to X'$ is dominant and induces a bijection of irreducible components.

Proof. Let $U \subset X$ be an affine open. Consider the inverse image U' of U in X'. Set $V = f^{-1}(U)$. By Lemma 29.53.6 we $V \to U' \to U$ is the normalization of U in V. Say $U = \mathrm{Spec}(A)$. Then V is quasi-compact, and hence has a finite number of irreducible components by assumption. Hence $V = \coprod_{i=1,\ldots n} V_i$ is a finite disjoint union of normal integral schemes by Properties, Lemma 28.7.5. By Lemma 29.53.10 we see that $U' = \coprod_{i=1,\ldots,n} U'_i$, where U'_i is the normalization of U in V_i. By Properties, Lemma 28.7.9 we see that $B_i = \Gamma(V_i, \mathcal{O}_{V_i})$ is a normal domain. Note that $U'_i = \mathrm{Spec}(A'_i)$, where $A'_i \subset B_i$ is the integral closure of A in B_i, see Lemma 29.53.1. By Algebra, Lemma 10.37.2 we see that $A'_i \subset B_i$ is a normal domain. Hence $U' = \coprod U'_i$ is a finite union of normal integral schemes and hence is normal.

As X' has an open covering by the schemes U' we conclude from Properties, Lemma 28.7.2 that X' is normal. On the other hand, each U' is a finite disjoint union of irreducible schemes, hence every quasi-compact open of X' has finitely many irreducible components (by a topological argument which we omit). Thus X' is a disjoint union of normal integral schemes by Properties, Lemma 28.7.5. It is clear from the description of X' above that $Y \to X'$ is dominant and induces a bijection on irreducible components $V \to U'$ for every affine open $U \subset X$. The bijection of irreducible components for the morphism $Y \to X'$ follows from this by a topological argument (omitted). $\square$

0AVK Lemma 29.53.14. Let $f : X \to S$ be a morphism. Assume that

(1) S is a Nagata scheme,
(2) f is quasi-compact and quasi-separated,
(3) quasi-compact opens of X have finitely many irreducible components,
(4) if $x \in X$ is a generic point of an irreducible component, then the field extension $\kappa(x)/\kappa(f(x))$ is finitely generated, and
(5) X is reduced.

Then the normalization $\nu : S' \to S$ of S in X is finite.

Proof. There is an immediate reduction to the case $S = \text{Spec}(R)$ where R is a Nagata ring by assumption (1). We have to show that the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R. Since f is quasi-compact by assumption (2) we can write $X = \bigcup_{i=1,\ldots,n} U_i$ with each U_i affine. Say $U_i = \text{Spec}(B_i)$. Each B_i is reduced by assumption (5) and has finitely many minimal primes $\mathfrak{q}_{i1}, \ldots, \mathfrak{q}_{im_i}$ by assumption (3) and Algebra, Lemma 10.26.1. We have

$$\Gamma(X, \mathcal{O}_X) \subset B_1 \times \ldots \times B_n \subset \prod\nolimits_{i=1,\ldots,n} \prod\nolimits_{j=1,\ldots,m_i} (B_i)_{\mathfrak{q}_{ij}}$$

the second inclusion by Algebra, Lemma 10.25.2. We have $\kappa(\mathfrak{q}_{ij}) = (B_i)_{\mathfrak{q}_{ij}}$ by Algebra, Lemma 10.25.1. Hence the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is contained in the product of the integral closures A_{ij} of R in $\kappa(\mathfrak{q}_{ij})$. Since R is Noetherian it suffices to show that A_{ij} is a finite R-module for each i, j. Let $\mathfrak{p}_{ij} \subset R$ be the image of $\mathfrak{q}_{ij}$. As $\kappa(\mathfrak{q}_{ij})/\kappa(\mathfrak{p}_{ij})$ is a finitely generated field extension by assumption (4), we see that $R \to \kappa(\mathfrak{q}_{ij})$ is essentially of finite type. Thus $R \to A_{ij}$ is finite by Algebra, Lemma 10.162.2. $\square$

03GR Lemma 29.53.15. Let $f : X \to S$ be a morphism. Assume that

(1) S is a Nagata scheme,
(2) f is of finite type,
(3) X is reduced.

Then the normalization $\nu : S' \to S$ of S in X is finite.

Proof. This is a special case of Lemma 29.53.14. Namely, (2) holds as the finite type morphism f is quasi-compact by definition and quasi-separated by Lemma 29.15.7. Condition (3) holds because X is locally Noetherian by Lemma 29.15.6. Finally, condition (4) holds because a finite type morphism induces finitely generated residue field extensions. $\square$

0BXB Lemma 29.53.16. Let $f : Y \to X$ be a finite type morphism of schemes with Y reduced and X Nagata. Let X' be the normalization of X in Y. Let $x' \in X'$ be a point such that

(1) $\dim(\mathcal{O}_{X',x'}) = 1$, and
(2) the fibre of $Y \to X'$ over x' is empty.

Then $\mathcal{O}_{X',x'}$ is a discrete valuation ring.

Proof. We can replace X by an affine neighbourhood of the image of x'. Hence we may assume $X = \mathrm{Spec}(A)$ with A Nagata. By Lemma 29.53.15 the morphism $X' \to X$ is finite. Hence we can write $X' = \mathrm{Spec}(A')$ for a finite A-algebra A'. By Lemma 29.53.7 after replacing X by X' we reduce to the case described in the next paragraph.

The case $X = X' = \mathrm{Spec}(A)$ with A Noetherian. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to our point x'. Choose $g \in \mathfrak{p}$ not contained in any minimal prime of A (use prime avoidance and the fact that A has finitely many minimal primes, see Algebra, Lemmas 10.15.2 and 10.31.6). Set $Z = f^{-1}V(g) \subset Y$; it is a closed subscheme of Y. Then $f(Z)$ does not contain any generic point by choice of g and does not contain x' because x' is not in the image of f. The closure of $f(Z)$ is the set of specializations of points of $f(Z)$ by Lemma 29.6.5. Thus the closure of $f(Z)$ does not contain x' because the condition $\dim(\mathcal{O}_{X',x'}) = 1$ implies only the generic points of $X = X'$ specialize to x'. In other words, after replacing X by an affine open neighbourhood of x' we may assume that $f^{-1}V(g) = \emptyset$. Thus g maps to an invertible global function on Y and we obtain a factorization

$$A \to A_g \to \Gamma(Y, \mathcal{O}_Y)$$

Since $X = X'$ this implies that A is equal to the integral closure of A in A_g. By Algebra, Lemma 10.36.11 we conclude that $A_\mathfrak{p}$ is the integral closure of $A_\mathfrak{p}$ in $A_\mathfrak{p}[1/g]$. By our choice of g, since $\dim(A_\mathfrak{p}) = 1$ and since A is reduced we see that $A_\mathfrak{p}[1/g]$ is a finite product of fields (the product of the residue fields of the minimal primes contained in $\mathfrak{p}$). Hence $A_\mathfrak{p}$ is normal (Algebra, Lemma 10.37.16) and the proof is complete. Some details omitted. $\square$

29.54. Normalization

035E Next, we come to the normalization of a scheme X. We only define/construct it when X has locally finitely many irreducible components. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X. Let

035M (29.54.0.1) $$f : Y = \coprod\nolimits_{\eta \in X^{(0)}} \mathrm{Spec}(\kappa(\eta)) \longrightarrow X$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 26.13. Note that this morphism is quasi-compact by assumption and quasi-separated as Y is separated (see Schemes, Section 26.21).

035N Definition 29.54.1. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the normalization of X as the morphism

$$\nu : X^\nu \longrightarrow X$$

which is the normalization of X in the morphism $f : Y \to X$ (29.54.0.1) constructed above.

Any locally Noetherian scheme has a locally finite set of irreducible components and the definition applies to it. Usually the normalization is defined only for reduced schemes. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X.

035O Lemma 29.54.2. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \to X_{red}$ is the normalization of X_{red}.

Proof. Let $f : Y \to X$ be the morphism (29.54.0.1). We get a factorization $Y \to X_{red} \to X$ of f from Schemes, Lemma 26.12.7. By Lemma 29.53.4 we obtain a canonical morphism $X^\nu \to X_{red}$ and that X^ν is the normalization of X_{red} in Y. The lemma follows as $Y \to X_{red}$ is identical to the morphism (29.54.0.1) constructed for X_{red}. □

If X is reduced, then the normalization of X is the same as the relative spectrum of the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions $\mathcal{K}_X$ (see Divisors, Section 31.23). Namely, $\mathcal{K}_X = f_*\mathcal{O}_Y$ in this case, see Divisors, Lemma 31.25.1 and its proof. We describe this here explicitly.

035P Lemma 29.54.3. Let X be a reduced scheme such that every quasi-compact open has finitely many irreducible components. Let $\text{Spec}(A) = U \subset X$ be an affine open. Then

(1) A has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$,
(2) the total ring of fractions $Q(A)$ of A is $Q(A/\mathfrak{q}_1) \times \ldots \times Q(A/\mathfrak{q}_t)$,
(3) the integral closure A' of A in $Q(A)$ is the product of the integral closures of the domains $A/\mathfrak{q}_i$ in the fields $Q(A/\mathfrak{q}_i)$, and
(4) $\nu^{-1}(U)$ is identified with the spectrum of A' where $\nu : X^\nu \to X$ is the normalization morphism.

Proof. Minimal primes correspond to irreducible components (Algebra, Lemma 10.26.1), hence we have (1) by assumption. Then $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_t$ because A is reduced (Algebra, Lemma 10.17.2). Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 10.25.4 and 10.25.1. This proves (2). Part (3) follows from Algebra, Lemma 10.37.16, or Lemma 29.53.10. Part (4) holds because it is clear that $f^{-1}(U) \to U$ is the morphism

$$\text{Spec}\left(\prod \kappa(\mathfrak{q}_i)\right) \longrightarrow \text{Spec}(A)$$

where $f : Y \to X$ is the morphism (29.54.0.1). □

0C3B Lemma 29.54.4. Let X be a scheme such that every quasi-compact open has a finite number of irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$. Then the following are canonically isomorphic as $\mathcal{O}_{X,x}$-algebras

(1) the stalk $(\nu_*\mathcal{O}_{X^\nu})_x$,
(2) the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions of $(\mathcal{O}_{X,x})_{red}$,
(3) the integral closure of $\mathcal{O}_{X,x}$ in the product of the residue fields of the minimal primes of $\mathcal{O}_{X,x}$ (and there are finitely many of these).

Proof. After replacing X by an affine open neighbourhood of x we may assume that X has finitely many irreducible components and that x is contained in each of them. Then the stalk $(\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in the product L of the residue fields of the minimal primes of A. This follows from the construction of the normalization and Lemma 29.53.1. Alternatively, you can use Lemma 29.54.3 and the fact that normalization commutes with localization (Algebra, Lemma 10.36.11). Since A_{red} has finitely many minimal primes (because these correspond exactly to the generic points of the irreducible components of X passing through x) we see that L is the total ring of fractions of A_{red} (Algebra, Lemma 10.25.4). Thus our ring is also the integral closure of A in the total ring of fractions of A_{red}. □

035Q Lemma 29.54.5. Let X be a scheme such that every quasi-compact open has finitely many irreducible components.

(1) The normalization X^ν is a disjoint union of integral normal schemes.
(2) The morphism $\nu : X^\nu \to X$ is integral, surjective, and induces a bijection on irreducible components.
(3) For any integral morphism $\alpha : X' \to X$ such that for $U \subset X$ quasi-compact open the inverse image $\alpha^{-1}(U)$ has finitely many irreducible components and $\alpha|_{\alpha^{-1}(U)} : \alpha^{-1}(U) \to U$ is birational[18] there exists a factorization $X^\nu \to X' \to X$ and $X^\nu \to X'$ is the normalization of X'.
(4) For any morphism $Z \to X$ with Z a normal scheme such that each irreducible component of Z dominates an irreducible component of X there exists a unique factorization $Z \to X^\nu \to X$.

Proof. Let $f : Y \to X$ be as in (29.54.0.1). The scheme X^ν is a disjoint union of normal integral schemes because Y is normal and every affine open of Y has finitely many irreducible components, see Lemma 29.53.13. This proves (1). Alternatively one can deduce (1) from Lemmas 29.54.2 and 29.54.3.

The morphism ν is integral by Lemma 29.53.4. By Lemma 29.53.13 the morphism $Y \to X^\nu$ induces a bijection on irreducible components, and by construction of Y this implies that $X^\nu \to X$ induces a bijection on irreducible components. By construction $f : Y \to X$ is dominant, hence also ν is dominant. Since an integral morphism is closed (Lemma 29.44.7) this implies that ν is surjective. This proves (2).

Suppose that $\alpha : X' \to X$ is as in (3). It is clear that X' satisfies the assumptions under which the normalization is defined. Let $f' : Y' \to X'$ be the morphism (29.54.0.1) constructed starting with X'. As α is locally birational it is clear that $Y' = Y$ and $f = \alpha \circ f'$. Hence the factorization $X^\nu \to X' \to X$ exists and $X^\nu \to X'$ is the normalization of X' by Lemma 29.53.4. This proves (3).

Let $g : Z \to X$ be a morphism whose domain is a normal scheme and such that every irreducible component dominates an irreducible component of X. By Lemma 29.54.2 we have $X^\nu = X^\nu_{red}$ and by Schemes, Lemma 26.12.7 $Z \to X$ factors through X_{red}. Hence we may replace X by X_{red} and assume X is reduced. Moreover, as the factorization is unique it suffices to construct it locally on Z. Let $W \subset Z$ and $U \subset X$ be affine opens such that $g(W) \subset U$. Write $U = \text{Spec}(A)$ and $W = \text{Spec}(B)$, with $g|_W$ given by $\varphi : A \to B$. We will use the results of Lemma 29.54.3 freely. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes of A. As Z is normal, we see that B is a normal ring, in particular reduced. Moreover, by assumption any minimal prime $\mathfrak{q} \subset B$ we have that $\varphi^{-1}(\mathfrak{q})$ is a minimal prime of A. Hence if $x \in A$ is a nonzerodivisor,

[18] This awkward formulation is necessary as we've only defined what it means for a morphism to be birational if the source and target have finitely many irreducible components. It suffices if $X'_{red} \to X_{red}$ satisfies the condition.

i.e., $x \not\in \bigcup \mathfrak{p}_i$, then $\varphi(x)$ is a nonzerodivisor in B. Thus we obtain a canonical ring map $Q(A) \to Q(B)$. As B is normal it is equal to its integral closure in $Q(B)$ (see Algebra, Lemma 10.37.12). Hence we see that the integral closure $A' \subset Q(A)$ of A maps into B via the canonical map $Q(A) \to Q(B)$. Since $\nu^{-1}(U) = \text{Spec}(A')$ this gives the canonical factorization $W \to \nu^{-1}(U) \to U$ of $\nu|_W$. We omit the verification that it is unique. □

0CDV Lemma 29.54.6. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $Z_i \subset X$, $i \in I$ be the irreducible components of X endowed with the reduced induced structure. Let $Z_i^\nu \to Z_i$ be the normalization. Then $\coprod_{i \in I} Z_i^\nu \to X$ is the normalization of X.

Proof. We may assume X is reduced, see Lemma 29.54.2. Then the lemma follows either from the local description in Lemma 29.54.3 or from Lemma 29.54.5 part (3) because $\coprod Z_i \to X$ is integral and locally birational (as X is reduced and has locally finitely many irreducible components). □

0BXC Lemma 29.54.7. Let X be a reduced scheme with finitely many irreducible components. Then the normalization morphism $X^\nu \to X$ is birational.

Proof. The normalization induces a bijection of irreducible components by Lemma 29.54.5. Let $\eta \in X$ be a generic point of an irreducible component of X and let $\eta^\nu \in X^\nu$ be the generic point of the corresponding irreducible component of X^ν. Then $\eta^\nu \mapsto \eta$ and to finish the proof we have to show that $\mathcal{O}_{X,\eta} \to \mathcal{O}_{X^\nu,\eta^\nu}$ is an isomorphism, see Definition 29.50.1. Because X and X^ν are reduced, we see that both local rings are equal to their residue fields (Algebra, Lemma 10.25.1). On the other hand, by the construction of the normalization as the normalization of X in $Y = \coprod \text{Spec}(\kappa(\eta))$ we see that we have $\kappa(\eta) \subset \kappa(\eta^\nu) \subset \kappa(\eta)$ and the proof is complete. □

0AB1 Lemma 29.54.8. A finite (or even integral) birational morphism $f : X \to Y$ of integral schemes with Y normal is an isomorphism.

Proof. Let $V \subset Y$ be an affine open with inverse image $U \subset X$ which is an affine open too. Since f is a birational morphism of integral schemes, the homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is an injective map of domains which induces an isomorphism of fraction fields. As Y is normal, the ring $\mathcal{O}_Y(V)$ is integrally closed in the fraction field. Since f is finite (or integral) every element of $\mathcal{O}_X(U)$ is integral over $\mathcal{O}_Y(V)$. We conclude that $\mathcal{O}_Y(V) = \mathcal{O}_X(U)$. This proves that f is an isomorphism as desired. □

035R Lemma 29.54.9. Let X be an integral, Japanese scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.

Proof. Follows from the definition (Properties, Definition 28.13.1) and Lemma 29.54.3. Namely, in this case the lemma says that $\nu^{-1}(\text{Spec}(A))$ is the spectrum of the integral closure of A in its field of fractions. □

035S Lemma 29.54.10. Let X be a Nagata scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.

Proof. Note that a Nagata scheme is locally Noetherian, thus Definition 29.54.1 does apply. The lemma is now a special case of Lemma 29.53.14 but we can also prove it directly as follows. Write $X^\nu \to X$ as the composition $X^\nu \to X_{red} \to X$.

As $X_{red} \to X$ is a closed immersion it is finite. Hence it suffices to prove the lemma for a reduced Nagata scheme (by Lemma 29.44.5). Let $\text{Spec}(A) = U \subset X$ be an affine open. By Lemma 29.54.3 we have $\nu^{-1}(U) = \text{Spec}(\prod A'_i)$ where A'_i is the integral closure of $A/\mathfrak{q}_i$ in its fraction field. As A is a Nagata ring (see Properties, Lemma 28.13.6) each of the ring extensions $A/\mathfrak{q}_i \subset A'_i$ are finite. Hence $A \to \prod A'_i$ is a finite ring map and we win. □

0GIQ Lemma 29.54.11. Let X be an irreducible, geometrically unibranch scheme. The normalization morphism $\nu : X^\nu \to X$ is a universal homeomorphism.

Proof. We have to show that ν is integral, universally injective, and surjective, see Lemma 29.45.5. By Lemma 29.54.5 the morphism ν is integral. Let $x \in X$ and set $A = \mathcal{O}_{X,x}$. Since X is irreducible we see that A has a single minimal prime $\mathfrak{p}$ and $A_{red} = A/\mathfrak{p}$. By Lemma 29.54.4 the stalk $A' = (\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of A in the fraction field of A_{red}. By More on Algebra, Definition 15.106.1 we see that A' has a single prime $\mathfrak{m}'$ lying over $\mathfrak{m}_x \subset A$ and $\kappa(\mathfrak{m}')/\kappa(x)$ is purely inseparable. Hence ν is bijective (hence surjective) and universally injective by Lemma 29.10.2. □

29.55. Weak normalization

0H3I We will only define the weak normalization of a scheme when it locally has finitely many irreducible components; similar to the case of normalization.

0H3J Lemma 29.55.1. Let $A \to B$ be a ring map inducing a dominant morphism $\text{Spec}(B) \to \text{Spec}(A)$ of spectra. There exists an A-subalgebra $B' \subset B$ such that

(1) $\text{Spec}(B') \to \text{Spec}(A)$ is a universal homeomorphism,
(2) given a factorization $A \to C \to B$ such that $\text{Spec}(C) \to \text{Spec}(A)$ is a universal homeomorphism, the image of $C \to B$ is contained in B'.

Proof. We will use Lemma 29.45.6 without further mention. Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r] & B_{red} \\ A \arrow[u] \arrow[r] & A_{red} \arrow[u] \end{tikzcd}$$

For any factorization $A \to C \to B$ of $A \to B$ as in (2), we see that $A_{red} \to C_{red} \to B_{red}$ is a factorization of $A_{red} \to B_{red}$ as in (2). It follows that if the lemma holds for $A_{red} \to B_{red}$ and produces the A_{red}-subalgebra $B'_{red} \subset B_{red}$, then setting $B' \subset B$ equal to the inverse image of B'_{red} solves the lemma for $A \to B$. This reduces us to the case discussed in the next paragraph.

Assume A and B are reduced. In this case $A \subset B$ by Algebra, Lemma 10.30.6. Let $A \to C \to B$ be a factorization as in (2). Then we may apply Proposition 29.46.8 to $A \subset C$ to see that every element of C is contained in an extension $A[c_1, \ldots, c_n] \subset C$ such that for $i = 1, \ldots, n$ we have

(1) $c_i^2, c_i^3 \in A[c_1, \ldots, c_{i-1}]$, or
(2) there exists a prime number p with $pc_i, c_i^p \in A[c_1, \ldots, c_{i-1}]$.

Thus property (2) holds if we define $B' \subset B$ to be the subset of elements $b \in B$ which are contained in an extension $A[b_1, \ldots, b_n] \subset B$ such that (*) holds: for $i = 1, \ldots, n$ we have

(1) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, or
(2) there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.

There are only two things to check: (a) B' is an A-subalgebra, and (b) $\text{Spec}(B') \to \text{Spec}(A)$ is a universal homeomorphism. Part (a) follows because given $n \geq 0$ and $b_1, \ldots, b_n \in B$ satisfying (*) and $m \geq 0$ and $b'_1, \ldots, b'_m \in B$ satisfying (*), the integer $n + m$ and $b_1, \ldots, b_n, b'_1, \ldots, b'_m \in B$ also satisfies (*). Finally, part (b) holds by Proposition 29.46.8 and our construction of B'. $\square$

0H3K Lemma 29.55.2. Let $A \to B$ be a ring map inducing a dominant morphism $\text{Spec}(B) \to \text{Spec}(A)$ of spectra. Formation of the A-subalgebra $B' \subset B$ in Lemma 29.55.1 commutes with localization (see proof for explanation).

Proof. Let $S \subset A$ be a multiplicative subset. Then $S^{-1}A \to S^{-1}B$ is a ring map which induces a dominant morphism $\text{Spec}(S^{-1}B) \to \text{Spec}(S^{-1}A)$ as well (see Lemmas 29.8.4 and 29.25.9). Hence Lemma 29.55.1 produces an $S^{-1}A$-subalgebra $(S^{-1}B)' \subset S^{-1}B$. The statement means that $S^{-1}B' = (S^{-1}B)'$ as $S^{-1}A$-subalgebras of $S^{-1}B$.

To see this is true, we will use the construction of B' and $(S^{-1}B)'$ in the proof of Lemma 29.55.1. In the first step, we see that B' is the inverse image of the A_{red}-subalgebra $B'_{red} \subset B_{red}$ constructed for the ring map $A_{red} \to B_{red}$ and similarly for $(S^{-1}B)'$. Noting that $S^{-1}B_{red} = (S^{-1}B)_{red}$ this reduces us to the case discussed in the next paragraph.

If A and B are reduced, we have constructed B' as the union of the subalgebras $A[b_1, \ldots, b_n]$ such that for $i = 1, \ldots, n$ we have

(1) $b_i^2, b_i^3 \in A[b_1, \ldots, b_{i-1}]$, or
(2) there exists a prime number p with $pb_i, b_i^p \in A[b_1, \ldots, b_{i-1}]$.

Similarly for $(S^{-1}B)' \subset S^{-1}B$. Thus it is clear that the image of $B' \to B \to S^{-1}B$ is contained in $(S^{-1}B)'$. To show that the corresponding map $S^{-1}B' \to (S^{-1}B)'$ is surjective, one uses Lemma 29.46.3 to clear denominators successively; we omit the details. $\square$

0H3L Lemma 29.55.3. Let $A \to B$ be a ring map inducing a dominant morphism $\text{Spec}(B) \to \text{Spec}(A)$ of spectra. There exists an A-subalgebra $B' \subset B$ such that

(1) $\text{Spec}(B') \to \text{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields,
(2) given a factorization $A \to C \to B$ such that $\text{Spec}(C) \to \text{Spec}(A)$ is a universal homeomorphism inducing isomorphisms on residue fields, the image of $C \to B$ is contained in B'.

Proof. This proof is exactly the same as the proof of Lemma 29.55.1 except we use Proposition 29.46.7 in stead of Proposition 29.46.8 $\square$

0H3M Lemma 29.55.4. Let $A \to B$ be a ring map inducing a dominant morphism $\text{Spec}(B) \to \text{Spec}(A)$ of spectra. Formation of the A-subalgebra $B' \subset B$ in Lemma 29.55.3 commutes with localization (see proof for explanation).

Proof. The proof is the same as the proof of Lemma 29.55.2. $\square$

0H3N Lemma 29.55.5. Let $f : Y \to X$ be a quasi-compact, quasi-separated, and dominant morphism of schemes.

(1) The category of factorizations $Y \to X' \to X$ where $X' \to X$ is a universal homeomorphism has an initial object $Y \to X^{Y/wn} \to X$.
(2) The category of factorizations $Y \to X' \to X$ where $X' \to X$ is a universal homeomorphism inducing isomorphisms on residue fields has an initial object $Y \to X^{Y/sn} \to X$.

Moreover, formation of the factorization $Y \to X^{Y/wn} \to X$ and $Y \to X^{Y/sn} \to X$ commutes with base change to open subschemes of X.

Proof. We will prove (1) and omit the proof of (2); also the final assertion will follow from the construction of the factorization. We will use Lemma 29.45.5 without further mention. First, let $Y \to X^{Y/n} \to X$ be the normalization of X in Y, see Definition 29.53.3. For $Y \to X' \to X$ as in (1), we obtain a unique morphism $X^{Y/n} \to X'$ compatible with the given morphisms, see Lemma 29.53.4. Thus it suffices to prove the lemma with f replaced by $X^{Y/n} \to X$. This reduces us to the case studied in the next paragraph.

Assume f is integral (the rest of the proof works more generally if f is affine). Let $U = \mathrm{Spec}(A)$ be an affine open of X and let $V = f^{-1}(U) = \mathrm{Spec}(B)$ be the inverse image in Y. Then $A \to B$ is a ring map which induces a dominant morphism on spectra. By Lemma 29.55.1 we obtain an A-subalgebra $B' \subset B$ such that setting $U^{V/wn} = \mathrm{Spec}(B')$ the factorization $V \to U^{V/wn} \to U$ is initial in the category of factorizations $V \to U' \to U$ where $U' \to U$ is a universal homeomorphism.

If $U_1 \subset U_2 \subset X$ are affine opens, then setting $V_i = f^{-1}(U_i)$ we obtain a canonical morphism

$$\rho^{U_2}_{U_1} : U_1^{V_1/wn} \to U_1 \times_{U_2} U_2^{V_2/wn}$$

over U_1 by the universal property of $U_1^{V_1/wn}$. These morphisms satisfy a natural functoriality which we leave to the reader to formulate and prove. Furthermore, the morphism $\rho^{U_2}_{U_1}$ is an isomorphism; this follows from Lemma 29.55.2 provided that $U_1 \subset U_2$ is a standard open and in the general case can be reduced to this case by the functorial nature of these maps and Schemes, Lemma 26.11.5 (details omitted). Thus by relative glueing (Constructions, Lemma 27.2.1) we obtain a morphism $X^{Y/wn} \to X$ which restricts to $U^{V/wn} \to U$ over U compatibly with the $\rho^{U_2}_{U_1}$. Of course, the morphisms $V \to U^{V/wn}$ glue to a morphism $Y \to X^{Y/wn}$ (see Constructions, Remark 27.2.3) and we get our factorization $Y \to X^{Y/wn} \to X$ where the second morphism is a universal homeomorphism.

Finally, let $Y \to X' \to X$ be a factorization as in (1). With $V \to U^{V/wn} \to U \subset X$ as above, we obtain a factorization $V \to U \times_X X' \to U$ where the second arrow is a universal homeomorphism and we obtain a unique morphism $g_U : U^{V/wn} \to U \times_X X'$ over U by the universal property of $U^{V/wn}$. These g_U are compatible with the morphisms $\rho^{U_2}_{U_1}$; details omitted. Hence there is a unique morphism $g : X^{Y/wn} \to X'$ over X agreeing with g_U over U, see Constructions, Remark 27.2.3. This proves that $Y \to X^{Y/wn} \to X$ is initial in our category and the proof is complete. □

0H3P Definition 29.55.6. Let $f : Y \to X$ be a quasi-compact, quasi-separated, and dominant morphism of schemes.

(1) The factorization $Y \to X^{Y/sn} \to X$ constructed in Lemma 29.55.5 part (2) is the seminormalization of X in Y.

(2) The factorization $Y \to X^{Y/wn} \to X$ constructed in Lemma 29.55.5 part (1) is the weak normalization of X in Y.

Here is a way to reinterpret the seminormalization of a scheme which locally has finitely many irreducible components.

0H3Q Lemma 29.55.7. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Then the seminormalization of X in X^ν is is the seminormalization of X. In a formula: $X^{sn} = X^{X^\nu/sn}$.

Proof. Let $f : Y \to X$ be as in (29.54.0.1) so that X^ν is the normalization of X in Y. The seminormalization $X^{sn} \to X$ of X is the initial object in the category of universal homeomorphisms $X' \to X$ inducing isomorphisms on residue fields. Since Y is the disjoint union of the spectra of the residue fields at the generic points of irreducible components of X, we see that for any $X' \to X$ in this category we obtain a canonical lift $f' : Y \to X'$ of f. Then by Lemma 29.53.4 we obtain a canonical morphism $X^\nu \to X'$. Whence in turn a canonical morphism $X^{X^\nu/sn} \to X'$ by the universal property of $X^{X^\nu/sn}$. In this way we see that $X^{X^\nu/sn}$ satisfies the same universal property that X^{sn} has and we conclude. □

Lemma 29.55.7 motivates the following definition. Since we have only constructed the normalization in case X locally has finitely many irreducible components, we will also restrict ourselves to that case for the weak normalization.

0H3R Definition 29.55.8. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the weak normalization of X as the weak normalization

$$X^\nu \longrightarrow X^{wn} \longrightarrow X$$

of X in the normalization X^ν of X (Definition 29.54.1). In a formula: $X^{wn} = X^{X^\nu/wn}$.

Combined with Lemma 29.55.7 we see that for a scheme X which locally has finitely many irreducible components there are canonical morphisms

$$X^\nu \to X^{wn} \to X^{sn} \to X$$

Having made this definition, we can say what it means for a scheme to be weakly normal (provided it has locally finitely many irreducible components).

0H3S Definition 29.55.9. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We say X is weakly normal if the weak normalization $X^{wn} \to X$ is an isomorphism (Definition 29.55.8).

It follows immediately from the definitions that for a scheme X such that every quasi-compact open has finitely many irreducible components we have

$$X \text{ normal} \Rightarrow X \text{ weakly normal} \Rightarrow X \text{ seminormal}$$

We can work out the meaning of weak normality in the affine case as follows.

0H3T Lemma 29.55.10. Let $X = \mathrm{Spec}(A)$ be an affine scheme which has finitely many irreducible components. Then X is weakly normal if and only if

(1) A is seminormal (Definition 29.47.1),

(2) for a prime number p and $z, w \in A$ such that (a) z is a nonzerodivisor, (b) w^p is divisible by z^p, and (c) pw is divisible by z, then w is divisible by z.

Proof. Assume X is weakly normal. Since a weakly normal scheme is seminormal, we see that (1) holds (by our definition of weakly normal schemes). In particular A is reduced. Let p, z, w be as in (2). Choose $x, y \in A$ such that $z^p x = w^p$ and $zy = pw$. Then $p^p x = y^p$. The ring map $A \to C = A[t]/(t^p - x, pt - y)$ induces a universal homeomorphism on spectra. The normalization X^ν of X is the spectrum of the integral closure A' of A in the total ring of fractions of A, see Lemma 29.54.3. Note that $a = w/z \in A'$ because $a^p = x$. Hence we have an A-algebra homomorphism $A \to C \to A'$ sending t to a. At this point the defining property $X = X^{wn} = X^{X^\nu/wn}$ of being weakly normal tells us that $C \to A'$ maps into A. Thus we find $a \in A$ as desired.

Conversely, assume (1) and (2). Let A' be as in the previous paragraph. We have to show that $X^{X^\nu/wn} = X$. By construction in the proof of Lemma 29.55.1, the scheme $X^{X^\nu/wn}$ is the spectrum of the subring of A' which is the union of the subrings $A[a_1, \ldots, a_n] \subset A'$ such that for $i = 1, \ldots, n$ we have

(a) $a_i^2, a_i^3 \in A[a_1, \ldots, a_{i-1}]$, or
(b) there exists a prime number p with $pa_i, a_i^p \in A[a_1, \ldots, a_{i-1}]$.

Then we can use (1) and (2) to inductively see that $a_1, \ldots, a_n \in A$; we omit the details. Consequently, we have $X = X^{X^\nu/wn}$ and hence X is weakly normal. $\square$

Here is the obligatory lemma.

0H3U Lemma 29.55.11. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The following are equivalent:

(1) The scheme X is weakly normal.
(2) For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ satisfies conditions (1) and (2) of Lemma 29.55.10.
(3) There exists an affine open covering $X = \bigcup U_i$ such that each ring $\mathcal{O}_X(U_i)$ satisfies conditions (1) and (2) of Lemma 29.55.10.
(4) There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is weakly normal.

Moreover, if X is weakly normal then every open subscheme is weakly normal.

Proof. The condition to X be weakly normal is that the morphism $X^{wn} = X^{X^\nu/wn} \to X$ is an isomorphism. Since the construction of $X^\nu \to X$ commutes with base change to open subschemes and since the construction of $X^{X^\nu/wn}$ commutes with base change to open subschemes of X (Lemma 29.55.5) the lemma is clear. $\square$

29.56. Zariski's Main Theorem (algebraic version)

03GS This is the version you can prove using purely algebraic methods. Before we can prove more powerful versions (for non-affine morphisms) we need to develop more tools. See Cohomology of Schemes, Section 30.21 and More on Morphisms, Section 37.43.

03GT Theorem 29.56.1 (Algebraic version of Zariski's Main Theorem). Let $f : Y \to X$ be an affine morphism of schemes. Assume f is of finite type. Let X' be the normalization of X in Y. Picture:

$$\begin{tikzcd} Y \arrow[rr, "f'"] \arrow[dr, "f"'] & & X' \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

Then there exists an open subscheme $U' \subset X'$ such that

(1) $(f')^{-1}(U') \to U'$ is an isomorphism, and
(2) $(f')^{-1}(U') \subset Y$ is the set of points at which f is quasi-finite.

Proof. There is an immediate reduction to the case where X and hence Y are affine. Say $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(A)$. Then $X' = \mathrm{Spec}(A')$, where A' is the integral closure of R in A, see Definitions 29.53.2 and 29.53.3. By Algebra, Theorem 10.123.12 for every $y \in Y$ at which f is quasi-finite, there exists an open $U'_y \subset X'$ such that $(f')^{-1}(U'_y) \to U'_y$ is an isomorphism. Set $U' = \bigcup U'_y$ where $y \in Y$ ranges over all points where f is quasi-finite. It remains to show that f is quasi-finite at all points of $(f')^{-1}(U')$. If $y \in (f')^{-1}(U')$ with image $x \in X$, then we see that $Y_x \to X'_x$ is an isomorphism in a neighbourhood of y. Hence there is no point of Y_x which specializes to y, since this is true for $f'(y)$ in X'_x, see Lemma 29.44.8. By Lemma 29.20.6 part (3) this implies f is quasi-finite at y. $\square$

We can use the algebraic version of Zariski's Main Theorem to show that the set of points where a morphism is quasi-finite is open.

01TI Lemma 29.56.2. Let $f : X \to S$ be a morphism of schemes. The set of points of X where f is quasi-finite is an open $U \subset X$. The induced morphism $U \to S$ is locally quasi-finite.

Proof. Suppose f is quasi-finite at x. Let $x \in U = \mathrm{Spec}(A) \subset X$, $V = \mathrm{Spec}(R) \subset S$ be affine opens as in Definition 29.20.1. By either Theorem 29.56.1 above or Algebra, Lemma 10.123.13, the set of primes $\mathfrak{q}$ at which $R \to A$ is quasi-finite is open in $\mathrm{Spec}(A)$. Since these all correspond to points of X where f is quasi-finite we get the first statement. The second statement is obvious. $\square$

We will improve the following lemma to general quasi-finite separated morphisms later, see More on Morphisms, Lemma 37.43.3.

03GU Lemma 29.56.3. Let $f : Y \to X$ be a morphism of schemes. Assume

(1) X and Y are affine, and
(2) f is quasi-finite.

Then there exists a diagram

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with Z affine, π finite and j an open immersion.

Proof. This is Algebra, Lemma 10.123.14 reformulated in the language of schemes. $\square$

03J2 Lemma 29.56.4. Let $f : Y \to X$ be a quasi-finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a nowhere dense subset of X.

Proof. As in the proof of Lemma 29.48.7 this reduces immediately to the case where the base X is affine. In this case $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of affine opens (as f is quasi-compact). Since each $T \cap Y_i$ is nowhere dense, and since a finite union of nowhere dense sets is nowhere dense (see Topology, Lemma 5.21.2), it suffices to prove that the image $f(T \cap Y_i)$ is nowhere dense in X. This reduces us to the case where both X and Y are affine. At this point we apply Lemma 29.56.3 above to get a diagram

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

with Z affine, π finite and j an open immersion. Set $\overline{T} = \overline{j(T)} \subset Z$. By Topology, Lemma 5.21.3 we see $\overline{T}$ is nowhere dense in Z. Since $f(T) \subset \pi(\overline{T})$ the lemma follows from the corresponding result in the finite case, see Lemma 29.48.7. $\square$

29.57. Universally bounded fibres

03J3 Let X be a scheme over a field k. If X is finite over k, then $X = \mathrm{Spec}(A)$ where A is a finite k-algebra. Another way to say this is that X is finite locally free over $\mathrm{Spec}(k)$, see Definition 29.48.1. Hence $X \to \mathrm{Spec}(k)$ has a degree which is an integer $d \geq 0$, namely $d = \dim_k(A)$. We sometime call this the degree of the (finite) scheme X over k.

03J4 Definition 29.57.1. Let $f : X \to Y$ be a morphism of schemes.

(1) We say the integer n bounds the degrees of the fibres of f if for all $y \in Y$ the fibre X_y is a finite scheme over $\kappa(y)$ whose degree over $\kappa(y)$ is $\leq n$.
(2) We say the fibres of f are universally bounded[19] if there exists an integer n which bounds the degrees of the fibres of f.

Note that in particular the number of points in a fibre is bounded by n as well. (The converse does not hold, even if all fibres are finite reduced schemes.)

03J5 Lemma 29.57.2. Let $f : X \to Y$ be a morphism of schemes. Let $n \geq 0$. The following are equivalent:

(1) the integer n bounds the degrees of the fibres of f, and
(2) for every morphism $\mathrm{Spec}(k) \to Y$, where k is a field, the fibre product $X_k = \mathrm{Spec}(k) \times_Y X$ is finite over k of degree $\leq n$.

In this case the fibres of f are universally bounded and the schemes X_k have at most n points. More precisely, if $X_k = \{x_1, \ldots, x_t\}$, then we have

$$n \geq \sum\nolimits_{i=1,\ldots,t} [\kappa(x_i) : k]$$

Proof. The implication (2) $\Rightarrow$ (1) is trivial. The other implication holds because if the image of $\mathrm{Spec}(k) \to Y$ is y, then $X_k = \mathrm{Spec}(k) \times_{\mathrm{Spec}(\kappa(y))} X_y$. By definition the fibres of f being universally bounded means that some n exists. Finally, suppose

[19] This is probably nonstandard notation.

that $X_k = \mathrm{Spec}(A)$. Then $\dim_k A = n$. Hence A is Artinian, all prime ideals are maximal ideals $\mathfrak{m}_i$, and A is the product of the localizations at these maximal ideals. See Algebra, Lemmas 10.53.2 and 10.53.6. Then $\mathfrak{m}_i$ corresponds to x_i, we have $A_{\mathfrak{m}_i} = \mathcal{O}_{X_k,x_i}$ and hence there is a surjection $A \to \bigoplus \kappa(\mathfrak{m}_i) = \bigoplus \kappa(x_i)$ which implies the inequality in the statement of the lemma by linear algebra. □

0CC2 Lemma 29.57.3. If f is a finite locally free morphism of degree d, then d bounds the degree of the fibres of f.

Proof. This is true because any base change of f is finite locally free of degree d (Lemma 29.48.4) and hence the fibres of f all have degree d. □

03J6 Lemma 29.57.4. A composition of morphisms with universally bounded fibres is a morphism with universally bounded fibres. More precisely, assume that n bounds the degrees of the fibres of $f : X \to Y$ and m bounds the degrees of $g : Y \to Z$. Then nm bounds the degrees of the fibres of $g \circ f : X \to Z$.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ have universally bounded fibres. Say that $\deg(X_y/\kappa(y)) \leq n$ for all $y \in Y$, and that $\deg(Y_z/\kappa(z)) \leq m$ for all $z \in Z$. Let $z \in Z$ be a point. By assumption the scheme Y_z is finite over $\mathrm{Spec}(\kappa(z))$. In particular, the underlying topological space of Y_z is a finite discrete set. The fibres of the morphism $f_z : X_z \to Y_z$ are the fibres of f at the corresponding points of Y, which are finite discrete sets by the reasoning above. Hence we conclude that the underlying topological space of X_z is a finite discrete set as well. Thus X_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 28.29.1 applied to the set of all points of X_z). Write $X_z = \mathrm{Spec}(A)$, $Y_z = \mathrm{Spec}(B)$, and $k = \kappa(z)$. Then $k \to B \to A$ and we know that (a) $\dim_k(B) \leq m$, and (b) for every maximal ideal $\mathfrak{m} \subset B$ we have $\dim_{\kappa(\mathfrak{m})}(A/\mathfrak{m}A) \leq n$. We claim this implies that $\dim_k(A) \leq nm$. Note that B is the product of its localizations $B_\mathfrak{m}$, for example because Y_z is a disjoint union of 1-point schemes, or by Algebra, Lemmas 10.53.2 and 10.53.6. So we see that $\dim_k(B) = \sum_\mathfrak{m} \dim_k(B_\mathfrak{m})$ and $\dim_k(A) = \sum_\mathfrak{m} \dim_k(A_\mathfrak{m})$ where in both cases $\mathfrak{m}$ runs over the maximal ideals of B (not of A). By the above, and Nakayama's Lemma (Algebra, Lemma 10.20.1) we see that each $A_\mathfrak{m}$ is a quotient of $B_\mathfrak{m}^{\oplus n}$ as a $B_\mathfrak{m}$-module. Hence $\dim_k(A_\mathfrak{m}) \leq n \dim_k(B_\mathfrak{m})$. Putting everything together we see that

$$\dim_k(A) = \sum\nolimits_\mathfrak{m} \dim_t a(A_\mathfrak{m}) \leq \sum\nolimits_\mathfrak{m} n \dim_k(B_\mathfrak{m}) = n \dim_k(B) \leq nm$$

as desired. □

03J7 Lemma 29.57.5. A base change of a morphism with universally bounded fibres is a morphism with universally bounded fibres. More precisely, if n bounds the degrees of the fibres of $f : X \to Y$ and $Y' \to Y$ is any morphism, then the degrees of the fibres of the base change $f' : Y' \times_Y X \to Y'$ is also bounded by n.

Proof. This is clear from the result of Lemma 29.57.2. □

03J8 Lemma 29.57.6. Let $f : X \to Y$ be a morphism of schemes. Let $Y' \to Y$ be a morphism of schemes, and let $f' : X' = X_{Y'} \to Y'$ be the base change of f. If $Y' \to Y$ is surjective and f' has universally bounded fibres, then f has universally bounded fibres. More precisely, if n bounds the degree of the fibres of f', then also n bounds the degrees of the fibres of f.

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibres of f'. We claim that n works for f also. Namely, if $y \in Y$ is a point, then choose a point $y' \in Y'$ lying over y and observe that

$$X'_{y'} = \mathrm{Spec}(\kappa(y')) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Since $X'_{y'}$ is assumed finite of degree $\leq n$ over $\kappa(y')$ it follows that also X_y is finite of degree $\leq n$ over $\kappa(y)$. (Some details omitted.) $\square$

03J9 Lemma 29.57.7. An immersion has universally bounded fibres.

Proof. The integer $n = 1$ works in the definition. $\square$

03WU Lemma 29.57.8. Let $f : X \to Y$ be an étale morphism of schemes. Let $n \geq 0$. The following are equivalent

(1) the integer n bounds the degrees of the fibres,
(2) for every field k and morphism $\mathrm{Spec}(k) \to Y$ the base change $X_k = \mathrm{Spec}(k) \times_Y X$ has at most n points, and
(3) for every $y \in Y$ and every separable algebraic closure $\kappa(y) \subset \kappa(y)^{sep}$ the scheme $X_{\kappa(y)^{sep}}$ has at most n points.

Proof. This follows from Lemma 29.57.2 and the fact that the fibres X_y are disjoint unions of spectra of finite separable field extensions of $\kappa(y)$, see Lemma 29.36.7. $\square$

Having universally bounded fibres is an absolute notion and not a relative notion. This is why the condition in the following lemma is that X is quasi-compact, and not that f is quasi-compact.

03JA Lemma 29.57.9. Let $f : X \to Y$ be a morphism of schemes. Assume that

(1) f is locally quasi-finite, and
(2) X is quasi-compact.

Then f has universally bounded fibres.

Proof. Since X is quasi-compact, there exists a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$ and affine opens $V_i \subset Y$, $i = 1, \ldots, n$ such that $f(U_i) \subset V_i$. Because of the local nature of "local quasi-finiteness" (see Lemma 29.20.6 part (4)) we see that the morphisms $f|_{U_i} : U_i \to V_i$ are locally quasi-finite morphisms of affines, hence quasi-finite, see Lemma 29.20.9. For $y \in Y$ it is clear that $X_y = \bigcup_{y \in V_i} (U_i)_y$ is an open covering. Hence it suffices to prove the lemma for a quasi-finite morphism of affines (namely, if n_i works for the morphism $f|_{U_i} : U_i \to V_i$, then $\sum n_i$ works for f).

Assume $f : X \to Y$ is a quasi-finite morphism of affines. By Lemma 29.56.3 we can find a diagram

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & Z \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

with Z affine, π finite and j an open immersion. Since j has universally bounded fibres (Lemma 29.57.7) this reduces us to showing that π has universally bounded fibres (Lemma 29.57.4).

This reduces us to a morphism of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where $A \to B$ is finite. Say B is generated by $x_1, \ldots, x_n$ over A and say $P_i(T) \in A[T]$ is a monic polynomial of degree d_i such that $P_i(x_i) = 0$ in B (a finite ring extension is integral, see Algebra, Lemma 10.36.3). With these notations it is clear that

$$\bigoplus\nolimits_{0 \leq e_i < d_i, i=1,\ldots n} A \longrightarrow B, \quad (a_{(e_1,\ldots,e_n)}) \longmapsto \sum a_{(e_1,\ldots,e_n)} x_1^{e_1} \ldots x_n^{e_n}$$

is a surjective A-module map. Thus for any prime $\mathfrak p \subset A$ this induces a surjective map $\kappa(\mathfrak p)$-vector spaces

$$\kappa(\mathfrak p)^{\oplus d_1 \ldots d_n} \longrightarrow B \otimes_A \kappa(\mathfrak p)$$

In other words, the integer $d_1 \ldots d_n$ works in the definition of a morphism with universally bounded fibres. □

03JB Lemma 29.57.10. Consider a commutative diagram of morphisms of schemes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres. More precisely, if n bounds the degree of the fibres of g, then also n bounds the degree of the fibres of h.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say the degree of the fibres of g is bounded by $n \in \mathbf{N}$. We claim n also works for h. Let $z \in Z$. Consider the morphism of schemes $X_z \to Y_z$. It is flat and surjective. By assumption X_z is a finite scheme over $\kappa(z)$, in particular it is the spectrum of an Artinian ring (by Algebra, Lemma 10.53.2). By Lemma 29.11.13 the morphism $X_z \to Y_z$ is affine in particular quasi-compact. It follows from Lemma 29.25.12 that Y_z is a finite discrete as this holds for X_z. Hence Y_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 28.29.1 applied to the set of all points of Y_z). Write $Y_z = \mathrm{Spec}(B)$ and $X_z = \mathrm{Spec}(A)$. Then A is faithfully flat over B, so $B \subset A$. Hence $\dim_k(B) \leq \dim_k(A) \leq n$ as desired. □

29.58. Miscellany

0H1L Results which do not fit elsewhere.

0H1M Lemma 29.58.1. Let $f : Y \to X$ be a morphism of schemes. Let $x \in X$ be a point. Assume that Y is reduced and $f(Y)$ is set-theoretically contained in $\{x\}$. Then f factors through the canonical morphism $x = \mathrm{Spec}(\kappa(x)) \to X$.

Proof. Omitted. Hints: working affine locally one reduces to a commutative algebra lemma. Given a ring map $A \to B$ with B reduced such that there exists a unique prime ideal $\mathfrak p \subset A$ in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, then $A \to B$ factors through $\kappa(\mathfrak p)$. This is a nice exercise. □

0H1N Lemma 29.58.2. Let $f : Y \to X$ be a morphism of schemes. Let $E \subset X$. Assume X is locally Noetherian, there are no nontrivial specializations among the elements of E, Y is reduced, and $f(Y) \subset E$. Then f factors through $\coprod_{x \in E} x \to X$.

Proof. When E is a singleton this follows from Lemma 29.58.1. If E is finite, then E (with the induced topology of X) is a finite discrete space by our assumption on specializations. Hence this case reduces to the singleton case. In general, there is a reduction to the case where X and Y are affine schemes. Say $f : Y \to X$

corresponds to the ring map $\varphi : A \to B$. Denote $A' \subset B$ the image of φ. Let $E' \subset \mathrm{Spec}(A') \subset \mathrm{Spec}(A)$ be the set of minimal primes of A'. By Algebra, Lemma 10.30.5 the set E' is contained in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A') \subset \mathrm{Spec}(A)$. We conclude that $E' \subset E$. Since A' is Noetherian we have E' is finite by Algebra, Lemma 10.31.6. Since any other point in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a specialization of an element of E' and in E, we conclude that the image is contained in E' (by our assumption on specializations between points of E). Thus we reduce to the case where E is finite which we dealt with above. □

29.59. Other chapters

CHAPTER 30

Cohomology of Schemes

01X6

30.1. Introduction

01X7 In this chapter we first prove a number of results on the cohomology of quasi-coherent sheaves. A fundamental reference is [DG67]. Having done this we will elaborate on cohomology of coherent sheaves in the Noetherian setting. See [Ser55b].

30.2. Čech cohomology of quasi-coherent sheaves

01X8 Let X be a scheme. Let $U \subset X$ be an affine open. Recall that a standard open covering of U is a covering of the form $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ where $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_X)$ generate the unit ideal, see Schemes, Definition 26.5.2.

01X9 Lemma 30.2.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ be a standard open covering of an affine open of X. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

Proof. Write $U = \text{Spec}(A)$ for some ring A. In other words, $f_1, \ldots, f_n$ are elements of A which generate the unit ideal of A. Write $\mathcal{F}|_U = \widetilde{M}$ for some A-module M. Clearly the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is identified with the complex

$$\prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod\nolimits_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots$$

We are asked to show that the extended complex

01XA (30.2.1.1) $$0 \to M \to \prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod\nolimits_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots$$

(whose truncation we have studied in Algebra, Lemma 10.24.1) is exact. It suffices to show that (30.2.1.1) is exact after localizing at a prime $\mathfrak{p}$, see Algebra, Lemma 10.23.1. In fact we will show that the extended complex localized at $\mathfrak{p}$ is homotopic to zero.

There exists an index i such that $f_i \not\in \mathfrak{p}$. Choose and fix such an element i_{fix}. Note that $M_{f_{i_{\text{fix}}}, \mathfrak{p}} = M_\mathfrak{p}$. Similarly for a localization at a product $f_{i_0} \ldots f_{i_p}$ and $\mathfrak{p}$ we can drop any f_{i_j} for which $i_j = i_{\text{fix}}$. Let us define a homotopy

$$h : \prod\nolimits_{i_0 \ldots i_{p+1}} M_{f_{i_0} \ldots f_{i_{p+1}}, \mathfrak{p}} \longrightarrow \prod\nolimits_{i_0 \ldots i_p} M_{f_{i_0} \ldots f_{i_p}, \mathfrak{p}}$$

by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i_{\text{fix}} i_0 \ldots i_p}$$

(This is “dual” to the homotopy in the proof of Cohomology, Lemma 20.10.4.) In other words, $h : \prod_{i_0} M_{f_{i_0}, \mathfrak{p}} \to M_\mathfrak{p}$ is projection onto the factor $M_{f_{i_{\text{fix}}}, \mathfrak{p}} = M_\mathfrak{p}$ and in general the map h equal projection onto the factors $M_{f_{i_{\text{fix}}} f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}} = M_{f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}}$. We compute

$$\begin{aligned} (dh + hd)(s)_{i_0 \ldots i_p} &= \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}} i_0 \ldots i_p} \\ &= \sum\nolimits_{j=0}^{p} (-1)^j s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum\nolimits_{j=0}^{p} (-1)^{j+1} s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} \\ &= s_{i_0 \ldots i_p} \end{aligned}$$

This proves the identity map is homotopic to zero as desired. ☐

The following lemma says in particular that for any affine scheme X and any quasi-coherent sheaf $\mathcal{F}$ on X we have

$$H^p(X, \mathcal{F}) = 0$$

for all $p > 0$.

01XB Lemma 30.2.2. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.

Proof. We are going to apply Cohomology, Lemma 20.11.9. As our basis $\mathcal{B}$ for the topology of X we are going to use the affine opens of X. As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X. Next we check that conditions (1), (2) and (3) of Cohomology, Lemma 20.11.9 hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings of any element of $\mathcal{B}$, see Schemes, Lemma 26.5.1. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma 30.2.1. ☐

Here is a relative version of the vanishing of cohomology of quasi-coherent sheaves on affines.

01XC Lemma 30.2.3. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is affine then $R^i f_* \mathcal{F} = 0$ for all $i > 0$.

Proof. According to Cohomology, Lemma 20.7.3 the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. By assumption, whenever V is affine we have that $f^{-1}(V)$ is affine, see Morphisms, Definition 29.11.1. By Lemma 30.2.2 we conclude that $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) = 0$ whenever V is affine. Since S has a basis consisting of affine opens we win. ☐

089W Lemma 30.2.4. Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F}) = H^i(S, f_* \mathcal{F})$ for all $i \geq 0$.

Proof. Follows from Lemma 30.2.3 and the Leray spectral sequence. See Cohomology, Lemma 20.13.6. ☐

The following two lemmas explain when Čech cohomology can be used to compute cohomology of quasi-coherent modules.

0BDX Lemma 30.2.5. Let X be a scheme. The following are equivalent

(1) X has affine diagonal $\Delta : X \to X \times X$,
(2) for $U, V \subset X$ affine open, the intersection $U \cap V$ is affine, and
(3) there exists an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that $U_{i_0 \ldots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \ldots, i_p \in I$.

In particular this holds if X is separated.

Proof. Assume X has affine diagonal. Let $U, V \subset X$ be affine opens. Then $U \cap V = \Delta^{-1}(U \times V)$ is affine. Thus (2) holds. It is immediate that (2) implies (3). Conversely, if there is a covering of X as in (3), then $X \times X = \bigcup U_i \times U_{i'}$ is an affine open covering, and we see that $\Delta^{-1}(U_i \times U_{i'}) = U_i \cap U_{i'}$ is affine. Then Δ is an affine morphism by Morphisms, Lemma 29.11.3. The final assertion follows from Schemes, Lemma 26.21.7. □

01XD Lemma 30.2.6. Let X be a scheme. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering such that $U_{i_0 \ldots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. In this case for any quasi-coherent sheaf $\mathcal{F}$ we have

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$$

as $\Gamma(X, \mathcal{O}_X)$-modules for all p.

Proof. In view of Lemma 30.2.2 this is a special case of Cohomology, Lemma 20.11.6. □

30.3. Vanishing of cohomology

01XE We have seen that on an affine scheme the higher cohomology groups of any quasi-coherent sheaf vanish (Lemma 30.2.2). It turns out that this also characterizes affine schemes. We give two versions.

01XF Lemma 30.3.1. Let X be a scheme. Assume that [Ser57], [DG67, II, Theorem 5.2.1 (d') and IV (1.7.17)]

(1) X is quasi-compact,
(2) for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we have $H^1(X, \mathcal{I}) = 0$.

Then X is affine.

Proof. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x. Write $U = \mathrm{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 26.12.4 there are quasi-coherent sheaves of ideals $\mathcal{I}$, resp. $\mathcal{I}'$ cutting out the reduced closed subschemes Z, resp. Z'. Consider the short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0.$$

Since x is a closed point of X and $x \notin Z$ we see that $\mathcal{I}/\mathcal{I}'$ is supported at x. In fact, the restriction of $\mathcal{I}/\mathcal{I}'$ to U corresponds to the A-module $A/\mathfrak{m}$. Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}') = A/\mathfrak{m}$. Since by assumption $H^1(X, \mathcal{I}') = 0$ we see there exists a global section $f \in \Gamma(X, \mathcal{I})$ which maps to the element $1 \in A/\mathfrak{m}$ as a section of $\mathcal{I}/\mathcal{I}'$. Clearly we have $x \in X_f \subset U$. This implies that $X_f = D(f_A)$ where f_A is the image of f in $A = \Gamma(U, \mathcal{O}_X)$. In particular X_f is affine.

Consider the union $W = \bigcup X_f$ over all $f \in \Gamma(X, \mathcal{O}_X)$ such that X_f is affine. Obviously W is open in X. By the arguments above every closed point of X is contained in W. The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 5.12.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 5.12.8). This would contradict the fact that all closed points are in W. Hence we conclude $X = W$.

Choose finitely many $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that $X = X_{f_1} \cup \ldots \cup X_{f_n}$ and such that each X_{f_i} is affine. This is possible as we've seen above. By Properties, Lemma 28.27.3 to finish the proof it suffices to show that $f_1, \ldots, f_n$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \xrightarrow{f_1,\ldots,f_n} \mathcal{O}_X \longrightarrow 0$$

The arrow defined by $f_1, \ldots, f_n$ is surjective since the opens X_{f_i} cover X. We let $\mathcal{F}$ be the kernel of this surjective map. Observe that $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

so that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to a quasi-coherent sheaf of ideals. Namely we can take $\mathcal{F}_i$ to be the intersection of $\mathcal{F}$ with the first i direct summands of $\mathcal{O}_X^{\oplus n}$. The assumption of the lemma implies that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for all i. This implies that $H^1(X, \mathcal{F}_2) = 0$ because it is sandwiched between $H^1(X, \mathcal{F}_1)$ and $H^1(X, \mathcal{F}_2/\mathcal{F}_1)$. Continuing like this we deduce that $H^1(X, \mathcal{F}) = 0$. Therefore we conclude that the map

$$\bigoplus\nolimits_{i=1,\ldots,n} \Gamma(X, \mathcal{O}_X) \xrightarrow{f_1,\ldots,f_n} \Gamma(X, \mathcal{O}_X)$$

is surjective as desired. $\square$

Note that if X is a Noetherian scheme then every quasi-coherent sheaf of ideals is automatically a coherent sheaf of ideals and a finite type quasi-coherent sheaf of ideals. Hence the preceding lemma and the next lemma both apply in this case.

01XG Lemma 30.3.2. Let X be a scheme. Assume that [Ser57], [DG67, II, Theorem 5.2.1]

(1) X is quasi-compact,
(2) X is quasi-separated, and
(3) $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals $\mathcal{I}$ of finite type.

Then X is affine.

Proof. By Properties, Lemma 28.22.3 every quasi-coherent sheaf of ideals is a directed colimit of quasi-coherent sheaves of ideals of finite type. By Cohomology, Lemma 20.19.1 taking cohomology on X commutes with directed colimits. Hence we see that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals on X. In other words we see that Lemma 30.3.1 applies. $\square$

We can use the arguments given above to find a sufficient condition to see when an invertible sheaf is ample. However, we warn the reader that this condition is not necessary.

0B5P Lemma 30.3.3. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that

(1) X is quasi-compact,
(2) for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ there exists an $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$.

Then $\mathcal{L}$ is ample.

Proof. This is proved in exactly the same way as Lemma 30.3.1. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Write $U = \mathrm{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 26.12.4 there are quasi-coherent sheaves of ideals $\mathcal{I}$, resp. $\mathcal{I}'$ cutting out the reduced closed subschemes Z, resp. Z'. Consider the short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0.$$

For every $n \geq 1$ we obtain a short exact sequence

$$0 \to \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to 0.$$

By our assumption we may pick n such that $H^1(X, \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$. Since x is a closed point of X and $x \notin Z$ we see that $\mathcal{I}/\mathcal{I}'$ is supported at x. In fact, the restriction of $\mathcal{I}/\mathcal{I}'$ to U corresponds to the A-module $A/\mathfrak{m}$. Since $\mathcal{L}$ is trivial on U we see that the restriction of $\mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ to U also corresponds to the A-module $A/\mathfrak{m}$. Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = A/\mathfrak{m}$. By our choice of n we see there exists a global section $s \in \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ which maps to the element $1 \in A/\mathfrak{m}$. Clearly we have $x \in X_s \subset U$ because s vanishes at points of Z. This implies that $X_s = D(f)$ where $f \in A$ is the image of s in $A \cong \Gamma(U, \mathcal{L}^{\otimes n})$. In particular X_s is affine.

Consider the union $W = \bigcup X_s$ over all $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ for $n \geq 1$ such that X_s is affine. Obviously W is open in X. By the arguments above every closed point of X is contained in W. The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 5.12.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 5.12.8). This would contradict the fact that all closed points are in W. Hence we conclude $X = W$. This means that $\mathcal{L}$ is ample by Properties, Definition 28.26.1. □

There is a variant of Lemma 30.3.3 with finite type ideal sheaves which we will formulate and prove here if we ever need it.

0F83 Lemma 30.3.4. Let $f : X \to Y$ be a quasi-compact morphism with X and Y quasi-separated. If $R^1f_*\mathcal{I} = 0$ for every quasi-coherent sheaf of ideals $\mathcal{I}$ on X, then f is affine.

Proof. Let $V \subset Y$ be an affine open subscheme. We have to show that $U = f^{-1}(V)$ is affine. The inclusion morphism $V \to Y$ is quasi-compact by Schemes, Lemma 26.21.14. Hence the base change $U \to X$ is quasi-compact, see Schemes, Lemma 26.19.3. Thus any quasi-coherent sheaf of ideals $\mathcal{I}$ on U extends to a quasi-coherent sheaf of ideals on X, see Properties, Lemma 28.22.1. Since the formation of R^1f_* is local on Y (Cohomology, Section 20.7) we conclude that $R^1(U \to V)_*\mathcal{I} = 0$ by the assumption in the lemma. Hence by the Leray Spectral sequence (Cohomology, Lemma 20.13.4) we conclude that $H^1(U, \mathcal{I}) = H^1(V, (U \to V)_*\mathcal{I})$. Since $(U \to V)_*\mathcal{I}$ is quasi-coherent by Schemes, Lemma 26.24.1, we have $H^1(V, (U \to V)_*\mathcal{I}) = 0$ by Lemma 30.2.2. Thus we find that U is affine by Lemma 30.3.1. □

30.4. Quasi-coherence of higher direct images

01XH We have seen that the higher cohomology groups of a quasi-coherent module on an affine are zero. For (quasi-)separated quasi-compact schemes X this implies vanishing of cohomology groups of quasi-coherent sheaves beyond a certain degree. However, it may not be the case that X has finite cohomological dimension, because that is defined in terms of vanishing of cohomology of all $\mathcal{O}_X$-modules.

08DR Lemma 30.4.1 (Induction Principle). Let X be a quasi-compact and quasi-separated scheme. Let P be a property of the quasi-compact opens of X. Assume that [BV03, Proposition 3.3.1]

(1) P holds for every affine open of X,
(2) if U is quasi-compact open, V affine open, P holds for U, V, and $U \cap V$, then P holds for $U \cup V$.

Then P holds for every quasi-compact open of X and in particular for X.

Proof. First we argue by induction that P holds for separated quasi-compact opens $W \subset X$. Namely, such an open can be written as $W = U_1 \cup \ldots \cup U_n$ and we can do induction on n using property (2) with $U = U_1 \cup \ldots \cup U_{n-1}$ and $V = U_n$. This is allowed because $U \cap V = (U_1 \cap U_n) \cup \ldots \cup (U_{n-1} \cap U_n)$ is also a union of $n-1$ affine open subschemes by Schemes, Lemma 26.21.7 applied to the affine opens U_i and U_n of W. Having said this, for any quasi-compact open $W \subset X$ we can do induction on the number of affine opens needed to cover W using the same trick as before and using that the quasi-compact open $U_i \cap U_n$ is separated as an open subscheme of the affine scheme U_n. $\square$

01XI Lemma 30.4.2. Let X be a quasi-compact scheme with affine diagonal (for example if X is separated). Let $t = t(X)$ be the minimal number of affine opens needed to cover X. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq t$ and all quasi-coherent sheaves $\mathcal{F}$.

Proof. First proof. By induction on t. If $t = 1$ the result follows from Lemma 30.2.2. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \ldots \cup U_{t-1}$ a union of $t-1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \ldots (U_{t-1} \cap V)$ is also a union of $t-1$ affine open subschemes. Namely, since the diagonal is affine, the intersection of two affine opens is affine, see Lemma 30.2.5. We apply the Mayer-Vietoris long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

see Cohomology, Lemma 20.8.2. By induction we see that the groups $H^i(U, \mathcal{F})$, $H^i(V, \mathcal{F})$, $H^i(U \cap V, \mathcal{F})$ are zero for $i \geq t-1$. It follows immediately that $H^i(X, \mathcal{F})$ is zero for $i \geq t$.

Second proof. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$ be a finite affine open covering. Since X is has affine diagonal the multiple intersections $U_{i_0 \ldots i_p}$ are all affine, see Lemma 30.2.5. By Lemma 30.2.6 the Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ agree with the cohomology groups. By Cohomology, Lemma 20.23.6 the Čech cohomology groups may be computed using the alternating Čech complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$. As the covering consists of t elements we see immediately that $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = 0$ for all $p \geq t$. Hence the result follows. $\square$

0BDY Lemma 30.4.3. Let X be a quasi-compact scheme with affine diagonal (for example if X is separated). Then

(1) given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an embedding $\mathcal{F} \to \mathcal{F}'$ of quasi-coherent $\mathcal{O}_X$-modules such that $H^p(X, \mathcal{F}') = 0$ for all $p \geq 1$, and
(2) $\{H^n(X, -)\}_{n \geq 0}$ is a universal δ-functor from $QCoh(\mathcal{O}_X)$ to Ab.

Proof. Let $X = \bigcup U_i$ be a finite affine open covering. Set $U = \coprod U_i$ and denote $j : U \to X$ the morphism inducing the given open immersions $U_i \to X$. Since U is an affine scheme and X has affine diagonal, the morphism j is affine, see Morphisms, Lemma 29.11.11. For every $\mathcal{O}_X$-module $\mathcal{F}$ there is a canonical map $\mathcal{F} \to j_*j^*\mathcal{F}$. This map is injective as can be seen by checking on stalks: if $x \in U_i$, then we have a factorization

$$\mathcal{F}_x \to (j_*j^*\mathcal{F})_x \to (j^*\mathcal{F})_{x'} = \mathcal{F}_x$$

where $x' \in U$ is the point x viewed as a point of $U_i \subset U$. Now if $\mathcal{F}$ is quasi-coherent, then $j^*\mathcal{F}$ is quasi-coherent on the affine scheme U hence has vanishing higher cohomology by Lemma 30.2.2. Then $H^p(X, j_*j^*\mathcal{F}) = 0$ for $p > 0$ by Lemma 30.2.4 as j is affine. This proves (1). Finally, we see that the map $H^p(X, \mathcal{F}) \to H^p(X, j_*j^*\mathcal{F})$ is zero and part (2) follows from Homology, Lemma 12.12.4. $\square$

071L Lemma 30.4.4. Let X be a quasi-compact quasi-separated scheme. Let $X = U_1 \cup \ldots \cup U_t$ be an open covering with each U_i quasi-compact and separated (for example affine). Set

$$d = \max_{I \subset \{1,\ldots,t\}} \left(|I| + t(\bigcap\nolimits_{i \in I} U_i) - 1\right)$$

where $t(U)$ is the minimal number of affines needed to cover the scheme U. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq d$ and all quasi-coherent sheaves $\mathcal{F}$.

Proof. Note that since X is quasi-separated and U_i quasi-compact the numbers $t(\bigcap_{i \in I} U_i)$ are finite. Proof using induction on t. If $t = 1$ then the result follows from Lemma 30.4.2. If $t > 1$, write $X = U \cup V$ with $U = U_1 \cup \ldots \cup U_{t-1}$ and $V = U_t$. We apply the Mayer-Vietoris long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

see Cohomology, Lemma 20.8.2. Since V is affine, we have $H^i(V, \mathcal{F}) = 0$ for $i \geq 0$. By induction hypothesis we have $H^i(U, \mathcal{F}) = 0$ for

$$i \geq \max_{I \subset \{1,\ldots,t-1\}} \left(|I| + t(\bigcap\nolimits_{i \in I} U_i) - 1\right)$$

and the bound on the right is less than the bound in the statement of the lemma. Finally we may use our induction hypothesis for the open $U \cap V = (U_1 \cap U_t) \cup \ldots \cup (U_{t-1} \cap U_t)$ to get the vanishing of $H^i(U \cap V, \mathcal{F}) = 0$ for

$$i \geq \max_{I \subset \{1,\ldots,t-1\}} \left(|I| + t(U_t \cap \bigcap\nolimits_{i \in I} U_i) - 1\right)$$

Since the bound on the right is at least 1 less than the bound in the statement of the lemma, the lemma follows. $\square$

01XJ Lemma 30.4.5. Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.

(1) For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the higher direct images $R^pf_*\mathcal{F}$ are quasi-coherent on S.
(2) If S is quasi-compact, there exists an integer $n = n(X, S, f)$ such that $R^pf_*\mathcal{F} = 0$ for all $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}$ on X.
(3) In fact, if S is quasi-compact we can find $n = n(X, S, f)$ such that for every morphism of schemes $S' \to S$ we have $R^p(f')_*\mathcal{F}' = 0$ for $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}'$ on X'. Here $f' : X' = S' \times_S X \to S'$ is the base change of f.

Proof. We first prove (1). Note that under the hypotheses of the lemma the sheaf $R^0f_*\mathcal{F} = f_*\mathcal{F}$ is quasi-coherent by Schemes, Lemma 26.24.1. Using Cohomology, Lemma 20.7.4 we see that forming higher direct images commutes with restriction to open subschemes. Since being quasi-coherent is local on S we reduce to the case discussed in the next paragraph.

Proof of (1) in case S is affine. We will use the induction principle. Since f quasi-compact and quasi-separated we see that X is quasi-compact and quasi-separated.

For $U \subset X$ quasi-compact open and $a = f|_U$ we let $P(U)$ be the property that $R^p a_* \mathcal{F}$ is quasi-coherent on S for all quasi-coherent modules $\mathcal{F}$ on U and all $p \geq 0$. Since $P(X)$ is (1), it suffices the prove conditions (1) and (2) of Lemma 30.4.1 hold. If U is affine, then $P(U)$ holds because $R^p a_* \mathcal{F} = 0$ for $p \geq 1$ (by Lemma 30.2.3 and Morphisms, Lemma 29.11.12) and we've already observed the result holds for $p = 0$ in the first paragraph. Next, let $U \subset X$ be a quasi-compact open, $V \subset X$ an affine open, and assume $P(U)$, $P(V)$, $P(U \cap V)$ hold. Let $a = f|_U$, $b = f|_V$, $c = f|_{U \cap V}$, and $g = f|_{U \cup V}$. Then for any quasi-coherent $\mathcal{O}_{U \cup V}$-module $\mathcal{F}$ we have the relative Mayer-Vietoris sequence

$$0 \to g_* \mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 g_* \mathcal{F} \to \ldots$$

see Cohomology, Lemma 20.8.3. By $P(U)$, $P(V)$, $P(U \cap V)$ we see that $R^p a_*(\mathcal{F}|_U)$, $R^p b_*(\mathcal{F}|_V)$ and $R^p c_*(\mathcal{F}|_{U \cap V})$ are all quasi-coherent. Using the results on quasi-coherent sheaves in Schemes, Section 26.24 this implies that each of the sheaves $R^p g_* \mathcal{F}$ is quasi-coherent since it sits in the middle of a short exact sequence with a cokernel of a map between quasi-coherent sheaves on the left and a kernel of a map between quasi-coherent sheaves on the right. Whence $P(U \cup V)$ and the proof of (1) is complete.

Next, we prove (3) and a fortiori (2). Choose a finite affine open covering $S = \bigcup_{j=1,\ldots m} S_j$. For each j choose a finite affine open covering $f^{-1}(S_j) = \bigcup_{i=1,\ldots t_j} U_{ji}$. Let

$$d_j = \max_{I \subset \{1,\ldots,t_j\}} \left(|I| + t(\bigcap\nolimits_{i \in I} U_{ji})\right)$$

be the integer found in Lemma 30.4.4. We claim that $n(X, S, f) = \max d_j$ works.

Namely, let $S' \to S$ be a morphism of schemes and let $\mathcal{F}'$ be a quasi-coherent sheaf on $X' = S' \times_S X$. We want to show that $R^p f'_* \mathcal{F}' = 0$ for $p \geq n(X, S, f)$. Since this question is local on S' we may assume that S' is affine and maps into S_j for some j. Then $X' = S' \times_{S_j} f^{-1}(S_j)$ is covered by the open affines $S' \times_{S_j} U_{ji}$, $i = 1, \ldots t_j$ and the intersections

$$\bigcap\nolimits_{i \in I} S' \times_{S_j} U_{ji} = S' \times_{S_j} \bigcap\nolimits_{i \in I} U_{ji}$$

are covered by the same number of affines as before the base change. Applying Lemma 30.4.4 we get $H^p(X', \mathcal{F}') = 0$. By the first part of the proof we already know that each $R^q f'_* \mathcal{F}'$ is quasi-coherent hence has vanishing higher cohomology groups on our affine scheme S', thus we see that $H^0(S', R^p f'_* \mathcal{F}') = H^p(X', \mathcal{F}') = 0$ by Cohomology, Lemma 20.13.6. Since $R^p f'_* \mathcal{F}'$ is quasi-coherent we conclude that $R^p f'_* \mathcal{F}' = 0$. $\square$

01XK Lemma 30.4.6. Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact. Assume S is affine. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

$$H^q(X, \mathcal{F}) = H^0(S, R^q f_* \mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Consider the Leray spectral sequence $E_2^{p,q} = H^p(S, R^q f_* \mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology, Lemma 20.13.4. By Lemma 30.4.5 we see that the sheaves $R^q f_* \mathcal{F}$ are quasi-coherent. By Lemma 30.2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. See also Cohomology, Lemma 20.13.6 (2) for the general principle. $\square$

30.5. Cohomology and base change, I

02KE Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : S' \to S$ is any morphism of schemes. Denote $X' = X_{S'} = S' \times_S X$ the base change of X and denote $f' : X' \to S'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^*\mathcal{F}$. Here is a diagram representing the situation:

02KF (30.5.0.1)
$$\begin{tikzcd} \mathcal{F}' = (g')^*\mathcal{F} & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] & \mathcal{F} \\ Rf'_*\mathcal{F}' & S' \arrow[r, "g"] & S & Rf_*\mathcal{F} \end{tikzcd}$$

Here is the simplest case of the base change property we have in mind.

02KG **Lemma 30.5.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is affine. In this case $f_*\mathcal{F} \cong Rf_*\mathcal{F}$ is a quasi-coherent sheaf, and for every base change diagram (30.5.0.1) we have

$$g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}.$$

Proof. The vanishing of higher direct images is Lemma 30.2.3. The statement is local on S and S'. Hence we may assume $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$ and $\mathcal{F} = \widetilde{M}$ for some A-module M. We use Schemes, Lemma 26.7.3 to describe pullbacks and pushforwards of $\mathcal{F}$. Namely, $X' = \mathrm{Spec}(R' \otimes_R A)$ and $\mathcal{F}'$ is the quasi-coherent sheaf associated to $(R' \otimes_R A) \otimes_A M$. Thus we see that the lemma boils down to the equality

$$(R' \otimes_R A) \otimes_A M = R' \otimes_R M$$

as R'-modules. $\square$

In many situations it is sufficient to know about the following special case of cohomology and base change. It follows immediately from the stronger results in Section 30.7, but since it is so important it deserves its own proof.

02KH **Lemma 30.5.2** (Flat base change)**.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module with pullback $\mathcal{F}' = (g')^*\mathcal{F}$. Assume that g is flat and that f is quasi-compact and quasi-separated. For any $i \geq 0$

(1) the base change map of Cohomology, Lemma 20.17.1 is an isomorphism
$$g^*R^if_*\mathcal{F} \longrightarrow R^if'_*\mathcal{F}',$$
(2) if $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(X', \mathcal{F}')$.

Proof. Using Cohomology, Lemma 20.17.1 in (1) is allowed since g' is flat by Morphisms, Lemma 29.25.8. Having said this, part (1) follows from part (2). Namely, part (1) is local on S' and hence we may assume S and S' are affine. In other words, we have $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ as in (2). Then since $R^if_*\mathcal{F}$ is quasi-coherent (Lemma 30.4.5), it is the quasi-coherent $\mathcal{O}_S$-module associated to the A-module $H^0(S, R^if_*\mathcal{F}) = H^i(X, \mathcal{F})$ (equality by Lemma 30.4.6). Similarly, $R^if'_*\mathcal{F}'$ is the quasi-coherent $\mathcal{O}_{S'}$-module associated to the B-module $H^i(X', \mathcal{F}')$. Since pullback by g corresponds to $- \otimes_A B$ on modules (Schemes, Lemma 26.7.3) we see that it suffices to prove (2).

Let $A \to B$ be a flat ring homomorphism. Let X be a quasi-compact and quasi-separated scheme over A. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $X_B = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ and denote $\mathcal{F}_B$ the pullback of $\mathcal{F}$. We are trying to show that the map

$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X_B, \mathcal{F}_B)$$

(given by the reference in the statement of the lemma) is an isomorphism.

In case X is separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 30.2.6. If $\mathcal{U}_B : X_B = (U_1)_B \cup \ldots \cup (U_t)_B$ we obtain by base change, then it is still the case that each $(U_i)_B$ is affine and that X_B is separated. Thus we obtain

$$\check{H}^p(\mathcal{U}_B, \mathcal{F}_B) = H^p(X_B, \mathcal{F}_B).$$

We have the following relation between the Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}_B, \mathcal{F}_B) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B$$

as follows from Lemma 30.5.1. Since $A \to B$ is flat, the same thing remains true on taking cohomology.

In case X is quasi-separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 20.11.5. The reader who wishes to avoid this spectral sequence can use Mayer-Vietoris and induction on t as in the proof of Lemma 30.4.5. The spectral sequence has E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B))$ which converges to $H^{p+q}(X_B, \mathcal{F}_B)$. Since the intersections $U_{i_0 \ldots i_p}$ are quasi-compact and separated, the result of the second paragraph of the proof gives $\check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B)) = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B$. Using that $A \to B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \to H^i(X_B, \mathcal{F}_B)$ is an isomorphism for all i and we win. □

0CKW Lemma 30.5.3 (Finite locally free base change). Consider a cartesian diagram of schemes

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module with pullback $\mathcal{G} = h^*\mathcal{F}$. If B is a finite locally free A-module, then $H^i(X, \mathcal{F}) \otimes_A B = H^i(Y, \mathcal{G})$.

Warning: Do not use this lemma unless you understand the difference between this and Lemma 30.5.2.

Proof. In case X is separated, choose an affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 30.2.6. Let $\mathcal{V} : Y = \bigcup_{i \in I} g^{-1}(U_i)$ be the corresponding affine open covering of Y. The opens $V_i = g^{-1}(U_i) = U_i \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ are affine and Y is separated. Thus we obtain

$$\check{H}^p(\mathcal{V}, \mathcal{G}) = H^p(Y, \mathcal{G}).$$

We claim the map of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$$

is an isomorphism. Namely, as B is finitely presented as an A-module we see that tensoring with B over A commutes with products, see Algebra, Proposition 10.89.3. Thus it suffices to show that the maps $\Gamma(U_{i_0 \ldots i_p}, \mathcal{F}) \otimes_A B \to \Gamma(V_{i_0 \ldots i_p}, \mathcal{G})$ are isomorphisms which follows from Lemma 30.5.1. Since $A \to B$ is flat, the same thing remains true on taking cohomology.

In the general case we argue in exactly the same way using affine open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and the corresponding covering $\mathcal{V} : Y = \bigcup_{i \in I} V_i$ with $V_i = g^{-1}(U_i)$ as above. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 20.11.5. The spectral sequence has E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{V}, \underline{H}^q(\mathcal{G}))$ which converges to $H^{p+q}(Y, \mathcal{G})$. Since the intersections $U_{i_0 \ldots i_p}$ are separated, the result of the previous paragraph gives isomorphisms $\Gamma(U_{i_0 \ldots i_p}, \underline{H}^q(\mathcal{F})) \otimes_A B \to \Gamma(V_{i_0 \ldots i_p}, \underline{H}^q(\mathcal{G}))$. Using that $- \otimes_A B$ commutes with products and is exact, we conclude that $\check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B \to \check{H}^p(\mathcal{V}, \underline{H}^q(\mathcal{G}))$ is an isomorphism. Using that $A \to B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \to H^i(Y, \mathcal{G})$ is an isomorphism for all i and we win. □

30.6. Colimits and higher direct images

07TA General results of this nature can be found in Cohomology, Section 20.19, Sheaves, Lemma 6.29.1, and Modules, Lemma 17.22.8.

07TB Lemma 30.6.1. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ be a filtered colimit of quasi-coherent sheaves on X. Then for any $p \geq 0$ we have

$$R^p f_* \mathcal{F} = \mathrm{colim}\, R^p f_* \mathcal{F}_i.$$

Proof. Recall that $R^p f_* \mathcal{F}$ is the sheaf associated to $U \mapsto H^p(f^{-1}U, \mathcal{F})$, see Cohomology, Lemma 20.7.3. Recall that the colimit is the sheaf associated to the presheaf colimit (taking colimits over opens). Hence we can apply Cohomology, Lemma 20.19.1 to $H^p(f^{-1}U, -)$ where U is affine to conclude. (Because the basis of affine opens in $f^{-1}U$ satisfies the assumptions of that lemma.) □

30.7. Cohomology and base change, II

071M Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is quasi-compact and quasi-separated we would like to represent $Rf_* \mathcal{F}$ by a complex of quasi-coherent sheaves on S. This follows from the fact that the sheaves $R^i f_* \mathcal{F}$ are quasi-coherent if S is quasi-compact and has affine diagonal, using that $D_{QCoh}(S)$ is equivalent to $D(QCoh(\mathcal{O}_S))$, see Derived Categories of Schemes, Proposition 36.7.5.

In this section we will use a different approach which produces an explicit complex having a good base change property. The construction is particularly easy if f and S are separated, or more generally have affine diagonal. Since this is the case which by far the most often used we treat it separately.

01XL Lemma 30.7.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume X is quasi-compact and X and S have affine diagonal (e.g., if X and S are separated). In this case we can compute $Rf_*\mathcal{F}$ as follows:

(1) Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$.
(2) For $i_0, \ldots, i_p \in \{1, \ldots, n\}$ denote $f_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to S$ the restriction of f to the intersection $U_{i_0\ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$.
(3) Set $\mathcal{F}_{i_0\ldots i_p}$ equal to the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$.
(4) Set
$$\check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) = \bigoplus\nolimits_{i_0\ldots i_p} f_{i_0\ldots i_p*}\mathcal{F}_{i_0\ldots i_p}$$
and define differentials $d : \check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U}, f, \mathcal{F})$ as in Cohomology, Equation (20.9.0.1).

Then the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ is a complex of quasi-coherent sheaves on S which comes equipped with an isomorphism
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf_*\mathcal{F}$$
in $D^+(S)$. This isomorphism is functorial in the quasi-coherent sheaf $\mathcal{F}$.

Proof. Consider the resolution $\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of Cohomology, Lemma 20.24.1. We have an equality of complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of quasi-coherent $\mathcal{O}_S$-modules. The morphisms $j_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to X$ and the morphisms $f_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to S$ are affine by Morphisms, Lemma 29.11.11 and Lemma 30.2.5. Hence $R^q j_{i_0\ldots i_p*}\mathcal{F}_{i_0\ldots i_p}$ as well as $R^q f_{i_0\ldots i_p*}\mathcal{F}_{i_0\ldots i_p}$ are zero for $q > 0$ (Lemma 30.2.3). Using $f \circ j_{i_0\ldots i_p} = f_{i_0\ldots i_p}$ and the spectral sequence of Cohomology, Lemma 20.13.8 we conclude that $R^q f_*(j_{i_0\ldots i_p*}\mathcal{F}_{i_0\ldots i_p}) = 0$ for $q > 0$. Since the terms of the complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ are finite direct sums of the sheaves $j_{i_0\ldots i_p*}\mathcal{F}_{i_0\ldots i_p}$ we conclude using Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) that
$$Rf_*\mathcal{F} = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$
as desired. □

Next, we are going to consider what happens if we do a base change.

01XM Lemma 30.7.2. With notation as in diagram (30.5.0.1). Assume $f : X \to S$ and $\mathcal{F}$ satisfy the hypotheses of Lemma 30.7.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup U_i$ of X. There is a canonical isomorphism
$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf'_*\mathcal{F}'$$
in $D^+(S')$. Moreover, if $S' \to S$ is affine, then in fact
$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$
with $\mathcal{U}' : X' = \bigcup U'_i$ where $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ is also affine.

Proof. In fact we may define $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ no matter whether S' is affine over S or not. Let $\mathcal{U}' : X' = \bigcup U'_i$ be the induced covering of X'. In this case we claim that

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$ defined in exactly the same manner as in Lemma 30.7.1. This is clear from the case of affine morphisms (Lemma 30.5.1) by working locally on S'. Moreover, exactly as in the proof of Lemma 30.7.1 one sees that there is an isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}') \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$ since the morphisms $U'_i \to X'$ and $U'_i \to S'$ are still affine (being base changes of affine morphisms). Details omitted. $\square$

The lemma above says that the complex

$$\mathcal{K}^\bullet = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$

is a bounded below complex of quasi-coherent sheaves on S which universally computes the higher direct images of $f : X \to S$. This is something about this particular complex and it is not preserved by replacing $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ by a quasi-isomorphic complex in general! In other words, this is not a statement that makes sense in the derived category. The reason is that the pullback $g^*\mathcal{K}^\bullet$ is not equal to the derived pullback $Lg^*\mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ in general!

Here is a more general case where we can prove this statement. We remark that the condition of S being separated is harmless in most applications, since this is usually used to prove some local property of the total derived image. The proof is significantly more involved and uses hypercoverings; it is a nice example of how you can use them sometimes.

01XN Lemma 30.7.3. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume that f is quasi-compact and quasi-separated and that S is quasi-compact and separated. There exists a bounded below complex $\mathcal{K}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules with the following property: For every morphism $g : S' \to S$ the complex $g^*\mathcal{K}^\bullet$ is a representative for $Rf'_*\mathcal{F}'$ with notation as in diagram (30.5.0.1).

Proof. (If f is separated as well, please see Lemma 30.7.2.) The assumptions imply in particular that X is quasi-compact and quasi-separated as a scheme. Let $\mathcal{B}$ be the set of affine opens of X. By Hypercoverings, Lemma 25.11.4 we can find a hypercovering $K = (I, \{U_i\})$ such that each I_n is finite and each U_i is an affine open of X. By Hypercoverings, Lemma 25.5.3 there is a spectral sequence with E_2-page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Note that $\check{H}^p(K, \underline{H}^q(\mathcal{F}))$ is the pth cohomology group of the complex

$$\prod\nolimits_{i \in I_0} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_1} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_2} H^q(U_i, \mathcal{F}) \to \ldots$$

Since each U_i is affine we see that this is zero unless $q = 0$ in which case we obtain

$$\prod\nolimits_{i \in I_0} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_1} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_2} \mathcal{F}(U_i) \to \ldots$$

Thus we conclude that $R\Gamma(X, \mathcal{F})$ is computed by this complex.

For any n and $i \in I_n$ denote $f_i : U_i \to S$ the restriction of f to U_i. As S is separated and U_i is affine this morphism is affine. Consider the complex of quasi-coherent sheaves

$$\mathcal{K}^\bullet = (\prod\nolimits_{i \in I_0} f_{i,*}\mathcal{F}|_{U_i} \to \prod\nolimits_{i \in I_1} f_{i,*}\mathcal{F}|_{U_i} \to \prod\nolimits_{i \in I_2} f_{i,*}\mathcal{F}|_{U_i} \to \ldots)$$

on S. As in Hypercoverings, Lemma 25.5.3 we obtain a map $\mathcal{K}^\bullet \to Rf_*\mathcal{F}$ in $D(\mathcal{O}_S)$ by choosing an injective resolution of $\mathcal{F}$ (details omitted). Consider any affine scheme V and a morphism $g : V \to S$. Then the base change X_V has a hypercovering $K_V = (I, \{U_{i,V}\})$ obtained by base change. Moreover, $g^*f_{i,*}\mathcal{F} = f_{i,V,*}(g')^*\mathcal{F}|_{U_{i,V}}$. Thus the arguments above prove that $\Gamma(V, g^*\mathcal{K}^\bullet)$ computes $R\Gamma(X_V, (g')^*\mathcal{F})$. This finishes the proof of the lemma as it suffices to prove the equality of complexes Zariski locally on S'. $\square$

The following lemma is a variant to flat base change.

0GN5 **Lemma 30.7.4.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_{S'}$-module flat over S. Assume f is quasi-compact and quasi-separated. For any $i \geq 0$ there is an identification

$$\mathcal{G} \otimes_{\mathcal{O}_{S'}} g^*R^if_*\mathcal{F} = R^if'_*\left((f')^*\mathcal{G} \otimes_{\mathcal{O}_{X'}} (g')^*\mathcal{F}\right)$$

Proof. Let us construct a map from left to right. First, we have the base change map $Lg^*Rf_*\mathcal{F} \to Rf'_*L(g')^*\mathcal{F}$. There is also the adjunction map $\mathcal{G} \to Rf'_*L(f')^*\mathcal{G}$. Using the relative cup product We obtain

$$\begin{aligned} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{S'}} Lg^*Rf_*\mathcal{F} &\to Rf'_*L(f')^*\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{S'}} Rf'_*L(g')^*\mathcal{F} \\ &\to Rf'_*\left(L(f')^*\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^*\mathcal{F}\right) \\ &\to Rf'_*\left((f')^*\mathcal{G} \otimes_{\mathcal{O}_{X'}} (g')^*\mathcal{F}\right) \end{aligned}$$

where for the middle arrow we used the relative cup product, see Cohomology, Remark 20.28.7. The source of the composition is

$$\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{S'}} Lg^*Rf_*\mathcal{F} = \mathcal{G} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{O}_S} g^{-1}Rf_*\mathcal{F}$$

by Cohomology, Lemma 20.27.4. Since $\mathcal{G}$ is flat as a sheaf of $g^{-1}\mathcal{O}_S$-modules and since g^{-1} is an exact functor, this is a complex whose ith cohomology sheaf is $\mathcal{G} \otimes_{g^{-1}\mathcal{O}_S} g^{-1}R^if_*\mathcal{F} = \mathcal{G} \otimes_{\mathcal{O}_{S'}} g^*R^if_*\mathcal{F}$. In this way we obtain global maps from left to right in the equality of the lemma. To show this map is an isomorphism we may work locally on S'. Thus we may and do assume that S and S' are affine schemes.

Proof in case S and S' are affine. Say $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ and say $\mathcal{G}$ corresponds to the B-module N which is assumed to be A-flat. Since S is affine, X is quasi-compact and quasi-separated. We will use a hypercovering argument to finish the proof; if X is separated or has affine diagonal, then you can use a Čech covering. Let $\mathcal{B}$ be the set of affine opens of X. By Hypercoverings, Lemma 25.11.4 we can find a hypercovering $K = (I, \{U_i\})$ of X such that each I_n is finite and each

U_i is an affine open of X. By Hypercoverings, Lemma 25.5.3 there is a spectral sequence with E_2-page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Since each U_i is affine and $\mathcal{F}$ is quasi-coherent the value of $\underline{H}^q(\mathcal{F})$ is zero on U_i for $q > 0$. Thus the spectral sequence degenerates and we conclude that the cohomology modules $H^q(X, \mathcal{F})$ are computed by

$$\prod\nolimits_{i \in I_0} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_1} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_2} \mathcal{F}(U_i) \to \ldots$$

Next, note that the base change of our hypercovering to S' is a hypercovering of $X' = S' \times_S X$. The schemes $S' \times_S U_i$ are affine too and we have

$$\left((f')^*\mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^*\mathcal{F}\right)(S' \times_S U_i) = N \otimes_A \mathcal{F}(U_i)$$

In this way we conclude that the cohomology modules $H^q(X', (f')^*\mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^*\mathcal{F})$ are computed by

$$N \otimes_A \left(\prod\nolimits_{i \in I_0} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_1} \mathcal{F}(U_i) \to \prod\nolimits_{i \in I_2} \mathcal{F}(U_i) \to \ldots\right)$$

Since N is flat over A, we conclude that

$$H^q(X', (f')^*\mathcal{G} \otimes_{\mathcal{O}_{S'}} (g')^*\mathcal{F}) = N \otimes_A H^q(X, \mathcal{F})$$

Since this is the translation into algebra of the statement we had to show the proof is complete. □

30.8. Cohomology of projective space

01XS In this section we compute the cohomology of the twists of the structure sheaf on $\mathbf{P}^n_S$ over a scheme S. Recall that $\mathbf{P}^n_S$ was defined as the fibre product $\mathbf{P}^n_S = S \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{P}^n_{\mathbf{Z}}$ in Constructions, Definition 27.13.2. It was shown to be equal to

$$\mathbf{P}^n_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n])$$

in Constructions, Lemma 27.21.5. In particular, projective space is a particular case of a projective bundle. If $S = \mathrm{Spec}(R)$ is affine then we have

$$\mathbf{P}^n_S = \mathbf{P}^n_R = \mathrm{Proj}(R[T_0, \ldots, T_n]).$$

All these identifications are compatible and compatible with the constructions of the twisted structure sheaves $\mathcal{O}_{\mathbf{P}^n_S}(d)$.

Before we state the result we need some notation. Let R be a ring. Recall that $R[T_0, \ldots, T_n]$ is a graded R-algebra where each T_i is homogeneous of degree 1. Denote $(R[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite free R-module of rank $\binom{n+d}{d}$ when $d \geq 0$ and zero else. It has a basis consisting of monomials $T_0^{e_0} \ldots T_n^{e_n}$ with $\sum e_i = d$. We will also use the following notation: $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ denotes the $\mathbf{Z}$-graded ring with $\frac{1}{T_i}$ in degree -1. In particular the $\mathbf{Z}$-graded $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ module

$$\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$$

which shows up in the statement below is zero in degrees $\geq -n$, is free on the generator $\frac{1}{T_0 \ldots T_n}$ in degree $-n-1$ and is free of rank $(-1)^n\binom{n+d}{d}$ for $d \leq -n-1$.

01XT Lemma 30.8.1. Let R be a ring. Let $n \geq 0$ be an integer. We have [DG67, III Proposition 2.1.12]

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \left\{ \begin{matrix} (R[T_0, \ldots, T_n])_d & \text{if} & q = 0 \\ 0 & \text{if} & q \neq 0, n \\ \left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d & \text{if} & q = n \end{matrix} \right.$$

as R-modules.

Proof. We will use the standard affine open covering

$$\mathcal{U} : \mathbf{P}^n_R = \bigcup\nolimits_{i=0}^n D_+(T_i)$$

to compute the cohomology using the Čech complex. This is permissible by Lemma 30.2.6 since any intersection of finitely many affine $D_+(T_i)$ is also a standard affine open (see Constructions, Section 27.8). In fact, we can use the alternating or ordered Čech complex according to Cohomology, Lemmas 20.23.3 and 20.23.6.

The ordering we will use on $\{0, \ldots, n\}$ is the usual one. Hence the complex we are looking at has terms

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{i_0 < \ldots < i_p} (R[T_0, \ldots, T_n, \frac{1}{T_{i_0} \ldots T_{i_p}}])_d$$

Moreover, the maps are given by the usual formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$$

see Cohomology, Section 20.23. Note that each term of this complex has a natural $\mathbf{Z}^{n+1}$-grading. Namely, we get this by declaring a monomial $T_0^{e_0} \ldots T_n^{e_n}$ to be homogeneous with weight $(e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. It is clear that the differential given above respects the grading. In a formula we have

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1}} \check{\mathcal{C}}^\bullet(\vec{e})$$

where not all summands on the right hand side occur (see below). Hence in order to compute the cohomology modules of the complex it suffices to compute the cohomology of the graded pieces and take the direct sum at the end.

Fix $\vec{e} = (e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. In order for this weight to occur in the complex above we need to assume $e_0 + \ldots + e_n = d$ (if not then it occurs for a different twist of the structure sheaf of course). Assuming this, set

$$NEG(\vec{e}) = \{i \in \{0, \ldots, n\} \mid e_i < 0\}.$$

With this notation the weight $\vec{e}$ summand $\check{\mathcal{C}}^\bullet(\vec{e})$ of the Čech complex above has the following terms

$$\check{\mathcal{C}}^p(\vec{e}) = \bigoplus\nolimits_{i_0 < \ldots < i_p,\ NEG(\vec{e}) \subset \{i_0, \ldots, i_p\}} R \cdot T_0^{e_0} \ldots T_n^{e_n}$$

In other words, the terms corresponding to $i_0 < \ldots < i_p$ such that $NEG(\vec{e})$ is not contained in $\{i_0 \ldots i_p\}$ are zero. The differential of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is still given by the exact same formula as above.

Suppose that $NEG(\vec{e}) = \{0, \ldots, n\}$, i.e., that all exponents e_i are negative. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has only one term, namely $\check{\mathcal{C}}^n(\vec{e}) = R \cdot \frac{1}{T_0^{-e_0} \ldots T_n^{-e_n}}$. Hence in this case

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot \frac{1}{T_0^{-e_0} \ldots T_n^{-e_n}} & \text{if} \quad q = n \\ 0 & \text{if} \quad \text{else} \end{cases}$$

The direct sum of all of these terms clearly gives the value

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d$$

in degree n as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

Assume $NEG(\vec{e}) = \emptyset$. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has a summand R corresponding to all $i_0 < \ldots < i_p$. Let us compare the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ to another complex. Namely, consider the affine open covering

$$\mathcal{V} : \text{Spec}(R) = \bigcup\nolimits_{i \in \{0, \ldots, n\}} V_i$$

where $V_i = \text{Spec}(R)$ for all i. Consider the alternating Čech complex

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})$$

By the same reasoning as above this computes the cohomology of the structure sheaf on $\text{Spec}(R)$. Hence we see that $H^p(\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})) = R$ if $p = 0$ and is 0 whenever $p > 0$. For these facts, see Lemma 30.2.1 and its proof. Note that also $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\text{Spec}(R)})$ has a summand R for every $i_0 < \ldots < i_p$ and has exactly the same differential as $\check{\mathcal{C}}^\bullet(\vec{e})$. In other words these complexes are isomorphic complexes and hence have the same cohomology. We conclude that

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot T_0^{e_0} \ldots T_n^{e_n} & \text{if} \quad q = 0 \\ 0 & \text{if} \quad \text{else} \end{cases}$$

in the case that $NEG(\vec{e}) = \emptyset$. The direct sum of all of these terms clearly gives the value

$$(R[T_0, \ldots, T_n])_d$$

in degree 0 as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

To finish the proof of the lemma we have to show that the complexes $\check{\mathcal{C}}^\bullet(\vec{e})$ are acyclic when $NEG(\vec{e})$ is neither empty nor equal to $\{0, \ldots, n\}$. Pick an index $i_{\text{fix}} \not\in NEG(\vec{e})$ (such an index exists). Consider the map

$$h : \check{\mathcal{C}}^{p+1}(\vec{e}) \to \check{\mathcal{C}}^p(\vec{e})$$

given by the rule that for $i_0 < \ldots < i_p$ we have

$$h(s)_{i_0 \ldots i_p} = \begin{cases} 0 & \text{if} \quad p \not\in \{0, \ldots, n-1\} \\ 0 & \text{if} \quad i_{\text{fix}} \in \{i_0, \ldots, i_p\} \\ s_{i_{\text{fix}} i_0 \ldots i_p} & \text{if} \quad i_{\text{fix}} < i_0 \\ (-1)^a s_{i_0 \ldots i_{a-1} i_{\text{fix}} i_a \ldots i_p} & \text{if} \quad i_{a-1} < i_{\text{fix}} < i_a \\ (-1)^p s_{i_0 \ldots i_p} & \text{if} \quad i_p < i_{\text{fix}} \end{cases}$$

Please compare with the proof of Lemma 30.2.1. This makes sense because we have

$$NEG(\vec{e}) \subset \{i_0, \ldots, i_p\} \Leftrightarrow NEG(\vec{e}) \subset \{i_{\text{fix}}, i_0, \ldots, i_p\}$$

The exact same (combinatorial) computation[1] as in the proof of Lemma 30.2.1 shows that

$$(hd + dh)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

Hence we see that the identity map of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is homotopic to zero which implies that it is acyclic. $\square$

In the following lemma we are going to use the pairing of free R-modules

$$R[T_0, \ldots, T_n] \times \frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \longrightarrow R$$

which is defined by the rule

$$(f, g) \longmapsto \text{coefficient of } \frac{1}{T_0 \ldots T_n} \text{ in } fg.$$

In other words, the basis element $T_0^{e_0} \ldots T_n^{e_n}$ pairs with the basis element $T_0^{d_0} \ldots T_n^{d_n}$ to give 1 if and only if $e_i + d_i = -1$ for all i, and pairs to zero in all other cases. Using this pairing we get an identification

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \right)_d = \mathrm{Hom}_R((R[T_0, \ldots, T_n])_{-n-1-d}, R)$$

Thus we can reformulate the result of Lemma 30.8.1 as saying that

01XU (30.8.1.1)
$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \left\{ \begin{matrix} (R[T_0, \ldots, T_n])_d & \text{if} & q = 0 \\ 0 & \text{if} & q \neq 0, n \\ \mathrm{Hom}_R((R[T_0, \ldots, T_n])_{-n-1-d}, R) & \text{if} & q = n \end{matrix} \right.$$

01XV **Lemma 30.8.2.** The identifications of Equation (30.8.1.1) are compatible with base change w.r.t. ring maps $R \to R'$. Moreover, for any $f \in R[T_0, \ldots, T_n]$ homogeneous of degree m the map multiplication by f

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d + m)$$

induces the map on the cohomology group via the identifications of Equation (30.8.1.1) which is multiplication by f for H^0 and the contragredient of multiplication by f

$$(R[T_0, \ldots, T_n])_{-n-1-(d+m)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

on H^n.

---

[1]For example, suppose that $i_0 < \ldots < i_p$ is such that $i_{\text{fix}} \notin \{i_0, \ldots, i_p\}$ and that $i_{a-1} < i_{\text{fix}} < i_a$ for some $1 \le a \le p$. Then we have

$$\begin{aligned} & (dh + hd)(s)_{i_0 \ldots i_p} \\ & = \sum\nolimits_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + (-1)^a d(s)_{i_0 \ldots i_{a-1} i_{\text{fix}} i_a \ldots i_p} \\ & = \sum\nolimits_{j=0}^{a-1} (-1)^{j+a-1} s_{i_0 \ldots \hat{i}_j \ldots i_{a-1} i_{\text{fix}} i_a \ldots i_p} + \sum\nolimits_{j=a}^{p} (-1)^{j+a} s_{i_0 \ldots i_{a-1} i_{\text{fix}} i_a \ldots \hat{i}_j \ldots i_p} + \\ & \sum\nolimits_{j=0}^{a-1} (-1)^{a+j} s_{i_0 \ldots \hat{i}_j \ldots i_{a-1} i_{\text{fix}} i_a \ldots i_p} + (-1)^{2a} s_{i_0 \ldots i_p} + \sum\nolimits_{j=a}^{p} (-1)^{a+j+1} s_{i_0 \ldots i_{a-1} i_{\text{fix}} i_a \ldots \hat{i}_j \ldots i_p} \\ & = s_{i_0 \ldots i_p} \end{aligned}$$

as desired. The other cases are similar.

Proof. Suppose that $R \to R'$ is a ring map. Let $\mathcal{U}$ be the standard affine open covering of $\mathbf{P}^n_R$, and let $\mathcal{U}'$ be the standard affine open covering of $\mathbf{P}^n_{R'}$. Note that $\mathcal{U}'$ is the pullback of the covering $\mathcal{U}$ under the canonical morphism $\mathbf{P}^n_{R'} \to \mathbf{P}^n_R$. Hence there is a map of Čech complexes

$$\gamma : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}', \mathcal{O}_{\mathbf{P}_{R'}}(d))$$

which is compatible with the map on cohomology by Cohomology, Lemma 20.15.1. It is clear from the computations in the proof of Lemma 30.8.1 that this map of Čech complexes is compatible with the identifications of the cohomology groups in question. (Namely the basis elements for the Čech complex over R simply map to the corresponding basis elements for the Čech complex over R'.) Whence the first statement of the lemma.

Now fix the ring R and consider two homogeneous polynomials $f, g \in R[T_0, \ldots, T_n]$ both of the same degree m. Since cohomology is an additive functor, it is clear that the map induced by multiplication by $f + g$ is the same as the sum of the maps induced by multiplication by f and the map induced by multiplication by g. Moreover, since cohomology is a functor, a similar result holds for multiplication by a product fg where f, g are both homogeneous (but not necessarily of the same degree). Hence to verify the second statement of the lemma it suffices to prove this when $f = x \in R$ or when $f = T_i$. In the case of multiplication by an element $x \in R$ the result follows since every cohomology groups or complex in sight has the structure of an R-module or complex of R-modules. Finally, we consider the case of multiplication by T_i as a $\mathcal{O}_{\mathbf{P}^n_R}$-linear map

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d + 1)$$

The statement on H^0 is clear. For the statement on H^n consider multiplication by T_i as a map on Čech complexes

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d + 1))$$

We are going to use the notation introduced in the proof of Lemma 30.8.1. We consider the effect of multiplication by T_i in terms of the decompositions

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1},\ \sum e_i = d} \check{\mathcal{C}}^\bullet(\vec{e})$$

and

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d + 1)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1},\ \sum e_i = d+1} \check{\mathcal{C}}^\bullet(\vec{e})$$

It is clear that it maps the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e})$ to the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e} + \vec{b}_i)$ where $\vec{b}_i = (0, \ldots, 0, 1, 0, \ldots, 0))$ the ith basis vector. In other words, it maps the summand of H^n corresponding to $\vec{e}$ with $e_i < 0$ and $\sum e_i = d$ to the summand of H^n corresponding to $\vec{e} + \vec{b}_i$ (which is zero if $e_i + b_i \geq 0$). It is easy to see that this corresponds exactly to the action of the contragredient of multiplication by T_i as a map

$$(R[T_0, \ldots, T_n])_{-n-1-(d+1)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

This proves the lemma. □

Before we state the relative version we need some notation. Namely, recall that $\mathcal{O}_S[T_0, \ldots, T_n]$ is a graded $\mathcal{O}_S$-module where each T_i is homogeneous of degree 1. Denote $(\mathcal{O}_S[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S.

01XW Lemma 30.8.3. Let S be a scheme. Let $n \geq 0$ be an integer. Consider the structure morphism

$$f : \mathbf{P}^n_S \longrightarrow S.$$

We have

$$R^q f_*(\mathcal{O}_{\mathbf{P}^n_S}(d)) = \begin{cases} (\mathcal{O}_S[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}((\mathcal{O}_S[T_0, \ldots, T_n])_{-n-1-d}, \mathcal{O}_S) & \text{if} \quad q = n \end{cases}$$

Proof. Omitted. Hint: This follows since the identifications in (30.8.1.1) are compatible with affine base change by Lemma 30.8.2. $\square$

Next we state the version for projective bundles associated to finite locally free sheaves. Let S be a scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$, see Modules, Section 17.14. In this case we think of $\text{Sym}(\mathcal{E})$ as a graded $\mathcal{O}_S$-module where $\mathcal{E}$ is the graded part of degree 1. And $\text{Sym}^d(\mathcal{E})$ is the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S. Recall that our normalization is that

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E})) \longrightarrow S$$

and that there are natural maps $\text{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$.

01XX Lemma 30.8.4. Let S be a scheme. Let $n \geq 1$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$. Consider the structure morphism

$$\pi : \mathbf{P}(\mathcal{E}) \longrightarrow S.$$

We have

$$R^q \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) = \begin{cases} \text{Sym}^d(\mathcal{E}) & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}(\text{Sym}^{-n-1-d}(\mathcal{E}) \otimes_{\mathcal{O}_S} \wedge^{n+1}\mathcal{E}, \mathcal{O}_S) & \text{if} \quad q = n \end{cases}$$

These identifications are compatible with base change and isomorphism between locally free sheaves.

Proof. Consider the canonical map

$$\pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

and twist down by 1 to get

$$\pi^*(\mathcal{E})(-1) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}$$

This is a surjective map from a locally free rank $n+1$ sheaf onto the structure sheaf. Hence the corresponding Koszul complex is exact (More on Algebra, Lemma 15.28.5). In other words there is an exact complex

$$0 \to \pi^*(\wedge^{n+1}\mathcal{E})(-n-1) \to \ldots \to \pi^*(\wedge^i\mathcal{E})(-i) \to \ldots \to \pi^*\mathcal{E}(-1) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})} \to 0$$

We will think of the term $\pi^*(\wedge^i\mathcal{E})(-i)$ as being in degree $-i$. We are going to compute the higher direct images of this acyclic complex using the first spectral sequence of Derived Categories, Lemma 13.21.3. Namely, we see that there is a spectral sequence with terms

$$E_1^{p,q} = R^q\pi_*\left(\pi^*(\wedge^{-p}\mathcal{E})(p)\right)$$

converging to zero! By the projection formula (Cohomology, Lemma 20.54.2) we have

$$E_1^{p,q} = \wedge^{-p}\mathcal{E} \otimes_{\mathcal{O}_S} R^q\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(p)\right).$$

Note that locally on S the sheaf $\mathcal{E}$ is trivial, i.e., isomorphic to $\mathcal{O}_S^{\oplus n+1}$, hence locally on S the morphism $\mathbf{P}(\mathcal{E}) \to S$ can be identified with $\mathbf{P}^n_S \to S$. Hence locally on S we can use the result of Lemmas 30.8.1, 30.8.2, or 30.8.3. It follows that $E_1^{p,q} = 0$ unless (p, q) is $(0, 0)$ or $(-n-1, n)$. The nonzero terms are

$$\begin{aligned} E_1^{0,0} &= \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})} = \mathcal{O}_S \\ E_1^{-n-1,n} &= R^n\pi_*\left(\pi^*(\wedge^{n+1}\mathcal{E})(-n-1)\right) = \wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right) \end{aligned}$$

Hence there can only be one nonzero differential in the spectral sequence namely the map $d_{n+1}^{-n-1,n} : E_{n+1}^{-n-1,n} \to E_{n+1}^{0,0}$ which has to be an isomorphism (because the spectral sequence converges to the 0 sheaf). Thus $E_1^{p,q} = E_{n+1}^{p,q}$ and we obtain a canonical isomorphism

$$\wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right) = R^n\pi_*\left(\pi^*(\wedge^{n+1}\mathcal{E})(-n-1)\right) \xrightarrow{d_{n+1}^{-n-1,n}} \mathcal{O}_S$$

Since $\wedge^{n+1}\mathcal{E}$ is an invertible sheaf, this implies that $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$ is invertible as well and canonically isomorphic to the inverse of $\wedge^{n+1}\mathcal{E}$. In other words we have proved the case $d = -n-1$ of the lemma.

Working locally on S we see immediately from the computation of cohomology in Lemmas 30.8.1, 30.8.2, or 30.8.3 the statements on vanishing of the lemma. Moreover the result on $R^0\pi_*$ is clear as well, since there are canonical maps $\text{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ for all d. It remains to show that the description of $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ is correct for $d < -n-1$. In order to do this we consider the map

$$\pi^*(\text{Sym}^{-d-n-1}(\mathcal{E})) \otimes_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$$

Applying $R^n\pi_*$ and the projection formula (see above) we get a map

$$\text{Sym}^{-d-n-1}(\mathcal{E}) \otimes_{\mathcal{O}_S} R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) \longrightarrow R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1) = (\wedge^{n+1}\mathcal{E})^{\otimes -1}$$

(the last equality we have shown above). Again by the local calculations of Lemmas 30.8.1, 30.8.2, or 30.8.3 it follows that this map induces a perfect pairing between $R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d))$ and $\text{Sym}^{-d-n-1}(\mathcal{E}) \otimes \wedge^{n+1}(\mathcal{E})$ as desired. □

30.9. Coherent sheaves on locally Noetherian schemes

01XY We have defined the notion of a coherent module on any ringed space in Modules, Section 17.12. Although it is possible to consider coherent sheaves on non-Noetherian schemes we will always assume the base scheme is locally Noetherian when we consider coherent sheaves. Here is a characterization of coherent sheaves on locally Noetherian schemes.

01XZ **Lemma 30.9.1.** Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent

1. $\mathcal{F}$ is coherent,
2. $\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module,
3. $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module,
4. for any affine open $\text{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module, and

(5) there exists an affine open covering $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ such that each $\mathcal{F}|_{U_i} = \widetilde{M_i}$ with M_i a finite A_i-module.

In particular $\mathcal{O}_X$ is coherent, any invertible $\mathcal{O}_X$-module is coherent, and more generally any finite locally free $\mathcal{O}_X$-module is coherent.

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) hold in general, see Modules, Lemma 17.12.2. If $\mathcal{F}$ is finitely presented then $\mathcal{F}$ is quasi-coherent, see Modules, Lemma 17.11.2. Hence also (3) $\Rightarrow$ (2).

Assume $\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module. By Properties, Lemma 28.16.1 we see that on any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module. Since A is Noetherian we see that M has a finite resolution

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0.$$

Hence $\mathcal{F}$ is of finite presentation by Properties, Lemma 28.16.2. In other words (2) $\Rightarrow$ (3).

By Modules, Lemma 17.12.5 it suffices to show that $\mathcal{O}_X$ is coherent in order to show that (3) implies (1). Thus we have to show: given any open $U \subset X$ and any finite collection of sections $f_i \in \mathcal{O}_X(U)$, $i = 1, \ldots, n$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$ is of finite type. Since being of finite type is a local property it suffices to check this in a neighbourhood of any $x \in U$. Thus we may assume $U = \mathrm{Spec}(A)$ is affine. In this case $f_1, \ldots, f_n \in A$ are elements of A. Since A is Noetherian, see Properties, Lemma 28.5.2 the kernel K of the map $\bigoplus_{i=1,\ldots,n} A \to A$ is a finite A-module. See for example Algebra, Lemma 10.51.1. As the functor $\widetilde{\ }$ is exact, see Schemes, Lemma 26.5.4 we get an exact sequence

$$\widetilde{K} \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$$

and by Properties, Lemma 28.16.1 again we see that $\widetilde{K}$ is of finite type. We conclude that (1), (2) and (3) are all equivalent.

It follows from Properties, Lemma 28.16.1 that (2) implies (4). It is trivial that (4) implies (5). The discussion in Schemes, Section 26.24 show that (5) implies that $\mathcal{F}$ is quasi-coherent and it is clear that (5) implies that $\mathcal{F}$ is of finite type. Hence (5) implies (2) and we win. $\square$

01Y0 Lemma 30.9.2. Let X be a locally Noetherian scheme. The category of coherent $\mathcal{O}_X$-modules is abelian. More precisely, the kernel and cokernel of a map of coherent $\mathcal{O}_X$-modules are coherent. Any extension of coherent sheaves is coherent.

Proof. This is a restatement of Modules, Lemma 17.12.4 in a particular case. $\square$

The following lemma does not always hold for the category of coherent $\mathcal{O}_X$-modules on a general ringed space X.

01Y1 Lemma 30.9.3. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Any quasi-coherent submodule of $\mathcal{F}$ is coherent. Any quasi-coherent quotient module of $\mathcal{F}$ is coherent.

Proof. We may assume that X is affine, say $X = \mathrm{Spec}(A)$. Properties, Lemma 28.5.2 implies that A is Noetherian. Lemma 30.9.1 turns this into algebra. The algebraic counter part of the lemma is that a quotient, or a submodule of a finite A-module is a finite A-module, see for example Algebra, Lemma 10.51.1. $\square$

01Y2 Lemma 30.9.4. Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.

Proof. It is shown in Modules, Lemma 17.22.6 that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent. The result for tensor products is Modules, Lemma 17.16.6 □

01Y3 Lemma 30.9.5. Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$.

(1) If $\mathcal{F}_x = 0$ then there exists an open neighbourhood $U \subset X$ of x such that $\mathcal{F}|_U = 0$.
(2) If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is injective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is injective.
(3) If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is surjective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is surjective.
(4) If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is bijective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is an isomorphism.

Proof. See Modules, Lemmas 17.9.4, 17.9.5, and 17.12.6. □

01Y4 Lemma 30.9.6. Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $x \in X$. Suppose $\psi : \mathcal{G}_x \to \mathcal{F}_x$ is a map of $\mathcal{O}_{X,x}$-modules. Then there exists an open neighbourhood $U \subset X$ of x and a map $\varphi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi_x = \psi$.

Proof. In view of Lemma 30.9.1 this is a reformulation of Modules, Lemma 17.22.4. □

01Y5 Lemma 30.9.7. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\text{Supp}(\mathcal{F})$ is closed, and $\mathcal{F}$ comes from a coherent sheaf on the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 29.5.5.

Proof. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and let $\mathcal{G}$ be the finite type quasi-coherent sheaf on Z such that $i_*\mathcal{G} \cong \mathcal{F}$. Since $Z = \text{Supp}(\mathcal{F})$ we see that the support is closed. The scheme Z is locally Noetherian by Morphisms, Lemmas 29.15.5 and 29.15.6. Finally, $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module by Lemma 30.9.1 □

087T Lemma 30.9.8. Let $i : Z \to X$ be a closed immersion of locally Noetherian schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor i_* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ and the category of coherent $\mathcal{O}_Z$-modules.

Proof. The functor is fully faithful by Morphisms, Lemma 29.4.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}$. By Morphisms, Lemma 29.4.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. By Modules, Lemma 17.13.3 we see that $\mathcal{G}$ is of finite type. Hence $\mathcal{G}$ is coherent by Lemma 30.9.1. Thus the functor is also essentially surjective as desired. □

01Y6 Lemma 30.9.9. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^pf_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent if $\mathcal{F}$ is coherent.

Proof. The higher direct images vanish by Lemma 30.2.3 and because a finite morphism is affine (by definition). Note that the assumptions imply that also X is locally Noetherian (see Morphisms, Lemma 29.15.6) and hence the statement makes sense. Let $\mathrm{Spec}(A) = V \subset Y$ be an affine open subset. By Morphisms, Definition 29.44.1 we see that $f^{-1}(V) = \mathrm{Spec}(B)$ with $A \to B$ finite. Lemma 30.9.1 turns the statement of the lemma into the following algebra fact: If M is a finite B-module, then M is also finite viewed as a A-module, see Algebra, Lemma 10.7.2. □

In the situation of the lemma also the higher direct images are coherent since they vanish. We will show that this is always the case for a proper morphism between locally Noetherian schemes (Proposition 30.19.1).

0B3J Lemma 30.9.10. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf with $\dim(\mathrm{Supp}(\mathcal{F})) \leq 0$. Then $\mathcal{F}$ is generated by global sections and $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Proof. By Lemma 30.9.7 we see that $\mathcal{F} = i_*\mathcal{G}$ where $i : Z \to X$ is the inclusion of the scheme theoretic support of $\mathcal{F}$ and where $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. Since the dimension of Z is 0, we see Z is a disjoint union of affines (Properties, Lemma 28.10.5). Hence $\mathcal{G}$ is globally generated and the higher cohomology groups of $\mathcal{G}$ are zero (Lemma 30.2.2). Hence $\mathcal{F} = i_*\mathcal{G}$ is globally generated. Since the cohomologies of $\mathcal{F}$ and $\mathcal{G}$ agree (Lemma 30.2.4 applies as a closed immersion is affine) we conclude that the higher cohomology groups of $\mathcal{F}$ are zero. □

0CYJ Lemma 30.9.11. Let X be a scheme. Let $j : U \to X$ be the inclusion of an open. Let $T \subset X$ be a closed subset contained in U. If $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module with $\mathrm{Supp}(\mathcal{F}) \subset T$, then $j_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module.

Proof. Consider the open covering $X = U \cup (X \setminus T)$. Then $j_*\mathcal{F}|_U = \mathcal{F}$ is coherent and $j_*\mathcal{F}|_{X\setminus T} = 0$ is also coherent. Hence $j_*\mathcal{F}$ is coherent. □

30.10. Coherent sheaves on Noetherian schemes

01Y7 In this section we mention some properties of coherent sheaves on Noetherian schemes.

01Y8 Lemma 30.10.1. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, given any sequence

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose a finite affine open covering. On each member of the covering we get stabilization by Algebra, Lemma 10.51.1. Hence the lemma follows. □

01Y9 Lemma 30.10.2. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subscheme $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $\mathrm{Supp}(\mathcal{F}) \subset Z$ (set theoretically).

Proof. This follows immediately from Algebra, Lemma 10.62.4 because X has a finite covering by spectra of Noetherian rings. □

01YA Lemma 30.10.3 (Artin-Rees). Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F} \cap \mathcal{G}$$

Proof. This follows immediately from Algebra, Lemma 10.51.2 because X has a finite covering by spectra of Noetherian rings. □

0GN6 Lemma 30.10.4. Let X be a Noetherian scheme. Every quasi-coherent $\mathcal{O}_X$-module is the filtered colimit of its coherent submodules.

Proof. This is a reformulation of Properties, Lemma 28.22.3 in view of the fact that a finite type quasi-coherent $\mathcal{O}_X$-module is coherent by Lemma 30.9.1. □

01YB Lemma 30.10.5. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subscheme and set $U = X \setminus Z$. There is a canonical isomorphism

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. We first prove the second map is an isomorphism. It is injective by Properties, Lemma 28.25.3. Since $\mathcal{F}$ is the union of its coherent submodules, see Properties, Lemma 28.22.3 (and Lemma 30.9.1) we may and do assume that $\mathcal{F}$ is coherent to prove surjectivity. Let $\mathcal{F}_n$ denote the quasi-coherent subsheaf of $\mathcal{F}$ consisting of sections annihilated by $\mathcal{I}^n$, see Properties, Lemma 28.25.3. Since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ we see that $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$ by Lemma 30.10.1. Set $\mathcal{H} = \mathcal{F}_n$ for this n. By Artin-Rees (Lemma 30.10.3) there exists an $c \geq 0$ such that $\mathcal{I}^m\mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^{m-c}\mathcal{H}$. Picking $m = n + c$ we get $\mathcal{I}^m\mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^n\mathcal{H} = 0$. Thus if we set $\mathcal{F}' = \mathcal{I}^m\mathcal{F}$ then we see that $\mathcal{F}' \cap \mathcal{F}_n = 0$ and $\mathcal{F}'|_U = \mathcal{F}|_U$. Note in particular that the subsheaf $(\mathcal{F}')_N$ of sections annihilated by $\mathcal{I}^N$ is zero for all $N \geq 0$. Hence by Properties, Lemma 28.25.3 we deduce that the top horizontal arrow in the following commutative diagram is a bijection:

$$\begin{tikzcd}
\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}') \arrow[r] \arrow[d] & \Gamma(U, \mathcal{F}') \arrow[d] \\
\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \arrow[r] & \Gamma(U, \mathcal{F})
\end{tikzcd}$$

Since also the right vertical arrow is a bijection we conclude that the bottom horizontal arrow is surjective as desired.

Next, we prove the first arrow of the lemma is a bijection. By Lemma 30.9.1 the sheaf $\mathcal{G}$ is of finite presentation and hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is quasi-coherent, see Schemes, Section 26.24. By definition we have

$$\mathcal{H}(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U)$$

Pick a ψ in the right hand side of the first arrow of the lemma, i.e., $\psi \in \mathcal{H}(U)$. The result just proved applies to $\mathcal{H}$ and hence there exists an $n \geq 0$ and an $\varphi : \mathcal{I}^n \to \mathcal{H}$ which recovers ψ on restriction to U. By Modules, Lemma 17.22.1 φ corresponds to a map

$$\varphi : \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}.$$

This is almost what we want except that the source of the arrow is the tensor product of $\mathcal{I}^n$ and $\mathcal{G}$ and not the product. We will show that, at the cost of increasing n, the difference is irrelevant. Consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{I}^n\mathcal{G} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Note that $\mathcal{I}^n\mathcal{K} = 0$ (proof omitted). By Artin-Rees again we see that

$$\mathcal{K} \cap \mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$$

for some m large enough. In other words we see that

$$\mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) \longrightarrow \mathcal{I}^{n+m}\mathcal{G}$$

is an isomorphism. Let φ' be the restriction of φ to this submodule thought of as a map $\mathcal{I}^{m+n}\mathcal{G} \to \mathcal{F}$. Then φ' gives an element of the left hand side of the first arrow of the lemma which maps to ψ via the arrow. In other words we have proved surjectivity of the arrow. We omit the proof of injectivity. $\square$

0FD0 Lemma 30.10.6. Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $U \subset X$ be open and let $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ be an $\mathcal{O}_U$-module map. Then there exists a coherent submodule $\mathcal{F}' \subset \mathcal{F}$ agreeing with $\mathcal{F}$ over U such that φ extends to $\varphi' : \mathcal{F}' \to \mathcal{G}$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals cutting out the reduced induced scheme structure on $X \setminus U$. If X is Noetherian, then Lemma 30.10.5 tells us that we can take $\mathcal{F}' = \mathcal{I}^n\mathcal{F}$ for some n. The general case will follow from this using Zorn's lemma.

Consider the set of triples $(U', \mathcal{F}', \varphi')$ where $U \subset U' \subset X$ is open, $\mathcal{F}' \subset \mathcal{F}|_{U'}$ is a coherent subsheaf agreeing with $\mathcal{F}$ over U, and $\varphi' : \mathcal{F}' \to \mathcal{G}|_{U'}$ restricts to φ over U. We say $(U'', \mathcal{F}'', \varphi'') \geq (U', \mathcal{F}', \varphi')$ if and only if $U'' \supset U'$, $\mathcal{F}''|_{U'} = \mathcal{F}'$, and $\varphi''|_{U'} = \varphi'$. It is clear that if we have a totally ordered collection of triples $(U_i, \mathcal{F}_i, \varphi_i)$, then we can glue the $\mathcal{F}_i$ to a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ over $U' = \bigcup U_i$ and extend φ to a map $\varphi' : \mathcal{F}' \to \mathcal{G}|_{U'}$. Hence any totally ordered subset of triples has an upper bound. Finally, suppose that $(U', \mathcal{F}', \varphi')$ is any triple but $U' \neq X$. Then we can choose an affine open $W \subset X$ which is not contained in U'. By the result of the first paragraph we can extend the subsheaf $\mathcal{F}'|_{W \cap U'}$ and the restriction $\varphi'|_{W \cap U'}$ to some subsheaf $\mathcal{F}'' \subset \mathcal{F}|_W$ and map $\varphi'' : \mathcal{F}'' \to \mathcal{G}|_W$. Of course the agreement between $(\mathcal{F}', \varphi')$ and $(\mathcal{F}'', \varphi'')$ over $W \cap U'$ exactly means that we can extend this to a triple $(U' \cup W, \mathcal{F}''', \varphi''')$. Hence any maximal triple $(U', \mathcal{F}', \varphi')$ (which exist by Zorn's lemma) must have $U' = X$ and the proof is complete. $\square$

30.11. Depth

0340 In this section we talk a little bit about depth and property (S_k) for coherent modules on locally Noetherian schemes. Note that we have already discussed this notion for locally Noetherian schemes in Properties, Section 28.12.

0341 Definition 30.11.1. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $k \geq 0$ be an integer.

(1) We say $\mathcal{F}$ has depth k at a point x of X if $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = k$.
(2) We say X has depth k at a point x of X if $\text{depth}(\mathcal{O}_{X,x}) = k$.

(3) We say $\mathcal{F}$ has property (S_k) if

$$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \min(k, \dim(\text{Supp}(\mathcal{F}_x)))$$

for all $x \in X$.

(4) We say X has property (S_k) if $\mathcal{O}_X$ has property (S_k).

Any coherent sheaf satisfies condition (S_0). Condition (S_1) is equivalent to having no embedded associated points, see Divisors, Lemma 31.4.3.

0EBC Lemma 30.11.2. Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules and $x \in X$.

(1) If $\mathcal{G}_x$ has depth ≥ 1, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ has depth ≥ 1.
(2) If $\mathcal{G}_x$ has depth ≥ 2, then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ has depth ≥ 2.

Proof. Observe that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{O}_X$-module by Lemma 30.9.4. Coherent modules are of finite presentation (Lemma 30.9.1) hence taking stalks commutes with taking $\mathcal{H}om$ and Hom, see Modules, Lemma 17.22.4. Thus we reduce to the case of finite modules over local rings which is More on Algebra, Lemma 15.23.10. □

0AXQ Lemma 30.11.3. Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules.

(1) If $\mathcal{G}$ has property (S_1), then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has property (S_1).
(2) If $\mathcal{G}$ has property (S_2), then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has property (S_2).

Proof. Follows immediately from Lemma 30.11.2 and the definitions. □

We have seen in Properties, Lemma 28.12.3 that a locally Noetherian scheme is Cohen-Macaulay if and only if (S_k) holds for all k. Thus it makes sense to introduce the following definition, which is equivalent to the condition that all stalks are Cohen-Macaulay modules.

0343 Definition 30.11.4. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is Cohen-Macaulay if and only if (S_k) holds for all $k \geq 0$.

0B3K Lemma 30.11.5. Let X be a regular scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is Cohen-Macaulay and $\text{Supp}(\mathcal{F}) = X$,
(2) $\mathcal{F}$ is finite locally free of rank > 0.

Proof. Let $x \in X$. If (2) holds, then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank > 0. Hence $\text{depth}(\mathcal{F}_x) = \dim(\mathcal{O}_{X,x})$ because a regular local ring is Cohen-Macaulay (Algebra, Lemma 10.106.3). Conversely, if (1) holds, then $\mathcal{F}_x$ is a maximal Cohen-Macaulay module over $\mathcal{O}_{X,x}$ (Algebra, Definition 10.103.8). Hence $\mathcal{F}_x$ is free by Algebra, Lemma 10.106.6. □

30.12. Devissage of coherent sheaves

01YC Let X be a Noetherian scheme. Consider an integral closed subscheme $i : Z \to X$. It is often convenient to consider coherent sheaves of the form $i_*\mathcal{G}$ where $\mathcal{G}$ is a coherent sheaf on Z. In particular we are interested in these sheaves when $\mathcal{G}$ is a torsion free rank 1 sheaf. For example $\mathcal{G}$ could be a nonzero sheaf of ideals on Z, or even more specifically $\mathcal{G} = \mathcal{O}_Z$.

Throughout this section we will use that a coherent sheaf is the same thing as a finite type quasi-coherent sheaf and that a quasi-coherent subquotient of a coherent sheaf is coherent, see Section 30.9. The support of a coherent sheaf is closed, see Modules, Lemma 17.9.6.

01YD Lemma 30.12.1. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\text{Supp}(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$

with $\text{Supp}(\mathcal{G}') \subset Z'$ and $\text{Supp}(\mathcal{G}) \subset Z$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subscheme structure on Z, see Schemes, Lemma 26.12.4. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence

$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$

For every point x of $Z' \setminus Z$ we have $\mathcal{I}_x = \mathcal{O}_{X,x}$ and hence $\mathcal{G}_{n,x} = 0$. Thus we see that $\text{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian scheme. Hence by Lemma 30.10.2 there exists an n such that $\mathcal{G}'_n|_{X \setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X \setminus Z'} = 0$. For such an n we see that $\text{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. $\square$

01YE Lemma 30.12.2. Let X be a Noetherian scheme. Let $i : Z \to X$ be an integral closed subscheme. Let $\xi \in Z$ be the generic point. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\mathcal{F}_\xi$ is annihilated by $\mathfrak{m}_\xi$. Then there exist an integer $r \geq 0$ and a coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ and an injective map of coherent sheaves

$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

which is an isomorphism in a neighbourhood of ξ.

Proof. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of local sections of $\mathcal{F}$ which are annihilated by $\mathcal{J}$. It is a quasi-coherent sheaf by Properties, Lemma 28.24.2. Moreover, $\mathcal{F}'_\xi = \mathcal{F}_\xi$ because $\mathcal{J}_\xi = \mathfrak{m}_\xi$ and part (3) of Properties, Lemma 28.24.2. By Lemma 30.9.5 we see that $\mathcal{F}' \to \mathcal{F}$ induces an isomorphism in a neighbourhood of ξ. Hence we may replace $\mathcal{F}$ by $\mathcal{F}'$ and assume that $\mathcal{F}$ is annihilated by $\mathcal{J}$.

Assume $\mathcal{J}\mathcal{F} = 0$. By Lemma 30.9.8 we can write $\mathcal{F} = i_*\mathcal{G}$ for some coherent sheaf $\mathcal{G}$ on Z. Suppose we can find a morphism $\mathcal{I}^{\oplus r} \to \mathcal{G}$ which is an isomorphism in a neighbourhood of the generic point ξ of Z. Then applying i_* (which is left exact) we get the result of the lemma. Hence we have reduced to the case $X = Z$.

Suppose $Z = X$ is an integral Noetherian scheme with generic point ξ. Note that $\mathcal{O}_{X,\xi} = \kappa(\xi)$ is the function field of X in this case. Since $\mathcal{F}_\xi$ is a finite $\mathcal{O}_\xi$-module we see that $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is finite. Hence the sheaves $\mathcal{O}_X^{\oplus r}$ and $\mathcal{F}$ have isomorphic stalks at ξ. By Lemma 30.9.6 there exists a nonempty open $U \subset X$ and a morphism $\psi : \mathcal{O}_X^{\oplus r}|_U \to \mathcal{F}|_U$ which is an isomorphism at ξ, and hence an isomorphism in a neighbourhood of ξ by Lemma 30.9.5. By Schemes, Lemma 26.12.4 there exists a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose associated closed subscheme $Z \subset X$ is the complement of U. By Lemma 30.10.5 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \to \mathcal{F}$ which recovers our ψ over U. Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective because X is integral and it is injective at the generic point of X (easy proof omitted). □

01YF Lemma 30.12.3. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exist an integral closed subscheme $Z_j \subset X$ and a nonzero coherent sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{matrix} Z \subset X \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \mathrm{Supp}(\mathcal{F}) = Z \text{ for which the lemma is wrong} \end{matrix} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because X is Noetherian we can choose a minimal element $Z \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is Z and for which the lemma does not hold. Clearly $Z \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If Z is not irreducible, then we can write $Z = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than Z. Then we can apply Lemma 30.12.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of Z each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that Z is irreducible.

Suppose Z is irreducible. Let $\mathcal{J}$ be the sheaf of ideals cutting out the reduced induced closed subscheme structure of Z, see Schemes, Lemma 26.12.4. By Lemma 30.10.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{J}^n\mathcal{F} \subset \mathcal{J}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{J}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

In the case where Z is irreducible and $\mathcal{J}\mathcal{F} = 0$ we can apply Lemma 30.12.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined as the quotient. Since $\mathcal{Q}$ is zero in a neighbourhood of ξ by the lemma just cited we see that the support of $\mathcal{Q}$ is strictly smaller than Z. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of Z. But then clearly $\mathcal{F}$ does too, which is our final contradiction. □

01YG Lemma 30.12.4. Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume

(1) For any short exact sequence of coherent sheaves
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
(2) For every integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$
by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 30.12.3 we can filter any $\mathcal{F}$ with successive subquotients as in (2). Hence the lemma follows. □

01YH Lemma 30.12.5. Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X with support contained in Z_0 such that

(1) For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.
(2) For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.
(3) There exists some coherent sheaf $\mathcal{G}$ on X such that
 (a) $\text{Supp}(\mathcal{G}) = Z_0$,
 (b) $\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,
 (c) $\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and
 (d) property $\mathcal{P}$ holds for $\mathcal{G}$.

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with support contained in Z_0 with a filtration
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$
by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. Or, if $\mathcal{F}$ has property $\mathcal{P}$ and all but one of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$ then so does the last one. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 30.12.3) whose subquotients have property $\mathcal{P}$ according to (2).

Let $\mathcal{G}$ be as in (3). By Lemma 30.12.2 there exist a sheaf of ideals $\mathcal{I}$ on Z_0, an integer $r \geq 1$, and a short exact sequence
$$0 \to ((Z_0 \to X)_*\mathcal{I})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$
where the support of $\mathcal{Q}$ is strictly contained in Z_0. By (3)(c) we see that $r = 1$. Since $\mathcal{Q}$ has property $\mathcal{P}$ too we conclude that $(Z_0 \to X)_*\mathcal{I}$ has property $\mathcal{P}$.

Next, suppose that $\mathcal{I}' \neq 0$ is another quasi-coherent sheaf of ideals on Z_0. Then we can consider the intersection $\mathcal{I}'' = \mathcal{I}' \cap \mathcal{I}$ and we get two short exact sequences
$$0 \to (Z_0 \to X)_*\mathcal{I}'' \to (Z_0 \to X)_*\mathcal{I} \to \mathcal{Q} \to 0$$

and

$$0 \to (Z_0 \to X)_*\mathcal{I}'' \to (Z_0 \to X)_*\mathcal{I}' \to \mathcal{Q}' \to 0.$$

Note that the support of the coherent sheaves $\mathcal{Q}$ and $\mathcal{Q}'$ are strictly contained in Z_0. Hence $\mathcal{Q}$ and $\mathcal{Q}'$ have property $\mathcal{P}$ (see above). Hence we conclude using (1) that $(Z_0 \to X)_*\mathcal{I}''$ and $(Z_0 \to X)_*\mathcal{I}'$ both have $\mathcal{P}$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 30.12.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. □

01YI Lemma 30.12.6. Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that

1. For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.
2. For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that
   (a) $\text{Supp}(\mathcal{G}) = Z$,
   (b) $\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,
   (c) $\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and
   (d) property $\mathcal{P}$ holds for $\mathcal{G}$.

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. According to Lemma 30.12.4 it suffices to show that for all integral closed subschemes $Z \subset X$ and all quasi-coherent ideal sheaves $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$. If this fails, then since X is Noetherian there is a minimal integral closed subscheme $Z_0 \subset X$ such that $\mathcal{P}$ fails for $(Z_0 \to X)_*\mathcal{I}_0$ for some quasi-coherent sheaf of ideals $\mathcal{I}_0 \subset \mathcal{O}_{Z_0}$, but $\mathcal{P}$ does hold for $(Z \to X)_*\mathcal{I}$ for all integral closed subschemes $Z \subset Z_0$, $Z \neq Z_0$ and quasi-coherent ideal sheaves $\mathcal{I} \subset \mathcal{O}_Z$. Since we have the existence of $\mathcal{G}$ for Z_0 by part (2), according to Lemma 30.12.5 this cannot happen. □

01YL Lemma 30.12.7. Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that

1. For any short exact sequence of coherent sheaves
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
2. If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.
3. For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.
4. There exists some coherent sheaf $\mathcal{G}$ such that
   (a) $\text{Supp}(\mathcal{G}) = Z_0$,
   (b) $\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and
   (c) for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. Note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 30.12.3) whose subquotients have property $\mathcal{P}$ according to (3).

Let us denote $i : Z_0 \to X$ the closed immersion. Consider a coherent sheaf $\mathcal{G}$ as in (4). By Lemma 30.12.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to i_*\mathcal{I}^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z_0. Let $\mathcal{I}' \subset \mathcal{I}$ be any nonzero quasi-coherent sheaf of ideals on Z_0 contained in $\mathcal{I}$. Then we also get a short exact sequence

$$0 \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{G} \to \mathcal{Q}' \to 0$$

where $\mathcal{Q}'$ has support properly contained in Z_0. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals cutting out the support of $\mathcal{Q}'$ (for example the ideal corresponding to the reduced induced closed subscheme structure on the support of $\mathcal{Q}'$). Then $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. By Lemma 30.10.2 we see that $\mathcal{J}^n\mathcal{Q}' = 0$ for some n. Hence $\mathcal{J}^n\mathcal{G} \subset i_*(\mathcal{I}')^{\oplus r}$. By assumption (4)(c) of the lemma we see there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}^n\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ for which property $\mathcal{P}$ holds. Hence we get a short exact sequence

$$0 \to \mathcal{G}' \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{Q}'' \to 0$$

where $\mathcal{Q}''$ has support properly contained in Z_0. Thus by our initial remarks and property (1) of the lemma we conclude that $i_*(\mathcal{I}')^{\oplus r}$ satisfies $\mathcal{P}$. Hence we see that $i_*\mathcal{I}'$ satisfies $\mathcal{P}$ by (2). Finally, for an arbitrary quasi-coherent sheaf of ideals $\mathcal{I}'' \subset \mathcal{O}_{Z_0}$ we can set $\mathcal{I}' = \mathcal{I}'' \cap \mathcal{I}$ and we get a short exact sequence

$$0 \to i_*(\mathcal{I}') \to i_*(\mathcal{I}'') \to \mathcal{Q}''' \to 0$$

where $\mathcal{Q}'''$ has support properly contained in Z_0. Hence we conclude that property $\mathcal{P}$ holds for $i_*\mathcal{I}''$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 30.12.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. $\square$

01YM Lemma 30.12.8. Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that

(1) For any short exact sequence of coherent sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) If $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r}$ for some $r \geq 1$, then it holds for $\mathcal{F}$.

(3) For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that
   (a) $\text{Supp}(\mathcal{G}) = Z$,
   (b) $\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and

(c) for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. Follows from Lemma 30.12.7 in exactly the same way that Lemma 30.12.6 follows from Lemma 30.12.5. □

30.13. Finite morphisms and affines

01YN In this section we use the results of the preceding sections to show that the image of a Noetherian affine scheme under a finite morphism is affine. We will see later that this result holds more generally (see Limits, Lemma 32.11.1 and Proposition 32.11.2).

01YO Lemma 30.13.1. Let $f : Y \to X$ be a morphism of schemes. Assume f is finite, surjective and X locally Noetherian. Let $Z \subset X$ be an integral closed subscheme with generic point ξ. Then there exists a coherent sheaf $\mathcal{F}$ on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$.

Proof. Note that Y is locally Noetherian by Morphisms, Lemma 29.15.6. Because f is surjective the fibre Y_ξ is not empty. Pick $\xi' \in Y$ mapping to ξ. Let $Z' = \overline{\{\xi'\}}$. We may think of $Z' \subset Y$ as a reduced closed subscheme, see Schemes, Lemma 26.12.4. Hence the sheaf $\mathcal{F} = (Z' \to Y)_*\mathcal{O}_{Z'}$ is a coherent sheaf on Y (see Lemma 30.9.9). Look at the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

We see that $f_*\mathcal{F} = i_*f'_*\mathcal{O}_{Z'}$. Hence the stalk of $f_*\mathcal{F}$ at ξ is the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ. Note that since Z' is integral with generic point ξ' we have that ξ' is the only point of Z' lying over ξ, see Algebra, Lemmas 10.36.3 and 10.36.20. Hence the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ equal $\mathcal{O}_{Z',\xi'} = \kappa(\xi')$. In particular the stalk of $f_*\mathcal{F}$ at ξ is not zero. This combined with the fact that $f_*\mathcal{F}$ is of the form $i_*f'_*(\text{something})$ implies the lemma. □

01YP Lemma 30.13.2. Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. If the morphism f is affine then $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$.

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ this acts on $\mathcal{F}$. Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of $\mathcal{F}$. It is a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{F}$ because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is local and hence we may assume X is affine. Since f is affine we see that Y is affine too. Thus we may write $Y = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(A)$, $\mathcal{F} = \widetilde{M}$, and $\mathcal{I} = \widetilde{I}$. The assertion of the lemma in this case boils down to the statement that

$$I(M_A) = ((IB)M)_A$$

where M_A indicates the A-module associated to the B-module M. □

01YQ Lemma 30.13.3. Let $f : Y \to X$ be a morphism of schemes. Assume

(1) f finite,
(2) f surjective,
(3) Y affine, and
(4) X Noetherian.

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This will in particular imply that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Then it follows that X is affine from either Lemma 30.3.1 or Lemma 30.3.2.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 30.12.8. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ. Let $\mathcal{F}$ be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$, see Lemma 30.13.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 30.12.8. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 30.13.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 30.13.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Since Y is affine we see that $H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$, see Lemma 30.2.2. Since f is finite, hence affine, we see that

$$H^1(X, \mathcal{J}\mathcal{G}) = H^1(X, f_*(f^{-1}\mathcal{J}\mathcal{F})) = H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$$

by Lemma 30.2.4. Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies $\mathcal{P}$. This verifies property (3)(c) of Lemma 30.12.8 as desired. □

30.14. Coherent sheaves on Proj, I

01YR In this section we discuss coherent sheaves on $\text{Proj}(A)$ where A is a Noetherian graded ring generated by A_1 over A_0. In the next section we discuss what happens if A is not generated by degree 1 elements. First, we formulate an all-in-one result for projective space over a Noetherian ring.

01YS Lemma 30.14.1. Let R be a Noetherian ring. Let $n \geq 0$ be an integer. For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have the following:

(1) There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection
$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \longrightarrow \mathcal{F}.$$
(2) We have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ unless $0 \leq i \leq n$.
(3) For any i the cohomology group $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module.
(4) If $i > 0$, then $H^i(\mathbf{P}^n_R, \mathcal{F}(d)) = 0$ for all d large enough.

(5) For any $k \in \mathbf{Z}$ the graded $R[T_0, \ldots, T_n]$-module
$$\bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$
is a finite $R[T_0, \ldots, T_n]$-module.

Proof. We will use that $\mathcal{O}_{\mathbf{P}^n_R}(1)$ is an ample invertible sheaf on the scheme $\mathbf{P}^n_R$. This follows directly from the definition since $\mathbf{P}^n_R$ covered by the standard affine opens $D_+(T_i)$. Hence by Properties, Proposition 28.26.13 every finite type quasi-coherent $\mathcal{O}_{\mathbf{P}^n_R}$-module is a quotient of a finite direct sum of tensor powers of $\mathcal{O}_{\mathbf{P}^n_R}(1)$. On the other hand coherent sheaves and finite type quasi-coherent sheaves are the same thing on projective space over R by Lemma 30.9.1. Thus we see (1).

Projective n-space $\mathbf{P}^n_R$ is covered by $n+1$ affines, namely the standard opens $D_+(T_i)$, $i = 0, \ldots, n$, see Constructions, Lemma 27.13.3. Hence we see that for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ for $i \geq n+1$, see Lemma 30.4.2. Hence (2) holds.

Let us prove (3) and (4) simultaneously for all coherent sheaves on $\mathbf{P}^n_R$ by descending induction on i. Clearly the result holds for $i \geq n+1$ by (2). Suppose we know the result for $i+1$ and we want to show the result for i. (If $i = 0$, then part (4) is vacuous.) Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_R$. Choose a surjection as in (1) and denote $\mathcal{G}$ the kernel so that we have a short exact sequence
$$0 \to \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \to \mathcal{F} \to 0$$
By Lemma 30.9.2 we see that $\mathcal{G}$ is coherent. The long exact cohomology sequence gives an exact sequence
$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j)) \to H^i(\mathbf{P}^n_R, \mathcal{F}) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}).$$
By induction assumption the right R-module is finite and by Lemma 30.8.1 the left R-module is finite. Since R is Noetherian it follows immediately that $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module. This proves the induction step for assertion (3). Since $\mathcal{O}_{\mathbf{P}^n_R}(d)$ is invertible we see that twisting on $\mathbf{P}^n_R$ is an exact functor (since you get it by tensoring with an invertible sheaf, see Constructions, Definition 27.10.1). This means that for all $d \in \mathbf{Z}$ the sequence
$$0 \to \mathcal{G}(d) \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d) \to \mathcal{F}(d) \to 0$$
is short exact. The resulting cohomology sequence is
$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^i(\mathbf{P}^n_R, \mathcal{F}(d)) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}(d)).$$
By induction assumption we see the module on the right is zero for $d \gg 0$ and by the computation in Lemma 30.8.1 the module on the left is zero as soon as $d \geq -\min\{d_j\}$ and $i \geq 1$. Hence the induction step for assertion (4). This concludes the proof of (3) and (4).

In order to prove (5) note that for all sufficiently large d the map
$$H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$
is surjective by the vanishing of $H^1(\mathbf{P}^n_R, \mathcal{G}(d))$ we just proved. In other words, the module
$$M_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is for k large enough a quotient of the corresponding module

$$N_k = \bigoplus_{d \geq k} H^0(\mathbf{P}^n_R, \bigoplus_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d))$$

When k is sufficiently small (e.g. $k < -d_j$ for all j) then

$$N_k = \bigoplus_{j=1,\ldots,r} R[T_0, \ldots, T_n](d_j)$$

by our computations in Section 30.8. In particular it is finitely generated. Suppose $k \in \mathbf{Z}$ is arbitrary. Choose $k_- \ll k \ll k_+$. Consider the diagram

$$\begin{tikzcd} N_{k_-} & N_{k_+} \arrow[l] \arrow[d] \\ M_k & M_{k_+} \arrow[l] \end{tikzcd}$$

where the vertical arrow is the surjective map above and the horizontal arrows are the obvious inclusion maps. By what was said above we see that N_{k_-} is a finitely generated $R[T_0, \ldots, T_n]$-module. Hence N_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module because it is a submodule of a finitely generated module and the ring $R[T_0, \ldots, T_n]$ is Noetherian. Since the vertical arrow is surjective we conclude that M_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module. The quotient M_k/M_{k_+} is finite as an R-module since it is a finite direct sum of the finite R-modules $H^0(\mathbf{P}^n_R, \mathcal{F}(d))$ for $k \leq d < k_+$. Note that we use part (3) for $i = 0$ here. Hence M_k/M_{k_+} is a fortiori a finite $R[T_0, \ldots, T_n]$-module. In other words, we have sandwiched M_k between two finite $R[T_0, \ldots, T_n]$-modules and we win. □

0AG6 Lemma 30.14.2. Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Set $X = \text{Proj}(A)$. Then X is a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection

$$\bigoplus_{j=1,\ldots,r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}.$$

(2) For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite A_0-module.
(3) If $p > 0$, then $H^p(X, \mathcal{F}(d)) = 0$ for all d large enough.
(4) For any $k \in \mathbf{Z}$ the graded A-module

$$\bigoplus_{d \geq k} H^0(X, \mathcal{F}(d))$$

is a finite A-module.

Proof. By assumption there exists a surjection of graded A_0-algebras

$$A_0[T_0, \ldots, T_n] \longrightarrow A$$

where $\deg(T_j) = 1$ for $j = 0, \ldots, n$. By Constructions, Lemma 27.11.5 this defines a closed immersion $i : X \to \mathbf{P}^n_{A_0}$ such that $i^*\mathcal{O}_{\mathbf{P}^n_{A_0}}(1) = \mathcal{O}_X(1)$. In particular, X is Noetherian as a closed subscheme of the Noetherian scheme $\mathbf{P}^n_{A_0}$. We claim that the results of the lemma for $\mathcal{F}$ follow from the corresponding results of Lemma 30.14.1 for the coherent sheaf $i_*\mathcal{F}$ (Lemma 30.9.8) on $\mathbf{P}^n_{A_0}$. For example, by this lemma there exists a surjection

$$\bigoplus_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_{A_0}}(d_j) \longrightarrow i_*\mathcal{F}.$$

By adjunction this corresponds to a map $\bigoplus_{j=1,\ldots,r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}$ which is surjective as well. The statements on cohomology follow from the fact that $H^p(X, \mathcal{F}(d)) = H^p(\mathbf{P}^n_{A_0}, i_*\mathcal{F}(d))$ by Lemma 30.2.4. $\square$

0AG7 Lemma 30.14.3. Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Let M be a finite graded A-module. Set $X = \text{Proj}(A)$ and let $\widetilde{M}$ be the quasi-coherent $\mathcal{O}_X$-module on X associated to M. The maps

$$M_n \longrightarrow \Gamma(X, \widetilde{M}(n))$$

from Constructions, Lemma 27.10.3 are isomorphisms for all sufficiently large n.

Proof. Because M is a finite A-module we see that $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module, i.e., a coherent $\mathcal{O}_X$-module. Set $N = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M}(n))$. We have to show that the map $M \to N$ of graded A-modules is an isomorphism in all sufficiently large degrees. By Properties, Lemma 28.28.5 we have a canonical isomorphism $\widetilde{N} \to \widetilde{M}$ such that the induced maps $N_n \to N_n = \Gamma(X, \widetilde{M}(n))$ are the identity maps. Thus we have maps $\widetilde{M} \to \widetilde{N} \to \widetilde{M}$ such that for all n the diagram

$$\begin{tikzcd} M_n \arrow[r] \arrow[d] & N_n \arrow[d] \arrow[dr, equal] & \\ \Gamma(X, \widetilde{M}(n)) \arrow[r] & \Gamma(X, \widetilde{N}(n)) \arrow[r, "\cong"] & \Gamma(X, \widetilde{M}(n)) \end{tikzcd}$$

is commutative. This means that the composition

$$M_n \to \Gamma(X, \widetilde{M}(n)) \to \Gamma(X, \widetilde{N}(n)) \to \Gamma(X, \widetilde{M}(n))$$

is equal to the canonical map $M_n \to \Gamma(X, \widetilde{M}(n))$. Clearly this implies that the composition $\widetilde{M} \to \widetilde{N} \to \widetilde{M}$ is the identity. Hence $\widetilde{M} \to \widetilde{N}$ is an isomorphism. Let $K = \text{Ker}(M \to N)$ and $Q = \text{Coker}(M \to N)$. Recall that the functor $M \mapsto \widetilde{M}$ is exact, see Constructions, Lemma 27.8.4. Hence we see that $\widetilde{K} = 0$ and $\widetilde{Q} = 0$. Recall that A is a Noetherian ring, M is a finitely generated A-module, and N is a graded A-module such that $N' = \bigoplus_{n \geq 0} N_n$ is finitely generated by the last part of Lemma 30.14.2. Hence $K' = \bigoplus_{n \geq 0} K_n$ and $Q' = \bigoplus_{n \geq 0} Q_n$ are finite A-modules. Observe that $\widetilde{Q} = \widetilde{Q'}$ and $\widetilde{K} = \widetilde{K'}$. Thus to finish the proof it suffices to show that a finite A-module K with $\widetilde{K} = 0$ has only finitely many nonzero homogeneous parts K_d with $d \geq 0$. To do this, let $x_1, \ldots, x_r \in K$ be homogeneous generators say sitting in degrees $d_1, \ldots, d_r$. Let $f_1, \ldots, f_n \in A_1$ be elements generating A over A_0. For each i and j there exists an $n_{ij} \geq 0$ such that $f_i^{n_{ij}} x_j = 0$ in $K_{d_j + n_{ij}}$: if not then $x_i/f_i^{d_i} \in K_{(f_i)}$ would not be zero, i.e., $\widetilde{K}$ would not be zero. Then we see that K_d is zero for $d > \max_j(d_j + \sum_i n_{ij})$ as every element of K_d is a sum of terms where each term is a monomials in the f_i times one of the x_j of total degree d. $\square$

Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Recall that $A_+ = \bigoplus_{n>0} A_n$ is the irrelevant ideal. Let M be a graded A-module. Recall that M is an A_+-power torsion module if for all $x \in M$ there is an $n \geq 1$ such that $(A_+)^n x = 0$, see More on Algebra, Definition 15.88.1. If M is finitely generated, then we see that this is equivalent to $M_n = 0$ for $n \gg 0$. Sometimes A_+-power torsion modules are called torsion modules. Sometimes a graded A-module M is called torsion free if $x \in M$ with $(A_+)^n x = 0$, $n > 0$ implies $x = 0$. Denote Mod_A the category of graded A-modules, Mod_A^{fg} the full subcategory of finitely generated ones, and $\text{Mod}_{A,torsion}^{fg}$ the full subcategory of modules M such that $M_n = 0$ for $n \gg 0$.

0BXD Proposition 30.14.4. Let A be a graded ring such that A_0 is Noetherian and A is generated by finitely many elements of A_1 over A_0. Set $X = \text{Proj}(A)$. The functor $M \mapsto \widetilde{M}$ induces an equivalence

$$\text{Mod}_A^{fg}/\text{Mod}_{A,torsion}^{fg} \longrightarrow \text{Coh}(\mathcal{O}_X)$$

whose quasi-inverse is given by $\mathcal{F} \longmapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$.

Proof. The subcategory $\text{Mod}_{A,torsion}^{fg}$ is a Serre subcategory of Mod_A^{fg}, see Homology, Definition 12.10.1. This is clear from the description of objects given above but it also follows from More on Algebra, Lemma 15.88.5. Hence the quotient category on the left of the arrow is defined in Homology, Lemma 12.10.6. To define the functor of the proposition, it suffices to show that the functor $M \mapsto \widetilde{M}$ sends torsion modules to 0. This is clear because for any $f \in A_+$ homogeneous the module M_f is zero and hence the value $M_{(f)}$ of $\widetilde{M}$ on $D_+(f)$ is zero too.

By Lemma 30.14.2 the proposed quasi-inverse makes sense. Namely, the lemma shows that $\mathcal{F} \longmapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$ is a functor $\text{Coh}(\mathcal{O}_X) \to \text{Mod}_A^{fg}$ which we can compose with the quotient functor $\text{Mod}_A^{fg} \to \text{Mod}_A^{fg}/\text{Mod}_{A,torsion}^{fg}$.

By Lemma 30.14.3 the composite left to right to left is isomorphic to the identity functor.

Finally, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ viewed as a graded A-module, so that our functor sends $\mathcal{F}$ to $M_{\geq 0} = \bigoplus_{n \geq 0} M_n$. By Properties, Lemma 28.28.5 the canonical map $\widetilde{M} \to \mathcal{F}$ is an isomorphism. Since the inclusion map $M_{\geq 0} \to M$ defines an isomorphism $\widetilde{M_{\geq 0}} \to \widetilde{M}$ we conclude that the composite right to left to right is isomorphic to the identity functor as well. □

30.15. Coherent sheaves on Proj, II

0BXE In this section we discuss coherent sheaves on $\text{Proj}(A)$ where A is a Noetherian graded ring. Most of the results will be deduced by sleight of hand from the corresponding result in the previous section where we discussed what happens if A is generated by degree 1 elements.

0B5Q Lemma 30.15.1. Let A be a Noetherian graded ring. Set $X = \text{Proj}(A)$. Then X is a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection
$$\bigoplus_{j=1,\ldots,r} \mathcal{O}_X(d_j) \longrightarrow \mathcal{F}.$$
(2) For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite A_0-module.
(3) If $p > 0$, then $H^p(X, \mathcal{F}(d)) = 0$ for all d large enough.
(4) For any $k \in \mathbf{Z}$ the graded A-module
$$\bigoplus_{d \geq k} H^0(X, \mathcal{F}(d))$$
is a finite A-module.

Proof. We will prove this by reducing the statement to Lemma 30.14.2. By Algebra, Lemmas 10.58.2 and 10.58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements $f_1, \ldots, f_r$ homogeneous of positive degree. Let d be a sufficiently divisible integer. Set $A' = A^{(d)}$ with notation as in Algebra, Section 10.56. Then A' is generated over $A'_0 = A_0$ by elements of degree 1, see Algebra, Lemma 10.56.2. Thus Lemma 30.14.2 applies to $X' = \text{Proj}(A')$.

By Constructions, Lemma 27.11.8 there exist an isomorphism of schemes $i : X \to X'$ and isomorphisms $\mathcal{O}_X(nd) \to i^*\mathcal{O}_{X'}(n)$ compatible with the map $A' \to A$ and the maps $A_n \to H^0(X, \mathcal{O}_X(n))$ and $A'_n \to H^0(X', \mathcal{O}_{X'}(n))$. Thus Lemma 30.14.2 implies X is Noetherian and that (1) and (2) hold. To see (3) and (4) we can use that for any fixed k, p, and q we have

$$\bigoplus\nolimits_{dn+q \geq k} H^p(X, \mathcal{F}(dn + q)) = \bigoplus\nolimits_{dn+q \geq k} H^p(X', (i_*\mathcal{F}(q))(n)$$

by the compatibilities above. If $p > 0$, we have the vanishing of the right hand side for k depending on q large enough by Lemma 30.14.2. Since there are only a finite number of congruence classes of integers modulo d, we see that (3) holds for $\mathcal{F}$ on X. If $p = 0$, then we have that the right hand side is a finite A'-module by Lemma 30.14.2. Using the finiteness of congruence classes once more, we find that $\bigoplus_{n \geq k} H^0(X, \mathcal{F}(n))$ is a finite A'-module too. Since the A'-module structure comes from the A-module structure (by the compatibilities mentioned above), we conclude it is finite as an A-module as well. $\square$

0B5R Lemma 30.15.2. Let A be a Noetherian graded ring and let d be the lcm of generators of A over A_0. Let M be a finite graded A-module. Set $X = \text{Proj}(A)$ and let $\widetilde{M}$ be the quasi-coherent $\mathcal{O}_X$-module on X associated to M. Let $k \in \mathbf{Z}$.

(1) $N' = \bigoplus_{n \geq k} H^0(X, \widetilde{M(n)})$ is a finite A-module,
(2) $N = \bigoplus_{n \geq k} H^0(X, \widetilde{M}(n))$ is a finite A-module,
(3) there is a canonical map $N \to N'$,
(4) if k is small enough there is a canonical map $M \to N'$,
(5) the map $M_n \to N'_n$ is an isomorphism for $n \gg 0$,
(6) $N_n \to N'_n$ is an isomorphism for $d|n$.

Proof. The map $N \to N'$ in (3) comes from Constructions, Equation (27.10.1.5) by taking global sections.

By Constructions, Equation (27.10.1.6) there is a map of graded A-modules $M \to \bigoplus_{n \in \mathbf{Z}} H^0(X, \widetilde{M(n)})$. If the generators of M sit in degrees $\geq k$, then the image is contained in the submodule $N' \subset \bigoplus_{n \in \mathbf{Z}} H^0(X, \widetilde{M(n)})$ and we get the map in (4).

By Algebra, Lemmas 10.58.2 and 10.58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements $f_1, \ldots, f_r$ homogeneous of positive degree. Let $d = \text{lcm}(\deg(f_i))$. Then we see that (6) holds for example by Constructions, Lemma 27.10.4.

Because M is a finite A-module we see that $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module, i.e., a coherent $\mathcal{O}_X$-module. Thus part (2) follows from Lemma 30.15.1.

We will deduce (1) from (2) using a trick. For $q \in \{0, \ldots, d - 1\}$ write

$${}^qN = \bigoplus\nolimits_{n+q \geq k} H^0(X, \widetilde{M(q)}(n))$$

By part (2) these are finite A-modules. The Noetherian ring A is finite over $A^{(d)} = \bigoplus_{n \geq 0} A_{dn}$, because it is generated by f_i over $A^{(d)}$ and $f_i^d \in A^{(d)}$. Hence ${}^qN$ is a finite $A^{(d)}$-module. Moreover, $A^{(d)}$ is Noetherian (follows from Algebra, Lemma 10.57.9). It follows that the $A^{(d)}$-submodule ${}^qN^{(d)} = \bigoplus_{n \in \mathbf{Z}} {}^qN_{dn}$ is a finite module over $A^{(d)}$. Using the isomorphisms $\widetilde{M(dn+q)} = \widetilde{M(q)}(dn)$ we can write

$$N' = \bigoplus\nolimits_{q \in \{0, \ldots, d-1\}} \bigoplus\nolimits_{dn+q \geq k} H^0(X, \widetilde{M(q)}(dn)) = \bigoplus\nolimits_{q \in \{0, \ldots, d-1\}} {}^qN^{(d)}$$

Thus N' is finite over $A^{(d)}$ and a fortiori finite over A. Thus (1) is true.

Let K be a finite A-module such that $\widetilde{K} = 0$. We claim that $K_n = 0$ for $d|n$ and $n \gg 0$. Arguing as above we see that $K^{(d)}$ is a finite $A^{(d)}$-module. Let $x_1, \ldots, x_m \in K$ be homogeneous generators of $K^{(d)}$ over $A^{(d)}$, say sitting in degrees $d_1, \ldots, d_m$ with $d|d_j$. For each i and j there exists an $n_{ij} \geq 0$ such that $f_i^{n_{ij}} x_j = 0$ in $K_{d_j + n_{ij}}$: if not then $x_j / f_i^{d_i/\deg(f_i)} \in K_{(f_i)}$ would not be zero, i.e., $\widetilde{K}$ would not be zero. Here we use that $\deg(f_i)|d|d_j$ for all i, j. We conclude that K_n is zero for n with $d|n$ and $n > \max_j(d_j + \sum_i n_{ij} \deg(f_i))$ as every element of K_n is a sum of terms where each term is a monomials in the f_i times one of the x_j of total degree n.

To finish the proof, we have to show that $M \to N'$ is an isomorphism in all sufficiently large degrees. The map $N \to N'$ induces an isomorphism $\widetilde{N} \to \widetilde{N'}$ because on the affine opens $D_+(f_i) = D_+(f_i^d)$ the corresponding modules are isomorphic: $N_{(f_i)} \cong N_{(f_i^d)} \cong N'_{(f_i^d)} \cong N'_{(f_i)}$ by property (6). By Properties, Lemma 28.28.5 we have a canonical isomorphism $\widetilde{N} \to \widetilde{M}$. The composition $\widetilde{N} \to \widetilde{M} \to \widetilde{N'}$ is the isomorphism above (proof omitted; hint: look on standard affine opens to check this). Thus the map $M \to N'$ induces an isomorphism $\widetilde{M} \to \widetilde{N'}$. Let $K = \mathrm{Ker}(M \to N')$ and $Q = \mathrm{Coker}(M \to N')$. Recall that the functor $M \mapsto \widetilde{M}$ is exact, see Constructions, Lemma 27.8.4. Hence we see that $\widetilde{K} = 0$ and $\widetilde{Q} = 0$. By the result of the previous paragraph we see that $K_n = 0$ and $Q_n = 0$ for $d|n$ and $n \gg 0$. At this point we finally see the advantage of using N' over N: the functor $M \rightsquigarrow N'$ is compatible with shifts (immediate from the construction). Thus, repeating the whole argument with M replaced by $M(q)$ we find that $K_n = 0$ and $Q_n = 0$ for $n \equiv q \bmod d$ and $n \gg 0$. Since there are only finitely many congruence classes modulo n the proof is finished. $\square$

Let A be a Noetherian graded ring. Recall that $A_+ = \bigoplus_{n > 0} A_n$ is the irrelevant ideal. By Algebra, Lemmas 10.58.2 and 10.58.1 the ring A_0 is Noetherian and A is generated over A_0 by finitely many elements $f_1, \ldots, f_r$ homogeneous of positive degree. Let $d = \mathrm{lcm}(\deg(f_i))$. Let M be a graded A-module. In this situation we say a homogeneous element $x \in M$ is irrelevant[2] if

$$(A_+ x)_{nd} = 0 \text{ for all } n \gg 0$$

If $x \in M$ is homogeneous and irrelevant and $f \in A$ is homogeneous, then fx is irrelevant too. Hence the set of irrelevant elements generate a graded submodule $M_{irrelevant} \subset M$. We will say M is irrelevant if every homogeneous element of M is irrelevant, i.e., if $M_{irrelevant} = M$. If M is finitely generated, then we see that

[2] This is nonstandard notation.

this is equivalent to $M_{nd} = 0$ for $n \gg 0$. Denote Mod_A the category of graded A-modules, Mod_A^{fg} the full subcategory of finitely generated ones, and $\text{Mod}_{A,irrelevant}^{fg}$ the full subcategory of irrelevant modules.

0BXF Proposition 30.15.3. Let A be a Noetherian graded ring. Set $X = \text{Proj}(A)$. The functor $M \mapsto \widetilde{M}$ induces an equivalence

$$\text{Mod}_A^{fg}/\text{Mod}_{A,irrelevant}^{fg} \longrightarrow \text{Coh}(\mathcal{O}_X)$$

whose quasi-inverse is given by $\mathcal{F} \longmapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$.

Proof. We urge the reader to read the proof in the case where A is generated in degree 1 first, see Proposition 30.14.4. Let $f_1, \ldots, f_r \in A$ be homogeneous elements of positive degree which generate A over A_0. Let d be the lcm of the degrees d_i of f_i. Let M be a finite A-module. Let us show that $\widetilde{M}$ is zero if and only if M is an irrelevant graded A-module (as defined above the statement of the proposition). Namely, let $x \in M$ be a homogeneous element. Choose $k \in \mathbf{Z}$ sufficiently small and let $N \to N'$ and $M \to N'$ be as in Lemma 30.15.2. We may also pick l sufficiently large such that $M_n \to N_n$ is an isomorphism for $n \geq l$. If $\widetilde{M}$ is zero, then $N = 0$. Thus for any $f \in A_+$ homogeneous with $\deg(f) + \deg(x) = nd$ and $nd > l$ we see that fx is zero because $N_{nd} \to N'_{nd}$ and $M_{nd} \to N'_{nd}$ are isomorphisms. Hence x is irrelevant. Conversely, assume M is irrelevant. Then M_{nd} is zero for $n \gg 0$ (see discussion above proposition). Clearly this implies that $M_{(f_i)} = M_{(f_i^{d/\deg(f_i)})} = 0$, whence $\widetilde{M} = 0$ by construction.

It follows that the subcategory $\text{Mod}_{A,irrelevant}^{fg}$ is a Serre subcategory of Mod_A^{fg} as the kernel of the exact functor $M \mapsto \widetilde{M}$, see Homology, Lemma 12.10.4 and Constructions, Lemma 27.8.4. Hence the quotient category on the left of the arrow is defined in Homology, Lemma 12.10.6. To define the functor of the proposition, it suffices to show that the functor $M \mapsto \widetilde{M}$ sends irrelevant modules to 0 which we have shown above.

By Lemma 30.15.1 the proposed quasi-inverse makes sense. Namely, the lemma shows that $\mathcal{F} \longmapsto \bigoplus_{n \geq 0} \Gamma(X, \mathcal{F}(n))$ is a functor $\text{Coh}(\mathcal{O}_X) \to \text{Mod}_A^{fg}$ which we can compose with the quotient functor $\text{Mod}_A^{fg} \to \text{Mod}_A^{fg}/\text{Mod}_{A,irrelevant}^{fg}$.

By Lemma 30.15.2 the composite left to right to left is isomorphic to the identity functor. Namely, let M be a finite graded A-module and let $k \in \mathbf{Z}$ sufficiently small and let $N \to N'$ and $M \to N'$ be as in Lemma 30.15.2. Then the kernel and cokernel of $M \to N'$ are nonzero in only finitely many degrees, hence are irrelevant. Moreover, the kernel and cokernel of the map $N \to N'$ are zero in all sufficiently large degrees divisible by d, hence these are irrelevant modules too. Thus $M \to N'$ and $N \to N'$ are both isomorphisms in the quotient category, as desired.

Finally, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ viewed as a graded A-module, so that our functor sends $\mathcal{F}$ to $M_{\geq 0} = \bigoplus_{n \geq 0} M_n$. By Properties, Lemma 28.28.5 the canonical map $\widetilde{M} \to \mathcal{F}$ is an isomorphism. Since the inclusion map $M_{\geq 0} \to M$ defines an isomorphism $\widetilde{M_{\geq 0}} \to \widetilde{M}$ we conclude that the composite right to left to right is isomorphic to the identity functor as well. $\square$

30.16. Higher direct images along projective morphisms

0B5S We first state and prove a result for when the base is affine and then we deduce some results for projective morphisms.

0B5T **Lemma 30.16.1.** Let R be a Noetherian ring. Let $X \to \mathrm{Spec}(R)$ be a proper morphism. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) The graded ring $A = \bigoplus_{d \geq 0} H^0(X, \mathcal{L}^{\otimes d})$ is a finitely generated R-algebra.
(2) There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection
$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{L}^{\otimes d_j} \longrightarrow \mathcal{F}.$$
(3) For any p the cohomology group $H^p(X, \mathcal{F})$ is a finite R-module.
(4) If $p > 0$, then $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all d large enough.
(5) For any $k \in \mathbf{Z}$ the graded A-module
$$\bigoplus\nolimits_{d \geq k} H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$$
is a finite A-module.

Proof. By Morphisms, Lemma 29.39.4 there exists a $d > 0$ and an immersion $i : X \to \mathbf{P}^n_R$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_R}(1)$. Since X is proper over R the morphism i is a closed immersion (Morphisms, Lemma 29.41.7). Thus we have $H^i(X, \mathcal{G}) = H^i(\mathbf{P}^n_R, i_*\mathcal{G})$ for any quasi-coherent sheaf $\mathcal{G}$ on X (by Lemma 30.2.4 and the fact that closed immersions are affine, see Morphisms, Lemma 29.11.9). Moreover, if $\mathcal{G}$ is coherent, then $i_*\mathcal{G}$ is coherent as well (Lemma 30.9.8). We will use these facts without further mention.

Proof of (1). Set $S = R[T_0, \ldots, T_n]$ so that $\mathbf{P}^n_R = \mathrm{Proj}(S)$. Observe that A is an S-algebra (but the ring map $S \to A$ is not a homomorphism of graded rings because S_n maps into A_{dn}). By the projection formula (Cohomology, Lemma 20.54.2) we have
$$i_*(\mathcal{L}^{\otimes nd+q}) = i_*(\mathcal{L}^{\otimes q}) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(n)$$
for all $n \in \mathbf{Z}$. We conclude that $\bigoplus_{n \geq 0} A_{nd+q}$ is a finite graded S-module by Lemma 30.14.1. Since $A = \bigoplus_{q \in \{0,\ldots,d-1} \bigoplus_{n \geq 0} A_{nd+q}$ we see that A is finite as an S-algebra, hence (1) is true.

Proof of (2). This follows from Properties, Proposition 28.26.13.

Proof of (3). Apply Lemma 30.14.1 and use $H^p(X, \mathcal{F}) = H^p(\mathbf{P}^n_R, i_*\mathcal{F})$.

Proof of (4). Fix $p > 0$. By the projection formula we have
$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes nd+q}) = i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes q}) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(n)$$
for all $n \in \mathbf{Z}$. By Lemma 30.14.1 we conclude that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{nd+q}) = 0$ for $n \gg 0$. Since there are only finitely many congruence classes of integers modulo d this proves (4).

Proof of (5). Fix an integer k. Set $M = \bigoplus_{n \geq k} H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. Arguing as above we conclude that $\bigoplus_{nd+q \geq k} M_{nd+q}$ is a finite graded S-module. Since $M = \bigoplus_{q \in \{0,\ldots,d-1\}} \bigoplus_{nd+q \geq k} M_{nd+q}$ we see that M is finite as an S-module. Since the S-module structure factors through the ring map $S \to A$, we conclude that M is finite as an A-module. $\square$

02O1 Lemma 30.16.2. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that

(1) S is Noetherian,
(2) f is proper,
(3) $\mathcal{F}$ is coherent, and
(4) $\mathcal{L}$ is relatively ample on X/S.

Then there exists an n_0 such that for all $n \geq n_0$ we have

$$R^p f_* \left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}\right) = 0$$

for all $p > 0$.

Proof. Choose a finite affine open covering $S = \bigcup V_j$ and set $X_j = f^{-1}(V_j)$. Clearly, if we solve the question for each of the finitely many systems $(X_j \to V_j, \mathcal{L}|_{X_j}, \mathcal{F}|_{V_j})$ then the result follows. Thus we may assume S is affine. In this case the vanishing of $R^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is equivalent to the vanishing of $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$, see Lemma 30.4.6. Thus the required vanishing follows from Lemma 30.16.1 (which applies because $\mathcal{L}$ is ample on X by Morphisms, Lemma 29.39.4). $\square$

02O4 Lemma 30.16.3. Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally projective morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_* \mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.

Proof. We first remark that a locally projective morphism is proper (Morphisms, Lemma 29.43.5) and hence of finite type. In particular X is locally Noetherian (Morphisms, Lemma 29.15.6) and hence the statement makes sense. Moreover, by Lemma 30.4.5 the sheaves $R^p f_* \mathcal{F}$ are quasi-coherent.

Having said this the statement is local on S (for example by Cohomology, Lemma 20.7.4). Hence we may assume $S = \mathrm{Spec}(R)$ is the spectrum of a Noetherian ring, and X is a closed subscheme of $\mathbf{P}^n_R$ for some n, see Morphisms, Lemma 29.43.4. In this case, the sheaves $R^p f_* \mathcal{F}$ are the quasi-coherent sheaves associated to the R-modules $H^p(X, \mathcal{F})$, see Lemma 30.4.6. Hence it suffices to show that R-modules $H^p(X, \mathcal{F})$ are finite R-modules (Lemma 30.9.1). This follows from Lemma 30.16.1 (because the restriction of $\mathcal{O}_{\mathbf{P}^n_R}(1)$ to X is ample on X). $\square$

30.17. Ample invertible sheaves and cohomology

01XO Here is a criterion for ampleness on proper schemes over affine bases in terms of vanishing of cohomology after twisting.

0B5U Lemma 30.17.1. Let R be a Noetherian ring. Let $f : X \to \mathrm{Spec}(R)$ be a proper morphism. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent [DG67, III Proposition 2.6.1]

(1) $\mathcal{L}$ is ample on X (this is equivalent to many other things, see Properties, Proposition 28.26.13 and Morphisms, Lemma 29.39.4),
(2) for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an $n_0 \geq 0$ such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$, and
(3) for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, there exists an $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 30.16.1. The implication (2) $\Rightarrow$ (3) is trivial. The implication (3) $\Rightarrow$ (1) is Lemma 30.3.3. $\square$

0B5V Lemma 30.17.2. Let R be a Noetherian ring. Let $f : Y \to X$ be a morphism of schemes proper over R. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume f is finite and surjective. Then $\mathcal{L}$ is ample if and only if $f^*\mathcal{L}$ is ample.

Proof. The pullback of an ample invertible sheaf by a quasi-affine morphism is ample, see Morphisms, Lemma 29.37.7. This proves one of the implications as a finite morphism is affine by definition.

Assume that $f^*\mathcal{L}$ is ample. Let P be the following property on coherent $\mathcal{O}_X$-modules $\mathcal{F}$: there exists an n_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $p > 0$. We will prove that P holds for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, which implies $\mathcal{L}$ is ample by Lemma 30.17.1. We are going to apply Lemma 30.12.8. Thus we have to verify (1), (2) and (3) of that lemma for P. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves and the fact that tensoring with an invertible sheaf is an exact functor. Property (2) follows since $H^p(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ. Let $\mathcal{F}$ be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$, see Lemma 30.13.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 30.12.8. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 30.13.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 30.13.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module. As $\mathcal{L}$ is ample we see from Lemma 30.17.1 that there exists an n_0 such that

$$H^p(Y, f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0,$$

for $n \geq n_0$ and $p > 0$. Since f is finite, hence affine, we see that

$$\begin{aligned} H^p(X, \mathcal{J}\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) &= H^p(X, f_*(f^{-1}\mathcal{J}\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \\ &= H^p(X, f_*(f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n})) \\ &= H^p(Y, f^{-1}\mathcal{J}\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{L}^{\otimes n}) = 0 \end{aligned}$$

Here we have used the projection formula (Cohomology, Lemma 20.54.2) and Lemma 30.2.4. Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies P. This verifies property (3)(c) of Lemma 30.12.8 as desired. □

Cohomology is functorial. In particular, given a ringed space X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a section $s \in \Gamma(X, \mathcal{L})$ we get maps

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}), \quad \xi \longmapsto s\xi$$

induced by the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ which is multiplication by s. We set $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ as a graded ring, see Modules, Definition 17.25.7. Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and an integer $p \geq 0$ we set

$$H^p_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

This is a graded $\Gamma_*(X, \mathcal{L})$-module by the multiplication defined above. Warning: the notation $H^p_*(X, \mathcal{L}, \mathcal{F})$ is nonstandard.

09MR Lemma 30.17.3. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is quasi-compact and quasi-separated, the canonical map

$$H^p_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(X_s, \mathcal{F})$$

which maps ξ/s^n to $s^{-n}\xi$ is an isomorphism.

Proof. Note that for $p = 0$ this is Properties, Lemma 28.17.2. We will prove the statement using the induction principle (Lemma 30.4.1) where for $U \subset X$ quasi-compact open we let $P(U)$ be the property: for all $p \geq 0$ the map

$$H^p_*(U, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(U_s, \mathcal{F})$$

is an isomorphism.

If U is affine, then both sides of the arrow displayed above are zero for $p > 0$ by Lemma 30.2.2 and Properties, Lemma 28.26.4 and the statement is true. If P is true for U, V, and $U \cap V$, then we can use the Mayer-Vietoris sequences (Cohomology, Lemma 20.8.2) to obtain a map of long exact sequences

$$\begin{tikzcd}
H^{p-1}_*(U \cap V, \mathcal{L}, \mathcal{F})_{(s)} \arrow[r] \arrow[d] & H^p_*(U \cup V, \mathcal{L}, \mathcal{F})_{(s)} \arrow[r] \arrow[d] & H^p_*(U, \mathcal{L}, \mathcal{F})_{(s)} \oplus H^p_*(V, \mathcal{L}, \mathcal{F})_{(s)} \arrow[d] \\
H^{p-1}(U_s \cap V_s, \mathcal{F}) \arrow[r] & H^p(U_s \cup V_s, \mathcal{F}) \arrow[r] & H^p(U_s, \mathcal{F}) \oplus H^p(V_s, \mathcal{F})
\end{tikzcd}$$

(only a snippet shown). Observe that $U_s \cap V_s = (U \cap V)_s$ and that $U_s \cup V_s = (U \cup V)_s$. Thus the left and right vertical maps are isomorphisms (as well as one more to the right and one more to the left which are not shown in the diagram). We conclude that $P(U \cup V)$ holds by the 5-lemma (Homology, Lemma 12.5.20). This finishes the proof. □

01XR Lemma 30.17.4. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume that

(1) X is quasi-compact and quasi-separated, and
(2) X_s is affine.

Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every $p > 0$ and all $\xi \in H^p(X, \mathcal{F})$ there exists an $n \geq 0$ such that $s^n\xi = 0$ in $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$.

Proof. Recall that $H^p(X_s, \mathcal{G})$ is zero for every quasi-coherent module $\mathcal{G}$ by Lemma 30.2.2. Hence the lemma follows from Lemma 30.17.3. □

For a more general version of the following lemma see Limits, Lemma 32.11.4.

09MS Lemma 30.17.5. Let $i : Z \to X$ be a closed immersion of Noetherian schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. Then $i^*\mathcal{L}$ is ample on Z, if and only if $\mathcal{L}$ is ample on X.

Proof. If $\mathcal{L}$ is ample, then $i^*\mathcal{L}$ is ample for example by Morphisms, Lemma 29.37.7. Assume $i^*\mathcal{L}$ is ample. We have to show that $\mathcal{L}$ is ample on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals cutting out the closed subscheme Z. Since $i(Z) = X$ set theoretically we see that $\mathcal{I}^n = 0$ for some n by Lemma 30.10.2. Consider the sequence

$$X = Z_n \supset Z_{n-1} \supset Z_{n-2} \supset \ldots \supset Z_1 = Z$$

of closed subschemes cut out by $0 = \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I}$. Then each of the closed immersions $Z_i \to Z_{i-1}$ is defined by a coherent sheaf of ideals of square zero. In this way we reduce to the case that $\mathcal{I}^2 = 0$.

Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

of quasi-coherent $\mathcal{O}_X$-modules. Tensoring with $\mathcal{L}^{\otimes n}$ we obtain short exact sequences

0B8T (30.17.5.1)
$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n} \to i_*i^*\mathcal{L}^{\otimes n} \to 0$$

As $\mathcal{I}^2 = 0$, we can use Morphisms, Lemma 29.4.1 to think of $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_Z$-module and then $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} = \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n}$ with obvious abuse of notation. Moreover, the cohomology of this sheaf over Z is canonically the same as the cohomology of this sheaf over X (as i is a homeomorphism).

Let $x \in X$ be a point and denote $z \in Z$ the corresponding point. Because $i^*\mathcal{L}$ is ample there exists an n and a section $s \in \Gamma(Z, i^*\mathcal{L}^{\otimes n})$ with $z \in Z_s$ and with Z_s affine. The obstruction to lifting s to a section of $\mathcal{L}^{\otimes n}$ over X is the boundary

$$\xi = \partial s \in H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n})$$

coming from the short exact sequence of sheaves (30.17.5.1). If we replace s by s^{e+1} then ξ is replaced by $\partial(s^{e+1}) = (e+1)s^e\xi$ in $H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes (e+1)n})$ because the boundary map for

$$0 \to \bigoplus\nolimits_{m \geq 0} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \to \bigoplus\nolimits_{m \geq 0} \mathcal{L}^{\otimes m} \to \bigoplus\nolimits_{m \geq 0} i_*i^*\mathcal{L}^{\otimes m} \to 0$$

is a derivation by Cohomology, Lemma 20.25.5. By Lemma 30.17.4 we see that $s^e\xi$ is zero for e large enough. Hence, after replacing s by a power, we can assume s is the image of a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n})$. Then $X_{s'}$ is an open subscheme and $Z_s \to X_{s'}$ is a surjective closed immersion of Noetherian schemes with Z_s affine. Hence X_s is affine by Lemma 30.13.3 and we conclude that $\mathcal{L}$ is ample. $\square$

For a more general version of the following lemma see Limits, Lemma 32.11.5.

0B7K **Lemma 30.17.6.** Let $i : Z \to X$ be a closed immersion of Noetherian schemes inducing a homeomorphism of underlying topological spaces. Then X is quasi-affine if and only if Z is quasi-affine.

Proof. Recall that a scheme is quasi-affine if and only if the structure sheaf is ample, see Properties, Lemma 28.27.1. Hence if Z is quasi-affine, then $\mathcal{O}_Z$ is ample, hence $\mathcal{O}_X$ is ample by Lemma 30.17.5, hence X is quasi-affine. A proof of the converse, which can also be seen in an elementary way, is gotten by reading the argument just given backwards. $\square$

0EBD **Lemma 30.17.7.** Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let n_0 be an integer. If $H^p(X, \mathcal{L}^{\otimes -n}) = 0$ for $n \geq n_0$ and $p > 0$, then X is affine.

Proof. We claim $H^p(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module and $p > 0$. Since X is quasi-compact by Properties, Definition 28.26.1 the claim finishes the proof by Lemma 30.3.1. The scheme X is separated by Properties, Lemma 28.26.8. Say X is covered by $e + 1$ affine opens. Then $H^p(X, \mathcal{F}) = 0$ for $p > e$, see Lemma 30.4.2. Thus we may use descending induction on p to prove the claim. Writing $\mathcal{F}$ as a filtered colimit of finite type quasi-coherent modules (Properties, Lemma 28.22.3) and using Cohomology, Lemma 20.19.1 we may assume $\mathcal{F}$ is of finite type. Then we can choose $n > n_0$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated, see Properties, Proposition 28.26.13. This means there is a short exact sequence

$$0 \to \mathcal{F}' \to \bigoplus\nolimits_{i \in I} \mathcal{L}^{\otimes -n} \to \mathcal{F} \to 0$$

for some set I (in fact we can choose I finite). By induction hypothesis we have $H^{p+1}(X, \mathcal{F}') = 0$ and by assumption (combined with the already used commutation of cohomology with colimits) we have $H^p(X, \bigoplus_{i \in I} \mathcal{L}^{\otimes -n}) = 0$. From the long exact cohomology sequence we conclude that $H^p(X, \mathcal{F}) = 0$ as desired. $\square$

0EBE Lemma 30.17.8. Let X be a quasi-affine scheme. If $H^p(X, \mathcal{O}_X) = 0$ for $p > 0$, then X is affine.

Proof. Since $\mathcal{O}_X$ is ample by Properties, Lemma 28.27.1 this follows from Lemma 30.17.7. $\square$

30.18. Chow's Lemma

02O2 In this section we prove Chow's lemma in the Noetherian case (Lemma 30.18.1). In Limits, Section 32.12 we prove some variants for the non-Noetherian case.

0200 Lemma 30.18.1. Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exist an $n \geq 0$ and a diagram [DG67, II Theorem 5.6.1(a)]

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, we may arrange it such that there exists a dense open subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism.

Proof. All of the schemes we will encounter during the rest of the proof are going to be of finite type over the Noetherian scheme S and hence Noetherian (see Morphisms, Lemma 29.15.6). All morphisms between them will automatically be quasi-compact, locally of finite type and quasi-separated, see Morphisms, Lemma 29.15.8 and Properties, Lemmas 28.5.4 and 28.5.8.

The scheme X has only finitely many irreducible components (Properties, Lemma 28.5.7). Say $X = X_1 \cup \ldots \cup X_r$ is the decomposition of X into irreducible components. Let $\eta_i \in X_i$ be the generic point. For every point $x \in X$ there exists an affine open $U_x \subset X$ which contains x and each of the generic points η_i. See Properties, Lemma 28.29.4. Since X is quasi-compact, we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_r$. In particular we conclude that the open $U = U_1 \cap \ldots \cap U_m \subset X$ is a dense open. This and the fact that the U_i are affine opens covering X are all that we will use below.

Let $X^* \subset X$ be the scheme theoretic closure of $U \to X$, see Morphisms, Definition 29.6.2. Let $U_i^* = X^* \cap U_i$. Note that U_i^* is a closed subscheme of U_i. Hence U_i^* is affine. Since U is dense in X the morphism $X^* \to X$ is a surjective closed immersion. It is an isomorphism over U. Hence we may replace X by X^* and U_i by U_i^* and assume that U is scheme theoretically dense in X, see Morphisms, Definition 29.7.1.

By Morphisms, Lemma 29.39.3 we can find an immersion $j_i : U_i \to \mathbf{P}^{n_i}_S$ for each i. By Morphisms, Lemma 29.7.7 we can find closed subschemes $Z_i \subset \mathbf{P}^{n_i}_S$ such that $j_i : U_i \to Z_i$ is a scheme theoretically dense open immersion. Note that $Z_i \to S$ is proper, see Morphisms, Lemma 29.43.5. Consider the morphism

$$j = (j_1|_U, \ldots, j_m|_U) : U \longrightarrow \mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_m}_S.$$

By the lemma cited above we can find a closed subscheme Z of $\mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_m}_S$ such that $j : U \to Z$ is an open immersion and such that U is scheme theoretically dense in Z. The morphism $Z \to S$ is proper. Consider the ith projection

$$\mathrm{pr}_i|_Z : Z \longrightarrow \mathbf{P}^{n_i}_S.$$

This morphism factors through Z_i (see Morphisms, Lemma 29.6.6). Denote $p_i : Z \to Z_i$ the induced morphism. This is a proper morphism, see Morphisms, Lemma 29.41.7 for example. At this point we have that $U \subset U_i \subset Z_i$ are scheme theoretically dense open immersions. Moreover, we can think of Z as the scheme theoretic image of the "diagonal" morphism $U \to Z_1 \times_S \ldots \times_S Z_m$.

Set $V_i = p_i^{-1}(U_i)$. Note that $p_i|_{V_i} : V_i \to U_i$ is proper. Set $X' = V_1 \cup \ldots \cup V_m$. By construction X' has an immersion into the scheme $\mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_m}_S$. Thus by the Segre embedding (see Constructions, Lemma 27.13.6) we see that X' has an immersion into a projective space over S.

We claim that the morphisms $p_i|_{V_i} : V_i \to U_i$ glue to a morphism $X' \to X$. Namely, it is clear that $p_i|_U$ is the identity map from U to U. Since $U \subset X'$ is scheme theoretically dense by construction, it is also scheme theoretically dense in the open subscheme $V_i \cap V_j$. Thus we see that $p_i|_{V_i \cap V_j} = p_j|_{V_i \cap V_j}$ as morphisms into the separated S-scheme X, see Morphisms, Lemma 29.7.10. We denote the resulting morphism $\pi : X' \to X$.

We claim that $\pi^{-1}(U_i) = V_i$. Since $\pi|_{V_i} = p_i|_{V_i}$ it follows that $V_i \subset \pi^{-1}(U_i)$. Consider the diagram

$$\begin{tikzcd} V_i \arrow[r] \arrow[rd, "p_i|_{V_i}"'] & \pi^{-1}(U_i) \arrow[d] \\ & U_i \end{tikzcd}$$

Since $V_i \to U_i$ is proper we see that the image of the horizontal arrow is closed, see Morphisms, Lemma 29.41.7. Since $V_i \subset \pi^{-1}(U_i)$ is scheme theoretically dense (as it contains U) we conclude that $V_i = \pi^{-1}(U_i)$ as claimed.

This shows that $\pi^{-1}(U_i) \to U_i$ is identified with the proper morphism $p_i|_{V_i} : V_i \to U_i$. Hence we see that X has a finite affine covering $X = \bigcup U_i$ such that the restriction of π is proper on each member of the covering. Thus by Morphisms, Lemma 29.41.3 we see that π is proper.

Finally we have to show that $\pi^{-1}(U) = U$. To see this we argue in the same way as above using the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[rd] & \pi^{-1}(U) \arrow[d] \\ & U \end{tikzcd}$$

and using that $\mathrm{id}_U : U \to U$ is proper and that U is scheme theoretically dense in $\pi^{-1}(U)$. □

0201 Remark 30.18.2. In the situation of Chow's Lemma 30.18.1:

(1) The morphism π is actually H-projective (hence projective, see Morphisms, Lemma 29.43.3) since the morphism $X' \to \mathbf{P}^n_S \times_S X = \mathbf{P}^n_X$ is a closed immersion (use the fact that π is proper, see Morphisms, Lemma 29.41.7).
(2) We may assume that $\pi^{-1}(U)$ is scheme theoretically dense in X'. Namely, we can simply replace X' by the scheme theoretic closure of $\pi^{-1}(U)$. In this case we can think of U as a scheme theoretically dense open subscheme of X'. See Morphisms, Section 29.6.
(3) If X is reduced then we may choose X' reduced. This is clear from (2).

30.19. Higher direct images of coherent sheaves

02O3 In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent.

02O5 Proposition 30.19.1. Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^if_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$. [DG67, III Theorem 3.2.1]

Proof. Since the problem is local on S we may assume that S is a Noetherian scheme. Since a proper morphism is of finite type we see that in this case X is a Noetherian scheme also. Consider the property $\mathcal{P}$ of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow R^pf_*\mathcal{F} \text{ is coherent for all } p \geq 0$$

We are going to use the result of Lemma 30.12.6 to prove that $\mathcal{P}$ holds for every coherent sheaf on X.

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1}f_*\mathcal{F}_3 \to R^pf_*\mathcal{F}_1 \to R^pf_*\mathcal{F}_2 \to R^pf_*\mathcal{F}_3 \to R^{p+1}f_*\mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemma 30.9.2 and 30.9.3. Hence property $\mathcal{P}$ holds for the third as well.

Let $Z \subset X$ be an integral closed subscheme. We have to find a coherent sheaf $\mathcal{F}$ on X whose support is contained in Z, whose stalk at the generic point ξ of Z is a 1-dimensional vector space over $\kappa(\xi)$ such that $\mathcal{P}$ holds for $\mathcal{F}$. Denote $g = f|_Z : Z \to S$ the restriction of f. Suppose we can find a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{G}_\xi$ is a 1-dimensional vector space over $\kappa(\xi)$, (b) $R^pg_*\mathcal{G} = 0$ for $p > 0$, and (c) $g_*\mathcal{G}$ is coherent. Then we can consider $\mathcal{F} = (Z \to X)_*\mathcal{G}$. As $Z \to X$ is a closed immersion we see that $(Z \to X)_*\mathcal{G}$ is coherent on X and $R^p(Z \to X)_*\mathcal{G} = 0$ for $p > 0$ (Lemma 30.9.9). Hence by the relative Leray spectral sequence (Cohomology, Lemma 20.13.8) we will have $R^pf_*\mathcal{F} = R^pg_*\mathcal{G} = 0$ for $p > 0$ and $f_*\mathcal{F} = g_*\mathcal{G}$ is coherent. Finally $\mathcal{F}_\xi = ((Z \to X)_*\mathcal{G})_\xi = \mathcal{G}_\xi$ which verifies the condition on the stalk at ξ. Hence everything depends on finding a coherent sheaf $\mathcal{G}$ on Z which has properties (a), (b), and (c).

We can apply Chow's Lemma 30.18.1 to the morphism $Z \to S$. Thus we get a diagram

$$\begin{tikzcd} Z \arrow[rd, "g"'] & Z' \arrow[l, "\pi"'] \arrow[r, "i"] \arrow[d, "g'"] & \mathbf{P}^m_S \arrow[ld] \\ & S & \end{tikzcd}$$

as in the statement of Chow's lemma. Also, let $U \subset Z$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. By the discussion in Remark 30.18.2 we see that $i' = (i, \pi) : Z' \to \mathbf{P}^m_Z$ is a closed immersion. Hence

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^m_S}(1) \cong (i')^*\mathcal{O}_{\mathbf{P}^m_Z}(1)$$

is g'-relatively ample and π-relatively ample (for example by Morphisms, Lemma 29.39.7). Hence by Lemma 30.16.2 there exists an $n \geq 0$ such that both $R^p\pi_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$. Property (a) holds because $\pi_*\mathcal{L}^{\otimes n}|_U$ is an invertible sheaf (as $\pi^{-1}(U) \to U$ is an isomorphism). Properties (b) and (c) hold because by the relative Leray spectral sequence (Cohomology, Lemma 20.13.8) we have

$$E_2^{p,q} = R^pg_*R^q\pi_*\mathcal{L}^{\otimes n} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes n}$$

and by choice of n the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes n}$ are those with $p = q = 0$. This implies that $R^pg_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$. Finally, applying the previous Lemma 30.16.3 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$ is coherent as desired. □

02O6 Lemma 30.19.2. Let $S = \text{Spec}(A)$ with A a Noetherian ring. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is a finite A-module for all $i \geq 0$.

Proof. This is just the affine case of Proposition 30.19.1. Namely, by Lemmas 30.4.5 and 30.4.6 we know that $R^if_*\mathcal{F}$ is the quasi-coherent sheaf associated to the A-module $H^i(X, \mathcal{F})$ and by Lemma 30.9.1 this is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. □

0897 Lemma 30.19.3. Let A be a Noetherian ring. Let B be a finitely generated graded A-algebra. Let $f : X \to \text{Spec}(A)$ be a proper morphism. Set $\mathcal{B} = f^*\widetilde{B}$. Let $\mathcal{F}$ be a quasi-coherent graded $\mathcal{B}$-module of finite type.

(1) For every $p \geq 0$ the graded B-module $H^p(X, \mathcal{F})$ is a finite B-module.
(2) If $\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module, then there exists an integer d_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$ and $d \geq d_0$.

Proof. To prove this we consider the fibre product diagram

$$\begin{tikzcd} X' = \text{Spec}(B) \times_{\text{Spec}(A)} X \arrow[r, "\pi"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \text{Spec}(B) \arrow[r] & \text{Spec}(A) \end{tikzcd}$$

Note that f' is a proper morphism, see Morphisms, Lemma 29.41.5. Also, B is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 10.31.1). This implies that X' is a Noetherian scheme (Morphisms, Lemma 29.15.6). Note that X' is the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ by Constructions, Lemma 27.4.6. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{B}$-module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_*\mathcal{F}' = \mathcal{F}$, see Morphisms, Lemma 29.11.6 Since $\mathcal{F}$ is finite type as a $\mathcal{B}$-module we conclude that $\mathcal{F}'$ is a finite type $\mathcal{O}_{X'}$-module (details omitted). In other words, $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module (Lemma 30.9.1). Since the morphism $\pi : X' \to X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 30.2.4. Thus (1) follows from Lemma 30.19.2. Given $\mathcal{L}$ as in (2) we set $\mathcal{L}' = \pi^*\mathcal{L}$. Note that $\mathcal{L}'$ is ample on X' by Morphisms, Lemma 29.37.7. By the projection formula (Cohomology, Lemma 20.54.2) we have $\pi_*(\mathcal{F}' \otimes \mathcal{L}') = \mathcal{F} \otimes \mathcal{L}$. Thus part (2) follows by the same reasoning as above from Lemma 30.16.2. □

30.20. The theorem on formal functions

02O7 In this section we study the behaviour of cohomology of sequences of sheaves either of the form $\{I^n\mathcal{F}\}_{n \geq 0}$ or of the form $\{\mathcal{F}/I^n\mathcal{F}\}_{n \geq 0}$ as n varies.

Here and below we use the following notation. Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$ we denote $\mathcal{I}^n\mathcal{F}$ the quasi-coherent subsheaf generated by products of local sections of $f^{-1}(\mathcal{I}^n)$ and $\mathcal{F}$. In a formula

$$\mathcal{I}^n\mathcal{F} = \operatorname{Im}\left(f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}\right).$$

Note that there are natural maps

$$f^{-1}(\mathcal{I}^n) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{I}^m\mathcal{F} \longrightarrow f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{I}^m\mathcal{F} \longrightarrow \mathcal{I}^{n+m}\mathcal{F}$$

Hence a section of $\mathcal{I}^n$ will give rise to a map $R^pf_*(\mathcal{I}^m\mathcal{F}) \to R^pf_*(\mathcal{I}^{n+m}\mathcal{F})$ by functoriality of higher direct images. Localizing and then sheafifying we see that there are $\mathcal{O}_Y$-module maps

$$\mathcal{I}^n \otimes_{\mathcal{O}_Y} R^pf_*(\mathcal{I}^m\mathcal{F}) \longrightarrow R^pf_*(\mathcal{I}^{n+m}\mathcal{F}).$$

In other words we see that $\bigoplus_{n \geq 0} R^pf_*(\mathcal{I}^n\mathcal{F})$ is a graded $\bigoplus_{n \geq 0} \mathcal{I}^n$-module.

If $Y = \operatorname{Spec}(A)$ and $\mathcal{I} = \widetilde{I}$ we denote $\mathcal{I}^n\mathcal{F}$ simply $I^n\mathcal{F}$. The maps introduced above give $M = \bigoplus H^p(X, I^n\mathcal{F})$ the structure of a graded $S = \bigoplus I^n$-module. If f is proper, A is Noetherian and $\mathcal{F}$ is coherent, then this turns out to be a module of finite type.

02O8 **Lemma 30.20.1.** Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $B = \bigoplus_{n \geq 0} I^n$. Let $f : X \to \operatorname{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ the graded B-module $\bigoplus_{n \geq 0} H^p(X, I^n\mathcal{F})$ is a finite B-module. [DG67, III Cor 3.3.2]

Proof. Let $\mathcal{B} = \bigoplus I^n\mathcal{O}_X = f^*\widetilde{B}$. Then $\bigoplus I^n\mathcal{F}$ is a finite type graded $\mathcal{B}$-module. Hence the result follows from Lemma 30.19.3 part (1). □

02O9 **Lemma 30.20.2.** Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$. Assume Y locally Noetherian, f proper, and $\mathcal{F}$ coherent. Then

$$\mathcal{M} = \bigoplus\nolimits_{n \geq 0} R^pf_*(\mathcal{I}^n\mathcal{F})$$

is a graded $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$-module which is quasi-coherent and of finite type.

Proof. The statement is local on Y, hence this reduces to the case where Y is affine. In the affine case the result follows from Lemma 30.20.1. Details omitted. □

02OA Lemma 30.20.3. Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ there exists an integer $c \geq 0$ such that

(1) the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$,
(2) the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-e}H^p(X, I^n\mathcal{F})$ where $e = \max(0, c-n)$ for $n+m \geq c$, $n, m \geq 0$,
(3) we have
$$\mathrm{Ker}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F})) = \mathrm{Ker}(H^p(X, I^n\mathcal{F}) \to H^p(X, I^{n-c}\mathcal{F}))$$
for $n \geq c$,
(4) there are maps $I^nH^p(X, \mathcal{F}) \to H^p(X, I^{n-c}\mathcal{F})$ for $n \geq c$ such that the compositions
$$H^p(X, I^n\mathcal{F}) \to I^{n-c}H^p(X, \mathcal{F}) \to H^p(X, I^{n-2c}\mathcal{F})$$
and
$$I^nH^p(X, \mathcal{F}) \to H^p(X, I^{n-c}\mathcal{F}) \to I^{n-2c}H^p(X, \mathcal{F})$$
for $n \geq 2c$ are the canonical ones, and
(5) the inverse systems $(H^p(X, I^n\mathcal{F}))$ and $(I^nH^p(X, \mathcal{F}))$ are pro-isomorphic.

Proof. Write $M_n = H^p(X, I^n\mathcal{F})$ for $n \geq 1$ and $M_0 = H^p(X, \mathcal{F})$ so that we have maps $\ldots \to M_3 \to M_2 \to M_1 \to M_0$. Setting $B = \bigoplus_{n \geq 0} I^n$, then $M = \bigoplus_{n \geq 0} M_n$ is a finite graded B-module, see Lemma 30.20.1. Observe that the products $B_n \otimes M_m \to M_{m+n}$, $a \otimes m \mapsto a \cdot m$ are compatible with the maps in our inverse system in the sense that the diagrams

$$\begin{tikzcd} B_n \otimes_A M_m \arrow[r] \arrow[d] & M_{n+m} \arrow[d] \\ B_n \otimes_A M_{m'} \arrow[r] & M_{n+m'} \end{tikzcd}$$

commute for $n, m' \geq 0$ and $m \geq m'$.

Proof of (1). Choose $d_1, \ldots, d_t \geq 0$ and $x_i \in M_{d_i}$ such that M is generated by $x_1, \ldots, x_t$ over B. For any $c \geq \max\{d_i\}$ we conclude that $B_{n-c} \cdot M_c = M_n$ for $n \geq c$ and we conclude (1) is true.

Proof of (2). Let c be as in the proof of (1). Let $n + m \geq c$. We have $M_{n+m} = B_{n+m-c} \cdot M_c$. If $c > n$ then we use $M_c \to M_n$ and the compatibility of products with transition maps pointed out above to conclude that the image of $M_{n+m} \to M_n$ is contained in $I^{n+m-c}M_n$. If $c \leq n$, then we write $M_{n+m} = B_m \cdot B_{n-c} \cdot M_c = B_m \cdot M_n$ to see that the image is contained in $I^m M_n$. This proves (2).

Let $K_n \subset M_n$ be the kernel of the map $M_n \to M_0$. The compatibility of products with transition maps pointed out above shows that $K = \bigoplus K_n \subset M$ is a graded B-submodule. As B is Noetherian and M is a finitely generated graded B-module, this shows that K is a finitely generated graded B-module. Choose $d'_1, \ldots, d'_{t'} \geq 0$

and $y_i \in K_{d'_i}$ such that K is generated by $y_1, \ldots, y_{t'}$ over B. Set $c = \max(d'_i, d'_j)$. Since $y_i \in \text{Ker}(M_{d'_i} \to M_0)$ we see that $B_n \cdot y_i \subset \text{Ker}(M_{n+d'_i} \to M_n)$. In this way we see that $K_n = \text{Ker}(M_n \to M_{n-c})$ for $n \geq c$. This proves (3).

Consider the following commutative solid diagram

$$\begin{tikzcd} I^n \otimes_A M_0 \arrow[r] \arrow[d] & I^n M_0 \arrow[r] \arrow[d, dotted] & M_0 \arrow[d] \\ M_n \arrow[r] & M_{n-c} \arrow[r] & M_0 \end{tikzcd}$$

Since the kernel of the surjective arrow $I^n \otimes_A M_0 \to I^n M_0$ maps into K_n by the above we obtain the dotted arrow and the composition $I^n M_0 \to M_{n-c} \to M_0$ is the canonical map. Then clearly the composition $I^n M_0 \to M_{n-c} \to I^{n-2c} M_0$ is the canonical map for $n \geq 2c$. Consider the composition $M_n \to I^{n-c} M_0 \to M_{n-2c}$. The first map sends an element of the form $a \cdot m$ with $a \in I^{n-c}$ and $m \in M_c$ to am' where m' is the image of m in M_0. Then the second map sends this to $a \cdot m'$ in M_{n-2c} and we see (4) is true.

Part (5) is an immediate consequence of (4) and the definition of morphisms of pro-objects. □

In the situation of Lemmas 30.20.1 and 30.20.3 consider the inverse system

$$\mathcal{F}/I\mathcal{F} \leftarrow \mathcal{F}/I^2\mathcal{F} \leftarrow \mathcal{F}/I^3\mathcal{F} \leftarrow \ldots$$

We would like to know what happens to the cohomology groups. Here is a first result.

02OB **Lemma 30.20.4.** Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \text{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$. There exists a $c \geq 0$ such that

(1) for all $n \geq c$ we have
$$\text{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c}H^p(X, \mathcal{F}).$$
(2) the inverse system
$$(H^p(X, \mathcal{F}/I^n\mathcal{F}))_{n \in \mathbf{N}}$$
satisfies the Mittag-Leffler condition (see Homology, Definition 12.31.2), and
(3) we have
$$\text{Im}(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \text{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$
for all $k \geq n + c$.

Proof. Let $c = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 30.20.3 for H^p and H^{p+1}.

Let us prove part (1). Consider the short exact sequence

$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

From the long exact cohomology sequence we see that

$$\text{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \text{Im}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F}))$$

Hence by Lemma 30.20.3 part (2) we see that this is contained in $I^{n-c}H^p(X, \mathcal{F})$ for $n \geq c$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler systems.

Let us prove part (3). Fix an n. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I^n\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}/I^n\mathcal{F} \arrow[r] & 0 \\ 0 \arrow[r] & I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & \mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}/I^{n+m}\mathcal{F} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

This gives rise to the following commutative diagram

$$\begin{tikzcd} H^p(X, \mathcal{F}) \arrow[r] & H^p(X, \mathcal{F}/I^n\mathcal{F}) \arrow[r, "\delta"'] & H^{p+1}(X, I^n\mathcal{F}) \arrow[r] & H^{p+1}(X, \mathcal{F}) \\ H^p(X, \mathcal{F}) \arrow[r] \arrow[u, "1"] & H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \arrow[r] \arrow[u, "\gamma"] & H^{p+1}(X, I^{n+m}\mathcal{F}) \arrow[r, "\beta"] \arrow[u, "\alpha"] & H^{p+1}(X, \mathcal{F}) \arrow[u, "1"'] \end{tikzcd}$$

with exact rows. By Lemma 30.20.3 part (4) the kernel of β is equal to the kernel of α for $m \geq c$. By a diagram chase this shows that the image of γ is contained in the kernel of δ which shows that part (3) is true (set $k = n + m$ to get it). $\square$

02OC Theorem 30.20.5 (Theorem on formal functions). Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$. The system of maps

$$H^p(X, \mathcal{F})/I^nH^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X, \mathcal{F})$ with respect to the ideal I, see Algebra, Section 10.96. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. This follows from Lemma 30.20.4 as follows. Set $M = H^p(X, \mathcal{F})$, $M_n = H^p(X, \mathcal{F}/I^n\mathcal{F})$, and denote $N_n = \mathrm{Im}(M \to M_n)$. By Lemma 30.20.4 parts (2) and (3) we see that (M_n) is a Mittag-Leffler system with $N_n \subset M_n$ equal to the image of M_k for all $k \gg n$. It follows that $\lim M_n = \lim N_n$ as topological modules (with limit topologies). On the other hand, the N_n form an inverse system of quotients of the module M and hence $\lim N_n$ is the completion of M with respect to the topology given by the kernels $K_n = \mathrm{Ker}(M \to N_n)$. By Lemma 30.20.4 part (1) we have $K_n \subset I^{n-c}M$ and since $N_n \subset M_n$ is annihilated by I^n we have $I^nM \subset K_n$. Thus the topology defined using the submodules K_n as a fundamental system of open neighbourhoods of 0 is the same as the I-adic topology and we find that the induced map $M^\wedge = \lim M/I^nM \to \lim N_n = \lim M_n$ is an isomorphism of topological modules[3]. $\square$

[3]To be sure, the limit topology on $M^\wedge$ is the same as its I-adic topology as follows from Algebra, Lemma 10.96.3. See More on Algebra, Section 15.36.

087U Lemma 30.20.6. Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then

$$H^p(X, \mathcal{F}) = \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 30.20.5) in the case of a complete Noetherian base ring. Namely, in this case the A-module $H^p(X, \mathcal{F})$ is finite (Lemma 30.19.2) hence I-adically complete (Algebra, Lemma 10.97.1) and we see that completion on the left hand side is not necessary. $\square$

02OD Lemma 30.20.7. Given a morphism of schemes $f : X \to Y$ and a quasi-coherent sheaf $\mathcal{F}$ on X. Assume

(1) Y locally Noetherian,
(2) f proper, and
(3) $\mathcal{F}$ coherent.

Let $y \in Y$ be a point. Consider the infinitesimal neighbourhoods

$$\begin{tikzcd} X_n = \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \times_Y X \arrow[r, "i_n"'] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r, "c_n"] & Y \end{tikzcd}$$

of the fibre $X_1 = X_y$ and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$(R^pf_*\mathcal{F})_y^\wedge \cong \lim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}_{Y,y}^\wedge$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 30.20.5. Let us spell it out. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (26.13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^*R^pf_*\mathcal{F} = R^pf'_*\mathcal{F}'$ by Lemma 30.5.2. Moreover, the infinitesimal neighbourhoods of the fibre X_y and X'_y are identified (verification omitted; hint: the morphisms c_n factor through c).

Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and that $y \in Y$ corresponds to the closed point (i.e., to $\mathfrak{m}$). In particular it follows that

$$(R^pf_*\mathcal{F})_y = \Gamma(Y, R^pf_*\mathcal{F}) = H^p(X, \mathcal{F}).$$

In this case also, the morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 30.9.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}_n) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. $\square$

Here is a lemma which we will generalize later to fibres of dimension > 0, namely the next lemma.

02OE **Lemma 30.20.8.** Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume

(1) Y locally Noetherian,
(2) f is proper, and
(3) $f^{-1}(\{y\})$ is finite.

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_y = 0$ for all $p > 0$.

Proof. The fibre X_y is finite, and by Morphisms, Lemma 29.20.7 it is a finite discrete space. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same. Hence each of the schemes X_n is affine according to Schemes, Lemma 26.11.8. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^pf_*\mathcal{F})^\wedge_y = 0$ by Lemma 30.20.7. Note that $R^pf_*\mathcal{F}$ is coherent by Proposition 30.19.1 and hence $R^pf_*\mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Nakayama's lemma (Algebra, Lemma 10.20.1) if the completion of a finite module over a local ring is zero, then the module is zero. Whence $(R^pf_*\mathcal{F})_y = 0$. □

02V7 **Lemma 30.20.9.** Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume

(1) Y locally Noetherian,
(2) f is proper, and
(3) $\dim(X_y) = d$.

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^pf_*\mathcal{F})_y = 0$ for all $p > d$.

Proof. The fibre X_y is of finite type over $\text{Spec}(\kappa(y))$. Hence X_y is a Noetherian scheme by Morphisms, Lemma 29.15.6. Hence the underlying topological space of X_y is Noetherian, see Properties, Lemma 28.5.5. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same as that of X_y. Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Cohomology, Proposition 20.20.7. Hence we see that $(R^pf_*\mathcal{F})^\wedge_y = 0$ by Lemma 30.20.7 for $p > d$. Note that $R^pf_*\mathcal{F}$ is coherent by Proposition 30.19.1 and hence $R^pf_*\mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Nakayama's lemma (Algebra, Lemma 10.20.1) if the completion of a finite module over a local ring is zero, then the module is zero. Whence $(R^pf_*\mathcal{F})_y = 0$. □

30.21. Applications of the theorem on formal functions

02OF We will add more here as needed. For the moment we need the following characterization of finite morphisms in the Noetherian case.

02OG **Lemma 30.21.1.** (For a more general version see More on Morphisms, Lemma 37.44.1.) Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. The following are equivalent

(1) f is finite, and
(2) f is proper with finite fibres.

Proof. A finite morphism is proper according to Morphisms, Lemma 29.44.11. A finite morphism is quasi-finite according to Morphisms, Lemma 29.44.10. A quasi-finite morphism has finite fibres, see Morphisms, Lemma 29.20.10. Hence a finite morphism is proper and has finite fibres.

Assume f is proper with finite fibres. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by

Morphisms, Lemma 29.44.7 whereupon it follows from Morphisms, Lemma 29.44.4 that f is finite.

To show that f is affine we may assume that S is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. Hence we may use the criterion of Lemma 30.3.2 to prove that X is affine. To see this let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type ideal sheaf. In particular $\mathcal{I}$ is a coherent sheaf on X. By Lemma 30.20.8 we conclude that $R^1 f_*\mathcal{I}_s = 0$ for all $s \in S$. In other words, $R^1 f_*\mathcal{I} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{I}) = H^1(S, f_*\mathcal{I})$. Since S is affine, and $f_*\mathcal{I}$ is quasi-coherent (Schemes, Lemma 26.24.1) we conclude $H^1(S, f_*\mathcal{I}) = 0$ from Lemma 30.2.2 as desired. Hence $H^1(X, \mathcal{I}) = 0$ as desired. $\square$

As a consequence we have the following useful result.

02OH Lemma 30.21.2. (For a more general version see More on Morphisms, Lemma 37.44.2.) Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume

(1) S is locally Noetherian,
(2) f is proper, and
(3) $f^{-1}(\{s\})$ is a finite set.

Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 29.20.7. By Morphisms, Lemma 29.56.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 29.20.9), hence has finite fibres (Morphisms, Lemma 29.20.10), hence is finite by Lemma 30.21.1. $\square$

0D2M Lemma 30.21.3. Let $f : X \to Y$ be a proper morphism of schemes with Y Noetherian. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that $\mathcal{L}_y$ is ample on X_y. Then there exists a d_0 such that for all $d \geq d_0$ we have

$$R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = 0 \text{ for } p > 0$$

and the map

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y \longrightarrow H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$$

is surjective.

Proof. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \text{Spec}(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (26.13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to \text{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^*R^p f_*\mathcal{F} = R^p f'_*\mathcal{F}'$ by Lemma 30.5.2. Moreover, the fibres X_y and X'_y are identified. Hence we may assume that $Y = \text{Spec}(A)$ is the spectrum of a Noetherian local ring $(A, \mathfrak{m}, \kappa)$ and $y \in Y$ corresponds to $\mathfrak{m}$. In this case $R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})_y = H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ for all $p \geq 0$. Denote $f_y : X_y \to \text{Spec}(\kappa)$ the projection.

Let $B = \text{Gr}_\mathfrak{m}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Consider the sheaf $\mathcal{B} = f_y^*\widetilde{B}$ of quasi-coherent graded $\mathcal{O}_{X_y}$-algebras. We will use notation as in Section 30.20 with

I replaced by $\mathfrak{m}$. Since X_y is the closed subscheme of X cut out by $\mathfrak{m}\mathcal{O}_X$ we may think of $\mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ as a coherent $\mathcal{O}_{X_y}$-module, see Lemma 30.9.8. Then $\bigoplus_{n\geq 0}\mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F}$ is a quasi-coherent graded $\mathcal{B}$-module of finite type because it is generated in degree zero over $\mathcal{B}$ and because the degree zero part is $\mathcal{F}_y = \mathcal{F}/\mathfrak{m}\mathcal{F}$ which is a coherent $\mathcal{O}_{X_y}$-module. Hence by Lemma 30.19.3 part (2) we see that

$$H^p(X_y, \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d}) = 0$$

for all $p > 0$, $d \geq d_0$, and $n \geq 0$. By Lemma 30.2.4 this is the same as the statement that $H^p(X, \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$.

Consider the short exact sequences

$$0 \to \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \to \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \to 0$$

of coherent $\mathcal{O}_X$-modules. Tensoring with $\mathcal{L}^{\otimes d}$ is an exact functor and we obtain short exact sequences

$$0 \to \mathfrak{m}^n\mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to \mathcal{F}/\mathfrak{m}^{n+1}\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d} \to 0$$

Using the long exact cohomology sequence and the vanishing above we conclude (using induction) that

(1) $H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$, $d \geq d_0$, and $n \geq 0$, and
(2) $H^0(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \to H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective for all $d \geq d_0$ and $n \geq 1$.

By the theorem on formal functions (Theorem 30.20.5) we find that the $\mathfrak{m}$-adic completion of $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. Since $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is a finite A-module by Lemma 30.19.2 it follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$ is zero for all $d \geq d_0$ and $p > 0$. For $p = 0$ we deduce from Lemma 30.20.4 part (3) that $H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) \to H^0(X_y, \mathcal{F}_y \otimes_{\mathcal{O}_{X_y}} \mathcal{L}_y^{\otimes d})$ is surjective, which gives the final statement of the lemma. □

0D2N Lemma 30.21.4. (For a more general version see More on Morphisms, Lemma 37.50.3.) Let $f : X \to Y$ be a proper morphism of schemes with Y Noetherian. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that $\mathcal{L}_y$ is ample on X_y. Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$.

Proof. Pick d_0 as in Lemma 30.21.3 for $\mathcal{F} = \mathcal{O}_X$. Pick $d \geq d_0$ so that we can find $r \geq 0$ and sections $s_{y,0}, \ldots, s_{y,r} \in H^0(X_y, \mathcal{L}_y^{\otimes d})$ which define a closed immersion

$$\varphi_y = \varphi_{\mathcal{L}_y^{\otimes d}, (s_{y,0}, \ldots, s_{y,r})} : X_y \to \mathbf{P}^r_{\kappa(y)}.$$

This is possible by Morphisms, Lemma 29.39.4 but we also use Morphisms, Lemma 29.41.7 to see that φ_y is a closed immersion and Constructions, Section 27.13 for the description of morphisms into projective space in terms of invertible sheaves and sections. By our choice of d_0, after replacing Y by an open neighbourhood of y, we can choose $s_0, \ldots, s_r \in H^0(X, \mathcal{L}^{\otimes d})$ mapping to $s_{y,0}, \ldots, s_{y,r}$. Let $X_{s_i} \subset X$ be the open subset where s_i is a generator of $\mathcal{L}^{\otimes d}$. Since the $s_{y,i}$ generate $\mathcal{L}_y^{\otimes d}$ we see that $X_y \subset U = \bigcup X_{s_i}$. Since $X \to Y$ is closed, we see that there is an open neighbourhood $y \in V \subset Y$ such that $f^{-1}(V) \subset U$. After replacing Y by V we may assume that the s_i generate $\mathcal{L}^{\otimes d}$. Thus we obtain a morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes d}, (s_0, \ldots, s_r)} : X \longrightarrow \mathbf{P}^r_Y$$

with $\mathcal{L}^{\otimes d} \cong \varphi^*\mathcal{O}_{\mathbf{P}^r_Y}(1)$ whose base change to y gives φ_y.

We will finish the proof by a sleight of hand; the "correct" proof proceeds by directly showing that φ is a closed immersion after base changing to an open neighbourhood of y. Namely, by Lemma 30.21.2 we see that φ is a finite over an open neighbourhood of the fibre $\mathbf{P}^r_{\kappa(y)}$ of $\mathbf{P}^r_Y \to Y$ above y. Using that $\mathbf{P}^r_Y \to Y$ is closed, after shrinking Y we may assume that φ is finite. Then $\mathcal{L}^{\otimes d} \cong \varphi^*\mathcal{O}_{\mathbf{P}^r_Y}(1)$ is ample by the very general Morphisms, Lemma 29.37.7. $\square$

30.22. Cohomology and base change, III

07VJ In this section we prove the simplest case of a very general phenomenon that will be discussed in Derived Categories of Schemes, Section 36.22. Please see Remark 30.22.2 for a translation of the following lemma into algebra.

07VK Lemma 30.22.1. Let A be a Noetherian ring and set $S = \mathrm{Spec}(A)$. Let $f : X \to S$ be a proper morphism of schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over S. Then

(1) $R\Gamma(X, \mathcal{F})$ is a perfect object of $D(A)$, and
(2) for any ring map $A \to A'$ the base change map

$$R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} A' \longrightarrow R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

is an isomorphism.

Proof. Choose a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. By Lemmas 30.7.1 and 30.7.2 the Čech complex $K^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$ satisfies

$$K^\bullet \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

for all ring maps $A \to A'$. Let $K^\bullet_{alt} = \check{C}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ be the alternating Čech complex. By Cohomology, Lemma 20.23.6 there is a homotopy equivalence $K^\bullet_{alt} \to K^\bullet$ of A-modules. In particular, we have

$$K^\bullet_{alt} \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

as well. Since $\mathcal{F}$ is flat over A we see that each K^n_{alt} is flat over A (see Morphisms, Lemma 29.25.2). Since moreover $K^\bullet_{alt}$ is bounded above (this is why we switched to the alternating Čech complex) $K^\bullet_{alt} \otimes_A A' = K^\bullet_{alt} \otimes_A^{\mathbf{L}} A'$ by the definition of derived tensor products (see More on Algebra, Section 15.59). By Lemma 30.19.2 the cohomology groups $H^i(K^\bullet_{alt})$ are finite A-modules. As $K^\bullet_{alt}$ is bounded, we conclude that $K^\bullet_{alt}$ is pseudo-coherent, see More on Algebra, Lemma 15.64.17. Given any A-module M set $A' = A \oplus M$ where M is a square zero ideal, i.e., $(a, m) \cdot (a', m') = (aa', am' + a'm)$. By the above we see that $K^\bullet_{alt} \otimes_A^{\mathbf{L}} A'$ has cohomology in degrees $0, \ldots, n$. Hence $K^\bullet_{alt} \otimes_A^{\mathbf{L}} M$ has cohomology in degrees $0, \ldots, n$. Hence $K^\bullet_{alt}$ has finite Tor dimension, see More on Algebra, Definition 15.66.1. We win by More on Algebra, Lemma 15.74.2. $\square$

07VL Remark 30.22.2. A consequence of Lemma 30.22.1 is that there exists a finite complex of finite projective A-modules $M^\bullet$ such that we have

$$H^i(X_{A'}, \mathcal{F}_{A'}) = H^i(M^\bullet \otimes_A A')$$

functorially in A'. The condition that $\mathcal{F}$ is flat over A is essential, see [Har98].

30.23. Coherent formal modules

0EHN As we do not yet have the theory of formal schemes to our disposal, we develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal scheme".

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. We will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism of such inverse systems is defined as usual. Let us denote the category of these inverse systems with $\mathrm{Coh}(X, \mathcal{I})$. We are going to proceed by proving a bunch of lemmas about objects in this category. In fact, most of the lemmas that follow are straightforward consequences of the following description of the category in the affine case.

087W **Lemma 30.23.1.** If $X = \mathrm{Spec}(A)$ is the spectrum of a Noetherian ring and $\mathcal{I}$ is the quasi-coherent sheaf of ideals associated to the ideal $I \subset A$, then $\mathrm{Coh}(X, \mathcal{I})$ is equivalent to the category of finite $A^\wedge$-modules where $A^\wedge$ is the completion of A with respect to I.

Proof. Let $\mathrm{Mod}^{fg}_{A,I}$ be the category of inverse systems (M_n) of finite A-modules satisfying: (1) M_n is annihilated by I^n and (2) $M_{n+1}/I^nM_{n+1} = M_n$. By the correspondence between coherent sheaves on X and finite A-modules (Lemma 30.9.1) it suffices to show $\mathrm{Mod}^{fg}_{A,I}$ is equivalent to the category of finite $A^\wedge$-modules. To see this it suffices to prove that given an object (M_n) of $\mathrm{Mod}^{fg}_{A,I}$ the module

$$M = \lim M_n$$

is a finite $A^\wedge$-module and that $M/I^nM = M_n$. As the transition maps are surjective, we see that $M \to M_1$ is surjective. Pick $x_1, \ldots, x_t \in M$ which map to generators of M_1. This induces a map of systems $(A/I^n)^{\oplus t} \to M_n$. By Nakayama's lemma (Algebra, Lemma 10.20.1) these maps are surjective. Let $K_n \subset (A/I^n)^{\oplus t}$ be the kernel. Property (2) implies that $K_{n+1} \to K_n$ is surjective, in particular the system (K_n) satisfies the Mittag-Leffler condition. By Homology, Lemma 12.31.3 we obtain an exact sequence $0 \to K \to (A^\wedge)^{\oplus t} \to M \to 0$ with $K = \lim K_n$. Hence M is a finite $A^\wedge$-module. As $K \to K_n$ is surjective it follows that

$$M/I^nM = \mathrm{Coker}(K \to (A/I^n)^{\oplus t}) = (A/I^n)^{\oplus t}/K_n = M_n$$

as desired. □

087X **Lemma 30.23.2.** Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.

(1) The category $\mathrm{Coh}(X, \mathcal{I})$ is abelian.
(2) For $U \subset X$ open the restriction functor $\mathrm{Coh}(X, \mathcal{I}) \to \mathrm{Coh}(U, \mathcal{I}|_U)$ is exact.
(3) Exactness in $\mathrm{Coh}(X, \mathcal{I})$ may be checked by restricting to the members of an open covering of X.

Proof. Let $\alpha = (\alpha_n) : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $\mathrm{Coh}(X, \mathcal{I})$. The cokernel of α is the inverse system $(\mathrm{Coker}(\alpha_n))$ (details omitted). To describe the kernel let

$$\mathcal{K}'_{l,m} = \mathrm{Im}(\mathrm{Ker}(\alpha_l) \to \mathcal{F}_m)$$

for $l \geq m$. We claim:

(a) the inverse system $(\mathcal{K}'_{l,m})_{l \geq m}$ is eventually constant, say with value $\mathcal{K}'_m$,
(b) the system $(\mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m)_{m \geq n}$ is eventually constant, say with value $\mathcal{K}_n$,
(c) the system $(\mathcal{K}_n)$ forms an object of $\text{Coh}(X, \mathcal{I})$, and
(d) this object is the kernel of α.

To see (a), (b), and (c) we may work affine locally, say $X = \text{Spec}(A)$ and $\mathcal{I}$ corresponds to the ideal $I \subset A$. By Lemma 30.23.1 α corresponds to a map $f : M \to N$ of finite $A^\wedge$-modules. Denote $K = \text{Ker}(f)$. Note that $A^\wedge$ is a Noetherian ring (Algebra, Lemma 10.97.6). Choose an integer $c \geq 0$ such that $K \cap I^n M \subset I^{n-c}K$ for $n \geq c$ (Algebra, Lemma 10.51.2) and which satisfies Algebra, Lemma 10.51.3 for the map f and the ideal $I^\wedge = IA^\wedge$. Then $\mathcal{K}'_{l,m}$ corresponds to the A-module

$$K'_{l,m} = \frac{a^{-1}(I^l N) + I^m M}{I^m M} = \frac{K + I^{l-c}f^{-1}(I^c N) + I^m M}{I^m M} = \frac{K + I^m M}{I^m M}$$

where the last equality holds if $l \geq m + c$. So $\mathcal{K}'_m$ corresponds to the A-module $K/K \cap I^m M$ and $\mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m$ corresponds to

$$\frac{K}{K \cap I^m M + I^n K} = \frac{K}{I^n K}$$

for $m \geq n + c$ by our choice of c above. Hence $\mathcal{K}_n$ corresponds to K/I^nK.

We prove (d). It is clear from the description on affines above that the composition $(\mathcal{K}_n) \to (\mathcal{F}_n) \to (\mathcal{G}_n)$ is zero. Let $\beta : (\mathcal{H}_n) \to (\mathcal{F}_n)$ be a morphism such that $\alpha \circ \beta = 0$. Then $\mathcal{H}_l \to \mathcal{F}_l$ maps into $\text{Ker}(\alpha_l)$. Since $\mathcal{H}_m = \mathcal{H}_l/\mathcal{I}^m\mathcal{H}_l$ for $l \geq m$ we obtain a system of maps $\mathcal{H}_m \to \mathcal{K}'_{l,m}$. Thus a map $\mathcal{H}_m \to \mathcal{K}'_m$. Since $\mathcal{H}_n = \mathcal{H}_m/\mathcal{I}^n\mathcal{H}_m$ we obtain a system of maps $\mathcal{H}_n \to \mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m$ and hence a map $\mathcal{H}_n \to \mathcal{K}_n$ as desired.

To finish the proof of (1) we still have to show that Coim = Im in $\text{Coh}(X, \mathcal{I})$. We have seen above that taking kernels and cokernels commutes, over affines, with the description of $\text{Coh}(X, \mathcal{I})$ as a category of modules. Since Im = Coim holds in the category of modules this gives Coim = Im in $\text{Coh}(X, \mathcal{I})$. Parts (2) and (3) of the lemma are immediate from our construction of kernels and cokernels. ☐

087Y **Lemma 30.23.3.** Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $\text{Coh}(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.

Proof. Omitted. Hint: Look on affine opens, use Lemma 30.23.1, and use Algebra, Lemma 10.20.1. ☐

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

0880 (30.23.3.1) $$\text{Coh}(\mathcal{O}_X) \longrightarrow \text{Coh}(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $\text{Coh}(X, \mathcal{I})$.

0881 **Lemma 30.23.4.** The functor (30.23.3.1) is exact.

Proof. It suffices to check this locally on X. Hence we may assume X is affine, i.e., we have a situation as in Lemma 30.23.1. The functor is the functor $\text{Mod}^{fg}_A \to \text{Mod}^{fg}_{A^\wedge}$ which associates to a finite A-module M the completion $M^\wedge$. Thus the result follows from Algebra, Lemma 10.97.2. ☐

0882 **Lemma 30.23.5.** Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. Then

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{\mathrm{Coh}(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge).$$

Proof. To prove this we may work affine locally on X. Hence we may assume $X = \mathrm{Spec}(A)$ and $\mathcal{F}$, $\mathcal{G}$ given by finite A-module M and N. Then $\mathcal{H}$ corresponds to the finite A-module $H = \mathrm{Hom}_A(M, N)$. The statement of the lemma becomes the statement

$$H^\wedge = \mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

via the equivalence of Lemma 30.23.1. By Algebra, Lemma 10.97.2 (used 3 times) we have

$$H^\wedge = \mathrm{Hom}_A(M, N) \otimes_A A^\wedge = \mathrm{Hom}_{A^\wedge}(M \otimes_A A^\wedge, N \otimes_A A^\wedge) = \mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

where the second equality uses that $A^\wedge$ is flat over A (see More on Algebra, Lemma 15.65.4). The lemma follows. $\square$

Let X be a Noetherian scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. We say an object $(\mathcal{F}_n)$ of $\mathrm{Coh}(X, \mathcal{I})$ is *$\mathcal{I}$-power torsion* or is *annihilated by a power of $\mathcal{I}$* if there exists a $c \geq 1$ such that $\mathcal{F}_n = \mathcal{F}_c$ for all $n \geq c$. If this is the case we will say that $(\mathcal{F}_n)$ is annihilated by $\mathcal{I}^c$. If $X = \mathrm{Spec}(A)$ is affine, then, via the equivalence of Lemma 30.23.1, these objects corresponds exactly to the finite A-modules annihilated by a power of I or by I^c.

0889 **Lemma 30.23.6.** Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Let $(\mathcal{F}_n)$ an object of $\mathrm{Coh}(X, \mathcal{I})$.

(1) If $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ is a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$, then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where
   (a) $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,
   (b) $a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,
   (c) $\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and
   (d) $\alpha = a^\wedge \circ \beta$.
(2) If $\alpha : \mathcal{G}^\wedge \to (\mathcal{F}_n)$ is a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$, then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where
   (a) $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,
   (b) $a : \mathcal{G} \to \mathcal{F}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,
   (c) $\beta : \mathcal{F}^\wedge \to (\mathcal{F}_n)$ is an isomorphism, and
   (d) $\alpha = \beta \circ a^\wedge$.

Proof. Proof of (1). The uniqueness implies it suffices to construct $(\mathcal{F}, a, \beta)$ Zariski locally on X. Thus we may assume $X = \mathrm{Spec}(A)$ and $\mathcal{I}$ corresponds to the ideal $I \subset A$. In this situation Lemma 30.23.1 applies. Let M' be the finite $A^\wedge$-module corresponding to $(\mathcal{F}_n)$. Let N be the finite A-module corresponding to $\mathcal{G}$. Then α corresponds to a map

$$\varphi : M' \longrightarrow N^\wedge$$

whose kernel and cokernel are annihilated by I^t for some t. Recall that $N^\wedge = N \otimes_A A^\wedge$ (Algebra, Lemma 10.97.1). By More on Algebra, Lemma 15.89.16 there is an A-module map $\psi : M \to N$ whose kernel and cokernel are I-power torsion and an isomorphism $M \otimes_A A^\wedge = M'$ compatible with φ. As N and M' are finite modules, we conclude that M is a finite A-module, see More on Algebra, Remark 15.89.19. Hence $M \otimes_A A^\wedge = M^\wedge$. We omit the verification that the triple $(M, N \to M, M^\wedge \to M')$ so obtained is unique up to unique isomorphism.

The proof of (2) is exactly the same and we omit it. $\square$

0EHP Lemma 30.23.7. Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Any object of $\text{Coh}(X, \mathcal{I})$ which is annihilated by a power of $\mathcal{I}$ is in the essential image of (30.23.3.1). Moreover, if $\mathcal{F}$, $\mathcal{G}$ are in $\text{Coh}(\mathcal{O}_X)$ and either $\mathcal{F}$ or $\mathcal{G}$ is annihilated by a power of $\mathcal{I}$, then the maps

$$\begin{array}{ccc} \text{Hom}_X(\mathcal{F}, \mathcal{G}) & & \text{Ext}_X(\mathcal{F}, \mathcal{G}) \\ \downarrow & & \downarrow \\ \text{Hom}_{\text{Coh}(X, \mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge) & & \text{Ext}_{\text{Coh}(X, \mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge) \end{array}$$

are isomorphisms.

Proof. Suppose $(\mathcal{F}_n)$ is an object of $\text{Coh}(X, \mathcal{I})$ which is annihilated by $\mathcal{I}^c$ for some $c \geq 1$. Then $\mathcal{F}_n \to \mathcal{F}_c$ is an isomorphism for $n \geq c$. Hence if we set $\mathcal{F} = \mathcal{F}_c$, then we see that $\mathcal{F}^\wedge \cong (\mathcal{F}_n)$. This proves the first assertion.

Let $\mathcal{F}$, $\mathcal{G}$ be objects of $\text{Coh}(\mathcal{O}_X)$ such that either $\mathcal{F}$ or $\mathcal{G}$ is annihilated by $\mathcal{I}^c$ for some $c \geq 1$. Then $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}^c$. Hence we see that

$$\text{Hom}_X(\mathcal{G}, \mathcal{F}) = H^0(X, \mathcal{H}) = \lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \text{Mor}_{\text{Coh}(X, \mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge).$$

see Lemma 30.23.5. This proves the statement on homomorphisms.

The notation Ext refers to extensions as defined in Homology, Section 12.6. The injectivity of the map on Ext's follows immediately from the bijectivity of the map on Hom's. For surjectivity, assume $\mathcal{F}$ is annihilated by a power of I. Then part (1) of Lemma 30.23.6 shows that given an extension

$$0 \to \mathcal{G}^\wedge \to (\mathcal{E}_n) \to \mathcal{F}^\wedge \to 0$$

in $\text{Coh}(U, I\mathcal{O}_U)$ the morphism $\mathcal{G}^\wedge \to (\mathcal{E}_n)$ is isomorphic to $\mathcal{G} \to \mathcal{E}^\wedge$ for some $\mathcal{G} \to \mathcal{E}$ in $\text{Coh}(\mathcal{O}_U)$. Similarly in the other case. $\square$

087Z Lemma 30.23.8. Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If $(\mathcal{F}_n)$ is an object of $\text{Coh}(X, \mathcal{I})$ then $\bigoplus \text{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ is a finite type, graded, quasi-coherent $\bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$-module.

Proof. The question is local on X hence we may assume X is affine, i.e., we have a situation as in Lemma 30.23.1. In this case, if $(\mathcal{F}_n)$ corresponds to the finite $A^\wedge$ module M, then $\bigoplus \text{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ corresponds to $\bigoplus I^nM/I^{n+1}M$ which is clearly a finite module over $\bigoplus I^n/I^{n+1}$. $\square$

0887 Lemma 30.23.9. Let $f : X \to Y$ be a morphism of Noetherian schemes. Let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals and set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Then there is a right exact functor

$$f^* : \text{Coh}(Y, \mathcal{J}) \longrightarrow \text{Coh}(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. If f is flat, then f^* is an exact functor.

Proof. Since $f^* : \mathrm{Coh}(\mathcal{O}_Y) \to \mathrm{Coh}(\mathcal{O}_X)$ is right exact we have

$$f^*\mathcal{G}_n = f^*(\mathcal{G}_{n+1}/\mathcal{I}^n\mathcal{G}_{n+1}) = f^*\mathcal{G}_{n+1}/f^{-1}\mathcal{I}^n f^*\mathcal{G}_{n+1} = f^*\mathcal{G}_{n+1}/\mathcal{J}^n f^*\mathcal{G}_{n+1}$$

hence the pullback of a system is a system. The construction of cokernels in the proof of Lemma 30.23.2 shows that $f^* : \mathrm{Coh}(Y, \mathcal{J}) \to \mathrm{Coh}(X, \mathcal{I})$ is always right exact. If f is flat, then $f^* : \mathrm{Coh}(\mathcal{O}_Y) \to \mathrm{Coh}(\mathcal{O}_X)$ is an exact functor. It follows from the construction of kernels in the proof of Lemma 30.23.2 that in this case $f^* : \mathrm{Coh}(Y, \mathcal{J}) \to \mathrm{Coh}(X, \mathcal{I})$ also transforms kernels into kernels. □

0EHQ Lemma 30.23.10. Let $f : X' \to X$ be a morphism of Noetherian schemes. Let $Z \subset X$ be a closed subscheme and denote $Z' = f^{-1}Z$ the scheme theoretic inverse image. Let $\mathcal{I} \subset \mathcal{O}_X$, $\mathcal{I}' \subset \mathcal{O}_{X'}$ be the corresponding quasi-coherent sheaves of ideals. If f is flat and the induced morphism $Z' \to Z$ is an isomorphism, then the pullback functor $f^* : \mathrm{Coh}(X, \mathcal{I}) \to \mathrm{Coh}(X', \mathcal{I}')$ (Lemma 30.23.9) is an equivalence.

Proof. If X and X' are affine, then this follows immediately from More on Algebra, Lemma 15.89.3. To prove it in general we let $Z_n \subset X$, $Z'_n \subset X'$ be the nth infinitesimal neighbourhoods of Z, Z'. The induced morphism $Z_n \to Z'_n$ is a homeomorphism on underlying topological spaces. On the other hand, if $z' \in Z'$ maps to $z \in Z$, then the ring map $\mathcal{O}_{X,z} \to \mathcal{O}_{X',z'}$ is flat and induces an isomorphism $\mathcal{O}_{X,z}/\mathcal{I}_z \to \mathcal{O}_{X',z'}/\mathcal{I}'_{z'}$. Hence it induces an isomorphism $\mathcal{O}_{X,z}/\mathcal{I}_z^n \to \mathcal{O}_{X',z'}/(\mathcal{I}'_{z'})^n$ for all $n \geq 1$ for example by More on Algebra, Lemma 15.89.2. Thus $Z'_n \to Z_n$ is an isomorphism of schemes. Thus f^* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}^n$ and the category of coherent $\mathcal{O}_{X'}$-modules annihilated by $(\mathcal{I}')^n$, see Lemma 30.9.8. This clearly implies the lemma. □

0EHR Lemma 30.23.11. Let X be a Noetherian scheme. Let $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. If $V(\mathcal{I}) = V(\mathcal{J})$ is the same closed subset of X, then $\mathrm{Coh}(X, \mathcal{I})$ and $\mathrm{Coh}(X, \mathcal{J})$ are equivalent.

Proof. First, assume $X = \mathrm{Spec}(A)$ is affine. Let $I, J \subset A$ be the ideals corresponding to $\mathcal{I}, \mathcal{J}$. Then $V(I) = V(J)$ implies we have $I^c \subset J$ and $J^d \subset I$ for some $c, d \geq 1$ by elementary properties of the Zariski topology (see Algebra, Section 10.17 and Lemma 10.32.5). Hence the I-adic and J-adic completions of A agree, see Algebra, Lemma 10.96.9. Thus the equivalence follows from Lemma 30.23.1 in this case.

In general, using what we said above and the fact that X is quasi-compact, to choose $c, d \geq 1$ such that $\mathcal{I}^c \subset \mathcal{J}$ and $\mathcal{J}^d \subset \mathcal{I}$. Then given an object $(\mathcal{F}_n)$ in $\mathrm{Coh}(X, \mathcal{I})$ we claim that the inverse system

$$(\mathcal{F}_{cn}/\mathcal{J}^n\mathcal{F}_{cn})$$

is in $\mathrm{Coh}(X, \mathcal{J})$. This may be checked on the members of an affine covering; we omit the details. In the same manner we can construct an object of $\mathrm{Coh}(X, \mathcal{I})$ starting with an object of $\mathrm{Coh}(X, \mathcal{J})$. We omit the verification that these constructions define mutually quasi-inverse functors. □

30.24. Grothendieck's existence theorem, I

087V In this section we discuss Grothendieck's existence theorem for the projective case. We will use the notion of coherent formal modules developed in Section 30.23. The reader who is familiar with formal schemes is encouraged to read the statement and proof of the theorem in [DG67].

0883 Lemma 30.24.1. Let A be Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (30.23.3.1) is fully faithful.

Proof. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Then $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is a coherent $\mathcal{O}_X$-module, see Modules, Lemma 17.22.6. By Lemma 30.23.5 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \mathrm{Mor}_{\mathrm{Coh}(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Hence fully faithfulness of (30.23.3.1) follows from the theorem on formal functions (Lemma 30.20.6) for the coherent sheaf $\mathcal{H}$. □

0884 Lemma 30.24.2. Let A be Noetherian ring and $I \subset A$ an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism and let $\mathcal{L}$ be an f-ample invertible sheaf. Let $\mathcal{I} = I\mathcal{O}_X$. Let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(X, \mathcal{I})$. Then there exists an integer d_0 such that

$$H^1(X, \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n) \otimes \mathcal{L}^{\otimes d}) = 0$$

for all $n \geq 0$ and all $d \geq d_0$.

Proof. Set $B = \bigoplus I^n/I^{n+1}$ and $\mathcal{B} = \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1} = f^*\widetilde{B}$. By Lemma 30.23.8 the graded quasi-coherent $\mathcal{B}$-module $\mathcal{G} = \bigoplus \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ is of finite type. Hence the lemma follows from Lemma 30.19.3 part (2). □

0885 Lemma 30.24.3. Let A be Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a projective morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (30.23.3.1) is an equivalence.

Proof. We have already seen that (30.23.3.1) is fully faithful in Lemma 30.24.1. Thus it suffices to show that the functor is essentially surjective.

We first show that every object $(\mathcal{F}_n)$ of $\mathrm{Coh}(X, \mathcal{I})$ is the quotient of an object in the image of (30.23.3.1). Let $\mathcal{L}$ be an f-ample invertible sheaf on X. Choose d_0 as in Lemma 30.24.2. Choose a $d \geq d_0$ such that $\mathcal{F}_1 \otimes \mathcal{L}^{\otimes d}$ is globally generated by some sections $s_{1,1}, \ldots, s_{t,1}$. Since the transition maps of the system

$$H^0(X, \mathcal{F}_{n+1} \otimes \mathcal{L}^{\otimes d}) \longrightarrow H^0(X, \mathcal{F}_n \otimes \mathcal{L}^{\otimes d})$$

are surjective by the vanishing of H^1 we can lift $s_{1,1}, \ldots, s_{t,1}$ to a compatible system of global sections $s_{1,n}, \ldots, s_{t,n}$ of $\mathcal{F}_n \otimes \mathcal{L}^{\otimes d}$. These determine a compatible system of maps

$$(s_{1,n}, \ldots, s_{t,n}) : (\mathcal{L}^{\otimes -d})^{\oplus t} \longrightarrow \mathcal{F}_n$$

Using Lemma 30.23.3 we deduce that we have a surjective map

$$\left((\mathcal{L}^{\otimes -d})^{\oplus t}\right)^\wedge \longrightarrow (\mathcal{F}_n)$$

as desired.

The result of the previous paragraph and the fact that $\mathrm{Coh}(X, \mathcal{I})$ is abelian (Lemma 30.23.2) implies that every object of $\mathrm{Coh}(X, \mathcal{I})$ is a cokernel of a map between objects coming from $\mathrm{Coh}(\mathcal{O}_X)$. As (30.23.3.1) is fully faithful and exact by Lemmas 30.24.1 and 30.23.4 we conclude. □

30.25. Grothendieck's existence theorem, II

0886 In this section we discuss Grothendieck's existence theorem in the proper case. Before we give the statement and proof, we need to develop a bit more theory regarding the categories $\mathrm{Coh}(X, \mathcal{I})$ of coherent formal modules introduced in Section 30.23.

0888 Remark 30.25.1. Let X be a Noetherian scheme and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $\mathrm{Coh}(X, \mathcal{I})$. Given an affine open $\mathrm{Spec}(A) = U \subset X$ with $\mathcal{I}|_U, \mathcal{K}|_U$ corresponding to ideals $I, K \subset A$ denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Lemma 30.23.1. We claim the following are equivalent

(1) there exists an integer $t \geq 1$ such that $\mathrm{Ker}(\alpha_n)$ and $\mathrm{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,
(2) for any affine open $\mathrm{Spec}(A) = U \subset X$ as above the modules $\mathrm{Ker}(\alpha_U)$ and $\mathrm{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$, and
(3) there exists a finite affine open covering $X = \bigcup U_i$ such that the conclusion of (2) holds for α_{U_i}.

If these equivalent conditions hold we will say that α is a map whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. To see the equivalence we use the following commutative algebra fact: suppose given an exact sequence

$$0 \to T \to M \to N \to Q \to 0$$

of A-modules with T and Q annihilated by K^t for some ideal $K \subset A$. Then for every $f, g \in K^t$ there exists a canonical map "fg" $: N \to M$ such that $M \to N \to M$ is equal to multiplication by fg. Namely, for $y \in N$ we can pick $x \in M$ mapping to fy in N and then we can set "fg"$(y) = gx$. Thus it is clear that $\mathrm{Ker}(M/JM \to N/JN)$ and $\mathrm{Coker}(M/JM \to N/JN)$ are annihilated by K^{2t} for any ideal $J \subset A$.

Applying the commutative algebra fact to α_{U_i} and $J = I^n$ we see that (3) implies (1). Conversely, suppose (1) holds and $M \to N$ is equal to α_U. Then there is a $t \geq 1$ such that $\mathrm{Ker}(M/I^nM \to N/I^nN)$ and $\mathrm{Coker}(M/I^nM \to N/I^nN)$ are annihilated by K^t for all n. We obtain maps "fg" $: N/I^nN \to M/I^nM$ which in the limit induce a map $N \to M$ as N and M are I-adically complete. Since the composition with $N \to M \to N$ is multiplication by fg we conclude that fg annihilates T and Q. In other words T and Q are annihilated by K^{2t} as desired.

088A Lemma 30.25.2. Let X be a Noetherian scheme. Let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $X_e \subset X$ be the closed subscheme cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(X, \mathcal{I})$. Assume

(1) the functor $\mathrm{Coh}(\mathcal{O}_{X_e}) \to \mathrm{Coh}(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and
(2) there exists a coherent sheaf $\mathcal{H}$ on X and a map $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Then $(\mathcal{F}_n)$ is in the essential image of (30.23.3.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Lemma 30.9.8. In particular we may identify $\mathrm{Coh}(\mathcal{O}_{X_e})$ with the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{K}^e$ and $\mathrm{Coh}(X_e, \mathcal{I}_e)$ as the full subcategory of

$\text{Coh}(X, \mathcal{I})$ of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (30.23.3.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $\text{Coh}(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. Hence $(\mathcal{G}_e)$ is an object of $\text{Coh}(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$, see Remark 30.25.1. Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

see Remark 30.25.1. We apply Lemma 30.23.6 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $\text{Coh}(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $\text{Coh}(X, \mathcal{K})$. Thus the uniqueness in Lemma 30.23.6 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight. This finishes the proof of the lemma. □

088B Lemma 30.25.3. Let Y be a Noetherian scheme. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Let $f : X \to Y$ be a proper morphism which is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $\text{Coh}(Y, \mathcal{J})$, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $\text{Coh}(X, \mathcal{I})$. Then there exists a map

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $\text{Coh}(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Proposition 30.19.1). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness we may assume that $Y = \text{Spec}(B)$ is affine. Let $J \subset B$ corresponds to the ideal $\mathcal{J}$. Set

$$M_n = H^0(X, \mathcal{F}/\mathcal{I}^n\mathcal{F}) \quad \text{and} \quad M = H^0(X, \mathcal{F})$$

By Lemma 30.20.4 and Theorem 30.20.5 the inverse limit of the modules M_n equals the completion $M^\wedge = \lim M/J^nM$. Set $N_n = H^0(Y, \mathcal{G}_n)$ and $N = \lim N_n$. Via the equivalence of categories of Lemma 30.23.1 the finite $B^\wedge$ modules N and $M^\wedge$ correspond to $(\mathcal{G}_n)$ and $f_*\mathcal{F}^\wedge$. It follows from this that α has to be the morphism of $\text{Coh}(Y, \mathcal{J})$ corresponding to the homomorphism

$$\lim \gamma_n : N = \lim_n N_n \longrightarrow \lim M_n = M^\wedge$$

of finite $B^\wedge$-modules.

We still have to show that the kernel and cokernel of α are annihilated by a power of $\mathcal{K}$. Set $Y' = \mathrm{Spec}(B^\wedge)$ and $X' = Y' \times_Y X$. Let $\mathcal{K}'$, $\mathcal{J}'$, $\mathcal{G}'_n$ and $\mathcal{I}'$, $\mathcal{F}'$ be the pullback of $\mathcal{K}$, $\mathcal{J}$, $\mathcal{G}_n$ and $\mathcal{I}$, $\mathcal{F}$, to Y' and X'. The projection morphism $f' : X' \to Y'$ is the base change of f by $Y' \to Y$. Note that $Y' \to Y$ is a flat morphism of schemes as $B \to B^\wedge$ is flat by Algebra, Lemma 10.97.2. Hence $f'_*\mathcal{F}'$, resp. $f'_*(f')^*\mathcal{G}'_n$ is the pullback of $f_*\mathcal{F}$, resp. $f_*f^*\mathcal{G}_n$ to Y' by Lemma 30.5.2. The uniqueness of our construction shows the pullback of α to Y' is the corresponding map α' constructed for the situation on Y'. Moreover, to check that the kernel and cokernel of α are annihilated by $\mathcal{K}^t$ it suffices to check that the kernel and cokernel of α' are annihilated by $(\mathcal{K}')^t$. Namely, to see this we need to check this for kernels and cokernels of the maps α_n and α'_n (see Remark 30.25.1) and the ring map $B \to B^\wedge$ induces an equivalence of categories between modules annihilated by J^n and $(J')^n$, see More on Algebra, Lemma 15.89.3. Thus we may assume B is complete with respect to J.

Assume $Y = \mathrm{Spec}(B)$ is affine, $\mathcal{J}$ corresponds to the ideal $J \subset B$, and B is complete with respect to J. In this case $(\mathcal{G}_n)$ is in the essential image of the functor $\mathrm{Coh}(\mathcal{O}_Y) \to \mathrm{Coh}(Y, \mathcal{J})$. Say $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module such that $(\mathcal{G}_n) = \mathcal{G}^\wedge$. Note that $f^*(\mathcal{G}^\wedge) = (f^*\mathcal{G})^\wedge$. Hence Lemma 30.24.1 tells us that β comes from an isomorphism $b : f^*\mathcal{G} \to \mathcal{F}$ and α is the completion functor applied to

$$\mathcal{G} \to f_*f^*\mathcal{G} \cong f_*\mathcal{F}$$

Hence we are trying to verify that the kernel and cokernel of the adjunction map $c : \mathcal{G} \to f_*f^*\mathcal{G}$ are annihilated by a power of $\mathcal{K}$. However, since the restriction $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an isomorphism we see that $c|_V$ is an isomorphism. Thus the coherent sheaves $\mathrm{Ker}(c)$ and $\mathrm{Coker}(c)$ are supported on $V(\mathcal{K})$ hence are annihilated by a power of $\mathcal{K}$ (Lemma 30.10.2) as desired. □

The following proposition is the form of Grothendieck's existence theorem which is most often used in practice.

088C Proposition 30.25.4. Let A be a Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. Then the functor (30.23.3.1) is an equivalence.

Proof. We have already seen that (30.23.3.1) is fully faithful in Lemma 30.24.1. Thus it suffices to show that the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that every object $(\mathcal{F}_n)$ annihilated by $\mathcal{K}$ is in the essential image. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Lemma 30.10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. (This uses the correspondence by coherent modules annihilated by $\mathcal{K}$ and coherent modules on the closed subscheme corresponding to $\mathcal{K}$, see Lemma 30.9.8.) Let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}(X, \mathcal{I})$. We will show that this object is in the essential image of the functor (30.23.3.1), thereby completion the proof of the proposition.

Apply Chow's lemma (Lemma 30.18.1) to find a proper surjective morphism $f : X' \to X$ which is an isomorphism over a dense open $U \subset X$ such that X' is projective over A. Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By the projective case of Grothendieck's existence theorem (Lemma 30.24.3) there exists a coherent module $\mathcal{F}'$ on X' such that $(\mathcal{F}')^\wedge \cong (f^*\mathcal{F}_n)$. By Proposition 30.19.1 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 30.25.3 there exists a morphism $(\mathcal{F}_n) \to \mathcal{H}^\wedge$ of $\mathrm{Coh}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. The powers $\mathcal{K}^e$ are all in Ξ so that (30.23.3.1) is an equivalence for the closed subschemes $X_e = V(\mathcal{K}^e)$. We conclude by Lemma 30.25.2. $\square$

30.26. Being proper over a base

0CYK This is just a short section to point out some useful features of closed subsets proper over a base and finite type, quasi-coherent modules with support proper over a base.

0CYL Lemma 30.26.1. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $Z \subset X$ be a closed subset. The following are equivalent

(1) the morphism $Z \to S$ is proper if Z is endowed with the reduced induced closed subscheme structure (Schemes, Definition 26.12.5),
(2) for some closed subscheme structure on Z the morphism $Z \to S$ is proper,
(3) for any closed subscheme structure on Z the morphism $Z \to S$ is proper.

Proof. The implications (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) are immediate. Thus it suffices to prove that (2) implies (3). We urge the reader to find their own proof of this fact. Let Z' and Z'' be closed subscheme structures on Z such that $Z' \to S$ is proper. We have to show that $Z'' \to S$ is proper. Let $Z''' = Z' \cup Z''$ be the scheme theoretic union, see Morphisms, Definition 29.4.4. Then Z''' is another closed subscheme structure on Z. This follows for example from the description of scheme theoretic unions in Morphisms, Lemma 29.4.6. Since $Z'' \to Z'''$ is a closed immersion it suffices to prove that $Z''' \to S$ is proper (see Morphisms, Lemmas 29.41.6 and 29.41.4). The morphism $Z' \to Z'''$ is a bijective closed immersion and in particular surjective and universally closed. Then the fact that $Z' \to S$ is separated implies that $Z''' \to S$ is separated, see Morphisms, Lemma 29.41.11. Moreover $Z''' \to S$ is locally of finite type as $X \to S$ is locally of finite type (Morphisms, Lemmas 29.15.5 and 29.15.3). Since $Z' \to S$ is quasi-compact and $Z' \to Z'''$ is a homeomorphism we see that $Z''' \to S$ is quasi-compact. Finally, since $Z' \to S$ is universally closed, we see that the same thing is true for $Z''' \to S$ by Morphisms, Lemma 29.41.9. This finishes the proof. $\square$

0CYM Definition 30.26.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $Z \subset X$ be a closed subset. We say Z is proper over S if the equivalent conditions of Lemma 30.26.1 are satisfied.

The lemma used in the definition above is false if the morphism $f : X \to S$ is not locally of finite type. Therefore we urge the reader not to use this terminology if f is not locally of finite type.

0CYN Lemma 30.26.3. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $Y \subset Z \subset X$ be closed subsets. If Z is proper over S, then the same is true for Y.

Proof. Omitted. □

0CYP Lemma 30.26.4. Consider a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with f locally of finite type. If Z is a closed subset of X proper over S, then $(g')^{-1}(Z)$ is a closed subset of X' proper over S'.

Proof. Observe that the statement makes sense as f' is locally of finite type by Morphisms, Lemma 29.15.4. Endow Z with the reduced induced closed subscheme structure. Denote $Z' = (g')^{-1}(Z)$ the scheme theoretic inverse image (Schemes, Definition 26.17.7). Then $Z' = X' \times_X Z = (S' \times_S X) \times_X Z = S' \times_S Z$ is proper over S' as a base change of Z over S (Morphisms, Lemma 29.41.5). □

0CYQ Lemma 30.26.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes which are locally of finite type over S.

(1) If Y is separated over S and $Z \subset X$ is a closed subset proper over S, then $f(Z)$ is a closed subset of Y proper over S.
(2) If f is universally closed and $Z \subset X$ is a closed subset proper over S, then $f(Z)$ is a closed subset of Y proper over S.
(3) If f is proper and $Z \subset Y$ is a closed subset proper over S, then $f^{-1}(Z)$ is a closed subset of X proper over S.

Proof. Proof of (1). Assume Y is separated over S and $Z \subset X$ is a closed subset proper over S. Endow Z with the reduced induced closed subscheme structure and apply Morphisms, Lemma 29.41.10 to $Z \to Y$ over S to conclude.

Proof of (2). Assume f is universally closed and $Z \subset X$ is a closed subset proper over S. Endow Z and $Z' = f(Z)$ with their reduced induced closed subscheme structures. We obtain an induced morphism $Z \to Z'$. Denote $Z'' = f^{-1}(Z')$ the scheme theoretic inverse image (Schemes, Definition 26.17.7). Then $Z'' \to Z'$ is universally closed as a base change of f (Morphisms, Lemma 29.41.5). Hence $Z \to Z'$ is universally closed as a composition of the closed immersion $Z \to Z''$ and $Z'' \to Z'$ (Morphisms, Lemmas 29.41.6 and 29.41.4). We conclude that $Z' \to S$ is separated by Morphisms, Lemma 29.41.11. Since $Z \to S$ is quasi-compact and $Z \to Z'$ is surjective we see that $Z' \to S$ is quasi-compact. Since $Z' \to S$ is the composition of $Z' \to Y$ and $Y \to S$ we see that $Z' \to S$ is locally of finite type (Morphisms, Lemmas 29.15.5 and 29.15.3). Finally, since $Z \to S$ is universally closed, we see that the same thing is true for $Z' \to S$ by Morphisms, Lemma 29.41.9. This finishes the proof.

Proof of (3). Assume f is proper and $Z \subset Y$ is a closed subset proper over S. Endow Z with the reduced induced closed subscheme structure. Denote $Z' = f^{-1}(Z)$ the scheme theoretic inverse image (Schemes, Definition 26.17.7). Then $Z' \to Z$ is proper as a base change of f (Morphisms, Lemma 29.41.5). Whence $Z' \to S$ is proper as the composition of $Z' \to Z$ and $Z \to S$ (Morphisms, Lemma 29.41.4). This finishes the proof. □

0CYR Lemma 30.26.6. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $Z_i \subset X$, $i = 1, \ldots, n$ be closed subsets. If Z_i, $i = 1, \ldots, n$ are proper over S, then the same is true for $Z_1 \cup \ldots \cup Z_n$.

Proof. Endow Z_i with their reduced induced closed subscheme structures. The morphism

$$Z_1 \amalg \ldots \amalg Z_n \longrightarrow X$$

is finite by Morphisms, Lemmas 29.44.12 and 29.44.13. As finite morphisms are universally closed (Morphisms, Lemma 29.44.11) and since $Z_1 \amalg \ldots \amalg Z_n$ is proper over S we conclude by Lemma 30.26.5 part (2) that the image $Z_1 \cup \ldots \cup Z_n$ is proper over S. □

Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Then the support $\text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a closed subset of X, see Morphisms, Lemma 29.5.3. Hence it makes sense to say "the support of $\mathcal{F}$ is proper over S".

0CYS Lemma 30.26.7. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. The following are equivalent

(1) the support of $\mathcal{F}$ is proper over S,
(2) the scheme theoretic support of $\mathcal{F}$ (Morphisms, Definition 29.5.5) is proper over S, and
(3) there exists a closed subscheme $Z \subset X$ and a finite type, quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that (a) $Z \to S$ is proper, and (b) $(Z \to X)_*\mathcal{G} = \mathcal{F}$.

Proof. The support $\text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a closed subset of X, see Morphisms, Lemma 29.5.3. Hence we can apply Definition 30.26.2. Since the scheme theoretic support of $\mathcal{F}$ is a closed subscheme whose underlying closed subset is $\text{Supp}(\mathcal{F})$ we see that (1) and (2) are equivalent by Definition 30.26.2. It is clear that (2) implies (3). Conversely, if (3) is true, then $\text{Supp}(\mathcal{F}) \subset Z$ (an inclusion of closed subsets of X) and hence $\text{Supp}(\mathcal{F})$ is proper over S for example by Lemma 30.26.3. □

0CYT Lemma 30.26.8. Consider a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with f locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. If the support of $\mathcal{F}$ is proper over S, then the support of $(g')^*\mathcal{F}$ is proper over S'.

Proof. Observe that the statement makes sense because $(g') * \mathcal{F}$ is of finite type by Modules, Lemma 17.9.2. We have $\text{Supp}((g')^*\mathcal{F}) = (g')^{-1}(\text{Supp}(\mathcal{F}))$ by Morphisms, Lemma 29.5.3. Thus the lemma follows from Lemma 30.26.4. □

0CYU Lemma 30.26.9. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$, $\mathcal{G}$ be finite type, quasi-coherent $\mathcal{O}_X$-module.

(1) If the supports of $\mathcal{F}$, $\mathcal{G}$ are proper over S, then the same is true for $\mathcal{F} \oplus \mathcal{G}$, for any extension of $\mathcal{G}$ by $\mathcal{F}$, for $\text{Im}(u)$ and $\text{Coker}(u)$ given any $\mathcal{O}_X$-module map $u : \mathcal{F} \to \mathcal{G}$, and for any quasi-coherent quotient of $\mathcal{F}$ or $\mathcal{G}$.

(2) If S is locally Noetherian, then the category of coherent $\mathcal{O}_X$-modules with support proper over S is a Serre subcategory (Homology, Definition 12.10.1) of the abelian category of coherent $\mathcal{O}_X$-modules.

Proof. Proof of (1). Let Z, Z' be the support of $\mathcal{F}$ and $\mathcal{G}$. Then all the sheaves mentioned in (1) have support contained in $Z \cup Z'$. Thus the assertion itself is clear from Lemmas 30.26.3 and 30.26.6 provided we check that these sheaves are finite type and quasi-coherent. For quasi-coherence we refer the reader to Schemes, Section 26.24. For "finite type" we suggest the reader take a look at Modules, Section 17.9.

Proof of (2). The proof is the same as the proof of (1). Note that the assertions make sense as X is locally Noetherian by Morphisms, Lemma 29.15.6 and by the description of the category of coherent modules in Section 30.9. $\square$

08DS Lemma 30.26.10. Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with support proper over S. Then $R^pf_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $p \geq 0$.

Proof. By Lemma 30.26.7 there exists a closed immersion $i : Z \to X$ and a finite type, quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that (a) $g = f \circ i : Z \to S$ is proper, and (b) $i_*\mathcal{G} = \mathcal{F}$. We see that $R^pg_*\mathcal{G}$ is coherent on S by Proposition 30.19.1. On the other hand, $R^qi_*\mathcal{G} = 0$ for $q > 0$ (Lemma 30.9.9). By Cohomology, Lemma 20.13.8 we get $R^pf_*\mathcal{F} = R^pg_*\mathcal{G}$ which concludes the proof. $\square$

0CYV Lemma 30.26.11. Let S be a Noetherian scheme. Let $f : X \to S$ be a finite type morphism. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. The following are Serre subcategories of $\text{Coh}(X, \mathcal{I})$

(1) the full subcategory of $\text{Coh}(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the support of $\mathcal{F}_1$ is proper over S,
(2) the full subcategory of $\text{Coh}(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that there exists a closed subscheme $Z \subset X$ proper over S with $\mathcal{I}_Z\mathcal{F}_n = 0$ for all $n \geq 1$.

Proof. We will use the criterion of Homology, Lemma 12.10.2. Moreover, we will use that if $0 \to (\mathcal{G}_n) \to (\mathcal{F}_n) \to (\mathcal{H}_n) \to 0$ is a short exact sequence of $\text{Coh}(X, \mathcal{I})$, then (a) $\mathcal{G}_n \to \mathcal{F}_n \to \mathcal{H}_n \to 0$ is exact for all $n \geq 1$ and (b) $\mathcal{G}_n$ is a quotient of $\text{Ker}(\mathcal{F}_m \to \mathcal{H}_m)$ for some $m \geq n$. See proof of Lemma 30.23.2.

Proof of (1). Let $(\mathcal{F}_n)$ be an object of $\text{Coh}(X, \mathcal{I})$. Then $\text{Supp}(\mathcal{F}_n) = \text{Supp}(\mathcal{F}_1)$ for all $n \geq 1$. Hence by remarks (a) and (b) above we see that for any short exact sequence $0 \to (\mathcal{G}_n) \to (\mathcal{F}_n) \to (\mathcal{H}_n) \to 0$ of $\text{Coh}(X, \mathcal{I})$ we have $\text{Supp}(\mathcal{G}_1) \cup \text{Supp}(\mathcal{H}_1) = \text{Supp}(\mathcal{F}_1)$. This proves that the category defined in (1) is a Serre subcategory of $\text{Coh}(X, \mathcal{I})$.

Proof of (2). Here we argue the same way. Let $0 \to (\mathcal{G}_n) \to (\mathcal{F}_n) \to (\mathcal{H}_n) \to 0$ be a short exact sequence of $\text{Coh}(X, \mathcal{I})$. If $Z \subset X$ is a closed subscheme and $\mathcal{I}_Z$ annihilates $\mathcal{F}_n$ for all n, then $\mathcal{I}_Z$ annihilates $\mathcal{G}_n$ and $\mathcal{H}_n$ for all n by (a) and (b) above. Hence if $Z \to S$ is proper, then we conclude that the category defined in (2) is closed under taking sub and quotient objects inside of $\text{Coh}(X, \mathcal{I})$. Finally, suppose that $Z \subset X$ and $Y \subset X$ are closed subschemes proper over S such that $\mathcal{I}_Z\mathcal{G}_n = 0$ and $\mathcal{I}_Y\mathcal{H}_n = 0$ for all $n \geq 1$. Then it follows from (a) above that

$\mathcal{I}_{Z \cup Y} = \mathcal{I}_Z \cdot \mathcal{I}_Y$ annihilates $\mathcal{F}_n$ for all n. By Lemma 30.26.6 (and via Definition 30.26.2 which tells us we may choose an arbitrary scheme structure used on the union) we see that $Z \cup Y \to S$ is proper and the proof is complete. □

30.27. Grothendieck's existence theorem, III

0CYW To state the general version of Grothendieck's existence theorem we introduce a bit more notation. Let A be a Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a separated finite type morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. In this situation we let

$$\mathrm{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $\mathrm{Coh}(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose support is proper over $\mathrm{Spec}(A)$. This is a Serre subcategory of $\mathrm{Coh}(\mathcal{O}_X)$, see Lemma 30.26.9. Similarly, we let

$$\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $\mathrm{Coh}(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the support of $\mathcal{F}_1$ is proper over $\mathrm{Spec}(A)$. This is a Serre subcategory of $\mathrm{Coh}(X, \mathcal{I})$ by Lemma 30.26.11 part (1). Since the support of a quotient module is contained in the support of the module, it follows that (30.23.3.1) induces a functor

088D (30.27.0.1) $$\mathrm{Coh}_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow \mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

We are now ready to state the main theorem of this section.

088E **Theorem 30.27.1** (Grothendieck's existence theorem). Let A be a Noetherian ring complete with respect to an ideal I. Let X be a separated, finite type scheme over A. Then the functor (30.27.0.1) [DG67, III Theorem 5.1.5]

$$\mathrm{Coh}_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow \mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$$

is an equivalence.

Proof. We will use the equivalence of categories of Lemma 30.9.8 without further mention. For a closed subscheme $Z \subset X$ proper over A in this proof we will say a coherent module on X is "supported on Z" if it is annihilated by the ideal sheaf of Z or equivalently if it is the pushforward of a coherent module on Z. By Proposition 30.25.4 we know that the result is true for the functor between coherent modules and systems of coherent modules supported on Z. Hence it suffices to show that every object of $\mathrm{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$ and every object of $\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ is supported on a closed subscheme $Z \subset X$ proper over A. This holds by definition for objects of $\mathrm{Coh}_{\text{support proper over } A}(\mathcal{O}_X)$. We will prove this statement for objects of $\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ using the method of proof of Proposition 30.25.4. We urge the reader to read that proof first.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Lemma 30.10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is supported on a closed subscheme $Z \subset X$ proper over A, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 30.18.1) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 29.43.11. Observe that

$$\mathrm{Supp}(f^*\mathcal{F}_n) = f^{-1}\mathrm{Supp}(\mathcal{F}_n) = f^{-1}\mathrm{Supp}(\mathcal{F}_1)$$

The first equality by Morphisms, Lemma 29.5.3. By assumption and Lemma 30.26.5 part (3) we see that $f^{-1}\mathrm{Supp}(\mathcal{F}_1)$ is proper over A. Hence the image of $f^{-1}\mathrm{Supp}(\mathcal{F}_1)$ under j is closed in Y' by Lemma 30.26.5 part (1). Thus $\mathcal{F}'_n = j_*f^*\mathcal{F}_n$ is coherent on Y' by Lemma 30.9.11. It follows that $(\mathcal{F}'_n)$ is an object of $\mathrm{Coh}(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Lemma 30.24.3) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $\mathrm{Coh}(Y', I\mathcal{O}_{Y'})$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that

$$\mathrm{Supp}(\mathcal{F}') \cap V(I\mathcal{O}_{Y'}) = \mathrm{Supp}(\mathcal{F}'_1) = j(f^{-1}\mathrm{Supp}(\mathcal{F}_1))$$

The structure morphism $p' : Y' \to \mathrm{Spec}(A)$ is proper, hence $p'(\mathrm{Supp}(\mathcal{F}') \setminus j(Y))$ is closed in $\mathrm{Spec}(A)$. A nonempty closed subset of $\mathrm{Spec}(A)$ contains a point of $V(I)$ as I is contained in the Jacobson radical of A by Algebra, Lemma 10.96.6. The displayed equation shows that $\mathrm{Supp}(\mathcal{F}') \cap (p')^{-1}V(I) \subset j(Y)$ hence we conclude that $\mathrm{Supp}(\mathcal{F}') \subset j(Y)$. Thus $\mathcal{F}'|_Y = j^*\mathcal{F}'$ is supported on a closed subscheme Z' of Y proper over A and $(\mathcal{F}'|_Y)^\wedge = (f^*\mathcal{F}_n)$.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Proposition 30.19.1 the $\mathcal{O}_X$-module $\mathcal{H} = f_*(\mathcal{F}'|_Y)$ is coherent and by Lemma 30.25.3 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ of $\mathrm{Coh}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power $\mathcal{K}^t$ of $\mathcal{K}$. We obtain an exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to (\mathcal{F}_n) \to \mathcal{H}^\wedge \to \mathrm{Coker}(\alpha) \to 0$$

in $\mathrm{Coh}(X, \mathcal{I})$. If $Z_0 \subset X$ is the scheme theoretic support of $\mathcal{H}$, then it is clear that $Z_0 \subset f(Z')$ set-theoretically. Hence Z_0 is proper over A by Lemma 30.26.3 and Lemma 30.26.5 part (2). Hence $\mathcal{H}^\wedge$ is in the subcategory defined in Lemma 30.26.11 part (2) and a fortiori in $\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$. We conclude that $\mathrm{Ker}(\alpha)$ and $\mathrm{Coker}(\alpha)$ are in $\mathrm{Coh}_{\text{support proper over } A}(X, \mathcal{I})$ by Lemma 30.26.11 part (1). By induction hypothesis, more precisely because $\mathcal{K}^t$ is in Ξ, we see that $\mathrm{Ker}(\alpha)$ and $\mathrm{Coker}(\alpha)$ are in the subcategory defined in Lemma 30.26.11 part (2). Since this is a Serre subcategory by the lemma, we conclude that the same is true for $(\mathcal{F}_n)$ which is what we wanted to show. □

088F Remark 30.27.2 (Unwinding Grothendieck's existence theorem). Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots & X \arrow[d] \\
S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots & S
\end{tikzcd}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and

(3) $\text{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 30.27.1 says that the completion functor

$$\begin{matrix} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A \end{matrix} \longrightarrow \begin{matrix} \text{systems } (\mathcal{F}_n) \\ \text{as above} \end{matrix}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

30.28. Grothendieck's algebraization theorem

0898 Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

0899 **Lemma 30.28.1.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d] & \ldots \\ X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots \end{tikzcd}$$

of schemes with cartesian squares. Assume that

(1) $Z_1 \to X_1$ is a closed immersion, and
(2) $Z_1 \to S_1$ is proper.

Then there exists a closed immersion of schemes $Z \to X$ such that $Z_n = Z \times_S S_n$. Moreover, Z is proper over S.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Morphisms, Lemma 29.45.7 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 30.27.2, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is exact (Lemma 30.23.4) we see that the cokernel $\mathcal{Q}$ of $\mathcal{O}_X \to \mathcal{F}$ has vanishing completion. Since $\mathcal{F}$ has support proper over S and so does $\mathcal{Q}$ this implies that $\mathcal{Q} = 0$ for example because the functor (30.27.0.1) is an equivalence by Grothendieck's existence theorem. Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. $\square$

In the following lemma it is actually enough to assume that $Y_1 \to X_1$ is finite as it will imply that $Y_n \to X_n$ is finite too (see More on Morphisms, Lemma 37.3.3).

09ZT **Lemma 30.28.2.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \text{Spec}(A)$ and $S_n = \text{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative

diagram

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & \ldots \\ X_1 \arrow[r, "i_1"] & X_2 \arrow[r, "i_2"] & X_3 \arrow[r] & \ldots \end{tikzcd}$$

of schemes with cartesian squares. Assume that

(1) $Y_n \to X_n$ is a finite morphism, and
(2) $Y_1 \to S_1$ is proper.

Then there exists a finite morphism of schemes $Y \to X$ such that $Y_n = Y \times_S S_n$. Moreover, Y is proper over S.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. This is a coherent $\mathcal{O}_{X_n}$-module as f_n is finite (Lemma 30.9.9). Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 30.27.2, tells us there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. Moreover, the support of $\mathcal{F}$ is proper over S. As the completion functor is fully faithful (Theorem 30.27.1) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\mathrm{Spec}}_X(\mathcal{F})$ finishes the proof. $\square$

0A42 Lemma 30.28.3. Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let X, Y be schemes over S. For $n \geq 1$ we set $X_n = X \times_S S_n$ and $Y_n = Y \times_S S_n$. Suppose given a compatible system of commutative diagrams

$$\begin{tikzcd} & X_{n+1} \arrow[rr, "g_{n+1}"] \arrow[dr] & & Y_{n+1} \arrow[dl] \\ X_n \arrow[ur] \arrow[r, "g_n"] \arrow[dr] & Y_n \arrow[urr] \arrow[d] & S_{n+1} & \\ & S_n \arrow[ur] & & \end{tikzcd}$$

Assume that

(1) $X \to S$ is proper, and
(2) $Y \to S$ is separated of finite type.

Then there exists a unique morphism of schemes $g : X \to Y$ over S such that g_n is the base change of g to S_n.

Proof. The morphisms $(1, g_n) : X_n \to X_n \times_S Y_n$ are closed immersions because $Y_n \to S_n$ is separated (Schemes, Lemma 26.21.11). Thus by Lemma 30.28.1 there exists a closed subscheme $Z \subset X \times_S Y$ proper over S whose base change to S_n recovers $X_n \subset X_n \times_S Y_n$. The first projection $p : Z \to X$ is a proper morphism (as Z is proper over S, see Morphisms, Lemma 29.41.7) whose base change to S_n is an isomorphism for all n. In particular, $p : Z \to X$ is finite over an open neighbourhood of X_0 by Lemma 30.21.2. As X is proper over S this open neighbourhood is all of X and we conclude $p : Z \to X$ is finite. Applying the equivalence of Proposition 30.25.4 we see that $p_*\mathcal{O}_Z = \mathcal{O}_X$ as this is true modulo I^n for all n. Hence p is an isomorphism and we obtain the morphism g as the composition $X \cong Z \to Y$. We omit the proof of uniqueness. $\square$

In order to prove an "abstract" algebraization theorem we need to assume we have an ample invertible sheaf, as the result is false without such an assumption.

089A Theorem 30.28.4 (Grothendieck's algebraization theorem). Let A be a Noetherian ring complete with respect to an ideal I. Set $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Consider a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "i_1"'] \arrow[d] & X_2 \arrow[r, "i_2"'] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots \\ S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \ldots \end{tikzcd}$$

of schemes with cartesian squares. Suppose given $(\mathcal{L}_n, \varphi_n)$ where each $\mathcal{L}_n$ is an invertible sheaf on X_n and $\varphi_n : i_n^*\mathcal{L}_{n+1} \to \mathcal{L}_n$ is an isomorphism. If

(1) $X_1 \to S_1$ is proper, and
(2) $\mathcal{L}_1$ is ample on X_1

then there exists a proper morphism of schemes $X \to S$ and an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and isomorphisms $X_n \cong X \times_S S_n$ and $\mathcal{L}_n \cong \mathcal{L}|_{X_n}$ compatible with the morphisms i_n and φ_n.

Proof. Since the squares in the diagram are cartesian and since the morphisms $S_n \to S_{n+1}$ are closed immersions, we see that the morphisms i_n are closed immersions too. In particular we may think of X_m as a closed subscheme of X_n for $m < n$. In fact X_m is the closed subscheme cut out by the quasi-coherent sheaf of ideals $I^m\mathcal{O}_{X_n}$. Moreover, the underlying topological spaces of the schemes $X_1, X_2, X_3, \ldots$ are all identified, hence we may (and do) think of sheaves $\mathcal{O}_{X_n}$ as living on the same underlying topological space; similarly for coherent $\mathcal{O}_{X_n}$-modules. Set

$$\mathcal{F}_n = \mathrm{Ker}(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$$

so that we obtain short exact sequences

$$0 \to \mathcal{F}_n \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0$$

By the above we have $\mathcal{F}_n = I^n\mathcal{O}_{X_{n+1}}$. It follows $\mathcal{F}_n$ is a coherent sheaf on X_{n+1} annihilated by I, hence we may (and do) think of it as a coherent module $\mathcal{O}_{X_1}$-module. Observe that for $m > n$ the sheaf

$$I^n\mathcal{O}_{X_m}/I^{n+1}\mathcal{O}_{X_m}$$

maps isomorphically to $\mathcal{F}_n$ under the map $\mathcal{O}_{X_m} \to \mathcal{O}_{X_{n+1}}$. Hence given $n_1, n_2 \geq 0$ we can pick an $m > n_1 + n_2$ and consider the multiplication map

$$I^{n_1}\mathcal{O}_{X_m} \times I^{n_2}\mathcal{O}_{X_m} \longrightarrow I^{n_1+n_2}\mathcal{O}_{X_m} \to \mathcal{F}_{n_1+n_2}$$

This induces an $\mathcal{O}_{X_1}$-bilinear map

$$\mathcal{F}_{n_1} \times \mathcal{F}_{n_2} \longrightarrow \mathcal{F}_{n_1+n_2}$$

which in turn defines the structure of a graded $\mathcal{O}_{X_1}$-algebra on $\mathcal{F} = \bigoplus_{n \geq 0} \mathcal{F}_n$.

Set $B = \bigoplus I^n/I^{n+1}$; this is a finitely generated graded A/I-algebra. Set $\mathcal{B} = (X_1 \to S_1)^*\widetilde{B}$. The discussion above provides us with a canonical surjection

$$\mathcal{B} \longrightarrow \mathcal{F}$$

of graded $\mathcal{O}_{X_1}$-algebras. In particular we see that $\mathcal{F}$ is a finite type quasi-coherent graded $\mathcal{B}$-module. By Lemma 30.19.3 we can find an integer d_0 such that $H^1(X_1, \mathcal{F}\otimes$

$\mathcal{L}^{\otimes d}) = 0$ for all $d \geq d_0$. Pick a $d \geq d_0$ such that there exist sections $s_{0,1}, \ldots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ which induce an immersion

$$\psi_1 : X_1 \to \mathbf{P}^N_{S_1}$$

over S_1, see Morphisms, Lemma 29.39.4. As X_1 is proper over S_1 we see that ψ_1 is a closed immersion, see Morphisms, Lemma 29.41.7 and Schemes, Lemma 26.10.4. We are going to "lift" ψ_1 to a compatible system of closed immersions of X_n into $\mathbf{P}^N$.

Upon tensoring the short exact sequences of the first paragraph of the proof by $\mathcal{L}_{n+1}^{\otimes d}$ we obtain short exact sequences

$$0 \to \mathcal{F}_n \otimes \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to 0$$

Using the isomorphisms φ_n we obtain isomorphisms $\mathcal{L}_{n+1} \otimes \mathcal{O}_{X_l} = \mathcal{L}_l$ for $l \leq n$. Whence the sequence above becomes

$$0 \to \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_n^{\otimes d} \to 0$$

The vanishing of $H^1(X, \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d})$ implies we can inductively lift $s_{0,1}, \ldots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ to sections $s_{0,n}, \ldots, s_{N,n} \in \Gamma(X_n, \mathcal{L}_n^{\otimes d})$. Thus we obtain a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"'] \arrow[d, "\psi_1"'] & X_2 \arrow[r, "i_2"'] \arrow[d, "\psi_2"'] & X_3 \arrow[r] \arrow[d, "\psi_3"'] & \ldots \\
\mathbf{P}^N_{S_1} \arrow[r] & \mathbf{P}^N_{S_2} \arrow[r] & \mathbf{P}^N_{S_3} \arrow[r] & \ldots
\end{tikzcd}$$

where $\psi_n = \varphi_{(\mathcal{L}_n, (s_{0,n}, \ldots, s_{N,n}))}$ in the notation of Constructions, Section 27.13. As the squares in the statement of the theorem are cartesian we see that the squares in the above diagram are cartesian. We win by applying Lemma 30.28.1. $\square$

30.29. Other chapters

CHAPTER 31

Divisors

01WO 31.1. Introduction

01WP In this chapter we study some very basic questions related to defining divisors, etc. A basic reference is [DG67].

31.2. Associated points

02OI Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is associated to M if there exists an element of M whose annihilator is $\mathfrak{p}$. See Algebra, Definition 10.63.1. Here is the definition of associated points for quasi-coherent sheaves on schemes as given in [DG67, IV Definition 3.1.1].

02OJ Definition 31.2.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is associated to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\text{Ass}(\mathcal{F})$ or $\text{Ass}_X(\mathcal{F})$ the set of associated points of $\mathcal{F}$.
(3) The associated points of X are the associated points of $\mathcal{O}_X$.

These definitions are most useful when X is locally Noetherian and $\mathcal{F}$ of finite type. For example it may happen that a generic point of an irreducible component of X is not associated to X, see Example 31.2.7. In the non-Noetherian case it may be more convenient to use weakly associated points, see Section 31.5. Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for locally Noetherian schemes.

02OK Lemma 31.2.2. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\text{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime.

(1) If $\mathfrak{p}$ is associated to M, then x is associated to $\mathcal{F}$.
(2) If $\mathfrak{p}$ is finitely generated, then the converse holds as well.

In particular, if X is locally Noetherian, then the equivalence

$$\mathfrak{p} \in \text{Ass}(M) \Leftrightarrow x \in \text{Ass}(\mathcal{F})$$

holds for all pairs $(\mathfrak{p}, x)$ as above.

Proof. This follows from Algebra, Lemma 10.63.15. But we can also argue directly as follows. Suppose $\mathfrak{p}$ is associated to M. Then there exists an $m \in M$ whose annihilator is $\mathfrak{p}$. Since localization is exact we see that $\mathfrak{p}A_\mathfrak{p}$ is the annihilator of $m/1 \in M_\mathfrak{p}$. Since $M_\mathfrak{p} = \mathcal{F}_x$ (Schemes, Lemma 26.5.4) we conclude that x is associated to $\mathcal{F}$.

Conversely, assume that x is associated to $\mathcal{F}$, and $\mathfrak{p}$ is finitely generated. As x is associated to $\mathcal{F}$ there exists an element $m' \in M_\mathfrak{p}$ whose annihilator is $\mathfrak{p}A_\mathfrak{p}$. Write

$m' = m/f$ for some $f \in A$, $f \notin \mathfrak{p}$. The annihilator I of m is an ideal of A such that $IA_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. Hence $I \subset \mathfrak{p}$, and $(\mathfrak{p}/I)_\mathfrak{p} = 0$. Since $\mathfrak{p}$ is finitely generated, there exists a $g \in A$, $g \notin \mathfrak{p}$ such that $g(\mathfrak{p}/I) = 0$. Hence the annihilator of gm is $\mathfrak{p}$ and we win.

If X is locally Noetherian, then A is Noetherian (Properties, Lemma 28.5.2) and $\mathfrak{p}$ is always finitely generated. ☐

05AD Lemma 31.2.3. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\text{Ass}(\mathcal{F}) \subset \text{Supp}(\mathcal{F})$.

Proof. This is immediate from the definitions. ☐

05AE Lemma 31.2.4. Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $\text{Ass}(\mathcal{F}_2) \subset \text{Ass}(\mathcal{F}_1) \cup \text{Ass}(\mathcal{F}_3)$ and $\text{Ass}(\mathcal{F}_1) \subset \text{Ass}(\mathcal{F}_2)$.

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 10.63.3. ☐

05AF Lemma 31.2.5. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\text{Ass}(\mathcal{F}) \cap U$ is finite for every quasi-compact open $U \subset X$.

Proof. This is true because the set of associated primes of a finite module over a Noetherian ring is finite, see Algebra, Lemma 10.63.5. To translate from schemes to algebra use that U is a finite union of affine opens, each of these opens is the spectrum of a Noetherian ring (Properties, Lemma 28.5.2), $\mathcal{F}$ corresponds to a finite module over this ring (Cohomology of Schemes, Lemma 30.9.1), and finally use Lemma 31.2.2. ☐

05AG Lemma 31.2.6. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then

$$\mathcal{F} = 0 \Leftrightarrow \text{Ass}(\mathcal{F}) = \emptyset.$$

Proof. If $\mathcal{F} = 0$, then $\text{Ass}(\mathcal{F}) = \emptyset$ by definition. Conversely, if $\text{Ass}(\mathcal{F}) = \emptyset$, then $\mathcal{F} = 0$ by Algebra, Lemma 10.63.7. To translate from schemes to algebra, restrict to any affine and use Lemma 31.2.2. ☐

05AI Example 31.2.7. Let k be a field. The ring $R = k[x_1, x_2, x_3, \ldots]/(x_i^2)$ is local with locally nilpotent maximal ideal $\mathfrak{m}$. There exists no element of R which has annihilator $\mathfrak{m}$. Hence $\text{Ass}(R) = \emptyset$, and $X = \text{Spec}(R)$ is an example of a scheme which has no associated points.

0B3L Lemma 31.2.8. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $U \subset X$ is open and $\text{Ass}(\mathcal{F}) \subset U$, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective.

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U. Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \to \mathcal{F}$ defined by s. Then $\text{Supp}(\mathcal{F}') \cap U = \emptyset$. On the other hand, $\text{Ass}(\mathcal{F}') \subset \text{Ass}(\mathcal{F})$ by Lemma 31.2.4. Since also $\text{Ass}(\mathcal{F}') \subset \text{Supp}(\mathcal{F}')$ (Lemma 31.2.3) we conclude $\text{Ass}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 31.2.6. ☐

05AH Lemma 31.2.9. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in \text{Supp}(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $\text{Supp}(\mathcal{F})$. Then $x \in \text{Ass}(\mathcal{F})$. In particular, any generic point of an irreducible component of X is an associated point of X.

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{Ass}(\mathcal{F}_x) \subset \text{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 10.63.7. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{Ass}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 10.63.2) we see that $\mathfrak{m}_x$ is associated to $\mathcal{F}_x$ and we win. $\square$

The following lemma is the analogue of More on Algebra, Lemma 15.23.12.

0AVL Lemma 31.2.10. Let X be a locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume that for every $x \in X$ at least one of the following happens

(1) $\mathcal{F}_x \to \mathcal{G}_x$ is injective, or
(2) $x \notin \text{Ass}(\mathcal{F})$.

Then φ is injective.

Proof. The assumptions imply that $\text{Ass}(\text{Ker}(\varphi)) = \emptyset$ and hence $\text{Ker}(\varphi) = 0$ by Lemma 31.2.6. $\square$

0AVM Lemma 31.2.11. Let X be a locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is coherent and that for every $x \in X$ one of the following happens

(1) $\mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, or
(2) $\text{depth}(\mathcal{F}_x) \geq 2$ and $x \notin \text{Ass}(\mathcal{G})$.

Then φ is an isomorphism.

Proof. This is a translation of More on Algebra, Lemma 15.23.13 into the language of schemes. $\square$

31.3. Morphisms and associated points

05DA Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $s \in S$ is a point, then it is often convenient to denote $\mathcal{F}_s$ the $\mathcal{O}_{X_s}$-module one gets by pulling back $\mathcal{F}$ by the morphism $i_s : X_s \to X$. Here X_s is the scheme theoretic fibre of f over s. In a formula

$$\mathcal{F}_s = i_s^*\mathcal{F}$$

Of course, this notation clashes with the already existing notation for the stalk of $\mathcal{F}$ at a point $x \in X$ if $f = \text{id}_X$. However, the notation is often convenient, as in the formulation of the following lemma.

05DB Lemma 31.3.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have

$$\text{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \supset \bigcup\nolimits_{s \in \text{Ass}_S(\mathcal{G})} \text{Ass}_{X_s}(\mathcal{F}_s)$$

and equality holds if S is locally Noetherian (for the notation $\mathcal{F}_s$ see above).

Proof. Let $x \in X$ and let $s = f(x) \in S$. Set $B = \mathcal{O}_{X,x}$, $A = \mathcal{O}_{S,s}$, $N = \mathcal{F}_x$, and $M = \mathcal{G}_s$. Note that the stalk of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ at x is equal to the B-module $M \otimes_A N$. Hence $x \in \text{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if $\mathfrak{m}_B$ is in $\text{Ass}_B(M \otimes_A N)$. Similarly $s \in \text{Ass}_S(\mathcal{G})$ and $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $\mathfrak{m}_A \in \text{Ass}_A(M)$ and $\mathfrak{m}_B/\mathfrak{m}_A B \in \text{Ass}_{B \otimes \kappa(\mathfrak{m}_A)}(N \otimes \kappa(\mathfrak{m}_A))$. Thus the lemma follows from Algebra, Lemma 10.65.5. $\square$

31.4. Embedded points

05AJ Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is an embedded associated prime of M if it is an associated prime of M which is not minimal among the associated primes of M. See Algebra, Definition 10.67.1. Here is the definition of embedded associated points for quasi-coherent sheaves on schemes as given in [DG67, IV Definition 3.1.1].

05AK **Definition 31.4.1.** Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) An embedded associated point of $\mathcal{F}$ is an associated point which is not maximal among the associated points of $\mathcal{F}$, i.e., it is the specialization of another associated point of $\mathcal{F}$.
(2) A point x of X is called an embedded point if x is an embedded associated point of $\mathcal{O}_X$.
(3) An embedded component of X is an irreducible closed subset $Z = \overline{\{x\}}$ where x is an embedded point of X.

In the Noetherian case when $\mathcal{F}$ is coherent we have the following.

05AL **Lemma 31.4.2.** Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then

(1) the generic points of irreducible components of $\mathrm{Supp}(\mathcal{F})$ are associated points of $\mathcal{F}$, and
(2) an associated point of $\mathcal{F}$ is embedded if and only if it is not a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F})$.

In particular an embedded point of X is an associated point of X which is not a generic point of an irreducible component of X.

Proof. Recall that in this case $Z = \mathrm{Supp}(\mathcal{F})$ is closed, see Morphisms, Lemma 29.5.3 and that the generic points of irreducible components of Z are associated points of $\mathcal{F}$, see Lemma 31.2.9. Finally, we have $\mathrm{Ass}(\mathcal{F}) \subset Z$, by Lemma 31.2.3. These results, combined with the fact that Z is a sober topological space and hence every point of Z is a specialization of a generic point of Z, imply (1) and (2). $\square$

0346 **Lemma 31.4.3.** Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Then the following are equivalent:

(1) $\mathcal{F}$ has no embedded associated points, and
(2) $\mathcal{F}$ has property (S_1).

Proof. This is Algebra, Lemma 10.157.2, combined with Lemma 31.2.2 above. $\square$

0BXG **Lemma 31.4.4.** Let X be a locally Noetherian scheme of dimension ≤ 1. The following are equivalent

(1) X is Cohen-Macaulay, and
(2) X has no embedded points.

Proof. Follows from Lemma 31.4.3 and the definitions. $\square$

083P **Lemma 31.4.5.** Let X be a locally Noetherian scheme. Let $U \subset X$ be an open subscheme. The following are equivalent

(1) U is scheme theoretically dense in X (Morphisms, Definition 29.7.1),
(2) U is dense in X and U contains all embedded points of X.

Proof. The question is local on X, hence we may assume that $X = \mathrm{Spec}(A)$ where A is a Noetherian ring. Then U is quasi-compact (Properties, Lemma 28.5.3) hence $U = D(f_1) \cup \ldots \cup D(f_n)$ (Algebra, Lemma 10.29.1). In this situation U is scheme theoretically dense in X if and only if $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective, see Morphisms, Example 29.7.4. Condition (2) translated into algebra means that for every associated prime $\mathfrak p$ of A there exists an i with $f_i \not\in \mathfrak p$.

Assume (1), i.e., $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. If $x \in A$ has annihilator a prime $\mathfrak p$, then x maps to a nonzero element of A_{f_i} for some i and hence $f_i \not\in \mathfrak p$. Thus (2) holds. Assume (2), i.e., every associated prime $\mathfrak p$ of A corresponds to a prime of A_{f_i} for some i. Then $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective because $A \to \prod_{\mathfrak p \in \mathrm{Ass}(A)} A_{\mathfrak p}$ is injective by Algebra, Lemma 10.63.19. □

02OL Lemma 31.4.6. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. The set of coherent subsheaves

$$\{\mathcal{K} \subset \mathcal{F} \mid \mathrm{Supp}(\mathcal{K}) \text{ is nowhere dense in } \mathrm{Supp}(\mathcal{F})\}$$

has a maximal element $\mathcal{K}$. Setting $\mathcal{F}' = \mathcal{F}/\mathcal{K}$ we have the following

(1) $\mathrm{Supp}(\mathcal{F}') = \mathrm{Supp}(\mathcal{F})$,
(2) $\mathcal{F}'$ has no embedded associated points, and
(3) there exists a dense open $U \subset X$ such that $U \cap \mathrm{Supp}(\mathcal{F})$ is dense in $\mathrm{Supp}(\mathcal{F})$ and $\mathcal{F}'|_U \cong \mathcal{F}|_U$.

Proof. This follows from Algebra, Lemmas 10.67.2 and 10.67.3. Note that U can be taken as the complement of the closure of the set of embedded associated points of $\mathcal{F}$. □

02OM Lemma 31.4.7. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module without embedded associated points. Set

$$\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})).$$

This is a coherent sheaf of ideals which defines a closed subscheme $Z \subset X$ without embedded points. Moreover there exists a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{F} = (Z \to X)_*\mathcal{G}$, (b) $\mathcal{G}$ has no associated embedded points, and (c) $\mathrm{Supp}(\mathcal{G}) = Z$ (as sets).

Proof. Some of the statements we have seen in the proof of Cohomology of Schemes, Lemma 30.9.7. The others follow from Algebra, Lemma 10.67.4. □

31.5. Weakly associated points

056K Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak p \subset R$ is weakly associated to M if there exists an element m of M such that $\mathfrak p$ is minimal among the primes containing the annihilator of m. See Algebra, Definition 10.66.1. If R is a local ring with maximal ideal $\mathfrak m$, then $\mathfrak m$ is weakly associated to M if and only if there exists an element $m \in M$ whose annihilator has radical $\mathfrak m$, see Algebra, Lemma 10.66.2.

056L Definition 31.5.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is weakly associated to $\mathcal{F}$ if the maximal ideal $\mathfrak m_x$ is weakly associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\mathrm{WeakAss}(\mathcal{F})$ the set of weakly associated points of $\mathcal{F}$.

(3) The weakly associated points of X are the weakly associated points of $\mathcal{O}_X$.

In this case, on any affine open, this corresponds exactly to the weakly associated primes as defined above. Here is the precise statement.

056M Lemma 31.5.2. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent

(1) $\mathfrak{p}$ is weakly associated to M, and
(2) x is weakly associated to $\mathcal{F}$.

Proof. This follows from Algebra, Lemma 10.66.2. $\square$

05AM Lemma 31.5.3. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then

$$\mathrm{Ass}(\mathcal{F}) \subset \mathrm{WeakAss}(\mathcal{F}) \subset \mathrm{Supp}(\mathcal{F}).$$

Proof. This is immediate from the definitions. $\square$

05AN Lemma 31.5.4. Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $\mathrm{WeakAss}(\mathcal{F}_2) \subset \mathrm{WeakAss}(\mathcal{F}_1) \cup \mathrm{WeakAss}(\mathcal{F}_3)$ and $\mathrm{WeakAss}(\mathcal{F}_1) \subset \mathrm{WeakAss}(\mathcal{F}_2)$.

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 10.66.4. $\square$

05AP Lemma 31.5.5. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then

$$\mathcal{F} = (0) \Leftrightarrow \mathrm{WeakAss}(\mathcal{F}) = \emptyset$$

Proof. Follows from Lemma 31.5.2 and Algebra, Lemma 10.66.5 $\square$

0B3M Lemma 31.5.6. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $U \subset X$ is open and $\mathrm{WeakAss}(\mathcal{F}) \subset U$, then $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is injective.

Proof. Let $s \in \Gamma(X, \mathcal{F})$ be a section which restricts to zero on U. Let $\mathcal{F}' \subset \mathcal{F}$ be the image of the map $\mathcal{O}_X \to \mathcal{F}$ defined by s. Then $\mathrm{Supp}(\mathcal{F}') \cap U = \emptyset$. On the other hand, $\mathrm{WeakAss}(\mathcal{F}') \subset \mathrm{WeakAss}(\mathcal{F})$ by Lemma 31.5.4. Since also $\mathrm{WeakAss}(\mathcal{F}') \subset \mathrm{Supp}(\mathcal{F}')$ (Lemma 31.5.3) we conclude $\mathrm{WeakAss}(\mathcal{F}') = \emptyset$. Hence $\mathcal{F}' = 0$ by Lemma 31.5.5. $\square$

05AQ Lemma 31.5.7. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in \mathrm{Supp}(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $\mathrm{Supp}(\mathcal{F})$. Then $x \in \mathrm{WeakAss}(\mathcal{F})$. In particular, any generic point of an irreducible component of X is weakly associated to $\mathcal{O}_X$.

Proof. Since $x \in \mathrm{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\mathrm{WeakAss}(\mathcal{F}_x) \subset \mathrm{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 10.66.5. On the other hand, by assumption $\mathrm{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\mathrm{WeakAss}(\mathcal{F}_x) \subset \mathrm{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 10.66.6) we see that $\mathfrak{m}_x$ is weakly associated to $\mathcal{F}_x$ and we win. $\square$

05AR Lemma 31.5.8. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathfrak{m}_x$ is a finitely generated ideal of $\mathcal{O}_{X,x}$, then

$$x \in \mathrm{Ass}(\mathcal{F}) \Leftrightarrow x \in \mathrm{WeakAss}(\mathcal{F}).$$

In particular, if X is locally Noetherian, then $\mathrm{Ass}(\mathcal{F}) = \mathrm{WeakAss}(\mathcal{F})$.

Proof. See Algebra, Lemma 10.66.9. ☐

0AVN Lemma 31.5.9. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point which is not in the image of f. Then s is not weakly associated to $f_*\mathcal{F}$.

Proof. Consider the base change $f' : X' \to \text{Spec}(\mathcal{O}_{S,s})$ of f by the morphism $g : \text{Spec}(\mathcal{O}_{S,s}) \to S$ and denote $g' : X' \to X$ the other projection. Then

$$(f_*\mathcal{F})_s = (g^*f_*\mathcal{F})_s = (f'_*(g')^*\mathcal{F})_s$$

The first equality because g induces an isomorphism on local rings at s and the second by flat base change (Cohomology of Schemes, Lemma 30.5.2). Of course $s \in \text{Spec}(\mathcal{O}_{S,s})$ is not in the image of f'. Thus we may assume S is the spectrum of a local ring $(A, \mathfrak{m})$ and s corresponds to $\mathfrak{m}$. By Schemes, Lemma 26.24.1 the sheaf $f_*\mathcal{F}$ is quasi-coherent, say corresponding to the A-module M. As s is not in the image of f we see that $X = \bigcup_{a \in \mathfrak{m}} f^{-1}D(a)$ is an open covering. Since X is quasi-compact we can find $a_1, \ldots, a_n \in \mathfrak{m}$ such that $X = f^{-1}D(a_1) \cup \ldots \cup f^{-1}D(a_n)$. It follows that

$$M \to M_{a_1} \oplus \ldots \oplus M_{a_r}$$

is injective. Hence for any nonzero element m of the stalk $M_\mathfrak{p}$ there exists an i such that $a_i^n m$ is nonzero for all $n \geq 0$. Thus $\mathfrak{m}$ is not weakly associated to M. ☐

0AVP Lemma 31.5.10. Let X be a scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume that for every $x \in X$ at least one of the following happens

(1) $\mathcal{F}_x \to \mathcal{G}_x$ is injective, or
(2) $x \notin \text{WeakAss}(\mathcal{F})$.

Then φ is injective.

Proof. The assumptions imply that $\text{WeakAss}(\text{Ker}(\varphi)) = \emptyset$ and hence $\text{Ker}(\varphi) = 0$ by Lemma 31.5.5. ☐

0E9I Lemma 31.5.11. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $j : U \to X$ be an open subscheme such that for $x \in X \setminus U$ we have $\text{depth}(\mathcal{F}_x) \geq 2$. Then

$$\mathcal{F} \longrightarrow j_*(\mathcal{F}|_U)$$

is an isomorphism and consequently $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is an isomorphism too.

Proof. We claim Lemma 31.2.11 applies to the map displayed in the lemma. Let $x \in X$. If $x \in U$, then the map is an isomorphism on stalks as $j_*(\mathcal{F}|_U)|_U = \mathcal{F}|_U$. If $x \in X \setminus U$, then $x \notin \text{Ass}(j_*(\mathcal{F}|_U))$ (Lemmas 31.5.9 and 31.5.3). Since we've assumed $\text{depth}(\mathcal{F}_x) \geq 2$ this finishes the proof. ☐

0EME Lemma 31.5.12. Let X be a reduced scheme. Then the weakly associated points of X are exactly the generic points of the irreducible components of X.

Proof. Follows from Algebra, Lemma 10.66.3. ☐

31.6. Morphisms and weakly associated points

05EW

05EX Lemma 31.6.1. Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have

$$\text{WeakAss}_S(f_*\mathcal{F}) \subset f(\text{WeakAss}_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. Then $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 31.5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 10.66.11. □

05EY Lemma 31.6.2. Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is locally Noetherian, then we have

$$f(\text{Ass}_X(\mathcal{F})) = \text{Ass}_S(f_*\mathcal{F}) = \text{WeakAss}_S(f_*\mathcal{F}) = f(\text{WeakAss}_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. As X is locally Noetherian the ring B is Noetherian, see Properties, Lemma 28.5.2. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 31.2.2 the associated points of $\mathcal{F}$ correspond exactly to the associated primes of M, and any associated prime of M as an A-module is an associated points of $f_*\mathcal{F}$. Hence the inclusion

$$f(\text{Ass}_X(\mathcal{F})) \subset \text{Ass}_S(f_*\mathcal{F})$$

follows from Algebra, Lemma 10.63.13. We have the inclusion

$$\text{Ass}_S(f_*\mathcal{F}) \subset \text{WeakAss}_S(f_*\mathcal{F})$$

by Lemma 31.5.3. We have the inclusion

$$\text{WeakAss}_S(f_*\mathcal{F}) \subset f(\text{WeakAss}_X(\mathcal{F}))$$

by Lemma 31.6.1. The outer sets are equal by Lemma 31.5.8 hence we have equality everywhere. □

05EZ Lemma 31.6.3. Let $f : X \to S$ be a finite morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\text{WeakAss}(f_*\mathcal{F}) = f(\text{WeakAss}(\mathcal{F}))$.

Proof. We may assume X and S affine, so $X \to S$ comes from a finite ring map $A \to B$. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 31.5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 10.66.13. □

05F0 Lemma 31.6.4. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $x \in X$ with $s = f(x)$. If f is flat at x, the point x is a generic point of the fibre X_s, and $s \in \text{WeakAss}_S(\mathcal{G})$, then $x \in \text{WeakAss}(f^*\mathcal{G})$.

Proof. Let $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$, and $M = \mathcal{G}_s$. Let $m \in M$ be an element whose annihilator $I = \{a \in A \mid am = 0\}$ has radical $\mathfrak{m}_A$. Then $m \otimes 1$ has annihilator IB as $A \to B$ is faithfully flat. Thus it suffices to see that $\sqrt{IB} = \mathfrak{m}_B$. This follows from the fact that the maximal ideal of $B/\mathfrak{m}_A B$ is locally nilpotent (see Algebra, Lemma 10.25.1) and the assumption that $\sqrt{I} = \mathfrak{m}_A$. Some details omitted. □

0CUC Lemma 31.6.5. Let K/k be a field extension. Let X be a scheme over k. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $y \in X_K$ with image $x \in X$. If y is a weakly associated point of the pullback $\mathcal{F}_K$, then x is a weakly associated point of $\mathcal{F}$.

Proof. This is the translation of Algebra, Lemma 10.66.19 into the language of schemes. $\square$

Here is a simple lemma where we find that pushforwards often have depth at least 2.

0EY0 Lemma 31.6.6. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$.

(1) If $s \notin f(X)$, then s is not weakly associated to $f_*\mathcal{F}$.
(2) If $s \notin f(X)$ and $\mathcal{O}_{S,s}$ is Noetherian, then s is not associated to $f_*\mathcal{F}$.
(3) If $s \notin f(X)$, $(f_*\mathcal{F})_s$ is a finite $\mathcal{O}_{S,s}$-module, and $\mathcal{O}_{S,s}$ is Noetherian, then $\text{depth}((f_*\mathcal{F})_s) \geq 2$.
(4) If $\mathcal{F}$ is flat over S and $a \in \mathfrak{m}_s$ is a nonzerodivisor, then a is a nonzerodivisor on $(f_*\mathcal{F})_s$.
(5) If $\mathcal{F}$ is flat over S and $a, b \in \mathfrak{m}_s$ is a regular sequence, then a is a nonzerodivisor on $(f_*\mathcal{F})_s$ and b is a nonzerodivisor on $(f_*\mathcal{F})_s/a(f_*\mathcal{F})_s$.
(6) If $\mathcal{F}$ is flat over S and $(f_*\mathcal{F})_s$ is a finite $\mathcal{O}_{S,s}$-module, then $\text{depth}((f_*\mathcal{F})_s) \geq \min(2, \text{depth}(\mathcal{O}_{S,s}))$.

Proof. Part (1) is Lemma 31.5.9. Part (2) follows from (1) and Lemma 31.5.8.

Proof of part (3). To show the depth is ≥ 2 it suffices to show that $\text{Hom}_{\mathcal{O}_{S,s}}(\kappa(s), (f_*\mathcal{F})_s) = 0$ and $\text{Ext}^1_{\mathcal{O}_{S,s}}(\kappa(s), (f_*\mathcal{F})_s) = 0$, see Algebra, Lemma 10.72.5. Using the exact sequence $0 \to \mathfrak{m}_s \to \mathcal{O}_{S,s} \to \kappa(s) \to 0$ it suffices to prove that the map

$$\text{Hom}_{\mathcal{O}_{S,s}}(\mathcal{O}_{S,s}, (f_*\mathcal{F})_s) \to \text{Hom}_{\mathcal{O}_{S,s}}(\mathfrak{m}_s, (f_*\mathcal{F})_s)$$

is an isomorphism. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we may replace S by $\text{Spec}(\mathcal{O}_{S,s})$ and X by $\text{Spec}(\mathcal{O}_{S,s}) \times_S X$. Denote $\mathfrak{m} \subset \mathcal{O}_S$ the ideal sheaf of s. Then we see that

$$\text{Hom}_{\mathcal{O}_{S,s}}(\mathfrak{m}_s, (f_*\mathcal{F})_s) = \text{Hom}_{\mathcal{O}_S}(\mathfrak{m}, f_*\mathcal{F}) = \text{Hom}_{\mathcal{O}_X}(f^*\mathfrak{m}, \mathcal{F})$$

the first equality because S is local with closed point s and the second equality by adjunction for f^*, f_* on quasi-coherent modules. However, since $s \notin f(X)$ we see that $f^*\mathfrak{m} = \mathcal{O}_X$. Working backwards through the arguments we get the desired equality.

For the proof of (4), (5), and (6) we use flat base change (Cohomology of Schemes, Lemma 30.5.2) to reduce to the case where S is the spectrum of $\mathcal{O}_{S,s}$. Then a nonzerodivisor $a \in \mathcal{O}_{S,s}$ deterimines a short exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{a} \mathcal{O}_S \to \mathcal{O}_S/a\mathcal{O}_S \to 0$$

Since $\mathcal{F}$ is flat over S, we obtain an exact sequence

$$0 \to \mathcal{F} \xrightarrow{a} \mathcal{F} \to \mathcal{F}/a\mathcal{F} \to 0$$

Pushing forward we obtain an exact sequence

$$0 \to f_*\mathcal{F} \xrightarrow{a} f_*\mathcal{F} \to f_*(\mathcal{F}/a\mathcal{F})$$

This proves (4) and it shows that $f_*\mathcal{F}/af_*\mathcal{F} \subset f_*(\mathcal{F}/a\mathcal{F})$. If b is a nonzerodivisor on $\mathcal{O}_{S,s}/a\mathcal{O}_{S,s}$, then the exact same argument shows $b : \mathcal{F}/a\mathcal{F} \to \mathcal{F}/a\mathcal{F}$ is injective. Pushing forward we conclude

$$b : f_*(\mathcal{F}/a\mathcal{F}) \to f_*(\mathcal{F}/a\mathcal{F})$$

is injective and hence also $b : f_*\mathcal{F}/af_*\mathcal{F} \to f_*\mathcal{F}/af_*\mathcal{F}$ is injective. This proves (5). Part (6) follows from (4) and (5) and the definitions. □

31.7. Relative assassin

05AS Let $A \to B$ be a ring map. Let N be a B-module. Recall that a prime $\mathfrak{q} \subset B$ is said to be in the relative assassin of N over B/A if $\mathfrak{q}$ is an associated prime of $N \otimes_A \kappa(\mathfrak{p})$. Here $\mathfrak{p} = A \cap \mathfrak{q}$. See Algebra, Definition 10.65.2. Here is the definition of the relative assassin for quasi-coherent sheaves over a morphism of schemes.

05AT Definition 31.7.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The relative assassin of $\mathcal{F}$ in X over S is the set

$$\mathrm{Ass}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{Ass}_{X_s}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Again there is a caveat that this is best used when the fibres of f are locally Noetherian and $\mathcal{F}$ is of finite type. In the general case we should probably use the relative weak assassin (defined in the next section). Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for morphisms of schemes whose fibres are locally Noetherian.

0CU5 Lemma 31.7.2. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $U \subset X$ and $V \subset S$ be affine opens with $f(U) \subset V$. Write $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. Then

$$\mathfrak{p} \in \mathrm{Ass}_{A/R}(M) \Rightarrow x \in \mathrm{Ass}_{X/S}(\mathcal{F})$$

If all fibres X_s of f are locally Noetherian, then $\mathfrak{p} \in \mathrm{Ass}_{A/R}(M) \Leftrightarrow x \in \mathrm{Ass}_{X/S}(\mathcal{F})$ for all pairs $(\mathfrak{p}, x)$ as above.

Proof. The set $\mathrm{Ass}_{A/R}(M)$ is defined in Algebra, Definition 10.65.2. Choose a pair $(\mathfrak{p}, x)$. Let $s = f(x)$. Let $\mathfrak{r} \subset R$ be the prime lying under $\mathfrak{p}$, i.e., the prime corresponding to s. Let $\mathfrak{p}' \subset A \otimes_R \kappa(\mathfrak{r})$ be the prime whose inverse image is $\mathfrak{p}$, i.e., the prime corresponding to x viewed as a point of its fibre X_s. Then $\mathfrak{p} \in \mathrm{Ass}_{A/R}(M)$ if and only if $\mathfrak{p}'$ is an associated prime of $M \otimes_R \kappa(\mathfrak{r})$, see Algebra, Lemma 10.65.1. Note that the ring $A \otimes_R \kappa(\mathfrak{r})$ corresponds to U_s and the module $M \otimes_R \kappa(\mathfrak{r})$ corresponds to the quasi-coherent sheaf $\mathcal{F}_s|_{U_s}$. Hence x is an associated point of $\mathcal{F}_s$ by Lemma 31.2.2. The reverse implication holds if $\mathfrak{p}'$ is finitely generated which is how the last sentence is seen to be true. □

05DC Lemma 31.7.3. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $g : S' \to S$ be a morphism of schemes. Consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

and set $\mathcal{F}' = (g')^*\mathcal{F}$. Let $x' \in X'$ be a point with images $x \in X$, $s' \in S'$ and $s \in S$. Assume f locally of finite type. Then $x' \in \text{Ass}_{X'/S'}(\mathcal{F}')$ if and only if $x \in \text{Ass}_{X/S}(\mathcal{F})$ and x' corresponds to a generic point of an irreducible component of $\text{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$.

Proof. Consider the morphism $X'_{s'} \to X_s$ of fibres. As $X_{s'} = X_s \times_{\text{Spec}(\kappa(s))} \text{Spec}(\kappa(s'))$ this is a flat morphism. Moreover $\mathcal{F}'_{s'}$ is the pullback of $\mathcal{F}_s$ via this morphism. As X_s is locally of finite type over the Noetherian scheme $\text{Spec}(\kappa(s))$ we have that X_s is locally Noetherian, see Morphisms, Lemma 29.15.6. Thus we may apply Lemma 31.3.1 and we see that

$$\text{Ass}_{X'_{s'}}(\mathcal{F}'_{s'}) = \bigcup\nolimits_{x \in \text{Ass}(\mathcal{F}_s)} \text{Ass}((X'_{s'})_x).$$

Thus to prove the lemma it suffices to show that the associated points of the fibre $(X'_{s'})_x$ of the morphism $X'_{s'} \to X_s$ over x are its generic points. Note that $(X'_{s'})_x = \text{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$ as schemes. By Algebra, Lemma 10.167.1 the ring $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ is a Noetherian Cohen-Macaulay ring. Hence its associated primes are its minimal primes, see Algebra, Proposition 10.63.6 (minimal primes are associated) and Algebra, Lemma 10.157.2 (no embedded primes). □

05KL Remark 31.7.4. With notation and assumptions as in Lemma 31.7.3 we see that it is always the case that $(g')^{-1}(\text{Ass}_{X/S}(\mathcal{F})) \supset \text{Ass}_{X'/S'}(\mathcal{F}')$. If the morphism $S' \to S$ is locally quasi-finite, then we actually have

$$(g')^{-1}(\text{Ass}_{X/S}(\mathcal{F})) = \text{Ass}_{X'/S'}(\mathcal{F}')$$

because in this case the field extensions $\kappa(s')/\kappa(s)$ are always finite. In fact, this holds more generally for any morphism $g : S' \to S$ such that all the field extensions $\kappa(s')/\kappa(s)$ are algebraic, because in this case all prime ideals of $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ are maximal (and minimal) primes, see Algebra, Lemma 10.36.19.

31.8. Relative weak assassin

05AU

05AV Definition 31.8.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The relative weak assassin of $\mathcal{F}$ in X over S is the set

$$\text{WeakAss}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \text{WeakAss}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

05F2 Lemma 31.8.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\text{WeakAss}_{X/S}(\mathcal{F}) = \text{Ass}_{X/S}(\mathcal{F})$.

Proof. This is true because the fibres of f are locally Noetherian schemes, and associated and weakly associated points agree on locally Noetherian schemes, see Lemma 31.5.8. □

0CUD Lemma 31.8.3. Let $f : X \to S$ be a morphism of schemes. Let $i : Z \to X$ be a finite morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $\text{WeakAss}_{X/S}(i_*\mathcal{F}) = i(\text{WeakAss}_{Z/S}(\mathcal{F}))$.

Proof. Let $i_s : Z_s \to X_s$ be the induced morphism between fibres. Then $(i_*\mathcal{F})_s = i_{s,*}(\mathcal{F}_s)$ by Cohomology of Schemes, Lemma 30.5.1 and the fact that i is affine. Hence we may apply Lemma 31.6.3 to conclude. □

31.9. Fitting ideals

0C3C This section is the continuation of the discussion in More on Algebra, Section 15.8. Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. In this situation we can construct the Fitting ideals

$$0 = \text{Fit}_{-1}(\mathcal{F}) \subset \text{Fit}_0(\mathcal{F}) \subset \text{Fit}_1(\mathcal{F}) \subset \ldots \subset \mathcal{O}_S$$

as the sequence of quasi-coherent ideals characterized by the following property: for every affine open $U = \text{Spec}(A)$ of S if $\mathcal{F}|_U$ corresponds to the A-module M, then $\text{Fit}_i(\mathcal{F})|_U$ corresponds to the ideal $\text{Fit}_i(M) \subset A$. This is well defined and a quasi-coherent sheaf of ideals because if $f \in A$, then the ith Fitting ideal of M_f over A_f is equal to $\text{Fit}_i(M)A_f$ by More on Algebra, Lemma 15.8.4.

Alternatively, we can construct the Fitting ideals in terms of local presentations of $\mathcal{F}$. Namely, if $U \subset X$ is open, and

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

is a presentation of $\mathcal{F}$ over U, then $\text{Fit}_r(\mathcal{F})|_U$ is generated by the $(n-r) \times (n-r)$-minors of the matrix defining the first arrow of the presentation. This is compatible with the construction above because this is how the Fitting ideal of a module over a ring is actually defined. Some details omitted.

0C3D **Lemma 31.9.1.** Let $f : T \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Then $f^{-1}\text{Fit}_i(\mathcal{F}) \cdot \mathcal{O}_T = \text{Fit}_i(f^*\mathcal{F})$.

Proof. Follows immediately from More on Algebra, Lemma 15.8.4 part (3). ☐

0C3E **Lemma 31.9.2.** Let S be a scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_S$-module. Then $\text{Fit}_r(\mathcal{F})$ is a quasi-coherent ideal of finite type.

Proof. Follows immediately from More on Algebra, Lemma 15.8.4 part (4). ☐

0CYX **Lemma 31.9.3.** Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $Z_0 \subset S$ be the closed subscheme cut out by $\text{Fit}_0(\mathcal{F})$. Let $Z \subset S$ be the scheme theoretic support of $\mathcal{F}$. Then

(1) $Z \subset Z_0 \subset S$ as closed subschemes,
(2) $Z = Z_0 = \text{Supp}(\mathcal{F})$ as closed subsets,
(3) there exists a finite type, quasi-coherent $\mathcal{O}_{Z_0}$-module $\mathcal{G}_0$ with

$$(Z_0 \to X)_*\mathcal{G}_0 = \mathcal{F}.$$

Proof. Recall that Z is locally cut out by the annihilator of $\mathcal{F}$, see Morphisms, Definition 29.5.5 (which uses Morphisms, Lemma 29.5.4 to define Z). Hence we see that $Z \subset Z_0$ scheme theoretically by More on Algebra, Lemma 15.8.4 part (6). On the other hand we have $Z = \text{Supp}(\mathcal{F})$ set theoretically by Morphisms, Lemma 29.5.4 and we have $Z_0 = Z$ set theoretically by More on Algebra, Lemma 15.8.4 part (7). Finally, to get $\mathcal{G}_0$ as in part (3) we can either use that we have $\mathcal{G}$ on Z as in Morphisms, Lemma 29.5.4 and set $\mathcal{G}_0 = (Z \to Z_0)_*\mathcal{G}$ or we can use Morphisms, Lemma 29.4.1 and the fact that $\text{Fit}_0(\mathcal{F})$ annihilates $\mathcal{F}$ by More on Algebra, Lemma 15.8.4 part (6). ☐

0C3F **Lemma 31.9.4.** Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $s \in S$. Then $\mathcal{F}$ can be generated by r elements in a neighbourhood of s if and only if $\text{Fit}_r(\mathcal{F})_s = \mathcal{O}_{S,s}$.

Proof. Follows immediately from More on Algebra, Lemma 15.8.6. □

0C3G Lemma 31.9.5. Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $r \geq 0$. The following are equivalent

(1) $\mathcal{F}$ is finite locally free of rank r
(2) $\text{Fit}_{r-1}(\mathcal{F}) = 0$ and $\text{Fit}_r(\mathcal{F}) = \mathcal{O}_S$, and
(3) $\text{Fit}_k(\mathcal{F}) = 0$ for $k < r$ and $\text{Fit}_k(\mathcal{F}) = \mathcal{O}_S$ for $k \geq r$.

Proof. Follows immediately from More on Algebra, Lemma 15.8.7. □

05P8 Lemma 31.9.6. Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. The closed subschemes

$$S = Z_{-1} \supset Z_0 \supset Z_1 \supset Z_2 \ldots$$

defined by the Fitting ideals of $\mathcal{F}$ have the following properties

(1) The intersection $\bigcap Z_r$ is empty.
(2) The functor $(Sch/S)^{opp} \to \text{Sets}$ defined by the rule

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is locally generated by } \leq r \text{ sections} \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the open subscheme $S \setminus Z_r$.
(3) The functor $F_r : (Sch/S)^{opp} \to \text{Sets}$ defined by the rule

$$T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by the locally closed subscheme $Z_{r-1} \setminus Z_r$ of S.

If $\mathcal{F}$ is of finite presentation, then $Z_r \to S$, $S \setminus Z_r \to S$, and $Z_{r-1} \setminus Z_r \to S$ are of finite presentation.

Proof. Part (1) is true because over every affine open U there is an integer n such that $\text{Fit}_n(\mathcal{F})|_U = \mathcal{O}_U$. Namely, we can take n to be the number of generators of $\mathcal{F}$ over U, see More on Algebra, Section 15.8.

For any morphism $g : T \to S$ we see from Lemmas 31.9.1 and 31.9.4 that $\mathcal{F}_T$ is locally generated by $\leq r$ sections if and only if $\text{Fit}_r(\mathcal{F}) \cdot \mathcal{O}_T = \mathcal{O}_T$. This proves (2).

For any morphism $g : T \to S$ we see from Lemmas 31.9.1 and 31.9.5 that $\mathcal{F}_T$ is free of rank r if and only if $\text{Fit}_r(\mathcal{F}) \cdot \mathcal{O}_T = \mathcal{O}_T$ and $\text{Fit}_{r-1}(\mathcal{F}) \cdot \mathcal{O}_T = 0$. This proves (3).

Assume $\mathcal{F}$ is of finite presentation. Then each of the morphisms $Z_r \to S$ is of finite presentation as $\text{Fit}_r(\mathcal{F})$ is of finite type (Lemma 31.9.2 and Morphisms, Lemma 29.21.7). This implies that $Z_{r-1} \setminus Z_r$ is a retrocompact open in Z_r (Properties, Lemma 28.24.1) and hence the morphism $Z_{r-1} \setminus Z_r \to Z_r$ is of finite presentation as well. □

Lemma 31.9.6 notwithstanding the following lemma does not hold if $\mathcal{F}$ is a finite type quasi-coherent module. Namely, the stratification still exists but it isn't true that it represents the functor F_{flat} in general.

05P9 Lemma 31.9.7. Let S be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_S$-module of finite presentation. Let $S = Z_{-1} \supset Z_0 \supset Z_1 \supset \ldots$ be as in Lemma 31.9.6. Set $S_r = Z_{r-1} \setminus Z_r$. Then $S' = \coprod_{r \geq 0} S_r$ represents the functor

$$F_{flat} : Sch/S \longrightarrow \text{Sets}, \qquad T \longmapsto \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ flat over } T \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, $\mathcal{F}|_{S_r}$ is locally free of rank r and the morphisms $S_r \to S$ and $S' \to S$ are of finite presentation.

Proof. Suppose that $g : T \to S$ is a morphism of schemes such that the pullback $\mathcal{F}_T = g^*\mathcal{F}$ is flat. Then $\mathcal{F}_T$ is a flat $\mathcal{O}_T$-module of finite presentation. Hence $\mathcal{F}_T$ is finite locally free, see Properties, Lemma 28.20.2. Thus $T = \coprod_{r \geq 0} T_r$, where $\mathcal{F}_T|_{T_r}$ is locally free of rank r. This implies that

$$F_{flat} = \coprod\nolimits_{r \geq 0} F_r$$

in the category of Zariski sheaves on Sch/S where F_r is as in Lemma 31.9.6. It follows that F_{flat} is represented by $\coprod_{r \geq 0}(Z_{r-1} \setminus Z_r)$ where Z_r is as in Lemma 31.9.6. The other statements also follow from the lemma. $\square$

0FJ0 Example 31.9.8. Let $R = \prod_{n \in \mathbf{N}} \mathbf{F}_2$. Let $I \subset R$ be the ideal of elements $a = (a_n)_{n \in \mathbf{N}}$ almost all of whose components are zero. Let $\mathfrak{m}$ be a maximal ideal containing I. Then $M = R/\mathfrak{m}$ is a finite flat R-module, because R is absolutely flat (More on Algebra, Lemma 15.104.6). Set $S = \text{Spec}(R)$ and $\mathcal{F} = \widetilde{M}$. The closed subschemes of Lemma 31.9.6 are $S = Z_{-1}$, $Z_0 = \text{Spec}(R/\mathfrak{m})$, and $Z_i = \emptyset$ for $i > 0$. But $\text{id} : S \to S$ does not factor through $(S \setminus Z_0) \amalg Z_0$ because $\mathfrak{m}$ is a nonisolated point of S. Thus Lemma 31.9.7 does not hold for finite type modules.

31.10. The singular locus of a morphism

0C3H Let $f : X \to S$ be a finite type morphism of schemes. The set U of points where f is smooth is an open of X (by Morphisms, Definition 29.34.1). In many situations it is useful to have a canonical closed subscheme $\text{Sing}(f) \subset X$ whose complement is U and whose formation commutes with arbitrary change of base.

If f is of finite presentation, then one choice would be to consider the closed subscheme Z cut out by functions which are affine locally "strictly standard" in the sense of Smoothing Ring Maps, Definition 16.2.3. It follows from Smoothing Ring Maps, Lemma 16.2.7 that if $f' : X' \to S'$ is the base change of f by a morphism $S' \to S$, then $Z' \subset S' \times_S Z$ where Z' is the closed subscheme of X' cut out by functions which are affine locally strictly standard. However, equality isn't clear. The notion of a strictly standard element was useful in the chapter on Popescu's theorem. The closed subscheme defined by these elements is (as far as we know) not used in the literature[1].

If f is flat, of finite presentation, and the fibres of f all are equidimensional of dimension d, then the dth fitting ideal of $\Omega_{X/S}$ is used to get a good closed subscheme. For any morphism of finite type the closed subschemes of X defined by the fitting ideals of $\Omega_{X/S}$ define a stratification of X in terms of the rank of $\Omega_{X/S}$ whose formation commutes with base change. This can be helpful; it is related to embedding dimensions of fibres, see Varieties, Section 33.46.

0C3I Lemma 31.10.1. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $X = Z_{-1} \supset Z_0 \supset Z_1 \supset \ldots$ be the closed subschemes defined by the fitting ideals of $\Omega_{X/S}$. Then the formation of Z_i commutes with arbitrary base change.

[1] If f is a local complete intersection morphism (More on Morphisms, Definition 37.62.2) then the closed subscheme cut out by the locally strictly standard elements is the correct thing to look at.

Proof. Observe that $\Omega_{X/S}$ is a finite type quasi-coherent $\mathcal{O}_X$-module (Morphisms, Lemma 29.32.12) hence the fitting ideals are defined. If $f' : X' \to S'$ is the base change of f by $g : S' \to S$, then $\Omega_{X'/S'} = (g')^*\Omega_{X/S}$ where $g' : X' \to X$ is the projection (Morphisms, Lemma 29.32.10). Hence $(g')^{-1}\text{Fit}_i(\Omega_{X/S}) \cdot \mathcal{O}_{X'} = \text{Fit}_i(\Omega_{X'/S'})$. This means that

$$Z'_i = (g')^{-1}(Z_i) = Z_i \times_X X'$$

scheme theoretically and this is the meaning of the statement of the lemma. □

The 0th fitting ideal of Ω cuts out the "ramified locus" of the morphism.

0C3J **Lemma 31.10.2.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The closed subscheme $Z \subset X$ cut out by the 0th fitting ideal of $\Omega_{X/S}$ is exactly the set of points where f is not unramified.

Proof. By Lemma 31.9.3 the complement of Z is exactly the locus where $\Omega_{X/S}$ is zero. This is exactly the set of points where f is unramified by Morphisms, Lemma 29.35.2. □

0C3K **Lemma 31.10.3.** Let $f : X \to S$ be a morphism of schemes. Let $d \geq 0$ be an integer. Assume

(1) f is flat,
(2) f is locally of finite presentation, and
(3) every nonempty fibre of f is equidimensional of dimension d.

Let $Z \subset X$ be the closed subscheme cut out by the dth fitting ideal of $\Omega_{X/S}$. Then Z is exactly the set of points where f is not smooth.

Proof. By Lemma 31.9.6 the complement of Z is exactly the locus where $\Omega_{X/S}$ can be generated by at most d elements. Hence the lemma follows from Morphisms, Lemma 29.34.14. □

31.11. Torsion free modules

0AVQ This section is the analogue of More on Algebra, Section 15.22 for quasi-coherent modules.

0AXR **Lemma 31.11.1.** Let X be an integral scheme with generic point η. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be nonempty open and $s \in \mathcal{F}(U)$. The following are equivalent

(1) for some $x \in U$ the image of s in $\mathcal{F}_x$ is torsion,
(2) for all $x \in U$ the image of s in $\mathcal{F}_x$ is torsion,
(3) the image of s in $\mathcal{F}_\eta$ is zero,
(4) the image of s in $j_*\mathcal{F}_\eta$ is zero, where $j : \eta \to X$ is the inclusion morphism.

Proof. Omitted. □

0AVR **Definition 31.11.2.** Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) We say a local section of $\mathcal{F}$ is *torsion* if it satisfies the equivalent conditions of Lemma 31.11.1.
(2) We say $\mathcal{F}$ is *torsion free* if every torsion section of $\mathcal{F}$ is 0.

Here is the obligatory lemma comparing this to the usual algebraic notion.

0AXS Lemma 31.11.3. Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is torsion free,
(2) for $U \subset X$ affine open $\mathcal{F}(U)$ is a torsion free $\mathcal{O}(U)$-module.

Proof. Omitted. $\square$

0AXT Lemma 31.11.4. Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The torsion sections of $\mathcal{F}$ form a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{F}_{tors} \subset \mathcal{F}$. The quotient module $\mathcal{F}/\mathcal{F}_{tors}$ is torsion free.

Proof. Omitted. See More on Algebra, Lemma 15.22.2 for the algebraic analogue. $\square$

0AXU Lemma 31.11.5. Let X be an integral scheme. Any flat quasi-coherent $\mathcal{O}_X$-module is torsion free.

Proof. Omitted. See More on Algebra, Lemma 15.22.9. $\square$

0AXV Lemma 31.11.6. Let $f : X \to Y$ be a flat morphism of integral schemes. Let $\mathcal{G}$ be a torsion free quasi-coherent $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a torsion free $\mathcal{O}_X$-module.

Proof. Omitted. See More on Algebra, Lemma 15.23.7 for the algebraic analogue. $\square$

0BCM Lemma 31.11.7. Let $f : X \to Y$ be a flat morphism of schemes. If Y is integral and the generic fibre of f is integral, then X is integral.

Proof. The algebraic analogue is this: let A be a domain with fraction field K and let B be a flat A-algebra such that $B \otimes_A K$ is a domain. Then B is a domain. This is true because B is torsion free by More on Algebra, Lemma 15.22.9 and hence $B \subset B \otimes_A K$. $\square$

0AXW Lemma 31.11.8. Let X be an integral scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is torsion free if and only if $\mathcal{F}_x$ is a torsion free $\mathcal{O}_{X,x}$-module for all $x \in X$.

Proof. Omitted. See More on Algebra, Lemma 15.22.6. $\square$

0AXX Lemma 31.11.9. Let X be an integral scheme. Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{F}''$ are torsion free, then $\mathcal{F}'$ is torsion free.

Proof. Omitted. See More on Algebra, Lemma 15.22.5 for the algebraic analogue. $\square$

0AXY Lemma 31.11.10. Let X be a locally Noetherian integral scheme with generic point η. Let $\mathcal{F}$ be a nonzero coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is torsion free,
(2) η is the only associated prime of $\mathcal{F}$,
(3) η is in the support of $\mathcal{F}$ and $\mathcal{F}$ has property (S_1), and
(4) η is in the support of $\mathcal{F}$ and $\mathcal{F}$ has no embedded associated prime.

Proof. This is a translation of More on Algebra, Lemma 15.22.8 into the language of schemes. We omit the translation. $\square$

0CC4 **Lemma 31.11.11.** Let X be an integral regular scheme of dimension ≤ 1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is torsion free,
(2) $\mathcal{F}$ is finite locally free.

Proof. It is clear that a finite locally free module is torsion free. For the converse, we will show that if $\mathcal{F}$ is torsion free, then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$. This is enough by Algebra, Lemma 10.78.2 and the fact that $\mathcal{F}$ is coherent. If $\dim(\mathcal{O}_{X,x}) = 0$, then $\mathcal{O}_{X,x}$ is a field and the statement is clear. If $\dim(\mathcal{O}_{X,x}) = 1$, then $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 10.119.7) and $\mathcal{F}_x$ is torsion free. Hence $\mathcal{F}_x$ is free by More on Algebra, Lemma 15.22.11. □

0AXZ **Lemma 31.11.12.** Let X be an integral scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is torsion free and $\mathcal{F}$ is of finite presentation, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is torsion free.

Proof. The statement makes sense because $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent by Schemes, Section 26.24. To see the statement is true, see More on Algebra, Lemma 15.22.12. Some details omitted. □

0AVS **Lemma 31.11.13.** Let X be an integral locally Noetherian scheme. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is coherent, $\mathcal{G}$ is torsion free, and that for every $x \in X$ one of the following happens

(1) $\mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, or
(2) $\text{depth}(\mathcal{F}_x) \geq 2$.

Then φ is an isomorphism.

Proof. This is a translation of More on Algebra, Lemma 15.23.14 into the language of schemes. □

31.12. Reflexive modules

0AVT This section is the analogue of More on Algebra, Section 15.23 for coherent modules on locally Noetherian schemes. The reason for working with coherent modules is that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent for every pair of coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, see Modules, Lemma 17.22.6.

0AVU **Definition 31.12.1.** Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The *reflexive hull* of $\mathcal{F}$ is the $\mathcal{O}_X$-module

$$\mathcal{F}^{**} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

We say $\mathcal{F}$ is *reflexive* if the natural map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is an isomorphism.

It follows from Lemma 31.12.8 that the reflexive hull is a reflexive $\mathcal{O}_X$-module. You can use the same definition to define reflexive modules in more general situations, but this does not seem to be very useful. Here is the obligatory lemma comparing this to the usual algebraic notion.

0AY0 **Lemma 31.12.2.** Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is reflexive,
(2) for $U \subset X$ affine open $\mathcal{F}(U)$ is a reflexive $\mathcal{O}(U)$-module.

Proof. Omitted. □

0AY1 Remark 31.12.3. If X is a scheme of finite type over a field, then sometimes a different notion of reflexive modules is used (see for example [HL97, bottom of page 5 and Definition 1.1.9]). This other notion uses $R\mathcal{H}om$ into a dualizing complex $\omega_X^\bullet$ instead of into $\mathcal{O}_X$ and should probably have a different name because it can be different when X is not Gorenstein. For example, if $X = \text{Spec}(k[t^3, t^4, t^5])$, then a computation shows the dualizing sheaf ω_X is not reflexive in our sense, but it is reflexive in the other sense as $\omega_X \to \mathcal{H}om(\mathcal{H}om(\omega_X, \omega_X), \omega_X)$ is an isomorphism.

0AY2 Lemma 31.12.4. Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

(1) If $\mathcal{F}$ is reflexive, then $\mathcal{F}$ is torsion free.
(2) The map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is injective if and only if $\mathcal{F}$ is torsion free.

Proof. Omitted. See More on Algebra, Lemma 15.23.2. □

0AY3 Lemma 31.12.5. Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is reflexive,
(2) $\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module for all $x \in X$,
(3) $\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module for all closed points $x \in X$.

Proof. By Modules, Lemma 17.22.4 we see that (1) and (2) are equivalent. Since every point of X specializes to a closed point (Properties, Lemma 28.5.9) we see that (2) and (3) are equivalent. □

0EBF Lemma 31.12.6. Let $f : X \to Y$ be a flat morphism of integral locally Noetherian schemes. Let $\mathcal{G}$ be a coherent reflexive $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a coherent reflexive $\mathcal{O}_X$-module.

Proof. Omitted. See More on Algebra, Lemma 15.22.4 for the algebraic analogue. □

0EBG Lemma 31.12.7. Let X be an integral locally Noetherian scheme. Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ be an exact sequence of coherent $\mathcal{O}_X$-modules. If $\mathcal{F}'$ is reflexive and $\mathcal{F}''$ is torsion free, then $\mathcal{F}$ is reflexive.

Proof. Omitted. See More on Algebra, Lemma 15.23.5. □

0AY4 Lemma 31.12.8. Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. If $\mathcal{G}$ is reflexive, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is reflexive.

Proof. The statement makes sense because $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent by Cohomology of Schemes, Lemma 30.9.4. To see the statement is true, see More on Algebra, Lemma 15.23.8. Some details omitted. □

0EBH Remark 31.12.9. Let X be an integral locally Noetherian scheme. Thanks to Lemma 31.12.8 we know that the reflexive hull $\mathcal{F}^{**}$ of a coherent $\mathcal{O}_X$-module is coherent reflexive. Consider the category $\mathcal{C}$ of coherent reflexive $\mathcal{O}_X$-modules. Taking reflexive hulls gives a left adjoint to the inclusion functor $\mathcal{C} \to \text{Coh}(\mathcal{O}_X)$. Observe that $\mathcal{C}$ is an additive category with kernels and cokernels. Namely, given $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}$, the usual kernel $\text{Ker}(\varphi)$ is reflexive (Lemma 31.12.7) and the reflexive hull $\text{Coker}(\varphi)^{**}$ of the usual cokernel is the cokernel in $\mathcal{C}$. Moreover $\mathcal{C}$ inherits a tensor product

$$\mathcal{F} \otimes_{\mathcal{C}} \mathcal{G} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{**}$$

which is associative and symmetric. There is an internal Hom in the sense that for any three objects $\mathcal{F}, \mathcal{G}, \mathcal{H}$ of $\mathcal{C}$ we have the identity

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F} \otimes_{\mathcal{C}} \mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

see Modules, Lemma 17.22.1. In $\mathcal{C}$ every object $\mathcal{F}$ has a dual object $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. Without further conditions on X it can happen that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \not\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{C}} \mathcal{G} \quad \text{and} \quad \mathcal{F} \otimes_{\mathcal{C}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \not\cong \mathcal{O}_X$$

for $\mathcal{F}, \mathcal{G}$ of rank 1 in $\mathcal{C}$. To make an example let $X = \mathrm{Spec}(R)$ where R is as in More on Algebra, Example 15.23.17 and let $\mathcal{F}, \mathcal{G}$ be the modules corresponding to M. Computation omitted.

0AY5 **Lemma 31.12.10.** Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is reflexive,
(2) for each $x \in X$ one of the following happens
   (a) $\mathcal{F}_x$ is a reflexive $\mathcal{O}_{X,x}$-module, or
   (b) $\mathrm{depth}(\mathcal{F}_x) \geq 2$.

Proof. Omitted. See More on Algebra, Lemma 15.23.15. $\square$

0EBI **Lemma 31.12.11.** Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent reflexive $\mathcal{O}_X$-module. Let $x \in X$.

(1) If $\mathrm{depth}(\mathcal{O}_{X,x}) \geq 2$, then $\mathrm{depth}(\mathcal{F}_x) \geq 2$.
(2) If X is (S_2), then $\mathcal{F}$ is (S_2).

Proof. Omitted. See More on Algebra, Lemma 15.23.16. $\square$

0EBJ **Lemma 31.12.12.** Let X be an integral locally Noetherian scheme. Let $j : U \to X$ be an open subscheme with complement Z. Assume $\mathcal{O}_{X,z}$ has depth ≥ 2 for all $z \in Z$. Then j^* and j_* define an equivalence of categories between the category of coherent reflexive $\mathcal{O}_X$-modules and the category of coherent reflexive $\mathcal{O}_U$-modules.

Proof. Let $\mathcal{F}$ be a coherent reflexive $\mathcal{O}_X$-module. For $z \in Z$ the stalk $\mathcal{F}_z$ has depth ≥ 2 by Lemma 31.12.11. Thus $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism by Lemma 31.5.11. Conversely, let $\mathcal{G}$ be a coherent reflexive $\mathcal{O}_U$-module. It suffices to show that $j_*\mathcal{G}$ is a coherent reflexive $\mathcal{O}_X$-module. To prove this we may assume X is affine. By Properties, Lemma 28.22.5 there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\mathcal{G} = j^*\mathcal{F}$. After replacing $\mathcal{F}$ by its reflexive hull, we may assume $\mathcal{F}$ is reflexive (see discussion above and in particular Lemma 31.12.8). By the above $j_*\mathcal{G} = j_*j^*\mathcal{F} = \mathcal{F}$ as desired. $\square$

If the scheme is normal, then reflexive is the same thing as torsion free and (S_2).

0AY6 **Lemma 31.12.13.** Let X be an integral locally Noetherian normal scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is reflexive,
(2) $\mathcal{F}$ is torsion free and has property (S_2), and
(3) there exists an open subscheme $j : U \to X$ such that
   (a) every irreducible component of $X \setminus U$ has codimension ≥ 2 in X,
   (b) $j^*\mathcal{F}$ is finite locally free, and
   (c) $\mathcal{F} = j_*j^*\mathcal{F}$.

Proof. Using Lemma 31.12.2 the equivalence of (1) and (2) follows from More on Algebra, Lemma 15.23.18. Let $U \subset X$ be as in (3). By Properties, Lemma 28.12.5 we see that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$. Since a finite locally free module is reflexive, we conclude (3) implies (1) by Lemma 31.12.12.

Assume (1). Let $U \subset X$ be the maximal open subscheme such that $j^*\mathcal{F} = \mathcal{F}|_U$ is finite locally free. So (3)(b) holds. Let $x \in X$ be a point. If $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, then $x \in U$, see Modules, Lemma 17.11.6. If $\dim(\mathcal{O}_{X,x}) \leq 1$, then $\mathcal{O}_{X,x}$ is either a field or a discrete valuation ring (Properties, Lemma 28.12.5) and hence $\mathcal{F}_x$ is free (More on Algebra, Lemma 15.22.11). Thus $x \notin U \Rightarrow \dim(\mathcal{O}_{X,x}) \geq 2$. Then Properties, Lemma 28.10.3 shows (3)(a) holds. By the already used Properties, Lemma 28.12.5 we also see that $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ for $x \notin U$ and hence (3)(c) follows from Lemma 31.12.12. $\square$

0AY7 Lemma 31.12.14. Let X be an integral locally Noetherian normal scheme with generic point η. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $T : \mathcal{G}_\eta \to \mathcal{F}_\eta$ be a linear map. Then T extends to a map $\mathcal{G} \to \mathcal{F}^{**}$ of $\mathcal{O}_X$-modules if and only if

(*) for every $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 1$ we have
$$T\left(\text{Im}(\mathcal{G}_x \to \mathcal{G}_\eta)\right) \subset \text{Im}(\mathcal{F}_x \to \mathcal{F}_\eta).$$

Proof. Because $\mathcal{F}^{**}$ is torsion free and $\mathcal{F}_\eta = \mathcal{F}^{**}_\eta$ an extension, if it exists, is unique. Thus it suffices to prove the lemma over the members of an open covering of X, i.e., we may assume X is affine. In this case we are asking the following algebra question: Let R be a Noetherian normal domain with fraction field K, let M, N be finite R-modules, let $T : M \otimes_R K \to N \otimes_R K$ be a K-linear map. When does T extend to a map $N \to M^{**}$? By More on Algebra, Lemma 15.23.19 this happens if and only if $N_\mathfrak{p}$ maps into $(M/M_{tors})_\mathfrak{p}$ for every height 1 prime $\mathfrak{p}$ of R. This is exactly condition $(*)$ of the lemma. $\square$

0B3N Lemma 31.12.15. Let X be a regular scheme of dimension ≤ 2. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The following are equivalent

(1) $\mathcal{F}$ is reflexive,
(2) $\mathcal{F}$ is finite locally free.

Proof. It is clear that a finite locally free module is reflexive. For the converse, we will show that if $\mathcal{F}$ is reflexive, then $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$. This is enough by Algebra, Lemma 10.78.2 and the fact that $\mathcal{F}$ is coherent. If $\dim(\mathcal{O}_{X,x}) = 0$, then $\mathcal{O}_{X,x}$ is a field and the statement is clear. If $\dim(\mathcal{O}_{X,x}) = 1$, then $\mathcal{O}_{X,x}$ is a discrete valuation ring (Algebra, Lemma 10.119.7) and $\mathcal{F}_x$ is torsion free. Hence $\mathcal{F}_x$ is free by More on Algebra, Lemma 15.22.11. If $\dim(\mathcal{O}_{X,x}) = 2$, then $\mathcal{O}_{X,x}$ is a regular local ring of dimension 2. By More on Algebra, Lemma 15.23.18 we see that $\mathcal{F}_x$ has depth ≥ 2. Hence $\mathcal{F}$ is free by Algebra, Lemma 10.106.6. $\square$

31.13. Effective Cartier divisors

01WQ We define the notion of an effective Cartier divisor before any other type of divisor.

01WR Definition 31.13.1. Let S be a scheme.

(1) A locally principal closed subscheme of S is a closed subscheme whose sheaf of ideals is locally generated by a single element.
(2) An effective Cartier divisor on S is a closed subscheme $D \subset S$ whose ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_S$ is an invertible $\mathcal{O}_S$-module.

Thus an effective Cartier divisor is a locally principal closed subscheme, but the converse is not always true. Effective Cartier divisors are closed subschemes of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is a nonzerodivisor. In particular they are nowhere dense.

01WS Lemma 31.13.2. Let S be a scheme. Let $D \subset S$ be a closed subscheme. The following are equivalent:

(1) The subscheme D is an effective Cartier divisor on S.
(2) For every $x \in D$ there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset S$ of x such that $U \cap D = \mathrm{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor.

Proof. Assume (1). For every $x \in D$ there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset S$ of x such that $\mathcal{I}_D|_U \cong \mathcal{O}_U$. In other words, there exists a section $f \in \Gamma(U, \mathcal{I}_D)$ which freely generates the restriction $\mathcal{I}_D|_U$. Hence $f \in A$, and the multiplication map $f : A \to A$ is injective. Also, since $\mathcal{I}_D$ is quasi-coherent we see that $D \cap U = \mathrm{Spec}(A/(f))$.

Assume (2). Let $x \in D$. By assumption there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset S$ of x such that $U \cap D = \mathrm{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor. Then $\mathcal{I}_D|_U \cong \mathcal{O}_U$ since it is equal to $\widetilde{(f)} \cong \widetilde{A} \cong \mathcal{O}_U$. Of course $\mathcal{I}_D$ restricted to the open subscheme $S \setminus D$ is isomorphic to $\mathcal{O}_{S \setminus D}$. Hence $\mathcal{I}_D$ is an invertible $\mathcal{O}_S$-module. $\square$

07ZT Lemma 31.13.3. Let S be a scheme. Let $Z \subset S$ be a locally principal closed subscheme. Let $U = S \setminus Z$. Then $U \to S$ is an affine morphism.

Proof. The question is local on S, see Morphisms, Lemmas 29.11.3. Thus we may assume $S = \mathrm{Spec}(A)$ and $Z = V(f)$ for some $f \in A$. In this case $U = D(f) = \mathrm{Spec}(A_f)$ is affine hence $U \to S$ is affine. $\square$

07ZU Lemma 31.13.4. Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $U = S \setminus D$. Then $U \to S$ is an affine morphism and U is scheme theoretically dense in S.

Proof. Affineness is Lemma 31.13.3. The density question is local on S, see Morphisms, Lemma 29.7.5. Thus we may assume $S = \mathrm{Spec}(A)$ and D corresponding to the nonzerodivisor $f \in A$, see Lemma 31.13.2. Thus $A \subset A_f$ which implies that $U \subset S$ is scheme theoretically dense, see Morphisms, Example 29.7.4. $\square$

056N Lemma 31.13.5. Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $s \in D$. If $\dim_s(S) < \infty$, then $\dim_s(D) < \dim_s(S)$.

Proof. Assume $\dim_s(S) < \infty$. Let $U = \mathrm{Spec}(A) \subset S$ be an affine open neighbourhood of s such that $\dim(U) = \dim_s(S)$ and such that $D = V(f)$ for some nonzerodivisor $f \in A$ (see Lemma 31.13.2). Recall that $\dim(U)$ is the Krull dimension of the ring A and that $\dim(U \cap D)$ is the Krull dimension of the ring $A/(f)$. Then f is not contained in any minimal prime of A. Hence any maximal chain of primes in $A/(f)$, viewed as a chain of primes in A, can be extended by adding a minimal prime. $\square$

01WT Definition 31.13.6. Let S be a scheme. Given effective Cartier divisors D_1, D_2 on S we set $D = D_1 + D_2$ equal to the closed subscheme of S corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the sum of the effective Cartier divisors D_1 and D_2.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

01WU Lemma 31.13.7. The sum of two effective Cartier divisors is an effective Cartier divisor.

Proof. Omitted. Locally $f_1, f_2 \in A$ are nonzerodivisors, then also $f_1 f_2 \in A$ is a nonzerodivisor. $\square$

02ON Lemma 31.13.8. Let X be a scheme. Let D, D' be two effective Cartier divisors on X. If $D \subset D'$ (as closed subschemes of X), then there exists an effective Cartier divisor D'' such that $D' = D + D''$.

Proof. Omitted. $\square$

07ZV Lemma 31.13.9. Let X be a scheme. Let Z, Y be two closed subschemes of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.

Proof. Applying Lemma 31.13.2 we obtain the following algebra situation: A is a ring, $I, J \subset A$ ideals and $f \in A$ a nonzerodivisor such that $IJ = (f)$. Thus the result follows from Algebra, Lemma 10.120.16. $\square$

0C4R Lemma 31.13.10. Let X be a scheme. Let $D, D' \subset X$ be effective Cartier divisors such that the scheme theoretic intersection $D \cap D'$ is an effective Cartier divisor on D'. Then $D + D'$ is the scheme theoretic union of D and D'.

Proof. See Morphisms, Definition 29.4.4 for the definition of scheme theoretic intersection and union. To prove the lemma working locally (using Lemma 31.13.2) we obtain the following algebra problem: Given a ring A and nonzerodivisors $f_1, f_2 \in A$ such that f_1 maps to a nonzerodivisor in A/f_2A, show that $f_1A \cap f_2A = f_1f_2A$. We omit the straightforward argument. $\square$

Recall that we have defined the inverse image of a closed subscheme under any morphism of schemes in Schemes, Definition 26.17.7.

053P Lemma 31.13.11. Let $f : S' \to S$ be a morphism of schemes. Let $Z \subset S$ be a locally principal closed subscheme. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subscheme of S'.

Proof. Omitted. $\square$

01WV Definition 31.13.12. Let $f : S' \to S$ be a morphism of schemes. Let $D \subset S$ be an effective Cartier divisor. We say the pullback of D by f is defined if the closed subscheme $f^{-1}(D) \subset S'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the pullback of the effective Cartier divisor.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice. Here is an example lemma.

02OO Lemma 31.13.13. Let $f : X \to Y$ be a morphism of schemes. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:

(1) $f(x) \notin D$ for any weakly associated point x of X,
(2) X, Y integral and f dominant,

(3) X reduced and $f(\xi) \notin D$ for any generic point ξ of any irreducible component of X,
(4) X is locally Noetherian and $f(x) \notin D$ for any associated point x of X,
(5) X is locally Noetherian, has no embedded points, and $f(\xi) \notin D$ for any generic point ξ of an irreducible component of X,
(6) f is flat, and
(7) add more here as needed.

Proof. The question is local on X, and hence we reduce to the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$, f is given by $\varphi : R \to A$ and $D = \mathrm{Spec}(R/(t))$ where $t \in R$ is a nonzerodivisor. The goal in each case is to show that $\varphi(t) \in A$ is a nonzerodivisor.

In case (1) this follows from Algebra, Lemma 10.66.7. Case (4) is a special case of (1) by Lemma 31.5.8. Case (5) follows from (4) and the definitions. Case (3) is a special case of (1) by Lemma 31.5.12. Case (2) is a special case of (3). If $R \to A$ is flat, then $t : R \to R$ being injective shows that $t : A \to A$ is injective. This proves (6). □

01WW Lemma 31.13.14. Let $f : S' \to S$ be a morphism of schemes. Let D_1, D_2 be effective Cartier divisors on S. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.

Proof. Omitted. □

31.14. Effective Cartier divisors and invertible sheaves

0C4S Since an effective Cartier divisor has an invertible ideal sheaf (Definition 31.13.1) the following definition makes sense.

01WX Definition 31.14.1. Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor with ideal sheaf $\mathcal{I}_D$.

(1) The invertible sheaf $\mathcal{O}_S(D)$ associated to D is defined by
$$\mathcal{O}_S(D) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_D, \mathcal{O}_S) = \mathcal{I}_D^{\otimes -1}.$$
(2) The canonical section, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_S(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_S$.
(3) We write $\mathcal{O}_S(-D) = \mathcal{O}_S(D)^{\otimes -1} = \mathcal{I}_D$.
(4) Given a second effective Cartier divisor $D' \subset S$ we define $\mathcal{O}_S(D - D') = \mathcal{O}_S(D) \otimes_{\mathcal{O}_S} \mathcal{O}_S(-D')$.

Some comments. We will see below that the assignment $D \mapsto \mathcal{O}_S(D)$ turns addition of effective Cartier divisors (Definition 31.13.6) into addition in the Picard group of S (Lemma 31.14.4). However, the expression $D - D'$ in the definition above does not have any geometric meaning. More precisely, we can think of the set of effective Cartier divisors on S as a commutative monoid $\mathrm{EffCart}(S)$ whose zero element is the empty effective Cartier divisor. Then the assignment $(D, D') \mapsto \mathcal{O}_S(D - D')$ defines a group homomorphism
$$\mathrm{EffCart}(S)^{gp} \longrightarrow \mathrm{Pic}(S)$$
where the left hand side is the group completion of $\mathrm{EffCart}(S)$. In other words, when we write $\mathcal{O}_S(D - D')$ we may think of $D - D'$ as an element of $\mathrm{EffCart}(S)^{gp}$.

0B3P Lemma 31.14.2. Let S be a scheme and let $D \subset S$ be an effective Cartier divisor. Then the conormal sheaf is $\mathcal{C}_{D/S} = \mathcal{I}_D|_D = \mathcal{O}_S(-D)|_D$ and the normal sheaf is $\mathcal{N}_{D/S} = \mathcal{O}_S(D)|_D$.

Proof. This follows from Morphisms, Lemma 29.31.2. □

0C4T Lemma 31.14.3. Let X be a scheme. Let $D, C \subset X$ be effective Cartier divisors with $C \subset D$ and let $D' = D + C$. Then there is a short exact sequence

$$0 \to \mathcal{O}_X(-D)|_C \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$

of $\mathcal{O}_X$-modules.

Proof. In the statement of the lemma and in the proof we use the equivalence of Morphisms, Lemma 29.4.1 to think of quasi-coherent modules on closed subschemes of X as quasi-coherent modules on X. Let $\mathcal{I}$ be the ideal sheaf of D in D'. Then there is a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$$

because $D \to D'$ is a closed immersion. There is a canonical surjection $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2 = \mathcal{C}_{D/D'}$. We have $\mathcal{C}_{D/X} = \mathcal{O}_X(-D)|_D$ by Lemma 31.14.2 and there is a canonical surjective map

$$\mathcal{C}_{D/X} \longrightarrow \mathcal{C}_{D/D'}$$

see Morphisms, Lemmas 29.31.3 and 29.31.4. Thus it suffices to show: (a) $\mathcal{I}^2 = 0$ and (b) $\mathcal{I}$ is an invertible $\mathcal{O}_C$-module. Both (a) and (b) can be checked locally, hence we may assume $X = \text{Spec}(A)$, $D = \text{Spec}(A/fA)$ and $C = \text{Spec}(A/gA)$ where $f, g \in A$ are nonzerodivisors (Lemma 31.13.2). Since $C \subset D$ we see that $f \in gA$. Then $I = fA/fgA$ has square zero and is invertible as an A/gA-module as desired. □

02OP Lemma 31.14.4. Let S be a scheme. Let D_1, D_2 be effective Cartier divisors on S. Let $D = D_1 + D_2$. Then there is a unique isomorphism

$$\mathcal{O}_S(D_1) \otimes_{\mathcal{O}_S} \mathcal{O}_S(D_2) \longrightarrow \mathcal{O}_S(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. □

0C4U Lemma 31.14.5. Let $f : S' \to S$ be a morphism of schemes. Let D be a effective Cartier divisors on S. If the pullback of D is defined then $f^*\mathcal{O}_S(D) = \mathcal{O}_{S'}(f^*D)$ and the canonical section 1_D pulls back to the canonical section 1_{f^*D}.

Proof. Omitted. □

01WY Definition 31.14.6. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a regular section if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

01WZ Lemma 31.14.7. Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:

(1) f is a regular section, and
(2) for any $x \in X$ the image $f \in \mathcal{O}_{X,x}$ is a nonzerodivisor.

If X is a scheme these are also equivalent to

(3) for any affine open $\text{Spec}(A) = U \subset X$ the image $f \in A$ is a nonzerodivisor,

(4) there exists an affine open covering $X = \bigcup \mathrm{Spec}(A_i)$ such that the image of f in A_i is a nonzerodivisor for all i.

Proof. Omitted. $\square$

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules, Definition 17.25.6 for the definition of the dual invertible sheaf.)

02OQ Definition 31.14.8. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$ be a global section. The zero scheme of s is the closed subscheme $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

02OR Lemma 31.14.9. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$.

(1) Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L})$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.
(2) For any morphism of schemes $f : Y \to X$ we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.
(3) The zero scheme $Z(s)$ is a locally principal closed subscheme.
(4) The zero scheme $Z(s)$ is an effective Cartier divisor if and only if s is a regular section of $\mathcal{L}$.

Proof. Omitted. $\square$

01X0 Lemma 31.14.10. Let X be a scheme.

(1) If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.
(2) Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\text{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{matrix}\text{isomorphism classes of pairs } (\mathcal{L}, s) \\ \text{consisting of an invertible } \mathcal{O}_X\text{-module} \\ \mathcal{L} \text{ and a regular global section } s\end{matrix}\right\}$$

Proof. Omitted. $\square$

0C6K Remark 31.14.11. Let X be a scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and s a regular section of $\mathcal{L}$. Then the zero scheme $D = Z(s)$ is an effective Cartier divisor on X and there are short exact sequences

$$0 \to \mathcal{O}_X \to \mathcal{L} \to i_*(\mathcal{L}|_D) \to 0 \quad \text{and} \quad 0 \to \mathcal{L}^{\otimes -1} \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0.$$

Given an effective Cartier divisor $D \subset X$ using Lemmas 31.14.10 and 31.14.2 we get

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*(\mathcal{N}_{D/X}) \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*(\mathcal{O}_D) \to 0$$

31.15. Effective Cartier divisors on Noetherian schemes

0B3Q In the locally Noetherian setting most of the discussion of effective Cartier divisors and regular sections simplifies somewhat.

0AYL **Lemma 31.15.1.** Let X be a locally Noetherian scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Then s is a regular section if and only if s does not vanish in the associated points of X.

Proof. Omitted. Hint: reduce to the affine case and $\mathcal{L}$ trivial and then use Lemma 31.14.7 and Algebra, Lemma 10.63.9. $\square$

0AG8 **Lemma 31.15.2.** Let X be a locally Noetherian scheme. Let $D \subset X$ be a closed subscheme corresponding to the quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$.

(1) If for every $x \in D$ the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ can be generated by one element, then D is locally principal.
(2) If for every $x \in D$ the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ can be generated by a single nonzerodivisor, then D is an effective Cartier divisor.

Proof. Let $\text{Spec}(A)$ be an affine neighbourhood of a point $x \in D$. Let $\mathfrak{p} \subset A$ be the prime corresponding to x. Let $I \subset A$ be the ideal defining the trace of D on $\text{Spec}(A)$. Since A is Noetherian (as X is locally Noetherian) the ideal I is generated by finitely many elements, say $I = (f_1, \ldots, f_r)$. Under the assumption of (1) we have $I_\mathfrak{p} = (f)$ for some $f \in A_\mathfrak{p}$. Then $f_i = g_i f$ for some $g_i \in A_\mathfrak{p}$. Write $g_i = a_i/h_i$ and $f = f'/h$ for some $a_i, h_i, f', h \in A$, $h_i, h \not\in \mathfrak{p}$. Then $I_{h_1 \ldots h_r h} \subset A_{h_1 \ldots h_r h}$ is principal, because it is generated by f'. This proves (1). For (2) we may assume $I = (f)$. The assumption implies that the image of f in $A_\mathfrak{p}$ is a nonzerodivisor. Then f is a nonzerodivisor on a neighbourhood of x by Algebra, Lemma 10.68.6. This proves (2). $\square$

0BCN **Lemma 31.15.3.** Let X be a locally Noetherian scheme.

(1) Let $D \subset X$ be a locally principal closed subscheme. Let $\xi \in D$ be a generic point of an irreducible component of D. Then $\dim(\mathcal{O}_{X,\xi}) \leq 1$.
(2) Let $D \subset X$ be an effective Cartier divisor. Let $\xi \in D$ be a generic point of an irreducible component of D. Then $\dim(\mathcal{O}_{X,\xi}) = 1$.

Proof. Proof of (1). By assumption we may assume $X = \text{Spec}(A)$ and $D = \text{Spec}(A/(f))$ where A is a Noetherian ring and $f \in A$. Let ξ correspond to the prime ideal $\mathfrak{p} \subset A$. The assumption that ξ is a generic point of an irreducible component of D signifies $\mathfrak{p}$ is minimal over (f). Thus $\dim(A_\mathfrak{p}) \leq 1$ by Algebra, Lemma 10.60.11.

Proof of (2). By part (1) we see that $\dim(\mathcal{O}_{X,\xi}) \leq 1$. On the other hand, the local equation f is a nonzerodivisor in $A_\mathfrak{p}$ by Lemma 31.13.2 which implies the dimension is at least 1 (because there must be a prime in $A_\mathfrak{p}$ not containing f by the elementary Algebra, Lemma 10.17.2). $\square$

0AG9 **Lemma 31.15.4.** Let X be a Noetherian scheme. Let $D \subset X$ be an integral closed subscheme which is also an effective Cartier divisor. Then the local ring of X at the generic point of D is a discrete valuation ring.

Proof. By Lemma 31.13.2 we may assume $X = \text{Spec}(A)$ and $D = \text{Spec}(A/(f))$ where A is a Noetherian ring and $f \in A$ is a nonzerodivisor. The assumption that

D is integral signifies that (f) is prime. Hence the local ring of X at the generic point is $A_{(f)}$ which is a Noetherian local ring whose maximal ideal is generated by a nonzerodivisor. Thus it is a discrete valuation ring by Algebra, Lemma 10.119.7. □

0B3R Lemma 31.15.5. Let X be a locally Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. If X is (S_k), then D is (S_{k-1}).

Proof. Let $x \in D$. Then $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/(f)$ where $f \in \mathcal{O}_{X,x}$ is a nonzerodivisor. By assumption we have $\text{depth}(\mathcal{O}_{X,x}) \geq \min(\dim(\mathcal{O}_{X,x}), k)$. By Algebra, Lemma 10.72.7 we have $\text{depth}(\mathcal{O}_{D,x}) = \text{depth}(\mathcal{O}_{X,x}) - 1$ and by Algebra, Lemma 10.60.13 $\dim(\mathcal{O}_{D,x}) = \dim(\mathcal{O}_{X,x}) - 1$. It follows that $\text{depth}(\mathcal{O}_{D,x}) \geq \min(\dim(\mathcal{O}_{D,x}), k-1)$ as desired. □

0B3S Lemma 31.15.6. Let X be a locally Noetherian normal scheme. Let $D \subset X$ be an effective Cartier divisor. Then D is (S_1).

Proof. By Properties, Lemma 28.12.5 we see that X is (S_2). Thus we conclude by Lemma 31.15.5. □

0AGA Lemma 31.15.7. Let X be a Noetherian scheme. Let $D \subset X$ be an integral closed subscheme. Assume that

(1) D has codimension 1 in X, and
(2) $\mathcal{O}_{X,x}$ is a UFD for all $x \in D$.

Then D is an effective Cartier divisor.

Proof. Let $x \in D$ and set $A = \mathcal{O}_{X,x}$. Let $\mathfrak{p} \subset A$ correspond to the generic point of D. Then $A_\mathfrak{p}$ has dimension 1 by assumption (1). Thus $\mathfrak{p}$ is a prime ideal of height 1. Since A is a UFD this implies that $\mathfrak{p} = (f)$ for some $f \in A$. Of course f is a nonzerodivisor and we conclude by Lemma 31.15.2. □

0AGB Lemma 31.15.8. Let X be a Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Assume there exist integral effective Cartier divisors $D_i \subset X$ and a closed subset $Z' \subset X$ of codimension ≥ 2 such that $Z \subset Z' \cup \bigcup D_i$ set-theoretically. Then there exists an effective Cartier divisor of the form

$$D = \sum a_i D_i \subset Z$$

such that $D \to Z$ is an isomorphism away from codimension 2 in X. The existence of the D_i is guaranteed if $\mathcal{O}_{X,x}$ is a UFD for all $x \in Z$ or if X is regular.

Proof. Let $\xi_i \in D_i$ be the generic point and let $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ be the local ring which is a discrete valuation ring by Lemma 31.15.4. Let $a_i \geq 0$ be the minimal valuation of an element of $\mathcal{I}_{Z,\xi_i} \subset \mathcal{O}_i$. We claim that the effective Cartier divisor $D = \sum a_i D_i$ works.

Namely, suppose that $x \in X$. Let $A = \mathcal{O}_{X,x}$. Let $D_1, \ldots, D_n$ be the pairwise distinct divisors D_i such that $x \in D_i$. For $1 \leq i \leq n$ let $f_i \in A$ be a local equation for D_i. Then f_i is a prime element of A and $\mathcal{O}_i = A_{(f_i)}$. Let $I = \mathcal{I}_{Z,x} \subset A$ be the stalk of the ideal sheaf of Z. By our choice of a_i we have $IA_{(f_i)} = f_i^{a_i} A_{(f_i)}$. We claim that $I \subset (\prod_{i=1,\ldots,n} f_i^{a_i})$.

Proof of the claim. The localization map $\varphi : A/(f_i) \to A_{(f_i)}/f_i A_{(f_i)}$ is injective as the prime ideal (f_i) is the inverse image of the maximal ideal $f_i A_{(f_i)}$. By induction on n we deduce that $\varphi_n : A/(f_i^n) \to A_{(f_i)}/f_i^n A_{(f_i)}$ is also injective. Since $\varphi_{a_i}(I) =$

0, we have $I \subset (f_i^{a_i})$. Thus, for any $x \in I$, we may write $x = f_1^{a_1} x_1$ for some $x_1 \in A$. Since $D_1, \ldots, D_n$ are pairwise distinct, f_i is a unit in $A_{(f_j)}$ for $i \neq j$. Comparing x and x_1 at $A_{(f_i)}$ for $n \geq i > 1$, we still have $x_1 \in (f_i^{a_i})$. Repeating the previous process, we inductively write $x_i = f_{i+1}^{a_{i+1}} x_{i+1}$ for any $n > i \geq 1$. In conclusion, $x \in (\prod_{i=1,\ldots n} f_i^{a_i})$ for any $x \in I$ as desired.

The claim shows that $\mathcal{I}_Z \subset \mathcal{I}_D$, i.e., that $D \subset Z$. Moreover, we also see that D and Z agree at the ξ_i, which proves that $D \to Z$ is an isomorphism away from codimension 2 on X.

To see the final statements we argue as follows. A regular local ring is a UFD (More on Algebra, Lemma 15.121.2) hence it suffices to argue in the UFD case. In that case, let D_i be the irreducible components of Z which have codimension 1 in X. By Lemma 31.15.7 each D_i is an effective Cartier divisor. □

0BXH Lemma 31.15.9. Let $Z \subset X$ be a closed subscheme of a Noetherian scheme. Assume

(1) Z has no embedded points,
(2) every irreducible component of Z has codimension 1 in X,
(3) every local ring $\mathcal{O}_{X,x}$, $x \in Z$ is a UFD or X is regular.

Then Z is an effective Cartier divisor.

Proof. Let $D = \sum a_i D_i$ be as in Lemma 31.15.8 where $D_i \subset Z$ are the irreducible components of Z. If $D \to Z$ is not an isomorphism, then $\mathcal{O}_Z \to \mathcal{O}_D$ has a nonzero kernel sitting in codimension ≥ 2. This would mean that Z has embedded points, which is forbidden by assumption (1). Hence $D \cong Z$ as desired. □

0BXI Lemma 31.15.10. Let R be a Noetherian UFD. Let $I \subset R$ be an ideal such that R/I has no embedded primes and such that every minimal prime over I has height 1. Then $I = (f)$ for some $f \in R$.

Proof. By Lemma 31.15.9 the ideal sheaf $\tilde{I}$ is invertible on $\text{Spec}(R)$. By More on Algebra, Lemma 15.117.3 it is generated by a single element. □

0BCP Lemma 31.15.11. Let X be a Noetherian scheme. Let $D \subset X$ be an effective Cartier divisor. Assume that there exist integral effective Cartier divisors $D_i \subset X$ such that $D \subset \bigcup D_i$ set theoretically. Then $D = \sum a_i D_i$ for some $a_i \geq 0$. The existence of the D_i is guaranteed if $\mathcal{O}_{X,x}$ is a UFD for all $x \in D$ or if X is regular.

Proof. Choose a_i as in Lemma 31.15.8 and set $D' = \sum a_i D_i$. Then $D' \to D$ is an inclusion of effective Cartier divisors which is an isomorphism away from codimension 2 on X. Pick $x \in X$. Set $A = \mathcal{O}_{X,x}$ and let $f, f' \in A$ be the nonzerodivisor generating the ideal of D, D' in A. Then $f = gf'$ for some $g \in A$. Moreover, for every prime $\mathfrak{p}$ of height ≤ 1 of A we see that g maps to a unit of $A_{\mathfrak{p}}$. This implies that g is a unit because the minimal primes over (g) have height 1 (Algebra, Lemma 10.60.11). □

0AYM Lemma 31.15.12. Let X be a Noetherian scheme which has an ample invertible sheaf. Then every invertible $\mathcal{O}_X$-module is isomorphic to

$$\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D')^{\otimes -1}$$

for some effective Cartier divisors D, D' in X. Moreover, given a finite subset $E \subset X$ we may choose D, D' such that $E \cap D = \emptyset$ and $E \cap D' = \emptyset$. If X is quasi-affine, then we may choose $D' = \emptyset$.

Proof. Let $x_1, \ldots, x_n$ be the associated points of X (Lemma 31.2.5).

If X is quasi-affine and $\mathcal{N}$ is any invertible $\mathcal{O}_X$-module, then we can pick a section t of $\mathcal{N}$ which does not vanish at any of the points of $E \cup \{x_1, \ldots, x_n\}$, see Properties, Lemma 28.29.7. Then t is a regular section of $\mathcal{N}$ by Lemma 31.15.1. Hence $\mathcal{N} \cong \mathcal{O}_X(D)$ where $D = Z(t)$ is the effective Cartier divisor corresponding to t, see Lemma 31.14.10. Since $E \cap D = \emptyset$ by construction we are done in this case.

Returning to the general case, let $\mathcal{L}$ be an ample invertible sheaf on X. There exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and such that $E \cup \{x_1, \ldots, x_n\} \subset X_s$ (Properties, Lemma 28.29.6).

Let $\mathcal{N}$ be an arbitrary invertible $\mathcal{O}_X$-module. By the quasi-affine case, we can find a section $t \in \mathcal{N}(X_s)$ which does not vanish at any point of $E \cup \{x_1, \ldots, x_n\}$. By Properties, Lemma 28.17.2 we see that for some $e \geq 0$ the section $s^e|_{X_s} t$ extends to a global section τ of $\mathcal{L}^{\otimes e} \otimes \mathcal{N}$. Thus both $\mathcal{L}^{\otimes e} \otimes \mathcal{N}$ and $\mathcal{L}^{\otimes e}$ are invertible sheaves which have global sections which do not vanish at any point of $E \cup \{x_1, \ldots, x_n\}$. Thus these are regular sections by Lemma 31.15.1. Hence $\mathcal{L}^{\otimes e} \otimes \mathcal{N} \cong \mathcal{O}_X(D)$ and $\mathcal{L}^{\otimes e} \cong \mathcal{O}_X(D')$ for some effective Cartier divisors D and D', see Lemma 31.14.10. By construction $E \cap D = \emptyset$ and $E \cap D' = \emptyset$ and the proof is complete. □

0B3T Lemma 31.15.13. Let X be an integral regular scheme of dimension 2. Let $i : D \to X$ be the immersion of an effective Cartier divisor. Let $\mathcal{F} \to \mathcal{F}' \to i_*\mathcal{G} \to 0$ be an exact sequence of coherent $\mathcal{O}_X$-modules. Assume

(1) $\mathcal{F}, \mathcal{F}'$ are locally free of rank r on a nonempty open of X,
(2) D is an integral scheme,
(3) $\mathcal{G}$ is a finite locally free $\mathcal{O}_D$-module of rank s.

Then $\mathcal{L} = (\wedge^r \mathcal{F})^{**}$ and $\mathcal{L}' = (\wedge^r \mathcal{F}')^{**}$ are invertible $\mathcal{O}_X$-modules and $\mathcal{L}' \cong \mathcal{L}(kD)$ for some $k \in \{0, \ldots, \min(s, r)\}$.

Proof. The first statement follows from Lemma 31.12.15 as assumption (1) implies that $\mathcal{L}$ and $\mathcal{L}'$ have rank 1. Taking $\wedge^r$ and double duals are functors, hence we obtain a canonical map $\sigma : \mathcal{L} \to \mathcal{L}'$ which is an isomorphism over the nonempty open of (1), hence nonzero. To finish the proof, it suffices to see that σ viewed as a global section of $\mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ does not vanish at any codimension point of X, except at the generic point of D and there with vanishing order at most $\min(s, r)$.

Translated into algebra, we arrive at the following problem: Let $(A, \mathfrak{m}, \kappa)$ be a discrete valuation ring with fraction field K. Let $M \to M' \to N \to 0$ be an exact sequence of finite A-modules with $\dim_K(M \otimes K) = \dim_K(M' \otimes K) = r$ and with $N \cong \kappa^{\oplus s}$. Show that the induced map $L = \wedge^r(M)^{**} \to L' = \wedge^r(M')^{**}$ vanishes to order at most $\min(s, r)$. We will use the structure theorem for modules over A, see More on Algebra, Lemma 15.124.3 or 15.124.9. Dividing out a finite A-module by a torsion submodule does not change the double dual. Thus we may replace M by M/M_{tors} and M' by $M'/\mathrm{Im}(M_{tors} \to M')$ and assume that M is torsion free. Then $M \to M'$ is injective and $M'_{tors} \to N$ is injective. Hence we may replace M' by M'/M'_{tors} and N by N/M'_{tors}. Thus we reduce to the case where M and M' are free of rank r and $N \cong \kappa^{\oplus s}$. In this case σ is the determinant of $M \to M'$ and vanishes to order s for example by Algebra, Lemma 10.121.7. □

31.16. Complements of affine opens

0BCQ In this section we discuss the result that the complement of an affine open in a variety has pure codimension 1.

0BCR Lemma 31.16.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring. The punctured spectrum $U = \text{Spec}(A) \setminus \{\mathfrak{m}\}$ of A is affine if and only if $\dim(A) \leq 1$.

Proof. If $\dim(A) = 0$, then U is empty hence affine (equal to the spectrum of the 0 ring). If $\dim(A) = 1$, then we can choose an element $f \in \mathfrak{m}$ not contained in any of the finite number of minimal primes of A (Algebra, Lemmas 10.31.6 and 10.15.2). Then $U = \text{Spec}(A_f)$ is affine.

The converse is more interesting. We will give a somewhat nonstandard proof and discuss the standard argument in a remark below. Assume $U = \text{Spec}(B)$ is affine. Since affineness and dimension are not affecting by going to the reduction we may replace A by the quotient by its ideal of nilpotent elements and assume A is reduced. Set $Q = B/A$ viewed as an A-module. The support of Q is $\{\mathfrak{m}\}$ as $A_\mathfrak{p} = B_\mathfrak{p}$ for all nonmaximal primes $\mathfrak{p}$ of A. We may assume $\dim(A) \geq 1$, hence as above we can pick $f \in \mathfrak{m}$ not contained in any of the minimal ideals of A. Since A is reduced this implies that f is a nonzerodivisor. In particular $\dim(A/fA) = \dim(A) - 1$, see Algebra, Lemma 10.60.13. Applying the snake lemma to multiplication by f on the short exact sequence $0 \to A \to B \to Q \to 0$ we obtain

$$0 \to Q[f] \to A/fA \to B/fB \to Q/fQ \to 0$$

where $Q[f] = \text{Ker}(f : Q \to Q)$. This implies that $Q[f]$ is a finite A-module. Since the support of $Q[f]$ is $\{\mathfrak{m}\}$ we see $l = \text{length}_A(Q[f]) < \infty$ (Algebra, Lemma 10.62.3). Set $l_n = \text{length}_A(Q[f^n])$. The exact sequence

$$0 \to Q[f^n] \to Q[f^{n+1}] \xrightarrow{f^n} Q[f]$$

shows inductively that $l_n < \infty$ and that $l_n \leq l_{n+1}$. Considering the exact sequence

$$0 \to Q[f] \to Q[f^{n+1}] \xrightarrow{f} Q[f^n] \to Q/fQ$$

and we see that the image of $Q[f^n]$ in Q/fQ has length $l_n - l_{n+1} + l \leq l$. Since $Q = \bigcup Q[f^n]$ we find that the length of Q/fQ is at most l, i.e., bounded. Thus Q/fQ is a finite A-module. Hence $A/fA \to B/fB$ is a finite ring map, in particular induces a closed map on spectra (Algebra, Lemmas 10.36.22 and 10.41.6). On the other hand $\text{Spec}(B/fB)$ is the punctured spectrum of $\text{Spec}(A/fA)$. This is a contradiction unless $\text{Spec}(B/fB) = \emptyset$ which means that $\dim(A/fA) = 0$ as desired. □

0BCS Remark 31.16.2. If $(A, \mathfrak{m})$ is a Noetherian local normal domain of dimension ≥ 2 and U is the punctured spectrum of A, then $\Gamma(U, \mathcal{O}_U) = A$. This algebraic version of Hartogs's theorem follows from the fact that $A = \bigcap_{\text{height}(\mathfrak{p})=1} A_\mathfrak{p}$ we've seen in Algebra, Lemma 10.157.6. Thus in this case U cannot be affine (since it would force $\mathfrak{m}$ to be a point of U). This is often used as the starting point of the proof of Lemma 31.16.1. To reduce the case of a general Noetherian local ring to this case, we first complete (to get a Nagata local ring), then replace A by $A/\mathfrak{q}$ for a suitable minimal prime, and then normalize. Each of these steps does not change the dimension and we obtain a contradiction. You can skip the completion step, but then the normalization in general is not a Noetherian domain. However, it is still a Krull domain of the same dimension (this is proved using Krull-Akizuki) and one can apply the same argument.

0BCT Remark 31.16.3. It is not clear how to characterize the non-Noetherian local rings $(A, \mathfrak{m})$ whose punctured spectrum is affine. Such a ring has a finitely generated ideal I with $\mathfrak{m} = \sqrt{I}$. Of course if we can take I generated by 1 element, then A has an affine puncture spectrum; this gives lots of non-Noetherian examples. Conversely, it follows from the argument in the proof of Lemma 31.16.1 that such a ring cannot possess a nonzerodivisor $f \in \mathfrak{m}$ with $H^0_I(A/fA) = 0$ (so A cannot have a regular sequence of length 2). Moreover, the same holds for any ring A' which is the target of a local homomorphism of local rings $A \to A'$ such that $\mathfrak{m}_{A'} = \sqrt{\mathfrak{m}A'}$.

0BCU Lemma 31.16.4. Let X be a locally Noetherian scheme. Let $U \subset X$ be an open subscheme such that the inclusion morphism $U \to X$ is affine. For every generic point ξ of an irreducible component of $X \setminus U$ the local ring $\mathcal{O}_{X,\xi}$ has dimension ≤ 1. If U is dense or if ξ is in the closure of U, then $\dim(\mathcal{O}_{X,\xi}) = 1$. [GD67, EGA IV, Corollaire 21.12.7]

Proof. Since ξ is a generic point of $X \setminus U$, we see that

$$U_\xi = U \times_X \mathrm{Spec}(\mathcal{O}_{X,\xi}) \subset \mathrm{Spec}(\mathcal{O}_{X,\xi})$$

is the punctured spectrum of $\mathcal{O}_{X,\xi}$ (hint: use Schemes, Lemma 26.13.2). As $U \to X$ is affine, we see that $U_\xi \to \mathrm{Spec}(\mathcal{O}_{X,\xi})$ is affine (Morphisms, Lemma 29.11.8) and we conclude that U_ξ is affine. Hence $\dim(\mathcal{O}_{X,\xi}) \leq 1$ by Lemma 31.16.1. If $\xi \in \overline{U}$, then there is a specialization $\eta \to \xi$ where $\eta \in U$ (just take η a generic point of an irreducible component of $\overline{U}$ which contains ξ; since $\overline{U}$ is locally Noetherian, hence locally has finitely many irreducible components, we see that $\eta \in U$). Then $\eta \in \mathrm{Spec}(\mathcal{O}_{X,\xi})$ and we see that the dimension cannot be 0. $\square$

0BCV Lemma 31.16.5. Let X be a separated locally Noetherian scheme. Let $U \subset X$ be an affine open. For every generic point ξ of an irreducible component of $X \setminus U$ the local ring $\mathcal{O}_{X,\xi}$ has dimension ≤ 1. If U is dense or if ξ is in the closure of U, then $\dim(\mathcal{O}_{X,\xi}) = 1$.

Proof. This follows from Lemma 31.16.4 because the morphism $U \to X$ is affine by Morphisms, Lemma 29.11.11. $\square$

The following lemma can sometimes be used to produce effective Cartier divisors.

0BCW Lemma 31.16.6. Let X be a Noetherian separated scheme. Let $U \subset X$ be a dense affine open. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.

Proof. Since X is Noetherian, the complement $X \setminus U$ has finitely many irreducible components $D_1, \ldots, D_r$ (Properties, Lemma 28.5.7 applied to the reduced induced subscheme structure on $X \setminus U$). Each $D_i \subset X$ has codimension 1 by Lemma 31.16.5 (and Properties, Lemma 28.10.3). Thus D_i is an effective Cartier divisor by Lemma 31.15.7. Hence we can take $D = D_1 + \ldots + D_r$. $\square$

0EGJ Lemma 31.16.7. Let X be a Noetherian scheme with affine diagonal. Let $U \subset X$ be a dense affine open. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.

Proof. Since X is Noetherian, the complement $X \setminus U$ has finitely many irreducible components $D_1, \ldots, D_r$ (Properties, Lemma 28.5.7 applied to the reduced induced subscheme structure on $X \setminus U$). We view D_i as a reduced closed subscheme of X. Let $X = \bigcup_{j \in J} X_j$ be an affine open covering of X. For all j in J, set $U_j = U \cap X_j$. Since X has affine diagonal, the scheme

$$U_j = X \times_{(X \times X)} (U \times X_j)$$

is affine. Therefore, as X_j is separated, it follows from Lemma 31.16.6 and its proof that for all $j \in J$ and $1 \leq i \leq r$ the intersection $D_i \cap X_j$ is either empty or an effective Cartier divisor in X_j. Thus $D_i \subset X$ is an effective Cartier divisor (as this is a local property). Hence we can take $D = D_1 + \ldots + D_r$. $\square$

0GML Lemma 31.16.8. Let X be a quasi-compact, regular scheme with affine diagonal. Then X has an ample family of invertible modules (Morphisms, Definition 29.12.1.

Proof. Observe that X is a finite disjoint union of integral schemes (Properties, Lemmas 28.9.4 and 28.7.6). Thus we may assume that X is integral as well as Noetherian, regular, and having affine diagonal. Let $x \in X$. Choose an affine open neighbourhood $U \subset X$ of x. Since X is integral, U is dense in X. By More on Algebra, Lemma 15.121.2 the local rings of X are UFDs. Hence by Lemma 31.16.7 we can find an effective Cartier divisor $D \subset X$ whose complement is U. Then the canonical section $s = 1_D$ of $\mathcal{L} = \mathcal{O}_X(D)$, see Definition 31.14.1, vanishes exactly along D hence $U = X_s$. Thus both conditions in Morphisms, Definition 29.12.1 hold and we are done. $\square$

31.17. Norms

0BCX Let $\pi : X \to Y$ be a finite morphism of schemes and let $d \geq 1$ be an integer. Let us say there exists a norm of degree d for π[2] if there exists a multiplicative map

$$\mathrm{Norm}_\pi : \pi_*\mathcal{O}_X \to \mathcal{O}_Y$$

of sheaves such that

(1) the composition $\mathcal{O}_Y \xrightarrow{\pi^\sharp} \pi_*\mathcal{O}_X \xrightarrow{\mathrm{Norm}_\pi} \mathcal{O}_Y$ equals $g \mapsto g^d$, and
(2) for $V \subset Y$ open if $f \in \mathcal{O}_X(\pi^{-1}V)$ is zero at $x \in \pi^{-1}(V)$, then $\mathrm{Norm}_\pi(f)$ is zero at $\pi(x)$.

We observe that condition (1) forces π to be surjective. Since Norm_π is multiplicative it sends units to units hence, given $y \in Y$, if f is a regular function on X defined at but nonvanishing at any $x \in X$ with $\pi(x) = y$, then $\mathrm{Norm}_\pi(f)$ is defined and does not vanish at y. This holds without requiring (2); in fact, the constructions in this section will only require condition (1) and only certain vanishing properties (which are used in particular in the proof of Lemma 31.17.4) will require property (2).

0BUT Lemma 31.17.1. Let $\pi : X \to Y$ be a finite morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$. There exists an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{\pi^{-1}(V)}$ is trivial.

Proof. Clearly we may assume Y and hence X affine. Since π is finite the fibre $\pi^{-1}(\{y\})$ over y is finite. Since X is affine, we can pick $s \in \Gamma(X, \mathcal{L})$ not vanishing in any point of $\pi^{-1}(\{y\})$. This follows from Properties, Lemma 28.29.7 but we also

[2]This is nonstandard notation.

give a direct argument. Namely, we can pick a finite set $E \subset X$ of closed points such that every $x \in \pi^{-1}(\{y\})$ specializes to some point of E. For $x \in E$ denote $i_x : x \to X$ the closed immersion. Then $\mathcal{L} \to \bigoplus_{x \in E} i_{x,*}i_x^*\mathcal{L}$ is a surjective map of quasi-coherent $\mathcal{O}_X$-modules, and hence the map

$$\Gamma(X, \mathcal{L}) \to \bigoplus\nolimits_{x \in E} \mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x$$

is surjective (as taking global sections is an exact functor on the category of quasi-coherent $\mathcal{O}_X$-modules, see Schemes, Lemma 26.7.5). Thus we can find an $s \in \Gamma(X, \mathcal{L})$ not vanishing at any point specializing to a point of E. Then $X_s \subset X$ is an open neighbourhood of $\pi^{-1}(\{y\})$. Since π is finite, hence closed, we conclude that there is an open neighbourhood $V \subset Y$ of y whose inverse image is contained in X_s as desired. □

0BCY Lemma 31.17.2. Let $\pi : X \to Y$ be a finite morphism of schemes. If there exists a norm of degree d for π, then there exists a homomorphism of abelian groups

$$\mathrm{Norm}_\pi : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

such that $\mathrm{Norm}_\pi(\pi^*\mathcal{N}) \cong \mathcal{N}^{\otimes d}$ for all invertible $\mathcal{O}_Y$-modules $\mathcal{N}$.

Proof. We will use the correspondence between isomorphism classes of invertible $\mathcal{O}_X$-modules and elements of $H^1(X, \mathcal{O}_X^*)$ given in Cohomology, Lemma 20.6.1 without further mention. We explain how to take the norm of an invertible $\mathcal{O}_X$-module $\mathcal{L}$. Namely, by Lemma 31.17.1 there exists an open covering $Y = \bigcup V_j$ such that $\mathcal{L}|_{\pi^{-1}V_j}$ is trivial. Choose a generating section $s_j \in \mathcal{L}(\pi^{-1}V_j)$ for each j. On the overlaps $\pi^{-1}V_j \cap \pi^{-1}V_{j'}$ we can write

$$s_j = u_{jj'}s_{j'}$$

for a unique $u_{jj'} \in \mathcal{O}_X^*(\pi^{-1}V_j \cap \pi^{-1}V_{j'})$. Thus we can consider the elements

$$v_{jj'} = \mathrm{Norm}_\pi(u_{jj'}) \in \mathcal{O}_Y^*(V_j \cap V_{j'})$$

These elements satisfy the cocycle condition (because the $u_{jj'}$ do and Norm_π is multiplicative) and therefore define an invertible $\mathcal{O}_Y$-module. We omit the verification that: this is well defined, additive on Picard groups, and satisfies the property $\mathrm{Norm}_\pi(\pi^*\mathcal{N}) \cong \mathcal{N}^{\otimes d}$ for all invertible $\mathcal{O}_Y$-modules $\mathcal{N}$. □

0BCZ Lemma 31.17.3. Let $\pi : X \to Y$ be a finite morphism of schemes. Assume there exists a norm of degree d for π. For any $\mathcal{O}_X$-linear map $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible $\mathcal{O}_X$-modules there is an $\mathcal{O}_Y$-linear map

$$\mathrm{Norm}_\pi(\varphi) : \mathrm{Norm}_\pi(\mathcal{L}) \longrightarrow \mathrm{Norm}_\pi(\mathcal{L}')$$

with $\mathrm{Norm}_\pi(\mathcal{L})$, $\mathrm{Norm}_\pi(\mathcal{L}')$ as in Lemma 31.17.2. Moreover, for $y \in Y$ the following are equivalent

(1) φ is zero at a point of $x \in X$ with $\pi(x) = y$, and
(2) $\mathrm{Norm}_\pi(\varphi)$ is zero at y.

Proof. We choose an open covering $Y = \bigcup V_j$ such that $\mathcal{L}$ and $\mathcal{L}'$ are trivial over the opens $\pi^{-1}V_j$. This is possible by Lemma 31.17.1. Choose generating sections s_j and s'_j of $\mathcal{L}$ and $\mathcal{L}'$ over the opens $\pi^{-1}V_j$. Then $\varphi(s_j) = f_js'_j$ for some $f_j \in \mathcal{O}_X(\pi^{-1}V_j)$. Define $\mathrm{Norm}_\pi(\varphi)$ to be multiplication by $\mathrm{Norm}_\pi(f_j)$ on V_j. An simple calculation involving the cocycles used to construct $\mathrm{Norm}_\pi(\mathcal{L})$, $\mathrm{Norm}_\pi(\mathcal{L}')$ in the proof of Lemma 31.17.2 shows that this defines a map as stated in the lemma. The final statement follows from condition (2) in the definition of a norm map of degree d. Some details omitted. $\square$

0BD0 Lemma 31.17.4. Let $\pi : X \to Y$ be a finite morphism of schemes. Assume X has an ample invertible sheaf and there exists a norm of degree d for π. Then Y has an ample invertible sheaf.

Proof. Let $\mathcal{L}$ be the ample invertible sheaf on X given to us by assumption. We will prove that $\mathcal{N} = \text{Norm}_\pi(\mathcal{L})$ is ample on Y.

Since X is quasi-compact (Properties, Definition 28.26.1) and $X \to Y$ surjective (by the existence of Norm_π) we see that Y is quasi-compact. Let $y \in Y$ be a point. To finish the proof we will show that there exists a section t of some positive tensor power of $\mathcal{N}$ which does not vanish at y such that Y_t is affine. To do this, choose an affine open neighbourhood $V \subset Y$ of y. Choose $n \gg 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that

$$\pi^{-1}(\{y\}) \subset X_s \subset \pi^{-1}V$$

by Properties, Lemma 28.29.6. Then $t = \text{Norm}_\pi(s)$ is a section of $\mathcal{N}^{\otimes n}$ which does not vanish at x and with $Y_t \subset V$, see Lemma 31.17.3. Then Y_t is affine by Properties, Lemma 28.26.4. $\square$

0BD1 Lemma 31.17.5. Let $\pi : X \to Y$ be a finite morphism of schemes. Assume X is quasi-affine and there exists a norm of degree d for π. Then Y is quasi-affine.

Proof. By Properties, Lemma 28.27.1 we see that $\mathcal{O}_X$ is an ample invertible sheaf on X. The proof of Lemma 31.17.4 shows that $\text{Norm}_\pi(\mathcal{O}_X) = \mathcal{O}_Y$ is an ample invertible $\mathcal{O}_Y$-module. Hence Properties, Lemma 28.27.1 shows that Y is quasi-affine. $\square$

0BD2 Lemma 31.17.6. Let $\pi : X \to Y$ be a finite locally free morphism of degree $d \geq 1$. Then there exists a canonical norm of degree d whose formation commutes with arbitrary base change.

Proof. Let $V \subset Y$ be an affine open such that $(\pi_*\mathcal{O}_X)|_V$ is finite free of rank d. Choosing a basis we obtain an isomorphism

$$\mathcal{O}_V^{\oplus d} \cong (\pi_*\mathcal{O}_X)|_V$$

For every $f \in \pi_*\mathcal{O}_X(V) = \mathcal{O}_X(\pi^{-1}(V))$ multiplication by f defines a $\mathcal{O}_V$-linear endomorphism m_f of the displayed free vector bundle. Thus we get a $d \times d$ matrix $M_f \in \text{Mat}(d \times d, \mathcal{O}_Y(V))$ and we can set

$$\text{Norm}_\pi(f) = \det(M_f)$$

Since the determinant of a matrix is independent of the choice of the basis chosen we see that this is well defined which also means that this construction will glue to a global map as desired. Compatibility with base change is straightforward from the construction.

Property (1) follows from the fact that the determinant of a $d \times d$ diagonal matrix with entries $g, g, \ldots, g$ is g^d. To see property (2) we may base change and assume that Y is the spectrum of a field k. Then $X = \text{Spec}(A)$ with A a k-algebra with $\dim_k(A) = d$. If there exists an $x \in X$ such that $f \in A$ vanishes at x, then there exists a map $A \to \kappa$ into a field such that f maps to zero in κ. Then $f : A \to A$ cannot be surjective, hence $\det(f : A \to A) = 0$ as desired. $\square$

0BD3 **Lemma 31.17.7.** Let $\pi : X \to Y$ be a finite surjective morphism with X and Y integral and Y normal. Then there exists a norm of degree $[R(X) : R(Y)]$ for π.

Proof. Let $\mathrm{Spec}(B) \subset Y$ be an affine open subset and let $\mathrm{Spec}(A) \subset X$ be its inverse image. Then A and B are domains. Let K be the fraction field of A and L the fraction field of B. Picture:

$$\begin{array}{ccc} L & \longrightarrow & K \\ \uparrow & & \uparrow \\ B & \longrightarrow & A \end{array}$$

Since K/L is a finite extension, there is a norm map $\mathrm{Norm}_{K/L} : K^* \to L^*$ of degree $d = [K : L]$; this is given by mapping $f \in K$ to $\det_L(f : K \to K)$ as in the proof of Lemma 31.17.6. Observe that the characteristic polynomial of $f : K \to K$ is a power of the minimal polynomial of f over L; in particular $\mathrm{Norm}_{K/L}(f)$ is a power of the constant coefficient of the minimal polynomial of f over L. Hence by Algebra, Lemma 10.38.6 $\mathrm{Norm}_{K/L}$ maps A into B. This determines a compatible system of maps on sections over affines and hence a global norm map Norm_π of degree d.

Property (1) is immediate from the construction. To see property (2) let $f \in A$ be contained in the prime ideal $\mathfrak{p} \subset A$. Let $f^m + b_1 f^{m-1} + \ldots + b_m$ be the minimal polynomial of f over L. By Algebra, Lemma 10.38.6 we have $b_i \in B$. Hence $b_0 \in B \cap \mathfrak{p}$. Since $\mathrm{Norm}_{K/L}(f) = b_0^{d/m}$ (see above) we conclude that the norm vanishes in the image point of $\mathfrak{p}$. □

0BDZ **Lemma 31.17.8.** Let X be a Noetherian scheme. Let p be a prime number such that $p\mathcal{O}_X = 0$. Then for some $e > 0$ there exists a norm of degree p^e for $X_{red} \to X$ where X_{red} is the reduction of X.

Proof. Let A be a Noetherian ring with $pA = 0$. Let $I \subset A$ be the ideal of nilpotent elements. Then $I^n = 0$ for some n (Algebra, Lemma 10.32.5). Pick e such that $p^e \geq n$. Then

$$A/I \longrightarrow A, \quad f \bmod I \longmapsto f^{p^e}$$

is well defined. This produces a norm of degree p^e for $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. Now if X is obtained by glueing some affine schemes $\mathrm{Spec}(A_i)$ then for some $e \gg 0$ these maps glue to a norm map for $X_{red} \to X$. Details omitted. □

0BD4 **Proposition 31.17.9.** Let $\pi : X \to Y$ be a finite surjective morphism of schemes. Assume that X has an ample invertible $\mathcal{O}_X$-module. If

(1) π is finite locally free, or
(2) Y is an integral normal scheme, or
(3) Y is Noetherian, $p\mathcal{O}_Y = 0$, and $X = Y_{red}$,

then Y has an ample invertible $\mathcal{O}_Y$-module.

Proof. Case (1) follows from a combination of Lemmas 31.17.6 and 31.17.4. Case (3) follows from a combination of Lemmas 31.17.8 and 31.17.4. In case (2) we first replace X by an irreducible component of X which dominates Y (viewed as a reduced closed subscheme of X). Then we can apply Lemma 31.17.7. □

0BD5 **Lemma 31.17.10.** Let $\pi : X \to Y$ be a finite surjective morphism of schemes. Assume that X is quasi-affine. If either

(1) π is finite locally free, or
(2) Y is an integral normal scheme

then Y is quasi-affine.

Proof. Case (1) follows from a combination of Lemmas 31.17.6 and 31.17.5. In case (2) we first replace X by an irreducible component of X which dominates Y (viewed as a reduced closed subscheme of X). Then we can apply Lemma 31.17.7. □

31.18. Relative effective Cartier divisors

056P The following lemma shows that an effective Cartier divisor which is flat over the base is really a "family of effective Cartier divisors" over the base. For example the restriction to any fibre is an effective Cartier divisor.

056Q Lemma 31.18.1. Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a closed subscheme. Assume

(1) D is an effective Cartier divisor, and
(2) $D \to S$ is a flat morphism.

Then for every morphism of schemes $g : S' \to S$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = S' \times_S X$ where $g' : X' \to X$ is the projection.

Proof. Using Lemma 31.13.2 we translate this as follows into algebra. Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor and that B/hB is flat over A. Then

$$0 \to B \xrightarrow{h} B \to B/hB \to 0$$

is a short exact sequence of A-modules with B/hB flat over A. By Algebra, Lemma 10.39.12 this sequence remains exact on tensoring over A with any module, in particular with any A-algebra A'. □

This lemma is the motivation for the following definition.

062T Definition 31.18.2. Let $f : X \to S$ be a morphism of schemes. A relative effective Cartier divisor on X/S is an effective Cartier divisor $D \subset X$ such that $D \to S$ is a flat morphism of schemes.

We warn the reader that this may be nonstandard notation. In particular, in [DG67, IV, Section 21.15] the notion of a relative divisor is discussed only when $X \to S$ is flat and locally of finite presentation. Our definition is a bit more general. However, it turns out that if $x \in D$ then $X \to S$ is flat at x in many cases (but not always).

0B8U Lemma 31.18.3. Let $f : X \to S$ be a morphism of schemes. If $D_1, D_2 \subset X$ are relative effective Cartier divisor on X/S then so is $D_1 + D_2$ (Definition 31.13.6).

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h_1, h_2 \in B$. Assume the h_i are nonzerodivisors and that B/h_iB is flat over A. Then h_1h_2 is a nonzerodivisor and B/h_1h_2B is flat over A. The reason is that we have a short exact sequence

$$0 \to B/h_1B \to B/h_1h_2B \to B/h_2B \to 0$$

where the first arrow is given by multiplication by h_2. Since the outer two are flat modules over A, so is the middle one, see Algebra, Lemma 10.39.13. □

0B8V Lemma 31.18.4. Let $f : X \to S$ be a morphism of schemes. If $D_1, D_2 \subset X$ are relative effective Cartier divisor on X/S and $D_1 \subset D_2$ as closed subschemes, then the effective Cartier divisor D such that $D_2 = D_1 + D$ (Lemma 31.13.8) is a relative effective Cartier divisor on X/S.

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h_1, h_2 \in B$. Assume the h_i are nonzerodivisors, that B/h_iB is flat over A, and that $(h_2) \subset (h_1)$. Then we can write $h_2 = hh_1$ where $h \in B$ is a nonzerodivisor. We get a short exact sequence

$$0 \to B/hB \to B/h_2B \to B/h_1B \to 0$$

where the first arrow is given by multiplication by h_1. Since the right two are flat modules over A, so is the middle one, see Algebra, Lemma 10.39.13. □

062U Lemma 31.18.5. Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If $x \in D$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.

Proof. Set $A = \mathcal{O}_{S,f(x)}$ and $B = \mathcal{O}_{X,x}$. Let $h \in B$ be an element which generates the ideal of D. Then h is a nonzerodivisor in B such that B/hB is a flat local A-algebra. Let $I \subset A$ be a finitely generated ideal. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "h"'] & B \arrow[r] & B/hB \arrow[r] & 0 \\ 0 \arrow[r] & B \otimes_A I \arrow[r, "h"] \arrow[u] & B \otimes_A I \arrow[r] \arrow[u] & B/hB \otimes_A I \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The lower sequence is short exact as B/hB is flat over A, see Algebra, Lemma 10.39.12. The right vertical arrow is injective as B/hB is flat over A, see Algebra, Lemma 10.39.5. Hence multiplication by h is surjective on the kernel K of the middle vertical arrow. By Nakayama's lemma, see Algebra, Lemma 10.20.1 we conclude that $K = 0$. Hence B is flat over A, see Algebra, Lemma 10.39.5. □

The following lemma relies on the algebraic version of openness of the flat locus. The scheme theoretic version can be found in More on Morphisms, Section 37.15.

062V Lemma 31.18.6. Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor. If f is locally of finite presentation, then there exists an open subscheme $U \subset X$ such that $D \subset U$ and such that $f|_U : U \to S$ is flat.

Proof. Pick $x \in D$. It suffices to find an open neighbourhood $U \subset X$ of x such that $f|_U$ is flat. Hence the lemma reduces to the case that $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$ are affine and that D is given by a nonzerodivisor $h \in B$. By assumption B is a finitely presented A-algebra and B/hB is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume h is the image of $h' \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $h', g_1, \ldots, g_m$ are in A_0. Then we can set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and h_0 the image of h' in B_0. Then $B = B_0 \otimes_{A_0} A$ and $B/hB = B_0/h_0B_0 \otimes_{A_0} A$. By Algebra, Lemma 10.168.1 we may, after enlarging A_0, assume that B_0/h_0B_0 is flat over A_0. Let $K_0 = \text{Ker}(h_0 : B_0 \to B_0)$. As B_0 is of finite type over $\mathbf{Z}$ we see that K_0 is a finitely generated ideal. Let $A_1 \subset A$ be a finite type $\mathbf{Z}$-subalgebra containing A_0 and denote B_1, h_1, K_1 the corresponding objects over A_1. By More on Algebra, Lemma 15.31.3 the map $K_0 \otimes_{A_0} A_1 \to K_1$ is surjective. On the other hand, the kernel of $h : B \to B$ is zero by assumption. Hence every element of K_0 maps to zero in K_1 for sufficiently large subrings $A_1 \subset A$. Since K_0 is finitely generated, we conclude that $K_1 = 0$ for a suitable choice of A_1.

Set $f_1 : X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$. Set $D_1 = \text{Spec}(B_1/h_1B_1)$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, it now suffices to prove the lemma for $X_1 \to S_1$ and the relative effective Cartier divisor D_1, see Morphisms, Lemma 29.25.7. Hence we have reduced to the case where A is a Noetherian ring. In this case we know that the ring map $A \to B$ is flat at every prime $\mathfrak{q}$ of $V(h)$ by Lemma 31.18.5. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 10.129.4 we win. $\square$

There is also the following lemma (whose idea is apparently due to Michael Artin, see [Nob77]) which needs no finiteness assumptions at all.

062W Lemma 31.18.7. Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If f is flat at all points of $X \setminus D$, then f is flat.

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor, that B/hB is flat over A, and that the localization B_h is flat over A. Then B is flat over A. The reason is that we have a short exact sequence

$$0 \to B \to B_h \to \text{colim}_n (1/h^n)B/B \to 0$$

and that the second and third terms are flat over A, which implies that B is flat over A (see Algebra, Lemma 10.39.13). Note that a filtered colimit of flat modules is flat (see Algebra, Lemma 10.39.3) and that by induction on n each $(1/h^n)B/B \cong B/h^nB$ is flat over A since it fits into the short exact sequence

$$0 \to B/h^{n-1}B \xrightarrow{h} B/h^nB \to B/hB \to 0$$

Some details omitted. $\square$

062X Example 31.18.8. Here is an example of a relative effective Cartier divisor D where the ambient scheme is not flat in a neighbourhood of D. Namely, let $A = k[t]$ and

$$B = k[t, x, y, x^{-1}y, x^{-2}y, \ldots]/(ty, tx^{-1}y, tx^{-2}y, \ldots)$$

Then B is not flat over A but $B/xB \cong A$ is flat over A. Moreover x is a nonzerodivisor and hence defines a relative effective Cartier divisor in $\text{Spec}(B)$ over $\text{Spec}(A)$.

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative effective Cartier divisor in terms of its fibres. See also More on Morphisms, Lemma 37.23.1 for a slightly different take on this lemma.

062Y Lemma 31.18.9. Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $Z \subset X$ be a closed subscheme. Let $x \in Z$ with image $s \in S$.

(1) If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, then there exists an open $U \subset X$ and a relative effective Cartier divisor $D \subset U$ such that $Z \cap U \subset D$ and $Z_s \cap U = D_s$.

(2) If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, the morphism $Z \to X$ is of finite presentation, and $Z \to S$ is flat at x, then we can choose U and D such that $Z \cap U = D$.
(3) If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x and Z is a locally principal closed subscheme of X in a neighbourhood of x, then we can choose U and D such that $Z \cap U = D$.

In particular, if $Z \to S$ is locally of finite presentation and flat and all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor. Similarly, if Z is a locally principal closed subscheme of X such that all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor.

Proof. Choose affine open neighbourhoods $\text{Spec}(A)$ of s and $\text{Spec}(B)$ of x such that $\varphi(\text{Spec}(B)) \subset \text{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to Z. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\text{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a single element which is a nonzerodivisor in $B \otimes_A \kappa(\mathfrak{p})$. Say $f \in I$ maps to this generator. We claim that after inverting an element $g \in B$, $g \notin \mathfrak{q}$ the closed subscheme $D = V(f) \subset \text{Spec}(B_g)$ is a relative effective Cartier divisor.

By Algebra, Lemma 10.168.1 we can find a flat finite type ring map $A_0 \to B_0$ of Noetherian rings, an element $f_0 \in B_0$, a ring map $A_0 \to A$ and an isomorphism $A \otimes_{A_0} B_0 \cong B$. If $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$ then we see that

$$B \otimes_A \kappa(\mathfrak{p}) = (B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)) \otimes_{\kappa(\mathfrak{p}_0))} \kappa(\mathfrak{p})$$

hence f_0 is a nonzerodivisor in $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$. By Algebra, Lemma 10.99.2 we see that f_0 is a nonzerodivisor in $(B_0)_{\mathfrak{q}_0}$ where $\mathfrak{q}_0 = B_0 \cap \mathfrak{q}$ and that $(B_0/f_0B_0)_{\mathfrak{q}_0}$ is flat over A_0. Hence by Algebra, Lemma 10.68.6 and Algebra, Theorem 10.129.4 there exists a $g_0 \in B_0$, $g_0 \notin \mathfrak{q}_0$ such that f_0 is a nonzerodivisor in $(B_0)_{g_0}$ and such that $(B_0/f_0B_0)_{g_0}$ is flat over A_0. Hence we see that $D_0 = V(f_0) \subset \text{Spec}((B_0)_{g_0})$ is a relative effective Cartier divisor. Since we know that this property is preserved under base change, see Lemma 31.18.1, we obtain the claim mentioned above with g equal to the image of g_0 in B.

At this point we have proved (1). To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 10.128.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 10.126.6. To see (3), after possibly shrinking U we may assume that the ideal of D is generated by a single nonzerodivisor f and the ideal of Z is generated by an element g. Then $f = gh$. But $g|_{U_s}$ and $f|_{U_s}$ cut out the same effective Cartier divisor in a neighbourhood of x. Hence $h|_{X_s}$ is a unit in $\mathcal{O}_{X_s,x}$, hence h is a unit in $\mathcal{O}_{X,x}$ hence h is a unit in an open neighbourhood of x. I.e., $Z \cap U = D$ after shrinking U.

The final statements of the lemma follow immediately from parts (2) and (3), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 29.21.3 and 29.21.11. $\square$

31.19. The normal cone of an immersion

062Z Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms, Lemma 29.4.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the conormal algebra of Z in X and is often simply denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms, Section 29.4.

Let $f : Z \to X$ be an immersion. We define the conormal algebra of f as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

0630 **Definition 31.19.1.** Let $f : Z \to X$ be an immersion. The conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X or the conormal algebra of f is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

0631 (31.19.1.1)
$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

0632 (31.19.1.2)
$$\text{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

0633 **Lemma 31.19.2.** Let $i : Z \to X$ be an immersion. The conormal algebra of i has the following properties:

(1) Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then
$$\mathcal{C}_{Z/X,*} = i^*\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) = i^{-1}\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

(2) For any affine open $\text{Spec}(R) = U \subset X$ such that $Z \cap U = \text{Spec}(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X,*}) = \bigoplus_{n \geq 0} I^n/I^{n+1}$.

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I^n/I^{n+1} = I^n \otimes_R R/I$. Details omitted. $\square$

0634 **Lemma 31.19.3.** Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

characterized by the following property: For every pair of affine opens $(\mathrm{Spec}(R) = U \subset X, \mathrm{Spec}(R') = U' \subset X')$ with $f(U) \subset U'$ such that $Z \cap U = \mathrm{Spec}(R/I)$ and $Z' \cap U' = \mathrm{Spec}(R'/I')$ the induced map

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X',*}) = \bigoplus (I')^n/(I')^{n+1} \longrightarrow \bigoplus\nolimits_{n \geq 0} I^n/I^{n+1} = \Gamma(Z \cap U, \mathcal{C}_{Z/X,*})$$

is the one induced by the ring map $f^\sharp : R' \to R$ which has the property $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \big(g^{-1}(\partial Z') \cup \partial Z\big)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 26.4.6 and 26.4.7. Hence we get an induced map of quasi-coherent sheaves $g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back by i gives $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to i^*(\mathcal{I}^n/\mathcal{I}^{n+1})$. Note that $i^*(\mathcal{I}^n/\mathcal{I}^{n+1}) = \mathcal{C}_{Z/X,n}$. On the other hand, $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*(i')^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*\mathcal{C}_{Z'/X',n}$. This gives the desired map.

Checking that the map is locally described as the given map $(I')^n/(I')^{n+1} \to I^n/I^{n+1}$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). □

0635 Lemma 31.19.4. Let

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, "i'"] & X' \end{tikzcd}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 31.19.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $(I')^n/(I')^{n+1} \otimes_{R'} R \to I^n/I^{n+1}$ is surjective. If $R' \to R$ is flat, then $I^n = (I')^n \otimes_{R'} R$ and we see the map is an isomorphism. □

0636 Definition 31.19.5. Let $i : Z \to X$ be an immersion of schemes. The normal cone C_ZX of Z in X is

$$C_ZX = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Constructions, Definitions 27.7.1 and 27.7.2. The normal bundle of Z in X is the vector bundle

$$N_ZX = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

see Constructions, Definitions 27.6.1 and 27.6.2.

Thus $C_ZX \to Z$ is a cone over Z and $N_ZX \to Z$ is a vector bundle over Z (recall that in our terminology this does not imply that the conormal sheaf is a finite locally free sheaf). Moreover, the canonical surjection (31.19.1.2) of graded algebras defines a canonical closed immersion

0637 (31.19.5.1) $$C_ZX \longrightarrow N_ZX$$

of cones over Z.

31.20. Regular ideal sheaves

067M In this section we generalize the notion of an effective Cartier divisor to higher codimension. Recall that a sequence of elements $f_1, \ldots, f_r$ of a ring R is a regular sequence if for each $i = 1, \ldots, r$ the element f_i is a nonzerodivisor on $R/(f_1, \ldots, f_{i-1})$ and $R/(f_1, \ldots, f_r) \neq 0$, see Algebra, Definition 10.68.1. There are three closely related weaker conditions that we can impose. The first is to assume that $f_1, \ldots, f_r$ is a Koszul-regular sequence, i.e., that $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for $i > 0$, see More on Algebra, Definition 15.30.1. The sequence is called an H_1-regular sequence if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$. Another condition we can impose is that with $J = (f_1, \ldots, f_r)$, the map

$$R/J[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

which maps T_i to $f_i \bmod J^2$ is an isomorphism. In this case we say that $f_1, \ldots, f_r$ is a quasi-regular sequence, see Algebra, Definition 10.69.1. Given an R-module M there is also a notion of M-regular and M-quasi-regular sequence.

We can generalize this to the case of ringed spaces as follows. Let X be a ringed space and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We say that $f_1, \ldots, f_r$ is a regular sequence if for each $i = 1, \ldots, r$ the map

0639 (31.20.0.1) $$f_i : \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_X/(f_1, \ldots, f_{i-1})$$

is an injective map of sheaves. We say that $f_1, \ldots, f_r$ is a Koszul-regular sequence if the Koszul complex

063A (31.20.0.2) $$K_\bullet(\mathcal{O}_X, f_\bullet),$$

see Modules, Definition 17.24.2, is acyclic in degrees > 0. We say that $f_1, \ldots, f_r$ is a H_1-regular sequence if the Koszul complex $K_\bullet(\mathcal{O}_X, f_\bullet)$ is exact in degree 1. Finally, we say that $f_1, \ldots, f_r$ is a quasi-regular sequence if the map

063B (31.20.0.3) $$\mathcal{O}_X/\mathcal{J}[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{J}^d/\mathcal{J}^{d+1}$$

is an isomorphism of sheaves where $\mathcal{J} \subset \mathcal{O}_X$ is the sheaf of ideals generated by $f_1, \ldots, f_r$. (There is also a notion of $\mathcal{F}$-regular and $\mathcal{F}$-quasi-regular sequence for a given $\mathcal{O}_X$-module $\mathcal{F}$ which we will introduce here if we ever need it.)

063C **Lemma 31.20.1.** Let X be a ringed space. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We have the following implications $f_1, \ldots, f_r$ is a regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a Koszul-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is an H_1-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a quasi-regular sequence.

Proof. Since we may check exactness at stalks, a sequence $f_1, \ldots, f_r$ is a regular sequence if and only if the maps

$$f_i : \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1})$$

are injective for all $x \in X$. In other words, the image of the sequence $f_1, \ldots, f_r$ in the ring $\mathcal{O}_{X,x}$ is a regular sequence for all $x \in X$. The other types of regularity can be checked stalkwise as well (details omitted). Hence the implications follow from More on Algebra, Lemmas 15.30.2, 15.30.3, and 15.30.6. $\square$

063D **Definition 31.20.2.** Let X be a ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals.

The concept of a Koszul-regular ideal sheaf was introduced in [BGI71, Expose VII, Definition 1.4] where it was called a regular ideal sheaf.

(1) We say $\mathcal{J}$ is regular if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(2) We say $\mathcal{J}$ is Koszul-regular if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a Koszul-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(3) We say $\mathcal{J}$ is H_1-regular if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a H_1-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
(4) We say $\mathcal{J}$ is quasi-regular if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.

Many properties of this notion immediately follow from the corresponding notions for regular and quasi-regular sequences in rings.

063E Lemma 31.20.3. Let X be a ringed space. Let $\mathcal{J}$ be a sheaf of ideals. We have the following implications: $\mathcal{J}$ is regular $\Rightarrow$ $\mathcal{J}$ is Koszul-regular $\Rightarrow$ $\mathcal{J}$ is H_1-regular $\Rightarrow$ $\mathcal{J}$ is quasi-regular.

Proof. The lemma immediately reduces to Lemma 31.20.1. $\square$

063H Lemma 31.20.4. Let X be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Then $\mathcal{J}$ is quasi-regular if and only if the following conditions are satisfied:

(1) $\mathcal{J}$ is an $\mathcal{O}_X$-module of finite type,
(2) $\mathcal{J}/\mathcal{J}^2$ is a finite locally free $\mathcal{O}_X/\mathcal{J}$-module, and
(3) the canonical maps
$$\text{Sym}^n_{\mathcal{O}_X/\mathcal{J}}(\mathcal{J}/\mathcal{J}^2) \longrightarrow \mathcal{J}^n/\mathcal{J}^{n+1}$$
are isomorphisms for all $n \geq 0$.

Proof. It is clear that if $U \subset X$ is an open such that $\mathcal{J}|_U$ is generated by a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ then $\mathcal{J}|_U$ is of finite type, $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free with basis $f_1, \ldots, f_r$, and the maps in (3) are isomorphisms because they are coordinate free formulation of the degree n part of (31.20.0.3). Hence it is clear that being quasi-regular implies conditions (1), (2), and (3).

Conversely, suppose that (1), (2), and (3) hold. Pick a point $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$. Then there exists a neighbourhood $U \subset X$ of x such that $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free of rank r over $\mathcal{O}_U/\mathcal{J}|_U$. After possibly shrinking U we may assume there exist $f_1, \ldots, f_r \in \mathcal{J}(U)$ which map to a basis of $\mathcal{J}|_U/\mathcal{J}^2|_U$ as an $\mathcal{O}_U/\mathcal{J}|_U$-module. In particular we see that the images of $f_1, \ldots, f_r$ in $\mathcal{J}_x/\mathcal{J}_x^2$ generate. Hence by Nakayama's lemma (Algebra, Lemma 10.20.1) we see that $f_1, \ldots, f_r$ generate the stalk $\mathcal{J}_x$. Hence, since $\mathcal{J}$ is of finite type, by Modules, Lemma 17.9.4 after shrinking U we may assume that $f_1, \ldots, f_r$ generate $\mathcal{J}$. Finally, from (3) and the isomorphism $\mathcal{J}|_U/\mathcal{J}^2|_U = \bigoplus \mathcal{O}_U/\mathcal{J}|_U f_i$ it is clear that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ is a quasi-regular sequence. $\square$

067N Lemma 31.20.5. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Let $x \in X$ and $f_1, \ldots, f_r \in \mathcal{J}_x$ whose images give a basis for the $\kappa(x)$-vector space $\mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$.

(1) If $\mathcal{J}$ is quasi-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form a quasi-regular sequence generating $\mathcal{J}|_U$.
(2) If $\mathcal{J}$ is H_1-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an H_1-regular sequence generating $\mathcal{J}|_U$.
(3) If $\mathcal{J}$ is Koszul-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an Koszul-regular sequence generating $\mathcal{J}|_U$.

Proof. First assume that $\mathcal{J}$ is quasi-regular. We may choose an open neighbourhood $U \subset X$ of x and a quasi-regular sequence $g_1, \ldots, g_s \in \mathcal{O}_X(U)$ which generates $\mathcal{J}|_U$. Note that this implies that $\mathcal{J}/\mathcal{J}^2$ is free of rank s over $\mathcal{O}_U/\mathcal{J}|_U$ (see Lemma 31.20.4 and its proof) and hence $r = s$. We may shrink U and assume $f_1, \ldots, f_r \in \mathcal{J}(U)$. Thus we may write

$$f_i = \sum a_{ij} g_j$$

for some $a_{ij} \in \mathcal{O}_X(U)$. By assumption the matrix $A = (a_{ij})$ maps to an invertible matrix over $\kappa(x)$. Hence, after shrinking U once more, we may assume that (a_{ij}) is invertible. Thus we see that $f_1, \ldots, f_r$ give a basis for $(\mathcal{J}/\mathcal{J}^2)|_U$ which proves that $f_1, \ldots, f_r$ is a quasi-regular sequence over U.

Note that in order to prove (2) and (3) we may, because the assumptions of (2) and (3) are stronger than the assumption in (1), already assume that $f_1, \ldots, f_r \in \mathcal{J}(U)$ and $f_i = \sum a_{ij} g_j$ with (a_{ij}) invertible as above, where now $g_1, \ldots, g_r$ is a H_1-regular or Koszul-regular sequence. Since the Koszul complex on $f_1, \ldots, f_r$ is isomorphic to the Koszul complex on $g_1, \ldots, g_r$ via the matrix (a_{ij}) (see More on Algebra, Lemma 15.28.4) we conclude that $f_1, \ldots, f_r$ is H_1-regular or Koszul-regular as desired. $\square$

063F Lemma 31.20.6. Any regular, Koszul-regular, H_1-regular, or quasi-regular sheaf of ideals on a scheme is a finite type quasi-coherent sheaf of ideals.

Proof. This follows as such a sheaf of ideals is locally generated by finitely many sections. And any sheaf of ideals locally generated by sections on a scheme is quasi-coherent, see Schemes, Lemma 26.10.1. $\square$

063G Lemma 31.20.7. Let X be a scheme. Let $\mathcal{J}$ be a sheaf of ideals. Then $\mathcal{J}$ is regular (resp. Koszul-regular, H_1-regular, quasi-regular) if and only if for every $x \in \text{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an affine open neighbourhood $x \in U \subset X$, $U = \text{Spec}(A)$ such that $\mathcal{J}|_U = \widetilde{I}$ and such that I is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in A$.

Proof. By assumption we can find an open neighbourhood U of x over which $\mathcal{J}$ is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$. After shrinking U we may assume that U is affine, say $U = \text{Spec}(A)$. Since $\mathcal{J}$ is quasi-coherent by Lemma 31.20.6 we see that $\mathcal{J}|_U = \widetilde{I}$ for some ideal $I \subset A$. Now we can use the fact that

$$\widetilde{\ }: \text{Mod}_A \longrightarrow QCoh(\mathcal{O}_U)$$

is an equivalence of categories which preserves exactness. For example the fact that the functions f_i generate $\mathcal{J}$ means that the f_i, seen as elements of A generate I. The fact that (31.20.0.1) is injective (resp. (31.20.0.2) is exact, (31.20.0.2) is exact in degree 1, (31.20.0.3) is an isomorphism) implies the corresponding property of the map $A/(f_1, \ldots, f_{i-1}) \to A/(f_1, \ldots, f_{i-1})$ (resp. the complex $K_\bullet(A, f_1, \ldots, f_r)$, the map $A/I[T_1, \ldots, T_r] \to \bigoplus I^n/I^{n+1}$). Thus $f_1, \ldots, f_r \in A$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence of the ring A. $\square$

063I Lemma 31.20.8. Let X be a locally Noetherian scheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let x be a point of the support of $\mathcal{O}_X/\mathcal{J}$. The following are equivalent

(1) $\mathcal{J}_x$ is generated by a regular sequence in $\mathcal{O}_{X,x}$,
(2) $\mathcal{J}_x$ is generated by a Koszul-regular sequence in $\mathcal{O}_{X,x}$,
(3) $\mathcal{J}_x$ is generated by an H_1-regular sequence in $\mathcal{O}_{X,x}$,
(4) $\mathcal{J}_x$ is generated by a quasi-regular sequence in $\mathcal{O}_{X,x}$,
(5) there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a regular sequence in A, and
(6) there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a Koszul-regular sequence in A, and
(7) there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by an H_1-regular sequence in A, and
(8) there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a quasi-regular sequence in A,
(9) there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is regular, and
(10) there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is Koszul-regular, and
(11) there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is H_1-regular, and
(12) there exists a neighbourhood U of x such that $\mathcal{J}|_U$ is quasi-regular.

In particular, on a locally Noetherian scheme the notions of regular, Koszul-regular, H_1-regular, or quasi-regular ideal sheaf all agree.

Proof. It follows from Lemma 31.20.7 that (5) $\Leftrightarrow$ (9), (6) $\Leftrightarrow$ (10), (7) $\Leftrightarrow$ (11), and (8) $\Leftrightarrow$ (12). It is clear that (5) $\Rightarrow$ (1), (6) $\Rightarrow$ (2), (7) $\Rightarrow$ (3), and (8) $\Rightarrow$ (4). We have (1) $\Rightarrow$ (5) by Algebra, Lemma 10.68.6. We have (9) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) by Lemma 31.20.3. Finally, (4) $\Rightarrow$ (1) by Algebra, Lemma 10.69.6. Now all 12 statements are equivalent. $\square$

31.21. Regular immersions

0638 Let $i : Z \to X$ be an immersion of schemes. By definition this means there exists an open subscheme $U \subset X$ such that Z is identified with a closed subscheme of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the corresponding quasi-coherent sheaf of ideals. Suppose $U' \subset X$ is a second such open subscheme, and denote $\mathcal{I}' \subset \mathcal{O}_{U'}$ the corresponding quasi-coherent sheaf of ideals. Then $\mathcal{I}|_{U \cap U'} = \mathcal{I}'|_{U \cap U'}$. Moreover, the support of $\mathcal{O}_U/\mathcal{I}$ is Z which is contained in $U \cap U'$ and is also the support of $\mathcal{O}_{U'}/\mathcal{I}'$. Hence it follows from Definition 31.20.2 that $\mathcal{I}$ is a regular ideal if and only if $\mathcal{I}'$ is a regular ideal. Similarly for being Koszul-regular, H_1-regular, or quasi-regular.

063J Definition 31.21.1. Let $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U and denote $\mathcal{I} \subset \mathcal{O}_U$ the corresponding quasi-coherent sheaf of ideals.

(1) We say i is a regular immersion if $\mathcal{I}$ is regular.
(2) We say i is a Koszul-regular immersion if $\mathcal{I}$ is Koszul-regular.
(3) We say i is a H_1-regular immersion if $\mathcal{I}$ is H_1-regular.
(4) We say i is a quasi-regular immersion if $\mathcal{I}$ is quasi-regular.

The concept of a Koszul-regular immersion was introduced in [BGI71, Expose VII, Definition 1.4] where it was called a regular immersion.

The discussion above shows that this is independent of the choice of U. The conditions are listed in decreasing order of strength, see Lemma 31.21.2. A Koszul-regular closed immersion is smooth locally a regular immersion, see Lemma 31.21.11. In the locally Noetherian case all four notions agree, see Lemma 31.20.8.

063K Lemma 31.21.2. Let $i : Z \to X$ be an immersion of schemes. We have the following implications: i is regular $\Rightarrow$ i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.

Proof. The lemma immediately reduces to Lemma 31.20.3. $\square$

063L Lemma 31.21.3. Let $i : Z \to X$ be an immersion of schemes. Assume X is locally Noetherian. Then i is regular $\Leftrightarrow$ i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.

Proof. Follows immediately from Lemma 31.21.2 and Lemma 31.20.8. $\square$

067P Lemma 31.21.4. Let $i : Z \to X$ be a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion. Let $X' \to X$ be a flat morphism. Then the base change $i' : Z \times_X X' \to X'$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion.

Proof. Via Lemma 31.20.7 this translates into the algebraic statements in Algebra, Lemmas 10.68.5 and 10.69.3 and More on Algebra, Lemma 15.30.5. $\square$

063M Lemma 31.21.5. Let $i : Z \to X$ be an immersion of schemes. Then i is a quasi-regular immersion if and only if the following conditions are satisfied

(1) i is locally of finite presentation,
(2) the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and
(3) the map (31.19.1.2) is an isomorphism.

Proof. An open immersion is locally of finite presentation. Hence we may replace X by an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U, i.e., we may assume that i is a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Recall, see Morphisms, Lemma 29.21.7 that $\mathcal{I}$ is of finite type if and only if i is locally of finite presentation. Hence the equivalence follows from Lemma 31.20.4 and unwinding the definitions. $\square$

063N Lemma 31.21.6. Let $Z \to Y \to X$ be immersions of schemes. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Morphisms, Lemma 29.31.5

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 31.21.5 and Lemma 31.20.3) it suffices to prove that the sequence is exact. By what was proven in Morphisms, Lemma 29.31.5 it suffices to show that the first map is injective. Working affine locally this reduces to the following question: Suppose that we have a ring A and ideals $I \subset J \subset A$. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Does this imply that $I/I^2 \otimes_A A/J \to J/J^2$ is injective? Note that $I/I^2 \otimes_A A/J = I/IJ$. Hence we are trying to prove that $I \cap J^2 = IJ$. This is the result of More on Algebra, Lemma 15.30.9. $\square$

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 10.69.8. The other types of regular immersions are preserved under composition.

067Q Lemma 31.21.7. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes.

(1) If i and j are regular immersions, so is $j \circ i$.
(2) If i and j are Koszul-regular immersions, so is $j \circ i$.
(3) If i and j are H_1-regular immersions, so is $j \circ i$.
(4) If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.

Proof. The algebraic version of (1) is Algebra, Lemma 10.68.7. The algebraic version of (2) is More on Algebra, Lemma 15.30.13. The algebraic version of (3) is More on Algebra, Lemma 15.30.11. The algebraic version of (4) is More on Algebra, Lemma 15.30.10. □

068Z Lemma 31.21.8. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume that the sequence

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Morphisms, Lemma 29.31.5 is exact and locally split.

(1) If $j \circ i$ is a quasi-regular immersion, so is i.
(2) If $j \circ i$ is a H_1-regular immersion, so is i.
(3) If both j and $j \circ i$ are Koszul-regular immersions, so is i.

Proof. After shrinking Y and X we may assume that i and j are closed immersions. Denote $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of Y and $\mathcal{J} \subset \mathcal{O}_X$ the ideal sheaf of Z. The conormal sequence is $0 \to \mathcal{I}/\mathcal{I}\mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/(\mathcal{I} + \mathcal{J}^2) \to 0$. Let $z \in Z$ and set $y = i(z)$, $x = j(y) = j(i(z))$. Choose $f_1, \ldots, f_n \in \mathcal{I}_x$ which map to a basis of $\mathcal{I}_x/\mathfrak{m}_z\mathcal{I}_x$. Extend this to $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{J}_x$ which map to a basis of $\mathcal{J}_x/\mathfrak{m}_z\mathcal{J}_x$. This is possible as we have assumed that the sequence of conormal sheaves is split in a neighbourhood of z, hence $\mathcal{I}_x/\mathfrak{m}_x\mathcal{I}_x \to \mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$ is injective.

Proof of (1). By Lemma 31.20.5 we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a quasi-regular sequence generating $\mathcal{J}$. Hence by Algebra, Lemma 10.69.5 we see that $g_1, \ldots, g_m$ induces a quasi-regular sequence on $Y \cap U$ cutting out Z.

Proof of (2). Exactly the same as the proof of (1) except using More on Algebra, Lemma 15.30.12.

Proof of (3). By Lemma 31.20.5 (applied twice) we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n$ forms a Koszul-regular sequence generating $\mathcal{I}$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a Koszul-regular sequence generating $\mathcal{J}$. Hence by More on Algebra, Lemma 15.30.14 we see that $g_1, \ldots, g_m$ induces a Koszul-regular sequence on $Y \cap U$ cutting out Z. □

0690 Lemma 31.21.9. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Pick $z \in Z$ and denote $y \in Y$, $x \in X$ the corresponding points. Assume X is locally Noetherian. The following are equivalent

(1) i is a regular immersion in a neighbourhood of z and j is a regular immersion in a neighbourhood of y,
(2) i and $j \circ i$ are regular immersions in a neighbourhood of z,

(3) $j \circ i$ is a regular immersion in a neighbourhood of z and the conormal sequence
$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$
is split exact in a neighbourhood of z.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Lemma 31.20.8. The implication (1) $\Rightarrow$ (2) is Lemma 31.21.7. The implication (2) $\Rightarrow$ (3) is Lemma 31.21.6. Thus it suffices to prove that (3) implies (1).

Assume (3). Set $A = \mathcal{O}_{X,x}$. Denote $I \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ and denote $J \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$. Note that any minimal sequence of elements generating J in A is a quasi-regular hence regular sequence, see Lemma 31.20.5. By assumption the conormal sequence
$$0 \to I/IJ \to J/J^2 \to J/(I + J^2) \to 0$$
is split exact as a sequence of A/J-modules. Hence we can pick a minimal system of generators $f_1, \ldots, f_n, g_1, \ldots, g_m$ of J with $f_1, \ldots, f_n \in I$ a minimal system of generators of I. As pointed out above $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A. It follows directly from the definition of a regular sequence that $f_1, \ldots, f_n$ is a regular sequence in A and $\overline{g}_1, \ldots, \overline{g}_m$ is a regular sequence in A/I. Thus j is a regular immersion at y and i is a regular immersion at z. □

0691 Remark 31.21.10. In the situation of Lemma 31.21.9 parts (1), (2), (3) are not equivalent to "$j \circ i$ and j are regular immersions at z and y". An example is $X = \mathbf{A}^1_k = \mathrm{Spec}(k[x])$, $Y = \mathrm{Spec}(k[x]/(x^2))$ and $Z = \mathrm{Spec}(k[x]/(x))$.

0692 Lemma 31.21.11. Let $i : Z \to X$ be a Koszul regular closed immersion. Then there exists a surjective smooth morphism $X' \to X$ such that the base change $i' : Z \times_X X' \to X'$ of i is a regular immersion.

Proof. We may assume that X is affine and the ideal of Z generated by a Koszul-regular sequence by replacing X by the members of a suitable affine open covering (affine opens as in Lemma 31.20.7). The affine case is More on Algebra, Lemma 15.30.17. □

0E9J Lemma 31.21.12. Let $i : Z \to X$ be an immersion. If Z and X are regular schemes, then i is a regular immersion.

Proof. Let $z \in Z$. By Lemma 31.20.8 it suffices to show that the kernel of $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is generated by a regular sequence. This follows from Algebra, Lemmas 10.106.4 and 10.106.3. □

31.22. Relative regular immersions

063P In this section we consider the base change property for regular immersions. The following lemma does not hold for regular immersions or for Koszul immersions, see Examples, Lemma 110.14.2.

063R Lemma 31.22.1. Let $f : X \to S$ be a morphism of schemes. Let $i : Z \subset X$ be an immersion. Assume

(1) i is an H_1-regular (resp. quasi-regular) immersion, and

(2) $Z \to S$ is a flat morphism.

Then for every morphism of schemes $g : S' \to S$ the base change $Z' = S' \times_S Z \to X' = S' \times_S X$ is an H_1-regular (resp. quasi-regular) immersion.

Proof. Unwinding the definitions and using Lemma 31.20.7 this translates into More on Algebra, Lemma 15.31.4. □

This lemma is the motivation for the following definition.

063S Definition 31.22.2. Let $f : X \to S$ be a morphism of schemes. Let $i : Z \to X$ be an immersion.

(1) We say i is a relative quasi-regular immersion if $Z \to S$ is flat and i is a quasi-regular immersion.
(2) We say i is a relative H_1-regular immersion if $Z \to S$ is flat and i is an H_1-regular immersion.

We warn the reader that this may be nonstandard notation. Lemma 31.22.1 guarantees that relative quasi-regular (resp. H_1-regular) immersions are preserved under any base change. A relative H_1-regular immersion is a relative quasi-regular immersion, see Lemma 31.21.2. Please take a look at Lemma 31.22.6 (or Lemma 31.22.4) which shows that if $Z \to X$ is a relative H_1-regular (or quasi-regular) immersion and the ambient scheme is (flat and) locally of finite presentation over S, then $Z \to X$ is actually a regular immersion and the same remains true after any base change.

063T Lemma 31.22.3. Let $f : X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative quasi-regular immersion. If $x \in Z$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.

Proof. Let $f_1, \ldots, f_r \in \mathcal{O}_{X,x}$ be a quasi-regular sequence cutting out the ideal of Z at x. By Algebra, Lemma 10.69.6 we know that $f_1, \ldots, f_r$ is a regular sequence. Hence f_r is a nonzerodivisor on $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ such that the quotient is a flat $\mathcal{O}_{S,f(x)}$-module. By Lemma 31.18.5 we conclude that $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ is a flat $\mathcal{O}_{S,f(x)}$-module. Continuing by induction we find that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module. □

063U Lemma 31.22.4. Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be an immersion. Assume

(1) $X \to S$ is flat and locally of finite presentation,
(2) $Z \to X$ is a relative quasi-regular immersion.

Then $Z \to X$ is a regular immersion and the same remains true after any base change.

Proof. Pick $x \in Z$ with image $s \in S$. To prove this it suffices to find an affine neighbourhood of x contained in U such that the result holds on that affine open. Hence we may assume that X is affine and there exist a quasi-regular sequence $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that $Z = V(f_1, \ldots, f_r)$. By More on Algebra, Lemma 15.31.4 the sequence $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a quasi-regular sequence in $\Gamma(X_s, \mathcal{O}_{X_s})$. Since X_s is Noetherian, this implies, possibly after shrinking X a bit, that $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a regular sequence, see Algebra, Lemmas 10.69.6 and 10.68.6. By Lemma 31.18.9 it follows that $Z_1 = V(f_1) \subset X$ is a relative effective Cartier divisor, again after possibly shrinking X a bit. Applying the same lemma again, but now to

$Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until on reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Since being a relative effective Cartier divisor is preserved under arbitrary base change, see Lemma 31.18.1, we also see that the final statement of the lemma holds. $\square$

0FUD Remark 31.22.5. The codimension of a relative quasi-regular immersion, if it is constant, does not change after a base change. In fact, if we have a ring map $A \to B$ and a quasi-regular sequence $f_1, \ldots, f_r \in B$ such that $B/(f_1, \ldots, f_r)$ is flat over A, then for any ring map $A \to A'$ we have a quasi-regular sequence $f_1 \otimes 1, \ldots, f_r \otimes 1$ in $B' = B \otimes_A A'$ by More on Algebra, Lemma 15.31.4 (which was used in the proof of Lemma 31.22.1 above). Now the proof of Lemma 31.22.4 shows that if $A \to B$ is flat and locally of finite presentation, then for every prime ideal $\mathfrak q' \subset B'$ the sequence $f_1 \otimes 1, \ldots, f_r \otimes 1$ is even a regular sequence in the local ring $B'_{\mathfrak q'}$.

063V Lemma 31.22.6. Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative H_1-regular immersion. Assume $X \to S$ is locally of finite presentation. Then

1. there exists an open subscheme $U \subset X$ such that $Z \subset U$ and such that $U \to S$ is flat, and
2. $Z \to X$ is a regular immersion and the same remains true after any base change.

Proof. Pick $x \in Z$. To prove (1) suffices to find an open neighbourhood $U \subset X$ of x such that $U \to S$ is flat. Hence the lemma reduces to the case that $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$ are affine and that Z is given by an H_1-regular sequence $f_1, \ldots, f_r \in B$. By assumption B is a finitely presented A-algebra and $B/(f_1, \ldots, f_r)B$ is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume f_i is the image of $f'_i \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $f'_1, \ldots, f'_r, g_1, \ldots, g_m$ are in A_0. We set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and we denote $f_{i,0}$ the image of f'_i in B_0. Then $B = B_0 \otimes_{A_0} A$ and

$$B/(f_1, \ldots, f_r) = B_0/(f_{0,1}, \ldots, f_{0,r}) \otimes_{A_0} A.$$

By Algebra, Lemma 10.168.1 we may, after enlarging A_0, assume that $B_0/(f_{0,1}, \ldots, f_{0,r})$ is flat over A_0. It may not be the case at this point that the Koszul cohomology group $H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ is zero. On the other hand, as B_0 is Noetherian, it is a finitely generated B_0-module. Let $\xi_1, \ldots, \xi_n \in H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ be generators. Let $A_0 \subset A_1 \subset A$ be a larger finite type $\mathbf{Z}$-subalgebra of A. Denote $f_{1,i}$ the image of $f_{0,i}$ in $B_1 = B_0 \otimes_{A_0} A_1$. By More on Algebra, Lemma 15.31.3 the map

$$H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r})) \otimes_{A_0} A_1 \longrightarrow H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$$

is surjective. Furthermore, it is clear that the colimit (over all choices of A_1 as above) of the complexes $K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})$ is the complex $K_\bullet(B, f_1, \ldots, f_r)$ which is acyclic in degree 1. Hence

$$\text{colim}_{A_0 \subset A_1 \subset A} H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})) = 0$$

by Algebra, Lemma 10.8.8. Thus we can find a choice of A_1 such that $\xi_1, \ldots, \xi_n$ all map to zero in $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$. In other words, the Koszul cohomology group $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$ is zero.

Consider the morphism of affine schemes $X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$ and $Z_1 = \text{Spec}(B_1/(f_{1,1}, \ldots, f_{1,r}))$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X =$

$X_1 \times_{S_1} S$, and similarly $Z = Z_1 \times_S S_1$, it now suffices to prove (1) for $X_1 \to S_1$ and the relative H_1-regular immersion $Z_1 \to X_1$, see Morphisms, Lemma 29.25.7. Hence we have reduced to the case where $X \to S$ is a finite type morphism of Noetherian schemes. In this case we know that $X \to S$ is flat at every point of Z by Lemma 31.22.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 10.129.4 we see that (1) holds. Part (2) then follows from an application of Lemma 31.22.4. □

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative quasi-regular immersion in terms of its fibres.

063W Lemma 31.22.7. Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $T \subset X$ be a closed subscheme. Let $x \in T$ with image $s \in S$.

(1) If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, then there exists an open $U \subset X$ and a relative quasi-regular immersion $Z \subset U$ such that $Z_s = T_s \cap U_s$ and $T \cap U \subset Z$.
(2) If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, the morphism $T \to X$ is of finite presentation, and $T \to S$ is flat at x, then we can choose U and Z as in (1) such that $T \cap U = Z$.
(3) If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, and T is cut out by c equations in a neighbourhood of x, where $c = \dim_x(X_s) - \dim_x(T_s)$, then we can choose U and Z as in (1) such that $T \cap U = Z$.

In each case $Z \to U$ is a regular immersion by Lemma 31.22.4. In particular, if $T \to S$ is locally of finite presentation and flat and all fibres $T_s \subset X_s$ are quasi-regular immersions, then $T \to X$ is a relative quasi-regular immersion.

Proof. Choose affine open neighbourhoods $\text{Spec}(A)$ of s and $\text{Spec}(B)$ of x such that $\varphi(\text{Spec}(B)) \subset \text{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to T. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\text{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a quasi-regular sequence of elements. After possibly localizing B at some $g \in B$, $g \notin \mathfrak{q}$ we may assume there exist $f_1, \ldots, f_r \in I$ which map to a quasi-regular sequence in $B \otimes_A \kappa(\mathfrak{p})$ which generates $I(B \otimes_A \kappa(\mathfrak{p}))$. By Algebra, Lemmas 10.69.6 and 10.68.6 we may assume after another localization that $f_1, \ldots, f_r \in I$ form a regular sequence in $B \otimes_A \kappa(\mathfrak{p})$. By Lemma 31.18.9 it follows that $Z_1 = V(f_1) \subset \text{Spec}(B)$ is a relative effective Cartier divisor, again after possibly localizing B. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until one reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Then $Z \to \text{Spec}(B)$ is a regular immersion and Z is flat over S, in particular $Z \to \text{Spec}(B)$ is a relative quasi-regular immersion over $\text{Spec}(A)$. This proves (1).

To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 10.128.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 10.126.6.

To see (3), after possibly shrinking U we may assume that the ideal of Z is generated by a regular sequence $f_1, \ldots, f_r$ (see our construction of Z above) and the ideal of

T is generated by $g_1, \ldots, g_c$. We claim that $c = r$. Namely,

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(T_s) &= \dim(\mathcal{O}_{T_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{T_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 10.116.3 and the second by r times applying Algebra, Lemma 10.60.13. As $T \subset Z$ we see that $f_i = \sum b_{ij} g_j$. But the ideals of Z and T cut out the same quasi-regular closed subscheme of X_s in a neighbourhood of x. Hence the matrix $(b_{ij}) \bmod \mathfrak{m}_x$ is invertible (some details omitted). Hence (b_{ij}) is invertible in an open neighbourhood of x. In other words, $T \cap U = Z$ after shrinking U.

The final statements of the lemma follow immediately from part (2), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 29.21.3 and 29.21.11. □

The following lemma is an enhancement of Morphisms, Lemma 29.34.20.

067R Lemma 31.22.8. Let $f : X \to S$ be a smooth morphism of schemes. Let $\sigma : S \to X$ be a section of f. Then σ is a regular immersion.

Proof. By Schemes, Lemma 26.21.10 the morphism σ is an immersion. After replacing X by an open neighbourhood of $\sigma(S)$ we may assume that σ is a closed immersion. Let $T = \sigma(S)$ be the corresponding closed subscheme of X. Since $T \to S$ is an isomorphism it is flat and of finite presentation. Also a smooth morphism is flat and locally of finite presentation, see Morphisms, Lemmas 29.34.9 and 29.34.8. Thus, according to Lemma 31.22.7, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. This follows immediately from Morphisms, Lemma 29.34.20 but we can also see it directly as follows. Let k be a field and let A be a smooth k-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal whose residue field is k. Then $\mathfrak{m}$ is generated by a quasi-regular sequence, possibly after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$. In Algebra, Lemma 10.140.3 we proved that $A_\mathfrak{m}$ is a regular local ring, hence $\mathfrak{m}A_\mathfrak{m}$ is generated by a regular sequence. This does indeed imply that $\mathfrak{m}$ is generated by a regular sequence (after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$), see Algebra, Lemma 10.68.6. □

The following lemma has a kind of converse, see Lemma 31.22.12.

067S Lemma 31.22.9. Let

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "j"'] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If j is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, then so is i.

Proof. We can write i as the composition

$$Y \to Y \times_S X \to X$$

By Lemma 31.22.8 the first arrow is a regular immersion. The second arrow is a flat base change of $Y \to S$, hence is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, see Lemma 31.21.4. We conclude by an application of Lemma 31.21.7. ☐

067T Lemma 31.22.10. Let

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is syntomic, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. After replacing X by an open neighbourhood of $i(Y)$ we may assume that i is a closed immersion. Let $T = i(Y)$ be the corresponding closed subscheme of X. Since $T \cong Y$ the morphism $T \to S$ is flat and of finite presentation (Morphisms, Lemmas 29.30.6 and 29.30.7). Also a smooth morphism is flat and locally of finite presentation (Morphisms, Lemmas 29.34.9 and 29.34.8). Thus, according to Lemma 31.22.7, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. As X_s is locally of finite type over a field, it is Noetherian (Morphisms, Lemma 29.15.6). Thus we can check that $T_s \subset X_s$ is a quasi-regular immersion at points, see Lemma 31.20.8. Take $t \in T_s$. By Morphisms, Lemma 29.30.9 the local ring $\mathcal{O}_{T_s,t}$ is a local complete intersection over $\kappa(s)$. The local ring $\mathcal{O}_{X_s,t}$ is regular, see Algebra, Lemma 10.140.3. By Algebra, Lemma 10.135.7 we see that the kernel of the surjection $\mathcal{O}_{X_s,t} \to \mathcal{O}_{T_s,t}$ is generated by a regular sequence, which is what we had to show. ☐

067U Lemma 31.22.11. Let

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is smooth, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. This is a special case of Lemma 31.22.10 because a smooth morphism is syntomic, see Morphisms, Lemma 29.34.7. ☐

0693 Lemma 31.22.12. Let

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "j"'] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth and i and j immersions. If i is a Koszul-regular (resp. H_1-regular, quasi-regular) immersion, then so is j.

Proof. We will use Lemma 31.21.2 without further mention. Let $y \in Y$ be any point. Set $x = i(y)$ and set $s = j(y)$. It suffices to prove the result after replacing X and S by open neighbourhoods U and V of x and s and Y by an open neighbourhood of y in $i^{-1}(U) \cap j^{-1}(V)$.

We first prove the result for $X = \mathbf{A}^n_S$. After replacing S by an affine open V and replacing Y by $j^{-1}(V)$ we may assume that j is a closed immersions and S is affine. Write $S = \text{Spec}(A)$. Then $j : Y \to S$ defines an isomorphism of Y to the closed subscheme $\text{Spec}(A/I)$ for some ideal $I \subset A$. The map $i : Y = \text{Spec}(A/I) \to \mathbf{A}^n_S = \text{Spec}(A[x_1, \ldots, x_n])$ corresponds to an A-algebra homomorphism $i^\sharp : A[x_1, \ldots, x_n] \to A/I$. Choose $a_i \in A$ which map to $i^\sharp(x_i)$ in A/I. Observe that the ideal of the closed immersion i is

$$J = (x_1 - a_1, \ldots, x_n - a_n) + IA[x_1, \ldots, x_n].$$

Set $K = (x_1 - a_1, \ldots, x_n - a_n)$. We claim the sequence

$$0 \to K/KJ \to J/J^2 \to J/(K + J^2) \to 0$$

is split exact. To see this note that K/K^2 is free with basis $x_i - a_i$ over the ring $A[x_1, \ldots, x_n]/K \cong A$. Hence K/KJ is free with the same basis over the ring $A[x_1, \ldots, x_n]/J \cong A/I$. On the other hand, taking derivatives gives a map

$$\text{d}_{A[x_1,\ldots,x_n]/A} : J/J^2 \longrightarrow \Omega_{A[x_1,\ldots,x_n]/A} \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]/J$$

which maps the generators $x_i - a_i$ to the basis elements $\text{d}x_i$ of the free module on the right. The claim follows. Moreover, note that $x_1 - a_1, \ldots, x_n - a_n$ is a regular sequence in $A[x_1, \ldots, x_n]$ with quotient ring $A[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \cong A$. Thus we have a factorization

$$Y \to V(x_1 - a_1, \ldots, x_n - a_n) \to \mathbf{A}^n_S$$

of our closed immersion i where the composition is Koszul-regular (resp. H_1-regular, quasi-regular), the second arrow is a regular immersion, and the associated conormal sequence is split. Now the result follows from Lemma 31.21.8.

Next, we prove the result holds if i is H_1-regular or quasi-regular. Namely, shrinking as in the first paragraph of the proof, we may assume that Y, X, and S are affine. In this case we can choose a closed immersion $h : X \to \mathbf{A}^n_S$ over S for some n. Note that h is a regular immersion by Lemma 31.22.11. Hence $h \circ i$ is a H_1-regular or quasi-regular immersion, see Lemma 31.21.7 (note that this step does not work in the "quasi-regular case"). Thus we reduce to the case $X = \mathbf{A}^n_S$ and S affine we proved above.

Finally, assume i is quasi-regular. After shrinking as in the first paragraph of the proof, we may use Morphisms, Lemma 29.36.20 to factor f as $X \to \mathbf{A}^n_S \to S$ where the first morphism $X \to \mathbf{A}^n_S$ is étale. This reduces the problem to the the two cases (a) $X = \mathbf{A}^n_S$ and (b) f is étale. Case (a) was handled in the second paragraph of the proof. Case (b) is handled by the next paragraph.

Assume f is étale. After shrinking we may assume X, Y, and S affine i and j closed immersions (small detail omitted). Say $S = \text{Spec}(A)$, $X = \text{Spec}(B)$ and $Y = \text{Spec}(B/J) = \text{Spec}(A/I)$. Shrinking further we may assume J is generated by a quasi-regular sequence. The ring map $A \to B$ is étale, hence formally étale (Algebra, Lemma 10.150.2). Thus $\bigoplus I^n/I^{n+1} \cong \bigoplus J^n/J^{n+1}$ by Algebra, Lemma 10.150.5. Since J is generated by a quasi-regular sequence, so is I. This finishes the proof. $\square$

31.23. Meromorphic functions and sections

01X1 This section contains only the general definitions and some elementary results. See [Kle79] for some possible pitfalls[3].

[3]Danger, Will Robinson!

Let $(X, \mathcal{O}_X)$ be a locally ringed space. For any open $U \subset X$ we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of $\mathcal{O}_X$ over U, see Definition 31.14.6. The restriction of a regular section to a smaller open is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of $\mathcal{O}_X$. We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X. Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U. Hence we may consider the presheaf of rings

$$U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U),$$

see Modules, Lemma 17.27.1.

01X2 Definition 31.23.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. The sheaf of meromorphic functions on X is the sheaf $\mathcal{K}_X$ associated to the presheaf displayed above. A meromorphic function on X is a global section of $\mathcal{K}_X$.

Since each element of each $\mathcal{S}(U)$ is a nonzerodivisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \to \mathcal{K}_X$ is injective.

01X3 Example 31.23.2. Let $A = \mathbf{C}[x, \{y_\alpha\}_{\alpha \in \mathbf{C}}]/((x-\alpha)y_\alpha, y_\alpha y_\beta)$. Any element of A can be written uniquely as $f(x) + \sum \lambda_\alpha y_\alpha$ with $f(x) \in \mathbf{C}[x]$ and $\lambda_\alpha \in \mathbf{C}$. Let $X = \mathrm{Spec}(A)$. In this case $\mathcal{O}_X = \mathcal{K}_X$, since on any affine open $D(f)$ the ring A_f any nonzerodivisor is a unit (proof omitted).

Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules, Lemma 17.27.2.

01X4 Definition 31.23.3. Let X be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of $\mathcal{K}_X$-modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).
(2) A meromorphic section of $\mathcal{F}$ is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have

$$\mathcal{K}_X(\mathcal{F})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_{X,x} = \mathcal{S}_x^{-1}\mathcal{F}_x$$

for any point $x \in X$. However, one has to be careful since it may not be the case that $\mathcal{S}_x$ is the set of nonzerodivisors in the local ring $\mathcal{O}_{X,x}$. Namely, there is always an injective map

$$\mathcal{K}_{X,x} \longrightarrow Q(\mathcal{O}_{X,x})$$

to the total quotient ring. It is also surjective if and only if $\mathcal{S}_x$ is the set of nonzerodivisors in $\mathcal{O}_{X,x}$. The sheaves of meromorphic sections aren't quasi-coherent modules in general, but they do have some properties in common with quasi-coherent modules.

02OT Definition 31.23.4. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. We say that pullbacks of meromorphic functions are defined for f if for every pair of open $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, and any section $s \in \Gamma(V, \mathcal{S}_Y)$ the pullback $f^\sharp(s) \in \Gamma(U, \mathcal{O}_X)$ is an element of $\Gamma(U, \mathcal{S}_X)$.

In this case there is an induced map $f^\sharp : f^{-1}\mathcal{K}_Y \to \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed spaces

$$\begin{tikzcd} (X, \mathcal{K}_X) \arrow[r] \arrow[d, "f"] & (X, \mathcal{O}_X) \arrow[d, "f"] \\ (Y, \mathcal{K}_Y) \arrow[r] & (Y, \mathcal{O}_Y) \end{tikzcd}$$

We sometimes denote $f^*(s) = f^\sharp(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

02OU Lemma 31.23.5. Let $f : X \to Y$ be a morphism of schemes. In each of the following cases pullbacks of meromorphic functions are defined.

(1) every weakly associated point of X maps to a generic point of an irreducible component of Y,
(2) X, Y are integral and f is dominant,
(3) X is integral and the generic point of X maps to a generic point of an irreducible component of Y,
(4) X is reduced and every generic point of every irreducible component of X maps to the generic point of an irreducible component of Y,
(5) X is locally Noetherian, and any associated point of X maps to a generic point of an irreducible component of Y,
(6) X is locally Noetherian, has no embedded points and any generic point of an irreducible component of X maps to the generic point of an irreducible component of Y, and
(7) f is flat.

Proof. The question is local on X and Y. Hence we reduce to the case where $X = \text{Spec}(A)$, $Y = \text{Spec}(R)$ and f is given by a ring map $\varphi : R \to A$. By the characterization of regular sections of the structure sheaf in Lemma 31.14.7 we have to show that $R \to A$ maps nonzerodivisors to nonzerodivisors. Let $t \in R$ be a nonzerodivisor.

If $R \to A$ is flat, then $t : R \to R$ being injective shows that $t : A \to A$ is injective. This proves (7).

In the other cases we note that t is not contained in any of the minimal primes of R (because every element of a minimal prime in a ring is a zerodivisor). Hence in case (1) we see that $\varphi(t)$ is not contained in any weakly associated prime of A. Thus this case follows from Algebra, Lemma 10.66.7. Case (5) is a special case of (1) by Lemma 31.5.8. Case (6) follows from (5) and the definitions. Case (4) is a special case of (1) by Lemma 31.5.12. Cases (2) and (3) are special cases of (4). □

0EMF Lemma 31.23.6. Let X be a scheme such that

(a) every weakly associated point of X is a generic point of an irreducible component of X, and
(b) any quasi-compact open has a finite number of irreducible components.

Let X^0 be the set of generic points of irreducible components of X. Then we have

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta} = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta}$$

where $j_\eta : \text{Spec}(\mathcal{O}_{X,\eta}) \to X$ is the canonical map of Schemes, Section 26.13. Moreover

(1) $\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,
(2) for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf

$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$

of meromorphic sections of $\mathcal{F}$ is quasi-coherent,
(3) $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors for any $x \in X$,
(4) $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$ for any $x \in X$,
(5) $\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$ for any affine open $U \subset X$,
(6) the ring of rational functions of X (Morphisms, Definition 29.49.3) is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.

Proof. Observe that a locally finite direct sum of sheaves of modules is equal to the product since you can check this on stalks for example. Then since $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ we see that (2) follows from the other statements. Also, observe that part (6) follows from the initial statement of the lemma and Morphisms, Lemma 29.49.5 when X^0 is finite; the general case of (6) follows from this by glueing (argument omitted).

Let $j : Y = \coprod_{\eta \in X^0} \mathrm{Spec}(\mathcal{O}_{X,\eta}) \to X$ be the product of the morphisms j_η. We have to show that $\mathcal{K}_X = j_*\mathcal{O}_Y$. First note that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of local rings of dimension 0: in a local ring of dimension zero any nonzerodivisor is a unit. Next, note that pullbacks of meromorphic functions are defined for j by Lemma 31.23.5. This gives a map

$$\mathcal{K}_X \longrightarrow j_*\mathcal{O}_Y.$$

Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then A is a ring with finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ and every weakly associated prime of A is one of the $\mathfrak{q}_i$. We obtain $Q(A) = \prod A_{\mathfrak{q}_i}$ by Algebra, Lemmas 10.25.4 and 10.66.7. In other words, already the value of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$ agrees with $j_*\mathcal{O}_Y(U)$ on our affine open U. Hence the displayed map is an isomorphism which proves the first displayed equality in the statement of the lemma.

Finally, we prove (1), (3), (4), and (5). Part (5) we saw during the course of the proof that $\mathcal{K}_X = j_*\mathcal{O}_Y$. The morphism j is quasi-compact by our assumption that the set of irreducible components of X is locally finite. Hence j is quasi-compact and quasi-separated (as Y is separated). By Schemes, Lemma 26.24.1 $j_*\mathcal{O}_Y$ is quasi-coherent. This proves (1). Let $x \in X$. We may choose an affine open neighbourhood $U = \mathrm{Spec}(A)$ of x all of whose irreducible components pass through x. Then $A \subset A_\mathfrak{p}$ because every weakly associated prime of A is contained in $\mathfrak{p}$ hence elements of $A \setminus \mathfrak{p}$ are nonzerodivisors by Algebra, Lemma 10.66.7. It follows easily that any nonzerodivisor of $A_\mathfrak{p}$ is the image of a nonzerodivisor on a (possibly smaller) affine open neighbourhood of x. This proves (3). Part (4) follows from part (3) by computing stalks. □

02OX Definition 31.23.7. Let X be a locally ringed space. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A meromorphic section s of $\mathcal{L}$ is said to be regular if the induced map $\mathcal{K}_X \to \mathcal{K}_X(\mathcal{L})$ is injective. In other words, s is a regular section of the invertible $\mathcal{K}_X$-module $\mathcal{K}_X(\mathcal{L})$, see Definition 31.14.6.

Let us spell out when (regular) meromorphic sections can be pulled back.

02OY Lemma 31.23.8. Let $f : X \to Y$ be a morphism of locally ringed spaces. Assume that pullbacks of meromorphic functions are defined for f (see Definition 31.23.4).

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \to \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of $\mathcal{F}$.
(2) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A regular meromorphic section s of $\mathcal{L}$ pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.

Proof. Omitted. □

02P0 Lemma 31.23.9. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let us denote $\mathcal{I} \subset \mathcal{O}_X$ the sheaf of ideals defined by the rule

$$\mathcal{I}(V) = \{f \in \mathcal{O}_X(V) \mid fs \in \mathcal{L}(V)\}.$$

The formula makes sense since $\mathcal{L}(V) \subset \mathcal{K}_X(\mathcal{L})(V)$. Then $\mathcal{I}$ is a quasi-coherent sheaf of ideals and we have injective maps

$$1 : \mathcal{I} \longrightarrow \mathcal{O}_X, \quad s : \mathcal{I} \longrightarrow \mathcal{L}$$

whose cokernels are supported on closed nowhere dense subsets of X.

Proof. The question is local on X. Hence we may assume that $X = \text{Spec}(A)$, and $\mathcal{L} = \mathcal{O}_X$. After shrinking further we may assume that $s = a/b$ with $a, b \in A$ both nonzerodivisors in A. Set $I = \{x \in A \mid x(a/b) \in A\}$.

To show that $\mathcal{I}$ is quasi-coherent we have to show that $I_f = \{x \in A_f \mid x(a/b) \in A_f\}$ for every $f \in A$. If $c/f^n \in A_f$, $(c/f^n)(a/b) \in A_f$, then we see that $f^m c(a/b) \in A$ for some m, hence $c/f^n \in I_f$. Conversely it is easy to see that I_f is contained in $\{x \in A_f \mid x(a/b) \in A_f\}$. This proves quasi-coherence.

Let us prove the final statement. It is clear that $(b) \subset I$. Hence $V(I) \subset V(b)$ is a nowhere dense subset as b is a nonzerodivisor. Thus the cokernel of 1 is supported in a nowhere dense closed set. The same argument works for the cokernel of s since $s(b) = (a) \subset sI \subset A$. □

02P1 Definition 31.23.10. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. The sheaf of ideals $\mathcal{I}$ constructed in Lemma 31.23.9 is called the ideal sheaf of denominators of s.

31.24. Meromorphic functions and sections; Noetherian case

0EMG For locally Noetherian schemes we can prove some results about the sheaf of meromorphic functions. However, there is an example in [Kle79] showing that $\mathcal{K}_X$ need not be quasi-coherent for a Noetherian scheme X.

08I7 Lemma 31.24.1. Let X be a quasi-compact scheme. Let $h \in \Gamma(X, \mathcal{O}_X)$ and $f \in \Gamma(X, \mathcal{K}_X)$ such that f restricts to zero on X_h. Then $h^n f = 0$ for some $n \gg 0$.

Proof. We can find a covering of X by affine opens U such that $f|_U = s^{-1}a$ with $a \in \mathcal{O}_X(U)$ and $s \in \mathcal{S}(U)$. Since X is quasi-compact we can cover it by finitely many affine opens of this form. Thus it suffices to prove the lemma when $X = \text{Spec}(A)$ and $f = s^{-1}a$. Note that $s \in A$ is a nonzerodivisor hence it suffices to prove the result when $f = a$. The condition $f|_{X_h} = 0$ implies that a maps to zero in $A_h = \mathcal{O}_X(X_h)$ as $\mathcal{O}_X \subset \mathcal{K}_X$. Thus $h^n a = 0$ for some $n > 0$ as desired. □

02OV Lemma 31.24.2. Let X be a locally Noetherian scheme.

(1) For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors, and hence $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.
(2) For any affine open $U \subset X$ the ring $\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$.

Proof. To prove this lemma we may assume X is the spectrum of a Noetherian ring A. Say $x \in X$ corresponds to $\mathfrak{p} \subset A$.

Proof of (1). It is clear that $\mathcal{S}_x$ is contained in the set of nonzerodivisors of $\mathcal{O}_{X,x} = A_\mathfrak{p}$. For the converse, let $f, g \in A$, $g \notin \mathfrak{p}$ and assume f/g is a nonzerodivisor in $A_\mathfrak{p}$. Let $I = \{a \in A \mid af = 0\}$. Then we see that $I_\mathfrak{p} = 0$ by exactness of localization. Since A is Noetherian we see that I is finitely generated and hence that $g'I = 0$ for some $g' \in A$, $g' \notin \mathfrak{p}$. Hence f is a nonzerodivisor in $A_{g'}$, i.e., in a Zariski open neighbourhood of $\mathfrak{p}$. Thus f/g is an element of $\mathcal{S}_x$.

Proof of (2). Let $f \in \Gamma(X, \mathcal{K}_X)$ be a meromorphic function. Set $I = \{a \in A \mid af \in A\}$. Fix a prime $\mathfrak{p} \subset A$ corresponding to the point $x \in X$. By (1) we can write the image of f in the stalk at $\mathfrak{p}$ as a/b, $a, b \in A_\mathfrak{p}$ with $b \in A_\mathfrak{p}$ not a zerodivisor. Write $b = c/d$ with $c, d \in A$, $d \notin \mathfrak{p}$. Then $ad - cf$ is a section of $\mathcal{K}_X$ which vanishes in an open neighbourhood of x. Say it vanishes on $D(e)$ with $e \in A$, $e \notin \mathfrak{p}$. Then $e^n(ad - cf) = 0$ for some $n \gg 0$ by Lemma 31.24.1. Thus $e^n c \in I$ and $e^n c$ maps to a nonzerodivisor in $A_\mathfrak{p}$. Let $\text{Ass}(A) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ be the associated primes of A. By looking at $IA_{\mathfrak{q}_i}$ and using Algebra, Lemma 10.63.15 the above says that $I \not\subset \mathfrak{q}_i$ for each i. By Algebra, Lemma 10.15.2 there exists an element $x \in I$, $x \notin \bigcup \mathfrak{q}_i$. By Algebra, Lemma 10.63.9 we see that x is not a zerodivisor on A. Hence $f = (xf)/x$ is an element of the total ring of fractions of A. This proves (2). □

0EMH Lemma 31.24.3. Let X be a locally Noetherian scheme having no embedded points. Let X^0 be the set of generic points of irreducible components of X. Then we have

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta} = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{O}_{X,\eta}$$

where $j_\eta : \text{Spec}(\mathcal{O}_{X,\eta}) \to X$ is the canonical map of Schemes, Section 26.13. Moreover

(1) $\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,
(2) for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf
$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$
of meromorphic sections of $\mathcal{F}$ is quasi-coherent, and
(3) the ring of rational functions of X is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.

Proof. This lemma is a special case of Lemma 31.23.6 because in the locally Noetherian case weakly associated points are the same thing as associated points by Lemma 31.5.8. □

0EMI Lemma 31.24.4. Let X be a locally Noetherian scheme having no embedded points. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ has a regular meromorphic section.

Proof. For each generic point η of X pick a generator s_η of the free rank 1 module $\mathcal{L}_\eta$ over the artinian local ring $\mathcal{O}_{X,\eta}$. It follows immediately from the description of $\mathcal{K}_X$ and $\mathcal{K}_X(\mathcal{L})$ in Lemma 31.24.3 that $s = \prod s_\eta$ is a regular meromorphic section of $\mathcal{L}$. □

02P2 **Lemma 31.24.5.** Suppose given

1. X a locally Noetherian scheme,
2. $\mathcal{L}$ an invertible $\mathcal{O}_X$-module,
3. s a regular meromorphic section of $\mathcal{L}$, and
4. $\mathcal{F}$ coherent on X without embedded associated points and $\text{Supp}(\mathcal{F}) = X$.

Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s. Let $T \subset X$ be the union of the supports of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{L}/s(\mathcal{I})$ which is a nowhere dense closed subset $T \subset X$ according to Lemma 31.23.9. Then there are canonical injective maps

$$1 : \mathcal{I}\mathcal{F} \to \mathcal{F}, \quad s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$$

whose cokernels are supported on T.

Proof. Reduce to the affine case with $\mathcal{L} \cong \mathcal{O}_X$, and $s = a/b$ with $a, b \in A$ both nonzerodivisors. Proof of reduction step omitted. Write $\mathcal{F} = \widetilde{M}$. Let $I = \{x \in A \mid x(a/b) \in A\}$ so that $\mathcal{I} = \widetilde{I}$ (see proof of Lemma 31.23.9). Note that $T = V(I) \cup V((a/b)I)$. For any A-module M consider the map $1 : IM \to M$; this is the map that gives rise to the map 1 of the lemma. Consider on the other hand the map $\sigma : IM \to M_b, x \mapsto ax/b$. Since b is not a zerodivisor in A, and since M has support $\text{Spec}(A)$ and no embedded primes we see that b is a nonzerodivisor on M also. Hence $M \subset M_b$. By definition of I we have $\sigma(IM) \subset M$ as submodules of M_b. Hence we get an A-module map $s : IM \to M$ (namely the unique map such that $s(z)/1 = \sigma(z)$ in M_b for all $z \in IM$). It is injective because a is a nonzerodivisor also (on both A and M). It is clear that M/IM is annihilated by I and that $M/s(IM)$ is annihilated by $(a/b)I$. Thus the lemma follows. □

31.25. Meromorphic functions and sections; reduced case

0EMJ For a scheme which is reduced and which locally has finitely many irreducible components, the sheaf of meromorphic functions is quasi-coherent.

02OW **Lemma 31.25.1.** Let X be a reduced scheme such that any quasi-compact open has a finite number of irreducible components. Let X^0 be the set of generic points of irreducible components of X. Then we have

$$\mathcal{K}_X = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\kappa(\eta) = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\kappa(\eta)$$

where $j_\eta : \text{Spec}(\kappa(\eta)) \to X$ is the canonical map of Schemes, Section 26.13. Moreover

1. $\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras,
2. for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf
$$\mathcal{K}_X(\mathcal{F}) = \bigoplus\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta = \prod\nolimits_{\eta \in X^0} j_{\eta,*}\mathcal{F}_\eta$$
of meromorphic sections of $\mathcal{F}$ is quasi-coherent,
3. $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors for any $x \in X$,
4. $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$ for any $x \in X$,
5. $\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$ for any affine open $U \subset X$,
6. the ring of rational functions of X is the ring of meromorphic functions on X, in a formula: $R(X) = \Gamma(X, \mathcal{K}_X)$.

Proof. This lemma is a special case of Lemma 31.23.6 because on a reduced scheme the weakly associated points are the generic points by Lemma 31.5.12. □

035T Lemma 31.25.2. Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components. Then the normalization morphism $\nu : X^\nu \to X$ is the morphism

$$\underline{\mathrm{Spec}}_X(\mathcal{O}') \longrightarrow X$$

where $\mathcal{O}' \subset \mathcal{K}_X$ is the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions.

Proof. Compare the definition of the normalization morphism $\nu : X^\nu \to X$ (see Morphisms, Definition 29.54.1) with the description of $\mathcal{K}_X$ in Lemma 31.25.1 above. □

01X5 Lemma 31.25.3. Let X be an integral scheme with generic point η. We have

(1) the sheaf of meromorphic functions is isomorphic to the constant sheaf with value the function field (see Morphisms, Definition 29.49.6) of X.
(2) for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaf $\mathcal{K}_X(\mathcal{F})$ is isomorphic to the constant sheaf with value $\mathcal{F}_\eta$.

Proof. Omitted. □

In some cases we can show regular meromorphic sections exist.

02OZ Lemma 31.25.4. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. In each of the following cases $\mathcal{L}$ has a regular meromorphic section:

(1) X is integral,
(2) X is reduced and any quasi-compact open has a finite number of irreducible components,
(3) X is locally Noetherian and has no embedded points.

Proof. In case (1) let $\eta \in X$ be the generic point. We have seen in Lemma 31.25.3 that $\mathcal{K}_X$, resp. $\mathcal{K}_X(\mathcal{L})$ is the constant sheaf with value $\kappa(\eta)$, resp. $\mathcal{L}_\eta$. Since $\dim_{\kappa(\eta)} \mathcal{L}_\eta = 1$ we can pick a nonzero element $s \in \mathcal{L}_\eta$. Clearly s is a regular meromorphic section of $\mathcal{L}$. In case (2) pick $s_\eta \in \mathcal{L}_\eta$ nonzero for all generic points η of X; this is possible as $\mathcal{L}_\eta$ is a 1-dimensional vector space over $\kappa(\eta)$. It follows immediately from the description of $\mathcal{K}_X$ and $\mathcal{K}_X(\mathcal{L})$ in Lemma 31.25.1 that $s = \prod s_\eta$ is a regular meromorphic section of $\mathcal{L}$. Case (3) is Lemma 31.24.4. □

31.26. Weil divisors

0BE0 We will introduce Weil divisors and rational equivalence of Weil divisors for locally Noetherian integral schemes. Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining Weil divisors. We have to allow infinite sums of prime divisors because a rational function may have infinitely many poles for example. For quasi-compact schemes our Weil divisors are finite sums as usual. Here is a basic lemma we will often use to prove collections of closed subschemes are locally finite.

0BE1 Lemma 31.26.1. Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. The collection of irreducible components of Z is locally finite in X.

Proof. Let $U \subset X$ be a quasi-compact open subscheme. Then U is a Noetherian scheme, and hence has a Noetherian underlying topological space (Properties, Lemma 28.5.5). Hence every subspace is Noetherian and has finitely many irreducible components (see Topology, Lemma 5.9.2). □

Recall that if Z is an irreducible closed subset of a scheme X, then the codimension of Z in X is equal to the dimension of the local ring $\mathcal{O}_{X,\xi}$, where $\xi \in Z$ is the generic point. See Properties, Lemma 28.10.3.

0BE2 Definition 31.26.2. Let X be a locally Noetherian integral scheme.

(1) A prime divisor is an integral closed subscheme $Z \subset X$ of codimension 1.
(2) A Weil divisor is a formal sum $D = \sum n_Z Z$ where the sum is over prime divisors of X and the collection $\{Z \mid n_Z \neq 0\}$ is locally finite (Topology, Definition 5.28.4).

The group of all Weil divisors on X is denoted $\text{Div}(X)$.

Our next task is to define the Weil divisor associated to a rational function. In order to do this we use the order of vanishing of a rational function along a prime divisor which is defined as follows.

02RJ Definition 31.26.3. Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. For every prime divisor $Z \subset X$ we define the order of vanishing of f along Z as the integer

$$\text{ord}_Z(f) = \text{ord}_{\mathcal{O}_{X,\xi}}(f)$$

where the right hand side is the notion of Algebra, Definition 10.121.2 and ξ is the generic point of Z.

Note that for $f, g \in R(X)^*$ we have

$$\text{ord}_Z(fg) = \text{ord}_Z(f) + \text{ord}_Z(g).$$

Of course it can happen that $\text{ord}_Z(f) < 0$. In this case we say that f has a pole along Z and that $-\text{ord}_Z(f) > 0$ is the order of pole of f along Z. It is important to note that the condition $\text{ord}_Z(f) \geq 0$ is not equivalent to the condition $f \in \mathcal{O}_{X,\xi}$ unless the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring.

02RL Lemma 31.26.4. Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. Then the collections

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } f \text{ not in } \mathcal{O}_{X,\xi}\}$$

and

$$\{Z \subset X \mid Z \text{ a prime divisor and } \text{ord}_Z(f) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subscheme $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$. Hence the prime divisors which can occur in the sets of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 31.26.1 gives the desired result. $\square$

This lemma allows us to make the following definition.

0BE3 Definition 31.26.5. Let X be a locally Noetherian integral scheme. Let $f \in R(X)^*$. The principal Weil divisor associated to f is the Weil divisor

$$\text{div}(f) = \text{div}_X(f) = \sum \text{ord}_Z(f)[Z]$$

where the sum is over prime divisors and $\text{ord}_Z(f)$ is as in Definition 31.26.3. This makes sense by Lemma 31.26.4.

02RP Lemma 31.26.6. Let X be a locally Noetherian integral scheme. Let $f, g \in R(X)^*$. Then

$$\text{div}_X(fg) = \text{div}_X(f) + \text{div}_X(g)$$

as Weil divisors on X.

Proof. This is clear from the additivity of the ord functions. $\square$

We see from the lemma above that the collection of principal Weil divisors form a subgroup of the group of all Weil divisors. This leads to the following definition.

0BE4 Definition 31.26.7. Let X be a locally Noetherian integral scheme. The Weil divisor class group of X is the quotient of the group of Weil divisors by the subgroup of principal Weil divisors. Notation: $\text{Cl}(X)$.

By construction we obtain an exact complex

0BE5 (31.26.7.1)
$$R(X)^* \xrightarrow{\text{div}} \text{Div}(X) \to \text{Cl}(X) \to 0$$

which we can think of as a presentation of $\text{Cl}(X)$. Our next task is to relate the Weil divisor class group to the Picard group.

31.27. The Weil divisor class associated to an invertible module

02SE In this section we go through exactly the same progression as in Section 31.26 to define a canonical map $\text{Pic}(X) \to \text{Cl}(X)$ on a locally Noetherian integral scheme.

Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\xi \in X$ be a point. If $s_\xi, s'_\xi \in \mathcal{L}_\xi$ generate $\mathcal{L}_\xi$ as $\mathcal{O}_{X,\xi}$-module, then there exists a unit $u \in \mathcal{O}_{X,\xi}^*$ such that $s_\xi = us'_\xi$. The stalk of the sheaf of meromorphic sections $\mathcal{K}_X(\mathcal{L})$ of $\mathcal{L}$ at x is equal to $\mathcal{K}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$. Thus the image of any meromorphic section s of $\mathcal{L}$ in the stalk at x can be written as $s = fs_\xi$ with $f \in \mathcal{K}_{X,x}$. Below we will abbreviate this by saying $f = s/s_\xi$. Also, if X is integral we have $\mathcal{K}_{X,x} = R(X)$ is equal to the function field of X, so $s/s_\xi \in R(X)$. If s is a regular meromorphic section, then actually $s/s_\xi \in R(X)^*$. On an integral scheme a regular meromorphic section is the same thing as a nonzero meromorphic section. Finally, we see that s/s_ξ is independent of the choice of s_ξ up to multiplication by a unit of the local ring $\mathcal{O}_{X,x}$. Putting everything together we see the following definition makes sense.

02SF Definition 31.27.1. Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every prime divisor $Z \subset X$ we define the order of vanishing of s along Z as the integer

$$\text{ord}_{Z,\mathcal{L}}(s) = \text{ord}_{\mathcal{O}_{X,\xi}}(s/s_\xi)$$

where the right hand side is the notion of Algebra, Definition 10.121.2, $\xi \in Z$ is the generic point, and $s_\xi \in \mathcal{L}_\xi$ is a generator.

As in the case of principal divisors we have the following lemma.

02SG Lemma 31.27.2. Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the sets

$$\{Z \subset X \mid Z \text{ a prime divisor with generic point } \xi \text{ and } s \text{ not in } \mathcal{L}_\xi\}$$

and

$$\{Z \subset X \mid Z \text{ is a prime divisor and } \text{ord}_{Z,\mathcal{L}}(s) \neq 0\}$$

are locally finite in X.

Proof. There exists a nonempty open subscheme $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the prime divisors which can occur in the sets of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 31.26.1. gives the desired result. $\square$

02SH Lemma 31.27.3. Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have

$$\sum \text{ord}_{Z,\mathcal{L}}(s)[Z] = \sum \text{ord}_{Z,\mathcal{L}}(s')[Z] + \text{div}(f)$$

as Weil divisors.

Proof. This is clear from the definitions. Note that Lemma 31.27.2 guarantees that the sums are indeed Weil divisors. $\square$

0BE6 Definition 31.27.4. Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the Weil divisor associated to s as
$$\text{div}_{\mathcal{L}}(s) = \sum \text{ord}_{Z,\mathcal{L}}(s)[Z] \in \text{Div}(X)$$
where the sum is over prime divisors.
(2) We define Weil divisor class associated to $\mathcal{L}$ as the image of $\text{div}_{\mathcal{L}}(s)$ in $\text{Cl}(X)$ where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 31.27.3.

As expected this construction is additive in the invertible module.

02SL Lemma 31.27.5. Let X be a locally Noetherian integral scheme. Let $\mathcal{L}$, $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$, and

$$\text{div}_{\mathcal{L}\otimes\mathcal{N}}(st) = \text{div}_{\mathcal{L}}(s) + \text{div}_{\mathcal{N}}(t)$$

in $\text{Div}(X)$. In particular, the Weil divisor class of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sum of the Weil divisor classes of $\mathcal{L}$ and $\mathcal{N}$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$. Let $Z \subset X$ be a prime divisor. Let $\xi \in Z$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$. Then $s_\xi t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$. Hence we see that

$$\text{div}_{\mathcal{L}\otimes\mathcal{N},Z}(st) = \text{div}_{\mathcal{L},Z}(s) + \text{div}_{\mathcal{N},Z}(t)$$

by the additivity of the ord_Z function. $\square$

In this way we obtain a homomorphism of abelian groups

0BE7 (31.27.5.1)
$$\text{Pic}(X) \longrightarrow \text{Cl}(X)$$

which assigns to an invertible module its Weil divisor class.

0BE8 Lemma 31.27.6. Let X be a locally Noetherian integral scheme. If X is normal, then the map (31.27.5.1) $\text{Pic}(X) \to \text{Cl}(X)$ is injective.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose associated Weil divisor class is trivial. Let s be a regular meromorphic section of $\mathcal{L}$. The assumption means that $\mathrm{div}_\mathcal{L}(s) = \mathrm{div}(f)$ for some $f \in R(X)^*$. Then we see that $t = f^{-1}s$ is a regular meromorphic section of $\mathcal{L}$ with $\mathrm{div}_\mathcal{L}(t) = 0$, see Lemma 31.27.3. We will show that t defines a trivialization of $\mathcal{L}$ which finishes the proof of the lemma. In order to prove this we may work locally on X. Hence we may assume that $X = \mathrm{Spec}(A)$ is affine and that $\mathcal{L}$ is trivial. Then A is a Noetherian normal domain and t is an element of its fraction field such that $\mathrm{ord}_{A_\mathfrak{p}}(t) = 0$ for all height 1 primes $\mathfrak{p}$ of A. Our goal is to show that t is a unit of A. Since $A_\mathfrak{p}$ is a discrete valuation ring for height one primes of A (Algebra, Lemma 10.157.4), the condition signifies that $t \in A_\mathfrak{p}^*$ for all primes $\mathfrak{p}$ of height 1. This implies $t \in A$ and $t^{-1} \in A$ by Algebra, Lemma 10.157.6 and the proof is complete. □

0BE9 Lemma 31.27.7. Let X be a locally Noetherian integral scheme. Consider the map (31.27.5.1) $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$. The following are equivalent

(1) the local rings of X are UFDs, and
(2) X is normal and $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ is surjective.

In this case $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ is an isomorphism.

Proof. If (1) holds, then X is normal by Algebra, Lemma 10.120.11. Hence the map (31.27.5.1) is injective by Lemma 31.27.6. Moreover, every prime divisor $D \subset X$ is an effective Cartier divisor by Lemma 31.15.7. In this case the canonical section 1_D of $\mathcal{O}_X(D)$ (Definition 31.14.1) vanishes exactly along D and we see that the class of D is the image of $\mathcal{O}_X(D)$ under the map (31.27.5.1). Thus the map is surjective as well.

Assume (2) holds. Pick a prime divisor $D \subset X$. Since (31.27.5.1) is surjective there exists an invertible sheaf $\mathcal{L}$, a regular meromorphic section s, and $f \in R(X)^*$ such that $\mathrm{div}_\mathcal{L}(s) + \mathrm{div}(f) = [D]$. In other words, $\mathrm{div}_\mathcal{L}(fs) = [D]$. Let $x \in X$ and let $A = \mathcal{O}_{X,x}$. Thus A is a Noetherian local normal domain with fraction field $K = R(X)$. Every height 1 prime of A corresponds to a prime divisor on X and every invertible $\mathcal{O}_X$-module restricts to the trivial invertible module on $\mathrm{Spec}(A)$. It follows that for every height 1 prime $\mathfrak{p} \subset A$ there exists an element $f \in K$ such that $\mathrm{ord}_{A_\mathfrak{p}}(f) = 1$ and $\mathrm{ord}_{A_{\mathfrak{p}'}}(f) = 0$ for every other height one prime $\mathfrak{p}'$. Then $f \in A$ by Algebra, Lemma 10.157.6. Arguing in the same fashion we see that every element $g \in \mathfrak{p}$ is of the form $g = af$ for some $a \in A$. Thus we see that every height one prime ideal of A is principal and A is a UFD by Algebra, Lemma 10.120.6. □

31.28. More on invertible modules

0BD6 In this section we discuss some properties of invertible modules.

0BD7 Lemma 31.28.1. Let $\varphi : X \to Y$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume that

(1) X is locally Noetherian,
(2) Y is locally Noetherian, integral, and normal,
(3) φ is flat with integral (hence nonempty) fibres,
(4) φ is either quasi-compact or locally of finite type,
(5) $\mathcal{L}$ is trivial when restricted to the generic fibre of φ.

Then $\mathcal{L} \cong \varphi^*\mathcal{N}$ for some invertible $\mathcal{O}_Y$-module $\mathcal{N}$.

Proof. Let $\xi \in Y$ be the generic point. Let X_ξ be the scheme theoretic fibre of φ over ξ. Denote $\mathcal{L}_\xi$ the pullback of $\mathcal{L}$ to X_ξ. Assumption (5) means that $\mathcal{L}_\xi$ is trivial. Choose a trivializing section $s \in \Gamma(X_\xi, \mathcal{L}_\xi)$. Observe that X is integral by Lemma 31.11.7. Hence we can think of s as a regular meromorphic section of $\mathcal{L}$. Pullbacks of meromorphic functions are defined for φ by Lemma 31.23.5. Let $\mathcal{N} \subset \mathcal{K}_Y$ be the $\mathcal{O}_Y$-module whose sections over an open $V \subset Y$ are those meromorphic functions $g \in \mathcal{K}_Y(V)$ such that $\varphi^*(g)s \in \mathcal{L}(\varphi^{-1}V)$. A priori $\varphi^*(g)s$ is a section of $\mathcal{K}_X(\mathcal{L})$ over $\varphi^{-1}V$. We claim that $\mathcal{N}$ is an invertible $\mathcal{O}_Y$-module and that the map

$$\varphi^*\mathcal{N} \longrightarrow \mathcal{L}, \quad g \longmapsto gs$$

is an isomorphism.

We first prove the claim in the following situation: X and Y are affine and $\mathcal{L}$ trivial. Say $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$ and s given by the element $s \in A \otimes_R K$ where K is the fraction field of R. We can write $s = a/r$ for some nonzero $r \in R$ and $a \in A$. Since s generates $\mathcal{L}$ on the generic fibre we see that there exists an $s' \in A \otimes_R K$ such that $ss' = 1$. Thus we see that $s = r'/a'$ for some nonzero $r' \in R$ and $a' \in A$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset R$ be the minimal primes over rr'. Each $R_{\mathfrak{p}_i}$ is a discrete valuation ring (Algebra, Lemmas 10.60.11 and 10.157.4). By assumption $\mathfrak{q}_i = \mathfrak{p}_i A$ is a prime. Hence $\mathfrak{q}_i A_{\mathfrak{q}_i}$ is generated by a single element and we find that $A_{\mathfrak{q}_i}$ is a discrete valuation ring as well (Algebra, Lemma 10.119.7). Of course $R_{\mathfrak{p}_i} \to A_{\mathfrak{q}_i}$ has ramification index 1. Let $e_i, e_i' \geq 0$ be the valuation of a, a' in $A_{\mathfrak{q}_i}$. Then $e_i + e_i'$ is the valuation of rr' in $R_{\mathfrak{p}_i}$. Note that

$$\mathfrak{p}_1^{(e_1+e_1')} \cap \ldots \cap \mathfrak{p}_i^{(e_n+e_n')} = (rr')$$

in R by Algebra, Lemma 10.157.6. Set

$$I = \mathfrak{p}_1^{(e_1)} \cap \ldots \cap \mathfrak{p}_i^{(e_n)} \quad \text{and} \quad I' = \mathfrak{p}_1^{(e_1')} \cap \ldots \cap \mathfrak{p}_i^{(e_n')}$$

so that $II' \subset (rr')$. Observe that

$$IA = (\mathfrak{p}_1^{(e_1)} \cap \ldots \cap \mathfrak{p}_i^{(e_n)})A = (\mathfrak{p}_1 A)^{(e_1)} \cap \ldots \cap (\mathfrak{p}_i A)^{(e_n)}$$

by Algebra, Lemmas 10.64.3 and 10.39.2. Similarly for $I'A$. Hence $a \in IA$ and $a' \in I'A$. We conclude that $IA \otimes_A I'A \to rr'A$ is surjective. By faithful flatness of $R \to A$ we find that $I \otimes_R I' \to (rr')$ is surjective as well. It follows that $II' = (rr')$ and I and I' are finite locally free of rank 1, see Algebra, Lemma 10.120.16. Thus Zariski locally on R we can write $I = (g)$ and $I' = (g')$ with $gg' = rr'$. Then $a = ug$ and $a' = u'g'$ for some $u, u' \in A$. We conclude that u, u' are units. Thus Zariski locally on R we have $s = ug/r$ and the claim follows in this case.

Let $y \in Y$ be a point. Pick $x \in X$ mapping to y. We may apply the result of the previous paragraph to $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$. We conclude there exists an element $g \in R(Y)^*$ well defined up to multiplication by an element of $\mathcal{O}_{Y,y}^*$ such that $\varphi^*(g)s$ generates $\mathcal{L}_x$. Hence $\varphi^*(g)s$ generates $\mathcal{L}$ in a neighbourhood U of x. Suppose x' is a second point lying over y and $g' \in R(Y)^*$ is such that $\varphi^*(g')s$ generates $\mathcal{L}$ in an open neighbourhood U' of x'. Then we can choose a point x'' in $U \cap U' \cap \varphi^{-1}(\{y\})$ because the fibre is irreducible. By the uniqueness for the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x''}$ we find that g and g' differ (multiplicatively) by an element in $\mathcal{O}_{Y,y}^*$. Hence we see that $\varphi^*(g)s$ is a generator for $\mathcal{L}$ on an open neighbourhood of $\varphi^{-1}(y)$. Let $Z \subset X$ be the set of points $z \in X$ such that $\varphi^*(g)s$ does not generate $\mathcal{L}_z$. The arguments above show that Z is closed and that $Z = \varphi^{-1}(T)$ for some subset $T \subset Y$ with $y \notin T$. If we can show that T is closed, then g will be a generator for $\mathcal{N}$ as an $\mathcal{O}_Y$-module in the open neighbourhood $Y \setminus T$ of y thereby finishing the proof (some details omitted).

If φ is quasi-compact, then T is closed by Morphisms, Lemma 29.25.12. If φ is locally of finite type, then φ is open by Morphisms, Lemma 29.25.10. Then $Y \setminus T$ is open as the image of the open $X \setminus Z$. $\square$

0BD8 Lemma 31.28.2. Let X be a locally Noetherian scheme. Let $U \subset X$ be an open and let $D \subset U$ be an effective Cartier divisor. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D' \subset X$ with $D = U \cap D'$.

Proof. Let $D' \subset X$ be the scheme theoretic image of the morphism $D \to X$. Since X is locally Noetherian the morphism $D \to X$ is quasi-compact, see Properties, Lemma 28.5.3. Hence the formation of D' commutes with passing to opens in X by Morphisms, Lemma 29.6.3. Thus we may assume $X = \mathrm{Spec}(A)$ is affine. Let $I \subset A$ be the ideal corresponding to D'. Let $\mathfrak{p} \subset A$ be a prime ideal corresponding to a point of $X \setminus U$. To finish the proof it is enough to show that $I_\mathfrak{p}$ is generated by one element, see Lemma 31.15.2. Thus we may replace X by $\mathrm{Spec}(A_\mathfrak{p})$, see Morphisms, Lemma 29.25.16. In other words, we may assume that X is the spectrum of a local UFD A. Then all local rings of A are UFD's. It follows that $D = \sum a_i D_i$ with $D_i \subset U$ an integral effective Cartier divisor, see Lemma 31.15.11. The generic points ξ_i of D_i correspond to prime ideals $\mathfrak{p}_i \subset A$ of height 1, see Lemma 31.15.3. Then $\mathfrak{p}_i = (f_i)$ for some prime element $f_i \in A$ and we conclude that D' is cut out by $\prod f_i^{a_i}$ as desired. $\square$

0BD9 Lemma 31.28.3. Let X be a locally Noetherian scheme. Let $U \subset X$ be an open and let $\mathcal{L}$ be an invertible $\mathcal{O}_U$-module. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}'$ with $\mathcal{L} \cong \mathcal{L}'|_U$.

Proof. Choose $x \in X$, $x \notin U$. We will show there exists an affine open neighbourhood $W \subset X$, such that $\mathcal{L}|_{W \cap U}$ extends to an invertible sheaf on W. This implies by glueing of sheaves (Sheaves, Section 6.33) that we can extend $\mathcal{L}$ to the strictly bigger open $U \cup W$. Let $W = \mathrm{Spec}(A)$ be an affine open neighbourhood. Since $U \cap W$ is quasi-affine, we see that we can write $\mathcal{L}|_{W \cap U}$ as $\mathcal{O}(D_1) \otimes \mathcal{O}(D_2)^{\otimes -1}$ for some effective Cartier divisors $D_1, D_2 \subset W \cap U$, see Lemma 31.15.12. Then D_1 and D_2 extend to effective Cartier divisors of W by Lemma 31.28.2 which gives us the extension of the invertible sheaf.

If X is Noetherian (which is the case most used in practice), the above combined with Noetherian induction finishes the proof. In the general case we argue as follows. First, because every local ring of a point outside of U is a domain and X is locally Noetherian, we see that the closure of U in X is open. Thus we may assume that $U \subset X$ is dense and schematically dense. Now we consider the set T of triples $(U', \mathcal{L}', \alpha)$ where $U \subset U' \subset X$ is an open subscheme, $\mathcal{L}'$ is an invertible $\mathcal{O}_{U'}$-module, and $\alpha : \mathcal{L}'|_U \to \mathcal{L}$ is an isomorphism. We endow T with a partial ordering $\leq$ defined by the rule $(U', \mathcal{L}', \alpha) \leq (U'', \mathcal{L}'', \alpha')$ if and only if $U' \subset U''$ and there exists an isomorphism $\beta : \mathcal{L}''|_{U'} \to \mathcal{L}'$ compatible with α and α'. Observe that β is unique (if it exists) because $U \subset X$ is dense. The first part of the proof shows that for any element $t = (U', \mathcal{L}', \alpha)$ of T with $U' \neq X$ there exists a $t' \in T$ with $t' > t$. Hence to finish the proof it suffices to show that Zorn's lemma applies. Thus consider a totally ordered subset $I \subset T$. If $i \in I$ corresponds to the triple

$(U_i, \mathcal{L}_i, \alpha_i)$, then we can construct an invertible module $\mathcal{L}'$ on $U' = \bigcup U_i$ as follows. For $W \subset U'$ open and quasi-compact we see that $W \subset U_i$ for some i and we set

$$\mathcal{L}'(W) = \mathcal{L}_i(W)$$

For the transition maps we use the β's (which are unique and hence compose correctly). This defines an invertible $\mathcal{O}$-module $\mathcal{L}'$ on the basis of quasi-compact opens of U' which is sufficient to define an invertible module (Sheaves, Section 6.30). We omit the details. $\square$

0BDA Lemma 31.28.4. Let R be a UFD. The Picard groups of the following are trivial.

(1) $\mathrm{Spec}(R)$ and any open subscheme of it.
(2) $\mathbf{A}^n_R = \mathrm{Spec}(R[x_1, \ldots, x_n])$ and any open subscheme of it.

In particular, the Picard group of any open subscheme of affine n-space $\mathbf{A}^n_k$ over a field k is trivial.

Proof. Since R is a UFD so is any localization of it and any polynomial ring over it (Algebra, Lemma 10.120.10). Thus if $U \subset \mathbf{A}^n_R$ is open, then the map $\mathrm{Pic}(\mathbf{A}^n_R) \to \mathrm{Pic}(U)$ is surjective by Lemma 31.28.3. The vanishing of $\mathrm{Pic}(\mathbf{A}^n_R)$ is equivalent to the vanishing of the picard group of the UFD $R[x_1, \ldots, x_n]$ which is proved in More on Algebra, Lemma 15.117.3. $\square$

0BXJ Lemma 31.28.5. Let R be a UFD. The Picard group of $\mathbf{P}^n_R$ is $\mathbf{Z}$. More precisely, there is an isomorphism

$$\mathbf{Z} \longrightarrow \mathrm{Pic}(\mathbf{P}^n_R), \quad m \longmapsto \mathcal{O}_{\mathbf{P}^n_R}(m)$$

In particular, the Picard group of $\mathbf{P}^n_k$ of projective space over a field k is $\mathbf{Z}$.

Proof. Observe that the local rings of $X = \mathbf{P}^n_R$ are UFDs because X is covered by affine pieces isomorphic to $\mathbf{A}^n_R$ and $R[x_1, \ldots, x_n]$ is a UFD (Algebra, Lemma 10.120.10). Hence X is an integral Noetherian scheme all of whose local rings are UFDs and we see that $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ by Lemma 31.27.7.

The displayed map is a group homomorphism by Constructions, Lemma 27.10.3. The map is injective because H^0 of $\mathcal{O}_X$ and $\mathcal{O}_X(m)$ are non-isomorphic R-modules if $m > 0$, see Cohomology of Schemes, Lemma 30.8.1. Let $\mathcal{L}$ be an invertible module on X. Consider the open $U = D_+(T_0) \cong \mathbf{A}^n_R$. The complement $H = X \setminus U$ is a prime divisor because it is isomorphic to $\mathrm{Proj}(R[T_1, \ldots, T_n])$ which is integral by the discussion in the previous paragraph. In fact H is the zero scheme of the regular global section T_0 of $\mathcal{O}_X(1)$ hence $\mathcal{O}_X(1)$ maps to the class of H in $\mathrm{Cl}(X)$. By Lemma 31.28.4 we see that $\mathcal{L}|_U \cong \mathcal{O}_U$. Let $s \in \mathcal{L}(U)$ be a trivializing section. Then we can think of s as a regular meromorphic section of $\mathcal{L}$ and we see that necessarily $\mathrm{div}_\mathcal{L}(s) = m[H]$ for some $m \in \mathbf{Z}$ as H is the only prime divisor of X not meeting U. In other words, we see that $\mathcal{L}$ and $\mathcal{O}_X(m)$ map to the same element of $\mathrm{Cl}(X)$ and hence $\mathcal{L} \cong \mathcal{O}_X(m)$ as desired. $\square$

31.29. Weil divisors on normal schemes

0EBK First we discuss properties of reflexive modules.

0EBL Lemma 31.29.1. Let X be an integral locally Noetherian normal scheme. For $\mathcal{F}$ and $\mathcal{G}$ coherent reflexive $\mathcal{O}_X$-modules the map

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{G})^{**} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism. The rule $\mathcal{F}, \mathcal{G} \mapsto (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{**}$ defines an abelian group law on the set of isomorphism classes of rank 1 coherent reflexive $\mathcal{O}_X$-modules.

Proof. Although not strictly necessary, we recommend reading Remark 31.12.9 before proceeding with the proof. Choose an open subscheme $j : U \to X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and such that $j^*\mathcal{F}$ and $j^*\mathcal{G}$ are finite locally free, see Lemma 31.12.13. The map

$$\mathcal{H}om_{\mathcal{O}_U}(j^*\mathcal{F}, \mathcal{O}_U) \otimes_{\mathcal{O}_U} j^*\mathcal{G} \to \mathcal{H}om_{\mathcal{O}_U}(j^*\mathcal{F}, j^*\mathcal{G})$$

is an isomorphism, because we may check it locally and it is clear when the modules are finite free. Observe that j^* applied to the displayed arrow of the lemma gives the arrow we've just shown is an isomorphism (small detail omitted). Since j^* defines an equivalence between coherent reflexive modules on U and coherent reflexive modules on X (by Lemma 31.12.12 and Serre's criterion Properties, Lemma 28.12.5), we conclude that the arrow of the lemma is an isomorphism too. If $\mathcal{F}$ has rank 1, then $j^*\mathcal{F}$ is an invertible $\mathcal{O}_U$-module and the reflexive module $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ restricts to its inverse. It follows in the same manner as before that $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee)^{**} = \mathcal{O}_X$. In this way we see that we have inverses for the group law given in the statement of the lemma. $\square$

0EBM Lemma 31.29.2. Let X be an integral locally Noetherian normal scheme. The group of rank 1 coherent reflexive $\mathcal{O}_X$-modules is isomorphic to the Weil divisor class group $\mathrm{Cl}(X)$ of X.

Proof. Let $\mathcal{F}$ be a rank 1 coherent reflexive $\mathcal{O}_X$-module. Choose an open $U \subset X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and such that $\mathcal{F}|_U$ is invertible, see Lemma 31.12.13. Observe that $\mathrm{Cl}(U) = \mathrm{Cl}(X)$ as the Weil divisor class group of X only depends on its field of rational functions and the points of codimension 1 and their local rings. Thus we can define the Weil divisor class of $\mathcal{F}$ to be the Weil divisor class of $\mathcal{F}|_U$ in $\mathrm{Cl}(U)$. We omit the verification that this is independent of the choice of U.

Denote $\mathrm{Cl}'(X)$ the set of isomorphism classes of rank 1 coherent reflexive $\mathcal{O}_X$-modules. The construction above gives a group homorphism

$$\mathrm{Cl}'(X) \longrightarrow \mathrm{Cl}(X)$$

because for any pair $\mathcal{F}, \mathcal{G}$ of elements of $\mathrm{Cl}'(X)$ we can choose a U which works for both and the assignment (31.27.5.1) sending an invertible module to its Weil divisor class is a homorphism. If $\mathcal{F}$ is in the kernel of this map, then we find that $\mathcal{F}|_U$ is trivial (Lemma 31.27.6) and hence $\mathcal{F}$ is trivial too by Lemma 31.12.12 and Serre's criterion Properties, Lemma 28.12.5. To finish the proof it suffices to check the map is surjective.

Let $D = \sum n_Z Z$ be a Weil divisor on X. We claim that there is an open $U \subset X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X and such that $Z|_U$ is an effective Cartier divisor for $n_Z \neq 0$. To prove the claim we may assume X is affine. Then we may assume $D = n_1 Z_1 + \ldots + n_r Z_r$ is a finite sum with $Z_1, \ldots, Z_r$ pairwise distinct. After throwing out $Z_i \cap Z_j$ for $i \neq j$ we may assume $Z_1, \ldots, Z_r$ are pairwise disjoint. This reduces us to the case of a single prime divisor Z on X. As X is (R_1) by Properties, Lemma 28.12.5 the local ring $\mathcal{O}_{X,\xi}$ at the generic point ξ of Z is a discrete valuation ring. Let $f \in \mathcal{O}_{X,\xi}$ be a uniformizer. Let $V \subset X$ be an open neighbourhood of ξ such that f is the image of an element $f \in \mathcal{O}_X(V)$. After shrinking V we may assume that $Z \cap V = V(f)$ scheme theoretically, since this is true in the local ring at ξ. In this case taking

$$U = X \setminus (Z \setminus V) = (X \setminus Z) \cup V$$

gives the desired open, thereby proving the claim.

In order to show that the divisor class of D is in the image, we may write $D = \sum_{n_Z<0} n_Z Z - \sum_{n_Z>0}(-n_Z)Z$. By additivity of the map constructed above, we may and do assume $n_Z \leq 0$ for all prime divisors Z (this step may be avoided if the reader so desires). Let $U \subset X$ be as in the claim above. If U is quasi-compact, then we write $D|_U = -n_1 Z_1 - \ldots - n_r Z_r$ for pairwise distinct prime divisors Z_i and $n_i > 0$ and we consider the invertible $\mathcal{O}_U$-module

$$\mathcal{L} = \mathcal{I}_1^{n_1} \ldots \mathcal{I}_r^{n_r} \subset \mathcal{O}_U$$

where $\mathcal{I}_i$ is the ideal sheaf of Z_i. This is invertible by our choice of U and Lemma 31.13.7. Also $\text{div}_\mathcal{L}(1) = D|_U$. Since $\mathcal{L} = \mathcal{F}|_U$ for some rank 1 coherent reflexive $\mathcal{O}_X$-module $\mathcal{F}$ by Lemma 31.12.12 we find that D is in the image of our map.

If U is not quasi-compact, then we define $\mathcal{L} \subset \mathcal{O}_U$ locally by the displayed formula above. The reader shows that the construction glues and finishes the proof exactly as before. Details omitted. □

0EBN Lemma 31.29.3. Let X be an integral locally Noetherian normal scheme. Let $\mathcal{F}$ be a rank 1 coherent reflexive $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{F})$. Let

$$U = \{x \in X \mid s : \mathcal{O}_{X,x} \to \mathcal{F}_x \text{ is an isomorphism}\}$$

Then $j : U \to X$ is an open subscheme of X and

$$j_*\mathcal{O}_U = \text{colim}(\mathcal{O}_X \xrightarrow{s} \mathcal{F} \xrightarrow{s} \mathcal{F}^{[2]} \xrightarrow{s} \mathcal{F}^{[3]} \xrightarrow{s} \ldots)$$

where $\mathcal{F}^{[1]} = \mathcal{F}$ and inductively $\mathcal{F}^{[n+1]} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^{[n]})^{**}$.

Proof. The set U is open by Modules, Lemmas 17.9.4 and 17.12.6. Observe that j is quasi-compact by Properties, Lemma 28.5.3. To prove the final statement it suffices to show for every quasi-compact open $W \subset X$ there is an isomorphism

$$\text{colim}\, \Gamma(W, \mathcal{F}^{[n]}) \longrightarrow \Gamma(U \cap W, \mathcal{O}_U)$$

of $\mathcal{O}_X(W)$-modules compatible with restriction maps. We will omit the verification of compatibilities. After replacing X by W and rewriting the above in terms of homs, we see that it suffices to construct an isomorphism

$$\text{colim}\, \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}^{[n]}) \longrightarrow \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U)$$

Choose an open $V \subset X$ such that every irreducible component of $X \setminus V$ has codimension ≥ 2 in X and such that $\mathcal{F}|_V$ is invertible, see Lemma 31.12.13. Then restriction defines an equivalence of categories between rank 1 coherent reflexive modules on X and V and between rank 1 coherent reflexive modules on U and $V \cap U$. See Lemma 31.12.12 and Serre's criterion Properties, Lemma 28.12.5. Thus it suffices to construct an isomorphism

$$\text{colim}\, \Gamma(V, (\mathcal{F}|_V)^{\otimes n}) \longrightarrow \Gamma(V \cap U, \mathcal{O}_U)$$

Since $\mathcal{F}|_V$ is invertible and since $U \cap V$ is equal to the set of points where $s|_V$ generates this invertible module, this is a special case of Properties, Lemma 28.17.2 (there is an explicit formula for the map as well). □

0EBP Lemma 31.29.4. Assumptions and notation as in Lemma 31.29.3. If s is nonzero, then every irreducible component of $X \setminus U$ has codimension 1 in X.

Proof. Let $\xi \in X$ be a generic point of an irreducible component Z of $X \setminus U$. After replacing X by an open neighbourhood of ξ we may assume that $Z = X \setminus U$ is irreducible. Since $s : \mathcal{O}_U \to \mathcal{F}|_U$ is an isomorphism, if the codimension of Z in X is ≥ 2, then $s : \mathcal{O}_X \to \mathcal{F}$ is an isomorphism by Lemma 31.12.12 and Serre's criterion Properties, Lemma 28.12.5. This would mean that $Z = \emptyset$, a contradiction. □

0EBQ Remark 31.29.5. Let A be a Noetherian normal domain. Let M be a rank 1 finite reflexive A-module. Let $s \in M$ be nonzero. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height 1 primes of A in the support of M/As. Then the open U of Lemma 31.29.3 is

$$U = \text{Spec}(A) \setminus (V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_r))$$

by Lemma 31.29.4. Moreover, if $M^{[n]}$ denotes the reflexive hull of $M \otimes_A \ldots \otimes_A M$ (n-factors), then

$$\Gamma(U, \mathcal{O}_U) = \text{colim}\, M^{[n]}$$

according to Lemma 31.29.3.

0EBR Lemma 31.29.6. Assumptions and notation as in Lemma 31.29.3. The following are equivalent

(1) the inclusion morphism $j : U \to X$ is affine, and
(2) for every $x \in X \setminus U$ there is an $n > 0$ such that $s^n \in \mathfrak{m}_x \mathcal{F}_x^{[n]}$.

Proof. Assume (1). Then for $x \in X \setminus U$ the inverse image U_x of U under the canonical morphism $f_x : \text{Spec}(\mathcal{O}_{X,x}) \to X$ is affine and does not contain x. Thus $\mathfrak{m}_x \Gamma(U_x, \mathcal{O}_{U_x})$ is the unit ideal. In particular, we see that we can write

$$1 = \sum f_i g_i$$

with $f_i \in \mathfrak{m}_x$ and $g_i \in \Gamma(U_x, \mathcal{O}_{U_x})$. By Lemma 31.29.3 we have $\Gamma(U_x, \mathcal{O}_{U_x}) = \text{colim}\, \mathcal{F}_x^{[n]}$ with transition maps given by multiplication by s. Hence for some $n > 0$ we have

$$s^n = \sum f_i t_i$$

for some $t_i = s^n g_i \in \mathcal{F}_x^{[n]}$. Thus (2) holds.

Conversely, assume that (2) holds. To prove j is affine is local on X, see Morphisms, Lemma 29.11.3. Thus we may and do assume that X is affine. Our goal is to show that U is affine. By Cohomology of Schemes, Lemma 30.17.8 it suffices to show that $H^p(U, \mathcal{O}_U) = 0$ for $p > 0$. Since $H^p(U, \mathcal{O}_U) = H^0(X, R^p j_* \mathcal{O}_U)$ (Cohomology of Schemes, Lemma 30.4.6) and since $R^p j_* \mathcal{O}_U$ is quasi-coherent (Cohomology of Schemes, Lemma 30.4.5) it is enough to show the stalk $(R^p j_* \mathcal{O}_U)_x$ at a point $x \in X$ is zero. Consider the base change diagram

$$\begin{tikzcd} U_x \arrow[r] \arrow[d, "j_x"'] & U \arrow[d, "j"] \\ \text{Spec}(\mathcal{O}_{X,x}) \arrow[r] & X \end{tikzcd}$$

By Cohomology of Schemes, Lemma 30.5.2 we have $(R^p j_* \mathcal{O}_U)_x = R^p j_{x,*} \mathcal{O}_{U_x}$. Hence we may assume X is local with closed point x and we have to show U is affine (because this is equivalent to the desired vanishing by the reference given above). In particular $d = \dim(X)$ is finite (Algebra, Proposition 10.60.9). If $x \in U$, then $U = X$ and the result is clear. If $d = 0$ and $x \notin U$, then $U = \emptyset$ and the result is clear. Now assume $d > 0$ and $x \notin U$. Since $j_*\mathcal{O}_U = \operatorname{colim} \mathcal{F}^{[n]}$ our assumption means that we can write

$$1 = \sum f_i g_i$$

for some $n > 0$, $f_i \in \mathfrak{m}_x$, and $g_i \in \mathcal{O}(U)$. By induction on d we know that $D(f_i) \cap U$ is affine for all i: going through the whole argument just given with X replaced by $D(f_i)$ we end up with Noetherian local rings whose dimension is strictly smaller than d. Hence U is affine by Properties, Lemma 28.27.3 as desired. $\square$

31.30. Relative Proj

07ZW Some results on relative Proj. First some very basic results. Recall that a relative Proj is always separated over the base, see Constructions, Lemma 27.16.9.

07ZX **Lemma 31.30.1.** Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If one of the following holds

(1) $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,
(2) $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,
(3) there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,

then p is quasi-compact.

Proof. The question is local on the base, see Schemes, Lemma 26.19.2. Thus we may assume S is affine. Say $S = \text{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. Then $X = \text{Proj}(A)$, see Constructions, Section 27.15. In case (1) we may after possibly localizing more assume that A is generated by homogeneous elements $f_1, \ldots, f_n \in A_+$ over A_0. Then $A_+ = (f_1, \ldots, f_n)$ by Algebra, Lemma 10.58.1. In case (3) we see that $\mathcal{F} = \widetilde{M}$ for some finite type A_0-module $M \subset A_+$. Say $M = \sum A_0 f_i$. Say $f_i = \sum f_{i,j}$ is the decomposition into homogeneous pieces. The condition in (3) signifies that $A_+ \subset \sqrt{(f_{i,j})}$. Thus in both cases we conclude that $\text{Proj}(A)$ is quasi-compact by Constructions, Lemma 27.8.9. Finally, (2) follows from (1). $\square$

07ZY **Lemma 31.30.2.** Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_S$-algebras, then p is of finite type and $\mathcal{O}_X(d)$ is a finite type $\mathcal{O}_X$-module.

Proof. The assumption implies that p is quasi-compact, see Lemma 31.30.1. Hence it suffices to show that p is locally of finite type. Thus the question is local on the base and target, see Morphisms, Lemma 29.15.2. Say $S = \text{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. After further localizing on S we may assume that A is a finite type R-algebra. The scheme X is constructed out of glueing the spectra of the rings $A_{(f)}$ for $f \in A_+$ homogeneous. Each of these is of finite type over R by Algebra, Lemma 10.57.9 part (1). Thus $\text{Proj}(A)$ is of finite type over R. To see the statement on $\mathcal{O}_X(d)$ use part (2) of Algebra, Lemma 10.57.9. $\square$

07ZZ Lemma 31.30.3. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{O}_S \to \mathcal{A}_0$ is an integral algebra map[4] and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then p is universally closed.

Proof. The question is local on the base. Thus we may assume that $X = \text{Spec}(R)$ is affine. Let $\mathcal{A}$ be the quasi-coherent $\mathcal{O}_X$-algebra associated to the graded R-algebra A. The assumption is that $R \to A_0$ is integral and A is of finite type over A_0. Write $X \to \text{Spec}(R)$ as the composition $X \to \text{Spec}(A_0) \to \text{Spec}(R)$. Since $R \to A_0$ is an integral ring map, we see that $\text{Spec}(A_0) \to \text{Spec}(R)$ is universally closed, see Morphisms, Lemma 29.44.7. The quasi-compact (see Constructions, Lemma 27.8.9) morphism

$$X = \text{Proj}(A) \to \text{Spec}(A_0)$$

satisfies the existence part of the valuative criterion by Constructions, Lemma 27.8.11 and hence it is universally closed by Schemes, Proposition 26.20.6. Thus $X \to \text{Spec}(R)$ is universally closed as a composition of universally closed morphisms. □

0800 Lemma 31.30.4. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. The following conditions are equivalent

(1) $\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,
(2) $\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_S$-algebra

If these conditions hold, then p is locally projective and in particular proper.

Proof. Assume that $\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module. Choose an affine open $U = \text{Spec}(R) \subset X$ such that $\mathcal{A}$ corresponds to a graded R-algebra A with A_0 a finite R-module. Condition (1) means that (after possibly localizing further on S) that A is a finite type A_0-algebra and condition (2) means that (after possibly localizing further on S) that A is a finite type R-algebra. Thus these conditions imply each other by Algebra, Lemma 10.6.2.

A locally projective morphism is proper, see Morphisms, Lemma 29.43.5. Thus we may now assume that $S = \text{Spec}(R)$ and $X = \text{Proj}(A)$ and that A_0 is finite over R and A of finite type over R. We will show that $X = \text{Proj}(A) \to \text{Spec}(R)$ is projective. We urge the reader to prove this for themselves, by directly constructing a closed immersion of X into a projective space over R, instead of reading the argument we give below.

By Lemma 31.30.2 we see that X is of finite type over $\text{Spec}(R)$. Constructions, Lemma 27.10.6 tells us that $\mathcal{O}_X(d)$ is ample on X for some $d \geq 1$ (see Properties, Section 28.26). Hence $X \to \text{Spec}(R)$ is quasi-projective (by Morphisms, Definition 29.40.1). By Morphisms, Lemma 29.43.12 we conclude that X is isomorphic to an open subscheme of a scheme projective over $\text{Spec}(R)$. Therefore, to finish the proof, it suffices to show that $X \to \text{Spec}(R)$ is universally closed (use Morphisms, Lemma 29.41.7). This follows from Lemma 31.30.3. □

0B3U Lemma 31.30.5. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$ and $\mathcal{A}_1$ is a finite type $\mathcal{O}_S$-module, then p is projective.

[4] In other words, the integral closure of $\mathcal{O}_S$ in $\mathcal{A}_0$, see Morphisms, Definition 29.53.2, equals $\mathcal{A}_0$.

Proof. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\text{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \to \mathbf{P}(\mathcal{A}_1)$, see Constructions, Lemma 27.18.5. $\square$

0D4C Lemma 31.30.6. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}_d$ is a flat $\mathcal{O}_S$-module for $d \gg 0$, then p is flat and $\mathcal{O}_X(d)$ is flat over S.

Proof. Affine locally flatness of X over S reduces to the following statement: Let R be a ring, let A be a graded R-algebra with A_d flat over R for $d \gg 0$, let $f \in A_d$ for some $d > 0$, then $A_{(f)}$ is flat over R. Since $A_{(f)} = \text{colim}\, A_{nd}$ where the transition maps are given by multiplication by f, this follows from Algebra, Lemma 10.39.3. Argue similarly to get flatness of $\mathcal{O}_X(d)$ over S. $\square$

0D4D Lemma 31.30.7. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is a finitely presented $\mathcal{O}_S$-algebra, then p is of finite presentation and $\mathcal{O}_X(d)$ is an $\mathcal{O}_X$-module of finite presentation.

Proof. Affine locally this reduces to the following statement: Let R be a ring and let A be a finitely presented graded R-algebra. Then $\text{Proj}(A) \to \text{Spec}(R)$ is of finite presentation and $\mathcal{O}_{\text{Proj}(A)}(d)$ is a $\mathcal{O}_{\text{Proj}(A)}$-module of finite presentation. The finite presentation condition implies we can choose a presentation

$$A = R[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$$

where $R[X_1, \ldots, X_n]$ is a polynomial ring graded by giving weights d_i to X_i and $F_1, \ldots, F_m$ are homogeneous polynomials of degree e_j. Let $R_0 \subset R$ be the subring generated by the coefficients of the polynomials $F_1, \ldots, F_m$. Then we set $A_0 = R_0[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$. By construction $A = A_0 \otimes_{R_0} R$. Thus by Constructions, Lemma 27.11.6 it suffices to prove the result for $X_0 = \text{Proj}(A_0)$ over R_0. By Lemma 31.30.2 we know X_0 is of finite type over R_0 and $\mathcal{O}_{X_0}(d)$ is a quasi-coherent $\mathcal{O}_{X_0}$-module of finite type. Since R_0 is Noetherian (as a finitely generated $\mathbf{Z}$-algebra) we see that X_0 is of finite presentation over R_0 (Morphisms, Lemma 29.21.9) and $\mathcal{O}_{X_0}(d)$ is of finite presentation by Cohomology of Schemes, Lemma 30.9.1. This finishes the proof. $\square$

31.31. Closed subschemes of relative proj

084M Some auxiliary lemmas about closed subschemes of relative proj.

0801 Lemma 31.31.1. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map

$$\mathcal{A} \longrightarrow \bigoplus\nolimits_{d \geq 0} p_*\left((i_*\mathcal{O}_Z)(d)\right).$$

If p is quasi-compact, then there is an isomorphism $Z = \underline{\text{Proj}}_S(\mathcal{A}/\mathcal{I})$.

Proof. The morphism p is separated by Constructions, Lemma 27.16.9. As p is quasi-compact, p_* transforms quasi-coherent modules into quasi-coherent modules, see Schemes, Lemma 26.24.1. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_S$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. The functoriality morphism

$Z' = \underline{\text{Proj}}_S(\mathcal{B}) \to \underline{\text{Proj}}_S(\mathcal{A})$ is everywhere defined and a closed immersion, see Constructions, Lemma 27.18.3. Hence it suffices to prove $Z = Z'$ as closed subschemes of X.

Having said this, the question is local on the base and we may assume that $S = \text{Spec}(R)$ and that $X = \text{Proj}(A)$ for some graded R-algebra A. Assume $\mathcal{I} = \widetilde{I}$ for $I \subset A$ a graded ideal. By Constructions, Lemma 27.8.9 there exist $f_0, \ldots, f_n \in A_+$ such that $A_+ \subset \sqrt{(f_0, \ldots, f_n)}$ in other words $X = \bigcup D_+(f_i)$. Therefore, it suffices to check that $Z \cap D_+(f_i) = Z' \cap D_+(f_i)$ for each i. By renumbering we may assume $i = 0$. Say $Z \cap D_+(f_0)$, resp. $Z' \cap D_+(f_0)$ is cut out by the ideal J, resp. J' of $A_{(f_0)}$.

The inclusion $J' \subset J$. Let d be the least common multiple of $\deg(f_0), \ldots, \deg(f_n)$. Note that each of the twists $\mathcal{O}_X(nd)$ is invertible, trivialized by $f_i^{nd/\deg(f_i)}$ over $D_+(f_i)$, and that for any quasi-coherent module $\mathcal{F}$ on X the multiplication maps $\mathcal{O}_X(nd) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \to \mathcal{F}(nd + m)$ are isomorphisms, see Constructions, Lemma 27.10.2. Observe that J' is the ideal generated by the elements g/f_0^e where $g \in I$ is homogeneous of degree $e \deg(f_0)$ (see proof of Constructions, Lemma 27.11.3). Of course, by replacing g by $f_0^l g$ for suitable l we may always assume that $d|e$. Then, since g vanishes as a section of $\mathcal{O}_X(e\deg(f_0))$ restricted to Z we see that g/f_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $g/f_0^e \in J$. Again we may assume $d|e$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(f_i)$ is cut out by some ideal $J_i \subset A_{(f_i)}$. Moreover,

$$J \cdot A_{(f_0 f_i)} = J_i \cdot A_{(f_0 f_i)}.$$

The right hand side is the localization of J_i with respect to $f_0^{\deg(f_i)}/f_i^{\deg(f_0)}$. It follows that

$$f_0^{e_i} g / f_i^{(e_i + e)\deg(f_0)/\deg(f_i)} \in J_i$$

for some $e_i \gg 0$ sufficiently divisible. This proves that $f_0^{\max(e_i)} g$ is an element of I, because its restriction to each affine open $D_+(f_i)$ vanishes on the closed subscheme $Z \cap D_+(f_i)$. Hence $g/f_0^e \in J'$ and we conclude $J \subset J'$ as desired. $\square$

0BXK Example 31.31.2. Let A be a graded ring. Let $X = \text{Proj}(A)$ and $S = \text{Spec}(A_0)$. Given a graded ideal $I \subset A$ we obtain a closed subscheme $V_+(I) = \text{Proj}(A/I) \to X$ by Constructions, Lemma 27.11.3. Translating the result of Lemma 31.31.1 we see that if X is quasi-compact, then any closed subscheme Z is of the form $V_+(I(Z))$ where the graded ideal $I(Z) \subset A$ is given by the rule

$$I(Z) = \text{Ker}(A \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(Z, \mathcal{O}_Z(n)))$$

Then we can ask the following two natural questions:

(1) Which ideals I are of the form $I(Z)$?
(2) Can we describe the operation $I \mapsto I(V_+(I))$?

We will answer this when A is Noetherian.

First, assume that A is generated by A_1 over A_0. In this case, for any ideal $I \subset A$ the kernel of the map $A/I \to \bigoplus \Gamma(\text{Proj}(A/I), \mathcal{O})$ is the set of torsion elements of A/I, see Cohomology of Schemes, Proposition 30.14.4. Hence we conclude that

$$I(V_+(I)) = \{x \in A \mid A_n x \subset I \text{ for some } n \geq 0\}$$

The ideal on the right is sometimes called the saturation of I. This answers (2) and the answer to (1) is that an ideal is of the form $I(Z)$ if and only if it is saturated, i.e., equal to its own saturation.

If A is a general Noetherian graded ring, then we use Cohomology of Schemes, Proposition 30.15.3. Thus we see that for d equal to the lcm of the degrees of generators of A over A_0 we get

$$I(V_+(I)) = \{x \in A \mid (Ax)_{nd} \subset I \text{ for all } n \gg 0\}$$

This can be different from the saturation of I if $d \neq 1$. For example, suppose that $A = \mathbf{Q}[x, y]$ with $\deg(x) = 2$ and $\deg(y) = 3$. Then $d = 6$. Let $I = (y^2)$. Then we see $y \in I(V_+(I))$ because for any homogeneous $f \in A$ such that $6 | \deg(fy)$ we have $y|f$, hence $fy \in I$. It follows that $I(V_+(I)) = (y)$ but $x^n y \notin I$ for all n hence $I(V_+(I))$ is not equal to the saturation.

0BXL Lemma 31.31.3. Let R be a UFD. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme which has no embedded points such that every irreducible component of Z has codimension 1 in $\mathbf{P}^n_R$. Then the ideal $I(Z) \subset R[T_0, \ldots, T_n]$ corresponding to Z is principal.

Proof. Observe that the local rings of $X = \mathbf{P}^n_R$ are UFDs because X is covered by affine pieces isomorphic to $\mathbf{A}^n_R$ and $R[x_1, \ldots, x_n]$ is a UFD (Algebra, Lemma 10.120.10). Thus Z is an effective Cartier divisor by Lemma 31.15.9. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to Z. Choose an isomorphism $\mathcal{O}(m) \to \mathcal{I}$ for some $m \in \mathbf{Z}$, see Lemma 31.28.5. Then the composition

$$\mathcal{O}_X(m) \to \mathcal{I} \to \mathcal{O}_X$$

is nonzero. We conclude that $m \leq 0$ and that the corresponding section of $\mathcal{O}_X(m)^{\otimes -1} = \mathcal{O}_X(-m)$ is given by some $F \in R[T_0, \ldots, T_n]$ of degree $-m$, see Cohomology of Schemes, Lemma 30.8.1. Thus on the ith standard open $U_i = D_+(T_i)$ the closed subscheme $Z \cap U_i$ is cut out by the ideal

$$(F(T_0/T_i, \ldots, T_n/T_i)) \subset R[T_0/T_i, \ldots, T_n/T_i]$$

Thus the homogeneous elements of the graded ideal $I(Z) = \text{Ker}(R[T_0, \ldots, T_n] \to \bigoplus \Gamma(\mathcal{O}_Z(m)))$ is the set of homogeneous polynomials G such that

$$G(T_0/T_i, \ldots, T_n/T_i) \in (F(T_0/T_i, \ldots, T_n/T_i))$$

for $i = 0, \ldots, n$. Clearing denominators, we see there exist $e_i \geq 0$ such that

$$T_i^{e_i} G \in (F)$$

for $i = 0, \ldots, n$. As R is a UFD, so is $R[T_0, \ldots, T_n]$. Then $F | T_0^{e_0} G$ and $F | T_1^{e_1} G$ implies $F|G$ as $T_0^{e_0}$ and $T_1^{e_1}$ have no factor in common. Thus $I(Z) = (F)$. $\square$

In case the closed subscheme is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

0802 Lemma 31.31.4. Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. If p is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{\text{Proj}}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$.

Proof. By Lemma 31.31.1 we know there exists a quasi-coherent graded sheaf of ideals $\mathcal{I} \subset \mathcal{A}$ such that $Z = \underline{\text{Proj}}(\mathcal{A}/\mathcal{I})$. Since S is quasi-compact we can choose a finite affine open covering $S = U_1 \cup \ldots \cup U_n$. Say $U_i = \text{Spec}(R_i)$. Let $\mathcal{A}|_{U_i}$ correspond to the graded R_i-algebra A_i and $\mathcal{I}|_{U_i}$ to the graded ideal $I_i \subset A_i$. Note that $p^{-1}(U_i) = \text{Proj}(A_i)$ as schemes over R_i. Since p is quasi-compact we can choose finitely many homogeneous elements $f_{i,j} \in A_{i,+}$ such that $p^{-1}(U_i) = D_+(f_{i,j})$. The condition on $Z \to X$ means that the ideal sheaf of Z in $\mathcal{O}_X$ is of finite type, see Morphisms, Lemma 29.21.7. Hence we can find finitely many homogeneous elements $h_{i,j,k} \in I_i \cap A_{i,+}$ such that the ideal of $Z \cap D_+(f_{i,j})$ is generated by the elements $h_{i,j,k}/f_{i,j}^{e_{i,j,k}}$. Choose $d > 0$ to be a common multiple of all the integers $\deg(f_{i,j})$ and $\deg(h_{i,j,k})$. By Properties, Lemma 28.22.3 there exists a finite type quasi-coherent $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d - \deg(h_{i,j,k}))/\deg(f_{i,j})}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. $\square$

The following version of Lemma 31.31.4 will be used in the proof of Lemma 31.34.2.

0803 Lemma 31.31.5. Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{\text{Proj}}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Let $U \subset X$ be an open. Assume that

(1) p is quasi-compact,
(2) i of finite presentation,
(3) $U \cap p(i(Z)) = \emptyset$,
(4) U is quasi-compact,
(5) $\mathcal{A}_n$ is a finite type $\mathcal{O}_S$-module for all n.

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{\text{Proj}}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. Let $\mathcal{I} \subset \mathcal{A}$ be the sheaf of quasi-coherent graded ideals constructed in Lemma 31.31.1. Let U_i, R_i, A_i, I_i, $f_{i,j}$, $h_{i,j,k}$, and d be as constructed in the proof of Lemma 31.31.4. Since $U \cap p(i(Z)) = \emptyset$ we see that $\mathcal{I}_d|_U = \mathcal{A}_d|_U$ (by our construction of $\mathcal{I}$ as a kernel). Since U is quasi-compact we can choose a finite affine open covering $U = W_1 \cup \ldots \cup W_m$. Since $\mathcal{A}_d$ is of finite type we can find finitely many sections $g_{t,s} \in \mathcal{A}_d(W_t)$ which generate $\mathcal{A}_d|_{W_t} = \mathcal{I}_d|_{W_t}$ as an $\mathcal{O}_{W_t}$-module. To finish the proof, note that by Properties, Lemma 28.22.3 there exists a finite type $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d - \deg(h_{i,j,k}))/\deg(f_{i,j})} \quad \text{and} \quad g_{t,s}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. $\square$

0B3V Lemma 31.31.6. Let X be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. There is a bijection

$$\left\{ \begin{matrix} \text{sections } \sigma \text{ of the} \\ \text{morphism } \mathbf{P}(\mathcal{E}) \to X \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{surjections } \mathcal{E} \to \mathcal{L} \text{ where} \\ \mathcal{L} \text{ is an invertible } \mathcal{O}_X\text{-module} \end{matrix} \right\}$$

In this case σ is a closed immersion and there is a canonical isomorphism

$$\text{Ker}(\mathcal{E} \to \mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \longrightarrow \mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}$$

Both the bijection and isomorphism are compatible with base change.

Proof. Recall that $\pi : \mathbf{P}(\mathcal{E}) \to X$ is the relative proj of the symmetric algebra on $\mathcal{E}$, see Constructions, Definition 27.21.1. Hence the descriptions of sections σ follows immediately from the description of the functor of points of $\mathbf{P}(\mathcal{E})$ in Constructions, Lemma 27.16.11. Since π is separated, any section is a closed immersion (Constructions, Lemma 27.16.9 and Schemes, Lemma 26.21.11). Let $U \subset X$ be an affine open and $k \in \mathcal{E}(U)$ and $s \in \mathcal{E}(U)$ be local sections such that k maps to zero in $\mathcal{L}$ and s maps to a generator $\overline{s}$ of $\mathcal{L}$. Then $f = k/s$ is a section of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$ defined in an open neighbourhood $D_+(s)$ of $s(U)$ in $\pi^{-1}(U)$. Moreover, since k maps to zero in $\mathcal{L}$ we see that f is a section of the ideal sheaf of $s(U)$ in $\pi^{-1}(U)$. Thus we can take the image $\overline{f}$ of f in $\mathcal{C}_{\sigma(X)/\mathbf{P}(\mathcal{E})}(U)$. We claim (1) that the image $\overline{f}$ depends only on the sections k and $\overline{s}$ and not on the choice of s and (2) that we get an isomorphism over U in this manner (see below). However, once (1) and (2) are established, we see that the construction is compatible with base change by $U' \to U$ where U' is affine, which proves that these local maps glue and are compatible with arbitrary base change.

To prove (1) and (2) we make explicit what is going on. Namely, say $U = \text{Spec}(A)$ and say $\mathcal{E} \to \mathcal{L}$ corresponds to the map of A-modules $M \to N$. Then $k \in K = \text{Ker}(M \to N)$ and $s \in M$ maps to a generator $\overline{s}$ of N. Hence $M = K \oplus As$. Thus

$$\text{Sym}(M) = \text{Sym}(K)[s]$$

Consider the identification $\text{Sym}(K) \to \text{Sym}(M)_{(s)}$ via the rule $g \mapsto g/s^n$ for $g \in \text{Sym}^n(K)$. This gives an isomorphism $D_+(s) = \text{Spec}(\text{Sym}(K))$ such that σ corresponds to the ring map $\text{Sym}(K) \to A$ mapping K to zero. Via this isomorphism we see that the quasi-coherent module corresponding to K is identified with $\mathcal{C}_{\sigma(U)/D_+(s)}$ proving (2). Finally, suppose that $s' = k' + s$ for some $k' \in K$. Then

$$k/s' = (k/s)(s/s') = (k/s)(s'/s)^{-1} = (k/s)(1 + k'/s)^{-1}$$

in an open neighbourhood of $\sigma(U)$ in $D_+(s)$. Thus we see that s'/s restricts to 1 on $\sigma(U)$ and we see that k/s' maps to the same element of the conormal sheaf as does k/s thereby proving (1). □

31.32. Blowing up

01OF Blowing up is an important tool in algebraic geometry.

01OG Definition 31.32.1. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Schemes, Definition 26.10.2. The blowing up of X along Z, or the blowing up of X in the ideal sheaf $\mathcal{I}$ is the morphism

$$b : \underline{\text{Proj}}_X\left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n\right) \longrightarrow X$$

The exceptional divisor of the blowup is the inverse image $b^{-1}(Z)$. Sometimes Z is called the center of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the "smallest" scheme over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blowup of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Constructions, Lemma 27.16.11. Note that $\mathcal{O}_{X'}(1)$ is b-relatively very ample, even though b need not be of finite type or even quasi-compact, because X' comes equipped with a closed immersion into $\mathbf{P}(\mathcal{I})$, see Morphisms, Example 29.38.3.

0804 Lemma 31.32.2. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \mathrm{Spec}(A)$ be an affine open subscheme of X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $b : X' \to X$ is the blowup of X in $\mathcal{I}$, then there is a canonical isomorphism

$$b^{-1}(U) = \mathrm{Proj}(\bigoplus\nolimits_{d \geq 0} I^d)$$

of $b^{-1}(U)$ with the homogeneous spectrum of the Rees algebra of I in A. Moreover, $b^{-1}(U)$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. The first statement is clear from the construction of the relative Proj via glueing, see Constructions, Section 27.15. For $a \in I$ denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra $\bigoplus_{n \geq 0} I^n$. Since these elements generate the Rees algebra over A we see that $\mathrm{Proj}(\bigoplus_{d \geq 0} I^d)$ is covered by the affine opens $D_+(a^{(1)})$. The affine scheme $D_+(a^{(1)})$ is the spectrum of the affine blowup algebra $A' = A[\frac{I}{a}]$, see Algebra, Definition 10.70.1. This finishes the proof. $\square$

0805 Lemma 31.32.3. Let $X_1 \to X_2$ be a flat morphism of schemes. Let $Z_2 \subset X_2$ be a closed subscheme. Let Z_1 be the inverse image of Z_2 in X_1. Let X'_i be the blowup of Z_i in X_i. Then there exists a cartesian diagram

$$\begin{tikzcd} X'_1 \arrow[r] \arrow[d] & X'_2 \arrow[d] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

of schemes.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given morphism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^*\mathcal{I}_2 \to \mathcal{O}_{X_1}$ (by definition of the inverse image, see Schemes, Definition 26.17.7). By Constructions, Lemma 27.16.10 we see that $X_1 \times_{X_2} X'_2$ is the relative Proj of $\bigoplus_{n \geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X'_2 = X'_1$. $\square$

02OS Lemma 31.32.4. Let X be a scheme. Let $Z \subset X$ be a closed subscheme. The blowing up $b : X' \to X$ of Z in X has the following properties:

(1) $b|_{b^{-1}(X \setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,
(2) the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',
(3) there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$

Proof. As blowing up commutes with restrictions to open subschemes (Lemma 31.32.3) the first statement just means that $X' = X$ if $Z = \emptyset$. In this case we are blowing up in the ideal sheaf $\mathcal{I} = \mathcal{O}_X$ and the result follows from Constructions, Example 27.8.14.

The second statement is local on X, hence we may assume X affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. By Lemma 31.32.2 we see that X' is covered by the spectra of the affine blowup algebras $A' = A[\frac{I}{a}]$. Then $IA' = aA'$ and a maps to a nonzerodivisor in A' according to Algebra, Lemma 10.70.2. This proves the lemma as the inverse image of Z in $\text{Spec}(A')$ corresponds to $\text{Spec}(A'/IA') \subset \text{Spec}(A')$.

Consider the canonical map $\psi_{univ,1} : b^*\mathcal{I} \to \mathcal{O}_{X'}(1)$, see discussion following Constructions, Definition 27.16.7. We claim that this factors through an isomorphism $\mathcal{I}_E \to \mathcal{O}_{X'}(1)$ (which proves the final assertion). Namely, on the affine open corresponding to the blowup algebra $A' = A[\frac{I}{a}]$ mentioned above $\psi_{univ,1}$ corresponds to the A'-module map

$$I \otimes_A A' \longrightarrow \left(\left(\bigoplus\nolimits_{d \geq 0} I^d\right)_{a^{(1)}}\right)_1$$

where $a^{(1)}$ is as in Algebra, Definition 10.70.1. We omit the verification that this is the map $I \otimes_A A' \to IA' = aA'$. □

0806 Lemma 31.32.5 (Universal property blowing up). Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{C}$ be the full subcategory of (Sch/X) consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 31.32.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By the material in Constructions, Section 27.16 the triple $(1, f : Y \to X, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 31.13.4. Thus the morphism $Y \to X'$ is unique by Morphisms, Lemma 29.7.10 (also b is separated by Constructions, Lemma 27.16.9). □

0BFL Lemma 31.32.6. Let $b : X' \to X$ be the blowing up of the scheme X along a closed subscheme Z. Let $U = \text{Spec}(A)$ be an affine open of X and let $I \subset A$ be the ideal corresponding to $Z \cap U$. Let $a \in I$ and let $x' \in X'$ be a point mapping to a point of U. Then x' is a point of the affine open $U' = \text{Spec}(A[\frac{I}{a}])$ if and only if the image of a in $\mathcal{O}_{X',x'}$ cuts out the exceptional divisor.

Proof. Since the exceptional divisor over U' is cut out by the image of a in $A' = A[\frac{I}{a}]$ one direction is clear. Conversely, assume that the image of a in $\mathcal{O}_{X',x'}$ cuts out E. Since every element of I maps to an element of the ideal defining E over $b^{-1}(U)$ we see that elements of I become divisible by a in $\mathcal{O}_{X',x'}$. Thus for $f \in I^n$ we can write $f = \psi(f)a^n$ for some $\psi(f) \in \mathcal{O}_{X',x'}$. Observe that since a maps to a nonzerodivisor of $\mathcal{O}_{X',x'}$ the element $\psi(f)$ is uniquely characterized by this. Then we define

$$A' \longrightarrow \mathcal{O}_{X',x'}, \quad f/a^n \longmapsto \psi(f)$$

Here we use the description of blowup algebras given following Algebra, Definition 31.32.1. The uniqueness mentioned above shows that this is an A-algebra homomorphism. This gives a morphism $\text{Spec}(\mathcal{O}_{X',x''}) \to \text{Spec}(A') = U'$. By the universal property of blowing up (Lemma 31.32.5) this is a morphism over X', which of course implies that $x' \in U'$. □

0807 **Lemma 31.32.7.** Let X be a scheme. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.

Proof. Immediate from the universal property of blowups (Lemma 31.32.5). $\square$

0808 **Lemma 31.32.8.** Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blowup X' of X in $\mathcal{I}$ is reduced.

Proof. Combine Lemma 31.32.2 with Algebra, Lemma 10.70.9. $\square$

02ND **Lemma 31.32.9.** Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a nonzero quasi-coherent sheaf of ideals. If X is integral, then the blowup X' of X in $\mathcal{I}$ is integral.

Proof. Combine Lemma 31.32.2 with Algebra, Lemma 10.70.10. $\square$

0BFM **Lemma 31.32.10.** Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $b : X' \to X$ be the blowing up of X along Z. Then b induces an bijective map from the set of generic points of irreducible components of X' to the set of generic points of irreducible components of X which are not in Z.

Proof. The exceptional divisor $E \subset X'$ is an effective Cartier divisor and $X' \setminus E \to X \setminus Z$ is an isomorphism, see Lemma 31.32.4. Thus it suffices to show the following: given an effective Cartier divisor $D \subset S$ of a scheme S none of the generic points of irreducible components of S are contained in D. To see this, we may replace S by the members of an affine open covering. Hence by Lemma 31.13.2 we may assume $S = \text{Spec}(A)$ and $D = V(f)$ where $f \in A$ is a nonzerodivisor. Then we have to show f is not contained in any minimal prime ideal $\mathfrak{p} \subset A$. If so, then f would map to a nonzerodivisor contained in the maximal ideal of $R_\mathfrak{p}$ which is a contradiction with Algebra, Lemma 10.25.1. $\square$

0809 **Lemma 31.32.11.** Let X be a scheme. Let $b : X' \to X$ be a blowup of X in a closed subscheme. The pullback $b^{-1}D$ is defined for all effective Cartier divisors $D \subset X$ and pullbacks of meromorphic functions are defined for b (Definitions 31.13.12 and 31.23.4).

Proof. By Lemmas 31.32.2 and 31.13.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. $\square$

080A **Lemma 31.32.12.** Let X be a scheme. Let $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 31.32.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 31.32.11. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 31.32.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 31.13.9. Thus a morphism $c' : Y \to X'$ over X

by Lemma 31.32.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. $\square$

02NS Lemma 31.32.13. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$. If $\mathcal{I}$ is of finite type, then

(1) $b : X' \to X$ is a projective morphism, and
(2) $\mathcal{O}_{X'}(1)$ is a b-relatively ample invertible sheaf.

Proof. The surjection of graded $\mathcal{O}_X$-algebras

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{I}) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{I}^d$$

defines via Constructions, Lemma 27.18.5 a closed immersion

$$X' = \underline{\mathrm{Proj}}_X(\bigoplus\nolimits_{d \geq 0} \mathcal{I}^d) \longrightarrow \mathbf{P}(\mathcal{I}).$$

Hence b is projective, see Morphisms, Definition 29.43.1. The second statement follows for example from the characterization of relatively ample invertible sheaves in Morphisms, Lemma 29.37.4. Some details omitted. $\square$

080B Lemma 31.32.14. Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subscheme of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subscheme of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subscheme $Y \subset X$ of finite presentation, such that

(1) $Y = Z \cup b(Z')$ set theoretically, and
(2) the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms, Lemma 29.21.7. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\mathrm{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open of X by Properties, Lemma 28.24.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism (Lemma 31.32.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open in X. By Lemma 31.31.5 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $X \setminus U$ we see that $V(\mathcal{J})$ is contained in $X \setminus U$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $Z \subset V(\mathcal{J})$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $V(\mathcal{J})$ equals $Z \cup b(Z')$. It follows that also $V(\mathcal{I}\mathcal{J}) = Z \cup b(Z')$ set theoretically. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blowup of X in $\mathcal{I}\mathcal{J}$ is the same as the blowup of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 31.32.12. Hence it suffices to show that the blowup of X' in

$b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blowup of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d \mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 31.32.7 and 31.32.12.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 31.32.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\text{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \text{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. □

31.33. Strict transform

080C In this section we briefly discuss strict transform under blowing up. Let S be a scheme and let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of S in Z and denote $E \subset S'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider a scheme X over S and form the cartesian diagram

$$\begin{tikzcd} \text{pr}_{S'}^{-1}E \arrow[r] \arrow[d] & X \times_S S' \arrow[r, "\text{pr}_X"'] \arrow[d, "\text{pr}_{S'}"'] & X \arrow[d, "f"] \\ E \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

Since E is an effective Cartier divisor (Lemma 31.32.4) we see that $\text{pr}_{S'}^{-1}E \subset X \times_S S'$ is locally principal (Lemma 31.13.11). Thus the complement of $\text{pr}_{S'}^{-1}E$ in $X \times_S S'$ is retrocompact (Lemma 31.13.3). Consequently, for a quasi-coherent $\mathcal{O}_{X\times_S S'}$-module $\mathcal{G}$ the subsheaf of sections supported on $\text{pr}_{S'}^{-1}E$ is a quasi-coherent submodule, see Properties, Lemma 28.24.5. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X\times_S S'}$, then this subsheaf is an ideal of $\mathcal{G}$.

080D Definition 31.33.1. With $Z \subset S$ and $f : X \to S$ as above.

(1) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform of $\mathcal{F}$ with respect to the blowup of S in Z is the quotient $\mathcal{F}'$ of $\text{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $\text{pr}_{S'}^{-1}E$.
(2) The strict transform of X is the closed subscheme $X' \subset X \times_S S'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X\times_S S'}$ supported on $\text{pr}_{S'}^{-1}E$.

Note that taking the strict transform along a blowup depends on the closed subscheme used for the blowup (and not just on the morphism $S' \to S$). This notion is often used for closed subschemes of S. It turns out that the strict transform of X is a blowup of X.

080E Lemma 31.33.2. In the situation of Definition 31.33.1.

(1) The strict transform X' of X is the blowup of X in the closed subscheme $f^{-1}Z$ of X.
(2) For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_S S'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 31.32.5) there exists a commutative diagram

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

whence a morphism $X'' \to X \times_S S'$. Thus the first assertion is that this morphism is a closed immersion with image X'. The question is local on X. Thus we may assume X and S are affine. Say that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and Z is cut out by the ideal $I \subset A$. Set $J = IB$. The map $B \otimes_A \bigoplus_{n \geq 0} I^n \to \bigoplus_{n \geq 0} J^n$ defines a closed immersion $X'' \to X \times_S S'$, see Constructions, Lemmas 27.11.6 and 27.11.5. We omit the verification that this morphism is the same as the one constructed above from the universal property. Pick $a \in I$ corresponding to the affine open $\mathrm{Spec}(A[\frac{I}{a}]) \subset S'$, see Lemma 31.32.2. The inverse image of $\mathrm{Spec}(A[\frac{I}{a}])$ in the strict transform X' of X is the spectrum of

$$B' = (B \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 28.24.5. On the other hand, letting $b \in J$ be the image of a we see that $\mathrm{Spec}(B[\frac{J}{b}])$ is the inverse image of $\mathrm{Spec}(A[\frac{I}{a}])$ in X''. By Algebra, Lemma 10.70.3 the open $\mathrm{Spec}(B[\frac{J}{b}])$ maps isomorphically to the open subscheme $\mathrm{pr}_{S'}^{-1}(\mathrm{Spec}(A[\frac{I}{a}]))$ of X'. Thus $X'' \to X'$ is an isomorphism.

In the notation above, let $\mathcal{F}$ correspond to the B-module N. The strict transform of $\mathcal{F}$ corresponds to the $B \otimes_A A[\frac{I}{a}]$-module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 28.24.5. The strict transform of $\mathcal{F}$ relative to the blowup of X in $f^{-1}Z$ corresponds to the $B[\frac{J}{b}]$-module $N \otimes_B B[\frac{J}{b}]/b$-power-torsion. In exactly the same way as above one proves that these two modules are isomorphic. Details omitted. □

080F **Lemma 31.33.3.** In the situation of Definition 31.33.1.

(1) If X is flat over S at all points lying over Z, then the strict transform of X is equal to the base change $X \times_S S'$.
(2) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over S at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $\mathrm{pr}_X^*\mathcal{F}$.

Proof. We will prove part (2) as it implies part (1) by the definition of the strict transform of a scheme over S. The question is local on X. Thus we may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and that $\mathcal{F}$ corresponds to the B-module N. Then $\mathcal{F}'$ over the open $\mathrm{Spec}(B \otimes_A A[\frac{I}{a}])$ of $X \times_S S'$ corresponds to the module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 28.24.5. Thus we have to show that the a-power-torsion of $N \otimes_A A[\frac{I}{a}]$ is zero. Let $y \in N \otimes_A A[\frac{I}{a}]$ with $a^n y = 0$. If $\mathfrak{q} \subset B$ is a prime and $a \notin \mathfrak{q}$, then y maps to zero in $(N \otimes_A A[\frac{I}{a}])_\mathfrak{q}$. on the other hand, if $a \in \mathfrak{q}$, then $N_\mathfrak{q}$ is a flat A-module and we see that $N_\mathfrak{q} \otimes_A A[\frac{I}{a}] = (N \otimes_A A[\frac{I}{a}])_\mathfrak{q}$ has no a-power torsion (as

$A[\frac{I}{a}]$ doesn't). Hence y maps to zero in this localization as well. We conclude that y is zero by Algebra, Lemma 10.23.1. ☐

080G Lemma 31.33.4. Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of Z in S. Let $g : X \to Y$ be an affine morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_S S' \to Y \times_S S'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.

Proof. Observe that $g'_*\text{pr}_X^*\mathcal{F} = \text{pr}_Y^*g_*\mathcal{F}$ by Cohomology of Schemes, Lemma 30.5.1. Let $\mathcal{K} \subset \text{pr}_X^*\mathcal{F}$ be the subsheaf of sections supported in the inverse image of Z in $X \times_S S'$. By Properties, Lemma 28.24.7 the pushforward $g'_*\mathcal{K}$ is the subsheaf of sections of $\text{pr}_Y^*g_*\mathcal{F}$ supported in the inverse image of Z in $Y \times_S S'$. As g' is affine (Morphisms, Lemma 29.11.8) we see that g'_* is exact, hence we conclude. ☐

080H Lemma 31.33.5. Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $D \subset S$ be an effective Cartier divisor. Let $Z' \subset S$ be the closed subscheme cut out by the product of the ideal sheaves of Z and D. Let $S' \to S$ be the blowup of S in Z.

(1) The blowup of S in Z' is isomorphic to $S' \to S$.
(2) Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ has no nonzero local sections supported in $f^{-1}D$, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of S in Z'.

Proof. The first statement follows on combining Lemmas 31.32.12 and 31.32.7. Using Lemma 31.32.2 the second statement translates into the following algebra problem. Let A be a ring, $I \subset A$ an ideal, $x \in A$ a nonzerodivisor, and $a \in I$. Let M be an A-module whose x-torsion is zero. To show: the a-power torsion in $M \otimes_A A[\frac{I}{a}]$ is equal to the xa-power torsion. The reason for this is that the kernel and cokernel of the map $A \to A[\frac{I}{a}]$ is a-power torsion, so this map becomes an isomorphism after inverting a. Hence the kernel and cokernel of $M \to M \otimes_A A[\frac{I}{a}]$ are a-power torsion too. This implies the result. ☐

080I Lemma 31.33.6. Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up with center Z. Let $Z' \subset S'$ be a closed subscheme. Let $S'' \to S'$ be the blowing up with center Z'. Let $Y \subset S$ be a closed subscheme such that $Y = Z \cup b(Z')$ set theoretically and the composition $S'' \to S$ is isomorphic to the blowing up of S in Y. In this situation, given any scheme X over S and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have

(1) the strict transform of $\mathcal{F}$ with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z, and
(2) the strict transform of X with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of X with respect to the blowup $S' \to S$ of S in Z.

Proof. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z. Let $\mathcal{F}''$ be the strict transform of $\mathcal{F}'$ with respect to the blowup $S'' \to S'$ of S' in Z'. Let $\mathcal{G}$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S'' \to S$

of S in Y. We also label the morphisms

$$\begin{tikzcd} X \times_S S'' \arrow[r, "q"'] \arrow[d, "f''"] & X \times_S S' \arrow[r, "p"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

By definition there is a surjection $p^*\mathcal{F} \to \mathcal{F}'$ and a surjection $q^*\mathcal{F}' \to \mathcal{F}''$ which combine by right exactness of q^* to a surjection $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$. Also we have the surjection $(p \circ q)^*\mathcal{F} \to \mathcal{G}$. Thus it suffices to prove that these two surjections have the same kernel.

The kernel of the surjection $p^*\mathcal{F} \to \mathcal{F}'$ is supported on $(f \circ p)^{-1}Z$, so this map is an isomorphism at points in the complement. Hence the kernel of $q^*p^*\mathcal{F} \to q^*\mathcal{F}'$ is supported on $(f \circ p \circ q)^{-1}Z$. The kernel of $q^*\mathcal{F}' \to \mathcal{F}''$ is supported on $(f' \circ q)^{-1}Z'$. Combined we see that the kernel of $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$ is supported on $(f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z' = (f \circ p \circ q)^{-1}Y$. By construction of $\mathcal{G}$ we see that we obtain a factorization $(p \circ q)^*\mathcal{F} \to \mathcal{F}'' \to \mathcal{G}$. To finish the proof it suffices to show that $\mathcal{F}''$ has no nonzero (local) sections supported on $(f \circ p \circ q)^{-1}(Y) = (f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z'$. This follows from Lemma 31.33.5 applied to $\mathcal{F}'$ on $X \times_S S'$ over S', the closed subscheme Z' and the effective Cartier divisor $b^{-1}Z$. □

080W Lemma 31.33.7. In the situation of Definition 31.33.1. Suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to S$. Then the strict transforms of $\mathcal{F}'_i$ relative to any blowup $S' \to S$ form a short exact sequence $0 \to \mathcal{F}'_1 \to \mathcal{F}'_2 \to \mathcal{F}'_3 \to 0$ too.

Proof. We may localize on S and X and assume both are affine. Then we may push $\mathcal{F}_i$ to S, see Lemma 31.33.4. We may assume that our blowup is the morphism $1 : S \to S$ associated to an effective Cartier divisor $D \subset S$. Then the translation into algebra is the following: Suppose that A is a ring and $0 \to M_1 \to M_2 \to M_3 \to 0$ is a universally exact sequence of A-modules. Let $a \in A$. Then the sequence

$$0 \to M_1/a\text{-power torsion} \to M_2/a\text{-power torsion} \to M_3/a\text{-power torsion} \to 0$$

is exact too. Namely, surjectivity of the last map and injectivity of the first map are immediate. The problem is exactness in the middle. Suppose that $x \in M_2$ maps to zero in M_3/a-power torsion. Then $y = a^n x \in M_1$ for some n. Then y maps to zero in M_2/a^nM_2. Since $M_1 \to M_2$ is universally injective we see that y maps to zero in M_1/a^nM_1. Thus $y = a^n z$ for some $z \in M_1$. Thus $a^n(x - y) = 0$. Hence y maps to the class of x in M_2/a-power torsion as desired. □

31.34. Admissible blowups

080J To have a bit more control over our blowups we introduce the following standard terminology.

080K Definition 31.34.1. Let X be a scheme. Let $U \subset X$ be an open subscheme. A morphism $X' \to X$ is called a U-admissible blowup if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blowup of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms, Lemma 29.21.7. In particular, a U-admissible blowup is a projective morphism, see Lemma 31.32.13. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 31.32.4) we will often abuse notation and think of U as an open subscheme of X' as well.

080L Lemma 31.34.2. Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.

Proof. Immediate from the more precise Lemma 31.32.14. □

080M Lemma 31.34.3. Let X be a quasi-compact and quasi-separated scheme. Let $U, V \subset X$ be quasi-compact open subschemes. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blowup of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$ (see Sheaves, Section 6.33). By Properties, Lemma 28.22.2 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. □

080N Lemma 31.34.4. Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blowup of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 31.32.12. □

080P Lemma 31.34.5. Let X be a quasi-compact and quasi-separated scheme. Let U, V be quasi-compact disjoint open subschemes of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Properties, Lemma 28.24.1. Then $V(\mathcal{I}\mathcal{J}) = X$ set theoretically, hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I} + \mathcal{J}$. This is $U \cup V$-admissible as $V(\mathcal{I} + \mathcal{J}) = X \setminus U \cup V$. We will show that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ such that $b^{-1}\mathcal{I}|_{X'_2} = 0$ and $b^{-1}\mathcal{J}|_{X'_1} = 0$ which will prove the lemma.

We will use the description of the blowing up in Lemma 31.32.2. Suppose that $U = \mathrm{Spec}(A) \subset X$ is an affine open such that $\mathcal{I}|_U$, resp. $\mathcal{J}|_U$ corresponds to the finitely generated ideal $I \subset A$, resp. $J \subset A$. Then

$$b^{-1}(U) = \mathrm{Proj}(A \oplus (I + J) \oplus (I + J)^2 \oplus \ldots)$$

This is covered by the affine open subsets $A[\frac{I+J}{x}]$ and $A[\frac{I+J}{y}]$ with $x \in I$ and $y \in J$. Since $x \in I$ is a nonzerodivisor in $A[\frac{I+J}{x}]$ and $IJ = 0$ we see that $JA[\frac{I+J}{x}] = 0$. Since $y \in J$ is a nonzerodivisor in $A[\frac{I+J}{y}]$ and $IJ = 0$ we see that $IA[\frac{I+J}{y}] = 0$. Moreover,

$$\mathrm{Spec}(A[\tfrac{I+J}{x}]) \cap \mathrm{Spec}(A[\tfrac{I+J}{y}]) = \mathrm{Spec}(A[\tfrac{I+J}{xy}]) = \emptyset$$

because xy is both a nonzerodivisor and zero. Thus $b^{-1}(U)$ is the disjoint union of the open subscheme U_1 defined as the union of the standard opens $\mathrm{Spec}(A[\frac{I+J}{x}])$ for $x \in I$ and the open subscheme U_2 which is the union of the affine opens $\mathrm{Spec}(A[\frac{I+J}{y}])$ for $y \in J$. We have seen that $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_2 and $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_1. We omit the verification that these open subschemes glue to global open subschemes X'_1 and X'_2. □

0ESL Lemma 31.34.6. Let X be a locally Noetherian scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let $U \subset X$ be the maximal open subscheme such that s corresponds to a section of $\mathcal{L}$ over U. The blowup $b : X' \to X$ in the ideal of denominators of s is U-admissible. There exists an effective Cartier divisor $D \subset X'$ and an isomorphism

$$b^*\mathcal{L} = \mathcal{O}_{X'}(D - E),$$

where $E \subset X'$ is the exceptional divisor such that the meromorphic section b^*s corresponds, via the isomorphism, to the meromorphic section $1_D \otimes (1_E)^{-1}$.

Proof. From the definition of the ideal of denominators in Definition 31.23.10 we immediately see that b is a U-admissible blowup. For the notation 1_D, 1_E, and $\mathcal{O}_{X'}(D - E)$ please see Definition 31.14.1. The pullback b^*s is defined by Lemmas 31.32.11 and 31.23.8. Thus the statement of the lemma makes sense. We can reinterpret the final assertion as saying that b^*s is a global regular section of $b^*\mathcal{L}(E)$ whose zero scheme is D. This uniquely defines D hence to prove the lemma we may work affine locally on X and X'. Assume $X = \mathrm{Spec}(A)$ is affine and $\mathcal{L} = \mathcal{O}_X$. Then s is a regular meromorphic function and shrinking further we may assume $s = a'/a$ with $a', a \in A$ nonzerodivisors. Then the ideal of denominators of s corresponds to the ideal $I = \{x \in A \mid xa' \in aA\}$. Recall that X' is covered by spectra of affine blowup algebras $A' = A[\frac{I}{x}]$ with $x \in I$ (Lemma 31.32.2). Fix $x \in I$ and write $xa' = aa''$ for some $a'' \in A$. The divisor $E \subset X'$ is cut out by $x \in A'$ over the spectrum of A' and hence $1/x$ is a generator of $\mathcal{O}_{X'}(E)$ over $\mathrm{Spec}(A')$. Finally, in the total quotient ring of A' we have $a'/a = a''/x$. Hence $b^*s = a'/a$ restricts to a regular section of $\mathcal{O}_{X'}(E)$ which is over $\mathrm{Spec}(A')$ given by a''/x. This finishes the proof. (The divisor $D \cap \mathrm{Spec}(A')$ is cut out by the image of a'' in A'.) □

31.35. Blowing up and flatness

0F84 We continue the discussion started in More on Algebra, Section 15.26. We will prove further results in More on Flatness, Section 38.30.

0CZP Lemma 31.35.1. Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $\text{Fit}_k(\mathcal{F})$, see Section 31.9. Let $S' \to S$ be the blowup of S in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ can locally be generated by $\leq k$ sections.

Proof. Recall that $\mathcal{F}'$ can locally be generated by $\leq k$ sections if and only if $\text{Fit}_k(\mathcal{F}') = \mathcal{O}_{S'}$, see Lemma 31.9.4. Hence this lemma is a translation of More on Algebra, Lemma 15.26.3. □

0CZQ Lemma 31.35.2. Let S be a scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. Let $Z_k \subset S$ be the closed subscheme cut out by $\text{Fit}_k(\mathcal{F})$, see Section 31.9. Assume that $\mathcal{F}$ is locally free of rank k on $S \setminus Z_k$. Let $S' \to S$ be the blowup of S in Z_k and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$. Then $\mathcal{F}'$ is locally free of rank k.

Proof. Translation of More on Algebra, Lemma 15.26.4. □

0ESN Lemma 31.35.3. Let X be a scheme. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $U \subset X$ be a scheme theoretically dense open such that $\mathcal{F}|_U$ is finite locally free of constant rank r. Then

(1) the blowup $b : X' \to X$ of X in the rth Fitting ideal of $\mathcal{F}$ is U-admissible,
(2) the strict transform $\mathcal{F}'$ of $\mathcal{F}$ with respect to b is locally free of rank r,
(3) the kernel $\mathcal{K}$ of the surjection $b^*\mathcal{F} \to \mathcal{F}'$ is finitely presented and $\mathcal{K}|_U = 0$,
(4) $b^*\mathcal{F}$ and $\mathcal{K}$ are perfect $\mathcal{O}_{X'}$-modules of tor dimension ≤ 1.

Proof. The ideal $\text{Fit}_r(\mathcal{F})$ is of finite type by Lemma 31.9.2 and its restriction to U is equal to $\mathcal{O}_U$ by Lemma 31.9.5. Hence $b : X' \to X$ is U-admissible, see Definition 31.34.1.

By Lemma 31.9.5 the restriction of $\text{Fit}_{r-1}(\mathcal{F})$ to U is zero, and since U is scheme theoretically dense we conclude that $\text{Fit}_{r-1}(\mathcal{F}) = 0$ on all of X. Thus it follows from Lemma 31.9.5 that $\mathcal{F}$ is locally free of rank r on the complement of subscheme cut out by the rth Fitting ideal of $\mathcal{F}$ (this complement may be bigger than U which is why we had to do this step in the argument). Hence by Lemma 31.35.2 the strict transform

$$b^*\mathcal{F} \longrightarrow \mathcal{F}'$$

is locally free of rank r. The kernel $\mathcal{K}$ of this map is supported on the exceptional divisor of the blowup b and hence $\mathcal{K}|_U = 0$. Finally, since $\mathcal{F}'$ is finite locally free and since the displayed arrow is surjective, we can locally on X' write $b^*\mathcal{F}$ as the direct sum of $\mathcal{K}$ and $\mathcal{F}'$. Since $b^*\mathcal{F}'$ is finitely presented (Modules, Lemma 17.11.4) the same is true for $\mathcal{K}$.

The statement on tor dimension follows from More on Algebra, Lemma 15.8.9. □

31.36. Modifications

0AYN In this section we will collect results of the type: after a modification such and such are true. We will later see that a modification can be dominated by a blowup (More on Flatness, Lemma 38.31.4).

0AYP Lemma 31.36.1. Let X be an integral scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. There exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible $\mathcal{O}_{X'}$-modules.

Proof. We prove this by induction on the rank r of $\mathcal{E}$. If $r = 1$ or $r = 0$ the lemma is obvious. Assume $r > 1$. Let $P = \mathbf{P}(\mathcal{E})$ with structure morphism $\pi : P \to X$, see Constructions, Section 27.21. Then π is proper (Lemma 31.30.4). There is a canonical surjection

$$\pi^*\mathcal{E} \to \mathcal{O}_P(1)$$

whose kernel is finite locally free of rank $r-1$. Choose a nonempty open subscheme $U \subset X$ such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$. Then $P_U = \pi^{-1}(U)$ is isomorphic to $\mathbf{P}^{r-1}_U$. In particular, there exists a section $s : U \to P_U$ of π. Let $X' \subset P$ be the scheme theoretic image of the morphism $U \to P_U \to P$. Then X' is integral (Morphisms, Lemma 29.6.7), the morphism $f = \pi|_{X'} : X' \to X$ is proper (Morphisms, Lemmas 29.41.6 and 29.41.4), and $f^{-1}(U) \to U$ is an isomorphism. Hence f is a modification (Morphisms, Definition 29.51.11). By construction the pullback $f^*\mathcal{E}$ has a two step filtration whose quotient is invertible because it is equal to $\mathcal{O}_P(1)|_{X'}$ and whose sub $\mathcal{E}'$ is locally free of rank $r-1$. By induction we can find a modification $g : X'' \to X'$ such that $g^*\mathcal{E}'$ has a filtration as in the statement of the lemma. Thus $f \circ g : X'' \to X$ is the required modification. $\square$

0C4V Lemma 31.36.2. Let S be a scheme. Let X, Y be schemes over S. Assume X is Noetherian and Y is proper over S. Given an S-rational map $f : U \to Y$ from X to Y there exists a morphism $p : X' \to X$ and an S-morphism $f' : X' \to Y$ such that

(1) p is proper and $p^{-1}(U) \to U$ is an isomorphism,
(2) $f'|_{p^{-1}(U)}$ is equal to $f \circ p|_{p^{-1}(U)}$.

Proof. Denote $j : U \to X$ the inclusion morphism. Let $X' \subset Y \times_S X$ be the scheme theoretic image of $(f, j) : U \to Y \times_S X$ (Morphisms, Definition 29.6.2). The projection $g : Y \times_S X \to X$ is proper (Morphisms, Lemma 29.41.5). The composition $p : X' \to X$ of $X' \to Y \times_S X$ and g is proper (Morphisms, Lemmas 29.41.6 and 29.41.4). Since g is separated and $U \subset X$ is retrocompact (as X is Noetherian) we conclude that $p^{-1}(U) \to U$ is an isomorphism by Morphisms, Lemma 29.6.8. On the other hand, the composition $f' : X' \to Y$ of $X' \to Y \times_S X$ and the projection $Y \times_S X \to Y$ agrees with f on $p^{-1}(U)$. $\square$

31.37. Other chapters

CHAPTER 32

Limits of Schemes

01YT 32.1. Introduction

01YU In this chapter we put material related to limits of schemes. We mostly study limits of inverse systems over directed sets (Categories, Definition 4.21.1) with affine transition maps. We discuss absolute Noetherian approximation. We characterize schemes locally of finite presentation over a base as those whose associated functor of points is limit preserving. As an application of absolute Noetherian approximation we prove that the image of an affine under an integral morphism is affine. Moreover, we prove some very general variants of Chow's lemma. A basic reference is [DG67].

32.2. Directed limits of schemes with affine transition maps

01YV In this section we construct the limit.

01YW Lemma 32.2.1. Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the schemes S_i are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. In fact S is affine and $S = \text{Spec}(\text{colim}_i R_i)$ with $R_i = \Gamma(S_i, \mathcal{O})$.

Proof. Just define $S = \text{Spec}(\text{colim}_i R_i)$. It follows from Schemes, Lemma 26.6.4 that S is the limit even in the category of locally ringed spaces. $\square$

01YX Lemma 32.2.2. Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. Moreover,

(1) each of the morphisms $f_i : S \to S_i$ is affine,
(2) for an element $0 \in I$ and any open subscheme $U_0 \subset S_0$ we have

$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$$

in the category of schemes.

Proof. Choose an element $0 \in I$. Note that I is nonempty as the limit is directed. For every $i \geq 0$ consider the quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras $\mathcal{A}_i = f_{i0,*}\mathcal{O}_{S_i}$. Recall that $S_i = \underline{\text{Spec}}_{S_0}(\mathcal{A}_i)$, see Morphisms, Lemma 29.11.3. Set $\mathcal{A} = \text{colim}_{i \geq 0} \mathcal{A}_i$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras, see Schemes, Section 26.24. Set $S = \underline{\text{Spec}}_{S_0}(\mathcal{A})$. By Morphisms, Lemma 29.11.5 we get for $i \geq 0$ morphisms $f_i : S \to S_i$ compatible with the transition morphisms. Note that the morphisms f_i are affine by Morphisms, Lemma 29.11.11 for example. By Lemma 32.2.1 above we see that for any affine open $U_0 \subset S_0$ the inverse image $U = f_0^{-1}(U_0) \subset S$ is the limit of the system of opens $U_i = f_{i0}^{-1}(U_0)$, $i \geq 0$ in the category of schemes.

Let T be a scheme. Let $g_i : T \to S_i$ be a compatible system of morphisms. To show that $S = \lim_i S_i$ we have to prove there is a unique morphism $g : T \to S$

with $g_i = f_i \circ g$ for all $i \in I$. For every $t \in T$ there exists an affine open $U_0 \subset S_0$ containing $g_0(t)$. Let $V \subset g_0^{-1}(U_0)$ be an affine open neighbourhood containing t. By the remarks above we obtain a unique morphism $g_V : V \to U = f_0^{-1}(U_0)$ such that $f_i \circ g_V = g_i|_{U_i}$ for all i. The open sets $V \subset T$ so constructed form a basis for the topology of T. The morphisms g_V glue to a morphism $g : T \to S$ because of the uniqueness property. This gives the desired morphism $g : T \to S$.

The final statement is clear from the construction of the limit above. □

01YZ Lemma 32.2.3. Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, Let $S = \lim_i S_i$. Let $0 \in I$. Suppose that T is a scheme over S_0. Then

$$T \times_{S_0} S = \lim_{i \geq 0} T \times_{S_0} S_i$$

Proof. The right hand side is a scheme by Lemma 32.2.2. The equality is formal, see Categories, Lemma 4.14.10. □

32.3. Infinite products

0CNH Infinite products of schemes usually do not exist. For example in Examples, Section 110.55 it is shown that an infinite product of copies of $\mathbf{P}^1$ is not even an algebraic space.

On the other hand, infinite products of affine schemes do exist and are affine. Using Schemes, Lemma 26.6.4 this corresponds to the fact that in the category of rings we have infinite coproducts: if I is a set and R_i is a ring for each i, then we can consider the ring

$$R = \otimes R_i = \text{colim}_{\{i_1, \ldots, i_n\} \subset I} R_{i_1} \otimes_{\mathbf{Z}} \ldots \otimes_{\mathbf{Z}} R_{i_n}$$

Given another ring A a map $R \to A$ is the same thing as a collection of ring maps $R_i \to A$ for all $i \in I$ as follows from the corresponding property of finite tensor products.

0CNI Lemma 32.3.1. Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be an affine morphism. Then the product $T = \prod T_i$ exists in the category of schemes over S. In fact, we have

$$T = \lim_{\{i_1, \ldots, i_n\} \subset I} T_{i_1} \times_S \ldots \times_S T_{i_n}$$

and the projection morphisms $T \to T_{i_1} \times_S \ldots \times_S T_{i_n}$ are affine.

Proof. Omitted. Hint: Argue as in the discussion preceding the lemma and use Lemma 32.2.2 for existence of the limit. □

0CNJ Lemma 32.3.2. Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be a surjective affine morphism. Then the product $T = \prod T_i$ in the category of schemes over S (Lemma 32.3.1) maps surjectively to S.

Proof. Let $s \in S$. Choose $t_i \in T_i$ mapping to s. Choose a huge field extension $K/\kappa(s)$ such that $\kappa(s_i)$ embeds into K for each i. Then we get morphisms $\text{Spec}(K) \to T_i$ with image s_i agreeing as morphisms to S. Whence a morphism $\text{Spec}(K) \to T$ which proves there is a point of T mapping to s. □

0CNK Lemma 32.3.3. Let S be a scheme. Let I be a set and for each $i \in I$ let $f_i : T_i \to S$ be an integral morphism. Then the product $T = \prod T_i$ in the category of schemes over S (Lemma 32.3.1) is integral over S.

Proof. Omitted. Hint: On affine pieces this reduces to the following algebra fact: if $A \to B_i$ is integral for all i, then $A \to \otimes_A B_i$ is integral. $\square$

32.4. Descending properties

081A First some basic lemmas describing the topology of a limit.

0CUE Lemma 32.4.1. Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 32.2.2). Then $S_{set} = \lim_i S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.

Proof. Pick $i \in I$. Take $U_i \subset S_i$ an affine open. Denote $U_{i'} = f_{i'i}^{-1}(U_i)$ and $U = f_i^{-1}(U_i)$. Here $f_{i'i} : S_{i'} \to S_i$ is the transition morphism and $f_i : S \to S_i$ is the projection. By Lemma 32.2.2 we have $U = \lim_{i' \geq i} U_i$. Suppose we can show that $U_{set} = \lim_{i' \geq i} U_{i',set}$. Then the lemma follows by a simple argument using an affine covering of S_i. Hence we may assume all S_i and S affine. This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \text{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Then

$$\text{Spec}(A) = \lim_i \text{Spec}(A_i)$$

Namely, suppose that we are given primes $\mathfrak{p}_i \subset A_i$ such that $\mathfrak{p}_i = \varphi_{ii'}^{-1}(\mathfrak{p}_{i'})$ for all $i' \geq i$. Then we simply set

$$\mathfrak{p} = \{x \in A \mid \exists i, x_i \in \mathfrak{p}_i \text{ with } \varphi_i(x_i) = x\}$$

It is clear that this is an ideal and has the property that $\varphi_i^{-1}(\mathfrak{p}) = \mathfrak{p}_i$. Then it follows easily that it is a prime ideal as well. $\square$

0CUF Lemma 32.4.2. Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 32.2.2). Then $S_{top} = \lim_i S_{i,top}$ where S_{top} indicates the underlying topological space of the scheme S. [DG67, IV, Proposition 8.2.9]

Proof. We will use the criterion of Topology, Lemma 5.14.3. We have seen that $S_{set} = \lim_i S_{i,set}$ in Lemma 32.4.1. The maps $f_i : S \to S_i$ are morphisms of schemes hence continuous. Thus $f_i^{-1}(U_i)$ is open for each open $U_i \subset S_i$. Finally, let $s \in S$ and let $s \in V \subset S$ be an open neighbourhood. Choose $0 \in I$ and choose an affine open neighbourhood $U_0 \subset S_0$ of the image of s. Then $f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$, see Lemma 32.2.2. Then $f_0^{-1}(U_0)$ and $f_{i0}^{-1}(U_0)$ are affine and

$$\mathcal{O}_S(f_0^{-1}(U_0)) = \text{colim}_{i \geq 0} \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$$

either by the proof of Lemma 32.2.2 or by Lemma 32.2.1. Choose $a \in \mathcal{O}_S(f_0^{-1}(U_0))$ such that $s \in D(a) \subset V$. This is possible because the principal opens form a basis for the topology on the affine scheme $f_0^{-1}(U_0)$. Then we can pick an $i \geq 0$ and $a_i \in \mathcal{O}_{S_i}(f_{i0}^{-1}(U_0))$ mapping to a. It follows that $D(a_i) \subset f_{i0}^{-1}(U_0) \subset S_i$ is an open subset whose inverse image in S is $D(a)$. This finishes the proof. $\square$

01Z2 Lemma 32.4.3. Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 32.2.2). If all the schemes S_i are nonempty and quasi-compact, then the limit $S = \lim_i S_i$ is nonempty.

Proof. Choose $0 \in I$. Note that I is nonempty as the limit is directed. Choose an affine open covering $S_0 = \bigcup_{j=1,\ldots,m} U_j$. Since I is directed there exists a $j \in \{1, \ldots, m\}$ such that $f_{i0}^{-1}(U_j) \neq \emptyset$ for all $i \geq 0$. Hence $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is not empty since a directed colimit of nonzero rings is nonzero (because $1 \neq 0$). As $\lim_{i \geq 0} f_{i0}^{-1}(U_j)$ is an open subscheme of the limit we win. $\square$

0CUG Lemma 32.4.4. Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms (Lemma 32.2.2). Let $s \in S$ with images $s_i \in S_i$. Then

(1) $s = \lim s_i$ as schemes, i.e., $\kappa(s) = \mathrm{colim}\, \kappa(s_i)$,
(2) $\overline{\{s\}} = \lim \overline{\{s_i\}}$ as sets, and
(3) $\overline{\{s\}} = \lim \overline{\{s_i\}}$ as schemes where $\overline{\{s\}}$ and $\overline{\{s_i\}}$ are endowed with the reduced induced scheme structure.

Proof. Choose $0 \in I$ and an affine open covering $S_0 = \bigcup_{j \in J} U_{0,j}$. For $i \geq 0$ let $U_{i,j} = f_{i,0}^{-1}(U_{0,j})$ and set $U_j = f_0^{-1}(U_{0,j})$. Here $f_{i'i} : S_{i'} \to S_i$ is the transition morphism and $f_i : S \to S_i$ is the projection. For $j \in J$ the following are equivalent: (a) $s \in U_j$, (b) $s_0 \in U_{0,j}$, (c) $s_i \in U_{i,j}$ for all $i \geq 0$. Let $J' \subset J$ be the set of indices for which (a), (b), (c) are true. Then $\overline{\{s\}} = \bigcup_{j \in J'} (\overline{\{s\}} \cap U_j)$ and similarly for $\overline{\{s_i\}}$ for $i \geq 0$. Note that $\overline{\{s\}} \cap U_j$ is the closure of the set $\{s\}$ in the topological space U_j. Similarly for $\overline{\{s_i\}} \cap U_{i,j}$ for $i \geq 0$. Hence it suffices to prove the lemma in the case S and S_i affine for all i. This reduces us to the algebra question considered in the next paragraph.

Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \mathrm{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Let $\mathfrak{p} \subset A$ be a prime and set $\mathfrak{p}_i = \varphi_i^{-1}(\mathfrak{p})$. Then

$$V(\mathfrak{p}) = \lim_i V(\mathfrak{p}_i)$$

This follows from Lemma 32.4.1 because $A/\mathfrak{p} = \mathrm{colim}\, A_i/\mathfrak{p}_i$. This equality of rings also shows the final statement about reduced induced scheme structures holds true. The equality $\kappa(\mathfrak{p}) = \mathrm{colim}\, \kappa(\mathfrak{p}_i)$ follows from the statement as well. $\square$

In the rest of this section we work in the following situation.

086P Situation 32.4.5. Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$ (Lemma 32.2.2). We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also choose an element $0 \in I$.

In this situation the morphism $S \to S_0$ is affine. It follows that S is quasi-compact and quasi-separated[1]. The type of result we are looking for is the following: If we have an object over S, then for some i there is a similar object over S_i.

01YY Lemma 32.4.6. In Situation 32.4.5.

(1) We have $S_{set} = \lim_i S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.
(2) We have $S_{top} = \lim_i S_{i,top}$ where S_{top} indicates the underlying topological space of the scheme S.

[1] Follows from Morphisms, Lemma 29.11.2, Topology, Definition 5.12.1, and Schemes, Lemma 26.21.12.

(3) If $s, s' \in S$ and s' is not a specialization of s then for some $i \in I$ the image $s'_i \in S_i$ of s' is not a specialization of the image $s_i \in S_i$ of s.
(4) Add more easy facts on topology of S here. (Requirement: whatever is added should be easy in the affine case.)

Proof. Part (1) is a special case of Lemma 32.4.1.

Part (2) is a special case of Lemma 32.4.2.

Part (3) is a special case of Lemma 32.4.4. □

01Z0 Lemma 32.4.7. In Situation 32.4.5. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on S_0. Set $\mathcal{F}_i = f_{i0}^*\mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^*\mathcal{F}_0$. Then

$$\Gamma(S, \mathcal{F}) = \mathrm{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i)$$

Proof. Write $\mathcal{A}_j = f_{i0,*}\mathcal{O}_{S_i}$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras (see Morphisms, Lemma 29.11.5) and S_i is the relative spectrum of $\mathcal{A}_i$ over S_0. In the proof of Lemma 32.2.2 we constructed S as the relative spectrum of $\mathcal{A} = \mathrm{colim}_{i \geq 0} \mathcal{A}_i$ over S_0. Set

$$\mathcal{M}_i = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}_i$$

and

$$\mathcal{M} = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}.$$

Then we have $f_{i0,*}\mathcal{F}_i = \mathcal{M}_i$ and $f_{0,*}\mathcal{F} = \mathcal{M}$. Since $\mathcal{A}$ is the colimit of the sheaves $\mathcal{A}_i$ and since tensor product commutes with directed colimits, we conclude that $\mathcal{M} = \mathrm{colim}_{i \geq 0} \mathcal{M}_i$. Since S_0 is quasi-compact and quasi-separated we see that

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \Gamma(S_0, \mathcal{M}) \\ &= \Gamma(S_0, \mathrm{colim}_{i \geq 0} \mathcal{M}_i) \\ &= \mathrm{colim}_{i \geq 0} \Gamma(S_0, \mathcal{M}_i) \\ &= \mathrm{colim}_{i \geq 0} \Gamma(S_i, \mathcal{F}_i) \end{aligned}$$

see Sheaves, Lemma 6.29.1 and Topology, Lemma 5.27.1 for the middle equality. □

01Z3 Lemma 32.4.8. In Situation 32.4.5. Suppose for each i we are given a nonempty closed subset $Z_i \subset S_i$ with $f_{i'i}(Z_{i'}) \subset Z_i$ for all $i' \geq i$. Then there exists a point $s \in S$ with $f_i(s) \in Z_i$ for all i.

Proof. Let $Z_i \subset S_i$ also denote the reduced closed subscheme associated to Z_i, see Schemes, Definition 26.12.5. A closed immersion is affine, and a composition of affine morphisms is affine (see Morphisms, Lemmas 29.11.9 and 29.11.7), and hence $Z_{i'} \to S_i$ is affine when $i' \geq i$. We conclude that the morphism $f_{i'i} : Z_{i'} \to Z_i$ is affine by Morphisms, Lemma 29.11.11. Each of the schemes Z_i is quasi-compact as a closed subscheme of a quasi-compact scheme. Hence we may apply Lemma 32.4.3 to see that $Z = \lim_i Z_i$ is nonempty. Since there is a canonical morphism $Z \to S$ we win. □

05F3 Lemma 32.4.9. In Situation 32.4.5. Suppose we are given an i and a morphism $T \to S_i$ such that

(1) $T \times_{S_i} S = \emptyset$, and
(2) T is quasi-compact.

Then $T \times_{S_i} S_{i'} = \emptyset$ for all sufficiently large i'.

Proof. By Lemma 32.2.3 we see that $T \times_{S_i} S = \lim_{i' \geq i} T \times_{S_i} S_{i'}$. Hence the result follows from Lemma 32.4.3. $\square$

05F4 Lemma 32.4.10. In Situation 32.4.5. Suppose we are given an i and a locally constructible subset $E \subset S_i$ such that $f_i(S) \subset E$. Then $f_{i'i}(S_{i'}) \subset E$ for all sufficiently large i'.

Proof. Writing S_i as a finite union of open affine subschemes reduces the question to the case that S_i is affine and E is constructible, see Lemma 32.2.2 and Properties, Lemma 28.2.1. In this case the complement $S_i \setminus E$ is constructible too. Hence there exists an affine scheme T and a morphism $T \to S_i$ whose image is $S_i \setminus E$, see Algebra, Lemma 10.29.4. By Lemma 32.4.9 we see that $T \times_{S_i} S_{i'}$ is empty for all sufficiently large i', and hence $f_{i'i}(S_{i'}) \subset E$ for all sufficiently large i'. $\square$

01Z4 Lemma 32.4.11. In Situation 32.4.5 we have the following:

(1) Given any quasi-compact open $V \subset S = \lim_i S_i$ there exists an $i \in I$ and a quasi-compact open $V_i \subset S_i$ such that $f_i^{-1}(V_i) = V$.
(2) Given $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$ there exists an index $i'' \geq i, i'$ such that $f_{i''i}^{-1}(V_i) = f_{i''i'}^{-1}(V_{i'})$.
(3) If $V_{1,i}, \ldots, V_{n,i} \subset S_i$ are quasi-compact opens and $S = f_i^{-1}(V_{1,i}) \cup \ldots \cup f_i^{-1}(V_{n,i})$ then $S_{i'} = f_{i'i}^{-1}(V_{1,i}) \cup \ldots \cup f_{i'i}^{-1}(V_{n,i})$ for some $i' \geq i$.

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = U_{1,0} \cup \ldots \cup U_{m,0}$. Denote $U_{j,i} \subset S_i$ the inverse image of $U_{j,0}$ under the transition morphism for $i \geq 0$. Denote U_j the inverse image of $U_{j,0}$ in S. Note that $U_j = \lim_i U_{j,i}$ is a limit of affine schemes.

We first prove the uniqueness statement: Let $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$. It suffices to show that $f_{i''i}^{-1}(V_i \cap U_{j,i''})$ and $f_{i''i'}^{-1}(V_{i'} \cap U_{j,i''})$ become equal for i'' large enough. Hence we reduce to the case of a limit of affine schemes. In this case write $S = \mathrm{Spec}(R)$ and $S_i = \mathrm{Spec}(R_i)$ for all $i \in I$. We may write $V_i = S_i \setminus V(h_1, \ldots, h_m)$ and $V_{i'} = S_{i'} \setminus V(g_1, \ldots, g_n)$. The assumption means that the ideals $\sum g_j R$ and $\sum h_j R$ have the same radical in R. This means that $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ for some $N \gg 0$ and $a_{jj'}$ and $b_{jj'}$ in R. Since $R = \mathrm{colim}_i R_i$ we can chose an index $i'' \geq i$ such that the equations $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ hold in $R_{i''}$ for some $a_{jj'}$ and $b_{jj'}$ in $R_{i''}$. This implies that the ideals $\sum g_j R_{i''}$ and $\sum h_j R_{i''}$ have the same radical in $R_{i''}$ as desired.

We prove existence: If S_0 is affine, then $S_i = \mathrm{Spec}(R_i)$ for all $i \geq 0$ and $S = \mathrm{Spec}(R)$ with $R = \mathrm{colim}\, R_i$. Then $V = S \setminus V(g_1, \ldots, g_n)$ for some $g_1, \ldots, g_n \in R$. Choose any i large enough so that each of the g_j comes from an element $g_{j,i} \in R_i$ and take $V_i = S_i \setminus V(g_{1,i}, \ldots, g_{n,i})$. If S_0 is general, then the opens $V \cap U_j$ are quasi-compact because S is quasi-separated. Hence by the affine case we see that for each $j = 1, \ldots, m$ there exists an $i_j \in I$ and a quasi-compact open $V_{i_j} \subset U_{j,i_j}$ whose inverse image in U_j is $V \cap U_j$. Set $i = \max(i_1, \ldots, i_m)$ and let $V_i = \bigcup f_{ii_j}^{-1}(V_{i_j})$.

The statement on coverings follows from the uniqueness statement for the opens $V_{1,i} \cup \ldots \cup V_{n,i}$ and S_i of S_i. $\square$

01Z5 Lemma 32.4.12. In Situation 32.4.5 if S is quasi-affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are quasi-affine.

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Let $s \in S$. We may choose an affine open $U_0 \subset S_0$ containing $f_0(s)$. Since S is quasi-affine we may choose an element $a \in \Gamma(S, \mathcal{O}_S)$ such that $s \in D(a) \subset f_0^{-1}(U_0)$, and such that $D(a)$ is affine. By Lemma 32.4.7 there exists an $i \geq 0$ such that a comes from an element $a_i \in \Gamma(S_i, \mathcal{O}_{S_i})$. For any index $j \geq i$ we denote a_j the image of a_i in the global sections of the structure sheaf of S_j. Consider the opens $D(a_j) \subset S_j$ and $U_j = f_{j0}^{-1}(U_0)$. Note that U_j is affine and $D(a_j)$ is a quasi-compact open of S_j, see Properties, Lemma 28.26.4 for example. Hence we may apply Lemma 32.4.11 to the opens U_j and $U_j \cup D(a_j)$ to conclude that $D(a_j) \subset U_j$ for some $j \geq i$. For such an index j we see that $D(a_j) \subset S_j$ is an affine open (because $D(a_j)$ is a standard affine open of the affine open U_j) containing the image $f_j(s)$.

We conclude that for every $s \in S$ there exist an index $i \in I$, and a global section $a \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that $D(a) \subset S_i$ is an affine open containing $f_i(s)$. Because S is quasi-compact we may choose a single index $i \in I$ and global sections $a_1, \ldots, a_m \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that each $D(a_j) \subset S_i$ is affine open and such that $f_i : S \to S_i$ has image contained in the union $W_i = \bigcup_{j=1,\ldots,m} D(a_j)$. For $i' \geq i$ set $W_{i'} = f_{i'i}^{-1}(W_i)$. Since $f_i^{-1}(W_i)$ is all of S we see (by Lemma 32.4.11 again) that for a suitable $i' \geq i$ we have $S_{i'} = W_{i'}$. Thus we may replace i by i' and assume that $S_i = \bigcup_{j=1,\ldots,m} D(a_j)$. This implies that $\mathcal{O}_{S_i}$ is an ample invertible sheaf on S_i (see Properties, Definition 28.26.1) and hence that S_i is quasi-affine, see Properties, Lemma 28.27.1. Hence we win. $\square$

01Z6 Lemma 32.4.13. In Situation 32.4.5 if S is affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are affine.

Proof. By Lemma 32.4.12 we may assume that S_0 is quasi-affine for some $0 \in I$. Set $R_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. Then S_0 is a quasi-compact open of $T_0 = \text{Spec}(R_0)$. Denote $j_0 : S_0 \to T_0$ the corresponding quasi-compact open immersion. For $i \geq 0$ set $\mathcal{A}_i = f_{i0,*}\mathcal{O}_{S_i}$. Since f_{i0} is affine we see that $S_i = \underline{\text{Spec}}_{S_0}(\mathcal{A}_i)$. Set $T_i = \underline{\text{Spec}}_{T_0}(j_{0,*}\mathcal{A}_i)$. Then $T_i \to T_0$ is affine, hence T_i is affine. Thus T_i is the spectrum of

$$R_i = \Gamma(T_0, j_{0,*}\mathcal{A}_i) = \Gamma(S_0, \mathcal{A}_i) = \Gamma(S_i, \mathcal{O}_{S_i}).$$

Write $S = \text{Spec}(R)$. We have $R = \text{colim}_i R_i$ by Lemma 32.4.7. Hence also $S = \lim_i T_i$. As formation of the relative spectrum commutes with base change, the inverse image of the open $S_0 \subset T_0$ in T_i is S_i. Let $Z_0 = T_0 \setminus S_0$ and let $Z_i \subset T_i$ be the inverse image of Z_0. As $S_i = T_i \setminus Z_i$, it suffices to show that Z_i is empty for some i. Assume Z_i is nonempty for all i to get a contradiction. By Lemma 32.4.8 there exists a point s of $S = \lim T_i$ which maps to a point of Z_i for every i. But $S = \lim_i S_i$, and hence we arrive at a contradiction by Lemma 32.4.6. $\square$

086Q Lemma 32.4.14. In Situation 32.4.5 if S is separated, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are separated.

Proof. Choose a finite affine open covering $S_0 = U_{0,1} \cup \ldots \cup U_{0,m}$. Set $U_{i,j} \subset S_i$ and $U_j \subset S$ equal to the inverse image of $U_{0,j}$. Note that $U_{i,j}$ and U_j are affine. As S is separated the intersections $U_{j_1} \cap U_{j_2}$ are affine. Since $U_{j_1} \cap U_{j_2} = \lim_{i \geq 0} U_{i,j_1} \cap U_{i,j_2}$

we see that $U_{i,j_1} \cap U_{i,j_2}$ is affine for large i by Lemma 32.4.13. To show that S_i is separated for large i it now suffices to show that

$$\mathcal{O}_{S_i}(U_{i,j_1}) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{S_i}(U_{i,j_2}) \longrightarrow \mathcal{O}_{S_i}(U_{i,j_1} \cap U_{i,j_2})$$

is surjective for large i (Schemes, Lemma 26.21.7).

To get rid of the annoying indices, assume we have affine opens $U, V \subset S_0$ such that $U \cap V$ is affine too. Let $U_i, V_i \subset S_i$, resp. $U, V \subset S$ be the inverse images. We have to show that $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i) \to \mathcal{O}(U_i \cap V_i)$ is surjective for i large enough and we know that $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective. Note that $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0) \to \mathcal{O}(U_0 \cap V_0)$ is of finite type, as the diagonal morphism $S_i \to S_i \times S_i$ is an immersion (Schemes, Lemma 26.21.2) hence locally of finite type (Morphisms, Lemmas 29.15.2 and 29.15.5). Thus we can choose elements $f_{0,1}, \ldots, f_{0,n} \in \mathcal{O}(U_0 \cap V_0)$ which generate $\mathcal{O}(U_0 \cap V_0)$ over $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0)$. Observe that for $i \geq 0$ the diagram of schemes

$$\begin{tikzcd} U_i \cap V_i \arrow[r] \arrow[d] & U_i \arrow[d] \\ U_0 \cap V_0 \arrow[r] & U_0 \end{tikzcd}$$

is cartesian. Thus we see that the images $f_{i,1}, \ldots, f_{i,n} \in \mathcal{O}(U_i \cap V_i)$ generate $\mathcal{O}(U_i \cap V_i)$ over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_0)$ and a fortiori over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$. By assumption the images $f_1, \ldots, f_n \in \mathcal{O}(U \otimes V)$ are in the image of the map $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$. Since $\mathcal{O}(U) \otimes \mathcal{O}(V) = \operatorname{colim} \mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$ we see that they are in the image of the map at some finite level and the lemma is proved. □

09MT Lemma 32.4.15. In Situation 32.4.5 let $\mathcal{L}_0$ be an invertible sheaf of modules on S_0. If the pullback $\mathcal{L}$ to S is ample, then for some $i \in I$ the pullback $\mathcal{L}_i$ to S_i is ample.

Proof. The assumption means there are finitely many sections $s_1, \ldots, s_m \in \Gamma(S, \mathcal{L})$ such that S_{s_j} is affine and such that $S = \bigcup S_{s_j}$, see Properties, Definition 28.26.1. By Lemma 32.4.7 we can find an $i \in I$ and sections $s_{i,j} \in \Gamma(S_i, \mathcal{L}_i)$ mapping to s_j. By Lemma 32.4.13 we may, after increasing i, assume that $(S_i)_{s_{i,j}}$ is affine for $j = 1, \ldots, m$. By Lemma 32.4.11 we may, after increasing i a last time, assume that $S_i = \bigcup (S_i)_{s_{i,j}}$. Then $\mathcal{L}_i$ is ample by definition. □

081B Lemma 32.4.16. Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over S.

(1) If $Y \to X$ is a closed immersion, X_i quasi-compact, and Y locally of finite type over S, then $Y \to X_i$ is a closed immersion for i large enough.
(2) If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to S$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.
(3) If $Y \to X$ is an isomorphism, X_i quasi-compact, $X_i \to S$ locally of finite type, the transition morphisms $X_{i'} \to X_i$ are closed immersions, and $Y \to S$ is locally of finite presentation, then $Y \to X_i$ is an isomorphism for i large enough.

Proof. Proof of (1). Choose $0 \in I$ and a finite affine open covering $X_0 = U_{0,1} \cup \ldots \cup U_{0,m}$ with the property that $U_{0,j}$ maps into an affine open $W_j \subset S$. Let $V_j \subset Y$, resp. $U_{i,j} \subset X_i$, $i \geq 0$, resp. $U_j \subset X$ be the inverse image of $U_{0,j}$. It suffices to prove that $V_j \to U_{i,j}$ is a closed immersion for i sufficiently large and we know that $V_j \to U_j$ is a closed immersion. Thus we reduce to the following algebra fact: If $A = \operatorname{colim} A_i$ is a directed colimit of R-algebras, $A \to B$ is a surjection of R-algebras, and B is a finitely generated R-algebra, then $A_i \to B$ is surjective for i sufficiently large.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open $X_0' \subset X_0$ such that $Y \to X_0$ factors through X_0'. After replacing X_i by the inverse image of X_0' for $i \geq 0$ we may assume all X_i' are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 32.4.11 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds.

Proof of (3). Working affine locally on X_0 for some $0 \in I$ as in the proof of (1) we reduce to the following algebra fact: If $A = \lim A_i$ is a directed colimit of R-algebras with surjective transition maps and A of finite presentation over A_0, then $A = A_i$ for some i. Namely, write $A = A_0/(f_1, \ldots, f_n)$. Pick i such that $f_1, \ldots, f_n$ map to zero under the surjective map $A_0 \to A_i$. $\square$

01ZH Lemma 32.4.17. Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume

(1) S quasi-separated,
(2) X_i quasi-compact and quasi-separated,
(3) $X \to S$ separated.

Then $X_i \to S$ is separated for all i large enough.

Proof. Let $0 \in I$. Note that I is nonempty as the limit is directed. As X_0 is quasi-compact we can find finitely many affine opens $U_1, \ldots, U_n \subset S$ such that $X_0 \to S$ maps into $U_1 \cup \ldots \cup U_n$. Denote $h_i : X_i \to S$ the structure morphism. It suffices to check that for some $i \geq 0$ the morphisms $h_i^{-1}(U_j) \to U_j$ are separated for $j = 1, \ldots, n$. Since S is quasi-separated the morphisms $U_j \to S$ are quasi-compact. Hence $h_i^{-1}(U_j)$ is quasi-compact and quasi-separated. In this way we reduce to the case S affine. In this case we have to show that X_i is separated and we know that X is separated. Thus the lemma follows from Lemma 32.4.14. $\square$

09ZM Lemma 32.4.18. Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume

(1) S quasi-compact and quasi-separated,
(2) X_i quasi-compact and quasi-separated,
(3) $X \to S$ affine.

Then $X_i \to S$ is affine for i large enough.

Proof. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. For each j the scheme $f^{-1}(V_j) = \lim_i f_i^{-1}(V_j)$ is affine (as a finite morphism is affine by definition). Hence by Lemma 32.4.13 there exists an $i \in I$ such that each $f_i^{-1}(V_j)$ is affine. In other words, $f_i : X_i \to S$ is affine for i large enough, see Morphisms, Lemma 29.11.3. $\square$

09ZN Lemma 32.4.19. Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume

(1) S quasi-compact and quasi-separated,
(2) X_i quasi-compact and quasi-separated,
(3) the transition morphisms $X_{i'} \to X_i$ are finite,
(4) $X_i \to S$ locally of finite type
(5) $X \to S$ integral.

Then $X_i \to S$ is finite for i large enough.

Proof. By Lemma 32.4.18 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is finite over V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 29.44.3). Namely, for $i' \geq i$ the composition $X_{i'} \to X_i \to S$ will be finite as a composition of finite morphisms (Morphisms, Lemma 29.44.5). This reduces us to the affine case: Let R be a ring and $A = \operatorname{colim} A_i$ with $R \to A$ integral and $A_i \to A_{i'}$ finite for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that A_i is finite over R for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since A is integral over R we can find monic polynomials $P_j \in R[T]$ such that $P_j(x_j) = 0$ in A. Thus there exists an $i' \geq i$ such that $P_j(x_j) = 0$ in $A_{i'}$ for $j = 1, \ldots, m$. Then the image A_i' of A_i in $A_{i'}$ is finite over R by Algebra, Lemma 10.36.5. Since $A_i' \subset A_{i'}$ is finite too we conclude that $A_{i'}$ is finite over R by Algebra, Lemma 10.7.3. □

0A0N Lemma 32.4.20. Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume

(1) S quasi-compact and quasi-separated,
(2) X_i quasi-compact and quasi-separated,
(3) the transition morphisms $X_{i'} \to X_i$ are closed immersions,
(4) $X_i \to S$ locally of finite type
(5) $X \to S$ a closed immersion.

Then $X_i \to S$ is a closed immersion for i large enough.

Proof. By Lemma 32.4.18 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is a closed subscheme of V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 29.2.1). This reduces us to the affine case: Let R be a ring and $A = \operatorname{colim} A_i$ with $R \to A$ surjective and $A_i \to A_{i'}$ surjective for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that $R \to A_i$ is surjective for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since $R \to A$ is surjective we can find $r_j \in R$ such that r_j maps to x_j in A. Thus there exists an $i' \geq i$ such that r_j maps to the image of x_j in $A_{i'}$ for $j = 1, \ldots, m$. Since $A_i \to A_{i'}$ is surjective this implies that $R \to A_{i'}$ is surjective. □

0GIH Lemma 32.4.21. Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume

(1) S quasi-separated,
(2) X_i quasi-compact and quasi-separated,
(3) the transition morphisms $X_{i'} \to X_i$ are closed immersions,
(4) $X_i \to S$ locally of finite type, and

(5) $X \to S$ an immersion.

Then $X_i \to S$ is an immersion for i large enough.

Proof. Choose an open subscheme $U \subset S$ such that $X \to S$ factors as a closed immersion $X \to U$ composed with the inclusion morphism $U \to S$. Since X is quasi-compact, we may shrink U and assume U is quasi-compact. Denote $V_i \subset X_i$ the inverse image of U. Since V_i pulls back to X we see that $V_i = X_i$ for all i large enough by Lemma 32.4.11. Thus we may assume $X = \lim X_i$ in the category of schemes over U. Then we see that $X_i \to U$ is a closed immersion for i large enough by Lemma 32.4.20. This proves the lemma. □

32.5. Absolute Noetherian Approximation

01Z1 A nice reference for this section is Appendix C of the article by Thomason and Trobaugh [TT90]. See Categories, Section 4.21 for our conventions regarding directed systems. We will use the existence result and properties of the limit from Section 32.2 without further mention.

01Z7 Lemma 32.5.1. Let W be a quasi-affine scheme of finite type over $\mathbf{Z}$. Suppose $W \to \text{Spec}(R)$ is an open immersion into an affine scheme. There exists a finite type $\mathbf{Z}$-algebra $A \subset R$ which induces an open immersion $W \to \text{Spec}(A)$. Moreover, R is the directed colimit of such subalgebras.

Proof. Choose an affine open covering $W = \bigcup_{i=1,\ldots,n} W_i$ such that each W_i is a standard affine open in $\text{Spec}(R)$. In other words, if we write $W_i = \text{Spec}(R_i)$ then $R_i = R_{f_i}$ for some $f_i \in R$. Choose finitely many $x_{ij} \in R_i$ which generate R_i over $\mathbf{Z}$. Pick an $N \gg 0$ such that each $f_i^N x_{ij}$ comes from an element of R, say $y_{ij} \in R$. Set A equal to the $\mathbf{Z}$-algebra generated by the f_i and the y_{ij} and (optionally) finitely many additional elements of R. Then A works. Details omitted. □

01Z9 Lemma 32.5.2. Suppose given a cartesian diagram of rings

$$\begin{tikzcd} B \arrow[r, "s"] & R \\ B' \arrow[u] \arrow[r] & R' \arrow[u, "t"'] \end{tikzcd}$$

Let $W' \subset \text{Spec}(R')$ be an open of the form $W' = D(f_1) \cup \ldots \cup D(f_n)$ such that $t(f_i) = s(g_i)$ for some $g_i \in B$ and $B_{g_i} \cong R_{s(g_i)}$. Then $B' \to R'$ induces an open immersion of W' into $\text{Spec}(B')$.

Proof. Set $h_i = (g_i, f_i) \in B'$. More on Algebra, Lemma 15.5.3 shows that $(B')_{h_i} \cong (R')_{f_i}$ as desired. □

The following lemma is a precise statement of Noetherian approximation.

07RN Lemma 32.5.3. Let S be a quasi-compact and quasi-separated scheme. Let $V \subset S$ be a quasi-compact open. Let I be a directed set and let $(V_i, f_{ii'})$ be an inverse system of schemes over I with affine transition maps, with each V_i of finite type over $\mathbf{Z}$, and with $V = \lim V_i$. Then there exist

(1) a directed set J,
(2) an inverse system of schemes $(S_j, g_{jj'})$ over J,
(3) an order preserving map $\alpha : J \to I$,

(4) open subschemes $V'_j \subset S_j$, and
(5) isomorphisms $V'_j \to V_{\alpha(j)}$

such that

(1) the transition morphisms $g_{jj'} : S_j \to S_{j'}$ are affine,
(2) each S_j is of finite type over $\mathbf{Z}$,
(3) $g_{jj'}^{-1}(V'_{j'}) = V'_j$,
(4) $S = \lim S_j$ and $V = \lim V'_j$, and
(5) the diagrams

$$\begin{tikzcd} V \arrow[d] \arrow[dr] & \\ V'_j \arrow[r] & V_{\alpha(j)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V'_j \arrow[r] \arrow[d] & V_{\alpha(j)} \arrow[d] \\ V'_{j'} \arrow[r] & V_{\alpha(j')} \end{tikzcd}$$

are commutative.

Proof. Set $Z = S \setminus V$. Choose affine opens $U_1, \ldots, U_m \subset S$ such that $Z \subset \bigcup_{l=1,\ldots,m} U_l$. Consider the opens

$$V \subset V \cup U_1 \subset V \cup U_1 \cup U_2 \subset \ldots \subset V \cup \bigcup\nolimits_{l=1,\ldots,m} U_l = S$$

If we can prove the lemma successively for each of the cases

$$V \cup U_1 \cup \ldots \cup U_l \subset V \cup U_1 \cup \ldots \cup U_{l+1}$$

then the lemma will follow for $V \subset S$. In each case we are adding one affine open. Thus we may assume

(1) $S = U \cup V$,
(2) U affine open in S,
(3) V quasi-compact open in S, and
(4) $V = \lim_i V_i$ with $(V_i, f_{ii'})$ an inverse system over a directed set I, each $f_{ii'}$ affine and each V_i of finite type over $\mathbf{Z}$.

Denote $f_i : V \to V_i$ the projections. Set $W = U \cap V$. As S is quasi-separated, this is a quasi-compact open of V. By Lemma 32.4.11 (and after shrinking I) we may assume that there exist opens $W_i \subset V_i$ such that $f_{ii'}^{-1}(W_{i'}) = W_i$ and such that $f_i^{-1}(W_i) = W$. Since W is a quasi-compact open of U it is quasi-affine. Hence we may assume (after shrinking I again) that W_i is quasi-affine for all i, see Lemma 32.4.12.

Write $U = \text{Spec}(B)$. Set $R = \Gamma(W, \mathcal{O}_W)$, and $R_i = \Gamma(W_i, \mathcal{O}_{W_i})$. By Lemma 32.4.7 we have $R = \text{colim}_i R_i$. Now we have the maps of rings

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ & R_i \arrow[u, "t_i"'] \end{tikzcd}$$

We set $B_i = \{(b, r) \in B \times R_i \mid s(b) = t_i(t)\}$ so that we have a cartesian diagram

$$\begin{tikzcd} B \arrow[r, "s"'] & R \\ B_i \arrow[u] \arrow[r] & R_i \arrow[u, "t_i"'] \end{tikzcd}$$

for each i. The transition maps $R_i \to R_{i'}$ induce maps $B_i \to B_{i'}$. It is clear that $B = \mathrm{colim}_i B_i$. In the next paragraph we show that for all sufficiently large i the composition $W_i \to \mathrm{Spec}(R_i) \to \mathrm{Spec}(B_i)$ is an open immersion.

As W is a quasi-compact open of $U = \mathrm{Spec}(B)$ we can find a finitely many elements $g_l \in B$, $l = 1, \ldots, m$ such that $D(g_l) \subset W$ and such that $W = \bigcup_{l=1,\ldots,m} D(g_l)$. Note that this implies $D(g_l) = W_{s(g_l)}$ as open subsets of U, where $W_{s(g_l)}$ denotes the largest open subset of W on which $s(g_l)$ is invertible. Hence

$$B_{g_l} = \Gamma(D(g_l), \mathcal{O}_U) = \Gamma(W_{s(g_l)}, \mathcal{O}_W) = R_{s(g_l)},$$

where the last equality is Properties, Lemma 28.17.1. Since $W_{s(g_l)}$ is affine this also implies that $D(s(g_l)) = W_{s(g_l)}$ as open subsets of $\mathrm{Spec}(R)$. Since $R = \mathrm{colim}_i R_i$ we can (after shrinking I) assume there exist $g_{l,i} \in R_i$ for all $i \in I$ such that $s(g_l) = t_i(g_{l,i})$. Of course we choose the $g_{l,i}$ such that $g_{l,i}$ maps to $g_{l,i'}$ under the transition maps $R_i \to R_{i'}$. Then, by Lemma 32.4.11 we can (after shrinking I again) assume the corresponding opens $D(g_{l,i}) \subset \mathrm{Spec}(R_i)$ are contained in W_i for $l = 1, \ldots, m$ and cover W_i. We conclude that the morphism $W_i \to \mathrm{Spec}(R_i) \to \mathrm{Spec}(B_i)$ is an open immersion, see Lemma 32.5.2.

By Lemma 32.5.1 we can write B_i as a directed colimit of subalgebras $A_{i,p} \subset B_i$, $p \in P_i$ each of finite type over $\mathbf{Z}$ and such that W_i is identified with an open subscheme of $\mathrm{Spec}(A_{i,p})$. Let $S_{i,p}$ be the scheme obtained by glueing V_i and $\mathrm{Spec}(A_{i,p})$ along the open W_i, see Schemes, Section 26.14. Here is the resulting commutative diagram of schemes:

$$\begin{tikzcd}
 & & V \arrow[dll] \arrow[d] & W \arrow[l] \arrow[dll] \arrow[d] \\
V_i \arrow[d] & W_i \arrow[l] \arrow[d] & S \arrow[dll] & U \arrow[l] \arrow[dll] \\
S_{i,p} & \mathrm{Spec}(A_{i,p}) \arrow[l] & &
\end{tikzcd}$$

The morphism $S \to S_{i,p}$ arises because the upper right square is a pushout in the category of schemes. Note that $S_{i,p}$ is of finite type over $\mathbf{Z}$ since it has a finite affine open covering whose members are spectra of finite type $\mathbf{Z}$-algebras. We define a preorder on $J = \coprod_{i \in I} P_i$ by the rule $(i', p') \geq (i, p)$ if and only if $i' \geq i$ and the map $B_i \to B_{i'}$ maps $A_{i,p}$ into $A_{i',p'}$. This is exactly the condition needed to define a morphism $S_{i',p'} \to S_{i,p}$: namely make a commutative diagram as above using the transition morphisms $V_{i'} \to V_i$ and $W_{i'} \to W_i$ and the morphism $\mathrm{Spec}(A_{i',p'}) \to \mathrm{Spec}(A_{i,p})$ induced by the ring map $A_{i,p} \to A_{i',p'}$. The relevant commutativities have been built into the constructions. We claim that S is the directed limit of the schemes $S_{i,p}$. Since by construction the schemes V_i have limit V this boils down to the fact that B is the limit of the rings $A_{i,p}$ which is true by construction. The map $\alpha : J \to I$ is given by the rule $j = (i,p) \mapsto i$. The open subscheme V'_j is just the image of $V_i \to S_{i,p}$ above. The commutativity of the diagrams in (5) is clear from the construction. This finishes the proof of the lemma. $\square$

01ZA Proposition 32.5.4. Let S be a quasi-compact and quasi-separated scheme. There exist a directed set I and an inverse system of schemes $(S_i, f_{ii'})$ over I such that

(1) the transition morphisms $f_{ii'}$ are affine
(2) each S_i is of finite type over $\mathbf{Z}$, and
(3) $S = \lim_i S_i$.

Proof. This is a special case of Lemma 32.5.3 with $V = \emptyset$. □

32.6. Limits and morphisms of finite presentation

01ZB The following is a generalization of Algebra, Lemma 10.127.3.

01ZC Proposition 32.6.1. Let $f : X \to S$ be a morphism of schemes. The following are equivalent: [DG67, IV, Proposition 8.14.2]

(1) The morphism f is locally of finite presentation.
(2) For any directed set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each T_i affine, we have

$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X)$$

(3) For any directed set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each $f_{ii'}$ affine and every T_i quasi-compact and quasi-separated as a scheme, we have

$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X)$$

Proof. It is clear that (3) implies (2).

Let us prove that (2) implies (1). Assume (2). Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \mathrm{colim}_i A_i$. According to Algebra, Lemma 10.127.3 we have to show that

$$\mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A) = \mathrm{colim}_i \mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i)$$

Consider the schemes $T_i = \mathrm{Spec}(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := \mathrm{Spec}(A) = \lim_i T_i$. The formula above becomes in terms of morphism sets of schemes

$$\mathrm{Mor}_V(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_V(T_i, U).$$

We first observe that $\mathrm{Mor}_V(T_i, U) = \mathrm{Mor}_S(T_i, U)$ and $\mathrm{Mor}_V(T, U) = \mathrm{Mor}_S(T, U)$. Hence we have to show that

$$\mathrm{Mor}_S(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, U)$$

and we are given that

$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X).$$

Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 32.4.8 there exists a point t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction.

Finally, let us prove that (1) implies (3). Assume (1). Let an inverse directed system $(T_i, f_{ii'})$ of S-schemes be given. Assume the morphisms $f_{ii'}$ are affine and each T_i is quasi-compact and quasi-separated as a scheme. Let $T = \lim_i T_i$. Denote $f_i : T \to T_i$ the projection morphisms. We have to show:

(a) Given morphisms $g_i, g'_i : T_i \to X$ over S such that $g_i \circ f_i = g'_i \circ f_i$, then there exists an $i' \geq i$ such that $g_i \circ f_{i'i} = g'_i \circ f_{i'i}$.
(b) Given any morphism $g : T \to X$ over S there exists an $i \in I$ and a morphism $g_i : T_i \to X$ such that $g = f_i \circ g_i$.

First let us prove the uniqueness part (a). Let $g_i, g'_i : T_i \to X$ be morphisms such that $g_i \circ f_i = g'_i \circ f_i$. For any $i' \geq i$ we set $g_{i'} = g_i \circ f_{i'i}$ and $g'_{i'} = g'_i \circ f_{i'i}$. We also set $g = g_i \circ f_i = g'_i \circ f_i$. Consider the morphism $(g_i, g'_i) : T_i \to X \times_S X$. Set

$$W = \bigcup\nolimits_{U \subset X \text{ affine open}, V \subset S \text{ affine open}, f(U) \subset V} U \times_V U.$$

This is an open in $X \times_S X$, with the property that the morphism $\Delta_{X/S}$ factors through a closed immersion into W, see the proof of Schemes, Lemma 26.21.2. Note that the composition $(g_i, g'_i) \circ f_i : T \to X \times_S X$ is a morphism into W because it factors through the diagonal by assumption. Set $Z_{i'} = (g_{i'}, g'_{i'})^{-1}(X \times_S X \setminus W)$. If each $Z_{i'}$ is nonempty, then by Lemma 32.4.8 there exists a point $t \in T$ which maps to $Z_{i'}$ for all $i' \geq i$. This is a contradiction with the fact that T maps into W. Hence we may increase i and assume that $(g_i, g'_i) : T_i \to X \times_S X$ is a morphism into W. By construction of W, and since T_i is quasi-compact we can find a finite affine open covering $T_i = T_{1,i} \cup \ldots \cup T_{n,i}$ such that $(g_i, g'_i)|_{T_{j,i}}$ is a morphism into $U \times_V U$ for some pair (U, V) as in the definition of W above. Since it suffices to prove that $g_{i'}$ and $g'_{i'}$ agree on each of the $f_{i'i}^{-1}(T_{j,i})$ this reduces us to the affine case. The affine case follows from Algebra, Lemma 10.127.3 and the fact that the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation (see Morphisms, Lemma 29.21.2).

Finally, we prove the existence part (b). Let $g : T \to X$ be a morphism of schemes over S. We can find a finite affine open covering $T = W_1 \cup \ldots \cup W_n$ such that for each $j \in \{1, \ldots, n\}$ there exist affine opens $U_j \subset X$ and $V_j \subset S$ with $f(U_j) \subset V_j$ and $g(W_j) \subset U_j$. By Lemmas 32.4.11 and 32.4.13 (after possibly shrinking I) we may assume that there exist affine open coverings $T_i = W_{1,i} \cup \ldots \cup W_{n,i}$ compatible with transition maps such that $W_j = \lim_i W_{j,i}$. We apply Algebra, Lemma 10.127.3 to the rings corresponding to the affine schemes U_j, V_j, $W_{j,i}$ and W_j using that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_j)$ is of finite presentation (see Morphisms, Lemma 29.21.2). Thus we can find for each j an index $i_j \in I$ and a morphism $g_{j,i_j} : W_{j,i_j} \to X$ such that $g_{j,i_j} \circ f_i|_{W_j} : W_j \to W_{j,i} \to X$ equals $g|_{W_j}$. By part (a) proved above, using the quasi-compactness of $W_{j_1,i} \cap W_{j_2,i}$ which follows as T_i is quasi-separated, we can find an index $i' \in I$ larger than all i_j such that

$$g_{j_1,i_{j_1}} \circ f_{i'i_{j_1}}|_{W_{j_1,i'} \cap W_{j_2,i'}} = g_{j_2,i_{j_2}} \circ f_{i'i_{j_2}}|_{W_{j_1,i'} \cap W_{j_2,i'}}$$

for all $j_1, j_2 \in \{1, \ldots, n\}$. Hence the morphisms $g_{j,i_j} \circ f_{i'i_j}|_{W_{j,i'}}$ glue to given the desired morphism $T_{i'} \to X$. □

05LX Remark 32.6.2. Let S be a scheme. Let us say that a functor $F : (Sch/S)^{opp} \to$ Sets is limit preserving if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \mathrm{colim}_i F(T_i)$. Let X be a scheme over S, and let $h_X : (Sch/S)^{opp} \to$ Sets be its functor of points, see Schemes, Section 26.15. In this terminology Proposition 32.6.1 says that a scheme X is locally of finite presentation over S if and only if h_X is limit preserving.

0CM0 Lemma 32.6.3. Let $f : X \to S$ be a morphism of schemes. If for every directed limit $T = \lim_{i \in I} T_i$ of affine schemes over S the map

$$\mathop{\mathrm{colim}} \mathrm{Mor}_S(T_i, X) \longrightarrow \mathrm{Mor}_S(T, X)$$

is surjective, then f is locally of finite presentation. In other words, in Proposition 32.6.1 parts (2) and (3) it suffices to check surjectivity of the map.

Proof. The proof is exactly the same as the proof of the implication "(2) implies (1)" in Proposition 32.6.1. Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \mathrm{colim}_i A_i$. According to Algebra, Lemma 10.127.3 it suffices to show that

$$\mathrm{colim}_i \mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i) \to \mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A)$$

is surjective. Consider the schemes $T_i = \mathrm{Spec}(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := \mathrm{Spec}(A) = \lim_i T_i$. The formula above becomes in terms of morphism sets of schemes

$$\mathrm{colim}_i \mathrm{Mor}_V(T_i, U) \to \mathrm{Mor}_V(\lim_i T_i, U)$$

We first observe that $\mathrm{Mor}_V(T_i, U) = \mathrm{Mor}_S(T_i, U)$ and $\mathrm{Mor}_V(T, U) = \mathrm{Mor}_S(T, U)$. Hence we have to show that

$$\mathrm{colim}_i \mathrm{Mor}_S(T_i, U) \to \mathrm{Mor}_S(\lim_i T_i, U)$$

is surjective and we are given that

$$\mathrm{colim}_i \mathrm{Mor}_S(T_i, X) \to \mathrm{Mor}_S(\lim_i T_i, X)$$

is surjective. Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 32.4.8 there exists a point t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction. $\square$

The following is an example application of Proposition 32.6.1.

0GWT Lemma 32.6.4. Let S be a scheme. Let X and Y be schemes over S. Assume Y is locally of finite presentation over S. Let $x \in X$ be a closed point such that $U = X \setminus \{x\} \to X$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{X,x}) \setminus \{x\}$ there is a bijection

$$\left\{\text{morphisms } X \to Y \text{ over } S\right\} \longrightarrow \left\{\begin{matrix} (a, b) \text{ where } a : U \to Y \text{ and } b : \mathrm{Spec}(\mathcal{O}_{X,x}) \to Y \\ \text{are morphisms over } S \text{ which agree over } V \end{matrix}\right\}$$

Proof. Let $W \subset X$ be an open neighbourhood of x. By glueing of schemes, see Schemes, Section 26.14 the result holds if we consider pairs of morphisms $a : U \to Y$ and $c : W \to Y$ which agree over $U \cap W$. We have $\mathcal{O}_{X,x} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of x in X. Hence $\mathrm{Spec}(\mathcal{O}_{X,x}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus by Proposition 32.6.1 any morphism $b : \mathrm{Spec}(\mathcal{O}_{X,x}) \to Y$ over S comes from a morphism $c : W \to Y$ for some W as above (and c is unique up to further shrinking W). For every affine open $x \in W$ we see that $U \cap W$ is quasi-compact as $U \to X$ is quasi-compact. Hence $V = \lim W \cap U = \lim W \setminus \{x\}$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 32.2.2). Thus if a and b agree over V, then after shrinking

W we see that a and c agree over $U \cap W$ (by the same proposition). The lemma follows. □

32.7. Relative approximation

09MU We discuss variants of Proposition 32.5.4 over a base.

0GS1 **Lemma 32.7.1.** Let $f : X \to S$ be a morphism of quasi-compact and quasi-separated schemes. Then there exists a direct set I and an inverse system $(f_i : X_i \to S_i)$ of morphisms schemes over I, such that the transition morphisms $X_i \to X_{i'}$ and $S_i \to S_{i'}$ are affine, such that X_i and S_i are of finite type over $\mathbf{Z}$, and such that $(X \to S) = \lim(X_i \to S_i)$.

Proof. Write $X = \lim_{a \in A} X_a$ and $S = \lim_{b \in B} S_b$ as in Proposition 32.5.4, i.e., with X_a and S_b of finite type over $\mathbf{Z}$ and with affine transition morphisms.

Fix $b \in B$. By Proposition 32.6.1 applied to S_b and $X = \lim X_a$ over $\mathbf{Z}$ we find there exists an $a \in A$ and a morphism $f_{a,b} : X_a \to S_b$ making the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \arrow[d] \\ X_a \arrow[r] & S_b \end{tikzcd}$$

commute. Let I be the set of triples $(a, b, f_{a,b})$ we obtain in this manner.

Let $(a, b, f_{a,b})$ and $(a', b', f_{a',b'})$ be in I. Let $b'' \leq \min(b, b')$. By Proposition 32.6.1 again, there exists an $a'' \geq \max(a, a')$ such that the compositions $X_{a''} \to X_a \to S_b \to S_{b''}$ and $X_{a''} \to X_{a'} \to S_{b'} \to S_{b''}$ are equal. We endow I with the preorder

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b \geq b',\ \text{and } g_{b,b'} \circ f_{a,b} = f_{a',b'} \circ h_{a,a'}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b,b'} : S_b \to S_{b'}$ are the transition morphisms. The remarks above show that I is directed and that the maps $I \to A$, $(a, b, f_{a,b}) \mapsto a$ and $I \to B$, $(a, b, f_{a,b})$ are cofinal. If for $i = (a, b, f_{a,b})$ we set $X_i = X_a$, $S_i = S_b$, and $f_i = f_{a,b}$, then we get an inverse system of morphisms over I and we have

$$\lim_{i \in I} X_i = \lim_{a \in A} X_a = X \quad \text{and} \quad \lim_{i \in I} S_i = \lim_{b \in B} S_b = S$$

by Categories, Lemma 4.17.4 (recall that limits over I are really limits over the opposite category associated to I and hence cofinal turns into initial). This finishes the proof. □

09MV **Lemma 32.7.2.** Let $f : X \to S$ be a morphism of schemes. Assume that

(1) X is quasi-compact and quasi-separated, and
(2) S is quasi-separated.

Then $X = \lim X_i$ is a limit of a directed system of schemes X_i of finite presentation over S with affine transition morphisms over S.

Proof. Since $f(X)$ is quasi-compact we may replace S by a quasi-compact open containing $f(X)$. Hence we may assume S is quasi-compact. By Lemma 32.7.1 we can write $(X \to S) = \lim(X_i \to S_i)$ for some directed inverse system of morphisms of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. Since limits commute with limits (Categories, Lemma 4.14.10) we have $X = \lim X_i \times_{S_i} S$. Let $i \geq i'$ in I. The morphism $X_i \times_{S_i} S \to X_{i'} \times_{S_{i'}} S$ is affine as the composition

$$X_i \times_{S_i} S \to X_i \times_{S_{i'}} S \to X_{i'} \times_{S_{i'}} S$$

where the first morphism is a closed immersion (by Schemes, Lemma 26.21.9) and the second is a base change of an affine morphism (Morphisms, Lemma 29.11.8) and the composition of affine morphisms is affine (Morphisms, Lemma 29.11.7). The morphisms f_i are of finite presentation (Morphisms, Lemmas 29.21.9 and 29.21.11) and hence the base changes $X_i \times_{f_i, S_i} S \to S$ are of finite presentation (Morphisms, Lemma 29.21.4). □

09YZ Lemma 32.7.3. Let $X \to S$ be an integral morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ finite and of finite presentation.

Proof. Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 26.24.1. Combining Properties, Lemma 28.22.13 we can write $\mathcal{A} = \text{colim}_i \mathcal{A}_i$ as a filtered colimit of finite and finitely presented $\mathcal{O}_S$-algebras. Then

$$X_i = \underline{\text{Spec}}_S(\mathcal{A}_i) \longrightarrow S$$

is a finite and finitely presented morphism of schemes. By construction $X = \lim_i X_i$ which proves the lemma. □

32.8. Descending properties of morphisms

081C This section is the analogue of Section 32.4 for properties of morphisms over S. We will work in the following situation.

081D Situation 32.8.1. Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms (Lemma 32.2.2). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S.

01ZN Lemma 32.8.2. Notation and assumptions as in Situation 32.8.1. If f is affine, then there exists an index $i \geq 0$ such that f_i is affine.

Proof. Let $Y_0 = \bigcup_{j=1,\ldots,m} V_{j,0}$ be a finite affine open covering. Set $U_{j,0} = f_0^{-1}(V_{j,0})$. For $i \geq 0$ we denote $V_{j,i}$ the inverse image of $V_{j,0}$ in Y_i and $U_{j,i} = f_i^{-1}(V_{j,i})$. Similarly we have $U_j = f^{-1}(V_j)$. Then $U_j = \lim_{i \geq 0} U_{j,i}$ (see Lemma 32.2.2). Since U_j is affine by assumption we see that each $U_{j,i}$ is affine for i large enough, see Lemma 32.4.13. As there are finitely many j we can pick an i which works for all j. Thus f_i is affine for i large enough, see Morphisms, Lemma 29.11.3. □

01ZO Lemma 32.8.3. Notation and assumptions as in Situation 32.8.1. If

(1) f is a finite morphism, and
(2) f_0 is locally of finite type,

then there exists an $i \geq 0$ such that f_i is finite.

Proof. A finite morphism is affine, see Morphisms, Definition 29.44.1. Hence by Lemma 32.8.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement follows from Algebra, Lemma 10.168.3. □

0C4W Lemma 32.8.4. Notation and assumptions as in Situation 32.8.1. If

(1) f is unramified, and

(2) f_0 is locally of finite type,

then there exists an $i \geq 0$ such that f_i is unramified.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being unramified is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i\in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite type. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is unramified, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is unramified. This follows from Algebra, Lemma 10.168.5. $\square$

01ZP Lemma 32.8.5. Notation and assumptions as in Situation 32.8.1. If

(1) f is a closed immersion, and
(2) f_0 is locally of finite type,

then there exists an $i \geq 0$ such that f_i is a closed immersion.

Proof. A closed immersion is affine, see Morphisms, Lemma 29.11.9. Hence by Lemma 32.8.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement is a consequence of Algebra, Lemma 10.168.4. $\square$

01ZQ Lemma 32.8.6. Notation and assumptions as in Situation 32.8.1. If f is separated, then f_i is separated for some $i \geq 0$.

Proof. Apply Lemma 32.8.5 to the diagonal morphism $\Delta_{X_0/S_0} : X_0 \to X_0 \times_{S_0} X_0$. (This is permissible as diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{S_0} X_0$ is quasi-compact and quasi-separated, see Schemes, Lemma 26.21.2, Morphisms, Lemma 29.15.5, and Schemes, Remark 26.21.18. $\square$

04AI Lemma 32.8.7. Notation and assumptions as in Situation 32.8.1. If

(1) f is flat,
(2) f_0 is locally of finite presentation,

then f_i is flat for some $i \geq 0$.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being flat is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i\in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is flat, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is flat. This follows from Algebra, Lemma 10.168.1 part (3). $\square$

06AC Lemma 32.8.8. Notation and assumptions as in Situation 32.8.1. If

(1) f is finite locally free (of degree d),
(2) f_0 is locally of finite presentation,

then f_i is finite locally free (of degree d) for some $i \geq 0$.

Proof. By Lemmas 32.8.7 and 32.8.3 we find an i such that f_i is flat and finite. On the other hand, f_i is locally of finite presentation. Hence f_i is finite locally free by Morphisms, Lemma 29.48.2. If moreover f is finite locally free of degree d, then the image of $Y \to Y_i$ is contained in the open and closed locus $W_d \subset Y_i$ over which f_i has degree d. By Lemma 32.4.10 we see that for some $i' \geq i$ the image of $Y_{i'} \to Y_i$ is contained in W_d. Then $f_{i'}$ will be finite locally free of degree d. □

0C0C Lemma 32.8.9. Notation and assumptions as in Situation 32.8.1. If

(1) f is smooth,
(2) f_0 is locally of finite presentation,

then f_i is smooth for some $i \geq 0$.

Proof. Being smooth is local on the source and the target (Morphisms, Lemma 29.34.2) hence we may assume S_0, X_0, Y_0 affine (details omitted). The corresponding algebra fact is Algebra, Lemma 10.168.8. □

07RP Lemma 32.8.10. Notation and assumptions as in Situation 32.8.1. If

(1) f is étale,
(2) f_0 is locally of finite presentation,

then f_i is étale for some $i \geq 0$.

Proof. Being étale is local on the source and the target (Morphisms, Lemma 29.36.2) hence we may assume S_0, X_0, Y_0 affine (details omitted). The corresponding algebra fact is Algebra, Lemma 10.168.7. □

081E Lemma 32.8.11. Notation and assumptions as in Situation 32.8.1. If

(1) f is an isomorphism, and
(2) f_0 is locally of finite presentation,

then f_i is an isomorphism for some $i \geq 0$.

Proof. By Lemmas 32.8.10 and 32.8.5 we can find an i such that f_i is flat and a closed immersion. Then f_i identifies X_i with an open and closed subscheme of Y_i, see Morphisms, Lemma 29.26.2. By assumption the image of $Y \to Y_i$ maps into $f_i(X_i)$. Thus by Lemma 32.4.10 we find that $Y_{i'}$ maps into $f_i(X_i)$ for some $i' \geq i$. It follows that $X_{i'} \to Y_{i'}$ is surjective and we win. □

0EUU Lemma 32.8.12. Notation and assumptions as in Situation 32.8.1. If

(1) f is an open immersion, and
(2) f_0 is locally of finite presentation,

then f_i is an open immersion for some $i \geq 0$.

Proof. By Lemma 32.8.10 we can find an i such that f_i is étale. Then $V_i = f_i(X_i)$ is a quasi-compact open subscheme of Y_i (Morphisms, Lemma 29.36.13). let V and $V_{i'}$ for $i' \geq i$ be the inverse image of V_i in Y and $Y_{i'}$. Then $f : X \to V$ is an isomorphism (namely it is a surjective open immersion). Hence by Lemma 32.8.11 we see that $X_{i'} \to V_{i'}$ is an isomorphism for some $i' \geq i$ as desired. □

0GTB Lemma 32.8.13. Notation and assumptions as in Situation 32.8.1. If

(1) f is an immersion, and
(2) f_0 is locally of finite type,

then f_i is an immersion for some $i \geq 0$.

Proof. There exists an open $V \subset Y$ such that the morphism f factors as $X \to V \to Y$ and such that $X \to V$ is a closed immersion, see discussion in Schemes, Section 26.10. Since X is quasi-compact, we may and do assume V is a quasi-compact open of Y. By Lemma 32.4.11 after increasing 0 we can find a quasi-compact open $V_0 \subset Y_0$ such that V is the inverse image of V_0. Then the inverse image of V_0 in X_0 is a quasi-compact open whose inverse image in X is X. Hence by the same lemma applied to $X = \lim X_i$ we may assume after increasing 0 that we have the factorization $X_0 \to V_0 \to Y_0$. Then for large enough $i \geq 0$ the morphism $X_i \to V_i$ where $V_i = Y_i \times_{Y_0} V_0$ is a closed immersion by Lemma 32.8.5 and the proof is complete. □

07RQ Lemma 32.8.14. Notation and assumptions as in Situation 32.8.1. If

(1) f is a monomorphism, and
(2) f_0 is locally of finite type,

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism of schemes $V \to W$ is a monomorphism if and only if the diagonal $V \to V \times_W V$ is an isomorphism (Schemes, Lemma 26.23.2). The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms, Lemma 29.21.12. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated (Schemes, Remark 26.21.18) we conclude from Lemma 32.8.11 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. □

07RR Lemma 32.8.15. Notation and assumptions as in Situation 32.8.1. If

(1) f is surjective, and
(2) f_0 is locally of finite presentation,

then there exists an $i \geq 0$ such that f_i is surjective.

Proof. The morphism f_0 is of finite presentation. Hence $E = f_0(X_0)$ is a constructible subset of Y_0, see Morphisms, Lemma 29.22.2. Since f_i is the base change of f_0 by $Y_i \to Y_0$ we see that the image of f_i is the inverse image of E in Y_i. Moreover, we know that $Y \to Y_0$ maps into E. Hence we win by Lemma 32.4.10. □

0C3L Lemma 32.8.16. Notation and assumptions as in Situation 32.8.1. If

(1) f is syntomic, and
(2) f_0 is locally of finite presentation,

then there exists an $i \geq 0$ such that f_i is syntomic.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being syntomic is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \mathrm{colim}_{i\in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is syntomic, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is syntomic. This follows from Algebra, Lemma 10.168.9. $\square$

32.9. Finite type closed in finite presentation

01ZD A result of this type is [Kie72, Satz 2.10]. Another reference is [Con07b].

01ZE **Lemma 32.9.1.** Let $f : X \to S$ be a morphism of schemes. Assume:

(1) The morphism f is locally of finite type.
(2) The scheme X is quasi-compact and quasi-separated.

Then there exists a morphism of finite presentation $f' : X' \to S$ and an immersion $X \to X'$ of schemes over S.

Proof. By Proposition 32.5.4 we can write $X = \lim_i X_i$ with each X_i of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[rd] & X_{i,S} \arrow[r] \arrow[d] & X_i \arrow[d] \\ & S \arrow[r] & \mathrm{Spec}(\mathbf{Z}) \end{tikzcd}$$

Note that X_i is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$, see Morphisms, Lemma 29.21.9. Hence the base change $X_{i,S} \to S$ is of finite presentation by Morphisms, Lemma 29.21.4. Thus it suffices to show that the arrow $X \to X_{i,S}$ is an immersion for i sufficiently large.

To do this we choose a finite affine open covering $X = V_1 \cup \ldots \cup V_n$ such that f maps each V_j into an affine open $U_j \subset S$. Let $h_{j,a} \in \mathcal{O}_X(V_j)$ be a finite set of elements which generate $\mathcal{O}_X(V_j)$ as an $\mathcal{O}_S(U_j)$-algebra, see Morphisms, Lemma 29.15.2. By Lemmas 32.4.11 and 32.4.13 (after possibly shrinking I) we may assume that there exist affine open coverings $X_i = V_{1,i} \cup \ldots \cup V_{n,i}$ compatible with transition maps such that $V_j = \lim_i V_{j,i}$. By Lemma 32.4.7 we can choose i so large that each $h_{j,a}$ comes from an element $h_{j,a,i} \in \mathcal{O}_{X_i}(V_{j,i})$. Thus the arrow in

$$V_j \longrightarrow U_j \times_{\mathrm{Spec}(\mathbf{Z})} V_{j,i} = (V_{j,i})_{U_j} \subset (V_{j,i})_S \subset X_{i,S}$$

is a closed immersion. Since $\bigcup (V_{j,i})_{U_j}$ forms an open of $X_{i,S}$ and since the inverse image of $(V_{j,i})_{U_j}$ in X is V_j it follows that $X \to X_{i,S}$ is an immersion. $\square$

01ZF **Remark 32.9.2.** We cannot do better than this if we do not assume more on S and the morphism $f : X \to S$. For example, in general it will not be possible to find a closed immersion $X \to X'$ as in the lemma. The reason is that this would imply that f is quasi-compact which may not be the case. An example is to take S to be infinite dimensional affine space with 0 doubled and X to be one of the two infinite dimensional affine spaces.

01ZG **Lemma 32.9.3.** Let $f : X \to S$ be a morphism of schemes. Assume:

(1) The morphism f is of locally of finite type.
(2) The scheme X is quasi-compact and quasi-separated, and

(3) The scheme S is quasi-separated.

Then there exists a morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. By Lemma 32.9.1 above there exists a morphism $Y \to S$ of finite presentation and an immersion $i : X \to Y$ of schemes over S. For every point $x \in X$, there exists an affine open $V_x \subset Y$ such that $i^{-1}(V_x) \to V_x$ is a closed immersion. Since X is quasi-compact we can find finitely may affine opens $V_1, \ldots, V_n \subset Y$ such that $i(X) \subset V_1 \cup \ldots \cup V_n$ and $i^{-1}(V_j) \to V_j$ is a closed immersion. In other words such that $i : X \to X' = V_1 \cup \ldots \cup V_n$ is a closed immersion of schemes over S. Since S is quasi-separated and Y is quasi-separated over S we deduce that Y is quasi-separated, see Schemes, Lemma 26.21.12. Hence the open immersion $X' = V_1 \cup \ldots \cup V_n \to Y$ is quasi-compact. This implies that $X' \to Y$ is of finite presentation, see Morphisms, Lemma 29.21.6. We conclude since then $X' \to Y \to S$ is a composition of morphisms of finite presentation, and hence of finite presentation (see Morphisms, Lemma 29.21.3). $\square$

09ZP Lemma 32.9.4. Let $X \to Y$ be a closed immersion of schemes. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ of schemes over Y where $X_i \to Y$ is a closed immersion of finite presentation.

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Properties, Lemma 28.22.3 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \text{colim}_{i \in I} \mathcal{I}_i$ of its quasi-coherent sheaves of ideals of finite type. Let $X_i \subset Y$ be the closed subscheme defined by $\mathcal{I}_i$. These form an inverse system of schemes indexed by I. The transition morphisms $X_i \to X_{i'}$ are affine because they are closed immersions. Each X_i is quasi-compact and quasi-separated since it is a closed subscheme of Y and Y is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_i X_i$ as follows directly from the fact that $\mathcal{I} = \text{colim}_{i \in I} \mathcal{I}_a$. Each of the morphisms $X_i \to Y$ is of finite presentation, see Morphisms, Lemma 29.21.7. $\square$

09ZQ Lemma 32.9.5. Let $f : X \to S$ be a morphism of schemes. Assume

(1) The morphism f is of locally of finite type.
(2) The scheme X is quasi-compact and quasi-separated, and
(3) The scheme S is quasi-separated.

Then $X = \lim X_i$ where the $X_i \to S$ are of finite presentation, the X_i are quasi-compact and quasi-separated, and the transition morphisms $X_{i'} \to X_i$ are closed immersions (which implies that $X \to X_i$ are closed immersions for all i).

Proof. By Lemma 32.9.3 there is a closed immersion $X \to Y$ with $Y \to S$ of finite presentation. Then Y is quasi-separated by Schemes, Lemma 26.21.12. Since X is quasi-compact, we may assume Y is quasi-compact by replacing Y with a quasi-compact open containing X. We see that $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation by Lemma 32.9.4. The morphisms $X_i \to S$ are of finite presentation by Morphisms, Lemma 29.21.3. $\square$

01ZJ Proposition 32.9.6. Let $f : X \to S$ be a morphism of schemes. Assume

(1) f is of finite type and separated, and
(2) S is quasi-compact and quasi-separated.

Then there exists a separated morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. Apply Lemma 32.9.5 and note that $X_i \to S$ is separated for large i by Lemma 32.4.17 as we have assumed that $X \to S$ is separated. $\square$

01ZK Lemma 32.9.7. Let $f : X \to S$ be a morphism of schemes. Assume

(1) f is finite, and
(2) S is quasi-compact and quasi-separated.

Then there exists a morphism which is finite and of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. We may write $X = \lim X_i$ as in Lemma 32.9.5. Applying Lemma 32.4.19 we see that $X_i \to S$ is finite for large enough i. $\square$

09YY Lemma 32.9.8. Let $f : X \to S$ be a morphism of schemes. Assume

(1) f is finite, and
(2) S quasi-compact and quasi-separated.

Then X is a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over S.

Proof. We may write $X = \lim X_i$ as in Lemma 32.9.5. Applying Lemma 32.4.19 we see that $X_i \to S$ is finite for large enough i. $\square$

32.10. Descending relative objects

01ZL The following lemma is typical of the type of results in this section. We write out the "standard" proof completely. It may be faster to convince yourself that the result is true than to read this proof.

01ZM Lemma 32.10.1. Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume

(1) the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,
(2) the schemes S_i are quasi-compact and quasi-separated.

Let $S = \lim_i S_i$. Then we have the following:

(1) For any morphism of finite presentation $X \to S$ there exists an index $i \in I$ and a morphism of finite presentation $X_i \to S_i$ such that $X \cong X_{i,S}$ as schemes over S.
(2) Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i, and a morphism $\varphi : X_{i,S} \to Y_{i,S}$ over S, there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ whose base change to S is φ.
(3) Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$ whose base changes $\varphi_{i,S} = \psi_{i,S}$ are equal, there exists an index $i' \geq i$ such that $\varphi_{i,S_{i'}} = \psi_{i,S_{i'}}$.

In other words, the category of schemes of finite presentation over S is the colimit over I of the categories of schemes of finite presentation over S_i.

Proof. In case each of the schemes S_i is affine, and we consider only affine schemes of finite presentation over S_i, resp. S this lemma is equivalent to Algebra, Lemma 10.127.8. We claim that the affine case implies the lemma in general.

Let us prove (3). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$. Assume that the base changes are equal: $\varphi_{i,S} = \psi_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma

32.2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Additionally we denote $\varphi_{i'}$ and $\psi_{i'}$ (resp. φ and ψ) the base change of φ_i and ψ_i to $S_{i'}$ (resp. S). So our assumption means that $\varphi = \psi$. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 29.21.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_{j,i'} \to Y_{i'}$ are quasi-compact, and the inverse images $U_{j,i'} = \varphi_i^{-1}(V_{j,i'})$ and $U'_{j,i'} = \psi_i^{-1}(V_{j,i'})$ are quasi-compact opens of $X_{i'}$. By assumption the inverse images of V_j under φ and ψ in X are equal. Hence by Lemma 32.4.11 there exists an index $i' \geq i$ such that of $U_{j,i'} = U'_{j,i'}$ in $X_{i'}$. Choose an finite affine open covering $U_{j,i'} = U'_{j,i'} = \bigcup W_{j,k,i'}$ which induce coverings $U_{j,i''} = U'_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$. By the affine case there exists an index i'' such that $\varphi_{i''}|_{W_{j,k,i''}} = \psi_{i''}|_{W_{j,k,i''}}$ for all j, k. Then i'' is an index such that $\varphi_{i''} = \psi_{i''}$ and (3) is proved.

Let us prove (2). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a morphism $\varphi : X_{i,S} \to Y_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 32.2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 29.21.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_j \to Y$ are quasi-compact, and the inverse images $U_j = \varphi^{-1}(V_j)$ are quasi-compact opens of X. Hence by Lemma 32.4.11 there exists an index $i' \geq i$ and quasi-compact opens $U_{j,i'}$ of $X_{i'}$ whose inverse image in X is U_j. Choose an finite affine open covering $U_{j,i'} = \bigcup W_{j,k,i'}$ which induce affine open coverings $U_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$ and an affine open covering $U_j = \bigcup W_{j,k}$. By the affine case there exists an index i'' and morphisms $\varphi_{j,k,i''} : W_{j,k,i''} \to V_{j,i''}$ such that $\varphi|_{W_{j,k}} = \varphi_{j,k,i'',S}$ for all j, k. By part (3) proved above, there is a further index $i''' \geq i''$ such that

$$\varphi_{j_1,k_1,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}} = \varphi_{j_2,k_2,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}}$$

for all j_1, j_2, k_1, k_2. Then i''' is an index such that there exists a morphism $\varphi_{i'''} : X_{i'''} \to Y_{i'''}$ whose base change to S gives φ. Hence (2) holds.

Let us prove (1). Suppose given a scheme X of finite presentation over S. Since X is of finite presentation over S, and since S is quasi-compact and quasi-separated, also X is quasi-compact and quasi-separated (see Morphisms, Lemma 29.21.10). Choose a finite affine open covering $X = \bigcup U_j$ such that each U_j maps into an affine open $V_j \subset S$. Denote $U_{j_1 j_2} = U_{j_1} \cap U_{j_2}$ and $U_{j_1 j_2 j_3} = U_{j_1} \cap U_{j_2} \cap U_{j_3}$. By Lemmas 32.4.11 and 32.4.13 we can find an index i_1 and affine opens $V_{j,i_1} \subset S_{i_1}$ such that each V_j is the inverse of this in S. Let $V_{j,i}$ be the inverse image of V_{j,i_1} in S_i for $i \geq i_1$. By the affine case we may find an index $i_2 \geq i_1$ and affine schemes $U_{j,i_2} \to V_{j,i_2}$ such that $U_j = S \times_{S_{i_2}} U_{j,i_2}$ is the base change. Denote $U_{j,i} = S_i \times_{S_{i_2}} U_{j,i_2}$ for $i \geq i_2$. By Lemma 32.4.11 there exists an index $i_3 \geq i_2$ and open subschemes $W_{j_1,j_2,i_3} \subset U_{j_1,i_3}$ whose base change to S is equal to $U_{j_1 j_2}$. Denote $W_{j_1,j_2,i} = S_i \times_{S_{i_3}} W_{j_1,j_2,i_3}$ for $i \geq i_3$. By part (2) shown above there exists an index $i_4 \geq i_3$ and morphisms $\varphi_{j_1,j_2,i_4} : W_{j_1,j_2,i_4} \to W_{j_2,j_1,i_4}$ whose base change to S gives the identity morphism $U_{j_1j_2} = U_{j_2j_1}$ for all j_1, j_2. For all $i \geq i_4$ denote $\varphi_{j_1,j_2,i} = \mathrm{id}_S \times \varphi_{j_1,j_2,i_4}$ the base change. We claim that for some $i_5 \geq i_4$ the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ forms a glueing datum as in Schemes, Section 26.14. In order to see this we have to verify that for i large enough we have

$$\varphi_{j_1,j_2,i}^{-1}(W_{j_1,j_2,i} \cap W_{j_1,j_3,i}) = W_{j_1,j_2,i} \cap W_{j_1,j_3,i}$$

and that for large enough i the cocycle condition holds. The first condition follows from Lemma 32.4.11 and the fact that $U_{j_2j_1j_3} = U_{j_1j_2j_3}$. The second from part (1) of the lemma proved above and the fact that the cocycle condition holds for the maps $\mathrm{id} : U_{j_1j_2} \to U_{j_2j_1}$. Ok, so now we can use Schemes, Lemma 26.14.2 to glue the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ to get a scheme $X_{i_5} \to S_{i_5}$. By construction the base change of X_{i_5} to S is formed by glueing the open affines U_j along the opens $U_{j_1} \leftarrow U_{j_1j_2} \to U_{j_2}$. Hence $S \times_{S_{i_5}} X_{i_5} \cong X$ as desired. □

01ZR Lemma 32.10.2. Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume

(1) all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,
(2) all the schemes S_i are quasi-compact and quasi-separated.

Let $S = \lim_i S_i$. Then we have the following:

(1) For any sheaf of $\mathcal{O}_S$-modules $\mathcal{F}$ of finite presentation there exists an index $i \in I$ and a sheaf of $\mathcal{O}_{S_i}$-modules of finite presentation $\mathcal{F}_i$ such that $\mathcal{F} \cong f_i^*\mathcal{F}_i$.
(2) Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a morphism $\varphi : f_i^*\mathcal{F}_i \to f_i^*\mathcal{G}_i$ over S. Then there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : f_{i'i}^*\mathcal{F}_i \to f_{i'i}^*\mathcal{G}_i$ whose base change to S is φ.
(3) Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a pair of morphisms $\varphi_i, \psi_i : \mathcal{F}_i \to \mathcal{G}_i$. Assume that the base changes are equal: $f_i^*\varphi_i = f_i^*\psi_i$. Then there exists an index $i' \geq i$ such that $f_{i'i}^*\varphi_i = f_{i'i}^*\psi_i$.

In other words, the category of modules of finite presentation over S is the colimit over I of the categories modules of finite presentation over S_i.

Proof. We sketch two proofs, but we omit the details.

First proof. If S and S_i are affine schemes, then this lemma is equivalent to Algebra, Lemma 10.127.6. In the general case, use Zariski glueing to deduce it from the affine case.

Second proof. We use

(1) there is an equivalence of categories between quasi-coherent $\mathcal{O}_S$-modules and vector bundles over S, see Constructions, Section 27.6, and
(2) a vector bundle $\mathbf{V}(\mathcal{F}) \to S$ is of finite presentation over S if and only if $\mathcal{F}$ is an $\mathcal{O}_S$-module of finite presentation.

Having said this, we can use Lemma 32.10.1 to show that the category of vector bundles of finite presentation over S is the colimit over I of the categories of vector bundles over S_i. □

0B8W Lemma 32.10.3. Let $S = \lim S_i$ be the limit of a directed system of quasi-compact and quasi-separated schemes S_i with affine transition morphisms. Then

(1) any finite locally free $\mathcal{O}_S$-module is the pullback of a finite locally free $\mathcal{O}_{S_i}$-module for some i,
(2) any invertible $\mathcal{O}_S$-module is the pullback of an invertible $\mathcal{O}_{S_i}$-module for some i, and
(3) any finite type quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$ is of the form $\mathcal{I}_i \cdot \mathcal{O}_S$ for some i and some finite type quasi-coherent ideal $\mathcal{I}_i \subset \mathcal{O}_{S_i}$.

Proof. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module. Since finite locally free modules are of finite presentation we can find an i and an $\mathcal{O}_{S_i}$-module $\mathcal{E}_i$ of finite presentation such that $f_i^*\mathcal{E}_i \cong \mathcal{E}$, see Lemma 32.10.2. After increasing i we may assume $\mathcal{E}_i$ is a flat $\mathcal{O}_{S_i}$-module, see Algebra, Lemma 10.168.1. (Using this lemma is not necessary, but it is convenient.) Then $\mathcal{E}_i$ is finite locally free by Algebra, Lemma 10.78.2.

If $\mathcal{L}$ is an invertible $\mathcal{O}_S$-module, then by the above we can find an i and finite locally free $\mathcal{O}_{S_i}$-modules $\mathcal{L}_i$ and $\mathcal{N}_i$ pulling back to $\mathcal{L}$ and $\mathcal{L}^{\otimes -1}$. After possible increasing i we see that the map $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$ descends to a map $\mathcal{L}_i \otimes_{\mathcal{O}_{S_i}} \mathcal{N}_i \to \mathcal{O}_{S_i}$. And after increasing i further, we may assume it is an isomorphism. It follows that $\mathcal{L}_i$ is an invertible module (Modules, Lemma 17.25.2) and the proof of (2) is complete.

Given $\mathcal{I}$ as in (3) we see that $\mathcal{O}_S \to \mathcal{O}_S/\mathcal{I}$ is a map of finitely presented $\mathcal{O}_S$-modules. Hence by Lemma 32.10.2 this is the pullback of some map $\mathcal{O}_{S_i} \to \mathcal{F}_i$ of finitely presented $\mathcal{O}_{S_i}$-modules. After increasing i we may assume this map is surjective (details omitted; hint: use Algebra, Lemma 10.127.5 on affine open cover). Then the kernel of $\mathcal{O}_{S_i} \to \mathcal{F}_i$ is a finite type quasi-coherent ideal in $\mathcal{O}_{S_i}$ whose pullback gives $\mathcal{I}$. □

05LY Lemma 32.10.4. With notation and assumptions as in Lemma 32.10.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i and that $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module of finite presentation. If the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S$ is flat over $Y_i \times_{S_i} S$, then there exists an index $i' \geq i$ such that the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S_{i'}$ is flat over $Y_i \times_{S_i} S_{i'}$.

Proof. (This lemma is the analogue of Lemma 32.8.7 for modules.) For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$, $\mathcal{F}_{i'} = (X_{i'} \to X_i)^*\mathcal{F}_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of φ_i to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, $\mathcal{F} = (X \to X_i)^*\mathcal{F}_i$ and φ the base change of φ_i to S. Let $Y_i = \bigcup_{j=1,\ldots,m} V_{j,i}$ be a finite affine open covering such that each $V_{j,i}$ maps into some affine open of S_i. For each $j = 1, \ldots m$ let $\varphi_i^{-1}(V_{j,i}) = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ be a finite affine open covering. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and $U_{k,j,i'}$ the inverse image of $U_{k,j,i}$ in $X_{i'}$. Similarly we have $U_{k,j} \subset X$ and $V_j \subset Y$. Then $U_{k,j} = \lim_{i' \geq i} U_{k,j,i'}$ and $V_j = \lim_{i' \geq i} V_j$ (see Lemma 32.2.2). Since $X_{i'} = \bigcup_{k,j} U_{k,j,i'}$ is a finite open covering it suffices to prove the lemma for each of the morphisms $U_{k,j,i} \to V_{j,i}$ and the sheaf $\mathcal{F}_i|_{U_{k,j,i}}$. Hence we see that the lemma reduces to the case that X_i and Y_i are affine and map into an affine open of S_i, i.e., we may also assume that S is affine.

In the affine case we reduce to the following algebra result. Suppose that $R = \text{colim}_{i \in I} R_i$. For some $i \in I$ suppose given a map $A_i \to B_i$ of finitely presented R_i-algebras. Let N_i be a finitely presented B_i-module. Then, if $R \otimes_{R_i} N_i$ is flat over $R \otimes_{R_i} A_i$, then for some $i' \geq i$ the module $R_{i'} \otimes_{R_i} N_i$ is flat over $R_{i'} \otimes_{R_i} A$. This is exactly the result proved in Algebra, Lemma 10.168.1 part (3). □

0EY1 Lemma 32.10.5. For a scheme T denote $\mathcal{C}_T$ the full subcategory of schemes W over T such that W is quasi-compact and quasi-separated and such that the structure morphism $W \to T$ is locally of finite presentation. Let $S = \lim S_i$ be a directed limit of schemes with affine transition morphisms. Then there is an equivalence of categories

$$\operatorname{colim} \mathcal{C}_{S_i} \longrightarrow \mathcal{C}_S$$

given by the base change functors.

Warning: do not use this lemma if you do not understand the difference between this lemma and Lemma 32.10.1.

Proof. Fully faithfulness. Suppose we have $i \in I$ and objects X_i, Y_i of $\mathcal{C}_{S_i}$. Denote $X = X_i \times_{S_i} S$ and $Y = Y_i \times_{S_i} S$. Suppose given a morphism $f : X \to Y$ over S. We can choose a finite affine open covering $Y_i = V_{i,1} \cup \ldots \cup V_{i,m}$ such that $V_{i,j} \to Y_i \to S_i$ maps into an affine open $W_{i,j}$ of S_i. Denote $Y = V_1 \cup \ldots \cup V_m$ the induced affine open covering of Y. Since $f : X \to Y$ is quasi-compact (Schemes, Lemma 26.21.14) after increasing i we may assume that there is a finite open covering $X_i = U_{i,1} \cup \ldots \cup U_{i,m}$ by quasi-compact opens such that the inverse image of $U_{i,j}$ in Y is $f^{-1}(V_j)$, see Lemma 32.4.11. By Lemma 32.10.1 applied to $f|_{f^{-1}(V_j)}$ over W_j we may assume, after increasing i, that there is a morphism $f_{i,j} : V_{i,j} \to U_{i,j}$ over S whose base change to S is $f|_{f^{-1}(V_j)}$. Increasing i more we may assume $f_{i,j}$ and $f_{i,j'}$ agree on the quasi-compact open $U_{i,j} \cap U_{i,j'}$. Then we can glue these morphisms to get the desired morphism $f_i : X_i \to Y_i$. This morphism is unique (up to increasing i) because this is true for the morphisms $f_{i,j}$.

To show that the functor is essentially surjective we argue in exactly the same way. Namely, suppose that X is an object of $\mathcal{C}_S$. Pick $i \in I$. We can choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that $U_j \to X \to S \to S_i$ factors through an affine open $W_{i,j} \subset S_i$. Set $W_j = W_{i,j} \times_{S_i} S$. This is an affine open of S. By Lemma 32.10.1, after increasing i, we may assume there exist $U_{i,j} \to W_{i,j}$ of finite presentation whose base change to W_j is U_j. After increasing i we may assume there exist quasi-compact opens $U_{i,j,j'} \subset U_{i,j}$ whose base changes to S are equal to $U_j \cap U_{j'}$. Claim: after increasing i we may assume the image of the morphism $U_{i,j,j'} \to U_{i,j} \to W_{i,j}$ ends up in $W_{i,j} \cap W_{i,j'}$. Namely, because the complement of $W_{i,j} \cap W_{i,j'}$ is closed in the affine scheme $W_{i,j}$ it is affine. Since $U_j \cap U_{j'} = \lim U_{i,j,j'}$ does map into $W_{i,j} \cap W_{i,j'}$ we can apply Lemma 32.4.9 to get the claim. Thus we can view both

$$U_{i,j,j'} \quad \text{and} \quad U_{i,j',j}$$

as schemes over $W_{i,j'}$ whose base changes to $W_{j'}$ recover $U_j \cap U_{j'}$. Hence after increasing i, using Lemma 32.10.1, we may assume there are isomorphisms $U_{i,j,j'} \to U_{i,j',j}$ over $W_{i,j'}$ and hence over S_i. Increasing i further (details omitted) we may assume these isomorphisms satisfy the cocycle condition mentioned in Schemes, Section 26.14. Applying Schemes, Lemma 26.14.1 we obtain an object X_i of $\mathcal{C}_{S_i}$ whose base change to S is isomorphic to X; we omit some of the verifications. $\square$

32.11. Characterizing affine schemes

01ZS If $f : X \to S$ is a surjective integral morphism of schemes such that X is an affine scheme then S is affine too. See [Con07b, A.2]. Our proof relies on the Noetherian case which we stated and proved in Cohomology of Schemes, Lemma 30.13.3. See also [DG67, II 6.7.1].

01ZT Lemma 32.11.1. Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and finite, and assume that X is affine. Then S is affine.

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Since X is separated and f is surjective and universally closed (Morphisms, Lemma 29.44.7), we see that S is separated (Morphisms, Lemma 29.41.11).

By Lemma 32.9.8 we can write $X = \lim_a X_a$ with $X_a \to S$ finite and of finite presentation. By Lemma 32.4.13 we see that X_a is affine for some $a \in A$. Replacing X by X_a we may assume that $X \to S$ is surjective, finite, of finite presentation and that X is affine.

By Proposition 32.5.4 we may write $S = \lim_{i \in I} S_i$ as a directed limits of schemes of finite type over $\mathbf{Z}$. By Lemma 32.10.1 we can after shrinking I assume there exist schemes $X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_S S_{i'}$ for $i' \geq i$ and such that $X = \lim_i X_i$. By Lemma 32.8.3 we may assume that $X_i \to S_i$ is finite for all $i \in I$ as well. By Lemma 32.4.13 once again we may assume that X_i is affine for all $i \in I$. Hence the result follows from the Noetherian case, see Cohomology of Schemes, Lemma 30.13.3. $\square$

05YU Proposition 32.11.2. Let $f : X \to S$ be a morphism of schemes. Assume X is affine and that f is surjective and universally closed[2]. Then S is affine.

Proof. By Morphisms, Lemma 29.41.11 the scheme S is separated. Then by Morphisms, Lemma 29.11.11 we find that f is affine. Whereupon by Morphisms, Lemma 29.44.7 we see that f is integral.

By the preceding paragraph, we may assume $f : X \to S$ is surjective and integral, X is affine, and S is separated. Since f is surjective and X is quasi-compact we also deduce that S is quasi-compact.

By Lemma 32.7.3 we can write $X = \lim_i X_i$ with $X_i \to S$ finite. By Lemma 32.4.13 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to S$ factors through each X_i we see that $X_i \to S$ is surjective. Hence we conclude that S is affine by Lemma 32.11.1. $\square$

09NL Lemma 32.11.3. Let X be a scheme which is set theoretically the union of finitely many affine closed subschemes. Then X is affine.

Proof. Let $Z_i \subset X$, $i = 1, \ldots, n$ be affine closed subschemes such that $X = \bigcup Z_i$ set theoretically. Then $\coprod Z_i \to X$ is surjective and integral with affine source. Hence X is affine by Proposition 32.11.2. $\square$

09MW Lemma 32.11.4. Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. Then $i^*\mathcal{L}$ is ample on Z, if and only if $\mathcal{L}$ is ample on X.

Proof. If $\mathcal{L}$ is ample, then $i^*\mathcal{L}$ is ample for example by Morphisms, Lemma 29.37.7. Assume $i^*\mathcal{L}$ is ample. Then Z is quasi-compact (Properties, Definition 28.26.1) and separated (Properties, Lemma 28.26.8). Since i is surjective, we see that X is quasi-compact. Since i is universally closed and surjective, we see that X is separated (Morphisms, Lemma 29.41.11).

[2] An integral morphism is universally closed, see Morphisms, Lemma 29.44.7.

By Proposition 32.5.4 we can write $X = \lim X_i$ as a directed limit of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. We can find an i and an invertible sheaf $\mathcal{L}_i$ on X_i whose pullback to X is isomorphic to $\mathcal{L}$, see Lemma 32.10.2.

For each i let $Z_i \subset X_i$ be the scheme theoretic image of the morphism $Z \to X_i$. If $\text{Spec}(A_i) \subset X_i$ is an affine open subscheme with inverse image of $\text{Spec}(A)$ in X and if $Z \cap \text{Spec}(A)$ is defined by the ideal $I \subset A$, then $Z_i \cap \text{Spec}(A_i)$ is defined by the ideal $I_i \subset A_i$ which is the inverse image of I in A_i under the ring map $A_i \to A$, see Morphisms, Example 29.6.4. Since $\text{colim}\, A_i/I_i = A/I$ it follows that $\lim Z_i = Z$. By Lemma 32.4.15 we see that $\mathcal{L}_i|_{Z_i}$ is ample for some i. Since Z and hence X maps into Z_i set theoretically, we see that $X_{i'} \to X_i$ maps into Z_i set theoretically for some $i' \geq i$, see Lemma 32.4.10. (Observe that since X_i is Noetherian, every closed subset of X_i is constructible.) Let $T \subset X_{i'}$ be the scheme theoretic inverse image of Z_i in $X_{i'}$. Observe that $\mathcal{L}_{i'}|_T$ is the pullback of $\mathcal{L}_i|_{Z_i}$ and hence ample by Morphisms, Lemma 29.37.7 and the fact that $T \to Z_i$ is an affine morphism. Thus we see that $\mathcal{L}_{i'}$ is ample on $X_{i'}$ by Cohomology of Schemes, Lemma 30.17.5. Pulling back to X (using the same lemma as above) we find that $\mathcal{L}$ is ample. □

0B7L Lemma 32.11.5. Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Then X is quasi-affine if and only if Z is quasi-affine.

Proof. Recall that a scheme is quasi-affine if and only if the structure sheaf is ample, see Properties, Lemma 28.27.1. Hence if Z is quasi-affine, then $\mathcal{O}_Z$ is ample, hence $\mathcal{O}_X$ is ample by Lemma 32.11.4, hence X is quasi-affine. A proof of the converse, which can also be seen in an elementary way, is gotten by reading the argument just given backwards. □

The following lemma does not really belong in this section.

0E21 Lemma 32.11.6. Let X be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on X. Assume we have morphisms of schemes

$$\text{Spec}(k) \leftarrow \text{Spec}(A) \to W \subset X$$

where k is a field, A is an integral k-algebra, W is open in X. Then there exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine, $X_s \subset W$, and $\text{Spec}(A) \to W$ factors through X_s

Proof. Since $\text{Spec}(A)$ is quasi-compact, we may replace W by a quasi-compact open still containing the image of $\text{Spec}(A) \to X$. Recall that X is quasi-separated and quasi-compact by dint of having an ample invertible sheaf, see Properties, Definition 28.26.1 and Lemma 28.26.7. By Proposition 32.5.4 we can write $X = \lim X_i$ as a limit of a directed system of schemes of finite type over $\mathbf{Z}$ with affine transition morphisms. For some i the ample invertible sheaf $\mathcal{L}$ on X descends to an ample invertible sheaf $\mathcal{L}_i$ on X_i and the open W is the inverse image of a quasi-compact open $W_i \subset X_i$, see Lemmas 32.4.15, 32.10.3, and 32.4.11. We may replace $X, W, \mathcal{L}$ by $X_i, W_i, \mathcal{L}_i$ and assume X is of finite presentation over $\mathbf{Z}$. Write $A = \text{colim}\, A_j$ as the colimit of its finite k-subalgebras. Then for some j the morphism $\text{Spec}(A) \to X$ factors through a morphism $\text{Spec}(A_j) \to X$, see Proposition 32.6.1. Since $\text{Spec}(A_j)$ is finite this reduces the lemma to Properties, Lemma 28.29.6. □

32.12. Variants of Chow's Lemma

01ZZ In this section we prove a number of variants of Chow's lemma. The most interesting version is probably just the Noetherian case, which we stated and proved in Cohomology of Schemes, Section 30.18.

0202 **Lemma 32.12.1.** Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram

$$\begin{tikzcd} X \arrow[dr] & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective.

Proof. By Proposition 32.9.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Clearly, if we prove the assertion for Y, then the result follows for X. Hence we may assume that X is of finite presentation over S.

Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 32.5.4. By Lemma 32.10.1 we can find an index $i \in I$ and a scheme $X_i \to S_i$ of finite presentation so that $X = S \times_{S_i} X_i$. By Lemma 32.8.6 we may assume that $X_i \to S_i$ is separated. Clearly, if we prove the assertion for X_i over S_i, then the assertion holds for X. The case $X_i \to S_i$ is treated by Cohomology of Schemes, Lemma 30.18.1. $\square$

0GII **Remark 32.12.2.** In the situation of Chow's Lemma 32.12.1:

(1) The morphism π is actually H-projective (hence projective, see Morphisms, Lemma 29.43.3) since the morphism $X' \to \mathbf{P}^n_S \times_S X = \mathbf{P}^n_X$ is a closed immersion (use the fact that π is proper, see Morphisms, Lemma 29.41.7).
(2) We may assume that X' is reduced as we can replace X' by its reduction without changing the other assertions of the lemma.
(3) We may assume that $X' \to X$ is of finite presentation without changing the other assertions of the lemma. This can be deduced from the proof of Lemma 32.12.1 but we can also prove this directly as follows. By (1) we have a closed immersion $X' \to \mathbf{P}^n_X$. By Lemma 32.9.4 we can write $X' = \lim X'_i$ where $X'_i \to \mathbf{P}^n_X$ is a closed immersion of finite presentation. In particular $X'_i \to X$ is of finite presentation, proper, and surjective. For large enough i the morphism $X'_i \to \mathbf{P}^n_S$ is an immersion by Lemma 32.4.16. Replacing X' by X'_i we get what we want.

Of course in general we can't simultaneously achieve both (2) and (3).

Here is a variant of Chow's lemma where we assume the scheme on top has finitely many irreducible components.

0203 **Lemma 32.12.3.** Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Assume that X has finitely many irreducible components. Then there exists an $n \geq 0$ and a diagram

$$\begin{tikzcd} X & X' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S \arrow[from=1-1, to=2-2, crossing over] & \end{tikzcd}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, there exists an open dense subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism of schemes.

Proof. Let $X = Z_1 \cup \ldots \cup Z_n$ be the decomposition of X into irreducible components. Let $\eta_j \in Z_j$ be the generic point.

There are (at least) two ways to proceed with the proof. The first is to redo the proof of Cohomology of Schemes, Lemma 30.18.1 using the general Properties, Lemma 28.29.4 to find suitable affine opens in X. (This is the "standard" proof.) The second is to use absolute Noetherian approximation as in the proof of Lemma 32.12.1 above. This is what we will do here.

By Proposition 32.9.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 32.5.4. By Lemma 32.10.1 we can find an index $i \in I$ and a scheme $Y_i \to S_i$ of finite presentation so that $Y = S \times_{S_i} Y_i$. By Lemma 32.8.6 we may assume that $Y_i \to S_i$ is separated. We have the following diagram

$$\begin{tikzcd} \eta_j \in Z_j \arrow[r] & X \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d] & Y_i \arrow[d] \\ & & S \arrow[r] & S_i \end{tikzcd}$$

Denote $h : X \to Y_i$ the composition.

For $i' \geq i$ write $Y_{i'} = S_{i'} \times_{S_i} Y_i$. Then $Y = \lim_{i' \geq i} Y_{i'}$, see Lemma 32.2.3. Choose $j, j' \in \{1, \ldots, n\}$, $j \neq j'$. Note that η_j is not a specialization of $\eta_{j'}$. By Lemma 32.4.6 we can replace i by a bigger index and assume that $h(\eta_j)$ is not a specialization of $h(\eta_{j'})$ for all pairs (j, j') as above. For such an index, let $Y' \subset Y_i$ be the scheme theoretic image of $h : X \to Y_i$, see Morphisms, Definition 29.6.2. The morphism h is quasi-compact as the composition of the quasi-compact morphisms $X \to Y$ and $Y \to Y_i$ (which is affine). Hence by Morphisms, Lemma 29.6.3 the morphism $X \to Y'$ is dominant. Thus the generic points of Y' are all contained in the set $\{h(\eta_1), \ldots, h(\eta_n)\}$, see Morphisms, Lemma 29.8.3. Since none of the $h(\eta_j)$ is the specialization of another we see that the points $h(\eta_1), \ldots, h(\eta_n)$ are pairwise distinct and are each a generic point of Y'.

We apply Cohomology of Schemes, Lemma 30.18.1 above to the morphism $Y' \to S_i$. This gives a diagram

$$\begin{tikzcd} Y' \arrow[dr] & Y^* \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[dl] \\ & S_i & \end{tikzcd}$$

such that π is proper and surjective and an isomorphism over a dense open subscheme $V \subset Y'$. By our choice of i above we know that $h(\eta_1), \ldots, h(\eta_n) \in V$. Consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, equal] & X \times_{Y'} Y^* \arrow[r] \arrow[d] & Y^* \arrow[r] \arrow[d] & \mathbf{P}^n_{S_i} \arrow[ddl] \\ & X \arrow[r] \arrow[d] & Y' \arrow[d] & \\ & S \arrow[r] & S_i & \end{tikzcd}$$

Note that $X' \to X$ is an isomorphism over the open subscheme $U = h^{-1}(V)$ which contains each of the η_j and hence is dense in X. We conclude $X \leftarrow X' \to \mathbf{P}^n_S$ is a solution to the problem posed in the lemma. $\square$

32.13. Applications of Chow's lemma

0204 Here is a first application of Chow's lemma.

081F **Lemma 32.13.1.** Assumptions and notation as in Situation 32.8.1. If

1. f is proper, and
2. f_0 is locally of finite type,

then there exists an i such that f_i is proper.

Proof. By Lemma 32.8.6 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact, see Schemes, Lemma 26.21.14. By Lemma 32.12.1 we can choose a diagram

$$\begin{tikzcd} X_0 \arrow[dr] & X_0' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_{Y_0} \arrow[dl] \\ & Y_0 & \end{tikzcd}$$

where $X_0' \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X_0' \to X_0$ is proper and surjective. Introduce $X' = X_0' \times_{Y_0} Y$ and $X_i' = X_0' \times_{Y_0} Y_i$. By Morphisms, Lemmas 29.41.4 and 29.41.5 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms, Lemma 29.41.7). By Morphisms, Lemma 29.41.9 it suffices to prove that $X_i' \to Y_i$ is proper for some i. By Lemma 32.8.5 we find that $X_i' \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X_i' \to Y_i$ is proper and we win. $\square$

09ZR **Lemma 32.13.2.** Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ is a directed limit of schemes X_i proper and of finite presentation over S such that all transition morphisms and the morphisms $X \to X_i$ are closed immersions.

Proof. By Proposition 32.9.6 we can find a closed immersion $X \to Y$ with Y separated and of finite presentation over S. By Lemma 32.12.1 we can find a

diagram

$$\begin{tikzcd} Y \arrow[dr] & Y' \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $Y' \to \mathbf{P}^n_S$ is an immersion, and $\pi : Y' \to Y$ is proper and surjective. By Lemma 32.9.4 we can write $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation. Denote $X'_i \subset Y'$, resp. $X' \subset Y'$ the scheme theoretic inverse image of $X_i \subset Y$, resp. $X \subset Y$. Then $\lim X'_i = X'$. Since $X' \to S$ is proper (Morphisms, Lemmas 29.41.4), we see that $X' \to \mathbf{P}^n_S$ is a closed immersion (Morphisms, Lemma 29.41.7). Hence for i large enough we find that $X'_i \to \mathbf{P}^n_S$ is a closed immersion by Lemma 32.4.20. Thus X'_i is proper over S. For such i the morphism $X_i \to S$ is proper by Morphisms, Lemma 29.41.9. □

0A0P Lemma 32.13.3. Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then there exists a directed set I, an inverse system $(f_i : X_i \to S_i)$ of morphisms of schemes over I, such that the transition morphisms $X_i \to X_{i'}$ and $S_i \to S_{i'}$ are affine, such that f_i is proper, such that S_i is of finite type over $\mathbf{Z}$, and such that $(X \to S) = \lim(X_i \to S_i)$.

Proof. By Lemma 32.13.2 we can write $X = \lim_{k \in K} X_k$ with $X_k \to S$ proper and of finite presentation. Next, by absolute Noetherian approximation (Proposition 32.5.4) we can write $S = \lim_{j \in J} S_j$ with S_j of finite type over $\mathbf{Z}$. For each k there exists a j and a morphism $X_{k,j} \to S_j$ of finite presentation with $X_k \cong S \times_{S_j} X_{k,j}$ as schemes over S, see Lemma 32.10.1. After increasing j we may assume $X_{k,j} \to S_j$ is proper, see Lemma 32.13.1. The set I will be consist of these pairs (k, j) and the corresponding morphism is $X_{k,j} \to S_j$. For every $k' \geq k$ we can find a $j' \geq j$ and a morphism $X_{j',k'} \to X_{j,k}$ over $S_{j'} \to S_j$ whose base change to S gives the morphism $X_{k'} \to X_k$ (follows again from Lemma 32.10.1). These morphisms form the transition morphisms of the system. Some details omitted. □

0EX1 Lemma 32.13.4. Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over S. If $Y \to X$ is proper, X_i quasi-compact and quasi-separated, and Y locally of finite type over S, then $Y \to X_i$ is proper for i large enough.

Proof. Choose a closed immersion $Y \to Y'$ with Y' proper and of finite presentation over X, see Lemma 32.13.2. Then choose an i and a proper morphism $Y'_i \to X_i$ such that $Y' = X \times_{X_i} Y'_i$. This is possible by Lemmas 32.10.1 and 32.13.1. Then after replacing i by a larger index we have that $Y \to Y'_i$ is a closed immersion, see Lemma 32.4.16. □

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms, Definition 29.5.5.

081G Lemma 32.13.5. Assumptions and notation as in Situation 32.8.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume

(1) f_0 is locally of finite type,
(2) $\mathcal{F}_0$ is of finite type,
(3) the scheme theoretic support of $\mathcal{F}$ is proper over Y.

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms, Lemma 29.5.3 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 29.41.9. By Lemma 32.13.1 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms, Lemmas 29.41.6 and 29.41.4. □

32.14. Universally closed morphisms

05JW In this section we discuss when a quasi-compact (but not necessarily separated) morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

05BD Lemma 32.14.1. Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $g : T \to S$ be a morphism of schemes. Let $t \in T$ be a point and $Z \subset X_T$ be a closed subscheme such that $Z \cap X_t = \emptyset$. Then there exists an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

Moreover, we may assume V and T' are affine.

Proof. Let $s = g(t)$. During the proof we may always replace T by an open neighbourhood of t. Hence we may also replace S by an open neighbourhood of s. Thus we may and do assume that T and S are affine. Say $S = \mathrm{Spec}(A)$, $T = \mathrm{Spec}(B)$, g is given by the ring map $A \to B$, and t correspond to the prime ideal $\mathfrak{q} \subset B$.

As $X \to S$ is quasi-compact and S is affine we may write $X = \bigcup_{i=1,\ldots,n} U_i$ as a finite union of affine opens. Write $U_i = \mathrm{Spec}(C_i)$. In particular we have $X_T = \bigcup_{i=1,\ldots,n} U_{i,T} = \bigcup_{i=1,\ldots n} \mathrm{Spec}(C_i \otimes_A B)$. Let $I_i \subset C_i \otimes_A B$ be the ideal corresponding to the closed subscheme $Z \cap U_{i,T}$. The condition that $Z \cap X_t = \emptyset$ signifies that I_i generates the unit ideal in the ring

$$C_i \otimes_A \kappa(\mathfrak{q}) = (B \setminus \mathfrak{q})^{-1} \left(C_i \otimes_A B/\mathfrak{q}C_i \otimes_A B\right)$$

Since $I_i(B \setminus \mathfrak{q})^{-1}(C_i \otimes_A B) = (B \setminus \mathfrak{q})^{-1} I_i$ this means that $1 = x_i/g_i$ for some $x_i \in I_i$ and $g_i \in B$, $g_i \notin \mathfrak{q}$. Thus, clearing denominators we can find a relation of the form

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = g_i$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$, and $g_i \in B$, $g_i \notin \mathfrak{q}$. After replacing B by $B_{g_1 \ldots g_n}$, i.e., after replacing T by a smaller affine neighbourhood of t, we may assume the equations read

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = 1$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$.

To finish the argument write B as a colimit of finitely presented A-algebras B_λ over a directed set Λ. For each λ set $\mathfrak{q}_\lambda = (B_\lambda \to B)^{-1}(\mathfrak{q})$. For sufficiently large $\lambda \in \Lambda$ we can find

(1) an element $x_{i,\lambda} \in C_i \otimes_A B_\lambda$ which maps to x_i,
(2) elements $f_{i,j,\lambda} \in \mathfrak{q}_{i,\lambda}$ mapping to $f_{i,j}$, and
(3) elements $c_{i,j,\lambda} \in C_i \otimes_A B_\lambda$ mapping to $c_{i,j}$.

After increasing λ a bit more the equation

$$x_{i,\lambda} + \sum\nolimits_j f_{i,j,\lambda} c_{i,j,\lambda} = 1$$

will hold. Fix such a λ and set $T' = \mathrm{Spec}(B_\lambda)$. Then $t' \in T'$ is the point corresponding to the prime $\mathfrak{q}_\lambda$. Finally, let $Z' \subset X_{T'}$ be the scheme theoretic image of $Z \to X_T \to X_{T'}$. As $X_T \to X_{T'}$ is affine, we can compute Z' on the affine open pieces $U_{i,T'}$ as the closed subscheme associated to $\mathrm{Ker}(C_i \otimes_A B_\lambda \to C_i \otimes_A B/I_i)$, see Morphisms, Example 29.6.4. Hence $x_{i,\lambda}$ is in the ideal defining Z'. Thus the last displayed equation shows that $Z' \cap X_{t'}$ is empty. $\square$

05JX Lemma 32.14.2. Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent

(1) f is universally closed,
(2) for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is closed, and
(3) for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.

Proof. It is clear that (1) implies (2). Let us prove that (2) implies (1). Suppose that the base change $X_T \to T$ is not closed for some scheme T over S. By Schemes, Lemma 26.19.8 this means that there exists some specialization $t_1 \leadsto t$ in T and a point $\xi \in X_T$ mapping to t_1 such that ξ does not specialize to a point in the fibre over t. Set $Z = \overline{\{\xi\}} \subset X_T$. Then $Z \cap X_t = \emptyset$. Apply Lemma 32.14.1. We find an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

Clearly this means that $X_{T'} \to T'$ maps the closed subset Z' to a subset of T' which contains $a(t_1)$ but not $t' = a(t)$. Since $a(t_1) \leadsto a(t) = t'$ we conclude that $X_{T'} \to T'$ is not closed. Hence we have shown that $X \to S$ not universally closed implies that $X_{T'} \to T'$ is not closed for some $T' \to S$ which is locally of finite presentation. In order words (2) implies (1).

Assume that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed for every integer n. We want to prove that $X_T \to T$ is closed for every scheme T which is locally of finite presentation over S. We may of course assume that T is affine and maps into an affine open V of S (since $X_T \to T$ being a closed is local on T). In this case there exists a closed immersion

$T \to \mathbf{A}^n \times V$ because $\mathcal{O}_T(T)$ is a finitely presented $\mathcal{O}_S(V)$-algebra, see Morphisms, Lemma 29.21.2. Then $T \to \mathbf{A}^n \times S$ is a locally closed immersion. Hence we get a cartesian diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "f_T"'] & \mathbf{A}^n \times X \arrow[d, "f_n"] \\ T \arrow[r] & \mathbf{A}^n \times S \end{tikzcd}$$

of schemes where the horizontal arrows are locally closed immersions. Hence any closed subset $Z \subset X_T$ can be written as $X_T \cap Z'$ for some closed subset $Z' \subset \mathbf{A}^n \times X$. Then $f_T(Z) = T \cap f_n(Z')$ and we see that if f_n is closed, then also f_T is closed. $\square$

0205 Lemma 32.14.3. Let S be a scheme. Let $f : X \to S$ be a separated morphism of finite type. The following are equivalent:

(1) The morphism f is proper.
(2) For any morphism $S' \to S$ which is locally of finite type the base change $X_{S'} \to S'$ is closed.
(3) For every $n \geq 0$ the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.

First proof. In view of the fact that a proper morphism is the same thing as a separated, finite type, and universally closed morphism, this lemma is a special case of Lemma 32.14.2. $\square$

Second proof. Clearly (1) implies (2), and (2) implies (3), so we just need to show (3) implies (1). First we reduce to the case when S is affine. Assume that (3) implies (1) when the base is affine. Now let $f : X \to S$ be a separated morphism of finite type. Being proper is local on the base (see Morphisms, Lemma 29.41.3), so if $S = \bigcup_\alpha S_\alpha$ is an open affine cover, and if we denote $X_\alpha := f^{-1}(S_\alpha)$, then it is enough to show that $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is proper for all α. Since S_α is affine, if the map $f|_{X_\alpha}$ satisfies (3), then it will satisfy (1) by assumption, and will be proper. To finish the reduction to the case S is affine, we must show that if $f : X \to S$ is separated of finite type satisfying (3), then $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is separated of finite type satisfying (3). Separatedness and finite type are clear. To see (3), notice that $\mathbf{A}^n \times X_\alpha$ is the open preimage of $\mathbf{A}^n \times S_\alpha$ under the map $1 \times f$. Fix a closed set $Z \subset \mathbf{A}^n \times X_\alpha$. Let $\bar{Z}$ denote the closure of Z in $\mathbf{A}^n \times X$. Then for topological reasons,

$$1 \times f(\bar{Z}) \cap \mathbf{A}^n \times S_\alpha = 1 \times f(Z).$$

Hence $1 \times f(Z)$ is closed, and we have reduced the proof of (3) $\Rightarrow$ (1) to the affine case.

Assume S affine, and $f : X \to S$ separated of finite type. We can apply Chow's Lemma 32.12.1 to get $\pi : X' \to X$ proper surjective and $X' \to \mathbf{P}^n_S$ an immersion. If X is proper over S, then $X' \to S$ is proper (Morphisms, Lemma 29.41.4). Since $\mathbf{P}^n_S \to S$ is separated, we conclude that $X' \to \mathbf{P}^n_S$ is proper (Morphisms, Lemma 29.41.7) and hence a closed immersion (Schemes, Lemma 26.10.4). Conversely, assume $X' \to \mathbf{P}^n_S$ is a closed immersion. Consider the diagram:

05LZ (32.14.3.1)
$$\begin{tikzcd} X' \arrow[r] \arrow[d, two heads, "\pi"'] & \mathbf{P}^n_S \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

All maps are a priori proper except for $X \to S$. Hence we conclude that $X \to S$ is proper by Morphisms, Lemma 29.41.9. Therefore, we have shown that $X \to S$ is proper if and only if $X' \to \mathbf{P}^n_S$ is a closed immersion.

Assume S is affine and (3) holds, and let n, X', π be as above. Since being a closed morphism is local on the base, the map $X \times \mathbf{P}^n \to S \times \mathbf{P}^n$ is closed since by (3) $X \times \mathbf{A}^n \to S \times \mathbf{A}^n$ is closed and since projective space is covered by copies of affine n-space, see Constructions, Lemma 27.13.3. By Morphisms, Lemma 29.41.5 the morphism

$$X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n$$

is proper. Since $\mathbf{P}^n$ is separated, the projection

$$X' \times_S \mathbf{P}^n_S = \mathbf{P}^n_{X'} \to X'$$

will be separated as it is just a base change of a separated morphism. Therefore, the map $X' \to X' \times_S \mathbf{P}^n_S$ is proper, since it is a section to a separated map (see Schemes, Lemma 26.21.11). Composing these morphisms

$$X' \to X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n \to S \times \mathbf{P}^n = \mathbf{P}^n_S$$

we find that the immersion $X' \to \mathbf{P}^n_S$ is closed, and hence a closed immersion. $\square$

32.15. Noetherian valuative criterion

0CM1 If the base is Noetherian we can show that the valuative criterion holds using only discrete valuation rings.

Many of the results in this section can (and perhaps should) be proved by appealing to the following lemma, although we have not always done so.

0CM2 Lemma 32.15.1. Let $f : X \to Y$ be a morphism of schemes. Assume f finite type and Y locally Noetherian. Let $y \in Y$ be a point in the closure of the image of f. Then there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

where A is a discrete valuation ring and K is its field of fractions mapping the closed point of $\mathrm{Spec}(A)$ to y. Moreover, we can assume that the image point of $\mathrm{Spec}(K) \to X$ is a generic point η of an irreducible component of X and that $K = \kappa(\eta)$.

Proof. By the non-Noetherian version of this lemma (Morphisms, Lemma 29.6.5) there exists a point $x \in X$ such that $f(x)$ specializes to y. We may replace x by any point specializing to x, hence we may assume that x is a generic point of an irreducible component of X. This produces a ring map $\mathcal{O}_{Y,y} \to \kappa(x)$ (see Schemes, Section 26.13). Let $R \subset \kappa(x)$ be the image. Then R is Noetherian as a quotient of the Noetherian local ring $\mathcal{O}_{Y,y}$. On the other hand, the extension $\kappa(x)$ is a finitely generated extension of the fraction field of R as f is of finite type. Thus there exists a discrete valuation ring $A \subset \kappa(x)$ with fraction field $\kappa(x)$ dominating R by Algebra, Lemma 10.119.13. Then

$$\begin{tikzcd} \mathrm{Spec}(\kappa(x)) \arrow[rrr] \arrow[d] & & & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(\mathcal{O}_{Y,y}) \arrow[r] & Y \end{tikzcd}$$

gives the desired diagram. □

First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of schemes having the following shape

0206 (32.15.1.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & S \end{tikzcd}$$

with A a valuation ring and K its field of fractions.

0207 **Lemma 32.15.2.** Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. The following are equivalent:

(1) The morphism f is separated.
(2) For any diagram (32.15.1.1) there is at most one dotted arrow.
(3) For all diagrams (32.15.1.1) with A a discrete valuation ring there is at most one dotted arrow.
(4) For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (32.15.1.1) such that the morphism $\mathrm{Spec}(K) \to X$ is the canonical one (see Schemes, Section 26.13) there is at most one dotted arrow.

Proof. Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to show (4) implies (1). Assume (4). We begin by reducing to S affine. Being separated is a local on the base (see Schemes, Lemma 26.21.7). Hence, if we can show that whenever $X \to S$ has (4) that the restriction $X_\alpha \to S_\alpha$ has (4) where $S_\alpha \subset S$ is an (affine) open subset and $X_\alpha := f^{-1}(S_\alpha)$, then we will be done. The generic points of the irreducible components of X_α will be the generic points of irreducible components of X, since X_α is open in X. Therefore, any two distinct dotted arrows in the diagram

05M0 (32.15.2.1)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & S_\alpha \end{tikzcd}$$

would then give two distinct arrows in diagram (32.15.1.1) via the maps $X_\alpha \to X$ and $S_\alpha \to S$, which is a contradiction. Thus we have reduced to the case S is affine. We remark that in the course of this reduction, we prove that if $X \to S$ has (4) then the restriction $U \to V$ has (4) for opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$.

We next wish to reduce to the case $X \to S$ is finite type. Assume that we know (4) implies (1) when X is finite type. Since S is Noetherian and X is locally of finite type over S we see X is locally Noetherian as well (see Morphisms, Lemma 29.15.6). Thus, $X \to S$ is quasi-separated (see Properties, Lemma 28.5.4), and therefore we may apply the valuative criterion to check whether X is separated (see Schemes, Lemma 26.22.2). Let $X = \bigcup_\alpha X_\alpha$ be an affine open cover of X. Given any two dotted arrows, in a diagram (32.15.1.1), the image of the closed points of $\mathrm{Spec}\, A$ will fall in two sets X_α and X_β. Since $X_\alpha \cup X_\beta$ is open, for topological reasons it must contain the image of $\mathrm{Spec}(A)$ under both maps. Therefore, the two dotted arrows factor through $X_\alpha \cup X_\beta \to X$, which is a scheme of finite type over S. Since $X_\alpha \cup X_\beta$ is an open subset of X, by our previous remark, $X_\alpha \cup X_\beta$ satisfies (4), so by assumption, is separated. This implies the two given dotted arrows are the same. Therefore, we have reduced to $X \to S$ is finite type.

Assume $X \to S$ of finite type and assume (4). Since $X \to S$ is finite type, and S is an affine Noetherian scheme, X is also Noetherian (see Morphisms, Lemma 29.15.6). Therefore, $X \to X \times_S X$ will be a quasi-compact immersion of Noetherian schemes. We proceed by contradiction. Assume that $X \to X \times_S X$ is not closed. Then, there is some $y \in X \times_S X$ in the closure of the image that is not in the image. As X is Noetherian it has finitely many irreducible components. Therefore, y is in the closure of the image of one of the irreducible components $X_0 \subset X$. Give X_0 the reduced induced structure. The composition $X_0 \to X \to X \times_S X$ factors through the closed subscheme $X_0 \times_S X_0 \subset X \times_S X$. Denote the closure of $\Delta(X_0)$ in $X_0 \times_S X_0$ by $\bar{X}_0$ (again as a reduced closed subscheme). Thus $y \in \bar{X}_0$. Since $X_0 \to X_0 \times_S X_0$ is an immersion, the image of X_0 will be open in $\bar{X}_0$. Hence X_0 and $\bar{X}_0$ are birational. Since $\bar{X}_0$ is a closed subscheme of a Noetherian scheme, it is Noetherian. Thus, the local ring $\mathcal{O}_{\bar{X}_0,y}$ is a local Noetherian domain with fraction field K equal to the function field of X_0. By the Krull-Akizuki theorem (see Algebra, Lemma 10.119.13), there exists a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}_0,y}$ with fraction field K. This allows to construct a diagram:

05M1 (32.15.2.2)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X_0 \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & X_0 \times_S X_0 \end{tikzcd}$$

which sends $\mathrm{Spec}\, K$ to the generic point of $\Delta(X_0)$ and the closed point of A to $y \in X_0 \times_S X_0$ (use the material in Schemes, Section 26.13 to construct the arrows). There cannot even exist a set theoretic dotted arrow, since y is not in the image of Δ by our choice of y. By categorical means, the existence of the dotted arrow in the above diagram is equivalent to the uniqueness of the dotted arrow in the following diagram:

05M2 (32.15.2.3)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & S \end{tikzcd}$$

Therefore, we have non-uniqueness in this latter diagram by the nonexistence in the first. Therefore, X_0 does not satisfy uniqueness for discrete valuation rings, and since X_0 is an irreducible component of X, we have that $X \to S$ does not satisfy (4). Therefore, we have shown (4) implies (1). $\square$

0208 **Lemma 32.15.3.** Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. The following are equivalent:

(1) The morphism f is proper.
(2) For any diagram (32.15.1.1) there exists exactly one dotted arrow.
(3) For all diagrams (32.15.1.1) with A a discrete valuation ring there exists exactly one dotted arrow.
(4) For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (32.15.1.1) such that the morphism $\mathrm{Spec}(K) \to X$ is the canonical one (see Schemes, Section 26.13) there exists exactly one dotted arrow.

Proof. (1) implies (2) implies (3) implies (4). We will now show (4) implies (1). As in the proof of Lemma 32.15.2, we can reduce to the case S is affine, since properness is local on the base, and if $X \to S$ satisfies (4), then $X_\alpha \to S_\alpha$ does as well for open $S_\alpha \subset S$ and $X_\alpha = f^{-1}(S_\alpha)$.

Now S is a Noetherian scheme, and so X is as well, since $X \to S$ is of finite type. Now we may use Chow's lemma (Cohomology of Schemes, Lemma 30.18.1) to get a surjective, proper, birational $X' \to X$ and an immersion $X' \to \mathbf{P}^n_S$. We wish to show $X \to S$ is universally closed. As in the proof of Lemma 32.14.3, it is enough to check that $X' \to \mathbf{P}^n_S$ is a closed immersion. For the sake of contradiction, assume that $X' \to \mathbf{P}^n_S$ is not a closed immersion. Then there is some $y \in \mathbf{P}^n_S$ that is in the closure of the image of X', but is not in the image. So y is in the closure of the image of an irreducible component X'_0 of X', but not in the image. Let $\bar{X}'_0 \subset \mathbf{P}^n_S$ be the closure of the image of X'_0. As $X' \to \mathbf{P}^n_S$ is an immersion of Noetherian schemes, the morphism $X'_0 \to \bar{X}'_0$ is open and dense. By Algebra, Lemma 10.119.13 or Properties, Lemma 28.5.10 we can find a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}'_0,y}$ and with identical field of fractions K. It is clear that K is the residue field at the generic point of X'_0. Thus the solid commutative diagram

05M3 (32.15.3.1)
$$\begin{tikzcd}
\mathrm{Spec}\, K \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & \mathbf{P}^n_S \arrow[d] \\
\mathrm{Spec}\, A \arrow[r, dashed] \arrow[ru, dashed] \arrow[rru] & X \arrow[r] & S
\end{tikzcd}$$

Note that the closed point of A maps to $y \in \mathbf{P}^n_S$. By construction, there does not exist a set theoretic lift to X'. As $X' \to X$ is birational, the image of X'_0 in X is an irreducible component X_0 of X and K is also identified with the function field of X_0. Hence, as $X \to S$ is assumed to satisfy (4), the dotted arrow $\mathrm{Spec}(A) \to X$ exists. Since $X' \to X$ is proper, the dotted arrow lifts to the dotted arrow $\mathrm{Spec}(A) \to X'$ (use Schemes, Proposition 26.20.6). We can compose this with the immersion $X' \to \mathbf{P}^n_S$ to obtain another morphism (not depicted in the diagram) from $\mathrm{Spec}(A) \to \mathbf{P}^n_S$. Since $\mathbf{P}^n_S$ is proper over S, it satisfies (2), and so these two morphisms agree. This is a contradiction, for we have constructed the forbidden lift of our original map $\mathrm{Spec}(A) \to \mathbf{P}^n_S$ to X'. $\square$

05JY **Lemma 32.15.4.** Let $f : X \to S$ be a finite type morphism of schemes. Assume S is locally Noetherian. Then the following are equivalent

(1) f is universally closed,

(2) for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed,
(3) for any diagram (32.15.1.1) there exists some dotted arrow,
(4) for all diagrams (32.15.1.1) with A a discrete valuation ring there exists some dotted arrow.

Proof. The equivalence of (1) and (2) is a special case of Lemma 32.14.2. The equivalence of (1) and (3) is a special case of Schemes, Proposition 26.20.6. Trivially (3) implies (4). Thus all we have to do is prove that (4) implies (2). We will prove that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed by the criterion of Schemes, Lemma 26.19.8. Pick n and a specialization $z \rightsquigarrow z'$ of points in $\mathbf{A}^n \times S$ and a point $y \in \mathbf{A}^n \times X$ lying over z. Note that $\kappa(y)$ is a finitely generated field extension of $\kappa(z)$ as $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is of finite type. Hence by Properties, Lemma 28.5.10 or Algebra, Lemma 10.119.13 implies that there exists a discrete valuation ring $A \subset \kappa(y)$ with fraction field $\kappa(z)$ dominating the image of $\mathcal{O}_{\mathbf{A}^n \times S, z'}$ in $\kappa(z)$. This gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(y)) \arrow[r] \arrow[d] & \mathbf{A}^n \times X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & \mathbf{A}^n \times S \arrow[r] & S
\end{tikzcd}$$

Now property (4) implies that there exists a morphism $\mathrm{Spec}(A) \to X$ which fits into this diagram. Since we already have the morphism $\mathrm{Spec}(A) \to \mathbf{A}^n$ from the left lower horizontal arrow we also get a morphism $\mathrm{Spec}(A) \to \mathbf{A}^n \times X$ fitting into the left square. Thus the image $y' \in \mathbf{A}^n \times X$ of the closed point is a specialization of y lying over z'. This proves that specializations lift along $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ and we win. $\square$

32.16. Refined Noetherian valuative criteria

0H1P One usually does not have to consider all possible diagrams with valuation rings when checking valuative criteria. An example is given by Morphisms, Lemma 29.42.2. In the Noetherian setting, we have also seen this in Lemmas 32.15.2 and 32.15.3. Here is another variant.

0CM3 Lemma 32.16.1. Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f and h are of finite type, that f is separated, and that $h(U)$ is dense in X. If given any commutative solid diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & S
\end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a dotted arrow making the diagram commute, then f is proper.

Proof. There is an immediate reduction to the case where S is affine. Then U is quasi-compact. Let $U = U_1 \cup \ldots \cup U_n$ be an affine open covering. We may replace U by $U_1 \amalg \ldots \amalg U_n$ without changing the assumptions, hence we may assume U is affine. Thus we can find an open immersion $U \to Y$ over X with Y proper over X. (First put U inside $\mathbf{A}^n_X$ using Morphisms, Lemma 29.39.2 and then take the closure inside $\mathbf{P}^n_X$, or you can directly use Morphisms, Lemma 29.43.12.) We can assume

U is dense in Y (replace Y by the scheme theoretic closure of U if necessary, see Morphisms, Section 29.7). Note that $g : Y \to X$ is surjective as the image is closed and contains the dense subset $h(U)$. We will show that $Y \to S$ is proper. This will imply that $X \to S$ is proper by Morphisms, Lemma 29.41.9 thereby finishing the proof. To show that $Y \to S$ is proper we will use part (4) of Lemma 32.15.3. To do this consider a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "y"'] \arrow[d] & Y \arrow[d, "f \circ g"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dotted] & S \end{tikzcd}$$

where A is a discrete valuation ring with fraction field K and where $y : \mathrm{Spec}(K) \to Y$ is the inclusion of a generic point. We have to show there exists a unique dotted arrow. Uniqueness holds by the converse to the valuative criterion for separatedness (Schemes, Lemma 26.22.1) since $Y \to S$ is separated as the composition of the separated morphisms $Y \to X$ and $X \to S$ (Schemes, Lemma 26.21.12). Existence can be seen as follows. As y is a generic point of Y, it is contained in U. By assumption of the lemma there exists a morphism $a : \mathrm{Spec}(A) \to X$ such that

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "y"'] \arrow[d] & U \arrow[r] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, "a"] & & S \end{tikzcd}$$

is commutative. Then since $Y \to X$ is proper, we can apply the valuative criterion for properness (Morphisms, Lemma 29.42.1) to find a morphism $b : \mathrm{Spec}(A) \to Y$ such that

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "y"'] \arrow[d] & Y \arrow[d, "g"] \\ \mathrm{Spec}(A) \arrow[r, "a"] \arrow[ru, "b"] & X \end{tikzcd}$$

is commutative. This finishes the proof since b can serve as the dotted arrow above. $\square$

0CM4 Lemma 32.16.2. Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f is locally of finite type, that h is of finite type, and that $h(U)$ is dense in X. If given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[rru, dashed] & & S \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists at most one dotted arrow making the diagram commute, then f is separated.

Proof. We will apply Lemma 32.16.1 to the morphisms $U \to X$ and $\Delta : X \to X \times_S X$. We check the conditions. Observe that Δ is quasi-compact by Properties, Lemma 28.5.4 (and Schemes, Lemma 26.21.13). Of course Δ is locally of finite type and separated (true for any diagonal morphism). Finally, suppose given a commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[rr, "{(a,b)}"] \arrow[urr, dashed] & & X \times_S X \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K. Then a and b give two dotted arrows in the diagram of the lemma and have to be equal. Hence as dotted arrow we can use $a = b$ which gives existence. This finishes the proof. □

0CM5 Lemma 32.16.3. Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that S is locally Noetherian, that f and h are of finite type, and that $h(U)$ is dense in X. If given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & U \arrow[r, "h"] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[rr] \arrow[urr, dashed] & & S \end{tikzcd}$$

where A is a discrete valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is proper.

Proof. Combine Lemmas 32.16.2 and 32.16.1. □

32.17. Valuative criteria over a Nagata base

0GWU When working with schemes locally of finite type over a Nagata base we can reduce to discrete valuation rings which are essentially of finite type over the base. The following are just some example results one can get.

0GWV Lemma 32.17.1. Let S be a Nagata scheme (and in particular locally Noetherian). Let $f : X \to Y$ be a quasi-compact morphism of schemes locally of finite type over S. The following are equivalent

1. f is universally closed,
2. for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times Y$ is closed,
3. for any commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ C \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

of schemes over S such that
   (a) C is a normal integral scheme of finite type over S,
   (b) $U = C \setminus \{c\}$ for some closed point $c \in C$,
   (c) $A = \mathcal{O}_{C,c}$ has dimension 1[3]

[3]It follows that A is a discrete valuation ring, see Algebra, Lemma 10.119.7. Moreover, c maps to a finite type point $s \in S$ and A is essentially of finite type over $\mathcal{O}_{S,s}$.

then in the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dashed] & Y \end{tikzcd}$$

where $K = \mathrm{Frac}(A)$ some dotted arrow exists[4] making the diagram commute.

Proof. We have seen the equivalence of (1) and (2) and the fact that these imply (3) in Lemma 32.15.4. Thus it suffices to prove that (3) implies (2). Observe that if condition (3) holds for $f : X \to Y$, then condition (3) holds for $1 \times f : \mathbf{A}^n \times X \to \mathbf{A}^n \times Y$ (see argument in the proof of Lemma 32.15.4). Hence it suffices to show that (3) implies that f is closed.

Reduction to the case where Y and S are affine; we suggest skipping this paragraph. Let $S' \subset S$ be an affine open and let $Y' \subset Y$ be an affine open mapping into S'. Set $X' = f^{-1}(Y')$. Then we claim that the restriction $f' : X' \to Y'$ of f viewed as a morphism of schemes over S' has property (3) also. We omit the details. Now if we can prove that f' is closed for all choices of S' and Y', then it follows that f is closed. This reduces us to the case discussed in the next paragraph.

Assume S and Y affine. Let $Z \subset X$ be a closed subset. We may and do view Z as a reduced closed subscheme of X. We have to show that $E = f(Z)$ is closed. Pick $y \in Y$ a closed point contained in the closure of $f(Z)$. It suffices to show $y \in E$. We assume $y \notin E$ to get a contradiction. The image $s \in S$ of y is a finite type point of S, see Morphisms, Lemma 29.16.5. Recall that E is constructible (Morphisms, Lemma 29.22.2). Consider the intersection $\mathrm{Spec}(\mathcal{O}_{Y,y}) \cap E$. This is a constructible subset of the spectrum (Morphisms, Lemma 29.22.1) which doesn't contain the closed point. Since the punctured spectrum $\mathrm{Spec}(\mathcal{O}_{Y,y}) \setminus \{y\}$ is Jacobson (Morphisms, Lemma 29.16.10), we find a closed point $t \in \mathrm{Spec}(\mathcal{O}_{Y,y}) \setminus \{y\}$ with $t \in E$ (see Topology, Lemma 5.18.5). In other words, $t \in E$ is a point of Y which has an immediate specialization $t \rightsquigarrow y$. As $t \in E$ the scheme theoretic fibre Z_t is nonempty. Choose a closed point $x \in Z_t$. In particular we have $[\kappa(x) : \kappa(t)] < \infty$ by the Hilbert Nullstellensatz (Morphisms, Lemma 29.20.3).

Denote $T = \overline{\{t\}} \subset Y$ the integral closed subscheme whose underlying topological space is as indicated (Schemes, Definition 26.12.5). Then $t \in T$ is the generic point. Denote $C \to T$ the normalization of T in $\kappa(x)$, see Morphisms, Section 29.53 (more precisely, $C \to T$ is the normalization of T in x where we view $x = \mathrm{Spec}(\kappa(x)) \to T$ as a scheme over T). Since S is a Nagata scheme, so is T (Morphisms, Lemma 29.18.1). Hence we see that $C \to T$ is finite (Morphisms, Lemma 29.53.14). As t is in the image we see that $C \to T$ is surjective (because the image is closed and T is the closure of t in Y). Choose a point $c \in C$ mapping to $y \in T$. Since y is a closed point of T we see that c is a closed point of C. Since $\dim(\mathcal{O}_{T,y}) = 1$ we see that $\dim(\mathcal{O}_{C,c}) = 1$ (the dimension is at least 1 as c is not the generic point of C and at most 1 as $C \to T$ is finite). As the function field of C is $\kappa(x)$ and as x is a point of X, we have a Y-rational map from C to X (see for example Morphisms, Lemma

[4]By Lemma 32.6.4 this is equivalent to asking for the existence of dotted arrow making the first commutative diagram commute.

29.49.2). Let $C \supset U \to X$ be a representative (in particular U is nonempty). We may assume $c \notin U$ (replace U by $U \setminus \{c\}$). Since c is a closed point of codimension 1 in the integral scheme C we have $C = U \amalg \{c\} \amalg \Sigma$ for some proper closed subset $\Sigma \subset C$. After replacing C by $C \setminus \Sigma$ we have constructed a commutative diagram as in part (3). By the 2nd footnote in the statement of the lemma, the existence of the dotted arrow produces an extension of the rational map to all of C and we get the contradiction because the image of c will be a point of Z mapping to y. $\square$

0GWW Lemma 32.17.2. Let S be a Nagata scheme (and in particular locally Noetherian). Let $f : X \to Y$ be a morphism of schemes locally of finite type over S. The following are equivalent

(1) f separated,
(2) for any commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ C \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

of schemes over S such that
 (a) C is a normal integral scheme of finite type over S,
 (b) $U = C \setminus \{c\}$ for some closed point $c \in C$,
 (c) $A = \mathcal{O}_{C,c}$ has dimension 1[5]
then in the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $K = \mathrm{Frac}(A)$ there exists at most one dotted arrow[6] making the diagram commute.

Proof. By Lemma 32.15.2 we see that (1) implies (2). Assume (2). In order to show that f is separated, we have to show that $\Delta : X \to X \times_Y X$ is closed. By Morphisms, Lemma 29.15.7 the morphism Δ is quasi-compact. By Lemma 32.17.1 it suffices to show: for any commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ C \arrow[r, "{(a_1,a_2)}"] \arrow[ur, dotted] & X \times_Y X \end{tikzcd}$$

of schemes over S such that

(1) C is a normal integral scheme of finite type over S,
(2) $U = C \setminus \{c\}$ for some closed point $c \in C$,
(3) $A = \mathcal{O}_{C,c}$ has dimension 1.

[5]It follows that A is a discrete valuation ring, see Algebra, Lemma 10.119.7. Moreover, c maps to a finite type point $s \in S$ and A is essentially of finite type over $\mathcal{O}_{S,s}$.

[6]By Lemma 32.6.4 this is equivalent to asking there to be at most one dotted arrow making the first commutative diagram commute.

then in the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & X \times_Y X \end{tikzcd}$$

where $K = \mathrm{Frac}(A)$ there exists some dotted arrow making the diagram commute. By Lemma 32.6.4 the existence of the dotted arrow in the second diagram is equivalent to the existence of the dotted arrow in the first diagram. Moreover, the existence there is the same as asking $a_1 = a_2$. However $a_1|_U = a_2|_U$, so by the uniqueness assumption (2) we see that this is true and the proof is complete. $\square$

0GWX Lemma 32.17.3. Let S be a Nagata scheme (and in particular locally Noetherian). Let $f : X \to Y$ be a quasi-compact morphism of schemes locally of finite type over S. The following are equivalent

(1) f proper,
(2) for any commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ C \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

of schemes over S such that
(a) C is a normal integral scheme of finite type over S,
(b) $U = C \setminus \{c\}$ for some closed point $c \in C$,
(c) $A = \mathcal{O}_{C,c}$ has dimension 1[7]
then in the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $K = \mathrm{Frac}(A)$ there exists exactly one dotted arrow[8] making the diagram commute.

Proof. This is formal from Lemmas 32.17.1 and 32.17.2 and the definition of proper morphisms as being finite type, separated, and universally closed. $\square$

32.18. Limits and dimensions of fibres

05M4 The following lemma is most often used in the situation of Lemma 32.10.1 to assure that if the fibres of the limit have dimension $\leq d$, then the fibres at some finite stage have dimension $\leq d$.

05M5 Lemma 32.18.1. Let I be a directed set. Let $(f_i : X_i \to S_i)$ be an inverse system of morphisms of schemes over I. Assume

[7] It follows that A is a discrete valuation ring, see Algebra, Lemma 10.119.7. Moreover, c maps to a finite type point $s \in S$ and A is essentially of finite type over $\mathcal{O}_{S,s}$.

[8] By Lemma 32.6.4 this is equivalent to asking for the existence and uniqueness of the dotted arrow making the first commutative diagram commute.

(1) all the morphisms $S_{i'} \to S_i$ are affine,
(2) all the schemes S_i are quasi-compact and quasi-separated,
(3) the morphisms f_i are of finite type, and
(4) the morphisms $X_{i'} \to X_i \times_{S_i} S_{i'}$ are closed immersions.

Let $f : X = \lim_i X_i \to S = \lim_i S_i$ be the limit. Let $d \geq 0$. If every fibre of f has dimension $\leq d$, then for some i every fibre of f_i has dimension $\leq d$.

Proof. For each i let $U_i = \{x \in X_i \mid \dim_x((X_i)_{f_i(x)}) \leq d\}$. This is an open subset of X_i, see Morphisms, Lemma 29.28.4. Set $Z_i = X_i \setminus U_i$ (with reduced induced scheme structure). We have to show that $Z_i = \emptyset$ for some i. If not, then $Z = \lim Z_i \neq \emptyset$, see Lemma 32.4.3. Say $z \in Z$ is a point. Note that $Z \subset X$ is a closed subscheme. Set $s = f(z)$. For each i let $s_i \in S_i$ be the image of s. We remark that Z_s is the limit of the schemes $(Z_i)_{s_i}$ and Z_s is also the limit of the schemes $(Z_i)_{s_i}$ base changed to $\kappa(s)$. Moreover, all the morphisms

$$Z_s \longrightarrow (Z_{i'})_{s_{i'}} \times_{\mathrm{Spec}(\kappa(s_{i'}))} \mathrm{Spec}(\kappa(s)) \longrightarrow (Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s)) \longrightarrow X_s$$

are closed immersions by assumption (4). Hence Z_s is the scheme theoretic intersection of the closed subschemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ in X_s. Since all the irreducible components of the schemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ have dimension $> d$ and contain z we conclude that Z_s contains an irreducible component of dimension $> d$ passing through z which contradicts the fact that $Z_s \subset X_s$ and $\dim(X_s) \leq d$. $\square$

094M Lemma 32.18.2. Notation and assumptions as in Situation 32.8.1. If

(1) f is a quasi-finite morphism, and
(2) f_0 is locally of finite type,

then there exists an $i \geq 0$ such that f_i is quasi-finite.

Proof. Follows immediately from Lemma 32.18.1. $\square$

0H3V Lemma 32.18.3. Assumptions and notation as in Situation 32.8.1. Let $d \geq 0$. If

(1) f has relative dimension $\leq d$ (Morphisms, Definition 29.29.1), and
(2) f_0 is locally of finite type,

then there exists an i such that f_i has relative dimension $\leq d$.

Proof. Follows immediately from Lemma 32.18.1. $\square$

0EY2 Lemma 32.18.4. Notation and assumptions as in Situation 32.8.1. If

(1) f has relative dimension d, and
(2) f_0 is locally of finite presentation,

then there exists an $i \geq 0$ such that f_i has relative dimension d.

Proof. By Lemma 32.18.1 we may assume all fibres of f_0 have dimension $\leq d$. By Morphisms, Lemma 29.28.6 the set $U_0 \subset X_0$ of points $x \in X_0$ such that the dimension of the fibre of $X_0 \to Y_0$ at x is $\leq d - 1$ is open and retrocompact in X_0. Hence the complement $E = X_0 \setminus U_0$ is constructible. Moreover the image of $X \to X_0$ is contained in E by Morphisms, Lemma 29.28.3. Thus for $i \gg 0$ we have that the image of $X_i \to X_0$ is contained in E (Lemma 32.4.10). Then all fibres of $X_i \to Y_i$ have dimension d by the aforementioned Morphisms, Lemma 29.28.3. $\square$

05M6 **Lemma 32.18.5.** Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $d \geq 0$ be an integer. If $Z \subset X$ be a closed subscheme such that $\dim(Z_s) \leq d$ for all $s \in S$, then there exists a closed subscheme $Z' \subset X$ such that

(1) $Z \subset Z'$,
(2) $Z' \to X$ is of finite presentation, and
(3) $\dim(Z'_s) \leq d$ for all $s \in S$.

Proof. By Proposition 32.5.4 we can write $S = \lim S_i$ as the limit of a directed inverse system of Noetherian schemes with affine transition maps. By Lemma 32.10.1 we may assume that there exist a system of morphisms $f_i : X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_{S_i} S_{i'}$ for all $i' \geq i$ and such that $X = X_i \times_{S_i} S$. Let $Z_i \subset X_i$ be the scheme theoretic image of $Z \to X \to X_i$. Then for $i' \geq i$ the morphism $X_{i'} \to X_i$ maps $Z_{i'}$ into Z_i and the induced morphism $Z_{i'} \to Z_i \times_{S_i} S_{i'}$ is a closed immersion. By Lemma 32.18.1 we see that the dimension of the fibres of $Z_i \to S_i$ all have dimension $\leq d$ for a suitable $i \in I$. Fix such an i and set $Z' = Z_i \times_{S_i} S \subset X$. Since S_i is Noetherian, we see that X_i is Noetherian, and hence the morphism $Z_i \to X_i$ is of finite presentation. Therefore also the base change $Z' \to X$ is of finite presentation. Moreover, the fibres of $Z' \to S$ are base changes of the fibres of $Z_i \to S_i$ and hence have dimension $\leq d$. $\square$

32.19. Base change in top degree

0EX2 For a proper morphism and a finite type quasi-coherent module the base change map is an isomorphism in top degree.

0EX3 **Lemma 32.19.1.** Let $f : X \to Y$ be a morphism of schemes. Let $d \geq 0$. Assume

(1) X and Y are quasi-compact and quasi-separated, and
(2) $R^i f_* \mathcal{F} = 0$ for $i > d$ and every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.

Then we have

(a) for any base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

we have $R^i f'_* \mathcal{F}' = 0$ for $i > d$ and any quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$,
(b) $R^d f'_*(\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^* \mathcal{G}') = R^d f'_* \mathcal{F}' \otimes_{\mathcal{O}_{Y'}} \mathcal{G}'$ for any quasi-coherent $\mathcal{O}_{Y'}$-module $\mathcal{G}'$,
(c) formation of $R^d f'_* \mathcal{F}'$ commutes with arbitrary further base change (see proof for explanation).

Proof. Before giving the proofs, we explain the meaning of (c). Suppose we have an additional cartesian square

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"'] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

tacked onto our given diagram. If (a) holds, then there is a canonical map $\gamma : h^*R^df'_*\mathcal{F}' \to R^df''_*(h')^*\mathcal{F}'$. Namely, γ is the map on degree d cohomology sheaves induced by the composition

$$Lh^*Rf'_*\mathcal{F}' \longrightarrow Rf''_*L(h')^*\mathcal{F}' \longrightarrow Rf''_*(h')^*\mathcal{F}'$$

Here the first arrow is the base change map (Cohomology, Remark 20.28.3) and the second arrow complex from the canonical map $L(g')^*\mathcal{F} \to (g')^*\mathcal{F}$. Similarly, since $Rf'_*\mathcal{F}$ has no nonzero cohomology sheaves in degrees $> d$ by (a) we have $H^d(Lh^*Rf_*\mathcal{F}') = h^*R^df_*\mathcal{F}$. The content of (c) is that γ is an isomorphism.

Having said this, we can check (a), (b), and (c) locally on Y' and Y''. Suppose that $V \subset Y$ is a quasi-compact open subscheme. Then we claim (1) and (2) hold for $f|_{f^{-1}(V)} : f^{-1}(V) \to V$. Namely, (1) is immediate and (2) follows because any quasi-coherent module on $f^{-1}(V)$ is the restriction of a quasi-coherent module on X (Properties, Lemma 28.22.1) and formation of higher direct images commutes with restriction to opens. Thus we may also work locally on Y. In other words, we may assume Y'', Y', and Y are affine schemes.

Proof of (a) when Y' and Y are affine. In this case the morphisms g and g' are affine. Thus $g_* = Rg_*$ and $g'_* = Rg'_*$ (Cohomology of Schemes, Lemma 30.2.3) and g_* is identified with the restriction functor on modules (Schemes, Lemma 26.7.3). Then

$$g_*(R^if'_*\mathcal{F}') = H^i(Rg_*Rf'_*\mathcal{F}') = H^i(Rf_*Rg'_*\mathcal{F}') = H^i(Rf_*g'_*\mathcal{F}') = Rf^i_*g'_*\mathcal{F}'$$

which is zero by assumption (2). Hence (a) by our description of g_*.

Proof of (b) when Y' is affine, say $Y' = \mathrm{Spec}(R')$. By part (a) we have $H^{d+1}(X', \mathcal{F}') = 0$ for any quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$, see Cohomology of Schemes, Lemma 30.4.6. Consider the functor F on R'-modules defined by the rule

$$F(M) = H^d(X', \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M})$$

By Cohomology, Lemma 20.19.1 this functor commutes with direct sums (this is where we use that X and hence X' is quasi-compact and quasi-separated). On the other hand, if $M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then

$$\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M_1} \to \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M_2} \to \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M_3} \to 0$$

is an exact sequence of quasi-coherent modules on X' and by the vanishing of higher cohomology given above we get an exact sequence

$$F(M_1) \to F(M_2) \to F(M_3) \to 0$$

In other words, F is right exact. Any right exact R'-linear functor $F : \mathrm{Mod}_{R'} \to \mathrm{Mod}_{R'}$ which commutes with direct sums is given by tensoring with an R'-module (omitted; left as exercise for the reader). Thus we obtain $F(M) = H^d(X', \mathcal{F}') \otimes_{R'} M$. Since $R^d(f')_*\mathcal{F}'$ and $R^d(f')_*(\mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{M})$ are quasi-coherent (Cohomology of Schemes, Lemma 30.4.5), the fact that $F(M) = H^d(X', \mathcal{F}') \otimes_{R'} M$ translates into the statement given in (b).

Proof of (c) when $Y'' \to Y' \to Y$ are morphisms of affine schemes. Say $Y'' = \mathrm{Spec}(R'')$ and $Y' = \mathrm{Spec}(R')$. Then we see that $R^df''_*(h')^*\mathcal{F}'$ is the quasi-coherent module on Y' associated to the R''-module $H^d(X'', (h')^*\mathcal{F}')$. Now $h' : X'' \to X'$ is affine hence $H^d(X'', (h')^*\mathcal{F}') = H^d(X, h'_*(h')^*\mathcal{F}')$ by the already used Cohomology of Schemes, Lemma 30.2.4. We have

$$h'_*(h')^*\mathcal{F}' = \mathcal{F}' \otimes_{\mathcal{O}_{X'}} (f')^*\widetilde{R''}$$

as the reader sees by checking on an affine open covering. Thus $H^d(X'', (h')^*\mathcal{F}') = H^d(X', \mathcal{F}') \otimes_{R'} R''$ by part (b) applied to f' and the proof is complete. □

0E7D Lemma 32.19.2. Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume f is proper and $\dim(X_y) = d$. Then

(1) for $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have $(R^i f_*\mathcal{F})_y = 0$ for all $i > d$,
(2) there is an affine open neighbourhood $V \subset Y$ of y such that $f^{-1}(V) \to V$ and d satisfy the assumptions and conclusions of Lemma 32.19.1.

Proof. By Morphisms, Lemma 29.28.4 and the fact that f is closed, we can find an affine open neighbourhood V of y such that the fibres over points of V all have dimension $\leq d$. Thus we may assume $X \to Y$ is a proper morphism all of whose fibres have dimension $\leq d$ with Y affine. We will show that (2) holds, which will immediately imply (1) for all $y \in Y$.

By Lemma 32.13.2 we can write $X = \lim X_i$ as a cofiltered limit with $X_i \to Y$ proper and of finite presentation and such that both $X \to X_i$ and transition morphisms are closed immersions. For some i we have that $X_i \to Y$ has fibres of dimension $\leq d$, see Lemma 32.18.1. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $R^p f_*\mathcal{F} = R^p f_{i,*}(X \to X_i)_*\mathcal{F}$ by Cohomology of Schemes, Lemma 30.2.3 and Leray (Cohomology, Lemma 20.13.8). Thus we may replace X by X_i and reduce to the case discussed in the next paragraph.

Assume Y is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$. It suffices to show that $H^p(X, \mathcal{F}) = 0$ for $p > d$. Namely, by Cohomology of Schemes, Lemma 30.4.6 we have $H^p(X, \mathcal{F}) = H^0(Y, R^p f_*\mathcal{F})$. On the other hand, $R^p f_*\mathcal{F}$ is quasi-coherent on Y by Cohomology of Schemes, Lemma 30.4.5, hence vanishing of global sections implies vanishing. Write $Y = \lim_{i\in I} Y_i$ as a cofiltered limit of affine schemes with Y_i the spectrum of a Noetherian ring (for example a finite type $\mathbf{Z}$-algebra). We can choose an element $0 \in I$ and a finite type morphism $X_0 \to Y_0$ such that $X \cong Y \times_{Y_0} X_0$, see Lemma 32.10.1. After increasing 0 we may assume $X_0 \to Y_0$ is proper (Lemma 32.13.1) and that the fibres of $X_0 \to Y_0$ have dimension $\leq d$ (Lemma 32.18.1). Since $X \to X_0$ is affine, we find that $H^p(X, \mathcal{F}) = H^p(X_0, (X \to X_0)_*\mathcal{F})$ by Cohomology of Schemes, Lemma 30.2.4. This reduces us to the case discussed in the next paragraph.

Assume Y is affine Noetherian and $f : X \to Y$ is proper and all fibres have dimension $\leq d$. In this case we can write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of coherent $\mathcal{O}_X$-modules, see Properties, Lemma 28.22.7. Then $H^p(X, \mathcal{F}) = \operatorname{colim} H^p(X, \mathcal{F}_i)$ by Cohomology, Lemma 20.19.1. Thus we may assume $\mathcal{F}$ is coherent. In this case we see that $(R^p f_*\mathcal{F})_y = 0$ for all $y \in Y$ by Cohomology of Schemes, Lemma 30.20.9. Thus $R^p f_*\mathcal{F} = 0$ and therefore $H^p(X, \mathcal{F}) = 0$ (see above) and we win. □

0EX4 Lemma 32.19.3. Let $f : X \to Y$ be a morphism of schemes. Let $d \geq 0$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume

(1) f is a proper morphism all of whose fibres have dimension $\leq d$,
(2) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type.

Then $R^d f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type.

Proof. The module $R^d f_*\mathcal{F}$ is quasi-coherent by Cohomology of Schemes, Lemma 30.4.5. The question is local on Y hence we may assume Y is affine. Say $Y = \mathrm{Spec}(R)$. Then it suffices to prove that $H^d(X, \mathcal{F})$ is a finite R-module.

By Lemma 32.13.2 we can write $X = \lim X_i$ as a cofiltered limit with $X_i \to Y$ proper and of finite presentation and such that both $X \to X_i$ and transition morphisms are closed immersions. For some i we have that $X_i \to Y$ has fibres of dimension $\leq d$, see Lemma 32.18.1. We have $R^p f_*\mathcal{F} = R^p f_{i,*}(X \to X_i)_*\mathcal{F}$ by Cohomology of Schemes, Lemma 30.2.3 and Leray (Cohomology, Lemma 20.13.8). Thus we may replace X by X_i and reduce to the case discussed in the next paragraph.

Assume Y is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$. We can write $\mathcal{F}$ as a quotient of a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$, see Properties, Lemma 28.22.8. The map $H^d(X, \mathcal{F}') \to H^d(X, \mathcal{F})$ is surjective, as we have $H^{d+1}(X, \mathrm{Ker}(\mathcal{F}' \to \mathcal{F})) = 0$ by the vanishing of higher cohomology seen in Lemma 32.19.2 (or its proof). Thus we reduce to the case discussed in the next paragraph.

Assume $Y = \mathrm{Spec}(R)$ is affine and $f : X \to Y$ is proper and of finite presentation and all fibres have dimension $\leq d$ and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation. Write $Y = \lim_{i \in I} Y_i$ as a cofiltered limit of affine schemes with $Y_i = \mathrm{Spec}(R_i)$ the spectrum of a Noetherian ring (for example a finite type $\mathbf{Z}$-algebra). We can choose an element $0 \in I$ and a finite type morphism $X_0 \to Y_0$ such that $X \cong Y \times_{Y_0} X_0$, see Lemma 32.10.1. After increasing 0 we may assume $X_0 \to Y_0$ is proper (Lemma 32.13.1) and that the fibres of $X_0 \to Y_0$ have dimension $\leq d$ (Lemma 32.18.1). After increasing 0 we can assume there is a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ which pulls back to $\mathcal{F}$, see Lemma 32.10.2. By Lemma 32.19.1 we have

$$H^d(X, \mathcal{F}) = H^d(X_0, \mathcal{F}_0) \otimes_{R_0} R$$

This finishes the proof because the cohomology module $H^d(X_0, \mathcal{F}_0)$ is finite by Cohomology of Schemes, Lemma 30.19.2. □

0EX5 Lemma 32.19.4. Let $f : X \to Y$ be a morphism of schemes. Let $d \geq 0$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume

(1) f is a proper morphism of finite presentation all of whose fibres have dimension $\leq d$,
(2) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.

Then $R^d f_*\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.

Proof. The proof is exactly the same as the proof of Lemma 32.19.3 except that the third paragraph can be skipped. We omit the details. □

32.20. Glueing in closed fibres

0E8P Applying our theory above to the spectrum of a local ring we obtain the following pleasing glueing result for relative schemes.

0BPA Lemma 32.20.1. Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ there is an equivalence of categories

$$\left\{X \to S \text{ of finite presentation}\right\} \longrightarrow \left\{ \begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ U & V \arrow[l] \arrow[r] & \mathrm{Spec}(\mathcal{O}_{S,s}) \end{tikzcd} \right\}$$

where on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

Proof. Let $W \subset S$ be an open neighbourhood of s. By glueing of relative schemes, see Constructions, Section 27.2, the functor

$$\left\{X \to S \text{ of finite presentation}\right\} \longrightarrow \left\{ \begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ U & W \setminus \{s\} \arrow[l] \arrow[r] & W \end{tikzcd} \right\}$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of schemes of finite presentation over $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of schemes of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 32.10.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U = \lim W \setminus \{s\}$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 32.2.2). Thus also the category of schemes of finite presentation over V is the limit of the categories of schemes of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. $\square$

0F21 Lemma 32.20.2. Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ there is an equivalence of categories

$$\{\mathcal{O}_S\text{-modules } \mathcal{F} \text{ of finite presentation}\} \longrightarrow \{(\mathcal{G}, \mathcal{H}, \alpha)\}$$

where on the right hand side we consider triples consisting of a $\mathcal{O}_U$-module $\mathcal{G}$ of finite presentation, a $\mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{S,s})}$-module $\mathcal{H}$ of finite presentation, and an isomorphism $\alpha : \mathcal{G}|_V \to \mathcal{H}|_V$ of $\mathcal{O}_V$-modules.

Proof. You can either prove this by redoing the proof of Lemma 32.20.1 using Lemma 32.10.2 or you can deduce it from Lemma 32.20.1 using the equivalence between quasi-coherent modules and "vector bundles" from Constructions, Section 27.6. We omit the details. $\square$

0BQ5 Lemma 32.20.3. Let S be a scheme. Let $U \subset S$ be a retrocompact open. Let $s \in S$ be a point in the complement of U. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \cap U$ there is an equivalence of categories

$$\mathrm{colim}_{s \in U' \supset U \text{ open}} \left\{ \begin{tikzcd} X \arrow[d] \\ U' \end{tikzcd} \right\} \longrightarrow \left\{ \begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ U & V \arrow[l] \arrow[r] & \mathrm{Spec}(\mathcal{O}_{S,s}) \end{tikzcd} \right\}$$

where on the left hand side the vertical arrow is of finite presentation and on the right hand side we consider commutative diagrams whose squares are cartesian and whose vertical arrows are of finite presentation.

Proof. Let $W \subset S$ be an open neighbourhood of s. By glueing of relative schemes, see Constructions, Section 27.2, the functor

$$\{X \to U' = U \cup W \text{ of finite presentation}\} \longrightarrow \left\{ \begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ U & W \cap U \arrow[l] \arrow[r] & W \end{tikzcd} \right\}$$

is an equivalence of categories. We have $\mathcal{O}_{S,s} = \mathrm{colim}\, \mathcal{O}_W(W)$ where W runs over the affine open neighbourhoods of s. Hence $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s. Thus the category of schemes of finite presentation over $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the category of schemes of finite presentation over W where W runs over the affine open neighbourhoods of s, see Lemma 32.10.1. For every affine open $s \in W$ we see that $U \cap W$ is quasi-compact as $U \to S$ is quasi-compact. Hence $V = \lim W \cap U$ is a limit of quasi-compact and quasi-separated schemes (see Lemma 32.2.2). Thus also the category of schemes of finite presentation over V is the limit of the categories of schemes of finite presentation over $W \cap U$ where W runs over the affine open neighbourhoods of s. The lemma follows formally from a combination of these results. □

0EY3 Lemma 32.20.4. Notation and assumptions as in Lemma 32.20.3. Let $U \subset U' \subset X$ be an open containing s.

(1) Let $f' : X \to U'$ correspond to $f : X' \to U$ and $g : Y \to \mathrm{Spec}(\mathcal{O}_{S,s})$ via the equivalence. If f and g are separated, proper, finite, étale, then after possibly shrinking U' the morphism f' has the same property.
(2) Let $a : X_1 \to X_2$ be a morphism of schemes of finite presentation over U' with base change $a' : X_1' \to X_2'$ over U and $b : Y_1 \to Y_2$ over $\mathrm{Spec}(\mathcal{O}_{S,s})$. If a' and b are separated, proper, finite, étale, then after possibly shrinking U' the morphism a has the same property.

Proof. Proof of (1). Recall that $\mathrm{Spec}(\mathcal{O}_{S,s})$ is the limit of the affine open neighbourhoods of s in S. Since g has the property in question, then the restriction of f' to one of these affine open neighbourhoods does too, see Lemmas 32.8.6, 32.13.1, 32.8.3, and 32.8.10. Since f' has the given property over U as f does, we conclude as one can check the property locally on the base.

Proof of (2). If we write $\mathrm{Spec}(\mathcal{O}_{S,s}) = \lim W$ where W runs over the affine open neighbourhoods of s in S, then we have $Y_i = \lim W \times_S X_i$. Thus we can use exactly the same arguments as in the proof of (1). □

0E8Q Lemma 32.20.5. Let S be a scheme. Let $s_1, \ldots, s_n \in S$ be pairwise distinct closed points such that $U = S \setminus \{s_1, \ldots, s_n\} \to S$ is quasi-compact. With $S_i = \mathrm{Spec}(\mathcal{O}_{S,s_i})$ and $U_i = S_i \setminus \{s_i\}$ there is an equivalence of categories

$$FP_S \longrightarrow FP_U \times_{(FP_{U_1} \times \ldots \times FP_{U_n})} (FP_{S_1} \times \ldots \times FP_{S_n})$$

where FP_T is the category of schemes of finite presentation over the scheme T.

Proof. For $n = 1$ this is Lemma 32.20.1. For $n > 1$ the lemma can be proved in exactly the same way or it can be deduced from it. For example, suppose that $f_i : X_i \to S_i$ are objects of FP_{S_i} and $f : X \to U$ is an object of FP_U and we're given isomorphisms $X_i \times_{S_i} U_i = X \times_U U_i$. By Lemma 32.20.1 we can find a morphism $f' : X' \to U' = S \setminus \{s_1, \ldots, s_{n-1}\}$ which is of finite presentation, which is isomorphic to X_i over S_i, which is isomorphic to X over U, and these isomorphisms are compatible with the given isomorphism $X_i \times_{S_n} U_n = X \times_U U_n$. Then we can apply induction to $f_i : X_i \to S_i$, $i \leq n-1$, $f' : X' \to U'$, and the induced isomorphisms $X_i \times_{S_i} U_i = X' \times_{U'} U_i$, $i \leq n-1$. This shows essential surjectivity. We omit the proof of fully faithfulness. $\square$

32.21. Application to modifications

0B3W Using the results from Section 32.20 we can describe the category of modifications of a scheme over a closed point in terms of the local ring.

0B3X Lemma 32.21.1. Let S be a scheme. Let $s \in S$ be a closed point such that $U = S \setminus \{s\} \to S$ is quasi-compact. With $V = \mathrm{Spec}(\mathcal{O}_{S,s}) \setminus \{s\}$ the base change functor

$$\left\{ \begin{matrix} f : X \to S \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} g : Y \to \mathrm{Spec}(\mathcal{O}_{S,s}) \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism} \end{matrix} \right\}$$

is an equivalence of categories.

Proof. This is a special case of Lemma 32.20.1. $\square$

0BFN Lemma 32.21.2. Notation and assumptions as in Lemma 32.21.1. Let $f : X \to S$ correspond to $g : Y \to \mathrm{Spec}(\mathcal{O}_{S,s})$ via the equivalence. Then f is separated, proper, finite, étale and add more here if and only if g is so.

Proof. The property of being separated, proper, integral, finite, etc is stable under base change. See Schemes, Lemma 26.21.12 and Morphisms, Lemmas 29.41.5 and 29.44.6. Hence if f has the property, then so does g. The converse follows from Lemma 32.20.4 but we also give a direct proof here. Namely, if g has to property, then f does in a neighbourhood of s by Lemmas 32.8.6, 32.13.1, 32.8.3, and 32.8.10. Since f clearly has the given property over $S \setminus \{s\}$ we conclude as one can check the property locally on the base. $\square$

0B3Y Remark 32.21.3. The lemma above can be generalized as follows. Let S be a scheme and let $T \subset S$ be a closed subset. Assume there exists a cofinal system of open neighbourhoods $T \subset W_i$ such that (1) $W_i \setminus T$ is quasi-compact and (2) $W_i \subset W_j$ is an affine morphism. Then $W = \lim W_i$ is a scheme which contains T as a closed subscheme. Set $U = X \setminus T$ and $V = W \setminus T$. Then the base change functor

$$\left\{ \begin{matrix} f : X \to S \text{ of finite presentation} \\ f^{-1}(U) \to U \text{ is an isomorphism} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} g : Y \to W \text{ of finite presentation} \\ g^{-1}(V) \to V \text{ is an isomorphism} \end{matrix} \right\}$$

is an equivalence of categories. If we ever need this we will change this remark into a lemma and provide a detailed proof.

32.22. Descending finite type schemes

0CNL This section continues the theme of Section 32.9 in the spirit of the results discussed in Section 32.10.

0CNM **Situation 32.22.1.** Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of Noetherian schemes with affine transition morphisms $S_{i'} \to S_i$ for $i' \geq i$.

0CNN **Lemma 32.22.2.** In Situation 32.22.1. Let $X \to S$ be quasi-separated and of finite type. Then there exists an $i \in I$ and a diagram

0CNP (32.22.2.1)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ S \arrow[r] & S_i \end{tikzcd}$$

such that $W \to S_i$ is of finite type and such that the induced morphism $X \to S \times_{S_i} W$ is a closed immersion.

Proof. By Lemma 32.9.3 we can find a closed immersion $X \to X'$ over S where X' is a scheme of finite presentation over S. By Lemma 32.10.1 we can find an i and a morphism of finite presentation $X'_i \to S_i$ whose pull back is X'. Set $W = X'_i$. $\square$

0CNQ **Lemma 32.22.3.** In Situation 32.22.1. Let $X \to S$ be quasi-separated and of finite type. Given $i \in I$ and a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ S \arrow[r] & S_i \end{tikzcd}$$

as in (32.22.2.1) for $i' \geq i$ let $X_{i'}$ be the scheme theoretic image of $X \to S_{i'} \times_{S_i} W$. Then $X = \lim_{i' \geq i} X_{i'}$.

Proof. Since X is quasi-compact and quasi-separated formation of the scheme theoretic image of $X \to S_{i'} \times_{S_i} W$ commutes with restriction to open subschemes (Morphisms, Lemma 29.6.3). Hence we may and do assume W is affine and maps into an affine open U_i of S_i. Let $U \subset S$, $U_{i'} \subset S_{i'}$ be the inverse image of U_i. Then U, $U_{i'}$, $S_{i'} \times_{S_i} W = U_{i'} \times_{U_i} W$, and $S \times_{S_i} W = U \times_{U_i} W$ are all affine. This implies X is affine because $X \to S \times_{S_i} W$ is a closed immersion. This also shows the ring map

$$\mathcal{O}(U) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W) \to \mathcal{O}(X)$$

is surjective. Let I be the kernel. Then we see that $X_{i'}$ is the spectrum of the ring

$$\mathcal{O}(X_{i'}) = \mathcal{O}(U_{i'}) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(W)/I_{i'}$$

where $I_{i'}$ is the inverse image of the ideal I (see Morphisms, Example 29.6.4). Since $\mathcal{O}(U) = \operatorname{colim} \mathcal{O}(U_{i'})$ we see that $I = \operatorname{colim} I_{i'}$ and we conclude that $\operatorname{colim} \mathcal{O}(X_{i'}) = \mathcal{O}(X)$. $\square$

0CNR **Lemma 32.22.4.** In Situation 32.22.1. Let $f : X \to Y$ be a morphism of schemes quasi-separated and of finite type over S. Let

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ S \arrow[r] & S_{i_1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] & V \arrow[d] \\ S \arrow[r] & S_{i_2} \end{tikzcd}$$

be diagrams as in (32.22.2.1). Let $X = \lim_{i \geq i_1} X_i$ and $Y = \lim_{i \geq i_2} Y_i$ be the corresponding limit descriptions as in Lemma 32.22.3. Then there exists an $i_0 \geq \max(i_1, i_2)$ and a morphism

$$(f_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$$

of inverse systems over $(S_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} f_i$. If $(g_i)_{i \geq i_0} : (X_i)_{i \geq i_0} \to (Y_i)_{i \geq i_0}$ is a second morphism of inverse systems over $(S_i)_{i \geq i_0}$ such that such that $f = \lim_{i \geq i_0} g_i$ then $f_i = g_i$ for all $i \gg i_0$.

Proof. Since $V \to S_{i_2}$ is of finite presentation and $X = \lim_{i \geq i_1} X_i$ we can appeal to Proposition 32.6.1 to find an $i_0 \geq \max(i_1, i_2)$ and a morphism $h : X_{i_0} \to V$ over S_{i_2} such that $X \to X_{i_0} \to V$ is equal to $X \to Y \to V$. For $i \geq i_0$ we get a commutative solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d, dotted] \arrow[dd, bend right] & X_{i_0} \arrow[d, "h"] \\ Y \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & V \arrow[d] \\ S \arrow[r] & S_i \arrow[r] & S_{i_0} \end{tikzcd}$$

Since $X \to X_i$ has scheme theoretically dense image and since Y_i is the scheme theoretic image of $Y \to S_i \times_{S_{i_2}} V$ we find that the morphism $X_i \to S_i \times_{S_{i_2}} V$ induced by the diagram factors through Y_i (Morphisms, Lemma 29.6.6). This proves existence.

Uniqueness. Let $E_i \subset X_i$ be the equalizer of f_i and g_i for $i \geq i_0$. By Schemes, Lemma 26.21.5 E_i is a locally closed subscheme of X_i. Since X_i is a closed subscheme of $S_i \times_{S_{i_0}} X_{i_0}$ and similarly for Y_i we see that

$$E_i = X_i \times_{(S_i \times_{S_{i_0}} X_{i_0})} (S_i \times_{S_{i_0}} E_{i_0})$$

Thus to finish the proof it suffices to show that $X_i \to X_{i_0}$ factors through E_{i_0} for some $i \geq i_0$. To do this we will use that $X \to X_{i_0}$ factors through E_{i_0} as both f_{i_0} and g_{i_0} are compatible with f. Since X_i is Noetherian, we see that the underlying topological space $|E_{i_0}|$ is a constructible subset of $|X_{i_0}|$ (Topology, Lemma 5.16.1). Hence $X_i \to X_{i_0}$ factors through E_{i_0} set theoretically for large enough i by Lemma 32.4.10. For such an i the scheme theoretic inverse image $(X_i \to X_{i_0})^{-1}(E_{i_0})$ is a closed subscheme of X_i through which X factors and hence equal to X_i since $X \to X_i$ has scheme theoretically dense image by construction. This concludes the proof. $\square$

0CNS **Remark 32.22.5.** In Situation 32.22.1 Lemmas 32.22.2, 32.22.3, and 32.22.4 tell us that the category of schemes quasi-separated and of finite type over S is equivalent to certain types of inverse systems of schemes over $(S_i)_{i \in I}$, namely the ones produced by applying Lemma 32.22.3 to a diagram of the form (32.22.2.1). For example, given $X \to S$ finite type and quasi-separated if we choose two different diagrams $X \to V_1 \to S_{i_1}$ and $X \to V_2 \to S_{i_2}$ as in (32.22.2.1), then applying Lemma 32.22.4 to id_X (in two directions) we see that the corresponding limit descriptions of X are canonically isomorphic (up to shrinking the directed set I). And so on and so forth.

0CNT Lemma 32.22.6. Notation and assumptions as in Lemma 32.22.4. If f is flat and of finite presentation, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is flat, $X_i = Y_i \times_{Y_{i_3}} X_{i_3}$, and $X = Y \times_{Y_{i_3}} X_{i_3}$.

Proof. By Lemma 32.10.1 we can choose an $i \geq i_2$ and a morphism $U \to Y_i$ of finite presentation such that $X = Y \times_{Y_i} U$ (this is where we use that f is of finite presentation). After increasing i we may assume that $U \to Y_i$ is flat, see Lemma 32.8.7. As discussed in Remark 32.22.5 we may and do replace the initial diagram used to define the system $(X_i)_{i \geq i_1}$ by the system corresponding to $X \to U \to S_i$. Thus $X_{i'}$ for $i' \geq i$ is defined as the scheme theoretic image of $X \to S_{i'} \times_{S_i} U$.

Because $U \to Y_i$ is flat (this is where we use that f is flat), because $X = Y \times_{Y_i} U$, and because the scheme theoretic image of $Y \to Y_i$ is Y_i, we see that the scheme theoretic image of $X \to U$ is U (Morphisms, Lemma 29.25.16). Observe that $Y_{i'} \to S_{i'} \times_{S_i} Y_i$ is a closed immersion for $i' \geq i$ by construction of the system of Y_j. Then the same argument as above shows that the scheme theoretic image of $X \to S_{i'} \times_{S_i} U$ is equal to the closed subscheme $Y_{i'} \times_{Y_i} U$. Thus we see that $X_{i'} = Y_{i'} \times_{Y_i} U$ for all $i' \geq i$ and hence the lemma holds with $i_3 = i$. $\square$

0CNU Lemma 32.22.7. Notation and assumptions as in Lemma 32.22.4. If f is smooth, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is smooth.

Proof. Combine Lemmas 32.22.6 and 32.8.9. $\square$

0CNV Lemma 32.22.8. Notation and assumptions as in Lemma 32.22.4. If f is proper, then there exists an $i_3 \geq i_0$ such that for $i \geq i_3$ we have f_i is proper.

Proof. By the discussion in Remark 32.22.5 the choice of i_1 and W fitting into a diagram as in (32.22.2.1) is immaterial for the truth of the lemma. Thus we choose W as follows. First we choose a closed immersion $X \to X'$ with $X' \to S$ proper and of finite presentation, see Lemma 32.13.2. Then we choose an $i_3 \geq i_2$ and a proper morphism $W \to Y_{i_3}$ such that $X' = Y \times_{Y_{i_3}} W$. This is possible because $Y = \lim_{i \geq i_2} Y_i$ and Lemmas 32.10.1 and 32.13.1. With this choice of W it is immediate from the construction that for $i \geq i_3$ the scheme X_i is a closed subscheme of $Y_i \times_{Y_{i_3}} W \subset S_i \times_{S_{i_3}} W$ and hence proper over Y_i. $\square$

0CNW Lemma 32.22.9. In Situation 32.22.1 suppose that we have a cartesian diagram

$$\begin{tikzcd} X^1 \arrow[r, "p"'] \arrow[d, "q"'] & X^3 \arrow[d, "a"] \\ X^2 \arrow[r, "b"] & X^4 \end{tikzcd}$$

of schemes quasi-separated and of finite type over S. For each $j = 1, 2, 3, 4$ choose $i_j \in I$ and a diagram

$$\begin{tikzcd} X^j \arrow[r] \arrow[d] & W^j \arrow[d] \\ S \arrow[r] & S_{i_j} \end{tikzcd}$$

as in (32.22.2.1). Let $X^j = \lim_{i \geq i_j} X_i^j$ be the corresponding limit descriptions as in Lemma 32.22.4. Let $(a_i)_{i \geq i_5}$, $(b_i)_{i \geq i_6}$, $(p_i)_{i \geq i_7}$, and $(q_i)_{i \geq i_8}$ be the corresponding morphisms of systems contructed in Lemma 32.22.4. Then there exists an $i_9 \geq \max(i_5, i_6, i_7, i_8)$ such that for $i \geq i_9$ we have $a_i \circ p_i = b_i \circ q_i$ and such that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion. If a and b are flat and of finite presentation, then there exists an $i_{10} \geq \max(i_5, i_6, i_7, i_8, i_9)$ such that for $i \geq i_{10}$ the last displayed morphism is an isomorphism.

Proof. According to the discussion in Remark 32.22.5 the choice of W^1 fitting into a diagram as in (32.22.2.1) is immaterial for the truth of the lemma. Thus we may choose $W^1 = W^2 \times_{W^4} W^3$. Then it is immediate from the construction of X_i^1 that $a_i \circ p_i = b_i \circ q_i$ and that

$$(q_i, p_i) : X_i^1 \longrightarrow X_i^2 \times_{b_i, X_i^4, a_i} X_i^3$$

is a closed immersion.

If a and b are flat and of finite presentation, then so are p and q as base changes of a and b. Thus we can apply Lemma 32.22.6 to each of a, b, p, q, and $a \circ p = b \circ q$. It follows that there exists an $i_9 \in I$ such that

$$(q_i, p_i) : X_i^1 \to X_i^2 \times_{X_i^4} X_i^3$$

is the base change of (q_{i_9}, p_{i_9}) by the morphism by the morphism $X_i^4 \to X_{i_9}^4$ for all $i \geq i_9$. We conclude that (q_i, p_i) is an isomorphism for all sufficiently large i by Lemma 32.8.11. ☐

32.23. Other chapters

(27) Constructions of Schemes
(28) Properties of Schemes
(29) Morphisms of Schemes
(30) Cohomology of Schemes
(31) Divisors
(32) Limits of Schemes
(33) Varieties
(34) Topologies on Schemes
(35) Descent
(36) Derived Categories of Schemes
(37) More on Morphisms
(38) More on Flatness
(39) Groupoid Schemes
(40) More on Groupoid Schemes
(41) Étale Morphisms of Schemes

Topics in Scheme Theory

(42) Chow Homology
(43) Intersection Theory
(44) Picard Schemes of Curves
(45) Weil Cohomology Theories
(46) Adequate Modules
(47) Dualizing Complexes
(48) Duality for Schemes
(49) Discriminants and Differents
(50) de Rham Cohomology
(51) Local Cohomology
(52) Algebraic and Formal Geometry
(53) Algebraic Curves
(54) Resolution of Surfaces
(55) Semistable Reduction
(56) Functors and Morphisms
(57) Derived Categories of Varieties
(58) Fundamental Groups of Schemes
(59) Étale Cohomology
(60) Crystalline Cohomology
(61) Pro-étale Cohomology
(62) Relative Cycles
(63) More Étale Cohomology
(64) The Trace Formula

Algebraic Spaces

(65) Algebraic Spaces
(66) Properties of Algebraic Spaces
(67) Morphisms of Algebraic Spaces
(68) Decent Algebraic Spaces
(69) Cohomology of Algebraic Spaces
(70) Limits of Algebraic Spaces
(71) Divisors on Algebraic Spaces
(72) Algebraic Spaces over Fields
(73) Topologies on Algebraic Spaces
(74) Descent and Algebraic Spaces
(75) Derived Categories of Spaces
(76) More on Morphisms of Spaces
(77) Flatness on Algebraic Spaces
(78) Groupoids in Algebraic Spaces
(79) More on Groupoids in Spaces
(80) Bootstrap
(81) Pushouts of Algebraic Spaces

Topics in Geometry

(82) Chow Groups of Spaces
(83) Quotients of Groupoids
(84) More on Cohomology of Spaces
(85) Simplicial Spaces
(86) Duality for Spaces
(87) Formal Algebraic Spaces
(88) Algebraization of Formal Spaces
(89) Resolution of Surfaces Revisited

Deformation Theory

(90) Formal Deformation Theory
(91) Deformation Theory
(92) The Cotangent Complex
(93) Deformation Problems

Algebraic Stacks

(94) Algebraic Stacks
(95) Examples of Stacks
(96) Sheaves on Algebraic Stacks
(97) Criteria for Representability
(98) Artin's Axioms
(99) Quot and Hilbert Spaces
(100) Properties of Algebraic Stacks
(101) Morphisms of Algebraic Stacks
(102) Limits of Algebraic Stacks
(103) Cohomology of Algebraic Stacks
(104) Derived Categories of Stacks
(105) Introducing Algebraic Stacks
(106) More on Morphisms of Stacks
(107) The Geometry of Stacks

Topics in Moduli Theory

(108) Moduli Stacks
(109) Moduli of Curves

CHAPTER 33

Varieties

0209 33.1. Introduction

020A In this chapter we start studying varieties and more generally schemes over a field. A fundamental reference is [DG67].

33.2. Notation

020B Throughout this chapter we use the letter k to denote the ground field.

33.3. Varieties

020C In the Stacks project we will use the following as our definition of a variety.

020D **Definition 33.3.1.** Let k be a field. A *variety* is a scheme X over k such that X is integral and the structure morphism $X \to \mathrm{Spec}(k)$ is separated and of finite type.

This definition has the following drawback. Suppose that k'/k is an extension of fields. Suppose that X is a variety over k. Then the base change $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is not necessarily a variety over k'. This phenomenon (in greater generality) will be discussed in detail in the following sections. The product of two varieties need not be a variety (this is really the same phenomenon). Here is an example.

020G **Example 33.3.2.** Let $k = \mathbf{Q}$. Let $X = \mathrm{Spec}(\mathbf{Q}(i))$ and $Y = \mathrm{Spec}(\mathbf{Q}(i))$. Then the product $X \times_{\mathrm{Spec}(k)} Y$ of the varieties X and Y is not a variety, since it is reducible. (It is isomorphic to the disjoint union of two copies of X.)

If the ground field is algebraically closed however, then the product of varieties is a variety. This follows from the results in the algebra chapter, but there we treat much more general situations. There is also a simple direct proof of it which we present here.

05P3 **Lemma 33.3.3.** Let k be an algebraically closed field. Let X, Y be varieties over k. Then $X \times_{\mathrm{Spec}(k)} Y$ is a variety over k.

Proof. The morphism $X \times_{\mathrm{Spec}(k)} Y \to \mathrm{Spec}(k)$ is of finite type and separated because it is the composition of the morphisms $X \times_{\mathrm{Spec}(k)} Y \to Y \to \mathrm{Spec}(k)$ which are separated and of finite type, see Morphisms, Lemmas 29.15.4 and 29.15.3 and Schemes, Lemma 26.21.12. To finish the proof it suffices to show that $X \times_{\mathrm{Spec}(k)} Y$ is integral. Let $X = \bigcup_{i=1,\ldots,n} U_i$, $Y = \bigcup_{j=1,\ldots,m} V_j$ be finite affine open coverings. If we can show that each $U_i \times_{\mathrm{Spec}(k)} V_j$ is integral, then we are done by Properties, Lemmas 28.3.2, 28.3.3, and 28.3.4. This reduces us to the affine case.

The affine case translates into the following algebra statement: Suppose that A, B are integral domains and finitely generated k-algebras. Then $A \otimes_k B$ is an integral domain. To get a contradiction suppose that

$$(\sum\nolimits_{i=1,\ldots,n} a_i \otimes b_i)(\sum\nolimits_{j=1,\ldots,m} c_j \otimes d_j) = 0$$

in $A \otimes_k B$ with both factors nonzero in $A \otimes_k B$. We may assume that $b_1, \ldots, b_n$ are k-linearly independent in B, and that $d_1, \ldots, d_m$ are k-linearly independent in B. Of course we may also assume that a_1 and c_1 are nonzero in A. Hence $D(a_1c_1) \subset \text{Spec}(A)$ is nonempty. By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) we can find a maximal ideal $\mathfrak m \subset A$ contained in $D(a_1c_1)$ and $A/\mathfrak m = k$ as k is algebraically closed. Denote $\overline{a}_i, \overline{c}_j$ the residue classes of a_i, c_j in $A/\mathfrak m = k$. The equation above becomes

$$(\sum\nolimits_{i=1,\ldots,n} \overline{a}_i b_i)(\sum\nolimits_{j=1,\ldots,m} \overline{c}_j d_j) = 0$$

which is a contradiction with $\mathfrak m \in D(a_1c_1)$, the linear independence of $b_1, \ldots, b_n$ and $d_1, \ldots, d_m$, and the fact that B is a domain. $\square$

33.4. Varieties and rational maps

0BXM Let k be a field. Let X and Y be varieties over k. We will use the phrase rational map of varieties from X to Y to mean a $\text{Spec}(k)$-rational map from the scheme X to the scheme Y as defined in Morphisms, Definition 29.49.1. As is customary, the phrase "rational map of varieties" does not refer to the (common) base field of the varieties, even though for general schemes we make the distinction between rational maps and rational maps over a given base.

The title of this section refers to the following fundamental theorem.

0BXN **Theorem 33.4.1.** Let k be a field. The category of varieties and dominant rational maps is equivalent to the category of finitely generated field extensions K/k.

Proof. Let X and Y be varieties with generic points $x \in X$ and $y \in Y$. Recall that dominant rational maps from X to Y are exactly those rational maps which map x to y (Morphisms, Definition 29.49.10 and discussion following). Thus given a dominant rational map $X \supset U \to Y$ we obtain a map of function fields

$$k(Y) = \kappa(y) = \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x} = \kappa(x) = k(X)$$

Conversely, such a k-algebra map (which is automatically local as the source and target are fields) determines (uniquely) a dominant rational map by Morphisms, Lemma 29.49.2. In this way we obtain a fully faithful functor. To finish the proof it suffices to show that every finitely generated field extension K/k is in the essential image. Since K/k is finitely generated, there exists a finite type k-algebra $A \subset K$ such that K is the fraction field of A. Then $X = \text{Spec}(A)$ is a variety whose function field is K. $\square$

Let k be a field. Let X and Y be varieties over k. We will use the phrase X and Y are birational varieties to mean X and Y are $\text{Spec}(k)$-birational as defined in Morphisms, Definition 29.50.1. As is customary, the phrase "birational varieties" does not refer to the (common) base field of the varieties, even though for general irreducible schemes we make the distinction between being birational and being birational over a given base.

0BXP Lemma 33.4.2. Let X and Y be varieties over a field k. The following are equivalent

(1) X and Y are birational varieties,
(2) the function fields $k(X)$ and $k(Y)$ are isomorphic,
(3) there exist nonempty opens of X and Y which are isomorphic as varieties,
(4) there exists an open $U \subset X$ and a birational morphism $U \to Y$ of varieties.

Proof. This is a special case of Morphisms, Lemma 29.50.6. $\square$

33.5. Change of fields and local rings

0C4X Some preliminary results on what happens to local rings under an extension of ground fields.

0C4Y Lemma 33.5.1. Let K/k be an extension of fields. Let X be scheme over k and set $Y = X_K$. If $y \in Y$ with image $x \in X$, then

(1) $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a faithfully flat local ring homomorphism,
(2) with $\mathfrak{p}_0 = \mathrm{Ker}(\kappa(x) \otimes_k K \to \kappa(y))$ we have $\kappa(y) = \kappa(\mathfrak{p}_0)$,
(3) $\mathcal{O}_{Y,y} = (\mathcal{O}_{X,x} \otimes_k K)_{\mathfrak{p}}$ where $\mathfrak{p} \subset \mathcal{O}_{X,x} \otimes_k K$ is the inverse image of $\mathfrak{p}_0$.
(4) we have $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y} = (\kappa(x) \otimes_k K)_{\mathfrak{p}_0}$

Proof. We may assume $X = \mathrm{Spec}(A)$ is affine. Then $Y = \mathrm{Spec}(A \otimes_k K)$. Since K is flat over k, we see that $A \to A \otimes_k K$ is flat. Hence $Y \to X$ is flat and we get the first statement if we also use Algebra, Lemma 10.39.17. The second statement follows from Schemes, Lemma 26.17.5. Now y corresponds to a prime ideal $\mathfrak{q} \subset A \otimes_k K$ and x to $\mathfrak{r} = A \cap \mathfrak{q}$. Then $\mathfrak{p}_0$ is the kernel of the induced map $\kappa(\mathfrak{r}) \otimes_k K \to \kappa(\mathfrak{q})$. The map on local rings is

$$A_{\mathfrak{r}} \longrightarrow (A \otimes_k K)_{\mathfrak{q}}$$

We can factor this map through $A_{\mathfrak{r}} \otimes_k K = (A \otimes_k K)_{\mathfrak{r}}$ to get

$$A_{\mathfrak{r}} \longrightarrow A_{\mathfrak{r}} \otimes_k K \longrightarrow (A \otimes_k K)_{\mathfrak{q}}$$

and then the second arrow is a localization at some prime. This prime ideal is the inverse image of $\mathfrak{p}_0$ (details omitted) and this proves (3). To see (4) use (3) and that localization and $- \otimes_k K$ are exact functors. $\square$

0C4Z Lemma 33.5.2. Notation as in Lemma 33.5.1. Assume X is locally of finite type over k. Then

$$\dim(\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}) = \mathrm{trdeg}_k(\kappa(x)) - \mathrm{trdeg}_K(\kappa(y)) = \dim(\mathcal{O}_{Y,y}) - \dim(\mathcal{O}_{X,x})$$

Proof. This is a restatement of Algebra, Lemma 10.116.7. $\square$

0C50 Lemma 33.5.3. Notation as in Lemma 33.5.1. Assume X is locally of finite type over k, that $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ and that $\kappa(x) \otimes_k K$ is reduced (for example if $\kappa(x)/k$ is separable or K/k is separable). Then $\mathfrak{m}_x\mathcal{O}_{Y,y} = \mathfrak{m}_y$.

Proof. (The parenthetical statement follows from Algebra, Lemma 10.43.6.) Combining Lemmas 33.5.1 and 33.5.2 we see that $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ has dimension 0 and is reduced. Hence it is a field. $\square$

33.6. Geometrically reduced schemes

035U If X is a reduced scheme over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced schemes.

035V Definition 33.6.1. Let k be a field. Let X be a scheme over k.

(1) Let $x \in X$ be a point. We say X is geometrically reduced at x if for any field extension k'/k and any point $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is reduced.
(2) We say X is geometrically reduced over k if X is geometrically reduced at every point of X.

This may seem a little mysterious at first, but it is really the same thing as the notion discussed in the algebra chapter. Here are some basic results explaining the connection.

035W Lemma 33.6.2. Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent

(1) X is geometrically reduced at x, and
(2) the ring $\mathcal{O}_{X,x}$ is geometrically reduced over k (see Algebra, Definition 10.43.1).

Proof. Assume (1). This in particular implies that $\mathcal{O}_{X,x}$ is reduced. Let k'/k be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.18.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a reduced ring. Hence we deduce (2) by Algebra, Lemma 10.44.3.

Assume (2). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 29.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is reduced by assumption and (1) is proved. $\square$

The notion isn't interesting in characteristic zero.

020I Lemma 33.6.3. Let X be a scheme over a perfect field k (e.g. k has characteristic zero). Let $x \in X$. If $\mathcal{O}_{X,x}$ is reduced, then X is geometrically reduced at x. If X is reduced, then X is geometrically reduced over k.

Proof. The first statement follows from Lemma 33.6.2 and Algebra, Lemma 10.43.6 and the definition of a perfect field (Algebra, Definition 10.45.1). The second statement follows from the first. $\square$

035X Lemma 33.6.4. Let k be a field of characteristic $p > 0$. Let X be a scheme over k. The following are equivalent

(1) X is geometrically reduced,
(2) $X_{k'}$ is reduced for every field extension k'/k,
(3) $X_{k'}$ is reduced for every finite purely inseparable field extension k'/k,
(4) $X_{k^{1/p}}$ is reduced,
(5) $X_{k^{perf}}$ is reduced,
(6) $X_{\bar{k}}$ is reduced,

(7) for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically reduced (see Algebra, Definition 10.43.1).

Proof. Assume (1). Then for every field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is reduced. In other words $X_{k'}$ is reduced. Hence (2).

Assume (2). Let $U \subset X$ be an affine open. Then for every field extension k'/k the scheme $X_{k'}$ is reduced, hence $U_{k'} = \text{Spec}(\mathcal{O}(U) \otimes_k k')$ is reduced, hence $\mathcal{O}(U) \otimes_k k'$ is reduced (see Properties, Section 28.3). In other words $\mathcal{O}(U)$ is geometrically reduced, so (7) holds.

Assume (7). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 26.17). Hence $X_{k'}$ is reduced. So (1) holds.

This proves that (1), (2), and (7) are equivalent. These are equivalent to (3), (4), (5), and (6) because we can apply Algebra, Lemma 10.44.3 to $\mathcal{O}_X(U)$ for $U \subset X$ affine open. $\square$

035Y Lemma 33.6.5. Let k be a field of characteristic $p > 0$. Let X be a scheme over k. Let $x \in X$. The following are equivalent

(1) X is geometrically reduced at x,
(2) $\mathcal{O}_{X_{k'},x'}$ is reduced for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ the unique point lying over x,
(3) $\mathcal{O}_{X_{k^{1/p}},x'}$ is reduced for $x' \in X_{k^{1/p}}$ the unique point lying over x, and
(4) $\mathcal{O}_{X_{k^{perf}},x'}$ is reduced for $x' \in X_{k^{perf}}$ the unique point lying over x.

Proof. Note that if k'/k is purely inseparable, then $X_{k'} \to X$ induces a homeomorphism on underlying topological spaces, see Algebra, Lemma 10.46.7. Whence the uniqueness of x' lying over x mentioned in the statement. Moreover, in this case $\mathcal{O}_{X_{k'},x'} = \mathcal{O}_{X,x} \otimes_k k'$. Hence the lemma follows from Lemma 33.6.2 above and Algebra, Lemma 10.44.3. $\square$

0384 Lemma 33.6.6. Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent

(1) X is geometrically reduced at x,
(2) $X_{k'}$ is geometrically reduced at x'.

In particular, X is geometrically reduced over k if and only if $X_{k'}$ is geometrically reduced over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension k'''/k (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is reduced. Thus by Algebra, Lemma 10.164.2 we conclude that $\mathcal{O}_{X_{k''},x''}$ is reduced. Thus by Lemma 33.6.5 we conclude that X is geometrically reduced at x. $\square$

035Z Lemma 33.6.7. Let k be a field. Let X, Y be schemes over k.

(1) If X is geometrically reduced at x, and Y reduced, then $X \times_k Y$ is reduced at every point lying over x.
(2) If X geometrically reduced over k and Y reduced. Then $X \times_k Y$ is reduced.

Proof. Combine, Lemmas 33.6.2 and 33.6.4 and Algebra, Lemma 10.43.5. $\square$

04KS Lemma 33.6.8. Let k be a field. Let X be a scheme over k.

(1) If $x' \rightsquigarrow x$ is a specialization and X is geometrically reduced at x, then X is geometrically reduced at x'.
(2) If $x \in X$ such that (a) $\mathcal{O}_{X,x}$ is reduced, and (b) for each specialization $x' \rightsquigarrow x$ where x' is a generic point of an irreducible component of X the scheme X is geometrically reduced at x', then X is geometrically reduced at x.
(3) If X is reduced and geometrically reduced at all generic points of irreducible components of X, then X is geometrically reduced.

Proof. Part (1) follows from Lemma 33.6.2 and the fact that if A is a geometrically reduced k-algebra, then $S^{-1}A$ is a geometrically reduced k-algebra for any multiplicative subset S of A, see Algebra, Lemma 10.43.3.

Let $A = \mathcal{O}_{X,x}$. The assumptions (a) and (b) of (2) imply that A is reduced, and that $A_{\mathfrak q}$ is geometrically reduced over k for every minimal prime $\mathfrak q$ of A. Hence A is geometrically reduced over k, see Algebra, Lemma 10.43.7. Thus X is geometrically reduced at x, see Lemma 33.6.2.

Part (3) follows trivially from part (2). $\square$

0360 Lemma 33.6.9. Let k be a field. Let X be a scheme over k. Let $x \in X$. Assume X locally Noetherian and geometrically reduced at x. Then there exists an open neighbourhood $U \subset X$ of x which is geometrically reduced over k.

Proof. Assume X locally Noetherian and geometrically reduced at x. By Properties, Lemma 28.29.8 we can find an affine open neighbourhood $U \subset X$ of x such that $R = \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is injective. By Lemma 33.6.2 the assumption means that $\mathcal{O}_{X,x}$ is geometrically reduced over k. By Algebra, Lemma 10.43.2 this implies that R is geometrically reduced over k, which in turn implies that U is geometrically reduced. $\square$

020F Example 33.6.10. Let $k = \mathbf{F}_p(s,t)$, i.e., a purely transcendental extension of the prime field. Consider the variety $X = \text{Spec}(k[x,y]/(1 + sx^p + ty^p))$. Let k'/k be any extension such that both s and t have a pth root in k'. Then the base change $X_{k'}$ is not reduced. Namely, the ring $k'[x,y]/(1 + sx^p + ty^p)$ contains the element $1 + s^{1/p}x + t^{1/p}y$ whose pth power is zero but which is not zero (since the ideal $(1 + sx^p + ty^p)$ certainly does not contain any nonzero element of degree $< p$).

04KT Lemma 33.6.11. Let k be a field. Let $X \to \text{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite purely inseparable extension k'/k such that $(X_{k'})_{red}$ is geometrically reduced over k'.

Proof. To prove this lemma we may replace X by its reduction X_{red}. Hence we may assume that X is reduced and locally of finite type over k. Let $x_1, \ldots, x_n \in X$ be the generic points of the irreducible components of X. Note that for every purely inseparable algebraic extension k'/k the morphism $(X_{k'})_{red} \to X$ is a homeomorphism, see Algebra, Lemma 10.46.7. Hence the points $x'_1, \ldots, x'_n$ lying over $x_1, \ldots, x_n$ are the generic points of the irreducible components of $(X_{k'})_{red}$. As X is reduced the local rings $K_i = \mathcal{O}_{X,x_i}$ are fields, see Algebra, Lemma 10.25.1. As X is locally of finite type over k the field extensions K_i/k are finitely generated field extensions. Finally, the local rings $\mathcal{O}_{(X_{k'})_{red},x'_i}$ are the fields $(K_i \otimes_k k')_{red}$. By Algebra, Lemma 10.45.3 we can find a finite purely inseparable extension k'/k such that $(K_i \otimes_k k')_{red}$ are separable field extensions of k'. In particular each $(K_i \otimes_k k')_{red}$ is geometrically reduced over k' by Algebra, Lemma 10.44.1. At this point Lemma 33.6.8 part (3) implies that $(X_{k'})_{red}$ is geometrically reduced. □

33.7. Geometrically connected schemes

0361 If X is a connected scheme over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected schemes.

0362 Definition 33.7.1. Let X be a scheme over the field k. We say X is geometrically connected over k if the scheme $X_{k'}$ is connected for every field extension k' of k.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected schemes are nonempty. Here is an example of a variety which is not geometrically connected.

020E Example 33.7.2. Let $k = \mathbf{Q}$. The scheme $X = \text{Spec}(\mathbf{Q}(i))$ is a variety over $\text{Spec}(\mathbf{Q})$. But the base change $X_{\mathbf{C}}$ is the spectrum of $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{Q}(i) \cong \mathbf{C} \times \mathbf{C}$ which is the disjoint union of two copies of $\text{Spec}(\mathbf{C})$. So in fact, this is an example of a non-geometrically connected variety.

054N Lemma 33.7.3. Let X be a scheme over the field k. Let k'/k be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, note that for any field extension k''/k there exists a common field extension k'''/k' and k'''/k''. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. □

0385 Lemma 33.7.4. Let k be a field. Let X, Y be schemes over k. Assume X is geometrically connected over k. Then the projection morphism

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. The scheme theoretic fibres of p are connected, since they are base changes of the geometrically connected scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 26.18.5. By Morphisms, Lemma 29.23.4 the map p is open. Thus we may apply Topology, Lemma 5.7.6 to conclude. □

0386 Lemma 33.7.5. Let k be a field. Let A be a k-algebra. Then $X = \mathrm{Spec}(A)$ is geometrically connected over k if and only if A is geometrically connected over k (see Algebra, Definition 10.48.3).

Proof. Immediate from the definitions. $\square$

0363 Lemma 33.7.6. Let k'/k be an extension of fields. Let X be a scheme over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 10.48.4. Hence $Z = \mathrm{Spec}(k')$ is geometrically connected over k by Lemma 33.7.5 above. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 33.7.4. $\square$

0387 Lemma 33.7.7. Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.

Proof. Assume $X_{\overline{k}}$ is connected. Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 33.7.6 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. $\square$

0388 Lemma 33.7.8. Let k be a field. Let X be a scheme over k. Let A be a k-algebra. Let $V \subset X_A$ be a quasi-compact open. Then there exists a finitely generated k-subalgebra $A' \subset A$ and a quasi-compact open $V' \subset X_{A'}$ such that $V = V'_A$.

Proof. We remark that if X is also quasi-separated this follows from Limits, Lemma 32.4.11. Let $U_1, \ldots, U_n$ be finitely many affine opens of X such that $V \subset \bigcup U_{i,A}$. Say $U_i = \mathrm{Spec}(R_i)$. Since V is quasi-compact we can find finitely many $f_{ij} \in R_i \otimes_k A$, $j = 1, \ldots, n_i$ such that $V = \bigcup_i \bigcup_{j=1,\ldots,n_i} D(f_{ij})$ where $D(f_{ij}) \subset U_{i,A}$ is the corresponding standard open. (We do not claim that $V \cap U_{i,A}$ is the union of the $D(f_{ij})$, $j = 1, \ldots, n_i$.) It is clear that we can find a finitely generated k-subalgebra $A' \subset A$ such that f_{ij} is the image of some $f'_{ij} \in R_i \otimes_k A'$. Set $V' = \bigcup D(f'_{ij})$ which is a quasi-compact open of $X_{A'}$. Denote $\pi : X_A \to X_{A'}$ the canonical morphism. We have $\pi(V) \subset V'$ as $\pi(D(f_{ij})) \subset D(f'_{ij})$. If $x \in X_A$ with $\pi(x) \in V'$, then $\pi(x) \in D(f'_{ij})$ for some i, j and we see that $x \in D(f_{ij})$ as f'_{ij} maps to f_{ij}. Thus we see that $V = \pi^{-1}(V')$ as desired. $\square$

Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$. Note that $\mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\tau) = \mathrm{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$$

of the opposite group on the scheme $\mathrm{Spec}(\overline{k})$. Let X be a scheme over k. Since $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$ by definition we see that the action above induces a canonical action

038A (33.7.8.1) $$\mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}.$$

04KU Lemma 33.7.9. Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then

(1) there exists a finite subextension $\overline{k}/k'/k$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,
(2) there exists an open subgroup $H \subset \text{Gal}(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.

Proof. By Lemma 33.7.8 there exists a finite subextension $k'/k \subset \overline{k}$ and an open $V' \subset X_{k'}$ which pulls back to V. This proves (1). Since $\text{Gal}(\overline{k}/k')$ is open in $\text{Gal}(\overline{k}/k)$ part (2) is clear as well. $\square$

038B Lemma 33.7.10. Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. Let X be a scheme over k. Let $\overline{T} \subset X_{\overline{k}}$ have the following properties

(1) $\overline{T}$ is a closed subset of $X_{\overline{k}}$,
(2) for every $\sigma \in \text{Gal}(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.

Then there exists a closed subset $T \subset X$ whose inverse image in $X_{\overline{k}}$ is $\overline{T}$.

Proof. This lemma immediately reduces to the case where $X = \text{Spec}(A)$ is affine. In this case, let $\overline{I} \subset A \otimes_k \overline{k}$ be the radical ideal corresponding to $\overline{T}$. Assumption (2) implies that $\sigma(\overline{I}) = \overline{I}$ for all $\sigma \in \text{Gal}(\overline{k}/k)$. Pick $x \in \overline{I}$. There exists a finite Galois extension k'/k contained in $\overline{k}$ such that $x \in A \otimes_k k'$. Set $G = \text{Gal}(k'/k)$. Set

$$P(T) = \prod\nolimits_{\sigma \in G} (T - \sigma(x)) \in (A \otimes_k k')[T]$$

It is clear that $P(T)$ is monic and is actually an element of $(A \otimes_k k')^G[T] = A[T]$ (by basic Galois theory). Moreover, if we write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_d$ the we see that $a_i \in I := A \cap \overline{I}$. Combining $P(x) = 0$ and $a_i \in I$ we find $x^d = -a_1 x^{d-1} - \ldots - a_d \in I(A \otimes_k \overline{k})$. Thus x is contained in the radical of $I(A \otimes_k \overline{k})$. Hence $\overline{I}$ is the radical of $I(A \otimes_k \overline{k})$ and setting $T = V(I)$ is a solution. $\square$

0389 Lemma 33.7.11. Let k be a field. Let X be a scheme over k. The following are equivalent

(1) X is geometrically connected,
(2) for every finite separable field extension k'/k the scheme $X_{k'}$ is connected.

Proof. It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \overline{k}$ be a separable algebraic closure of k. By Lemma 33.7.7 it follows that $X_{\overline{k}}$ is disconnected. Say $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty.

Suppose that $W \subset X$ is any quasi-compact open. Then $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are open and closed in $W_{\overline{k}}$. In particular $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 33.7.9 both $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\text{Gal}(\overline{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open such that both $W_{0,\overline{k}} \cap \overline{U}$ and $W_{0,\overline{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $\overline{k}/k'/k$ and a decomposition $W_{0,k'} = U'_0 \amalg V'_0$ into open and closed subsets such that $W_{0,\overline{k}} \cap \overline{U} = (U'_0)_{\overline{k}}$ and $W_{0,\overline{k}} \cap \overline{V} = (V'_0)_{\overline{k}}$.

Let $H = \text{Gal}(\overline{k}/k') \subset \text{Gal}(\overline{k}/k)$. In particular $\sigma(W_{0,\overline{k}} \cap \overline{U}) = W_{0,\overline{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open $W \subset X$ we set

$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{V}).$$

Now, since $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are fixed by an open subgroup of $\text{Gal}(\overline{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed. Also, by construction $W_{\bar{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W_{\overline{k}} \cap U_{W'} = U_W$ and $W_{\overline{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\overline{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\bar{k}}$ are closed and H-invariant by construction. Hence by Lemma 33.7.10 we have $U = (U')_{\bar{k}}$, and $V = (V')_{\bar{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. $\square$

038C Lemma 33.7.12. Let k be a field. Let $\overline{k}/k$ be a (possibly infinite) Galois extension. Let $f : T \to X$ be a morphism of schemes over k. Assume $T_{\overline{k}}$ connected and $X_{\overline{k}}$ disconnected. Then X is disconnected.

Proof. Write $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open and closed. Denote $\overline{f} : T_{\overline{k}} \to X_{\overline{k}}$ the base change of f. Since $T_{\overline{k}}$ is connected we see that $T_{\overline{k}}$ is contained in either $\overline{f}^{-1}(\overline{U})$ or $\overline{f}^{-1}(\overline{V})$. Say $T_{\overline{k}} \subset \overline{f}^{-1}(\overline{U})$.

Fix a quasi-compact open $W \subset X$. There exists a finite Galois subextension $\overline{k}/k'/k$ such that $\overline{U} \cap W_{\overline{k}}$ and $\overline{V} \cap W_{\overline{k}}$ come from quasi-compact opens $U', V' \subset W_{k'}$. Then also $W_{k'} = U' \amalg V'$. Consider

$$U'' = \bigcap\nolimits_{\sigma \in \text{Gal}(k'/k)} \sigma(U'), \quad V'' = \bigcup\nolimits_{\sigma \in \text{Gal}(k'/k)} \sigma(V').$$

These are Galois invariant, open and closed, and $W_{k'} = U'' \amalg V''$. By Lemma 33.7.10 we get open and closed subsets $U_W, V_W \subset W$ such that $U'' = (U_W)_{k'}$, $V'' = (V_W)_{k'}$ and $W = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W \cap U_{W'} = U_W$ and $W \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X = U \amalg V$. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $f(T)$ by construction. $\square$

056R Lemma 33.7.13. Let k be a field. Let $T \to X$ be a morphism of schemes over k. Assume T is geometrically connected and X connected. Then X is geometrically connected. [DG67, IV Corollary 4.5.13.1(i)]

Proof. This is a reformulation of Lemma 33.7.12. $\square$

04KV Lemma 33.7.14. Let k be a field. Let X be a scheme over k. Assume X is connected and has a point x such that k is algebraically closed in $\kappa(x)$. Then X is geometrically connected. In particular, if X has a k-rational point and X is connected, then X is geometrically connected.

Proof. Set $T = \text{Spec}(\kappa(x))$. Let $\overline{k}$ be a separable algebraic closure of k. The assumption on $\kappa(x)/k$ implies that $T_{\overline{k}}$ is irreducible, see Algebra, Lemma 10.47.8. Hence by Lemma 33.7.13 we see that $X_{\overline{k}}$ is connected. By Lemma 33.7.7 we conclude that X is geometrically connected. $\square$

04PY Lemma 33.7.15. Let K/k be an extension of fields. Let X be a scheme over k. For every connected component T of X the inverse image $T_K \subset X_K$ is a union of connected components of X_K.

Proof. This is a purely topological statement. Denote $p : X_K \to X$ the projection morphism. Let $T \subset X$ be a connected component of X. Let $t \in T_K = p^{-1}(T)$. Let $C \subset X_K$ be a connected component containing t. Then $p(C)$ is a connected subset of X which meets T, hence $p(C) \subset T$. Hence $C \subset T_K$. $\square$

The following lemma will be superseded by the stronger Lemma 33.7.17 below.

07VM Lemma 33.7.16. Let K/k be a finite extension of fields and let X be a scheme over k. Denote by $p : X_K \to X$ the projection morphism. For every connected component T of X_K the image $p(T)$ is a connected component of X.

Proof. The image $p(T)$ is contained in some connected component X' of X. Consider X' as a closed subscheme of X in any way. Then T is also a connected component of $X'_K = p^{-1}(X')$ and we may therefore assume that X is connected. The morphism p is open (Morphisms, Lemma 29.23.4), closed (Morphisms, Lemma 29.44.7) and the fibers of p are finite sets (Morphisms, Lemma 29.44.10). Thus we may apply Topology, Lemma 5.7.7 to conclude. $\square$

04PZ Lemma 33.7.17 (Gabber). Let K/k be an extension of fields. Let X be a scheme over k. Denote $p : X_K \to X$ the projection morphism. Let $\overline{T} \subset X_K$ be a connected component. Then $p(\overline{T})$ is a connected component of X. Email from Ofer Gabber dated June 4, 2016

Proof. When K/k is finite this is Lemma 33.7.16. In general the proof is more difficult.

Let $T \subset X$ be the connected component of X containing the image of $\overline{T}$. We may replace X by T (with the induced reduced subscheme structure). Thus we may assume X is connected. Let $A = H^0(X, \mathcal{O}_X)$. Let $L \subset A$ be the maximal weakly étale k-subalgebra, see More on Algebra, Lemma 15.105.2. Since A does not have any nontrivial idempotents we see that L is a field and a separable algebraic extension of k by More on Algebra, Lemma 15.105.1. Observe that L is also the maximal weakly étale L-subalgebra of A (because any weakly étale L-algebra is weakly étale over k by More on Algebra, Lemma 15.104.9). By Schemes, Lemma 26.6.4 we obtain a factorization $X \to \text{Spec}(L) \to \text{Spec}(k)$ of the structure morphism.

Let L'/L be a finite separable extension. By Cohomology of Schemes, Lemma 30.5.3 we have

$$A \otimes_L L' = H^0(X \times_{\text{Spec}(L)} \text{Spec}(L'), \mathcal{O}_{X \times_{\text{Spec}(L)} \text{Spec}(L')})$$

The maximal weakly étale L'-subalgebra of $A \otimes_L L'$ is $L \otimes_L L' = L'$ by More on Algebra, Lemma 15.105.4. In particular $A \otimes_L L'$ does not have nontrivial idempotents (such an idempotent would generate a weakly étale subalgebra) and we conclude that $X \times_{\text{Spec}(L)} \text{Spec}(L')$ is connected. By Lemma 33.7.11 we conclude that X is geometrically connected over L.

Let's give $\overline{T}$ the reduced induced scheme structure and consider the composition

$$\overline{T} \xrightarrow{i} X_K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \xrightarrow{\pi} \mathrm{Spec}(L \otimes_k K)$$

The image is contained in a connected component of $\mathrm{Spec}(L \otimes_k K)$. Since $K \to L \otimes_k K$ is integral we see that the connected components of $\mathrm{Spec}(L \otimes_k K)$ are points and all points are closed, see Algebra, Lemma 10.36.19. Thus we get a quotient field $L \otimes_k K \to E$ such that $\overline{T}$ maps into $\mathrm{Spec}(E) \subset \mathrm{Spec}(L \otimes_k K)$. Hence $i(\overline{T}) \subset \pi^{-1}(\mathrm{Spec}(E))$. But

$$\pi^{-1}(\mathrm{Spec}(E)) = (X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)) \times_{\mathrm{Spec}(L\otimes_k K)} \mathrm{Spec}(E) = X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(E)$$

which is connected because X is geometrically connected over L. Then we get the equality $\overline{T} = X \times_{\mathrm{Spec}(L)} \mathrm{Spec}(E)$ (set theoretically) and we conclude that $\overline{T} \to X$ is surjective as desired. □

Let X be a scheme. We denote $\pi_0(X)$ the set of connected components of X.

038D Lemma 33.7.18. Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \pi_0(X_{\overline{k}}) \longrightarrow \pi_0(X_{\overline{k}})$$

with the following properties:

(1) An element $\overline{T} \in \pi_0(X_{\overline{k}})$ is fixed by the action if and only if there exists a connected component $T \subset X$, which is geometrically connected over k, such that $T_{\overline{k}} = \overline{T}$.
(2) For any field extension k'/k with separable algebraic closure $\overline{k}'$ the diagram

$$\begin{tikzcd} \mathrm{Gal}(\overline{k}'/k') \times \pi_0(X_{\overline{k}'}) \arrow[r] \arrow[d] & \pi_0(X_{\overline{k}'}) \arrow[d] \\ \mathrm{Gal}(\overline{k}/k) \times \pi_0(X_{\overline{k}}) \arrow[r] & \pi_0(X_{\overline{k}}) \end{tikzcd}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 33.7.6).

Proof. The action (33.7.8.1) of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its connected components. Connected components are always closed (Topology, Lemma 5.7.3). Hence if $\overline{T}$ is as in (1), then by Lemma 33.7.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically connected over k, see Lemma 33.7.7. To see that T is a connected component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is a connected component of X. In this case $T'_{\overline{k}'}$ strictly contains $\overline{T}$ and hence is disconnected. By Lemma 33.7.12 this means that T' is disconnected! Contradiction.

We omit the proof of the functoriality in (2). □

038E Lemma 33.7.19. Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. Assume

(1) X is quasi-compact, and
(2) the connected components of $X_{\overline{k}}$ are open.

Then

(a) $\pi_0(X_{\overline{k}})$ is finite, and
(b) the action of $\text{Gal}(\overline{k}/k)$ on $\pi_0(X_{\overline{k}})$ is continuous.

Moreover, assumptions (1) and (2) are satisfied when X is of finite type over k.

Proof. Since the connected components are open, cover $X_{\overline{k}}$ (Topology, Lemma 5.7.3) and $X_{\overline{k}}$ is quasi-compact, we conclude that there are only finitely many of them. Thus (a) holds. By Lemma 33.7.8 these connected components are each defined over a finite subextension of $\overline{k}/k$ and we get (b). If X is of finite type over k, then $X_{\overline{k}}$ is of finite type over $\overline{k}$ (Morphisms, Lemma 29.15.4). Hence $X_{\overline{k}}$ is a Noetherian scheme (Morphisms, Lemma 29.15.6). Thus $X_{\overline{k}}$ has finitely many irreducible components (Properties, Lemma 28.5.7) and a fortiori finitely many connected components (which are therefore open). $\square$

33.8. Geometrically irreducible schemes

0364 If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

0365 Definition 33.8.1. Let X be a scheme over the field k. We say X is geometrically irreducible over k if the scheme $X_{k'}$ is irreducible[1] for any field extension k' of k.

054P Lemma 33.8.2. Let X be a scheme over the field k. Let k'/k be a field extension. Then X is geometrically irreducible over k if and only if $X_{k'}$ is geometrically irreducible over k'.

Proof. If X is geometrically irreducible over k, then it is clear that $X_{k'}$ is geometrically irreducible over k'. For the converse, note that for any field extension k''/k there exists a common field extension k'''/k' and k'''/k''. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the irreducibility of $X_{k'''}$ implies the irreducibility of $X_{k''}$. Thus if $X_{k'}$ is geometrically irreducible over k' then X is geometrically irreducible over k. $\square$

020J Lemma 33.8.3. Let X be a scheme over a separably closed field k. If X is irreducible, then X_K is irreducible for any field extension K/k. I.e., X is geometrically irreducible over k.

Proof. Use Properties, Lemma 28.3.3 and Algebra, Lemma 10.47.2. $\square$

038F Lemma 33.8.4. Let k be a field. Let X, Y be schemes over k. Assume X is geometrically irreducible over k. Then the projection morphism

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between irreducible components.

Proof. First, note that the scheme theoretic fibres of p are irreducible, since they are base changes of the geometrically irreducible scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 26.18.5. By Morphisms, Lemma 29.23.4 the map p is open. Thus we may apply Topology, Lemma 5.8.15 to conclude. $\square$

[1] An irreducible space is nonempty.

038G Lemma 33.8.5. Let k be a field. Let X be a scheme over k. The following are equivalent

(1) X is geometrically irreducible over k,
(2) for every nonempty affine open U the k-algebra $\mathcal{O}_X(U)$ is geometrically irreducible over k (see Algebra, Definition 10.47.4),
(3) X is irreducible and there exists an affine open covering $X = \bigcup U_i$ such that each k-algebra $\mathcal{O}_X(U_i)$ is geometrically irreducible, and
(4) there exists an open covering $X = \bigcup_{i \in I} X_i$ with $I \neq \emptyset$ such that X_i is geometrically irreducible for each i and such that $X_i \cap X_j \neq \emptyset$ for all $i, j \in I$.

Moreover, if X is geometrically irreducible so is every nonempty open subscheme of X.

Proof. An affine scheme $\text{Spec}(A)$ over k is geometrically irreducible if and only if A is geometrically irreducible over k; this is immediate from the definitions. Recall that if a scheme is irreducible so is every nonempty open subscheme of X, any two nonempty open subsets have a nonempty intersection. Also, if every affine open is irreducible then the scheme is irreducible, see Properties, Lemma 28.3.3. Hence the final statement of the lemma is clear, as well as the implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4). If (4) holds, then for any field extension k'/k the scheme $X_{k'}$ has a covering by irreducible opens which pairwise intersect. Hence $X_{k'}$ is irreducible. Hence (4) implies (1). □

054Q Lemma 33.8.6. Let X be an irreducible scheme over the field k. Let $\xi \in X$ be its generic point. The following are equivalent

(1) X is geometrically irreducible over k, and
(2) $\kappa(\xi)$ is geometrically irreducible over k.

Proof. Assume (1). Recall that $\mathcal{O}_{X,\xi}$ is the filtered colimit of $\mathcal{O}_X(U)$ where U runs over the nonempty open affine subschemes of X. Combining Lemma 33.8.5 and Algebra, Lemma 10.47.6 we see that $\mathcal{O}_{X,\xi}$ is geometrically irreducible over k. Since $\mathcal{O}_{X,\xi} \to \kappa(\xi)$ is a surjection with locally nilpotent kernel (see Algebra, Lemma 10.25.1) it follows that $\kappa(\xi)$ is geometrically irreducible, see Algebra, Lemma 10.46.7.

Assume (2). We may assume that X is reduced. Let $U \subset X$ be a nonempty affine open. Then $U = \text{Spec}(A)$ where A is a domain with fraction field $\kappa(\xi)$. Thus A is a k-subalgebra of a geometrically irreducible k-algebra. Hence by Algebra, Lemma 10.47.6 we see that A is geometrically irreducible over k. By Lemma 33.8.5 we conclude that X is geometrically irreducible over k. □

038H Lemma 33.8.7. Let k'/k be an extension of fields. Let X be a scheme over k. Set $X' = X_{k'}$. Assume k separably algebraically closed. Then the morphism $X' \to X$ induces a bijection of irreducible components.

Proof. Since k is separably algebraically closed we see that k' is geometrically irreducible over k, see Algebra, Lemma 10.47.5. Hence $Z = \text{Spec}(k')$ is geometrically irreducible over k. by Lemma 33.8.5 above. Since $X' = Z \times_k X$ the result is a special case of Lemma 33.8.4. □

038I Lemma 33.8.8. Let k be a field. Let X be a scheme over k. The following are equivalent:

(1) X is geometrically irreducible over k,
(2) for every finite separable field extension k'/k the scheme $X_{k'}$ is irreducible, and
(3) $X_{\overline{k}}$ is irreducible, where $k \subset \overline{k}$ is a separable algebraic closure of k.

Proof. Assume $X_{\overline{k}}$ is irreducible, i.e., assume (3). Let k'/k be a field extension. There exists a field extension $\overline{k}'/\overline{k}$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 33.8.7 we see that $X_{\overline{k}'}$ is irreducible. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is irreducible. Hence (1) holds.

Let $k \subset \overline{k}$ be a separable algebraic closure of k. Assume not (3), i.e., assume $X_{\overline{k}}$ is reducible. Our goal is to show that also $X_{k'}$ is reducible for some finite subextension $\overline{k}/k'/k$. Let $X = \bigcup_{i \in I} U_i$ be an affine open covering with U_i not empty. If for some i the scheme U_i is reducible, or if for some pair $i \neq j$ the intersection $U_i \cap U_j$ is empty, then X is reducible (Properties, Lemma 28.3.3) and we are done. In particular we may assume that $U_{i,\overline{k}} \cap U_{j,\overline{k}}$ for all $i, j \in I$ is nonempty and we conclude that $U_{i,\overline{k}}$ has to be reducible for some i. According to Algebra, Lemma 10.47.3 this means that $U_{i,k'}$ is reducible for some finite separable field extension k'/k. Hence also $X_{k'}$ is reducible. Thus we see that (2) implies (3).

The implication (1) $\Rightarrow$ (2) is immediate. This proves the lemma. $\square$

04KW Lemma 33.8.9. Let K/k be an extension of fields. Let X be a scheme over k. For every irreducible component T of X the inverse image $T_K \subset X_K$ is a union of irreducible components of X_K.

Proof. Let $T \subset X$ be an irreducible component of X. The morphism $T_K \to T$ is flat, so generalizations lift along $T_K \to T$. Hence every $\xi \in T_K$ which is a generic point of an irreducible component of T_K maps to the generic point η of T. If $\xi' \rightsquigarrow \xi$ is a specialization in X_K then ξ' maps to η since there are no points specializing to η in X. Hence $\xi' \in T_K$ and we conclude that $\xi = \xi'$. In other words ξ is the generic point of an irreducible component of X_K. This means that the irreducible components of T_K are all irreducible components of X_K. $\square$

For a scheme X we denote $\text{IrredComp}(X)$ the set of irreducible components of X.

04KX Lemma 33.8.10. Let K/k be an extension of fields. Let X be a scheme over k. For every irreducible component $\overline{T} \subset X_K$ the image of $\overline{T}$ in X is an irreducible component in X. This defines a canonical map

$$\text{IrredComp}(X_K) \longrightarrow \text{IrredComp}(X)$$

which is surjective.

Proof. Consider the diagram

$$\begin{tikzcd} X_K \arrow[d] & X_{\overline{K}} \arrow[l] \arrow[d] \\ X & X_{\overline{k}} \arrow[l] \end{tikzcd}$$

where $\overline{K}$ is the separable algebraic closure of K, and where $\overline{k}$ is the separable algebraic closure of k. By Lemma 33.8.7 the morphism $X_{\overline{K}} \to X_{\overline{k}}$ induces a bijection between irreducible components. Hence it suffices to show the lemma for the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{K}} \to X_K$. In other words we may assume that $K = \overline{k}$.

The morphism $p : X_{\overline{k}} \to X$ is integral, flat and surjective. Flatness implies that generalizations lift along p, see Morphisms, Lemma 29.25.9. Hence generic points of irreducible components of $X_{\overline{k}}$ map to generic points of irreducible components of X. Integrality implies that p is universally closed, see Morphisms, Lemma 29.44.7. Hence we conclude that the image $p(\overline{T})$ of an irreducible component is a closed irreducible subset which contains a generic point of an irreducible component of X, hence $p(\overline{T})$ is an irreducible component of X. This proves the first assertion. If $T \subset X$ is an irreducible component, then $p^{-1}(T) = T_K$ is a nonempty union of irreducible components, see Lemma 33.8.9. Each of these necessarily maps onto T by the first part. Hence the map is surjective. □

0G69 Lemma 33.8.11. Let k be a field. Let X be a scheme over k. If X is irreducible and has a dense set of k-rational points, then X is geometrically irreducible.

Proof. Let k'/k be a finite extension of fields and let $Z, Z' \subset X_{k'}$ be irreducible components. It suffices to show $Z = Z'$, see Lemma 33.8.8. By Lemma 33.8.10 we have $p(Z) = p(Z') = X$ where $p : X_{k'} \to X$ is the projection. If $Z \neq Z'$ then $Z \cap Z'$ is nowhere dense in $X_{k'}$ and hence $p(Z \cap Z')$ is not dense by Morphisms, Lemma 29.48.7; here we also use that p is a finite morphism as the base change of the finite morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, see Morphisms, Lemma 29.44.6. Thus we can pick a k-rational point $x \in X$ with $x \notin p(Z \cap Z')$. Since the residue field of x is k we see that $p^{-1}(\{x\}) = \{x'\}$ where $x' \in X_{k'}$ is a point whose residue field is k'. Since $x \in p(Z) = p(Z')$ we conclude that $x' \in Z \cap Z'$ which is the contradiction we were looking for. □

038J Lemma 33.8.12. Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{IrredComp}(X_{\overline{k}}) \longrightarrow \mathrm{IrredComp}(X_{\overline{k}})$$

with the following properties:

(1) An element $\overline{T} \in \mathrm{IrredComp}(X_{\overline{k}})$ is fixed by the action if and only if there exists an irreducible component $T \subset X$, which is geometrically irreducible over k, such that $T_{\overline{k}} = \overline{T}$.
(2) For any field extension k'/k with separable algebraic closure $\overline{k}'$ the diagram

$$\begin{tikzcd} \mathrm{Gal}(\overline{k}'/k') \times \mathrm{IrredComp}(X_{\overline{k}'}) \arrow[r] \arrow[d] & \mathrm{IrredComp}(X_{\overline{k}'}) \arrow[d] \\ \mathrm{Gal}(\overline{k}/k) \times \mathrm{IrredComp}(X_{\overline{k}}) \arrow[r] & \mathrm{IrredComp}(X_{\overline{k}}) \end{tikzcd}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 33.8.7).

Proof. The action (33.7.8.1) of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its irreducible components. Irreducible components are always closed (Topology, Lemma 5.7.3). Hence if $\overline{T}$ is as in (1), then by Lemma 33.7.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically irreducible over k, see Lemma

33.8.8. To see that T is an irreducible component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is an irreducible component of X. Let $\overline{\eta}$ be the generic point of $\overline{T}$. It maps to the generic point η of T. Then the generic point $\xi \in T'$ specializes to η. As $X_{\overline{k}} \to X$ is flat there exists a point $\overline{\xi} \in X_{\overline{k}}$ which maps to ξ and specializes to $\overline{\eta}$. It follows that the closure of the singleton $\{\overline{\xi}\}$ is an irreducible closed subset of $X_{\overline{\xi}}$ which strictly contains $\overline{T}$. This is the desired contradiction.

We omit the proof of the functoriality in (2). $\square$

04KY Lemma 33.8.13. Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. The fibres of the map

$$\text{IrredComp}(X_{\overline{k}}) \longrightarrow \text{IrredComp}(X)$$

of Lemma 33.8.10 are exactly the orbits of $\text{Gal}(\overline{k}/k)$ under the action of Lemma 33.8.12.

Proof. Let $T \subset X$ be an irreducible component of X. Let $\eta \in T$ be its generic point. By Lemmas 33.8.9 and 33.8.10 the generic points of irreducible components of $\overline{T}$ which map into T map to η. By Algebra, Lemma 10.47.14 the Galois group acts transitively on all of the points of $X_{\overline{k}}$ mapping to η. Hence the lemma follows. $\square$

04KZ Lemma 33.8.14. Let k be a field. Assume $X \to \text{Spec}(k)$ locally of finite type. In this case

(1) the action

$$\text{Gal}(\overline{k}/k)^{opp} \times \text{IrredComp}(X_{\overline{k}}) \longrightarrow \text{IrredComp}(X_{\overline{k}})$$

is continuous if we give $\text{IrredComp}(X_{\overline{k}})$ the discrete topology,

(2) every irreducible component of $X_{\overline{k}}$ can be defined over a finite extension of k, and

(3) given any irreducible component $T \subset X$ the scheme $T_{\overline{k}}$ is a finite union of irreducible components of $X_{\overline{k}}$ which are all in the same $\text{Gal}(\overline{k}/k)$-orbit.

Proof. Let $\overline{T}$ be an irreducible component of $X_{\overline{k}}$. We may choose an affine open $U \subset X$ such that $\overline{T} \cap U_{\overline{k}}$ is not empty. Write $U = \text{Spec}(A)$, so A is a finite type k-algebra, see Morphisms, Lemma 29.15.2. Hence $A_{\overline{k}}$ is a finite type $\overline{k}$-algebra, and in particular Noetherian. Let $\mathfrak{p} = (f_1, \ldots, f_n)$ be the prime ideal corresponding to $\overline{T} \cap U_{\overline{k}}$. Since $A_{\overline{k}} = A \otimes_k \overline{k}$ we see that there exists a finite subextension $\overline{k}/k'/k$ such that each $f_i \in A_{k'}$. It is clear that $\text{Gal}(\overline{k}/k')$ fixes $\overline{T}$, which proves (1).

Part (2) follows by applying Lemma 33.8.12 (1) to the situation over k' which implies the irreducible component $\overline{T}$ is of the form $T'_{\overline{k}}$ for some irreducible $T' \subset X_{k'}$.

To prove (3), let $T \subset X$ be an irreducible component. Choose an irreducible component $\overline{T} \subset X_{\overline{k}}$ which maps to T, see Lemma 33.8.10. By the above the orbit of $\overline{T}$ is finite, say it is $\overline{T}_1, \ldots, \overline{T}_n$. Then $\overline{T}_1 \cup \ldots \cup \overline{T}_n$ is a $\text{Gal}(\overline{k}/k)$-invariant closed subset of $X_{\overline{k}}$ hence of the form $W_{\overline{k}}$ for some $W \subset X$ closed by Lemma 33.7.10. Clearly $W = T$ and we win. $\square$

054R Lemma 33.8.15. Let k be a field. Let $X \to \text{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite separable extension k'/k such that every irreducible component of $X_{k'}$ is geometrically irreducible over k'.

Proof. Let $\overline{k}$ be a separable algebraic closure of k. The assumption that X has finitely many irreducible components combined with Lemma 33.8.14 (3) shows that $X_{\overline{k}}$ has finitely many irreducible components $\overline{T}_1, \ldots, \overline{T}_n$. By Lemma 33.8.14 (2) there exists a finite extension $\overline{k}/k'/k$ and irreducible components $T_i \subset X_{k'}$ such that $\overline{T}_i = T_{i,\overline{k}}$ and we win. □

054S Lemma 33.8.16. Let X be a scheme over the field k. Assume X has finitely many irreducible components which are all geometrically irreducible. Then X has finitely many connected components each of which is geometrically connected.

Proof. This is clear because a connected component is a union of irreducible components. Details omitted. □

33.9. Geometrically integral schemes

0366 If X is an integral scheme over a field, then it can happen that X becomes either nonreduced or reducible after extending the ground field. This does not happen for geometrically integral schemes.

020H Definition 33.9.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is geometrically pointwise integral at x if for every field extension k'/k and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is integral.
(2) We say X is geometrically pointwise integral if X is geometrically pointwise integral at every point.
(3) We say X is geometrically integral over k if the scheme $X_{k'}$ is integral for every field extension k' of k.

The distinction between notions (2) and (3) is necessary. For example if $k = \mathbf{R}$ and $X = \mathrm{Spec}(\mathbf{C}[x])$, then X is geometrically pointwise integral over $\mathbf{R}$ but of course not geometrically integral.

038K Lemma 33.9.2. Let k be a field. Let X be a scheme over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.

Proof. See Properties, Lemma 28.3.4. □

0BUG Lemma 33.9.3. Let k be a field. Let X be a proper scheme over k.

(1) $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra,
(2) $A = \prod_{i=1,\ldots,n} A_i$ is a product of Artinian local k-algebras, one factor for each connected component of X,
(3) if X is reduced, then $A = \prod_{i=1,\ldots,n} k_i$ is a product of fields, each a finite extension of k,
(4) if X is geometrically reduced, then k_i is finite separable over k,
(5) if X is geometrically connected, then A is geometrically irreducible over k,
(6) if X is geometrically irreducible, then A is geometrically irreducible over k,
(7) if X is geometrically reduced and connected, then $A = k$, and
(8) if X is geometrically integral, then $A = k$.

Proof. By Cohomology of Schemes, Lemma 30.19.2 we see that $A = H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra. This proves (1).

Then A is a product of local Artinian k-algebras by Algebra, Lemma 10.53.2 and Proposition 10.60.7. If $X = Y \amalg Z$ with Y and Z open in X, then we obtain an idempotent $e \in A$ by taking the section of $\mathcal{O}_X$ which is 1 on Y and 0 on Z. Conversely, if $e \in A$ is an idempotent, then we get a corresponding decomposition of X. Finally, as X has a Noetherian underlying topological space its connected components are open. Hence the connected components of X correspond 1-to-1 with primitive idempotents of A. This proves (2).

If X is reduced, then A is reduced. Hence the local rings $A_i = k_i$ are reduced and therefore fields (for example by Algebra, Lemma 10.25.1). This proves (3).

If X is geometrically reduced, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ (equality by Cohomology of Schemes, Lemma 30.5.2) is reduced. This implies that $k_i \otimes_k \overline{k}$ is a product of fields and hence k_i/k is separable for example by Algebra, Lemmas 10.44.1 and 10.44.3. This proves (4).

If X is geometrically connected, then $A \otimes_k \overline{k} = H^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$ is a zero dimensional local ring by part (2) and hence its spectrum has one point, in particular it is irreducible. Thus A is geometrically irreducible. This proves (5). Of course (5) implies (6).

If X is geometrically reduced and connected, then $A = k_1$ is a field and the extension k_1/k is finite separable and geometrically irreducible. However, then $k_1 \otimes_k \overline{k}$ is a product of $[k_1 : k]$ copies of $\overline{k}$ and we conclude that $k_1 = k$. This proves (7). Of course (7) implies (8). $\square$

Here is a baby version of Stein factorization; actual Stein factorization will be discussed in More on Morphisms, Section 37.53.

0FD1 Lemma 33.9.4. Let X be a proper scheme over a field k. Set $A = H^0(X, \mathcal{O}_X)$. The fibres of the canonical morphism $X \to \mathrm{Spec}(A)$ are geometrically connected.

Proof. Set $S = \mathrm{Spec}(A)$. The canonical morphism $X \to S$ is the morphism corresponding to $\Gamma(S, \mathcal{O}_S) = A = \Gamma(X, \mathcal{O}_X)$ via Schemes, Lemma 26.6.4. The k-algebra A is a finite product $A = \prod A_i$ of local Artinian k-algebras finite over k, see Lemma 33.9.3. Denote $s_i \in S$ the point corresponding to the maximal ideal of A_i. Choose an algebraic closure $\overline{k}$ of k and set $\overline{A} = A \otimes_k \overline{k}$. Choose an embedding $\kappa(s_i) \to \overline{k}$ over k; this determines a $\overline{k}$-algebra map

$$\sigma_i : \overline{A} = A \otimes_k \overline{k} \to \kappa(s_i) \otimes_k \overline{k} \to \overline{k}$$

Consider the base change

$$\begin{tikzcd} \overline{X} \arrow[r] \arrow[d] & X \arrow[d] \\ \overline{S} \arrow[r] & S \end{tikzcd}$$

of X to $\overline{S} = \mathrm{Spec}(\overline{A})$. By Cohomology of Schemes, Lemma 30.5.2 we have $\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}) = \overline{A}$. If $\overline{s}_i \in \mathrm{Spec}(\overline{A})$ denotes the $\overline{k}$-rational point corresponding to σ_i, then we see that $\overline{s}_i$ maps to $s_i \in S$ and $\overline{X}_{\overline{s}_i}$ is the base change of X_{s_i} by $\mathrm{Spec}(\sigma_i)$. Thus we see that it suffices to prove the lemma in case k is algebraically closed.

Assume k is algebraically closed. In this case $\kappa(s_i)$ is algebraically closed and we have to show that X_{s_i} is connected. The product decomposition $A = \prod A_i$ corresponds to a disjoint union decomposition $\mathrm{Spec}(A) = \coprod \mathrm{Spec}(A_i)$, see Algebra, Lemma 10.21.2. Denote X_i the inverse image of $\mathrm{Spec}(A_i)$. It follows from Lemma 33.9.3 part (2) that $A_i = \Gamma(X_i, \mathcal{O}_{X_i})$. Observe that $X_{s_i} \to X_i$ is a closed immersion inducing an isomorphism on underlying topological spaces (because $\mathrm{Spec}(A_i)$ is a singleton). Hence if X_{s_i} isn't connected, then neither is X_i. So either X_i is empty and $A_i = 0$ or X_i can be written as $U \amalg V$ with U and V open and nonempty which would imply that A_i has a nontrivial idempotent. Since A_i is local this is a contradiction and the proof is complete. □

0FD2 Lemma 33.9.5. Let k be a field. Let X be a proper geometrically reduced scheme over k. The following are equivalent

(1) $H^0(X, \mathcal{O}_X) = k$, and
(2) X is geometrically connected.

Proof. By Lemma 33.9.4 we have (1) $\Rightarrow$ (2). By Lemma 33.9.3 we have (2) $\Rightarrow$ (1). □

33.10. Geometrically normal schemes

038L In Properties, Definition 28.7.1 we have defined the notion of a normal scheme. This notion is defined even for non-Noetherian schemes. Hence, contrary to our discussion of "geometrically regular" schemes we consider all field extensions of the ground field.

038M Definition 33.10.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is geometrically normal at x if for every field extension k'/k and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'}, x'}$ is normal.
(2) We say X is geometrically normal over k if X is geometrically normal at every $x \in X$.

038N Lemma 33.10.2. Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent

(1) X is geometrically normal at x,
(2) for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'}, x'}$ is normal, and
(3) the ring $\mathcal{O}_{X,x}$ is geometrically normal over k (see Algebra, Definition 10.165.2).

Proof. It is clear that (1) implies (2). Assume (2). Let k'/k be a finite purely inseparable field extension (for example $k = k'$). Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.18.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'}, x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a normal ring. Hence we deduce (3) by Algebra, Lemma 10.165.1.

Assume (3). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 29.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'}, x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is normal by assumption and (1) is proved. □

038O Lemma 33.10.3. Let k be a field. Let X be a scheme over k. The following are equivalent

(1) X is geometrically normal,
(2) $X_{k'}$ is a normal scheme for every field extension k'/k,
(3) $X_{k'}$ is a normal scheme for every finitely generated field extension k'/k,
(4) $X_{k'}$ is a normal scheme for every finite purely inseparable field extension k'/k,
(5) for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically normal (see Algebra, Definition 10.165.2), and
(6) $X_{k^{perf}}$ is a normal scheme.

Proof. Assume (1). Then for every field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is normal. By definition this means that $X_{k'}$ is normal. Hence (2).

It is clear that (2) implies (3) implies (4).

Assume (4) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a normal scheme for any finite purely inseparable extension k'/k (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a normal ring for all finite purely inseparable extensions k'/k. Hence $\mathcal{O}(U)$ is a geometrically normal k-algebra by definition. Hence (4) implies (5).

Assume (5). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 26.17). Hence $X_{k'}$ is normal. So (1) holds.

The equivalence of (5) and (6) follows from the definition of geometrically normal algebras and the equivalence (just proved) of (3) and (4). $\square$

038P Lemma 33.10.4. Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent

(1) X is geometrically normal at x,
(2) $X_{k'}$ is geometrically normal at x'.

In particular, X is geometrically normal over k if and only if $X_{k'}$ is geometrically normal over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension k'''/k (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''}, x''} \longrightarrow \mathcal{O}_{X_{k'''}, x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is normal. Thus by Algebra, Lemma 10.164.3 we conclude that $\mathcal{O}_{X_{k''}, x''}$ is normal. By Lemma 33.10.2 we see that X is geometrically normal at x. $\square$

06DG Lemma 33.10.5. Let k be a field. Let X be a geometrically normal scheme over k and let Y be a normal scheme over k. Then $X \times_k Y$ is a normal scheme.

Proof. This reduces to Algebra, Lemma 10.165.5 by Lemma 33.10.3. $\square$

0C3M Lemma 33.10.6. Let k be a field. Let X be a normal scheme over k. Let K/k be a separable field extension. Then X_K is a normal scheme.

Proof. Follows from Lemma 33.10.5 and Algebra, Lemma 10.165.4. $\square$

0FD3 Lemma 33.10.7. Let k be a field. Let X be a proper geometrically normal scheme over k. The following are equivalent

(1) $H^0(X, \mathcal{O}_X) = k$,
(2) X is geometrically connected,
(3) X is geometrically irreducible, and
(4) X is geometrically integral.

Proof. By Lemma 33.9.5 we have the equivalence of (1) and (2). A locally Noetherian normal scheme (such as $X_{\overline{k}}$) is a disjoint union of its irreducible components (Properties, Lemma 28.7.6). Thus we see that (2) and (3) are equivalent. Since $X_{\overline{k}}$ is assumed reduced, we see that (3) and (4) are equivalent too. $\square$

33.11. Change of fields and locally Noetherian schemes

038Q Let X a locally Noetherian scheme over a field k. It is not always that case that $X_{k'}$ is locally Noetherian too. For example if $X = \text{Spec}(\overline{\mathbf{Q}})$ and $k = \mathbf{Q}$, then $X_{\overline{\mathbf{Q}}}$ is the spectrum of $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ which is not Noetherian. (Hint: It has too many idempotents). But if we only base change using finitely generated field extensions then the Noetherian property is preserved. (Or if X is locally of finite type over k, since this property is preserved under base change.)

038R Lemma 33.11.1. Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Then X is locally Noetherian if and only if $X_{k'}$ is locally Noetherian.

Proof. Using Properties, Lemma 28.5.2 we reduce to the case where X is affine, say $X = \text{Spec}(A)$. In this case we have to prove that A is Noetherian if and only if $A_{k'}$ is Noetherian. Since $A \to A_{k'} = k' \otimes_k A$ is faithfully flat, we see that if $A_{k'}$ is Noetherian, then so is A, by Algebra, Lemma 10.164.1. Conversely, if A is Noetherian then $A_{k'}$ is Noetherian by Algebra, Lemma 10.31.8. $\square$

33.12. Geometrically regular schemes

038S A geometrically regular scheme over a field k is a locally Noetherian scheme over k which remains regular upon suitable changes of base field. A finite type scheme over k is geometrically regular if and only if it is smooth over k (see Lemma 33.12.6). The notion of geometric regularity is most interesting in situations where smoothness cannot be used such as formal fibres (insert future reference here).

In the following definition we restrict ourselves to locally Noetherian schemes, since the property of being a regular local ring is only defined for Noetherian local rings. By Lemma 33.11.1 above, if we restrict ourselves to finitely generated field extensions then this property is preserved under change of base field. This comment will be used without further reference in this section. In particular the following definition makes sense.

038T Definition 33.12.1. Let k be a field. Let X be a locally Noetherian scheme over k.

(1) Let $x \in X$. We say X is geometrically regular at x over k if for every finitely generated field extension k'/k and any $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular.
(2) We say X is geometrically regular over k if X is geometrically regular at all of its points.

A similar definition works to define geometrically Cohen-Macaulay, (R_k), and (S_k) schemes over a field. We will add a section for these separately as needed.

038U Lemma 33.12.2. Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$. The following are equivalent

(1) X is geometrically regular at x,
(2) for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular, and
(3) the ring $\mathcal{O}_{X,x}$ is geometrically regular over k (see Algebra, Definition 10.166.2).

Proof. It is clear that (1) implies (2). Assume (2). This in particular implies that $\mathcal{O}_{X,x}$ is a regular local ring. Let k'/k be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.46.7 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.18.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a regular ring. Hence we deduce (3) from the definition of a geometrically regular ring.

Assume (3). Let k'/k be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 29.9.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is regular by assumption and (1) is proved. $\square$

038V Lemma 33.12.3. Let k be a field. Let X be a locally Noetherian scheme over k. The following are equivalent

(1) X is geometrically regular,
(2) $X_{k'}$ is a regular scheme for every finitely generated field extension k'/k,
(3) $X_{k'}$ is a regular scheme for every finite purely inseparable field extension k'/k,
(4) for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically regular (see Algebra, Definition 10.166.2), and
(5) there exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is geometrically regular over k.

Proof. Assume (1). Then for every finitely generated field extension k'/k and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is regular. By Properties, Lemma 28.9.2 this means that $X_{k'}$ is regular. Hence (2).

It is clear that (2) implies (3).

Assume (3) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a regular scheme for any finite purely inseparable extension k'/k (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a regular ring for all finite purely inseparable extensions k'/k. Hence $\mathcal{O}(U)$ is a geometrically regular k-algebra and we see that (4) holds.

It is clear that (4) implies (5). Let $X = \bigcup U_i$ be an affine open covering as in (5). For any field extension k'/k the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U_i) \otimes_k k'$ (see Schemes, Section 26.17). Hence $X_{k'}$ is regular. So (1) holds. $\square$

038W Lemma 33.12.4. Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent

(1) X is geometrically regular at x,
(2) $X_{k'}$ is geometrically regular at x'.

In particular, X is geometrically regular over k if and only if $X_{k'}$ is geometrically regular over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let k''/k be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common, finitely generated, field extension k'''/k (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x''''}.$$

This is a flat local ring homomorphism of Noetherian local rings and hence faithfully flat. By (2) we see that the local ring on the right is regular. Thus by Algebra, Lemma 10.110.9 we conclude that $\mathcal{O}_{X_{k''},x''}$ is regular. By Lemma 33.12.2 we see that X is geometrically regular at x. $\square$

The following lemma is a geometric variant of Algebra, Lemma 10.166.3.

05AW Lemma 33.12.5. Let k be a field. Let $f : X \to Y$ be a morphism of locally Noetherian schemes over k. Let $x \in X$ be a point and set $y = f(x)$. If X is geometrically regular at x and f is flat at x then Y is geometrically regular at y. In particular, if X is geometrically regular over k and f is flat and surjective, then Y is geometrically regular over k.

Proof. Let k' be finite purely inseparable extension of k. Let $f' : X_{k'} \to Y_{k'}$ be the base change of f. Let $x' \in X_{k'}$ be the unique point lying over x. If we show that $Y_{k'}$ is regular at $y' = f'(x')$, then Y is geometrically regular over k at y', see Lemma 33.12.3. By Morphisms, Lemma 29.25.7 the morphism $X_{k'} \to Y_{k'}$ is flat at x'. Hence the ring map

$$\mathcal{O}_{Y_{k'},y'} \longrightarrow \mathcal{O}_{X_{k'},x'}$$

is a flat local homomorphism of local Noetherian rings with right hand side regular by assumption. Hence the left hand side is a regular local ring by Algebra, Lemma 10.110.9. $\square$

038X Lemma 33.12.6. Let k be a field. Let X be a scheme locally of finite type over k. Let $x \in X$. Then X is geometrically regular at x if and only if $X \to \mathrm{Spec}(k)$ is smooth at x (Morphisms, Definition 29.34.1).

Proof. The question is local around x, hence we may assume that $X = \mathrm{Spec}(A)$ for some finite type k-algebra. Let x correspond to the prime $\mathfrak{p}$.

If A is smooth over k at $\mathfrak{p}$, then we may localize A and assume that A is smooth over k. In this case $k' \otimes_k A$ is smooth over k' for all extension fields k'/k, and each of these Noetherian rings is regular by Algebra, Lemma 10.140.3.

Assume X is geometrically regular at x. Consider the residue field $K := \kappa(x) = \kappa(\mathfrak p)$ of x. It is a finitely generated extension of k. By Algebra, Lemma 10.45.3 there exists a finite purely inseparable extension k'/k such that the compositum $k'K$ is a separable field extension of k'. Let $\mathfrak p' \subset A' = k' \otimes_k A$ be a prime ideal lying over $\mathfrak p$. It is the unique prime lying over $\mathfrak p$, see Algebra, Lemma 10.46.7. Hence the residue field $K' := \kappa(\mathfrak p')$ is the compositum $k'K$. By assumption the local ring $(A')_{\mathfrak p'}$ is regular. Hence by Algebra, Lemma 10.140.5 we see that $k' \to A'$ is smooth at $\mathfrak p'$. This in turn implies that $k \to A$ is smooth at $\mathfrak p$ by Algebra, Lemma 10.137.19. The lemma is proved. □

038Y Example 33.12.7. Let $k = \mathbf{F}_p(t)$. It is quite easy to give an example of a regular variety V over k which is not geometrically reduced. For example we can take $\text{Spec}(k[x]/(x^p - t))$. In fact, there exists an example of a regular variety V which is geometrically reduced, but not even geometrically normal. Namely, take for $p > 2$ the scheme $V = \text{Spec}(k[x, y]/(y^2 - x^p + t))$. This is a variety as the polynomial $y^2 - x^p + t \in k[x, y]$ is irreducible. The morphism $V \to \text{Spec}(k)$ is smooth at all points except at the point $v_0 \in V$ corresponding to the maximal ideal $(y, x^p - t)$ (because $2y$ is invertible). In particular we see that V is (geometrically) regular at all points, except possibly v_0. The local ring

$$\mathcal{O}_{V,v_0} = \big(k[x, y]/(y^2 - x^p + t)\big)_{(y,x^p-t)}$$

is a domain of dimension 1. Its maximal ideal is generated by 1 element, namely y. Hence it is a discrete valuation ring and regular. Let $k' = k[t^{1/p}]$. Denote $t' = t^{1/p} \in k'$, $V' = V_{k'}$, $v_0' \in V'$ the unique point lying over v_0. Over k' we can write $x^p - t = (x - t')^p$, but the polynomial $y^2 - (x - t')^p$ is still irreducible and V' is still a variety. But the element

$$\frac{y}{x - t'} \in (\text{fraction field of } \mathcal{O}_{V',v_0'})$$

is integral over $\mathcal{O}_{V',v_0'}$ (just compute its square) and not contained in it, so V' is not normal at v_0'. This concludes the example.

33.13. Change of fields and the Cohen-Macaulay property

045O The following lemma says that it does not make sense to define geometrically Cohen-Macaulay schemes, since these would be the same as Cohen-Macaulay schemes.

045P Lemma 33.13.1. Let X be a locally Noetherian scheme over the field k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have

$$\mathcal{O}_{X,x} \text{ is Cohen-Macaulay} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Cohen-Macaulay}$$

If X is locally of finite type over k, the same holds for any field extension k'/k.

Proof. The first case of the lemma follows from Algebra, Lemma 10.167.2. The second case of the lemma is equivalent to Algebra, Lemma 10.130.6. □

33.14. Change of fields and the Jacobson property

0477 A scheme locally of finite type over a field has plenty of closed points, namely it is Jacobson. Moreover, the residue fields are finite extensions of the ground field.

0478 Lemma 33.14.1. Let X be a scheme which is locally of finite type over k. Then

(1) for any closed point $x \in X$ the extension $\kappa(x)/k$ is algebraic, and
(2) X is a Jacobson scheme (Properties, Definition 28.6.1).

Proof. A scheme is Jacobson if and only if it has an affine open covering by Jacobson schemes, see Properties, Lemma 28.6.3. The property on residue fields at closed points is also local on X. Hence we may assume that X is affine. In this case the result is a consequence of the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1. It also follows from a combination of Morphisms, Lemmas 29.16.8, 29.16.9, and 29.16.10. $\square$

It turns out that if X is not locally of finite type, then we can achieve the same result after making a suitably large base field extension.

0479 Lemma 33.14.2. Let X be a scheme over a field k. For any field extension K/k whose cardinality is large enough we have

(1) for any closed point $x \in X_K$ the extension $\kappa(x)/K$ is algebraic, and
(2) X_K is a Jacobson scheme (Properties, Definition 28.6.1).

Proof. Choose an affine open covering $X = \bigcup U_i$. By Algebra, Lemma 10.35.12 and Properties, Lemma 28.6.2 there exist cardinals κ_i such that $U_{i,K}$ has the desired properties over K if $\#(K) \geq \kappa_i$. Set $\kappa = \max\{\kappa_i\}$. Then if the cardinality of K is larger than κ we see that each $U_{i,K}$ satisfies the conclusions of the lemma. Hence X_K is Jacobson by Properties, Lemma 28.6.3. The statement on residue fields at closed points of X_K follows from the corresponding statements for residue fields of closed points of the $U_{i,K}$. $\square$

33.15. Change of fields and ample invertible sheaves

0BDB The following result is typical for the results in this section.

0BDC Lemma 33.15.1. Let k be a field. Let X be a scheme over k. If there exists an ample invertible sheaf on X_K for some field extension K/k, then X has an ample invertible sheaf.

Proof. Let K/k be a field extension such that X_K has an ample invertible sheaf $\mathcal{L}$. The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 28.26.1). Since X_K is quasi-separated (by Properties, Lemma 28.26.7) we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 26.21.6 applies.

Write $K = \text{colim}\, A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\text{Spec}(k)} \text{Spec}(A_i)$. Since $X_K = \lim X_i$ we find an i and an invertible sheaf $\mathcal{L}_i$ on X_i whose pullback to X_K is $\mathcal{L}$ (Limits, Lemma 32.10.3; here and below we use that X is quasi-compact and quasi-separated as just shown). By Limits, Lemma 32.4.15 we may assume $\mathcal{L}_i$ is ample after possibly increasing i. Fix such an i and let $\mathfrak{m} \subset A_i$ be a maximal ideal. By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. Hence $X_{k'} \subset X_i$ is a closed subscheme hence has an ample invertible sheaf (Properties, Lemma 28.26.3). Since $X_{k'} \to X$ is finite locally free we conclude that X has an ample invertible sheaf by Divisors, Proposition 31.17.9. $\square$

0BDD Lemma 33.15.2. Let k be a field. Let X be a scheme over k. If X_K is quasi-affine for some field extension K/k, then X is quasi-affine.

Proof. Let K/k be a field extension such that X_K is quasi-affine. The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 28.18.1). Since X_K is quasi-separated (as an open subscheme of an affine scheme) we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 26.21.6 applies.

Write $K = \operatorname{colim} A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A_i)$. Since $X_K = \lim X_i$ we find an i such that X_i is quasi-affine (Limits, Lemma 32.4.12; here we use that X is quasi-compact and quasi-separated as just shown). By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. Hence $X_{k'} \subset X_i$ is a closed subscheme hence is quasi-affine (Properties, Lemma 28.27.2). Since $X_{k'} \to X$ is finite locally free we conclude by Divisors, Lemma 31.17.10. $\square$

0BDE Lemma 33.15.3. Let k be a field. Let X be a scheme over k. If X_K is quasi-projective over K for some field extension K/k, then X is quasi-projective over k.

Proof. By definition a morphism of schemes $g : Y \to T$ is quasi-projective if it is locally of finite type, quasi-compact, and there exists a g-ample invertible sheaf on Y. Let K/k be a field extension such that X_K is quasi-projective over K. Let $\mathrm{Spec}(A) \subset X$ be an affine open. Then U_K is an affine open subscheme of X_K, hence A_K is a K-algebra of finite type. Then A is a k-algebra of finite type by Algebra, Lemma 10.126.1. Hence $X \to \mathrm{Spec}(k)$ is locally of finite type. Since $X_K \to \mathrm{Spec}(K)$ is quasi-compact, we see that X_K is quasi-compact, hence X is quasi-compact, hence $X \to \mathrm{Spec}(k)$ is of finite type. By Morphisms, Lemma 29.39.4 we see that X_K has an ample invertible sheaf. Then X has an ample invertible sheaf by Lemma 33.15.1. Hence $X \to \mathrm{Spec}(k)$ is quasi-projective by Morphisms, Lemma 29.39.4. $\square$

The following lemma is a special case of Descent, Lemma 35.23.14.

0BDF Lemma 33.15.4. Let k be a field. Let X be a scheme over k. If X_K is proper over K for some field extension K/k, then X is proper over k.

Proof. Let K/k be a field extension such that X_K is proper over K. Recall that this implies X_K is separated and quasi-compact (Morphisms, Definition 29.41.1). The morphism $X_K \to X$ is surjective. Hence X is quasi-compact as the image of a quasi-compact scheme (Properties, Definition 28.26.1). Since X_K is separated we see that X is quasi-separated: If $U, V \subset X$ are affine open, then $(U \cap V)_K = U_K \cap V_K$ is quasi-compact and $(U \cap V)_K \to U \cap V$ is surjective. Thus Schemes, Lemma 26.21.6 applies.

Write $K = \operatorname{colim} A_i$ as the colimit of the subalgebras of K which are of finite type over k. Denote $X_i = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A_i)$. By Limits, Lemma 32.13.1 there exists an i such that $X_i \to \mathrm{Spec}(A_i)$ is proper. Here we use that X is quasi-compact and quasi-separated as just shown. Choose a maximal ideal $\mathfrak{m} \subset A_i$. By the Hilbert Nullstellensatz (Algebra, Theorem 10.34.1) the residue field $k' = A_i/\mathfrak{m}$ is a finite extension of k. The base change $X_{k'} \to \mathrm{Spec}(k')$ is proper (Morphisms, Lemma 29.41.5). Since k'/k is finite both $X_{k'} \to X$ and the composition $X_{k'} \to \mathrm{Spec}(k)$ are proper as well (Morphisms, Lemmas 29.44.11, 29.41.5, and 29.41.4).

The first implies that X is separated over k as $X_{k'}$ is separated (Morphisms, Lemma 29.41.11). The second implies that $X \to \mathrm{Spec}(k)$ is proper by Morphisms, Lemma 29.41.9. ☐

0BDG Lemma 33.15.5. Let k be a field. Let X be a scheme over k. If X_K is projective over K for some field extension K/k, then X is projective over k.

Proof. A scheme over k is projective over k if and only if it is quasi-projective and proper over k. See Morphisms, Lemma 29.43.13. Thus the lemma follows from Lemmas 33.15.3 and 33.15.4. ☐

33.16. Tangent spaces

0B28 In this section we define the tangent space of a morphism of schemes at a point of the source using points with values in dual numbers.

0B29 Definition 33.16.1. For any ring R the dual numbers over R is the R-algebra denoted $R[\epsilon]$. As an R-module it is free with basis 1, ϵ and the R-algebra structure comes from setting $\epsilon^2 = 0$.

Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s = f(x)$ in S. Consider the solid commutative diagram

0B2A (33.16.1.1)
$$\begin{tikzcd} \mathrm{Spec}(\kappa(x)) \arrow[r] \arrow[rr, bend left] \arrow[dr] & \mathrm{Spec}(\kappa(x)[\epsilon]) \arrow[r, dotted] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd}$$

with the curved arrow being the canonical morphism of $\mathrm{Spec}(\kappa(x))$ into X.

0B2B Lemma 33.16.2. The set of dotted arrows making (33.16.1.1) commute has a canonical $\kappa(x)$-vector space structure.

Proof. Set $\kappa = \kappa(x)$. Observe that we have a pushout in the category of schemes

$$\mathrm{Spec}(\kappa[\epsilon]) \amalg_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa[\epsilon]) = \mathrm{Spec}(\kappa[\epsilon_1, \epsilon_2])$$

where $\kappa[\epsilon_1, \epsilon_2]$ is the κ-algebra with basis $1, \epsilon_1, \epsilon_2$ and $\epsilon_1^2 = \epsilon_1\epsilon_2 = \epsilon_2^2 = 0$. This follows immediately from the corresponding result for rings and the description of morphisms from spectra of local rings to schemes in Schemes, Lemma 26.13.1. Given two arrows $\theta_1, \theta_2 : \mathrm{Spec}(\kappa[\epsilon]) \to X$ we can consider the morphism

$$\theta_1 + \theta_2 : \mathrm{Spec}(\kappa[\epsilon]) \to \mathrm{Spec}(\kappa[\epsilon_1, \epsilon_2]) \xrightarrow{\theta_1, \theta_2} X$$

where the first arrow is given by $\epsilon_i \mapsto \epsilon$. On the other hand, given $\lambda \in \kappa$ there is a self map of $\mathrm{Spec}(\kappa[\epsilon])$ corresponding to the κ-algebra endomorphism of $\kappa[\epsilon]$ which sends ϵ to $\lambda\epsilon$. Precomposing $\theta : \mathrm{Spec}(\kappa[\epsilon]) \to X$ by this selfmap gives $\lambda\theta$. The reader can verify the axioms of a vector space by verifying the existence of suitable commutative diagrams of schemes. We omit the details. (An alternative proof would be to express everything in terms of local rings and then verify the vector space axioms on the level of ring maps.) ☐

0B2C Definition 33.16.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. The set of dotted arrows making (33.16.1.1) commute with its canonical $\kappa(x)$-vector space structure is called the tangent space of X over S at x and we denote it $T_{X/S,x}$. An element of this space is called a tangent vector of X/S at x.

Since tangent vectors at $x \in X$ live in the scheme theoretic fibre X_s of $f : X \to S$ over $s = f(x)$, we get a canonical identification

0BEA (33.16.3.1)
$$T_{X/S,x} = T_{X_s/s,x}$$

This pleasing definition involving the functor of points has the following algebraic description, which suggests defining the cotangent space of X over S at x as the $\kappa(x)$-vector space

$$T^*_{X/S,x} = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

simply because it is canonically $\kappa(x)$-dual to the tangent space of X over S at x.

0B2D Lemma 33.16.4. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. There is a canonical isomorphism

$$T_{X/S,x} = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\Omega_{X/S,x}, \kappa(x))$$

of vector spaces over $\kappa(x)$.

Proof. Set $\kappa = \kappa(x)$. Given $\theta \in T_{X/S,x}$ we obtain a map

$$\theta^*\Omega_{X/S} \to \Omega_{\mathrm{Spec}(\kappa[\epsilon])/\mathrm{Spec}(\kappa(s))} \to \Omega_{\mathrm{Spec}(\kappa[\epsilon])/\mathrm{Spec}(\kappa)}$$

Taking sections we obtain an $\mathcal{O}_{X,x}$-linear map $\xi_\theta : \Omega_{X/S,x} \to \kappa \mathrm{d}\epsilon$, i.e., an element of the right hand side of the formula of the lemma. To show that $\theta \mapsto \xi_\theta$ is an isomorphism we can replace S by s and X by the scheme theoretic fibre X_s. Indeed, both sides of the formula only depend on the scheme theoretic fibre; this is clear for $T_{X/S,x}$ and for the RHS see Morphisms, Lemma 29.32.10. We may also replace X by the spectrum of $\mathcal{O}_{X,x}$ as this does not change $T_{X/S,x}$ (Schemes, Lemma 26.13.1) nor $\Omega_{X/S,x}$ (Modules, Lemma 17.28.7).

Let $(A, \mathfrak{m}, \kappa)$ be a local ring over a field k. To finish the proof we have to show that any A-linear map $\xi : \Omega_{A/k} \to \kappa$ comes from a unique k-algebra map $\varphi : A \to \kappa[\epsilon]$ agreeing with the canonical map $c : A \to \kappa$ modulo ϵ. Write $\varphi(a) = c(a) + D(a)\epsilon$ the reader sees that $a \mapsto D(a)$ is a k-derivation. Using the universal property of $\Omega_{A/k}$ we see that each D corresponds to a unique ξ and vice versa. This finishes the proof. $\square$

0B2E Lemma 33.16.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point and let $s = f(x) \in S$. Assume that $\kappa(x) = \kappa(s)$. Then there are canonical isomorphisms

$$\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

and

$$T_{X/S,x} = \mathrm{Hom}_{\kappa(x)}(\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}), \kappa(x))$$

This works more generally if $\kappa(x)/\kappa(s)$ is a separable algebraic extension.

Proof. The second isomorphism follows from the first by Lemma 33.16.4. For the first, we can replace S by s and X by X_s, see Morphisms, Lemma 29.32.10. We may also replace X by the spectrum of $\mathcal{O}_{X,x}$, see Modules, Lemma 17.28.7. Thus we have to show the following algebra fact: let $(A, \mathfrak{m}, \kappa)$ be a local ring over a field k such that κ/k is separable algebraic. Then the canonical map

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{A/k} \otimes \kappa$$

is an isomorphism. Observe that $\mathfrak{m}/\mathfrak{m}^2 = H_1(NL_{\kappa/A})$. By Algebra, Lemma 10.134.4 it suffices to show that $\Omega_{\kappa/k} = 0$ and $H_1(NL_{\kappa/k}) = 0$. Since κ is the union of its finite separable extensions in k it suffices to prove this when κ is a finite separable extension of k (Algebra, Lemma 10.134.9). In this case the ring map $k \to \kappa$ is étale and hence $NL_{\kappa/k} = 0$ (more or less by definition, see Algebra, Section 10.143). □

0B2F Lemma 33.16.6. Let $f : X \to Y$ be a morphism of schemes over a base scheme S. Let $x \in X$ be a point. Set $y = f(x)$. If $\kappa(y) = \kappa(x)$, then f induces a natural linear map

$$\mathrm{d}f : T_{X/S,x} \longrightarrow T_{Y/S,y}$$

which is dual to the linear map $\Omega_{Y/S,y} \otimes \kappa(y) \to \Omega_{X/S,x}$ via the identifications of Lemma 33.16.4.

Proof. Omitted. □

0BEB Lemma 33.16.7. Let X, Y be schemes over a base S. Let $x \in X$ and $y \in Y$ with the same image point $s \in S$ such that $\kappa(s) = \kappa(x)$ and $\kappa(s) = \kappa(y)$. There is a canonical isomorphism

$$T_{X \times_S Y/S,(x,y)} = T_{X/S,x} \oplus T_{Y/S,y}$$

The map from left to right is induced by the maps on tangent spaces coming from the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$. The map from right to left is induced by the maps $1 \times y : X_s \to X_s \times_s Y_s$ and $x \times 1 : Y_s \to X_s \times_s Y_s$ via the identification (33.16.3.1) of tangent spaces with tangent spaces of fibres.

Proof. The direct sum decomposition follows from Morphisms, Lemma 29.32.11 via Lemma 33.16.5. Compatibility with the maps comes from Lemma 33.16.6. □

0B2G Lemma 33.16.8. Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base scheme S. Let $x \in X$ be a point. Set $y = f(x)$ and assume that $\kappa(y) = \kappa(x)$. Then the following are equivalent

(1) $\mathrm{d}f : T_{X/S,x} \longrightarrow T_{Y/S,y}$ is injective, and
(2) f is unramified at x.

Proof. The morphism f is locally of finite type by Morphisms, Lemma 29.15.8. The map $\mathrm{d}f$ is injective, if and only if $\Omega_{Y/S,y} \otimes \kappa(y) \to \Omega_{X/S,x} \otimes \kappa(x)$ is surjective (Lemma 33.16.6). The exact sequence $f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$ (Morphisms, Lemma 29.32.9) then shows that this happens if and only if $\Omega_{X/Y,x} \otimes \kappa(x) = 0$. Hence the result follows from Morphisms, Lemma 29.35.14. □

33.17. Generically finite morphisms

0AB5 In this section we revisit the notion of a generically finite morphism of schemes as studied in Morphisms, Section 29.51.

0AB6 Lemma 33.17.1. Let $f : X \to Y$ be locally of finite type. Let $y \in Y$ be a point such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Assume in addition one of the following conditions is satisfied

(1) for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),
(2) for every generic point η of an irreducible component of X such that $f(\eta) \rightsquigarrow y$ the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),
(3) f is quasi-finite at every generic point of an irreducible component of X,
(4) Y is locally Noetherian and f is quasi-finite at a dense set of points of X,

(5) add more here.

Then f is quasi-finite at every point of X lying over y.

Proof. Condition (4) implies X is locally Noetherian (Morphisms, Lemma 29.15.6). The set of points at which morphism is quasi-finite is open (Morphisms, Lemma 29.56.2). A dense open of a locally Noetherian scheme contains all generic point of irreducible components, hence (4) implies (3). Condition (3) implies condition (1) by Morphisms, Lemma 29.20.5. Condition (1) implies condition (2). Thus it suffices to prove the lemma in case (2) holds.

Assume (2) holds. Recall that $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is the set of points of Y specializing to y, see Schemes, Lemma 26.13.2. Combined with Morphisms, Lemma 29.20.13 this shows we may replace Y by $\mathrm{Spec}(\mathcal{O}_{Y,y})$. Thus we may assume $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring of dimension ≤ 1 and y is the closed point.

Let $X = \bigcup X_i$ be the irreducible components of X viewed as reduced closed subschemes. If we can show each fibre $X_{i,y}$ is a discrete space, then $X_y = \bigcup X_{i,y}$ is discrete as well and we conclude that $X \to Y$ is quasi-finite at all points of X_y by Morphisms, Lemma 29.20.6. Thus we may assume X is an integral scheme.

If $X \to Y$ maps the generic point η of X to y, then X is the spectrum of a finite extension of $\kappa(y)$ and the result is true. Assume that X maps η to a point corresponding to a minimal prime $\mathfrak{q}$ of B different from $\mathfrak{m}_B$. We obtain a factorization $X \to \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(B)$. Let $x \in X$ be a point lying over y. By the dimension formula (Morphisms, Lemma 29.52.1) we have

$$\dim(\mathcal{O}_{X,x}) \leq \dim(B/\mathfrak{q}) + \mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X)) - \mathrm{trdeg}_{\kappa(y)}\kappa(x)$$

We know that $\dim(B/\mathfrak{q}) = 1$, that the generic point of X is not equal to x and specializes to x and that $R(X)$ is algebraic over $\kappa(\mathfrak{q})$. Thus we get

$$1 \leq 1 - \mathrm{trdeg}_{\kappa(y)}\kappa(x)$$

Hence every point x of X_y is closed in X_y by Morphisms, Lemma 29.20.2 and hence $X \to Y$ is quasi-finite at every point x of X_y by Morphisms, Lemma 29.20.6 (which also implies that X_y is a discrete topological space). $\square$

0AB7 Lemma 33.17.2. Let $f : X \to Y$ be a proper morphism. Let $y \in Y$ be a point such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Assume in addition one of the following conditions is satisfied

(1) for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),
(2) for every generic point η of an irreducible component of X such that $f(\eta) \rightsquigarrow y$ the field extension $\kappa(\eta)/\kappa(f(\eta))$ is finite (or algebraic),
(3) f is quasi-finite at every generic point of X,
(4) Y is locally Noetherian and f is quasi-finite at a dense set of points of X,
(5) add more here.

Then there exists an open neighbourhood $V \subset Y$ of y such that $f^{-1}(V) \to V$ is finite.

Proof. By Lemma 33.17.1 the morphism f is quasi-finite at every point of the fibre X_y. Hence X_y is a discrete topological space (Morphisms, Lemma 29.20.6). As f is proper the fibre X_y is quasi-compact, i.e., finite. Thus we can apply Cohomology of Schemes, Lemma 30.21.2 to conclude. $\square$

0BFP **Lemma 33.17.3.** Let X be a Noetherian scheme. Let $f : Y \to X$ be a birational proper morphism of schemes with Y reduced. Let $U \subset X$ be the maximal open over which f is an isomorphism. Then U contains

(1) every point of codimension 0 in X,
(2) every $x \in X$ of codimension 1 on X such that $\mathcal{O}_{X,x}$ is a discrete valuation ring,
(3) every $x \in X$ such that the fibre of $Y \to X$ over x is finite and such that $\mathcal{O}_{X,x}$ is normal, and
(4) every $x \in X$ such that f is quasi-finite at some $y \in Y$ lying over x and $\mathcal{O}_{X,x}$ is normal.

Proof. Part (1) follows from Morphisms, Lemma 29.51.6. Part (2) follows from part (3) and Lemma 33.17.2 (and the fact that finite morphisms have finite fibres).

Part (3) follows from part (4) and Morphisms, Lemma 29.20.7 but we will also give a direct proof. Let $x \in X$ be as in (3). By Cohomology of Schemes, Lemma 30.21.2 we may assume f is finite. We may assume X affine. This reduces us to the case of a finite birational morphism of Noetherian affine schemes $Y \to X$ and $x \in X$ such that $\mathcal{O}_{X,x}$ is a normal domain. Since $\mathcal{O}_{X,x}$ is a domain and X is Noetherian, we may replace X by an affine open of x which is integral. Then, since $Y \to X$ is birational and Y is reduced we see that Y is integral. Writing $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ we see that $A \subset B$ is a finite inclusion of domains having the same field of fractions. If $\mathfrak{p} \subset A$ is the prime corresponding to x, then $A_\mathfrak{p}$ being normal implies that $A_\mathfrak{p} \subset B_\mathfrak{p}$ is an equality. Since B is a finite A-module, we see there exists an $a \in A$, $a \notin \mathfrak{p}$ such that $A_a \to B_a$ is an isomorphism.

Let $x \in X$ and $y \in Y$ be as in (4). After replacing X by an affine open neighbourhood we may assume $X = \text{Spec}(A)$ and $A \subset \mathcal{O}_{X,x}$, see Properties, Lemma 28.29.8. Then A is a domain and hence X is integral. Since f is birational and Y is reduced it follows that Y is integral too. Consider the ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$. This is a ring map which is essentially of finite type, the residue field extension is finite, and $\dim(\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}) = 0$ (to see this trace through the definitions of quasi-finite maps in Morphisms, Definition 29.20.1 and Algebra, Definition 10.122.3). By Algebra, Lemma 10.124.2 $\mathcal{O}_{Y,y}$ is the localization of a finite $\mathcal{O}_{X,x}$-algebra B. Of course we may replace B by the image of B in $\mathcal{O}_{Y,y}$ and assume that B is a domain with the same fraction field as $\mathcal{O}_{Y,y}$. Then $\mathcal{O}_{X,x} \subset B$ have the same fraction field as f is birational. Since $\mathcal{O}_{X,x}$ is normal, we conclude that $\mathcal{O}_{X,x} = B$ (because finite implies integral), in particular, we see that $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$. By Morphisms, Lemma 29.42.4 after shrinking X we may assume there is a section $X \to Y$ of f mapping x to y and inducing the given isomorphism on local rings. Since $X \to Y$ is closed (by Schemes, Lemma 26.21.11) necessarily maps the generic point of X to the generic point of Y it follows that the image of $X \to Y$ is Y. Then $Y = X$ and we've proved what we wanted to show. $\square$

33.18. Variants of Noether normalization

0CBG Noether normalization is the statement that if k is a field and A is a finite type k algebra of dimension d, then there exists a finite injective k-algebra homomorphism $k[x_1, \ldots, x_d] \to A$. See Algebra, Lemma 10.115.4. Geometrically this means there is a finite surjective morphism $\text{Spec}(A) \to \mathbf{A}^d_k$ over $\text{Spec}(k)$.

0CBH **Lemma 33.18.1.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $V \subset S$ be an affine open neighbourhood of s. If f is locally of finite type and $\dim_x(X_s) = d$, then there exists an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization

$$U \xrightarrow{\pi} \mathbf{A}^d_V \to V$$

of $f|_U : U \to V$ such that π is quasi-finite.

Proof. This follows from Algebra, Lemma 10.125.2. $\square$

0CBI **Lemma 33.18.2.** Let $f : X \to S$ be a finite type morphism of affine schemes. Let $s \in S$. If $\dim(X_s) = d$, then there exists a factorization

$$X \xrightarrow{\pi} \mathbf{A}^d_S \to S$$

of f such that the morphism $\pi_s : X_s \to \mathbf{A}^d_{\kappa(s)}$ of fibres over s is finite.

Proof. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and let $A \to B$ be the ring map corresponding to f. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. We can choose a surjection $A[x_1, \ldots, x_r] \to B$. By Algebra, Lemma 10.115.4 there exist elements $y_1, \ldots, y_d \in A$ in the $\mathbf{Z}$-subalgebra of A generated by $x_1, \ldots, x_r$ such that the A-algebra homomorphism $A[t_1, \ldots, t_d] \to B$ sending t_i to y_i induces a finite $\kappa(\mathfrak{p})$-algebra homomorphism $\kappa(\mathfrak{p})[t_1, \ldots, t_d] \to B \otimes_A \kappa(\mathfrak{p})$. This proves the lemma. $\square$

0CBJ **Lemma 33.18.3.** Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Let $V = \mathrm{Spec}(A)$ be an affine open neighbourhood of $f(x)$ in S. If f is unramified at x, then there exist exists an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[r, "j"] \arrow[dr] & \mathrm{Spec}(A[t]_{g'}/(g)) \arrow[r] \arrow[d] & \mathrm{Spec}(A[t]) = \mathbf{A}^1_V \arrow[dl] \\
Y & & V \arrow[ll] &
\end{tikzcd}$$

where j is an immersion, $g \in A[t]$ is a monic polynomial, and g' is the derivative of g with respect to t. If f is étale at x, then we may choose the diagram such that j is an open immersion.

Proof. The unramified case is a translation of Algebra, Proposition 10.152.1. In the étale case this is a translation of Algebra, Proposition 10.144.4 or equivalently it follows from Morphisms, Lemma 29.36.14 although the statements differ slightly. $\square$

0CBK **Lemma 33.18.4.** Let $f : X \to S$ be a finite type morphism of affine schemes. Let $x \in X$ with image $s \in S$. Let

$$r = \dim_{\kappa(x)} \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \dim_{\kappa(x)} \Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \dim_{\kappa(x)} T_{X/S,x}$$

Then there exists a factorization

$$X \xrightarrow{\pi} \mathbf{A}^r_S \to S$$

of f such that π is unramified at x.

Proof. By Morphisms, Lemma 29.32.12 the first dimension is finite. The first equality follows as the restriction of $\Omega_{X/S}$ to the fibre is the module of differentials from Morphisms, Lemma 29.32.10. The last equality follows from Lemma 33.16.4. Thus we see that the statement makes sense.

To prove the lemma write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and let $A \to B$ be the ring map corresponding to f. Let $\mathfrak q \subset B$ be the prime ideal corresponding to x. Choose a surjection of A-algebras $A[x_1, \ldots, x_t] \to B$. Since $\Omega_{B/A}$ is generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_t$ we see that their images in $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ generate this as a $\kappa(x)$-vector space. After renumbering we may assume that $\mathrm{d}x_1, \ldots, \mathrm{d}x_r$ map to a basis of $\Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$. We claim that $P = A[x_1, \ldots, x_r] \to B$ is unramified at $\mathfrak q$. To see this it suffices to show that $\Omega_{B/P,\mathfrak q} = 0$ (Algebra, Lemma 10.151.3). Note that $\Omega_{B/P}$ is the quotient of $\Omega_{B/A}$ by the submodule generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_r$. Hence $\Omega_{B/P,\mathfrak q} \otimes_{B_{\mathfrak q}} \kappa(\mathfrak q) = 0$ by our choice of $x_1, \ldots, x_r$. By Nakayama's lemma, more precisely Algebra, Lemma 10.20.1 part (2) which applies as $\Omega_{B/P}$ is finite (see reference above), we conclude that $\Omega_{B/P,\mathfrak q} = 0$. $\square$

0CBL Lemma 33.18.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $V \subset S$ be an affine open neighbourhood of s. If f is locally of finite type and

$$r = \dim_{\kappa(x)} \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \dim_{\kappa(x)} \Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \dim_{\kappa(x)} T_{X/S,x}$$

then there exist

(1) an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization
$$U \xrightarrow{j} \mathbf{A}^{r+1}_V \to V$$
of $f|_U$ such that j is an immersion, or
(2) an affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ and a factorization
$$U \xrightarrow{j} D \to V$$
of $f|_U$ such that j is a closed immersion and $D \to V$ is smooth of relative dimension r.

Proof. Pick any affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$. Apply Lemma 33.18.4 to $U \to V$ to get $U \to \mathbf{A}^r_V \to V$ as in the statement of that lemma. By Lemma 33.18.3 we get a factorization

$$U \xrightarrow{j} D \xrightarrow{j'} \mathbf{A}^{r+1}_V \xrightarrow{p} \mathbf{A}^r_V \to V$$

where j and j' are immersions, p is the projection, and $p \circ j'$ is standard étale. Thus we see in particular that (1) and (2) hold. $\square$

33.19. Dimension of fibres

0B2H We have already seen that dimension of fibres of finite type morphisms typically jump up. In this section we discuss the phenomenon that in codimension 1 this does not happen. More generally, we discuss how much the dimension of a fibre can jump. Here is a list of related results:

(1) For a finite type morphism $X \to S$ the set of $x \in X$ with $\dim_x(X_{f(x)}) \leq d$ is open, see Algebra, Lemma 10.125.6 and Morphisms, Lemma 29.28.4.
(2) We have the dimension formula, see Algebra, Lemma 10.113.1 and Morphisms, Lemma 29.52.1.

(3) Constant fibre dimension for an integral finite type scheme dominating a valuation ring, see Algebra, Lemma 10.125.9.
(4) If $X \to S$ is of finite type and is quasi-finite at every generic point of X, then $X \to S$ is quasi-finite in codimension 1, see Algebra, Lemma 10.113.2 and Lemma 33.17.1.

The last result mentioned above generalizes as follows.

0B2I Lemma 33.19.1. Let $f : X \to Y$ be locally of finite type. Let $x \in X$ be a point with image $y \in Y$ such that $\mathcal{O}_{Y,y}$ is Noetherian of dimension ≤ 1. Let $d \geq 0$ be an integer such that for every generic point η of an irreducible component of X which contains x, we have $\dim_\eta(X_{f(\eta)}) = d$. Then $\dim_x(X_y) = d$.

Proof. Recall that $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is the set of points of Y specializing to y, see Schemes, Lemma 26.13.2. Thus we may replace Y by $\mathrm{Spec}(\mathcal{O}_{Y,y})$ and assume $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring of dimension ≤ 1 and y is the closed point. We may also replace X by an affine neighbourhood of x.

Let $X = \bigcup X_i$ be the irreducible components of X viewed as reduced closed subschemes. If we can show each fibre $X_{i,y}$ has dimension d, then $X_y = \bigcup X_{i,y}$ has dimension d as well. Thus we may assume X is an integral scheme.

If $X \to Y$ maps the generic point η of X to y, then X is a scheme over $\kappa(y)$ and the result is true by assumption. Assume that X maps η to a point $\xi \in Y$ corresponding to a minimal prime $\mathfrak{q}$ of B different from $\mathfrak{m}_B$. We obtain a factorization $X \to \mathrm{Spec}(B/\mathfrak{q}) \to \mathrm{Spec}(B)$. By the dimension formula (Morphisms, Lemma 29.52.1) we have

$$\dim(\mathcal{O}_{X,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq \dim(B/\mathfrak{q}) + \mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X))$$

We have $\dim(B/\mathfrak{q}) = 1$. We have $\mathrm{trdeg}_{\kappa(\mathfrak{q})}(R(X)) = d$ by our assumption that $\dim_\eta(X_\xi) = d$, see Morphisms, Lemma 29.28.1. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_s,x}$ has a kernel (as $\eta \mapsto \xi \neq y$) and since $\mathcal{O}_{X,x}$ is a Noetherian domain we see that $\dim(\mathcal{O}_{X,x}) > \dim(\mathcal{O}_{X_y,x})$. We conclude that

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq d$$

(Morphisms, Lemma 29.28.1). On the other hand, we have $\dim_x(X_s) \geq \dim_\eta(X_{f(\eta)}) = d$ by Morphisms, Lemma 29.28.4. □

0B2J Lemma 33.19.2. Let $f : X \to \mathrm{Spec}(R)$ be a morphism from an irreducible scheme to the spectrum of a valuation ring. If f is locally of finite type and surjective, then the special fibre is equidimensional of dimension equal to the dimension of the generic fibre.

Proof. We may replace X by its reduction because this does not change the dimension of X or of the special fibre. Then X is integral and the lemma follows from Algebra, Lemma 10.125.9. □

The following lemma generalizes Lemma 33.19.1.

0B2K Lemma 33.19.3. Let $f : X \to Y$ be locally of finite type. Let $x \in X$ be a point with image $y \in Y$ such that $\mathcal{O}_{Y,y}$ is Noetherian. Let $d \geq 0$ be an integer such that for every generic point η of an irreducible component of X which contains x, we have $f(\eta) \neq y$ and $\dim_\eta(X_{f(\eta)}) = d$. Then $\dim_x(X_y) \leq d + \dim(\mathcal{O}_{Y,y}) - 1$.

Proof. Exactly as in the proof of Lemma 33.19.1 we reduce to the case $X = \mathrm{Spec}(A)$ with A a domain and $Y = \mathrm{Spec}(B)$ where B is a Noetherian local ring whose maximal ideal corresponds to y. After replacing B by $B/\mathrm{Ker}(B \to A)$ we may assume that B is a domain and that $B \subset A$. Then we use the dimension formula (Morphisms, Lemma 29.52.1) to get

$$\dim(\mathcal{O}_{X,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) \leq \dim(B) + \mathrm{trdeg}_B(A)$$

We have $\mathrm{trdeg}_B(A) = d$ by our assumption that $\dim_\eta(X_\xi) = d$, see Morphisms, Lemma 29.28.1. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_y,x}$ has a kernel (as $f(\eta) \neq y$) and since $\mathcal{O}_{X,x}$ is a Noetherian domain we see that $\dim(\mathcal{O}_{X,x}) > \dim(\mathcal{O}_{X_y,x})$. We conclude that

$$\dim_x(X_y) = \dim(\mathcal{O}_{X_y,x}) + \mathrm{trdeg}_{\kappa(y)}\kappa(x) < \dim(B) + d$$

(equality by Morphisms, Lemma 29.28.1) which proves what we want. □

33.20. Algebraic schemes

06LF The following definition is taken from [DG67, I Definition 6.4.1].

06LG Definition 33.20.1. Let k be a field. An algebraic k-scheme is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is of finite type. A locally algebraic k-scheme is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is locally of finite type.

Note that every (locally) algebraic k-scheme is (locally) Noetherian, see Morphisms, Lemma 29.15.6. The category of algebraic k-schemes has all products and fibre products (unlike the category of varieties over k). Similarly for the category of locally algebraic k-schemes.

06LH Lemma 33.20.2. Let k be a field. Let X be a locally algebraic k-scheme of dimension 0. Then X is a disjoint union of spectra of local Artinian k-algebras A with $\dim_k(A) < \infty$. If X is an algebraic k-scheme of dimension 0, then in addition X is affine and the morphism $X \to \mathrm{Spec}(k)$ is finite.

Proof. Let X be a locally algebraic k-scheme of dimension 0. Let $U = \mathrm{Spec}(A) \subset X$ be an affine open subscheme. Since $\dim(X) = 0$ we see that $\dim(A) = 0$. By Noether normalization, see Algebra, Lemma 10.115.4 we see that there exists a finite injection $k \to A$, i.e., $\dim_k(A) < \infty$. Hence A is Artinian, see Algebra, Lemma 10.53.2. This implies that $A = A_1 \times \ldots \times A_r$ is a product of finitely many Artinian local rings, see Algebra, Lemma 10.53.6. Of course $\dim_k(A_i) < \infty$ for each i as the sum of these dimensions equals $\dim_k(A)$.

The arguments above show that X has an open covering whose members are finite discrete topological spaces. Hence X is a discrete topological space. It follows that X is isomorphic to the disjoint union of its connected components each of which is a singleton. Since a singleton scheme is affine we conclude (by the results of the paragraph above) that each of these singletons is the spectrum of a local Artinian k-algebra A with $\dim_k(A) < \infty$.

Finally, if X is an algebraic k-scheme of dimension 0, then X is quasi-compact hence is a finite disjoint union $X = \mathrm{Spec}(A_1) \amalg \ldots \amalg \mathrm{Spec}(A_r)$ hence affine (see Schemes, Lemma 26.6.8) and we have seen the finiteness of $X \to \mathrm{Spec}(k)$ in the first paragraph of the proof. □

The following lemma collects some statements on dimension theory for locally algebraic schemes.

0A21 **Lemma 33.20.3.** Let k be a field. Let X be a locally algebraic k-scheme.

0B17 (1) The topological space of X is catenary (Topology, Definition 5.11.4).
0B18 (2) For $x \in X$ closed, we have $\dim_x(X) = \dim(\mathcal{O}_{X,x})$.
0B19 (3) For X irreducible we have $\dim(X) = \dim(U)$ for any nonempty open $U \subset X$ and $\dim(X) = \dim_x(X)$ for any $x \in X$.
0B1A (4) For X irreducible any chain of irreducible closed subsets can be extended to a maximal chain and all maximal chains of irreducible closed subsets have length equal to $\dim(X)$.
0B1B (5) For $x \in X$ we have $\dim_x(X) = \max \dim(Z) = \min \dim(\mathcal{O}_{X,x'})$ where the maximum is over irreducible components $Z \subset X$ containing x and the minimum is over specializations $x \rightsquigarrow x'$ with x' closed in X.
0B1C (6) If X is irreducible with generic point x, then $\dim(X) = \text{trdeg}_k(\kappa(x))$.
0B1D (7) If $x \rightsquigarrow x'$ is an immediate specialization of points of X, then we have $\text{trdeg}_k(\kappa(x)) = \text{trdeg}_k(\kappa(x')) + 1$.
0B1E (8) The dimension of X is the supremum of the numbers $\text{trdeg}_k(\kappa(x))$ where x runs over the generic points of the irreducible components of X.
0B1F (9) If $x \rightsquigarrow x'$ is a nontrivial specialization of points of X, then
   (a) $\dim_x(X) \leq \dim_{x'}(X)$,
   (b) $\dim(\mathcal{O}_{X,x}) < \dim(\mathcal{O}_{X,x'})$,
   (c) $\text{trdeg}_k(\kappa(x)) > \text{trdeg}_k(\kappa(x'))$, and
   (d) any maximal chain of nontrivial specializations $x = x_0 \rightsquigarrow x_1 \rightsquigarrow \ldots \rightsquigarrow x_n = x$ has length $n = \text{trdeg}_k(\kappa(x)) - \text{trdeg}_k(\kappa(x'))$.
0B1G (10) For $x \in X$ we have $\dim_x(X) = \text{trdeg}_k(\kappa(x)) + \dim(\mathcal{O}_{X,x})$.
0B1H (11) If $x \rightsquigarrow x'$ is an immediate specialization of points of X and X is irreducible or equidimensional, then $\dim(\mathcal{O}_{X,x'}) = \dim(\mathcal{O}_{X,x}) + 1$.

Proof. Instead on relying on the more general results proved earlier we will reduce the statements to the corresponding statements for finite type k-algebras and cite results from the chapter on commutative algebra.

Proof of (1). This is local on X by Topology, Lemma 5.11.5. Thus we may assume $X = \text{Spec}(A)$ where A is a finite type k-algebra. We have to show that A is catenary (Algebra, Lemma 10.105.2). We can reduce to $k[x_1, \ldots, x_n]$ using Algebra, Lemma 10.105.7 and then apply Algebra, Lemma 10.114.3. Alternatively, this holds because k is Cohen-Macaulay (trivially) and Cohen-Macaulay rings are universally catenary (Algebra, Lemma 10.105.9).

Proof of (2). Choose an affine neighbourhood $U = \text{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. Hence we reduce to the affine case, which is Algebra, Lemma 10.114.6.

Proof of (3). It suffices to show that any two nonempty affine opens $U, U' \subset X$ have the same dimension (any finite chain of irreducible subsets meets an affine open). Pick a closed point x of X with $x \in U \cap U'$. This is possible because X is irreducible, hence $U \cap U'$ is nonempty, hence there is such a closed point because X is Jacobson by Lemma 33.14.1. Then $\dim(U) = \dim(\mathcal{O}_{X,x}) = \dim(U')$ by Algebra, Lemma 10.114.4 (strictly speaking you have to replace X by its reduction before applying the lemma).

Proof of (4). Given a chain of irreducible closed subsets we can find an affine open $U \subset X$ which meets the smallest one. Thus the statement follows from Algebra, Lemma 10.114.4 and $\dim(U) = \dim(X)$ which we have seen in (3).

Proof of (5). Choose an affine neighbourhood $U = \mathrm{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. The rule $Z \mapsto Z \cap U$ is a bijection between irreducible components of X passing through x and irreducible components of U passing through x. Also, $\dim(Z \cap U) = \dim(Z)$ for such Z by (3). Hence the statement follows from Algebra, Lemma 10.114.5.

Proof of (6). By (3) this reduces to the case where $X = \mathrm{Spec}(A)$ is affine. In this case it follows from Algebra, Lemma 10.116.1 applied to A_{red}.

Proof of (7). Let $Z = \overline{\{x\}} \supset Z' = \overline{\{x'\}}$. Then it follows from (4) that $Z \supset Z'$ is the start of a maximal chain of irreducible closed subschemes in Z and consequently $\dim(Z) = \dim(Z') + 1$. We conclude by (6).

Proof of (8). A simple topological argument shows that $\dim(X) = \sup \dim(Z)$ where the supremum is over the irreducible components of X (hint: use Topology, Lemma 5.8.3). Thus this follows from (6).

Proof of (9). Part (a) follows from the fact that any open $U \subset X$ containing x' also contains x. Part (b) follows because $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,x'}$ hence any chain of primes in $\mathcal{O}_{X,x}$ corresponds to a chain of primes in $\mathcal{O}_{X,x'}$ which can be extended by adding $\mathfrak{m}_{x'}$ at the end. Both (c) and (d) follow formally from (7).

Proof of (10). Choose an affine neighbourhood $U = \mathrm{Spec}(A)$ of x. Then $\dim_x(X) = \dim_x(U)$. Hence we reduce to the affine case, which is Algebra, Lemma 10.116.3.

Proof of (11). If X is equidimensional (Topology, Definition 5.10.5) then $\dim(X)$ is equal to the dimension of every irreducible component of X, whence $\dim_x(X) = \dim(X) = \dim_{x'}(X)$ by (5). Thus this follows from (7). □

0B2L Lemma 33.20.4. Let k be a field. Let $f : X \to Y$ be a morphism of locally algebraic k-schemes.

(1) For $y \in Y$, the fibre X_y is a locally algebraic scheme over $\kappa(y)$ hence all the results of Lemma 33.20.3 apply.
(2) Assume X is irreducible. Set $Z = \overline{f(X)}$ and $d = \dim(X) - \dim(Z)$. Then
    (a) $\dim_x(X_{f(x)}) \geq d$ for all $x \in X$,
    (b) the set of $x \in X$ with $\dim_x(X_{f(x)}) = d$ is dense open,
    (c) if $\dim(\mathcal{O}_{Z,f(x)}) \geq 1$, then $\dim_x(X_{f(x)}) \leq d + \dim(\mathcal{O}_{Z,f(x)}) - 1$,
    (d) if $\dim(\mathcal{O}_{Z,f(x)}) = 1$, then $\dim_x(X_{f(x)}) = d$,
(3) For $x \in X$ with $y = f(x)$ we have $\dim_x(X_y) \geq \dim_x(X) - \dim_y(Y)$.

Proof. The morphism f is locally of finite type by Morphisms, Lemma 29.15.8. Hence the base change $X_y \to \mathrm{Spec}(\kappa(y))$ is locally of finite type. This proves (1). In the rest of the proof we will freely use the results of Lemma 33.20.3 for X, Y, and the fibres of f.

Proof of (2). Let $\eta \in X$ be the generic point and set $\xi = f(\eta)$. Then $Z = \overline{\{\xi\}}$. Hence

$$d = \dim(X) - \dim(Z) = \mathrm{trdeg}_k \kappa(\eta) - \mathrm{trdeg}_k \kappa(\xi) = \mathrm{trdeg}_{\kappa(\xi)} \kappa(\eta) = \dim_\eta(X_\xi)$$

Thus parts (2)(a) and (2)(b) follow from Morphisms, Lemma 29.28.4. Parts (2)(c) and (2)(d) follow from Lemmas 33.19.3 and 33.19.1.

Proof of (3). Let $x \in X$. Let $X' \subset X$ be a irreducible component of X passing through x of dimension $\dim_x(X)$. Then (2) implies that $\dim_x(X_y) \geq \dim(X') - \dim(Z')$ where $Z' \subset Y$ is the closure of the image of X'. This proves (3). $\square$

0B2M Lemma 33.20.5. Let k be a field. Let X, Y be locally algebraic k-schemes.

(1) For $z \in X \times Y$ lying over (x, y) we have $\dim_z(X \times Y) = \dim_x(X) + \dim_y(Y)$.
(2) We have $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Proof. Proof of (1). Consider the factorization

$$X \times Y \longrightarrow Y \longrightarrow \mathrm{Spec}(k)$$

of the structure morphism. The first morphism $p : X \times Y \to Y$ is flat as a base change of the flat morphism $X \to \mathrm{Spec}(k)$ by Morphisms, Lemma 29.25.8. Moreover, we have $\dim_z(p^{-1}(y)) = \dim_x(X)$ by Morphisms, Lemma 29.28.3. Hence $\dim_z(X \times Y) = \dim_x(X) + \dim_y(Y)$ by Morphisms, Lemma 29.28.2. Part (2) is a direct consequence of (1). $\square$

33.21. Complete local rings

0C51 Some results on complete local rings of schemes over fields.

0C52 Lemma 33.21.1. Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$ be a point with residue field κ. There is an isomorphism

0C53 (33.21.1.1)
$$\kappa[[x_1, \ldots, x_n]]/I \longrightarrow \mathcal{O}_{X,x}^\wedge$$

inducing the identity on residue fields. In general we cannot choose (33.21.1.1) to be a k-algebra isomorphism. However, if the extension κ/k is separable, then we can choose (33.21.1.1) to be an isomorphism of k-algebras.

Proof. The existence of the isomorphism is an immediate consequence of the Cohen structure theorem[2] (Algebra, Theorem 10.160.8).

Let p be an odd prime number, let $k = \mathbf{F}_p(t)$, and $A = k[x, y]/(y^2 + x^p - t)$. Then the completion $A^\wedge$ of A in the maximal ideal $\mathfrak{m} = (y)$ is isomorphic to $k(t^{1/p})[[z]]$ as a ring but not as a k-algebra. The reason is that $A^\wedge$ does not contain an element whose pth power is t (as the reader can see by computing modulo y^2). This also shows that any isomorphism (33.21.1.1) cannot be a k-algebra isomorphism.

If κ/k is separable, then there is a k-algebra homomorphism $\kappa \to \mathcal{O}_{X,x}^\wedge$ inducing the identity on residue fields by More on Algebra, Lemma 15.38.3. Let $f_1, \ldots, f_n \in \mathfrak{m}_x$ be generators. Consider the map

$$\kappa[[x_1, \ldots, x_n]] \longrightarrow \mathcal{O}_{X,x}^\wedge, \quad x_i \longmapsto f_i$$

Since both sides are $(x_1, \ldots, x_n)$-adically complete (the right hand side by Algebra, Lemmas 10.96.3) this map is surjective by Algebra, Lemma 10.96.1 as it is surjective modulo $(x_1, \ldots, x_n)$ by construction. $\square$

[2]Note that if κ has characteristic p, then the theorem just says we get a surjection $\Lambda[[x_1, \ldots, x_n]] \to \mathcal{O}_{X,x}^\wedge$ where Λ is a Cohen ring for κ. But of course in this case the map factors through $\Lambda/p\Lambda[[x_1, \ldots, x_n]]$ and $\Lambda/p\Lambda = \kappa$.

0C54 **Lemma 33.21.2.** Let K/k be an extension of fields. Let X be a locally algebraic k-scheme. Set $Y = X_K$. Let $y \in Y$ be a point with image $x \in X$. Assume that $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$ and that $\kappa(x)/k$ is separable. Choose an isomorphism

$$\kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \longrightarrow \mathcal{O}^\wedge_{X,x}$$

of k-algebras as in (33.21.1.1). Then we have an isomorphism

$$\kappa(y)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \longrightarrow \mathcal{O}^\wedge_{Y,y}$$

of K-algebras as in (33.21.1.1). Here we use $\kappa(x) \to \kappa(y)$ to view g_j as a power series over $\kappa(y)$.

Proof. The local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ induces a local ring map $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{Y,y}$. The induced map

$$\kappa(x) \to \kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \to \mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{Y,y}$$

composed with the projection to $\kappa(y)$ is the canonical homomorphism $\kappa(x) \to \kappa(y)$. By Lemma 33.5.1 the residue field $\kappa(y)$ is a localization of $\kappa(x) \otimes_k K$ at the kernel $\mathfrak{p}_0$ of $\kappa(x) \otimes_k K \to \kappa(y)$. On the other hand, by Lemma 33.5.3 the local ring $(\kappa(x) \otimes_k K)_{\mathfrak{p}_0}$ is equal to $\kappa(y)$. Hence the map

$$\kappa(x) \otimes_k K \to \mathcal{O}^\wedge_{Y,y}$$

factors canonically through $\kappa(y)$. We obtain a commutative diagram

$$\begin{tikzcd}
\kappa(y) \arrow[rr] & & \mathcal{O}^\wedge_{Y,y} \\
\kappa(x) \arrow[u] \arrow[r] & \kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \arrow[r] & \mathcal{O}^\wedge_{X,x} \arrow[u]
\end{tikzcd}$$

Let $f_i \in \mathfrak{m}^\wedge_x \subset \mathcal{O}^\wedge_{X,x}$ be the image of x_i. Observe that $\mathfrak{m}^\wedge_x = (f_1, \ldots, f_n)$ as the map is surjective. Consider the map

$$\kappa(y)[[x_1, \ldots, x_n]] \longrightarrow \mathcal{O}^\wedge_{Y,y}, \quad x_i \longmapsto f_i$$

where here f_i really means the image of f_i in $\mathfrak{m}^\wedge_y$. Since $\mathfrak{m}_x\mathcal{O}_{Y,y} = \mathfrak{m}_y$ by Lemma 33.5.3 we see that the right hand side is complete with respect to $(x_1, \ldots, x_n)$ (use Algebra, Lemma 10.96.3 to see that it is a complete local ring). Since both sides are $(x_1, \ldots, x_n)$-adically complete our map is surjective by Algebra, Lemma 10.96.1 as it is surjective modulo $(x_1, \ldots, x_n)$. Of course the power series $g_1, \ldots, g_m$ are mapped to zero under this map, as they already map to zero in $\mathcal{O}^\wedge_{X,x}$. Thus we have the commutative diagram

$$\begin{tikzcd}
\kappa(y)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \arrow[r] & \mathcal{O}^\wedge_{Y,y} \\
\kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m) \arrow[u] \arrow[r] & \mathcal{O}^\wedge_{X,x} \arrow[u]
\end{tikzcd}$$

We still need to show that the top horizontal arrow is an isomorphism. We already know that it is surjective. We know that $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat (Lemma 33.5.1), which implies that $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{Y,y}$ is flat (More on Algebra, Lemma 15.43.8). Thus we may apply Algebra, Lemma 10.99.1 with $R = \kappa(x)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m)$, with $S = \kappa(y)[[x_1, \ldots, x_n]]/(g_1, \ldots, g_m)$, with $M = \mathcal{O}^\wedge_{Y,y}$, and with $N = S$ to conclude that the map is injective. $\square$

33.22. Global generation

0B5W Some lemmas related to global generation of quasi-coherent modules.

0B57 Lemma 33.22.1. Let $X \to \mathrm{Spec}(A)$ be a morphism of schemes. Let $A \subset A'$ be a faithfully flat ring map. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is globally generated if and only if the base change $\mathcal{F}_{A'}$ is globally generated.

Proof. More precisely, set $X_{A'} = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$. Let $\mathcal{F}_{A'} = p^*\mathcal{F}$ where $p : X_{A'} \to X$ is the projection. By Cohomology of Schemes, Lemma 30.5.2 we have $H^0(X_{k'}, \mathcal{F}_{A'}) = H^0(X, \mathcal{F}) \otimes_A A'$. Thus if s_i, $i \in I$ are generators for $H^0(X, \mathcal{F})$ as an A-module, then their images in $H^0(X_{A'}, \mathcal{F}_{A'})$ are generators for $H^0(X_{A'}, \mathcal{F}_{A'})$ as an A'-module. Thus we have to show that the map $\alpha : \bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$, $(f_i) \mapsto \sum f_i s_i$ is surjective if and only if $p^*\alpha$ is surjective. This we may check over an affine open $U = \mathrm{Spec}(B)$ of X. Then $\mathcal{F}|_U$ corresponds to a B-module M and $s_i|_U$ to elements $x_i \in M$. Thus we have to show that $\bigoplus_{i \in I} B \to M$ is surjective if and only if the base change $\bigoplus_{i \in I} B \otimes_A A' \to M \otimes_A A'$ is surjective. This is true because $A \to A'$ is faithfully flat. $\square$

0B58 Lemma 33.22.2. Let k be an infinite field. Let X be a scheme of finite type over k. Let $\mathcal{L}$ be a very ample invertible sheaf on X. Let $n \geq 0$ and $x, x_1, \ldots, x_n \in X$ be points with x a k-rational point, i.e., $\kappa(x) = k$, and $x \neq x_i$ for $i = 1, \ldots, n$. Then there exists an $s \in H^0(X, \mathcal{L})$ which vanishes at x but not at x_i.

Proof. If $n = 0$ the result is trivial, hence we assume $n > 0$. By definition of a very ample invertible sheaf, the lemma immediately reduces to the case where $X = \mathbf{P}^r_k$ for some $r > 0$ and $\mathcal{L} = \mathcal{O}_X(1)$. Write $\mathbf{P}^r_k = \mathrm{Proj}(k[T_0, \ldots, T_r])$. Set $V = H^0(X, \mathcal{L}) = kT_0 \oplus \ldots \oplus kT_r$. Since x is a k-rational point, we see that the set $s \in V$ which vanish at x is a codimension 1 subspace $W \subset V$ and that W generates the homogeneous prime ideal corresponding to x. Since $x_i \neq x$ the corresponding homogeneous prime $\mathfrak{p}_i \subset k[T_0, \ldots, T_r]$ does not contain W. Since k is infinite, we then see that $W \neq \bigcup W \cap \mathfrak{q}_i$ and the proof is complete. $\square$

0B3Z Lemma 33.22.3. Let k be an infinite field. Let X be an algebraic k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $V \to \Gamma(X, \mathcal{L})$ be a linear map of k-vector spaces whose image generates $\mathcal{L}$. Then there exists a subspace $W \subset V$ with $\dim_k(W) \leq \dim(X) + 1$ which generates $\mathcal{L}$.

Proof. Throughout the proof we will use that for every $x \in X$ the linear map

$$\psi_x : V \to \Gamma(X, \mathcal{L}) \to \mathcal{L}_x \to \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is nonzero. The proof is by induction on $\dim(X)$.

The base case is $\dim(X) = 0$. In this case X has finitely many points $X = \{x_1, \ldots, x_n\}$ (see for example Lemma 33.20.2). Since k is infinite there exists a vector $v \in V$ such that $\psi_{x_i}(v) \neq 0$ for all i. Then $W = k \cdot v$ does the job.

Assume $\dim(X) > 0$. Let $X_i \subset X$ be the irreducible components of dimension equal to $\dim(X)$. Since X is Noetherian there are only finitely many of these. For each i pick a point $x_i \in X_i$. As above choose $v \in V$ such that $\psi_{x_i}(v) \neq 0$ for all i. Let $Z \subset X$ be the zero scheme of the image of v in $\Gamma(X, \mathcal{L})$, see Divisors, Definition 31.14.8. By construction $\dim(Z) < \dim(X)$. By induction we can find $W \subset V$ with $\dim(W) \leq \dim(X)$ such that W generates $\mathcal{L}|_Z$. Then $W + k \cdot v$ generates $\mathcal{L}$. $\square$

33.23. Separating points and tangent vectors

0E8R This is just the following result.

0E8S **Lemma 33.23.1.** Let k be an algebraically closed field. Let X be a proper k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $V \subset H^0(X, \mathcal{L})$ be a k-subvector space. If

(1) for every pair of distinct closed points $x, y \in X$ there is a section $s \in V$ which vanishes at x but not at y, and
(2) for every closed point $x \in X$ and nonzero tangent vector $\theta \in T_{X/k,x}$ there exists a section $s \in V$ which vanishes at x but whose pullback by θ is nonzero,

then $\mathcal{L}$ is very ample and the canonical morphism $\varphi_{\mathcal{L},V} : X \to \mathbf{P}(V)$ is a closed immersion.

Proof. Condition (1) implies in particular that the elements of V generate $\mathcal{L}$ over X. Hence we get a canonical morphism

$$\varphi = \varphi_{\mathcal{L},V} : X \longrightarrow \mathbf{P}(V)$$

by Constructions, Example 27.21.2. The morphism φ is proper by Morphisms, Lemma 29.41.7. By (1) the map φ is injective on closed points (computation omitted). In particular, the fibre over any closed point of $\mathbf{P}(V)$ is a singleton (small detail omitted). Thus we see that φ is finite, for example use Cohomology of Schemes, Lemma 30.21.2. To finish the proof it suffices to show that the map

$$\varphi^\sharp : \mathcal{O}_{\mathbf{P}(V)} \longrightarrow \varphi_*\mathcal{O}_X$$

is surjective. This we may check on stalks at closed points. Let $x \in X$ be a closed point with image the closed point $p = \varphi(x) \in \mathbf{P}(V)$. Since $\varphi^{-1}(\{p\}) = \{x\}$ by (1) and since φ is proper (hence closed), we see that $\varphi^{-1}(U)$ runs through a fundamental system of open neighbourhoods of x as U runs through a fundamental system of open neighbourhoods of p. We conclude that on stalks at p we obtain the map

$$\varphi^\sharp_x : \mathcal{O}_{\mathbf{P}(V),p} \longrightarrow \mathcal{O}_{X,x}$$

In particular, $\mathcal{O}_{X,x}$ is a finite $\mathcal{O}_{\mathbf{P}(V),p}$-module. Moreover, the residue fields of x and p are equal to k (as k is algebraically closed – use the Hilbert Nullstellensatz). Finally, condition (2) implies that the map

$$T_{X/k,x} \longrightarrow T_{\mathbf{P}(V)/k,p}$$

is injective since any nonzero θ in the kernel of this map couldn't possibly satisfy the conclusion of (2). In terms of the map of local rings above this means that

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$$

is surjective, see Lemma 33.16.5. Now the proof is finished by applying Algebra, Lemma 10.20.3. □

0E8T **Lemma 33.23.2.** Let k be an algebraically closed field. Let X be a proper k-scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Suppose that for every closed subscheme $Z \subset X$ of dimension 0 and degree 2 over k the map

$$H^0(X, \mathcal{L}) \longrightarrow H^0(Z, \mathcal{L}|_Z)$$

is surjective. Then $\mathcal{L}$ is very ample on X over k.

Proof. This is a reformulation of Lemma 33.23.1. Namely, given distinct closed points $x, y \in X$ taking $Z = x \cup y$ (viewed as closed subscheme) we get condition (1) of the lemma. And given a nonzero tangent vector $\theta \in T_{X/k,x}$ the morphism $\theta : \mathrm{Spec}(k[\epsilon]) \to X$ is a closed immersion. Setting $Z = \mathrm{Im}(\theta)$ we obtain condition (2) of the lemma. □

33.24. Closures of products

047A Some results on the relation between closure and products.

047B Lemma 33.24.1. Let k be a field. Let X, Y be schemes over k, and let $A \subset X$, $B \subset Y$ be subsets. Set

$$AB = \{z \in X \times_k Y \mid \mathrm{pr}_X(z) \in A,\ \mathrm{pr}_Y(z) \in B\} \subset X \times_k Y$$

Then set theoretically we have

$$\overline{A} \times_k \overline{B} = \overline{AB}$$

Proof. The inclusion $\overline{AB} \subset \overline{A} \times_k \overline{B}$ is immediate. We may replace X and Y by the reduced closed subschemes $\overline{A}$ and $\overline{B}$. Let $W \subset X \times_k Y$ be a nonempty open subset. By Morphisms, Lemma 29.23.4 the subset $U = \mathrm{pr}_X(W)$ is nonempty open in X. Hence $A \cap U$ is nonempty. Pick $a \in A \cap U$. Denote $Y_{\kappa(a)} = \{a\} \times_k Y$ the fibre of $\mathrm{pr}_X : X \times_k Y \to X$ over a. By Morphisms, Lemma 29.23.4 again the morphism $Y_a \to Y$ is open as $\mathrm{Spec}(\kappa(a)) \to \mathrm{Spec}(k)$ is universally open. Hence the nonempty open subset $W_a = W \times_{X \times_k Y} Y_a$ maps to a nonempty open subset of Y. We conclude there exists a $b \in B$ in the image. Hence $AB \cap W \neq \emptyset$ as desired. □

04Q0 Lemma 33.24.2. Let k be a field. Let $f : A \to X$, $g : B \to Y$ be morphisms of schemes over k. Then set theoretically we have

$$\overline{f(A)} \times_k \overline{g(B)} = \overline{(f \times g)(A \times_k B)}$$

Proof. This follows from Lemma 33.24.1 as the image of $f \times g$ is $f(A)g(B)$ in the notation of that lemma. □

04Q1 Lemma 33.24.3. Let k be a field. Let $f : A \to X$, $g : B \to Y$ be quasi-compact morphisms of schemes over k. Let $Z \subset X$ be the scheme theoretic image of f, see Morphisms, Definition 29.6.2. Similarly, let $Z' \subset Y$ be the scheme theoretic image of g. Then $Z \times_k Z'$ is the scheme theoretic image of $f \times g$.

Proof. Recall that Z is the smallest closed subscheme of X through which f factors. Similarly for Z'. Let $W \subset X \times_k Y$ be the scheme theoretic image of $f \times g$. As $f \times g$ factors through $Z \times_k Z'$ we see that $W \subset Z \times_k Z'$.

To prove the other inclusion let $U \subset X$ and $V \subset Y$ be affine opens. By Morphisms, Lemma 29.6.3 the scheme $Z \cap U$ is the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$, and similarly for $Z' \cap V$ and $W \cap U \times_k V$. Hence we may assume X and Y affine. As f and g are quasi-compact this implies that $A = \bigcup U_i$ is a finite union of affines and $B = \bigcup V_j$ is a finite union of affines. Then we may replace A by $\coprod U_i$ and B by $\coprod V_j$, i.e., we may assume that A and B are affine as well. In this case Z is cut out by $\mathrm{Ker}(\Gamma(X, \mathcal{O}_X) \to \Gamma(A, \mathcal{O}_A))$ and similarly for Z' and W. Hence the result follows from the equality

$$\Gamma(A \times_k B, \mathcal{O}_{A \times_k B}) = \Gamma(A, \mathcal{O}_A) \otimes_k \Gamma(B, \mathcal{O}_B)$$

which holds as A and B are affine. Details omitted. □

33.25. Schemes smooth over fields

04QM Here are two lemmas characterizing smooth schemes over fields.

04QN Lemma 33.25.1. Let k be a field. Let X be a scheme over k. Assume

(1) X is locally of finite type over k,
(2) $\Omega_{X/k}$ is locally free, and
(3) k has characteristic zero.

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. This follows from Algebra, Lemma 10.140.7. □

In positive characteristic there exist nonreduced schemes of finite type whose sheaf of differentials is free, for example $\mathrm{Spec}(\mathbf{F}_p[t]/(t^p))$ over $\mathrm{Spec}(\mathbf{F}_p)$. If the ground field k is nonperfect of characteristic p, there exist reduced schemes X/k with free $\Omega_{X/k}$ which are nonsmooth, for example $\mathrm{Spec}(k[t]/(t^p - a)$ where $a \in k$ is not a pth power.

04QP Lemma 33.25.2. Let k be a field. Let X be a scheme over k. Assume

(1) X is locally of finite type over k,
(2) $\Omega_{X/k}$ is locally free,
(3) X is reduced, and
(4) k is perfect.

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. Let $x \in X$ be a point. As X is locally Noetherian (see Morphisms, Lemma 29.15.6) there are finitely many irreducible components $X_1, \ldots, X_n$ passing through x (see Properties, Lemma 28.5.5 and Topology, Lemma 5.9.2). Let $\eta_i \in X_i$ be the generic point. As X is reduced we have $\mathcal{O}_{X,\eta_i} = \kappa(\eta_i)$, see Algebra, Lemma 10.25.1. Moreover, $\kappa(\eta_i)$ is a finitely generated field extension of the perfect field k hence separably generated over k (see Algebra, Section 10.42). It follows that $\Omega_{X/k,\eta_i} = \Omega_{\kappa(\eta_i)/k}$ is free of rank the transcendence degree of $\kappa(\eta_i)$ over k. By Morphisms, Lemma 29.28.1 we conclude that $\dim_{\eta_i}(X_i) = \mathrm{rank}_{\eta_i}(\Omega_{X/k})$. Since $x \in X_1 \cap \ldots \cap X_n$ we see that

$$\mathrm{rank}_x(\Omega_{X/k}) = \mathrm{rank}_{\eta_i}(\Omega_{X/k}) = \dim(X_i).$$

Therefore $\dim_x(X) = \mathrm{rank}_x(\Omega_{X/k})$, see Algebra, Lemma 10.114.5. It follows that $X \to \mathrm{Spec}(k)$ is smooth at x for example by Algebra, Lemma 10.140.3. □

056S Lemma 33.25.3. Let $X \to \mathrm{Spec}(k)$ be a smooth morphism where k is a field. Then X is a regular scheme.

Proof. (See also Lemma 33.12.6.) By Algebra, Lemma 10.140.3 every local ring $\mathcal{O}_{X,x}$ is regular. And because X is locally of finite type over k it is locally Noetherian. Hence X is regular by Properties, Lemma 28.9.2. □

056T Lemma 33.25.4. Let $X \to \mathrm{Spec}(k)$ be a smooth morphism where k is a field. Then X is geometrically regular, geometrically normal, and geometrically reduced over k.

Proof. (See also Lemma 33.12.6.) Let k' be a finite purely inseparable extension of k. It suffices to prove that $X_{k'}$ is regular, normal, reduced, see Lemmas 33.12.3, 33.10.3, and 33.6.5. By Morphisms, Lemma 29.34.5 the morphism $X_{k'} \to \mathrm{Spec}(k')$

is smooth too. Hence it suffices to show that a scheme X smooth over a field is regular, normal, and reduced. We see that X is regular by Lemma 33.25.3. Hence Properties, Lemma 28.9.4 guarantees that X is normal. $\square$

055T Lemma 33.25.5. Let k be a field. Let $d \geq 0$. Let $W \subset \mathbf{A}^d_k$ be nonempty open. Then there exists a closed point $w \in W$ such that $k \subset \kappa(w)$ is finite separable.

Proof. After possible shrinking W we may assume that $W = \mathbf{A}^d_k \setminus V(f)$ for some $f \in k[x_1, \ldots, x_d]$. If the lemma is wrong then $f(a_1, \ldots, a_d) = 0$ for all $(a_1, \ldots, a_d) \in (k^{sep})^d$. This is absurd as k^{sep} is an infinite field. $\square$

056U Lemma 33.25.6. Let k be a field. If X is smooth over $\mathrm{Spec}(k)$ then the set

$$\{x \in X \text{ closed such that } k \subset \kappa(x) \text{ is finite separable}\}$$

is dense in X.

Proof. It suffices to show that given a nonempty smooth X over k there exists at least one closed point whose residue field is finite separable over k. To see this, choose a diagram

$$X \longleftarrow U \xrightarrow{\pi} \mathbf{A}^d_k$$

with π étale, see Morphisms, Lemma 29.36.20. The morphism $\pi : U \to \mathbf{A}^d_k$ is open, see Morphisms, Lemma 29.36.13. By Lemma 33.25.5 we may choose a closed point $w \in \pi(U)$ whose residue field is finite separable over k. Pick any $x \in U$ with $\pi(x) = w$. By Morphisms, Lemma 29.36.7 the field extension $\kappa(x)/\kappa(w)$ is finite separable. Hence $\kappa(x)/k$ is finite separable. The point x is a closed point of X by Morphisms, Lemma 29.20.2. $\square$

056V Lemma 33.25.7. Let X be a scheme over a field k. If X is locally of finite type and geometrically reduced over k then X contains a dense open which is smooth over k.

Proof. The problem is local on X, hence we may assume X is quasi-compact. Let $X = X_1 \cup \ldots \cup X_n$ be the irreducible components of X. Then $Z = \bigcup_{i \neq j} X_i \cap X_j$ is nowhere dense in X. Hence we may replace X by $X \setminus Z$. As $X \setminus Z$ is a disjoint union of irreducible schemes, this reduces us to the case where X is irreducible. As X is irreducible and reduced, it is integral, see Properties, Lemma 28.3.4. Let $\eta \in X$ be its generic point. Then the function field $K = k(X) = \kappa(\eta)$ is geometrically reduced over k, hence separable over k, see Algebra, Lemma 10.44.1. Let $U = \mathrm{Spec}(A) \subset X$ be any nonempty affine open so that $K = A_{(0)}$ is the fraction field of A. Apply Algebra, Lemma 10.140.5 to conclude that A is smooth at (0) over k. By definition this means that some principal localization of A is smooth over k and we win. $\square$

0B8X Lemma 33.25.8. Let k be a perfect field. Let X be a locally algebraic reduced k-scheme, for example a variety over k. Then we have

$$\{x \in X \mid X \to \mathrm{Spec}(k) \text{ is smooth at } x\} = \{x \in X \mid \mathcal{O}_{X,x} \text{ is regular}\}$$

and this is a dense open subscheme of X.

Proof. The equality of the two sets follows immediately from Algebra, Lemma 10.140.5 and the definitions (see Algebra, Definition 10.45.1 for the definition of a perfect field). The set is open because the set of points where a morphism of schemes is smooth is open, see Morphisms, Definition 29.34.1. Finally, we give two arguments to see that it is dense: (1) The generic points of X are in the set as the local rings at generic points are fields (Algebra, Lemma 10.25.1) hence regular. (2) We use that X is geometrically reduced by Lemma 33.6.3 and hence Lemma 33.25.7 applies. $\square$

05AX Lemma 33.25.9. Let k be a field. Let $f : X \to Y$ be a morphism of schemes locally of finite type over k. Let $x \in X$ be a point and set $y = f(x)$. If $X \to \mathrm{Spec}(k)$ is smooth at x and f is flat at x then $Y \to \mathrm{Spec}(k)$ is smooth at y. In particular, if X is smooth over k and f is flat and surjective, then Y is smooth over k.

Proof. It suffices to show that Y is geometrically regular at y, see Lemma 33.12.6. This follows from Lemma 33.12.5 (and Lemma 33.12.6 applied to (X, x)). $\square$

0CDW Lemma 33.25.10. Let k be a field. Let X be a variety over k which has a k-rational point x such that X is smooth at x. Then X is geometrically integral over k.

Proof. Let $U \subset X$ be the smooth locus of X. By assumption U is nonempty and hence dense and scheme theoretically dense. Then $U_{\overline{k}} \subset X_{\overline{k}}$ is dense and scheme theoretically dense as well (some details omitted). Thus it suffices to show that U is geometrically integral. Because U has a k-rational point it is geometrically connected by Lemma 33.7.14. On the other hand, $U_{\overline{k}}$ is reduced and normal (Lemma 33.25.4. Since a connected normal Noetherian scheme is integral (Properties, Lemma 28.7.6) the proof is complete. $\square$

0H3W Lemma 33.25.11. Let X be a scheme of finite type over a field k. There exists a finite purely inseparable extension k'/k, an integer $t \geq 0$, and closed subschemes

$$X_{k'} \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_0 = (X_{k'})_{red}$ and $Z_i \setminus Z_{i+1}$ is smooth over k' for all i.

Proof. We may use induction on $\dim(X)$. By Lemma 33.6.11 we can find a finite purely inseparable extension k'/k such that $(X_{k'})_{red}$ is geometrically reduced over k'. By Lemma 33.25.7 there is a nowhere dense closed subscheme $X' \subset (X_{k'})_{red}$ such that $(X_{k'})_{red} \setminus X'$ is smooth over k'. Then $\dim(X') < \dim(X)$. By induction hypothesis there exists a finite purely inseparable extension k''/k', an integer $t' \geq 0$, and closed subschemes

$$X'_{k''} \supset Y_0 \supset Y_1 \supset \ldots \supset Y_{t'} = \emptyset$$

such that $Y_0 = (X'_{k''})_{red}$ and $Y_i \setminus Y_{i+1}$ is smooth over k'' for all i. Then we let $t = t' + 1$ and we consider

$$X_{k''} \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

given by $Z_0 = (X_{k''})_{red}$ and $Z_i = Y_{i-1}$ for $i > 0$; this makes sense as $X'_{k''}$ is a closed subscheme of $X_{k''}$. We omit the verification that all the stated properties hold. $\square$

33.26. Types of varieties

04L0 Short section discussion some elementary global properties of varieties.

04L1 Definition 33.26.1. Let k be a field. Let X be a variety over k.

(1) We say X is an affine variety if X is an affine scheme. This is equivalent to requiring X to be isomorphic to a closed subscheme of $\mathbf{A}^n_k$ for some n.

(2) We say X is a projective variety if the structure morphism $X \to \text{Spec}(k)$ is projective. By Morphisms, Lemma 29.43.4 this is true if and only if X is isomorphic to a closed subscheme of $\mathbf{P}^n_k$ for some n.
(3) We say X is a quasi-projective variety if the structure morphism $X \to \text{Spec}(k)$ is quasi-projective. By Morphisms, Lemma 29.40.6 this is true if and only if X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ for some n.
(4) A proper variety is a variety such that the morphism $X \to \text{Spec}(k)$ is proper.
(5) A smooth variety is a variety such that the morphism $X \to \text{Spec}(k)$ is smooth.

Note that a projective variety is a proper variety, see Morphisms, Lemma 29.43.5. Also, an affine variety is quasi-projective as $\mathbf{A}^n_k$ is isomorphic to an open subscheme of $\mathbf{P}^n_k$, see Constructions, Lemma 27.13.3.

04L2 Lemma 33.26.2. Let X be a proper variety over k. Then

(1) $K = H^0(X, \mathcal{O}_X)$ is a field which is a finite extension of the field k,
(2) if X is geometrically reduced, then K/k is separable,
(3) if X is geometrically irreducible, then K/k is purely inseparable,
(4) if X is geometrically integral, then $K = k$.

Proof. This is a special case of Lemma 33.9.3. $\square$

33.27. Normalization

0BXQ Some issues associated to normalization.

0BXR Lemma 33.27.1. Let k be a field. Let X be a locally algebraic scheme over k. Let $\nu : X^\nu \to X$ be the normalization morphism, see Morphisms, Definition 29.54.1. Then

(1) ν is finite, dominant, and X^ν is a disjoint union of normal irreducible locally algebraic schemes over k,
(2) ν factors as $X^\nu \to X_{red} \to X$ and the first morphism is the normalization morphism of X_{red},
(3) if X is a reduced algebraic scheme, then ν is birational,
(4) if X is a variety, then X^ν is a variety and ν is a finite birational morphism of varieties.

Proof. Since X is locally of finite type over a field, we see that X is locally Noetherian (Morphisms, Lemma 29.15.6) hence every quasi-compact open has finitely many irreducible components (Properties, Lemma 28.5.7). Thus Morphisms, Definition 29.54.1 applies. The normalization X^ν is always a disjoint union of normal integral schemes and the normalization morphism ν is always dominant, see Morphisms, Lemma 29.54.5. Since X is universally Nagata (Morphisms, Lemma 29.18.2) we see that ν is finite (Morphisms, Lemma 29.54.10). Hence X^ν is locally algebraic too. At this point we have proved (1).

Part (2) is Morphisms, Lemma 29.54.2.

Part (3) is Morphisms, Lemma 29.54.7.

Part (4) follows from (1), (2), (3), and the fact that X^ν is separated as a scheme finite over a separated scheme. $\square$

0GK4 **Lemma 33.27.2.** Let k be a field. Let X be a proper scheme over k. Let $\nu : X^\nu \to X$ be the normalization morphism, see Morphisms, Definition 29.54.1. Then X^ν is proper over k. If X is projective over k, then X^ν is projective over k.

Proof. By Lemma 33.27.1 the morphism ν is finite. Hence X^ν is proper over k by Morphisms, Lemmas 29.44.11 and 29.41.4. The morphism ν is projective by Morphisms, Lemma 29.44.16 and hence if X is projective over k, then X^ν is projective over k by Morphisms, Lemma 29.43.14. $\square$

0BXS **Lemma 33.27.3.** Let k be a field. Let $f : Y \to X$ be a quasi-compact morphism of locally algebraic schemes over k. Let X' be the normalization of X in Y. If Y is reduced, then $X' \to X$ is finite.

Proof. Since Y is quasi-separated (by Properties, Lemma 28.5.4 and Morphisms, Lemma 29.15.6) the morphism f is quasi-separated (Schemes, Lemma 26.21.13). Hence Morphisms, Definition 29.53.3 applies. The result follows from Morphisms, Lemma 29.53.14. This uses that locally algebraic schemes are locally Noetherian (hence have locally finitely many irreducible components) and that locally algebraic schemes are Nagata (Morphisms, Lemma 29.18.2). Some small details omitted. $\square$

0BXT **Lemma 33.27.4.** Let k be a field. Let X be an algebraic k-scheme. Then there exists a finite purely inseparable extension k'/k such that the normalization Y of $X_{k'}$ is geometrically normal over k'.

Proof. Let $K = k^{perf}$ be the perfect closure. Let Y_K be the normalization of X_K, see Lemma 33.27.1. By Limits, Lemma 32.10.1 there exists a finite sub extension $K/k'/k$ and a morphism $\nu : Y \to X_{k'}$ of finite presentation whose base change to K is the normalization morphism $\nu_K : Y_K \to X_K$. Observe that Y is geometrically normal over k' (Lemma 33.10.3). After increasing k' we may assume $Y \to X_{k'}$ is finite (Limits, Lemma 32.8.3). Since $\nu_K : Y_K \to X_K$ is the normalization morphism, it induces a birational morphism $Y_K \to (X_K)_{red}$. Hence there is a dense open $V_K \subset X_K$ such that $\nu_K^{-1}(V_K) \to V_K$ is a closed immersion (inducing an isomorphism of $\nu_K^{-1}(V_K)$ with $V_{K,red}$, see for example Morphisms, Lemma 29.51.6). After increasing k' we find V_K is the base change of a dense open $V \subset Y$ and the morphism $\nu^{-1}(V) \to V$ is a closed immersion (Limits, Lemmas 32.4.11 and 32.8.5). It follows readily from this that ν is the normalization morphism and the proof is complete. $\square$

0C3N **Lemma 33.27.5.** Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $\nu : X^\nu \to X$ be the normalization of X and let $Y^\nu \to X_K$ be the normalization of the base change. Then the canonical morphism

$$Y^\nu \longrightarrow X^\nu \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$$

is an isomorphism if K/k is separable and a universal homeomorphism in general.

Proof. Set $Y = X_K$. Let $X^{(0)}$, resp. $Y^{(0)}$ be the set of generic points of irreducible components of X, resp. Y. Then the projection morphism $\pi : Y \to X$ satisfies $\pi(Y^{(0)}) = X^{(0)}$. This is true because π is surjective, open, and generizing, see Morphisms, Lemmas 29.23.4 and 29.23.5. If we view $X^{(0)}$, resp. $Y^{(0)}$ as (reduced) schemes, then X^ν, resp. Y^ν is the normalization of X, resp. Y in $X^{(0)}$, resp. $Y^{(0)}$. Thus Morphisms, Lemma 29.53.5 gives a canonical morphism $Y^\nu \to X^\nu$

over $Y \to X$ which in turn gives the canonical morphism of the lemma by the universal property of the fibre product.

To prove this morphism has the properties stated in the lemma we may assume $X = \mathrm{Spec}(A)$ is affine. Let $Q(A_{red})$ be the total ring of fractions of A_{red}. Then X^ν is the spectrum of the integral closure A' of A in $Q(A_{red})$, see Morphisms, Lemmas 29.54.2 and 29.54.3. Similarly, Y^ν is the spectrum of the integral closure B' of $A \otimes_k K$ in $Q((A \otimes_k K)_{red})$. There is a canonical map $Q(A_{red}) \to Q((A \otimes_k K)_{red})$, a canonical map $A' \to B'$, and the morphism of the lemma corresponds to the induced map

$$A' \otimes_k K \longrightarrow B'$$

of K-algebras. The kernel consists of nilpotent elements as the kernel of $Q(A_{red}) \otimes_k K \to Q((A \otimes_k K)_{red})$ is the set of nilpotent elements.

If K/k is separable, then $A' \otimes_k K$ is normal by Lemma 33.10.6. In particular it is reduced, whence $Q((A \otimes_k K)_{red}) = Q(A' \otimes_k K)$ and $B' = A' \otimes_k K$ by Algebra, Lemma 10.37.16.

Assume K/k is not separable. Then the characteristic of k is $p > 0$. We will show that for every $b \in B'$ there is a power q of p such that b^q is in the image of $A' \otimes_k K$. This will prove that the displayed map is a universal homeomorphism by Algebra, Lemma 10.46.7. For a given b there is a subfield $F \subset K$ with F/k finitely generated such that b is contained in $Q((A \otimes_k F)_{red})$ and is integral over $A \otimes_k F$. Choose a monic polynomial $P = T^d + \alpha_1 T^{d-1} + \ldots + \alpha_d$ with $P(b) = 0$ and $\alpha_i \in A \otimes_k F$. Choose a transcendence basis $t_1, \ldots, t_r$ for F over k. Let $F/F'/k(t_1, \ldots, t_r)$ be the maximal separable subextension (Fields, Lemma 9.14.6). Since F/F' is finite purely inseparable, there is a q such that $\lambda^q \in F'$ for all $\lambda \in F$. Then b^q is in $Q((A \otimes_k F')_{red})$ and satisfies the polynomial $T^d + \alpha_1^q T^{d-1} + \ldots + \alpha_d^q$ with $\alpha_i^q \in A \otimes_k F'$. By the separable case we see that $b^q \in A' \otimes_k F'$ and the proof is complete. □

0C3P Lemma 33.27.6. Let k be a field. Let X be a locally algebraic k-scheme. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$ be a point such that (a) $\mathcal{O}_{X,x}$ is reduced, (b) $\dim(\mathcal{O}_{X,x}) = 1$, and (c) for every $x' \in X^\nu$ with $\nu(x') = x$ the extension $\kappa(x')/k$ is separable. Then X is geometrically reduced at x and X^ν is geometrically regular at x' with $\nu(x') = x$.

Proof. We will use the results of Lemma 33.27.1 without further mention. Let $x' \in X^\nu$ be a point over x. By dimension theory (Section 33.20) we have $\dim(\mathcal{O}_{X^\nu,x'}) = 1$. Since X^ν is normal, we see that $\mathcal{O}_{X^\nu,x'}$ is a discrete valuation ring (Properties, Lemma 28.12.5). Thus $\mathcal{O}_{X^\nu,x'}$ is a regular local k-algebra whose residue field is separable over k. Hence $k \to \mathcal{O}_{X^\nu,x'}$ is formally smooth in the $\mathfrak{m}_{x'}$-adic topology, see More on Algebra, Lemma 15.38.5. Then $\mathcal{O}_{X^\nu,x'}$ is geometrically regular over k by More on Algebra, Theorem 15.40.1. Thus X^ν is geometrically regular at x' by Lemma 33.12.2.

Since $\mathcal{O}_{X,x}$ is reduced, the family of maps $\mathcal{O}_{X,x} \to \mathcal{O}_{X^\nu,x'}$ is injective. Since $\mathcal{O}_{X^\nu,x'}$ is a geometrically reduced k-algebra, it follows immediately that $\mathcal{O}_{X,x}$ is a geometrically reduced k-algebra. Hence X is geometrically reduced at x by Lemma 33.6.2. □

33.28. Groups of invertible functions

04L3 It is often (but not always) the case that $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group if X is a variety over k. We show this by a series of lemmas. Everything rests on the following special case.

04L4 **Lemma 33.28.1.** Let k be an algebraically closed field. Let $\overline{X}$ be a proper variety over k. Let $X \subset \overline{X}$ be an open subscheme. Assume X is normal. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.

Proof. Since the statement only concerns X, we may replace $\overline{X}$ by a different proper variety over k. Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism. By Lemma 33.27.1 we have that ν is finite and $\overline{X}^\nu$ is a variety. Since X is normal, we see that $\nu^{-1}(X) \to X$ is an isomorphism (tiny detail omitted). Finally, we see that $\overline{X}^\nu$ is proper over k as a finite morphism is proper (Morphisms, Lemma 29.44.11) and compositions of proper morphisms are proper (Morphisms, Lemma 29.41.4). Thus we may and do assume $\overline{X}$ is normal.

We will use without further mention that for any affine open U of $\overline{X}$ the ring $\mathcal{O}(U)$ is a finitely generated k-algebra, which is Noetherian, a domain and normal, see Algebra, Lemma 10.31.1, Properties, Definition 28.3.1, Properties, Lemmas 28.5.2 and 28.7.2, Morphisms, Lemma 29.15.2.

Let $\xi_1, \ldots, \xi_r$ be the generic points of the complement of X in $\overline{X}$. There are finitely many since $\overline{X}$ has a Noetherian underlying topological space (see Morphisms, Lemma 29.15.6, Properties, Lemma 28.5.5, and Topology, Lemma 5.9.2). For each i the local ring $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ is a normal Noetherian local domain (as a localization of a Noetherian normal domain). Let $J \subset \{1, \ldots, r\}$ be the set of indices i such that $\dim(\mathcal{O}_i) = 1$. For $j \in J$ the local ring $\mathcal{O}_j$ is a discrete valuation ring, see Algebra, Lemma 10.119.7. Hence we obtain a valuation

$$v_j : k(\overline{X})^* \longrightarrow \mathbf{Z}$$

with the property that $v_j(f) \geq 0 \Leftrightarrow f \in \mathcal{O}_j$.

Think of $\mathcal{O}(X)$ as a sub k-algebra of $k(X) = k(\overline{X})$. We claim that the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod\nolimits_{j \in J} \mathbf{Z}, \quad f \longmapsto \prod v_j(f)$$

is k^*. It is clear that this claim proves the lemma. Namely, suppose that $f \in \mathcal{O}(X)$ is an element of the kernel. Let $U = \text{Spec}(B) \subset \overline{X}$ be any affine open. Then B is a Noetherian normal domain. For every height one prime $\mathfrak{q} \subset B$ with corresponding point $\xi \in X$ we see that either $\xi = \xi_j$ for some $j \in J$ or that $\xi \in X$. The reason is that $\text{codim}(\overline{\{\xi\}}, \overline{X}) = 1$ by Properties, Lemma 28.10.3 and hence if $\xi \in \overline{X} \setminus X$ it must be a generic point of $\overline{X} \setminus X$, hence equal to some ξ_j, $j \in J$. We conclude that $f \in \mathcal{O}_{X,\xi} = B_\mathfrak{q}$ in either case as f is in the kernel of the map. Thus $f \in \bigcap_{\text{ht}(\mathfrak{q})=1} B_\mathfrak{q} = B$, see Algebra, Lemma 10.157.6. In other words, we see that $f \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$. But since k is algebraically closed we conclude that $f \in k$ by Lemma 33.26.2. $\square$

Next, we generalize the case above by some elementary arguments, still keeping the field algebraically closed.

04L5 **Lemma 33.28.2.** Let k be an algebraically closed field. Let X be an integral scheme locally of finite type over k. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.

Proof. As X is integral the restriction mapping $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective for any nonempty open subscheme $U \subset X$. Hence we may assume that X is affine. Choose a closed immersion $X \to \mathbf{A}^n_k$ and denote $\overline{X}$ the closure of X in $\mathbf{P}^n_k$ via the usual immersion $\mathbf{A}^n_k \to \mathbf{P}^n_k$. Thus we may assume that X is an affine open of a projective variety $\overline{X}$.

Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism, see Morphisms, Definition 29.54.1. We know that ν is finite, dominant, and that $\overline{X}^\nu$ is a normal irreducible scheme, see Morphisms, Lemmas 29.54.5, 29.54.9, and 29.18.2. It follows that $\overline{X}^\nu$ is a proper variety, because $\overline{X} \to \mathrm{Spec}(k)$ is proper as a composition of a finite and a proper morphism (see results in Morphisms, Sections 29.41 and 29.44). It also follows that ν is a surjective morphism, because the image of ν is closed and contains the generic point of $\overline{X}$. Hence setting $X^\nu = \nu^{-1}(X)$ we see that it suffices to prove the result for X^ν. In other words, we may assume that X is a nonempty open of a normal proper variety $\overline{X}$. This case is handled by Lemma 33.28.1. □

The preceding lemma implies the following slight generalization.

04L6 **Lemma 33.28.3.** Let k be an algebraically closed field. Let X be a connected reduced scheme which is locally of finite type over k with finitely many irreducible components. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.

Proof. Let $X = \bigcup X_i$ be the irreducible components. By Lemma 33.28.2 we see that $\mathcal{O}(X_i)^*/k^*$ is a finitely generated abelian group. Let $f \in \mathcal{O}(X)^*$ be in the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod \mathcal{O}(X_i)^*/k^*.$$

Then for each i there exists an element $\lambda_i \in k$ such that $f|_{X_i} = \lambda_i$. By restricting to $X_i \cap X_j$ we conclude that $\lambda_i = \lambda_j$ if $X_i \cap X_j \neq \emptyset$. Since X is connected we conclude that all λ_i agree and hence that $f \in k^*$. This proves that

$$\mathcal{O}(X)^*/k^* \subset \prod \mathcal{O}(X_i)^*/k^*$$

and the lemma follows as on the right we have a product of finitely many finitely generated abelian groups. □

04MI **Lemma 33.28.4.** Let k be a field. Let X be a scheme over k which is connected and reduced. Then the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is a field.

Proof. Let $k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k. Then $X \to \mathrm{Spec}(k)$ factors through $\mathrm{Spec}(k')$, see Schemes, Lemma 26.6.4. As X is reduced we see that k' has no nonzero nilpotent elements. As $k \to k'$ is integral we see that every prime ideal of k' is both a maximal ideal and a minimal prime, and $\mathrm{Spec}(k')$ is totally disconnected, see Algebra, Lemmas 10.36.20 and 10.26.5. As X is connected the morphism $X \to \mathrm{Spec}(k')$ is constant, say with image the point corresponding to $\mathfrak{p} \subset k'$. Then any $f \in k'$, $f \notin \mathfrak{p}$ maps to an invertible element of $\mathcal{O}_X$. By definition of k' this then forces f to be a unit of k'. Hence we see that k' is local with maximal ideal $\mathfrak{p}$, see Algebra, Lemma 10.18.2. Since we've already seen that k' is reduced this implies that k' is a field, see Algebra, Lemma 10.25.1. □

04L7 **Proposition 33.28.5.** Let k be a field. Let X be a scheme over k. Assume that X is locally of finite type over k, connected, reduced, and has finitely many irreducible components. Then $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group if in addition to the conditions above at least one of the following conditions is satisfied:

(1) the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is k,
(2) X has a k-rational point, or
(3) X is geometrically integral.

Proof. Let $\overline{k}$ be an algebraic closure of k. Let Y be a connected component of $(X_{\overline{k}})_{red}$. Note that the canonical morphism $p : Y \to X$ is open (by Morphisms, Lemma 29.23.4) and closed (by Morphisms, Lemma 29.44.7). Hence $p(Y) = X$ as X was assumed connected. In particular, as X is reduced this implies $\mathcal{O}(X) \subset \mathcal{O}(Y)$. By Lemma 33.8.14 we see that Y has finitely many irreducible components. Thus Lemma 33.28.3 applies to Y. This implies that if $\mathcal{O}(X)^*/k^*$ is not a finitely generated abelian group, then there exist elements $f \in \mathcal{O}(X)$, $f \notin k$ which map to an element of $\overline{k}$ via the map $\mathcal{O}(X) \to \mathcal{O}(Y)$. In this case f is algebraic over k, hence integral over k. Thus, if condition (1) holds, then this cannot happen. To finish the proof we show that conditions (2) and (3) imply (1).

Let $k \subset k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k in $\Gamma(X, \mathcal{O}_X)$. By Lemma 33.28.4 we see that k' is a field. If $e : \mathrm{Spec}(k) \to X$ is a k-rational point, then $e^\sharp : \Gamma(X, \mathcal{O}_X) \to k$ is a section to the inclusion map $k \to \Gamma(X, \mathcal{O}_X)$. In particular the restriction of $e^\sharp$ to k' is a field map $k' \to k$ over k, which clearly shows that (2) implies (1).

If the integral closure k' of k in $\Gamma(X, \mathcal{O}_X)$ is not trivial, then we see that X is either not geometrically connected (if k'/k is not purely inseparable) or that X is not geometrically reduced (if k'/k is nontrivial purely inseparable). Details omitted. Hence (3) implies (1). □

04L8 **Lemma 33.28.6.** Let k be a field. Let X be a variety over k. The group $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group provided at least one of the following conditions holds:

(1) k is integrally closed in $\Gamma(X, \mathcal{O}_X)$,
(2) k is algebraically closed in $k(X)$,
(3) X is geometrically integral over k, or
(4) k is the "intersection" of the field extensions $\kappa(x)/k$ where x runs over the closed points of x.

Proof. We see that (1) is enough by Proposition 33.28.5. We omit the verification that each of (2), (3), (4) implies (1). □

33.29. Künneth formula, I

0BEC In this section we prove the Künneth formula when the base is a field and we are considering cohomology of quasi-coherent modules. For a more general version, please see Derived Categories of Schemes, Section 36.23.

0BED **Lemma 33.29.1.** Let k be a field. Let X and Y be schemes over k and let $\mathcal{F}$, resp. $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module, resp. $\mathcal{O}_Y$-module. Then we have a canonical isomorphism

$$H^n(X \times_{\mathrm{Spec}(k)} Y, \mathrm{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_{\mathrm{Spec}(k)} Y}} \mathrm{pr}_2^*\mathcal{G}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G})$$

provided X and Y are quasi-compact and have affine diagonal[3] (for example if X and Y are separated).

Proof. In this proof unadorned products and tensor products are over k. As maps

$$H^p(X, \mathcal{F}) \otimes H^q(Y, \mathcal{G}) \longrightarrow H^n(X \times Y, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

we use functoriality of cohomology to get maps $H^p(X, \mathcal{F}) \to H^p(X \times Y, \text{pr}_1^*\mathcal{F})$ and $H^p(Y, \mathcal{G}) \to H^p(X \times Y, \text{pr}_2^*\mathcal{G})$ and then we use the cup product

$$\cup : H^p(X \times Y, \text{pr}_1^*\mathcal{F}) \otimes H^q(X \times Y, \text{pr}_2^*\mathcal{G}) \longrightarrow H^n(X \times Y, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

The result is true when X and Y are affine by the vanishing of higher cohomology groups on affines (Cohomology of Schemes, Lemma 30.2.2) and the definitions (of pullbacks of quasi-coherent modules and tensor products of quasi-coherent modules).

Choose finite affine open coverings $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$. This determines an affine open covering $\mathcal{W} : X \times Y = \bigcup_{(i,j) \in I \times J} U_i \times V_j$. Note that $\mathcal{W}$ is a refinement of $\text{pr}_1^{-1}\mathcal{U}$ and of $\text{pr}_2^{-1}\mathcal{V}$. Thus by Cohomology, Lemma 20.15.1 we obtain maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F}) \quad \text{and} \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_2^*\mathcal{G})$$

compatible with pullback maps on cohomology. In Cohomology, Equation (20.25.3.2) we have constructed a map of complexes

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_2^*\mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

defining the cup product on cohomology. Combining the above we obtain a map of complexes

0BEE
$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}) \tag{33.29.1.1}$$

We warn the reader that this map is not an isomorphism of complexes. Recall that we may compute the cohomologies of our quasi-coherent sheaves using our coverings (Cohomology of Schemes, Lemmas 30.2.5 and 30.2.6). Thus on cohomology (33.29.1.1) reproduces the map of the lemma.

Consider a short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of quasi-coherent modules. Since the construction of (33.29.1.1) is functorial in $\mathcal{F}$ and since the formation of the relevant Čech complexes is exact in the variable $\mathcal{F}$ (because we are taking sections over affine opens) we find a map between short exact sequence of complexes

$$\begin{tikzcd}
\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \arrow[r] \arrow[d] & \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}') \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \arrow[r] \arrow[d] & \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}'') \otimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \arrow[d] \\
\check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}) \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F}' \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}) \arrow[r] & \check{\mathcal{C}}^\bullet(\mathcal{W}, \text{pr}_1^*\mathcal{F}'' \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})
\end{tikzcd}$$

(we have dropped the outer zeros). Looking at long exact cohomology sequences we find that if the result of the lemma holds for 2-out-of-3 of $\mathcal{F}, \mathcal{F}', \mathcal{F}''$, then it holds for the third.

Observe that X has finite cohomological dimension for quasi-coherent modules, see Cohomology of Schemes, Lemma 30.4.2. Using induction on $d(\mathcal{F}) = \max\{d \mid H^d(X, \mathcal{F}) \neq 0\}$ we will reduce to the case $d(\mathcal{F}) = 0$. Assume $d(\mathcal{F}) > 0$. By

[3]The case where X and Y are quasi-separated will be discussed in Lemma 33.29.2 below.

Cohomology of Schemes, Lemma 30.4.3 we have seen that there exists an embedding $\mathcal{F} \to \mathcal{F}'$ such that $H^p(X, \mathcal{F}') = 0$ for all $p \geq 1$. Setting $\mathcal{F}'' = \text{Coker}(\mathcal{F} \to \mathcal{F}')$ we see that $d(\mathcal{F}'') < d(\mathcal{F})$. Then we can apply the result from the previous paragraph to see that it suffices to prove the lemma for $\mathcal{F}'$ and $\mathcal{F}''$ thereby proving the induction step.

Arguing in the same fashion for $\mathcal{G}$ we find that we may assume that both $\mathcal{F}$ and $\mathcal{G}$ have nonzero cohomology only in degree 0. Let $V \subset Y$ be an affine open. Consider the affine open covering $\mathcal{U}_V : X \times V = \bigcup_{i \in I} U_i \times V$. It is immediate that

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes \mathcal{G}(V) = \check{\mathcal{C}}^\bullet(\mathcal{U}_V, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$$

(equality of complexes). We conclude that

$$R\text{pr}_{2,*}(\text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes_k \mathcal{G} \cong \bigoplus\nolimits_{\alpha \in A} \mathcal{G}$$

on Y. Here A is a basis for the k-vector space $\Gamma(X, \mathcal{F})$. Cohomology on Y commutes with direct sums (Cohomology, Lemma 20.19.1). Using the Leray spectral sequence for pr_2 (via Cohomology, Lemma 20.13.6) we conclude that $H^n(X \times Y, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G})$ is zero for $n > 0$ and isomorphic to $H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$ for $n = 0$. This finishes the proof (except that we should check that the isomorphism is indeed given by cup product in degree 0; we omit the verification). □

0BEF Lemma 33.29.2. Let k be a field. Let X and Y be schemes over k and let $\mathcal{F}$, resp. $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module, resp. $\mathcal{O}_Y$-module. Then we have a canonical isomorphism

$$H^n(X \times_{\text{Spec}(k)} Y, \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_{\text{Spec}(k)} Y}} \text{pr}_2^*\mathcal{G}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G})$$

provided X and Y are quasi-compact and quasi-separated.

Proof. If X and Y are separated or more generally have affine diagonal, then please see Lemma 33.29.1 for "better" proof (the feature it has over this proof is that it identifies the maps as pullbacks followed by cup products). Let X', resp. Y' be the infinitesimal thickening of X, resp. Y whose structure sheaf is $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$, resp. $\mathcal{O}_{Y'} = \mathcal{O}_Y \oplus \mathcal{G}$ where $\mathcal{F}$, resp. $\mathcal{G}$ is an ideal of square zero. Then

$$\mathcal{O}_{X' \times Y'} = \mathcal{O}_{X \times Y} \oplus \text{pr}_1^*\mathcal{F} \oplus \text{pr}_2^*\mathcal{G} \oplus \text{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_2^*\mathcal{G}$$

as sheaves on $X \times Y$. In this way we see that it suffices to prove that

$$H^n(X \times Y, \mathcal{O}_{X \times Y}) = \bigoplus\nolimits_{p+q=n} H^p(X, \mathcal{O}_X) \otimes_k H^q(Y, \mathcal{O}_Y)$$

for any pair of quasi-compact and quasi-separated schemes over k. Some details omitted.

To prove this statement we use cohomology and base change in the form of Cohomology of Schemes, Lemma 30.7.3. This lemma tells us there exists a bounded below complex of k-vector spaces, i.e., a complex $\mathcal{K}^\bullet$ of quasi-coherent modules on $\text{Spec}(k)$, which universally computes the cohomology of Y over $\text{Spec}(k)$. In particular, we see that

$$R\text{pr}_{1,*}(\mathcal{O}_{X \times Y}) \cong (X \to \text{Spec}(k))^*\mathcal{K}^\bullet$$

in $D(\mathcal{O}_X)$. Up to homotopy the complex $\mathcal{K}^\bullet$ is isomorphic to $\bigoplus_{q \geq 0} H^q(Y, \mathcal{O}_Y)[-q]$ because this is true for every complex of vector spaces over a field. We conclude that

$$R\text{pr}_{1,*}(\mathcal{O}_{X \times Y}) \cong \bigoplus\nolimits_{q \geq 0} H^q(Y, \mathcal{O}_Y)[-q] \otimes_k \mathcal{O}_X$$

in $D(\mathcal{O}_X)$. Then we have

$$\begin{aligned} R\Gamma(X \times Y, \mathcal{O}_{X \times Y}) &= R\Gamma(X, R\text{pr}_{1,*}(\mathcal{O}_{X \times Y})) \\ &= R\Gamma(X, \bigoplus\nolimits_{q \geq 0} H^q(Y, \mathcal{O}_Y)[-q] \otimes_k \mathcal{O}_X) \\ &= \bigoplus\nolimits_{q \geq 0} R\Gamma(X, H^q(Y, \mathcal{O}_Y) \otimes \mathcal{O}_X)[-q] \\ &= \bigoplus\nolimits_{q \geq 0} R\Gamma(X, \mathcal{O}_X) \otimes_k H^q(Y, \mathcal{O}_Y)[-q] \\ &= \bigoplus\nolimits_{p, q \geq 0} H^p(X, \mathcal{O}_X)[-p] \otimes_k H^q(Y, \mathcal{O}_Y)[-q] \end{aligned}$$

as desired. The first equality by Leray for pr_1 (Cohomology, Lemma 20.13.1). The second by our decomposition of the total direct image given above. The third because cohomology always commutes with finite direct sums (and cohomology of Y vanishes in sufficiently large degree by Cohomology of Schemes, Lemma 30.4.4). The fourth because cohomology on X commutes with infinite direct sums by Cohomology, Lemma 20.19.1. The final equality by our remark on the derived category of a field above. $\square$

33.30. Picard groups of varieties

0BEG In this section we collect some elementary results on Picard groups of algebraic varieties.

0CDX **Lemma 33.30.1.** Let $A \to B$ be a faithfully flat ring map. Let X be a quasi-compact and quasi-separated scheme over A. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose pullback to X_B is trivial. Then $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{\otimes -1})$ are invertible $H^0(X, \mathcal{O}_X)$-modules and the multiplication map induces an isomorphism

$$H^0(X, \mathcal{L}) \otimes_{H^0(X, \mathcal{O}_X)} H^0(X, \mathcal{L}^{\otimes -1}) \longrightarrow H^0(X, \mathcal{O}_X)$$

Proof. Denote $\mathcal{L}_B$ the pullback of $\mathcal{L}$ to X_B. Choose an isomorphism $\mathcal{L}_B \to \mathcal{O}_{X_B}$. Set $R = H^0(X, \mathcal{O}_X)$, $M = H^0(X, \mathcal{L})$ and think of M as an R-module. For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ with pullback $\mathcal{F}_B$ on X_B there is a canonical isomorphism $H^0(X_B, \mathcal{F}_B) = H^0(X, \mathcal{F}) \otimes_A B$, see Cohomology of Schemes, Lemma 30.5.2. Thus we have

$$M \otimes_R (R \otimes_A B) = M \otimes_A B = H^0(X_B, \mathcal{L}_B) \cong H^0(X_B, \mathcal{O}_{X_B}) = R \otimes_A B$$

Since $R \to R \otimes_A B$ is faithfully flat (as the base change of the faithfully flat map $A \to B$), we conclude that M is an invertible R-module by Algebra, Proposition 10.83.3. Similarly $N = H^0(X, \mathcal{L}^{\otimes -1})$ is an invertible R-module. To see that the statement on tensor products is true, use that it is true after pulling back to X_B and faithful flatness of $R \to R \otimes_A B$. Some details omitted. $\square$

0CDY **Lemma 33.30.2.** Let $A \to B$ be a faithfully flat ring map. Let X be a scheme over A such that

1. X is quasi-compact and quasi-separated, and
2. $R = H^0(X, \mathcal{O}_X)$ is a semi-local ring.

Then the pullback map $\text{Pic}(X) \to \text{Pic}(X_B)$ is injective.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module whose pullback $\mathcal{L}'$ to X_B is trivial. Set $M = H^0(X, \mathcal{L})$ and $N = H^0(X, \mathcal{L}^{\otimes -1})$. By Lemma 33.30.1 the R-modules M and N are invertible. Since R is semi-local $M \cong R$ and $N \cong R$, see Algebra, Lemma 10.78.7. Choose generators $s \in M$ and $t \in N$. Then $st \in R = H^0(X, \mathcal{O}_X)$ is a unit by the last part of Lemma 33.30.1. We conclude that s and t define trivializations of $\mathcal{L}$ and $\mathcal{L}^{\otimes -1}$ over X. $\square$

0CC5 Lemma 33.30.3. Let k'/k be a field extension. Let X be a scheme over k such that

(1) X is quasi-compact and quasi-separated, and
(2) $R = H^0(X, \mathcal{O}_X)$ is semi-local, e.g., if $\dim_k R < \infty$.

Then the pullback map $\text{Pic}(X) \to \text{Pic}(X_{k'})$ is injective.

Proof. Special case of Lemma 33.30.2. If $\dim_k R < \infty$, then R is Artinian and hence semi-local (Algebra, Lemmas 10.53.2 and 10.53.3). $\square$

0CDP Example 33.30.4. Lemma 33.30.3 is not true without some condition on the scheme X over the field k. Here is an example. Let k be a field. Let $t \in \mathbf{P}^1_k$ be a closed point. Set $X = \mathbf{P}^1 \setminus \{t\}$. Then we have a surjection

$$\mathbf{Z} = \text{Pic}(\mathbf{P}^1_k) \longrightarrow \text{Pic}(X)$$

The first equality by Divisors, Lemma 31.28.5 and surjective by Divisors, Lemma 31.28.3 (as $\mathbf{P}^1_k$ is smooth of dimension 1 over k and hence all its local rings are discrete valuation rings). If $\mathcal{L}$ is in the kernel of the displayed map, then $\mathcal{L} \cong \mathcal{O}_{\mathbf{P}^1_k}(nt)$ for some $n \in \mathbf{Z}$. We leave it to the reader to show that $\mathcal{O}_{\mathbf{P}^1_k}(t) \cong \mathcal{O}_{\mathbf{P}^1_k}(d)$ where $d = [\kappa(t) : k]$. Hence

$$\text{Pic}(X) = \mathbf{Z}/d\mathbf{Z}$$

Thus if t is not a k-rational point, then $d > 1$ and this Picard group is nonzero. On the other hand, if we extend the ground field k to any field extension k' such that there exists a k-embedding $\kappa(t) \to k'$, then $\mathbf{P}^1_{k'} \setminus X_{k'}$ has a k'-rational point t'. Hence $\mathcal{O}_{\mathbf{P}^1_{k'}}(1) = \mathcal{O}_{\mathbf{P}^1_{k'}}(t')$ will be in the kernel of the map $\mathbf{Z} \to \text{Pic}(X_{k'})$ and it will follow in the same manner as above that $\text{Pic}(X_{k'}) = 0$.

The following lemma tells us that "rationally equivalence invertible modules" are isomorphic on normal varieties.

0BEH Lemma 33.30.5. Let k be a field. Let X be a normal variety over k. Let $U \subset \mathbf{A}^n_k$ be an open subscheme with k-rational points $p, q \in U(k)$. For every invertible module $\mathcal{L}$ on $X \times_{\text{Spec}(k)} U$ the restrictions $\mathcal{L}|_{X \times p}$ and $\mathcal{L}|_{X \times q}$ are isomorphic.

Proof. The fibres of $X \times_{\text{Spec}(k)} U \to X$ are open subschemes of affine n-space over fields. Hence these fibres have trivial Picard groups by Divisors, Lemma 31.28.4. Applying Divisors, Lemma 31.28.1 we see that $\mathcal{L}$ is the pullback of an invertible module $\mathcal{N}$ on X. $\square$

33.31. Uniqueness of base field

04MJ The phrase "let X be a scheme over k" means that X is a scheme which comes equipped with a morphism $X \to \text{Spec}(k)$. Now we can ask whether the field k is uniquely determined by the scheme X. Of course this is not the case, since for example $\mathbf{A}^1_{\mathbf{C}}$ which we ordinarily consider as a scheme over the field $\mathbf{C}$ of complex numbers, could also be considered as a scheme over $\mathbf{Q}$. But what if we ask that the morphism $X \to \mathrm{Spec}(k)$ does not factor as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ for any nontrivial field extension k'/k? In other words we ask that k is somehow maximal such that X lives over k.

An example to show that this still does not guarantee uniqueness of k is the scheme

$$X = \mathrm{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(y)}, P \in \mathbf{Q}[y], P \neq 0\right]\right)$$

At first sight this seems to be a scheme over $\mathbf{Q}(x)$, but on a second look it is clear that it is also a scheme over $\mathbf{Q}(y)$. Moreover, the fields $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are subfields of $R = \Gamma(X, \mathcal{O}_X)$ which are maximal among the subfields of R (details omitted). In particular, both $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are maximal in the sense above. Note that both morphisms $X \to \mathrm{Spec}(\mathbf{Q}(x))$ and $X \to \mathrm{Spec}(\mathbf{Q}(y))$ are "essentially of finite type" (i.e., the corresponding ring map is essentially of finite type). Hence X is a Noetherian scheme of finite dimension, i.e., it is not completely pathological.

Another issue that can prevent uniqueness is that the scheme X may be nonreduced. In that case there can be many different morphisms from X to the spectrum of a given field. As an explicit example consider the dual numbers $D = \mathbf{C}[y]/(y^2) = \mathbf{C} \oplus \epsilon\mathbf{C}$. Given any derivation $\theta : \mathbf{C} \to \mathbf{C}$ over $\mathbf{Q}$ we get a ring map

$$\mathbf{C} \longrightarrow D, \quad c \longmapsto c + \epsilon\theta(c).$$

The subfield of $\mathbf{C}$ on which all of these maps are the same is the algebraic closure of $\mathbf{Q}$. This means that taking the intersection of all the fields that X can live over may end up being a very small field if X is nonreduced.

One observation in this regard is the following: given a field k and two subfields k_1, k_2 of k such that k is finite over k_1 and over k_2, then in general it is not the case that k is finite over $k_1 \cap k_2$. An example is the field $k = \mathbf{Q}(t)$ and its subfields $k_1 = \mathbf{Q}(t^2)$ and $\mathbf{Q}((t+1)^2)$. Namely we have $k_1 \cap k_2 = \mathbf{Q}$ in this case. So in the following we have to be careful when taking intersections of fields.

Having said all of this we now show that if X is locally of finite type over a field, then some uniqueness holds. Here is the precise result.

04MK Proposition 33.31.1. Let X be a scheme. Let $a : X \to \mathrm{Spec}(k_1)$ and $b : X \to \mathrm{Spec}(k_2)$ be morphisms from X to spectra of fields. Assume a, b are locally of finite type, and X is reduced, and connected. Then we have $k_1' = k_2'$, where $k_i' \subset \Gamma(X, \mathcal{O}_X)$ is the integral closure of k_i in $\Gamma(X, \mathcal{O}_X)$.

Proof. First, assume the lemma holds in case X is quasi-compact (we will do the quasi-compact case below). As X is locally of finite type over a field, it is locally Noetherian, see Morphisms, Lemma 29.15.6. In particular this means that it is locally connected, connected components of open subsets are open, and intersections of quasi-compact opens are quasi-compact, see Properties, Lemma 28.5.5, Topology, Lemma 5.7.11, Topology, Section 5.9, and Topology, Lemma 5.16.1. Pick an open covering $X = \bigcup_{i \in I} U_i$ such that each U_i is quasi-compact and connected. For each i let $K_i \subset \mathcal{O}_X(U_i)$ be the integral closure of k_1 and of k_2. For each pair $i, j \in I$ we decompose

$$U_i \cap U_j = \coprod U_{i,j,l}$$

into its finitely many connected components. Write $K_{i,j,l} \subset \mathcal{O}(U_{i,j,l})$ for the integral closure of k_1 and of k_2. By Lemma 33.28.4 the rings K_i and $K_{i,j,l}$ are fields. Now we claim that k'_1 and k'_2 both equal the kernel of the map

$$\prod K_i \longrightarrow \prod K_{i,j,l}, \quad (x_i)_i \longmapsto x_i|_{U_{i,j,l}} - x_j|_{U_{i,j,l}}$$

which proves what we want. Namely, it is clear that k'_1 is contained in this kernel. On the other hand, suppose that $(x_i)_i$ is in the kernel. By the sheaf condition $(x_i)_i$ corresponds to $f \in \mathcal{O}(X)$. Pick some $i_0 \in I$ and let $P(T) \in k_1[T]$ be a monic polynomial with $P(x_{i_0}) = 0$. Then we claim that $P(f) = 0$ which proves that $f \in k_1$. To prove this we have to show that $P(x_i) = 0$ for all i. Pick $i \in I$. As X is connected there exists a sequence $i_0, i_1, \ldots, i_n = i \in I$ such that $U_{i_t} \cap U_{i_{t+1}} \neq \emptyset$. Now this means that for each t there exists an l_t such that x_{i_t} and $x_{i_{t+1}}$ map to the same element of the field $K_{i,j,l}$. Hence if $P(x_{i_t}) = 0$, then $P(x_{i_{t+1}}) = 0$. By induction, starting with $P(x_{i_0}) = 0$ we deduce that $P(x_i) = 0$ as desired.

To finish the proof of the lemma we prove the lemma under the additional hypothesis that X is quasi-compact. By Lemma 33.28.4 after replacing k_i by k'_i we may assume that k_i is integrally closed in $\Gamma(X, \mathcal{O}_X)$. This implies that $\mathcal{O}(X)^*/k_i^*$ is a finitely generated abelian group, see Proposition 33.28.5. Let $k_{12} = k_1 \cap k_2$ as a subring of $\mathcal{O}(X)$. Note that k_{12} is a field. Since

$$k_1^*/k_{12}^* \longrightarrow \mathcal{O}(X)^*/k_2^*$$

we see that k_1^*/k_{12}^* is a finitely generated abelian group as well. Hence there exist $\alpha_1, \ldots, \alpha_n \in k_1^*$ such that every element $\lambda \in k_1$ has the form

$$\lambda = c\alpha_1^{e_1} \ldots \alpha_n^{e_n}$$

for some $e_i \in \mathbf{Z}$ and $c \in k_{12}$. In particular, the ring map

$$k_{12}[x_1, \ldots, x_n, \frac{1}{x_1 \ldots x_n}] \longrightarrow k_1, \quad x_i \longmapsto \alpha_i$$

is surjective. By the Hilbert Nullstellensatz, Algebra, Theorem 10.34.1 we conclude that k_1 is a finite extension of k_{12}. In the same way we conclude that k_2 is a finite extension of k_{12}. In particular both k_1 and k_2 are contained in the integral closure k'_{12} of k_{12} in $\Gamma(X, \mathcal{O}_X)$. But since k'_{12} is a field by Lemma 33.28.4 and since we chose k_i to be integrally closed in $\Gamma(X, \mathcal{O}_X)$ we conclude that $k_1 = k_{12} = k_2$ as desired. □

33.32. Automorphisms

0GWY A section on automorphisms of schemes over fields. For some information on (infinitesimal) automorphisms of curves, see Algebraic Curves, Section 53.25 and Moduli of Curves, Section 109.7.

0G05 Lemma 33.32.1. Let X be a reduced scheme of finite type over a field k. Let $f : X \to X$ be an automorphism over k which induces the identity map on the underlying topological space of X. Then

(1) $f^*\mathcal{F} \cong \mathcal{F}$ for every coherent $\mathcal{O}_X$-module, and
(2) if $\dim(Z) > 0$ for every irreducible component $Z \subset X$, then f is the identity.

Proof. Part (1) follows from part (2) and the fact that the connected components of X of dimension 0 are spectra of fields.

Let $Z \subset X$ be an irreducible component viewed as an integral closed subscheme. Clearly $f(Z) \subset Z$ and $f|_Z : Z \to Z$ is an automorphism over k which induces the identity map on the underlying topological space of Z. Since X is reduced, it suffices to show that the arrows $f|_Z : Z \to Z$ are the identity. This reduces us to the case discussed in the next paragraph.

Assume X is irreducible of dimension > 0. Choose a nonempty affine open $U \subset X$. Since $f(U) \subset U$ and since $U \subset X$ is scheme theoretically dense it suffices to prove that $f|_U : U \to U$ is the identity.

Assume $X = \text{Spec}(A)$ is affine, irreducible, of dimension > 0 and k is an infinite field. Let $g \in A$ be nonconstant. The set

$$S = \bigcup\nolimits_{\lambda \in k} V(g - \lambda)$$

is dense in X because it is the inverse image of the dense subset $\mathbf{A}^1_k(k)$ by the nonconstant morphism $g : X \to \mathbf{A}^1_k$. If $x \in S$, then the image $g(x)$ of g in $\kappa(x)$ is in the image of $k \to \kappa(x)$. Hence $f^\sharp : \kappa(x) \to \kappa(x)$ fixes $g(x)$. Thus the image of $f^\sharp(g)$ in $\kappa(x)$ is equal to $g(x)$. We conclude that

$$S \subset V(g - f^\sharp(g))$$

and since X is reduced and S is dense we conclude $g = f^\sharp(g)$. This proves $f^\sharp = \text{id}_A$ as A is generated as a k-algebra by elements g as above (details omitted; hint: the set of constant functions is a finite dimensional k-subvector space of A). We conclude that $f = \text{id}_X$.

Assume $X = \text{Spec}(A)$ is affine, irreducible, of dimension > 0 and k is a finite field. If for every 1-dimensional integral closed subscheme $C \subset X$ the restriction $f|_C : C \to C$ is the identity, then f is the identity. This reduces us to the case where X is a curve. A curve over a finite field has a finite automorphism group (details omitted). Hence f has finite order, say n. Then we pick $g : X \to \mathbf{A}^1_k$ nonconstant as above and we consider

$$S = \{x \in X \text{ closed such that } [\kappa(g(x)) : k] \text{ is prime to } n\}$$

Arguing as before we find that S is dense in X. Since for $x \in X$ closed the map $f^\sharp : \kappa(x) \to \kappa(x)$ is an automorphism of order dividing n we see that for $x \in S$ this automorphism acts trivially on the subfield generated by the image of g in $\kappa(x)$. Thus we conclude that $S \subset V(g - f^\sharp(g))$ and we win as before. □

33.33. Euler characteristics

0BEI In this section we prove some elementary properties of Euler characteristics of coherent sheaves on schemes proper over fields.

0BEJ Definition 33.33.1. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. In this situation the Euler characteristic of $\mathcal{F}$ is the integer

$$\chi(X, \mathcal{F}) = \sum\nolimits_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

For justification of the formula see below.

In the situation of the definition only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Schemes, Lemma 30.4.5) and each of these spaces is finite dimensional (Cohomology of Schemes, Lemma 30.19.2). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair $(X, \mathcal{F})$.

08AA Lemma 33.33.2. Let k be a field. Let X be a proper scheme over k. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent modules on X. Then

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \ldots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \ldots$$

This immediately implies the lemma. $\square$

0AYT Lemma 33.33.3. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent sheaf with $\dim(\text{Supp}(\mathcal{F})) \leq 0$. Then

(1) $\mathcal{F}$ is generated by global sections,
(2) $H^0(X, \mathcal{F}) = \bigoplus_{x \in \text{Supp}(\mathcal{F})} \mathcal{F}_x$,
(3) $H^i(X, \mathcal{F}) = 0$ for $i > 0$,
(4) $\chi(X, \mathcal{F}) = \dim_k H^0(X, \mathcal{F})$, and
(5) $\chi(X, \mathcal{F} \otimes \mathcal{E}) = n\chi(X, \mathcal{F})$ for every locally free module $\mathcal{E}$ of rank n.

Proof. By Cohomology of Schemes, Lemma 30.9.7 we see that $\mathcal{F} = i_*\mathcal{G}$ where $i : Z \to X$ is the inclusion of the scheme theoretic support of $\mathcal{F}$ and where $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module. By definition of the scheme theoretic support the underlying topological space of Z is $\text{Supp}(\mathcal{F})$. Since the dimension of Z is 0, we see Z is affine (Properties, Lemma 28.10.5). Hence $\mathcal{G}$ is globally generated and the higher cohomology groups of $\mathcal{G}$ are zero (Cohomology of Schemes, Lemma 30.2.2). In fact, by Lemma 33.20.2 the scheme Z is a finite disjoint union of spectra of local Artinian rings. Thus correspondingly $H^0(Z, \mathcal{G}) = \bigoplus_{z \in Z} \mathcal{G}_z$. The cohomologies of $\mathcal{F}$ and $\mathcal{G}$ agree by Cohomology of Schemes, Lemma 30.2.4. Thus $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and $H^0(X, \mathcal{F}) = H^0(Z, \mathcal{G})$. In particular we have (3) is true. For $z \in Z$ corresponding to $x \in \text{Supp}(\mathcal{F})$ we have $\mathcal{G}_z = (i_*\mathcal{G})_x = \mathcal{F}_x$. We conclude that (2) holds. Of course (2) implies (1). We have (4) by definition of the Euler characteristic $\chi(X, \mathcal{F})$ and (3). By the projection formula (Cohomology, Lemma 20.54.2) we have

$$i_*(\mathcal{G} \otimes i^*\mathcal{E}) = \mathcal{F} \otimes \mathcal{E}.$$

Since Z has dimension 0 the locally free sheaf $i^*\mathcal{E}$ is isomorphic to $\mathcal{O}_Z^{\oplus n}$ and arguing as above we see that (5) holds. $\square$

08AB Lemma 33.33.4. Let k'/k be an extension of fields. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $X_{k'}$. Then $\chi(X, \mathcal{F}) = \chi(X', \mathcal{F}')$.

Proof. This is true because

$$H^i(X_{k'}, \mathcal{F}') = H^i(X, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 30.5.2. □

0BEK Lemma 33.33.5. Let k be a field. Let $f : Y \to X$ be a morphism of proper schemes over k. Let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. Then

$$\chi(Y, \mathcal{G}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{G})$$

Proof. The formula makes sense: the sheaves $R^i f_* \mathcal{G}$ are coherent and only a finite number of them are nonzero, see Cohomology of Schemes, Proposition 30.19.1 and Lemma 30.4.5. By Cohomology, Lemma 20.13.4 there is a spectral sequence with

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{G})$$

converging to $H^{p+q}(Y, \mathcal{G})$. By finiteness of cohomology on X we see that only a finite number of $E_2^{p,q}$ are nonzero and each $E_2^{p,q}$ is a finite dimensional vector space. It follows that the same is true for $E_r^{p,q}$ for $r \geq 2$ and that

$$\sum (-1)^{p+q} \dim_k E_r^{p,q}$$

is independent of r. Since for r large enough we have $E_r^{p,q} = E_\infty^{p,q}$ and since convergence means there is a filtration on $H^n(Y, \mathcal{G})$ whose graded pieces are $E_\infty^{p,q}$ with $p + q = n$ (this is the meaning of convergence of the spectral sequence), we conclude. Compare also with the more general Homology, Lemma 12.24.12. □

33.34. Projective space

0B2N Some results on projective space over a field.

0B2P Lemma 33.34.1. Let k be a field and $n \geq 0$. Then $\mathbf{P}^n_k$ is a smooth projective variety of dimension n over k.

Proof. Omitted. □

0B2Q Lemma 33.34.2. Let k be a field and $n \geq 0$. Let $X, Y \subset \mathbf{A}^n_k$ be closed subsets. Assume that X and Y are equidimensional, $\dim(X) = r$ and $\dim(Y) = s$. Then every irreducible component of $X \cap Y$ has dimension $\geq r + s - n$.

Proof. Consider the closed subscheme $X \times Y \subset \mathbf{A}^{2n}_k$ where we use coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then $X \cap Y = X \times Y \cap V(x_1 - y_1, \ldots, x_n - y_n)$. Let $t \in X \cap Y \subset X \times Y$ be a closed point. By Lemma 33.20.5 we have $\dim_t(X \times Y) = \dim(X) + \dim(Y)$. Thus $\dim(\mathcal{O}_{X \times Y, t}) = r + s$ by Lemma 33.20.3. By Algebra, Lemma 10.60.13 we conclude that

$$\dim(\mathcal{O}_{X \cap Y, t}) = \dim(\mathcal{O}_{X \times Y, t}/(x_1 - y_1, \ldots, x_n - y_n)) \geq r + s - n$$

This implies the result by Lemma 33.20.3. □

0B2R Lemma 33.34.3. Let k be a field and $n \geq 0$. Let $X, Y \subset \mathbf{P}^n_k$ be nonempty closed subsets. If $\dim(X) = r$ and $\dim(Y) = s$ and $r + s \geq n$, then $X \cap Y$ is nonempty and $\dim(X \cap Y) \geq r + s - n$.

Proof. Write $\mathbf{A}^n = \mathrm{Spec}(k[x_0, \ldots, x_n])$ and $\mathbf{P}^n = \mathrm{Proj}(k[T_0, \ldots, T_n])$. Consider the morphism $\pi : \mathbf{A}^{n+1} \setminus \{0\} \to \mathbf{P}^n$ which sends $(x_0, \ldots, x_n)$ to the point $[x_0 : \ldots : x_n]$. More precisely, it is the morphism associated to the pair $(\mathcal{O}_{\mathbf{A}^{n+1} \setminus \{0\}}, (x_0, \ldots, x_n))$, see Constructions, Lemma 27.13.1. Over the standard affine open $D_+(T_i)$ we get the morphism associated to the ring map

$$k\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] \longrightarrow k\left[T_0, \ldots, T_n, \frac{1}{T_i}\right] \cong k\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\left[T_i, \frac{1}{T_i}\right]$$

which is surjective and smooth of relative dimension 1 with irreducible fibres (details omitted). Hence $\pi^{-1}(X)$ and $\pi^{-1}(Y)$ are nonempty closed subsets of dimension $r+1$ and $s+1$. Choose an irreducible component $V \subset \pi^{-1}(X)$ of dimension $r+1$ and an irreducible component $W \subset \pi^{-1}(Y)$ of dimension $s+1$. Observe that this implies V and W contain every fibre of π they meet (since π has irreducible fibres of dimension 1 and since Lemma 33.20.4 says the fibres of $V \to \pi(V)$ and $W \to \pi(W)$ have dimension ≥ 1). Let $\overline{V}$ and $\overline{W}$ be the closure of V and W in $\mathbf{A}^{n+1}$. Since $0 \in \mathbf{A}^{n+1}$ is in the closure of every fibre of π we see that $0 \in \overline{V} \cap \overline{W}$. By Lemma 33.34.2 we have $\dim(\overline{V} \cap \overline{W}) \geq r+s-n+1$. Arguing as above using Lemma 33.20.4 again, we conclude that $\pi(V \cap W) \subset X \cap Y$ has dimension at least $r+s-n$ as desired. $\square$

0BXU Lemma 33.34.4. Let k be a field. Let $Z \subset \mathbf{P}^n_k$ be a closed subscheme which has no embedded points such that every irreducible component of Z has dimension $n-1$. Then the ideal $I(Z) \subset k[T_0, \ldots, T_n]$ corresponding to Z is principal.

Proof. This is a special case of Divisors, Lemma 31.31.3. $\square$

33.35. Coherent sheaves on projective space

089X In this section we prove some results on the cohomology of coherent sheaves on $\mathbf{P}^n$ over a field which can be found in [Mum66]. These will be useful later when discussing Quot and Hilbert schemes.

089Y 33.35.1. Preliminaries. Let k be a field, $n \geq 1$, $d \geq 1$, and let $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(d))$ be a nonzero section. In this section we will write $\mathcal{O}(d)$ for the dth twist of the structure sheaf on projective space (Constructions, Definitions 27.10.1 and 27.13.2). Since $\mathbf{P}^n_k$ is a variety this section is regular, hence s is a regular section of $\mathcal{O}(d)$ and defines an effective Cartier divisor $H = Z(s) \subset \mathbf{P}^n_k$, see Divisors, Section 31.13. Such a divisor H is called a hypersurface and if $d = 1$ it is called a hyperplane.

089Z Lemma 33.35.2. Let k be a field. Let $n \geq 1$. Let $i : H \to \mathbf{P}^n_k$ be a hyperplane. Then there exists an isomorphism

$$\varphi : \mathbf{P}^{n-1}_k \longrightarrow H$$

such that $i^*\mathcal{O}(1)$ pulls back to $\mathcal{O}(1)$.

Proof. We have $\mathbf{P}^n_k = \text{Proj}(k[T_0, \ldots, T_n])$. The section s corresponds to a homogeneous form in $T_0, \ldots, T_n$ of degree 1, see Cohomology of Schemes, Section 30.8. Say $s = \sum a_i T_i$. Constructions, Lemma 27.13.7 gives that $H = \text{Proj}(k[T_0, \ldots, T_n]/I)$ for the graded ideal I defined by setting I_d equal to the kernel of the map $\Gamma(\mathbf{P}^n_k, \mathcal{O}(d)) \to \Gamma(H, i^*\mathcal{O}(d))$. By our construction of $Z(s)$ in Divisors, Definition 31.14.8 we see that on $D_+(T_j)$ the ideal of H is generated by $\sum a_i T_i/T_j$ in the polynomial ring $k[T_0/T_j, \ldots, T_n/T_j]$. Thus it is clear that I is the ideal generated by $\sum a_i T_i$. Note that

$$k[T_0, \ldots, T_n]/I = k[T_0, \ldots, T_n]/(\sum a_i T_i) \cong k[S_0, \ldots, S_{n-1}]$$

as graded rings. For example, if $a_n \neq 0$, then mapping S_i equal to the class of T_i works. We obtain the desired isomorphism by functoriality of Proj. Equality of twists of structure sheaves follows for example from Constructions, Lemma 27.11.5. □

08A0 Lemma 33.35.3. Let k be an infinite field. Let $n \geq 1$. Let $\mathcal{F}$ be a coherent module on $\mathbf{P}^n_k$. Then there exist a nonzero section $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ and a short exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to i_*\mathcal{G} \to 0$$

where $i : H \to \mathbf{P}^n_k$ is the hyperplane H associated to s and $\mathcal{G} = i^*\mathcal{F}$.

Proof. The map $\mathcal{F}(-1) \to \mathcal{F}$ comes from Constructions, Equation (27.10.1.2) with $n = 1$, $m = -1$ and the section s of $\mathcal{O}(1)$. Let's work out what this map looks like if we restrict it to $D_+(T_0)$. Write $D_+(T_0) = \mathrm{Spec}(k[x_1, \ldots, x_n])$ with $x_i = T_i/T_0$. Identify $\mathcal{O}(1)|_{D_+(T_0)}$ with $\mathcal{O}$ using the section T_0. Hence if $s = \sum a_iT_i$ then $s|_{D_+(T_0)} = a_0 + \sum a_ix_i$ with the identification chosen above. Furthermore, suppose $\mathcal{F}|_{D_+(T_0)}$ corresponds to the finite $k[x_1, \ldots, x_n]$-module M. Via the identification $\mathcal{F}(-1) = \mathcal{F} \otimes \mathcal{O}(-1)$ and our chosen trivialization of $\mathcal{O}(1)$ we see that $\mathcal{F}(-1)$ corresponds to M as well. Thus restricting $\mathcal{F}(-1) \to \mathcal{F}$ to $D_+(T_0)$ gives the map

$$M \xrightarrow{a_0 + \sum a_ix_i} M$$

To see that the arrow is injective, it suffices to pick $a_0 + \sum a_ix_i$ outside any of the associated primes of M, see Algebra, Lemma 10.63.9. By Algebra, Lemma 10.63.5 the set $\mathrm{Ass}(M)$ of associated primes of M is finite. Note that for $\mathfrak{p} \in \mathrm{Ass}(M)$ the intersection $\mathfrak{p} \cap \{a_0 + \sum a_ix_i\}$ is a proper k-subvector space. We conclude that there is a finite family of proper sub vector spaces $V_1, \ldots, V_m \subset \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ such that if we take s outside of $\bigcup V_i$, then multiplication by s is injective over $D_+(T_0)$. Similarly for the restriction to $D_+(T_j)$ for $j = 1, \ldots, n$. Since k is infinite, a finite union of proper sub vector spaces is never equal to the whole space, hence we may choose s such that the map is injective. The cokernel of $\mathcal{F}(-1) \to \mathcal{F}$ is annihilated by $\mathrm{Im}(s : \mathcal{O}(-1) \to \mathcal{O})$ which is the ideal sheaf of H by Divisors, Definition 31.14.8. Hence we obtain $\mathcal{G}$ on H using Cohomology of Schemes, Lemma 30.9.8. □

08A1 Remark 33.35.4. Let k be an infinite field. Let $n \geq 1$. Given a finite number of coherent modules $\mathcal{F}_i$ on $\mathbf{P}^n_k$ we can choose a single $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ such that the statement of Lemma 33.35.3 works for each of them. To prove this, just apply the lemma to $\bigoplus \mathcal{F}_i$.

0EGK Remark 33.35.5. In the situation of Lemmas 33.35.2 and 33.35.3 we have $H \cong \mathbf{P}^{n-1}_k$ with Serre twists $\mathcal{O}_H(d) = i^*\mathcal{O}_{\mathbf{P}^n_k}(d)$. For every $d \in \mathbf{Z}$ we have a short exact sequence

$$0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*(\mathcal{G}(d)) \to 0$$

Namely, tensoring by $\mathcal{O}_{\mathbf{P}^n_k}(d)$ is an exact functor and by the projection formula (Cohomology, Lemma 20.54.2) we have $i_*(\mathcal{G}(d)) = i_*\mathcal{G} \otimes \mathcal{O}_{\mathbf{P}^n_k}(d)$. We obtain corresponding long exact sequences

$$H^i(\mathbf{P}^n_k, \mathcal{F}(d-1)) \to H^i(\mathbf{P}^n_k, \mathcal{F}(d)) \to H^i(H, \mathcal{G}(d)) \to H^{i+1}(\mathbf{P}^n_k, \mathcal{F}(d-1))$$

This follows from the above and the fact that we have $H^i(\mathbf{P}^n_k, i_*\mathcal{G}(d)) = H^i(H, \mathcal{G}(d))$ by Cohomology of Schemes, Lemma 30.2.4 (closed immersions are affine).

08A2 33.35.6. Regularity. Here is the definition.

08A3 Definition 33.35.7. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. We say $\mathcal{F}$ is m-regular if

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) = 0$$

for $i = 1, \ldots, n$.

Note that $\mathcal{F} = \mathcal{O}(d)$ is m-regular if and only if $d \geq -m$. This follows from the computation of cohomology groups in Cohomology of Schemes, Equation (30.8.1.1). Namely, we see that $H^n(\mathbf{P}^n_k, \mathcal{O}(d)) = 0$ if and only if $d \geq -n$.

08A4 Lemma 33.35.8. Let k'/k be an extension of fields. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $\mathbf{P}^n_{k'}$. Then $\mathcal{F}$ is m-regular if and only if $\mathcal{F}'$ is m-regular.

Proof. This is true because

$$H^i(\mathbf{P}^n_{k'}, \mathcal{F}') = H^i(\mathbf{P}^n_k, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 30.5.2. □

08A5 Lemma 33.35.9. In the situation of Lemma 33.35.3, if $\mathcal{F}$ is m-regular, then $\mathcal{G}$ is m-regular on $H \cong \mathbf{P}^{n-1}_k$.

Proof. Recall that $H^i(\mathbf{P}^n_k, i_*\mathcal{G}) = H^i(H, \mathcal{G})$ by Cohomology of Schemes, Lemma 30.2.4. Hence we see that for $i \geq 1$ we get

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) \to H^i(H, \mathcal{G}(m-i)) \to H^{i+1}(\mathbf{P}^n_k, \mathcal{F}(m-1-i))$$

by Remark 33.35.5. The lemma follows. □

08A6 Lemma 33.35.10. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is $(m+1)$-regular.

Proof. We prove this by induction on n. If $n = 0$ every sheaf is m-regular for all m and there is nothing to prove. By Lemma 33.35.8 we may replace k by an infinite overfield and assume k is infinite. Thus we may apply Lemma 33.35.3. By Lemma 33.35.9 we know that $\mathcal{G}$ is m-regular. By induction on n we see that $\mathcal{G}$ is $(m+1)$-regular. Considering the long exact cohomology sequence associated to the sequence

$$0 \to \mathcal{F}(m-i) \to \mathcal{F}(m+1-i) \to i_*\mathcal{G}(m+1-i) \to 0$$

as in Remark 33.35.5 the reader easily deduces for $i \geq 1$ the vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m+1-i))$ from the (known) vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m-i))$ and $H^i(\mathbf{P}^n_k, \mathcal{G}(m+1-i))$. □

08A7 Lemma 33.35.11. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then the multiplication map

$$H^0(\mathbf{P}^n_k, \mathcal{F}(m)) \otimes_k H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \longrightarrow H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$$

is surjective.

Proof. Let k'/k be an extension of fields. Let $\mathcal{F}'$ be as in Lemma 33.35.8. By Cohomology of Schemes, Lemma 30.5.2 the base change of the linear map of the lemma to k' is the same linear map for the sheaf $\mathcal{F}'$. Since $k \to k'$ is faithfully flat it suffices to prove the lemma over k', i.e., we may assume k is infinite.

Assume k is infinite. We prove the lemma by induction on n. The case $n = 0$ is trivial as $\mathcal{O}(1) \cong \mathcal{O}$ is generated by T_0. For $n > 0$ apply Lemma 33.35.3 and tensor the sequence by $\mathcal{O}(m+1)$ to get

$$0 \to \mathcal{F}(m) \xrightarrow{s} \mathcal{F}(m+1) \to i_*\mathcal{G}(m+1) \to 0$$

see Remark 33.35.5. Let $t \in H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$. By induction the image $\overline{t} \in H^0(H, \mathcal{G}(m+1))$ is the image of $\sum g_i \otimes \overline{s}_i$ with $\overline{s}_i \in \Gamma(H, \mathcal{O}(1))$ and $g_i \in H^0(H, \mathcal{G}(m))$. Since $\mathcal{F}$ is m-regular we have $H^1(\mathbf{P}^n_k, \mathcal{F}(m-1)) = 0$, hence long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{F}(m-1) \xrightarrow{s} \mathcal{F}(m) \to i_*\mathcal{G}(m) \to 0$$

shows we can lift g_i to $f_i \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$. We can also lift $\overline{s}_i$ to $s_i \in H^0(\mathbf{P}^n_k, \mathcal{O}(1))$ (see proof of Lemma 33.35.2 for example). After substracting the image of $\sum f_i \otimes s_i$ from t we see that we may assume $\overline{t} = 0$. But this exactly means that t is the image of $f \otimes s$ for some $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$ as desired. □

08A8 Lemma 33.35.12. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}(m)$ is globally generated.

Proof. For all $d \gg 0$ the sheaf $\mathcal{F}(d)$ is globally generated. This follows for example from the first part of Cohomology of Schemes, Lemma 30.14.1. Pick $d \geq m$ such that $\mathcal{F}(d)$ is globally generated. Choose a basis $f_1, \ldots, f_r \in H^0(\mathbf{P}^n_k, \mathcal{F})$. By Lemma 33.35.11 every element $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(d))$ can be written as $f = \sum P_i f_i$ for some $P_i \in k[T_0, \ldots, T_n]$ homogeneous of degree $d-m$. Since the sections f generate $\mathcal{F}(d)$ it follows that the sections f_i generate $\mathcal{F}(m)$. □

08A9 33.35.13. Hilbert polynomials. The following lemma will be made obsolete by the more general Lemma 33.45.1.

08AC Lemma 33.35.14. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function

$$d \longmapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$$

is a polynomial.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and $\mathcal{F}(d) = \mathcal{F}$. Hence in this case the function is constant, i.e., a polynomial of degree 0. Assume $n > 0$. By Lemma 33.33.4 we may assume k is infinite. Apply Lemma 33.35.3. Applying Lemma 33.33.2 to the twisted sequences $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*\mathcal{G}(d) \to 0$ we obtain

$$\chi(\mathbf{P}^n_k, \mathcal{F}(d)) - \chi(\mathbf{P}^n_k, \mathcal{F}(d-1)) = \chi(H, \mathcal{G}(d))$$

See Remark 33.35.5. Since $H \cong \mathbf{P}^{n-1}_k$ by induction the right hand side is a polynomial. The lemma is finished by noting that any function $f : \mathbf{Z} \to \mathbf{Z}$ with the property that the map $d \mapsto f(d) - f(d-1)$ is a polynomial, is itself a polynomial. We omit the proof of this fact (hint: compare with Algebra, Lemma 10.58.5). □

08AD Definition 33.35.15. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function $d \mapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$ is called the Hilbert polynomial of $\mathcal{F}$.

The Hilbert polynomial has coefficients in $\mathbf{Q}$ and not in general in $\mathbf{Z}$. For example the Hilbert polynomial of $\mathcal{O}_{\mathbf{P}^n_k}$ is

$$d \longmapsto \binom{d+n}{n} = \frac{d^n}{n!} + \ldots$$

This follows from the following lemma and the fact that

$$H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(d)) = k[T_0, \ldots, T_n]_d$$

(degree d part) whose dimension over k is $\binom{d+n}{n}$.

08AE Lemma 33.35.16. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$ with Hilbert polynomial $P \in \mathbf{Q}[t]$. Then

$$P(d) = \dim_k H^0(\mathbf{P}^n_k, \mathcal{F}(d))$$

for all $d \gg 0$.

Proof. This follows from the vanishing of cohomology of high enough twists of $\mathcal{F}$. See Cohomology of Schemes, Lemma 30.14.1. $\square$

08AF 33.35.17. Boundedness of quotients. In this subsection we bound the regularity of quotients of a given coherent sheaf on $\mathbf{P}^n$ in terms of the Hilbert polynomial.

08AG Lemma 33.35.18. Let k be a field. Let $n \geq 0$. Let $r \geq 1$. Let $P \in \mathbf{Q}[t]$. There exists an integer m depending on n, r, and P with the following property: if

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus r} \to \mathcal{F} \to 0$$

is a short exact sequence of coherent sheaves on $\mathbf{P}^n_k$ and $\mathcal{F}$ has Hilbert polynomial P, then $\mathcal{K}$ is m-regular.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \text{Spec}(k)$ and any coherent module is 0-regular and any surjective map is surjective on global sections. Assume $n > 0$. Consider an exact sequence as in the lemma. Let $P' \in \mathbf{Q}[t]$ be the polynomial $P'(t) = P(t) - P(t-1)$. Let m' be the integer which works for $n-1$, r, and P'. By Lemmas 33.35.8 and 33.33.4 we may replace k by a field extension, hence we may assume k is infinite. Apply Lemma 33.35.3 to the coherent sheaf $\mathcal{F}$. The Hilbert polynomial of $\mathcal{F}' = i^*\mathcal{F}$ is P' (see proof of Lemma 33.35.14). Since i^* is right exact we see that $\mathcal{F}'$ is a quotient of $\mathcal{O}_H^{\oplus r} = i^*\mathcal{O}^{\oplus r}$. Thus the induction hypothesis applies to $\mathcal{F}'$ on $H \cong \mathbf{P}^{n-1}_k$ (Lemma 33.35.2). Note that the map $\mathcal{K}(-1) \to \mathcal{K}$ is injective as $\mathcal{K} \subset \mathcal{O}^{\oplus r}$ and has cokernel $i_*\mathcal{H}$ where $\mathcal{H} = i^*\mathcal{K}$. By the snake lemma (Homology, Lemma 12.5.17) we obtain a commutative diagram with exact columns and rows

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{K}(-1) \arrow[r] \arrow[d] & \mathcal{O}^{\oplus r}(-1) \arrow[r] \arrow[d] & \mathcal{F}(-1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{O}^{\oplus r} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & i_*\mathcal{H} \arrow[r] \arrow[d] & i_*\mathcal{O}_H^{\oplus r} \arrow[r] \arrow[d] & i_*\mathcal{F}' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Thus the induction hypothesis applies to the exact sequence $0 \to \mathcal{H} \to \mathcal{O}_H^{\oplus r} \to \mathcal{F}' \to 0$ on $H \cong \mathbf{P}^{n-1}_k$ (Lemma 33.35.2) and $\mathcal{H}$ is m'-regular. Recall that this implies that $\mathcal{H}$ is d-regular for all $d \geq m'$ (Lemma 33.35.10).

Let $i \geq 2$ and $d \geq m'$. It follows from the long exact cohomology sequence associated to the left column of the diagram above and the vanishing of $H^{i-1}(H, \mathcal{H}(d))$ that the map

$$H^i(\mathbf{P}^n_k, \mathcal{K}(d-1)) \longrightarrow H^i(\mathbf{P}^n_k, \mathcal{K}(d))$$

is injective. As these groups are zero for $d \gg 0$ (Cohomology of Schemes, Lemma 30.14.1) we conclude $H^i(\mathbf{P}^n_k, \mathcal{K}(d))$ are zero for all $d \geq m'$ and $i \geq 2$.

We still have to control H^1. First we observe that all the maps

$$H^1(\mathbf{P}^n_k, \mathcal{K}(m'-1)) \to H^1(\mathbf{P}^n_k, \mathcal{K}(m')) \to H^1(\mathbf{P}^n_k, \mathcal{K}(m'+1)) \to \ldots$$

are surjective by the vanishing of $H^1(H, \mathcal{H}(d))$ for $d \geq m'$. Suppose $d > m'$ is such that

$$H^1(\mathbf{P}^n_k, \mathcal{K}(d-1)) \longrightarrow H^1(\mathbf{P}^n_k, \mathcal{K}(d))$$

is injective. Then $H^0(\mathbf{P}^n_k, \mathcal{K}(d)) \to H^0(H, \mathcal{H}(d))$ is surjective. Consider the commutative diagram

$$\begin{tikzcd} H^0(\mathbf{P}^n_k, \mathcal{K}(d)) \otimes_k H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \arrow[r] \arrow[d] & H^0(\mathbf{P}^n_k, \mathcal{K}(d+1)) \arrow[d] \\ H^0(H, \mathcal{H}(d)) \otimes_k H^0(H, \mathcal{O}_H(1)) \arrow[r] & H^0(H, \mathcal{H}(d+1)) \end{tikzcd}$$

By Lemma 33.35.11 we see that the bottom horizontal arrow is surjective. Hence the right vertical arrow is surjective. We conclude that

$$H^1(\mathbf{P}^n_k, \mathcal{K}(d)) \longrightarrow H^1(\mathbf{P}^n_k, \mathcal{K}(d+1))$$

is injective. By induction we see that

$$H^1(\mathbf{P}^n_k, \mathcal{K}(d-1)) \to H^1(\mathbf{P}^n_k, \mathcal{K}(d)) \to H^1(\mathbf{P}^n_k, \mathcal{K}(d+1)) \to \ldots$$

are all injective and we conclude that $H^1(\mathbf{P}^n_k, \mathcal{K}(d-1)) = 0$ because of the eventual vanishing of these groups. Thus the dimensions of the groups $H^1(\mathbf{P}^n_k, \mathcal{K}(d))$ for $d \geq m'$ are strictly decreasing until they become zero. It follows that the regularity of $\mathcal{K}$ is bounded by $m' + \dim_k H^1(\mathbf{P}^n_k, \mathcal{K}(m'))$. On the other hand, by the vanishing of the higher cohomology groups we have

$$\dim_k H^1(\mathbf{P}^n_k, \mathcal{K}(m')) = -\chi(\mathbf{P}^n_k, \mathcal{K}(m')) + \dim_k H^0(\mathbf{P}^n_k, \mathcal{K}(m'))$$

Note that the H^0 has dimension bounded by the dimension of $H^0(\mathbf{P}^n_k, \mathcal{O}^{\oplus r}(m'))$ which is at most $r\binom{n+m'}{n}$ if $m' > 0$ and zero if not. Finally, the term $\chi(\mathbf{P}^n_k, \mathcal{K}(m'))$ is equal to $r\binom{n+m'}{n} - P(m')$. This gives a bound of the desired type finishing the proof of the lemma. □

33.36. Frobenii

0CC6 Let p be a prime number. If X is a scheme, then we say "X has characteristic p", or "X is of characteristic p", or "X is in characteristic p" if p is zero in $\mathcal{O}_X$.

03SM Definition 33.36.1. Let p be a prime number. Let X be a scheme in characteristic p. The absolute frobenius of X is the morphism $F_X : X \to X$ given by the identity on the underlying topological space and with $F_X^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$.

This makes sense because for any ring A of characteristic p the map $F_A : A \to A$, $a \mapsto a^p$ is a ring endomorphism which induces the identity on $\text{Spec}(A)$. Moreover, if A is local, then F_A is a local homomorphism. In this way we see that the absolute frobenius of X is an endomorphism of X in the category of schemes. It turns out that the absolute frobenius defines a self map of the identity functor on the category of schemes in characteristic p.

0CC7 Lemma 33.36.2. Let $p > 0$ be a prime number. Let $f : X \to Y$ be a morphism of schemes in characteristic p. Then the diagram

$$\begin{tikzcd} X \arrow[r, "F_X"'] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

commutes.

Proof. This follows from the following trivial algebraic fact: if $\varphi : A \to B$ is a homomorphism of rings of characteristic p, then $\varphi(a^p) = \varphi(a)^p$. □

0CC8 Lemma 33.36.3. Let $p > 0$ be a prime number. Let X be a scheme in characteristic p. Then the absolute frobenius $F_X : X \to X$ is a universal homeomorphism, is integral, and induces purely inseparable residue field extensions.

Proof. This follows from the corresponding results for the frobenius endomorphism $F_A : A \to A$ of a ring A of characteristic $p > 0$. See the discussion in Algebra, Section 10.46, for example Lemma 10.46.7. □

If we are working with schemes over a fixed base, then there is a relative version of the frobenius morphism.

0CC9 Definition 33.36.4. Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. We define

$$X^{(p)} = X^{(p/S)} = X \times_{S, F_S} S$$

viewed as a scheme over S. Applying Lemma 33.36.2 we see there is a unique morphism $F_{X/S} : X \longrightarrow X^{(p)}$ over S fitting into the commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"'] \arrow[rr, bend left, "F_X"] \arrow[dr] & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

where the right square is cartesian. The morphism $F_{X/S}$ is called the relative Frobenius morphism of X/S.

Observe that $X \mapsto X^{(p)}$ is a functor; it is the base change functor for the absolute frobenius morphism $F_S : S \to S$. We have the same lemmas as before regarding the relative Frobenius morphism.

0CCA Lemma 33.36.5. Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let $f : X \to Y$ be a morphism of schemes over S . Then the diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"'] \arrow[d, "f"'] & X^{(p)} \arrow[d, "f^{(p)}"] \\ Y \arrow[r, "F_{Y/S}"] & Y^{(p)} \end{tikzcd}$$

commutes.

Proof. This follows from Lemma 33.36.2 and the definitions. □

0CCB Lemma 33.36.6. Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. Then the relative frobenius $F_{X/S} : X \to X^{(p)}$ is a universal homeomorphism, is integral, and induces purely inseparable residue field extensions.

Proof. By Lemma 33.36.3 the morphisms $F_X : X \to X$ and the base change $h : X^{(p)} \to X$ of F_S are universal homeomorphisms. Since $h \circ F_{X/S} = F_X$ we conclude that $F_{X/S}$ is a universal homeomorphism (Morphisms, Lemma 29.45.8). By Morphisms, Lemmas 29.45.5 and 29.10.2 we conclude that $F_{X/S}$ has the other properties as well. □

0CCC Lemma 33.36.7. Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. Then $\Omega_{X/S} = \Omega_{X/X^{(p)}}$.

Proof. This translates into the following algebra fact. Let $A \to B$ be a homomorphism of rings of characteristic p. Set $B' = B \otimes_{A,F_A} A$ and consider the ring map $F_{B/A} : B' \to B$, $b \otimes a \mapsto b^p a$. Then our assertion is that $\Omega_{B/A} = \Omega_{B/B'}$. This is true because $\mathrm{d}(b^p a) = 0$ if $\mathrm{d} : B \to \Omega_{B/A}$ is the universal derivation and hence d is a B'-derivation. □

0CCD Lemma 33.36.8. Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. If $X \to S$ is locally of finite type, then $F_{X/S}$ is finite.

Proof. This translates into the following algebra fact. Let $A \to B$ be a finite type homomorphism of rings of characteristic p. Set $B' = B \otimes_{A,F_A} A$ and consider the ring map $F_{B/A} : B' \to B$, $b \otimes a \mapsto b^p a$. Then our assertion is that $F_{B/A}$ is finite.

Namely, if $x_1, \ldots, x_n \in B$ are generators over A, then x_i is integral over B' because $x_i^p = F_{B/A}(x_i \otimes 1)$. Hence $F_{B/A} : B' \to B$ is finite by Algebra, Lemma 10.36.5. $\square$

0CCE Lemma 33.36.9. Let k be a field of characteristic $p > 0$. Let X be a scheme over k. Then X is geometrically reduced if and only if $X^{(p)}$ is reduced.

Proof. Consider the absolute frobenius $F_k : k \to k$. Then $F_k(k) = k^p$ in other words, $F_k : k \to k$ is isomorphic to the embedding of k into $k^{1/p}$. Thus the lemma follows from Lemma 33.6.4. $\square$

0CCF Lemma 33.36.10. Let k be a field of characteristic $p > 0$. Let X be a variety over k. The following are equivalent

(1) $X^{(p)}$ is reduced,
(2) X is geometrically reduced,
(3) there is a nonempty open $U \subset X$ smooth over k.

In this case $X^{(p)}$ is a variety over k and $F_{X/k} : X \to X^{(p)}$ is a finite dominant morphism of degree $p^{\dim(X)}$.

Proof. We have seen the equivalence of (1) and (2) in Lemma 33.36.9. We have seen that (2) implies (3) in Lemma 33.25.7. If (3) holds, then U is geometrically reduced (see for example Lemma 33.12.6) and hence X is geometrically reduced by Lemma 33.6.8. In this way we see that (1), (2), and (3) are equivalent.

Assume (1), (2), and (3) hold. Since $F_{X/k}$ is a homeomorphism (Lemma 33.36.6) we see that $X^{(p)}$ is a variety. Then $F_{X/k}$ is finite by Lemma 33.36.8. It is dominant as it is surjective. To compute the degree (Morphisms, Definition 29.51.8) it suffices to compute the degree of $F_{U/k} : U \to U^{(p)}$ (as $F_{U/k} = F_{X/k}|_U$ by Lemma 33.36.5). After shrinking U a bit we may assume there exists an étale morphism $h : U \to \mathbf{A}^n_k$, see Morphisms, Lemma 29.36.20. Of course $n = \dim(U)$ because $\mathbf{A}^n_k \to \mathrm{Spec}(k)$ is smooth of relative dimension n, the étale morphism h is smooth of relative dimension 0, and $U \to \mathrm{Spec}(k)$ is smooth of relative dimension $\dim(U)$ and relative dimensions add up correctly (Morphisms, Lemma 29.29.3). Observe that h is a generically finite dominant morphism of varieties, and hence $\deg(h)$ is defined. By Lemma 33.36.5 we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/k}"'] \arrow[d, "h"'] & X^{(p)} \arrow[d, "h^{(p)}"] \\ \mathbf{A}^n_k \arrow[r, "F_{\mathbf{A}^n_k/k}"] & (\mathbf{A}^n_k)^{(p)} \end{tikzcd}$$

Since $h^{(p)}$ is a base change of h it is étale as well and it follows that $h^{(p)}$ is a generically finite dominant morphism of varieties as well. The degree of $h^{(p)}$ is the degree of the extension $k(X^{(p)})/k((\mathbf{A}^n_k)^{(p)})$ which is the same as the degree of the extension $k(X)/k(\mathbf{A}^n_k)$ because $h^{(p)}$ is the base change of h (small detail omitted). By multiplicativity of degrees (Morphisms, Lemma 29.51.9) it suffices to show that the degree of $F_{\mathbf{A}^n_k/k}$ is p^n. To see this observe that $(\mathbf{A}^n_k)^{(p)} = \mathbf{A}^n_k$ and that $F_{\mathbf{A}^n_k/k}$ is given by the map sending the coordinates to their pth powers. $\square$

0CCG Remark 33.36.11. Let $p > 0$ be a prime number. Let S be a scheme in characteristic p. Let X be a scheme over S. For $n \geq 1$

$$X^{(p^n)} = X^{(p^n/S)} = X \times_{S, F_S^n} S$$

viewed as a scheme over S. Observe that $X \mapsto X^{(p^n)}$ is a functor. Applying Lemma 33.36.2 we see $F_{X/S,n} = (F_X^n, \text{id}_S) : X \longrightarrow X^{(p^n)}$ is a morphism over S fitting into the commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/S,n}"'] \arrow[rr, bend left, "F_X^n"] \arrow[dr] & X^{(p^n)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S^n"] & S \end{tikzcd}$$

where the right square is cartesian. The morphism $F_{X/S,n}$ is sometimes called the n-fold relative Frobenius morphism of X/S. This makes sense because we have the formula

$$F_{X/S,n} = F_{X^{(p^{n-1})}/S} \circ \ldots \circ F_{X^{(p)}/S} \circ F_{X/S}$$

which shows that $F_{X/S,n}$ is the composition of n relative Frobenii. Since we have

$$F_{X^{(p^m)}/S} = F^{(p)}_{X^{(p^{m-1})}/S} = \ldots = F^{(p^m)}_{X/S}$$

(details omitted) we get also that

$$F_{X/S,n} = F^{(p^{n-1})}_{X/S} \circ \ldots \circ F^{(p)}_{X/S} \circ F_{X/S}$$

33.37. Glueing dimension one rings

09MX This section contains some algebraic preliminaries to proving that a finite set of codimension 1 points of a separated scheme is contained in an affine open.

09MY Situation 33.37.1. Here we are given a commutative diagram of rings

$$\begin{tikzcd} A \arrow[r] & K \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where K is a field and A, B are subrings of K with fraction field K. Finally, $R = A \times_K B = A \cap B$.

09MZ Lemma 33.37.2. In Situation 33.37.1 assume that B is a valuation ring. Then for every unit u of A either $u \in R$ or $u^{-1} \in R$.

Proof. Namely, if the image c of u in K is in B, then $u \in R$. Otherwise, $c^{-1} \in B$ (Algebra, Lemma 10.50.4) and $u^{-1} \in R$. $\square$

The following lemma explains the meaning of the condition "$A \otimes B \to K$ is surjective" which comes up quite a bit in the following.

09N0 Lemma 33.37.3. In Situation 33.37.1 assume A is a Noetherian ring of dimension 1. The following are equivalent

(1) $A \otimes B \to K$ is not surjective,
(2) there exists a discrete valuation ring $\mathcal{O} \subset K$ containing both A and B.

Proof. It is clear that (2) implies (1). On the other hand, if $A \otimes B \to K$ is not surjective, then the image $C \subset K$ is not a field hence C has a nonzero maximal ideal $\mathfrak{m}$. Choose a valuation ring $\mathcal{O} \subset K$ dominating $C_\mathfrak{m}$. By Algebra, Lemma 10.119.12 applied to $A \subset \mathcal{O}$ the ring $\mathcal{O}$ is Noetherian. Hence $\mathcal{O}$ is a discrete valuation ring by Algebra, Lemma 10.50.18. $\square$

09N1 **Lemma 33.37.4.** In Situation 33.37.1 assume

(1) A is a Noetherian semi-local domain of dimension 1,
(2) B is a discrete valuation ring,

Then we have the following two possibilities

(a) If A^* is not contained in R, then $\text{Spec}(A) \to \text{Spec}(R)$ and $\text{Spec}(B) \to \text{Spec}(R)$ are open immersions covering $\text{Spec}(R)$ and $K = A \otimes_R B$.
(b) If A^* is contained in R, then B dominates one of the local rings of A at a maximal ideal and $A \otimes B \to K$ is not surjective.

Proof. Assumption (a) implies there is a unit u of A whose image in K lies in the maximal ideal of B. Then u is a nonzerodivisor of R and for every $a \in A$ there exists an n such that $u^n a \in R$. It follows that $A = R_u$.

Let $\mathfrak{m}_A$ be the Jacobson radical of A. Let $x \in \mathfrak{m}_A$ be a nonzero element. Since $\dim(A) = 1$ we see that $K = A_x$. After replacing x by $x^n u^m$ for some $n \geq 1$ and $m \in \mathbf{Z}$ we may assume x maps to a unit of B. We see that for every $b \in B$ we have that $x^n b$ in the image of R for some n. Thus $B = R_x$.

Let $z \in R$. If $z \notin \mathfrak{m}_A$ and z does not map to an element of $\mathfrak{m}_B$, then z is invertible. Thus $x + u$ is invertible in R. Hence $\text{Spec}(R) = D(x) \cup D(u)$. We have seen above that $D(u) = \text{Spec}(A)$ and $D(x) = \text{Spec}(B)$.

Case (b). If $x \in \mathfrak{m}_A$, then $1 + x$ is a unit and hence $1 + x \in R$, i.e, $x \in R$. Thus we see that $\mathfrak{m}_A \subset R \subset A$. In fact, in this case A is integral over R. Namely, write $A/\mathfrak{m}_A = \kappa_1 \times \ldots \times \kappa_n$ as a product of fields. Say $x = (c_1, \ldots, c_r, 0, \ldots, 0)$ is an element with $c_i \neq 0$. Then

$$x^2 - x(c_1, \ldots, c_r, 1, \ldots, 1) = 0$$

Since R contains all units we see that $A/\mathfrak{m}_A$ is integral over the image of R in it, and hence A is integral over R. It follows that $R \subset A \subset B$ as B is integrally closed. Moreover, if $x \in \mathfrak{m}_A$ is nonzero, then $K = A_x = \bigcup x^{-n} A = \bigcup x^{-n} R$. Hence $x^{-1} \notin B$, i.e., $x \in \mathfrak{m}_B$. We conclude $\mathfrak{m}_A \subset \mathfrak{m}_B$. Thus $A \cap \mathfrak{m}_B$ is a maximal ideal of A thereby finishing the proof. □

09N2 **Lemma 33.37.5.** Let B be a semi-local Noetherian domain of dimension 1. Let B' be the integral closure of B in its fraction field. Then B' is a semi-local Dedekind domain. Let x be a nonzero element of the Jacobson radical of B'. Then for every $y \in B'$ there exists an n such that $x^n y \in B$.

Proof. Let $\mathfrak{m}_B$ be the Jacobson radical of B. The structure of B' results from Algebra, Lemma 10.120.18. Given $x, y \in B'$ as in the statement of the lemma consider the subring $B \subset A \subset B'$ generated by x and y. Then A is finite over B (Algebra, Lemma 10.36.5). Since the fraction fields of B and A are the same we see that the finite module A/B is supported on the set of closed points of B. Thus $\mathfrak{m}_B^n A \subset B$ for a suitable n. Moreover, $\text{Spec}(B') \to \text{Spec}(A)$ is surjective (Algebra, Lemma 10.36.17), hence A is semi-local as well. It also follows that x is in the Jacobson radical $\mathfrak{m}_A$ of A. Note that $\mathfrak{m}_A = \sqrt{\mathfrak{m}_B A}$. Thus $x^m y \in \mathfrak{m}_B A$ for some m. Then $x^{nm} y \in B$. □

09N3 **Lemma 33.37.6.** In Situation 33.37.1 assume

(1) A is a Noetherian semi-local domain of dimension 1,
(2) B is a Noetherian semi-local domain of dimension 1,

(3) $A \otimes B \to K$ is surjective.

Then $\text{Spec}(A) \to \text{Spec}(R)$ and $\text{Spec}(B) \to \text{Spec}(R)$ are open immersions covering $\text{Spec}(R)$ and $K = A \otimes_R B$.

Proof. Special case: B is integrally closed in K. This means that B is a Dedekind domain (Algebra, Lemma 10.120.17) whence all of its localizations at maximal ideals are discrete valuation rings. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of B. We set

$$R_1 = A \times_K B_{\mathfrak{m}_1}$$

Observing that $A \otimes_{R_1} B_{\mathfrak{m}_1} \to K$ is surjective we conclude from Lemma 33.37.4 that A and $B_{\mathfrak{m}_1}$ define open subschemes covering $\text{Spec}(R_1)$ and that $K = A \otimes_{R_1} B_{\mathfrak{m}_1}$. In particular R_1 is a semi-local Noetherian ring of dimension 1. By induction we define

$$R_{i+1} = R_i \times_K B_{\mathfrak{m}_{i+1}}$$

for $i = 1, \ldots, r-1$. Observe that $R = R_n$ because $B = B_{\mathfrak{m}_1} \cap \ldots \cap B_{\mathfrak{m}_r}$ (see Algebra, Lemma 10.157.6). It follows from the inductive procedure that $R \to A$ defines an open immersion $\text{Spec}(A) \to \text{Spec}(R)$. On the other hand, the maximal ideals $\mathfrak{n}_i$ of R not in this open correspond to the maximal ideals $\mathfrak{m}_i$ of B and in fact the ring map $R \to B$ defines an isomorphisms $R_{\mathfrak{n}_i} \to B_{\mathfrak{m}_i}$ (details omitted; hint: in each step we added exactly one maximal ideal to $\text{Spec}(R_i)$). It follows that $\text{Spec}(B) \to \text{Spec}(R)$ is an open immersion as desired.

General case. Let $B' \subset K$ be the integral closure of B. See Lemma 33.37.5. Then the special case applies to $R' = A \times_K B'$. Pick $x \in R'$ which is not contained in the maximal ideals of A and is contained in the maximal ideals of B' (see Algebra, Lemma 10.15.4). By Lemma 33.37.5 there exists an integer n such that $x^n \in R = A \times_K B$. Replace x by x^n so $x \in R$. For every $y \in R'$ there exists an integer n such that $x^n y \in R$. On the other hand, it is clear that $R'_x = A$. Thus $R_x = A$. Exchanging the roles of A and B we also find an $y \in R$ such that $B = R_y$. Note that inverting both x and y leaves no primes except (0). Thus $K = R_{xy} = R_x \otimes_R R_y$. This finishes the proof. $\square$

09N4 Lemma 33.37.7. Let K be a field. Let $A_1, \ldots, A_r \subset K$ be Noetherian semi-local rings of dimension 1 with fraction field K. If $A_i \otimes A_j \to K$ is surjective for all $i \neq j$, then there exists a Noetherian semi-local domain $A \subset K$ of dimension 1 contained in $A_1, \ldots, A_r$ such that

(1) $A \to A_i$ induces an open immersion $j_i : \text{Spec}(A_i) \to \text{Spec}(A)$,
(2) $\text{Spec}(A)$ is the union of the opens $j_i(\text{Spec}(A_i))$,
(3) each closed point of $\text{Spec}(A)$ lies in exactly one of these opens.

Proof. Namely, we can take $A = A_1 \cap \ldots \cap A_r$. First we note that (3), once (1) and (2) have been proven, follows from the assumption that $A_i \otimes A_j \to K$ is surjective since if $\mathfrak{m} \in j_i(\text{Spec}(A_i)) \cap j_j(\text{Spec}(A_j))$, then $A_i \otimes A_j \to K$ ends up in $A_{\mathfrak{m}}$. To prove (1) and (2) we argue by induction on r. If $r > 1$ by induction we have the results (1) and (2) for $B = A_2 \cap \ldots \cap A_r$. Then we apply Lemma 33.37.6 to see they hold for $A = A_1 \cap B$. $\square$

09N5 Lemma 33.37.8. Let A be a domain with fraction field K. Let $B_1, \ldots, B_r \subset K$ be Noetherian 1-dimensional semi-local domains whose fraction fields are K. If $A \otimes B_i \to K$ are surjective for $i = 1, \ldots, r$, then there exists an $x \in A$ such that x^{-1} is in the Jacobson radical of B_i for $i = 1, \ldots, r$.

Proof. Let B_i' be the integral closure of B_i in K. Suppose we find a nonzero $x \in A$ such that x^{-1} is in the Jacobson radical of B_i' for $i = 1, \ldots, r$. Then by Lemma 33.37.5, after replacing x by a power we get $x^{-1} \in B_i$. Since $\mathrm{Spec}(B_i') \to \mathrm{Spec}(B_i)$ is surjective we see that x^{-1} is then also in the Jacobson radical of B_i. Thus we may assume that each B_i is a semi-local Dedekind domain.

If B_i is not local, then remove B_i from the list and add back the finite collection of local rings $(B_i)_\mathfrak{m}$. Thus we may assume that B_i is a discrete valuation ring for $i = 1, \ldots, r$.

Let $v_i : K \to \mathbf{Z}$, $i = 1, \ldots, r$ be the corresponding discrete valuations (see Algebra, Lemma 10.120.17). We are looking for a nonzero $x \in A$ with $v_i(x) < 0$ for $i = 1, \ldots, r$. We will prove this by induction on r.

If $r = 1$ and the result is wrong, then $A \subset B$ and the map $A \otimes B \to K$ is not surjective, contradiction.

If $r > 1$, then by induction we can find a nonzero $x \in A$ such that $v_i(x) < 0$ for $i = 1, \ldots, r - 1$. If $v_r(x) < 0$ then we are done, so we may assume $v_r(x) \geq 0$. By the base case we can find $y \in A$ nonzero such that $v_r(y) < 0$. After replacing x by a power we may assume that $v_i(x) < v_i(y)$ for $i = 1, \ldots, r - 1$. Then $x + y$ is the element we are looking for. $\square$

0AB2 Lemma 33.37.9. Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_\mathfrak{p}$ where the product is over the minimal primes of A. Let $a_1, a_2 \in \mathfrak{m}_A$ map to the same element of L. Then $a_1^n = a_2^n$ for some $n > 0$.

Proof. Write $a_1 = a_2 + x$. Then x maps to zero in L. Hence x is a nilpotent element of A because $\bigcap \mathfrak{p}$ is the radical of (0) and the annihilator I of x contains a power of the maximal ideal because $\mathfrak{p} \not\in V(I)$ for all minimal primes. Say $x^k = 0$ and $\mathfrak{m}^n \subset I$. Then

$$a_1^{k+n} = a_2^{k+n} + \binom{n+k}{1} a_2^{n+k-1} x + \binom{n+k}{2} a_2^{n+k-2} x^2 + \ldots + \binom{n+k}{k-1} a_2^{n+1} x^{k-1} = a_2^{n+k}$$

because $a_2 \in \mathfrak{m}_A$. $\square$

0AB3 Lemma 33.37.10. Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_\mathfrak{p}$ and $I = \bigcap \mathfrak{p}$ where the product and intersection are over the minimal primes of A. Let $f \in L$ be an element of the form $f = i + a$ where $a \in \mathfrak{m}_A$ and $i \in IL$. Then some power of f is in the image of $A \to L$.

Proof. Since A is Noetherian we have $I^t = 0$ for some $t > 0$. Suppose that we know that $f = a + i$ with $i \in I^k L$. Then $f^n = a^n + na^{n-1} i \bmod I^{k+1} L$. Hence it suffices to show that $na^{n-1} i$ is in the image of $I^k \to I^k L$ for some $n \gg 0$. To see this, pick a $g \in A$ such that $\mathfrak{m}_A = \sqrt{(g)}$ (Algebra, Lemma 10.60.8). Then $L = A_g$ for example by Algebra, Proposition 10.60.7. On the other hand, there is an n such that $a^n \in (g)$. Hence we can clear denominators for elements of L by multiplying by a high power of a. $\square$

0AB4 Lemma 33.37.11. Let A be a Noetherian local ring of dimension 1. Let $L = \prod A_\mathfrak{p}$ where the product is over the minimal primes of A. Let $K \to L$ be an integral ring map. Then there exist $a \in \mathfrak{m}_A$ and $x \in K$ which map to the same element of L such that $\mathfrak{m}_A = \sqrt{(a)}$.

Proof. By Lemma 33.37.10 we may replace A by $A/(\bigcap \mathfrak p)$ and assume that A is a reduced ring (some details omitted). We may also replace K by the image of $K \to L$. Then K is a reduced ring. The map $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is surjective and closed (details omitted). Hence $\mathrm{Spec}(K)$ is a finite discrete space. It follows that K is a finite product of fields.

Let $\mathfrak p_j$, $j = 1, \ldots, m$ be the minimal primes of A. Set L_j be the fraction field of A_j so that $L = \prod_{j=1,\ldots,m} L_j$. Let A_j be the normalization of $A/\mathfrak p_j$. Then A_j is a semi-local Dedekind domain with at least one maximal ideal, see Algebra, Lemma 10.120.18. Let n be the sum of the numbers of maximal ideals in $A_1, \ldots, A_m$. For such a maximal ideal $\mathfrak m \subset A_j$ we consider the function

$$v_{\mathfrak m} : L \to L_j \to \mathbf{Z} \cup \{\infty\}$$

where the second arrow is the discrete valuation corresponding to the discrete valuation ring $(A_j)_{\mathfrak m}$ extended by mapping 0 to ∞. In this way we obtain n functions $v_1, \ldots, v_n : L \to \mathbf{Z} \cup \{\infty\}$. We will find an element $x \in K$ such that $v_i(x) < 0$ for all $i = 1, \ldots, n$.

First we claim that for each i there exists an element $x \in K$ with $v_i(x) < 0$. Namely, suppose that v_i corresponds to $\mathfrak m \subset A_j$. If $v_i(x) \geq 0$ for all $x \in K$, then K maps into $(A_j)_{\mathfrak m}$ inside the fraction field L_j of A_j. The image of K in L_j is a field over L_j is algebraic by Algebra, Lemma 10.36.18. Combined we get a contradiction with Algebra, Lemma 10.50.8.

Suppose we have found an element $x \in K$ such that $v_1(x) < 0, \ldots, v_r(x) < 0$ for some $r < n$. If $v_{r+1}(x) < 0$, then x works for $r+1$. If not, then choose some $y \in K$ with $v_{r+1}(y) < 0$ as is possible by the result of the previous paragraph. After replacing x by x^n for some $n > 0$, we may assume $v_i(x) < v_i(y)$ for $i = 1, \ldots, r$. Then $v_j(x+y) = v_j(x) < 0$ for $j = 1, \ldots, r$ by properties of valuations and similarly $v_{r+1}(x+y) = v_{r+1}(y) < 0$. Arguing by induction, we find $x \in K$ with $v_i(x) < 0$ for $i = 1, \ldots, n$.

In particular, the element $x \in K$ has nonzero projection in each factor of K (recall that K is a finite product of fields and if some component of x was zero, then one of the values $v_i(x)$ would be ∞). Hence x is invertible and $x^{-1} \in K$ is an element with $\infty > v_i(x^{-1}) > 0$ for all i. It follows from Lemma 33.37.5 that for some $e < 0$ the element $x^e \in K$ maps to an element of $\mathfrak m_A/\mathfrak p_j \subset A/\mathfrak p_j$ for all $j = 1, \ldots, m$. Observe that the cokernel of the map $\mathfrak m_A \to \prod \mathfrak m_A/\mathfrak p_j$ is annihilated by a power of $\mathfrak m_A$. Hence after replacing e by a more negative e, we find an element $a \in \mathfrak m_A$ whose image in $\mathfrak m_A/\mathfrak p_j$ is equal to the image of x^e. The pair (a, x^e) satisfies the conclusions of the lemma. □

09N6 **Lemma 33.37.12.** Let A be a ring. Let $\mathfrak p_1, \ldots, \mathfrak p_r$ be a finite set of a primes of A. Let $S = A \setminus \bigcup \mathfrak p_i$. Then S is a multiplicative system and $S^{-1}A$ is a semi-local ring whose maximal ideals correspond to the maximal elements of the set $\{\mathfrak p_i\}$.

Proof. If $a, b \in A$ and $a, b \in S$, then $a, b \notin \mathfrak p_i$ hence $ab \notin \mathfrak p_i$, hence $ab \in S$. Also $1 \in S$. Thus S is a multiplicative subset of A. By the description of $\mathrm{Spec}(S^{-1}A)$ in Algebra, Lemma 10.17.5 and by Algebra, Lemma 10.15.2 we see that the primes of $S^{-1}A$ correspond to the primes of A contained in one of the $\mathfrak p_i$. Hence the maximal ideals of $S^{-1}A$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak p_1, \ldots, \mathfrak p_r\}$. □

33.38. One dimensional Noetherian schemes

09N7 The main result of this section is that a Noetherian separated scheme of dimension 1 has an ample invertible sheaf. See Proposition 33.38.12.

09N8 Lemma 33.38.1. Let X be a scheme all of whose local rings are Noetherian of dimension ≤ 1. Let $U \subset X$ be a retrocompact open. Denote $j : U \to X$ the inclusion morphism. Then $R^p j_* \mathcal{F} = 0$, $p > 0$ for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$.

Proof. We may check the vanishing of $R^p j_* \mathcal{F}$ at stalks. Formation of $R^q j_*$ commutes with flat base change, see Cohomology of Schemes, Lemma 30.5.2. Thus we may assume that X is the spectrum of a Noetherian local ring of dimension ≤ 1. In this case X has a closed point x and finitely many other points $x_1, \ldots, x_n$ which specialize to x but not each other (see Algebra, Lemma 10.31.6). If $x \in U$, then $U = X$ and the result is clear. If not, then $U = \{x_1, \ldots, x_r\}$ for some r after possibly renumbering the points. Then U is affine (Schemes, Lemma 26.11.8). Thus the result follows from Cohomology of Schemes, Lemma 30.2.3. $\square$

09N9 Lemma 33.38.2. Let X be an affine scheme all of whose local rings are Noetherian of dimension ≤ 1. Then any quasi-compact open $U \subset X$ is affine.

Proof. Denote $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. By Lemma 33.38.1 the higher direct images $R^p j_* \mathcal{F}$ are zero. The $\mathcal{O}_X$-module $j_* \mathcal{F}$ is quasi-coherent (Schemes, Lemma 26.24.1). Hence it has vanishing higher cohomology groups by Cohomology of Schemes, Lemma 30.2.2. By the Leray spectral sequence Cohomology, Lemma 20.13.6 we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$. Thus U is affine, for example by Cohomology of Schemes, Lemma 30.3.1. $\square$

09NA Lemma 33.38.3. Let X be a scheme. Let $U \subset X$ be an open. Assume

(1) U is a retrocompact open of X,
(2) $X \setminus U$ is discrete, and
(3) for $x \in X \setminus U$ the local ring $\mathcal{O}_{X,x}$ is Noetherian of dimension ≤ 1.

Then (1) there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$ and (2) the map $\text{Pic}(X) \to \text{Pic}(U)$ is surjective.

Proof. Let $X \setminus U = \{x_i; i \in I\}$. Choose affine opens $U_i \subset X$ with $x_i \in U_i$ and $x_j \notin U_i$ for $j \neq i$. This is possible by condition (2). Say $U_i = \text{Spec}(A_i)$. Let $\mathfrak{m}_i \subset A_i$ be the maximal ideal corresponding to x_i. By our assumption on the local rings there are only a finite number of prime ideals $\mathfrak{q} \subset \mathfrak{m}_i$, $\mathfrak{q} \neq \mathfrak{m}_i$ (see Algebra, Lemma 10.31.6). Thus by prime avoidance (Algebra, Lemma 10.15.2) we can find $f_i \in \mathfrak{m}_i$ not contained in any of those primes. Then $V(f_i) = \{\mathfrak{m}_i\} \amalg Z_i$ for some closed subset $Z_i \subset U_i$ because Z_i is a retrocompact open subset of $V(f_i)$ closed under specialization, see Algebra, Lemma 10.41.7. After shrinking U_i we may assume $V(f_i) = \{x_i\}$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f_i on $U \cap U_i$ and by f_i/f_j on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ such that the section s defined by 1 on U and f_i on U_i is as in the statement of the lemma.

Let $\mathcal{N}$ be an invertible $\mathcal{O}_U$-module. Let N_i be the invertible $(A_i)_{f_i}$ module such that $\mathcal{N}|_{U \cap U_i}$ is equal to $\tilde{N}_i$. Observe that $(A_{\mathfrak{m}_i})_{f_i}$ is an Artinian ring (as a dimension zero Noetherian ring, see Algebra, Lemma 10.60.5). Thus it is a product of local rings (Algebra, Lemma 10.53.6) and hence has trivial Picard group. Thus, after shrinking U_i (i.e., after replacing A_i by $(A_i)_g$ for some $g \in A_i$, $g \notin \mathfrak{m}_i$) we can assume that $N_i = (A_i)_{f_i}$, i.e., that $\mathcal{N}|_{U \cap U_i}$ is trivial. In this case it is clear how to extend $\mathcal{N}$ to an invertible sheaf over X (by extending it by a trivial invertible module over each U_i). $\square$

09NB Lemma 33.38.4. Let X be an integral separated scheme. Let $U \subset X$ be a nonempty affine open such that $X \setminus U$ is a finite set of points $x_1, \ldots, x_r$ with $\mathcal{O}_{X,x_i}$ Noetherian of dimension 1. Then there exists a globally generated invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$.

Proof. Say $U = \text{Spec}(A)$ and let K be the function field of X. Write $B_i = \mathcal{O}_{X,x_i}$ and $\mathfrak{m}_i = \mathfrak{m}_{x_i}$. Since $x_i \notin U$ we see that the open $U \times_X \text{Spec}(B_i)$ of $\text{Spec}(B_i)$ has only one point, i.e., $U \times_X \text{Spec}(B_i) = \text{Spec}(K)$. Since X is separated, we find that $\text{Spec}(K)$ is a closed subscheme of $U \times \text{Spec}(B_i)$, i.e., the map $A \otimes B_i \to K$ is a surjection. By Lemma 33.37.8 we can find a nonzero $f \in A$ such that $f^{-1} \in \mathfrak{m}_i$ for $i = 1, \ldots, r$. Pick opens $x_i \in U_i \subset X$ such that $f^{-1} \in \mathcal{O}(U_i)$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f on $U \cap U_i$ and by 1 on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ with two sections:

(1) a section s defined by 1 on U and f^{-1} on U_i is as in the statement of the lemma, and
(2) a section t defined by f on U and 1 on U_i.

Note that $X_t \supset U_1 \cup \ldots \cup U_r$. Hence s, t generate $\mathcal{L}$ and the lemma is proved. $\square$

09NC Lemma 33.38.5. Let X be a quasi-compact scheme. If for every $x \in X$ there exists a pair $(\mathcal{L}, s)$ consisting of a globally generated invertible sheaf $\mathcal{L}$ and a global section s such that $x \in X_s$ and X_s is affine, then X has an ample invertible sheaf.

Proof. Since X is quasi-compact we can find a finite collection $(\mathcal{L}_i, s_i)$, $i = 1, \ldots, n$ of pairs such that $\mathcal{L}_i$ is globally generated, X_{s_i} is affine and $X = \bigcup X_{s_i}$. Again because X is quasi-compact we can find, for each i, a finite collection of sections $t_{i,j}$ of $\mathcal{L}_i$, $j = 1, \ldots, m_i$ such that $X = \bigcup X_{t_{i,j}}$. Set $t_{i,0} = s_i$. Consider the invertible sheaf

$$\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}_n$$

and the global sections

$$\tau_J = t_{1,j_1} \otimes \ldots \otimes t_{n,j_n}$$

By Properties, Lemma 28.26.4 the open X_{τ_J} is affine as soon as $j_i = 0$ for some i. It is a simple matter to see that these opens cover X. Hence $\mathcal{L}$ is ample by definition. $\square$

09ND Lemma 33.38.6. Let X be a Noetherian integral separated scheme of dimension 1. Then X has an ample invertible sheaf.

Proof. Choose an affine open covering $X = U_1 \cup \ldots \cup U_n$. Since X is Noetherian, each of the sets $X \setminus U_i$ is finite. Thus by Lemma 33.38.4 we can find a pair $(\mathcal{L}_i, s_i)$ consisting of a globally generated invertible sheaf $\mathcal{L}_i$ and a global section s_i such that $U_i = X_{s_i}$. We conclude that X has an ample invertible sheaf by Lemma 33.38.5. $\square$

0C0T Lemma 33.38.7. Let $f : X \to Y$ be a finite morphism of schemes. Assume there exists an open $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism and $Y \setminus V$ is a discrete space. Then every invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_Y$-module.

Proof. We will use that $\text{Pic}(X) = H^1(X, \mathcal{O}_X^*)$, see Cohomology, Lemma 20.6.1. Consider the Leray spectral sequence for the abelian sheaf $\mathcal{O}_X^*$ and f, see Cohomology, Lemma 20.13.4. Consider the induced map

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^0(Y, R^1 f_* \mathcal{O}_X^*)$$

Divisors, Lemma 31.17.1 says exactly that this map is zero. Hence Leray gives $H^1(X, \mathcal{O}_X^*) = H^1(Y, f_* \mathcal{O}_X^*)$. Next we consider the map

$$f^\sharp : \mathcal{O}_Y^* \longrightarrow f_* \mathcal{O}_X^*$$

By assumption the kernel and cokernel of this map are supported on the closed subset $T = Y \setminus V$ of Y. Since T is a discrete topological space by assumption the higher cohomology groups of any abelian sheaf on Y supported on T is zero (follows from Cohomology, Lemma 20.20.1, Modules, Lemma 17.6.1, and the fact that $H^i(T, \mathcal{F}) = 0$ for any $i > 0$ and any abelian sheaf $\mathcal{F}$ on T). Breaking the displayed map into short exact sequences

$$0 \to \text{Ker}(f^\sharp) \to \mathcal{O}_Y^* \to \text{Im}(f^\sharp) \to 0, \quad 0 \to \text{Im}(f^\sharp) \to f_* \mathcal{O}_X^* \to \text{Coker}(f^\sharp) \to 0$$

we first conclude that $H^1(Y, \mathcal{O}_Y^*) \to H^1(Y, \text{Im}(f^\sharp))$ is surjective and then that $H^1(Y, \text{Im}(f^\sharp)) \to H^1(Y, f_* \mathcal{O}_X^*)$ is surjective. Combining all the above we find that $H^1(Y, \mathcal{O}_Y^*) \to H^1(X, \mathcal{O}_X^*)$ is surjective as desired. □

09NE Lemma 33.38.8. Let X be a scheme. Let $Z_1, \ldots, Z_n \subset X$ be closed subschemes. Let $\mathcal{L}_i$ be an invertible sheaf on Z_i. Assume that

(1) X is reduced,
(2) $X = \bigcup Z_i$ set theoretically, and
(3) $Z_i \cap Z_j$ is a discrete topological space for $i \neq j$.

Then there exists an invertible sheaf $\mathcal{L}$ on X whose restriction to Z_i is $\mathcal{L}_i$. Moreover, if we are given sections $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ which are nonvanishing at the points of $Z_i \cap Z_j$, then we can choose $\mathcal{L}$ such that there exists a $s \in \Gamma(X, \mathcal{L})$ with $s|_{Z_i} = s_i$ for all i.

Proof. The existence of $\mathcal{L}$ can be deduced from Lemma 33.38.7 but we will also give a direct proof and we will use the direct proof to see the statement about sections is true. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is reduced we have

$$X \setminus T = \bigcup (Z_i \setminus T)$$

as schemes. Assumption (3) implies T is a discrete subset of X. Thus for each $t \in T$ we can find an open $U_t \subset X$ with $t \in U_t$ but $t' \notin U_t$ for $t' \in T$, $t' \neq t$. By shrinking U_t if necessary, we may assume that there exist isomorphisms $\varphi_{t,i} : \mathcal{L}_i|_{U_t \cap Z_i} \to \mathcal{O}_{U_t \cap Z_i}$. Furthermore, for each i choose an open covering

$$Z_i \setminus T = \bigcup\nolimits_j U_{ij}$$

such that there exist isomorphisms $\varphi_{i,j} : \mathcal{L}_i|_{U_{ij}} \cong \mathcal{O}_{U_{ij}}$. Observe that

$$\mathcal{U} : X = \bigcup U_t \cup \bigcup U_{ij}$$

is an open covering of X. We claim that we can use the isomorphisms $\varphi_{t,i}$ and $\varphi_{i,j}$ to define a 2-cocycle with values in $\mathcal{O}_X^*$ for this covering that defines $\mathcal{L}$ as in the statement of the lemma.

Namely, if $i \neq i'$, then $U_{i,j} \cap U_{i',j'} = \emptyset$ and there is nothing to do. For $U_{i,j} \cap U_{i,j'}$ we have $\mathcal{O}_X(U_{i,j} \cap U_{i,j'}) = \mathcal{O}_{Z_i}(U_{i,j} \cap U_{i,j'})$ by the first remark of the proof. Thus the transition function for $\mathcal{L}_i$ (more precisely $\varphi_{i,j} \circ \varphi_{i,j'}^{-1}$) defines the value of our cocycle on this intersection. For $U_t \cap U_{i,j}$ we can do the same thing. Finally, for $t \neq t'$ we have

$$U_t \cap U_{t'} = \coprod (U_t \cap U_{t'}) \cap Z_i$$

and moreover the intersection $U_t \cap U_{t'} \cap Z_i$ is contained in $Z_i \setminus T$. Hence by the same reasoning as before we see that

$$\mathcal{O}_X(U_t \cap U_{t'}) = \prod \mathcal{O}_{Z_i}(U_t \cap U_{t'} \cap Z_i)$$

and we can use the transition functions for $\mathcal{L}_i$ (more precisely $\varphi_{t,i} \circ \varphi_{t',i}^{-1}$) to define the value of our cocycle on $U_t \cap U_{t'}$. This finishes the proof of existence of $\mathcal{L}$.

Given sections s_i as in the last assertion of the lemma, in the argument above, we choose U_t such that $s_i|_{U_t \cap Z_i}$ is nonvanishing and we choose $\varphi_{t,i}$ such that $\varphi_{t,i}(s_i|_{U_t \cap Z_i}) = 1$. Then using 1 over U_t and $\varphi_{i,j}(s_i|_{U_{i,j}})$ over $U_{i,j}$ will define a section of $\mathcal{L}$ which restricts to s_i over Z_i. □

09NW Remark 33.38.9. Let A be a reduced ring. Let I, J be ideals of A such that $V(I) \cup V(J) = \mathrm{Spec}(A)$. Set $B = A/J$. Then $I \to IB$ is an isomorphism of A-modules. Namely, we have $IB = I + J/J = I/(I \cap J)$ and $I \cap J$ is zero because A is reduced and $\mathrm{Spec}(A) = V(I) \cup V(J) = V(I \cap J)$. Thus for any projective A-module P we also have $IP = I(P/JP)$.

09NX Lemma 33.38.10. Let X be a Noetherian reduced separated scheme of dimension 1. Then X has an ample invertible sheaf.

Proof. Let Z_i, $i = 1, \ldots, n$ be the irreducible components of X. We view these as reduced closed subschemes of X. By Lemma 33.38.6 there exist ample invertible sheaves $\mathcal{L}_i$ on Z_i. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is Noetherian of dimension 1, the set T is finite and consists of closed points of X. For each i we may, possibly after replacing $\mathcal{L}_i$ by a power, choose $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ such that $(Z_i)_{s_i}$ is affine and contains $T \cap Z_i$, see Properties, Lemma 28.29.6.

By Lemma 33.38.8 we can find an invertible sheaf $\mathcal{L}$ on X and $s \in \Gamma(X, \mathcal{L})$ such that $(\mathcal{L}, s)|_{Z_i} = (\mathcal{L}_i, s_i)$. Observe that X_s contains T and is set theoretically equal to the affine closed subschemes $(Z_i)_{s_i}$. Thus it is affine by Limits, Lemma 32.11.3. To finish the proof, it suffices to find for every $x \in X$, $x \notin T$ an integer $m > 0$ and a section $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that X_t is affine and $x \in X_t$. Since $x \notin T$ we see that $x \in Z_i$ for some unique i, say $i = 1$. Let $Z \subset X$ be the reduced closed subscheme whose underlying topological space is $Z_2 \cup \ldots \cup Z_n$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Denote that $\mathcal{I}_1 \subset \mathcal{O}_{Z_1}$ the inverse image of this ideal sheaf under the inclusion morphism $Z_1 \to X$. Observe that

$$\Gamma(X, \mathcal{I}\mathcal{L}^{\otimes m}) = \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$$

see Remark 33.38.9. Thus it suffices to find $m > 0$ and $t \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ affine. Since $\mathcal{L}_1$ is ample and since x is not in $Z_1 \cap T = V(\mathcal{I}_1)$ we can find a section $t_1 \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m_1})$ with $x \in (Z_1)_{t_1}$, see Properties, Proposition 28.26.13. Since $\mathcal{L}_1$ is ample we can find a section $t_2 \in \Gamma(Z_1, \mathcal{L}_1^{\otimes m_2})$ with $x \in (Z_1)_{t_2}$ and $(Z_1)_{t_2}$ affine, see Properties, Definition 28.26.1. Set $m = m_1 + m_2$ and $t = t_1 t_2$. Then $t \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ by construction and $(Z_1)_t$ is affine by Properties, Lemma 28.26.4. □

09NY Lemma 33.38.11. Let $i : Z \to X$ be a closed immersion of schemes. If the underlying topological space of X is Noetherian and $\dim(X) \leq 1$, then $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is surjective.

Proof. Consider the short exact sequence

$$0 \to (1 + \mathcal{I}) \cap \mathcal{O}_X^* \to \mathcal{O}_X^* \to i_*\mathcal{O}_Z^* \to 0$$

of sheaves of abelian groups on X where $\mathcal{I}$ is the quasi-coherent sheaf of ideals corresponding to Z. Since $\dim(X) \leq 1$ we see that $H^2(X, \mathcal{F}) = 0$ for any abelian sheaf $\mathcal{F}$, see Cohomology, Proposition 20.20.7. Hence the map $H^1(X, \mathcal{O}_X^*) \to H^1(X, i_*\mathcal{O}_Z^*)$ is surjective. By Cohomology, Lemma 20.20.1 we have $H^1(X, i_*\mathcal{O}_Z^*) = H^1(Z, \mathcal{O}_Z^*)$. This proves the lemma by Cohomology, Lemma 20.6.1. □

09NZ Proposition 33.38.12. Let X be a Noetherian separated scheme of dimension 1. Then X has an ample invertible sheaf.

Proof. Let $Z \subset X$ be the reduction of X. By Lemma 33.38.10 the scheme Z has an ample invertible sheaf. Thus by Lemma 33.38.11 there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X whose restriction to Z is ample. Then $\mathcal{L}$ is ample by an application of Cohomology of Schemes, Lemma 30.17.5. □

09P0 Remark 33.38.13. In fact, if X is a scheme whose reduction is a Noetherian separated scheme of dimension 1, then X has an ample invertible sheaf. The argument to prove this is the same as the proof of Proposition 33.38.12 except one uses Limits, Lemma 32.11.4 instead of Cohomology of Schemes, Lemma 30.17.5.

The following lemma actually holds for quasi-finite separated morphisms as the reader can see by using Zariski's main theorem (More on Morphisms, Lemma 37.43.3) and Lemma 33.38.3.

0C0U Lemma 33.38.14. Let $f : X \to Y$ be a morphism of schemes. Assume Y is Noetherian of dimension ≤ 1, f is finite, and there exists a dense open $V \subset Y$ such that $f^{-1}(V) \to V$ is a closed immersion. Then every invertible $\mathcal{O}_X$-module is the pullback of an invertible $\mathcal{O}_Y$-module.

Proof. We factor f as $X \to Z \to Y$ where Z is the scheme theoretic image of f. Then $X \to Z$ is an isomorphism over $V \cap Z$ and Lemma 33.38.7 applies. On the other hand, Lemma 33.38.11 applies to $Z \to Y$. Some details omitted. □

33.39. The delta invariant

0C3Q In this section we define the δ-invariant of a singular point on a reduced 1-dimensional Nagata scheme.

0C3R Lemma 33.39.1. Let $(A, \mathfrak{m})$ be a Noetherian 1-dimensional local ring. Let $f \in \mathfrak{m}$. The following are equivalent

(1) $\mathfrak{m} = \sqrt{(f)}$,
(2) f is not contained in any minimal prime of A, and

(3) $A_f = \prod_{\mathfrak p \text{ minimal}} A_{\mathfrak p}$ as A-algebras.

Such an $f \in \mathfrak m$ exists. If $\text{depth}(A) = 1$ (for example A is reduced), then (1) – (3) are also equivalent to

(4) f is a nonzerodivisor,
(5) A_f is the total ring of fractions of A.

If A is reduced, then (1) – (5) are also equivalent to

(6) A_f is the product of the residue fields at the minimal primes of A.

Proof. The spectrum of A has finitely many primes $\mathfrak p_1, \ldots, \mathfrak p_n$ besides $\mathfrak m$ and these are all minimal, see Algebra, Lemma 10.31.6. Then the equivalence of (1) and (2) follows from Algebra, Lemma 10.17.2. Clearly, (3) implies (2). Conversely, if (2) is true, then the spectrum of A_f is the subset $\{\mathfrak p_1, \ldots, \mathfrak p_n\}$ of $\text{Spec}(A)$ with induced topology, see Algebra, Lemma 10.17.5. This is a finite discrete topological space. Hence $A_f = \prod_{\mathfrak p \text{ minimal}} A_{\mathfrak p}$ by Algebra, Proposition 10.60.7. The existence of an f is asserted in Algebra, Lemma 10.60.8.

Assume A has depth 1. (This is the maximum by Algebra, Lemma 10.72.3 and holds if A is reduced by Algebra, Lemma 10.157.3.) Then $\mathfrak m$ is not an associated prime of A. Every minimal prime of A is an associated prime (Algebra, Proposition 10.63.6). Hence the set of nonzerodivisors of A is exactly the set of elements not contained in any of the minimal primes by Algebra, Lemma 10.63.9. Thus (4) is equivalent to (2). Part (5) is equivalent to (3) by Algebra, Lemma 10.25.4.

Then $A_{\mathfrak p}$ is a field for $\mathfrak p \subset A$ minimal, see Algebra, Lemma 10.25.1. Hence (3) is equivalent ot (6). □

0C3S Lemma 33.39.2. Let $(A, \mathfrak m)$ be a reduced Nagata 1-dimensional local ring. Let A' be the integral closure of A in the total ring of fractions of A. Then A' is a normal Nagata ring, $A \to A'$ is finite, and A'/A has finite length as an A-module.

Proof. The total ring of fractions is essentially of finite type over A hence $A \to A'$ is finite because A is Nagata, see Algebra, Lemma 10.162.2. The ring A' is normal for example by Algebra, Lemma 10.37.16 and 10.31.6. The ring A' is Nagata for example by Algebra, Lemma 10.162.5. Choose $f \in \mathfrak m$ as in Lemma 33.39.1. As $A' \subset A_f$ it is clear that $A_f = A'_f$. Hence the support of the finite A-module A'/A is contained in $\{\mathfrak m\}$. It follows that it has finite length by Algebra, Lemma 10.62.3. □

0C3T Definition 33.39.3. Let A be a reduced Nagata local ring of dimension 1. The δ-invariant of A is $\text{length}_A(A'/A)$ where A' is as in Lemma 33.39.2.

We prove some lemmas about the behaviour of this invariant.

0C3U Lemma 33.39.4. Let A be a reduced Nagata local ring of dimension 1. The δ-invariant of A is 0 if and only if A is a discrete valuation ring.

Proof. If A is a discrete valuation ring, then A is normal and the ring A' is equal to A. Conversely, if the δ-invariant of A is 0, then A is integrally closed in its total ring of fractions which implies that A is normal (Algebra, Lemma 10.37.16) and this forces A to be a discrete valuation ring by Algebra, Lemma 10.119.7. □

0C3V Lemma 33.39.5. Let A be a reduced Nagata local ring of dimension 1. Let $A \to A'$ be as in Lemma 33.39.2. Let A^h, A^{sh}, resp. $A^\wedge$ be the henselization, strict henselization, resp. completion of A. Then A^h, A^{sh}, resp. $A^\wedge$ is a reduced Nagata local ring of dimension 1 and $A' \otimes_A A^h$, $A' \otimes_A A^{sh}$, resp. $A' \otimes_A A^\wedge$ is the integral closure of A^h, A^{sh}, resp. $A^\wedge$ in its total ring of fractions.

Proof. Observe that $A^\wedge$ is reduced, see More on Algebra, Lemma 15.43.6. The rings A^h and A^{sh} are reduced by More on Algebra, Lemma 15.45.4. The dimensions of A, A^h, A^{sh}, and $A^\wedge$ are the same by More on Algebra, Lemmas 15.43.1 and 15.45.7.

Recall that a Noetherian local ring is Nagata if and only if the formal fibres of A are geometrically reduced, see More on Algebra, Lemma 15.52.4. This property is inherited by A^h and A^{sh}, see the material in More on Algebra, Section 15.51 and especially Lemma 15.51.8. The completion is Nagata by Algebra, Lemma 10.162.8.

Now we come to the statement on integral closures. Before continuing let us pick $f \in \mathfrak m$ as in Lemma 33.39.1. Then the image of f in A^h, A^{sh}, and $A^\wedge$ clearly is an element satisfying properties (1) – (6) in that ring.

Since $A \to A'$ is finite we see that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ is the product of henselian local rings finite over A^h and A^{sh}, see Algebra, Lemma 10.153.4. Each of these local rings is the henselization of A' at a maximal ideal $\mathfrak m' \subset A'$ lying over $\mathfrak m$, see Algebra, Lemma 10.156.1 or 10.156.3. Hence these local rings are normal domains by More on Algebra, Lemma 15.45.6. It follows that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ are normal rings. Since $A^h \to A' \otimes_A A^h$ and $A^{sh} \to A' \otimes_A A^{sh}$ are finite (hence integral) and since $A' \otimes_A A^h \subset (A^h)_f = Q(A^h)$ and $A' \otimes_A A^{sh} \subset (A^{sh})_f = Q(A^{sh})$ we conclude that $A' \otimes_A A^h$ and $A' \otimes_A A^{sh}$ are the desired integral closures.

For the completion we argue in entirely the same manner. First, by Algebra, Lemma 10.97.8 we have

$$A' \otimes_A A^\wedge = (A')^\wedge = \prod (A'_{\mathfrak m'})^\wedge$$

The local rings $A'_{\mathfrak m'}$ are normal and have dimension 1 (by Algebra, Lemma 10.113.2 for example or the discussion in Algebra, Section 10.112). Thus $A'_{\mathfrak m'}$ is a discrete valuation ring, see Algebra, Lemma 10.119.7. Hence $(A'_{\mathfrak m'})^\wedge$ is a discrete valuation ring by More on Algebra, Lemma 15.43.5. It follows that $A' \otimes_A A^\wedge$ is a normal ring and we can conclude in exactly the same manner as before. □

0C3W Lemma 33.39.6. Let A be a reduced Nagata local ring of dimension 1. The δ-invariant of A is the same as the δ-invariant of the henselization, strict henselization, or the completion of A.

Proof. Let us do this in case of the completion $B = A^\wedge$; the other cases are proved in exactly the same manner. Let A', resp. B' be the integral closure of A, resp. B in its total ring of fractions. Then $B' = A' \otimes_A B$ by Lemma 33.39.5. Hence $B'/B = A'/A \otimes_A B$. The equality now follows from Algebra, Lemma 10.52.13 and the fact that $B \otimes_A \kappa_A = \kappa_B$. □

0C1T Definition 33.39.7. Let k be a field. Let X be a locally algebraic k-scheme. Let $x \in X$ be a point such that $\mathcal{O}_{X,x}$ is reduced and $\dim(\mathcal{O}_{X,x}) = 1$. The δ-invariant of X at x is the δ-invariant of $\mathcal{O}_{X,x}$ as defined in Definition 33.39.3.

This makes sense because the local ring of a locally algebraic scheme is Nagata by Algebra, Proposition 10.162.16. Of course, more generally we can make this definition whenever $x \in X$ is a point of a scheme such that the local ring $\mathcal{O}_{X,x}$ is reduced, Nagata of dimension 1. It follows from Lemma 33.39.6 that the δ-invariant of X at x is

$$\delta\text{-invariant of } X \text{ at } x = \delta\text{-invariant of } \mathcal{O}_{X,x}^h = \delta\text{-invariant of } \mathcal{O}_{X,x}^\wedge$$

We conclude that the δ-invariant is an invariant of the complete local ring of the point.

0C3X Lemma 33.39.8. Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be a field extension and set $Y = X_K$. Let $y \in Y$ with image $x \in X$. Assume X is geometrically reduced at x and $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) = 1$. Then

$$\delta\text{-invariant of } X \text{ at } x \leq \delta\text{-invariant of } Y \text{ at } y$$

Proof. Set $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}_{Y,y}$. By Lemma 33.6.2 we see that A is geometrically reduced. Hence B is a localization of $A \otimes_k K$. Let $A \to A'$ be as in Lemma 33.39.2. Then

$$B' = B \otimes_{(A \otimes_k K)} (A' \otimes_k K)$$

is finite over B and $B \to B'$ induces an isomorphism on total rings of fractions. Namely, pick $f \in \mathfrak{m}_A$ satisfying (1) – (6) of Lemma 33.39.1; since $\dim(B) = 1$ we see that $f \in \mathfrak{m}_B$ playes the same role for B and we see that $B_f = B'_f$ because $A_f = A'_f$. Let B'' be the integral closure of B in its total ring of fractions as in Lemma 33.39.2. Then $B' \subset B''$. Thus the δ-invariant of Y at y is $\text{length}_B(B''/B)$ and

$$\begin{aligned} \text{length}_B(B''/B) &\geq \text{length}_B(B'/B) \\ &= \text{length}_B((A'/A) \otimes_A B) \\ &= \text{length}_B(B/\mathfrak{m}_A B)\text{length}_A(A'/A) \end{aligned}$$

by Algebra, Lemma 10.52.13 since $A \to B$ is flat (as a localization of $A \to A \otimes_k K$). Since $\text{length}_A(A'/A)$ is the δ-invariant of X at x and since $\text{length}_B(B/\mathfrak{m}_A B) \geq 1$ the lemma is proved. $\square$

0C3Y Lemma 33.39.9. Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be a field extension and set $Y = X_K$. Let $y \in Y$ with image $x \in X$. Assume assumptions (a), (b), (c) of Lemma 33.27.6 hold for $x \in X$ and that $\dim(\mathcal{O}_{Y,y}) = 1$. Then the δ-invariant of X at x is δ-invariant of Y at y.

Proof. Set $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}_{Y,y}$. By Lemma 33.27.6 we see that A is geometrically reduced. Hence B is a localization of $A \otimes_k K$. Let $A \to A'$ be as in Lemma 33.39.2. By Lemma 33.27.6 we see that $A' \otimes_k K$ is normal. Hence

$$B' = B \otimes_{(A \otimes_k K)} (A' \otimes_k K)$$

is normal, finite over B, and $B \to B'$ induces an isomorphism on total rings of fractions. Namely, pick $f \in \mathfrak{m}_A$ satisfying (1) – (6) of Lemma 33.39.1; since $\dim(B) = 1$ we see that $f \in \mathfrak{m}_B$ playes the same role for B and we see that $B_f = B'_f$ because $A_f = A'_f$. It follows that $B \to B'$ is as in Lemma 33.39.2 for B. Thus we have to show that $\text{length}_A(A'/A) = \text{length}_B(B'/B) = \text{length}_B((A'/A) \otimes_A B)$. Since $A \to B$ is flat (as a localization of $A \to A \otimes_k K$) and since $\mathfrak{m}_B = \mathfrak{m}_A B$ (because $B/\mathfrak{m}_A B$ is zero dimensional by the remarks above and a localization of $K \otimes_k \kappa(x)$ which is reduced as $\kappa(x)$ is separable over k) we conclude by Algebra, Lemma 10.52.13. $\square$

33.40. The number of branches

0C3Z We have defined the number of branches of a scheme at a point in Properties, Section 28.15.

0C1S **Lemma 33.40.1.** Let X be a scheme. Assume every quasi-compact open of X has finitely many irreducible components. Let $\nu : X^\nu \to X$ be the normalization of X. Let $x \in X$.

(1) The number of branches of X at x is the number of inverse images of x in X^ν.
(2) The number of geometric branches of X at x is $\sum_{\nu(x^\nu)=x}[\kappa(x^\nu) : \kappa(x)]_s$.

Proof. First note that the assumption on X exactly means that the normalization is defined, see Morphisms, Definition 29.54.1. Then the stalk $A' = (\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in the total ring of fractions of A_{red}, see Morphisms, Lemma 29.54.4. Since ν is an integral morphism, we see that the points of X^ν lying over x correspond to the primes of A' lying over the maximal ideal $\mathfrak{m}$ of A. As $A \to A'$ is integral, this is the same thing as the maximal ideals of A' (Algebra, Lemmas 10.36.20 and 10.36.22). Thus the lemma now follows from its algebraic counterpart: More on Algebra, Lemma 15.106.7. $\square$

0C40 **Lemma 33.40.2.** Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $y \in X_K$ be a point with image x in X. Then the number of geometric branches of X at x is the number of geometric branches of X_K at y.

Proof. Write $Y = X_K$ and let X^ν, resp. Y^ν be the normalization of X, resp. Y. Consider the commutative diagram

$$\begin{tikzcd} Y^\nu \arrow[r] \arrow[d] & X^\nu_K \arrow[r] \arrow[d, "\nu_K"'] & X^\nu \arrow[d, "\nu"'] \\ Y \arrow[r, equal] & Y \arrow[r] & X \end{tikzcd}$$

By Lemma 33.27.5 we see that the left top horizontal arrow is a universal homeomorphism. Hence it induces purely inseparable residue field extensions, see Morphisms, Lemmas 29.45.5 and 29.10.2. Thus the number of geometric branches of Y at y is $\sum_{\nu_K(y')=y}[\kappa(y') : \kappa(y)]_s$ by Lemma 33.40.1. Similarly $\sum_{\nu(x')=x}[\kappa(x') : \kappa(x)]_s$ is the number of geometric branches of X at x. Using Schemes, Lemma 26.17.5 our statement follows from the following algebra fact: given a field extension l/κ and an algebraic field extension m/κ, then

$$\sum_{m\otimes_\kappa l\to m'}[m' : l']_s = [m : \kappa]_s$$

where the sum is over the quotient fields of $m \otimes_\kappa l$. One can prove this in an elementary way, or one can use Lemma 33.7.6 applied to

$$\mathrm{Spec}(m\otimes_\kappa l)\times_{\mathrm{Spec}(l)}\mathrm{Spec}(\bar{l}) = \mathrm{Spec}(m)\otimes_{\mathrm{Spec}(\kappa)}\mathrm{Spec}(\bar{l}) \longrightarrow \mathrm{Spec}(m)\times_{\mathrm{Spec}(\kappa)}\mathrm{Spec}(\overline{\kappa})$$

because one can interpret $[m : \kappa]_s$ as the number of connected components of the right hand side and the sum $\sum_{m\otimes_\kappa l\to m'}[m' : l']_s$ as the number of connected components of the left hand side. $\square$

0C55 **Lemma 33.40.3.** Let k be a field. Let X be a locally algebraic k-scheme. Let K/k be an extension of fields. Let $y \in X_K$ be a point with image x in X. Then X is geometrically unibranch at x if and only if X_K is geometrically unibranch at y.

Proof. Immediate from Lemma 33.40.2 and More on Algebra, Lemma 15.106.7. □

0C41 **Definition 33.40.4.** Let A and A_i, $1 \leq i \leq n$ be local rings. We say A is a *wedge of $A_1, \ldots, A_n$* if there exist isomorphisms

$$\kappa_{A_1} \to \kappa_{A_2} \to \ldots \to \kappa_{A_n}$$

and A is isomorphic to the ring consisting of n-tuples $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ which map to the same element of κ_{A_n}.

If we are given a base ring Λ and A and A_i are Λ-algebras, then we require $\kappa_{A_i} \to \kappa_{A_{i+1}}$ to be a Λ-algebra isomorphisms and A to be isomorphic as a Λ-algebra to the Λ-algebra consisting of n-tuples $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ which map to the same element of κ_{A_n}. In particular, if $\Lambda = k$ is a field and the maps $k \to \kappa_{A_i}$ are isomorphisms, then there is a unique choice for the isomorphisms $\kappa_{A_i} \to \kappa_{A_{i+1}}$ and we often speak of *the wedge of $A_1, \ldots, A_n$*.

0C42 **Lemma 33.40.5.** Let $(A, \mathfrak{m})$ be a strictly henselian 1-dimensional reduced Nagata local ring. Then

$$\delta\text{-invariant of } A \geq \text{number of geometric branches of } A - 1$$

If equality holds, then A is a wedge of $n \geq 1$ strictly henselian discrete valuation rings.

Proof. The number of geometric branches is equal to the number of branches of A (immediate from More on Algebra, Definition 15.106.6). Let $A \to A'$ be as in Lemma 33.39.2. Observe that the number of branches of A is the number of maximal ideals of A', see More on Algebra, Lemma 15.106.7. There is a surjection

$$A'/A \longrightarrow \left(\prod\nolimits_{\mathfrak{m}'} \kappa(\mathfrak{m}')\right)/\kappa(\mathfrak{m})$$

Since $\dim_{\kappa(\mathfrak{m})} \prod \kappa(\mathfrak{m}')$ is $\geq$ the number of branches, the inequality is obvious.

If equality holds, then $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m})$ for all $\mathfrak{m}' \subset A'$ and the displayed arrow above is an isomorphism. Since A is henselian and $A \to A'$ is finite, we see that A' is a product of local henselian rings, see Algebra, Lemma 10.153.4. The factors are the local rings $A'_{\mathfrak{m}'}$ and as A' is normal, these factors are discrete valuation rings (Algebra, Lemma 10.119.7). Since the displayed arrow is an isomorphism we see that A is indeed the wedge of these local rings. □

0C43 **Lemma 33.40.6.** Let $(A, \mathfrak{m})$ be a 1-dimensional reduced Nagata local ring. Then

$$\delta\text{-invariant of } A \geq \text{number of geometric branches of } A - 1$$

Proof. We may replace A by the strict henselization of A without changing the δ-invariant (Lemma 33.39.6) and without changing the number of geometric branches of A (this is immediate from the definition, see More on Algebra, Definition 15.106.6). Thus we may assume A is strictly henselian and we may apply Lemma 33.40.5. □

33.41. Normalization of one dimensional schemes

0C44 The normalization morphism of a Noetherian scheme of dimension 1 has unexpectedly good properties by the Krull-Akizuki result.

0C45 Lemma 33.41.1. Let X be a locally Noetherian scheme of dimension 1. Let $\nu : X^\nu \to X$ be the normalization. Then

(1) ν is integral, surjective, and induces a bijection on irreducible components,
(2) there is a factorization $X^\nu \to X_{red} \to X$ and the morphism $X^\nu \to X_{red}$ is the normalization of X_{red},
(3) $X^\nu \to X_{red}$ is birational,
(4) for every closed point $x \in X$ the stalk $(\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of $\mathcal{O}_{X,x}$ in the total ring of fractions of $(\mathcal{O}_{X,x})_{red} = \mathcal{O}_{X_{red},x}$,
(5) the fibres of ν are finite and the residue field extensions are finite,
(6) X^ν is a disjoint union of integral normal Noetherian schemes and each affine open is the spectrum of a finite product of Dedekind domains.

Proof. Many of the results are in fact general properties of the normalization morphism, see Morphisms, Lemmas 29.54.2, 29.54.4, 29.54.5, and 29.54.7. What is not clear is that the fibres are finite, that the induced residue field extensions are finite, and that X^ν locally looks like the spectrum of a Dedekind domain (and hence is Noetherian). To see this we may assume that $X = \mathrm{Spec}(A)$ is affine, Noetherian, dimension 1, and that A is reduced. Then we may use the description in Morphisms, Lemma 29.54.3 to reduce to the case where A is a Noetherian domain of dimension 1. In this case the desired properties follow from Krull-Akizuki in the form stated in Algebra, Lemma 10.120.18. $\square$

Of course there is a variant of the following lemma in case X is not reduced.

0C1R Lemma 33.41.2. Let X be a reduced Nagata scheme of dimension 1. Let $\nu : X^\nu \to X$ be the normalization. Let $x \in X$ denote a closed point. Then

(1) $\nu : X^\nu \to X$ is finite, surjective, and birational,
(2) $\mathcal{O}_X \subset \nu_*\mathcal{O}_{X^\nu}$ and $\nu_*\mathcal{O}_{X^\nu}/\mathcal{O}_X$ is a direct sum of skyscraper sheaves $\mathcal{Q}_x$ in the singular points x of X,
(3) $A' = (\nu_*\mathcal{O}_{X^\nu})_x$ is the integral closure of $A = \mathcal{O}_{X,x}$ in its total ring of fractions,
(4) $\mathcal{Q}_x = A'/A$ has finite length equal to the δ-invariant of X at x,
(5) A' is a semi-local ring which is a finite product of Dedekind domains,
(6) $A^\wedge$ is a reduced Noetherian complete local ring of dimension 1,
(7) $(A')^\wedge$ is the integral closure of $A^\wedge$ in its total ring of fractions,
(8) $(A')^\wedge$ is a finite product of complete discrete valuation rings, and
(9) $A'/A \cong (A')^\wedge/A^\wedge$.

Proof. We may and will use all the results of Lemma 33.41.1. Finiteness of ν follows from Morphisms, Lemma 29.54.10. Since X is reduced, Nagata, of dimension 1, we see that the regular locus is a dense open $U \subset X$ by More on Algebra, Proposition 15.48.7. Since a regular scheme is normal, this shows that ν is an isomorphism over U. Since $\dim(X) \leq 1$ this implies that ν is not an isomorphism over a discrete set of closed points $x \in X$. In particular we see that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_{X^\nu} \to \bigoplus\nolimits_{x \in X \setminus U} \mathcal{Q}_x \to 0$$

As we have the description of the stalks of $\nu_*\mathcal{O}_{X^\nu}$ by Lemma 33.41.1, we conclude that $Q_x = A'/A$ indeed has length equal to the δ-invariant of X at x. Note that $Q_x \neq 0$ exactly when x is a singular point for example by Lemma 33.39.4. The description of A' as a product of semi-local Dedekind domains follows from Lemma 33.41.1 as well. The relationship between A, A', and $(A')^\wedge$ we have see in Lemma 33.39.5 (and its proof). □

33.42. Finding affine opens

09NF We continue the discussion started in Properties, Section 28.29. It turns out that we can find affines containing a finite given set of codimension 1 points on a separated scheme. See Proposition 33.42.7.

We will improve on the following lemma in Descent, Lemma 35.25.4.

09NG **Lemma 33.42.1.** Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If

(1) f is separated,
(2) there is an open covering $X = \bigcup U_i$ such that $f|_{U_i} : U_i \to Y$ is an open immersion, and
(3) if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,

then f is an open immersion.

Proof. Suppose that $y = f(x) = f(x')$. Pick a specialization $y_0 \leadsto y$ where y_0 is a generic point of an irreducible component of Y. Since f is locally on the source an isomorphism we can pick specializations $x_0 \leadsto x$ and $x'_0 \leadsto x'$ mapping to $y_0 \leadsto y$. Note that $x_0, x'_0 \in X^0$. Hence $x_0 = x'_0$ by assumption (3). As f is separated we conclude that $x = x'$. Thus f is an open immersion. □

09NH **Lemma 33.42.2.** Let $X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. If

(1) $\mathcal{O}_{X,x} = \mathcal{O}_{S,s}$,
(2) X is reduced,
(3) $X \to S$ is of finite type, and
(4) S has finitely many irreducible components,

then there exists an open neighbourhood U of x such that $f|_U$ is an open immersion.

Proof. We may remove the (finitely many) irreducible components of S which do not contain s. We may replace S by an affine open neighbourhood of s. We may replace X by an affine open neighbourhood of x. Say $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime ideal corresponding to x, resp. s. As A is a reduced and all of the minimal primes of A are contained in $\mathfrak{p}$ we see that $A \subset A_\mathfrak{p}$. As $X \to S$ is of finite type, B is of finite type over A. Let $b_1, \ldots, b_n \in B$ be elements which generate B over A Since $A_\mathfrak{p} = B_\mathfrak{q}$ we can find $f \in A$, $f \notin \mathfrak{p}$ and $a_i \in A$ such that b_i and a_i/f have the same image in $B_\mathfrak{q}$. Thus we can find $g \in B$, $g \notin \mathfrak{q}$ such that $g(fb_i - a_i) = 0$ in B. It follows that the image of $A_f \to B_{fg}$ contains the images of $b_1, \ldots, b_n$, in particular also the image of g. Choose $n \geq 0$ and $f' \in A$ such that f'/f^n maps to the image of g in B_{fg}. Since $A_\mathfrak{p} = B_\mathfrak{q}$ we see that $f' \notin \mathfrak{p}$. We conclude that $A_{ff'} \to B_{fg}$ is surjective. Finally, as $A_{ff'} \subset A_\mathfrak{p} = B_\mathfrak{q}$ (see above) the map $A_{ff'} \to B_{fg}$ is injective, hence an isomorphism. □

09NI Lemma 33.42.3. Let $f : T \to X$ be a morphism of schemes. Let X^0, resp. T^0 denote the sets of generic points of irreducible components. Let $t_1, \ldots, t_m \in T$ be a finite set of points with images $x_j = f(t_j)$. If

(1) T is affine,
(2) X is quasi-separated,
(3) X^0 is finite
(4) $f(T^0) \subset X^0$ and $f : T^0 \to X^0$ is injective, and
(5) $\mathcal{O}_{X,x_j} = \mathcal{O}_{T,t_j}$,

then there exists an affine open of X containing $x_1, \ldots, x_r$.

Proof. Using Limits, Proposition 32.11.2 there is an immediate reduction to the case where X and T are reduced. Details omitted.

Assume X and T are reduced. We may write $T = \lim_{i \in I} T_i$ as a directed limit of schemes of finite presentation over X with affine transition morphisms, see Limits, Lemma 32.7.2. Pick $i \in I$ such that T_i is affine, see Limits, Lemma 32.4.13. Say $T_i = \text{Spec}(R_i)$ and $T = \text{Spec}(R)$. Let $R' \subset R$ be the image of $R_i \to R$. Then $T' = \text{Spec}(R')$ is affine, reduced, of finite type over X, and $T \to T'$ dominant. For $j = 1, \ldots, r$ let $t'_j \in T'$ be the image of t_j. Consider the local ring maps

$$\mathcal{O}_{X,x_j} \to \mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$$

Denote $(T')^0$ the set of generic points of irreducible components of T'. Let $\xi \rightsquigarrow t'_j$ be a specialization with $\xi \in (T')^0$. As $T \to T'$ is dominant we can choose $\eta \in T^0$ mapping to ξ (warning: a priori we do not know that η specializes to t_j). Assumption (3) applied to η tells us that the image θ of ξ in X corresponds to a minimal prime of $\mathcal{O}_{X,x_j}$. Lifting ξ via the isomorphism of (5) we obtain a specialization $\eta' \rightsquigarrow t_j$ with $\eta' \in T^0$ mapping to $\theta \rightsquigarrow x_j$. The injectivity of (4) shows that $\eta = \eta'$. Thus every minimal prime of $\mathcal{O}_{T',t'_j}$ lies below a minimal prime of $\mathcal{O}_{T,t_j}$. We conclude that $\mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$ is injective, hence both maps above are isomorphisms.

By Lemma 33.42.2 there exists an open $U \subset T'$ containing all the points t'_j such that $U \to X$ is a local isomorphism as in Lemma 33.42.1. By that lemma we see that $U \to X$ is an open immersion. Finally, by Properties, Lemma 28.29.5 we can find an open $W \subset U \subset T'$ containing all the t'_j. The image of W in X is the desired affine open. □

09NJ Lemma 33.42.4. Let X be an integral separated scheme. Let $x_1, \ldots, x_r \in X$ be a finite set of points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.

Proof. Let K be the field of rational functions of X. Set $A_i = \mathcal{O}_{X,x_i}$. Then $A_i \subset K$ and K is the fraction field of A_i. Since X is separated, and $x_i \neq x_j$ there cannot be a valuation ring $\mathcal{O} \subset K$ dominating both A_i and A_j. Namely, considering the diagram

$$\begin{tikzcd} \text{Spec}(\mathcal{O}) \arrow[r] \arrow[d] & \text{Spec}(A_1) \arrow[d] \\ \text{Spec}(A_2) \arrow[r] & X \end{tikzcd}$$

and applying the valuative criterion of separatedness (Schemes, Lemma 26.22.1) we would get $x_i = x_j$. Thus we see by Lemma 33.37.3 that $A_i \otimes A_j \to K$ is surjective for all $i \neq j$. By Lemma 33.37.7 we see that $A = A_1 \cap \ldots \cap A_r$ is a Noetherian semi-local ring with exactly r maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ such that $A_i = A_{\mathfrak{m}_i}$. Moreover,

$$\mathrm{Spec}(A) = \mathrm{Spec}(A_1) \cup \ldots \cup \mathrm{Spec}(A_r)$$

is an open covering and the intersection of any two pieces of this covering is $\mathrm{Spec}(K)$. Thus the given morphisms $\mathrm{Spec}(A_i) \to X$ glue to a morphism of schemes

$$\mathrm{Spec}(A) \longrightarrow X$$

mapping $\mathfrak{m}_i$ to x_i and inducing isomorphisms of local rings. Thus the result follows from Lemma 33.42.3. □

09NK Lemma 33.42.5. Let A be a ring, $I \subset A$ an ideal, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ primes of A, and $\overline{f} \in A/I$ an element. If $I \not\subset \mathfrak{p}_i$ for all i, then there exists an $f \in A$, $f \notin \mathfrak{p}_i$ which maps to $\overline{f}$ in A/I.

Proof. We may assume there are no inclusion relations among the $\mathfrak{p}_i$ (by removing the smaller primes). First pick any $f \in A$ lifting $\overline{f}$. Let S be the set $s \in \{1, \ldots, r\}$ such that $f \in \mathfrak{p}_s$. If S is empty we are done. If not, consider the ideal $J = I \prod_{i \notin S} \mathfrak{p}_i$. Note that J is not contained in $\mathfrak{p}_s$ for $s \in S$ because there are no inclusions among the $\mathfrak{p}_i$ and because I is not contained in any $\mathfrak{p}_i$. Hence we can choose $g \in J$, $g \notin \mathfrak{p}_s$ for $s \in S$ by Algebra, Lemma 10.15.2. Then $f + g$ is a solution to the problem posed by the lemma. □

09NM Lemma 33.42.6. Let X be a scheme. Let $T \subset X$ be finite set of points. Assume

(1) X has finitely many irreducible components $Z_1, \ldots, Z_t$, and
(2) $Z_i \cap T$ is contained in an affine open of the reduced induced subscheme corresponding to Z_i.

Then there exists an affine open subscheme of X containing T.

Proof. Using Limits, Proposition 32.11.2 there is an immediate reduction to the case where X is reduced. Details omitted. In the rest of the proof we endow every closed subset of X with the induced reduced closed subscheme structure.

We argue by induction that we can find an affine open $U \subset Z_1 \cup \ldots \cup Z_r$ containing $T \cap (Z_1 \cup \ldots \cup Z_r)$. For $r = 1$ this holds by assumption. Say $r > 1$ and let $U \subset Z_1 \cup \ldots \cup Z_{r-1}$ be an affine open containing $T \cap (Z_1 \cup \ldots \cup Z_{r-1})$. Let $V \subset X_r$ be an affine open containing $T \cap Z_r$ (exists by assumption). Then $U \cap V$ contains $T \cap (Z_1 \cup \ldots \cup Z_{r-1}) \cap Z_r$. Hence

$$\Delta = (U \cap Z_r) \setminus (U \cap V)$$

does not contain any element of T. Note that Δ is a closed subset of U. By prime avoidance (Algebra, Lemma 10.15.2), we can find a standard open U' of U containing $T \cap U$ and avoiding Δ, i.e., $U' \cap Z_r \subset U \cap V$. After replacing U by U' we may assume that $U \cap V$ is closed in U.

Using that by the same arguments as above also the set $\Delta' = (U \cap (Z_1 \cup \ldots \cup Z_{r-1})) \setminus (U \cap V)$ does not contain any element of T we find a $h \in \mathcal{O}(V)$ such that $D(h) \subset V$ contains $T \cap V$ and such that $U \cap D(h) \subset U \cap V$. Using that $U \cap V$ is closed in U we can use Lemma 33.42.5 to find an element $g \in \mathcal{O}(U)$ whose restriction to $U \cap V$ equals the restriction of h to $U \cap V$ and such that $T \cap U \subset D(g)$. Then we can replace U by $D(g)$ and V by $D(h)$ to reach the situation where $U \cap V$ is closed in both U and V. In this case the scheme $U \cup V$ is affine by Limits, Lemma 32.11.3. This proves the induction step and thereby the lemma. □

Here is a conclusion we can draw from the material above.

09NN Proposition 33.42.7. Let X be a separated scheme such that every quasi-compact open has a finite number of irreducible components. Let $x_1, \ldots, x_r \in X$ be points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.

Proof. We can replace X by a quasi-compact open containing $x_1, \ldots, x_r$ hence we may assume that X has finitely many irreducible components. By Lemma 33.42.6 we reduce to the case where X is integral. This case is Lemma 33.42.4. □

33.43. Curves

0A22 In the Stacks project we will use the following as our definition of a curve.

0A23 Definition 33.43.1. Let k be a field. A curve is a variety of dimension 1 over k.

Two standard examples of curves over k are the affine line $\mathbf{A}^1_k$ and the projective line $\mathbf{P}^1_k$. The scheme $X = \text{Spec}(k[x,y]/(f))$ is a curve if and only if $f \in k[x,y]$ is irreducible.

Our definition of a curve has the same problems as our definition of a variety, see the discussion following Definition 33.3.1. Moreover, it means that every curve comes with a specified field of definition. For example $X = \text{Spec}(\mathbf{C}[x])$ is a curve over $\mathbf{C}$ but we can also view it as a curve over $\mathbf{R}$. The scheme $\text{Spec}(\mathbf{Z})$ isn't a curve, even though the schemes $\text{Spec}(\mathbf{Z})$ and $\mathbf{A}^1_{\mathbf{F}_p}$ behave similarly in many respects.

0A24 Lemma 33.43.2. Let X be a separated, irreducible scheme of dimension > 0 over a field k. Let $x \in X$ be a closed point. The open subscheme $X \setminus \{x\}$ is not proper over k.

Proof. Since X is irreducible, $U = X \setminus \{x\}$ is not closed in X. In particular, the immersion $U \to X$ is not proper. By Morphisms, Lemma 29.41.7 (here we use X is separated), $U \to \text{Spec}(k)$ is not proper either. □

0A25 Lemma 33.43.3. Let X be a separated finite type scheme over a field k. If $\dim(X) \leq 1$ then X is H-quasi-projective over k.

Proof. By Proposition 33.38.12 the scheme X has an ample invertible sheaf $\mathcal{L}$. By Morphisms, Lemma 29.39.3 we see that X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ over $\text{Spec}(k)$. This is the definition of being H-quasi-projective over k, see Morphisms, Definition 29.40.1. □

0A26 Lemma 33.43.4. Let X be a proper scheme over a field k. If $\dim(X) \leq 1$ then X is H-projective over k.

Proof. By Lemma 33.43.3 we see that X is a locally closed subscheme of $\mathbf{P}^n_k$ for some field k. Since X is proper over k it follows that X is a closed subscheme of $\mathbf{P}^n_k$ (Morphisms, Lemma 29.41.7). □

0BXV Lemma 33.43.5. Let X be a separated scheme of finite type over k. If $\dim(X) \leq 1$, then there exists an open immersion $j : X \to \overline{X}$ with the following properties

(1) $\overline{X}$ is H-projective over k, i.e., $\overline{X}$ is a closed subscheme of $\mathbf{P}^d_k$ for some d,
(2) $j(X) \subset \overline{X}$ is dense and scheme theoretically dense,
(3) $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ for some closed points $x_i \in \overline{X}$.

Proof. By Lemma 33.43.3 we may assume X is a locally closed subscheme of $\mathbf{P}^d_k$ for some d. Let $\overline{X} \subset \mathbf{P}^d_k$ be the scheme theoretic image of $X \to \mathbf{P}^d_k$, see Morphisms, Definition 29.6.2. The description in Morphisms, Lemma 29.7.7 gives properties (1) and (2). Then $\dim(X) = 1 \Rightarrow \dim(\overline{X}) = 1$ for example by looking at generic points, see Lemma 33.20.3. As $\overline{X}$ is Noetherian, it then follows that $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ is a finite set of closed points. $\square$

0BXW Lemma 33.43.6. Let X be a separated scheme of finite type over k. If X is reduced and $\dim(X) \leq 1$, then there exists an open immersion $j : X \to \overline{X}$ such that

(1) $\overline{X}$ is H-projective over k, i.e., $\overline{X}$ is a closed subscheme of $\mathbf{P}^d_k$ for some d,
(2) $j(X) \subset \overline{X}$ is dense and scheme theoretically dense,
(3) $\overline{X} \setminus X = \{x_1, \ldots, x_n\}$ for some closed points $x_i \in \overline{X}$,
(4) the local rings $\mathcal{O}_{\overline{X},x_i}$ are discrete valuation rings for $i = 1, \ldots, n$.

Proof. Let $j : X \to \overline{X}$ be as in Lemma 33.43.5. Consider the normalization X' of $\overline{X}$ in X. By Lemma 33.27.3 the morphism $X' \to \overline{X}$ is finite. By Morphisms, Lemma 29.44.16 $X' \to \overline{X}$ is projective. By Morphisms, Lemma 29.43.16 we see that $X' \to \overline{X}$ is H-projective. By Morphisms, Lemma 29.43.7 we see that $X' \to \mathrm{Spec}(k)$ is H-projective. Let $\{x'_1, \ldots, x'_m\} \subset X'$ be the inverse image of $\{x_1, \ldots, x_n\} = \overline{X} \setminus X$. Then $\dim(\mathcal{O}_{X',x'_i}) = 1$ for all $1 \leq i \leq m$. Hence the local rings $\mathcal{O}_{X',x'}$ are discrete valuation rings by Morphisms, Lemma 29.53.16. Then $X \to X'$ and $\{x'_1, \ldots, x'_m\}$ is as desired. $\square$

0GK5 Lemma 33.43.7. Let X be a separated scheme of finite type over k with $\dim(X) \leq 1$. Then there exists a commutative diagram

$$\begin{tikzcd}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n \arrow[dr] & Y_1 \amalg \ldots \amalg Y_n \arrow[l, "j"'] \arrow[r, "\nu"] \arrow[d] & X_{k'} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
 & \mathrm{Spec}(k'_1) \amalg \ldots \amalg \mathrm{Spec}(k'_n) \arrow[r] & \mathrm{Spec}(k') \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

of schemes with the following properties:

(1) k'/k is a finite purely inseparable extension of fields,
(2) ν is the normalization of $X_{k'}$,
(3) j is an open immersion with dense image,
(4) k'_i/k' is a finite separable extension for $i = 1, \ldots, n$,
(5) $\overline{Y}_i$ is smooth, projective, geometrically irreducible dimension ≤ 1 over k'_i.

Proof. As we may replace X by its reduction, we may and do assume X is reduced. Choose $X \to \overline{X}$ as in Lemma 33.43.6. If we can show the lemma for $\overline{X}$, then the lemma follows for X (details omitted). Thus we may and do assume X is projective.

Choose k'/k finite purely inseparable such that the normalization of $X_{k'}$ is geometrically normal over k', see Lemma 33.27.4. Denote $Y = (X_{k'})^\nu$ the normalization; for properties of the normalization, see Section 33.27. Then Y is geometrically regular as normal and regular are the same in dimension ≤ 1, see Properties, Lemma 28.12.6. Hence Y is smooth over k' by Lemma 33.12.6. Let $Y = Y_1 \amalg \ldots \amalg Y_n$ be the decomposition of Y into irreducible components. Set $k'_i = \Gamma(Y_i, \mathcal{O}_{Y_i})$. These are finite separable extensions of k' by Lemma 33.9.3. The proof is finished by Lemma 33.9.4. $\square$

0B8Y Lemma 33.43.8. Let k be a field. Let X be a curve over k. Let $x \in X$ be a closed point. We think of x as a (reduced) closed subscheme of X with sheaf of ideals $\mathcal{I}$. The following are equivalent

(1) $\mathcal{O}_{X,x}$ is regular,
(2) $\mathcal{O}_{X,x}$ is normal,
(3) $\mathcal{O}_{X,x}$ is a discrete valuation ring,
(4) $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module,
(5) x is an effective Cartier divisor on X.

If k is perfect or if $\kappa(x)$ is separable over k, these are also equivalent to

(6) $X \to \mathrm{Spec}(k)$ is smooth at x.

Proof. Since X is a curve, the local ring $\mathcal{O}_{X,x}$ is a Noetherian local domain of dimension 1 (Lemma 33.20.3). Parts (4) and (5) are equivalent by definition and are equivalent to $\mathcal{I}_x = \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ having one generator (Divisors, Lemma 31.15.2). The equivalence of (1), (2), (3), (4), and (5) therefore follows from Algebra, Lemma 10.119.7. The final statement follows from Lemma 33.25.8 in case k is perfect. If $\kappa(x)/k$ is separable, then the equivalence follows from Algebra, Lemma 10.140.5. $\square$

0H1F Remark 33.43.9. Let k be a field. Let X be a regular curve over k. By Lemmas 33.43.8 and 33.43.6 there exists a nonsingular projective curve $\overline{X}$ which is a compactification of X, i.e., there exists an open immersion $j : X \to \overline{X}$ such that the complement consists of a finite number of closed points. If k is perfect, then X and $\overline{X}$ are smooth over k and $\overline{X}$ is a smooth projective compactification of X.

Observe that if an affine scheme X over k is proper over k then X is finite over k (Morphisms, Lemma 29.44.11) and hence has dimension 0 (Algebra, Lemma 10.53.2 and Proposition 10.60.7). Hence a scheme of dimension > 0 over k cannot be both affine and proper over k. Thus the possibilities in the following lemma are mutually exclusive.

0A27 Lemma 33.43.10. Let X be a curve over k. Then either X is an affine scheme or X is H-projective over k.

Proof. Choose $X \to \overline{X}$ with $\overline{X} \setminus X = \{x_1, \ldots, x_r\}$ as in Lemma 33.43.6. Then $\overline{X}$ is a curve as well. If $r = 0$, then $X = \overline{X}$ is H-projective over k. Thus we may assume $r \geq 1$ and our goal is to show that X is affine. By Lemma 33.38.2 it suffices to show that $\overline{X} \setminus \{x_1\}$ is affine. This reduces us to the claim stated in the next paragraph.

Let X be an H-projective curve over k. Let $x \in X$ be a closed point such that $\mathcal{O}_{X,x}$ is a discrete valuation ring. Claim: $U = X \setminus \{x\}$ is affine. By Lemma 33.43.8 the point x defines an effective Cartier divisor of X. For $n \geq 1$ denote $nx = x + \ldots + x$ the n-fold sum, see Divisors, Definition 31.13.6. Denote $\mathcal{O}_{nx}$ the structure sheaf of nx viewed as a coherent module on X. Since every invertible module on the local scheme nx is trivial the first short exact sequence of Divisors, Remark 31.14.11 reads

$$0 \to \mathcal{O}_X \xrightarrow{1} \mathcal{O}_X(nx) \to \mathcal{O}_{nx} \to 0$$

in our case. Note that $\dim_k H^0(X, \mathcal{O}_{nx}) \geq n$. Namely, by Lemma 33.33.3 we have $H^0(X, \mathcal{O}_{nx}) = \mathcal{O}_{X,x}/(\pi^n)$ where π in $\mathcal{O}_{X,x}$ is a uniformizer and the powers π^i map to k-linearly independent elements in $\mathcal{O}_{X,x}/(\pi^n)$ for $i = 0, 1, \ldots, n-1$. We have $\dim_k H^1(X, \mathcal{O}_X) < \infty$ by Cohomology of Schemes, Lemma 30.19.2. If $n > \dim_k H^1(X, \mathcal{O}_X)$ we conclude from the long exact cohomology sequence that there exists an $s \in \Gamma(X, \mathcal{O}_X(nx))$ which is not a section of $\mathcal{O}_X$. If we take n minimal with this property, then s will map to a generator of the stalk $(\mathcal{O}_X(nx))_x$ since otherwise it would define a section of $\mathcal{O}_X((n-1)x) \subset \mathcal{O}_X(nx)$. For this n we conclude that $s_0 = 1$ and $s_1 = s$ generate the invertible module $\mathcal{L} = \mathcal{O}_X(nx)$.

Consider the corresponding morphism $f = \varphi_{\mathcal{L},(s_0,s_1)} : X \to \mathbf{P}^1_k$ of Constructions, Section 27.13. Observe that the inverse image of $D_+(T_0)$ is $U = X \setminus \{x\}$ as the section s_0 of $\mathcal{L}$ only vanishes at x. In particular, f is non-constant, i.e., $\text{Im}(f)$ has more than one point. Hence f must map the generic point η of X to the generic point of $\mathbf{P}^1_k$. Hence if $y \in \mathbf{P}^1_k$ is a closed point, then $f^{-1}(\{y\})$ is a closed set of X not containing η, hence finite. Finally, f is proper[4]. By Cohomology of Schemes, Lemma 30.21.2[5] we conclude that f is finite. Hence $U = f^{-1}(D_+(T_0))$ is affine. $\square$

The following lemma combined with Lemma 33.43.2 tells us that given a separated scheme X of dimension 1 and of finite type over k, then $X \setminus Z$ is affine, whenever the closed subset Z meets every irreducible component of X.

0A28 Lemma 33.43.11. Let X be a separated scheme of finite type over k. If $\dim(X) \leq 1$ and no irreducible component of X is proper of dimension 1, then X is affine.

Proof. Let $X = \bigcup X_i$ be the decomposition of X into irreducible components. We think of X_i as an integral scheme (using the reduced induced scheme structure, see Schemes, Definition 26.12.5). In particular X_i is a singleton (hence affine) or a curve hence affine by Lemma 33.43.10. Then $\coprod X_i \to X$ is finite surjective and $\coprod X_i$ is affine. Thus we see that X is affine by Cohomology of Schemes, Lemma 30.13.3. $\square$

33.44. Degrees on curves

0AYQ We start defining the degree of an invertible sheaf and more generally a locally free sheaf on a proper scheme of dimension 1 over a field. In Section 33.33 we defined the Euler characteristic of a coherent sheaf $\mathcal{F}$ on a proper scheme X over a field k by the formula

$$\chi(X, \mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

0AYR Definition 33.44.1. Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The degree of $\mathcal{L}$ is defined by

$$\deg(\mathcal{L}) = \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X)$$

[4]Namely, a H-projective variety is a proper variety by Morphisms, Lemma 29.43.13. A morphism of varieties whose source is a proper variety is a proper morphism by Morphisms, Lemma 29.41.7.

[5]One can avoid using this lemma which relies on the theorem of formal functions. Namely, X is projective hence it suffices to show a proper morphism $f : X \to Y$ with finite fibres between quasi-projective schemes over k is finite. To do this, one chooses an affine open of X containing the fibre of f over a point y using that any finite set of points of a quasi-projective scheme over k is contained in an affine. Shrinking Y to a small affine neighbourhood of y one reduces to the case of a proper morphism between affines. Such a morphism is finite by Morphisms, Lemma 29.44.7.

More generally, if $\mathcal{E}$ is a locally free sheaf of rank n we define the degree of $\mathcal{E}$ by

$$\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - n\chi(X, \mathcal{O}_X)$$

Observe that this depends on the triple $\mathcal{E}/X/k$. If X is disconnected and $\mathcal{E}$ is finite locally free (but not of constant rank), then one can modify the definition by summing the degrees of the restriction of $\mathcal{E}$ to the connected components of X. If $\mathcal{E}$ is just a coherent sheaf, there are several different ways of extending the definition[6]. In a series of lemmas we show that this definition has all the properties one expects of the degree.

0B59 Lemma 33.44.2. Let k'/k be an extension of fields. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of constant rank n. Then the degree of $\mathcal{E}/X/k$ is equal to the degree of $\mathcal{E}_{k'}/X_{k'}/k'$.

Proof. More precisely, set $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. Let $\mathcal{E}_{k'} = p^*\mathcal{E}$ where $p : X_{k'} \to X$ is the projection. By Cohomology of Schemes, Lemma 30.5.2 we have $H^i(X_{k'}, \mathcal{E}_{k'}) = H^i(X, \mathcal{E}) \otimes_k k'$ and $H^i(X_{k'}, \mathcal{O}_{X_{k'}}) = H^i(X, \mathcal{O}_X) \otimes_k k'$. Hence we see that the Euler characteristics are unchanged, hence the degree is unchanged. □

0AYS Lemma 33.44.3. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ be a short exact sequence of locally free $\mathcal{O}_X$-modules each of finite constant rank. Then

$$\deg(\mathcal{E}_2) = \deg(\mathcal{E}_1) + \deg(\mathcal{E}_3)$$

Proof. Follows immediately from additivity of Euler characteristics (Lemma 33.33.2) and additivity of ranks. □

0AYU Lemma 33.44.4. Let k be a field. Let $f : X' \to X$ be a birational morphism of proper schemes of dimension ≤ 1 over k. Then

$$\deg(f^*\mathcal{E}) = \deg(\mathcal{E})$$

for every finite locally free sheaf of constant rank. More generally it suffices if f induces a bijection between irreducible components of dimension 1 and isomorphisms of local rings at the corresponding generic points.

Proof. The morphism f is proper (Morphisms, Lemma 29.41.7) and has fibres of dimension ≤ 0. Hence f is finite (Cohomology of Schemes, Lemma 30.21.2). Thus

$$Rf_*f^*\mathcal{E} = f_*f^*\mathcal{E} = \mathcal{E} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_{X'}$$

Since f induces an isomorphism on local rings at generic points of all irreducible components of dimension 1 we see that the kernel and cokernel

$$0 \to \mathcal{K} \to \mathcal{O}_X \to f_*\mathcal{O}_{X'} \to \mathcal{Q} \to 0$$

[6] If X is a proper curve and $\mathcal{F}$ is a coherent sheaf on X, then one often defines the degree as $\chi(X, \mathcal{F}) - r\chi(X, \mathcal{O}_X)$ where $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is the rank of $\mathcal{F}$ at the generic point ξ of X.

have supports of dimension ≤ 0. Note that tensoring this with $\mathcal{E}$ is still an exact sequence as $\mathcal{E}$ is locally free. We obtain

$$\begin{aligned}\chi(X, \mathcal{E}) - \chi(X', f^*\mathcal{E}) &= \chi(X, \mathcal{E}) - \chi(X, f_*f^*\mathcal{E}) \\ &= \chi(X, \mathcal{E}) - \chi(X, \mathcal{E} \otimes f_*\mathcal{O}_{X'}) \\ &= \chi(X, \mathcal{K} \otimes \mathcal{E}) - \chi(X, \mathcal{Q} \otimes \mathcal{E}) \\ &= n\chi(X, \mathcal{K}) - n\chi(X, \mathcal{Q}) \\ &= n\chi(X, \mathcal{O}_X) - n\chi(X, f_*\mathcal{O}_{X'}) \\ &= n\chi(X, \mathcal{O}_X) - n\chi(X', \mathcal{O}_{X'})\end{aligned}$$

which proves what we want. The first equality as f is finite, see Cohomology of Schemes, Lemma 30.2.4. The second equality by projection formula, see Cohomology, Lemma 20.54.2. The third by additivity of Euler characteristics, see Lemma 33.33.2. The fourth by Lemma 33.33.3. □

0AYV Lemma 33.44.5. Let k be a field. Let X be a proper curve over k with generic point ξ. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then

$$\chi(X, \mathcal{E} \otimes \mathcal{F}) = r \deg(\mathcal{E}) + n\chi(X, \mathcal{F})$$

where $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is the rank of $\mathcal{F}$.

Proof. Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X expressing that the formula of the lemma holds. We claim that the assumptions (1) and (2) of Cohomology of Schemes, Lemma 30.12.6 hold for $\mathcal{P}$. Namely, (1) holds because the Euler characteristic and the rank r are additive in short exact sequences of coherent sheaves. And (2) holds too: If $Z = X$ then we may take $\mathcal{G} = \mathcal{O}_X$ and $\mathcal{P}(\mathcal{O}_X)$ is true by the definition of degree. If $i : Z \to X$ is the inclusion of a closed point we may take $\mathcal{G} = i_*\mathcal{O}_Z$ and $\mathcal{P}$ holds by Lemma 33.33.3 and the fact that $r = 0$ in this case. □

Let k be a field. Let X be a finite type scheme over k of dimension ≤ 1. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1. We view C_i as a scheme by using the induced reduced scheme structure. Let $\xi_i \in C_i$ be the generic point. The multiplicity of C_i in X is defined as the length

$$m_i = \text{length}_{\mathcal{O}_{X,\xi_i}} \mathcal{O}_{X,\xi_i}$$

This makes sense because $\mathcal{O}_{X,\xi_i}$ is a zero dimensional Noetherian local ring and hence has finite length over itself (Algebra, Proposition 10.60.7). See Chow Homology, Section 42.9 for additional information. It turns out the degree of a locally free sheaf only depends on the restriction of the irreducible components.

0AYW Lemma 33.44.6. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n. Then

$$\deg(\mathcal{E}) = \sum m_i \deg(\mathcal{E}|_{C_i})$$

where $C_i \subset X$, $i = 1, \ldots, t$ are the irreducible components of dimension 1 with reduced induced scheme structure and m_i is the multiplicity of C_i in X.

Proof. Observe that the statement makes sense because $C_i \to \text{Spec}(k)$ is proper of dimension 1 (Morphisms, Lemmas 29.41.6 and 29.41.4). Consider the open subscheme $U_i = X \setminus (\bigcup_{j \neq i} C_j)$ and let $X_i \subset X$ be the scheme theoretic closure of U_i. Note that $X_i \cap U_i = U_i$ (scheme theoretically) and that $X_i \cap U_j = \emptyset$ (set theoretically) for $i \neq j$; this follows from the description of scheme theoretic closure in Morphisms, Lemma 29.7.7. Thus we may apply Lemma 33.44.4 to the morphism $X' = \bigcup X_i \to X$. Since it is clear that $C_i \subset X_i$ (scheme theoretically) and that the multiplicity of C_i in X_i is equal to the multiplicity of C_i in X, we see that we reduce to the case discussed in the following paragraph.

Assume X is irreducible with generic point ξ. Let $C = X_{red}$ have multiplicity m. We have to show that $\deg(\mathcal{E}) = m \deg(\mathcal{E}|_C)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal defining the closed subscheme C. Let $e \geq 0$ be minimal such that $\mathcal{I}^{e+1} = 0$ (Cohomology of Schemes, Lemma 30.10.2). We argue by induction on e. If $e = 0$, then $X = C$ and the result is immediate. Otherwise we set $\mathcal{F} = \mathcal{I}^e$ viewed as a coherent $\mathcal{O}_C$-module (Cohomology of Schemes, Lemma 30.9.8). Let $X' \subset X$ be the closed subscheme cut out by the coherent ideal $\mathcal{I}^e$ and let m' be the multiplicity of C in X'. Taking stalks at ξ of the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

we find (use Algebra, Lemmas 10.52.3, 10.52.6, and 10.52.5) that

$$m = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{X,\xi} = \dim_{\kappa(\xi)} \mathcal{F}_\xi + \text{length}_{\mathcal{O}_{X',\xi}} \mathcal{O}_{X',\xi} = r + m'$$

where r is the rank of $\mathcal{F}$ as a coherent sheaf on C. Tensoring with $\mathcal{E}$ we obtain a short exact sequence

$$0 \to \mathcal{E}|_C \otimes \mathcal{F} \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_{X'} \to 0$$

By induction we have $\chi(\mathcal{E} \otimes \mathcal{O}_{X'}) = m' \deg(\mathcal{E}|_C)$. By Lemma 33.44.5 we have $\chi(\mathcal{E}|_C \otimes \mathcal{F}) = r \deg(\mathcal{E}|_C) + n\chi(\mathcal{F})$. Putting everything together we obtain the result. $\square$

0AYX Lemma 33.44.7. Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{E}$, $\mathcal{V}$ be locally free $\mathcal{O}_X$-modules of constant finite rank. Then

$$\deg(\mathcal{E} \otimes \mathcal{V}) = \text{rank}(\mathcal{E}) \deg(\mathcal{V}) + \text{rank}(\mathcal{V}) \deg(\mathcal{E})$$

Proof. By Lemma 33.44.6 and elementary arithmetic, we reduce to the case of a proper curve. This case follows from Lemma 33.44.5. $\square$

0DJ5 Lemma 33.44.8. Let k be a field, let X be a proper scheme of dimension ≤ 1 over k, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n. Then

$$\deg(\mathcal{E}) = \deg(\wedge^n(\mathcal{E})) = \deg(\det(\mathcal{E}))$$

Proof. By Lemma 33.44.6 and elementary arithmetic, we reduce to the case of a proper curve. Then there exists a modification $f : X' \to X$ such that $f^*\mathcal{E}$ has a filtration whose successive quotients are invertible modules, see Divisors, Lemma 31.36.1. By Lemma 33.44.4 we may work on X'. Thus we may assume we have a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_n = \mathcal{E}$$

by locally free $\mathcal{O}_X$-modules with $\mathcal{L}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is invertible. By Modules, Lemma 17.26.1 and induction we find $\det(\mathcal{E}) = \mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n$. Thus the equality follows from Lemma 33.44.7 and additivity (Lemma 33.44.3). $\square$

0AYY **Lemma 33.44.9.** Let k be a field, let X be a proper scheme of dimension ≤ 1 over k. Let D be an effective Cartier divisor on X. Then D is finite over $\text{Spec}(k)$ of degree $\deg(D) = \dim_k \Gamma(D, \mathcal{O}_D)$. For a locally free sheaf $\mathcal{E}$ of rank n we have

$$\deg(\mathcal{E}(D)) = n \deg(D) + \deg(\mathcal{E})$$

where $\mathcal{E}(D) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$.

Proof. Since D is nowhere dense in X (Divisors, Lemma 31.13.4) we see that $\dim(D) \leq 0$. Hence D is finite over k by Lemma 33.20.2. Since k is a field, the morphism $D \to \text{Spec}(k)$ is finite locally free and hence has a degree (Morphisms, Definition 29.48.1), which is clearly equal to $\dim_k \Gamma(D, \mathcal{O}_D)$ as stated in the lemma. By Divisors, Definition 31.14.1 there is a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to i_*i^*\mathcal{O}_X(D) \to 0$$

where $i : D \to X$ is the closed immersion. Tensoring with $\mathcal{E}$ we obtain a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}(D) \to i_*i^*\mathcal{E}(D) \to 0$$

The equation of the lemma follows from additivity of the Euler characteristic (Lemma 33.33.2) and Lemma 33.33.3. $\square$

0C6P **Lemma 33.44.10.** Let k be a field. Let X be a proper scheme over k which is reduced and connected. Let $\kappa = H^0(X, \mathcal{O}_X)$. Then κ/k is a finite extension of fields and $w = [\kappa : k]$ divides

(1) $\deg(\mathcal{E})$ for all locally free $\mathcal{O}_X$-modules $\mathcal{E}$,
(2) $[\kappa(x) : k]$ for all closed points $x \in X$, and
(3) $\deg(D)$ for all closed subschemes $D \subset X$ of dimension zero.

Proof. See Lemma 33.9.3 for the assertions about κ. For every quasi-coherent $\mathcal{O}_X$-module, the k-vector spaces $H^i(X, \mathcal{F})$ are κ-vector spaces. The divisibilities easily follow from this statement and the definitions. $\square$

0AYZ **Lemma 33.44.11.** Let k be a field. Let $f : X \to Y$ be a nonconstant morphism of proper curves over k. Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module. Then

$$\deg(f^*\mathcal{E}) = \deg(X/Y) \deg(\mathcal{E})$$

Proof. The degree of X over Y is defined in Morphisms, Definition 29.51.8. Thus $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module of rank $\deg(X/Y)$, i.e., $\deg(X/Y) = \dim_{\kappa(\xi)}(f_*\mathcal{O}_X)_\xi$ where ξ is the generic point of Y. Thus we obtain

$$\begin{aligned} \chi(X, f^*\mathcal{E}) &= \chi(Y, f_*f^*\mathcal{E}) \\ &= \chi(Y, \mathcal{E} \otimes f_*\mathcal{O}_X) \\ &= \deg(X/Y) \deg(\mathcal{E}) + n\chi(Y, f_*\mathcal{O}_X) \\ &= \deg(X/Y) \deg(\mathcal{E}) + n\chi(X, \mathcal{O}_X) \end{aligned}$$

as desired. The first equality as f is finite, see Cohomology of Schemes, Lemma 30.2.4. The second equality by projection formula, see Cohomology, Lemma 20.54.2. The third equality by Lemma 33.44.5. $\square$

The following is a trivial but important consequence of the results on degrees above.

0B40 **Lemma 33.44.12.** Let k be a field. Let X be a proper curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) If $\mathcal{L}$ has a nonzero section, then $\deg(\mathcal{L}) \geq 0$.
(2) If $\mathcal{L}$ has a nonzero section s which vanishes at a point, then $\deg(\mathcal{L}) > 0$.
(3) If $\mathcal{L}$ and $\mathcal{L}^{-1}$ have nonzero sections, then $\mathcal{L} \cong \mathcal{O}_X$.
(4) If $\deg(\mathcal{L}) \leq 0$ and $\mathcal{L}$ has a nonzero section, then $\mathcal{L} \cong \mathcal{O}_X$.
(5) If $\mathcal{N} \to \mathcal{L}$ is a nonzero map of invertible $\mathcal{O}_X$-modules, then $\deg(\mathcal{L}) \geq \deg(\mathcal{N})$ and if equality holds then it is an isomorphism.

Proof. Let s be a nonzero section of $\mathcal{L}$. Since X is a curve, we see that s is a regular section. Hence there is an effective Cartier divisor $D \subset X$ and an isomorphism $\mathcal{L} \to \mathcal{O}_X(D)$ mapping s the canonical section 1 of $\mathcal{O}_X(D)$, see Divisors, Lemma 31.14.10. Then $\deg(\mathcal{L}) = \deg(D)$ by Lemma 33.44.9. As $\deg(D) \geq 0$ and $= 0$ if and only if $D = \emptyset$, this proves (1) and (2). In case (3) we see that $\deg(\mathcal{L}) = 0$ and $D = \emptyset$. Similarly for (4). To see (5) apply (1) and (4) to the invertible sheaf

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{L})$$

which has degree $\deg(\mathcal{L}) - \deg(\mathcal{N})$ by Lemma 33.44.7. ☐

0E22 Lemma 33.44.13. Let k be a field. Let X be a proper scheme over k which is reduced, connected, and equidimensional of dimension 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\deg(\mathcal{L}|_C) \leq 0$ for all irreducible components C of X, then either $H^0(X, \mathcal{L}) = 0$ or $\mathcal{L} \cong \mathcal{O}_X$.

Proof. Let $s \in H^0(X, \mathcal{L})$ be nonzero. Since X is reduced there exists an irreducible component C of X with $s|_C \neq 0$. But if $s|_C$ is nonzero, then s is nonwhere vanishing on C by Lemma 33.44.12. This in turn implies s is nowhere vanishing on every irreducible component of X meeting C. Since X is connected, we conclude that s vanishes nowhere and the lemma follows. ☐

0B5X Lemma 33.44.14. Let k be a field. Let X be a proper curve over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample if and only if $\deg(\mathcal{L}) > 0$.

Proof. If $\mathcal{L}$ is ample, then there exists an $n > 0$ and a section $s \in H^0(X, \mathcal{L}^{\otimes n})$ with X_s affine. Since X isn't affine (otherwise by Morphisms, Lemma 29.44.11 X would be finite), we see that s vanishes at some point. Hence $\deg(\mathcal{L}^{\otimes n}) > 0$ by Lemma 33.44.12. By Lemma 33.44.7 we conclude that $\deg(\mathcal{L}) = 1/n \deg(\mathcal{L}^{\otimes n}) > 0$.

Assume $\deg(\mathcal{L}) > 0$. Then

$$\dim_k H^0(X, \mathcal{L}^{\otimes n}) \geq \chi(X, \mathcal{L}^n) = n \deg(\mathcal{L}) + \chi(X, \mathcal{O}_X)$$

grows linearly with n. Hence for any finite collection of closed points $x_1, \ldots, x_t$ of X, we can find an n such that $\dim_k H^0(X, \mathcal{L}^{\otimes n}) > \sum \dim_k \kappa(x_i)$. (Recall that by Hilbert Nullstellensatz, the extension fields $\kappa(x_i)/k$ are finite, see for example Morphisms, Lemma 29.20.3). Hence we can find a nonzero $s \in H^0(X, \mathcal{L}^{\otimes n})$ vanishing in $x_1, \ldots, x_t$. In particular, if we choose $x_1, \ldots, x_t$ such that $X \setminus \{x_1, \ldots, x_t\}$ is affine, then X_s is affine too (for example by Properties, Lemma 28.26.4 although if we choose our finite set such that $\mathcal{L}|_{X \setminus \{x_1, \ldots, x_t\}}$ is trivial, then it is immediate). The conclusion is that we can find an $n > 0$ and a nonzero section $s \in H^0(X, \mathcal{L}^{\otimes n})$ such that X_s is affine.

We will show that for every quasi-coherent sheaf of ideals $\mathcal{I}$ there exists an $m > 0$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes m})$ is zero. This will finish the proof by Cohomology of Schemes, Lemma 30.17.1. To see this we consider the maps

$$\mathcal{I} \xrightarrow{s} \mathcal{I} \otimes \mathcal{L}^{\otimes n} \xrightarrow{s} \mathcal{I} \otimes \mathcal{L}^{\otimes 2n} \xrightarrow{s} \ldots$$

Since $\mathcal{I}$ is torsion free, these maps are injective and isomorphisms over X_s, hence the cokernels have vanishing H^1 (by Cohomology of Schemes, Lemma 30.9.10 for example). We conclude that the maps of vector spaces

$$H^1(X, \mathcal{I}) \to H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) \to H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes 2n}) \to \ldots$$

are surjective. On the other hand, the dimension of $H^1(X, \mathcal{I})$ is finite, and every element maps to zero eventually by Cohomology of Schemes, Lemma 30.17.4. Thus for some $e > 0$ we see that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes en})$ is zero. This finishes the proof. $\square$

0B5Y Lemma 33.44.15. Let k be a field. Let X be a proper scheme of dimension ≤ 1 over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $C_i \subset X$, $i = 1, \ldots, t$ be the irreducible components of dimension 1. The following are equivalent:

(1) $\mathcal{L}$ is ample, and
(2) $\deg(\mathcal{L}|_{C_i}) > 0$ for $i = 1, \ldots, t$.

Proof. Let $x_1, \ldots, x_r \in X$ be the isolated closed points. Think of $x_i = \mathrm{Spec}(\kappa(x_i))$ as a scheme. Consider the morphism of schemes

$$f : C_1 \amalg \ldots \amalg C_t \amalg x_1 \amalg \ldots \amalg x_r \longrightarrow X$$

This is a finite surjective morphism of schemes proper over k (details omitted). Thus $\mathcal{L}$ is ample if and only if $f^*\mathcal{L}$ is ample (Cohomology of Schemes, Lemma 30.17.2). Thus we conclude by Lemma 33.44.14. $\square$

0B8Z Lemma 33.44.16. Let k be an algebraically closed field. Let X be a proper curve over k. Then there exist

(1) an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\dim_k H^0(X, \mathcal{L}) = 1$ and $H^1(X, \mathcal{L}) = 0$, and
(2) an invertible $\mathcal{O}_X$-module $\mathcal{N}$ with $\dim_k H^0(X, \mathcal{N}) = 0$ and $H^1(X, \mathcal{N}) = 0$.

Proof. Choose a closed immersion $i : X \to \mathbf{P}^n_k$ (Lemma 33.43.4). Setting $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n}(d)$ for $d \gg 0$ we see that there exists an invertible sheaf $\mathcal{L}$ with $H^0(X, \mathcal{L}) \neq 0$ and $H^1(X, \mathcal{L}) = 0$ (see Cohomology of Schemes, Lemma 30.17.1 for vanishing and the references therein for nonvanishing). We will finish the proof of (1) by descending induction on $t = \dim_k H^0(X, \mathcal{L})$. The base case $t = 1$ is trivial. Assume $t > 1$.

Let $U \subset X$ be the nonempty open subset of nonsingular points studied in Lemma 33.25.8. Let $s \in H^0(X, \mathcal{L})$ be nonzero. There exists a closed point $x \in U$ such that s does not vanish in x. Let $\mathcal{I}$ be the ideal sheaf of $i : x \to X$ as in Lemma 33.43.8. Look at the short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{L} \to i_*i^*\mathcal{L} \to 0$$

Observe that $H^0(X, i_*i^*\mathcal{L}) = H^0(x, i^*\mathcal{L})$ has dimension 1 as x is a k-rational point (k is algebraically closed). Since s does not vanish at x we conclude that

$$H^0(X, \mathcal{L}) \longrightarrow H^0(X, i_*i^*\mathcal{L})$$

is surjective. Hence $\dim_k H^0(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}) = t - 1$. Finally, the long exact sequence of cohomology also shows that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}) = 0$ thereby finishing the proof of the induction step.

To get an invertible sheaf as in (2) take an invertible sheaf $\mathcal{L}$ as in (1) and do the argument in the previous paragraph one more time. $\square$

0B90 **Lemma 33.44.17.** Let k be an algebraically closed field. Let X be a proper curve over k. Set $g = \dim_k H^1(X, \mathcal{O}_X)$. For every invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\deg(\mathcal{L}) \geq 2g - 1$ we have $H^1(X, \mathcal{L}) = 0$.

Proof. Let $\mathcal{N}$ be the invertible module we found in Lemma 33.44.16 part (2). The degree of $\mathcal{N}$ is $\chi(X, \mathcal{N}) - \chi(X, \mathcal{O}_X) = 0 - (1 - g) = g - 1$. Hence the degree of $\mathcal{L} \otimes \mathcal{N}^{\otimes -1}$ is $\deg(\mathcal{L}) - (g-1) \geq g$. Hence $\chi(X, \mathcal{L} \otimes \mathcal{N}^{\otimes -1}) \geq g + 1 - g = 1$. Thus there is a nonzero global section s whose zero scheme is an effective Cartier divisor D of degree $\deg(\mathcal{L}) - (g-1)$. This gives a short exact sequence

$$0 \to \mathcal{N} \xrightarrow{s} \mathcal{L} \to i_*(\mathcal{L}|_D) \to 0$$

where $i : D \to X$ is the inclusion morphism. We conclude that $H^0(X, \mathcal{L})$ maps isomorphically to $H^0(D, \mathcal{L}|_D)$ which has dimension $\deg(\mathcal{L}) - (g-1)$. The result follows from the definition of degree. $\square$

33.45. Numerical intersections

0BEL In this section we play around with the Euler characteristic of coherent sheaves on proper schemes to obtain numerical intersection numbers for invertible modules. Our main tool will be the following lemma.

0BEM **Lemma 33.45.1.** Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. The map

$$(n_1, \ldots, n_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree at most the dimension of the support of $\mathcal{F}$.

Proof. We prove this by induction on $\dim(\text{Supp}(\mathcal{F}))$. If this number is zero, then the function is constant with value $\dim_k \Gamma(X, \mathcal{F})$ by Lemma 33.33.3. Assume $\dim(\text{Supp}(\mathcal{F})) > 0$.

If $\mathcal{F}$ has embedded associated points, then we can consider the short exact sequence $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$ constructed in Divisors, Lemma 31.4.6. Since the dimension of the support of $\mathcal{K}$ is strictly less, the result holds for $\mathcal{K}$ by induction hypothesis and with strictly smaller total degree. By additivity of the Euler characteristic (Lemma 33.33.2) it suffices to prove the result for $\mathcal{F}'$. Thus we may assume $\mathcal{F}$ does not have embedded associated points.

If $i : Z \to X$ is a closed immersion and $\mathcal{F} = i_*\mathcal{G}$, then we see that the result for X, $\mathcal{F}$, $\mathcal{L}_1, \ldots, \mathcal{L}_r$ is equivalent to the result for Z, $\mathcal{G}$, $i^*\mathcal{L}_1, \ldots, i^*\mathcal{L}_r$ (since the cohomologies agree, see Cohomology of Schemes, Lemma 30.2.4). Applying Divisors, Lemma 31.4.7 we may assume that X has no embedded components and $X = \text{Supp}(\mathcal{F})$.

Pick a regular meromorphic section s of $\mathcal{L}_1$, see Divisors, Lemma 31.25.4. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s and consider the maps

$$\mathcal{I}\mathcal{F} \to \mathcal{F}, \quad \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes \mathcal{L}_1$$

of Divisors, Lemma 31.24.5. These are injective and have cokernels $\mathcal{Q}$, $\mathcal{Q}'$ supported on nowhere dense closed subschemes of $X = \text{Supp}(\mathcal{F})$. Tensoring with the invertible module $\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$ is exact, hence using additivity again we see that

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1+1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$
$$= \chi(\mathcal{Q} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \chi(\mathcal{Q}' \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r})$$

Thus we see that the function $P(n_1, \ldots, n_r)$ of the lemma has the property that

$$P(n_1 + 1, n_2, \ldots, n_r) - P(n_1, \ldots, n_r)$$

is a numerical polynomial of total degree $<$ the dimension of the support of $\mathcal{F}$. Of course by symmetry the same thing is true for

$$P(n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots, n_r) - P(n_1, \ldots, n_r)$$

for any $i \in \{1, \ldots, r\}$. A simple arithmetic argument shows that P is a numerical polynomial of total degree at most $\dim(\text{Supp}(\mathcal{F}))$. □

The following lemma roughly shows that the leading coefficient only depends on the length of the coherent module in the generic points of its support.

0BEN Lemma 33.45.2. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible $\mathcal{O}_X$-modules. Let $d = \dim(\text{Supp}(\mathcal{F}))$. Let $Z_i \subset X$ be the irreducible components of $\text{Supp}(\mathcal{F})$ of dimension d. Let $\xi_i \in Z_i$ be the generic point and set $m_i = \text{length}_{\mathcal{O}_{X,\xi_i}}(\mathcal{F}_{\xi_i})$. Then

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum\nolimits_i m_i \ \chi(Z_i, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_{Z_i})$$

is a numerical polynomial in $n_1, \ldots, n_r$ of total degree $< d$.

Proof. Consider pairs (ξ, Z) where $Z \subset X$ is an integral closed subscheme of dimension d and ξ is its generic point. Then the finite $\mathcal{O}_{X,\xi}$-module $\mathcal{F}_\xi$ has support contained in $\{\xi\}$ hence the length $m_Z = \text{length}_{\mathcal{O}_{X,\xi}}(\mathcal{F}_\xi)$ is finite (Algebra, Lemma 10.62.3) and zero unless $Z = Z_i$ for some i. Thus the expression of the lemma can be written as

$$E(\mathcal{F}) = \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) - \sum m_Z \ \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_Z)$$

where the sum is over integral closed subschemes $Z \subset X$ of dimension d. The assignment $\mathcal{F} \mapsto E(\mathcal{F})$ is additive in short exact sequences $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of coherent $\mathcal{O}_X$-modules whose support has dimension $\leq d$. This follows from additivity of Euler characteristics (Lemma 33.33.2) and additivity of lengths (Algebra, Lemma 10.52.3). Let us apply Cohomology of Schemes, Lemma 30.12.3 to find a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists an integral closed subscheme $V_j \subset X$ and a nonzero sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{V_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (V_j \to X)_*\mathcal{I}_j$$

It follows that $V_j \subset \text{Supp}(\mathcal{F})$ and hence $\dim(V_j) \leq d$. By the additivity we remarked upon above it suffices to prove the result for each of the subquotients $\mathcal{F}_j/\mathcal{F}_{j-1}$. Thus it suffices to prove the result when $\mathcal{F} = (V \to X)_*\mathcal{I}$ where $V \subset X$ is an integral closed subscheme of dimension $\leq d$ and $\mathcal{I} \subset \mathcal{O}_V$ is a nonzero coherent sheaf of ideals. If $\dim(V) < d$ and more generally for $\mathcal{F}$ whose support has dimension $< d$, then the first term in $E(\mathcal{F})$ has total degree $< d$ by Lemma 33.45.1 and the second term is zero. If $\dim(V) = d$, then we can use the short exact sequence

$$0 \to (V \to X)_*\mathcal{I} \to (V \to X)_*\mathcal{O}_V \to (V \to X)_*(\mathcal{O}_V/\mathcal{I}) \to 0$$

The result holds for the middle sheaf because the only Z occurring in the sum is $Z = V$ with $m_Z = 1$ and because

$$H^i(X, ((V \to X)_*\mathcal{O}_V) \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}) = H^i(V, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}|_V)$$

by the projection formula (Cohomology, Section 20.54) and Cohomology of Schemes, Lemma 30.2.4; so in this case we actually have $E(\mathcal{F}) = 0$. The result holds for the sheaf on the right because its support has dimension $< d$. Thus the result holds for the sheaf on the left and the lemma is proved. □

0BEP Definition 33.45.3. Let k be a field. Let X be a proper scheme over k. Let $i : Z \to X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. We define the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ as the coefficient of $n_1 \ldots n_d$ in the numerical polynomial

$$\chi(X, i_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

In the special case that $\mathcal{L}_1 = \ldots = \mathcal{L}_d = \mathcal{L}$ we write $(\mathcal{L}^d \cdot Z)$.

The displayed equality in the definition follows from the projection formula (Cohomology, Section 20.54) and Cohomology of Schemes, Lemma 30.2.4. We prove a few lemmas for these intersection numbers.

0BEQ Lemma 33.45.4. In the situation of Definition 33.45.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is an integer.

Proof. Any numerical polynomial of degree e in $n_1, \ldots, n_d$ can be written uniquely as a $\mathbf{Z}$-linear combination of the functions $\binom{n_1}{k_1}\binom{n_2}{k_2} \ldots \binom{n_d}{k_d}$ with $k_1 + \ldots + k_d \leq e$. Apply this with $e = d$. Left as an exercise. □

0BER Lemma 33.45.5. In the situation of Definition 33.45.3 the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is additive: if $\mathcal{L}_i = \mathcal{L}_i' \otimes \mathcal{L}_i''$, then we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_i \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_1 \cdots \mathcal{L}_i' \cdots \mathcal{L}_d \cdot Z) + (\mathcal{L}_1 \cdots \mathcal{L}_i'' \cdots \mathcal{L}_d \cdot Z)$$

Proof. This is true because by Lemma 33.45.1 the function

$$(n_1, \ldots, n_{i-1}, n_i', n_i'', n_{i+1}, \ldots, n_d) \mapsto \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes (\mathcal{L}_i')^{\otimes n_i'} \otimes (\mathcal{L}_i'')^{\otimes n_i''} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_Z)$$

is a numerical polynomial of total degree at most d in $d + 1$ variables. □

0BES Lemma 33.45.6. In the situation of Definition 33.45.3 let $Z_i \subset Z$ be the irreducible components of dimension d. Let $m_i = \text{length}_{\mathcal{O}_{X,\xi_i}}(\mathcal{O}_{Z,\xi_i})$ where $\xi_i \in Z_i$ is the generic point. Then

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \sum m_i(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z_i)$$

Proof. Immediate from Lemma 33.45.2 and the definitions. □

0BET Lemma 33.45.7. Let k be a field. Let $f : Y \to X$ be a morphism of proper schemes over k. Let $Z \subset Y$ be an integral closed subscheme of dimension d and let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then

$$(f^*\mathcal{L}_1 \cdots f^*\mathcal{L}_d \cdot Z) = \deg(f|_Z : Z \to f(Z))(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ is as in Morphisms, Definition 29.51.8 or 0 if $\dim(f(Z)) < d$.

Proof. The left hand side is computed using the coefficient of $n_1 \ldots n_d$ in the function

$$\chi(Y, \mathcal{O}_Z \otimes f^*\mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes f^*\mathcal{L}_d^{\otimes n_d}) = \sum (-1)^i \chi(X, R^i f_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$$

The equality follows from Lemma 33.33.5 and the projection formula (Cohomology, Lemma 20.54.2). If $f(Z)$ has dimension $< d$, then the right hand side is a polynomial of total degree $< d$ by Lemma 33.45.1 and the result is true. Assume $\dim(f(Z)) = d$. Let $\xi \in f(Z)$ be the generic point. By dimension theory (see Lemmas 33.20.3 and 33.20.4) the generic point of Z is the unique point of Z mapping to ξ. Then $f : Z \to f(Z)$ is finite over a nonempty open of $f(Z)$, see Morphisms, Lemma 29.51.1. Thus $\deg(f : Z \to f(Z))$ is defined and in fact it is equal to the length of the stalk of $f_*\mathcal{O}_Z$ at ξ over $\mathcal{O}_{X,\xi}$. Moreover, the stalk of $R^i f_*\mathcal{O}_X$ at ξ is zero for $i > 0$ because we just saw that $f|_Z$ is finite in a neighbourhood of ξ (so that Cohomology of Schemes, Lemma 30.9.9 gives the vanishing). Thus the terms $\chi(X, R^i f_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ with $i > 0$ have total degree $< d$ and

$$\chi(X, f_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}) = \deg(f : Z \to f(Z))\chi(f(Z), \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_{f(Z)})$$

modulo a polynomial of total degree $< d$ by Lemma 33.45.2. The desired result follows. $\square$

0BEU Lemma 33.45.8. Let k be a field. Let X be proper over k. Let $Z \subset X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Assume there exists an effective Cartier divisor $D \subset Z$ such that $\mathcal{L}_1|_Z \cong \mathcal{O}_Z(D)$. Then

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot D)$$

Proof. We may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$. Thus we may assume $X = Z$ and $\mathcal{L}_1 = \mathcal{O}_X(D)$. Then $\mathcal{L}_1^{-1}$ is the ideal sheaf of D and we can consider the short exact sequence

$$0 \to \mathcal{L}_1^{\otimes -1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

Set $P(n_1, \ldots, n_d) = \chi(X, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d})$ and $Q(n_1, \ldots, n_d) = \chi(D, \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes n_d}|_D)$. We conclude from additivity that

$$P(n_1, \ldots, n_d) - P(n_1 - 1, n_2, \ldots, n_d) = Q(n_1, \ldots, n_d)$$

Because the total degree of P is at most d, we see that the coefficient of $n_1 \ldots n_d$ in P is equal to the coefficient of $n_2 \ldots n_d$ in Q. $\square$

0BEV Lemma 33.45.9. Let k be a field. Let X be proper over k. Let $Z \subset X$ be a closed subscheme of dimension d. If $\mathcal{L}_1, \ldots, \mathcal{L}_d$ are ample, then $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ is positive.

Proof. We will prove this by induction on d. The case $d = 0$ follows from Lemma 33.33.3. Assume $d > 0$. By Lemma 33.45.6 we may assume that Z is an integral closed subscheme. In fact, we may replace X by Z and $\mathcal{L}_i$ by $\mathcal{L}_i|_Z$ to reduce to the case $Z = X$ is a proper variety of dimension d. By Lemma 33.45.5 we may replace $\mathcal{L}_1$ by a positive tensor power. Thus we may assume there exists a nonzero section $s \in \Gamma(X, \mathcal{L}_1)$ such that X_s is affine (here we use the definition of ample invertible sheaf, see Properties, Definition 28.26.1). Observe that X is not affine because proper and affine implies finite (Morphisms, Lemma 29.44.11) which contradicts $d > 0$. It follows that s has a nonempty vanishing scheme $Z(s) \subset X$. Since X is a variety, s is a regular section of $\mathcal{L}_1$, so $Z(s)$ is an effective Cartier divisor, thus

$Z(s)$ has codimension 1 in X, and hence $Z(s)$ has dimension $d-1$ (here we use material from Divisors, Sections 31.13, 31.14, and 31.15 and from dimension theory as in Lemma 33.20.3). By Lemma 33.45.8 we have

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot X) = (\mathcal{L}_2 \cdots \mathcal{L}_d \cdot Z(s))$$

By induction the right hand side is positive and the proof is complete. □

0BEW Definition 33.45.10. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme the degree of Z with respect to $\mathcal{L}$, denoted $\deg_{\mathcal{L}}(Z)$, is the intersection number $(\mathcal{L}^d \cdot Z)$ where $d = \dim(Z)$.

By Lemma 33.45.9 the degree of a subscheme is always a positive integer. We note that $\deg_{\mathcal{L}}(Z) = d$ if and only if

$$\chi(Z, \mathcal{L}^{\otimes n}|_Z) = \frac{d}{\dim(Z)!} n^{\dim(Z)} + l.o.t$$

as can be seen using that

$$(n_1 + \ldots + n_{\dim(Z)})^{\dim(Z)} = \dim(Z)!\ n_1 \ldots n_{\dim(Z)} + \text{other terms}$$

0BEX Lemma 33.45.11. Let k be a field. Let $f : Y \to X$ be a finite dominant morphism of proper varieties over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Then

$$\deg_{f^*\mathcal{L}}(Y) = \deg(f) \deg_{\mathcal{L}}(X)$$

where $\deg(f)$ is as in Morphisms, Definition 29.51.8.

Proof. The statement makes sense because $f^*\mathcal{L}$ is ample by Morphisms, Lemma 29.37.7. Having said this the result is a special case of Lemma 33.45.7. □

Finally we relate the intersection number with a curve to the notion of degrees of invertible modules on curves introduced in Section 33.44.

0BEY Lemma 33.45.12. Let k be a field. Let X be a proper scheme over k. Let $Z \subset X$ be a closed subscheme of dimension ≤ 1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then

$$(\mathcal{L} \cdot Z) = \deg(\mathcal{L}|_Z)$$

where $\deg(\mathcal{L}|_Z)$ is as in Definition 33.44.1. If $\mathcal{L}$ is ample, then $\deg_{\mathcal{L}}(Z) = \deg(\mathcal{L}|_Z)$.

Proof. This follows from the fact that the function $n \mapsto \chi(Z, \mathcal{L}|_Z^{\otimes n})$ has degree 1 and hence the leading coefficient is the difference of consecutive values. □

0BJ8 Proposition 33.45.13 (Asymptotic Riemann-Roch). Let k be a field. Let X be a proper scheme over k of dimension d. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Then

$$\dim_k \Gamma(X, \mathcal{L}^{\otimes n}) \sim cn^d + l.o.t.$$

where $c = \deg_{\mathcal{L}}(X)/d!$ is a positive constant.

Proof. This follows from the definitions, Lemma 33.45.9, and the vanishing of higher cohomology in Cohomology of Schemes, Lemma 30.17.1. □

33.46. Embedding dimension

0C2G There are several ways to define the embedding dimension, but for closed points on algebraic schemes over algebraically closed fields all definitions are equivalent to the following.

0C1Q **Definition 33.46.1.** Let k be an algebraically closed field. Let X be a locally algebraic k-scheme and let $x \in X$ be a closed point. The *embedding dimension of X at x* is $\dim_k \mathfrak{m}_x/\mathfrak{m}_x^2$.

Facts about embedding dimension. Let k, X, x be as in Definition 33.46.1.

(1) The embedding dimension of X at x is the dimension of the tangent space $T_{X/k,x}$ (Definition 33.16.3) as a k-vector space.
(2) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists a surjection
$$k[[x_1, \ldots, x_d]] \longrightarrow \mathcal{O}_{X,x}^\wedge$$
of k-algebras.
(3) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and a closed immersion $U \to Y$ where Y is a smooth variety of dimension d over k.
(4) The embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and an unramified morphism $U \to \mathbf{A}^d_k$.
(5) If we are given a closed embedding $X \to Y$ with Y smooth over k, then the embedding dimension of X at x is the smallest integer $d \geq 0$ such that there exists a closed subscheme $Z \subset Y$ with $X \subset Z$, with $Z \to \mathrm{Spec}(k)$ smooth at x, and with $\dim_x(Z) = d$.

If we ever need these, we will formulate a precise result and provide a proof.

Non-algebraically closed ground fields or non-closed points. Let k be a field and let X be a locally algebraic k-scheme. If $x \in X$ is a point, then we have several options for the embedding dimension of X at x. Namely, we could use

(1) $\dim_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)$,
(2) $\dim_{\kappa(x)}(T_{X/k,x}) = \dim_{\kappa(x)}(\Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x))$ (Lemma 33.16.4),
(3) the smallest integer $d \geq 0$ such that there exists an open neighbourhood $U \subset X$ of x and a closed immersion $U \to Y$ where Y is a smooth variety of dimension d over k.

In characteristic zero (1) = (2) if x is a closed point; more generally this holds if $\kappa(x)$ is separable algebraic over k, see Lemma 33.16.5. It seems that the geometric definition (3) corresponds most closely to the geometric intuition the phrase "embedding dimension" invokes. Since one can show that (3) and (2) define the same number (this follows from Lemma 33.18.5) this is what we will use. In our terminology we will make clear that we are taking the embedding dimension relative to the ground field.

0C2H **Definition 33.46.2.** Let k be a field. Let X be a locally algebraic k-scheme. Let $x \in X$ be a point. The *embedding dimension of X/k at x* is $\dim_{\kappa(x)}(T_{X/k,x})$.

If $(A, \mathfrak{m}, \kappa)$ is a Noetherian local ring the embedding dimension of A is sometimes defined as the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over κ. Above we have seen that if A is given as an algebra over a field k, it may be preferable to use $\dim_\kappa(\Omega_{A/k} \otimes_A \kappa)$. Let us call this quantity the embedding dimension of A/k. With this terminology in place we have

$$\text{embed dim of } X/k \text{ at } x = \text{embed dim of } \mathcal{O}_{X,x}/k = \text{embed dim of } \mathcal{O}_{X,x}^\wedge/k$$

if k, X, x are as in Definition 33.46.2.

33.47. Bertini theorems

0FD4 In this section we prove results of the form: given a smooth projective variety X over a field k there exists an ample divisor $H \subset X$ which is smooth.

0FD5 Lemma 33.47.1. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a closed subscheme. Then there exists an integer n_0 such that for all $n \geq n_0$ the kernel V_n of $\Gamma(X, \mathcal{L}^{\otimes n}) \to \Gamma(Z, \mathcal{L}^{\otimes n}|_Z)$ generates $\mathcal{L}^{\otimes n}|_{X \setminus Z}$ and the canonical morphism

$$X \setminus Z \longrightarrow \mathbf{P}(V_n)$$

is an immersion of schemes over k.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of Z. Observe that via the inclusion $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \subset \mathcal{L}^{\otimes n}$ we have $V_n = \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$. Choose n_1 such that for $n \geq n_1$ the sheaf $\mathcal{I} \otimes \mathcal{L}^{\otimes n}$ is globally generated, see Properties, Proposition 28.26.13. It follows that V_n gererates $\mathcal{L}^{\otimes n}|_{X \setminus Z}$ for $n \geq n_1$.

For $n \geq n_1$ denote $\psi_n : V_n \to \Gamma(X \setminus Z, \mathcal{L}^{\otimes n}|_{X \setminus Z})$ the restriction map. We get a canonical morphism

$$\varphi = \varphi_{\mathcal{L}^{\otimes n}|_{X \setminus Z}, \psi_n} : X \setminus Z \longrightarrow \mathbf{P}(V_n)$$

by Constructions, Example 27.21.2. Choose n_2 such that for all $n \geq n_2$ the invertible sheaf $\mathcal{L}^{\otimes n}$ is very ample on X. We claim that $n_0 = n_1 + n_2$ works.

Proof of the claim. Say $n \geq n_0$ and write $n = n_1 + n'$. For $x \in X \setminus Z$ we can choose $s_1 \in V_1$ not vanishing at x. Set $V' = \Gamma(X, \mathcal{L}^{\otimes n'})$. By our choice of n and n' we see that the corresponding morphism $\varphi' : X \to \mathbf{P}(V')$ is a closed immersion. Thus if we choose $s' \in \Gamma(X, \mathcal{L}^{\otimes n'})$ not vanishing at x, then $X_{s'} = (\varphi')^{-1}(D_+(s'))$ (see Constructions, Lemma 27.14.1) is affine and $X_{s'} \to D_+(s')$ is a closed immersion. Then $s = s_1 \otimes s' \in V_n$ does not vanish at x. If $D_+(s) \subset \mathbf{P}(V_n)$ denotes the corresponding open affine space of our projective space, then $\varphi^{-1}(D_+(s)) = X_s \subset X \setminus Z$ (see reference above). The open $X_s = X_{s'} \cap X_{s_1}$ is affine, see Properties, Lemma 28.26.4. Consider the ring map

$$\text{Sym}(V)_{(s)} \longrightarrow \mathcal{O}_X(X_s)$$

defining the morphism $X_s \to D_+(s)$. Because $X_{s'} \to D_+(s')$ is a closed immersion, the images of the elements

$$\frac{s_1 \otimes t'}{s_1 \otimes s'}$$

where $t' \in V'$ generate the image of $\mathcal{O}_X(X_{s'}) \to \mathcal{O}_X(X_s)$. Since $X_s \to X_{s'}$ is an open immersion, this implies that $X_s \to D_+(s)$ is an immersion of affine schemes (see below). Thus φ_n is an immersion by Morphisms, Lemma 29.3.5.

Let $a : A' \to A$ and $c : B \to A$ be ring maps such that $\text{Spec}(a)$ is an immersion and $\text{Im}(a) \subset \text{Im}(c)$. Set $B' = A' \times_A B$ with projections $b : B' \to B$ and $c' : B' \to A'$.

By assumption c' is surjective and hence $\mathrm{Spec}(c')$ is a closed immersion. Whence $\mathrm{Spec}(c') \circ \mathrm{Spec}(a)$ is an immersion (Schemes, Lemma 26.24.3). Then $\mathrm{Spec}(c)$ has to be an immersion because it factors the immersion $\mathrm{Spec}(c') \circ \mathrm{Spec}(a) = \mathrm{Spec}(b) \circ \mathrm{Spec}(c)$, see Morphisms, Lemma 29.3.1. $\square$

0G47 Situation 33.47.2. Let k be a field, let X be a scheme over k, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, let V be a finite dimensional k-vector space, and let $\psi : V \to \Gamma(X, \mathcal{L})$ be a k-linear map. Say $\dim(V) = r$ and we have a basis $v_1, \ldots, v_r$ of V. Then we obtain a "universal divisor"

$$H_{univ} = Z(s_{univ}) \subset \mathbf{A}^r \times_k X$$

as the zero scheme (Divisors, Definition 31.14.8) of the section

$$s_{univ} = \sum\nolimits_{i=1,\ldots,r} x_i\psi(v_i) \in \Gamma(\mathbf{A}^r \times_k X, \mathrm{pr}_2^*\mathcal{L})$$

For a field extension k'/k the k'-points $v \in \mathbf{A}^r_k(k')$ correspond to vectors $(a_1, \ldots, a_r)$ of elements of k'. Thus we may on the one hand think of v as the element $v = \sum_{i=1,\ldots,r} a_i v_i \in V \otimes_k k'$ and on the other hand we may assign to v the section

$$\psi(v) = \sum\nolimits_{i=1,\ldots,r} a_i\psi(v_i) \in \Gamma(X_{k'}, \mathcal{L}|_{X_{k'}})$$

With this notation it is clear that the fibre of H_{univ} over $v \in V \otimes k'$ is the zero scheme of $\psi(v)$. In a formula:

$$H_v = H_{univ,v} = Z(\psi(v))$$

We will denote this common value by H_v as indicated. Finally, in this situation let P be a property of vectors $v \in V \otimes_k k'$ for k'/k an arbitrary field extension[7]. We say P holds for general $v \in V \otimes_k k'$ if there exists a nonempty Zariski open $U \subset \mathbf{A}^r_k$ such that if v corresponds to a k'-point of U for any k'/k then $P(v)$ holds.

0FD6 Lemma 33.47.3. In Situation 33.47.2 assume

(1) X is smooth over k,
(2) the image of $\psi : V \to \Gamma(X, \mathcal{L})$ generates $\mathcal{L}$,
(3) the corresponding morphism $\varphi_{\mathcal{L},\psi} : X \to \mathbf{P}(V)$ is an immersion.

Then for general $v \in V \otimes_k k'$ the scheme H_v is smooth over k'.

Proof. (We observe that X is separated and finite type as a locally closed subscheme of a projective space.) Let us use the notation introduced above the statement of the lemma. We consider the projections

$$\begin{tikzcd} \mathbf{A}^r_k \times_k X \arrow[d] & H_{univ} \arrow[l] \arrow[r] \arrow[dl, "p"] \arrow[dr, "q"] & \mathbf{A}^r_k \times_k X \arrow[d] \\ X & & \mathbf{A}^r_k \end{tikzcd}$$

Let $\Sigma \subset H_{univ}$ be the singular locus of the morphsm $q : H_{univ} \to \mathbf{A}^r_k$, i.e., the set of points where q is not smooth. Then Σ is closed because the smooth locus of a morphism is open by definition. Since the fibre of a smooth morphism is smooth, it suffices to prove $q(\Sigma)$ is contained in a proper closed subset of $\mathbf{A}^r_k$. Since Σ (with reduced induced scheme structure) is a finite type scheme over k it suffices to prove

[7]For example we could consider the condition that H_v is smooth over k', or geometrically irreducible over k'.

$\dim(\Sigma) < r$ This follows from Lemma 33.20.4. Since dimensions aren't changed by replacing k by a bigger field (Morphisms, Lemma 29.28.3), we may and do assume k is algebraically closed. By dimension theory (Lemma 33.20.4), it suffices to prove that for $x \in X \setminus Z$ closed we have $p^{-1}(\{x\}) \cap \Sigma$ has dimension $< r - \dim(X')$ where X' is the unique irreducible component of X containing x. As X is smooth over k and x is a closed point we have $\dim(X') = \dim \mathfrak{m}_x/\mathfrak{m}_x^2$ (Morphisms, Lemma 29.34.12 and Algebra, Lemma 10.140.1). Thus we win if

$$\dim p^{-1}(x) \cap \Sigma < r - \dim \mathfrak{m}_x/\mathfrak{m}_x^2$$

for all $x \in X$ closed.

Since V globally generated $\mathcal{L}$, for every irreducible component X' of X there is a nonempty Zariski open of $\mathbf{A}^r$ such that the fibres of q over this open do not contain X'. (For example, if $x' \in X'$ is a closed point, then we can take the open corresponding to those vectors $v \in V$ such that $\psi(v)$ does not vanish at x'. This open will be the complement of a hyperplane in $\mathbf{A}^r_k$.) Let $U \subset \mathbf{A}^r$ be the (nonempty) intersection of these opens. Then the fibres of $q^{-1}(U) \to U$ are effective Cartier divisors on the fibres of $U \times_k X \to U$ (because a nonvanishing section of an invertible module on an integral scheme is a regular section). Hence the morphism $q^{-1}(U) \to U$ is flat by Divisors, Lemma 31.18.9. Thus for $x \in X$ closed and $v \in V = \mathbf{A}^r_k(k)$, if $(x, v) \in H_{univ}$, i.e., if $x \in H_v$ then q is smooth at (x, v) if and only if the fibre H_v is smooth at x, see Morphisms, Lemma 29.34.14.

Consider the image $\psi(v)_x$ in the stalk $\mathcal{L}_x$ of the section corresponding to $v \in V$. We have

$$x \in H_v \Leftrightarrow \psi(v)_x \in \mathfrak{m}_x\mathcal{L}_x$$

If this is true, then we have

$$H_v \text{ singular at } x \Leftrightarrow \psi(v)_x \in \mathfrak{m}_x^2\mathcal{L}_x$$

Namely, $\psi(v)_x$ is not contained in $\mathfrak{m}_x^2\mathcal{L}_x \Leftrightarrow$ the local equation for $H_v \subset X$ at x is not contained in $\mathfrak{m}_x^2 \Leftrightarrow \mathcal{O}_{H_v,x}$ is regular (Algebra, Lemma 10.106.3) $\Leftrightarrow H_v$ is smooth at x over k (Algebra, Lemma 10.140.5). We conclude that the closed points of $p^{-1}(x) \cap \Sigma$ correspond to those $v \in V$ such that $\psi(v)_x \in \mathfrak{m}_x^2\mathcal{L}_x$. However, as $\varphi_{\mathcal{L},\psi}$ is an immersion the map

$$V \longrightarrow \mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$$

is surjective (small detail omitted). By the above, the closed points of the locus $p^{-1}(x) \cap \Sigma$ viewed as a subspace of V is the kernel of this map and hence has dimension $r - \dim \mathfrak{m}_x/\mathfrak{m}_x^2 - 1$ as desired. □

33.48. Enriques-Severi-Zariski

0FVD In this section we prove some results of the form: twisting by a "very negative" invertible module kills low degree cohomology. We also deduce the connectedness of a hypersurface section of a normal proper scheme of dimension ≥ 2.

0FD7 Lemma 33.48.1. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. If $\mathrm{Ass}(\mathcal{F})$ does not contain any closed points, then $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$ for $n \ll 0$.

Proof. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ let $\mathcal{P}(\mathcal{F})$ be the property: there exists an $n_0 \in \mathbf{Z}$ such that for $n \leq n_0$ every section s of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ has support consisting only of closed points. Since $\text{Ass}(\mathcal{F}) = \text{Ass}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ we see that it suffices to prove $\mathcal{P}$ holds for all coherent modules on X. To do this we will prove that conditions (1), (2), and (3) of Cohomology of Schemes, Lemma 30.12.8 are satisfied.

To see condition (1) suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

is a short exact sequence of coherent $\mathcal{O}_X$-modules such that we have $\mathcal{P}$ for $\mathcal{F}_i$, $i = 1, 2$. Let n_1, n_2 be the cutoffs we find. Let $\mathcal{F}_2' \subset \mathcal{F}_2$ be the maximal coherent submodule whose support is a finite set of closed points. Let $\mathcal{I} \subset \mathcal{O}_X$ be the annihilator of $\mathcal{F}_2'$. Since $\mathcal{L}$ is ample, we can find an $e > 0$ such that $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e}$ is globally generated. Set $n_0 = \min(n_2, n_1 - e)$. Let $n \leq n_0$ and let t be a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$. The image of t in $\mathcal{F}_2 \otimes \mathcal{L}^{\otimes n}$ falls into $\mathcal{F}_2' \otimes \mathcal{L}^{\otimes n}$ because $n \leq n_2$. Hence for any $s \in \Gamma(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes e})$ the product $t \otimes s$ lies in $\mathcal{F}_1 \otimes \mathcal{L}^{\otimes n+e}$. Thus $t \otimes s$ has support contained in the finite set of closed points in $\text{Ass}(\mathcal{F}_1)$ because $n + e \leq n_1$. Since by our choice of e we may choose s invertible in any point not in the support of $\mathcal{F}_2'$ we conclude that the support of t is contained in the union of the finite set of closed points in $\text{Ass}(\mathcal{F}_1)$ and the finite set of closed points in $\text{Ass}(\mathcal{F}_2)$. This finishes the proof of condition (1).

Condition (2) is immediate.

For condition (3) we choose $\mathcal{G} = \mathcal{O}_Z$. In this case, if Z is a closed point of X, then there is nothing the show. If $\dim(Z) > 0$, then we will show that $\Gamma(Z, \mathcal{L}^{\otimes n}|_Z) = 0$ for $n < 0$. Namely, let s be a nonzero section of a negative power of $\mathcal{L}|_Z$. Choose a nonzero section t of a positive power of $\mathcal{L}|_Z$ (this is possible as $\mathcal{L}$ is ample, see Properties, Proposition 28.26.13). Then $s^{\deg(t)} \otimes t^{\deg(s)}$ is a nonzero global section of $\mathcal{O}_Z$ (because Z is integral) and hence a unit (Lemma 33.9.3). This implies that t is a trivializing section of a positive power of $\mathcal{L}$. Thus the function $n \mapsto \dim_k \Gamma(X, \mathcal{L}^{\otimes n})$ is bounded on an infinite set of positive integers which contradicts asymptotic Riemann-Roch (Proposition 33.45.13) since $\dim(Z) > 0$. □

0FD8 **Lemma 33.48.2** (Enriques-Severi-Zariski). *Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that for $x \in X$ closed we have $\text{depth}(\mathcal{F}_x) \geq 2$. Then $H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) = 0$ for $m \ll 0$.*

Proof. Choose a closed immersion $i : X \to \mathbf{P}^n_k$ such that $i^*\mathcal{O}(1) \cong \mathcal{L}^{\otimes e}$ for some $e > 0$ (see Morphisms, Lemma 29.39.4). Then it suffices to prove the lemma for

$$\mathcal{G} = i_*(\mathcal{F} \oplus \mathcal{F} \otimes \mathcal{L} \oplus \ldots \oplus \mathcal{F} \otimes \mathcal{L}^{\otimes e-1}) \quad \text{and} \quad \mathcal{O}(1)$$

on $\mathbf{P}^n_k$. Namely, we have

$$H^1(\mathbf{P}^n_k, \mathcal{G}(m)) = \bigoplus\nolimits_{j=0,\ldots,e-1} H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes j+me})$$

by Cohomology of Schemes, Lemma 30.2.4. Also, if $y \in \mathbf{P}^n_k$ is a closed point then $\text{depth}(\mathcal{G}_y) = \infty$ if $y \notin i(X)$ and $\text{depth}(\mathcal{G}_y) = \text{depth}(\mathcal{F}_x)$ if $y = i(x)$ because in this case $\mathcal{G}_y \cong \mathcal{F}_x^{\oplus e}$ as a module over $\mathcal{O}_{\mathbf{P}^n_k, x}$ and we can use for example Algebra, Lemma 10.72.11 to get the equality.

Assume $X = \mathbf{P}^n_k$ and $\mathcal{L} = \mathcal{O}(1)$ and k is infinite. Choose $s \in H^0(\mathbf{P}^1_k, \mathcal{O}(1))$ which determines an exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{s} \mathcal{F} \to \mathcal{G} \to 0$$

as in Lemma 33.35.3. Since the map $\mathcal{F}(-1) \to \mathcal{F}$ is affine locally given by multiplying by a nonzerodivisor on $\mathcal{F}$ we see that for $x \in \mathbf{P}^n_k$ closed we have $\text{depth}(\mathcal{G}_x) \geq 1$, see Algebra, Lemma 10.72.7. Hence by Lemma 33.48.1 we have $H^0(\mathcal{G}(m)) = 0$ for $m \ll 0$. Looking at the long exact sequence of cohomology after twisting (see Remark 33.35.5) we find that the sequence of numbers

$$\dim H^1(\mathbf{P}^n_k, \mathcal{F}(m))$$

stabilizes for $m \leq m_0$ for some integer m_0. Let N be the common dimension of these spaces for $m \leq m_0$. We have to show $N = 0$.

For $d > 0$ and $m \leq m_0$ consider the bilinear map

$$H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m-d)) \longrightarrow H^1(\mathbf{P}^n_k, \mathcal{F}(m))$$

By linear algebra, there is a codimension $\leq N^2$ subspace $V_m \subset H^0(\mathbf{P}^n_k, \mathcal{O}(d))$ such that multiplication by $s' \in V_m$ annihilates $H^1(\mathbf{P}^n_k, \mathcal{F}(m-d))$. Observe that for $m' < m \leq m_0$ the diagram

$$\begin{tikzcd}
H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m'-d)) \arrow[r] \arrow[d, "1 \times s^{m'-m}"] & H^1(\mathbf{P}^n_k, \mathcal{F}(m')) \arrow[d, "s^{m'-m}"] \\
H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \times H^1(\mathbf{P}^n_k, \mathcal{F}(m-d)) \arrow[r] & H^1(\mathbf{P}^n_k, \mathcal{F}(m))
\end{tikzcd}$$

commutes with isomorphisms going vertically. Thus $V_m = V$ is independent of $m \leq m_0$. For $x \in \text{Ass}(\mathcal{F})$ set $Z = \overline{\{x\}}$. For d large enough the linear map

$$H^0(\mathbf{P}^n_k, \mathcal{O}(d)) \to H^0(Z, \mathcal{O}(d)|_Z)$$

has rank $> N^2$ because $\dim(Z) \geq 1$ (for example this follows from asymptotic Riemann-Roch and ampleness $\mathcal{O}(1)$; details omitted). Hence we can find $s' \in V$ such that s' does not vanish in any associated point of $\mathcal{F}$ (use that the set of associated points is finite). Then we obtain

$$0 \to \mathcal{F}(-d) \xrightarrow{s'} \mathcal{F} \to \mathcal{G}' \to 0$$

and as before we conclude as before that multiplication by s' on $H^1(\mathbf{P}^n_k, \mathcal{F}(m-d))$ is injective for $m \ll 0$. This contradicts the choice of s' unless $N = 0$ as desired.

We still have to treat the case where k is finite. In this case let K/k be any infinite algebraic field extension. Denote $\mathcal{F}_K$ and $\mathcal{L}_K$ the pullbacks of $\mathcal{F}$ and $\mathcal{L}$ to $X_K = \text{Spec}(K) \times_{\text{Spec}(k)} X$. We have

$$H^1(X_K, \mathcal{F}_K \otimes \mathcal{L}_K^{\otimes m}) = H^1(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \otimes_k K$$

by Cohomology of Schemes, Lemma 30.5.2. On the other hand, a closed point x_K of X_K maps to a closed point x of X because K/k is an algebraic extension. The ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_K,x_K}$ is flat (Lemma 33.5.1). Hence we have

$$\text{depth}(\mathcal{F}_{x_K}) = \text{depth}(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_K,x_K}) \geq \text{depth}(\mathcal{F}_x)$$

by Algebra, Lemma 10.163.1 (in fact equality holds here but we don't need it). Therefore the result over k follows from the result over the infinite field K and the proof is complete. $\square$

0FD9 **Lemma 33.48.3.** Let k be a field. Let X be a proper scheme over k. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. Assume

(1) s is a regular section (Divisors, Definition 31.14.6),
(2) for every closed point $x \in X$ we have $\operatorname{depth}(\mathcal{O}_{X,x}) \geq 2$, and
(3) X is connected.

Then the zero scheme $Z(s)$ of s is connected.

Proof. Since s is a regular section, so is $s^n \in \Gamma(X, \mathcal{L}^{\otimes n})$ for all $n > 1$. Moreover, the inclusion morphism $Z(s) \to Z(s^n)$ is a bijection on underlying topological spaces. Hence if $Z(s)$ is disconnected, so is $Z(s^n)$. Now consider the canonical short exact sequence

$$0 \to \mathcal{L}^{\otimes -n} \xrightarrow{s^n} \mathcal{O}_X \to \mathcal{O}_{Z(s^n)} \to 0$$

Consider the k-algebra $R_n = \Gamma(X, \mathcal{O}_{Z(s^n)})$. If $Z(s)$ is disconnected, i.e., $Z(s^n)$ is disconnected, then either R_n is zero in case $Z(s^n) = \emptyset$ or R_n contains a nontrivial idempotent in case $Z(s^n) = U \amalg V$ with $U, V \subset Z(s^n)$ open and nonempty (the reader may wish to consult Lemma 33.9.3). Thus the map $\Gamma(X, \mathcal{O}_X) \to R_n$ cannot be an isomorphism. It follows that either $H^0(X, \mathcal{L}^{\otimes -n})$ or $H^1(X, \mathcal{L}^{\otimes -n})$ is nonzero for infinitely many positive n. This contradicts Lemma 33.48.1 or 33.48.2 and the proof is complete. $\square$

33.49. Other chapters

CHAPTER 34

Topologies on Schemes

020K
34.1. Introduction

020L In this document we explain what the different topologies on the category of schemes are. Some references are [Gro71] and [BLR90]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 7.6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

34.2. The general procedure

020M In this section we explain a general procedure for producing the sites we will be working with. Suppose we want to study sheaves over schemes with respect to some topology τ. In order to get a site, as in Sites, Definition 7.6.2, of schemes with that topology we have to do some work. Namely, we cannot simply say "consider all schemes with the Zariski topology" since that would give a "big" category. Instead, in each section of this chapter we will proceed as follows:

(1) We define a class Cov_τ of coverings of schemes satisfying the axioms of Sites, Definition 7.6.2. It will always be the case that a Zariski open covering of a scheme is a covering for τ.
(2) We single out a notion of standard τ-covering within the category of affine schemes.
(3) We define what is an "absolute" big τ-site Sch_τ. These are the sites one gets by appropriately choosing a set of schemes and a set of coverings.
(4) For any object S of Sch_τ we define the big τ-site $(Sch/S)_\tau$ and for suitable τ the small[1] τ-site S_τ.
(5) In addition there is a site $(\text{Aff}/S)_\tau$ using the notion of standard τ-covering of affines[2] whose category of sheaves is equivalent to the category of sheaves on $(Sch/S)_\tau$.

The above is a little clumsy in that we do not end up with a canonical choice for the big τ-site of a scheme, or even the small τ-site of a scheme. If you are willing to ignore set theoretic difficulties, then you can work with classes and end up with canonical big and small sites...

[1] The words big and small here do not relate to bigness/smallness of the corresponding categories.

[2] In the case of the ph topology we deviate very slightly from this approach, see Definition 34.8.11 and the surrounding discussion.

34.3. The Zariski topology

020N

020O Definition 34.3.1. Let T be a scheme. A Zariski covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is an open immersion and such that $T = \bigcup f_i(T_i)$.

This defines a (proper) class of coverings. Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

020P Lemma 34.3.2. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a Zariski covering of T.
(2) If $\{T_i \to T\}_{i\in I}$ is a Zariski covering and for each i we have a Zariski covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a Zariski covering.
(3) If $\{T_i \to T\}_{i\in I}$ is a Zariski covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a Zariski covering.

Proof. Omitted. □

020Q Lemma 34.3.3. Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a Zariski covering of T. Then there exists a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is a standard open of T, see Schemes, Definition 26.5.2. Moreover, we may choose each U_j to be an open of one of the T_i.

Proof. Follows as T is quasi-compact and standard opens form a basis for its topology. This is also proved in Schemes, Lemma 26.5.1. □

Thus we define the corresponding standard coverings of affines as follows.

020R Definition 34.3.4. Compare Schemes, Definition 26.5.2. Let T be an affine scheme. A standard Zariski covering of T is a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ with each $U_j \to T$ inducing an isomorphism with a standard affine open of T.

020S Definition 34.3.5. A big Zariski site is any site Sch_{Zar} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of Zariski coverings Cov_0 among these schemes.
(2) As underlying category of Sch_{Zar} take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) As coverings of Sch_{Zar} choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of Zariski coverings, and the set Cov_0 chosen above.

It is shown in Sites, Lemma 7.8.8 that, after having chosen the category Sch_α, the category of sheaves on Sch_α does not depend on the choice of coverings chosen in (3) above. In other words, the topos $Sh(Sch_{Zar})$ only depends on the choice of the category Sch_α. It is shown in Sets, Lemma 3.9.9 that these categories are closed under many constructions of algebraic geometry, e.g., fibre products and taking open and closed subschemes. We can also show that the exact choice of Sch_α does not matter too much, see Section 34.12.

Another approach would be to assume the existence of a strongly inaccessible cardinal and to define Sch_{Zar} to be the category of schemes contained in a chosen universe with set of coverings the Zariski coverings contained in that same universe.

Before we continue with the introduction of the big Zariski site of a scheme S, let us point out that the topology on a big Zariski site Sch_{Zar} is in some sense induced from the Zariski topology on the category of all schemes.

03WV Lemma 34.3.6. Let Sch_{Zar} be a big Zariski site as in Definition 34.3.5. Let $T \in \mathrm{Ob}(Sch_{Zar})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary Zariski covering of T. There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{Zar} which is tautologically equivalent (see Sites, Definition 7.8.2) to $\{T_i \to T\}_{i \in I}$.

Proof. Since each $T_i \to T$ is an open immersion, we see by Sets, Lemma 3.9.9 that each T_i is isomorphic to an object V_i of Sch_{Zar}. The covering $\{V_i \to T\}_{i \in I}$ is tautologically equivalent to $\{T_i \to T\}_{i \in I}$ (using the identity map on I both ways). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.11.1. $\square$

020T Definition 34.3.7. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S.

(1) The big Zariski site of S, denoted $(Sch/S)_{Zar}$, is the site Sch_{Zar}/S introduced in Sites, Section 7.25.
(2) The small Zariski site of S, which we denote S_{Zar}, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are those U/S such that $U \to S$ is an open immersion. A covering of S_{Zar} is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$.
(3) The big affine Zariski site of S, denoted $(\mathrm{Aff}/S)_{Zar}$, is the full subcategory of $(Sch/S)_{Zar}$ consisting of objects U/S such that U is an affine scheme. A covering of $(\mathrm{Aff}/S)_{Zar}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}((\mathrm{Aff}/S)_{Zar})$ which is a standard Zariski covering.
(4) The small affine Zariski site of S, denoted $S_{affine,Zar}$, is the full subcategory of S_{Zar} whose objects are those U/S such that U is an affine scheme. A covering of $S_{affine,Zar}$ is any covering $\{U_i \to U\}$ of S_{Zar} with $U \in \mathrm{Ob}(S_{affine,Zar})$ which is a standard Zariski covering.

It is not completely clear that the small Zariski site, the big affine Zariski site, and the small affine Zariski site are sites. We check this now.

020U Lemma 34.3.8. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The structures S_{Zar}, $(\mathrm{Aff}/S)_{Zar}$, and $S_{affine,Zar}$ defined above are sites.

Proof. Let us show that S_{Zar} is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{Zar}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$ we also have $U_i \in \mathrm{Ob}(S_{Zar})$. This follows from the definitions as the composition of open immersions is an open immersion.

Let us show that $(\mathrm{Aff}/S)_{Zar}$ is a site. Reasoning as above, it suffices to show that the collection of standard Zariski coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. For each $i \in \{1, \ldots, n\}$ let $g_{i1}, \ldots, g_{in_i} \in R_{f_i}$ be elements generating the unit ideal of R_{f_i}. Write $g_{ij} = f_{ij}/f_i^{e_{ij}}$ which is possible. After replacing f_{ij} by $f_i f_{ij}$ if necessary, we have that $D(f_{ij}) \subset D(f_i) \cong \text{Spec}(R_{f_i})$ is equal to $D(g_{ij}) \subset \text{Spec}(R_{f_i})$. Hence we see that the family of morphisms $\{D(g_{ij}) \to \text{Spec}(R)\}$ is a standard Zariski covering. From these considerations it follows that (2) holds for standard Zariski coverings. We omit the verification of (1) and (3).

We omit the proof that $S_{affine,Zar}$ is a site. □

020V Lemma 34.3.9. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The underlying categories of the sites Sch_{Zar}, $(Sch/S)_{Zar}$, S_{Zar}, $(\text{Aff}/S)_{Zar}$, and $S_{affine,Zar}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{Zar}$, and S_{Zar} both have a final object, namely S/S.

Proof. For Sch_{Zar} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \text{Ob}(Sch_{Zar})$. The fibre product $V \times_U W$ in Sch_{Zar} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{Zar}$. This proves the result for $(Sch/S)_{Zar}$. If $U \to S$, $V \to U$ and $W \to U$ are open immersions then so is $V \times_U W \to S$ and hence we get the result for S_{Zar}. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(\text{Aff}/S)_{Zar}$ and $S_{affine,Zar}$. □

Next, we check that the big, resp. small affine site defines the same topos as the big, resp. small site.

020W Lemma 34.3.10. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $(\text{Aff}/S)_{Zar} \to (Sch/S)_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((\text{Aff}/S)_{Zar})$ to $Sh((Sch/S)_{Zar})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (\text{Aff}/S)_{Zar} \to (Sch/S)_{Zar}$. Being cocontinuous just means that any Zariski covering of T/S, T affine, can be refined by a standard Zariski covering of T. This is the content of Lemma 34.3.3. Hence (1) holds. We see u is continuous simply because a standard Zariski covering is a Zariski covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

0F1B Lemma 34.3.11. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $S_{affine,Zar} \to S_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh(S_{affine,Zar})$ to $Sh(S_{Zar})$.

Proof. Omitted. Hint: compare with the proof of Lemma 34.3.10. □

Let us check that the notion of a sheaf on the small Zariski site corresponds to notion of a sheaf on S.

020X Lemma 34.3.12. The category of sheaves on S_{Zar} is equivalent to the category of sheaves on the underlying topological space of S.

Proof. We will use repeatedly that for any object U/S of S_{Zar} the morphism $U \to S$ is an isomorphism onto an open subscheme. Let $\mathcal{F}$ be a sheaf on S. Then we define a sheaf on S_{Zar} by the rule $\mathcal{F}'(U/S) = \mathcal{F}(\text{Im}(U \to S))$. For the converse, we choose for every open subscheme $U \subset S$ an object $U'/S \in \text{Ob}(S_{Zar})$ with $\text{Im}(U' \to S) = U$ (here you have to use Sets, Lemma 3.9.9). Given a sheaf $\mathcal{G}$ on S_{Zar} we define a sheaf on S by setting $\mathcal{G}'(U) = \mathcal{G}(U'/S)$. To see that $\mathcal{G}'$ is a sheaf we use that for any open covering $U = \bigcup_{i \in I} U_i$ the covering $\{U_i \to U\}_{i \in I}$ is combinatorially equivalent to a covering $\{U'_j \to U'\}_{j \in J}$ in S_{Zar} by Sets, Lemma 3.11.1, and we use Sites, Lemma 7.8.4. Details omitted. $\square$

From now on we will not make any distinction between a sheaf on S_{Zar} or a sheaf on S. We will always use the procedures of the proof of the lemma to go between the two notions. Next, we establish some relationships between the topoi associated to these sites.

020Y Lemma 34.3.13. Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor $T_{Zar} \to (Sch/S)_{Zar}$ is cocontinuous and induces a morphism of topoi

$$i_f : Sh(T_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{Zar}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{Zar} \to (Sch/S)_{Zar}$. In other words, given and open immersion $j : U \to T$ corresponding to an object of T_{Zar} we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 34.3.9. Moreover, T_{Zar} has equalizers (as any two morphisms with the same source and target are the same) and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 7.21.5 and 7.21.6. $\square$

020Z Lemma 34.3.14. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The inclusion functor $S_{Zar} \to (Sch/S)_{Zar}$ satisfies the hypotheses of Sites, Lemma 7.21.8 and hence induces a morphism of sites

$$\pi_S : (Sch/S)_{Zar} \longrightarrow S_{Zar}$$

and a morphism of topoi

$$i_S : Sh(S_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

such that $\pi_S \circ i_S = \text{id}$. Moreover, $i_S = i_{\text{id}_S}$ with i_{id_S} as in Lemma 34.3.13. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{Zar} \to (Sch/S)_{Zar}$, in addition to the properties seen in the proof of Lemma 34.3.13 above, also is fully faithful and transforms the final object into the final object. The lemma follows. $\square$

04BS Definition 34.3.15. In the situation of Lemma 34.3.14 the functor $i_S^{-1} = \pi_{S,*}$ is often called the restriction to the small Zariski site, and for a sheaf $\mathcal{F}$ on the big Zariski site we denote $\mathcal{F}|_{S_{Zar}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\text{Mor}_{Sh(S_{Zar})}(\mathcal{F}|_{S_{Zar}}, \mathcal{G}) = \text{Mor}_{Sh((Sch/S)_{Zar})}(\mathcal{F}, i_{S,*}\mathcal{G})$$
$$\text{Mor}_{Sh(S_{Zar})}(\mathcal{G}, \mathcal{F}|_{S_{Zar}}) = \text{Mor}_{Sh((Sch/S)_{Zar})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$.

0210 **Lemma 34.3.16.** Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor

$$u : (Sch/T)_{Zar} \longrightarrow (Sch/S)_{Zar}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{Zar} \longrightarrow (Sch/T)_{Zar}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 34.3.13). Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0211 **Lemma 34.3.17.** Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}.

(1) We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 34.3.13 and i_T as in Lemma 34.3.14.
(2) The functor $S_{Zar} \to T_{Zar}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi
$$f_{small} : Sh(T_{Zar}) \longrightarrow Sh(S_{Zar}).$$
The functors f_{small}^{-1} and $f_{small,*}$ agree with the usual notions f^{-1} and f_* is we identify sheaves on T_{Zar}, resp. S_{Zar} with sheaves on T, resp. S via Lemma 34.3.12.
(3) We have a commutative diagram of morphisms of sites
$$\begin{tikzcd} T_{Zar} \arrow[d, "f_{small}"'] & (Sch/T)_{Zar} \arrow[l, "\pi_T"'] \arrow[d, "f_{big}"] \\ S_{Zar} & (Sch/S)_{Zar} \arrow[l, "\pi_S"] \end{tikzcd}$$
so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
(4) We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

Statement (2): See Sites, Example 7.14.2.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functor $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 34.3.15 we have: for $\mathcal{F}$ a sheaf on the big Zariski site of T

$$(f_{big,*}\mathcal{F})|_{S_{Zar}} = f_{small,*}(\mathcal{F}|_{T_{Zar}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small Zariski site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

0212 Lemma 34.3.18. Given schemes X, Y, Z in $(Sch/S)_{Zar}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.3.16. For the functors on the small sites this is Sheaves, Lemma 6.21.2 via the identification of Lemma 34.3.12. $\square$

0DD9 Lemma 34.3.19. Let Sch_{Zar} be a big Zariski site. Consider a cartesian diagram

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"'] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in Sch_{Zar}. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Sch/T)_{Zar}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on S'_{Zar} (use Lemmas 34.3.13 and 34.3.17). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 7.28.1. $\square$

We can think about a sheaf on the big Zariski site of S as a collection of "usual" sheaves on all schemes over S.

0213 Lemma 34.3.20. Let S be a scheme contained in a big Zariski site Sch_{Zar}. A sheaf $\mathcal{F}$ on the big Zariski site $(Sch/S)_{Zar}$ is given by the following data:

(1) for every $T/S \in \mathrm{Ob}((Sch/S)_{Zar})$ a sheaf $\mathcal{F}_T$ on T,
(2) for every $f : T' \to T$ in $(Sch/S)_{Zar}$ a map $c_f : f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.

These data are subject to the following conditions:

(a) given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{Zar}$ the composition $c_g \circ g^{-1}c_f$ is equal to $c_{f \circ g}$, and
(b) if $f : T' \to T$ in $(Sch/S)_{Zar}$ is an open immersion then c_f is an isomorphism.

Proof. This lemma follows from a purely sheaf theoretic statement discussed in Sites, Remark 7.26.7. We also give a direct proof in this case.

Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U'/S)$ for any open $U \subset T$, and $U' \to T$ an open immersion in $(Sch/T)_{Zar}$ with image U, see Lemmas 34.3.12 and 34.3.13. Hence given $f : T' \to T$ over S and $U, U' \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U'/S) \to \mathcal{F}(U' \times_T T'/S) = \mathcal{F}_{T'}(f^{-1}(U))$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U' \times_T T' \to U'$ over S. The collection of these maps are compatible with restrictions, and hence define an f-map

c_f from $\mathcal{F}_T$ to $\mathcal{F}_{T'}$, see Sheaves, Definition 6.21.7 and the discussion surrounding it. It is clear that $c_{f \circ g}$ is the composition of c_f and c_g, since composition of restriction maps of $\mathcal{F}$ gives restriction maps.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ (where we think of c_f as an f-map again). The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

34.4. The étale topology

0214 Let S be a scheme. We would like to define the étale-topology on the category of schemes over S. According to our general principle we first introduce the notion of an étale covering.

0215 Definition 34.4.1. Let T be a scheme. An étale covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is étale and such that $T = \bigcup f_i(T_i)$.

0216 Lemma 34.4.2. Any Zariski covering is an étale covering.

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism, see Morphisms, Lemma 29.36.9. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

0217 Lemma 34.4.3. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an étale covering of T.
(2) If $\{T_i \to T\}_{i \in I}$ is an étale covering and for each i we have an étale covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an étale covering.
(3) If $\{T_i \to T\}_{i \in I}$ is an étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an étale covering.

Proof. Omitted. □

0218 Lemma 34.4.4. Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an étale covering of T. Then there exists an étale covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.

Proof. Omitted. □

Thus we define the corresponding standard coverings of affines as follows.

0219 Definition 34.4.5. Let T be an affine scheme. A standard étale covering of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine and étale over T and $T = \bigcup f_j(U_j)$.

In the definition above we do not assume the morphisms f_j are standard étale. The reason is that if we did then the standard étale coverings would not define a site on Aff$/S$, for example because of Algebra, Lemma 10.144.2 part (4). On the other hand, an étale morphism of affines is automatically standard smooth, see Algebra, Lemma 10.143.2. Hence a standard étale covering is a standard smooth covering and a standard syntomic covering.

021A Definition 34.4.6. A big étale site is any site $Sch_{étale}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of étale coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of étale coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big étale site of a scheme S, let us point out that the topology on a big étale site $Sch_{étale}$ is in some sense induced from the étale topology on the category of all schemes.

03WW Lemma 34.4.7. Let $Sch_{étale}$ be a big étale site as in Definition 34.4.6. Let $T \in \text{Ob}(Sch_{étale})$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary étale covering of T.

(1) There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{étale}$ which refines $\{T_i \to T\}_{i \in I}$.
(2) If $\{T_i \to T\}_{i \in I}$ is a standard étale covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.
(3) If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 34.4.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an étale covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of $Sch_{étale}$. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of $Sch_{étale}$ by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{étale}$ by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j \in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{étale}$ by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. $\square$

021B Definition 34.4.8. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S.

(1) The big étale site of S, denoted $(Sch/S)_{étale}$, is the site $Sch_{étale}/S$ introduced in Sites, Section 7.25.
(2) The small étale site of S, which we denote $S_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are those U/S such that $U \to S$ is étale. A covering of $S_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \text{Ob}(S_{étale})$.
(3) The big affine étale site of S, denoted $(\text{Aff}/S)_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are those U/S such that U is an affine scheme. A covering of $(\text{Aff}/S)_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \text{Ob}((\text{Aff}/S)_{étale})$ which is a standard étale covering.

(4) The small affine étale site of S, denoted $S_{affine,étale}$, is the full subcategory of $S_{étale}$ whose objects are those U/S such that U is an affine scheme. A covering of $S_{affine,étale}$ is any covering $\{U_i \to U\}$ of $S_{étale}$ with $U \in \mathrm{Ob}(S_{affine,étale})$ which is a standard étale covering.

It is not completely clear that the big affine étale site, the small étale site, and the small affine étale site are sites. We check this now.

021C Lemma 34.4.9. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The structures $S_{étale}$, $(\mathrm{Aff}/S)_{étale}$, and $S_{affine,étale}$ are sites.

Proof. Let us show that $S_{étale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{étale}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(\mathrm{Sch}/S)_{étale}$ with $U \in \mathrm{Ob}(S_{étale})$ we also have $U_i \in \mathrm{Ob}(S_{étale})$. This follows from the definitions as the composition of étale morphisms is an étale morphism.

Let us show that $(\mathrm{Aff}/S)_{étale}$ is a site. Reasoning as above, it suffices to show that the collection of standard étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a standard étale covering $\{T_i \to T\}_{i\in I}$ and for each i we have a standard étale covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard étale covering because $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine.

We omit the proof that $S_{affine,étale}$ is a site. □

021D Lemma 34.4.10. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The underlying categories of the sites $Sch_{étale}$, $(Sch/S)_{étale}$, $S_{étale}$, $(\mathrm{Aff}/S)_{étale}$, and $S_{affine,étale}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{étale}$, and $S_{étale}$ both have a final object, namely S/S.

Proof. For $Sch_{étale}$ it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{étale})$. The fibre product $V \times_U W$ in $Sch_{étale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{étale}$. This proves the result for $(Sch/S)_{étale}$. If $U \to S$, $V \to U$ and $W \to U$ are étale then so is $V \times_U W \to S$ and hence we get the result for $S_{étale}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(\mathrm{Aff}/S)_{étale}$ and $S_{affine,étale}$. □

Next, we check that the big, resp. small affine site defines the same topos as the big, resp. small site.

021E Lemma 34.4.11. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $(\mathrm{Aff}/S)_{étale} \to (Sch/S)_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh((\mathrm{Aff}/S)_{étale})$ to $Sh((Sch/S)_{étale})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (\mathrm{Aff}/S)_{étale} \to (Sch/S)_{étale}$. Being cocontinuous just means that any étale covering of T/S, T affine, can be refined by a standard étale covering of T. This is the content of Lemma 34.4.4. Hence (1) holds. We see u is continuous simply because a standard étale covering is a étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

04HR Lemma 34.4.12. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $S_{affine,étale} \to S_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh(S_{affine,étale})$ to $Sh(S_{étale})$.

Proof. Omitted. Hint: compare with the proof of Lemma 34.4.11. □

Next, we establish some relationships between the topoi associated to these sites.

021F Lemma 34.4.13. Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor $T_{étale} \to (Sch/S)_{étale}$ is cocontinuous and induces a morphism of topoi

$$i_f : Sh(T_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{étale} \to (Sch/S)_{étale}$. In other words, given an étale morphism $j : U \to T$ corresponding to an object of $T_{étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 34.4.10. Let $a, b : U \to V$ be two morphisms in $T_{étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/T}, V \times_T V, (a,b)} U \times_T U$$

which is a fibre product of schemes étale over T, hence étale over T. Thus $T_{étale}$ has equalizers and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 7.21.5 and 7.21.6. □

021G Lemma 34.4.14. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The inclusion functor $S_{étale} \to (Sch/S)_{étale}$ satisfies the hypotheses of Sites, Lemma 7.21.8 and hence induces a morphism of sites

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

and a morphism of topoi

$$i_S : Sh(S_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

such that $\pi_S \circ i_S = \text{id}$. Moreover, $i_S = i_{\text{id}_S}$ with i_{id_S} as in Lemma 34.4.13. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{étale} \to (Sch/S)_{étale}$, in addition to the properties seen in the proof of Lemma 34.4.13 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 7.21.8. □

04BT Definition 34.4.15. In the situation of Lemma 34.4.14 the functor $i_S^{-1} = \pi_{S,*}$ is often called the restriction to the small étale site, and for a sheaf $\mathcal{F}$ on the big étale site we denote $\mathcal{F}|_{S_{étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the small site that

$$\mathrm{Mor}_{Sh(S_{\acute{e}tale})}(\mathcal{F}|_{S_{\acute{e}tale}}, \mathcal{G}) = \mathrm{Mor}_{Sh((Sch/S)_{\acute{e}tale})}(\mathcal{F}, i_{S,*}\mathcal{G})$$
$$\mathrm{Mor}_{Sh(S_{\acute{e}tale})}(\mathcal{G}, \mathcal{F}|_{S_{\acute{e}tale}}) = \mathrm{Mor}_{Sh((Sch/S)_{\acute{e}tale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{\acute{e}tale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{\acute{e}tale}} = \mathcal{G}$.

021H Lemma 34.4.16. Let $Sch_{\acute{e}tale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{\acute{e}tale}$. The functor

$$u : (Sch/T)_{\acute{e}tale} \longrightarrow (Sch/S)_{\acute{e}tale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{\acute{e}tale} \longrightarrow (Sch/T)_{\acute{e}tale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{\acute{e}tale}) \longrightarrow Sh((Sch/S)_{\acute{e}tale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 34.4.13). Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. $\square$

021I Lemma 34.4.17. Let $Sch_{\acute{e}tale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{\acute{e}tale}$.

(1) We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 34.4.13 and i_T as in Lemma 34.4.14.
(2) The functor $S_{\acute{e}tale} \to T_{\acute{e}tale}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of sites
$$f_{small} : T_{\acute{e}tale} \longrightarrow S_{\acute{e}tale}$$
We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.
(3) We have a commutative diagram of morphisms of sites
$$\begin{tikzcd} T_{\acute{e}tale} \arrow[d, "f_{small}"'] & (Sch/T)_{\acute{e}tale} \arrow[l, "\pi_T"'] \arrow[d, "f_{big}"] \\ S_{\acute{e}tale} & (Sch/S)_{\acute{e}tale} \arrow[l, "\pi_S"'] \end{tikzcd}$$
so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
(4) We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{\acute{e}tale} \to T_{\acute{e}tale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemma 34.4.3 (3) and 34.4.10. Moreover, both $S_{\acute{e}tale}$, $T_{\acute{e}tale}$ have final objects, namely S/S and T/T and $u(S/S) =$

T/T. Hence by Sites, Proposition 7.14.7 the functor u corresponds to a morphism of sites $T_{étale} \to S_{étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 7.15.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 34.4.15 we have: for $\mathcal{F}$ a sheaf on the big étale site of T

$$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

021J Lemma 34.4.18. Given schemes X, Y, Y in $Sch_{étale}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.4.16. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 34.4.17. □

0DDA Lemma 34.4.19. Let $Sch_{étale}$ be a big étale site. Consider a cartesian diagram

$$\begin{tikzcd} T' \arrow[r, "g'"'] \arrow[d, "f'"'] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in $Sch_{étale}$. Then $i_g^{-1} \circ f_{big,*} = f'_{small,*} \circ (i_{g'})^{-1}$ and $g_{big}^{-1} \circ f_{big,*} = f'_{big,*} \circ (g'_{big})^{-1}$.

Proof. Since the diagram is cartesian, we have for U'/S' that $U' \times_{S'} T' = U' \times_S T$. Hence both $i_g^{-1} \circ f_{big,*}$ and $f'_{small,*} \circ (i_{g'})^{-1}$ send a sheaf $\mathcal{F}$ on $(Sch/T)_{étale}$ to the sheaf $U' \mapsto \mathcal{F}(U' \times_{S'} T')$ on $S'_{étale}$ (use Lemmas 34.4.13 and 34.4.16). The second equality can be proved in the same manner or can be deduced from the very general Sites, Lemma 7.28.1. □

We can think about a sheaf on the big étale site of S as a collection of "usual" sheaves on all schemes over S.

021K Lemma 34.4.20. Let S be a scheme contained in a big étale site $Sch_{étale}$. A sheaf $\mathcal{F}$ on the big étale site $(Sch/S)_{étale}$ is given by the following data:

(1) for every $T/S \in \text{Ob}((Sch/S)_{étale})$ a sheaf $\mathcal{F}_T$ on $T_{étale}$,
(2) for every $f : T' \to T$ in $(Sch/S)_{étale}$ a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.

These data are subject to the following conditions:

(a) given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{étale}$ the composition $c_g \circ g_{small}^{-1} c_f$ is equal to $c_{f \circ g}$, and
(b) if $f : T' \to T$ in $(Sch/S)_{étale}$ is étale then c_f is an isomorphism.

Proof. This lemma follows from a purely sheaf theoretic statement discussed in Sites, Remark 7.26.7. We also give a direct proof in this case.

Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{étale})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U/S)$ for any $U \to T$ in $T_{étale}$ see Lemma 34.4.13. Hence given $f : T' \to T$ over S and $U \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U/S) \to \mathcal{F}(U \times_T T'/S) = \mathcal{F}_{T'}(U \times_T T')$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U \times_T T' \to U$ over S. The collection of these maps are compatible with restrictions, and hence define a map $c'_f : \mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ where $u : T_{étale} \to T'_{étale}$ is the base change functor associated to f. By adjunction of $f_{small,*}$ (see Sites, Section 7.13) with f_{small}^{-1} this is the same as a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$. It is clear that $c'_{f\circ g}$ is the composition of c'_f and $f_{small,*}c'_g$, since composition of restriction maps of $\mathcal{F}$ gives restriction maps, and this gives the desired relationship among c_f, c_g and $c_{f\circ g}$.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{étale})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ where we think of c_f as a map $\mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ again. The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

34.5. The smooth topology

021Y In this section we define the smooth topology. This is a bit pointless as it will turn out later (see More on Morphisms, Section 37.38) that this topology defines the same topos as the étale topology. But still it makes sense and it is used occasionally.

021Z Definition 34.5.1. Let T be a scheme. A smooth covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is smooth and such that $T = \bigcup f_i(T_i)$.

0220 Lemma 34.5.2. Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.

Proof. This is clear from the definitions, the fact that an étale morphism is smooth see Morphisms, Definition 29.36.1 and Lemma 34.4.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

0221 Lemma 34.5.3. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a smooth covering of T.
(2) If $\{T_i \to T\}_{i\in I}$ is a smooth covering and for each i we have a smooth covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a smooth covering.
(3) If $\{T_i \to T\}_{i\in I}$ is a smooth covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a smooth covering.

Proof. Omitted. □

0222 Lemma 34.5.4. Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a smooth covering of T. Then there exists a smooth covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard smooth, see Morphisms, Definition 29.34.1. Moreover, we may choose each U_j to be open affine in one of the T_i.

Proof. Omitted, but see Algebra, Lemma 10.137.10. □

Thus we define the corresponding standard coverings of affines as follows.

0223 Definition 34.5.5. Let T be an affine scheme. A standard smooth covering of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard smooth and $T = \bigcup f_j(U_j)$.

03WY Definition 34.5.6. A big smooth site is any site Sch_{smooth} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of smooth coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of smooth coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big smooth site of a scheme S, let us point out that the topology on a big smooth site Sch_{smooth} is in some sense induced from the smooth topology on the category of all schemes.

03WZ Lemma 34.5.7. Let Sch_{smooth} be a big smooth site as in Definition 34.5.6. Let $T \in \text{Ob}(Sch_{smooth})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary smooth covering of T.

(1) There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{smooth} which refines $\{T_i \to T\}_{i\in I}$.
(2) If $\{T_i \to T\}_{i\in I}$ is a standard smooth covering, then it is tautologically equivalent to a covering of Sch_{smooth}.
(3) If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{smooth}.

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 34.5.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a smooth covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{smooth}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{smooth} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{smooth} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{smooth} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

03X0 Definition 34.5.8. Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S.

(1) The big smooth site of S, denoted $(Sch/S)_{smooth}$, is the site Sch_{smooth}/S introduced in Sites, Section 7.25.
(2) The big affine smooth site of S, denoted $(\text{Aff}/S)_{smooth}$, is the full subcategory of $(Sch/S)_{smooth}$ whose objects are affine U/S. A covering of

$(\mathrm{Aff}/S)_{smooth}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{smooth}$ which is a standard smooth covering.

Next, we check that the big affine site defines the same topos as the big site.

06VC Lemma 34.5.9. Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S. The functor $(\mathrm{Aff}/S)_{smooth} \to (Sch/S)_{smooth}$ is special cocontinuous and induces an equivalence of topoi from $Sh((\mathrm{Aff}/S)_{smooth})$ to $Sh((Sch/S)_{smooth})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (\mathrm{Aff}/S)_{smooth} \to (Sch/S)_{smooth}$. Being cocontinuous just means that any smooth covering of T/S, T affine, can be refined by a standard smooth covering of T. This is the content of Lemma 34.5.4. Hence (1) holds. We see u is continuous simply because a standard smooth covering is a smooth covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

04HC Lemma 34.5.10. Let Sch_{smooth} be a big smooth site. Let $f : T \to S$ be a morphism in Sch_{smooth}. The functor

$$u : (Sch/T)_{smooth} \longrightarrow (Sch/S)_{smooth}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{smooth} \longrightarrow (Sch/T)_{smooth}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{smooth}) \longrightarrow Sh((Sch/S)_{smooth})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

34.6. The syntomic topology

0224 In this section we define the syntomic topology. This topology is quite interesting in that it often has the same cohomology groups as the fppf topology but is technically easier to deal with.

0225 Definition 34.6.1. Let T be a scheme. An syntomic covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is syntomic and such that $T = \bigcup f_i(T_i)$.

0226 Lemma 34.6.2. Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic, see Morphisms, Lemma 29.34.7 and Lemma 34.5.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

0227 Lemma 34.6.3. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a syntomic covering of T.
(2) If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and for each i we have a syntomic covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a syntomic covering.
(3) If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a syntomic covering.

Proof. Omitted. □

0228 Lemma 34.6.4. Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a syntomic covering of T. Then there exists a syntomic covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard syntomic, see Morphisms, Definition 29.30.1. Moreover, we may choose each U_j to be open affine in one of the T_i.

Proof. Omitted, but see Algebra, Lemma 10.136.15. □

Thus we define the corresponding standard coverings of affines as follows.

0229 Definition 34.6.5. Let T be an affine scheme. A standard syntomic covering of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard syntomic and $T = \bigcup f_j(U_j)$.

03X1 Definition 34.6.6. A big syntomic site is any site $Sch_{syntomic}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of syntomic coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of syntomic coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big syntomic site of a scheme S, let us point out that the topology on a big syntomic site $Sch_{syntomic}$ is in some sense induced from the syntomic topology on the category of all schemes.

03X2 Lemma 34.6.7. Let $Sch_{syntomic}$ be a big syntomic site as in Definition 34.6.6. Let $T \in \text{Ob}(Sch_{syntomic})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary syntomic covering of T.

(1) There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{syntomic}$ which refines $\{T_i \to T\}_{i\in I}$.
(2) If $\{T_i \to T\}_{i\in I}$ is a standard syntomic covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.
(3) If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.

Proof. For each i choose an affine open covering $T_i = \bigcup_{j \in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 34.6.3 the refinement $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a syntomic covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of $Sch_{syntomic}$. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of $Sch_{syntomic}$ by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i \in I}$ refines $\{T_i \to T\}_{i \in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i \in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j \in J}$ of T in the site $Sch_{syntomic}$ by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j \in J}$ is a covering as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{syntomic}$ by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

03X3 Definition 34.6.8. Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S.

(1) The big syntomic site of S, denoted $(Sch/S)_{syntomic}$, is the site $Sch_{syntomic}/S$ introduced in Sites, Section 7.25.
(2) The big affine syntomic site of S, denoted $(\text{Aff}/S)_{syntomic}$, is the full subcategory of $(Sch/S)_{syntomic}$ whose objects are affine U/S. A covering of $(\text{Aff}/S)_{syntomic}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{syntomic}$ which is a standard syntomic covering.

Next, we check that the big affine site defines the same topos as the big site.

06VD Lemma 34.6.9. Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S. The functor $(\text{Aff}/S)_{syntomic} \to (Sch/S)_{syntomic}$ is special cocontinuous and induces an equivalence of topoi from $Sh((\text{Aff}/S)_{syntomic})$ to $Sh((Sch/S)_{syntomic})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (\text{Aff}/S)_{syntomic} \to (Sch/S)_{syntomic}$. Being cocontinuous just means that any syntomic covering of T/S, T affine, can be refined by a standard syntomic covering of T. This is the content of Lemma 34.6.4. Hence (1) holds. We see u is continuous simply because a standard syntomic covering is a syntomic covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

04HD Lemma 34.6.10. Let $Sch_{syntomic}$ be a big syntomic site. Let $f : T \to S$ be a morphism in $Sch_{syntomic}$. The functor

$$u : (Sch/T)_{syntomic} \longrightarrow (Sch/S)_{syntomic}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{syntomic} \longrightarrow (Sch/T)_{syntomic}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{syntomic}) \longrightarrow Sh((Sch/S)_{syntomic})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

34.7. The fppf topology

021L Let S be a scheme. We would like to define the fppf-topology[3] on the category of schemes over S. According to our general principle we first introduce the notion of an fppf-covering.

021M Definition 34.7.1. Let T be a scheme. An fppf covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

021N Lemma 34.7.2. Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation, see Morphisms, Lemmas 29.30.6 and 29.30.7, and Lemma 34.6.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

021O Lemma 34.7.3. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fppf covering of T.
(2) If $\{T_i \to T\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering.
(3) If $\{T_i \to T\}_{i \in I}$ is an fppf covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fppf covering.

Proof. The first assertion is clear. The second follows as the composition of flat morphisms is flat (see Morphisms, Lemma 29.25.6) and the composition of morphisms of finite presentation is of finite presentation (see Morphisms, Lemma 29.21.3). The third follows as the base change of a flat morphism is flat (see Morphisms, Lemma 29.25.8) and the base change of a morphism of finite presentation is of finite presentation (see Morphisms, Lemma 29.21.4). Moreover, the base change of a surjective family of morphisms is surjective (proof omitted). □

021P Lemma 34.7.4. Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fppf covering of T. Then there exists an fppf covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.

Proof. This follows directly from the definitions using that a morphism which is flat and locally of finite presentation is open, see Morphisms, Lemma 29.25.10. □

Thus we define the corresponding standard coverings of affines as follows.

[3]The letters fppf stand for “fidèlement plat de présentation finie”.

021Q **Definition 34.7.5.** Let T be an affine scheme. A *standard fppf covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, flat and of finite presentation over T and $T = \bigcup f_j(U_j)$.

021R **Definition 34.7.6.** A *big fppf site* is any site Sch_{fppf} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of fppf coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big fppf site of a scheme S, let us point out that the topology on a big fppf site Sch_{fppf} is in some sense induced from the fppf topology on the category of all schemes.

03WX **Lemma 34.7.7.** Let Sch_{fppf} be a big fppf site as in Definition 34.7.6. Let $T \in \text{Ob}(Sch_{fppf})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary fppf covering of T.

(1) There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{fppf} which refines $\{T_i \to T\}_{i\in I}$.
(2) If $\{T_i \to T\}_{i\in I}$ is a standard fppf covering, then it is tautologically equivalent to a covering of Sch_{fppf}.
(3) If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{fppf}.

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 34.7.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an fppf covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{fppf}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{fppf} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{fppf} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{fppf} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. $\square$

021S **Definition 34.7.8.** Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S.

(1) The *big fppf site of S*, denoted $(Sch/S)_{fppf}$, is the site Sch_{fppf}/S introduced in Sites, Section 7.25.
(2) The *big affine fppf site of S*, denoted $(\text{Aff}/S)_{fppf}$, is the full subcategory of $(Sch/S)_{fppf}$ whose objects are affine U/S. A covering of $(\text{Aff}/S)_{fppf}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{fppf}$ which is a standard fppf covering.

It is not completely clear that the big affine fppf site is a site. We check this now.

021T Lemma 34.7.9. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. Then $(\mathit{Aff}/S)_{fppf}$ is a site.

Proof. Let us show that $(\mathit{Aff}/S)_{fppf}$ is a site. Reasoning as in the proof of Lemma 34.4.9 it suffices to show that the collection of standard fppf coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a standard fppf covering $\{T_i \to T\}_{i\in I}$ and for each i we have a standard fppf covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard fppf covering because $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine. $\square$

021U Lemma 34.7.10. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The underlying categories of the sites Sch_{fppf}, $(Sch/S)_{fppf}$, and $(\mathit{Aff}/S)_{fppf}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{fppf}$ has a final object, namely S/S.

Proof. For Sch_{fppf} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{fppf})$. The fibre product $V \times_U W$ in Sch_{fppf} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{fppf}$. This proves the result for $(Sch/S)_{fppf}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(\mathit{Aff}/S)_{fppf}$. $\square$

Next, we check that the big affine site defines the same topos as the big site.

021V Lemma 34.7.11. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The functor $(\mathit{Aff}/S)_{fppf} \to (Sch/S)_{fppf}$ is cocontinuous and induces an equivalence of topoi from $Sh((\mathit{Aff}/S)_{fppf})$ to $Sh((Sch/S)_{fppf})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (\mathit{Aff}/S)_{fppf} \to (Sch/S)_{fppf}$. Being cocontinuous just means that any fppf covering of T/S, T affine, can be refined by a standard fppf covering of T. This is the content of Lemma 34.7.4. Hence (1) holds. We see u is continuous simply because a standard fppf covering is a fppf covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. $\square$

Next, we establish some relationships between the topoi associated to these sites.

021W Lemma 34.7.12. Let Sch_{fppf} be a big fppf site. Let $f : T \to S$ be a morphism in Sch_{fppf}. The functor

$$u : (Sch/T)_{fppf} \longrightarrow (Sch/S)_{fppf}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{fppf} \longrightarrow (Sch/T)_{fppf}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. $\square$

021X Lemma 34.7.13. Given schemes X, Y, Y in $(Sch/S)_{fppf}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.7.12. $\square$

34.8. The ph topology

0DBC In this section we define the ph topology. This is the topology generated by Zariski coverings and proper surjective morphisms, see Lemma 34.8.15.

We borrow our notation/terminology from the paper [GL01] by Goodwillie and Lichtenbaum. These authors show that if we restrict to the subcategory of Noetherian schemes, then the ph topology is the same as the "h topology" as originally defined by Voevodsky: this is the topology generated by Zariski open coverings and finite type morphisms which are universally submersive. They also show that the two topologies do not agree on non-Noetherian schemes, see [GL01, Example 4.5]. We return to (our version of) the h topology in More on Flatness, Section 38.34.

Before we can define the coverings in our topology we need to do a bit of work.

0DBD Definition 34.8.1. Let T be an affine scheme. A standard ph covering is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ constructed from a proper surjective morphism $f : U \to T$ and an affine open covering $U = \bigcup_{j=1,\ldots,m} U_j$ by setting $f_j = f|_{U_j}$.

It follows immediately from Chow's lemma that we can refine a standard ph covering by a standard ph covering corresponding to a surjective projective morphism.

0DBE Lemma 34.8.2. Let $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ be a standard ph covering. Let $T' \to T$ be a morphism of affine schemes. Then $\{U_j \times_T T' \to T'\}_{j=1,\ldots,m}$ is a standard ph covering.

Proof. Let $f : U \to T$ be proper surjective and let an affine open covering $U = \bigcup_{j=1,\ldots,m} U_j$ be given as in Definition 34.8.1. Then $U \times_T T' \to T'$ is proper surjective (Morphisms, Lemmas 29.9.4 and 29.41.5). Also, $U \times_T T' = \bigcup_{j=1,\ldots,m} U_j \times_T T'$ is an affine open covering. This concludes the proof. $\square$

0DBF Lemma 34.8.3. Let T be an affine scheme. Each of the following types of families of maps with target T has a refinement by a standard ph covering:

(1) any Zariski open covering of T,
(2) $\{W_{ji} \to T\}_{j=1,\ldots,m,i=1,\ldots n_j}$ where $\{W_{ji} \to U_j\}_{i=1,\ldots,n_j}$ and $\{U_j \to T\}_{j=1,\ldots,m}$ are standard ph coverings.

Proof. Part (1) follows from the fact that any Zariski open covering of T can be refined by a finite affine open covering.

Proof of (3). Choose $U \to T$ proper surjective and $U = \bigcup_{j=1,\ldots,m} U_j$ as in Definition 34.8.1. Choose $W_j \to U_j$ proper surjective and $W_j = \bigcup W_{ji}$ as in Definition 34.8.1.

By Chow's lemma (Limits, Lemma 32.12.1) we can find $W'_j \to W_j$ proper surjective and closed immersions $W'_j \to \mathbf{P}^{e_j}_{U_j}$. Thus, after replacing W_j by W'_j and $W_j = \bigcup W_{ji}$ by a suitable affine open covering of W'_j, we may assume there is a closed immersion $W_j \subset \mathbf{P}^{e_j}_{U_j}$ for all $j = 1, \ldots, m$.

Let $\overline{W}_j \subset \mathbf{P}^{e_j}_U$ be the scheme theoretic closure of W_j. Then $W_j \subset \overline{W}_j$ is an open subscheme; in fact W_j is the inverse image of $U_j \subset U$ under the morphism $\overline{W}_j \to U$. (To see this use that $W_j \to \mathbf{P}^{e_j}_U$ is quasi-compact and hence formation of the scheme theoretic image commutes with restriction to opens, see Morphisms, Section 29.6.) Let $Z_j = U \setminus U_j$ with reduced induced closed subscheme structure. Then

$$V_j = \overline{W}_j \amalg Z_j \to U$$

is proper surjective and the open subscheme $W_j \subset V_j$ is the inverse image of U_j. Hence for $v \in V_j$, $v \notin W_j$ we can pick an affine open neighbourhood $v \in V_{j,v} \subset V_j$ which maps into $U_{j'}$ for some $1 \leq j' \leq m$.

To finish the proof we consider the proper surjective morphism

$$V = V_1 \times_U V_2 \times_U \ldots \times_U V_m \longrightarrow U \longrightarrow T$$

and the covering of V by the affine opens

$$V_{1,v_1} \times_U \ldots \times_U V_{j-1,v_{j-1}} \times_U W_{ji} \times_U V_{j+1,v_{j+1}} \times_U \ldots \times_U V_{m,v_m}$$

These do indeed form a covering, because each point of U is in some U_j and the inverse image of U_j in V is equal to $V_1 \times \ldots \times V_{j-1} \times W_j \times V_{j+1} \times \ldots \times V_m$. Observe that the morphism from the affine open displayed above to T factors through W_{ji} thus we obtain a refinement. Finally, we only need a finite number of these affine opens as V is quasi-compact (as a scheme proper over the affine scheme T). □

0DBG Definition 34.8.4. Let T be a scheme. A ph covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that f_i is locally of finite type and such that for every affine open $U \subset T$ there exists a standard ph covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining the family $\{T_i \times_T U \to U\}_{i \in I}$.

A standard ph covering is a ph covering by Lemma 34.8.2.

0DBH Lemma 34.8.5. A Zariski covering is a ph covering[4].

Proof. This is true because a Zariski covering of an affine scheme can be refined by a standard ph covering by Lemma 34.8.3. □

0DES Lemma 34.8.6. Let $f : Y \to X$ be a surjective proper morphism of schemes. Then $\{Y \to X\}$ is a ph covering.

Proof. Omitted. □

0ET9 Lemma 34.8.7. Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms such that f_i is locally of finite type for all i. The following are equivalent

(1) $\{T_i \to T\}_{i \in I}$ is a ph covering,
(2) there is a ph covering which refines $\{T_i \to T\}_{i \in I}$, and
(3) $\{\coprod_{i \in I} T_i \to T\}$ is a ph covering.

[4]We will see in More on Morphisms, Lemma 37.48.7 that fppf coverings (and hence syntomic, smooth, or étale coverings) are ph coverings as well.

Proof. The equivalence of (1) and (2) follows immediately from Definition 34.8.4 and the fact that a refinement of a refinement is a refinement. Because of the equivalence of (1) and (2) and since $\{T_i \to T\}_{i \in I}$ refines $\{\coprod_{i \in I} T_i \to T\}$ we see that (1) implies (3). Finally, assume (3) holds. Let $U \subset T$ be an affine open and let $\{U_j \to U\}_{j=1,\ldots,m}$ be a standard ph covering which refines $\{U \times_T \coprod_{i \in I} T_i \to U\}$. This means that for each j we have a morphism

$$h_j : U_j \longrightarrow U \times_T \coprod\nolimits_{i \in I} T_i = \coprod\nolimits_{i \in I} U \times_T T_i$$

over U. Since U_j is quasi-compact, we get disjoint union decompositions $U_j = \coprod_{i \in I} U_{j,i}$ by open and closed subschemes almost all of which are empty such that $h_j|_{U_{j,i}}$ maps $U_{j,i}$ into $U \times_T T_i$. It follows that

$$\{U_{j,i} \to U\}_{j=1,\ldots,m,\ i \in I,\ U_{j,i} \neq \emptyset}$$

is a standard ph covering (small detail omitted) refining $\{U \times_T T_i \to U\}_{i \in I}$. Thus (1) holds. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

0DBI Lemma 34.8.8. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a ph covering of T.
(2) If $\{T_i \to T\}_{i \in I}$ is a ph covering and for each i we have a ph covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a ph covering.
(3) If $\{T_i \to T\}_{i \in I}$ is a ph covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a ph covering.

Proof. Assertion (1) is clear.

Proof of (3). The base change $T_i \times_T T' \to T'$ is locally of finite type by Morphisms, Lemma 29.15.4. hence we only need to check the condition on affine opens. Let $U' \subset T'$ be an affine open subscheme. Since U' is quasi-compact we can find a finite affine open covering $U' = U'_1 \cup \ldots \cup U'$ such that $U'_j \to T$ maps into an affine open $U_j \subset T$. Choose a standard ph covering $\{U_{jl} \to U_j\}_{l=1,\ldots,n_j}$ refining $\{T_i \times_T U_j \to U_j\}$. By Lemma 34.8.2 the base change $\{U_{jl} \times_{U_j} U'_j \to U'_j\}$ is a standard ph covering. Note that $\{U'_j \to U'\}$ is a standard ph covering as well. By Lemma 34.8.3 the family $\{U_{jl} \times_{U_j} U'_j \to U'\}$ can be refined by a standard ph covering. Since $\{U_{jl} \times_{U_j} U'_j \to U'\}$ refines $\{T_i \times_T U' \to U'\}$ we conclude.

Proof of (2). Composition preserves being locally of finite type, see Morphisms, Lemma 29.15.3. Hence we only need to check the condition on affine opens. Let $U \subset T$ be affine open. First we pick a standard ph covering $\{U_k \to U\}_{k=1,\ldots,m}$ refining $\{T_i \times_T U \to U\}$. Say the refinement is given by morphisms $U_k \to T_{i_k}$ over T. Then

$$\{T_{i_k j} \times_{T_{i_k}} U_k \to U_k\}_{j \in J_{i_k}}$$

is a ph covering by part (3). As U_k is affine, we can find a standard ph covering $\{U_{ka} \to U_k\}_{a=1,\ldots,b_k}$ refining this family. Then we apply Lemma 34.8.3 to see that $\{U_{ka} \to U\}$ can be refined by a standard ph covering. Since $\{U_{ka} \to U\}$ refines $\{T_{ij} \times_T U \to U\}$ this finishes the proof. □

0DBJ Definition 34.8.9. A big ph site is any site Sch_{ph} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of ph coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of ph coverings, and the set Cov_0 chosen above.

See the remarks following Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big ph site of a scheme S, let us point out that the topology on a big ph site Sch_{ph} is in some sense induced from the ph topology on the category of all schemes.

0DBK Lemma 34.8.10. Let Sch_{ph} be a big ph site as in Definition 34.8.9. Let $T \in \text{Ob}(Sch_{ph})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary ph covering of T.

(1) There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{ph} which refines $\{T_i \to T\}_{i\in I}$.
(2) If $\{T_i \to T\}_{i\in I}$ is a standard ph covering, then it is tautologically equivalent to a covering of Sch_{ph}.
(3) If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{ph}.

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemmas 34.8.5 and 34.8.8 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a ph covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{ph}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{ph} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{ph} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{ph} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. $\square$

0DBL Definition 34.8.11. Let S be a scheme. Let Sch_{ph} be a big ph site containing S.

(1) The big ph site of S, denoted $(Sch/S)_{ph}$, is the site Sch_{ph}/S introduced in Sites, Section 7.25.
(2) The big affine ph site of S, denoted $(\text{Aff}/S)_{ph}$, is the full subcategory of $(Sch/S)_{ph}$ whose objects are affine U/S. A covering of $(\text{Aff}/S)_{ph}$ is any finite covering $\{U_i \to U\}$ of $(Sch/S)_{ph}$ with U_i and U affine.

Observe that the coverings in $(\text{Aff}/S)_{ph}$ are not given by standard ph coverings. The reason is simply that this would fail the second axiom of Sites, Definition 7.6.2. Rather, the coverings in $(\text{Aff}/S)_{ph}$ are those finite families $\{U_i \to U\}$ of finite type morphisms between affine objects of $(Sch/S)_{ph}$ which can be refined by a standard ph covering. We explicitly state and prove that the big affine ph site is a site.

0DBM Lemma 34.8.12. Let S be a scheme. Let Sch_{ph} be a big ph site containing S. Then $(\text{Aff}/S)_{ph}$ is a site.

Proof. Reasoning as in the proof of Lemma 34.4.9 it suffices to show that the collection of finite ph coverings $\{U_i \to U\}$ with U, U_i affine satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a finite ph covering $\{T_i \to T\}_{i\in I}$ with T_i, T affine, and for each i a finite ph covering $\{T_{ij} \to T_i\}_{j\in J_i}$ with T_{ij} affine , then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a ph covering (Lemma 34.8.8), $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine. □

0DBN Lemma 34.8.13. Let S be a scheme. Let Sch_{ph} be a big ph site containing S. The underlying categories of the sites Sch_{ph}, $(Sch/S)_{ph}$, and $(\mathrm{Aff}/S)_{ph}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{ph}$ has a final object, namely S/S.

Proof. For Sch_{ph} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{ph})$. The fibre product $V \times_U W$ in Sch_{ph} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{ph}$. This proves the result for $(Sch/S)_{ph}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(\mathrm{Aff}/S)_{ph}$. □

Next, we check that the big affine site defines the same topos as the big site.

0DBP Lemma 34.8.14. Let S be a scheme. Let Sch_{ph} be a big ph site containing S. The functor $(\mathrm{Aff}/S)_{ph} \to (Sch/S)_{ph}$ is cocontinuous and induces an equivalence of topoi from $Sh((\mathrm{Aff}/S)_{ph})$ to $Sh((Sch/S)_{ph})$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (\mathrm{Aff}/S)_{ph} \to (Sch/S)_{ph}$. Being cocontinuous follows because any ph covering of T/S, T affine, can be refined by a standard ph covering of T by definition. Hence (1) holds. We see u is continuous simply because a finite ph covering of an affine by affines is a ph covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering (which is a ph covering). □

0DBQ Lemma 34.8.15. Let $\mathcal{F}$ be a presheaf on $(Sch/S)_{ph}$. Then $\mathcal{F}$ is a sheaf if and only if

(1) $\mathcal{F}$ satisfies the sheaf condition for Zariski coverings, and
(2) if $f : V \to U$ is proper surjective, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.

Moreover, in the presence of (1) property (2) is equivalent to property

(2') the sheaf property for $\{V \to U\}$ as in (2) with U affine.

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a ph covering, i.e., a covering in $(Sch/S)_{ph}$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}(T)$ restricting to s_i over T_i. Let $T = \bigcup U_j$ be an affine open covering. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the ph coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ and a finite affine open covering $V_j = \bigcup V_{jk}$ such that the standard ph covering $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $V \to U$ is a morphism of $(Sch/S)_{ph}$ which is proper and surjective. Choose an affine open covering $U = \bigcup U_i$ and set $V_i = V \times_U U_i$. Then we see that $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective because we know $\mathcal{F}(U_i) \to \mathcal{F}(V_i)$ is injective by (2') and we know $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective by (1). Finally, suppose that we are given an $t \in \mathcal{F}(V)$ in the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$. Then $t|_{V_i}$ is in the equalizer of the two maps $\mathcal{F}(V_i) \to \mathcal{F}(V_i \times_{U_i} V_i)$ for all i. Hence we obtain a unique section $s_i \in \mathcal{F}(U_i)$ mapping to $t|_{V_i}$ for all i by (2'). We omit the verification that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j; this uses the uniqueness property just shown. By the sheaf property for the covering $U = \bigcup U_i$ we obtain a section $s \in \mathcal{F}(U)$. We omit the proof that s maps to t in $\mathcal{F}(V)$. □

Next, we establish some relationships between the topoi associated to these sites.

0DBR Lemma 34.8.16. Let Sch_{ph} be a big ph site. Let $f : T \to S$ be a morphism in Sch_{ph}. The functor

$$u : (Sch/T)_{ph} \longrightarrow (Sch/S)_{ph}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{ph} \longrightarrow (Sch/T)_{ph}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{ph}) \longrightarrow Sh((Sch/S)_{ph})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. □

0DBS Lemma 34.8.17. Given schemes X, Y, Y in $(Sch/S)_{ph}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 34.8.16. □

34.9. The fpqc topology

022A

022B Definition 34.9.1. Let T be a scheme. An fpqc covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. It is slightly harder to recognize an fpqc covering, hence we provide some lemmas to do so.

03L7 Lemma 34.9.2. Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. The following are equivalent

(1) $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering,
(2) each f_i is flat and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,
(3) each f_i is flat and there exists an affine open covering $T = \bigcup_{\alpha\in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.

If T is quasi-separated, these are also equivalent to

(4) each f_i is flat, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.

Proof. We omit the proof of the equivalence of (1), (2), and (3). From now on assume T is quasi-separated. We prove (4) implies (2). Let $U \subset T$ be an affine open. To prove (2) it suffices to show that for every $t \in U$ there exist finitely many quasi-compact opens $U_j \subset T_{i_j}$ such that $f_{i_j}(U_j) \subset U$ and such that $\bigcup f_{i_j}(U_j)$ is a neighbourhood of t in U. By assumption there do exist finitely many quasi-compact opens $U'_j \subset T_{i_j}$ such that such that $\bigcup f_{i_j}(U'_j)$ is a neighbourhood of t in T. Since T is quasi-separated we see that $U_j = U'_j \cap f_j^{-1}(U)$ is quasi-compact open as desired. Since it is clear that (2) implies (4) the proof is finished. □

040I Lemma 34.9.3. Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. The following are equivalent

(1) $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering, and
(2) setting $T' = \coprod_{i\in I} T_i$, and $f = \coprod_{i\in I} f_i$ the family $\{f : T' \to T\}$ is an fpqc covering.

Proof. Suppose that $U \subset T$ is an affine open. If (1) holds, then we find $i_1, \ldots, i_n \in I$ and affine opens $U_j \subset T_{i_j}$ such that $U = \bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$. Then $U_1 \amalg \ldots \amalg U_n \subset T'$ is a quasi-compact open surjecting onto U. Thus $\{f : T' \to T\}$ is an fpqc covering by Lemma 34.9.2. Conversely, if (2) holds then there exists a quasi-compact open $U' \subset T'$ with $U = f(U')$. Then $U_j = U' \cap T_j$ is quasi-compact open in T_j and empty for almost all j. By Lemma 34.9.2 we see that (1) holds. □

03L8 Lemma 34.9.4. Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms of schemes with target T. Assume that

(1) each f_i is flat, and
(2) the family $\{f_i : T_i \to T\}_{i\in I}$ can be refined by an fpqc covering of T.

Then $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering of T.

Proof. Let $\{g_j : X_j \to T\}_{j \in J}$ be an fpqc covering refining $\{f_i : T_i \to T\}$. Suppose that $U \subset T$ is affine open. Choose $j_1, \ldots, j_m \in J$ and $V_k \subset X_{j_k}$ affine open such that $U = \bigcup g_{j_k}(V_k)$. For each j pick $i_j \in I$ and a morphism $h_j : X_j \to T_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since $h_{j_k}(V_k)$ is quasi-compact we can find a quasi-compact open $h_{j_k}(V_k) \subset U_k \subset f_{i_{j_k}}^{-1}(U)$. Then $U = \bigcup f_{i_{j_k}}(U_k)$. We conclude that $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering by Lemma 34.9.2. □

03L9 Lemma 34.9.5. Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that

(1) each f_i is flat, and
(2) there exists an fpqc covering $\{g_j : S_j \to T\}_{j \in J}$ such that each $\{S_j \times_T T_i \to S_j\}_{i \in I}$ is an fpqc covering.

Then $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering of T.

Proof. We will use Lemma 34.9.2 without further mention. Let $U \subset T$ be an affine open. By (2) we can find quasi-compact opens $V_j \subset S_j$ for $j \in J$, almost all empty, such that $U = \bigcup g_j(V_j)$. Then for each j we can choose quasi-compact opens $W_{ij} \subset S_j \times_T T_i$ for $i \in I$, almost all empty, with $V_j = \bigcup_i \text{pr}_1(W_{ij})$. Thus $\{S_j \times_T T_i \to T\}$ is an fpqc covering. Since this covering refines $\{f_i : T_i \to T\}$ we conclude by Lemma 34.9.4. □

022C Lemma 34.9.6. Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 34.7.2. Let $\{f_i : U_i \to U\}_{i \in I}$ be an fppf covering. By definition this means that the f_i are flat which checks the first condition of Definition 34.9.1. To check the second, let $V \subset U$ be an affine open subset. Write $f_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since each f_i is open (Morphisms, Lemma 29.25.10), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} f_i(V_{ij})$ is an open covering of V. Since V is quasi-compact, this covering has a finite refinement. This finishes the proof. □

The fpqc[5] topology cannot be treated in the same way as the fppf topology[6]. Namely, suppose that R is a nonzero ring. We will see in Lemma 34.9.14 that there does not exist a set A of fpqc-coverings of Spec(R) such that every fpqc-covering can be refined by an element of A. If $R = k$ is a field, then the reason for this unboundedness is that there does not exist a field extension of k such that every field extension of k is contained in it.

If you ignore set theoretic difficulties, then you run into presheaves which do not have a sheafification, see [Wat75, Theorem 5.5]. A mildly interesting option is to consider only those faithfully flat ring extensions $R \to R'$ where the cardinality of R' is suitably bounded. (And if you consider all schemes in a fixed universe as in SGA4 then you are bounding the cardinality by a strongly inaccessible cardinal.) However, it is not so clear what happens if you change the cardinal to a bigger one.

For these reasons we do not introduce fpqc sites and we will not consider cohomology with respect to the fpqc-topology.

[5]The letters fpqc stand for "fidèlement plat quasi-compacte".

[6]A more precise statement would be that the analogue of Lemma 34.7.7 for the fpqc topology does not hold.

On the other hand, given a contravariant functor $F : Sch^{opp} \to \textit{Sets}$ it does make sense to ask whether F satisfies the sheaf property for the fpqc topology, see below. Moreover, we can wonder about descent of object in the fpqc topology, etc. Simply put, for certain results the correct generality is to work with fpqc coverings.

022D Lemma 34.9.7. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fpqc covering of T.
(2) If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fpqc covering.
(3) If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering.

Proof. Part (1) is immediate. Recall that the composition of flat morphisms is flat and that the base change of a flat morphism is flat (Morphisms, Lemmas 29.25.8 and 29.25.6). Thus we can apply Lemma 34.9.2 in each case to check that our families of morphisms are fpqc coverings.

Proof of (2). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{f_{ij} : T_{ij} \to T_i\}_{j \in J_i}$. Let $U \subset T$ be an affine open. We can find quasi-compact opens $U_i \subset T_i$ for $i \in I$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then for each i we can choose quasi-compact opens $W_{ij} \subset T_{ij}$ for $j \in J_i$, almost all empty, with $U_i = \bigcup_j f_{ij}(U_{ij})$. Thus $\{T_{ij} \to T\}$ is an fpqc covering.

Proof of (3). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes. Let $U' \subset T'$ be an affine open which maps into the affine open $U \subset T$. Choose quasi-compact opens $U_i \subset T_i$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then $U' \times_U U_i$ is a quasi-compact open of $T' \times_T T_i$ and $U' = \bigcup \text{pr}_1(U' \times_U U_i)$. Since T' can be covered by such affine opens $U' \subset T'$ we see that $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering by Lemma 34.9.2. $\square$

022E Lemma 34.9.8. Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of T. Then there exists an fpqc covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.

Proof. This follows directly from the definition. $\square$

022F Definition 34.9.9. Let T be an affine scheme. A standard fpqc covering of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,n}$ with each U_j is affine, flat over T and $T = \bigcup f_j(U_j)$.

Since we do not introduce the affine site we have to show directly that the collection of all standard fpqc coverings satisfies the axioms.

03LA Lemma 34.9.10. Let T be an affine scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a standard fpqc covering of T.
(2) If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and for each i we have a standard fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fpqc covering.
(3) If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and $T' \to T$ is a morphism of affine schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a standard fpqc covering.

Proof. This follows formally from the fact that compositions and base changes of flat morphisms are flat (Morphisms, Lemmas 29.25.8 and 29.25.6) and that fibre products of affine schemes are affine (Schemes, Lemma 26.17.2). $\square$

03LB **Lemma 34.9.11.** Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that

(1) each f_i is flat, and
(2) every affine scheme Z and morphism $h : Z \to T$ there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{T_i \times_T Z \to Z\}_{i \in I}$.

Then $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering of T.

Proof. Let $T = \bigcup U_\alpha$ be an affine open covering. For each α the pullback family $\{T_i \times_T U_\alpha \to U_\alpha\}$ can be refined by a standard fpqc covering, hence is an fpqc covering by Lemma 34.9.4. As $\{U_\alpha \to T\}$ is an fpqc covering we conclude that $\{T_i \to T\}$ is an fpqc covering by Lemma 34.9.5. $\square$

022G **Definition 34.9.12.** Let F be a contravariant functor on the category of schemes with values in sets.

(1) Let $\{U_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. We say that F satisfies the sheaf property for the given family if for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \times_T U_j} = \xi_j|_{U_i \times_T U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
(2) We say that F satisfies the sheaf property for the fpqc topology if it satisfies the sheaf property for any fpqc covering.

We try to avoid using the terminology "F is a sheaf" in this situation since we are not defining a category of fpqc sheaves as we explained above.

022H **Lemma 34.9.13.** Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies

(1) the sheaf property for every Zariski covering, and
(2) the sheaf property for any standard fpqc covering.

Moreover, in the presence of (1) property (2) is equivalent to property

(2') the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i. Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i.

By Lemma 34.9.8 we can find a standard fpqc covering $\{U_j \to U\}_{j=1,\ldots,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \to U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \text{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1, \ldots, n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard fpqc covering and in general this follows from (1) and the affine case by choosing an affine open covering of V. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j} \times_T T_i} = s_i|_{T_{i_j} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}$ is a standard fpqc covering, then $\coprod T_i \to T$ is a faithfully flat morphism of affine schemes. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_{i'})$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\coprod T_i \to T\}$ is the same. □

The following lemma is here just to point out set theoretical difficulties do indeed arise and should be ignored by most readers.

0BBK Lemma 34.9.14. Let R be a nonzero ring. There does not exist a set A of fpqc-coverings of $\mathrm{Spec}(R)$ such that every fpqc-covering can be refined by an element of A.

Proof. Let us first explain this when $R = k$ is a field. For any set I consider the purely transcendental field extension $k_I = k(\{t_i\}_{i \in I})/k$. Since $k \to k_I$ is faithfully flat we see that $\{\mathrm{Spec}(k_I) \to \mathrm{Spec}(k)\}$ is an fpqc covering. Let A be a set and for each $\alpha \in A$ let $\mathcal{U}_\alpha = \{S_{\alpha,j} \to \mathrm{Spec}(k)\}_{j \in J_\alpha}$ be an fpqc covering. If $\mathcal{U}_\alpha$ refines $\{\mathrm{Spec}(k_I) \to \mathrm{Spec}(k)\}$ then the morphisms $S_{\alpha,j} \to \mathrm{Spec}(k)$ factor through $\mathrm{Spec}(k_I)$. Since $\mathcal{U}_\alpha$ is a covering, at least some $S_{\alpha,j}$ is nonempty. Pick a point $s \in S_{\alpha,j}$. Since we have the factorization $S_{\alpha,j} \to \mathrm{Spec}(k_I) \to \mathrm{Spec}(k)$ we obtain a homomorphism of fields $k_I \to \kappa(s)$. In particular, we see that the cardinality of $\kappa(s)$ is at least the cardinality of I. Thus if we take I to be a set of cardinality bigger than the cardinalities of the residue fields of all the schemes $S_{\alpha,j}$, then such a factorization does not exist and the lemma holds for $R = k$.

General case. Since R is nonzero it has a maximal prime ideal $\mathfrak{m}$ with residue field κ. Let I be a set and consider $R_I = S_I^{-1}R[\{t_i\}_{i \in I}]$ where $S_I \subset R[\{t_i\}_{i \in I}]$ is the multiplicative subset of $f \in R[\{t_i\}_{i \in I}]$ such that f maps to a nonzero element of $R/\mathfrak{p}[\{t_i\}_{i \in I}]$ for all primes $\mathfrak{p}$ of R. Then R_I is a faithfully flat R-algebra and $\{\mathrm{Spec}(R_I) \to \mathrm{Spec}(R)\}$ is an fpqc covering. We leave it as an exercise to the reader to show that $R_I \otimes_R \kappa \cong \kappa(\{t_i\}_{i \in I}) = \kappa_I$ with notation as above (hint: use that $R \to \kappa$ is surjective and that any $f \in R[\{t_i\}_{i \in I}]$ one of whose monomials occurs with coefficient 1 is an element of S_I). Let A be a set and for each $\alpha \in A$ let $\mathcal{U}_\alpha = \{S_{\alpha,j} \to \mathrm{Spec}(R)\}_{j \in J_\alpha}$ be an fpqc covering. If $\mathcal{U}_\alpha$ refines $\{\mathrm{Spec}(R_I) \to \mathrm{Spec}(R)\}$, then by base change we conclude that $\{S_{\alpha,j} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa) \to \mathrm{Spec}(\kappa)\}$ refines $\{\mathrm{Spec}(\kappa_I) \to \mathrm{Spec}(\kappa)\}$. Hence by the result of the previous paragraph, there exists an I such that this is not the case and the lemma is proved. □

34.10. The V topology

0ETA The V topology is stronger than all other topologies in this chapter. Roughly speaking it is generated by Zariski coverings and by quasi-compact morphisms satisfying a lifting property for specializations (Lemma 34.10.13). However, the procedure we will use to define V coverings is a bit different. We will first define standard V coverings of affines and then use these to define V coverings in general. Typographical point: in the literature sometimes "v-covering" is used instead of "V covering".

0ETB Definition 34.10.1. Let T be an affine scheme. A standard V covering is a finite family $\{T_j \to T\}_{j=1,\ldots,m}$ with T_j affine such that for every morphism $g : \mathrm{Spec}(V) \to T$ where V is a valuation ring, there is an extension $V \subset W$ of valuation rings (More on Algebra, Definition 15.123.1), an index $1 \leq j \leq m$, and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & T_j \arrow[d] \\ \mathrm{Spec}(V) \arrow[r, "g"] & T \end{tikzcd}$$

We first prove a few basic lemmas about this notion.

0ETC Lemma 34.10.2. A standard fpqc covering is a standard V covering.

Proof. Let $\{X_i \to X\}_{i=1,\ldots,n}$ be a standard fpqc covering (Definition 34.9.9). Let $g : \mathrm{Spec}(V) \to X$ be a morphism where V is a valuation ring. Let $x \in X$ be the image of the closed point of $\mathrm{Spec}(V)$. Choose an i and a point $x_i \in X_i$ mapping to x. Then $\mathrm{Spec}(V) \times_X X_i$ has a point x_i' mapping to the closed point of $\mathrm{Spec}(V)$. Since $\mathrm{Spec}(V) \times_X X_i \to \mathrm{Spec}(V)$ is flat we can find a specialization $x_i'' \rightsquigarrow x_i'$ of points of $\mathrm{Spec}(V) \times_X X_i$ with x_i'' mapping to the generic point of $\mathrm{Spec}(V)$, see Morphisms, Lemma 29.25.9. By Schemes, Lemma 26.20.4 we can choose a valuation ring W and a morphism $h : \mathrm{Spec}(W) \to \mathrm{Spec}(V) \times_X X_i$ such that h maps the generic point of $\mathrm{Spec}(W)$ to x_i'' and the closed point of $\mathrm{Spec}(W)$ to x_i'. We obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & X_i \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & X \end{tikzcd}$$

where $V \to W$ is an extension of valuation rings. This proves the lemma. ☐

0ETD Lemma 34.10.3. A standard ph covering is a standard V covering.

Proof. Let T be an affine scheme. Let $f : U \to T$ be a proper surjective morphism. Let $U = \bigcup_{j=1,\ldots,m} U_j$ be a finite affine open covering. We have to show that $\{U_j \to T\}$ is a standard V covering, see Definition 34.8.1. Let $g : \mathrm{Spec}(V) \to T$ be a morphism where V is a valuation ring with fraction field K. Since $U \to T$ is surjective, we may choose a field extension L/K and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[rr] \arrow[d] & & U \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(V) \arrow[r, "g"] & T \end{tikzcd}$$

By Algebra, Lemma 10.50.2 we can choose a valuation ring $W \subset L$ dominating V. By the valuative criterion of properness (Morphisms, Lemma 29.42.1) we can then find the morphism h in the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r] \arrow[d] & \mathrm{Spec}(W) \arrow[r, "h"'] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(V) \arrow[r, "g"] & X \end{tikzcd}$$

Since $\operatorname{Spec}(W)$ has a unique closed point, we see that $\operatorname{Im}(h)$ is contained in U_j for some j. Thus $h : \operatorname{Spec}(W) \to U_j$ is the desired lift and we conclude $\{U_j \to T\}$ is a standard V covering. □

0ETE Lemma 34.10.4. Let $\{T_j \to T\}_{j=1,\ldots,m}$ be a standard V covering. Let $T' \to T$ be a morphism of affine schemes. Then $\{T_j \times_T T' \to T'\}_{j=1,\ldots,m}$ is a standard V covering.

Proof. Let $\operatorname{Spec}(V) \to T'$ be a morphism where V is a valuation ring. By assumption we can find an extension of valuation rings $V \subset W$, an i, and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(W) \arrow[r] \arrow[d] & T_i \arrow[d] \\ \operatorname{Spec}(V) \arrow[r] & T \end{tikzcd}$$

By the universal property of fibre products we obtain a morphism $\operatorname{Spec}(W) \to T' \times_T T_i$ as desired. □

0ETF Lemma 34.10.5. Let T be an affine scheme. Let $\{T_j \to T\}_{j=1,\ldots,m}$ be a standard V covering. Let $\{T_{ji} \to T_j\}_{i=1,\ldots n_j}$ be a standard V covering. Then $\{T_{ji} \to T\}_{i,j}$ is a standard V covering.

Proof. This follows formally from the observation that if $V \subset W$ and $W \subset \Omega$ are extensions of valuation rings, then $V \subset \Omega$ is an extension of valuation rings. □

0ETG Lemma 34.10.6. Let T be an affine scheme. Let $\{T_j \to T\}_{j=1,\ldots,m}$ be a family of morphisms with T_j affine for all j. The following are equivalent

(1) $\{T_j \to T\}_{j=1,\ldots,m}$ is a standard V covering,
(2) there is a standard V covering which refines $\{T_j \to T\}_{j=1,\ldots,m}$, and
(3) $\{\coprod_{j=1,\ldots,m} T_j \to T\}$ is a standard V covering.

Proof. Omitted. Hints: This follows almost immediately from the definition. The only slightly interesting point is that a morphism from the spectrum of a local ring into $\coprod_{j=1,\ldots,m} T_j$ must factor through some T_j. □

0ETH Definition 34.10.7. Let T be a scheme. A V covering of T is a family of morphisms $\{T_i \to T\}_{i\in I}$ of schemes such that for every affine open $U \subset T$ there exists a standard V covering $\{U_j \to U\}_{j=1,\ldots,m}$ refining the family $\{T_i \times_T U \to U\}_{i\in I}$.

The V topology has the same set theoretical problems as the fpqc topology. Thus we refrain from defining V sites and we will not consider cohomology with respect to the V topology. On the other hand, given a $F : \mathit{Sch}^{opp} \to \textit{Sets}$ it does make sense to ask whether F satisfies the sheaf property for the V topology, see below. Moreover, we can wonder about descent of object in the V topology, etc.

0ETI Lemma 34.10.8. Let T be a scheme. Let $\{f_i : T_i \to T\}_{i\in I}$ be a family of morphisms. The following are equivalent

(1) $\{T_i \to T\}_{i\in I}$ is a V covering,
(2) there is a V covering which refines $\{T_i \to T\}_{i\in I}$, and
(3) $\{\coprod_{i\in I} T_i \to T\}$ is a V covering.

Proof. Omitted. Hint: compare with the proof of Lemma 34.8.7. □

0ETJ **Lemma 34.10.9.** Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a V covering of T.
(2) If $\{T_i \to T\}_{i \in I}$ is a V covering and for each i we have a V covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a V covering.
(3) If $\{T_i \to T\}_{i \in I}$ is a V covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a V covering.

Proof. Assertion (1) is clear.

Proof of (3). Let $U' \subset T'$ be an affine open subscheme. Since U' is quasi-compact we can find a finite affine open covering $U' = U'_1 \cup \ldots \cup U'$ such that $U'_j \to T$ maps into an affine open $U_j \subset T$. Choose a standard V covering $\{U_{jl} \to U_j\}_{l=1,\ldots,n_j}$ refining $\{T_i \times_T U_j \to U_j\}$. By Lemma 34.10.4 the base change $\{U_{jl} \times_{U_j} U'_j \to U'_j\}$ is a standard V covering. Note that $\{U'_j \to U'\}$ is a standard V covering (for example by Lemma 34.10.2). By Lemma 34.10.5 the family $\{U_{jl} \times_{U_j} U'_j \to U'\}$ is a standard V covering. Since $\{U_{jl} \times_{U_j} U'_j \to U'\}$ refines $\{T_i \times_T U' \to U'\}$ we conclude.

Proof of (2). Let $U \subset T$ be affine open. First we pick a standard V covering $\{U_k \to U\}_{k=1,\ldots,m}$ refining $\{T_i \times_T U \to U\}$. Say the refinement is given by morphisms $U_k \to T_{i_k}$ over T. Then

$$\{T_{i_k j} \times_{T_{i_k}} U_k \to U_k\}_{j \in J_{i_k}}$$

is a V covering by part (3). As U_k is affine, we can find a standard V covering $\{U_{ka} \to U_k\}_{a=1,\ldots,b_k}$ refining this family. Then we apply Lemma 34.10.5 to see that $\{U_{ka} \to U\}$ is a standard V covering which refines $\{T_{ij} \times_T U \to U\}$. This finishes the proof. □

0ETK **Lemma 34.10.10.** Any fpqc covering is a V covering. A fortiori, any fppf, syntomic, smooth, étale or Zariski covering is a V covering. Also, a ph covering is a V covering.

Proof. An fpqc covering can affine locally be refined by a standard fpqc covering, see Lemmas 34.9.8. A standard fpqc covering is a standard V covering, see Lemma 34.10.2. Hence the first statement follows from our definition of V covers in terms of standard V coverings. The conclusion for fppf, syntomic, smooth, étale or Zariski coverings follows as these are fpqc coverings, see Lemma 34.9.6.

The statement on ph coverings follows from Lemma 34.10.3 in the same manner. □

0ETL **Definition 34.10.11.** Let F be a contravariant functor on the category of schemes with values in sets. We say that F satisfies the sheaf property for the V topology if it satisfies the sheaf property for any V covering (see Definition 34.9.12).

We try to avoid using the terminology "F is a sheaf" in this situation since we are not defining a category of V sheaves as we explained above.

0ETM **Lemma 34.10.12.** Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the V topology if and only if it satisfies

(1) the sheaf property for every Zariski covering, and
(2) the sheaf property for any standard V covering.

Moreover, in the presence of (1) property (2) is equivalent to property

(2') the sheaf property for a standard V covering of the form $\{V \to U\}$, i.e., consisting of a single arrow.

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i \in I}$ be a V covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i. Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i.

We can find a standard V covering $\{U_j \to U\}_{j=1,\ldots,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \to U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \text{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1, \ldots, n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard V covering (Lemma 34.10.4) and in general this follows from (1) and the affine case by choosing an affine open covering of V. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j} \times_T T_i} = s_i|_{T_{i_j} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}_{i=1,\ldots,n}$ is a standard V covering, then $\coprod_{i=1,\ldots,n} T_i \to T$ is a morphism of affine schemes which is clearly also a standard V covering. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_{i'})$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\coprod T_i \to T\}$ is the same. □

The following lemma shows that being a V covering is related to the possibility of lifting specializations.

0ETN Lemma 34.10.13. Let $X \to Y$ be a quasi-compact morphism of schemes. The following are equivalent

(1) $\{X \to Y\}$ is a V covering,
(2) for any valuation ring V and morphism $g : \text{Spec}(V) \to Y$ there exists an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{tikzcd} \text{Spec}(W) \arrow[r] \arrow[d] & X \arrow[d] \\ \text{Spec}(V) \arrow[r] & Y \end{tikzcd}$$

(3) for any morphism $Z \to Y$ and specialization $z' \rightsquigarrow z$ of points in Z, there is a specialization $w' \rightsquigarrow w$ of points in $Z \times_Y X$ mapping to $z' \rightsquigarrow z$.

Proof. Assume (1) and let $g : \text{Spec}(V) \to Y$ be as in (2). Since V is a local ring there is an affine open $U \subset Y$ such that g factors through U. By Definition 34.10.7 we can find a standard V covering $\{U_j \to U\}$ refining $\{X \times_Y U \to U\}$. By Definition 34.10.1 we can find a j, an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{tikzcd} \text{Spec}(W) \arrow[r] \arrow[d] & U_j \arrow[r, dotted] \arrow[d] & X \arrow[dl] \\ \text{Spec}(V) \arrow[r] & Y & \end{tikzcd}$$

We have the dotted arrow making the diagram commute by the refinement property of the covering and we see that (2) holds.

Assume (2) and let $Z \to Y$ and $z' \rightsquigarrow z$ be as in (3). By Schemes, Lemma 26.20.4 we can find a valuation ring V and a morphism $\mathrm{Spec}(V) \to Z$ such that the closed point of $\mathrm{Spec}(V)$ maps to z and the generic point of $\mathrm{Spec}(V)$ maps to z'. By (2) we can find an extension of valuation rings $V \subset W$ and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[rr] \arrow[d] & & X \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & Z \arrow[r] & Y \end{tikzcd}$$

The generic and closed points of $\mathrm{Spec}(W)$ map to points $w' \rightsquigarrow w$ in $Z \times_Y X$ via the induced morphism $\mathrm{Spec}(W) \to Z \times_Y X$. This shows that (3) holds.

Assume (3) holds and let $U \subset Y$ be an affine open. Choose a finite affine open covering $U \times_Y X = \bigcup_{j=1,\ldots,m} U_j$. This is possible as $X \to Y$ is quasi-compact. We claim that $\{U_j \to U\}$ is a standard V covering. The claim implies (1) is true and finishes the proof of the lemma. In order to prove the claim, let V be a valuation ring and let $g : \mathrm{Spec}(V) \to U$ be a morphism. By (3) we find a specialization $w' \rightsquigarrow w$ of points of

$$T = \mathrm{Spec}(V) \times_X Y = \mathrm{Spec}(V) \times_U (U \times_X Y)$$

such that w' maps to the generic point of $\mathrm{Spec}(V)$ and w maps to the closed point of $\mathrm{Spec}(V)$. By Schemes, Lemma 26.20.4 we can find a valuation ring W and a morphism $\mathrm{Spec}(W) \to T$ such that the generic point of $\mathrm{Spec}(W)$ maps to w' and the closed point of $\mathrm{Spec}(W)$ maps to w. The composition $\mathrm{Spec}(W) \to T \to \mathrm{Spec}(V)$ corresponds to an inclusion $V \subset W$ which presents W as an extension of the valuation ring V. Since $T = \bigcup \mathrm{Spec}(V) \times_U U_j$ is an open covering, we see that $\mathrm{Spec}(W) \to T$ factors through $\mathrm{Spec}(V) \times_U U_j$ for some j. Thus we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & U_j \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & U \end{tikzcd}$$

and the proof of the claim is complete. □

A V covering gives a universally submersive family of maps. The converse of this lemma is false, see Examples, Section 110.78.

0ETP Lemma 34.10.14. Let $\{f_i : X_i \to X\}_{i \in I}$ be a V covering. Then

$$\coprod\nolimits_{i \in I} f_i : \coprod\nolimits_{i \in I} X_i \longrightarrow X$$

is a universally submersive morphism of schemes (Morphisms, Definition 29.24.1).

Proof. We will use without further mention that the base change of a V covering is a V covering (Lemma 34.10.9). In particular it suffices to show that the morphism is submersive. Being submersive is clearly Zariski local on the base. Thus we may assume X is affine. Then $\{X_i \to X\}$ can be refined by a standard V covering $\{Y_j \to X\}$. If we can show that $\coprod Y_j \to X$ is submersive, then since there is a factorization $\coprod Y_j \to \coprod X_i \to X$ we conclude that $\coprod X_i \to X$ is submersive. Set $Y = \coprod Y_j$ and consider the morphism of affines $f : Y \to X$. By Lemma 34.10.13 we know that we can lift any specialization $x' \rightsquigarrow x$ in X to some specialization $y' \rightsquigarrow y$ in Y. Thus if $T \subset X$ is a subset such that $f^{-1}(T)$ is closed in Y, then $T \subset X$ is closed under specialization. Since $f^{-1}(T) \subset Y$ with the reduced induced closed subscheme structure is an affine scheme, we conclude that $T \subset X$ is closed by Algebra, Lemma 10.41.5. Hence f is submersive. $\square$

34.11. Change of topologies

03FE Let $f : X \to Y$ be a morphism of schemes over a base scheme S. In this case we have the following morphisms of sites[7] (with suitable choices of sites as in Remark 34.11.1 below):

(1) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$,
(2) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{syntomic}$,
(3) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{smooth}$,
(4) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{étale}$,
(5) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{Zar}$,
(6) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{syntomic}$,
(7) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{smooth}$,
(8) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{étale}$,
(9) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{Zar}$,
(10) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{smooth}$,
(11) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{étale}$,
(12) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{Zar}$,
(13) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{étale}$,
(14) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{Zar}$,
(15) $(Sch/X)_{Zar} \longrightarrow (Sch/Y)_{Zar}$,
(16) $(Sch/X)_{fppf} \longrightarrow Y_{étale}$,
(17) $(Sch/X)_{syntomic} \longrightarrow Y_{étale}$,
(18) $(Sch/X)_{smooth} \longrightarrow Y_{étale}$,
(19) $(Sch/X)_{étale} \longrightarrow Y_{étale}$,
(20) $(Sch/X)_{fppf} \longrightarrow Y_{Zar}$,
(21) $(Sch/X)_{syntomic} \longrightarrow Y_{Zar}$,
(22) $(Sch/X)_{smooth} \longrightarrow Y_{Zar}$,
(23) $(Sch/X)_{étale} \longrightarrow Y_{Zar}$,
(24) $(Sch/X)_{Zar} \longrightarrow Y_{Zar}$,
(25) $X_{étale} \longrightarrow Y_{étale}$,
(26) $X_{étale} \longrightarrow Y_{Zar}$,
(27) $X_{Zar} \longrightarrow Y_{Zar}$,

In each case the underlying continuous functor $Sch/Y \to Sch/X$, or $Y_\tau \to Sch/X$ is the functor $Y'/Y \mapsto X \times_Y Y'/X$. Namely, in the sections above we have seen the morphisms $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ and $f_{small} : X_\tau \to Y_\tau$ for τ as above. We also have seen the morphisms of sites $\pi_Y : (Sch/Y)_\tau \to Y_\tau$ for $\tau \in \{étale, Zariski\}$. On the other hand, it is clear that the identity functor $(Sch/X)_\tau \to (Sch/X)_{\tau'}$

[7]We have not included the comparison between the ph topology and the others; for this see More on Morphisms, Remark 37.48.8.

defines a morphism of sites when τ is a stronger topology than τ'. Hence composing these gives the list of possible morphisms above.

Because of the simple description of the underlying functor it is clear that given morphisms of schemes $X \to Y \to Z$ the composition of two of the morphisms of sites above, e.g.,

$$(Sch/X)_{\tau_0} \longrightarrow (Sch/Y)_{\tau_1} \longrightarrow (Sch/Z)_{\tau_2}$$

is the corresponding morphism of sites associated to the morphism of schemes $X \to Z$.

03FF Remark 34.11.1. Take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set of schemes $\{X, Y, S\}$. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained. Next, for $\tau \in \{Zariski, étale, smooth, syntomic\}$ let Sch_τ have the same underlying category as Sch_{fppf} with coverings $\text{Cov}_\tau \subset \text{Cov}_{fppf}$ simply the subset of τ-coverings. It is straightforward to check that this gives rise to a big site Sch_τ.

34.12. Change of big sites

022I In this section we explain what happens on changing the big Zariski/fppf/étale sites.

Let $\tau, \tau' \in \{Zariski, étale, smooth, syntomic, fppf\}$. Given two big sites Sch_τ and $Sch'_{\tau'}$ we say that Sch_τ is contained in $Sch'_{\tau'}$ if $\text{Ob}(Sch_\tau) \subset \text{Ob}(Sch'_{\tau'})$ and $\text{Cov}(Sch_\tau) \subset \text{Cov}(Sch'_{\tau'})$. In this case τ is stronger than τ', for example, no fppf site can be contained in an étale site.

022J Lemma 34.12.1. Any set of big Zariski sites is contained in a common big Zariski site. The same is true, mutatis mutandis, for big fppf and big étale sites.

Proof. This is true because the union of a set of sets is a set, and the constructions in Sets, Lemmas 3.9.2 and 3.11.1 allow one to start with any initially given set of schemes and coverings. $\square$

022K Lemma 34.12.2. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Suppose given big sites Sch_τ and Sch'_τ. Assume that Sch_τ is contained in Sch'_τ. The inclusion functor $Sch_\tau \to Sch'_\tau$ satisfies the assumptions of Sites, Lemma 7.21.8. There are morphisms of topoi

$$\begin{aligned} g : Sh(Sch_\tau) &\longrightarrow Sh(Sch'_\tau) \\ f : Sh(Sch'_\tau) &\longrightarrow Sh(Sch_\tau) \end{aligned}$$

such that $f \circ g \cong \text{id}$. For any object S of Sch_τ the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 7.21.8 also. Hence similarly we obtain morphisms

$$\begin{aligned} g : Sh((Sch/S)_\tau) &\longrightarrow Sh((Sch'/S)_\tau) \\ f : Sh((Sch'/S)_\tau) &\longrightarrow Sh((Sch/S)_\tau) \end{aligned}$$

with $f \circ g \cong \text{id}$.

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 7.21.8 are immediate for the functors $Sch_\tau \to Sch'_\tau$ and $(Sch/S)_\tau \to (Sch'/S)_\tau$. Property (a) holds by Lemma 34.3.6, 34.4.7, 34.5.7, 34.6.7, or 34.7.7. Property (d) holds because fibre products in the categories Sch_τ, Sch'_τ exist and are compatible with fibre products in the category of schemes. $\square$

Discussion: The functor $g^{-1} = f_*$ is simply the restriction functor which associates to a sheaf $\mathcal{G}$ on Sch'_τ the restriction $\mathcal{G}|_{Sch_\tau}$. Hence this lemma simply says that given any sheaf of sets $\mathcal{F}$ on Sch_τ there exists a canonical sheaf $\mathcal{F}'$ on Sch'_τ such that $\mathcal{F}|_{Sch'_\tau} = \mathcal{F}'$. In fact the sheaf $\mathcal{F}'$ has the following description: it is the sheafification of the presheaf

$$Sch'_\tau \longrightarrow \text{Sets}, \quad V \longmapsto \text{colim}_{V \to U} \mathcal{F}(U)$$

where U is an object of Sch_τ. This is true because $\mathcal{F}' = f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ according to Sites, Lemmas 7.21.5 and 7.21.8.

34.13. Extending functors

0EUV Let us start with a simple example which explains what we are doing. Let R be a ring. Suppose F is a functor defined on the category $\mathcal{C}$ of R-algebras of the form

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

for $n, m \geq 0$ integers and $f_1, \ldots, f_m \in R[x_1, \ldots, x_m]$ elements. Then for any R-algebra B we can define

$$F'(B) = \text{colim}_{A \to B,\ A \in \mathcal{C}} F(A)$$

It turns out F' is the unique functor on the category of all R-algebras which extends F and commutes with filtered colimits. The same procedure works in the category of schemes if we impose that our functor is a Zariski sheaf.

0EUW Lemma 34.13.1. Let S be a scheme. Let $\mathcal{C}$ be a full subcategory of the category Sch/S of all schemes over S. Assume

(1) if $X \to S$ is an object of $\mathcal{C}$ and $U \subset X$ is an affine open, then $U \to S$ is isomorphic to an object of $\mathcal{C}$,
(2) if V is an affine scheme lying over an affine open $U \subset S$ such that $V \to U$ is of finite presentation, then $V \to S$ is isomorphic to an object of $\mathcal{C}$.

Let $F : \mathcal{C}^{opp} \to \text{Sets}$ be a functor. Assume

(a) for any Zariski covering $\{f_i : X_i \to X\}_{i \in I}$ with X, X_i objects of $\mathcal{C}$ we have the sheaf condition for F and this family[8],
(b) if $X = \lim X_i$ is a directed limit of affine schemes over S with X, X_i objects of $\mathcal{C}$, then $F(X) = \text{colim}\, F(X_i)$.

Then there is a unique way to extend F to a functor $F' : (Sch/S)^{opp} \to \text{Sets}$ satisfying the analogues of (a) and (b), i.e., F' satisfies the sheaf condition for any Zariski covering and $F'(X) = \text{colim}\, F'(X_i)$ whenever $X = \lim X_i$ is a directed limit of affine schemes over S.

[8]As we do not know that $X_i \times_X X_j$ is in $\mathcal{C}$ this has to be interpreted as follows: by property (1) there exist Zariski coverings $\{U_{ijk} \to X_i \times_X X_j\}_{k \in K_{ij}}$ with U_{ijk} an object of $\mathcal{C}$. Then the sheaf condition says that $F(X)$ is the equalizer of the two maps from $\prod F(X_i)$ to $\prod F(U_{ijk})$.

Proof. The idea will be to first extend F to a sufficiently large collection of affine schemes over S and then use the Zariski sheaf property to extend to all schemes.

Suppose that V is an affine scheme over S whose structure morphism $V \to S$ factors through some affine open $U \subset S$. In this case we can write

$$V = \lim V_i$$

as a cofiltered limit with $V_i \to U$ of finite presentation and V_i affine. See Algebra, Lemma 10.127.2. By conditions (1) and (2) we may replace our V_i by objects of $\mathcal{C}$. Observe that $V_i \to S$ is locally of finite presentation (if S is quasi-separated, then these morphisms are actually of finite presentation). Then we set

$$F'(V) = \operatorname{colim} F(V_i)$$

Actually, we can give a more canonical expression, namely

$$F'(V) = \operatorname{colim}_{V \to V'} F(V')$$

where the colimit is over the category of morphisms $V \to V'$ over S where V' is an object of $\mathcal{C}$ whose structure morphism $V' \to S$ is locally of finite presentation. The reason this is the same as the first formula is that by Limits, Proposition 32.6.1 our inverse system V_i is cofinal in this category! Finally, note that if V were an object of $\mathcal{C}$, then $F'(V) = F(V)$ by assumption (b).

The second formula turns F' into a contravariant functor on the category formed by affine schemes V over S whose structure morphism factors through an affine open of S. Let V be such an affine scheme over S and suppose that $V = \bigcup_{k=1,\ldots,n} V_k$ is a finite open covering by affines. Then it makes sense to ask if the sheaf condition holds for F' and this open covering. This is true and easy to show: write $V = \lim V_i$ as in the previous paragraph. By Limits, Lemma 32.4.11 for all sufficiently large i we can find affine opens $V_{i,k} \subset V_i$ compatible with transition maps pulling back to V_k in V. Thus

$$F'(V_k) = \operatorname{colim} F(V_{i,k}) \quad \text{and} \quad F'(V_k \cap V_l) = \operatorname{colim} F(V_{i,k} \cap V_{i,l})$$

Strictly speaking in these formulas we need to replace $V_{i,k}$ and $V_{i,k} \cap V_{i,l}$ by isomorphic affine objects of $\mathcal{C}$ before applying the functor F. Since I is directed the colimits pass through equalizers. Hence the sheaf condition (b) for F and the Zariski coverings $\{V_{i,k} \to V_i\}$ implies the sheaf condition for F' and this covering.

Let X be a general scheme over S. Let $\mathcal{B}_X$ denote the collection of affine opens of X whose structure morphism to S maps into an affine open of S. It is clear that $\mathcal{B}_X$ is a basis for the topology of X. By the result of the previous paragraph and Sheaves, Lemma 6.30.4 we see that F' is a sheaf on $\mathcal{B}_X$. Hence F' restricted to $\mathcal{B}_X$ extends uniquely to a sheaf F'_X on X, see Sheaves, Lemma 6.30.6. If X is an object of $\mathcal{C}$ then we have a canonical identification $F'_X(X) = F(X)$ by the agreement of F' and F on the objects for which they are both defined and the fact that F satisfies the sheaf condition for Zariski coverings.

Let $f : X \to Y$ be a morphism of schemes over S. We get a unique f-map from F'_Y to F'_X compatible with the maps $F'(V) \to F'(U)$ for all $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ with $f(U) \subset V$, see Sheaves, Lemma 6.30.16. We omit the verification that these maps compose correctly given morphisms $X \to Y \to Z$ of schemes over S. We also omit the verification that if f is a morphism of $\mathcal{C}$, then the induced map $F'_Y(Y) \to F'_X(X)$ is the same as the map $F(Y) \to F(X)$ via the identifications $F'_X(X) = F(X)$ and

$F'_Y(Y) = F(Y)$ above. In this way we see that the desired extension of F is the functor which sends X/S to $F'_X(X)$.

Property (a) for the functor $X \mapsto F'_X(X)$ is almost immediate from the construction; we omit the details. Suppose that $X = \lim_{i \in I} X_i$ is a directed limit of affine schemes over S. We have to show that

$$F'_X(X) = \mathrm{colim}_{i \in I} F'_{X_i}(X_i)$$

First assume that there is some $i \in I$ such that $X_i \to S$ factors through an affine open $U \subset S$. Then F' is defined on X and on $X_{i'}$ for $i' \geq i$ and we see that $F'_{X_{i'}}(X_{i'}) = F'(X_{i'})$ for $i' \geq i$ and $F'_X(X) = F'(X)$. In this case every arrow $X \to V$ with V locally of finite presentation over S factors as $X \to X_{i'} \to V$ for some $i' \geq i$, see Limits, Proposition 32.6.1. Thus we have

$$\begin{aligned} F'_X(X) &= F'(X) \\ &= \mathrm{colim}_{X \to V} F(V) \\ &= \mathrm{colim}_{i' \geq i} \mathrm{colim}_{X_{i'} \to V} F(V) \\ &= \mathrm{colim}_{i' \geq i} F'(X_{i'}) \\ &= \mathrm{colim}_{i' \geq i} F'_{X_{i'}}(X_{i'}) \\ &= \mathrm{colim}_{i' \in I} F'_{X_{i'}}(X_{i'}) \end{aligned}$$

as desired. Finally, in general we pick any $i \in I$ and we choose a finite affine open covering $V_i = V_{i,1} \cup \ldots \cup V_{i,n}$ such that $V_{i,k} \to S$ factors through an affine open of S. Let $V_k \subset V$ and $V_{i',k}$ for $i' \geq i$ be the inverse images of $V_{i,k}$. By the previous case we see that

$$F'_{V_k}(V_k) = \mathrm{colim}_{i' \geq i} F'_{V_{i',k}}(V_{i',k})$$

and

$$F'_{V_k \cap V_l}(V_k \cap V_l) = \mathrm{colim}_{i' \geq i} F'_{V_{i',k} \cap V_{i',l}}(V_{i',k} \cap V_{i',l})$$

By the sheaf property and exactness of filtered colimits we find that $F'_X(X) = \mathrm{colim}_{i \in I} F'_{X_i}(X_i)$ also in this case. This finishes the proof of property (b) and hence finishes the proof of the lemma. □

049N Lemma 34.13.2. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let T be an affine scheme which is written as a limit $T = \lim_{i \in I} T_i$ of a directed inverse system of affine schemes.

(1) Let $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ be a standard τ-covering of T, see Definitions 34.3.4, 34.4.5, 34.5.5, 34.6.5, and 34.7.5. Then there exists an index i and a standard τ-covering $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ whose base change $T \times_{T_i} \mathcal{V}_i$ to T is isomorphic to $\mathcal{V}$.
(2) Let $\mathcal{V}_i, \mathcal{V}'_i$ be a pair of standard τ-coverings of T_i. If $f : T \times_{T_i} \mathcal{V}_i \to T \times_{T_i} \mathcal{V}'_i$ is a morphism of coverings of T, then there exists an index $i' \geq i$ and a morphism $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ whose base change to T is f.
(3) If $f, g : \mathcal{V} \to \mathcal{V}'_i$ are morphisms of standard τ-coverings of T_i whose base changes f_T, g_T to T are equal then there exists an index $i' \geq i$ such that $f_{T_{i'}} = g_{T_{i'}}$.

In other words, the category of standard τ-coverings of T is the colimit over I of the categories of standard τ-coverings of T_i.

Proof. Let us prove this for $\tau = fppf$. By Limits, Lemma 32.10.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemmas 32.8.2 and 32.8.7 the same is true for category of schemes which are affine, flat and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of flat finitely presented morphisms with $V_{j,i}$ affine, and the base change $\coprod_j T \times_{T_i} V_{j,i} \to T$ is surjective, then for some $i' \geq i$ the morphism $\coprod T_{i'} \times_{T_i} V_{j,i} \to T_{i'}$ is surjective. Denote $W_{i'} \subset T_{i'}$, resp. $W \subset T$ the image. Of course $W = T$ by assumption. Since the morphisms are flat and of finite presentation we see that W_i is a quasi-compact open of T_i, see Morphisms, Lemma 29.25.10. Moreover, $W = T \times_{T_i} W_i$ (formation of image commutes with base change). Hence by Limits, Lemma 32.4.11 we conclude that $W_{i'} = T_{i'}$ for some large enough i' and we win.

For $\tau \in \{Zariski, étale, smooth, syntomic\}$ a standard τ-covering is a standard fppf covering. Hence the fully faithfulness of the functor holds. The only issue is to show that given a standard fppf covering $\mathcal{V}_i$ for some i such that $\mathcal{V}_i \times_{T_i} T$ is a standard τ-covering, then $\mathcal{V}_i \times_{T_i} T_{i'}$ is a standard τ-covering for all $i' \gg i$. This follows immediately from Limits, Lemmas 32.8.12, 32.8.10, 32.8.9, and 32.8.16. □

0GDW Lemma 34.13.3. Let S, $\mathcal{C}$, F satisfy conditions (1), (2), (a), and (b) of Lemma 34.13.1 and denote $F' : (Sch/S)^{opp} \to Sets$ the unique extension constructed in the lemma. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Assume

(c) for any standard τ-covering $\{V_i \to V\}_{i=1,\ldots,n}$ of affines in Sch/S such that $V \to S$ factors through an affine open $U \subset S$ and $V \to U$ is of finite presentation, the sheaf condition hold for F and $\{V_i \to V\}_{i=1,\ldots,n}$[9].

Then F' satisfies the sheaf condition for all τ-coverings.

Proof. Let X be a scheme over S and let $\{X_i \to X\}_{i\in I}$ be a τ-covering. Let $s_i \in F'(X_i)$ be elements such that s_i and s_j map to the same element of $F'(X_i \times_X X_j)$ for all $i, j \in I$. We have to show that there is a unique element $s \in F'(X)$ restricting to $s_i \in F'(X_i)$ for all $i \in I$.

Special case: X is an affine such that the structure morphism maps into an affine open U of S and the covering $\{X_i \to X\}_{i\in I}$ is a standard τ-covering. In this case we can write

$$X = \lim V_k$$

as a cofiltered limit with $V_k \to U$ of finite presentation and V_k affine. See Algebra, Lemma 10.127.2. By Lemma 34.13.2 there exists a k and a standard τ-covering $\{V_{k,i} \to V_k\}_{i\in I}$ whose base change to X is the given covering. For $k' \geq k$ denote $\{V_{k',i} \to V_{k'}\}_{i\in I}$ the base change to $V_{k'}$ of our covering. Then we see that

$$\begin{aligned} F'(X) &= \mathrm{colim}_{k'\geq k} F(V_k) \\ &= \mathrm{colim}_{k'\geq k} \text{Equalizer}(\textstyle\prod F(V_{k',i}) \rightrightarrows \prod F(V_{k',i} \times_{V_{k'}} V_{k',j}) \\ &= \text{Equalizer}(\mathrm{colim}_{k'\geq k} \textstyle\prod F(V_{k',i}) \rightrightarrows \mathrm{colim}_{k'\geq k} \prod F(V_{k',i} \times_{V_{k'}} V_{k',j}) \\ &= \text{Equalizer}(\textstyle\prod F'(X_i) \rightrightarrows \prod F'(X_i \times_X X_j) \end{aligned}$$

The first equality holds by construction of F'. The second holds by assumption (c). The third holds because filtered colimits are exact. The fourth again holds by

[9]This makes sense as V, V_i, and $V_i \times_V V_j$ are isomorphic to objects of $\mathcal{C}$ by (2).

construction of F'. In this way we find that the sheaf property holds for F' with respect to $\{X_i \to X\}_{i\in I}$.

General case. Choose an affine open covering $X = \bigcup U_k$ such that each U_k maps into an affine open of S. For every k we may choose a standard τ-covering $\{V_{k,j} \to U_k\}_{j=1,\ldots,m_k}$ which refines $\{X_i \times_X U_k \to U_k\}_{i\in I}$. For each $j \in \{1, \ldots, m_k\}$ choose an index $i_{k,j} \in I$ and a morphism $g_{k,j} : V_{k,j} \to X_{i_{k,j}}$ over X. Let $s_{k,j}$ be the element of $F'(V_{k,j})$ we get by restricting $s_{i_{k,j}}$ via $g_{k,j}$. Observe that $s_{k,j}$ and $s_{k',j'}$ restrict to the same element of $F'(V_{k,j} \times_X V_{k',j'})$ for all k and k' and all $j \in \{1, \ldots, m_k\}$ and $j' \in \{1, \ldots, m_{k'}\}$; verification omitted. In particular, by the result of the previous paragraph there is a unique element $s_k \in F'(U_k)$ restricting to $s_{k,j}$ for all j. With this notation we are ready to finish the proof.

Proof of uniqueness of s: this is true because F' satisfies the sheaf property for Zariski coverings and $s|_{U_k}$ must be equal to s_k because both restrict to $s_{k,j}$ for all j. This uniqueness then shows that s_k and $s_{k'}$ must restrict to the same section of F' over (the non-affine scheme) $U_k \cap U_{k'}$ because these sections restrict to the same section over the τ-covering $\{V_{k,j} \times_X V_{k',j'} \to U_k \cap U_{k'}\}$. Thus by the sheaf property for Zariski coverings, there is a unique section s of F' over X whose restriction to U_k is s_k. We omit the verification (similar to the above) that s restricts to s_i over X_i. □

0EUX Lemma 34.13.4. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Let S be a scheme contained in a big site Sch_τ. Let $F : (Sch/S)_\tau^{opp} \to$ Sets be a τ-sheaf satisfying property (b) of Lemma 34.13.1 with $\mathcal{C} = (Sch/S)_\tau$. Then the extension F' of F to the category of all schemes over S satisfies the sheaf condition for all τ-coverings.

Proof. This follows from Lemma 34.13.3 applied with $\mathcal{C} = (Sch/S)_\tau$. Conditions (1), (2), (a), and (b) of Lemma 34.13.1 hold; we omit the details. Thus we get our unique extension F' to the category of all schemes over S. Finally, observe that any standard τ-covering is tautologically equivalent to a covering in $(Sch/S)_\tau$, see Sets, Lemma 3.9.9 as well as Lemmas 34.3.6, 34.4.7, 34.5.7, 34.6.7, and 34.7.7. By Sites, Lemma 7.8.4 the sheaf property passes through tautological equivalence of coverings. Hence the fact that F is a τ-sheaf implies that property (c) of Lemma 34.13.3 holds and we conclude. □

34.14. Other chapters

Preliminaries

CHAPTER 35

Descent

0238 ## 35.1. Introduction

0239 In the chapter on topologies on schemes (see Topologies, Section 34.1) we introduced Zariski, étale, fppf, smooth, syntomic and fpqc coverings of schemes. In this chapter we discuss what kind of structures over schemes can be descended through such coverings. See for example [Gro95a], [Gro95b], [Gro95e], [Gro95f], [Gro95c], and [Gro95d]. This is also meant to introduce the notions of descent, descent data, effective descent data, in the less formal setting of descent questions for quasi-coherent sheaves, schemes, etc. The formal notion, that of a stack over a site, is discussed in the chapter on stacks (see Stacks, Section 8.1).

35.2. Descent data for quasi-coherent sheaves

023A In this chapter we will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

023B Definition 35.2.1. Let S be a scheme. Let $\{f_i : S_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{S_i \times_S S_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram
$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"'] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & \mathrm{pr}_2^*\mathcal{F}_k \\ & \mathrm{pr}_1^*\mathcal{F}_j \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$
of $\mathcal{O}_{S_i \times_S S_j \times_S S_k}$-modules commutes. This is called the cocycle condition.

(2) A morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{S_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the diagrams
$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[r, "\varphi_{ij}"'] \arrow[d, "\mathrm{pr}_0^*\psi_i"'] & \mathrm{pr}_1^*\mathcal{F}_j \arrow[d, "\mathrm{pr}_1^*\psi_j"] \\ \mathrm{pr}_0^*\mathcal{F}'_i \arrow[r, "\varphi'_{ij}"] & \mathrm{pr}_1^*\mathcal{F}'_j \end{tikzcd}$$
commute.

A good example to keep in mind is the following. Suppose that $S = \bigcup S_i$ is an open covering. In that case we have seen descent data for sheaves of sets in Sheaves, Section 6.33 where we called them "glueing data for sheaves of sets with respect to the given covering". Moreover, we proved that the category of glueing data is equivalent to the category of sheaves on S. We will show the analogue in the setting above when $\{S_i \to S\}_{i \in I}$ is an fpqc covering.

In the extreme case where the covering $\{S \to S\}$ is given by id_S a descent datum is necessarily of the form $(\mathcal{F}, \mathrm{id}_\mathcal{F})$. The cocycle condition guarantees that the identity on $\mathcal{F}$ is the only permitted map in this case. The following lemma shows in particular that to every quasi-coherent sheaf of $\mathcal{O}_S$-modules there is associated a unique descent datum with respect to any given family.

023C Lemma 35.2.2. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be families of morphisms of schemes with fixed target. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j \in J}$. Then

(1) The system
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i \in I}$.
(2) This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.
(3) Given a second morphism $(g', \alpha' : I \to J, (g'_i))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g'_i)^*\mathcal{F}_{\alpha'(i)}, (g'_i \times g'_{i'})^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g'_i)^*\mathcal{F}_{\alpha'(i)}$ which give the isomorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g'_i) : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. □

Any family $\mathcal{U} = \{S_i \to S\}_{i \in I}$ is a refinement of the trivial covering $\{S \to S\}$ in a unique way. For a quasi-coherent sheaf $\mathcal{F}$ on S we denote simply $(\mathcal{F}|_{S_i}, can)$ the descent datum with respect to $\mathcal{U}$ obtained by the procedure above.

023D Definition 35.2.3. Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{S \to S\}$ the trivial descent datum.
(2) The pullback of the trivial descent datum to $\{S_i \to S\}$ is called the canonical descent datum. Notation: $(\mathcal{F}|_{S_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given covering is said to be effective if there exists a quasi-coherent sheaf $\mathcal{F}$ on S such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{S_i}, can)$.

023E Lemma 35.2.4. Let S be a scheme. Let $S = \bigcup U_i$ be an open covering. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.

Proof. This follows immediately from Sheaves, Section 6.33 and the fact that being quasi-coherent is a local property, see Modules, Definition 17.10.1. □

To prove more we first need to study the case of modules over rings.

35.3. Descent for modules

023F Let $R \to A$ be a ring map. By Simplicial, Example 14.5.5 this gives rise to a cosimplicial R-algebra

$$A \rightleftarrows A \otimes_R A \rightleftarrows A \otimes_R A \otimes_R A$$

Let us denote this $(A/R)_\bullet$ so that $(A/R)_n$ is the $(n+1)$-fold tensor product of A over R. Given a map $\varphi : [n] \to [m]$ the R-algebra map $(A/R)_\bullet(\varphi)$ is the map

$$a_0 \otimes \ldots \otimes a_n \longmapsto \prod\nolimits_{\varphi(i)=0} a_i \otimes \prod\nolimits_{\varphi(i)=1} a_i \otimes \ldots \otimes \prod\nolimits_{\varphi(i)=m} a_i$$

where we use the convention that the empty product is 1. Thus the first few maps, notation as in Simplicial, Section 14.5, are

$$\begin{matrix}
\delta^1_0 & : & a_0 & \mapsto & 1 \otimes a_0 \\
\delta^1_1 & : & a_0 & \mapsto & a_0 \otimes 1 \\
\sigma^0_0 & : & a_0 \otimes a_1 & \mapsto & a_0 a_1 \\
\delta^2_0 & : & a_0 \otimes a_1 & \mapsto & 1 \otimes a_0 \otimes a_1 \\
\delta^2_1 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes 1 \otimes a_1 \\
\delta^2_2 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes a_1 \otimes 1 \\
\sigma^1_0 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 a_1 \otimes a_2 \\
\sigma^1_1 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 \otimes a_1 a_2
\end{matrix}$$

and so on.

An R-module M gives rise to a cosimplicial $(A/R)_\bullet$-module $(A/R)_\bullet \otimes_R M$. In other words $M_n = (A/R)_n \otimes_R M$ and using the R-algebra maps $(A/R)_n \to (A/R)_m$ to define the corresponding maps on $M \otimes_R (A/R)_\bullet$.

The analogue to a descent datum for quasi-coherent sheaves in the setting of modules is the following.

023G Definition 35.3.1. Let $R \to A$ be a ring map.

(1) A descent datum (N, φ) for modules with respect to $R \to A$ is given by an A-module N and an isomorphism of $A \otimes_R A$-modules

$$\varphi : N \otimes_R A \to A \otimes_R N$$

such that the cocycle condition holds: the diagram of $A \otimes_R A \otimes_R A$-module maps

$$\begin{tikzcd}
N \otimes_R A \otimes_R A \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}"'] & & A \otimes_R A \otimes_R N \\
& A \otimes_R N \otimes_R A \arrow[ur, "\varphi_{12}"'] &
\end{tikzcd}$$

commutes (see below for notation).

(2) A morphism $(N, \varphi) \to (N', \varphi')$ of descent data is a morphism of A-modules $\psi : N \to N'$ such that the diagram

$$\begin{tikzcd}
N \otimes_R A \arrow[r, "\varphi"] \arrow[d, "\psi \otimes \mathrm{id}_A"'] & A \otimes_R N \arrow[d, "\mathrm{id}_A \otimes \psi"] \\
N' \otimes_R A \arrow[r, "\varphi'"] & A \otimes_R N'
\end{tikzcd}$$

is commutative.

In the definition we use the notation that $\varphi_{01} = \varphi \otimes \text{id}_A$, $\varphi_{12} = \text{id}_A \otimes \varphi$, and $\varphi_{02}(n \otimes 1 \otimes 1) = \sum a_i \otimes 1 \otimes n_i$ if $\varphi(n \otimes 1) = \sum a_i \otimes n_i$. All three are $A \otimes_R A \otimes_R A$-module homomorphisms. Equivalently we have

$$\varphi_{ij} = \varphi \otimes_{(A/R)_1,\ (A/R)_\bullet(\tau^2_{ij})} (A/R)_2$$

where $\tau^2_{ij} : [1] \to [2]$ is the map $0 \mapsto i$, $1 \mapsto j$. Namely, $(A/R)_\bullet(\tau^2_{02})(a_0 \otimes a_1) = a_0 \otimes 1 \otimes a_1$, and similarly for the others[1].

We need some more notation to be able to state the next lemma. Let (N, φ) be a descent datum with respect to a ring map $R \to A$. For $n \geq 0$ and $i \in [n]$ we set

$$N_{n,i} = A \otimes_R \ldots \otimes_R A \otimes_R N \otimes_R A \otimes_R \ldots \otimes_R A$$

with the factor N in the ith spot. It is an $(A/R)_n$-module. If we introduce the maps $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ then we see that

$$N_{n,i} = N \otimes_{(A/R)_0,\ (A/R)_\bullet(\tau^n_i)} (A/R)_n$$

For $0 \leq i \leq j \leq n$ we let $\tau^n_{ij} : [1] \to [n]$ be the map such that 0 maps to i and 1 to j. Similarly to the above the homomorphism φ induces isomorphisms

$$\varphi^n_{ij} = \varphi \otimes_{(A/R)_1,\ (A/R)_\bullet(\tau^n_{ij})} (A/R)_n : N_{n,i} \longrightarrow N_{n,j}$$

of $(A/R)_n$-modules when $i < j$. If $i = j$ we set $\varphi^n_{ij} = \text{id}$. Since these are all isomorphisms they allow us to move the factor N to any spot we like. And the cocycle condition exactly means that it does not matter how we do this (e.g., as a composition of two of these or at once). Finally, for any $\beta : [n] \to [m]$ we define the morphism

$$N_{\beta,i} : N_{n,i} \to N_{m,\beta(i)}$$

as the unique $(A/R)_\bullet(\beta)$-semi linear map such that

$$N_{\beta,i}(1 \otimes \ldots \otimes n \otimes \ldots \otimes 1) = 1 \otimes \ldots \otimes n \otimes \ldots \otimes 1$$

for all $n \in N$. This hints at the following lemma.

023H Lemma 35.3.2. Let $R \to A$ be a ring map. Given a descent datum (N, φ) we can associate to it a cosimplicial $(A/R)_\bullet$-module $N_\bullet$[2] by the rules $N_n = N_{n,n}$ and given $\beta : [n] \to [m]$ setting we define

$$N_\bullet(\beta) = (\varphi^m_{\beta(n)m}) \circ N_{\beta,n} : N_{n,n} \longrightarrow N_{m,m}.$$

This procedure is functorial in the descent datum.

Proof. Here are the first few maps where $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$

$$\begin{array}{lcccccc}
\delta^1_0 & : & N & \to & A \otimes N & n & \mapsto & 1 \otimes n \\
\delta^1_1 & : & N & \to & A \otimes N & n & \mapsto & \sum \alpha_i \otimes x_i \\
\sigma^0_0 & : & A \otimes N & \to & N & a_0 \otimes n & \mapsto & a_0 n \\
\delta^2_0 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & 1 \otimes a_0 \otimes n \\
\delta^2_1 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & a_0 \otimes 1 \otimes n \\
\delta^2_2 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & \sum a_0 \otimes \alpha_i \otimes x_i \\
\sigma^1_0 & : & A \otimes A \otimes N & \to & A \otimes N & a_0 \otimes a_1 \otimes n & \mapsto & a_0 a_1 \otimes n \\
\sigma^1_1 & : & A \otimes A \otimes N & \to & A \otimes N & a_0 \otimes a_1 \otimes n & \mapsto & a_0 \otimes a_1 n
\end{array}$$

[1]Note that $\tau^2_{ij} = \delta^2_k$, if $\{i, j, k\} = [2] = \{0, 1, 2\}$, see Simplicial, Definition 14.2.1.

[2]We should really write $(N, \varphi)_\bullet$.

with notation as in Simplicial, Section 14.5. We first verify the two properties $\sigma_0^0 \circ \delta_0^1 = \text{id}$ and $\sigma_0^0 \circ \delta_1^1 = \text{id}$. The first one, $\sigma_0^0 \circ \delta_0^1 = \text{id}$, is clear from the explicit description of the morphisms above. To prove the second relation we have to use the cocycle condition (because it does not hold for an arbitrary isomorphism $\varphi : N \otimes_R A \to A \otimes_R N$). Write $p = \sigma_0^0 \circ \delta_1^1 : N \to N$. By the description of the maps above we deduce that p is also equal to

$$p = \varphi \otimes \text{id} : N = (N \otimes_R A) \otimes_{(A \otimes_R A)} A \longrightarrow (A \otimes_R N) \otimes_{(A \otimes_R A)} A = N$$

Since φ is an isomorphism we see that p is an isomorphism. Write $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$ for certain $\alpha_i \in A$ and $x_i \in N$. Then $p(n) = \sum \alpha_i x_i$. Next, write $\varphi(x_i \otimes 1) = \sum \alpha_{ij} \otimes y_j$ for certain $\alpha_{ij} \in A$ and $y_j \in N$. Then the cocycle condition says that

$$\sum \alpha_i \otimes \alpha_{ij} \otimes y_j = \sum \alpha_i \otimes 1 \otimes x_i.$$

This means that $p(n) = \sum \alpha_i x_i = \sum \alpha_i \alpha_{ij} y_j = \sum \alpha_i p(x_i) = p(p(n))$. Thus p is a projector, and since it is an isomorphism it is the identity.

To prove fully that $N_\bullet$ is a cosimplicial module we have to check all 5 types of relations of Simplicial, Remark 14.5.3. The relations on composing σ's are obvious. The relations on composing δ's come down to the cocycle condition for φ. In exactly the same way as above one checks the relations $\sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1} = \text{id}$. Finally, the other relations on compositions of δ's and σ's hold for any φ whatsoever. □

Note that to an R-module M we can associate a canonical descent datum, namely $(M \otimes_R A, can)$ where $can : (M \otimes_R A) \otimes_R A \to A \otimes_R (M \otimes_R A)$ is the obvious map: $(m \otimes a) \otimes a' \mapsto a \otimes (m \otimes a')$.

023I Lemma 35.3.3. Let $R \to A$ be a ring map. Let M be an R-module. The cosimplicial $(A/R)_\bullet$-module associated to the canonical descent datum is isomorphic to the cosimplicial module $(A/R)_\bullet \otimes_R M$.

Proof. Omitted. □

023J Definition 35.3.4. Let $R \to A$ be a ring map. We say a descent datum (N, φ) is effective if there exists an R-module M and an isomorphism of descent data from $(M \otimes_R A, can)$ to (N, φ).

Let $R \to A$ be a ring map. Let (N, φ) be a descent datum. We may take the cochain complex $s(N_\bullet)$ associated with $N_\bullet$ (see Simplicial, Section 14.25). It has the following shape:

$$N \to A \otimes_R N \to A \otimes_R A \otimes_R N \to \ldots$$

We can describe the maps. The first map is the map

$$n \longmapsto 1 \otimes n - \varphi(n \otimes 1).$$

The second map on pure tensors has the values

$$a \otimes n \longmapsto 1 \otimes a \otimes n - a \otimes 1 \otimes n + a \otimes \varphi(n \otimes 1).$$

It is clear how the pattern continues.

In the special case where $N = A \otimes_R M$ we see that for any $m \in M$ the element $1 \otimes m$ is in the kernel of the first map of the cochain complex associated to the cosimplicial module $(A/R)_\bullet \otimes_R M$. Hence we get an extended cochain complex

023K (35.3.4.1) $$0 \to M \to A \otimes_R M \to A \otimes_R A \otimes_R M \to \ldots$$

Here we think of the 0 as being in degree -2, the module M in degree -1, the module $A \otimes_R M$ in degree 0, etc. Note that this complex has the shape

$$0 \to R \to A \to A \otimes_R A \to A \otimes_R A \otimes_R A \to \ldots$$

when $M = R$.

023L Lemma 35.3.5. Suppose that $R \to A$ has a section. Then for any R-module M the extended cochain complex (35.3.4.1) is exact.

Proof. By Simplicial, Lemma 14.28.5 the map $R \to (A/R)_\bullet$ is a homotopy equivalence of cosimplicial R-algebras (here R denotes the constant cosimplicial R-algebra). Hence $M \to (A/R)_\bullet \otimes_R M$ is a homotopy equivalence in the category of cosimplicial R-modules, because $\otimes_R M$ is a functor from the category of R-algebras to the category of R-modules, see Simplicial, Lemma 14.28.4. This implies that the induced map of associated complexes is a homotopy equivalence, see Simplicial, Lemma 14.28.6. Since the complex associated to the constant cosimplicial R-module M is the complex

$$M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} M \xrightarrow{1} M \ldots$$

we win (since the extended version simply puts an extra M at the beginning). □

023M Lemma 35.3.6. Suppose that $R \to A$ is faithfully flat, see Algebra, Definition 10.39.1. Then for any R-module M the extended cochain complex (35.3.4.1) is exact.

Proof. Suppose we can show there exists a faithfully flat ring map $R \to R'$ such that the result holds for the ring map $R' \to A' = R' \otimes_R A$. Then the result follows for $R \to A$. Namely, for any R-module M the cosimplicial module $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ is just the cosimplicial module $R' \otimes_R (M \otimes_R (A/R)_\bullet)$. Hence the vanishing of cohomology of the complex associated to $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ implies the vanishing of the cohomology of the complex associated to $M \otimes_R (A/R)_\bullet$ by faithful flatness of $R \to R'$. Similarly for the vanishing of cohomology groups in degrees -1 and 0 of the extended complex (proof omitted).

But we have such a faithful flat extension. Namely $R' = A$ works because the ring map $R' = A \to A' = A \otimes_R A$ has a section $a \otimes a' \mapsto aa'$ and Lemma 35.3.5 applies. □

Here is how the complex relates to the question of effectivity.

039W Lemma 35.3.7. Let $R \to A$ be a faithfully flat ring map. Let (N, φ) be a descent datum. Then (N, φ) is effective if and only if the canonical map

$$A \otimes_R H^0(s(N_\bullet)) \longrightarrow N$$

is an isomorphism.

Proof. If (N, φ) is effective, then we may write $N = A \otimes_R M$ with $\varphi = can$. It follows that $H^0(s(N_\bullet)) = M$ by Lemmas 35.3.3 and 35.3.6. Conversely, suppose the map of the lemma is an isomorphism. In this case set $M = H^0(s(N_\bullet))$. This is an R-submodule of N, namely $M = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\}$. The only thing to check is that via the isomorphism $A \otimes_R M \to N$ the canonical descent data agrees with φ. We omit the verification. □

039X Lemma 35.3.8. Let $R \to A$ be a faithfully flat ring map, and let $R \to R'$ be faithfully flat. Set $A' = R' \otimes_R A$. If all descent data for $R' \to A'$ are effective, then so are all descent data for $R \to A$.

Proof. Let (N, φ) be a descent datum for $R \to A$. Set $N' = R' \otimes_R N = A' \otimes_A N$, and denote $\varphi' = \mathrm{id}_{R'} \otimes \varphi$ the base change of the descent datum φ. Then (N', φ') is a descent datum for $R' \to A'$ and $H^0(s(N'_\bullet)) = R' \otimes_R H^0(s(N_\bullet))$. Moreover, the map $A' \otimes_{R'} H^0(s(N'_\bullet)) \to N'$ is identified with the base change of the A-module map $A \otimes_R H^0(s(N)) \to N$ via the faithfully flat map $A \to A'$. Hence we conclude by Lemma 35.3.7. □

Here is the main result of this section. Its proof may seem a little clumsy; for a more highbrow approach see Remark 35.3.11 below.

023N Proposition 35.3.9. Let $R \to A$ be a faithfully flat ring map. Then

(1) any descent datum on modules with respect to $R \to A$ is effective,
(2) the functor $M \mapsto (A \otimes_R M, can)$ from R-modules to the category of descent data is an equivalence, and
(3) the inverse functor is given by $(N, \varphi) \mapsto H^0(s(N_\bullet))$.

Proof. We only prove (1) and omit the proofs of (2) and (3). As $R \to A$ is faithfully flat, there exists a faithfully flat base change $R \to R'$ such that $R' \to A' = R' \otimes_R A$ has a section (namely take $R' = A$ as in the proof of Lemma 35.3.6). Hence, using Lemma 35.3.8 we may assume that $R \to A$ has a section, say $\sigma : A \to R$. Let (N, φ) be a descent datum relative to $R \to A$. Set

$$M = H^0(s(N_\bullet)) = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\} \subset N$$

By Lemma 35.3.7 it suffices to show that $A \otimes_R M \to N$ is an isomorphism.

Take an element $n \in N$. Write $\varphi(n \otimes 1) = \sum a_i \otimes x_i$ for certain $a_i \in A$ and $x_i \in N$. By Lemma 35.3.2 we have $n = \sum a_i x_i$ in N (because $\sigma_0^0 \circ \delta_1^1 = \mathrm{id}$ in any cosimplicial object). Next, write $\varphi(x_i \otimes 1) = \sum a_{ij} \otimes y_j$ for certain $a_{ij} \in A$ and $y_j \in N$. The cocycle condition means that

$$\sum a_i \otimes a_{ij} \otimes y_j = \sum a_i \otimes 1 \otimes x_i$$

in $A \otimes_R A \otimes_R N$. We conclude two things from this:

(1) applying σ to the first A we get $\sum \sigma(a_i)\varphi(x_i \otimes 1) = \sum \sigma(a_i) \otimes x_i$,
(2) applying σ to the middle A we get $\sum_i a_i \otimes \sum_j \sigma(a_{ij}) y_j = \sum a_i \otimes x_i$.

Part (1) shows that $\sum \sigma(a_i) x_i \in M$. Applying this to x_i we see that $\sum \sigma(a_{ij}) y_i \in M$ for all i. Multiplying out the equation in (2) we conclude that $\sum_i a_i (\sum_j \sigma(a_{ij}) y_j) = \sum a_i x_i = n$. Hence $A \otimes_R M \to N$ is surjective. Finally, suppose that $m_i \in M$ and $\sum a_i m_i = 0$. Then we see by applying φ to $\sum a_i m_i \otimes 1$ that $\sum a_i \otimes m_i = 0$. In other words $A \otimes_R M \to N$ is injective and we win. □

023O Remark 35.3.10. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. The ring $A = \prod_i R_{f_i}$ is a faithfully flat R-algebra. We remark that the cosimplicial ring $(A/R)_\bullet$ has the following ring in degree n:

$$\prod\nolimits_{i_0, \ldots, i_n} R_{f_{i_0} \ldots f_{i_n}}$$

Hence the results above recover Algebra, Lemmas 10.24.2, 10.24.1 and 10.24.5. But the results above actually say more because of exactness in higher degrees. Namely, it implies that Čech cohomology of quasi-coherent sheaves on affines is trivial. Thus we get a second proof of Cohomology of Schemes, Lemma 30.2.1.

039Y Remark 35.3.11. Let R be a ring. Let $A_\bullet$ be a cosimplicial R-algebra. In this setting a descent datum corresponds to an cosimplicial $A_\bullet$-module $M_\bullet$ with the property that for every $n, m \geq 0$ and every $\varphi : [n] \to [m]$ the map $M(\varphi) : M_n \to M_m$ induces an isomorphism

$$M_n \otimes_{A_n, A(\varphi)} A_m \longrightarrow M_m.$$

Let us call such a cosimplicial module a cartesian module. In this setting, the proof of Proposition 35.3.9 can be split in the following steps

(1) If $R \to R'$ and $R \to A$ are faithfully flat, then descent data for A/R are effective if descent data for $(R' \otimes_R A)/R'$ are effective.
(2) Let A be an R-algebra. Descent data for A/R correspond to cartesian $(A/R)_\bullet$-modules.
(3) If $R \to A$ has a section then $(A/R)_\bullet$ is homotopy equivalent to R, the constant cosimplicial R-algebra with value R.
(4) If $A_\bullet \to B_\bullet$ is a homotopy equivalence of cosimplicial R-algebras then the functor $M_\bullet \mapsto M_\bullet \otimes_{A_\bullet} B_\bullet$ induces an equivalence of categories between cartesian $A_\bullet$-modules and cartesian $B_\bullet$-modules.

For (1) see Lemma 35.3.8. Part (2) uses Lemma 35.3.2. Part (3) we have seen in the proof of Lemma 35.3.5 (it relies on Simplicial, Lemma 14.28.5). Moreover, part (4) is a triviality if you think about it right!

35.4. Descent for universally injective morphisms

08WE Numerous constructions in algebraic geometry are made using techniques of descent, such as constructing objects over a given space by first working over a somewhat larger space which projects down to the given space, or verifying a property of a space or a morphism by pulling back along a covering map. The utility of such techniques is of course dependent on identification of a wide class of effective descent morphisms. Early in the Grothendieckian development of modern algebraic geometry, the class of morphisms which are quasi-compact and faithfully flat was shown to be effective for descending objects, morphisms, and many properties thereof.

As usual, this statement comes down to a property of rings and modules. For a homomorphism $f : R \to S$ to be an effective descent morphism for modules, Grothendieck showed that it is sufficient for f to be faithfully flat. However, this excludes many natural examples: for instance, any split ring homomorphism is an effective descent morphism. One natural example of this even arises in the proof of faithfully flat descent: for $f : R \to S$ any ring homomorphism, $1_S \otimes f : S \to S \otimes_R S$ is split by the multiplication map whether or not it is flat.

One may then ask whether there is a natural ring-theoretic condition implying effective descent for modules which includes both the case of a faithfully flat morphism and that of a split ring homomorphism. It may surprise the reader (at least it surprised this author) to learn that a complete answer to this question has been known since around 1970! Namely, it is not hard to check that a necessary condition for $f : R \to S$ to be an effective descent morphism for modules is that f must be universally injective in the category of R-modules, that is, for any R-module M, the map $1_M \otimes f : M \to M \otimes_R S$ must be injective. This then turns out to be a sufficient condition as well. For example, if f is split in the category of R-modules (but not necessarily in the category of rings), then f is an effective descent morphism for modules.

The history of this result is a bit involved: it was originally asserted by Olivier [Oli70], who called universally injective morphisms pure, but without a clear indication of proof. One can extract the result from the work of Joyal and Tierney [JT84], but to the best of our knowledge, the first free-standing proof to appear in the literature is that of Mesablishvili [Mes00]. The first purpose of this section is to expose Mesablishvili's proof; this requires little modification of his original presentation aside from correcting typos, with the one exception that we make explicit the relationship between the customary definition of a descent datum in algebraic geometry and the one used in [Mes00]. The proof turns out to be entirely category-theoretic, and consequently can be put in the language of monads (and thus applied in other contexts); see [JT04].

The second purpose of this section is to collect some information about which properties of modules, algebras, and morphisms can be descended along universally injective ring homomorphisms. The cases of finite modules and flat modules were treated by Mesablishvili [Mes02].

08WF 35.4.1. Category-theoretic preliminaries. We start by recalling a few basic notions from category theory which will simplify the exposition. In this subsection, fix an ambient category.

For two morphisms $g_1, g_2 : B \to C$, recall that an equalizer of g_1 and g_2 is a morphism $f : A \to B$ which satisfies $g_1 \circ f = g_2 \circ f$ and is universal for this property. This second statement means that any commutative diagram

$$\begin{tikzcd} A' \arrow[d, dashed] \arrow[dr, "e"] \arrow[drr, bend left] & & \\ A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

without the dashed arrow can be uniquely completed. We also say in this situation that the diagram

08WG (35.4.1.1)
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

is an equalizer. Reversing arrows gives the definition of a coequalizer. See Categories, Sections 4.10 and 4.11.

Since it involves a universal property, the property of being an equalizer is typically not stable under applying a covariant functor. Just as for monomorphisms and epimorphisms, one can get around this in some cases by exhibiting splittings.

08WH Definition 35.4.2. A split equalizer is a diagram (35.4.1.1) with $g_1 \circ f = g_2 \circ f$ for which there exist auxiliary morphisms $h : B \to A$ and $i : C \to B$ such that

08WI (35.4.2.1)
$$h \circ f = 1_A, \quad f \circ h = i \circ g_1, \quad i \circ g_2 = 1_B.$$

The point is that the equalities among arrows force (35.4.1.1) to be an equalizer: the map e factors uniquely through f by writing $e = f \circ (h \circ e)$. Consequently, applying a covariant functor to a split equalizer gives a split equalizer; applying a contravariant functor gives a split coequalizer, whose definition is apparent.

08WJ 35.4.3. Universally injective morphisms. Recall that Rings denotes the category of commutative rings with 1. For an object R of Rings we denote Mod_R the category of R-modules.

08WK Remark 35.4.4. Any functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories which is exact and takes nonzero objects to nonzero objects reflects injections and surjections. Namely, exactness implies that F preserves kernels and cokernels (compare with Homology, Section 12.7). For example, if $f : R \to S$ is a faithfully flat ring homomorphism, then $\bullet \otimes_R S : \text{Mod}_R \to \text{Mod}_S$ has these properties.

Let R be a ring. Recall that a morphism $f : M \to N$ in Mod_R is universally injective if for all $P \in \text{Mod}_R$, the morphism $f \otimes 1_P : M \otimes_R P \to N \otimes_R P$ is injective. See Algebra, Definition 10.82.1.

08WL Definition 35.4.5. A ring map $f : R \to S$ is universally injective if it is universally injective as a morphism in Mod_R.

08WM Example 35.4.6. Any split injection in Mod_R is universally injective. In particular, any split injection in Rings is universally injective.

08WN Example 35.4.7. For a ring R and $f_1, \ldots, f_n \in R$ generating the unit ideal, the morphism $R \to R_{f_1} \oplus \ldots \oplus R_{f_n}$ is universally injective. Although this is immediate from Lemma 35.4.8, it is instructive to check it directly: we immediately reduce to the case where R is local, in which case some f_i must be a unit and so the map $R \to R_{f_i}$ is an isomorphism.

08WP Lemma 35.4.8. Any faithfully flat ring map is universally injective.

Proof. This is a reformulation of Algebra, Lemma 10.82.11. □

The key observation from [Mes00] is that universal injectivity can be usefully reformulated in terms of a splitting, using the usual construction of an injective cogenerator in Mod_R.

08WQ Definition 35.4.9. Let R be a ring. Define the contravariant functor $C : \text{Mod}_R \to \text{Mod}_R$ by setting

$$C(M) = \text{Hom}_{\text{Ab}}(M, \mathbf{Q}/\mathbf{Z}),$$

with the R-action on $C(M)$ given by $rf(s) = f(rs)$.

This functor was denoted $M \mapsto M^\vee$ in More on Algebra, Section 15.55.

08WR Lemma 35.4.10. For a ring R, the functor $C : \text{Mod}_R \to \text{Mod}_R$ is exact and reflects injections and surjections.

Proof. Exactness is More on Algebra, Lemma 15.55.6 and the other properties follow from this, see Remark 35.4.4. □

08WS Remark 35.4.11. We will use frequently the standard adjunction between Hom and tensor product, in the form of the natural isomorphism of contravariant functors

08WT (35.4.11.1) $$C(\bullet_1 \otimes_R \bullet_2) \cong \text{Hom}_R(\bullet_1, C(\bullet_2)) : \text{Mod}_R \times \text{Mod}_R \to \text{Mod}_R$$

taking $f : M_1 \otimes_R M_2 \to \mathbf{Q}/\mathbf{Z}$ to the map $m_1 \mapsto (m_2 \mapsto f(m_1 \otimes m_2))$. See Algebra, Lemma 10.14.5. A corollary of this observation is that if

$$\begin{tikzcd} C(M) \arrow[r, shift left] \arrow[r, shift right] & C(N) \arrow[r] & C(P) \end{tikzcd}$$

is a split coequalizer diagram in Mod_R, then so is

$$\begin{tikzcd} C(M \otimes_R Q) \arrow[r, shift left] \arrow[r, shift right] & C(N \otimes_R Q) \arrow[r] & C(P \otimes_R Q) \end{tikzcd}$$

for any $Q \in \text{Mod}_R$.

08WU Lemma 35.4.12. Let R be a ring. A morphism $f : M \to N$ in Mod_R is universally injective if and only if $C(f) : C(N) \to C(M)$ is a split surjection.

Proof. By (35.4.11.1), for any $P \in \text{Mod}_R$ we have a commutative diagram

$$\begin{tikzcd} \text{Hom}_R(P, C(N)) \arrow[r, "\text{Hom}_R(P, C(f))"'] \arrow[d, "\cong"] & \text{Hom}_R(P, C(M)) \arrow[d, "\cong"] \\ C(P \otimes_R N) \arrow[r, "C(1_P \otimes f)"] & C(P \otimes_R M). \end{tikzcd}$$

If f is universally injective, then $1_{C(M)} \otimes f : C(M) \otimes_R M \to C(M) \otimes_R N$ is injective, so both rows in the above diagram are surjective for $P = C(M)$. We may thus lift $1_{C(M)} \in \text{Hom}_R(C(M), C(M))$ to some $g \in \text{Hom}_R(C(N), C(M))$ splitting $C(f)$. Conversely, if $C(f)$ is a split surjection, then both rows in the above diagram are surjective, so by Lemma 35.4.10, $1_P \otimes f$ is injective. $\square$

08WV Remark 35.4.13. Let $f : M \to N$ be a universally injective morphism in Mod_R. By choosing a splitting g of $C(f)$, we may construct a functorial splitting of $C(1_P \otimes f)$ for each $P \in \text{Mod}_R$. Namely, by (35.4.11.1) this amounts to splitting $\text{Hom}_R(P, C(f))$ functorially in P, and this is achieved by the map $g \circ \bullet$.

08WW 35.4.14. Descent for modules and their morphisms. Throughout this subsection, fix a ring map $f : R \to S$. As seen in Section 35.3 we can use the language of cosimplicial algebras to talk about descent data for modules, but in this subsection we prefer a more down to earth terminology.

For $i = 1, 2, 3$, let S_i be the i-fold tensor product of S over R. Define the ring homomorphisms $\delta_0^1, \delta_1^1 : S_1 \to S_2$, $\delta_{01}^1, \delta_{02}^1, \delta_{12}^1 : S_1 \to S_3$, and $\delta_0^2, \delta_1^2, \delta_2^2 : S_2 \to S_3$ by the formulas

$$\begin{aligned} \delta_0^1(a_0) &= 1 \otimes a_0 \\ \delta_1^1(a_0) &= a_0 \otimes 1 \\ \delta_0^2(a_0 \otimes a_1) &= 1 \otimes a_0 \otimes a_1 \\ \delta_1^2(a_0 \otimes a_1) &= a_0 \otimes 1 \otimes a_1 \\ \delta_2^2(a_0 \otimes a_1) &= a_0 \otimes a_1 \otimes 1 \\ \delta_{01}^1(a_0) &= 1 \otimes 1 \otimes a_0 \\ \delta_{02}^1(a_0) &= 1 \otimes a_0 \otimes 1 \\ \delta_{12}^1(a_0) &= a_0 \otimes 1 \otimes 1. \end{aligned}$$

In other words, the upper index indicates the source ring, while the lower index indicates where to insert factors of 1. (This notation is compatible with the notation introduced in Section 35.3.)

Recall[3] from Definition 35.3.1 that for $M \in \text{Mod}_S$, a descent datum on M relative to f is an isomorphism

$$\theta : M \otimes_{S,\delta_0^1} S_2 \longrightarrow M \otimes_{S,\delta_1^1} S_2$$

of S_2-modules satisfying the cocycle condition

08WX (35.4.14.1) $$(\theta \otimes \delta_2^2) \circ (\theta \otimes \delta_2^0) = (\theta \otimes \delta_2^1) : M \otimes_{S,\delta_{01}^1} S_3 \to M \otimes_{S,\delta_{12}^1} S_3.$$

Let $DD_{S/R}$ be the category of S-modules equipped with descent data relative to f.

For example, for $M_0 \in \text{Mod}_R$ and a choice of isomorphism $M \cong M_0 \otimes_R S$ gives rise to a descent datum by identifying $M \otimes_{S,\delta_0^1} S_2$ and $M \otimes_{S,\delta_1^1} S_2$ naturally with $M_0 \otimes_R S_2$. This construction in particular defines a functor $f^* : \text{Mod}_R \to DD_{S/R}$.

08WY Definition 35.4.15. The functor $f^* : \text{Mod}_R \to DD_{S/R}$ is called base extension along f. We say that f is a descent morphism for modules if f^* is fully faithful. We say that f is an effective descent morphism for modules if f^* is an equivalence of categories.

Our goal is to show that for f universally injective, we can use θ to locate M_0 within M. This process makes crucial use of some equalizer diagrams.

08WZ Lemma 35.4.16. For $(M, \theta) \in DD_{S/R}$, the diagram
(35.4.16.1)

08X0 $$M \xrightarrow{\theta \circ (1_M \otimes \delta_0^1)} M \otimes_{S,\delta_1^1} S_2 \underset{1_{M \otimes S_2} \otimes \delta_1^2}{\overset{(\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2)}{\rightrightarrows}} M \otimes_{S,\delta_{12}^1} S_3$$

is a split equalizer.

Proof. Define the ring homomorphisms $\sigma_0^0 : S_2 \to S_1$ and $\sigma_0^1, \sigma_1^1 : S_3 \to S_2$ by the formulas

$$\sigma_0^0(a_0 \otimes a_1) = a_0 a_1$$
$$\sigma_0^1(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2$$
$$\sigma_1^1(a_0 \otimes a_1 \otimes a_2) = a_0 \otimes a_1 a_2.$$

We then take the auxiliary morphisms to be $1_M \otimes \sigma_0^0 : M \otimes_{S,\delta_1^1} S_2 \to M$ and $1_M \otimes \sigma_0^1 : M \otimes_{S,\delta_{12}^1} S_3 \to M \otimes_{S,\delta_1^1} S_2$. Of the compatibilities required in (35.4.2.1), the first follows from tensoring the cocycle condition (35.4.14.1) with σ_1^1 and the others are immediate. $\square$

08X1 Lemma 35.4.17. For $(M, \theta) \in DD_{S/R}$, the diagram
(35.4.17.1)

08X2 $$C(M \otimes_{S,\delta_{12}^1} S_3) \underset{C(1_{M \otimes S_2} \otimes \delta_1^2)}{\overset{C((\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2))}{\rightrightarrows}} C(M \otimes_{S,\delta_1^1} S_2) \xrightarrow{C(\theta \circ (1_M \otimes \delta_0^1))} C(M).$$

obtained by applying C to (35.4.16.1) is a split coequalizer.

Proof. Omitted. $\square$

[3]To be precise, our θ here is the inverse of φ from Definition 35.3.1.

08X3 **Lemma 35.4.18.** The diagram

08X4 (35.4.18.1)
$$\begin{tikzcd} S_1 \arrow[r, "\delta_1^1"] & S_2 \arrow[r, shift left, "\delta_2^2"] \arrow[r, shift right, "\delta_1^2"'] & S_3 \end{tikzcd}$$

is a split equalizer.

Proof. In Lemma 35.4.16, take $(M, \theta) = f^*(S)$. ☐

This suggests a definition of a potential quasi-inverse functor for f^*.

08X5 **Definition 35.4.19.** Define the functor $f_* : DD_{S/R} \to \text{Mod}_R$ by taking $f_*(M, \theta)$ to be the R-submodule of M for which the diagram
(35.4.19.1)

08X6
$$\begin{tikzcd} f_*(M,\theta) \arrow[r] & M \arrow[r, shift left, "\theta\circ(1_M\otimes\delta_0^1)"] \arrow[r, shift right, "1_M\otimes\delta_1^1"'] & M \otimes_{S,\delta_1^1} S_2 \end{tikzcd}$$

is an equalizer.

Using Lemma 35.4.16 and the fact that the restriction functor $\text{Mod}_S \to \text{Mod}_R$ is right adjoint to the base extension functor $\bullet \otimes_R S : \text{Mod}_R \to \text{Mod}_S$, we deduce that f_* is right adjoint to f^*.

We are ready for the key lemma. In the faithfully flat case this is a triviality (see Remark 35.4.21), but in the general case some argument is needed.

08X7 **Lemma 35.4.20.** If f is universally injective, then the diagram
(35.4.20.1)

08X8
$$\begin{tikzcd} f_*(M,\theta)\otimes_R S \arrow[r, "\theta\circ(1_M\otimes\delta_0^1)"] & M \otimes_{S,\delta_1^1} S_2 \arrow[r, shift left, "(\theta\otimes\delta_2^2)\circ(1_M\otimes\delta_0^2)"] \arrow[r, shift right, "1_{M\otimes S_2}\otimes\delta_1^2"'] & M \otimes_{S,\delta_{12}^1} S_3 \end{tikzcd}$$

obtained by tensoring (35.4.19.1) over R with S is an equalizer.

Proof. By Lemma 35.4.12 and Remark 35.4.13, the map $C(1_N \otimes f) : C(N \otimes_R S) \to C(N)$ can be split functorially in N. This gives the upper vertical arrows in the commutative diagram

$$\begin{tikzcd}
C(M\otimes_{S,\delta_1^1} S_2) \arrow[r, shift left, "C(\theta\circ(1_M\otimes\delta_0^1))"] \arrow[r, shift right, "C(1_M\otimes\delta_1^1)"'] \arrow[d] & C(M) \arrow[r] \arrow[d] & C(f_*(M,\theta)) \arrow[d, dashed] \\
C(M\otimes_{S,\delta_{12}^1} S_3) \arrow[r, shift left, "C((\theta\otimes\delta_2^2)\circ(1_M\otimes\delta_0^2))"] \arrow[r, shift right, "C(1_{M\otimes S_2}\otimes\delta_1^2)"'] \arrow[d] & C(M\otimes_{S,\delta_1^1} S_2) \arrow[r, "C(\theta\circ(1_M\otimes\delta_0^1))"] \arrow[d, "C(1_M\otimes\delta_1^1)"] & C(M) \arrow[d] \\
C(M\otimes_{S,\delta_1^1} S_2) \arrow[r, shift left, "C(\theta\circ(1_M\otimes\delta_0^1))"] \arrow[r, shift right, "C(1_M\otimes\delta_1^1)"'] & C(M) \arrow[r] \arrow[ur, equal] & C(f_*(M,\theta))
\end{tikzcd}$$

in which the compositions along the columns are identity morphisms. The second row is the coequalizer diagram (35.4.17.1); this produces the dashed arrow. From the top right square, we obtain auxiliary morphisms $C(f_*(M,\theta)) \to C(M)$ and $C(M) \to C(M \otimes_{S,\delta_1^1} S_2)$ which imply that the first row is a split coequalizer diagram.

By Remark 35.4.11, we may tensor with S inside C to obtain the split coequalizer diagram

$$C(M \otimes_{S,\delta_2^2\circ\delta_1^1} S_3) \underset{C(1_{M\otimes S_2}\otimes\delta_1^2)}{\overset{C((\theta\otimes\delta_2^2)\circ(1_M\otimes\delta_0^2))}{\rightrightarrows}} C(M\otimes_{S,\delta_1^1} S_2) \xrightarrow{C(\theta\circ(1_M\otimes\delta_0^1))} C(f_*(M,\theta)\otimes_R S).$$

By Lemma 35.4.10, we conclude (35.4.20.1) must also be an equalizer. $\square$

08X9 Remark 35.4.21. If f is a split injection in Mod_R, one can simplify the argument by splitting f directly, without using C. Things are even simpler if f is faithfully flat; in this case, the conclusion of Lemma 35.4.20 is immediate because tensoring over R with S preserves all equalizers.

08XA Theorem 35.4.22. The following conditions are equivalent.

(a) The morphism f is a descent morphism for modules.
(b) The morphism f is an effective descent morphism for modules.
(c) The morphism f is universally injective.

Proof. It is clear that (b) implies (a). We now check that (a) implies (c). If f is not universally injective, we can find $M \in \text{Mod}_R$ such that the map $1_M \otimes f : M \to M \otimes_R S$ has nontrivial kernel N. The natural projection $M \to M/N$ is not an isomorphism, but its image in $DD_{S/R}$ is an isomorphism. Hence f^* is not fully faithful.

We finally check that (c) implies (b). By Lemma 35.4.20, for $(M,\theta) \in DD_{S/R}$, the natural map $f^*f_*(M,\theta) \to M$ is an isomorphism of S-modules. On the other hand, for $M_0 \in \text{Mod}_R$, we may tensor (35.4.18.1) with M_0 over R to obtain an equalizer sequence, so $M_0 \to f_*f^*M$ is an isomorphism. Consequently, f_* and f^* are quasi-inverse functors, proving the claim. $\square$

08XB 35.4.23. Descent for properties of modules. Throughout this subsection, fix a universally injective ring map $f : R \to S$, an object $M \in \text{Mod}_R$, and a ring map $R \to A$. We now investigate the question of which properties of M or A can be checked after base extension along f. We start with some results from [Mes02].

08XC Lemma 35.4.24. If $M \in \text{Mod}_R$ is flat, then $C(M)$ is an injective R-module.

Proof. Let $0 \to N \to P \to Q \to 0$ be an exact sequence in Mod_R. Since M is flat,

$$0 \to N \otimes_R M \to P \otimes_R M \to Q \otimes_R M \to 0$$

is exact. By Lemma 35.4.10,

$$0 \to C(Q \otimes_R M) \to C(P \otimes_R M) \to C(N \otimes_R M) \to 0$$

is exact. By (35.4.11.1), this last sequence can be rewritten as

$$0 \to \text{Hom}_R(Q, C(M)) \to \text{Hom}_R(P, C(M)) \to \text{Hom}_R(N, C(M)) \to 0.$$

Hence $C(M)$ is an injective object of Mod_R. $\square$

08XD Theorem 35.4.25. If $M \otimes_R S$ has one of the following properties as an S-module

(a) finitely generated;
(b) finitely presented;
(c) flat;
(d) faithfully flat;

(e) finite projective;

then so does M as an R-module (and conversely).

Proof. To prove (a), choose a finite set $\{n_i\}$ of generators of $M \otimes_R S$ in Mod_S. Write each n_i as $\sum_j m_{ij} \otimes s_{ij}$ with $m_{ij} \in M$ and $s_{ij} \in S$. Let F be the finite free R-module with basis e_{ij} and let $F \to M$ be the R-module map sending e_{ij} to m_{ij}. Then $F \otimes_R S \to M \otimes_R S$ is surjective, so $\text{Coker}(F \to M) \otimes_R S$ is zero and hence $\text{Coker}(F \to M)$ is zero. This proves (a).

To see (b) assume $M \otimes_R S$ is finitely presented. Then M is finitely generated by (a). Choose a surjection $R^{\oplus n} \to M$ with kernel K. Then $K \otimes_R S \to S^{\oplus r} \to M \otimes_R S \to 0$ is exact. By Algebra, Lemma 10.5.3 the kernel of $S^{\oplus r} \to M \otimes_R S$ is a finite S-module. Thus we can find finitely many elements $k_1, \ldots, k_t \in K$ such that the images of $k_i \otimes 1$ in $S^{\oplus r}$ generate the kernel of $S^{\oplus r} \to M \otimes_R S$. Let $K' \subset K$ be the submodule generated by $k_1, \ldots, k_t$. Then $M' = R^{\oplus r}/K'$ is a finitely presented R-module with a morphism $M' \to M$ such that $M' \otimes_R S \to M \otimes_R S$ is an isomorphism. Thus $M' \cong M$ as desired.

To prove (c), let $0 \to M' \to M'' \to M \to 0$ be a short exact sequence in Mod_R. Since $\bullet \otimes_R S$ is a right exact functor, $M'' \otimes_R S \to M \otimes_R S$ is surjective. So by Lemma 35.4.10 the map $C(M \otimes_R S) \to C(M'' \otimes_R S)$ is injective. If $M \otimes_R S$ is flat, then Lemma 35.4.24 shows $C(M \otimes_R S)$ is an injective object of Mod_S, so the injection $C(M \otimes_R S) \to C(M'' \otimes_R S)$ is split in Mod_S and hence also in Mod_R. Since $C(M \otimes_R S) \to C(M)$ is a split surjection by Lemma 35.4.12, it follows that $C(M) \to C(M'')$ is a split injection in Mod_R. That is, the sequence

$$0 \to C(M) \to C(M'') \to C(M') \to 0$$

is split exact. For $N \in \text{Mod}_R$, by (35.4.11.1) we see that

$$0 \to C(M \otimes_R N) \to C(M'' \otimes_R N) \to C(M' \otimes_R N) \to 0$$

is split exact. By Lemma 35.4.10,

$$0 \to M' \otimes_R N \to M'' \otimes_R N \to M \otimes_R N \to 0$$

is exact. This implies M is flat over R. Namely, taking M' a free module surjecting onto M we conclude that $\text{Tor}_1^R(M, N) = 0$ for all modules N and we can use Algebra, Lemma 10.75.8. This proves (c).

To deduce (d) from (c), note that if $N \in \text{Mod}_R$ and $M \otimes_R N$ is zero, then $M \otimes_R S \otimes_S (N \otimes_R S) \cong (M \otimes_R N) \otimes_R S$ is zero, so $N \otimes_R S$ is zero and hence N is zero.

To deduce (e) at this point, it suffices to recall that M is finitely generated and projective if and only if it is finitely presented and flat. See Algebra, Lemma 10.78.2. $\square$

There is a variant for R-algebras.

08XE Theorem 35.4.26. If $A \otimes_R S$ has one of the following properties as an S-algebra

(a) of finite type;
(b) of finite presentation;
(c) formally unramified;
(d) unramified;
(e) étale;

then so does A as an R-algebra (and of course conversely).

Proof. To prove (a), choose a finite set $\{x_i\}$ of generators of $A \otimes_R S$ over S. Write each x_i as $\sum_j y_{ij} \otimes s_{ij}$ with $y_{ij} \in A$ and $s_{ij} \in S$. Let F be the polynomial R-algebra on variables e_{ij} and let $F \to M$ be the R-algebra map sending e_{ij} to y_{ij}. Then $F \otimes_R S \to A \otimes_R S$ is surjective, so $\mathrm{Coker}(F \to A) \otimes_R S$ is zero and hence $\mathrm{Coker}(F \to A)$ is zero. This proves (a).

To see (b) assume $A \otimes_R S$ is a finitely presented S-algebra. Then A is finite type over R by (a). Choose a surjection $R[x_1, \ldots, x_n] \to A$ with kernel I. Then $I \otimes_R S \to S[x_1, \ldots, x_n] \to A \otimes_R S \to 0$ is exact. By Algebra, Lemma 10.6.3 the kernel of $S[x_1, \ldots, x_n] \to A \otimes_R S$ is a finitely generated ideal. Thus we can find finitely many elements $y_1, \ldots, y_t \in I$ such that the images of $y_i \otimes 1$ in $S[x_1, \ldots, x_n]$ generate the kernel of $S[x_1, \ldots, x_n] \to A \otimes_R S$. Let $I' \subset I$ be the ideal generated by $y_1, \ldots, y_t$. Then $A' = R[x_1, \ldots, x_n]/I'$ is a finitely presented R-algebra with a morphism $A' \to A$ such that $A' \otimes_R S \to A \otimes_R S$ is an isomorphism. Thus $A' \cong A$ as desired.

To prove (c), recall that A is formally unramified over R if and only if the module of relative differentials $\Omega_{A/R}$ vanishes, see Algebra, Lemma 10.148.2 or [GD67, Proposition 17.2.1]. Since $\Omega_{(A \otimes_R S)/S} = \Omega_{A/R} \otimes_R S$, the vanishing descends by Theorem 35.4.22.

To deduce (d) from the previous cases, recall that A is unramified over R if and only if A is formally unramified and of finite type over R, see Algebra, Lemma 10.151.2.

To prove (e), recall that by Algebra, Lemma 10.151.8 or [GD67, Théorème 17.6.1] the algebra A is étale over R if and only if A is flat, unramified, and of finite presentation over R. □

08XF Remark 35.4.27. It would make things easier to have a faithfully flat ring homomorphism $g : R \to T$ for which $T \to S \otimes_R T$ has some extra structure. For instance, if one could ensure that $T \to S \otimes_R T$ is split in Rings, then it would follow that every property of a module or algebra which is stable under base extension and which descends along faithfully flat morphisms also descends along universally injective morphisms. An obvious guess would be to find g for which T is not only faithfully flat but also injective in Mod_R, but even for $R = \mathbf{Z}$ no such homomorphism can exist.

35.5. Fpqc descent of quasi-coherent sheaves

023R The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

023S Lemma 35.5.1. Let S be an affine scheme. Let $\mathcal{U} = \{f_i : U_i \to S\}_{i=1,\ldots,n}$ be a standard fpqc covering of S, see Topologies, Definition 34.9.9. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.

Proof. This is a restatement of Proposition 35.3.9 in terms of schemes. First, note that a descent datum ξ for quasi-coherent sheaves with respect to $\mathcal{U}$ is exactly the same as a descent datum ξ' for quasi-coherent sheaves with respect to the covering $\mathcal{U}' = \{\coprod_{i=1,\ldots,n} U_i \to S\}$. Moreover, effectivity for ξ is the same as effectivity for

ξ'. Hence we may assume $n = 1$, i.e., $\mathcal{U} = \{U \to S\}$ where U and S are affine. In this case descent data correspond to descent data on modules with respect to the ring map

$$\Gamma(S, \mathcal{O}) \longrightarrow \Gamma(U, \mathcal{O}).$$

Since $U \to S$ is surjective and flat, we see that this ring map is faithfully flat. In other words, Proposition 35.3.9 applies and we win. □

023T Proposition 35.5.2. Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}$ be an fpqc covering, see Topologies, Definition 34.9.1. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.

Proof. Let $S = \bigcup_{j \in J} V_j$ be an affine open covering. For $j, j' \in J$ we denote $V_{jj'} = V_j \cap V_{j'}$ the intersection (which need not be affine). For $V \subset S$ open we denote $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ which is a fpqc-covering (Topologies, Lemma 34.9.7). By definition of an fpqc covering, we can find for each $j \in J$ a finite set K_j, a map $\underline{i} : K_j \to I$, affine opens $U_{\underline{i}(k),k} \subset U_{\underline{i}(k)}$, $k \in K_j$ such that $\mathcal{V}_j = \{U_{\underline{i}(k),k} \to V_j\}_{k \in K_j}$ is a standard fpqc covering of V_j. And of course, $\mathcal{V}_j$ is a refinement of $\mathcal{U}_{V_j}$. Picture

$$\begin{tikzcd} \mathcal{V}_j \arrow[r] \arrow[d, squiggly] & \mathcal{U}_{V_j} \arrow[r] \arrow[d, squiggly] & \mathcal{U} \arrow[d, squiggly] \\ V_j \arrow[r, equal] & V_j \arrow[r] & S \end{tikzcd}$$

where the top horizontal arrows are morphisms of families of morphisms with fixed target (see Sites, Definition 7.8.1).

To prove the proposition you show successively the faithfulness, fullness, and essential surjectivity of the functor from quasi-coherent sheaves to descent data.

Faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a, b : \mathcal{F} \to \mathcal{G}$ be homomorphisms of $\mathcal{O}_S$-modules. Suppose $\varphi_i^*(a) = \varphi_i^*(b)$ for all i. Pick $s \in S$. Then $s = \varphi_i(u)$ for some $i \in I$ and $u \in U_i$. Since $\mathcal{O}_{S,s} \to \mathcal{O}_{U_i,u}$ is flat, hence faithfully flat (Algebra, Lemma 10.39.17) we see that $a_s = b_s : \mathcal{F}_s \to \mathcal{G}_s$. Hence $a = b$.

Fully faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a_i : \varphi_i^*\mathcal{F} \to \varphi_i^*\mathcal{G}$ be homomorphisms of $\mathcal{O}_{U_i}$-modules such that $\mathrm{pr}_0^* a_i = \mathrm{pr}_1^* a_j$ on $U_i \times_U U_j$. We can pull back these morphisms to get morphisms

$$a_k : \varphi_{i(k)}^*\mathcal{F}|_{U_{\underline{i}(k),k}} \longrightarrow \varphi_{i(k)}^*\mathcal{G}|_{U_{\underline{i}(k),k}}$$

$k \in K_j$ with notation as above. Moreover, Lemma 35.2.2 assures us that these define a morphism between (canonical) descent data on $\mathcal{V}_j$. Hence, by Lemma 35.5.1, we get correspondingly unique morphisms $a_j : \mathcal{F}|_{V_j} \to \mathcal{G}|_{V_j}$. To see that $a_j|_{V_{jj'}} = a_{j'}|_{V_{jj'}}$ we use that both a_j and $a_{j'}$ agree with the pullback of the morphism $(a_i)_{i \in I}$ of (canonical) descent data to any covering refining both $\mathcal{V}_{j,V_{jj'}}$ and $\mathcal{V}_{j',V_{jj'}}$, and using the faithfulness already shown. For example the covering $\mathcal{V}_{jj'} = \{V_k \times_S V_{k'} \to V_{jj'}\}_{k \in K_j, k' \in K_{j'}}$ will do.

Essential surjectivity. Let $\xi = (\mathcal{F}_i, \varphi_{ii'})$ be a descent datum for quasi-coherent sheaves relative to the covering $\mathcal{U}$. Pull back this descent datum to get descent data ξ_j for quasi-coherent sheaves relative to the coverings $\mathcal{V}_j$ of V_j. By Lemma 35.5.1 once again there exist quasi-coherent sheaves $\mathcal{F}_j$ on V_j whose associated canonical descent datum is isomorphic to ξ_j. By fully faithfulness (proved above) we see there are isomorphisms

$$\phi_{jj'} : \mathcal{F}_j|_{V_{jj'}} \longrightarrow \mathcal{F}_{j'}|_{V_{jj'}}$$

corresponding to the isomorphism of descent data between the pullback of ξ_j and $\xi_{j'}$ to $\mathcal{V}_{jj'}$. To see that these maps $\phi_{jj'}$ satisfy the cocycle condition we use faithfulness (proved above) over the triple intersections $V_{jj'j''}$. Hence, by Lemma 35.2.4 we see that the sheaves $\mathcal{F}_j$ glue to a quasi-coherent sheaf $\mathcal{F}$ as desired. We still have to verify that the canonical descent datum relative to $\mathcal{U}$ associated to $\mathcal{F}$ is isomorphic to the descent datum we started out with. This verification is omitted. $\square$

35.6. Galois descent for quasi-coherent sheaves

0CDQ Galois descent for quasi-coherent sheaves is just a special case of fpqc descent for quasi-coherent sheaves. In this section we will explain how to translate from a Galois descent to an fpqc descent and then apply earlier results to conclude.

Let k'/k be a field extension. Then $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ is an fpqc covering. Let X be a scheme over k. For a k-algebra A we set $X_A = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$. By Topologies, Lemma 34.9.7 we see that $\{X_{k'} \to X\}$ is an fpqc covering. Observe that

$$X_{k'} \times_X X_{k'} = X_{k' \otimes_k k'} \quad \text{and} \quad X_{k'} \times_X X_{k'} \times_X X_{k'} = X_{k' \otimes_k k' \otimes_k k'}$$

Thus a descent datum for quasi-coherent sheaves with respect to $\{X_{k'} \to X\}$ is given by a quasi-coherent sheaf $\mathcal{F}$ on $X_{k'}$, an isomorphism $\varphi : \mathrm{pr}_0^*\mathcal{F} \to \mathrm{pr}_1^*\mathcal{F}$ on $X_{k' \otimes_k k'}$ which satisfies an obvious cocycle condition on $X_{k' \otimes_k k' \otimes_k k'}$. We will work out what this means in the case of a Galois extension below.

Let k'/k be a finite Galois extension with Galois group $G = \mathrm{Gal}(k'/k)$. Then there are k-algebra isomorphisms

$$k' \otimes_k k' \longrightarrow \prod\nolimits_{\sigma \in G} k', \quad a \otimes b \longrightarrow \prod a\sigma(b)$$

and

$$k' \otimes_k k' \otimes_k k' \longrightarrow \prod\nolimits_{(\sigma,\tau) \in G \times G} k', \quad a \otimes b \otimes c \longrightarrow \prod a\sigma(b)\sigma(\tau(c))$$

The reason for choosing here $a\sigma(b)\sigma(\tau(c))$ and not $a\sigma(b)\tau(c)$ is that the formulas below simplify but it isn't strictly necessary. Given $\sigma \in G$ we denote

$$f_\sigma = \mathrm{id}_X \times \mathrm{Spec}(\sigma) : X_{k'} \longrightarrow X_{k'}$$

Please keep in mind that because $\mathrm{Spec}(-)$ is a contravariant functor we have $f_{\sigma\tau} = f_\tau \circ f_\sigma$ and not the other way around. Using the first isomorphism above we obtain an identification

$$X_{k' \otimes_k k'} = \coprod\nolimits_{\sigma \in G} X_{k'}$$

such that pr_0 corresponds to the map

$$\coprod\nolimits_{\sigma \in G} X_{k'} \xrightarrow{\coprod \mathrm{id}} X_{k'}$$

and such that pr_1 corresponds to the map

$$\coprod\nolimits_{\sigma \in G} X_{k'} \xrightarrow{\coprod f_\sigma} X_{k'}$$

Thus we see that a descent datum φ on $\mathcal{F}$ over $X_{k'}$ corresponds to a family of isomorphisms $\varphi_\sigma : \mathcal{F} \to f_\sigma^*\mathcal{F}$. To work out the cocycle condition we use the identification

$$X_{k' \otimes_k k' \otimes_k k'} = \coprod\nolimits_{(\sigma, \tau) \in G \times G} X_{k'}.$$

we get from our isomorphism of algebras above. Via this identification the map pr_{01} corresponds to the map

$$\coprod\nolimits_{(\sigma, \tau) \in G \times G} X_{k'} \longrightarrow \coprod\nolimits_{\sigma \in G} X_{k'}$$

which maps the summand with index (σ, τ) to the summand with index σ via the identity morphism. The map pr_{12} corresponds to the map

$$\coprod\nolimits_{(\sigma, \tau) \in G \times G} X_{k'} \longrightarrow \coprod\nolimits_{\sigma \in G} X_{k'}$$

which maps the summand with index (σ, τ) to the summand with index τ via the morphism f_σ. Finally, the map pr_{02} corresponds to the map

$$\coprod\nolimits_{(\sigma, \tau) \in G \times G} X_{k'} \longrightarrow \coprod\nolimits_{\sigma \in G} X_{k'}$$

which maps the summand with index (σ, τ) to the summand with index $\sigma\tau$ via the identity morphism. Thus the cocycle condition

$$\text{pr}_{02}^*\varphi = \text{pr}_{12}^*\varphi \circ \text{pr}_{01}^*\varphi$$

translates into one condition for each pair (σ, τ), namely

$$\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$$

as maps $\mathcal{F} \to f_{\sigma\tau}^*\mathcal{F}$. (Everything works out beautifully; for example the target of φ_σ is $f_\sigma^*\mathcal{F}$ and the source of $f_\sigma^*\varphi_\tau$ is $f_\sigma^*\mathcal{F}$ as well.)

0CDR Lemma 35.6.1. Let k'/k be a (finite) Galois extension with Galois group G. Let X be a scheme over k. The category of quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category of systems $(\mathcal{F}, (\varphi_\sigma)_{\sigma \in G})$ where

(1) $\mathcal{F}$ is a quasi-coherent module on $X_{k'}$,
(2) $\varphi_\sigma : \mathcal{F} \to f_\sigma^*\mathcal{F}$ is an isomorphism of modules,
(3) $\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ for all $\sigma, \tau \in G$.

Here $f_\sigma = \text{id}_X \times \text{Spec}(\sigma) : X_{k'} \to X_{k'}$.

Proof. As seen above a datum $(\mathcal{F}, (\varphi_\sigma)_{\sigma \in G})$ as in the lemma is the same thing as a descent datum for the fpqc covering $\{X_{k'} \to X\}$. Thus the lemma follows from Proposition 35.5.2. $\square$

A slightly more general case of the above is the following. Suppose we have a surjective finite étale morphism $X \to Y$ and a finite group G together with a group homomorphism $G^{opp} \to \text{Aut}_Y(X), \sigma \mapsto f_\sigma$ such that the map

$$G \times X \longrightarrow X \times_Y X, \quad (\sigma, x) \longmapsto (x, f_\sigma(x))$$

is an isomorphism. Then the same result as above holds.

0D1V Lemma 35.6.2. Let $X \to Y$, G, and $f_\sigma : X \to X$ be as above. The category of quasi-coherent $\mathcal{O}_Y$-modules is equivalent to the category of systems $(\mathcal{F}, (\varphi_\sigma)_{\sigma \in G})$ where

1. $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module,
2. $\varphi_\sigma : \mathcal{F} \to f_\sigma^*\mathcal{F}$ is an isomorphism of modules,
3. $\varphi_{\sigma\tau} = f_\sigma^*\varphi_\tau \circ \varphi_\sigma$ for all $\sigma, \tau \in G$.

Proof. Since $X \to Y$ is surjective finite étale $\{X \to Y\}$ is an fpqc covering. Since $G \times X \to X \times_Y X$, $(\sigma, x) \mapsto (x, f_\sigma(x))$ is an isomorphism, we see that $G \times G \times X \to X \times_Y X \times_Y X$, $(\sigma, \tau, x) \mapsto (x, f_\sigma(x), f_{\sigma\tau}(x))$ is an isomorphism too. Using these identifications, the category of data as in the lemma is the same as the category of descent data for quasi-coherent sheaves for the covering $\{x \to Y\}$. Thus the lemma follows from Proposition 35.5.2. □

35.7. Descent of finiteness properties of modules

05AY In this section we prove that one can check quasi-coherent module has a certain finiteness conditions by checking on the members of a covering.

05AZ Lemma 35.7.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.

Proof. Omitted. For the affine case, see Algebra, Lemma 10.83.2. □

09UB Lemma 35.7.2. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. If

1. f is open as a map of topological spaces,
2. f is surjective and flat, and
3. $f^*\mathcal{F}$ is of finite type,

then $\mathcal{F}$ is of finite type.

Proof. Let $y \in Y$ be a point. Choose a point $x \in X$ mapping to y. Choose an open $x \in U \subset X$ and elements $s_1, \ldots, s_n$ of $f^*\mathcal{F}(U)$ which generate $f^*\mathcal{F}$ over U. Since $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ we can after shrinking U assume $s_i = \sum t_{ij} \otimes a_{ij}$ with $t_{ij} \in f^{-1}\mathcal{F}(U)$ and $a_{ij} \in \mathcal{O}_X(U)$. After shrinking U further we may assume that t_{ij} comes from a section $s_{ij} \in \mathcal{F}(V)$ for some $V \subset Y$ open with $f(U) \subset V$. Let N be the number of sections s_{ij} and consider the map

$$\sigma = (s_{ij}) : \mathcal{O}_V^{\oplus N} \to \mathcal{F}|_V$$

By our choice of the sections we see that $f^*\sigma|_U$ is surjective. Hence for every $u \in U$ the map

$$\sigma_{f(u)} \otimes_{\mathcal{O}_{Y,f(u)}} \mathcal{O}_{X,u} : \mathcal{O}_{X,u}^{\oplus N} \longrightarrow \mathcal{F}_{f(u)} \otimes_{\mathcal{O}_{Y,f(u)}} \mathcal{O}_{X,u}$$

is surjective. As f is flat, the local ring map $\mathcal{O}_{Y,f(u)} \to \mathcal{O}_{X,u}$ is flat, hence faithfully flat (Algebra, Lemma 10.39.17). Hence $\sigma_{f(u)}$ is surjective. Since f is open, $f(U)$ is an open neighbourhood of y and the proof is done. □

05B0 Lemma 35.7.3. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.

Proof. Omitted. For the affine case, see Algebra, Lemma 10.83.2. □

082U Lemma 35.7.4. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is locally generated by r sections as an $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is locally generated by r sections as an $\mathcal{O}_X$-module.

Proof. By Lemma 35.7.1 we see that $\mathcal{F}$ is of finite type. Hence Nakayama's lemma (Algebra, Lemma 10.20.1) implies that $\mathcal{F}$ is generated by r sections in the neighbourhood of a point $x \in X$ if and only if $\dim_{\kappa(x)} \mathcal{F}_x \otimes \kappa(x) \leq r$. Choose an i and a point $x_i \in X_i$ mapping to x. Then $\dim_{\kappa(x)} \mathcal{F}_x \otimes \kappa(x) = \dim_{\kappa(x_i)} (f_i^*\mathcal{F})_{x_i} \otimes \kappa(x_i)$ which is $\leq r$ as $f_i^*\mathcal{F}$ is locally generated by r sections. $\square$

05B1 Lemma 35.7.5. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.

Proof. Omitted. For the affine case, see Algebra, Lemma 10.83.2. $\square$

05B2 Lemma 35.7.6. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.

Proof. This follows from the fact that a quasi-coherent sheaf is finite locally free if and only if it is of finite presentation and flat, see Algebra, Lemma 10.78.2. Namely, if each $f_i^*\mathcal{F}$ is flat and of finite presentation, then so is $\mathcal{F}$ by Lemmas 35.7.5 and 35.7.3. $\square$

The definition of a locally projective quasi-coherent sheaf can be found in Properties, Section 28.21.

05JZ Lemma 35.7.7. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.

Proof. Omitted. For Zariski coverings this is Properties, Lemma 28.21.2. For the affine case this is Algebra, Theorem 10.95.6. $\square$

05VF Remark 35.7.8. Being locally free is a property of quasi-coherent modules which does not descend in the fpqc topology. Namely, suppose that R is a ring and that M is a projective R-module which is a countable direct sum $M = \bigoplus L_n$ of rank 1 locally free modules, but not locally free, see Examples, Lemma 110.33.4. Then M becomes free on making the faithfully flat base change

$$R \longrightarrow \bigoplus\nolimits_{m \geq 1} \bigoplus\nolimits_{(i_1, \ldots, i_m) \in \mathbf{Z}^{\oplus m}} L_1^{\otimes i_1} \otimes_R \ldots \otimes_R L_m^{\otimes i_m}$$

But we don't know what happens for fppf coverings. In other words, we don't know the answer to the following question: Suppose $A \to B$ is a faithfully flat ring map of finite presentation. Let M be an A-module such that $M \otimes_A B$ is free. Is M a locally free A-module? It turns out that if A is Noetherian, then the answer is yes. This follows from the results of [Bas63]. But in general we don't know the answer. If you know the answer, or have a reference, please email stacks.project@gmail.com.

We also add here two results which are related to the results above, but are of a slightly different nature.

05B3 Lemma 35.7.9. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by a finite ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finite as a B-module if and only if M is finite as an A-module, see Algebra, Lemma 10.7.2. Combined with Properties, Lemma 28.16.1 this proves the lemma. □

05B4 Lemma 35.7.10. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by a finite and finitely presented ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finitely presented as a B-module if and only if M is finitely presented as an A-module, see Algebra, Lemma 10.36.23. Combined with Properties, Lemma 28.16.2 this proves the lemma. □

35.8. Quasi-coherent sheaves and topologies, I

03DR The results in this section say there is a natural equivalence between the category quasi-coherent modules on a scheme S and the category of quasi-coherent modules on many of the sites associated to S in the chapter on topologies.

Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Consider the functor

03DS (35.8.0.1)
$$(Sch/S)^{opp} \longrightarrow \mathrm{Ab}, \quad (f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F}).$$

03DT Lemma 35.8.1. Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf, fpqc\}$. The functor defined in (35.8.0.1) satisfies the sheaf condition with respect to any τ-covering $\{T_i \to T\}_{i \in I}$ of any scheme T over S.

Proof. For $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$ a τ-covering is also a fpqc-covering, see the results in Topologies, Lemmas 34.4.2, 34.5.2, 34.6.2, 34.7.2, and 34.9.6. Hence it suffices to prove the theorem for a fpqc covering. Assume that $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering where $f : T \to S$ is given. Suppose that we have a family of sections $s_i \in \Gamma(T_i, f_i^*f^*\mathcal{F})$ such that $s_i|_{T_i \times_T T_j} = s_j|_{T_i \times_T T_j}$. We have to find the correspond section $s \in \Gamma(T, f^*\mathcal{F})$. We can reinterpret the s_i as a family of maps $\varphi_i : f_i^*\mathcal{O}_T = \mathcal{O}_{T_i} \to f_i^*f^*\mathcal{F}$ compatible with the canonical descent data associated to the quasi-coherent sheaves $\mathcal{O}_T$ and $f^*\mathcal{F}$ on T. Hence by Proposition 35.5.2 we see that we may (uniquely) descend these to a map $\mathcal{O}_T \to f^*\mathcal{F}$ which gives us our section s. □

We may in particular make the following definition.

03DU Definition 35.8.2. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Let S be a scheme. Let Sch_τ be a big site containing S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module.

(1) The structure sheaf of the big site $(Sch/S)_\tau$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.

(2) If $\tau = Zariski$ or $\tau = \acute{e}tale$ the structure sheaf of the small site S_{Zar} or $S_{\acute{e}tale}$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.
(3) The sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$ on the big site $(Sch/S)_\tau$ is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).
(4) If $\tau = Zariski$ or $\tau = \acute{e}tale$ the sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$ on the small site S_{Zar} or $S_{\acute{e}tale}$ is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).

Note how we use the same notation $\mathcal{F}^a$ in each case. No confusion can really arise from this as by definition the rule that defines the sheaf $\mathcal{F}^a$ is independent of the site we choose to look at.

03FG Remark 35.8.3. In Topologies, Lemma 34.3.12 we have seen that the small Zariski site of a scheme S is equivalent to S as a topological space in the sense that the categories of sheaves are naturally equivalent. Now that S_{Zar} is also endowed with a structure sheaf $\mathcal{O}$ we see that sheaves of modules on the ringed site $(S_{Zar}, \mathcal{O})$ agree with sheaves of modules on the ringed space $(S, \mathcal{O}_S)$.

070R Remark 35.8.4. Let $f : T \to S$ be a morphism of schemes. Each of the morphisms of sites f_{sites} listed in Topologies, Section 34.11 becomes a morphism of ringed sites. Namely, each of these morphisms of sites $f_{sites} : (Sch/T)_\tau \to (Sch/S)_{\tau'}$, or $f_{sites} : (Sch/S)_\tau \to S_{\tau'}$ is given by the continuous functor $S'/S \mapsto T \times_S S'/S$. Hence, given S'/S we let

$$f_{sites}^\sharp : \mathcal{O}(S'/S) \longrightarrow f_{sites,*}\mathcal{O}(S'/S) = \mathcal{O}(S \times_S S'/T)$$

be the usual map $\text{pr}_{S'}^\sharp : \mathcal{O}(S') \to \mathcal{O}(T \times_S S')$. Similarly, the morphism $i_f : Sh(T_\tau) \to Sh((Sch/S)_\tau)$ for $\tau \in \{Zar, \acute{e}tale\}$, see Topologies, Lemmas 34.3.13 and 34.4.13, becomes a morphism of ringed topoi because $i_f^{-1}\mathcal{O} = \mathcal{O}$. Here are some special cases:

(1) The morphism of big sites $f_{big} : (Sch/X)_{fppf} \to (Sch/Y)_{fppf}$, becomes a morphism of ringed sites

$$(f_{big}, f_{big}^\sharp) : ((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow ((Sch/Y)_{fppf}, \mathcal{O}_Y)$$

as in Modules on Sites, Definition 18.6.1. Similarly for the big syntomic, smooth, étale and Zariski sites.
(2) The morphism of small sites $f_{small} : X_{\acute{e}tale} \to Y_{\acute{e}tale}$ becomes a morphism of ringed sites

$$(f_{small}, f_{small}^\sharp) : (X_{\acute{e}tale}, \mathcal{O}_X) \longrightarrow (Y_{\acute{e}tale}, \mathcal{O}_Y)$$

as in Modules on Sites, Definition 18.6.1. Similarly for the small Zariski site.

Let S be a scheme. It is clear that given an $\mathcal{O}$-module on (say) $(Sch/S)_{Zar}$ the pullback to (say) $(Sch/S)_{fppf}$ is just the fppf-sheafification. To see what happens when comparing big and small sites we have the following.

070S Lemma 35.8.5. Let S be a scheme. Denote

$$\begin{array}{lcll} \text{id}_{\tau,Zar} & : & (Sch/S)_\tau \to S_{Zar}, & \tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\} \\ \text{id}_{\tau,\acute{e}tale} & : & (Sch/S)_\tau \to S_{\acute{e}tale}, & \tau \in \{\acute{e}tale, smooth, syntomic, fppf\} \\ \text{id}_{small,\acute{e}tale,Zar} & : & S_{\acute{e}tale} \to S_{Zar}, & \end{array}$$

the morphisms of ringed sites of Remark 35.8.4. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules which we view a sheaf of $\mathcal{O}$-modules on S_{Zar}. Then

(1) $(\mathrm{id}_{\tau,Zar})^*\mathcal{F}$ is the τ-sheafification of the Zariski sheaf
$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$
on $(Sch/S)_\tau$, and

(2) $(\mathrm{id}_{small,\acute{e}tale,Zar})^*\mathcal{F}$ is the étale sheafification of the Zariski sheaf
$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$
on $S_{\acute{e}tale}$.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules on $S_{\acute{e}tale}$. Then

(3) $(\mathrm{id}_{\tau,\acute{e}tale})^*\mathcal{G}$ is the τ-sheafification of the étale sheaf
$$(f : T \to S) \longmapsto \Gamma(T, f_{small}^*\mathcal{G})$$
where $f_{small} : T_{\acute{e}tale} \to S_{\acute{e}tale}$ is the morphism of ringed small étale sites of Remark 35.8.4.

Proof. Proof of (1). We first note that the result is true when $\tau = Zar$ because in that case we have the morphism of topoi $i_f : Sh(T_{Zar}) \to Sh((Sch/S)_{Zar})$ such that $\mathrm{id}_{\tau,Zar} \circ i_f = f_{small}$ as morphisms $T_{Zar} \to S_{Zar}$, see Topologies, Lemmas 34.3.13 and 34.3.17. Since pullback is transitive (see Modules on Sites, Lemma 18.13.3) we see that $i_f^*(\mathrm{id}_{\tau,Zar})^*\mathcal{F} = f_{small}^*\mathcal{F}$ as desired. Hence, by the remark preceding this lemma we see that $(\mathrm{id}_{\tau,Zar})^*\mathcal{F}$ is the τ-sheafification of the presheaf $T \mapsto \Gamma(T, f^*\mathcal{F})$.

The proof of (3) is exactly the same as the proof of (1), except that it uses Topologies, Lemmas 34.4.13 and 34.4.17. We omit the proof of (2). $\square$

03FH Remark 35.8.6. Remark 35.8.4 and Lemma 35.8.5 have the following applications:

(1) Let S be a scheme. The construction $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback under the morphism of ringed sites $\mathrm{id}_{\tau,Zar} : ((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$ or the morphism $\mathrm{id}_{small,\acute{e}tale,Zar} : (S_{\acute{e}tale}, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$.

(2) Let $f : X \to Y$ be a morphism of schemes. For any of the morphisms f_{sites} of ringed sites of Remark 35.8.4 we have
$$(f^*\mathcal{F})^a = f_{sites}^*\mathcal{F}^a.$$
This follows from (1) and the fact that pullbacks are compatible with compositions of morphisms of ringed sites, see Modules on Sites, Lemma 18.13.3.

03DV Lemma 35.8.7. Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$.

(1) The sheaf $\mathcal{F}^a$ is a quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$, as defined in Modules on Sites, Definition 18.23.1.

(2) If $\tau = Zariski$ or $\tau = \acute{e}tale$, then the sheaf $\mathcal{F}^a$ is a quasi-coherent $\mathcal{O}$-module on S_{Zar} or $S_{\acute{e}tale}$ as defined in Modules on Sites, Definition 18.23.1.

Proof. Let $\{S_i \to S\}$ be a Zariski covering such that we have exact sequences
$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}_{S_i} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}_{S_i} \longrightarrow \mathcal{F} \longrightarrow 0$$

for some index sets K_i and J_i. This is possible by the definition of a quasi-coherent sheaf on a ringed space (See Modules, Definition 17.10.1).

Proof of (1). Let $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$. It is clear that $\mathcal{F}^a|_{(Sch/S_i)_\tau}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 18.23.3.

Proof of (2). Let $\tau = \acute{e}tale$. It is clear that $\mathcal{F}^a|_{(S_i)_{\acute{e}tale}}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}|_{(S_i)_{\acute{e}tale}} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}|_{(S_i)_{\acute{e}tale}} \longrightarrow \mathcal{F}^a|_{(S_i)_{\acute{e}tale}} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 18.23.3. The case $\tau = Zariski$ is similar (actually, it is really tautological since the corresponding ringed topoi agree). $\square$

0GN7 Lemma 35.8.8. Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Each of the functors $\mathcal{F} \mapsto \mathcal{F}^a$ of Definition 35.8.2

$$QCoh(\mathcal{O}_S) \to QCoh((Sch/S)_\tau, \mathcal{O}) \quad \text{or} \quad QCoh(\mathcal{O}_S) \to QCoh(S_\tau, \mathcal{O})$$

is fully faithful.

Proof. (By Lemma 35.8.7 we do indeed get functors as indicated.) We may and do identify $\mathcal{O}_S$-modules on S with modules on $(S_{Zar}, \mathcal{O}_S)$. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ on quasi-coherent modules $\mathcal{F}$ is given by pullback by a morphism f of ringed sites, see Remark 35.8.6. In each case the functor f_* is given by restriction along the inclusion functor $S_{Zar} \to S_\tau$ or $S_{Zar} \to (Sch/S)_\tau$ (see discussion of how these morphisms of sites are defined in Topologies, Section 34.11). Combining this with the description of $f^*\mathcal{F} = \mathcal{F}^a$ we see that $f_*f^*\mathcal{F} = \mathcal{F}$ provided that $\mathcal{F}$ is quasi-coherent. Then we see that

$$\mathrm{Hom}_\mathcal{O}(\mathcal{F}^a, \mathcal{G}^a) = \mathrm{Hom}_\mathcal{O}(f^*\mathcal{F}, f^*\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, f_*f^*\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$$

as desired. $\square$

03DX Proposition 35.8.9. Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$.

(1) The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories
$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O})$$
between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the big τ site of S.

(2) Let $\tau = Zariski$ or $\tau = \acute{e}tale$. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories
$$QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$
between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the small τ site of S.

Proof. We have seen in Lemma 35.8.7 that the functor is well defined. By Lemma 35.8.8 the functor is fully faithful. To finish the proof we will show that a quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ gives rise to a descent datum for quasi-coherent sheaves relative to a τ-covering of S. Having produced this descent datum we will appeal to Proposition 35.5.2 to get the corresponding quasi-coherent sheaf on S.

Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}$-modules on the big τ site of S. By Modules on Sites, Definition 18.23.1 there exists a τ-covering $\{S_i \to S\}_{i \in I}$ of S such that each of the restrictions $\mathcal{G}|_{(Sch/S_i)_\tau}$ has a global presentation

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau} \longrightarrow 0$$

for some index sets J_i and K_i. We claim that this implies that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is $\mathcal{F}_i^a$ for some quasi-coherent sheaf $\mathcal{F}_i$ on S_i. Namely, this is clear for the direct sums $\bigoplus_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau}$ and $\bigoplus_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau}$. Hence we see that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is a cokernel of a map $\varphi : \mathcal{K}_i^a \to \mathcal{L}_i^a$ for some quasi-coherent sheaves $\mathcal{K}_i$, $\mathcal{L}_i$ on S_i. By the fully faithfulness of $(\ )^a$ we see that $\varphi = \phi^a$ for some map of quasi-coherent sheaves $\phi : \mathcal{K}_i \to \mathcal{L}_i$ on S_i. Then it is clear that $\mathcal{G}|_{(Sch/S_i)_\tau} \cong \text{Coker}(\phi)^a$ as claimed.

Since $\mathcal{G}$ lives on all of the category $(Sch/S)_\tau$ we see that

$$(\text{pr}_0^*\mathcal{F}_i)^a \cong \mathcal{G}|_{(Sch/(S_i \times_S S_j))_\tau} \cong (\text{pr}_1^*\mathcal{F})^a$$

as $\mathcal{O}$-modules on $(Sch/(S_i \times_S S_j))_\tau$. Hence, using fully faithfulness again we get canonical isomorphisms

$$\phi_{ij} : \text{pr}_0^*\mathcal{F}_i \longrightarrow \text{pr}_1^*\mathcal{F}_j$$

of quasi-coherent modules over $S_i \times_S S_j$. We omit the verification that these satisfy the cocycle condition. Since they do we see by effectivity of descent for quasi-coherent sheaves and the covering $\{S_i \to S\}$ (Proposition 35.5.2) that there exists a quasi-coherent sheaf $\mathcal{F}$ on S with $\mathcal{F}|_{S_i} \cong \mathcal{F}_i$ compatible with the given descent data. In other words we are given $\mathcal{O}$-module isomorphisms

$$\phi_i : \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau}$$

which agree over $S_i \times_S S_j$. Hence, since $\mathcal{H}om_\mathcal{O}(\mathcal{F}^a, \mathcal{G})$ is a sheaf (Modules on Sites, Lemma 18.27.1), we conclude that there is a morphism of $\mathcal{O}$-modules $\mathcal{F}^a \to \mathcal{G}$ recovering the isomorphisms ϕ_i above. Hence this is an isomorphism and we win.

The case of the sites $S_{\acute{e}tale}$ and S_{Zar} is proved in the exact same manner. $\square$

05VG Lemma 35.8.10. Let S be a scheme. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Let $\mathcal{P}$ be one of the properties of modules[4] defined in Modules on Sites, Definitions 18.17.1, 18.23.1, and 18.28.1. The equivalences of categories

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O}) \quad \text{and} \quad QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 35.8.9 have the property

$$\mathcal{F} \text{ has } \mathcal{P} \Leftrightarrow \mathcal{F}^a \text{ has } \mathcal{P} \text{ as an } \mathcal{O}\text{-module}$$

except (possibly) when $\mathcal{P}$ is "locally free" or "coherent". If $\mathcal{P}$ ="coherent" the equivalence holds for $QCoh(\mathcal{O}_S) \to QCoh(S_\tau, \mathcal{O})$ when S is locally Noetherian and τ is Zariski or étale.

Proof. This is immediate for the global properties, i.e., those defined in Modules on Sites, Definition 18.17.1. For the local properties we can use Modules on Sites, Lemma 18.23.3 to translate "$\mathcal{F}^a$ has $\mathcal{P}$" into a property on the members of a covering of X. Hence the result follows from Lemmas 35.7.1, 35.7.3, 35.7.4, 35.7.5,

[4]The list is: free, finite free, generated by global sections, generated by r global sections, generated by finitely many global sections, having a global presentation, having a global finite presentation, locally free, finite locally free, locally generated by sections, locally generated by r sections, finite type, of finite presentation, coherent, or flat.

and 35.7.6. Being coherent for a quasi-coherent module is the same as being of finite type over a locally Noetherian scheme (see Cohomology of Schemes, Lemma 30.9.1) hence this reduces to the case of finite type modules (details omitted). $\square$

35.9. Cohomology of quasi-coherent modules and topologies

0GN8 In this section we prove that cohomology of quasi-coherent modules is independent of the choice of topology.

03FI **Lemma 35.9.1.** Let S be a scheme. Let

(a) $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$ and $\mathcal{C} = (Sch/S)_\tau$, or
(b) let $\tau = étale$ and $\mathcal{C} = S_{étale}$, or
(c) let $\tau = Zariski$ and $\mathcal{C} = S_{Zar}$.

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $U \in \text{Ob}(\mathcal{C})$ be affine. Let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a standard affine τ-covering in $\mathcal{C}$. Then

(1) $\mathcal{V} = \{\coprod_{i=1,\ldots,n} U_i \to U\}$ is a τ-covering of U,
(2) $\mathcal{U}$ is a refinement of $\mathcal{V}$, and
(3) the induced map on Čech complexes (Cohomology on Sites, Equation (21.8.2.1))
$$\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$
is an isomorphism of complexes.

Proof. This follows because
$$(\coprod\nolimits_{i_0=1,\ldots,n} U_{i_0}) \times_U \ldots \times_U (\coprod\nolimits_{i_p=1,\ldots,n} U_{i_p}) = \coprod\nolimits_{i_0,\ldots,i_p \in \{1,\ldots,n\}} U_{i_0} \times_U \ldots \times_U U_{i_p}$$
and the fact that $\mathcal{F}(\coprod_a V_a) = \prod_a \mathcal{F}(V_a)$ since disjoint unions are τ-coverings. $\square$

03FJ **Lemma 35.9.2.** Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let τ, $\mathcal{C}$, U, $\mathcal{U}$ be as in Lemma 35.9.1. Then there is an isomorphism of complexes
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a) \cong s((A/R)_\bullet \otimes_R M)$$
(see Section 35.3) where $R = \Gamma(U, \mathcal{O}_U)$, $M = \Gamma(U, \mathcal{F}^a)$ and $R \to A$ is a faithfully flat ring map. In particular
$$\check{H}^p(\mathcal{U}, \mathcal{F}^a) = 0$$
for all $p \geq 1$.

Proof. By Lemma 35.9.1 we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a)$ is isomorphic to $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^a)$ where $\mathcal{V} = \{V \to U\}$ with $V = \coprod_{i=1,\ldots n} U_i$ affine also. Set $A = \Gamma(V, \mathcal{O}_V)$. Since $\{V \to U\}$ is a τ-covering we see that $R \to A$ is faithfully flat. On the other hand, by definition of $\mathcal{F}^a$ we have that the degree p term $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{F}^a)$ is
$$\Gamma(V \times_U \ldots \times_U V, \mathcal{F}^a) = \Gamma(\text{Spec}(A \otimes_R \ldots \otimes_R A), \mathcal{F}^a) = A \otimes_R \ldots \otimes_R A \otimes_R M$$
We omit the verification that the maps of the Čech complex agree with the maps in the complex $s((A/R)_\bullet \otimes_R M)$. The vanishing of cohomology is Lemma 35.3.6. $\square$

03DW **Proposition 35.9.3.** Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$.

(1) There is a canonical isomorphism
$$H^q(S, \mathcal{F}) = H^q((Sch/S)_\tau, \mathcal{F}^a).$$

(2) There are canonical isomorphisms

$$H^q(S, \mathcal{F}) = H^q(S_{Zar}, \mathcal{F}^a) = H^q(S_{\acute{e}tale}, \mathcal{F}^a).$$

Proof. The result for $q = 0$ is clear from the definition of $\mathcal{F}^a$. Let $\mathcal{C} = (Sch/S)_\tau$, or $\mathcal{C} = S_{\acute{e}tale}$, or $\mathcal{C} = S_{Zar}$.

We are going to apply Cohomology on Sites, Lemma 21.10.9 with $\mathcal{F} = \mathcal{F}^a$, $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ the set of affine schemes in $\mathcal{C}$, and $\mathrm{Cov} \subset \mathrm{Cov}_\mathcal{C}$ the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Lemma 35.9.2. Hence we conclude that $H^p(U, \mathcal{F}^a) = 0$ for every affine object U of $\mathcal{C}$.

Next, let $U \in \mathrm{Ob}(\mathcal{C})$ be any separated object. Denote $f : U \to S$ the structure morphism. Let $U = \bigcup U_i$ be an affine open covering. We may also think of this as a τ-covering $\mathcal{U} = \{U_i \to U\}$ of U in $\mathcal{C}$. Note that $U_{i_0} \times_U \ldots \times_U U_{i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ is affine as we assumed U separated. By Cohomology on Sites, Lemma 21.10.7 and the result above we see that

$$H^p(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = H^p(U, f^*\mathcal{F})$$

the last equality by Cohomology of Schemes, Lemma 30.2.6. In particular, if S is separated we can take $U = S$ and $f = \mathrm{id}_S$ and the proposition is proved. We suggest the reader skip the rest of the proof (or rewrite it to give a clearer exposition).

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on S. Choose an injective resolution $\mathcal{F}^a \to \mathcal{J}^\bullet$ on $\mathcal{C}$. Denote $\mathcal{J}^n|_S$ the restriction of $\mathcal{J}^n$ to opens of S; this is a sheaf on the topological space S as open coverings are τ-coverings. We get a complex

$$0 \to \mathcal{F} \to \mathcal{J}^0|_S \to \mathcal{J}^1|_S \to \ldots$$

which is exact since its sections over any affine open $U \subset S$ is exact (by the vanishing of $H^p(U, \mathcal{F}^a)$, $p > 0$ seen above). Hence by Derived Categories, Lemma 13.18.6 there exists map of complexes $\mathcal{J}^\bullet|_S \to \mathcal{I}^\bullet$ which in particular induces a map

$$R\Gamma(\mathcal{C}, \mathcal{F}^a) = \Gamma(S, \mathcal{J}^\bullet) \longrightarrow \Gamma(S, \mathcal{I}^\bullet) = R\Gamma(S, \mathcal{F}).$$

Taking cohomology gives the map $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ which we have to prove is an isomorphism. Let $\mathcal{U} : S = \bigcup U_i$ be an affine open covering which we may think of as a τ-covering also. By the above we get a map of double complexes

$$\check{C}^\bullet(\mathcal{U}, \mathcal{J}) = \check{C}^\bullet(\mathcal{U}, \mathcal{J}|_S) \longrightarrow \check{C}^\bullet(\mathcal{U}, \mathcal{I}).$$

This map induces a map of spectral sequences

$${}^\tau E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a)) \longrightarrow E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

The first spectral sequence converges to $H^{p+q}(\mathcal{C}, \mathcal{F})$ and the second to $H^{p+q}(S, \mathcal{F})$. On the other hand, we have seen that the induced maps ${}^\tau E_2^{p,q} \to E_2^{p,q}$ are bijections (as all the intersections are separated being opens in affines). Whence also the maps $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ are isomorphisms, and we win. $\square$

03LC Proposition 35.9.4. Let $f : T \to S$ be a morphism of schemes.

(1) The equivalences of categories of Proposition 35.8.9 are compatible with pullback. More precisely, we have $f^*(\mathcal{G}^a) = (f^*\mathcal{G})^a$ for any quasi-coherent sheaf $\mathcal{G}$ on S.
(2) The equivalences of categories of Proposition 35.8.9 part (1) are not compatible with pushforward in general.

(3) If f is quasi-compact and quasi-separated, and $\tau \in \{Zariski, étale\}$ then f_* and $f_{small,*}$ preserve quasi-coherent sheaves and the diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_T) & \xrightarrow{f_*} & QCoh(\mathcal{O}_S) \\ \downarrow{\scriptstyle \mathcal{F}\mapsto\mathcal{F}^a} & & \downarrow{\scriptstyle \mathcal{G}\mapsto\mathcal{G}^a} \\ QCoh(T_\tau, \mathcal{O}) & \xrightarrow{f_{small,*}} & QCoh(S_\tau, \mathcal{O}) \end{array}$$

is commutative, i.e., $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$.

Proof. Part (1) follows from the discussion in Remark 35.8.6. Part (2) is just a warning, and can be explained in the following way: First the statement cannot be made precise since f_* does not transform quasi-coherent sheaves into quasi-coherent sheaves in general. Even if this is the case for f (and any base change of f), then the compatibility over the big sites would mean that formation of $f_*\mathcal{F}$ commutes with any base change, which does not hold in general. An explicit example is the quasi-compact open immersion $j : X = \mathbf{A}^2_k \setminus \{0\} \to \mathbf{A}^2_k = Y$ where k is a field. We have $j_*\mathcal{O}_X = \mathcal{O}_Y$ but after base change to $\text{Spec}(k)$ by the 0 map we see that the pushforward is zero.

Let us prove (3) in case $\tau = étale$. Note that f, and any base change of f, transforms quasi-coherent sheaves into quasi-coherent sheaves, see Schemes, Lemma 26.24.1. The equality $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$ means that for any étale morphism $g : U \to S$ we have $\Gamma(U, g^*f_*\mathcal{F}) = \Gamma(U \times_S T, (g')^*\mathcal{F})$ where $g' : U \times_S T \to T$ is the projection. This is true by Cohomology of Schemes, Lemma 30.5.2. □

071N Lemma 35.9.5. Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on T. For either the étale or Zariski topology, there are canonical isomorphisms $R^if_{small,*}(\mathcal{F}^a) = (R^if_*\mathcal{F})^a$.

Proof. We prove this for the étale topology; we omit the proof in the case of the Zariski topology. By Cohomology of Schemes, Lemma 30.4.5 the sheaves $R^if_*\mathcal{F}$ are quasi-coherent so that the assertion makes sense. The sheaf $R^if_{small,*}\mathcal{F}^a$ is the sheaf associated to the presheaf

$$U \longmapsto H^i(U \times_S T, \mathcal{F}^a)$$

where $g : U \to S$ is an object of $S_{étale}$, see Cohomology on Sites, Lemma 21.7.4. By our conventions the right hand side is the étale cohomology of the restriction of $\mathcal{F}^a$ to the localization $T_{étale}/U \times_S T$ which equals $(U \times_S T)_{étale}$. By Proposition 35.9.3 this is presheaf the same as the presheaf

$$U \longmapsto H^i(U \times_S T, (g')^*\mathcal{F}),$$

where $g' : U \times_S T \to T$ is the projection. If U is affine then this is the same as $H^0(U, R^if'_*(g')^*\mathcal{F})$, see Cohomology of Schemes, Lemma 30.4.6. By Cohomology of Schemes, Lemma 30.5.2 this is equal to $H^0(U, g^*R^if_*\mathcal{F})$ which is the value of $(R^if_*\mathcal{F})^a$ on U. Thus the values of the sheaves of modules $R^if_{small,*}(\mathcal{F}^a)$ and $(R^if_*\mathcal{F})^a$ on every affine object of $S_{étale}$ are canonically isomorphic which implies they are canonically isomorphic. □

35.10. Quasi-coherent sheaves and topologies, II

0GN9 We continue the discussion comparing quasi-coherent modules on a scheme S with quasi-coherent modules on any of the sites associated to S in the chapter on topologies.

0GNA Lemma 35.10.1. In Lemma 35.8.5 the morphism of ringed sites $\text{id}_{small,étale,Zar} : S_{étale} \to S_{Zar}$ is flat.

Proof. Let us denote $\epsilon = \text{id}_{small,étale,Zar}$ and $\mathcal{O}_{étale}$ and $\mathcal{O}_{Zar}$ the structure sheaves on $S_{étale}$ and S_{Zar}. We have to show that $\mathcal{O}_{étale}$ is a flat $\epsilon^{-1}\mathcal{O}_{Zar}$-module. Recall that étale morphisms are open, see Morphisms, Lemma 29.36.13. It follows (from the construction of pullback on sheaves) that $\epsilon^{-1}\mathcal{O}_{Zar}$ is the sheafification of the presheaf $\mathcal{O}'$ on $S_{étale}$ which sends an étale morphism $f : V \to S$ to $\mathcal{O}_S(f(V))$. If both V and $U = f(V) \subset S$ are affine, then $V \to U$ is an étale morphism of affines, hence corresponds to an étale ring map. Since étale ring maps are flat, we see that $\mathcal{O}_S(U) = \mathcal{O}'(V) \to \mathcal{O}_{étale}(V) = \mathcal{O}_V(V)$ is flat. Finally, for every étale morphism $f : V \to S$, i.e., object of $S_{étale}$, there is an affine open covering $V = \bigcup V_i$ such that $f(V_i)$ is an affine open in S for all i[5]. Thus the result by Modules on Sites, Lemma 18.28.4. □

06VE Lemma 35.10.2. Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. The functors

$$QCoh(\mathcal{O}_S) \longrightarrow \text{Mod}((Sch/S)_\tau, \mathcal{O}) \quad \text{and} \quad QCoh(\mathcal{O}_S) \longrightarrow \text{Mod}(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 35.8.9 are

(1) fully faithful,
(2) compatible with direct sums,
(3) compatible with colimits,
(4) right exact,
(5) exact as a functor $QCoh(\mathcal{O}_S) \to \text{Mod}(S_{étale}, \mathcal{O})$,
(6) not exact as a functor $QCoh(\mathcal{O}_S) \to \text{Mod}((Sch/S)_\tau, \mathcal{O})$ in general,
(7) given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$, $\mathcal{G}$ we have $(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G})^a = \mathcal{F}^a \otimes_{\mathcal{O}} \mathcal{G}^a$,
(8) if $\tau = étale$ or $\tau = Zariski$, given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation we have $(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}))^a = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a)$ in $\text{Mod}(S_\tau, \mathcal{O})$,
(9) given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$, $\mathcal{G}$ we do not have $(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}))^a = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a)$ in $\text{Mod}((Sch/S)_\tau, \mathcal{O})$ in general even if $\mathcal{F}$ is of finite presentation, and
(10) given a short exact sequence $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ of $\mathcal{O}$-modules then $\mathcal{E}$ is quasi-coherent[6], i.e., $\mathcal{E}$ is in the essential image of the functor.

Proof. Part (1) we saw in Proposition 35.8.9.

We have seen in Schemes, Section 26.24 that a colimit of quasi-coherent sheaves on a scheme is a quasi-coherent sheaf. Moreover, in Remark 35.8.6 we saw that $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for a morphism of ringed sites, hence commutes

[5] Namely, for $y \in V$, we pick an affine open $y \in V' \subset V$ with $f(V')$ contained in an affine open $U \subset S$. Then we pick an affine open $f(y) \in U' \subset f(V')$. Then $V'' = f^{-1}(U') \subset V'$ is affine as it is equal to $U' \times_U V'$ and $f(V'') = U'$ is affine too.

[6] Warning: This is misleading. See part (6).

with all colimits, see Modules on Sites, Lemma 18.14.3. Thus (3) and its special case (3) hold.

This also shows that the functor is right exact (i.e., commutes with finite colimits), hence (4).

The functor $QCoh(\mathcal{O}_S) \to QCoh(S_{\acute{e}tale}, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is left exact because an étale morphism is flat, see Morphisms, Lemma 29.36.12. This proves (5).

To see (6), suppose that $S = \text{Spec}(\mathbf{Z})$. Then $2 : \mathcal{O}_S \to \mathcal{O}_S$ is injective but the associated map of $\mathcal{O}$-modules on $(Sch/S)_\tau$ isn't injective because $2 : \mathbf{F}_2 \to \mathbf{F}_2$ isn't injective and $\text{Spec}(\mathbf{F}_2)$ is an object of $(Sch/S)_\tau$.

Part (7) holds because, as mentioned above, the functor $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for a morphism of ringed sites and such commute with tensor products by Modules on Sites, Lemma 18.26.2.

Part (8) is obvious if $\tau = Zariski$ because the category of $\mathcal{O}$-modules on S_{Zar} is the same as the category of $\mathcal{O}_S$-modules on the topological space S. If $\tau = \acute{e}tale$ then (8) holds because, as mentioned above, the functor $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for the flat morphism of ringed sites $(S_{\acute{e}tale}, \mathcal{O}) \to (S_{Zar}, \mathcal{O}_S)$, see Lemma 35.10.1. Pullback by flat morphisms of ringed sites commutes with taking internal hom out of a finitely presented module by Modules on Sites, Lemma 18.31.4.

To see (9), suppose that $S = \text{Spec}(\mathbf{Z})$. Let $\mathcal{F} = \text{Coker}(2 : \mathcal{O}_S \to \mathcal{O}_S)$ and $\mathcal{G} = \mathcal{O}_S$. Then $\mathcal{F}^a = \text{Coker}(2 : \mathcal{O} \to \mathcal{O})$ and $\mathcal{G}^a = \mathcal{O}$. Hence $\mathcal{H}om_\mathcal{O}(\mathcal{F}^a, \mathcal{G}^a) = \mathcal{O}[2]$ is equal to the 2-torsion in $\mathcal{O}$, which is not zero, see proof of (6). On the other hand, the module $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})$ is zero.

Proof of (10). Let $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_2$ quasi-coherent on S. Consider the restriction

$$0 \to \mathcal{F}_1 \to \mathcal{E}|_{S_{Zar}} \to \mathcal{F}_2$$

to S_{Zar}. By Proposition 35.9.3 we see that on any affine $U \subset S$ we have $H^1(U, \mathcal{F}_1^a) = H^1(U, \mathcal{F}_1) = 0$. Hence the sequence above is also exact on the right. By Schemes, Section 26.24 we conclude that $\mathcal{F} = \mathcal{E}|_{S_{Zar}}$ is quasi-coherent. Thus we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathcal{F}_1^a \arrow[r] \arrow[d] & \mathcal{F}^a \arrow[r] \arrow[d] & \mathcal{F}_2^a \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1^a \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F}_2^a \arrow[r] & 0
\end{tikzcd}$$

To finish the proof it suffices to show that the top row is also right exact. To do this, denote once more $U = \text{Spec}(A) \subset S$ an affine open of S. We have seen above that $0 \to \mathcal{F}_1(U) \to \mathcal{E}(U) \to \mathcal{F}_2(U) \to 0$ is exact. For any affine scheme V/U, $V = \text{Spec}(B)$ the map $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ is injective. We have $\mathcal{F}_1^a(V) = \mathcal{F}_1(U) \otimes_A B$ by definition. The injection $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ factors as

$$\mathcal{F}_1(U) \otimes_A B \to \mathcal{E}(U) \otimes_A B \to \mathcal{E}(U)$$

Considering A-algebras B of the form $B = A \oplus M$ we see that $\mathcal{F}_1(U) \to \mathcal{E}(U)$ is universally injective (see Algebra, Definition 10.82.1). Since $\mathcal{E}(U) = \mathcal{F}(U)$ we conclude that $\mathcal{F}_1 \to \mathcal{F}$ remains injective after any base change, or equivalently that $\mathcal{F}_1^a \to \mathcal{F}^a$ is injective. $\square$

0GNB Lemma 35.10.3. Let S be a scheme. The category $QCoh(S_{étale}, \mathcal{O})$ of quasi-coherent modules on $S_{étale}$ has the following properties:

(1) Any direct sum of quasi-coherent sheaves is quasi-coherent.
(2) Any colimit of quasi-coherent sheaves is quasi-coherent.
(3) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.
(4) Given a short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.
(5) Given two quasi-coherent $\mathcal{O}$-modules the tensor product is quasi-coherent.
(6) Given two quasi-coherent $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation. then the internal hom $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.

Proof. The corresponding facts hold for quasi-coherent modules on the scheme S, see Schemes, Section 26.24. The proof will be to use Lemma 35.10.2 to transfer these truths to $S_{étale}$.

Proof of (1). Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $QCoh(S_{étale}, \mathcal{O})$. Write $\mathcal{F}_i = \mathcal{G}_i^a$ for some quasi-coherent modules $\mathcal{G}_i$ on S. Then $\bigoplus \mathcal{F}_i = (\bigoplus \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (2). Let $\mathcal{I} \to QCoh(S_{étale}, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (3). Let $a : \mathcal{F} \to \mathcal{F}'$ be an arrow of $QCoh(S_{étale}, \mathcal{O})$. Write $a = b^a$ for some map $b : \mathcal{G} \to \mathcal{G}'$ of quasi-coherent modules on S. By the lemma cited we have $\text{Ker}(a) = \text{Ker}(b)^a$ and $\text{Coker}(a) = \text{Coker}(b)^a$ and we conclude.

Proof of (4). This follows from (3) except in the case when we know $\mathcal{F}_1$ and $\mathcal{F}_3$ are quasi-coherent. In this case write $\mathcal{F}_1 = \mathcal{G}_1^a$ and $\mathcal{F}_3 = \mathcal{G}_3^a$ with $\mathcal{G}_i$ quasi-coherent on S. By Lemma 35.10.2 part (10) we conclude.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{F}'$ be in $QCoh(S_{étale}, \mathcal{O})$. Write $\mathcal{F} = \mathcal{G}^a$ and $\mathcal{F}' = (\mathcal{G}')^a$ with $\mathcal{G}$ and $\mathcal{G}'$ quasi-coherent on S. By the lemma cited we have $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{F}' = (\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{G}')^a$ and we conclude.

Proof of (6). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh(S_{étale}, \mathcal{O})$ with $\mathcal{F}$ of finite presentation. Write $\mathcal{F} = \mathcal{H}^a$ and $\mathcal{G} = (\mathcal{I})^a$ with $\mathcal{H}$ and $\mathcal{I}$ quasi-coherent on S. By Lemma 35.8.10 we see that $\mathcal{H}$ is of finite presentation. By Lemma 35.10.2 part (8) we have $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = (\mathcal{H}om_{\mathcal{O}_S}(\mathcal{H}, \mathcal{I}))^a$ and we conclude. $\square$

0GNC Lemma 35.10.4. Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. The category $QCoh((Sch/S)_\tau, \mathcal{O})$ of quasi-coherent modules on $(Sch/S)_\tau$ has the following properties:

(1) Any direct sum of quasi-coherent sheaves is quasi-coherent.
(2) Any colimit of quasi-coherent sheaves is quasi-coherent.
(3) The cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.
(4) Given a short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if $\mathcal{F}_1$ and $\mathcal{F}_3$ are quasi-coherent so is $\mathcal{F}_2$.
(5) Given two quasi-coherent $\mathcal{O}$-modules the tensor product is quasi-coherent.
(6) Given two quasi-coherent $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is finite locally free, the internal hom $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.

Proof. The corresponding facts hold for quasi-coherent modules on the scheme S, see Schemes, Section 26.24. The proof will be to use Lemma 35.10.2 to transfer these truths to $(Sch/S)_\tau$.

Proof of (1). Let $\mathcal{F}_i$, $i \in I$ be a family of objects of $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $\mathcal{F}_i = \mathcal{G}_i^a$ for some quasi-coherent modules $\mathcal{G}_i$ on S. Then $\bigoplus \mathcal{F}_i = (\bigoplus \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (2). Let $\mathcal{I} \to QCoh((Sch/S)_\tau, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ by the lemma cited and we conclude.

Proof of (3). Let $a : \mathcal{F} \to \mathcal{F}'$ be an arrow of $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $a = b^a$ for some map $b : \mathcal{G} \to \mathcal{G}'$ of quasi-coherent modules on S. By the lemma cited we have $\operatorname{Coker}(a) = \operatorname{Coker}(b)^a$ (because a cokernel is a colimit) and we conclude.

Proof of (4). Write $\mathcal{F}_1 = \mathcal{G}_1^a$ and $\mathcal{F}_3 = \mathcal{G}_3^a$ with $\mathcal{G}_i$ quasi-coherent on S. By Lemma 35.10.2 part (10) we conclude.

Proof of (5). Let $\mathcal{F}$ and $\mathcal{F}'$ be in $QCoh((Sch/S)_\tau, \mathcal{O})$. Write $\mathcal{F} = \mathcal{G}^a$ and $\mathcal{F}' = (\mathcal{G}')^a$ with $\mathcal{G}$ and $\mathcal{G}'$ quasi-coherent on S. By the lemma cited we have $\mathcal{F} \otimes_\mathcal{O} \mathcal{F}' = (\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{G}')^a$ and we conclude.

Proof of (6). Write $\mathcal{F} = \mathcal{H}^a$ for some quasi-coherent $\mathcal{O}_S$-module. By Lemma 35.8.10 we see that $\mathcal{H}$ is finite locally free. The problem is Zariski local on S (details omitted) hence we may assume $\mathcal{H} = \mathcal{O}_S^{\oplus n}$ is finite free. Then $\mathcal{F} = \mathcal{O}^{\oplus n}$ and $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) = \mathcal{G}^{\oplus n}$ is quasi-coherent. ☐

0GND Example 35.10.5. Let S be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent modules on $(Sch/S)_\tau$ for one of the topologies τ considered in Lemma 35.10.4. In general it is not the case that $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$ is quasi-coherent even if $\mathcal{F}$ is of finite presentation. Namely, say $S = \operatorname{Spec}(\mathbf{Z})$, $\mathcal{F} = \operatorname{Coker}(2 : \mathcal{O} \to \mathcal{O})$, and $\mathcal{G} = \mathcal{O}$. Then $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G}) = \mathcal{O}[2]$ is equal to the 2-torsion in $\mathcal{O}$, which is not quasi-coherent.

0GNE Lemma 35.10.6. Let S be a scheme.

(1) The category $QCoh((Sch/S)_{fppf}, \mathcal{O})$ has colimits and they agree with colimits in the categories $\operatorname{Mod}((Sch/S)_{Zar}, \mathcal{O})$, $\operatorname{Mod}((Sch/S)_{étale}, \mathcal{O})$, and $\operatorname{Mod}((Sch/S)_{fppf}, \mathcal{O})$.
(2) Given $\mathcal{F}, \mathcal{G}$ in $QCoh((Sch/S)_{fppf}, \mathcal{O})$ the tensor products $\mathcal{F} \otimes_\mathcal{O} \mathcal{G}$ computed in $\operatorname{Mod}((Sch/S)_{Zar}, \mathcal{O})$, $\operatorname{Mod}((Sch/S)_{étale}, \mathcal{O})$, or $\operatorname{Mod}((Sch/S)_{fppf}, \mathcal{O})$ agree and the common value is an object of $QCoh((Sch/S)_{fppf}, \mathcal{O})$.
(3) Given $\mathcal{F}, \mathcal{G}$ in $QCoh((Sch/S)_{fppf}, \mathcal{O})$ with $\mathcal{F}$ finite locally free (in fppf, or equivalently étale, or equivalently Zariski topology) the internal homs $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$ computed in $\operatorname{Mod}((Sch/S)_{Zar}, \mathcal{O})$, $\operatorname{Mod}((Sch/S)_{étale}, \mathcal{O})$, or $\operatorname{Mod}((Sch/S)_{fppf}, \mathcal{O})$ agree and the common value is an object of $QCoh((Sch/S)_{fppf}, \mathcal{O})$.

Proof. This lemma collects the results shown above in a slightly different manner. First of all, by Lemma 35.10.4 we already know the output of the construction in (1), (2), or (3) ends up in $QCoh((Sch/S)_\tau, \mathcal{O})$. It remains to show in each case that the result is independent of the topology used. The key to this is that the equivalence $QCoh(\mathcal{O}_S) \to QCoh((Sch/S)_\tau, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ of Proposition 35.8.9 is given by the same formula independent of the choice of the topology $\tau \in \{Zariski, étale, fppf\}$.

Proof of (1). Let $\mathcal{I} \to QCoh((Sch/S)_{fppf}, \mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Write $\mathcal{F}_i = \mathcal{G}_i^a$ so we get a diagram $\mathcal{I} \to QCoh(\mathcal{O}_S)$. Then $\operatorname{colim} \mathcal{F}_i = (\operatorname{colim} \mathcal{G}_i)^a$ in $\text{Mod}((Sch/S)_\tau, \mathcal{O})$ for $\tau \in \{Zariski, \acute{e}tale, fppf\}$ by Lemma 35.10.2. This proves (1).

Proof of (2). Write $\mathcal{F} = \mathcal{H}^a$ and $\mathcal{G} = (\mathcal{I})^a$ with $\mathcal{H}$ and $\mathcal{I}$ quasi-coherent on S. Then $\mathcal{F} \otimes_\mathcal{O} \mathcal{G} = (\mathcal{H} \otimes_\mathcal{O} \mathcal{I})^a$ in $\text{Mod}((Sch/S)_\tau, \mathcal{O})$ for $\tau \in \{Zariski, \acute{e}tale, fppf\}$ by Lemma 35.10.2. This proves (2).

Proof of (3). Let $\mathcal{F}$ and $\mathcal{G}$ be in $QCoh((Sch/S)_{fppf}, \mathcal{O})$. Write $\mathcal{F} = \mathcal{H}^a$ with $\mathcal{H}$ quasi-coherent on S. By Lemma 35.8.10 we have

$$
\begin{aligned}
\mathcal{F} \text{ finite locally free in fppf topology} &\Leftrightarrow \mathcal{H} \text{ finite locally free on } S \\
&\Leftrightarrow \mathcal{F} \text{ finite locally free in étale topology} \\
&\Leftrightarrow \mathcal{H} \text{ finite locally free on } S \\
&\Leftrightarrow \mathcal{F} \text{ finite locally free in Zariski topology}
\end{aligned}
$$

This explains the parenthetical statement of part (3). Now, if these equivalent conditions hold, then $\mathcal{H}$ is finite locally free. The construction of $\mathcal{H}om_\mathcal{O}(\mathcal{F}, \mathcal{G})$ in Modules on Sites, Section 18.27 depends only on $\mathcal{F}$ and $\mathcal{G}$ as presheaves of modules (only whether the output $\mathcal{H}om$ is a sheaf depends on whether $\mathcal{F}$ and $\mathcal{G}$ are sheaves). □

35.11. Quasi-coherent modules and affines

0GZT Let S be a scheme[7]. Let $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Recall that $(\text{Aff}/S)_\tau$ is the full subcategory of $(Sch/S)_\tau$ whose objects are affine turned into a site by declaring the coverings to be the standard τ-coverings. By Topologies, Lemmas 34.3.10, 34.4.11, 34.5.9, 34.6.9, and 34.7.11 we have an equivalence of topoi $g : Sh((\text{Aff}/S)_\tau) \to Sh((Sch/S)_\tau)$ whose pullback functor is given by restriction. Recalling that $\mathcal{O}$ denotes the structure sheaf on $(Sch/S)_\tau$, let us temporarily and pedantically denote $\mathcal{O}_{\text{Aff}}$ the restriction of $\mathcal{O}$ to $(\text{Aff}/S)_\tau$. Then we obtain an equivalence

0GZU (35.11.0.1)
$$(Sh((\text{Aff}/S)_\tau), \mathcal{O}_{\text{Aff}}) \longrightarrow (Sh((Sch/S)_\tau), \mathcal{O})$$

of ringed topoi. Having said this we can compare quasi-coherent modules as well.

0GZV **Lemma 35.11.1.** *Let S be a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_{\text{Aff}}$-modules on $(\text{Aff}/S)_{fppf}$. The following are equivalent*

1. *for every morphism $U \to U'$ of $(\text{Aff}/S)_{fppf}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}(U')} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism,*
2. *$\mathcal{F}$ is a sheaf on $(\text{Aff}/S)_{Zar}$ and a quasi-coherent module on the ringed site $((\text{Aff}/S)_{Zar}, \mathcal{O}_{\text{Aff}})$ in the sense of Modules on Sites, Definition 18.23.1,*
3. *same as in (3) for the étale topology,*
4. *same as in (3) for the smooth topology,*
5. *same as in (3) for the syntomic topology,*
6. *same as in (3) for the fppf topology,*
7. *$\mathcal{F}$ corresponds to a quasi-coherent module on $(Sch/S)_{Zar}$, $(Sch/S)_{\acute{e}tale}$, $(Sch/S)_{smooth}$, $(Sch/S)_{syntomic}$, or $(Sch/S)_{fppf}$ via the equivalence (35.11.0.1),*

[7] In this section, as in Topologies, Section 34.11, we choose our sites $(Sch/S)_\tau$ to have the same underlying category for $\tau \in \{Zariski, \acute{e}tale, smooth, syntomic, fppf\}$. Then also the sites $(\text{Aff}/S)_\tau$ have the same underlying category.

(8) $\mathcal{F}$ comes from a unique quasi-coherent $\mathcal{O}_S$-module $\mathcal{G}$ by the procedure described in Section 35.8.

Proof. Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 18.23.2) we see that the equivalence (35.11.0.1) induces an equivalence between categories of quasi-coherent modules. Proposition 35.8.9 says the topology we use to study quasi-coherent modules on Sch/S does not matter and it also tells us that (8) is the same as (7). Hence we see that (2) – (8) are all equivalent.

Assume the equivalent conditions (2) – (8) hold and let $\mathcal{G}$ be as in (8). Let $h : U \to U' \to S$ be a morphism of Aff$/S$. Denote $f : U \to S$ and $f' : U' \to S$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 26.7.3.

Assume (1) holds. To finish the proof it suffices to prove (2). Let U be an object of $(\text{Aff}/S)_{Zar}$. Say $U = \text{Spec}(R)$. A standard open covering $U = U_1 \cup \ldots \cup U_n$ is given by $U_i = D(f_i)$ for some elements $f_1, \ldots, f_n \in R$ generating the unit ideal of R. By property (1) we see that

$$\mathcal{F}(U_i) = \mathcal{F}(U) \otimes_R R_{f_i} = \mathcal{F}(U)_{f_i}$$

and

$$\mathcal{F}(U_i \cap U_j) = \mathcal{F}(U) \otimes_R R_{f_i f_j} = \mathcal{F}(U)_{f_i f_j}$$

Thus we conclude from Algebra, Lemma 10.24.1 that $\mathcal{F}$ is a sheaf on $(\text{Aff}/S)_{Zar}$. Choose a presentation

$$\bigoplus\nolimits_{k \in K} R \longrightarrow \bigoplus\nolimits_{l \in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $(\text{Aff}/S)_{Zar}$ we obtain a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}_{Aff}(U') \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}_{Aff}(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $(\text{Aff}/S)_{Zar}/U$ has a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}_{Aff}|_{(\text{Aff}/S)_{Zar}/U} \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}_{Aff}|_{(\text{Aff}/S)_{Zar}/U} \longrightarrow \mathcal{F}|_{(\text{Aff}/S)_{Zar}/U} \longrightarrow 0$$

With apologies for the horrible notation, this finishes the proof. $\square$

We continue the discussion started in the introduction to this section. Let $\tau \in \{Zariski, étale\}$. Recall that $S_{affine,\tau}$ is the full subcategory of S_τ whose objects are affine turned into a site by declaring the coverings to be the standard τ coverings. See Topologies, Definitions 34.3.7 and 34.4.8. By Topologies, Lemmas 34.3.11, resp. 34.4.12 we have an equivalence of topoi $g : Sh(S_{affine,\tau}) \to Sh(S_\tau)$, whose pullback functor is given by restriction. Recalling that $\mathcal{O}$ denotes the structure sheaf on S_τ let us temporarily and pedantically denote $\mathcal{O}_{affine}$ the restriction of $\mathcal{O}$ to $S_{affine,\tau}$. Then we obtain an equivalence

0GZW (35.11.1.1) $$(Sh(S_{affine,\tau}), \mathcal{O}_{affine}) \longrightarrow (Sh(S_\tau), \mathcal{O})$$

of ringed topoi. Having said this we can compare quasi-coherent modules as well.

0GZX Lemma 35.11.2. Let S be a scheme. Let $\tau \in \{Zariski, étale\}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_{affine}$-modules on $S_{affine,\tau}$. The following are equivalent

(1) for every morphism $U \to U'$ of $S_{affine,\tau}$ the map $\mathcal{F}(U') \otimes_{\mathcal{O}(U')} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism,
(2) $\mathcal{F}$ is a sheaf on $S_{affine,\tau}$ and a quasi-coherent module on the ringed site $(S_{affine,\tau}, \mathcal{O}_{affine})$ in the sense of Modules on Sites, Definition 18.23.1,
(3) $\mathcal{F}$ corresponds to a quasi-coherent module on S_τ via the equivalence (35.11.1.1),
(4) $\mathcal{F}$ comes from a unique quasi-coherent $\mathcal{O}_S$-module $\mathcal{G}$ by the procedure described in Section 35.8.

Proof. Let us prove this in the case of the étale topology.

Assume (1) holds. To show that $\mathcal{F}$ is a sheaf, let $\mathcal{U} = \{U_i \to U\}_{i=1,\ldots,n}$ be a covering of $S_{affine,étale}$. The sheaf condition for $\mathcal{F}$ and $\mathcal{U}$, by our assumption on $\mathcal{F}$. reduces to showing that

$$0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i) \to \prod \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i \times_U U_j)$$

is exact. This is true because $\mathcal{O}(U) \to \prod \mathcal{O}(U_i)$ is faithfully flat (by Lemma 35.9.1 and the fact that coverings in $S_{affine,étale}$ are standard étale coverings) and we may apply Lemma 35.3.6. Next, we show that $\mathcal{F}$ is quasi-coherent on $S_{affine,étale}$. Namely, for U in $S_{affine,étale}$, set $R = \mathcal{O}(U)$ and choose a presentation

$$\bigoplus\nolimits_{k \in K} R \longrightarrow \bigoplus\nolimits_{l \in L} R \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

by free R-modules. By property (1) and the right exactness of tensor product we see that for every morphism $U' \to U$ in $S_{affine,étale}$ we obtain a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}(U') \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}(U') \longrightarrow \mathcal{F}(U') \longrightarrow 0$$

In other words, we see that the restriction of $\mathcal{F}$ to the localized category $S_{affine,etale}/U$ has a presentation

$$\bigoplus\nolimits_{k \in K} \mathcal{O}_{affine}|_{S_{affine,étale}/U} \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}_{affine}|_{S_{affine,étale}/U} \longrightarrow \mathcal{F}|_{S_{affine,étale}/U} \longrightarrow 0$$

as required to show that $\mathcal{F}$ is quasi-coherent. With apologies for the horrible notation, this finishes the proof that (1) implies (2).

Since the notion of a quasi-coherent module is intrinsic (Modules on Sites, Lemma 18.23.2) we see that the equivalence (35.11.1.1) induces an equivalence between categories of quasi-coherent modules. Thus we have the equivalence of (2) and (3).

The equivalence of (3) and (4) follows from Proposition 35.8.9.

Let us assume (4) and prove (1). Namely, let $\mathcal{G}$ be as in (4). Let $h : U \to U' \to S$ be a morphism of $S_{affine,étale}$. Denote $f : U \to S$ and $f' : U' \to S$ the structure morphisms, so that $f = f' \circ h$. We have $\mathcal{F}(U') = \Gamma(U', (f')^*\mathcal{G})$ and $\mathcal{F}(U) = \Gamma(U, f^*\mathcal{G}) = \Gamma(U, h^*(f')^*\mathcal{G})$. Hence (1) holds by Schemes, Lemma 26.7.3.

We omit the proof in the case of the Zariski topology. $\square$

35.12. Parasitic modules

07AF Parasitic modules are those which are zero when restricted to schemes flat over the base scheme. Here is the formal definition.

06ZL Definition 35.12.1. Let S be a scheme. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.

(1) $\mathcal{F}$ is called parasitic[8] if for every flat morphism $U \to S$ we have $\mathcal{F}(U) = 0$.
(2) $\mathcal{F}$ is called parasitic for the τ-topology if for every τ-covering $\{U_i \to S\}_{i\in I}$ we have $\mathcal{F}(U_i) = 0$ for all i.

If $\tau = fppf$ this means that $\mathcal{F}|_{U_{Zar}} = 0$ whenever $U \to S$ is flat and locally of finite presentation; similar for the other cases.

0755 Lemma 35.12.2. Let S be a scheme. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.

(1) If $\mathcal{G}$ is parasitic for the τ-topology, then $H^p_\tau(U, \mathcal{G}) = 0$ for every U open in S, resp. étale over S, resp. smooth over S, resp. syntomic over S, resp. flat and locally of finite presentation over S.
(2) If $\mathcal{G}$ is parasitic then $H^p_\tau(U, \mathcal{G}) = 0$ for every U flat over S.

Proof. Proof in case $\tau = fppf$; the other cases are proved in the exact same way. The assumption means that $\mathcal{G}(U) = 0$ for any $U \to S$ flat and locally of finite presentation. Apply Cohomology on Sites, Lemma 21.10.9 to the subset $\mathcal{B} \subset \mathrm{Ob}((Sch/S)_{fppf})$ consisting of $U \to S$ flat and locally of finite presentation and the collection Cov of all fppf coverings of elements of $\mathcal{B}$. $\square$

07AG Lemma 35.12.3. Let $f : T \to S$ be a morphism of schemes. For any parasitic $\mathcal{O}$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^if_*\mathcal{F}$ are parasitic $\mathcal{O}$-modules on $(Sch/S)_\tau$.

Proof. Recall that $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto H^i((Sch/U \times_S T)_\tau, \mathcal{F})$$

see Cohomology on Sites, Lemma 21.7.4. If $U \to S$ is flat, then $U \times_S T \to T$ is flat as a base change. Hence the displayed group is zero by Lemma 35.12.2. If $\{U_i \to U\}$ is a τ-covering then $U_i \times_S T \to T$ is also flat. Hence it is clear that the sheafification of the displayed presheaf is zero on schemes U flat over S. $\square$

0756 Lemma 35.12.4. Let S be a scheme. Let $\tau \in \{Zar, \acute{e}tale\}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules on $(Sch/S)_{fppf}$ such that

(1) $\mathcal{G}|_{S_\tau}$ is quasi-coherent, and
(2) for every flat, locally finitely presented morphism $g : U \to S$ the canonical map $g^*_{\tau,small}(\mathcal{G}|_{S_\tau}) \to \mathcal{G}|_{U_\tau}$ is an isomorphism.

Then $H^p(U, \mathcal{G}) = H^p(U, \mathcal{G}|_{U_\tau})$ for every U flat and locally of finite presentation over S.

Proof. Let $\mathcal{F}$ be the pullback of $\mathcal{G}|_{S_\tau}$ to the big fppf site $(Sch/S)_{fppf}$. Note that $\mathcal{F}$ is quasi-coherent. There is a canonical comparison map $\varphi : \mathcal{F} \to \mathcal{G}$ which by assumptions (1) and (2) induces an isomorphism $\mathcal{F}|_{U_\tau} \to \mathcal{G}|_{U_\tau}$ for all $g : U \to S$ flat and locally of finite presentation. Hence in the short exact sequences

$$0 \to \mathrm{Ker}(\varphi) \to \mathcal{F} \to \mathrm{Im}(\varphi) \to 0$$

and

$$0 \to \mathrm{Im}(\varphi) \to \mathcal{G} \to \mathrm{Coker}(\varphi) \to 0$$

the sheaves $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are parasitic for the fppf topology. By Lemma 35.12.2 we conclude that $H^p(U, \mathcal{F}) \to H^p(U, \mathcal{G})$ is an isomorphism for $g : U \to S$

[8] This may be nonstandard notation.

flat and locally of finite presentation. Since the result holds for $\mathcal{F}$ by Proposition 35.9.3 we win. $\square$

35.13. Fpqc coverings are universal effective epimorphisms

023P We apply the material above to prove an interesting result, namely Lemma 35.13.7. By Sites, Section 7.12 this lemma implies that the representable presheaves on any of the sites $(Sch/S)_\tau$ are sheaves for $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. First we prove a helper lemma.

02KI **Lemma 35.13.1.** For a scheme X denote $|X|$ the underlying set. Let $f : X \to S$ be a morphism of schemes. Then

$$|X \times_S X| \to |X| \times_{|S|} |X|$$

is surjective.

Proof. Follows immediately from the description of points on the fibre product in Schemes, Lemma 26.17.5. $\square$

0EUA **Lemma 35.13.2.** Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of affine schemes. The following are equivalent

(1) for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

$$\Gamma(X, \mathcal{F}) = \text{Equalizer}\left(\prod\nolimits_{i \in I} \Gamma(X_i, f_i^*\mathcal{F}) \rightrightarrows \prod\nolimits_{i, j \in I} \Gamma(X_i \times_X X_j, (f_i \times f_j)^*\mathcal{F}) \right)$$

(2) $\{f_i : X_i \to X\}_{i \in I}$ is a universal effective epimorphism (Sites, Definition 7.12.1) in the category of affine schemes.

Proof. Assume (2) holds and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the scheme (Constructions, Section 27.4)

$$X' = \underline{\text{Spec}}_X(\mathcal{O}_X \oplus \mathcal{F})$$

where $\mathcal{O}_X \oplus \mathcal{F}$ is an $\mathcal{O}_X$-algebra with multiplication $(f, s)(f', s') = (ff', fs' + f's)$. If $s_i \in \Gamma(X_i, f_i^*\mathcal{F})$ is a section, then s_i determines a unique element of

$$\Gamma(X' \times_X X_i, \mathcal{O}_{X' \times_X X_i}) = \Gamma(X_i, \mathcal{O}_{X_i}) \oplus \Gamma(X_i, f_i^*\mathcal{F})$$

Proof of equality omitted. If $(s_i)_{i \in I}$ is in the equalizer of (1), then, using the equality

$$\text{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$$

which holds for any scheme T, we see that these sections define a family of morphisms $h_i : X' \times_X X_i \to \mathbf{A}^1_{\mathbf{Z}}$ with $h_i \circ \text{pr}_1 = h_j \circ \text{pr}_2$ as morphisms $(X' \times_X X_i) \times_{X'} (X' \times_X X_j) \to \mathbf{A}^1_{\mathbf{Z}}$. Since we've assume (2) we obtain a morphism $h : X' \to \mathbf{A}^1_{\mathbf{Z}}$ compatible with the morphisms h_i which in turn determines an element $s \in \Gamma(X, \mathcal{F})$. We omit the verification that s maps to s_i in $\Gamma(X_i, f_i^*\mathcal{F})$.

Assume (1). Let T be an affine scheme and let $h_i : X_i \to T$ be a family of morphisms such that $h_i \circ \text{pr}_1 = h_j \circ \text{pr}_2$ on $X_i \times_X X_j$ for all $i, j \in I$. Then

$$\prod h_i^\sharp : \Gamma(T, \mathcal{O}_T) \to \prod \Gamma(X_i, \mathcal{O}_{X_i})$$

maps into the equalizer and we find that we get a ring map $\Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X)$ by the assumption of the lemma for $\mathcal{F} = \mathcal{O}_X$. This ring map corresponds to a morphism $h : X \to T$ such that $h_i = h \circ f_i$. Hence our family is an effective epimorphism.

Let $p : Y \to X$ be a morphism of affines. We will show the base changes $g_i : Y_i \to Y$ of f_i form an effective epimorphism by applying the result of the previous paragraph. Namely, if $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then

$$\Gamma(Y, \mathcal{G}) = \Gamma(X, p_*\mathcal{G}), \quad \Gamma(Y_i, g_i^*\mathcal{G}) = \Gamma(X, f_i^*p_*\mathcal{G}),$$

and

$$\Gamma(Y_i \times_Y Y_j, (g_i \times g_j)^*\mathcal{G}) = \Gamma(X, (f_i \times f_j)^*p_*\mathcal{G})$$

by the trivial base change formula (Cohomology of Schemes, Lemma 30.5.1). Thus we see property (1) lemma holds for the family g_i. □

0EUB Lemma 35.13.3. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes.

(1) If the family is universal effective epimorphism in the category of schemes, then $\coprod f_i$ is surjective.
(2) If X and X_i are affine and the family is a universal effective epimorphism in the category of affine schemes, then $\coprod f_i$ is surjective.

Proof. Omitted. Hint: perform base change by $\mathrm{Spec}(\kappa(x)) \to X$ to see that any $x \in X$ has to be in the image. □

0EUC Lemma 35.13.4. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes. If for every morphism $Y \to X$ with Y affine the family of base changes $g_i : Y_i \to Y$ forms an effective epimorphism, then the family of f_i forms a universal effective epimorphism in the category of schemes.

Proof. Let $Y \to X$ be a morphism of schemes. We have to show that the base changes $g_i : Y_i \to Y$ form an effective epimorphism. To do this, assume given a scheme T and morphisms $h_i : Y_i \to T$ with $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ on $Y_i \times_Y Y_j$. Choose an affine open covering $Y = \bigcup V_\alpha$. Set $V_{\alpha,i}$ equal to the inverse image of V_α in Y_i. Then we see that $V_{\alpha,i} \to V_\alpha$ is the base change of f_i by $V_\alpha \to X$. Thus by assumption the family of restrictions $h_i|_{V_{\alpha,i}}$ come from a morphism of schemes $h_\alpha : V_\alpha \to T$. We leave it to the reader to show that these agree on overlaps and define the desired morphism $Y \to T$. See discussion in Schemes, Section 26.14. □

0EUD Lemma 35.13.5. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of affine schemes. Assume the equivalent assumption of Lemma 35.13.2 hold and that moreover for any morphism of affines $Y \to X$ the map

$$\coprod X_i \times_X Y \longrightarrow Y$$

is a submersive map of topological spaces (Topology, Definition 5.6.3). Then our family of morphisms is a universal effective epimorphism in the category of schemes.

Proof. By Lemma 35.13.4 it suffices to base change our family of morphisms by $Y \to X$ with Y affine. Set $Y_i = X_i \times_X Y$. Let T be a scheme and let $h_i : Y_i \to Y$ be a family of morphisms such that $h_i \circ \mathrm{pr}_1 = h_j \circ \mathrm{pr}_2$ on $Y_i \times_Y Y_j$. Note that Y as a set is the coequalizer of the two maps from $\coprod Y_i \times_Y Y_j$ to $\coprod Y_i$. Namely, surjectivity by the affine case of Lemma 35.13.3 and injectivity by Lemma 35.13.1. Hence there is a set map of underlying sets $h : Y \to T$ compatible with the maps h_i. By the second condition of the lemma we see that h is continuous! Thus if $y \in Y$ and $U \subset T$ is an affine open neighbourhood of $h(y)$, then we can find an affine open $V \subset Y$ such that $h(V) \subset U$. Setting $V_i = Y_i \times_Y V = X_i \times_X V$ we can use the result proved in Lemma 35.13.2 to see that $h|_V : V \to U \subset T$ comes from a unique morphism of affine schemes $h_V : V \to U$ agreeing with $h_i|_{V_i}$ as morphisms of schemes for all i. Glueing these h_V (see Schemes, Section 26.14) gives a morphism $Y \to T$ as desired. $\square$

03N0 Lemma 35.13.6. Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering. Suppose that for each i we have an open subset $W_i \subset T_i$ such that for all $i, j \in I$ we have $\text{pr}_0^{-1}(W_i) = \text{pr}_1^{-1}(W_j)$ as open subsets of $T_i \times_T T_j$. Then there exists a unique open subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.

Proof. Apply Lemma 35.13.1 to the map $\coprod_{i \in I} T_i \to T$. It implies there exists a subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i, namely $W = \bigcup f_i(W_i)$. To see that W is open we may work Zariski locally on T. Hence we may assume that T is affine. Using Topologies, Definition 34.9.1 we may choose a standard fpqc covering $\{g_j : V_j \to T\}_{j \in J}$ which refines $\{T_i \to T\}_{i \in I}$. Let $\alpha : J \to I$ and $h_j : V_j \to T_{\alpha(j)}$ be as in Sites, Definition 7.8.1. Then $g_j^{-1}(W) = h_j^{-1}(W_{\alpha(j)})$. Thus we may assume $\{f_i : T_i \to T\}$ is a standard fpqc covering. In this case we may apply Morphisms, Lemma 29.25.12 to the morphism $\coprod T_i \to T$ to conclude that W is open. $\square$

023Q Lemma 35.13.7. Let $\{T_i \to T\}$ be an fpqc covering, see Topologies, Definition 34.9.1. Then $\{T_i \to T\}$ is a universal effective epimorphism in the category of schemes, see Sites, Definition 7.12.1. In other words, every representable functor on the category of schemes satisfies the sheaf condition for the fpqc topology, see Topologies, Definition 34.9.12.

Proof. Let S be a scheme. We have to show the following: Given morphisms $\varphi_i : T_i \to S$ such that $\varphi_i|_{T_i \times_T T_j} = \varphi_j|_{T_i \times_T T_j}$ there exists a unique morphism $T \to S$ which restricts to φ_i on each T_i. In other words, we have to show that the functor $h_S = \text{Mor}_{Sch}(-, S)$ satisfies the sheaf property for the fpqc topology.

If $\{T_i \to T\}$ is a Zariski covering, then this follows from Schemes, Lemma 26.14.1. Thus Topologies, Lemma 34.9.13 reduces us to the case of a covering $\{X \to Y\}$ given by a single surjective flat morphism of affines.

First proof. By Lemma 35.8.1 we have the sheaf condition for quasi-coherent modules for $\{X \to Y\}$. By Lemma 35.13.6 the morphism $X \to Y$ is universally submersive. Hence we may apply Lemma 35.13.5 to see that $\{X \to Y\}$ is a universal effective epimorphism.

Second proof. Let $R \to A$ be the faithfully flat ring map corresponding to our surjective flat morphism $\pi : X \to Y$. Let $f : X \to S$ be a morphism such that $f \circ \text{pr}_1 = f \circ \text{pr}_2$ as morphisms $X \times_Y X = \text{Spec}(A \otimes_R A) \to S$. By Lemma 35.13.1 we see that as a map on the underlying sets f is of the form $f = g \circ \pi$ for some (set theoretic) map $g : \text{Spec}(R) \to S$. By Morphisms, Lemma 29.25.12 and the fact that f is continuous we see that g is continuous.

Pick $y \in Y = \text{Spec}(R)$. Choose $U \subset S$ affine open containing $g(y)$. Say $U = \text{Spec}(B)$. By the above we may choose an $r \in R$ such that $y \in D(r) \subset g^{-1}(U)$. The restriction of f to $\pi^{-1}(D(r))$ into U corresponds to a ring map $B \to A_r$. The two induced ring maps $B \to A_r \otimes_{R_r} A_r = (A \otimes_R A)_r$ are equal by assumption on f. Note that $R_r \to A_r$ is faithfully flat. By Lemma 35.3.6 the equalizer of the two arrows $A_r \to A_r \otimes_{R_r} A_r$ is R_r. We conclude that $B \to A_r$ factors uniquely through a map $B \to R_r$. This map in turn gives a morphism of schemes $D(r) \to U \to S$, see Schemes, Lemma 26.6.4.

What have we proved so far? We have shown that for any prime $\mathfrak{p} \subset R$, there exists a standard affine open $D(r) \subset \text{Spec}(R)$ such that the morphism $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \to S$ factors uniquely through some morphism of schemes $D(r) \to S$. We omit the verification that these morphisms glue to the desired morphism $\text{Spec}(R) \to S$. $\square$

0BMN Lemma 35.13.8. Consider schemes X, Y, Z and morphisms $a, b : X \to Y$ and a morphism $c : Y \to Z$ with $c \circ a = c \circ b$. Set $d = c \circ a = c \circ b$. If there exists an fpqc covering $\{Z_i \to Z\}$ such that

(1) for all i the morphism $Y \times_{c,Z} Z_i \to Z_i$ is the coequalizer of $(a, 1) : X \times_{d,Z} Z_i \to Y \times_{c,Z} Z_i$ and $(b, 1) : X \times_{d,Z} Z_i \to Y \times_{c,Z} Z_i$, and
(2) for all i and i' the morphism $Y \times_{c,Z} (Z_i \times_Z Z_{i'}) \to (Z_i \times_Z Z_{i'})$ is the coequalizer of $(a, 1) : X \times_{d,Z} (Z_i \times_Z Z_{i'}) \to Y \times_{c,Z} (Z_i \times_Z Z_{i'})$ and $(b, 1) : X \times_{d,Z} (Z_i \times_Z Z_{i'}) \to Y \times_{c,Z} (Z_i \times_Z Z_{i'})$

then c is the coequalizer of a and b.

Proof. Namely, for a scheme T a morphism $Z \to T$ is the same thing as a collection of morphism $Z_i \to T$ which agree on overlaps by Lemma 35.13.7. $\square$

35.14. Descent of finiteness and smoothness properties of morphisms

02KJ In this section we show that several properties of morphisms (being smooth, locally of finite presentation, and so on) descend under faithfully flat morphisms. We start with an algebraic version. (The "Noetherian" reader should consult Lemma 35.14.2 instead of the next lemma.)

02KK Lemma 35.14.1. Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite presentation and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite presentation.

Proof. Consider the algebra $C = B \otimes_A B$ together with the pair of maps $p, q : B \to C$ given by $p(b) = b \otimes 1$ and $q(b) = 1 \otimes b$. Of course the two compositions $A \to B \to C$ are the same. Note that as $p : B \to C$ is flat and of finite presentation (base change of $A \to B$), the ring map $R \to C$ is of finite presentation (as the composite of $R \to B \to C$).

We are going to use the criterion Algebra, Lemma 10.127.3 to show that $R \to A$ is of finite presentation. Let S be any R-algebra, and suppose that $S = \text{colim}_{\lambda \in \Lambda} S_\lambda$ is written as a directed colimit of R-algebras. Let $A \to S$ be an R-algebra homomorphism. We have to show that $A \to S$ factors through one of the S_λ. Consider the rings $B' = S \otimes_A B$ and $C' = S \otimes_A C = B' \otimes_S B'$. As B is faithfully flat of finite presentation over A, also B' is faithfully flat of finite presentation over S. By Algebra, Lemma 10.168.1 part (2) applied to the pair $(S \to B', B')$ and the system (S_λ) there exists a $\lambda_0 \in \Lambda$ and a flat, finitely presented S_{λ_0}-algebra B_{λ_0} such that $B' = S \otimes_{S_{\lambda_0}} B_{\lambda_0}$. For $\lambda \geq \lambda_0$ set $B_\lambda = S_\lambda \otimes_{S_{\lambda_0}} B_{\lambda_0}$ and $C_\lambda = B_\lambda \otimes_{S_\lambda} B_\lambda$.

We interrupt the flow of the argument to show that $S_\lambda \to B_\lambda$ is faithfully flat for λ large enough. (This should really be a separate lemma somewhere else, maybe in the chapter on limits.) Since $\text{Spec}(B_{\lambda_0}) \to \text{Spec}(S_{\lambda_0})$ is flat and of finite presentation it is open (see Morphisms, Lemma 29.25.10). Let $I \subset S_{\lambda_0}$ be an ideal such that $V(I) \subset \text{Spec}(S_{\lambda_0})$ is the complement of the image. Note that formation of the image commutes with base change. Hence, since $\text{Spec}(B') \to \text{Spec}(S)$ is surjective, and $B' = B_{\lambda_0} \otimes_{S_{\lambda_0}} S$ we see that $IS = S$. Thus for some $\lambda \geq \lambda_0$ we have $IS_\lambda = S_\lambda$. For this and all greater λ the morphism $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(S_\lambda)$ is surjective.

By analogy with the notation in the first paragraph of the proof denote $p_\lambda, q_\lambda : B_\lambda \to C_\lambda$ the two canonical maps. Then $B' = \mathrm{colim}_{\lambda \geq \lambda_0} B_\lambda$ and $C' = \mathrm{colim}_{\lambda \geq \lambda_0} C_\lambda$. Since B and C are finitely presented over R there exist (by Algebra, Lemma 10.127.3 applied several times) a $\lambda \geq \lambda_0$ and an R-algebra maps $B \to B_\lambda$, $C \to C_\lambda$ such that the diagram

$$\begin{tikzcd} C \arrow[r] & C_\lambda \\ B \arrow[u, shift left, "q"'] \arrow[u, shift right, "p"] \arrow[r] & B_\lambda \arrow[u, shift left, "q_\lambda"'] \arrow[u, shift right, "p_\lambda"] \end{tikzcd}$$

is commutative. OK, and this means that $A \to B \to B_\lambda$ maps into the equalizer of p_λ and q_λ. By Lemma 35.3.6 we see that S_λ is the equalizer of p_λ and q_λ. Thus we get the desired ring map $A \to S_\lambda$ and we win. $\square$

Here is an easier version of this dealing with the property of being of finite type.

0367 **Lemma 35.14.2.** Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite type and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite type.

Proof. By Algebra, Lemma 10.168.2 there exists a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & A_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\ R \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$. Since $R \to B$ is of finite type by assumption, we may add some elements to A_0 and assume that the map $B_0 \to B$ is surjective! In this case, since $A_0 \to B_0$ is faithfully flat, we see that as

$$(A_0 \to A) \otimes_{A_0} B_0 \cong (B_0 \to B)$$

is surjective, also $A_0 \to A$ is surjective. Hence we win. $\square$

02KL **Lemma 35.14.3.** Let [DG67, IV, 17.7.5 (i) and (ii)].

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[rd, "p"'] & & Y \arrow[ld, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that f is surjective, flat and locally of finite presentation and assume that p is locally of finite presentation (resp. locally of finite type). Then q is locally of finite presentation (resp. locally of finite type).

Proof. The problem is local on S and Y. Hence we may assume that S and Y are affine. Since f is flat and locally of finite presentation, we see that f is open (Morphisms, Lemma 29.25.10). Hence, since Y is quasi-compact, there exist finitely many affine opens $X_i \subset X$ such that $Y = \bigcup f(X_i)$. Clearly we may replace X by $\coprod X_i$, and hence we may assume X is affine as well. In this case the lemma is equivalent to Lemma 35.14.1 (resp. Lemma 35.14.2) above. $\square$

We use this to improve some of the results on morphisms obtained earlier.

02KM Lemma 35.14.4. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) f is surjective, and syntomic (resp. smooth, resp. étale),
(2) p is syntomic (resp. smooth, resp. étale).

Then q is syntomic (resp. smooth, resp. étale).

Proof. Combine Morphisms, Lemmas 29.30.16, 29.34.19, and 29.36.19 with Lemma 35.14.3 above. $\square$

Actually we can strengthen this result as follows.

05B5 Lemma 35.14.5. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) f is surjective, flat, and locally of finite presentation,
(2) p is smooth (resp. étale).

Then q is smooth (resp. étale).

Proof. Assume (1) and that p is smooth. By Lemma 35.14.3 we see that q is locally of finite presentation. By Morphisms, Lemma 29.25.13 we see that q is flat. Hence now it suffices to show that the fibres of q are smooth, see Morphisms, Lemma 29.34.3. Apply Varieties, Lemma 33.25.9 to the flat surjective morphisms $X_s \to Y_s$ for $s \in S$ to conclude. We omit the proof of the étale case. $\square$

05B6 Remark 35.14.6. With the assumptions (1) and p smooth in Lemma 35.14.5 it is not automatically the case that $X \to Y$ is smooth. A counter example is $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(k[s])$, $Y = \mathrm{Spec}(k[t])$ and f given by $t \mapsto s^2$. But see also Lemma 35.14.7 for some information on the structure of f.

05B7 Lemma 35.14.7. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) f is surjective, flat, and locally of finite presentation,
(2) p is syntomic.

Then both q and f are syntomic.

Proof. By Lemma 35.14.3 we see that q is of finite presentation. By Morphisms, Lemma 29.25.13 we see that q is flat. By Morphisms, Lemma 29.30.10 it now suffices to show that the local rings of the fibres of $Y \to S$ and the fibres of $X \to Y$ are local complete intersection rings. To do this we may take the fibre of $X \to Y \to S$ at a point $s \in S$, i.e., we may assume S is the spectrum of a field. Pick a point $x \in X$ with image $y \in Y$ and consider the ring map

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

This is a flat local homomorphism of local Noetherian rings. The local ring $\mathcal{O}_{X,x}$ is a complete intersection. Thus may use Avramov's result, see Divided Power Algebra, Lemma 23.8.9, to conclude that both $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ are complete intersection rings. $\square$

The following type of lemma is occasionally useful.

06NB Lemma 35.14.8. Let $X \to Y \to Z$ be morphism of schemes. Let P be one of the following properties of morphisms of schemes: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $\{X \to Y\}$ can be refined by an fppf covering of Y. Then $Y \to Z$ is P.

Proof. Let $\text{Spec}(C) \subset Z$ be an affine open and let $\text{Spec}(B) \subset Y$ be an affine open which maps into $\text{Spec}(C)$. The assumption on $X \to Y$ implies we can find a standard affine fppf covering $\{\text{Spec}(B_j) \to \text{Spec}(B)\}$ and lifts $x_j : \text{Spec}(B_j) \to X$. Since $\text{Spec}(B_j)$ is quasi-compact we can find finitely many affine opens $\text{Spec}(A_i) \subset X$ lying over $\text{Spec}(B)$ such that the image of each x_j is contained in the union $\bigcup \text{Spec}(A_i)$. Hence after replacing each $\text{Spec}(B_j)$ by a standard affine Zariski coverings of itself we may assume we have a standard affine fppf covering $\{\text{Spec}(B_i) \to \text{Spec}(B)\}$ such that each $\text{Spec}(B_i) \to Y$ factors through an affine open $\text{Spec}(A_i) \subset X$ lying over $\text{Spec}(B)$. In other words, we have ring maps $C \to B \to A_i \to B_i$ for each i. Note that we can also consider

$$C \to B \to A = \prod A_i \to B' = \prod B_i$$

and that the ring map $B \to \prod B_i$ is faithfully flat and of finite presentation.

The case $P = flat$. In this case we know that $C \to A$ is flat and we have to prove that $C \to B$ is flat. Suppose that $N \to N' \to N''$ is an exact sequence of C-modules. We want to show that $N \otimes_C B \to N' \otimes_C B \to N'' \otimes_C B$ is exact. Let H be its cohomology and let H' be the cohomology of $N \otimes_C B' \to N' \otimes_C B' \to N'' \otimes_C B'$. As $B \to B'$ is flat we know that $H' = H \otimes_B B'$. On the other hand $N \otimes_C A \to N' \otimes_C A \to N'' \otimes_C A$ is exact hence has zero cohomology. Hence the map $H \to H'$ is zero (as it factors through the zero module). Thus $H' = 0$. As $B \to B'$ is faithfully flat we conclude that $H = 0$ as desired.

The case $P = locally\ finite\ type$. In this case we know that $C \to A$ is of finite type and we have to prove that $C \to B$ is of finite type. Because $B \to B'$ is of finite presentation (hence of finite type) we see that $A \to B'$ is of finite type, see Algebra, Lemma 10.6.2. Therefore $C \to B'$ is of finite type and we conclude by Lemma 35.14.2.

The case $P = locally\ finite\ presentation$. In this case we know that $C \to A$ is of finite presentation and we have to prove that $C \to B$ is of finite presentation. Because $B \to B'$ is of finite presentation and $B \to A$ of finite type we see that

$A \to B'$ is of finite presentation, see Algebra, Lemma 10.6.2. Therefore $C \to B'$ is of finite presentation and we conclude by Lemma 35.14.1. □

35.15. Local properties of schemes

0347 It often happens one can prove the members of a covering of a scheme have a certain property. In many cases this implies the scheme has the property too. For example, if S is a scheme, and $f : S' \to S$ is a surjective flat morphism such that S' is a reduced scheme, then S is reduced. You can prove this by looking at local rings and using Algebra, Lemma 10.164.2. We say that the property of being reduced descends through flat surjective morphisms. Some results of this type are collected in Algebra, Section 10.164 and for schemes in Section 35.19. Some analogous results on descending properties of morphisms are in Section 35.14.

On the other hand, there are examples of surjective flat morphisms $f : S' \to S$ with S reduced and S' not, for example the morphism $\text{Spec}(k[x]/(x^2)) \to \text{Spec}(k)$. Hence the property of being reduced does not ascend along flat morphisms. Having infinite residue fields is a property which does ascend along flat morphisms (but does not descend along surjective flat morphisms of course). Some results of this type are collected in Algebra, Section 10.163.

Finally, we say that a property is local for the flat topology if it ascends along flat morphisms and descends along flat surjective morphisms. A somewhat silly example is the property of having residue fields of a given characteristic. To be more precise, and to tie this in with the various topologies on schemes, we make the following formal definition.

0348 Definition 35.15.1. Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. We say $\mathcal{P}$ is local in the τ-topology if for any τ-covering $\{S_i \to S\}_{i \in I}$ (see Topologies, Section 34.2) we have

$$S \text{ has } \mathcal{P} \Leftrightarrow \text{each } S_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for S if and only if it holds for any scheme S' isomorphic to S. In fact, if $\tau = fpqc, fppf, syntomic, smooth, \acute{e}tale$, or $Zariski$, then if S has $\mathcal{P}$ and $S' \to S$ is flat, flat and locally of finite presentation, syntomic, smooth, étale, or an open immersion, then S' has $\mathcal{P}$. This is true because we can always extend $\{S' \to S\}$ to a τ-covering.

We have the following implications: $\mathcal{P}$ is local in the fpqc topology $\Rightarrow$ $\mathcal{P}$ is local in the fppf topology $\Rightarrow$ $\mathcal{P}$ is local in the syntomic topology $\Rightarrow$ $\mathcal{P}$ is local in the smooth topology $\Rightarrow$ $\mathcal{P}$ is local in the étale topology $\Rightarrow$ $\mathcal{P}$ is local in the Zariski topology. This follows from Topologies, Lemmas 34.4.2, 34.5.2, 34.6.2, 34.7.2, and 34.9.6.

0349 Lemma 35.15.2. Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. Assume that

(1) the property is local in the Zariski topology,
(2) for any morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S' if property $\mathcal{P}$ holds for S, and

(3) for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S if property $\mathcal{P}$ holds for S'.

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 34.9.1 34.7.1 34.4.1 34.5.1, or 34.6.1 and Topologies, Lemma 34.9.8, 34.7.4, 34.4.4, 34.5.4, or 34.6.4. Details omitted. □

034A Remark 35.15.3. In Lemma 35.15.2 above if $\tau = smooth$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = étale$.

35.16. Properties of schemes local in the fppf topology

034B In this section we find some properties of schemes which are local on the base in the fppf topology.

034C Lemma 35.16.1. The property $\mathcal{P}(S) =$"S is locally Noetherian" is local in the fppf topology.

Proof. We will use Lemma 35.15.2. First we note that "being locally Noetherian" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 28.5.1. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is locally Noetherian, then S' is locally Noetherian. This is Morphisms, Lemma 29.15.6. Finally, we have to show that if $S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is locally Noetherian, then S is locally Noetherian. This follows from Algebra, Lemma 10.164.1. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

0368 Lemma 35.16.2. The property $\mathcal{P}(S) =$"S is Jacobson" is local in the fppf topology.

Proof. We will use Lemma 35.15.2. First we note that "being Jacobson" is local in the Zariski topology. This is Properties, Lemma 28.6.3. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is Jacobson, then S' is Jacobson. This is Morphisms, Lemma 29.16.9. Finally, we have to show that if $f : S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is Jacobson, then S is Jacobson. Say $S = \text{Spec}(A)$ and $S' = \text{Spec}(B)$ and $S' \to S$ given by $A \to B$. Then $A \to B$ is finitely presented and faithfully flat. Moreover, the ring B is Jacobson, see Properties, Lemma 28.6.3.

By Algebra, Lemma 10.168.10 there exists a diagram

$$\begin{tikzcd} B \arrow[rr] & & B' \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

with $A \to B'$ finitely presented, faithfully flat and quasi-finite. In particular, $B \to B'$ is finite type, and we see from Algebra, Proposition 10.35.19 that B' is Jacobson. Hence we may assume that $A \to B$ is quasi-finite as well as faithfully flat and of finite presentation.

Assume A is not Jacobson to get a contradiction. According to Algebra, Lemma 10.35.5 there exists a nonmaximal prime $\mathfrak{p} \subset A$ and an element $f \in A$, $f \notin \mathfrak{p}$ such that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$.

This leads to a contradiction as follows. First let $\mathfrak{p} \subset \mathfrak{m}$ be a maximal ideal of A. Pick a prime $\mathfrak{m}' \subset B$ lying over $\mathfrak{m}$ (exists because $A \to B$ is faithfully flat, see Algebra, Lemma 10.39.16). As $A \to B$ is flat, by going down see Algebra, Lemma 10.39.19, we can find a prime $\mathfrak{q} \subset \mathfrak{m}'$ lying over $\mathfrak{p}$. In particular we see that $\mathfrak{q}$ is not maximal. Hence according to Algebra, Lemma 10.35.5 again the set $V(\mathfrak{q}) \cap D(f)$ is infinite (here we finally use that B is Jacobson). All points of $V(\mathfrak{q}) \cap D(f)$ map to $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$. Hence the fibre over $\mathfrak{p}$ is infinite. This contradicts the fact that $A \to B$ is quasi-finite (see Algebra, Lemma 10.122.4 or more explicitly Morphisms, Lemma 29.20.10). Thus the lemma is proved. □

0BAL Lemma 35.16.3. The property $\mathcal{P}(S)$ =“every quasi-compact open of S has a finite number of irreducible components” is local in the fppf topology.

Proof. We will use Lemma 35.15.2. First we note that $\mathcal{P}$ is local in the Zariski topology. Next, we show that if $T \to S$ is a flat, finitely presented morphism of affines and S has a finite number of irreducible components, then so does T. Namely, since $T \to S$ is flat, the generic points of T map to the generic points of S, see Morphisms, Lemma 29.25.9. Hence it suffices to show that for $s \in S$ the fibre T_s has a finite number of generic points. Note that T_s is an affine scheme of finite type over $\kappa(s)$, see Morphisms, Lemma 29.15.4. Hence T_s is Noetherian and has a finite number of irreducible components (Morphisms, Lemma 29.15.6 and Properties, Lemma 28.5.7). Finally, we have to show that if $T \to S$ is a surjective flat, finitely presented morphism of affines and T has a finite number of irreducible components, then so does S. This follows from Topology, Lemma 5.8.5. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

35.17. Properties of schemes local in the syntomic topology

0369 In this section we find some properties of schemes which are local on the base in the syntomic topology.

036A Lemma 35.17.1. The property $\mathcal{P}(S)$ =“S is locally Noetherian and (S_k)” is local in the syntomic topology.

Proof. We will check (1), (2) and (3) of Lemma 35.15.2. As a syntomic morphism is flat of finite presentation (Morphisms, Lemmas 29.30.7 and 29.30.6) we have already checked this for “being locally Noetherian” in the proof of Lemma 35.16.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Cohomology of Schemes, Definition 30.11.1. Next, we show that if $S' \to S$ is a syntomic morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemma 10.163.4 (use Morphisms, Lemma 29.30.2 and Algebra, Definition 10.136.1 and Lemma 10.135.3). Finally, we show that if $S' \to S$ is a surjective syntomic morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 10.164.5. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

036B Lemma 35.17.2. The property $\mathcal{P}(S)$ =“S is Cohen-Macaulay” is local in the syntomic topology.

Proof. This is clear from Lemma 35.17.1 above since a scheme is Cohen-Macaulay if and only if it is locally Noetherian and (S_k) for all $k \geq 0$, see Properties, Lemma 28.12.3. $\square$

35.18. Properties of schemes local in the smooth topology

034D In this section we find some properties of schemes which are local on the base in the smooth topology.

034E Lemma 35.18.1. The property $\mathcal{P}(S) =$"S is reduced" is local in the smooth topology.

Proof. We will use Lemma 35.15.2. First we note that "being reduced" is local in the Zariski topology. This is clear from the definition, see Schemes, Definition 26.12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is reduced, then S' is reduced. This is Algebra, Lemma 10.163.7. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is reduced, then S is reduced. This is Algebra, Lemma 10.164.2. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. $\square$

034F Lemma 35.18.2. The property $\mathcal{P}(S) =$"S is normal" is local in the smooth topology.

Proof. We will use Lemma 35.15.2. First we show "being normal" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 28.7.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is normal, then S' is normal. This is Algebra, Lemma 10.163.9. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is normal, then S is normal. This is Algebra, Lemma 10.164.3. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. $\square$

036C Lemma 35.18.3. The property $\mathcal{P}(S) =$"S is locally Noetherian and (R_k)" is local in the smooth topology.

Proof. We will check (1), (2) and (3) of Lemma 35.15.2. As a smooth morphism is flat of finite presentation (Morphisms, Lemmas 29.34.9 and 29.34.8) we have already checked this for "being locally Noetherian" in the proof of Lemma 35.16.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Properties, Definition 28.12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemmas 10.163.5 (use Morphisms, Lemma 29.34.2, Algebra, Lemmas 10.137.4 and 10.140.3). Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 10.164.6. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. $\square$

036D Lemma 35.18.4. The property $\mathcal{P}(S) =$"S is regular" is local in the smooth topology.

Proof. This is clear from Lemma 35.18.3 above since a locally Noetherian scheme is regular if and only if it is locally Noetherian and (R_k) for all $k \geq 0$. $\square$

036E Lemma 35.18.5. The property $\mathcal{P}(S) =$"S is Nagata" is local in the smooth topology.

Proof. We will check (1), (2) and (3) of Lemma 35.15.2. First we note that being Nagata is local in the Zariski topology. This is Properties, Lemma 28.13.6. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is Nagata, then S' is Nagata. This is Morphisms, Lemma 29.18.1. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is Nagata, then S is Nagata. This is Algebra, Lemma 10.164.7. Thus (1), (2) and (3) of Lemma 35.15.2 hold and we win. □

35.19. Variants on descending properties

06QL Sometimes one can descend properties, which are not local. We put results of this kind in this section. See also Section 35.14 on descending properties of morphisms, such as smoothness.

06QM Lemma 35.19.1. If $f : X \to Y$ is a flat and surjective morphism of schemes and X is reduced, then Y is reduced.

Proof. The result follows by looking at local rings (Schemes, Definition 26.12.1) and Algebra, Lemma 10.164.2. □

06QN Lemma 35.19.2. Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.

Proof. This lemma reduces to the following algebra statement: If $A \to B$ is a faithfully flat, finitely presented ring homomorphism with B Noetherian and regular, then A is Noetherian and regular. We see that A is Noetherian by Algebra, Lemma 10.164.1 and regular by Algebra, Lemma 10.110.9. □

35.20. Germs of schemes

04QQ

04QR Definition 35.20.1. Germs of schemes.

(1) A pair (X, x) consisting of a scheme X and a point $x \in X$ is called the germ of X at x.
(2) A morphism of germs $f : (X, x) \to (S, s)$ is an equivalence class of morphisms of schemes $f : U \to S$ with $f(x) = s$ where $U \subset X$ is an open neighbourhood of x. Two such f, f' are said to be equivalent if and only if f and f' agree in some open neighbourhood of x.
(3) We define the composition of morphisms of germs by composing representatives (this is well defined).

Before we continue we need one more definition.

04QS Definition 35.20.2. Let $f : (X, x) \to (S, s)$ be a morphism of germs. We say f is étale (resp. smooth) if there exists a representative $f : U \to S$ of f which is an étale morphism (resp. a smooth morphism) of schemes.

35.21. Local properties of germs

04QT

04N1 Definition 35.21.1. Let $\mathcal{P}$ be a property of germs of schemes. We say that $\mathcal{P}$ is étale local (resp. smooth local) if for any étale (resp. smooth) morphism of germs $(U', u') \to (U, u)$ we have $\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u')$.

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$. The following lemma tells us that the assertion $\dim_x(X) = d$ is an étale local property of germs.

04N4 Lemma 35.21.2. Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\dim_u(U) = \dim_v(V)$.

Proof. In the statement $\dim_u(U)$ is the dimension of U at u as defined in Topology, Definition 5.10.1 as the minimum of the Krull dimensions of open neighbourhoods of u in U. Similarly for $\dim_v(V)$.

Let us show that $\dim_v(V) \geq \dim_u(U)$. Let V' be an open neighbourhood of v in V. Then there exists an open neighbourhood U' of u in U contained in $f^{-1}(V')$ such that $\dim_u(U) = \dim(U')$. Suppose that $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ is a chain of irreducible closed subschemes of U'. If $\xi_i \in Z_i$ is the generic point then we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. This gives specializations $f(\xi_n) \rightsquigarrow f(\xi_{n-1}) \rightsquigarrow \ldots \rightsquigarrow f(\xi_0)$ in V'. Note that $f(\xi_j) \neq f(\xi_i)$ if $i \neq j$ as the fibres of f are discrete (see Morphisms, Lemma 29.36.7). Hence we see that $\dim(V') \geq n$. The inequality $\dim_v(V) \geq \dim_u(U)$ follows formally.

Let us show that $\dim_u(U) \geq \dim_v(V)$. Let U' be an open neighbourhood of u in U. Note that $V' = f(U')$ is an open neighbourhood of v by Morphisms, Lemma 29.25.10. Hence $\dim(V') \geq \dim_v(V)$. Pick a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subschemes of V'. Let $\xi_i \in Z_i$ be the generic point, so we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. Since $\xi_0 \in f(U')$ we can find a point $\eta_0 \in U'$ with $f(\eta_0) = \xi_0$. Consider the map of local rings

$$\mathcal{O}_{V', \xi_0} \longrightarrow \mathcal{O}_{U', \eta_0}$$

which is a flat local ring map by Morphisms, Lemma 29.36.12. Note that the points ξ_i correspond to primes of the ring on the left by Schemes, Lemma 26.13.2. Hence by going down (see Algebra, Section 10.41) for the displayed ring map we can find a sequence of specializations $\eta_n \rightsquigarrow \eta_{n-1} \rightsquigarrow \ldots \rightsquigarrow \eta_0$ in U' mapping to the sequence $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$ under f. This implies that $\dim_u(U) \geq \dim_v(V)$. □

Let (X, x) be a germ of a scheme. The isomorphism class of the local ring $\mathcal{O}_{X,x}$ is an invariant of the germ. The following lemma says that the property $\dim(\mathcal{O}_{X,x}) = d$ is an étale local property of germs.

04N8 Lemma 35.21.3. Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\dim(\mathcal{O}_{U,u}) = \dim(\mathcal{O}_{V,v})$.

Proof. The algebraic statement we are asked to prove is the following: If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q})$. This is More on Algebra, Lemma 15.44.2. □

Let (X, x) be a germ of a scheme. The isomorphism class of the local ring $\mathcal{O}_{X,x}$ is an invariant of the germ. The following lemma says that the property "$\mathcal{O}_{X,x}$ is regular" is an étale local property of germs.

0AH7 Lemma 35.21.4. Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\mathcal{O}_{U,u}$ is a regular local ring if and only if $\mathcal{O}_{V,v}$ is a regular local ring.

Proof. The algebraic statement we are asked to prove is the following: If $A \to B$ is an étale ring map and $\mathfrak q$ is a prime of B lying over $\mathfrak p \subset A$, then $A_{\mathfrak p}$ is regular if and only if $B_{\mathfrak q}$ is regular. This is More on Algebra, Lemma 15.44.3. □

35.22. Properties of morphisms local on the target

02KN Suppose that $f : X \to Y$ is a morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Let $f' : X' \to Y'$ be the base change of f by g:

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Let $\mathcal{P}$ be a property of morphisms of schemes. Then we can wonder if (a) $\mathcal{P}(f) \Rightarrow \mathcal{P}(f')$, and also whether the converse (b) $\mathcal{P}(f') \Rightarrow \mathcal{P}(f)$ is true. If (a) holds whenever g is flat, then we say $\mathcal{P}$ is preserved under flat base change. If (b) holds whenever g is surjective and flat, then we say $\mathcal{P}$ descends through flat surjective base changes. If $\mathcal{P}$ is preserved under flat base changes and descends through flat surjective base changes, then we say $\mathcal{P}$ is flat local on the target. Compare with the discussion in Section 35.15. This turns out to be a very important notion which we formalize in the following definition.

02KO Definition 35.22.1. Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. We say $\mathcal{P}$ is τ local on the base, or τ local on the target, or local on the base for the τ-topology if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ (see Topologies, Section 34.2) and any morphism of schemes $f : X \to Y$ over S we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

04QU Lemma 35.22.2. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. □

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subscheme $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

06QP Lemma 35.22.3. Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. For any morphism of schemes $f : X \to Y$ there exists a largest open $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,

(1) if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then $g(Y') \subset W(f)$,
(2) if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and
(3) if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.

Proof. Consider the union W of the images $g(Y') \subset Y$ of morphisms $g : Y' \to Y$ with the properties:

(1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
(2) the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms, Lemma 29.25.10) we see that $W \subset Y$ is an open subset of Y. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 35.22.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. $\square$

02KP Lemma 35.22.4. Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. Assume that

(1) the property is preserved under flat, flat and locally of finite presentation, étale, smooth, or syntomic base change depending on whether τ is fpqc, fppf, étale, smooth, or syntomic (compare with Schemes, Definition 26.18.3),
(2) the property is Zariski local on the base.
(3) for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, and any morphism of schemes $f : X \to S$ property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the base change $f' : X' = S' \times_S X \to S'$.

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 34.9.1 34.7.1 34.4.1 34.5.1, or 34.6.1 and Topologies, Lemma 34.9.8, 34.7.4, 34.4.4, 34.5.4, or 34.6.4. Details omitted. $\square$

034G Remark 35.22.5. (This is a repeat of Remark 35.15.3 above.) In Lemma 35.22.4 above if $\tau = smooth$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = \acute{e}tale$.

35.23. Properties of morphisms local in the fpqc topology on the target

02YJ In this section we find a large number of properties of morphisms of schemes which are local on the base in the fpqc topology. By contrast, in Examples, Section 110.64

we will show that the properties "projective" and "quasi-projective" are not local on the base even in the Zariski topology.

02KQ **Lemma 35.23.1.** *The property $\mathcal{P}(f)=$"f is quasi-compact" is fpqc local on the base.*

Proof. A base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 26.19.3. Being quasi-compact is Zariski local on the base, see Schemes, Lemma 26.19.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-compact. Then X' is quasi-compact, and $X' \to X$ is surjective. Hence X is quasi-compact. This implies that f is quasi-compact. Therefore Lemma 35.22.4 applies and we win. $\square$

02KR **Lemma 35.23.2.** *The property $\mathcal{P}(f)=$"f is quasi-separated" is fpqc local on the base.*

Proof. Any base change of a quasi-separated morphism is quasi-separated, see Schemes, Lemma 26.21.12. Being quasi-separated is Zariski local on the base (from the definition or by Schemes, Lemma 26.21.6). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is quasi-compact. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 35.23.1 this implies Δ is quasi-compact, and hence f is quasi-separated. Therefore Lemma 35.22.4 applies and we win. $\square$

02KS **Lemma 35.23.3.** *The property $\mathcal{P}(f)=$"f is universally closed" is fpqc local on the base.*

Proof. A base change of a universally closed morphism is universally closed by definition. Being universally closed is Zariski local on the base (from the definition or by Morphisms, Lemma 29.41.2). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally closed. Let $T \to S$ be any morphism. Consider the diagram

$$\begin{tikzcd} X' \arrow[d] & S' \times_S T \times_S X \arrow[l] \arrow[r] \arrow[d] & T \times_S X \arrow[d] \\ S' & S' \times_S T \arrow[l] \arrow[r] & T \end{tikzcd}$$

in which both squares are cartesian. Thus the assumption implies that the middle vertical arrow is closed. The right horizontal arrows are flat, quasi-compact and surjective (as base changes of $S' \to S$). Hence a subset of T is closed if and only if its inverse image in $S' \times_S T$ is closed, see Morphisms, Lemma 29.25.12. An easy diagram chase shows that the right vertical arrow is closed too, and we conclude $X \to S$ is universally closed. Therefore Lemma 35.22.4 applies and we win. $\square$

02KT **Lemma 35.23.4.** *The property $\mathcal{P}(f)=$"f is universally open" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 35.23.3. $\square$

0CEW **Lemma 35.23.5.** *The property $\mathcal{P}(f)=$"f is universally submersive" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 35.23.3 using that a quasi-compact flat surjective morphism is universally submersive by Morphisms, Lemma 29.25.12. ☐

02KU Lemma 35.23.6. The property $\mathcal{P}(f) =$"f is separated" is fpqc local on the base.

Proof. A base change of a separated morphism is separated, see Schemes, Lemma 26.21.12. Being separated is Zariski local on the base (from the definition or by Schemes, Lemma 26.21.7). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is a closed immersion, hence universally closed. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 35.23.3 this implies Δ is universally closed. Since it is an immersion (Schemes, Lemma 26.21.2) we conclude Δ is a closed immersion. Hence f is separated. Therefore Lemma 35.22.4 applies and we win. ☐

02KV Lemma 35.23.7. The property $\mathcal{P}(f) =$"f is surjective" is fpqc local on the base.

Proof. This is clear. ☐

02KW Lemma 35.23.8. The property $\mathcal{P}(f) =$"f is universally injective" is fpqc local on the base.

Proof. A base change of a universally injective morphism is universally injective (this is formal). Being universally injective is Zariski local on the base; this is clear from the definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally injective. Let K be a field, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $S' \to S$ is surjective and by the discussion in Schemes, Section 26.13 there exists a field extension K'/K and a morphism $\mathrm{Spec}(K') \to S'$ such that the following solid diagram commutes

$$
\begin{tikzcd}
\mathrm{Spec}(K') \arrow[dd] \arrow[dr, dashed, "a'{,}b'"'] \arrow[drr] & & \\
& X' \arrow[r] \arrow[d] & S' \arrow[d] \\
\mathrm{Spec}(K) \arrow[r, "a{,}b"] & X \arrow[r] & S
\end{tikzcd}
$$

As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to S'$ is universally injective we get $a' = b'$, by Morphisms, Lemma 29.10.2. Clearly this forces $a = b$ (by the discussion in Schemes, Section 26.13). Therefore Lemma 35.22.4 applies and we win.

An alternative proof would be to use the characterization of a universally injective morphism as one whose diagonal is surjective, see Morphisms, Lemma 29.10.2. The lemma then follows from the fact that the property of being surjective is fpqc local on the base, see Lemma 35.23.7. (Hint: use that the base change of the diagonal is the diagonal of the base change.) ☐

0CEX Lemma 35.23.9. The property $\mathcal{P}(f) =$"f is a universal homeomorphism" is fpqc local on the base.

Proof. This can be proved in exactly the same manner as Lemma 35.23.3. Alternatively, one can use that a map of topological spaces is a homeomorphism if and only if it is injective, surjective, and open. Thus a universal homeomorphism is the same thing as a surjective, universally injective, and universally open morphism. Thus the lemma follows from Lemmas 35.23.7, 35.23.8, and 35.23.4. $\square$

02KX Lemma 35.23.10. The property $\mathcal{P}(f) =$"f is locally of finite type" is fpqc local on the base.

Proof. Being locally of finite type is preserved under base change, see Morphisms, Lemma 29.15.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 29.15.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite type. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite type. We have to show that $R \to A$ is of finite type. This is the result of Algebra, Lemma 10.126.1. It follows that f is locally of finite type. Therefore Lemma 35.22.4 applies and we win. $\square$

02KY Lemma 35.23.11. The property $\mathcal{P}(f) =$"f is locally of finite presentation" is fpqc local on the base.

Proof. Being locally of finite presentation is preserved under base change, see Morphisms, Lemma 29.21.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 29.21.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite presentation. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite presentation. We have to show that $R \to A$ is of finite presentation. This is the result of Algebra, Lemma 10.126.2. It follows that f is locally of finite presentation. Therefore Lemma 35.22.4 applies and we win. $\square$

02KZ Lemma 35.23.12. The property $\mathcal{P}(f) =$"f is of finite type" is fpqc local on the base.

Proof. Combine Lemmas 35.23.1 and 35.23.10. $\square$

02L0 Lemma 35.23.13. The property $\mathcal{P}(f) =$"f is of finite presentation" is fpqc local on the base.

Proof. Combine Lemmas 35.23.1, 35.23.2 and 35.23.11. $\square$

02L1 Lemma 35.23.14. The property $\mathcal{P}(f) =$"f is proper" is fpqc local on the base.

Proof. The lemma follows by combining Lemmas 35.23.3, 35.23.6 and 35.23.12. $\square$

02L2 Lemma 35.23.15. The property $\mathcal{P}(f) =$"f is flat" is fpqc local on the base.

Proof. Being flat is preserved under arbitrary base change, see Morphisms, Lemma 29.25.8. Being flat is Zariski local on the base by definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is flat. Let $U \subset X$ be an affine open.

Then $U' = S' \times_S U$ is affine. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is flat. Goal: Show that $R \to A$ is flat. This follows immediately from Algebra, Lemma 10.39.8. Hence f is flat. Therefore Lemma 35.22.4 applies and we win. $\square$

02L3 Lemma 35.23.16. The property $\mathcal{P}(f) =$"f is an open immersion" is fpqc local on the base.

Proof. The property of being an open immersion is stable under base change, see Schemes, Lemma 26.18.2. The property of being an open immersion is Zariski local on the base (this is obvious).

Let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an open immersion. We claim that f is an open immersion. Then f' is universally open, and universally injective. Hence we conclude that f is universally open by Lemma 35.23.4, and universally injective by Lemma 35.23.8. In particular $f(X) \subset S$ is open. If for every affine open $U \subset f(X)$ we can prove that $f^{-1}(U) \to U$ is an isomorphism, then f is an open immersion and we're done. If $U' \subset S'$ denotes the inverse image of U, then $U' \to U$ is a faithfully flat morphism of affines and $(f')^{-1}(U') \to U'$ is an isomorphism (as $f'(X')$ contains U' by our choice of U). Thus we reduce to the case discussed in the next paragraph.

Let $S' \to S$ be a flat surjective morphism of affine schemes, let $f : X \to S$ be a morphism, and assume that the base change $f' : X' \to S'$ is an isomorphism. We have to show that f is an isomorphism also. It is clear that f is surjective, universally injective, and universally open (see arguments above for the last two). Hence f is bijective, i.e., f is a homeomorphism. Thus f is affine by Morphisms, Lemma 29.45.4. Since

$$\mathcal{O}(S') \to \mathcal{O}(X') = \mathcal{O}(S') \otimes_{\mathcal{O}(S)} \mathcal{O}(X)$$

is an isomorphism and since $\mathcal{O}(S) \to \mathcal{O}(S')$ is faithfully flat this implies that $\mathcal{O}(S) \to \mathcal{O}(X)$ is an isomorphism. Thus f is an isomorphism. This finishes the proof of the claim above. Therefore Lemma 35.22.4 applies and we win. $\square$

02L4 Lemma 35.23.17. The property $\mathcal{P}(f) =$"f is an isomorphism" is fpqc local on the base.

Proof. Combine Lemmas 35.23.7 and 35.23.16. $\square$

02L5 Lemma 35.23.18. The property $\mathcal{P}(f) =$"f is affine" is fpqc local on the base.

Proof. A base change of an affine morphism is affine, see Morphisms, Lemma 29.11.8. Being affine is Zariski local on the base, see Morphisms, Lemma 29.11.3. Finally, let $g : S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is affine. In other words, X' is affine, say $X' = \mathrm{Spec}(A')$. Also write $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. We have to show that X is affine.

By Lemmas 35.23.1 and 35.23.6 we see that $X \to S$ is separated and quasi-compact. Thus $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 26.24.1. Hence $f_*\mathcal{O}_X = \widetilde{A}$ for some R-algebra A. In fact $A = \Gamma(X, \mathcal{O}_X)$ of course. Also, by flat base change (see for example Cohomology of Schemes, Lemma 30.5.2) we have

$g^*f_*\mathcal{O}_X = f'_*\mathcal{O}_{X'}$. In other words, we have $A' = R' \otimes_R A$. Consider the canonical morphism

$$X \longrightarrow \mathrm{Spec}(A)$$

over S from Schemes, Lemma 26.6.4. By the above the base change of this morphism to S' is an isomorphism. Hence it is an isomorphism by Lemma 35.23.17. Therefore Lemma 35.22.4 applies and we win. $\square$

02L6 Lemma 35.23.19. The property $\mathcal{P}(f)$ ="f is a closed immersion" is fpqc local on the base.

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is a closed immersion. This implies that each f_i is affine, see Morphisms, Lemma 29.11.9. By Lemma 35.23.18 we conclude that f is affine. It remains to show that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. For every $y \in Y$ there exists an i and a point $y_i \in Y_i$ mapping to y. By Cohomology of Schemes, Lemma 30.5.2 the sheaf $f_{i,*}(\mathcal{O}_{Y_i \times_Y X})$ is the pullback of $f_*\mathcal{O}_X$. By assumption it is a quotient of $\mathcal{O}_{Y_i}$. Hence we see that

$$\Big(\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y\Big) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y_i,y_i}$$

is surjective. Since $\mathcal{O}_{Y_i,y_i}$ is faithfully flat over $\mathcal{O}_{Y,y}$ this implies the surjectivity of $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ as desired. $\square$

02L7 Lemma 35.23.20. The property $\mathcal{P}(f)$ ="f is quasi-affine" is fpqc local on the base.

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{g_i : Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is quasi-affine. This implies that each f_i is quasi-compact and separated. By Lemmas 35.23.1 and 35.23.6 this implies that f is quasi-compact and separated. Consider the sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} = f_*\mathcal{O}_X$. By Schemes, Lemma 26.24.1 it is a quasi-coherent $\mathcal{O}_Y$-algebra. Consider the canonical morphism

$$j : X \longrightarrow \underline{\mathrm{Spec}}_Y(\mathcal{A})$$

see Constructions, Lemma 27.4.7. By flat base change (see for example Cohomology of Schemes, Lemma 30.5.2) we have $g_i^*f_*\mathcal{O}_X = f_{i,*}\mathcal{O}_{X'}$ where $g_i : Y_i \to Y$ are the given flat maps. Hence the base change j_i of j by g_i is the canonical morphism of Constructions, Lemma 27.4.7 for the morphism f_i. By assumption and Morphisms, Lemma 29.13.3 all of these morphisms j_i are quasi-compact open immersions. Hence, by Lemmas 35.23.1 and 35.23.16 we see that j is a quasi-compact open immersion. Hence by Morphisms, Lemma 29.13.3 again we conclude that f is quasi-affine. $\square$

02L8 Lemma 35.23.21. The property $\mathcal{P}(f)$ ="f is a quasi-compact immersion" is fpqc local on the base.

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Write $X_i = Y_i \times_Y X$ and $f_i : X_i \to Y_i$ the base change of f. Also denote $q_i : Y_i \to Y$ the given flat morphisms. Assume each f_i is a quasi-compact immersion. By Schemes, Lemma 26.23.8 each f_i is separated. By Lemmas 35.23.1 and 35.23.6 this implies that f is quasi-compact and separated. Let $X \to Z \to Y$ be the factorization of f through its scheme theoretic image. By Morphisms, Lemma 29.6.3 the closed subscheme $Z \subset Y$ is cut out by the quasi-coherent sheaf of ideals

$\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ as f is quasi-compact. By flat base change (see for example Cohomology of Schemes, Lemma 30.5.2; here we use f is separated) we see $f_{i,*}\mathcal{O}_{X_i}$ is the pullback $q_i^* f_*\mathcal{O}_X$. Hence $Y_i \times_Y Z$ is cut out by the quasi-coherent sheaf of ideals $q_i^*\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{Y_i} \to f_{i,*}\mathcal{O}_{X_i})$. By Morphisms, Lemma 29.7.7 the morphisms $X_i \to Y_i \times_Y Z$ are open immersions. Hence by Lemma 35.23.16 we see that $X \to Z$ is an open immersion and hence f is a immersion as desired (we already saw it was quasi-compact). □

02L9 Lemma 35.23.22. The property $\mathcal{P}(f) =$"f is integral" is fpqc local on the base.

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms, Lemma 29.44.7. Hence the lemma follows on combining Lemmas 35.23.3 and 35.23.18. □

02LA Lemma 35.23.23. The property $\mathcal{P}(f) =$"f is finite" is fpqc local on the base.

Proof. An finite morphism is the same thing as an integral morphism which is locally of finite type. See Morphisms, Lemma 29.44.4. Hence the lemma follows on combining Lemmas 35.23.10 and 35.23.22. □

02VI Lemma 35.23.24. The properties $\mathcal{P}(f) =$"f is locally quasi-finite" and $\mathcal{P}(f) =$"f is quasi-finite" are fpqc local on the base.

Proof. Let $f : X \to S$ be a morphism of schemes, and let $\{S_i \to S\}$ be an fpqc covering such that each base change $f_i : X_i \to S_i$ is locally quasi-finite. We have already seen (Lemma 35.23.10) that "locally of finite type" is fpqc local on the base, and hence we see that f is locally of finite type. Then it follows from Morphisms, Lemma 29.20.13 that f is locally quasi-finite. The quasi-finite case follows as we have already seen that "quasi-compact" is fpqc local on the base (Lemma 35.23.1). □

02VJ Lemma 35.23.25. The property $\mathcal{P}(f) =$"f is locally of finite type of relative dimension d" is fpqc local on the base.

Proof. This follows immediately from the fact that being locally of finite type is fpqc local on the base and Morphisms, Lemma 29.28.3. □

02VK Lemma 35.23.26. The property $\mathcal{P}(f) =$"f is syntomic" is fpqc local on the base.

Proof. A morphism is syntomic if and only if it is locally of finite presentation, flat, and has locally complete intersections as fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 35.23.15, and 35.23.11). Hence the result follows for syntomic from Morphisms, Lemma 29.30.12. □

02VL Lemma 35.23.27. The property $\mathcal{P}(f) =$"f is smooth" is fpqc local on the base.

Proof. A morphism is smooth if and only if it is locally of finite presentation, flat, and has smooth fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 35.23.15, and 35.23.11). Hence the result follows for smooth from Morphisms, Lemma 29.34.15. □

02VM Lemma 35.23.28. The property $\mathcal{P}(f) =$"f is unramified" is fpqc local on the base. The property $\mathcal{P}(f) =$"f is G-unramified" is fpqc local on the base.

Proof. A morphism is unramified (resp. G-unramified) if and only if it is locally of finite type (resp. finite presentation) and its diagonal morphism is an open immersion (see Morphisms, Lemma 29.35.13). We have seen already that being locally of finite type (resp. locally of finite presentation) and an open immersion is fpqc local on the base (Lemmas 35.23.11, 35.23.10, and 35.23.16). Hence the result follows formally. $\square$

02VN Lemma 35.23.29. The property $\mathcal{P}(f) =$"f is étale" is fpqc local on the base.

Proof. A morphism is étale if and only if it flat and G-unramified. See Morphisms, Lemma 29.36.16. We have seen already that being flat and G-unramified are fpqc local on the base (Lemmas 35.23.15, and 35.23.28). Hence the result follows. $\square$

02VO Lemma 35.23.30. The property $\mathcal{P}(f) =$"f is finite locally free" is fpqc local on the base. Let $d \geq 0$. The property $\mathcal{P}(f) =$"f is finite locally free of degree d" is fpqc local on the base.

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms, Lemma 29.48.2). Hence this follows from Lemmas 35.23.23, 35.23.15, and 35.23.11. If $f : Z \to U$ is finite locally free, and $\{U_i \to U\}$ is a surjective family of morphisms such that each pullback $Z \times_U U_i \to U_i$ has degree d, then $Z \to U$ has degree d, for example because we can read off the degree in a point $u \in U$ from the fibre $(f_*\mathcal{O}_Z)_u \otimes_{\mathcal{O}_{U,u}} \kappa(u)$. $\square$

02YK Lemma 35.23.31. The property $\mathcal{P}(f) =$"f is a monomorphism" is fpqc local on the base.

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to S_i$ of f is a monomorphism. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : S_i \times_S T \to X_i$ are the base changes. In particular the compositions $S_i \times_S T \to T \to X$ are equal. Since $\coprod S_i \times_S T \to T$ is an epimorphism (see e.g. Lemma 35.13.7) we conclude $a = b$. $\square$

0694 Lemma 35.23.32. The properties

$$\begin{aligned} &\mathcal{P}(f) =\text{"}f \text{ is a Koszul-regular immersion"}, \\ &\mathcal{P}(f) =\text{"}f \text{ is an } H_1\text{-regular immersion", and} \\ &\mathcal{P}(f) =\text{"}f \text{ is a quasi-regular immersion"} \end{aligned}$$

are fpqc local on the base.

Proof. We will use the criterion of Lemma 35.22.4 to prove this. By Divisors, Definition 31.21.1 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is Zariski local on the base. By Divisors, Lemma 31.21.4 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is preserved under flat base change. The final hypothesis (3) of Lemma 35.22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is locally on $\text{Spec}(B)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in B, then $I \subset A$ is locally on $\text{Spec}(A)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in A. This is More on Algebra, Lemma 15.32.4. $\square$

35.24. Properties of morphisms local in the fppf topology on the target

02YL In this section we find some properties of morphisms of schemes for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

02YM **Lemma 35.24.1.** *The property $\mathcal{P}(f)$ ="f is an immersion" is fppf local on the base.*

Proof. The property of being an immersion is stable under base change, see Schemes, Lemma 26.18.2. The property of being an immersion is Zariski local on the base. Finally, let $\pi : S' \to S$ be a surjective morphism of affine schemes, which is flat and locally of finite presentation. Note that $\pi : S' \to S$ is open by Morphisms, Lemma 29.25.10. Let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an immersion. In particular we see that $f'(X') = \pi^{-1}(f(X))$ is locally closed. Hence by Topology, Lemma 5.6.4 we see that $f(X) \subset S$ is locally closed. Let $Z \subset S$ be the closed subset $Z = \overline{f(X)} \setminus f(X)$. By Topology, Lemma 5.6.4 again we see that $f'(X')$ is closed in $S' \setminus Z'$. Hence we may apply Lemma 35.23.19 to the fpqc covering $\{S' \setminus Z' \to S \setminus Z\}$ and conclude that $f : X \to S \setminus Z$ is a closed immersion. In other words, f is an immersion. Therefore Lemma 35.22.4 applies and we win. □

35.25. Application of fpqc descent of properties of morphisms

02LB The following lemma may seem a bit frivolous but turns out is a useful tool in studying étale and unramified morphisms.

06NC **Lemma 35.25.1.** *Let $f : X \to Y$ be a flat, quasi-compact, surjective monomorphism. Then f is an isomorphism.*

Proof. As f is a flat, quasi-compact, surjective morphism we see $\{X \to Y\}$ is an fpqc covering of Y. The diagonal $\Delta : X \to X \times_Y X$ is an isomorphism (Schemes, Lemma 26.23.2). This implies that the base change of f by f is an isomorphism. Hence we see f is an isomorphism by Lemma 35.23.17. □

We can use this lemma to show the following important result; we also give a proof avoiding fpqc descent. We will discuss this and related results in more detail in Étale Morphisms, Section 41.14.

02LC **Lemma 35.25.2.** *A universally injective étale morphism is an open immersion.*

First proof. Let $f : X \to Y$ be an étale morphism which is universally injective. Then f is open (Morphisms, Lemma 29.36.13) hence we can replace Y by $f(X)$ and we may assume that f is surjective. Then f is bijective and open hence a homeomorphism. Hence f is quasi-compact. Thus by Lemma 35.25.1 it suffices to show that f is a monomorphism. As $X \to Y$ is étale the morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion by Morphisms, Lemma 29.35.13 (and Morphisms, Lemma 29.36.16). As f is universally injective $\Delta_{X/Y}$ is also surjective, see Morphisms, Lemma 29.10.2. Hence $\Delta_{X/Y}$ is an isomorphism, i.e., $X \to Y$ is a monomorphism. □

Second proof. Let $f : X \to Y$ be an étale morphism which is universally injective. Then f is open (Morphisms, Lemma 29.36.13) hence we can replace Y by $f(X)$ and we may assume that f is surjective. Since the hypotheses remain satisfied after any base change, we conclude that f is a universal homeomorphism. Therefore f is integral, see Morphisms, Lemma 29.45.5. It follows that f is finite by Morphisms, Lemma 29.44.4. It follows that f is finite locally free by Morphisms, Lemma 29.48.2. To finish the proof, it suffices that f is finite locally free of degree 1 (a finite locally free morphism of degree 1 is an isomorphism). There is decomposition of Y into open and closed subschemes V_d such that $f^{-1}(V_d) \to V_d$ is finite locally free of degree d, see Morphisms, Lemma 29.48.5. If V_d is not empty, we can pick a morphism $\mathrm{Spec}(k) \to V_d \subset Y$ where k is an algebraically closed field (just take the algebraic closure of the residue field of some point of V_d). Then $\mathrm{Spec}(k) \times_Y X \to \mathrm{Spec}(k)$ is a disjoint union of copies of $\mathrm{Spec}(k)$, by Morphisms, Lemma 29.36.7 and the fact that k is algebraically closed. However, since f is universally injective, there can only be one copy and hence $d = 1$ as desired. □

We can reformulate the hypotheses in the lemma above a bit by using the following characterization of flat universally injective morphisms.

09NP Lemma 35.25.3. Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If

(1) f is flat and separated,
(2) for $\xi \in X^0$ we have $\kappa(f(\xi)) = \kappa(\xi)$, and
(3) if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,

then f is universally injective.

Proof. We have to show that $\Delta : X \to X \times_Y X$ is surjective, see Morphisms, Lemma 29.10.2. As $X \to Y$ is separated, the image of Δ is closed. Thus if Δ is not surjective, we can find a generic point $\eta \in X \times_S X$ of an irreducible component of $X \times_S X$ which is not in the image of Δ. The projection $\mathrm{pr}_1 : X \times_Y X \to X$ is flat as a base change of the flat morphism $X \to Y$, see Morphisms, Lemma 29.25.8. Hence generalizations lift along pr_1, see Morphisms, Lemma 29.25.9. We conclude that $\xi = \mathrm{pr}_1(\eta) \in X^0$. However, assumptions (2) and (3) guarantee that the scheme $(X \times_Y X)_{f(\xi)}$ has at most one point for every $\xi \in X^0$. In other words, we have $\Delta(\xi) = \eta$ a contradiction. □

Thus we can reformulate Lemma 35.25.2 as follows.

09NQ Lemma 35.25.4. Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If

(1) f is étale and separated,
(2) for $\xi \in X^0$ we have $\kappa(f(\xi)) = \kappa(\xi)$, and
(3) if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,

then f is an open immersion.

Proof. Immediate from Lemmas 35.25.3 and 35.25.2. □

0F4J Lemma 35.25.5. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let Z be a closed subset of X. If there exists an fpqc covering $\{Y_i \to Y\}$ such that the inverse image $Z_i \subset Y_i \times_Y X$ is proper over Y_i (Cohomology of Schemes, Definition 30.26.2) then Z is proper over Y.

Proof. Endow Z with the reduced induced closed subscheme structure, see Schemes, Definition 26.12.5. For every i the base change $Y_i \times_Y Z$ is a closed subscheme of $Y_i \times_Y X$ whose underlying closed subset is Z_i. By definition (via Cohomology of Schemes, Lemma 30.26.1) we conclude that the projections $Y_i \times_Y Z \to Y_i$ are proper morphisms. Hence $Z \to Y$ is a proper morphism by Lemma 35.23.14. Thus Z is proper over Y by definition. $\square$

0D2P Lemma 35.25.6. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\{g_i : S_i \to S\}_{i \in I}$ be an fpqc covering. Let $f_i : X_i \to S_i$ be the base change of f and let $\mathcal{L}_i$ be the pullback of $\mathcal{L}$ to X_i. The following are equivalent

(1) $\mathcal{L}$ is ample on X/S, and
(2) $\mathcal{L}_i$ is ample on X_i/S_i for every $i \in I$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Morphisms, Lemma 29.37.9. Assume $\mathcal{L}_i$ is ample on X_i/S_i for every $i \in I$. By Morphisms, Definition 29.37.1 this implies that $X_i \to S_i$ is quasi-compact and by Morphisms, Lemma 29.37.3 this implies $X_i \to S$ is separated. Hence f is quasi-compact and separated by Lemmas 35.23.1 and 35.23.6.

This means that $\mathcal{A} = \bigoplus_{d \geq 0} f_*\mathcal{L}^{\otimes d}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra (Schemes, Lemma 26.24.1). Moreover, the formation of $\mathcal{A}$ commutes with flat base change by Cohomology of Schemes, Lemma 30.5.2. In particular, if we set $\mathcal{A}_i = \bigoplus_{d \geq 0} f_{i,*}\mathcal{L}_i^{\otimes d}$ then we have $\mathcal{A}_i = g_i^*\mathcal{A}$. It follows that the natural maps $\psi_d : f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ of $\mathcal{O}_X$ pullback to give the natural maps $\psi_{i,d} : f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ of $\mathcal{O}_{X_i}$-modules. Since $\mathcal{L}_i$ is ample on X_i/S_i we see that for any point $x_i \in X_i$, there exists a $d \geq 1$ such that $f_i^*(\mathcal{A}_i)_d \to \mathcal{L}_i^{\otimes d}$ is surjective on stalks at x_i. This follows either directly from the definition of a relatively ample module or from Morphisms, Lemma 29.37.4. If $x \in X$, then we can choose an i and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is flat hence faithfully flat, we conclude that for every $x \in X$ there exists a $d \geq 1$ such that $f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ is surjective on stalks at x. This implies that the open subset $U(\psi) \subset X$ of Constructions, Lemma 27.19.1 corresponding to the map $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ of graded $\mathcal{O}_X$-algebras is equal to X. Consider the corresponding morphism

$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{\text{Proj}}_S(\mathcal{A})$$

It is clear from the above that the base change of $r_{\mathcal{L},\psi}$ to S_i is the morphism $r_{\mathcal{L}_i,\psi_i}$ which is an open immersion by Morphisms, Lemma 29.37.4. Hence $r_{\mathcal{L},\psi}$ is an open immersion by Lemma 35.23.16 and we conclude $\mathcal{L}$ is ample on X/S by Morphisms, Lemma 29.37.4. $\square$

35.26. Properties of morphisms local on the source

036F It often happens one can prove a morphism has a certain property after precomposing with some other morphism. In many cases this implies the morphism has the property too. We formalize this in the following definition.

036G Definition 35.26.1. Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{Zariski, fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. We say $\mathcal{P}$ is τ local on the source, or local on the source for the τ-topology if for any morphism of schemes $f : X \to Y$ over S, and any τ-covering $\{X_i \to X\}_{i \in I}$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

04QV Lemma 35.26.2. Let $\tau \in \{fpqc, fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. □

0CEY Lemma 35.26.3. Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the source. For any morphism of schemes $f : X \to Y$ there exists a largest open $W(f) \subset X$ such that the restriction $f|_{W(f)} : W(f) \to Y$ has $\mathcal{P}$. Moreover, if $g : X' \to X$ is flat and locally of finite presentation, syntomic, smooth, or étale and $f' = f \circ g : X' \to Y$, then $g^{-1}(W(f)) = W(f')$.

Proof. Consider the union W of the images $g(X') \subset X$ of morphisms $g : X' \to X$ with the properties:

(1) g is flat and locally of finite presentation, syntomic, smooth, or étale, and
(2) the composition $X' \to X \to Y$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms, Lemma 29.25.10) we see that $W \subset X$ is an open subset of X. Since $\mathcal{P}$ is local in the τ topology the restriction $f|_W : W \to Y$ has property $\mathcal{P}$ because we are given a τ covering $\{X' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence of $W(f)$. The compatibility stated in the last sentence follows immediately from the construction of $W(f)$. □

036H Lemma 35.26.4. Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{fpqc, fppf, \acute{e}tale, smooth, syntomic\}$. Assume that

(1) the property is preserved under precomposing with flat, flat locally of finite presentation, étale, smooth or syntomic morphisms depending on whether τ is fpqc, fppf, étale, smooth, or syntomic,
(2) the property is Zariski local on the source,
(3) the property is Zariski local on the target,
(4) for any morphism of affine schemes $f : X \to Y$, and any surjective morphism of affine schemes $X' \to X$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the composition $f' : X' \to Y$.

Then $\mathcal{P}$ is τ local on the source.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 34.9.1 34.7.1 34.4.1 34.5.1, or 34.6.1 and Topologies, Lemma 34.9.8, 34.7.4, 34.4.4, 34.5.4, or 34.6.4. Details omitted. (Hint: Use locality on the source and target to reduce the verification of property $\mathcal{P}$ to the case of a morphism between affines. Then apply (1) and (4).) □

036I Remark 35.26.5. (This is a repeat of Remarks 35.15.3 and 35.22.5 above.) In Lemma 35.26.4 above if $\tau = smooth$ then in condition (4) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = \acute{e}tale$.

35.27. Properties of morphisms local in the fpqc topology on the source

036J Here are some properties of morphisms that are fpqc local on the source.

036K **Lemma 35.27.1.** *The property $\mathcal{P}(f) =$"f is flat" is fpqc local on the source.*

Proof. Since flatness is defined in terms of the maps of local rings (Morphisms, Definition 29.25.1) what has to be shown is the following algebraic fact: Suppose $A \to B \to C$ are local homomorphisms of local rings, and assume $B \to C$ is flat. Then $A \to B$ is flat if and only if $A \to C$ is flat. If $A \to B$ is flat, then $A \to C$ is flat by Algebra, Lemma 10.39.4. Conversely, assume $A \to C$ is flat. Note that $B \to C$ is faithfully flat, see Algebra, Lemma 10.39.17. Hence $A \to B$ is flat by Algebra, Lemma 10.39.10. (Also see Morphisms, Lemma 29.25.13 for a direct proof.) □

036L **Lemma 35.27.2.** *Then property $\mathcal{P}(f : X \to Y) =$"for every $x \in X$ the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective" is fpqc local on the source.*

Proof. Omitted. This is just a (probably misguided) attempt to be playful. □

35.28. Properties of morphisms local in the fppf topology on the source

036M Here are some properties of morphisms that are fppf local on the source.

036N **Lemma 35.28.1.** *The property $\mathcal{P}(f) =$"f is locally of finite presentation" is fppf local on the source.*

Proof. Being locally of finite presentation is Zariski local on the source and the target, see Morphisms, Lemma 29.21.2. It is a property which is preserved under composition, see Morphisms, Lemma 29.21.3. This proves (1), (2) and (3) of Lemma 35.26.4. The final condition (4) is Lemma 35.14.1. Hence we win. □

036O **Lemma 35.28.2.** *The property $\mathcal{P}(f) =$"f is locally of finite type" is fppf local on the source.*

Proof. Being locally of finite type is Zariski local on the source and the target, see Morphisms, Lemma 29.15.2. It is a property which is preserved under composition, see Morphisms, Lemma 29.15.3, and a flat morphism locally of finite presentation is locally of finite type, see Morphisms, Lemma 29.21.8. This proves (1), (2) and (3) of Lemma 35.26.4. The final condition (4) is Lemma 35.14.2. Hence we win. □

036P **Lemma 35.28.3.** *The property $\mathcal{P}(f) =$"f is open" is fppf local on the source.*

Proof. Being an open morphism is clearly Zariski local on the source and the target. It is a property which is preserved under composition, see Morphisms, Lemma 29.23.3, and a flat morphism of finite presentation is open, see Morphisms, Lemma 29.25.10 This proves (1), (2) and (3) of Lemma 35.26.4. The final condition (4) follows from Morphisms, Lemma 29.25.12. Hence we win. □

036Q **Lemma 35.28.4.** *The property $\mathcal{P}(f) =$"f is universally open" is fppf local on the source.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering. Denote $f_i : X_i \to X$ the compositions. We have to show that f is universally open if and only if each f_i is universally open. If f is universally open, then also each f_i is universally open since the maps $X_i \to X$ are universally open and compositions of universally open morphisms are universally open (Morphisms, Lemmas 29.25.10 and 29.23.3). Conversely, assume each f_i is universally open. Let $Y' \to Y$ be a morphism of schemes. Denote $X' = Y' \times_Y X$ and $X'_i = Y' \times_Y X_i$. Note that $\{X'_i \to X'\}_{i \in I}$ is an fppf covering also. The morphisms $f'_i : X'_i \to Y'$ are open by assumption. Hence by the Lemma 35.28.3 above we conclude that $f' : X' \to Y'$ is open as desired. □

35.29. Properties of morphisms local in the syntomic topology on the source

036R Here are some properties of morphisms that are syntomic local on the source.

036S Lemma 35.29.1. The property $\mathcal{P}(f) =$"f is syntomic" is syntomic local on the source.

Proof. Combine Lemma 35.26.4 with Morphisms, Lemma 29.30.2 (local for Zariski on source and target), Morphisms, Lemma 29.30.3 (pre-composing), and Lemma 35.14.4 (part (4)). □

35.30. Properties of morphisms local in the smooth topology on the source

036T Here are some properties of morphisms that are smooth local on the source. Note also the (in some respects stronger) result on descending smoothness via flat morphisms, Lemma 35.14.5.

036U Lemma 35.30.1. The property $\mathcal{P}(f) =$"f is smooth" is smooth local on the source.

Proof. Combine Lemma 35.26.4 with Morphisms, Lemma 29.34.2 (local for Zariski on source and target), Morphisms, Lemma 29.34.4 (pre-composing), and Lemma 35.14.4 (part (4)). □

35.31. Properties of morphisms local in the étale topology on the source

036V Here are some properties of morphisms that are étale local on the source.

036W Lemma 35.31.1. The property $\mathcal{P}(f) =$"f is étale" is étale local on the source.

Proof. Combine Lemma 35.26.4 with Morphisms, Lemma 29.36.2 (local for Zariski on source and target), Morphisms, Lemma 29.36.3 (pre-composing), and Lemma 35.14.4 (part (4)). □

03X4 Lemma 35.31.2. The property $\mathcal{P}(f) =$"f is locally quasi-finite" is étale local on the source.

Proof. We are going to use Lemma 35.26.4. By Morphisms, Lemma 29.20.11 the property of being locally quasi-finite is local for Zariski on source and target. By Morphisms, Lemmas 29.20.12 and 29.36.6 we see the precomposition of a locally quasi-finite morphism by an étale morphism is locally quasi-finite. Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is locally quasi-finite. Then $X' \to Y$ is of finite type, and by Lemma 35.14.2 we see that $X \to Y$ is of finite type also. Moreover, by assumption $X' \to Y$ has finite fibres, and hence $X \to Y$ has finite fibres also. We conclude that $X \to Y$ is quasi-finite by Morphisms, Lemma 29.20.10. This proves the last assumption of Lemma 35.26.4 and finishes the proof. □

03YV Lemma 35.31.3. The property $\mathcal{P}(f) =$"f is unramified" is étale local on the source. The property $\mathcal{P}(f) =$"f is G-unramified" is étale local on the source.

Proof. We are going to use Lemma 35.26.4. By Morphisms, Lemma 29.35.3 the property of being unramified (resp. G-unramified) is local for Zariski on source and target. By Morphisms, Lemmas 29.35.4 and 29.36.5 we see the precomposition of an unramified (resp. G-unramified) morphism by an étale morphism is unramified (resp. G-unramified). Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $f : X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is unramified (resp. G-unramified). Then $X' \to Y$ is of finite type (resp. finite presentation), and by Lemma 35.14.2 (resp. Lemma 35.14.1) we see that $X \to Y$ is of finite type (resp. finite presentation) also. By Morphisms, Lemma 29.34.16 we have a short exact sequence

$$0 \to f^*\Omega_{X/Y} \to \Omega_{X'/Y} \to \Omega_{X'/X} \to 0.$$

As $X' \to Y$ is unramified we see that the middle term is zero. Hence, as f is faithfully flat we see that $\Omega_{X/Y} = 0$. Hence $X \to Y$ is unramified (resp. G-unramified), see Morphisms, Lemma 29.35.2. This proves the last assumption of Lemma 35.26.4 and finishes the proof. □

35.32. Properties of morphisms étale local on source-and-target

04QW Let $\mathcal{P}$ be a property of morphisms of schemes. There is an intuitive meaning to the phrase "$\mathcal{P}$ is étale local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both étale local on the source and étale local on the target. Before we discuss this further we give two silly examples.

04QX Example 35.32.1. Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(X \to Y) =$"Y is locally Noetherian". The reader can verify that this is étale local on the source and étale local on the target (omitted, see Lemma 35.16.1). But it is not true that if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$. Namely, f could be the identity on Y and g could be an open immersion of a locally Noetherian scheme Y into a non locally Noetherian scheme Z.

The following example is in some sense worse.

04QY Example 35.32.2. Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(f : X \to Y) =$"for every $y \in Y$ which is a specialization of some $f(x)$, $x \in X$ the local ring $\mathcal{O}_{Y,y}$ is Noetherian". Let us verify that this is étale local on the source and étale local on the target. We will freely use Schemes, Lemma 26.13.2.

Local on the target: Let $\{g_i : Y_i \to Y\}$ be an étale covering. Let $f_i : X_i \to Y_i$ be the base change of f, and denote $h_i : X_i \to X$ the projection. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y_i$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow g_i(y_i)$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,g_i(y_i)}$ is Noetherian. Also $\mathcal{O}_{Y,g_i(y_i)} \to \mathcal{O}_{Y_i,y_i}$ is a localization of an étale ring map. Hence $\mathcal{O}_{Y_i,y_i}$ is Noetherian by Algebra, Lemma 10.31.1. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $y_i \in Y_i$ mapping to y. Since x can be viewed as a point of $\mathrm{Spec}(\mathcal{O}_{Y,y}) \times_Y X$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y_i,y_i}$ is faithfully flat, there exists a point $x_i \in \mathrm{Spec}(\mathcal{O}_{Y_i,y_i}) \times_Y X$ mapping to x. Then $x_i \in X_i$, and $f_i(x_i)$ specializes to y_i. Thus we see that $\mathcal{O}_{Y_i,y_i}$ is Noetherian by $\mathcal{P}(f_i)$ which implies that $\mathcal{O}_{Y,y}$ is Noetherian by Algebra, Lemma 10.164.1.

Local on the source: Let $\{h_i : X_i \to X\}$ be an étale covering. Let $f_i : X_i \to Y$ be the composition $f \circ h_i$. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow y$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,y}$ is Noetherian. Thus $\mathcal{P}(f_i)$ holds. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $x_i \in X_i$ mapping to x. Then y is a specialization of $f_i(x_i) = f(x)$. Hence $\mathcal{P}(f_i)$ implies $\mathcal{O}_{Y,y}$ is Noetherian as desired.

We claim that there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with surjective étale vertical arrows, such that h has $\mathcal{P}$ and f does not have $\mathcal{P}$. Namely, let

$$Y = \mathrm{Spec}\left(\mathbf{C}[x_n; n \in \mathbf{Z}]/(x_n x_m; n \neq m)\right)$$

and let $X \subset Y$ be the open subscheme which is the complement of the point all of whose coordinates $x_n = 0$. Let $U = X$, let $V = X \amalg Y$, let a, b the obvious map, and let $h : U \to V$ be the inclusion of $U = X$ into the first summand of V. The claim above holds because U is locally Noetherian, but Y is not.

What should be the correct notion of a property which is étale local on the source-and-target? We think that, by analogy with Morphisms, Definition 29.14.1 it should be the following.

04QZ Definition 35.32.3. Let $\mathcal{P}$ be a property of morphisms of schemes. We say $\mathcal{P}$ is étale local on source-and-target if

(1) (stable under precomposing with étale maps) if $f : X \to Y$ is étale and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under étale base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is étale, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
  (a) f has $\mathcal{P}$,
  (b) for every $x \in X$ there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"'] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

  with étale vertical arrows and $u \in U$ with $a(u) = x$ such that h has $\mathcal{P}$.

It turns out this definition excludes the behavior seen in Examples 35.32.1 and 35.32.2. We will compare this to the definition in the paper [DM69] by Deligne and Mumford in Remark 35.32.8. Moreover, a property which is étale local on the source-and-target is étale local on the source and étale local on the target. Finally, the converse is almost true as we will see in Lemma 35.32.6.

04R0 Lemma 35.32.4. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Then

(1) $\mathcal{P}$ is étale local on the source,
(2) $\mathcal{P}$ is étale local on the target,
(3) $\mathcal{P}$ is stable under postcomposing with étale morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$, and

(4) $\mathcal{P}$ has a permanence property: given $f : X \to Y$ and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$, then f has $\mathcal{P}$.

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an étale covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $x \in X$ we can find an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, and $\mathrm{id}_Y : Y \to Y$ is an étale morphism, and h_i is as in part (3) of Definition 35.32.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 35.32.3 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}_{i \in I}$ be an étale covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in X$ we pick an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, $Y_i \to Y$ is étale, and h_i is as in part (3) of Definition 35.32.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 35.32.3 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale. For every $x \in X$ we can think of $(X, x) \to (X, x)$ as an étale morphism of germs, $Y \to Z$ is an étale morphism, and $h = f$ is as in part (3) of Definition 35.32.3. Thus we see that $g \circ f$ has $\mathcal{P}$.

Proof of (4). Let $f : X \to Y$ be a morphism and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$. Then by Definition 35.32.3 part (2) we see that $\mathrm{pr}_Y : Y \times_Z X \to Y$ has $\mathcal{P}$. But the morphism $(f, 1) : X \to Y \times_Z X$ is étale as a section to the étale projection $\mathrm{pr}_X : Y \times_Z X \to X$, see Morphisms, Lemma 29.36.18. Hence $f = \mathrm{pr}_Y \circ (f, 1)$ has $\mathcal{P}$ by Definition 35.32.3 part (1). $\square$

The following lemma is the analogue of Morphisms, Lemma 29.14.4.

04R1 Lemma 35.32.5. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

(a) f has property $\mathcal{P}$,
(b) for every $x \in X$ there exists an étale morphism of germs $a : (U, u) \to (X, x)$, an étale morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,
(c) for any commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b étale the morphism h has $\mathcal{P}$,
(d) for some diagram as in (c) with $a : U \to X$ surjective h has $\mathcal{P}$,
(e) there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,
(f) there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,

(g) there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ an étale covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.

Proof. The equivalence of (a) and (b) is part of Definition 35.32.3. The equivalence of (a) and (e) is Lemma 35.32.4 part (2). The equivalence of (a) and (f) is Lemma 35.32.4 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (a). If (a) holds, then for any diagram as in (c) the morphism $f \circ a$ has $\mathcal{P}$ by Definition 35.32.3 part (1), whereupon h has $\mathcal{P}$ by Lemma 35.32.4 part (4). Thus (a) and (c) are equivalent. It is clear that (c) implies (d). To see that (d) implies (a) assume we have a diagram as in (c) with $a : U \to X$ surjective and h having $\mathcal{P}$. Then $b \circ h$ has $\mathcal{P}$ by Lemma 35.32.4 part (3). Since $\{a : U \to X\}$ is an étale covering we conclude that f has $\mathcal{P}$ by Lemma 35.32.4 part (1). □

It seems that the result of the following lemma is not a formality, i.e., it actually uses something about the geometry of étale morphisms.

04R2 Lemma 35.32.6. Let $\mathcal{P}$ be a property of morphisms of schemes. Assume

(1) $\mathcal{P}$ is étale local on the source,
(2) $\mathcal{P}$ is étale local on the target, and
(3) $\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open subscheme then $X \to Z$ has $\mathcal{P}$.

Then $\mathcal{P}$ is étale local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of schemes which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 35.26.2 we see that $\mathcal{P}$ is stable under precomposing with étale morphisms. By Lemma 35.22.2 we see that $\mathcal{P}$ is stable under étale base change. Hence it suffices to prove part (3) of Definition 35.32.3 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of schemes which satisfies Definition 35.32.3 part (3)(b). In other words, for every $x \in X$ there exists an étale morphism $a_x : U_x \to X$, a point $u_x \in U_x$ mapping to x, an étale morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with a, b étale, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is étale local on the target. Because a is surjective and $\mathcal{P}$ is étale local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This reduces the lemma to proving that $\mathcal{P}$ is stable under postcomposing with an étale morphism.

During the rest of the proof we let $f : X \to Y$ be a morphism with property $\mathcal{P}$ and $g : Y \to Z$ is an étale morphism. Consider the following statements:

(-) With no additional assumptions $g \circ f$ has property $\mathcal{P}$.
(A) Whenever Z is affine $g \circ f$ has property $\mathcal{P}$.

(AA) Whenever X and Z are affine $g \circ f$ has property $\mathcal{P}$.
(AAA) Whenever X, Y, and Z are affine $g \circ f$ has property $\mathcal{P}$.

Once we have proved (-) the proof of the lemma will be complete.

Claim 1: (AAA) $\Rightarrow$ (AA). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with X, Z affine. As X is affine hence quasi-compact we can find finitely many affine open $Y_i \subset Y$, $i = 1, \ldots, n$ such that $X = \bigcup_{i=1,\ldots,n} f^{-1}(Y_i)$. Set $X_i = f^{-1}(Y_i)$. By Lemma 35.22.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. Hence $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ has $\mathcal{P}$ as $\mathcal{P}$ is étale local on the target. By (AAA) applied to $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ and the étale morphism $\coprod_{i=1,\ldots,n} Y_i \to Z$ we see that $\coprod_{i=1,\ldots,n} X_i \to Z$ has $\mathcal{P}$. Now $\{\coprod_{i=1,\ldots,n} X_i \to X\}$ is an étale covering, hence as $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$ as desired.

Claim 2: (AAA) $\Rightarrow$ (A). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with Z affine. Choose an affine open covering $X = \bigcup X_i$. As $\mathcal{P}$ is étale local on the source we see that each $f|_{X_i} : X_i \to Y$ has $\mathcal{P}$. By (AA), which follows from (AAA) according to Claim 1, we see that $X_i \to Z$ has $\mathcal{P}$ for each i. Since $\{X_i \to X\}$ is an étale covering and $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$.

Claim 3: (AAA) $\Rightarrow$ (-). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above. Choose an affine open covering $Z = \bigcup Z_i$. Set $Y_i = g^{-1}(Z_i)$ and $X_i = f^{-1}(Y_i)$. By Lemma 35.22.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. By (A), which follows from (AAA) according to Claim 2, we see that $X_i \to Z_i$ has $\mathcal{P}$ for each i. Since $\mathcal{P}$ is local on the target and $X_i = (g \circ f)^{-1}(Z_i)$ we conclude that $X \to Z$ has $\mathcal{P}$.

Thus to prove the lemma it suffices to prove (AAA). Let $f : X \to Y$ and $g : Y \to Z$ be as above X, Y, Z affine. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemma 29.36.6 and Lemma 29.57.9. Hence we can do induction on the integer n bounding the degree of the fibres of $Y \to Z$. See Morphisms, Lemma 29.57.8 for a description of this integer in the case of an étale morphism. If $n = 1$, then $Y \to Z$ is an open immersion, see Lemma 35.25.2, and the result follows from assumption (3) of the lemma. Assume $n > 1$.

Consider the following commutative diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f_Y"'] \arrow[d] & Y \times_Z Y \arrow[r, "\mathrm{pr}"'] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Note that we have a decomposition into open and closed subschemes $Y \times_Z Y = \Delta_{Y/Z}(Y) \amalg Y'$, see Morphisms, Lemma 29.35.13. As a base change the degrees of the fibres of the second projection $\mathrm{pr} : Y \times_Z Y \to Y$ are bounded by n, see Morphisms, Lemma 29.57.5. On the other hand, $\mathrm{pr}|_{\Delta(Y)} : \Delta(Y) \to Y$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 29.57.8 we conclude the degrees of the fibres of the restriction $\mathrm{pr}|_{Y'} : Y' \to Y$ are bounded by $n - 1$. Set $X' = f_Y^{-1}(Y')$. Picture

$$\begin{tikzcd} X \amalg X' \arrow[r, "f \amalg f'"'] \arrow[d, equal] & \Delta(Y) \amalg Y' \arrow[r] \arrow[d, equal] & Y \arrow[d, equal] \\ X \times_Z Y \arrow[r, "f_Y"] & Y \times_Z Y \arrow[r, "\mathrm{pr}"] & Y \end{tikzcd}$$

As $\mathcal{P}$ is étale local on the target and hence stable under étale base change (see Lemma 35.22.2) we see that f_Y has $\mathcal{P}$. Hence, as $\mathcal{P}$ is étale local on the source, $f' = f_Y|_{X'}$ has $\mathcal{P}$. By induction hypothesis we see that $X' \to Y$ has $\mathcal{P}$. As $\mathcal{P}$ is local on the source, and $\{X \to X \times_Z Y, X' \to X \times_Y Z\}$ is an étale covering, we conclude that $\mathrm{pr} \circ f_Y$ has $\mathcal{P}$. Note that $g \circ f$ can be viewed as a morphism $g \circ f : X \to g(Y)$. As $\mathrm{pr} \circ f_Y$ is the pullback of $g \circ f : X \to g(Y)$ via the étale covering $\{Y \to g(Y)\}$, and as $\mathcal{P}$ is étale local on the target, we conclude that $g \circ f : X \to g(Y)$ has property $\mathcal{P}$. Finally, applying assumption (3) of the lemma once more we conclude that $g \circ f : X \to Z$ has property $\mathcal{P}$. □

04R3 Remark 35.32.7. Using Lemma 35.32.6 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are étale local on the source-and-target. In each case we list the lemma which implies the property is étale local on the source and the lemma which implies the property is étale local on the target. In each case the third assumption of Lemma 35.32.6 is trivial to check, and we omit it. Here is the list:

(1) flat, see Lemmas 35.27.1 and 35.23.15,
(2) locally of finite presentation, see Lemmas 35.28.1 and 35.23.11,
(3) locally finite type, see Lemmas 35.28.2 and 35.23.10,
(4) universally open, see Lemmas 35.28.4 and 35.23.4,
(5) syntomic, see Lemmas 35.29.1 and 35.23.26,
(6) smooth, see Lemmas 35.30.1 and 35.23.27,
(7) étale, see Lemmas 35.31.1 and 35.23.29,
(8) locally quasi-finite, see Lemmas 35.31.2 and 35.23.24,
(9) unramified, see Lemmas 35.31.3 and 35.23.28,
(10) G-unramified, see Lemmas 35.31.3 and 35.23.28, and
(11) add more here as needed.

04R4 Remark 35.32.8. At this point we have three possible definitions of what it means for a property $\mathcal{P}$ of morphisms to be "étale local on the source and target":

(ST) $\mathcal{P}$ is étale local on the source and $\mathcal{P}$ is étale local on the target,
(DM) (the definition in the paper [DM69, Page 100] by Deligne and Mumford) for every diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "a"'] & V \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with surjective étale vertical arrows we have $\mathcal{P}(h) \Leftrightarrow \mathcal{P}(f)$, and
(SP) $\mathcal{P}$ is étale local on the source-and-target.

In this section we have seen that (SP) $\Rightarrow$ (DM) $\Rightarrow$ (ST). The Examples 35.32.1 and 35.32.2 show that neither implication can be reversed. Finally, Lemma 35.32.6 shows that the difference disappears when looking at properties of morphisms which are stable under postcomposing with open immersions, which in practice will always be the case.

0CEZ Lemma 35.32.9. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Given a commutative diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "f'"'] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \text{with points} \quad \begin{tikzcd} x' \arrow[r] \arrow[d] & y' \arrow[d] \\ x \arrow[r] & y \end{tikzcd}$$

such that g' is étale at x' and g is étale at y', then $x \in W(f) \Leftrightarrow x' \in W(f')$ where $W(-)$ is as in Lemma 35.26.3.

Proof. Lemma 35.26.3 applies since $\mathcal{P}$ is étale local on the source by Lemma 35.32.4.

Assume $x \in W(f)$. Let $U' \subset X'$ and $V' \subset Y'$ be open neighbourhoods of x' and y' such that $f'(U') \subset V'$, $g'(U') \subset W(f)$ and $g'|_{U'}$ and $g|_{V'}$ are étale. Then $f \circ g'|_{U'} = g \circ f'|_{U'}$ has $\mathcal{P}$ by property (1) of Definition 35.32.3. Then $f'|_{U'} : U' \to V'$ has property $\mathcal{P}$ by (4) of Lemma 35.32.4. Then by (3) of Lemma 35.32.4 we conclude that $f'_{U'} : U' \to Y'$ has $\mathcal{P}$. Hence $U' \subset W(f')$ by definition. Hence $x' \in W(f')$.

Assume $x' \in W(f')$. Let $U' \subset X'$ and $V' \subset Y'$ be open neighbourhoods of x' and y' such that $f'(U') \subset V'$, $U' \subset W(f')$ and $g'|_{U'}$ and $g|_{V'}$ are étale. Then $U' \to Y'$ has $\mathcal{P}$ by definition of $W(f')$. Then $U' \to V'$ has $\mathcal{P}$ by (4) of Lemma 35.32.4. Then $U' \to Y$ has $\mathcal{P}$ by (3) of Lemma 35.32.4. Let $U \subset X$ be the image of the étale (hence open) morphism $g'|_U' : U' \to X$. Then $\{U' \to U\}$ is an étale covering and we conclude that $U \to Y$ has $\mathcal{P}$ by (1) of Lemma 35.32.4. Thus $U \subset W(f)$ by definition. Hence $x \in W(f)$. □

0CF0 Lemma 35.32.10. Let k be a field. Let $n \geq 2$. For $1 \leq i, j \leq n$ with $i \neq j$ and $d \geq 0$ denote $T_{i,j,d}$ the automorphism of $\mathbf{A}^n_k$ given in coordinates by

$$(x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_{i-1}, x_i + x_j^d, x_{i+1}, \ldots, x_n)$$

Let $W \subset \mathbf{A}^n_k$ be a nonempty open subscheme such that $T_{i,j,d}(W) = W$ for all i, j, d as above. Then either $W = \mathbf{A}^n_k$ or the characteristic of k is $p > 0$ and $\mathbf{A}^n_k \setminus W$ is a finite set of closed points whose coordinates are algebraic over $\mathbf{F}_p$.

Proof. We may replace k by any extension field in order to prove this. Let Z be an irreducible component of $\mathbf{A}^n_k \setminus W$. Assume $\dim(Z) \geq 1$, to get a contradiction. Then there exists an extension field k'/k and a k'-valued point $\xi = (\xi_1, \ldots, \xi_n) \in (k')^n$ of $Z_{k'} \subset \mathbf{A}^n_{k'}$ such that at least one of $x_1, \ldots, x_n$ is transcendental over the prime field. Claim: the orbit of ξ under the group generated by the transformations $T_{i,j,d}$ is Zariski dense in $\mathbf{A}^n_{k'}$. The claim will give the desired contradiction.

If the characteristic of k' is zero, then already the operators $T_{i,j,0}$ will be enough since these transform ξ into the points

$$(\xi_1 + a_1, \ldots, \xi_n + a_n)$$

for arbitrary $(a_1, \ldots, a_n) \in \mathbf{Z}^n_{\geq 0}$. If the characteristic is $p > 0$, we may assume after renumbering that ξ_n is transcendental over $\mathbf{F}_p$. By successively applying the operators $T_{i,n,d}$ for $i < n$ we see the orbit of ξ contains the elements

$$(\xi_1 + P_1(\xi_n), \ldots, \xi_{n-1} + P_{n-1}(\xi_n), \xi_n)$$

for arbitrary $(P_1, \ldots, P_{n-1}) \in \mathbf{F}_p[t]$. Thus the Zariski closure of the orbit contains the coordinate hyperplane $x_n = \xi_n$. Repeating the argument with a different coordinate, we conclude that the Zariski closure contains $x_i = \xi_i + P(\xi_n)$ for any

$P \in \mathbf{F}_p[t]$ such that $\xi_i + P(\xi_n)$ is transcendental over $\mathbf{F}_p$. Since there are infinitely many such P the claim follows.

Of course the argument in the preceding paragraph also applies if $Z = \{z\}$ has dimension 0 and the coordinates of z in $\kappa(z)$ are not algebraic over $\mathbf{F}_p$. The lemma follows. $\square$

0CF1 **Lemma 35.32.11.** Let $\mathcal{P}$ be a property of morphisms of schemes. Assume

(1) $\mathcal{P}$ is étale local on the source,
(2) $\mathcal{P}$ is smooth local on the target,
(3) $\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open subscheme then $X \to Z$ has $\mathcal{P}$.

Given a commutative diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "f'"'] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \text{with points} \quad \begin{tikzcd} x' \arrow[r] \arrow[d] & y' \arrow[d] \\ x \arrow[r] & y \end{tikzcd}$$

such that g is smooth y' and $X' \to X \times_Y Y'$ is étale at x', then $x \in W(f) \Leftrightarrow x' \in W(f')$ where $W(-)$ is as in Lemma 35.26.3.

Proof. Since $\mathcal{P}$ is étale local on the source we see that $x \in W(f)$ if and only if the image of x in $X \times_Y Y'$ is in $W(X \times_Y Y' \to Y')$. Hence we may assume the diagram in the lemma is cartesian.

Assume $x \in W(f)$. Since $\mathcal{P}$ is smooth local on the target we see that $(g')^{-1}W(f) = W(f) \times_Y Y' \to Y'$ has $\mathcal{P}$. Hence $(g')^{-1}W(f) \subset W(f')$. We conclude $x' \in W(f')$.

Assume $x' \in W(f')$. For any open neighbourhood $V' \subset Y'$ of y' we may replace Y' by V' and X' by $U' = (f')^{-1}V'$ because $V' \to Y'$ is smooth and hence the base change $W(f') \cap U' \to V'$ of $W(f') \to Y'$ has property $\mathcal{P}$. Thus we may assume there exists an étale morphism $Y' \to \mathbf{A}^n_Y$ over Y, see Morphisms, Lemma 29.36.20. Picture

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ \mathbf{A}^n_X \arrow[r, "f_n"'] \arrow[d] & \mathbf{A}^n_Y \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

By Lemma 35.32.6 (and because étale coverings are smooth coverings) we see that $\mathcal{P}$ is étale local on the source-and-target. By Lemma 35.32.9 we see that $W(f')$ is the inverse image of the open $W(f_n) \subset \mathbf{A}^n_X$. In particular $W(f_n)$ contains a point lying over x. After replacing X by the image of $W(f_n)$ (which is open) we may assume $W(f_n) \to X$ is surjective. Claim: $W(f_n) = \mathbf{A}^n_X$. The claim implies f has $\mathcal{P}$ as $\mathcal{P}$ is local in the smooth topology and $\{\mathbf{A}^n_Y \to Y\}$ is a smooth covering.

Essentially, the claim follows as $W(f_n) \subset \mathbf{A}^n_X$ is a "translation invariant" open which meets every fibre of $\mathbf{A}^n_X \to X$. However, to produce an argument along these lines one has to do étale localization on Y to produce enough translations and it becomes a bit annoying. Instead we use the automorphisms of Lemma

35.32.10 and étale morphisms of affine spaces. We may assume $n \geq 2$. Namely, if $n = 0$, then we are done. If $n = 1$, then we consider the diagram

$$\begin{tikzcd} \mathbf{A}^2_X \arrow[r, "f_2"'] \arrow[d, "p"'] & \mathbf{A}^2_Y \arrow[d] \\ \mathbf{A}^1_X \arrow[r, "f_1"] & \mathbf{A}^1_Y \end{tikzcd}$$

We have $p^{-1}(W(f_1)) \subset W(f_2)$ (see first paragraph of the proof). Thus $W(f_2) \to X$ is still surjective and we may work with f_2. Assume $n \geq 2$.

For any $1 \leq i, j \leq n$ with $i \neq j$ and $d \geq 0$ denote $T_{i,j,d}$ the automorphism of $\mathbf{A}^n$ defined in Lemma 35.32.10. Then we get a commutative diagram

$$\begin{tikzcd} \mathbf{A}^n_X \arrow[r, "f_n"'] \arrow[d, "T_{i,j,d}"'] & \mathbf{A}^n_Y \arrow[d, "T_{i,j,d}"] \\ \mathbf{A}^n_X \arrow[r, "f_n"] & \mathbf{A}^n_Y \end{tikzcd}$$

whose vertical arrows are isomorphisms. We conclude that $T_{i,j,d}(W(f_n)) = W(f_n)$. Applying Lemma 35.32.10 we conclude for any $x \in X$ the fibre $W(f_n)_x \subset \mathbf{A}^n_x$ is either $\mathbf{A}^n_x$ (this is what we want) or $\kappa(x)$ has characteristic $p > 0$ and $W(f_n)_x$ is the complement of a finite set $Z_x \subset \mathbf{A}^n_x$ of closed points. The second possibility cannot occur. Namely, consider the morphism $T_p : \mathbf{A}^n \to \mathbf{A}^n$ given by

$$(x_1, \ldots, x_n) \mapsto (x_1 - x_1^p, \ldots, x_n - x_n^p)$$

As above we get a commutative diagram

$$\begin{tikzcd} \mathbf{A}^n_X \arrow[r, "f_n"'] \arrow[d, "T_p"'] & \mathbf{A}^n_Y \arrow[d, "T_p"] \\ \mathbf{A}^n_X \arrow[r, "f_n"] & \mathbf{A}^n_Y \end{tikzcd}$$

The morphism $T_p : \mathbf{A}^n_X \to \mathbf{A}^n_X$ is étale at every point lying over x and the morphism $T_p : \mathbf{A}^n_Y \to \mathbf{A}^n_Y$ is étale at every point lying over the image of x in Y. (Details omitted; hint: compute the derivatives.) We conclude that

$$T_p^{-1}(W) \cap \mathbf{A}^n_x = W \cap \mathbf{A}^n_x$$

by Lemma 35.32.9 (we've already seen $\mathcal{P}$ is étale local on the source-and-target). Since $T_p : \mathbf{A}^n_x \to \mathbf{A}^n_x$ is finite étale of degree $p^n > 1$ we see that if Z_x is not empty then it contains $T_p^{-1}(Z_x)$ which is bigger. This contradiction finishes the proof. □

35.33. Properties of morphisms of germs local on source-and-target

04R5 In this section we discuss the analogue of the material in Section 35.32 for morphisms of germs of schemes.

04NB Definition 35.33.1. Let $\mathcal{Q}$ be a property of morphisms of germs of schemes. We say $\mathcal{Q}$ is étale local on the source-and-target if for any commutative diagram

$$\begin{tikzcd} (U', u') \arrow[r, "h'"'] \arrow[d, "a"'] & (V', v') \arrow[d, "b"] \\ (U, u) \arrow[r, "h"] & (V, v) \end{tikzcd}$$

of germs with étale vertical arrows we have $\mathcal{Q}(h) \Leftrightarrow \mathcal{Q}(h')$.

04R6 Lemma 35.33.2. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs defined by the rule

$$\mathcal{Q}((X, x) \to (S, s)) \Leftrightarrow \text{there exists a representative } U \to S \text{ which has } \mathcal{P}$$

Then $\mathcal{Q}$ is étale local on the source-and-target as in Definition 35.33.1.

Proof. If a morphism of germs $(X, x) \to (S, s)$ has $\mathcal{Q}$, then there are arbitrarily small neighbourhoods $U \subset X$ of x and $V \subset S$ of s such that a representative $U \to V$ of $(X, x) \to (S, s)$ has $\mathcal{P}$. This follows from Lemma 35.32.4. Let

$$\begin{tikzcd} (U', u') \arrow[r, "h'"'] \arrow[d, "a"'] & (V', v') \arrow[d, "b"] \\ (U, u) \arrow[r, "h"] & (V, v) \end{tikzcd}$$

be as in Definition 35.33.1. Choose $U_1 \subset U$ and a representative $h_1 : U_1 \to V$ of h. Choose $V'_1 \subset V'$ and an étale representative $b_1 : V'_1 \to V$ of b (Definition 35.20.2). Choose $U'_1 \subset U'$ and representatives $a_1 : U'_1 \to U_1$ and $h'_1 : U'_1 \to V'_1$ of a and h' with a_1 étale. After shrinking U'_1 we may assume $h_1 \circ a_1 = b_1 \circ h'_1$. By the initial remark of the proof, we are trying to show $u' \in W(h'_1) \Leftrightarrow u \in W(h_1)$ where $W(-)$ is as in Lemma 35.26.3. Thus the lemma follows from Lemma 35.32.9. □

04R7 Lemma 35.33.3. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let Q be the associated property of morphisms of germs, see Lemma 35.33.2. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

(1) f has property $\mathcal{P}$, and
(2) for every $x \in X$ the morphism of germs $(X, x) \to (Y, f(x))$ has property $\mathcal{Q}$.

Proof. The implication (1) ⇒ (2) is direct from the definitions. The implication (2) ⇒ (1) also follows from part (3) of Definition 35.32.3. □

A morphism of germs $(X, x) \to (S, s)$ determines a well defined map of local rings. Hence the following lemma makes sense.

04ND Lemma 35.33.4. The property of morphisms of germs

$$\mathcal{P}((X, x) \to (S, s)) = \mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \text{ is flat}$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 35.33.1 we obtain the following diagram of local homomorphisms of local rings

$$\begin{tikzcd} \mathcal{O}_{U',u'} & \mathcal{O}_{V',v'} \arrow[l] \\ \mathcal{O}_{U,u} \arrow[u] & \mathcal{O}_{V,v} \arrow[l] \arrow[u] \end{tikzcd}$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are essentially of finite presentation, flat, and unramified (see Algebra, Section 10.143). In particular the vertical maps are faithfully flat, see Algebra, Lemma 10.39.17. Now, if the upper horizontal arrow is flat, then the lower horizontal arrow is flat by an application of Algebra, Lemma 10.39.10 with $R = \mathcal{O}_{V,v}$, $S = \mathcal{O}_{U,u}$ and $M = \mathcal{O}_{U',u'}$. If the lower horizontal arrow is flat, then the ring map

$$\mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \longleftarrow \mathcal{O}_{V',v'}$$

is flat by Algebra, Lemma 10.39.7. And the ring map

$$\mathcal{O}_{U',u'} \longleftarrow \mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u}$$

is a localization of a map between étale ring extensions of $\mathcal{O}_{U,u}$, hence flat by Algebra, Lemma 10.143.8. $\square$

04NI Lemma 35.33.5. Consider a commutative diagram of morphisms of schemes

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & V' \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

with étale vertical arrows and a point $v' \in V'$ mapping to $v \in V$. Then the morphism of fibres $U'_{v'} \to U_v$ is étale.

Proof. Note that $U'_v \to U_v$ is étale as a base change of the étale morphism $U' \to U$. The scheme U'_v is a scheme over V'_v. By Morphisms, Lemma 29.36.7 the scheme V'_v is a disjoint union of spectra of finite separable field extensions of $\kappa(v)$. One of these is $v' = \mathrm{Spec}(\kappa(v'))$. Hence $U'_{v'}$ is an open and closed subscheme of U'_v and it follows that $U'_{v'} \to U'_v \to U_v$ is étale (as a composition of an open immersion and an étale morphism, see Morphisms, Section 29.36). $\square$

Given a morphism of germs of schemes $(X, x) \to (S, s)$ we can define the fibre as the isomorphism class of germs (U_s, x) where $U \to S$ is any representative. We will often abuse notation and just write (X_s, x).

04NJ Lemma 35.33.6. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs

$$\mathcal{P}_d((X, x) \to (S, s)) = \text{the local ring } \mathcal{O}_{X_s,x} \text{ of the fibre has dimension } d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 35.33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 35.33.5. Hence the result follows from Lemma 35.21.3. $\square$

04NK Lemma 35.33.7. Let $r \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs

$$\mathcal{P}_r((X, x) \to (S, s)) \Leftrightarrow \text{trdeg}_{\kappa(s)}\kappa(x) = r$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 35.33.1 we obtain the following diagram of local homomorphisms of local rings

$$\begin{tikzcd} \mathcal{O}_{U',u'} & \mathcal{O}_{V',v'} \arrow[l] \\ \mathcal{O}_{U,u} \arrow[u] & \mathcal{O}_{V,v} \arrow[l] \arrow[u] \end{tikzcd}$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are unramified (see Algebra, Section 10.143). Hence $\kappa(u')/\kappa(u)$ and $\kappa(v')/\kappa(v)$ are finite separable field extensions. Thus we have $\text{trdeg}_{\kappa(v)}\kappa(u) = \text{trdeg}_{\kappa(v')}\kappa(u)$ which proves the lemma. $\square$

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$.

04NL Lemma 35.33.8. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs

$$\mathcal{P}_d((X, x) \to (S, s)) \Leftrightarrow \dim_x(X_s) = d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 35.33.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 35.33.5. Hence now the equality $\dim_u(U_v) = \dim_{u'}(U'_{v'})$ follows from Lemma 35.21.2. $\square$

35.34. Descent data for schemes over schemes

023U Most of the arguments in this section are formal relying only on the definition of a descent datum. In Simplicial Spaces, Section 85.27 we will examine the relationship with simplicial schemes which will somewhat clarify the situation.

023V Definition 35.34.1. Let $f : X \to S$ be a morphism of schemes.

(1) Let $V \to X$ be a scheme over X. A descent datum for $V/X/S$ is an isomorphism $\varphi : V \times_S X \to X \times_S V$ of schemes over $X \times_S X$ satisfying the cocycle condition that the diagram

$$\begin{tikzcd} V \times_S X \times_S X \arrow[rr, "\varphi_{02}"] \arrow[dr, "\varphi_{01}"'] & & X \times_S X \times_S V \\ & X \times_S V \times_S X \arrow[ur, "\varphi_{12}"'] & \end{tikzcd}$$

commutes (with obvious notation).

(2) We also say that the pair $(V/X, \varphi)$ is a descent datum relative to $X \to S$.

(3) A morphism $f : (V/X, \varphi) \to (V'/X, \varphi')$ of descent data relative to $X \to S$ is a morphism $f : V \to V'$ of schemes over X such that the diagram

$$\begin{tikzcd} V \times_S X \arrow[r, "\varphi"'] \arrow[d, "f \times \mathrm{id}_X"'] & X \times_S V \arrow[d, "\mathrm{id}_X \times f"] \\ V' \times_S X \arrow[r, "\varphi'"] & X \times_S V' \end{tikzcd}$$

commutes.

There are all kinds of "miraculous" identities which arise out of the definition above. For example the pullback of φ via the diagonal morphism $\Delta : X \to X \times_S X$ can be seen as a morphism $\Delta^*\varphi : V \to V$. This because $X \times_{\Delta, X \times_S X} (V \times_S X) = V$ and also $X \times_{\Delta, X \times_S X} (X \times_S V) = V$. In fact, $\Delta^*\varphi$ is equal to the identity. This is a good exercise if you are unfamiliar with this material.

02VP Remark 35.34.2. Let $X \to S$ be a morphism of schemes. Let $(V/X, \varphi)$ be a descent datum relative to $X \to S$. We may think of the isomorphism φ as an isomorphism

$$(X \times_S X) \times_{\mathrm{pr}_0, X} V \longrightarrow (X \times_S X) \times_{\mathrm{pr}_1, X} V$$

of schemes over $X \times_S X$. So loosely speaking one may think of φ as a map $\varphi : \mathrm{pr}_0^*V \to \mathrm{pr}_1^*V$[9]. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi = \mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

Here is the definition in case you have a family of morphisms with fixed target.

023W Definition 35.34.3. Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) A descent datum (V_i, φ_{ij}) relative to the family $\{X_i \to S\}$ is given by a scheme V_i over X_i for each $i \in I$, an isomorphism $\varphi_{ij} : V_i \times_S X_j \to X_i \times_S V_j$ of schemes over $X_i \times_S X_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{tikzcd} V_i \times_S X_j \times_S X_k \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & X_i \times_S X_j \times_S V_k \\ & X_i \times_S V_j \times_S X_k \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$

of schemes over $X_i \times_S X_j \times_S X_k$ commutes (with obvious notation).

(2) A morphism $\psi : (V_i, \varphi_{ij}) \to (V'_i, \varphi'_{ij})$ of descent data is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of X_i-schemes $\psi_i : V_i \to V'_i$ such that all the diagrams

$$\begin{tikzcd} V_i \times_S X_j \arrow[r, "\varphi_{ij}"'] \arrow[d, "\psi_i \times \mathrm{id}"'] & X_i \times_S V_j \arrow[d, "\mathrm{id} \times \psi_j"] \\ V'_i \times_S X_j \arrow[r, "\varphi'_{ij}"] & X_i \times_S V'_j \end{tikzcd}$$

commute.

[9]Unfortunately, we have chosen the "wrong" direction for our arrow here. In Definitions 35.34.1 and 35.34.3 we should have the opposite direction to what was done in Definition 35.2.1 by the general principle that "functions" and "spaces" are dual.

This is the notion that comes up naturally for example when the question arises whether the fibred category of relative curves is a stack in the fpqc topology (it isn't – at least not if you stick to schemes).

02VQ Remark 35.34.4. Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S. Let (V_i, φ_{ij}) be a descent datum relative to $\{X_i \to S\}$. We may think of the isomorphisms φ_{ij} as isomorphisms

$$(X_i \times_S X_j) \times_{\mathrm{pr}_0, X_i} V_i \longrightarrow (X_i \times_S X_j) \times_{\mathrm{pr}_1, X_j} V_j$$

of schemes over $X_i \times_S X_j$. So loosely speaking one may think of φ_{ij} as an isomorphism $\mathrm{pr}_0^* V_i \to \mathrm{pr}_1^* V_j$ over $X_i \times_S X_j$. The cocycle condition then says that $\mathrm{pr}_{02}^* \varphi_{ik} = \mathrm{pr}_{12}^* \varphi_{jk} \circ \mathrm{pr}_{01}^* \varphi_{ij}$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

The reason we will usually work with the version of a family consisting of a single morphism is the following lemma.

023X Lemma 35.34.5. Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be a family of morphisms with target S. Set $X = \coprod_{i \in I} X_i$, and consider it as an S-scheme. There is a canonical equivalence of categories

$$\begin{matrix} \text{category of descent data} \\ \text{relative to the family } \{X_i \to S\}_{i \in I} \end{matrix} \longrightarrow \begin{matrix} \text{category of descent data} \\ \text{relative to } X/S \end{matrix}$$

which maps (V_i, φ_{ij}) to (V, φ) with $V = \coprod_{i \in I} V_i$ and $\varphi = \coprod \varphi_{ij}$.

Proof. Observe that $X \times_S X = \coprod_{ij} X_i \times_S X_j$ and similarly for higher fibre products. Giving a morphism $V \to X$ is exactly the same as giving a family $V_i \to X_i$. And giving a descent datum φ is exactly the same as giving a family φ_{ij}. $\square$

023Y Lemma 35.34.6. Pullback of descent data for schemes over schemes.

(1) Let

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "a'"'] & X \arrow[d, "a"] \\ S' \arrow[r, "h"] & S \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. The construction

$$(V \to X, \varphi) \longmapsto f^*(V \to X, \varphi) = (V' \to X', \varphi')$$

where $V' = X' \times_X V$ and where φ' is defined as the composition

$$\begin{tikzcd} V' \times_{S'} X' \arrow[r, equal] & (X' \times_X V) \times_{S'} X' \arrow[r, equal] & (X' \times_{S'} X') \times_{X \times_S X} (V \times_S X) \arrow[d, "\mathrm{id} \times \varphi"] \\ X' \times_{S'} V' \arrow[r, equal] & X' \times_{S'} (X' \times_X V) \arrow[r, equal] & (X' \times_{S'} X') \times_{X \times_S X} (X \times_S V) \end{tikzcd}$$

defines a functor from the category of descent data relative to $X \to S$ to the category of descent data relative to $X' \to S'$.

(2) Given two morphisms $f_i : X' \to X$, $i = 0, 1$ making the diagram commute the functors f_0^* and f_1^* are canonically isomorphic.

Proof. We omit the proof of (1), but we remark that the morphism φ' is the morphism $(f \times f)^*\varphi$ in the notation introduced in Remark 35.34.2. For (2) we indicate which morphism $f_0^*V \to f_1^*V$ gives the functorial isomorphism. Namely, since f_0 and f_1 both fit into the commutative diagram we see there is a unique morphism $r : X' \to X \times_S X$ with $f_i = \text{pr}_i \circ r$. Then we take

$$\begin{aligned} f_0^*V &= X' \times_{f_0, X} V \\ &= X' \times_{\text{pr}_0 \circ r, X} V \\ &= X' \times_{r, X \times_S X} (X \times_S X) \times_{\text{pr}_0, X} V \\ &\xrightarrow{\varphi} X' \times_{r, X \times_S X} (X \times_S X) \times_{\text{pr}_1, X} V \\ &= X' \times_{\text{pr}_1 \circ r, X} V \\ &= X' \times_{f_1, X} V \\ &= f_1^*V \end{aligned}$$

We omit the verification that this works. □

02VR Definition 35.34.7. With $S, S', X, X', f, a, a', h$ as in Lemma 35.34.6 the functor

$$(V, \varphi) \longmapsto f^*(V, \varphi)$$

constructed in that lemma is called the pullback functor on descent data.

02VS Lemma 35.34.8 (Pullback of descent data for schemes over families). Let $\mathcal{U} = \{U_i \to S'\}_{i \in I}$ and $\mathcal{V} = \{V_j \to S\}_{j \in J}$ be families of morphisms with fixed target. Let $\alpha : I \to J$, $h : S' \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1.

(1) Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \to S'\}$. The system
$$\left(g_i^*Y_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
(with notation as in Remark 35.34.4) is a descent datum relative to $\mathcal{V}$.
(2) This construction defines a functor between descent data relative to $\mathcal{U}$ and descent data relative to $\mathcal{V}$.
(3) Given a second $\alpha' : I \to J$, $h' : S' \to S$ and $g_i' : U_i \to V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.
(4) These functors agree, via Lemma 35.34.5, with the pullback functors constructed in Lemma 35.34.6.

Proof. This follows from Lemma 35.34.6 via the correspondence of Lemma 35.34.5. □

02VT Definition 35.34.9. With $\mathcal{U} = \{U_i \to S'\}_{i \in I}$, $\mathcal{V} = \{V_j \to S\}_{j \in J}$, $\alpha : I \to J$, $h : S' \to S$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 35.34.8 the functor

$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^*Y_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')})$$

constructed in that lemma is called the pullback functor on descent data.

If $\mathcal{U}$ and $\mathcal{V}$ have the same target S, and if $\mathcal{U}$ refines $\mathcal{V}$ (see Sites, Definition 7.8.1) but no explicit pair (α, g_i) is given, then we can still talk about the pullback functor since we have seen in Lemma 35.34.8 that the choice of the pair does not matter (up to a canonical isomorphism).

023Z Definition 35.34.10. Let S be a scheme. Let $f : X \to S$ be a morphism of schemes.

(1) Given a scheme U over S we have the trivial descent datum of U relative to $\text{id} : S \to S$, namely the identity morphism on U.
(2) By Lemma 35.34.6 we get a canonical descent datum on $X \times_S U$ relative to $X \to S$ by pulling back the trivial descent datum via f. We often denote $(X \times_S U, can)$ this descent datum.
(3) A descent datum (V, φ) relative to X/S is called effective if (V, φ) is isomorphic to the canonical descent datum $(X \times_S U, can)$ for some scheme U over S.

Thus being effective means there exists a scheme U over S and an isomorphism $\psi : V \to X \times_S U$ of X-schemes such that φ is equal to the composition

$$V \times_S X \xrightarrow{\psi \times \text{id}_X} X \times_S U \times_S X = X \times_S X \times_S U \xrightarrow{\text{id}_X \times \psi^{-1}} X \times_S V$$

02VU Definition 35.34.11. Let S be a scheme. Let $\{X_i \to S\}$ be a family of morphisms with target S.

(1) Given a scheme U over S we have a canonical descent datum on the family of schemes $X_i \times_S U$ by pulling back the trivial descent datum for U relative to $\{\text{id} : S \to S\}$. We denote this descent datum $(X_i \times_S U, can)$.
(2) A descent datum (V_i, φ_{ij}) relative to $\{X_i \to S\}$ is called effective if there exists a scheme U over S such that (V_i, φ_{ij}) is isomorphic to $(X_i \times_S U, can)$.

35.35. Fully faithfulness of the pullback functors

02VV It turns out that the pullback functor between descent data for fpqc-coverings is fully faithful. In other words, morphisms of schemes satisfy fpqc descent. The goal of this section is to prove this. The reader is encouraged instead to prove this him/herself. The key is to use Lemma 35.13.7.

02VW Lemma 35.35.1. A surjective and flat morphism is an epimorphism in the category of schemes.

Proof. Suppose we have $h : X' \to X$ surjective and flat and $a, b : X \to Y$ morphisms such that $a \circ h = b \circ h$. As h is surjective we see that a and b agree on underlying topological spaces. Pick $x' \in X'$ and set $x = h(x')$ and $y = a(x) = b(x)$. Consider the local ring maps

$$a^\sharp_x, b^\sharp_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

These become equal when composed with the flat local homomorphism $h^\sharp_{x'} : \mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$. Since a flat local homomorphism is faithfully flat (Algebra, Lemma 10.39.17) we conclude that $h^\sharp_{x'}$ is injective. Hence $a^\sharp_x = b^\sharp_x$ which implies $a = b$ as desired. $\square$

02VX Lemma 35.35.2. Let $h : S' \to S$ be a surjective, flat morphism of schemes. The base change functor

$$Sch/S \longrightarrow Sch/S', \quad X \longmapsto S' \times_S X$$

is faithful.

Proof. Let X_1, X_2 be schemes over S. Let $\alpha, \beta : X_2 \to X_1$ be morphisms over S. If α, β base change to the same morphism then we get a commutative diagram as follows

$$\begin{tikzcd} X_2 & S' \times_S X_2 \arrow[l] \arrow[r] \arrow[d] & X_2 \arrow[d, "\beta"] \\ X_1 & S' \times_S X_1 \arrow[l] \arrow[r] & X_1 \arrow[from=1-1, to=2-1, "\alpha"] \end{tikzcd}$$

Hence it suffices to show that $S' \times_S X_2 \to X_2$ is an epimorphism. As the base change of a surjective and flat morphism it is surjective and flat (see Morphisms, Lemmas 29.9.4 and 29.25.8). Hence the lemma follows from Lemma 35.35.1. □

0240 Lemma 35.35.3. In the situation of Lemma 35.34.6 assume that $f : X' \to X$ is surjective and flat. Then the pullback functor is faithful.

Proof. Let (V_i, φ_i), $i = 1, 2$ be descent data for $X \to S$. Let $\alpha, \beta : V_1 \to V_2$ be morphisms of descent data. Suppose that $f^*\alpha = f^*\beta$. Our task is to show that $\alpha = \beta$. Note that α, β are morphisms of schemes over X, and that $f^*\alpha$, $f^*\beta$ are simply the base changes of α, β to morphisms over X'. Hence the lemma follows from Lemma 35.35.2. □

Here is the key lemma of this section.

0241 Lemma 35.35.4. In the situation of Lemma 35.34.6 assume

(1) $\{f : X' \to X\}$ is an fpqc covering (for example if f is surjective, flat, and quasi-compact), and
(2) $S = S'$.

Then the pullback functor is fully faithful.

Proof. Assumption (1) implies that f is surjective and flat. Hence the pullback functor is faithful by Lemma 35.35.3. Let (V, φ) and (W, ψ) be two descent data relative to $X \to S$. Set $(V', \varphi') = f^*(V, \varphi)$ and $(W', \psi') = f^*(W, \psi)$. Let $\alpha' : V' \to W'$ be a morphism of descent data for X' over S. We have to show there exists a morphism $\alpha : V \to W$ of descent data for X over S whose pullback is α'.

Recall that V' is the base change of V by f and that φ' is the base change of φ by $f \times f$ (see Remark 35.34.2). By assumption the diagram

$$\begin{tikzcd} V' \times_S X' \arrow[r, "\varphi'"'] \arrow[d, "\alpha' \times \mathrm{id}"'] & X' \times_S V' \arrow[d, "\mathrm{id} \times \alpha'"] \\ W' \times_S X' \arrow[r, "\psi'"] & X' \times_S W' \end{tikzcd}$$

commutes. We claim the two compositions

$$V' \times_V V' \xrightarrow{\mathrm{pr}_i} V' \xrightarrow{\alpha'} W' \longrightarrow W, \quad i = 0, 1$$

are the same. The reader is advised to prove this themselves rather than read the rest of this paragraph. (Please email if you find a nice clean argument.) Let v_0, v_1 be points of V' which map to the same point $v \in V$. Let $x_i \in X'$ be the image of v_i, and let x be the point of X which is the image of v in X. In other words, $v_i = (x_i, v)$ in $V' = X' \times_X V$. Write $\varphi(v, x) = (x, v')$ for some point v' of V. This is possible because φ is a morphism over $X \times_S X$. Denote $v_i' = (x_i, v')$ which is a point of V'. Then a calculation (using the definition of φ') shows that $\varphi'(v_i, x_j) = (x_i, v_j')$. Denote $w_i = \alpha'(v_i)$ and $w_i' = \alpha'(v_i')$. Now we may write $w_i = (x_i, u_i)$ for some point u_i of W, and $w_i' = (x_i, u_i')$ for some point u_i' of W. The claim is equivalent to the assertion: $u_0 = u_1$. A formal calculation using the definition of ψ' (see Lemma 35.34.6) shows that the commutativity of the diagram displayed above says that

$$((x_i, x_j), \psi(u_i, x)) = ((x_i, x_j), (x, u_j'))$$

as points of $(X' \times_S X') \times_{X \times_S X} (X \times_S W)$ for all $i, j \in \{0, 1\}$. This shows that $\psi(u_0, x) = \psi(u_1, x)$ and hence $u_0 = u_1$ by taking ψ^{-1}. This proves the claim because the argument above was formal and we can take scheme points (in other words, we may take $(v_0, v_1) = \mathrm{id}_{V' \times_V V'}$).

At this point we can use Lemma 35.13.7. Namely, $\{V' \to V\}$ is a fpqc covering as the base change of the morphism $f : X' \to X$. Hence, by Lemma 35.13.7 the morphism $\alpha' : V' \to W' \to W$ factors through a unique morphism $\alpha : V \to W$ whose base change is necessarily α'. Finally, we see the diagram

$$\begin{tikzcd} V \times_S X \arrow[r, "\varphi"'] \arrow[d, "\alpha \times \mathrm{id}"'] & X \times_S V \arrow[d, "\mathrm{id} \times \alpha"] \\ W \times_S X \arrow[r, "\psi"] & X \times_S W \end{tikzcd}$$

commutes because its base change to $X' \times_S X'$ commutes and the morphism $X' \times_S X' \to X \times_S X$ is surjective and flat (use Lemma 35.35.2). Hence α is a morphism of descent data $(V, \varphi) \to (W, \psi)$ as desired. □

The following two lemmas have been obsoleted by the improved exposition of the previous material. But they are still true!

0242 Lemma 35.35.5. Let $X \to S$ be a morphism of schemes. Let $f : X \to X$ be a selfmap of X over S. In this case pullback by f is isomorphic to the identity functor on the category of descent data relative to $X \to S$.

Proof. This is clear from Lemma 35.34.6 since it tells us that $f^* \cong \mathrm{id}^*$. □

0243 Lemma 35.35.6. Let $f : X' \to X$ be a morphism of schemes over a base scheme S. Assume there exists a morphism $g : X \to X'$ over S, for example if f has a section. Then the pullback functor of Lemma 35.34.6 defines an equivalence of categories between the category of descent data relative to X/S and X'/S.

Proof. Let $g : X \to X'$ be a morphism over S. Lemma 35.35.5 above shows that the functors $f^* \circ g^* = (g \circ f)^*$ and $g^* \circ f^* = (f \circ g)^*$ are isomorphic to the respective identity functors as desired. □

040J Lemma 35.35.7. Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $X \to S$ is surjective and flat. Then the pullback functor of Lemma 35.34.6 is a faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 35.35.6. The second morphism is surjective and flat, hence induces a faithful functor by Lemma 35.35.3. □

040K Lemma 35.35.8. Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $\{X \to S\}$ is an fpqc covering (for example if f is surjective, flat and quasi-compact). Then the pullback functor of Lemma 35.34.6 is a fully faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 35.35.6. The second morphism is an fpqc covering hence induces a fully faithful functor by Lemma 35.35.4. $\square$

02VZ Lemma 35.35.9. Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $\mathrm{id} : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i) = j}$ is an fpqc covering of V_j. Then the pullback functor

$$\text{descent data relative to } \mathcal{V} \longrightarrow \text{descent data relative to } \mathcal{U}$$

of Lemma 35.34.8 is fully faithful.

Proof. Consider the morphism of schemes

$$g : X = \coprod\nolimits_{i \in I} U_i \longrightarrow Y = \coprod\nolimits_{j \in J} V_j$$

over S which on the ith component maps into the $\alpha(i)$th component via the morphism $g_{\alpha(i)}$. We claim that $\{g : X \to Y\}$ is an fpqc covering of schemes. Namely, by Topologies, Lemma 34.9.3 for each j the morphism $\{\coprod_{\alpha(i)=j} U_i \to V_j\}$ is an fpqc covering. Thus for every affine open $V \subset V_j$ (which we may think of as an affine open of Y) we can find finitely many affine opens $W_1, \ldots, W_n \subset \coprod_{\alpha(i)=j} U_i$ (which we may think of as affine opens of X) such that $V = \bigcup_{i=1,\ldots,n} g(W_i)$. This provides enough affine opens of Y which can be covered by finitely many affine opens of X so that Topologies, Lemma 34.9.2 part (3) applies, and the claim follows. Let us write $DD(X/S)$, resp. $DD(\mathcal{U})$ for the category of descent data with respect to X/S, resp. $\mathcal{U}$, and similarly for Y/S and $\mathcal{V}$. Consider the diagram

$$\begin{tikzcd} DD(Y/S) \arrow[r] & DD(X/S) \\ DD(\mathcal{V}) \arrow[u, "\text{Lemma 35.34.5}"] \arrow[r] & DD(\mathcal{U}) \arrow[u, "\text{Lemma 35.34.5}"'] \end{tikzcd}$$

This diagram is commutative, see the proof of Lemma 35.34.8. The vertical arrows are equivalences. Hence the lemma follows from Lemma 35.35.4 which shows the top horizontal arrow of the diagram is fully faithful. $\square$

The next lemma shows that, in order to check effectiveness, we may always Zariski refine the given family of morphisms with target S.

02VY Lemma 35.35.10. Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $\mathrm{id} : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is a Zariski covering (see Topologies, Definition 34.3.1) of V_j. Then the pullback functor

$$\text{descent data relative to } \mathcal{V} \longrightarrow \text{descent data relative to } \mathcal{U}$$

of Lemma 35.34.8 is an equivalence of categories. In particular, the category of schemes over S is equivalent to the category of descent data relative to any Zariski covering of S.

Proof. The functor is faithful and fully faithful by Lemma 35.35.9. Let us indicate how to prove that it is essentially surjective. Let $(X_i, \varphi_{ii'})$ be a descent datum relative to $\mathcal{U}$. Fix $j \in J$ and set $I_j = \{i \in I \mid \alpha(i) = j\}$. For $i, i' \in I_j$ note that there is a canonical morphism

$$c_{ii'} : U_i \times_{g_i, V_j, g_{i'}} U_{i'} \to U_i \times_S U_{i'}.$$

Hence we can pullback $\varphi_{ii'}$ by this morphism and set $\psi_{ii'} = c_{ii'}^*\varphi_{ii'}$ for $i, i' \in I_j$. In this way we obtain a descent datum $(X_i, \psi_{ii'})$ relative to the Zariski covering $\{g_i : U_i \to V_j\}_{i \in I_j}$. Note that $\psi_{ii'}$ is an isomorphism from the open $X_{i, U_i \times_{V_j} U_{i'}}$ of X_i to the corresponding open of $X_{i'}$. It follows from Schemes, Section 26.14 that we may glue $(X_i, \psi_{ii'})$ into a scheme Y_j over V_j. Moreover, the morphisms $\varphi_{ii'}$ for $i \in I_j$ and $i' \in I_{j'}$ glue to a morphism $\varphi_{jj'} : Y_j \times_S V_{j'} \to V_j \times_S Y_{j'}$ satisfying the cocycle condition (details omitted). Hence we obtain the desired descent datum $(Y_j, \varphi_{jj'})$ relative to $\mathcal{V}$. □

02W0 Lemma 35.35.11. Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be fpqc-coverings of S. If $\mathcal{U}$ is a refinement of $\mathcal{V}$, then the pullback functor

$$\text{descent data relative to } \mathcal{V} \longrightarrow \text{descent data relative to } \mathcal{U}$$

is fully faithful. In particular, the category of schemes over S is identified with a full subcategory of the category of descent data relative to any fpqc-covering of S.

Proof. Consider the fpqc-covering $\mathcal{W} = \{U_i \times_S V_j \to S\}_{(i,j) \in I \times J}$ of S. It is a refinement of both $\mathcal{U}$ and $\mathcal{V}$. Hence we have a 2-commutative diagram of functors and categories

$$\begin{tikzcd} DD(\mathcal{V}) \arrow[rr] \arrow[dr] & & DD(\mathcal{U}) \arrow[dl] \\ & DD(\mathcal{W}) & \end{tikzcd}$$

Notation as in the proof of Lemma 35.35.9 and commutativity by Lemma 35.34.8 part (3). Hence clearly it suffices to prove the functors $DD(\mathcal{V}) \to DD(\mathcal{W})$ and $DD(\mathcal{U}) \to DD(\mathcal{W})$ are fully faithful. This follows from Lemma 35.35.9 as desired. □

040L Remark 35.35.12. Lemma 35.35.11 says that morphisms of schemes satisfy fpqc descent. In other words, given a scheme S and schemes X, Y over S the functor

$$(Sch/S)^{opp} \longrightarrow \text{Sets}, \quad T \longmapsto \text{Mor}_T(X_T, Y_T)$$

satisfies the sheaf condition for the fpqc topology. The simplest case of this is the following. Suppose that $T \to S$ is a surjective flat morphism of affines. Let $\psi_0 : X_T \to Y_T$ be a morphism of schemes over T which is compatible with the canonical descent data. Then there exists a unique morphism $\psi : X \to Y$ whose base change to T is ψ_0. In fact this special case follows in a straightforward manner from Lemma 35.35.4. And, in turn, that lemma is a formal consequence of the following two facts: (a) the base change functor by a faithfully flat morphism is faithful, see Lemma 35.35.2 and (b) a scheme satisfies the sheaf condition for the fpqc topology, see Lemma 35.13.7.

0AP4 **Lemma 35.35.13.** Let $X \to S$ be a surjective, quasi-compact, flat morphism of schemes. Let (V, φ) be a descent datum relative to X/S. Suppose that for all $v \in V$ there exists an open subscheme $v \in W \subset V$ such that $\varphi(W \times_S X) \subset X \times_S W$ and such that the descent datum $(W, \varphi|_{W \times_S X})$ is effective. Then (V, φ) is effective.

Proof. Let $V = \bigcup W_i$ be an open covering with $\varphi(W_i \times_S X) \subset X \times_S W_i$ and such that the descent datum $(W_i, \varphi|_{W_i \times_S X})$ is effective. Let $U_i \to S$ be a scheme and let $\alpha_i : (X \times_S U_i, can) \to (W_i, \varphi|_{W_i \times_S X})$ be an isomorphism of descent data. For each pair of indices (i, j) consider the open $\alpha_i^{-1}(W_i \cap W_j) \subset X \times_S U_i$. Because everything is compatible with descent data and since $\{X \to S\}$ is an fpqc covering, we may apply Lemma 35.13.6 to find an open $U_{ij} \subset U_j$ such that $\alpha_i^{-1}(W_i \cap W_j) = X \times_S U_{ij}$. Now the identity morphism on $W_i \cap W_j$ is compatible with descent data, hence comes from a unique morphism $\varphi_{ij} : U_{ij} \to U_{ji}$ over S (see Remark 35.35.12). Then $(U_i, U_{ij}, \varphi_{ij})$ is a glueing data as in Schemes, Section 26.14 (proof omitted). Thus we may assume there is a scheme U over S such that $U_i \subset U$ is open, $U_{ij} = U_i \cap U_j$ and $\varphi_{ij} = \mathrm{id}_{U_i \cap U_j}$, see Schemes, Lemma 26.14.1. Pulling back to X we can use the α_i to get the desired isomorphism $\alpha : X \times_S U \to V$. $\square$

35.36. Descending types of morphisms

02W1 In the following we study the question as to whether descent data for schemes relative to a fpqc-covering are effective. The first remark to make is that this is not always the case. We will see this in Algebraic Spaces, Example 65.14.2. Even projective morphisms do not always satisfy descent for fpqc-coverings, by Examples, Lemma 110.65.1.

On the other hand, if the schemes we are trying to descend are particularly simple, then it is sometime the case that for whole classes of schemes descent data are effective. We will introduce terminology here that describes this phenomenon abstractly, even though it may lead to confusion if not used correctly later on.

02W2 **Definition 35.36.1.** Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{Zariski, fpqc, fppf, étale, smooth, syntomic\}$. We say morphisms of type $\mathcal{P}$ *satisfy descent for τ-coverings* if for any τ-covering $\mathcal{U} : \{U_i \to S\}_{i \in I}$ (see Topologies, Section 34.2), any descent datum (X_i, φ_{ij}) relative to $\mathcal{U}$ such that each morphism $X_i \to U_i$ has property $\mathcal{P}$ is effective.

Note that in each of the cases we have already seen that the functor from schemes over S to descent data over $\mathcal{U}$ is fully faithful (Lemma 35.35.11 combined with the results in Topologies that any τ-covering is also a fpqc-covering). We have also seen that descent data are always effective with respect to Zariski coverings (Lemma 35.35.10). It may be prudent to only study the notion just introduced when $\mathcal{P}$ is either stable under any base change or at least local on the base in the τ-topology (see Definition 35.22.1) in order to avoid erroneous arguments (relying on $\mathcal{P}$ when descending halfway).

Here is the obligatory lemma reducing this question to the case of a covering given by a single morphism of affines.

02W3 **Lemma 35.36.2.** Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Suppose that

(1) $\mathcal{P}$ is stable under any base change (see Schemes, Definition 26.18.3),

(2) if $Y_j \to V_j$, $j = 1, \ldots, m$ have $\mathcal{P}$, then so does $\coprod Y_j \to \coprod V_j$, and
(3) for any surjective morphism of affines $X \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, any descent datum (V, φ) relative to X over S such that $\mathcal{P}$ holds for $V \to X$ is effective.

Then morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings.

Proof. Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}_{i \in I}$ be a τ-covering of S. Let $(X_i, \varphi_{ii'})$ be a descent datum relative to $\mathcal{U}$ and assume that each morphism $X_i \to U_i$ has property $\mathcal{P}$. We have to show there exists a scheme $X \to S$ such that $(X_i, \varphi_{ii'}) \cong (U_i \times_S X, can)$.

Before we start the proof proper we remark that for any family of morphisms $\mathcal{V} : \{V_j \to S\}$ and any morphism of families $\mathcal{V} \to \mathcal{U}$, if we pullback the descent datum $(X_i, \varphi_{ii'})$ to a descent datum $(Y_j, \varphi_{jj'})$ over $\mathcal{V}$, then each of the morphisms $Y_j \to V_j$ has property $\mathcal{P}$ also. This is true because of assumption (1) that $\mathcal{P}$ is stable under any base change and the definition of pullback (see Definition 35.34.9). We will use this without further mention.

First, let us prove the lemma when S is affine. By Topologies, Lemma 34.9.8, 34.7.4, 34.4.4, 34.5.4, or 34.6.4 there exists a standard τ-covering $\mathcal{V} : \{V_j \to S\}_{j=1,\ldots,m}$ which refines $\mathcal{U}$. The pullback functor $DD(\mathcal{U}) \to DD(\mathcal{V})$ between categories of descent data is fully faithful by Lemma 35.35.11. Hence it suffices to prove that the descent datum over the standard τ-covering $\mathcal{V}$ is effective. By assumption (2) we see that $\coprod Y_j \to \coprod V_j$ has property $\mathcal{P}$. By Lemma 35.34.5 this reduces us to the covering $\{\coprod_{j=1,\ldots,m} V_j \to S\}$ for which we have assumed the result in assumption (3) of the lemma. Hence the lemma holds when S is affine.

Assume S is general. Let $V \subset S$ be an affine open. By the properties of site the family $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ is a τ-covering of V. Denote $(X_i, \varphi_{ii'})_V$ the restriction (or pullback) of the given descent datum to $\mathcal{U}_V$. Hence by what we just saw we obtain a scheme X_V over V whose canonical descent datum with respect to $\mathcal{U}_V$ is isomorphic to $(X_i, \varphi_{ii'})_V$. Suppose that $V' \subset V$ is an affine open of V. Then both $X_{V'}$ and $V' \times_V X_V$ have canonical descent data isomorphic to $(X_i, \varphi_{ii'})_{V'}$. Hence, by Lemma 35.35.11 again we obtain a canonical morphism $\rho^V_{V'} : X_{V'} \to X_V$ over S which identifies $X_{V'}$ with the inverse image of V' in X_V. We omit the verification that given affine opens $V'' \subset V' \subset V$ of S we have $\rho^V_{V''} = \rho^V_{V'} \circ \rho^{V'}_{V''}$.

By Constructions, Lemma 27.2.1 the data $(X_V, \rho^V_{V'})$ glue to a scheme $X \to S$. Moreover, we are given isomorphisms $V \times_S X \to X_V$ which recover the maps $\rho^V_{V'}$. Unwinding the construction of the schemes X_V we obtain isomorphisms

$$V \times_S U_i \times_S X \longrightarrow V \times_S X_i$$

compatible with the maps $\varphi_{ii'}$ and compatible with restricting to smaller affine opens in X. This implies that the canonical descent datum on $U_i \times_S X$ is isomorphic to the given descent datum and we win. □

35.37. Descending affine morphisms

0244 In this section we show that "affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

0245 **Lemma 35.37.1.** Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 34.9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is affine, then the descent datum is effective.

Proof. Being affine is a property of morphisms of schemes which is local on the base and preserved under any base change, see Morphisms, Lemmas 29.11.3 and 29.11.8. Hence Lemma 35.36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $V \to X$ is affine. Then $V \to X$ being affine implies that $V = \text{Spec}(B)$ for some A-algebra B (see Morphisms, Definition 29.11.1). The isomorphism φ corresponds to an isomorphism of rings

$$\varphi^\sharp : B \otimes_R A \longleftarrow A \otimes_R B$$

as $A \otimes_R A$-algebras. The cocycle condition on φ says that

$$\begin{tikzcd} B \otimes_R A \otimes_R A & & A \otimes_R A \otimes_R B \arrow[ll] \arrow[dl] \\ & A \otimes_R B \otimes_R A \arrow[ul] & \end{tikzcd}$$

is commutative. Inverting these arrows we see that we have a descent datum for modules with respect to $R \to A$ as in Definition 35.3.1. Hence we may apply Proposition 35.3.9 to obtain an R-module $C = \text{Ker}(B \to A \otimes_R B)$ and an isomorphism $A \otimes_R C \cong B$ respecting descent data. Given any pair $c, c' \in C$ the product cc' in B lies in C since the map φ is an algebra homomorphism. Hence C is an R-algebra whose base change to A is isomorphic to B compatibly with descent data. Applying Spec we obtain a scheme U over S such that $(V, \varphi) \cong (X \times_S U, can)$ as desired. $\square$

03I0 **Lemma 35.37.2.** Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 34.9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is a closed immersion, then the descent datum is effective.

Proof. This is true because a closed immersion is an affine morphism (Morphisms, Lemma 29.11.9), and hence Lemma 35.37.1 applies. $\square$

35.38. Descending quasi-affine morphisms

0246 In this section we show that "quasi-affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

0247 **Lemma 35.38.1.** Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 34.9.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is quasi-affine, then the descent datum is effective.

Proof. Being quasi-affine is a property of morphisms of schemes which is preserved under any base change, see Morphisms, Lemmas 29.13.3 and 29.13.5. Hence Lemma 35.36.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let

(V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is quasi-affine.

According to Morphisms, Lemma 29.13.3 this means that

$$V \longrightarrow \underline{\mathrm{Spec}}_X(\pi_*\mathcal{O}_V) = W$$

is a quasi-compact open immersion of schemes over X. The projections $\mathrm{pr}_i : X \times_S X \to X$ are flat and hence we have

$$\mathrm{pr}_0^*\pi_*\mathcal{O}_V = (\pi \times \mathrm{id}_X)_*\mathcal{O}_{V \times_S X}, \quad \mathrm{pr}_1^*\pi_*\mathcal{O}_V = (\mathrm{id}_X \times \pi)_*\mathcal{O}_{X \times_S V}$$

by flat base change (Cohomology of Schemes, Lemma 30.5.2). Thus the isomorphism $\varphi : V \times_S X \to X \times_S V$ (which is an isomorphism over $X \times_S X$) induces an isomorphism of quasi-coherent sheaves of algebras

$$\varphi^\sharp : \mathrm{pr}_0^*\pi_*\mathcal{O}_V \longrightarrow \mathrm{pr}_1^*\pi_*\mathcal{O}_V$$

on $X \times_S X$. The cocycle condition for φ implies the cocycle condition for $\varphi^\sharp$. Another way to say this is that it produces a descent datum φ' on the affine scheme W relative to X over S, which moreover has the property that the morphism $V \to W$ is a morphism of descent data. Hence by Lemma 35.37.1 (or by effectivity of descent for quasi-coherent algebras) we obtain a scheme $U' \to S$ with an isomorphism $(W, \varphi') \cong (X \times_S U', can)$ of descent data. We note in passing that U' is affine by Lemma 35.23.18.

And now we can think of V as a (quasi-compact) open $V \subset X \times_S U'$ with the property that it is stable under the descent datum

$$can : X \times_S U' \times_S X \to X \times_S X \times_S U', (x_0, u', x_1) \mapsto (x_0, x_1, u').$$

In other words $(x_0, u') \in V \Rightarrow (x_1, u') \in V$ for any x_0, x_1, u' mapping to the same point of S. Because $X \to S$ is surjective we immediately find that V is the inverse image of a subset $U \subset U'$ under the morphism $X \times_S U' \to U'$. Because $X \to S$ is quasi-compact, flat and surjective also $X \times_S U' \to U'$ is quasi-compact flat and surjective. Hence by Morphisms, Lemma 29.25.12 this subset $U \subset U'$ is open and we win. $\square$

35.39. Descent data in terms of sheaves

02W4 Here is another way to think about descent data in case of a covering on a site.

02W5 **Lemma 35.39.1.** Let $\tau \in \{Zariski, fppf, \acute{e}tale, smooth, syntomic\}$[10]. Let Sch_τ be a big τ-site. Let $S \in \mathrm{Ob}(Sch_\tau)$. Let $\{S_i \to S\}_{i \in I}$ be a covering in the site $(Sch/S)_\tau$. There is an equivalence of categories

$$\left\{\begin{matrix}\text{descent data } (X_i, \varphi_{ii'}) \text{ such that} \\ \text{each } X_i \in \mathrm{Ob}((Sch/S)_\tau)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{sheaves } F \text{ on } (Sch/S)_\tau \text{ such that} \\ \text{each } h_{S_i} \times F \text{ is representable}\end{matrix}\right\}.$$

Moreover,

1. the objects representing $h_{S_i} \times F$ on the right hand side correspond to the schemes X_i on the left hand side, and
2. the sheaf F is representable if and only if the corresponding descent datum $(X_i, \varphi_{ii'})$ is effective.

[10]The fact that fpqc is missing is not a typo. See discussion in Topologies, Section 34.9.

Proof. We have seen in Section 35.13 that representable presheaves are sheaves on the site $(Sch/S)_\tau$. Moreover, the Yoneda lemma (Categories, Lemma 4.3.5) guarantees that maps between representable sheaves correspond one to one with maps between the representing objects. We will use these remarks without further mention during the proof.

Let us construct the functor from right to left. Let F be a sheaf on $(Sch/S)_\tau$ such that each $h_{S_i} \times F$ is representable. In this case let X_i be a representing object in $(Sch/S)_\tau$. It comes equipped with a morphism $X_i \to S_i$. Then both $X_i \times_S S_{i'}$ and $S_i \times_S X_{i'}$ represent the sheaf $h_{S_i} \times F \times h_{S_{i'}}$ and hence we obtain an isomorphism

$$\varphi_{ii'} : X_i \times_S S_{i'} \to S_i \times_S X_{i'}$$

It is straightforward to see that the maps $\varphi_{ii'}$ are morphisms over $S_i \times_S S_{i'}$ and satisfy the cocycle condition. The functor from right to left is given by this construction $F \mapsto (X_i, \varphi_{ii'})$.

Let us construct a functor from left to right. For each i denote F_i the sheaf h_{X_i}. The isomorphisms $\varphi_{ii'}$ give isomorphisms

$$\varphi_{ii'} : F_i \times h_{S_{i'}} \longrightarrow h_{S_i} \times F_{i'}$$

over $h_{S_i} \times h_{S_{i'}}$. Set F equal to the coequalizer in the following diagram

$$\coprod_{i,i'} F_i \times h_{S_{i'}} \underset{\mathrm{pr}_1 \circ \varphi_{ii'}}{\overset{\mathrm{pr}_0}{\rightrightarrows}} \coprod_i F_i \longrightarrow F$$

The cocycle condition guarantees that $h_{S_i} \times F$ is isomorphic to F_i and hence representable. The functor from left to right is given by this construction $(X_i, \varphi_{ii'}) \mapsto F$.

We omit the verification that these constructions are mutually quasi-inverse functors. The final statements (1) and (2) follow from the constructions. $\square$

02W6 Remark 35.39.2. In the statement of Lemma 35.39.1 the condition that $h_{S_i} \times F$ is representable is equivalent to the condition that the restriction of F to $(Sch/S_i)_\tau$ is representable.

35.40. Other chapters

CHAPTER 36

Derived Categories of Schemes

08CU 36.1. Introduction

08CV In this chapter we discuss derived categories of modules on schemes. Most of the material discussed here can be found in [TT90], [BN93], [BV03], and [LN07]. Of course there are many other references.

36.2. Conventions

08CW If $\mathcal{A}$ is an abelian category and M is an object of $\mathcal{A}$ then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A, then $K(A)$ denotes the homotopy category of complexes of A-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.

36.3. Derived category of quasi-coherent modules

06YZ In this section we discuss the relationship between quasi-coherent modules and all modules on a scheme X. A reference is [TT90, Appendix B]. By the discussion in Schemes, Section 26.24 the embedding $QCoh(\mathcal{O}_X) \subset \text{Mod}(\mathcal{O}_X)$ exhibits $QCoh(\mathcal{O}_X)$ as a weak Serre subcategory of the category of $\mathcal{O}_X$-modules. Denote

$$D_{QCoh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are quasi-coherent, see Derived Categories, Section 13.17. Thus we obtain a canonical functor

06VT (36.3.0.1)
$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.17.1.1).

08DT **Lemma 36.3.1.** *Let X be a scheme. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

Proof. By Injectives, Lemma 19.13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any Grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 26.24. $\square$

We will need some information on derived limits. We warn the reader that in the lemma below the derived limit will typically not be an object of D_{QCoh}.

0A0J Lemma 36.3.2. Let X be a scheme. Let (K_n) be an inverse system of $D_{QCoh}(\mathcal{O}_X)$ with derived limit $K = R\lim K_n$ in $D(\mathcal{O}_X)$. Assume $H^q(K_{n+1}) \to H^q(K_n)$ is surjective for all $q \in \mathbf{Z}$ and $n \geq 1$. Then

(1) $H^q(K) = \lim H^q(K_n)$,
(2) $R\lim H^q(K_n) = \lim H^q(K_n)$, and
(3) for every affine open $U \subset X$ we have $H^p(U, \lim H^q(K_n)) = 0$ for $p > 0$.

Proof. Let $\mathcal{B}$ be the set of affine opens of X. Since $H^q(K_n)$ is quasi-coherent we have $H^p(U, H^q(K_n)) = 0$ for $U \in \mathcal{B}$ by Cohomology of Schemes, Lemma 30.2.2. Moreover, the maps $H^0(U, H^q(K_{n+1})) \to H^0(U, H^q(K_n))$ are surjective for $U \in \mathcal{B}$ by Schemes, Lemma 26.7.5. Part (1) follows from Cohomology, Lemma 20.37.11 whose conditions we have just verified. Parts (2) and (3) follow from Cohomology, Lemma 20.37.4. $\square$

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

08D3 Lemma 36.3.3. Let X be a scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then the canonical map $E \to R\lim \tau_{\geq -n} E$ is an isomorphism[1].

Proof. Denote $\mathcal{H}^i = H^i(E)$ the ith cohomology sheaf of E. Let $\mathcal{B}$ be the set of affine open subsets of X. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Cohomology of Schemes, Lemma 30.2.2. Thus the lemma follows from Cohomology, Lemma 20.37.9. $\square$

08D4 Lemma 36.3.4. Let X be a scheme. Let $F : \text{Mod}(\mathcal{O}_X) \to \text{Ab}$ be an additive functor and $N \geq 0$ an integer. Assume that

(1) F commutes with countable direct products,
(2) $R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.

Then for $E \in D_{QCoh}(\mathcal{O}_X)$

(1) $H^i(RF(\tau_{\leq a}E)) \to H^i(RF(E))$ is an isomorphism for $i \leq a$,
(2) $H^i(RF(E)) \to H^i(RF(\tau_{\geq b-N+1}E))$ is an isomorphism for $i \geq b$,
(3) if $H^i(E) = 0$ for $i \notin [a, b]$ for some $-\infty \leq a \leq b \leq \infty$, then $H^i(RF(E)) = 0$ for $i \notin [a, b + N - 1]$.

Proof. Statement (1) is Derived Categories, Lemma 13.16.1.

Proof of statement (2). Write $E_n = \tau_{\geq -n}E$. We have $E = R\lim E_n$, see Lemma 36.3.3. Thus $RF(E) = R\lim RF(E_n)$ in $D(\text{Ab})$ by Injectives, Lemma 19.13.6. Thus for every $i \in \mathbf{Z}$ we have a short exact sequence

$$0 \to R^1\lim H^{i-1}(RF(E_n)) \to H^i(RF(E)) \to \lim H^i(RF(E_n)) \to 0$$

see More on Algebra, Remark 15.86.10. To prove (2) we will show that the term on the left is zero and that the term on the right equals $H^i(RF(E_{-b+N-1})$ for any b with $i \geq b$.

For every n we have a distinguished triangle

$$H^{-n}(E)[n] \to E_n \to E_{n-1} \to H^{-n}(E)[n+1]$$

[1]In particular, E has a K-injective representative as in Cohomology, Lemma 20.38.1.

(Derived Categories, Remark 13.12.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$H^i(RF(H^{-n}(E)[n])) = R^{i+n}F(H^{-n}(E)) = 0$$

for $i + n \geq N$ and

$$H^i(RF(H^{-n}(E)[n+1])) = R^{i+n+1}F(H^{-n}(E)) = 0$$

for $i + n + 1 \geq N$. We conclude that

$$H^i(RF(E_n)) \to H^i(RF(E_{n-1}))$$

is an isomorphism for $n \geq N - i$. Thus the systems $H^i(RF(E_n))$ all satisfy the ML condition and the $R^1 \lim$ term in our short exact sequence is zero (see discussion in More on Algebra, Section 15.86). Moreover, the system $H^i(RF(E_n))$ is constant starting with $n = N - i - 1$ as desired.

Proof of (3). Under the assumption on E we have $\tau_{\leq a-1}E = 0$ and we get the vanishing of $H^i(RF(E))$ for $i \leq a - 1$ from (1). Similarly, we have $\tau_{\geq b+1}E = 0$ and hence we get the vanishing of $H^i(RF(E))$ for $i \geq b + n$ from part (2). $\square$

The following lemma is the key ingredient to many of the results in this chapter.

06Z0 Lemma 36.3.5. Let $X = \mathrm{Spec}(A)$ be an affine scheme. All the functors in the diagram

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[rr, "(36.3.0.1)"'] & & D_{QCoh}(\mathcal{O}_X) \arrow[dl, "R\Gamma(X,-)"] \\ & D(A) \arrow[ul, "\sim"] & \end{tikzcd}$$

are equivalences of triangulated categories. Moreover, for E in $D_{QCoh}(\mathcal{O}_X)$ we have $H^0(X, E) = H^0(X, H^0(E))$.

Proof. The functor $R\Gamma(X, -)$ gives a functor $D(\mathcal{O}_X) \to D(A)$ and hence by restriction a functor

06VU (36.3.5.1) $$R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \longrightarrow D(A).$$

We will show this functor is quasi-inverse to (36.3.0.1) via the equivalence between quasi-coherent modules on X and the category of A-modules.

Elucidation. Denote $(Y, \mathcal{O}_Y)$ the one point space with sheaf of rings given by A. Denote $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ the obvious morphism of ringed spaces. Then $R\Gamma(X, -)$ can be identified with $R\pi_*$ and the functor (36.3.0.1) via the equivalence $\mathrm{Mod}(\mathcal{O}_Y) = \mathrm{Mod}_A = QCoh(\mathcal{O}_X)$ can be identified with $L\pi^* = \pi^* = \widetilde{\ }$(see Modules, Lemma 17.10.5 and Schemes, Lemmas 26.7.1 and 26.7.5). Thus the functors

$$D(A) \rightleftarrows D(\mathcal{O}_X)$$

are adjoint (by Cohomology, Lemma 20.28.1). In particular we obtain canonical adjunction mappings

$$a : \widetilde{R\Gamma(X, E)} \longrightarrow E$$

for E in $D(\mathcal{O}_X)$ and

$$b : M^\bullet \longrightarrow R\Gamma(X, \widetilde{M^\bullet})$$

for $M^\bullet$ a complex of A-modules.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. We may apply Lemma 36.3.4 to the functor $F(-) = \Gamma(X, -)$ with $N = 1$ by Cohomology of Schemes, Lemma 30.2.2. Hence

$$H^0(R\Gamma(X, E)) = H^0(R\Gamma(X, \tau_{\geq 0}E)) = \Gamma(X, H^0(E))$$

(the last equality by definition of the canonical truncation). Using this we will show that the adjunction mappings a and b induce isomorphisms $H^0(a)$ and $H^0(b)$. Thus a and b are quasi-isomorphisms (as the statement is invariant under shifts) and the lemma is proved.

In both cases we use that $\widetilde{\ }$ is an exact functor (Schemes, Lemma 26.5.4). Namely, this implies that

$$H^0\left(\widetilde{R\Gamma(X, E)}\right) = H^0(\widetilde{R\Gamma(X, E)}) = \Gamma(\widetilde{X, H^0}(E))$$

which is equal to $H^0(E)$ because $H^0(E)$ is quasi-coherent. Thus $H^0(a)$ is an isomorphism. For the other direction we have

$$H^0(R\Gamma(X, \widetilde{M^\bullet})) = \Gamma(X, H^0(\widetilde{M^\bullet})) = \Gamma(X, \widetilde{H^0(M^\bullet)}) = H^0(M^\bullet)$$

which proves that $H^0(b)$ is an isomorphism. □

08DV Lemma 36.3.6. Let $X = \mathrm{Spec}(A)$ be an affine scheme. If $K^\bullet$ is a K-flat complex of A-modules, then $\widetilde{K^\bullet}$ is a K-flat complex of $\mathcal{O}_X$-modules.

Proof. By More on Algebra, Lemma 15.59.3 we see that $K^\bullet \otimes_A A_\mathfrak{p}$ is a K-flat complex of $A_\mathfrak{p}$-modules for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Hence we conclude from Cohomology, Lemma 20.26.4 (and Schemes, Lemma 26.5.4) that $\widetilde{K^\bullet}$ is K-flat. □

0DJK Lemma 36.3.7. If $f : X \to Y$ is a morphism of affine schemes given by the ring map $A \to B$, then the diagram

$$\begin{tikzcd} D(B) \arrow[r] \arrow[d] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(A) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. Follows from Lemma 36.3.5 using that $R\Gamma(Y, Rf_*K) = R\Gamma(X, K)$ by Cohomology, Lemma 20.32.5. □

08DW Lemma 36.3.8. Let $f : Y \to X$ be a morphism of schemes.

(1) The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.
(2) If X and Y are affine and f is given by the ring map $A \to B$, then the diagram

$$\begin{tikzcd} D(B) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \\ D(A) \arrow[r] \arrow[u, "-\otimes_A^{\mathbf{L}} B"] & D_{QCoh}(\mathcal{O}_X) \arrow[u, "Lf^*"'] \end{tikzcd}$$

commutes.

Proof. We first prove the diagram

$$\begin{tikzcd} D(B) \arrow[r] & D(\mathcal{O}_Y) \\ D(A) \arrow[r] \arrow[u, "-\otimes_A^{\mathbf{L}} B"] & D(\mathcal{O}_X) \arrow[u, "Lf^*"'] \end{tikzcd}$$

commutes. This is clear from Lemma 36.3.6 and the constructions of the functors in question. To see (1) let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To see that Lf^*E has quasi-coherent cohomology sheaves we may work locally on X. Note that Lf^* is compatible with restricting to open subschemes. Hence we can assume that f is a morphism of affine schemes as in (2). Then we can apply Lemma 36.3.5 to see that E comes from a complex of A-modules. By the commutativity of the first diagram of the proof the same holds for Lf^*E and we conclude (1) is true. □

08DX **Lemma 36.3.9.** Let X be a scheme.

1. For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(\mathcal{O}_X)$.
2. If $X = \mathrm{Spec}(A)$ is affine then
$$\widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{K^\bullet} = \widetilde{M^\bullet \otimes^{\mathbf{L}}_A K^\bullet}$$
for any pair of complexes of A-modules $K^\bullet$, $M^\bullet$.

Proof. The equality of (2) follows immediately from Lemma 36.3.6 and the construction of the derived tensor product. To see (1) let K, L be objects of $D_{QCoh}(\mathcal{O}_X)$. To check that $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$ we may work locally on X, hence we may assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 36.3.5 we may represent K and L by complexes of A-modules. Then part (2) implies the result. □

36.4. Total direct image

08DY The following lemma is the analogue of Cohomology of Schemes, Lemma 30.4.5.

08D5 **Lemma 36.4.1.** Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.

1. The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_S)$.
2. If S is quasi-compact, there exists an integer $N = N(X, S, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m \geq N$.
3. In fact, if S is quasi-compact we can find $N = N(X, S, f)$ such that for every morphism of schemes $S' \to S$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to S'$ is the base change of f.

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. The question is local on S, hence we may assume S is quasi-compact. Pick $N = N(X, S, f)$ as in Cohomology of Schemes, Lemma 30.4.5. Thus $R^pf_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$ and the same remains true after base change.

First, assume E is bounded below. We will show (1) and (2) and (3) hold for such E with our choice of N. In this case we can for example use the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

(Derived Categories, Lemma 13.21.3), the quasi-coherence of $R^pf_*H^q(E)$, and the vanishing of $R^pf_*H^q(E)$ for $p \geq N$ to see that (1), (2), and (3) hold in this case.

Next we prove (2) and (3). Say $H^m(E) = 0$ for $m > 0$. Let $U \subset S$ be affine open. By Cohomology of Schemes, Lemma 30.4.6 and our choice of N we have $H^p(f^{-1}(U), \mathcal{F}) = 0$ for $p \geq N$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Hence we may apply Lemma 36.3.4 to the functor $\Gamma(f^{-1}(U), -)$ to see that

$$R\Gamma(U, Rf_*E) = R\Gamma(f^{-1}(U), E)$$

has vanishing cohomology in degrees $\geq N$. Since this holds for all $U \subset S$ affine open we conclude that $H^m(Rf_*E) = 0$ for $m \geq N$.

Next, we prove (1) in the general case. Recall that there is a distinguished triangle

$$\tau_{\leq -n-1}E \to E \to \tau_{\geq -n}E \to (\tau_{\leq -n-1}E)[1]$$

in $D(\mathcal{O}_X)$, see Derived Categories, Remark 13.12.4. By (2) we see that $Rf_*\tau_{\leq -n-1}E$ has vanishing cohomology sheaves in degrees $\geq -n+N$. Thus, given an integer q we see that R^qf_*E is equal to $R^qf_*\tau_{\geq -n}E$ for some n and the result above applies. □

0G9N Lemma 36.4.2. Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules each of which is right acyclic for f_*. Then $f_*\mathcal{F}^\bullet$ represents $Rf_*\mathcal{F}^\bullet$ in $D(\mathcal{O}_S)$.

Proof. There is always a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. Our task is to show that this is an isomorphism on cohomology sheaves. As the statement is invariant under shifts it suffices to show that $H^0(f_*(\mathcal{F}^\bullet)) \to R^0f_*\mathcal{F}^\bullet$ is an isomorphism. The statement is local on S hence we may assume S affine. By Lemma 36.4.1 we have $R^0f_*\mathcal{F}^\bullet = R^0f_*\tau_{\geq -n}\mathcal{F}^\bullet$ for all sufficiently large n. Thus we may assume $\mathcal{F}^\bullet$ bounded below. As each $\mathcal{F}^n$ is right f_*-acyclic by assumption we see that $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$ is a quasi-isomorphism by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7). □

0G9P Lemma 36.4.3. Let X be a quasi-separated and quasi-compact scheme. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules each of which is right acyclic for $\Gamma(X, -)$. Then $\Gamma(X, \mathcal{F}^\bullet)$ represents $R\Gamma(X, \mathcal{F}^\bullet)$ in $D(\Gamma(X, \mathcal{O}_X))$.

Proof. Apply Lemma 36.4.2 to the canonical morphism $X \to \text{Spec}(\Gamma(X, \mathcal{O}_X))$. Some details omitted. □

0G9Q Lemma 36.4.4. Let X be a quasi-separated and quasi-compact scheme. For any object K of $D_{QCoh}(\mathcal{O}_X)$ the spectral sequence

$$E_2^{i,j} = H^i(X, H^j(K)) \Rightarrow H^{i+j}(X, K)$$

of Cohomology, Example 20.29.3 is bounded and converges.

Proof. By the construction of the spectral sequence via Cohomology, Lemma 20.29.1 using the filtration given by $\tau_{\leq -p}K$, we see that suffices to show that given $n \in \mathbf{Z}$ we have

$$H^n(X, \tau_{\leq -p}K) = 0 \text{ for } p \gg 0$$

and

$$H^n(X, K) = H^n(X, \tau_{\leq -p}K) \text{ for } p \ll 0$$

The first follows from Lemma 36.3.4 applied with $F = \Gamma(X, -)$ and the bound in Cohomology of Schemes, Lemma 30.4.5. The second holds whenever $-p \leq n$ for any ringed space $(X, \mathcal{O}_X)$ and any $K \in D(\mathcal{O}_X)$. □

08DZ Lemma 36.4.5. Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ commutes with direct sums.

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer N as in Lemma 36.4.1. Then $R^0f_*E = R^0f_*\tau_{\geq -N}E$ and $R^0f_*E_i = R^0f_*\tau_{\geq -N}E_i$ by the lemma cited. Observe that $\tau_{\geq -N}E = \bigoplus \tau_{\geq -N}E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E \quad \text{and} \quad R^pf_*H^q(E_i) \Rightarrow R^{p+q}f_*E_i$$

(Derived Categories, Lemma 13.21.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Schemes, Lemma 30.6.1. □

36.5. Affine morphisms

0AVV In this section we collect some information about pushforward along an affine morphism of schemes.

0G9R Lemma 36.5.1. Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then $f_*\mathcal{F}^\bullet = Rf_*\mathcal{F}^\bullet$.

Proof. Combine Lemma 36.4.2 with Cohomology of Schemes, Lemma 30.2.3. An alternative proof is to work affine locally on S and use Lemma 36.3.7. □

08I8 Lemma 36.5.2. Let $f : X \to S$ be an affine morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ reflects isomorphisms.

Proof. The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is local on S. Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ given in Lemma 36.3.5. Some details omitted. □

08I9 Lemma 36.5.3. Let $f : X \to S$ be an affine morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_S)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_S} f_*\mathcal{O}_X$.

Proof. Since f is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Schemes, Lemma 30.2.3). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on S. Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ and the functor Lf^* given in Lemmas 36.3.5 and 36.3.8. Some details omitted. □

Let Y be a scheme. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_Y$-algebras. We will denote $D_{QCoh}(\mathcal{A})$ the inverse image of $D_{QCoh}(\mathcal{O}_X)$ under the restriction functor $D(\mathcal{A}) \to D(\mathcal{O}_X)$. In other words, $K \in D(\mathcal{A})$ is in $D_{QCoh}(\mathcal{A})$ if and only if its cohomology sheaves are quasi-coherent as $\mathcal{O}_X$-modules. If $\mathcal{A}$ is quasi-coherent itself this is the same as asking the cohomology sheaves to be quasi-coherent as $\mathcal{A}$-modules, see Morphisms, Lemma 29.11.6.

0AVW Lemma 36.5.4. Let $f : X \to Y$ be an affine morphism of schemes. Then f_* induces an equivalence

$$\Phi : D_{QCoh}(\mathcal{O}_X) \longrightarrow D_{QCoh}(f_*\mathcal{O}_X)$$

whose composition with $D_{QCoh}(f_*\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$.

Proof. Recall that Rf_* is computed on an object $K \in D_{QCoh}(\mathcal{O}_X)$ by choosing a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules representing K and taking $f_*\mathcal{I}^\bullet$. Thus we let $\Phi(K)$ be the complex $f_*\mathcal{I}^\bullet$ viewed as a complex of $f_*\mathcal{O}_X$-modules. Denote $g : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ the obvious morphism of ringed spaces. Then g is a flat morphism of ringed spaces (see below for a description of the stalks) and Φ is the restriction of Rg_* to $D_{QCoh}(\mathcal{O}_X)$. We claim that Lg^* is a quasi-inverse. First, observe that Lg^* sends $D_{QCoh}(f_*\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_X)$ because g^* transforms quasi-coherent modules into quasi-coherent modules (Modules, Lemma 17.10.4). To finish the proof it suffices to show that the adjunction mappings

$$Lg^*\Phi(K) = Lg^*Rg_*K \to K \quad \text{and} \quad M \to Rg_*Lg^*M = \Phi(Lg^*M)$$

are isomorphisms for $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D_{QCoh}(f_*\mathcal{O}_X)$. This is a local question, hence we may assume Y and therefore X are affine.

Assume $Y = \operatorname{Spec}(B)$ and $X = \operatorname{Spec}(A)$. Let $\mathfrak{p} = x \in \operatorname{Spec}(A) = X$ be a point mapping to $\mathfrak{q} = y \in \operatorname{Spec}(B) = Y$. Then $(f_*\mathcal{O}_X)_y = A_\mathfrak{q}$ and $\mathcal{O}_{X,x} = A_\mathfrak{p}$ hence g is flat. Hence g^* is exact and $H^i(Lg^*M) = g^*H^i(M)$ for any M in $D(f_*\mathcal{O}_X)$. For $K \in D_{QCoh}(\mathcal{O}_X)$ we see that

$$H^i(\Phi(K)) = H^i(Rf_*K) = f_*H^i(K)$$

by the vanishing of higher direct images (Cohomology of Schemes, Lemma 30.2.3) and Lemma 36.3.4 (small detail omitted). Thus it suffice to show that

$$g^*g_*\mathcal{F} \to \mathcal{F} \quad \text{and} \quad \mathcal{G} \to g_*g^*\mathcal{F}$$

are isomorphisms where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\mathcal{G}$ is a quasi-coherent $f_*\mathcal{O}_X$-module. This follows from Morphisms, Lemma 29.11.6. □

36.6. Cohomology with support in a closed subset

0G7F We elaborate on the material in Cohomology, Sections 20.21 and 20.34 for schemes and quasi-coherent modules.

08DA Definition 36.6.1. Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset X$ be a closed subset. We say E is supported on T if the cohomology sheaves $H^i(E)$ are supported on T.

We repeat some of the discussion from Cohomology, Section 20.34 in the situation of the definition. Let X be a scheme. Let $T \subset X$ be a closed subset. The category of $\mathcal{O}_X$-modules whose support is contained in T is a Serre subcategory of the category of all $\mathcal{O}_X$-modules, see Homology, Definition 12.10.1 and Modules, Lemma 17.5.2.

In the following we will denote $D_T(\mathcal{O}_X)$ the strictly full, saturated triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T, see Derived Categories, Section 13.17.

In the situation of Definition 36.6.1 denote $i : T \to X$ the inclusion map. Recall from Cohomology, Section 20.34 that in this situation we have a functor $R\mathcal{H}_T : D(\mathcal{O}_X) \to D(i^{-1}\mathcal{O}_X)$ which is right adjoint to $i_* : D(i^{-1}\mathcal{O}_X) \to D(\mathcal{O}_X)$.

0G7G Lemma 36.6.2. Let X be a scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is a retrocompact open of X. Let $i : T \to X$ be the inclusion.

(1) For E in $D_{QCoh}(\mathcal{O}_X)$ we have $i_*R\mathcal{H}_T(E)$ in $D_{QCoh,T}(\mathcal{O}_X)$.
(2) The functor $i_* \circ R\mathcal{H}_T : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh,T}(\mathcal{O}_X)$ is right adjoint to the inclusion functor $D_{QCoh,T}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_X)$.

Proof. Set $U = X \setminus T$ and denote $j : U \to X$ the inclusion. By Cohomology, Lemma 20.34.6 there is a distinguished triangle

$$i_*R\mathcal{H}_T(E) \to E \to Rj_*(E|_U) \to i_*R\mathcal{H}_Z(E)[1]$$

in $D(\mathcal{O}_X)$. By Lemma 36.4.1 the complex $Rj_*(E|_U)$ has quasi-coherent cohomology sheaves (this is where we use that U is retrocompact in X). Thus we see that (1) is true. Part (2) follows from this and the adjointness of functors in Cohomology, Lemma 20.34.2. $\square$

0G7H Lemma 36.6.3. Let X be a scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is a retrocompact open of X. Then for a family of objects E_i, $i \in I$ of $D_{QCoh}(\mathcal{O}_X)$ we have $R\mathcal{H}_T(\bigoplus E_i) = \bigoplus R\mathcal{H}_T(E_i)$.

Proof. Set $U = X \setminus T$ and denote $j : U \to X$ the inclusion. By Cohomology, Lemma 20.34.6 there is a distinguished triangle

$$i_*R\mathcal{H}_T(E) \to E \to Rj_*(E|_U) \to i_*R\mathcal{H}_Z(E)[1]$$

in $D(\mathcal{O}_X)$ for any E in $D(\mathcal{O}_X)$. The functor $E \mapsto Rj_*(E|_U)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.4.5. It follows that the same is true for the functor $i_* \circ R\mathcal{H}_T$ (details omitted). Since $i_* : D(i^{-1}\mathcal{O}_X) \to D_T(\mathcal{O}_X)$ is an equivalence (Cohomology, Lemma 20.34.2) we conclude. $\square$

0G7I Remark 36.6.4. Let X be a scheme. Let $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Denote $Z \subset X$ the closed subscheme cut out by $f_1, \ldots, f_c$. For $0 \leq p < c$ and $1 \leq i_0 < \ldots < i_p \leq c$ we denote $U_{i_0 \ldots i_p} \subset X$ the open subscheme where $f_{i_0} \ldots f_{i_p}$ is invertible. For any $\mathcal{O}_X$-module $\mathcal{F}$ we set

$$\mathcal{F}_{i_0 \ldots i_p} = (U_{i_0 \ldots i_p} \to X)_*(\mathcal{F}|_{U_{i_0 \ldots i_p}})$$

In this situation the extended alternating Čech complex is the complex of $\mathcal{O}_X$-modules

0G7J (36.6.4.1)
$$0 \to \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \to \ldots \to \bigoplus\nolimits_{i_0 < \ldots < i_p} \mathcal{F}_{i_0 \ldots i_p} \to \ldots \to \mathcal{F}_{1 \ldots c} \to 0$$

where $\mathcal{F}$ is put in degree 0. The maps are constructed as follows. Given $1 \leq i_0 < \ldots < i_{p+1} \leq c$ and $0 \leq j \leq p+1$ we have the canonical map

$$\mathcal{F}_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \to \mathcal{F}_{i_0 \ldots i_p}$$

coming from the inclusion $U_{i_0 \ldots i_p} \subset U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$. The differentials in the extended alternating complex use these canonical maps with sign $(-1)^j$.

0G7K Lemma 36.6.5. With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 36.6.4 the complex (36.6.4.1) restricts to an acyclic complex over $X \setminus Z$.

We remark that this lemma holds more generally for any extended alternating Čech complex defined as in Remark 36.6.4 starting with a finite open covering $X \setminus Z = U_1 \cup \ldots \cup U_c$.

Proof. Let $W \subset X \setminus Z$ be an open subset. Evaluating the complex of sheaves (36.6.4.1) on W we obtain the complex

$$\mathcal{F}(W) \to \bigoplus\nolimits_{i_0} \mathcal{F}(U_{i_0} \cap W) \to \bigoplus\nolimits_{i_0 < i_1} \mathcal{F}(U_{i_0 i_1} \cap W) \to \ldots$$

In other words, we obtain the extended ordered Čech complex for the covering $W = \bigcup U_i \cap W$ and the standard ordering on $\{1, \ldots, c\}$, see Cohomology, Section 20.23. By Cohomology, Lemma 20.23.7 this complex is homotopic to zero as soon as W is contained in $V(f_i)$ for some $1 \leq i \leq c$. This finishes the proof. $\square$

0G7L Remark 36.6.6. Let X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ be as in Remark 36.6.4. Denote $\mathcal{F}^\bullet$ the complex (36.6.4.1). By Lemma 36.6.5 the cohomology sheaves of $\mathcal{F}^\bullet$ are supported on Z hence $\mathcal{F}^\bullet$ is an object of $D_Z(\mathcal{O}_X)$. On the other hand, the equality $\mathcal{F}^0 = \mathcal{F}$ determines a canonical map $\mathcal{F}^\bullet \to \mathcal{F}$ in $D(\mathcal{O}_X)$. As $i_* \circ R\mathcal{H}_Z$ is a right adjoint to the inclusion functor $D_Z(\mathcal{O}_X) \to D(\mathcal{O}_X)$, see Cohomology, Lemma 20.34.2, we obtain a canonical commutative diagram

$$\begin{tikzcd} \mathcal{F}^\bullet \arrow[rr] \arrow[dr] & & \mathcal{F} \\ & i_* R\mathcal{H}_Z(\mathcal{F}) \arrow[ur] & \end{tikzcd}$$

in $D(\mathcal{O}_X)$ functorial in the $\mathcal{O}_X$-module $\mathcal{F}$.

0G7M Lemma 36.6.7. With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 36.6.4. If $\mathcal{F}$ is quasi-coherent, then the complex (36.6.4.1) represents $i_* R\mathcal{H}_Z(\mathcal{F})$ in $D_Z(\mathcal{O}_X)$.

Proof. Let us denote $\mathcal{F}^\bullet$ the complex (36.6.4.1). The statement of the lemma means that the map $\mathcal{F}^\bullet \to i_* R\mathcal{H}_Z(\mathcal{F})$ of Remark 36.6.6 is an isomorphism. Since $\mathcal{F}^\bullet$ is in $D_Z(\mathcal{O}_X)$ (see remark cited), we see that $i_* R\mathcal{H}_Z(\mathcal{F}^\bullet) = \mathcal{F}^\bullet$ by Cohomology, Lemma 20.34.2. The morphism $U_{i_0 \ldots i_p} \to X$ is affine as it is given over affine opens of X by inverting the function $f_{i_0} \ldots f_{i_p}$. Thus we see that

$$\mathcal{F}_{i_0 \ldots i_p} = (U_{i_0 \ldots i_p} \to X)_* \mathcal{F}|_{U_{i_0 \ldots i_p}} = R(U_{i_0 \ldots i_p} \to X)_* \mathcal{F}|_{U_{i_0 \ldots i_p}}$$

by Cohomology of Schemes, Lemma 30.2.3 and the assumption that $\mathcal{F}$ is quasi-coherent. We conclude that $R\mathcal{H}_Z(\mathcal{F}_{i_0 \ldots i_p}) = 0$ by Cohomology, Lemma 20.34.7. Thus $i_* R\mathcal{H}_Z(\mathcal{F}^p) = 0$ for $p > 0$. Putting everything together we obtain

$$\mathcal{F}^\bullet = i_* R\mathcal{H}_Z(\mathcal{F}^\bullet) = i_* R\mathcal{H}_Z(\mathcal{F})$$

as desired. $\square$

0G7N Lemma 36.6.8. Let X be a scheme. Let $T \subset X$ be a closed subset which can locally be cut out by at most c elements of the structure sheaf. Then $\mathcal{H}^i_Z(\mathcal{F}) = 0$ for $i > c$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.

Proof. This follows immediately from the local description of $R\mathcal{H}_T(\mathcal{F})$ given in Lemma 36.6.7. $\square$

0G7P Lemma 36.6.9. Let X be a scheme. Let $T \subset X$ be a closed subset which can locally be cut out by a Koszul regular sequence having c elements. Then $\mathcal{H}^i_Z(\mathcal{F}) = 0$ for $i \neq c$ for every flat, quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.

Proof. By the description of $R\mathcal{H}_Z(\mathcal{F})$ given in Lemma 36.6.7 this boils down to the following algebra statement: given a ring R, a Koszul regular sequence $f_1, \ldots, f_c \in R$, and a flat R-module M, the extended alternating Čech complex $M \to \bigoplus_{i_0} M_{f_{i_0}} \to \bigoplus_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \to \ldots \to M_{f_1 \ldots f_c}$ from More on Algebra, Section 15.29 only has cohomology in degree c. By More on Algebra, Lemma 15.31.1 we obtain the desired vanishing for the extended alternating Čech complex of R. Since the complex for M is obtained by tensoring this with the flat R-module M (More on Algebra, Lemma 15.29.2) we conclude. $\square$

0G7Q Remark 36.6.10. With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 36.6.4. There is a canonical $\mathcal{O}_X|_Z$-linear map

$$c_{f_1, \ldots, f_c} : i^*\mathcal{F} \longrightarrow \mathcal{H}^c_Z(\mathcal{F})$$

functorial in $\mathcal{F}$. Namely, denoting $\mathcal{F}^\bullet$ the extended alternating Čech complex (36.6.4.1) we have the canonical map $\mathcal{F}^\bullet \to i_*R\mathcal{H}_Z(\mathcal{F})$ of Remark 36.6.6. This determines a canonical map

$$\mathrm{Coker}\left(\bigoplus \mathcal{F}_{1 \ldots \hat{\imath} \ldots c} \to \mathcal{F}_{1 \ldots c}\right) \longrightarrow i_*\mathcal{H}^c_Z(\mathcal{F})$$

on cohomology sheaves in degree c. Given a local section s of $\mathcal{F}$ we can consider the local section

$$\frac{s}{f_1 \ldots f_c}$$

of $\mathcal{F}_{1 \ldots c}$. The class of this section in the cokernel displayed above depends only on s modulo the image of $(f_1, \ldots, f_c) : \mathcal{F}^{\oplus c} \to \mathcal{F}$. Since $i_*i^*\mathcal{F}$ is equal to the cokernel of $(f_1, \ldots, f_c) : \mathcal{F}^{\oplus c} \to \mathcal{F}$ we see that we get an $\mathcal{O}_X$-module map $i_*i^*\mathcal{F} \to i_*\mathcal{H}^c_Z(\mathcal{F})$. As i_* is fully faithful we get the map $c_{f_1, \ldots, f_c}$.

0G7R Example 36.6.11. Let $X = \mathrm{Spec}(A)$ be affine, $f_1, \ldots, f_c \in A$, and let $\mathcal{F} = \widetilde{M}$ for some A-module M. The map $c_{f_1, \ldots, f_c}$ of Remark 36.6.10 can be described as the map

$$M/(f_1, \ldots, f_c)M \longrightarrow \mathrm{Coker}\left(\bigoplus M_{f_1 \ldots \hat{f_i} \ldots f_c} \to M_{f_1 \ldots f_c}\right)$$

sending the class of $s \in M$ to the class of $s/f_1 \ldots f_c$ in the cokernel.

0G7S Lemma 36.6.12. With X, $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$, and $\mathcal{F}$ as in Remark 36.6.4. Let $a_{ji} \in \Gamma(X, \mathcal{O}_X)$ for $1 \leq i, j \leq c$ and set $g_j = \sum_{i=1, \ldots, c} a_{ji} f_i$. Assume $g_1, \ldots, g_c$ scheme theoretically cut out Z. If $\mathcal{F}$ is quasi-coherent, then

$$c_{f_1, \ldots, f_c} = \det(a_{ji}) c_{g_1, \ldots, g_c}$$

where $c_{f_1, \ldots, f_c}$ and $c_{g_1, \ldots, g_c}$ are as in Remark 36.6.10.

Proof. We will prove that $c_{f_1, \ldots, f_c}(s) = \det(a_{ij}) c_{g_1, \ldots, g_c}(s)$ as global sections of $\mathcal{H}_Z(\mathcal{F})$ for any $s \in \mathcal{F}(X)$. This is sufficient since we then obtain the same result for section over any open subscheme of X. To do this, for $1 \leq i_0 < \ldots < i_p \leq c$ and $1 \leq j_0 < \ldots < j_q \leq c$ we denote $U_{i_0 \ldots i_p} \subset X$, $V_{j_0 \ldots j_q} \subset X$, and $W_{i_0 \ldots i_p, j_0 \ldots j_q} \subset X$ the open subscheme where $f_{i_0} \ldots f_{i_p}$ is invertible, $g_{j_0} \ldots g_{j_q}$ is invertible, and where $f_{i_0} \ldots f_{i_p} g_{j_0} \ldots g_{j_q}$ is invertible. We denote $\mathcal{F}_{i_0 \ldots i_p}$, resp. $\mathcal{F}'_{j_0 \ldots j_q}$ $\mathcal{F}''_{i_0 \ldots i_p, j_0 \ldots j_q}$ the pushforward to X of the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$, resp. $V_{j_0\ldots j_q}$, resp. $W_{i_0\ldots i_p,j_0\ldots j_q}$. Then we obtain three extended alternating Čech complexes

$$\mathcal{F}^\bullet : \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \to \bigoplus\nolimits_{i_0<i_1} \mathcal{F}_{i_0i_1} \to \ldots$$

and

$$(\mathcal{F}')^\bullet : \mathcal{F} \to \bigoplus\nolimits_{j_0} \mathcal{F}'_{j_0} \to \bigoplus\nolimits_{j_0<j_1} \mathcal{F}'_{j_0j_1} \to \ldots$$

and

$$(\mathcal{F}'')^\bullet : \mathcal{F} \to \bigoplus\nolimits_{i_0} \mathcal{F}_{i_0} \oplus \bigoplus\nolimits_{j_0} \mathcal{F}'_{j_0} \to \bigoplus\nolimits_{i_0<i_1} \mathcal{F}_{i_0i_1} \oplus \bigoplus\nolimits_{i_0,j_0} \mathcal{F}''_{i_0,j_0} \oplus \bigoplus\nolimits_{j_0<j_1} \mathcal{F}'_{j_0j_1} \to \ldots$$

whose differentials are those used in defining (36.6.4.1). There are maps of complexes

$$(\mathcal{F}'')^\bullet \to \mathcal{F}^\bullet \quad \text{and} \quad (\mathcal{F}'')^\bullet \to (\mathcal{F}')^\bullet$$

given by the projection maps on the terms (and hence inducing the identity map in degree 0). Observe that by Lemma 36.6.7 each of these complexes represents $i_*R\mathcal{H}_Z(\mathcal{F})$ and these maps represent the identity on this object. Thus it suffices to find an element

$$\sigma \in H^c((\mathcal{F}'')^\bullet(X))$$

mapping to $c_{f_1,\ldots,f_c}(s)$ and $\det(a_{ji})c_{g_1,\ldots,g_c}(s)$ by these two maps. It turns out we can explicitly give a cocycle for σ. Namely, we take

$$\sigma_{1\ldots c} = \frac{s}{f_1 \ldots f_c} \in \mathcal{F}_{1\ldots c}(X) \quad \text{and} \quad \sigma'_{1\ldots c} = \frac{\det(a_{ji})s}{g_1 \ldots g_c} \in \mathcal{F}'_{1\ldots c}(X)$$

and we take

$$\sigma_{i_0\ldots i_p,j_0\ldots j_{c-p-2}} = \frac{\lambda(i_0 \ldots i_p, j_0 \ldots j_{c-p-2})s}{f_{i_0} \ldots f_{i_p} g_{j_0} \ldots g_{j_{c-p-2}}} \in \mathcal{F}''_{i_0\ldots i_p,j_0\ldots j_{c-p-2}}(X)$$

where $\lambda(i_0 \ldots i_p, j_0 \ldots j_{c-p-2})$ is the coefficient of $e_1 \wedge \ldots \wedge e_c$ in the formal expresssion

$$e_{i_0} \wedge \ldots \wedge e_{i_p} \wedge (a_{j_0 1}e_1 + \ldots + a_{j_0 c}e_c) \wedge \ldots \wedge (a_{j_{c-p-2}1}e_1 + \ldots + a_{j_{c-p-2}c}e_c)$$

To verify that σ is a cocycle, we have to show for $1 \leq i_0 < \ldots < i_p \leq c$ and $1 \leq j_0 < \ldots < j_{c-p-1} \leq c$ that we have

$$\begin{aligned} 0 = & \sum\nolimits_{a=0,\ldots,p} (-1)^a f_{i_a} \lambda(i_0 \ldots \hat{i}_a \ldots i_p, j_0 \ldots j_{c-p-1}) \\ & + \sum\nolimits_{b=0,\ldots,c-p-1} (-1)^{p+b+1} g_{j_b} \lambda(i_0 \ldots i_p, j_0 \ldots \hat{j}_b \ldots j_{c-p-1}) \end{aligned}$$

The easiest way to see this is perhaps to argue that the formal expression

$$\xi = e_{i_0} \wedge \ldots \wedge e_{i_p} \wedge (a_{j_0 1}e_1 + \ldots + a_{j_0 c}e_c) \wedge \ldots \wedge (a_{j_{c-p-1}1}e_1 + \ldots + a_{j_{c-p-1}c}e_c)$$

is 0 as it is an element of the $(c+1)$st wedge power of the free module on $e_1, \ldots, e_c$ and that the expression above is the image of ξ under the Koszul differential sending $e_i \to f_i$. Some details omitted. □

0G7T Lemma 36.6.13. Let X be a scheme. Let $Z \to X$ be a closed immersion of finite presentation whose conormal sheaf $\mathcal{C}_{Z/X}$ is locally free of rank c. Then there is a canonical map

$$c : \wedge^c(\mathcal{C}_{Z/X})^\vee \otimes_{\mathcal{O}_Z} i^*\mathcal{F} \longrightarrow \mathcal{H}^c_Z(\mathcal{F})$$

functorial in the quasi-coherent module $\mathcal{F}$.

Proof. Follows from the construction in Remark 36.6.10 and the independence of the choice of generators of the ideal sheaf shown in Lemma 36.6.12. Some details omitted. □

0G7U Remark 36.6.14. Let $g : X' \to X$ be a morphism of schemes. Let $f_1, \ldots, f_c \in \Gamma(X, \mathcal{O}_X)$. Set $f_i' = g^\sharp(f_i) \in \Gamma(X', \mathcal{O}_{X'})$. Denote $Z \subset X$, resp. $Z' \subset X'$ the closed subscheme cut out by $f_1, \ldots, f_c$, resp. $f_1', \ldots, f_c'$. Then $Z' = Z \times_X X'$. Denote $h : Z' \to Z$ the induced morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Set $\mathcal{F}' = g^*\mathcal{F}$. In this setting, if $\mathcal{F}$ is quasi-coherent, then the diagram

$$\begin{tikzcd}
(i')^{-1}\mathcal{O}_{X'} \otimes_{h^{-1}i^{-1}\mathcal{O}_X} h^{-1}\mathcal{H}^c_Z(\mathcal{F}) \arrow[r] & \mathcal{H}^c_{Z'}(\mathcal{F}') \\
h^*i^*\mathcal{F} \arrow[r] \arrow[u, "c_{f_1, \ldots, f_c}"] & (i')^*\mathcal{F}' \arrow[u, "c_{f_1', \ldots, f_c'}"']
\end{tikzcd}$$

is commutative where the top horizonal arrow is the map of Cohomology, Remark 20.34.12 on cohomology sheaves in degree c. Namely, denote $\mathcal{F}^\bullet$, resp. $(\mathcal{F}')^\bullet$ the extended alternating Čech complex constructed in Remark 36.6.4 using $\mathcal{F}, f_1, \ldots, f_c$, resp. $\mathcal{F}', f_1', \ldots, f_c'$. Note that $(\mathcal{F}')^\bullet = g^*\mathcal{F}^\bullet$. Then, without assuming $\mathcal{F}$ is quasi-coherent, the diagram

$$\begin{tikzcd}
i'_*L(g|_{Z'})^*R\mathcal{H}_Z(\mathcal{F}) \arrow[r] \arrow[d, equal] & i'_*R\mathcal{H}_{Z'}(Lg^*\mathcal{F}) \arrow[d] \\
Lg^*i_*R\mathcal{H}_Z(\mathcal{F}) & i'_*R\mathcal{H}_{Z'}(\mathcal{F}') \\
Lg^*(\mathcal{F}^\bullet) \arrow[r] \arrow[u] & (\mathcal{F}')^\bullet \arrow[u]
\end{tikzcd}$$

is commutative where $g|_{Z'} : (Z', (i')^{-1}\mathcal{O}_{X'}) \to (Z, i^{-1}\mathcal{O}_X)$ is the induced morphism of ringed spaces. Here the top horizontal arrow is given in Cohomology, Remark 20.34.12 as is the explanation for the equal sign. The arrows pointing up are from Remark 36.6.6. The lower horizonal arrow is the map $Lg^*\mathcal{F}^\bullet \to g^*\mathcal{F}^\bullet = (\mathcal{F}')^\bullet$ and the arrow pointing down is induced by $Lg^*\mathcal{F} \to g^*\mathcal{F} = \mathcal{F}'$. The diagram commutes because going around the diagram both ways we obtain two arrows $Lg^*\mathcal{F}^\bullet \to i'_*R\mathcal{H}_{Z'}(\mathcal{F}')$ whose composition with $i'_*R\mathcal{H}_{Z'}(\mathcal{F}') \to \mathcal{F}'$ is the canonical map $Lg^*\mathcal{F}^\bullet \to \mathcal{F}'$. Some details omitted. Now the commutativity of the first diagram follows by looking at this diagram on cohomology sheaves in degree c and using that the construction of the map $i^*\mathcal{F} \to \text{Coker}(\bigoplus \mathcal{F}_{1\ldots\hat{i}\ldots c} \to \mathcal{F}_{1\ldots c})$ used in Remark 36.6.10 is compatible with pullbacks.

36.7. The coherator

08D6 Let X be a scheme. The coherator is a functor

$$Q_X : \text{Mod}(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $QCoh(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$. It exists for any scheme X and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties, Proposition 28.23.4.

Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(QCoh(\mathcal{O}_X)).$$

Since Q_X is right adjoint to the inclusion functor $QCoh(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_X)$ we see that RQ_X is right adjoint to the canonical functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$ by Derived Categories, Lemma 13.30.3.

In this section we will study the functor RQ_X. In Section 36.21 we will study the (closely related) right adjoint to the inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ (when it exists).

08D7 Lemma 36.7.1. Let $f : X \to Y$ be an affine morphism of schemes. Then f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$. This functor has the property that

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "f_*"] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. The functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is exact, see Cohomology of Schemes, Lemma 30.2.3. Hence f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 13.16.9. The diagram commutes by Lemma 36.5.1. $\square$

08D8 Lemma 36.7.2. Let $f : X \to Y$ be a morphism of schemes. Assume f is quasi-compact, quasi-separated, and flat. Then, denoting

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ we have $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. We will prove this by showing that $RQ_Y \circ Rf_*$ and $\Phi \circ RQ_X$ are right adjoint to the same functor $D(QCoh(\mathcal{O}_Y)) \to D(\mathcal{O}_X)$.

Since f is quasi-compact and quasi-separated, we see that f_* preserves quasi-coherence, see Schemes, Lemma 26.24.1. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 28.23.4). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology, Lemma 20.28.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 13.30.3.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then

$$\begin{aligned} \text{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \text{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \text{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \text{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \text{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$

This implies what we want. $\square$

08D9 **Lemma 36.7.3.** Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then

(1) $Q_X : \mathrm{Mod}(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\Gamma(X, \mathcal{F})$,
(2) $RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ is the functor which sends E to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,
(3) restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor RQ_X defines a quasi-inverse to (36.3.0.1).

Proof. The functor Q_X is the functor

$$\mathcal{F} \mapsto \widetilde{\Gamma(X, \mathcal{F})}$$

by Schemes, Lemma 26.7.1. This immediately implies (1) and (2). The third assertion follows from (the proof of) Lemma 36.3.5. $\square$

At this point we are ready to prove a criterion for when the functor $D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence.

09T6 **Lemma 36.7.4.** Let X be a quasi-compact and quasi-separated scheme. Suppose that for every affine open $U \subset X$ the right derived functor

$$\Phi : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_X))$$

of the left exact functor $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{tikzcd}
D(QCoh(\mathcal{O}_U)) \arrow[r, "i_U"'] \arrow[d, "\Phi"'] & D_{QCoh}(\mathcal{O}_U) \arrow[d, "Rj_*"] \\
D(QCoh(\mathcal{O}_X)) \arrow[r, "i_X"] & D_{QCoh}(\mathcal{O}_X)
\end{tikzcd}$$

Then the functor (36.3.0.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ and let A be an object of $D(QCoh(\mathcal{O}_X))$. We have to show that the adjunction maps

$$RQ_X(i_X(A)) \to A \quad \text{and} \quad E \to i_X(RQ_X(E))$$

are isomorphisms. Consider the hypothesis H_n: the adjunction maps above are isomorphisms whenever E and $i_X(A)$ are supported (Definition 36.6.1) on a closed subset of X which is contained in the union of n affine opens of X. We will prove H_n by induction on n.

Base case: $n = 0$. In this case $E = 0$, hence the map $E \to i_X(RQ_X(E))$ is an isomorphism. Similarly $i_X(A) = 0$. Thus the cohomology sheaves of $i_X(A)$ are zero. Since the inclusion functor $QCoh(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$ is fully faithful and exact, we conclude that the cohomology objects of A are zero, i.e., $A = 0$ and $RQ_X(i_X(A)) \to A$ is an isomorphism as well.

Induction step. Suppose that E and $i_X(A)$ are supported on a closed subset T of X contained in $U_1 \cup \ldots \cup U_n$ with $U_i \subset X$ affine open. Set $U = U_n$. Consider the distinguished triangles

$$A \to \Phi(A|_U) \to A' \to A[1] \quad \text{and} \quad E \to Rj_*(E|_U) \to E' \to E[1]$$

where Φ is as in the statement of the lemma. Note that $E \to Rj_*(E|_U)$ is a quasi-isomorphism over $U = U_n$. Since $i_X \circ \Phi = Rj_* \circ i_U$ by assumption and since $i_X(A)|_U = i_U(A|_U)$ we see that $i_X(A) \to i_X(\Phi(A|_U))$ is a quasi-isomorphism over U. Hence $i_X(A')$ and E' are supported on the closed subset $T \setminus U$ of X which is contained in $U_1 \cup \ldots \cup U_{n-1}$. By induction hypothesis the statement is true for A' and E'. By Derived Categories, Lemma 13.4.3 it suffices to prove the maps

$$RQ_X(i_X(\Phi(A|_U))) \to \Phi(A|_U) \quad \text{and} \quad Rj_*(E|_U) \to i_X(RQ_X(Rj_*E|_U))$$

are isomorphisms. By assumption and by Lemma 36.7.2 (the inclusion morphism $j : U \to X$ is flat, quasi-compact, and quasi-separated) we have

$$RQ_X(i_X(\Phi(A|_U))) = RQ_X(Rj_*(i_U(A|_U))) = \Phi(RQ_U(i_U(A|_U)))$$

and

$$i_X(RQ_X(Rj_*(E|_U))) = i_X(\Phi(RQ_U(E|_U))) = Rj_*(i_U(RQ_U(E|_U)))$$

Finally, the maps

$$RQ_U(i_U(A|_U)) \to A|_U \quad \text{and} \quad E|_U \to i_U(RQ_U(E|_U))$$

are isomorphisms by Lemma 36.7.3. The result follows. □

08DB Proposition 36.7.5. Let X be a quasi-compact scheme with affine diagonal. Then the functor (36.3.0.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. Let $U \subset X$ be an affine open. Then the morphism $U \to X$ is affine by Morphisms, Lemma 29.11.11. Thus the assumption of Lemma 36.7.4 holds by Lemma 36.7.1 and we win. □

0CRX Lemma 36.7.6. Let $f : X \to Y$ be a morphism of schemes. Assume X and Y are quasi-compact and have affine diagonal. Then, denoting

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ the diagram

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

is commutative.

Proof. Observe that the horizontal arrows in the diagram are equivalences of categories by Proposition 36.7.5. Hence we can identify these categories (and similarly for other quasi-compact schemes with affine diagonal). The statement of the lemma is that the canonical map $\Phi(K) \to Rf_*(K)$ is an isomorphism for all K in $D(QCoh(\mathcal{O}_X))$. Note that if $K_1 \to K_2 \to K_3 \to K_1[1]$ is a distinguished triangle in $D(QCoh(\mathcal{O}_X))$ and the statement is true for two-out-of-three, then it is true for the third.

Let $U \subset X$ be an affine open. Since the diagonal of X is affine, the inclusion morphism $j : U \to X$ is affine (Morphisms, Lemma 29.11.11). Similarly, the composition $g = f \circ j : U \to Y$ is affine. Let $\mathcal{I}^\bullet$ be a K-injective complex in $QCoh(\mathcal{O}_U)$. Since $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ has an exact left adjoint $j^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ we see that $j_*\mathcal{I}^\bullet$ is a K-injective complex in $QCoh(\mathcal{O}_X)$, see Derived Categories, Lemma 13.31.9. It follows that

$$\Phi(j_*\mathcal{I}^\bullet) = f_*j_*\mathcal{I}^\bullet = g_*\mathcal{I}^\bullet$$

By Lemma 36.7.1 we see that $j_*\mathcal{I}^\bullet$ represents $Rj_*\mathcal{I}^\bullet$ and $g_*\mathcal{I}^\bullet$ represents $Rg_*\mathcal{I}^\bullet$. On the other hand, we have $Rf_* \circ Rj_* = Rg_*$. Hence $f_*j_*\mathcal{I}^\bullet$ represents $Rf_*(j_*\mathcal{I}^\bullet)$. We conclude that the lemma is true for any complex of the form $j_*\mathcal{G}^\bullet$ with $\mathcal{G}^\bullet$ a complex of quasi-coherent modules on U. (Note that if $\mathcal{G}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism, then $j_*\mathcal{G}^\bullet \to j_*\mathcal{I}^\bullet$ is a quasi-isomorphism as well since j_* is an exact functor on quasi-coherent modules.)

Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Let $T \subset X$ be a closed subset such that the support of $\mathcal{F}^p$ is contained in T for all p. We will use induction on the minimal number n of affine opens $U_1, \ldots, U_n$ such that $T \subset U_1 \cup \ldots \cup U_n$. The base case $n = 0$ is trivial. If $n \geq 1$, then set $U = U_1$ and denote $j : U \to X$ the open immersion as above. We consider the map of complexes $c : \mathcal{F}^\bullet \to j_*j^*\mathcal{F}^\bullet$. We obtain two short exact sequences of complexes:

$$0 \to \text{Ker}(c) \to \mathcal{F}^\bullet \to \text{Im}(c) \to 0$$

and

$$0 \to \text{Im}(c) \to j_*j^*\mathcal{F}^\bullet \to \text{Coker}(c) \to 0$$

The complexes $\text{Ker}(c)$ and $\text{Coker}(c)$ are supported on $T \setminus U \subset U_2 \cup \ldots \cup U_n$ and the result holds for them by induction. The result holds for $j_*j^*\mathcal{F}^\bullet$ by the discussion in the preceding paragraph. We conclude by looking at the distinguished triangles associated to the short exact sequences and using the initial remark of the proof. $\square$

0CRY Remark 36.7.7 (Warning). Let X be a quasi-compact scheme with affine diagonal. Even though we know that $D(QCoh(\mathcal{O}_X)) = D_{QCoh}(\mathcal{O}_X)$ by Proposition 36.7.5 strange things can happen and it is easy to make mistakes with this material. One pitfall is to carelessly assume that this equality means derived functors are the same. For example, suppose we have a quasi-compact open $U \subset X$. Then we can consider the higher right derived functors

$$R^i(QCoh)\Gamma(U, -) : QCoh(\mathcal{O}_X) \to \text{Ab}$$

of the left exact functor $\Gamma(U, -)$. Since this is a universal δ-functor, and since the functors $H^i(U, -)$ (defined for all abelian sheaves on X) restricted to $QCoh(\mathcal{O}_X)$ form a δ-functor, we obtain canonical tranformations

$$t^i : R^i(QCoh)\Gamma(U, -) \to H^i(U, -).$$

These transformations aren't in general isomorphisms even if $X = \text{Spec}(A)$ is affine! Namely, we have $R^1(QCoh)\Gamma(U, \widetilde{I}) = 0$ if I an injective A-module by construction of right derived functors and the equivalence of $QCoh(\mathcal{O}_X)$ and Mod_A. But Examples, Lemma 110.46.2 shows there exists A, I, and U such that $H^1(U, \widetilde{I}) \neq 0$.

36.8. The coherator for Noetherian schemes

09T1 In the case of Noetherian schemes we can use the following lemma.

09T2 **Lemma 36.8.1.** Let X be a Noetherian scheme. Let $\mathcal{J}$ be an injective object of $QCoh(\mathcal{O}_X)$. Then $\mathcal{J}$ is a flasque sheaf of $\mathcal{O}_X$-modules.

Proof. Let $U \subset X$ be an open subset and let $s \in \mathcal{J}(U)$ be a section. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals defining the reduced induced scheme structure on $X \setminus U$ (see Schemes, Definition 26.12.5). By Cohomology of Schemes, Lemma 30.10.5 the section s corresponds to a map $\sigma : \mathcal{I}^n \to \mathcal{J}$ for some n. As $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$ we can extend σ to a map $\tilde{s} : \mathcal{O}_X \to \mathcal{J}$. Then $\tilde{s}$ corresponds to a global section of $\mathcal{J}$ restricting to s. $\square$

09T3 **Lemma 36.8.2.** Let $f : X \to Y$ be a morphism of Noetherian schemes. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram

$$\begin{tikzcd} D(QCoh(\mathcal{O}_X)) \arrow[r] \arrow[d, "\Phi"] & D_{QCoh}(\mathcal{O}_X) \arrow[d, "Rf_*"] \\ D(QCoh(\mathcal{O}_Y)) \arrow[r] & D_{QCoh}(\mathcal{O}_Y) \end{tikzcd}$$

commutes.

Proof. Since X and Y are Noetherian schemes the morphism is quasi-compact and quasi-separated (see Properties, Lemma 28.5.4 and Schemes, Remark 26.21.18). Thus f_* preserve quasi-coherence, see Schemes, Lemma 26.24.1. Next, let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 28.23.4), we can represent K by a K-injective complex $\mathcal{I}^\bullet$ such that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_*\mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this by Lemma 36.4.2 it suffices to show that $\mathcal{I}^n$ is right f_*-acyclic for all $n \in \mathbf{Z}$. This is true because $\mathcal{I}^n$ is flasque by Lemma 36.8.1 and flasque modules are right f_*-acyclic by Cohomology, Lemma 20.12.5. $\square$

09T4 **Proposition 36.8.3.** Let X be a Noetherian scheme. Then the functor (36.3.0.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. This follows from Lemma 36.7.4 and Lemma 36.8.2. $\square$

36.9. Koszul complexes

08CX Let A be a ring and let $f_1, \ldots, f_r$ be a sequence of elements of A. We have defined the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ in More on Algebra, Definition 15.28.2. It is a chain complex sitting in degrees $r, \ldots, 0$. We turn this into a cochain complex $K^\bullet(f_1, \ldots, f_r)$ by setting $K^{-n}(f_1, \ldots, f_r) = K_n(f_1, \ldots, f_r)$ and using the same differentials. In the rest of this section all the complexes will be cochain complexes.

We define a complex $I^\bullet(f_1, \ldots, f_r)$ such that we have a distinguished triangle

$$I^\bullet(f_1, \ldots, f_r) \to A \to K^\bullet(f_1, \ldots, f_r) \to I^\bullet(f_1, \ldots, f_r)[1]$$

in $K(A)$. In other words, we set

$$I^i(f_1, \ldots, f_r) = \begin{cases} K^{i-1}(f_1, \ldots, f_r) & \text{if } i \leq 0 \\ 0 & \text{else} \end{cases}$$

and we use the negative of the differential on $K^\bullet(f_1, \ldots, f_r)$. The maps in the distinguished triangle are the obvious ones. Note that $I^0(f_1, \ldots, f_r) = A^{\oplus r} \to A$ is given by multiplication by f_i on the ith factor. Hence $I^\bullet(f_1, \ldots, f_r) \to A$ factors as

$$I^\bullet(f_1, \ldots, f_r) \to I \to A$$

where $I = (f_1, \ldots, f_r)$. In fact, there is a short exact sequence

$$0 \to H^{-1}(K^\bullet(f_1, \ldots, f_s)) \to H^0(I^\bullet(f_1, \ldots, f_s)) \to I \to 0$$

and for every $i < 0$ we have $H^i(I^\bullet(f_1, \ldots, f_r)) = H^{i-1}(K^\bullet(f_1, \ldots, f_r)$. Observe that given a second sequence $g_1, \ldots, g_r$ of elements of A there are canonical maps

$$I^\bullet(f_1g_1, \ldots, f_rg_r) \to I^\bullet(f_1, \ldots, f_r) \quad \text{and} \quad K^\bullet(f_1g_1, \ldots, f_rg_r) \to K^\bullet(f_1, \ldots, f_r)$$

compatible with the maps described above. The first of these maps is given by multiplication by g_i on the ith summand of $I^0(f_1g_1, \ldots, f_rg_r) = A^{\oplus r}$. In particular, given $f_1, \ldots, f_r$ we obtain an inverse system of complexes

08CY (36.9.0.1) $$I^\bullet(f_1, \ldots, f_r) \leftarrow I^\bullet(f_1^2, \ldots, f_r^2) \leftarrow I^\bullet(f_1^3, \ldots, f_r^3) \leftarrow \ldots$$

which will play an important role in that which is to follow. To easily formulate the following lemmas we fix some notation.

08CZ Situation 36.9.1. Here A is a ring and $f_1, \ldots, f_r$ is a sequence of elements of A. We set $X = \text{Spec}(A)$ and $U = D(f_1) \cup \ldots \cup D(f_r) \subset X$. We denote $\mathcal{U} : U = \bigcup_{i=1,\ldots,r} D(f_i)$ the given open covering of U.

Our first lemma is that the complexes above can be used to compute the cohomology of quasi-coherent sheaves on U. Suppose given a complex $I^\bullet$ of A-modules and an A-module M. Then we define $\text{Hom}_A(I^\bullet, M)$ to be the complex with nth term $\text{Hom}_A(I^{-n}, M)$ and differentials given as the contragredients of the differentials on $I^\bullet$.

08D0 Lemma 36.9.2. In Situation 36.9.1. Let M be an A-module and denote $\mathcal{F}$ the associated $\mathcal{O}_X$-module. Then there is a canonical isomorphism of complexes

$$\text{colim}_e \text{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M) \longrightarrow \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

functorial in M.

Proof. Recall that the alternating Čech complex is the subcomplex of the usual Čech complex given by alternating cochains, see Cohomology, Section 20.23. As usual we view a p-cochain in $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ as an alternating function s on $\{1, \ldots, r\}^{p+1}$ whose value $s_{i_0 \ldots i_p}$ at $(i_0, \ldots, i_p)$ lies in $M_{f_{i_0} \ldots f_{i_p}} = \mathcal{F}(U_{i_0 \ldots i_p})$. On the other hand, a p-cochain t in $\text{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M)$ is given by a map $t : \wedge^{p+1}(A^{\oplus r}) \to M$. Write $[i] \in A^{\oplus r}$ for the ith basis element and write

$$[i_0, \ldots, i_p] = [i_0] \wedge \ldots \wedge [i_p] \in \wedge^{p+1}(A^{\oplus r})$$

Then we send t as above to s with

$$s_{i_0 \ldots i_p} = \frac{t([i_0, \ldots, i_p])}{f_{i_0}^e \ldots f_{i_p}^e}$$

It is clear that s so defined is an alternating cochain. The construction of this map is compatible with the transition maps of the system as the transition map

$$I^\bullet(f_1^e, \ldots, f_r^e) \leftarrow I^\bullet(f_1^{e+1}, \ldots, f_r^{e+1}),$$

of the (36.9.0.1) sends $[i_0, \ldots, i_p]$ to $f_{i_0} \ldots f_{i_p}[i_0, \ldots, i_p]$. It is clear from the description of the localizations $M_{f_{i_0} \ldots f_{i_p}}$ in Algebra, Lemma 10.9.9 that these maps define an isomorphism of cochain modules in degree p in the limit. To finish the proof we have to show that the map is compatible with differentials. To see this recall that

$$\begin{aligned} d(s)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_p} \\ &= \sum\nolimits_{j=0}^{p+1} (-1)^j \frac{t([i_0, \ldots, \hat{i}_j, \ldots i_{p+1}])}{f_{i_0}^e \ldots \hat{f}_{i_j}^e \ldots f_{i_{p+1}}^e} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{d(t)([i_0, \ldots, i_{p+1}])}{f_{i_0}^e \ldots f_{i_{p+1}}^e} &= \frac{t(d[i_0, \ldots, i_{p+1}])}{f_{i_0}^e \ldots f_{i_{p+1}}^e} \\ &= \frac{\sum_j (-1)^j f_{i_j}^e t([i_0, \ldots, \hat{i}_j, \ldots i_{p+1}])}{f_{i_0}^e \ldots f_{i_{p+1}}^e} \end{aligned}$$

The two formulas agree by inspection. □

Suppose given a finite complex $I^\bullet$ of A-modules and a complex of A-modules $M^\bullet$. We obtain a double complex $H^{\bullet,\bullet} = \text{Hom}_A(I^\bullet, M^\bullet)$ where $H^{p,q} = \text{Hom}_A(I^p, M^q)$. The first differential comes from the differential on $\text{Hom}_A(I^\bullet, M^q)$ and the second from the differential on $M^\bullet$. Associated to this double complex is the total complex with degree n term given by

$$\bigoplus\nolimits_{p+q=n} \text{Hom}_A(I^p, M^q)$$

and differential as in Homology, Definition 12.18.3. As our complex $I^\bullet$ has only finitely many nonzero terms, the direct sum displayed above is finite. The conventions for taking the total complex associated to a Čech complex of a complex are as in Cohomology, Section 20.25.

08D1 **Lemma 36.9.3.** In Situation 36.9.1. Let $M^\bullet$ be a complex of A-modules and denote $\mathcal{F}^\bullet$ the associated complex of $\mathcal{O}_X$-modules. Then there is a canonical isomorphism of complexes

$$\operatorname{colim}_e \operatorname{Tot}(\operatorname{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet)) \longrightarrow \operatorname{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$$

functorial in $M^\bullet$.

Proof. Immediate from Lemma 36.9.2 and our conventions for taking associated total complexes. $\square$

08D2 **Lemma 36.9.4.** In Situation 36.9.1. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical isomorphism

$$\operatorname{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

in $D(A)$ functorial in $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{B}$ be the set of affine opens of U. Since the higher cohomology groups of a quasi-coherent module on an affine scheme are zero (Cohomology of Schemes, Lemma 30.2.2) this is a special case of Cohomology, Lemma 20.40.2. $\square$

In Situation 36.9.1 denote I_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A-modules $I^\bullet(f_1^e, \ldots, f_r^e)$ via the equivalence of Lemma 36.3.5. The maps (36.9.0.1) give a system

$$I_1 \leftarrow I_2 \leftarrow I_3 \leftarrow \ldots$$

Moreover, there is a compatible system of maps $I_e \to \mathcal{O}_X$ which become isomorphisms when restricted to U. Thus we see that for every object E of $D(\mathcal{O}_X)$ there is a canonical map

08DC (36.9.4.1)
$$\operatorname{colim}_e \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E) \longrightarrow H^0(U, E)$$

constructed by sending a map $I_e \to E$ to its restriction to U and using that $\operatorname{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_U, E|_U) = H^0(U, E)$.

08DD **Proposition 36.9.5.** In Situation 36.9.1. For every object E of $D_{QCoh}(\mathcal{O}_X)$ the map (36.9.4.1) is an isomorphism.

Proof. By Lemma 36.3.5 we may assume that E is given by a complex of quasi-coherent sheaves $\mathcal{F}^\bullet$. Let $M^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ be the corresponding complex of A-modules. By Lemmas 36.9.3 and 36.9.4 we have quasi-isomorphisms

$$\operatorname{colim}_e \operatorname{Tot}(\operatorname{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet)) \longrightarrow \operatorname{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

Taking H^0 on both sides we obtain

$$\operatorname{colim}_e \operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = H^0(U, E)$$

Since $\operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E)$ by Lemma 36.3.5 the lemma follows. $\square$

In Situation 36.9.1 denote K_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A-modules $K^\bullet(f_1^e, \ldots, f_r^e)$ via the equivalence of Lemma 36.3.5. Thus we have distinguished triangles

$$I_e \to \mathcal{O}_X \to K_e \to I_e[1]$$

and a system

$$K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \ldots$$

compatible with the system (I_e). Moreover, there is a compatible system of maps

$$K_e \to H^0(K_e) = \mathcal{O}_X/(f_1^e, \ldots, f_r^e)$$

08E3 Lemma 36.9.6. In Situation 36.9.1. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Assume that $H^i(E)|_U = 0$ for $i = -r+1, \ldots, 0$. Then given $s \in H^0(X, E)$ there exists an $e \geq 0$ and a morphism $K_e \to E$ such that s is in the image of $H^0(X, K_e) \to H^0(X, E)$.

Proof. Since U is covered by r affine opens we have $H^j(U, \mathcal{F}) = 0$ for $j \geq r$ and any quasi-coherent module (Cohomology of Schemes, Lemma 30.4.2). By Lemma 36.3.4 we see that $H^0(U, E)$ is equal to $H^0(U, \tau_{\geq -r+1}E)$. There is a spectral sequence

$$H^j(U, H^i(\tau_{\geq -r+1}E)) \Rightarrow H^{i+j}(U, \tau_{\geq -N}E)$$

see Derived Categories, Lemma 13.21.3. Hence $H^0(U, E) = 0$ by our assumed vanishing of cohomology sheaves of E. We conclude that $s|_U = 0$. Think of s as a morphism $\mathcal{O}_X \to E$ in $D(\mathcal{O}_X)$. By Proposition 36.9.5 the composition $I_e \to \mathcal{O}_X \to E$ is zero for some e. By the distinguished triangle $I_e \to \mathcal{O}_X \to K_e \to I_e[1]$ we obtain a morphism $K_e \to E$ such that s is the composition $\mathcal{O}_X \to K_e \to E$. □

36.10. Pseudo-coherent and perfect complexes

08E4 In this section we make the connection between the general notions defined in Cohomology, Sections 20.46, 20.47, 20.48, and 20.49 and the corresponding notions for complexes of modules in More on Algebra, Sections 15.64, 15.66, and 15.74.

08E5 Lemma 36.10.1. Let X be a scheme. If E is an m-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$ and $H^m(E)$ is a quotient of a quasi-coherent $\mathcal{O}_X$-module. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.

Proof. Locally on X there exists a strictly perfect complex $\mathcal{E}^\bullet$ such that $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ for $i > m$ and $H^m(E)$ is a quotient of $H^m(\mathcal{E}^\bullet)$. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. □

08E7 Lemma 36.10.2. Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is an m-pseudo-coherent (resp. pseudo-coherent) as an object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of A-modules.

Proof. It is immediate from the definitions that if $M^\bullet$ is m-pseudo-coherent, so is E. To prove the converse, assume E is m-pseudo-coherent. As $X = \mathrm{Spec}(A)$ is quasi-compact with a basis for the topology given by standard opens, we can find a standard open covering $X = D(f_1) \cup \ldots \cup D(f_n)$ and strictly perfect complexes $\mathcal{E}_i^\bullet$ on $D(f_i)$ and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ inducing isomorphisms on H^j for $j > m$ and surjections on H^m. By Cohomology, Lemma 20.46.8 after refining the open covering we may assume α_i is given by a map of complexes $\mathcal{E}_i^\bullet \to \widetilde{M^\bullet}|_{U_i}$ for each i. By Modules, Lemma 17.14.6 the terms $\mathcal{E}_i^n$ are finite locally free modules. Hence after refining the open covering we may assume each $\mathcal{E}_i^n$ is a finite free $\mathcal{O}_{U_i}$-module. From the definition it follows that $M_{f_i}^\bullet$ is an m-pseudo-coherent complex of A_{f_i}-modules. We conclude by applying More on Algebra, Lemma 15.64.14.

The case "pseudo-coherent" follows from the fact that E is pseudo-coherent if and only if E is m-pseudo-coherent for all m (by definition) and the same is true for $M^\bullet$ by More on Algebra, Lemma 15.64.5. $\square$

08E8 Lemma 36.10.3. Let X be a Noetherian scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent

(1) $H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and
(2) E is m-pseudo-coherent.

In particular, E is pseudo-coherent if and only if E is an object of $D^-_{\text{Coh}}(\mathcal{O}_X)$.

Proof. As X is quasi-compact we see that in both (1) and (2) the object E is bounded above. Thus the question is local on X and we may assume X is affine. Say $X = \text{Spec}(A)$ for some Noetherian ring A. In this case E corresponds to a complex of A-modules $M^\bullet$ by Lemma 36.3.5. By Lemma 36.10.2 we see that E is m-pseudo-coherent if and only if $M^\bullet$ is m-pseudo-coherent. On the other hand, $H^i(E)$ is coherent if and only if $H^i(M^\bullet)$ is a finite A-module (Properties, Lemma 28.16.1). Thus the result follows from More on Algebra, Lemma 15.64.17. $\square$

08E9 Lemma 36.10.4. Let $X = \text{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then

(1) E has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$.
(2) E has finite tor dimension if and only if $M^\bullet$ has finite tor dimension.

Proof. Part (2) follows trivially from part (1). In the proof of (1) we will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 36.3.5 without further mention. Assume $M^\bullet$ has tor amplitude in $[a, b]$. Then $K^\bullet$ is isomorphic in $D(A)$ to a complex $K^\bullet$ of flat A-modules with $K^i = 0$ for $i \notin [a, b]$, see More on Algebra, Lemma 15.66.3. Then E is isomorphic to $\widetilde{K^\bullet}$. Since each $\widetilde{K^i}$ is a flat $\mathcal{O}_X$-module, we see that E has tor amplitude in $[a, b]$ by Cohomology, Lemma 20.48.3.

Assume that E has tor amplitude in $[a, b]$. Then E is bounded whence $M^\bullet$ is in $K^-(A)$. Thus we may replace $M^\bullet$ by a bounded above complex of A-modules. We may even choose a projective resolution and assume that $M^\bullet$ is a bounded above complex of free A-modules. Then for any A-module N we have

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet \otimes_A N}$$

in $D(\mathcal{O}_X)$. Thus the vanishing of cohomology sheaves of the left hand side implies $M^\bullet$ has tor amplitude in $[a, b]$. $\square$

0DHY Lemma 36.10.5. Let $f : X \to S$ be a morphism of affine schemes corresponding to the ring map $R \to A$. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then

(1) E as an object of $D(f^{-1}\mathcal{O}_S)$ has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$ as an object of $D(R)$.
(2) E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ if and only if $M^\bullet$ has finite tor dimension as an object of $D(R)$.

Proof. Consider a prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. Let $x \in X$ and $s = f(x) \in S$ be the corresponding points. Then $(f^{-1}\mathcal{O}_S)_x = \mathcal{O}_{S,s} = R_\mathfrak{p}$ and $E_x = M^\bullet_\mathfrak{q}$. Keeping this in mind we can see the equivalence as follows.

If $M^\bullet$ has tor amplitude in $[a, b]$ as a complex of R-modules, then the same is true for the localization of $M^\bullet$ at any prime of A. Then we conclude by Cohomology, Lemma 20.48.5 that E has tor amplitude in $[a, b]$ as a complex of sheaves of $f^{-1}\mathcal{O}_S$-modules. Conversely, assume that E has tor amplitude in $[a, b]$ as an object of $D(f^{-1}\mathcal{O}_S)$. We conclude (using the last cited lemma) that $M^\bullet_\mathfrak{q}$ has tor amplitude in $[a, b]$ as a complex of $R_\mathfrak{p}$-modules for every prime $\mathfrak{q} \subset A$ lying over $\mathfrak{p} \subset R$. By More on Algebra, Lemma 15.66.15 we find that $M^\bullet$ has tor amplitude in $[a, b]$ as a complex of R-modules. This finishes the proof of (1).

Since X is quasi-compact, if E locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_S$-modules, then actually E has tor amplitude in $[a, b]$ for some a, b as a complex of $f^{-1}\mathcal{O}_S$-modules. Thus (2) follows from (1). $\square$

08EA Lemma 36.10.6. Let X be a quasi-separated scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent

(1) E has tor amplitude in $[a, b]$, and
(2) for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \notin [a, b]$.

Proof. It is clear that (1) implies (2). Assume (2). Let $U \subset X$ be an affine open. As X is quasi-separated the morphism $j : U \to X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Schemes, Lemma 26.24.1). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Write $U = \mathrm{Spec}(A)$ and let $M^\bullet$ be the complex of A-modules corresponding to $E|_U$ by Lemma 36.3.5. We have just shown that $M^\bullet \otimes^{\mathbf{L}}_A N$ has vanishing cohomology groups outside the range $[a, b]$, in other words $M^\bullet$ has tor amplitude in $[a, b]$. By Lemma 36.10.4 we conclude that $E|_U$ has tor amplitude in $[a, b]$. This proves the lemma. $\square$

08EB Lemma 36.10.7. Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is perfect as an object of $D(A)$.

Proof. This is a logical consequence of Lemmas 36.10.2 and 36.10.4, Cohomology, Lemma 20.49.5, and More on Algebra, Lemma 15.74.2. $\square$

As a consequence of our description of pseudo-coherent complexes on schemes we can prove certain internal homs are quasi-coherent.

0A6H Lemma 36.10.8. Let X be a scheme.

(1) If L is in $D^+_{QCoh}(\mathcal{O}_X)$ and K in $D(\mathcal{O}_X)$ is pseudo-coherent, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$ and locally bounded below.
(2) If L is in $D_{QCoh}(\mathcal{O}_X)$ and K in $D(\mathcal{O}_X)$ is perfect, then $R\mathcal{H}om(K, L)$ is in $D_{QCoh}(\mathcal{O}_X)$.
(3) If $X = \mathrm{Spec}(A)$ is affine and $K, L \in D(A)$ then

$$R\mathcal{H}om(\widetilde{K}, \widetilde{L}) = \widetilde{R\mathrm{Hom}_A(K, L)}$$

in the following two cases
(a) K is pseudo-coherent and L is bounded below,
(b) K is perfect and L arbitrary.

(4) If $X = \mathrm{Spec}(A)$ and K, L are in $D(A)$, then the nth cohomology sheaf of $R\mathcal{H}om(\widetilde{K}, \widetilde{L})$ is the sheaf associated to the presheaf

$$X \supset D(f) \longmapsto \mathrm{Ext}^n_{A_f}(K \otimes_A A_f, L \otimes_A A_f)$$

for $f \in A$.

Proof. The construction of the internal hom in the derived category of $\mathcal{O}_X$ commutes with localization (see Cohomology, Section 20.42). Hence to prove (1) and (2) we may replace X by an affine open. By Lemmas 36.3.5, 36.10.2, and 36.10.7 in order to prove (1) and (2) it suffices to prove (3).

Part (3) follows from the computation of the internal hom of Cohomology, Lemma 20.46.11 by representing K by a bounded above (resp. finite) complex of finite projective A-modules and L by a bounded below (resp. arbitrary) complex of A-modules.

To prove (4) recall that on any ringed space the nth cohomology sheaf of $R\mathcal{H}om(A, B)$ is the sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{D(U)}(A|_U, B|_U[n]) = \mathrm{Ext}^n_{D(\mathcal{O}_U)}(A|_U, B|_U)$$

See Cohomology, Section 20.42. On the other hand, the restriction of $\widetilde{K}$ to a principal open $D(f)$ is the image of $K \otimes_A A_f$ and similarly for L. Hence (4) follows from the equivalence of categories of Lemma 36.3.5. ☐

0ATN Lemma 36.10.9. Let X be a scheme. Let K, L, M be objects of $D_{QCoh}(\mathcal{O}_X)$. The map

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_X} R\mathcal{H}om(M, L) \longrightarrow R\mathcal{H}om(M, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

of Cohomology, Lemma 20.42.6 is an isomorphism in the following cases

(1) M perfect, or
(2) K is perfect, or
(3) M is pseudo-coherent, $L \in D^+(\mathcal{O}_X)$, and K has finite tor dimension.

Proof. Lemma 36.10.8 reduces cases (1) and (3) to the affine case which is treated in More on Algebra, Lemma 15.98.3. (You also have to use Lemmas 36.10.2, 36.10.7, and 36.10.4 to do the translation into algebra.) If K is perfect but no other assumptions are made, then we do not know that either side of the arrow is in $D_{QCoh}(\mathcal{O}_X)$ but the result is still true because we can work locally and reduce to the case that K is a finite complex of finite free modules in which case it is clear. ☐

36.11. Derived category of coherent modules

08E0 Let X be a locally Noetherian scheme. In this case the category $\mathrm{Coh}(\mathcal{O}_X) \subset \mathrm{Mod}(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 12.10 and Cohomology of Schemes, Lemma 30.9.2. Denote

$$D_{\mathrm{Coh}}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13.17. Thus we obtain a canonical functor

08E1 (36.11.0.1) $$D(\mathrm{Coh}(\mathcal{O}_X)) \longrightarrow D_{\mathrm{Coh}}(\mathcal{O}_X)$$

see Derived Categories, Equation (13.17.1.1).

0FDA Lemma 36.11.1. Let X be a Noetherian scheme. Then the functor

$$D^-(\mathrm{Coh}(\mathcal{O}_X)) \longrightarrow D^-_{\mathrm{Coh}(\mathcal{O}_X)}(QCoh(\mathcal{O}_X))$$

is an equivalence.

Proof. Observe that $\mathrm{Coh}(\mathcal{O}_X) \subset QCoh(\mathcal{O}_X)$ is a Serre subcategory, see Homology, Definition 12.10.1 and Lemma 12.10.2 and Cohomology of Schemes, Lemmas 30.9.2 and 30.9.3. On the other hand, if $\mathcal{G} \to \mathcal{F}$ is a surjection from a quasi-coherent $\mathcal{O}_X$-module to a coherent $\mathcal{O}_X$-module, then there exists a coherent submodule $\mathcal{G}' \subset \mathcal{G}$ which surjects onto $\mathcal{F}$. Namely, we can write $\mathcal{G}$ as the filtered union of its coherent submodules by Properties, Lemma 28.22.3 and then one of these will do the job. Thus the lemma follows from Derived Categories, Lemma 13.17.4. □

0FDB Proposition 36.11.2. Let X be a Noetherian scheme. Then the functors

$$D^-(\mathrm{Coh}(\mathcal{O}_X)) \longrightarrow D^-_{\mathrm{Coh}}(\mathcal{O}_X) \quad \text{and} \quad D^b(\mathrm{Coh}(\mathcal{O}_X)) \longrightarrow D^b_{\mathrm{Coh}}(\mathcal{O}_X)$$

are equivalences.

Proof. Consider the commutative diagram

$$\begin{tikzcd} D^-(\mathrm{Coh}(\mathcal{O}_X)) \arrow[r] \arrow[d] & D^-_{\mathrm{Coh}}(\mathcal{O}_X) \arrow[d] \\ D^-(QCoh(\mathcal{O}_X)) \arrow[r] & D^-_{QCoh}(\mathcal{O}_X) \end{tikzcd}$$

By Lemma 36.11.1 the left vertical arrow is fully faithful. By Proposition 36.8.3 the bottom arrow is an equivalence. By construction the right vertical arrow is fully faithful. We conclude that the top horizontal arrow is fully faithful. If K is an object of $D^-_{\mathrm{Coh}}(\mathcal{O}_X)$ then the object K' of $D^-(QCoh(\mathcal{O}_X))$ which corresponds to it by Proposition 36.8.3 will have coherent cohomology sheaves. Hence K' is in the essential image of the left vertical arrow by Lemma 36.11.1 and we find that the top horizontal arrow is essentially surjective. This finishes the proof for the bounded above case. The bounded case follows immediately from the bounded above case. □

08E2 Lemma 36.11.3. Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D^b_{\mathrm{Coh}}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^b_{\mathrm{Coh}}(\mathcal{O}_S)$.

Proof. Consider the spectral sequence

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E$$

see Derived Categories, Lemma 13.21.3. By assumption and Cohomology of Schemes, Lemma 30.26.10 the sheaves $R^pf_*H^q(E)$ are coherent. Hence $R^{p+q}f_*E$ is coherent, i.e., $Rf_*E \in D_{\mathrm{Coh}}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Schemes, Lemma 30.4.5 or from Lemma 36.4.1. □

0D0B Lemma 36.11.4. Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D^+_{\mathrm{Coh}}(\mathcal{O}_X)$ such that the support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^+_{\text{Coh}}(\mathcal{O}_S)$.

Proof. The proof is the same as the proof of Lemma 36.11.3. You can also deduce it from Lemma 36.11.3 by considering what the exact functor Rf_* does to the distinguished triangles $\tau_{\leq a}E \to E \to \tau_{\geq a+1}E \to \tau_{\leq a}E[1]$. $\square$

0D0C Lemma 36.11.5. Let X be a locally Noetherian scheme. If L is in $D^+_{\text{Coh}}(\mathcal{O}_X)$ and K in $D^-_{\text{Coh}}(\mathcal{O}_X)$, then $R\mathcal{H}om(K, L)$ is in $D^+_{\text{Coh}}(\mathcal{O}_X)$.

Proof. It suffices to prove this when X is the spectrum of a Noetherian ring A. By Lemma 36.10.3 we see that K is pseudo-coherent. Then we can use Lemma 36.10.8 to translate the problem into the following algebra problem: for $L \in D^+_{\text{Coh}}(A)$ and K in $D^-_{\text{Coh}}(A)$, then $R\operatorname{Hom}_A(K, L)$ is in $D^+_{\text{Coh}}(A)$. Since L is bounded below and K is bounded below there is a convergent spectral sequence

$$\operatorname{Ext}^p_A(K, H^q(L)) \Rightarrow \operatorname{Ext}^{p+q}_A(K, L)$$

and there are convergent spectral sequences

$$\operatorname{Ext}^i_A(H^{-j}(K), H^q(L)) \Rightarrow \operatorname{Ext}^{i+j}_A(K, H^q(L))$$

See Injectives, Remarks 19.13.9 and 19.13.11. This finishes the proof as the modules $\operatorname{Ext}^p_A(M, N)$ are finite for finite A-modules M, N by Algebra, Lemma 10.71.9. $\square$

0FXU Lemma 36.11.6. Let X be a Noetherian scheme. Let E in $D(\mathcal{O}_X)$ be perfect. Then

(1) E is in $D^b_{\text{Coh}}(\mathcal{O}_X)$,
(2) if L is in $D_{\text{Coh}}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D_{\text{Coh}}(\mathcal{O}_X)$,
(3) if L is in $D^b_{\text{Coh}}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^b_{\text{Coh}}(\mathcal{O}_X)$,
(4) if L is in $D^+_{\text{Coh}}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^+_{\text{Coh}}(\mathcal{O}_X)$,
(5) if L is in $D^-_{\text{Coh}}(\mathcal{O}_X)$ then $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ and $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ are in $D^-_{\text{Coh}}(\mathcal{O}_X)$.

Proof. Since X is quasi-compact, each of these statements can be checked over the members of any open covering of X. Thus we may assume E is represented by a bounded complex $\mathcal{E}^\bullet$ of finite free modules, see Cohomology, Lemma 20.49.3. In this case each of the statements is clear as both $R\mathcal{H}om_{\mathcal{O}_X}(E, L)$ and $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ can be computed on the level of complexes using $\mathcal{E}^\bullet$, see Cohomology, Lemmas 20.46.9 and 20.26.9. Some details omitted. $\square$

0D0D Lemma 36.11.7. Let A be a Noetherian ring. Let X be a proper scheme over A. For L in $D^+_{\text{Coh}}(\mathcal{O}_X)$ and K in $D^-_{\text{Coh}}(\mathcal{O}_X)$, the A-modules $\operatorname{Ext}^n_{\mathcal{O}_X}(K, L)$ are finite.

Proof. Recall that

$$\operatorname{Ext}^n_{\mathcal{O}_X}(K, L) = H^n(X, R\mathcal{H}om_{\mathcal{O}_X}(K, L)) = H^n(\operatorname{Spec}(A), Rf_*R\mathcal{H}om_{\mathcal{O}_X}(K, L))$$

see Cohomology, Lemma 20.42.1 and Cohomology, Section 20.13. Thus the result follows from Lemmas 36.11.5 and 36.11.4. $\square$

0FDC Lemma 36.11.8. Let X be a locally Noetherian regular scheme. Then every object of $D^b_{\text{Coh}}(\mathcal{O}_X)$ is perfect. If X is quasi-compact, i.e., Noetherian regular, then conversely every perfect object of $D(\mathcal{O}_X)$ is in $D^b_{\text{Coh}}(\mathcal{O}_X)$.

Proof. Let K be an object of $D^b_{\text{Coh}}(\mathcal{O}_X)$. To check that K is perfect, we may work affine locally on X (see Cohomology, Section 20.49). Then K is perfect by Lemma 36.10.7 and More on Algebra, Lemma 15.74.14. The converse is Lemma 36.11.6. $\square$

36.12. Descent finiteness properties of complexes

09UC This section is the analogue of Descent, Section 35.7 for objects of the derived category of a scheme. The easiest such result is probably the following.

09UD Lemma 36.12.1. Let $f : X \to Y$ be a surjective flat morphism of schemes (or more generally locally ringed spaces). Let $E \in D(\mathcal{O}_Y)$. Let $a, b \in \mathbf{Z}$. Then E has tor-amplitude in $[a, b]$ if and only if Lf^*E has tor-amplitude in $[a, b]$.

Proof. Pullback always preserves tor-amplitude, see Cohomology, Lemma 20.48.4. We may check tor-amplitude in $[a, b]$ on stalks, see Cohomology, Lemma 20.48.5. A flat local ring homomorphism is faithfully flat by Algebra, Lemma 10.39.17. Thus the result follows from More on Algebra, Lemma 15.66.17. $\square$

09UE Lemma 36.12.2. Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology, Lemma 20.47.3. Conversely, assume that Lf_i^*E is m-pseudo-coherent for all i. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fpqc covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of affine schemes. In this case the result follows from More on Algebra, Lemma 15.64.15 via Lemmas 36.3.5 and 36.10.2. $\square$

09UF Lemma 36.12.3. Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology, Lemma 20.47.3. Conversely, assume that Lf_i^*E is m-pseudo-coherent for all i. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fppf covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of finite presentation.

Since f is flat the derived functor Lf^* is just given by f^* and f^* is exact. Hence $H^i(Lf^*E) = f^*H^i(E)$. Since Lf^*E is m-pseudo-coherent, we see that $Lf^*E \in D^-(\mathcal{O}_Y)$. Since f is surjective and flat, we see that $E \in D^-(\mathcal{O}_X)$. Let $i \in \mathbf{Z}$ be the largest integer such that $H^i(E)$ is nonzero. If $i < m$, then we are done. Otherwise, $f^*H^i(E)$ is a finite type $\mathcal{O}_Y$-module by Cohomology, Lemma 20.47.9. Then by Descent, Lemma 35.7.2 the $\mathcal{O}_X$-module $H^i(E)$ is of finite type. Thus, after replacing X by the members of a finite affine open covering, we may assume there exists a map

$$\alpha : \mathcal{O}_X^{\oplus n}[-i] \longrightarrow E$$

such that $H^i(\alpha)$ is a surjection. Let C be the cone of α in $D(\mathcal{O}_X)$. Pulling back to Y and using Cohomology, Lemma 20.47.4 we find that Lf^*C is m-pseudo-coherent. Moreover $H^j(C) = 0$ for $j \geq i$. Thus by induction on i we see that C is m-pseudo-coherent. Using Cohomology, Lemma 20.47.4 again we conclude. $\square$

09UG Lemma 36.12.4. Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D(\mathcal{O}_X)$. Then E is perfect if and only if each Lf_i^*E is perfect.

Proof. Pullback always preserves perfect complexes, see Cohomology, Lemma 20.49.6. Conversely, assume that Lf_i^*E is perfect for all i. Then the cohomology sheaves of each Lf_i^*E are quasi-coherent, see Lemma 36.10.1 and Cohomology, Lemma 20.49.5. Since the morphisms f_i is flat we see that $H^p(Lf_i^*E) = f_i^*H^p(E)$. Thus the cohomology sheaves of E are quasi-coherent by Descent, Proposition 35.5.2. Having said this the lemma follows formally from Cohomology, Lemma 20.49.5 and Lemmas 36.12.1 and 36.12.2. □

09VA Lemma 36.12.5. Let $i : Z \to X$ be a morphism of ringed spaces such that i is a closed immersion of underlying topological spaces and such that $i_*\mathcal{O}_Z$ is pseudo-coherent as an $\mathcal{O}_X$-module. Let $E \in D(\mathcal{O}_Z)$. Then E is m-pseudo-coherent if and only if Ri_*E is m-pseudo-coherent.

Proof. Throughout this proof we will use that i_* is an exact functor, and hence that $Ri_* = i_*$, see Modules, Lemma 17.6.1.

Assume E is m-pseudo-coherent. Let $x \in X$. We will find a neighbourhood of x such that i_*E is m-pseudo-coherent on it. If $x \notin Z$ then this is clear. Thus we may assume $x \in Z$. We will use that $U \cap Z$ for $x \in U \subset X$ open form a fundamental system of neighbourhoods of x in Z. After shrinking X we may assume E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(E)$ is an $\mathcal{O}_Z$-module of finite type, see Cohomology, Lemma 20.47.9. Thus we may choose, after shrinking X, a map $\mathcal{O}_Z^{\oplus n}[-p] \to E$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(E)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$ and that C is m-pseudo-coherent by Cohomology, Lemma 20.47.4. By induction we see that i_*C is m-pseudo-coherent on X. Since $i_*\mathcal{O}_Z$ is m-pseudo-coherent on X as well, we conclude from the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

and Cohomology, Lemma 20.47.4 that i_*E is m-pseudo-coherent.

Assume that i_*E is m-pseudo-coherent. Let $z \in Z$. We will find a neighbourhood of z such that E is m-pseudo-coherent on it. We will use that $U \cap Z$ for $z \in U \subset X$ open form a fundamental system of neighbourhoods of z in Z. After shrinking X we may assume i_*E and hence E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(i_*E) = i_*H^p(E)$ is an $\mathcal{O}_X$-module of finite type, see Cohomology, Lemma 20.47.9. Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Z$-modules representing E. We may choose, after shrinking X, a map $\alpha : \mathcal{O}_X^{\oplus n}[-p] \to i_*\mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_X^{\oplus n} \to i_*H^p(\mathcal{E}^\bullet)$. By adjunction we find a map $\alpha : \mathcal{O}_Z^{\oplus n}[-p] \to \mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(\mathcal{E}^\bullet)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$. From the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

the fact that $i_*\mathcal{O}_Z$ is pseudo-coherent and Cohomology, Lemma 20.47.4 we conclude that i_*C is m-pseudo-coherent. By induction we conclude that C is m-pseudo-coherent. By Cohomology, Lemma 20.47.4 again we conclude that E is m-pseudo-coherent. □

09VB Lemma 36.12.6. Let $f : X \to Y$ be a finite morphism of schemes such that $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_Y$-module[2]. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Then E is m-pseudo-coherent if and only if Rf_*E is m-pseudo-coherent.

Proof. This is a translation of More on Algebra, Lemma 15.64.11 into the language of schemes. To do the translation, use Lemmas 36.3.5 and 36.10.2. □

36.13. Lifting complexes

08EC Let $U \subset X$ be an open subspace of a ringed space and denote $j : U \to X$ the inclusion morphism. The functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is essentially surjective as Rj_* is a right inverse to restriction. In this section we extend this to complexes with quasi-coherent cohomology sheaves, etc.

08ED Lemma 36.13.1. Let X be a scheme and let $j : U \to X$ be a quasi-compact open immersion. The functors

$$D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D^+_{QCoh}(\mathcal{O}_X) \to D^+_{QCoh}(\mathcal{O}_U)$$

are essentially surjective. If X is quasi-compact, then the functors

$$D^-_{QCoh}(\mathcal{O}_X) \to D^-_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D^b_{QCoh}(\mathcal{O}_X) \to D^b_{QCoh}(\mathcal{O}_U)$$

are essentially surjective.

Proof. The argument preceding the lemma applies for the first case because Rj_* maps $D_{QCoh}(\mathcal{O}_U)$ into $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.4.1. It is clear that Rj_* maps $D^+_{QCoh}(\mathcal{O}_U)$ into $D^+_{QCoh}(\mathcal{O}_X)$ which implies the statement on bounded below complexes. Finally, Lemma 36.4.1 guarantees that Rj_* maps $D^-_{QCoh}(\mathcal{O}_U)$ into $D^-_{QCoh}(\mathcal{O}_X)$ if X is quasi-compact. Combining these two we obtain the last statement. □

0G48 Lemma 36.13.2. Let X be a Noetherian scheme and let $j : U \to X$ be an open immersion. The functor $D^b_{\text{Coh}}(\mathcal{O}_X) \to D^b_{\text{Coh}}(\mathcal{O}_U)$ is essentially surjective.

Proof. Let K be an object of $D^b_{\text{Coh}}(\mathcal{O}_U)$. By Proposition 36.11.2 we can represent K by a bounded complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_U$-modules. Say $\mathcal{F}^i = 0$ for $i \not\in [a, b]$ for some $a \leq b$. Since j is quasi-compact and separated, the terms of the bounded complex $j_*\mathcal{F}^\bullet$ are quasi-coherent modules on X, see Schemes, Lemma 26.24.1. We inductively pick a coherent submodule $\mathcal{G}^i \subset j_*\mathcal{F}^i$ as follows. For $i = a$ we pick any coherent submodule $\mathcal{G}^a \subset j_*\mathcal{F}^a$ whose restriction to U is $\mathcal{F}^a$. This is possible by Properties, Lemma 28.22.2. For $i > a$ we first pick any coherent submodule $\mathcal{H}^i \subset j_*\mathcal{F}^i$ whose restriction to U is $\mathcal{F}^i$ and then we set $\mathcal{G}^i = \text{Im}(\mathcal{H}^i \oplus \mathcal{G}^{i-1} \to j_*\mathcal{F}^i)$. It is clear that $\mathcal{G}^\bullet \subset j_*\mathcal{F}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules whose restriction to U is $\mathcal{F}^\bullet$ as desired. □

08EE Lemma 36.13.3. Let X be an affine scheme and let $U \subset X$ be a quasi-compact open subscheme. For any pseudo-coherent object E of $D(\mathcal{O}_U)$ there exists a bounded above complex of finite free $\mathcal{O}_X$-modules whose restriction to U is isomorphic to E.

[2]This means that f is pseudo-coherent, see More on Morphisms, Lemma 37.60.8.

Proof. By Lemma 36.10.1 we see that E is an object of $D_{QCoh}(\mathcal{O}_U)$. By Lemma 36.13.1 we may assume $E = E'|U$ for some object E' of $D_{QCoh}(\mathcal{O}_X)$. Write $X = \mathrm{Spec}(A)$. By Lemma 36.3.5 we can find a complex $M^\bullet$ of A-modules whose associated complex of $\mathcal{O}_X$-modules is a representative of E'.

Choose $f_1, \ldots, f_r \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_r)$. By Lemma 36.10.2 the complexes $M^\bullet_{f_j}$ are pseudo-coherent complexes of A_{f_j}-modules. Let n be an integer. Assume we have a map of complexes $\alpha : F^\bullet \to M^\bullet$ where $F^\bullet$ is bounded above, $F^i = 0$ for $i < n$, each F^i is a finite free R-module, such that

$$H^i(\alpha_{f_j}) : H^i(F^\bullet_{f_j}) \to H^i(M^\bullet_{f_j})$$

is an isomorphism for $i > n$ and surjective for $i = n$. Picture

$$\begin{tikzcd} & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Since each $M^\bullet_{f_j}$ has vanishing cohomology in large degrees we can find such a map for $n \gg 0$. By induction on n we are going to extend this to a map of complexes $F^\bullet \to M^\bullet$ such that $H^i(\alpha_{f_j})$ is an isomorphism for all i. The lemma will follow by taking $\widetilde{F^\bullet}$.

The induction step will be to extend the diagram above by adding F^{n-1}. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 13.9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet_{f_j}) = 0$ for $i \geq n$. By More on Algebra, Lemma 15.64.2 we see that $C^\bullet_{f_j}$ is $(n-1)$-pseudo-coherent. By More on Algebra, Lemma 15.64.3 we see that $H^{-1}(C^\bullet_{f_j})$ is a finite A_{f_j}-module. Choose a finite free A-module F^{n-1} and an A-module $\beta : F^{n-1} \to C^{-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that $F^{n-1}_{f_j}$ surjects onto $H^{n-1}(C^\bullet_{f_j})$. (Some details omitted; hint: clear denominators.) Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{tikzcd} & F^{n-1} \arrow[r, "d^{n-1}"'] \arrow[d, "\alpha^{n-1}"] & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ \ldots \arrow[r] & M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{tikzcd} (F^n \to \ldots) \arrow[r] \arrow[d] & (F^{n-1} \to \ldots) \arrow[r] \arrow[d] & F^{n-1} \arrow[r] \arrow[d, "\beta"'] & (F^n \to \ldots)[1] \arrow[d] \\ (F^n \to \ldots) \arrow[r] & M^\bullet \arrow[r] & C^\bullet \arrow[r] & (F^n \to \ldots)[1] \end{tikzcd}$$

Hence our choice of β implies that the map of complexes $(F^{-1} \to \ldots) \to M^\bullet$ induces an isomorphism on cohomology localized at f_j in degrees $\geq n$ and a surjection in degree -1. This finishes the proof of the lemma. $\square$

08EF Lemma 36.13.4. Let X be a quasi-compact and quasi-separated scheme. Let $E \in D^b_{QCoh}(\mathcal{O}_X)$. There exists an integer $n_0 > 0$ such that $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = 0$ for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and every $n \geq n_0$.

Proof. Recall that $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n])$. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 20.33.3. Thus if $X = U \cup V$ and the result of the lemma holds for $E|_U$, $E|_V$, and $E|_{U\cap V}$ for some bound n_0, then the result holds for E with bound $n_0 + 1$. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 30.4.1.

Assume $X = \mathrm{Spec}(A)$ is affine. Choose a complex of A-modules $M^\bullet$ whose associated complex of quasi-coherent modules represents E, see Lemma 36.3.5. Write $\mathcal{E} = \widetilde{P}$ for some A-module P. Since $\mathcal{E}$ is finite locally free, we see that P is a finite projective A-module. We have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n]) &= \mathrm{Hom}_{D(A)}(P, M^\bullet[n]) \\ &= \mathrm{Hom}_{K(A)}(P, M^\bullet[n]) \\ &= \mathrm{Hom}_A(P, H^n(M^\bullet)) \end{aligned}$$

The first equality by Lemma 36.3.5, the second equality by Derived Categories, Lemma 13.19.8, and the final equality because $\mathrm{Hom}_A(P, -)$ is an exact functor. As E and hence $M^\bullet$ is bounded we get zero for all sufficiently large n. $\square$

08EG Lemma 36.13.5. Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object E of $D(\mathcal{O}_U)$ there exists an integer r and a finite locally free sheaf $\mathcal{F}$ on U such that $\mathcal{F}[-r] \oplus E$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.

Proof. Say $X = \mathrm{Spec}(A)$. Recall that a perfect complex is pseudo-coherent, see Cohomology, Lemma 20.49.5. By Lemma 36.13.3 we can find a bounded above complex $\mathcal{F}^\bullet$ of finite free A-modules such that E is isomorphic to $\mathcal{F}^\bullet|_U$ in $D(\mathcal{O}_U)$. By Cohomology, Lemma 20.49.5 and since U is quasi-compact, we see that E has finite tor dimension, say E has tor amplitude in $[a, b]$. Pick $r < a$ and set

$$\mathcal{F} = \mathrm{Ker}(\mathcal{F}^r \to \mathcal{F}^{r+1}) = \mathrm{Im}(\mathcal{F}^{r-1} \to \mathcal{F}^r).$$

Since E has tor amplitude in $[a, b]$ we see that $\mathcal{F}|_U$ is flat (Cohomology, Lemma 20.48.2). Hence $\mathcal{F}|_U$ is flat and of finite presentation, thus finite locally free (Properties, Lemma 28.20.2). It follows that

$$(\mathcal{F} \to \mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)|_U$$

is a strictly perfect complex on U representing E. We obtain a distinguished triangle

$$\mathcal{F}|_U[-r-1] \to E \to (\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)|_U \to \mathcal{F}|_U[-r]$$

Note that $(\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)$ is a perfect complex on X. To finish the proof it suffices to pick r such that the map $\mathcal{F}|_U[-r-1] \to E$ is zero in $D(\mathcal{O}_U)$, see Derived Categories, Lemma 13.4.11. By Lemma 36.13.4 this holds if $r \ll 0$. $\square$

08EH Lemma 36.13.6. Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. Let E, E' be objects of $D_{QCoh}(\mathcal{O}_X)$ with E perfect. For every map $\alpha : E|_U \to E'|_U$ there exist maps

$$E \xleftarrow{\beta} E_1 \xrightarrow{\gamma} E'$$

of perfect complexes on X such that $\beta : E_1 \to E$ restricts to an isomorphism on U and such that $\alpha = \gamma|_U \circ \beta|_U^{-1}$. Moreover we can assume $E_1 = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I$ for some perfect complex I on X.

Proof. Write $X = \mathrm{Spec}(A)$. Write $U = D(f_1) \cup \ldots \cup D(f_r)$. Choose finite complex of finite projective A-modules $M^\bullet$ representing E (Lemma 36.10.7). Choose a complex of A-modules $(M')^\bullet$ representing E' (Lemma 36.3.5). In this case the complex $H^\bullet = \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$ is a complex of A-modules whose associated complex of quasi-coherent $\mathcal{O}_X$-modules represents $R\mathcal{H}om(E, E')$, see Cohomology, Lemma 20.46.9. Then α determines an element s of $H^0(U, R\mathcal{H}om(E, E'))$, see Cohomology, Lemma 20.42.1. There exists an e and a map

$$\xi : I^\bullet(f_1^e, \ldots, f_r^e) \to \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$$

corresponding to s, see Proposition 36.9.5. Letting E_1 be the object corresponding to complex of quasi-coherent $\mathcal{O}_X$-modules associated to

$$\mathrm{Tot}(I^\bullet(f_1^e, \ldots, f_r^e) \otimes_A M^\bullet)$$

we obtain $E_1 \to E$ using the canonical map $I^\bullet(f_1^e, \ldots, f_r^e) \to A$ and $E_1 \to E'$ using ξ and Cohomology, Lemma 20.42.1. □

08EI Lemma 36.13.7. Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ the object $F \oplus F[1]$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.

Proof. By Lemma 36.13.5 we can find a perfect object E of $D(\mathcal{O}_X)$ such that $E|_U = \mathcal{F}[r] \oplus F$ for some finite locally free $\mathcal{O}_U$-module $\mathcal{F}$. By Lemma 36.13.6 we can find a morphism of perfect complexes $\alpha : E_1 \to E$ such that $(E_1)|_U \cong E|_U$ and such that $\alpha|_U$ is the map

$$\begin{pmatrix} \mathrm{id}_{\mathcal{F}[r]} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{F}[r] \oplus F \to \mathcal{F}[r] \oplus F$$

Then the cone on α is a solution. □

08EJ Lemma 36.13.8. Let X be a quasi-compact and quasi-separated scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. For any morphism $\alpha : E \to E'$ in $D_{QCoh}(\mathcal{O}_X)$ such that

(1) E is perfect, and
(2) E' is supported on $T = V(f)$

there exists an $n \geq 0$ such that $f^n \alpha = 0$.

Proof. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 20.33.3. Thus if $X = U \cup V$ and the result of the lemma holds for $f|_U$, $f|_V$, and $f|_{U \cap V}$, then the result holds for f. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 30.4.1.

Let $X = \mathrm{Spec}(A)$. Then $f \in A$. We will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 36.3.5 without further mention. Represent E by a finite complex of finite projective A-modules $P^\bullet$. This is possible by Lemma 36.10.7. Let t be the largest integer such that P^t is nonzero. The distinguished triangle

$$P^t[-t] \to P^\bullet \to \sigma_{\leq t-1} P^\bullet \to P^t[-t+1]$$

shows that by induction on the length of the complex $P^\bullet$ we can reduce to the case where $P^\bullet$ has a single nonzero term. This and the shift functor reduces us to the case where $P^\bullet$ consists of a single finite projective A-module P in degree 0. Represent E' by a complex $M^\bullet$ of A-modules. Then α corresponds to a map $P \to H^0(M^\bullet)$. Since the module $H^0(M^\bullet)$ is supported on $V(f)$ by assumption (2) we see that every element of $H^0(M^\bullet)$ is annihilated by a power of f. Since P is a finite A-module the map $f^n\alpha : P \to H^0(M^\bullet)$ is zero for some n as desired. □

08EK Lemma 36.13.9. Let X be an affine scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ supported on $T \cap U$ the object $F \oplus F[1]$ is the restriction of a perfect object E of $D(\mathcal{O}_X)$ supported in T.

Proof. Say $T = V(g_1, \ldots, g_s)$. After replacing g_j by a power we may assume multiplication by g_j is zero on F, see Lemma 36.13.8. Choose E as in Lemma 36.13.7. Note that $g_j : E \to E$ restricts to zero on U. Choose a distinguished triangle

$$E \xrightarrow{g_1} E \to C_1 \to E[1]$$

By Derived Categories, Lemma 13.4.11 the object C_1 restricts to $F \oplus F[1] \oplus F[1] \oplus F[2]$ on U. Moreover, $g_1 : C_1 \to C_1$ has square zero by Derived Categories, Lemma 13.4.5. Namely, the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "0"'] & C_1 \arrow[r] \arrow[d, "g_1"'] & E[1] \arrow[d, "0"'] \\ E \arrow[r] & C_1 \arrow[r] & E[1] \end{tikzcd}$$

is commutative since the compositions $E \xrightarrow{g_1} E \to C_1$ and $C_1 \to E[1] \xrightarrow{g_1} E[1]$ are zero. Continuing, setting C_{i+1} equal to the cone of the map $g_i : C_i \to C_i$ we obtain a perfect complex C_s on X supported on T whose restriction to U gives

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

Choose morphisms of perfect complexes $\beta : C' \to C_s$ and $\gamma : C' \to C_s$ as in Lemma 36.13.6 such that $\beta|_U$ is an isomorphism and such that $\gamma|_U \circ \beta|_U^{-1}$ is the morphism

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s] \to F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

which is the identity on all summands except for F where it is zero. By Lemma 36.13.6 we also have $C' = C_s \otimes^{\mathbf{L}} I$ for some perfect complex I on X. Hence the nullity of $g_j^2 \text{id}_{C_s}$ implies the same thing for C'. Thus C' is supported on T as well. Then $\text{Cone}(\gamma)$ is a solution. □

A special case of the following lemma can be found in [Nee96].

09IM Lemma 36.13.10. Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_U$ be a map where P is a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$. Then there exists a map $\beta : R \to E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_U$ in $D(\mathcal{O}_U)$ compatible α and $\beta|_U$.

Proof. Since X is quasi-compact there exists an integer m such that $X = U \cup V_1 \cup \ldots \cup V_m$ for some affine opens V_j of X. Arguing by induction on m we see that we may assume $m = 1$. In other words, we may assume that $X = U \cup V$ with V affine.

By Lemma 36.13.9 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U\cap V} \to (P \oplus P[1])|_{U\cap V}$. By Lemma 36.13.6 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $T \cap V$) and assume that the map

$$Q|_{U\cap V} \to (P \oplus P[1])|_{U\cap V} \longrightarrow P|_{U\cap V} \longrightarrow E|_{U\cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 20.45.1 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T as desired. □

09IN Remark 36.13.11. The proof of Lemma 36.13.10 shows that

$$R|_U = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_U$ equals that of P. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_U$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_U$ and P above any given degree.

36.14. Approximation by perfect complexes

08EL In this section we discuss the observation, due to Neeman and Lipman, that a pseudo-coherent complex can be "approximated" by perfect complexes.

08EM Definition 36.14.1. Let X be a scheme. Consider triples (T, E, m) where

(1) $T \subset X$ is a closed subset,
(2) E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
(3) $m \in \mathbf{Z}$.

We say approximation holds for the triple (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Namely, it is clear that if approximation holds for the triple (T, E, m), then

(1) E is m-pseudo-coherent, see Cohomology, Definition 20.47.1, and
(2) the cohomology sheaves $H^i(E)$ are supported on T for $i \geq m$.

Moreover, the "support" of a perfect complex is a closed subscheme whose complement is retrocompact in X (details omitted). Hence we cannot expect approximation to hold without this assumption on T. This partly explains the conditions in the following definition.

08EN Definition 36.14.2. Let X be a scheme. We say approximation by perfect complexes holds on X if for any closed subset $T \subset X$ with $X \setminus T$ retro-compact in X there exists an integer r such that for every triple (T, E, m) as in Definition 36.14.1 with

(1) E is $(m - r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

We will prove that approximation by perfect complexes holds for quasi-compact and quasi-separated schemes. It seems that the second condition is necessary for our method of proof. It is possible that the first condition may be weakened to "E is m-pseudo-coherent" by carefuly analyzing the arguments below.

08EP Lemma 36.14.3. Let X be a scheme. Let $U \subset X$ be an open subscheme. Let (T, E, m) be a triple as in Definition 36.14.1. If

(1) $T \subset U$,
(2) approximation holds for $(T, E|_U, m)$, and
(3) the sheaves $H^i(E)$ for $i \geq m$ are supported on T,

then approximation holds for (T, E, m).

Proof. Let $j : U \to X$ be the inclusion morphism. If $P \to E|_U$ is an approximation of the triple $(T, E|_U, m)$ over U, then $j_!P = Rj_*P \to j_!(E|_U) \to E$ is an approximation of (T, E, m) over X. See Cohomology, Lemmas 20.33.6 and 20.49.10. □

08EQ Lemma 36.14.4. Let X be an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 36.14.1 such that there exists an integer $r \geq 0$ with

(1) E is m-pseudo-coherent,
(2) $H^i(E)$ is supported on T for $i \geq m - r + 1$,
(3) $X \setminus T$ is the union of r affine opens.

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Say $X = \mathrm{Spec}(A)$. Write $T = V(f_1, \ldots, f_r)$. (The case $r = 0$, i.e., $T = X$ follows immediately from Lemma 36.10.2 and the definitions.) Let (T, E, m) be a triple as in the lemma. Let t be the largest integer such that $H^t(E)$ is nonzero. We will proceed by induction on t. The base case is $t < m$; in this case the result is trivial. Now suppose that $t \geq m$. By Cohomology, Lemma 20.47.9 the sheaf $H^t(E)$ is of finite type. Since it is quasi-coherent it is generated by finitely many sections (Properties, Lemma 28.16.1). For every $s \in \Gamma(X, H^t(E)) = H^t(X, E)$ (see proof of Lemma 36.3.5) we can find an $e > 0$ and a morphism $K_e[-t] \to E$ such that s is in the image of $H^0(K_e) = H^t(K_e[-t]) \to H^t(E)$, see Lemma 36.9.6. Taking a finite direct sum of these maps we obtain a map $P \to E$ where P is a perfect complex supported on T, where $H^i(P) = 0$ for $i > t$, and where $H^t(P) \to E$ is surjective. Choose a distinguished triangle

$$P \to E \to E' \to P[1]$$

Then E' is m-pseudo-coherent (Cohomology, Lemma 20.47.4), $H^i(E') = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r + 1$. By induction we find an approximation $P' \to E'$ of (T, E', m). Fit the composition $P' \to E' \to P[1]$ into a distinguished triangle $P \to P'' \to P' \to P[1]$ and extend the morphisms $P' \to E'$ and $P[1] \to P[1]$ into a morphism of distinguished triangles

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & P[1] \arrow[d] \\ P \arrow[r] & E \arrow[r] & E' \arrow[r] & P[1] \end{tikzcd}$$

using TR3. Then P'' is a perfect complex (Cohomology, Lemma 20.49.7) supported on T. An easy diagram chase shows that $P'' \to E$ is the desired approximation. □

08ER Lemma 36.14.5. Let X be a scheme. Let $X = U \cup V$ be an open covering with U quasi-compact, V affine, and $U \cap V$ quasi-compact. If approximation by perfect complexes holds on U, then approximation holds on X.

Proof. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X. Let r_U be the integer of Definition 36.14.2 adapted to the pair $(U, T \cap U)$. Set $T' = T \setminus U$. Note that $T' \subset V$ and that $V \setminus T' = (X \setminus T) \cap U \cap V$ is quasi-compact by our assumption on T. Let r' be the number of affines needed to cover $V \setminus T'$. We claim that $r = \max(r_U, r')$ works for the pair (X, T).

To see this choose a triple (T, E, m) such that E is $(m-r)$-pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m - r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that E can be approximated by induction on t.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 36.14.4 guarantees the existence of an approximation $P \to E|_V$ of $(T', E|_V, m)$ on V. Applying Lemma 36.14.3 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m).

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap U, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. By Lemma 36.13.9 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma 36.13.6 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V} \longrightarrow P|_{U \cap V} \longrightarrow E|_{U \cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 20.45.1 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \to H^t(E)$ is surjective on restriction to U. Choose a distinguished triangle

$$R \to E \to E' \to R[1]$$

Then E' is $(m-r)$-pseudo-coherent (Cohomology, Lemma 20.47.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of (T, E', m). Fit the composition $R' \to E' \to R[1]$ into a distinguished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$ and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R'' \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & R[1] \arrow[d] \\ R \arrow[r] & E \arrow[r] & E' \arrow[r] & R[1] \end{tikzcd}$$

using TR3. Then R'' is a perfect complex (Cohomology, Lemma 20.49.7) supported on T. An easy diagram chase shows that $R'' \to E$ is the desired approximation. $\square$

08ES Theorem 36.14.6. Let X be a quasi-compact and quasi-separated scheme. Then approximation by perfect complexes holds on X.

Proof. This follows from the induction principle of Cohomology of Schemes, Lemma 30.4.1 and Lemmas 36.14.5 and 36.14.4. $\square$

36.15. Generating derived categories

09IP In this section we prove that the derived category $D_{QCoh}(\mathcal{O}_X)$ of a quasi-compact and quasi-separated scheme can be generated by a single perfect object. We urge the reader to read the proof of this result in the wonderful paper by Bondal and van den Bergh, see [BV03].

09IQ **Lemma 36.15.1.** Let X be a quasi-compact and quasi-separated scheme. Let U be a quasi-compact open subscheme. Let P be a perfect object of $D(\mathcal{O}_U)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.

Proof. Special case of Lemma 36.13.10. $\square$

09IR **Lemma 36.15.2.** In Situation 36.9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. For $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent [BN93, Proposition 6.1]

(1) $E = Rj_*(E|_U)$, and
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$.

Proof. Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[n], E) = 0$$

for all n as $K|_U = 0$. Thus it suffices to prove the result for N. In other words, we may assume that E restricts to zero on U. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i+e''_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e''_i}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 15.28.11. Hence if $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$ then the same thing is true for the K replaced by K_e as in Lemma 36.9.6. Thus our lemma follows immediately from that one and the fact that E is determined by the complex of A-modules $R\Gamma(X, E)$, see Lemma 36.3.5. $\square$

09IS **Theorem 36.15.3.** Let X be a quasi-compact and quasi-separated scheme. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent

(1) $E = 0$, and
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ for all $n \in \mathbf{Z}$.

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

If X is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 36.3.5.

Assume that $X = U \cup V$ is an open covering with U quasi-compact such that the theorem holds for U and V is an affine open. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 36.15.1 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to U is a direct sum one of whose summands is P. Say $V = \mathrm{Spec}(A)$. Let $Z = X \setminus U$. This is a closed subset of V with $V \setminus Z$ quasi-compact. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on $Z \subset V$ closed, the pushforward $K' = R(V \to X)_*K$

is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 20.49.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Cohomology, Lemma 20.33.6 we have $K' = R(V \to X)_! K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 36.15.2 the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \cap V \to V)_* E|_{U \cap V}$. This implies that $E = R(U \to X)_* E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. $\square$

The following result is an strengthening of Theorem 36.15.3 proved using exactly the same methods. Recall that for a closed subset T of a scheme X we denote $D_T(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T (Definition 36.6.1). We similarly denote $D_{QCoh,T}(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of those complexes whose cohomology sheaves are quasi-coherent and are suppported on T.

0A9A **Lemma 36.15.4.** Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. With notation as above, the category $D_{QCoh,T}(\mathcal{O}_X)$ is generated by a single perfect object. [Rou08, Theorem 6.8]

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

Assume $X = \mathrm{Spec}(A)$ is affine. In this case there exist $f_1, \ldots, f_r \in A$ such that $T = V(f_1, \ldots, f_r)$. Let K be the Koszul complex on $f_1, \ldots, f_r$ as in Lemma 36.15.2. Then K is a perfect object with cohomology supported on T and hence a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$. On the other hand, if $E \in D_{QCoh,T}(\mathcal{O}_X)$ and $\mathrm{Hom}(K, E[n]) = 0$ for all n, then Lemma 36.15.2 tells us that $E = Rj_*(E|_{X \setminus T}) = 0$. Hence K generates $D_{QCoh,T}(\mathcal{O}_X)$, (by our definition of generators of triangulated categories in Derived Categories, Definition 13.36.3).

Assume that $X = U \cup V$ is an open covering with V affine and U quasi-compact such that the lemma holds for U. Let P be a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$ which is a generator for $D_{QCoh,T \cap U}(\mathcal{O}_U)$. Using Lemma 36.13.10 we may choose a perfect object Q of $D(\mathcal{O}_X)$ supported on T whose restriction to U is a direct sum one of whose summands is P. Write $V = \mathrm{Spec}(B)$. Let $Z = X \setminus U$. Then Z is a closed subset of V such that $V \setminus Z$ is quasi-compact. As X is quasi-separated, it follows that $Z \cap T$ is a closed subset of V such that $W = V \setminus (Z \cap T)$ is quasi-compact. Thus we can choose $g_1, \ldots, g_s \in B$ such that $Z \cap T = V(g_1, \ldots, g_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $g_1, \ldots, g_s$ over B. Note that since K is supported on $(Z \cap T) \subset V$ closed, the pushforward $K' = R(V \to X)_* K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 20.49.10). We claim that $Q \oplus K'$ is a generator for $D_{QCoh,T}(\mathcal{O}_X)$.

Let E be an object of $D_{QCoh,T}(\mathcal{O}_X)$ such that there are no nontrivial maps from any shift of $Q \oplus K'$ into E. By Cohomology, Lemma 20.33.6 we have $K' = R(V \to X)_! K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 36.15.2 we have $E|_V = Rj_* E|_W$ where $j : W \to V$ is the inclusion. Picture

$$\begin{tikzcd} W \arrow[r, "j"'] & V & Z \cap T \arrow[l] \arrow[d] \\ U \cap V \arrow[u, "j'"] \arrow[ur, "j''"'] & & Z \arrow[ul] \end{tikzcd}$$

Since E is supported on T we see that $E|_W$ is supported on $T \cap W = T \cap U \cap V$ which is closed in W. We conclude that

$$E|_V = Rj_*(E|_W) = Rj_*(Rj'_*(E|_{U\cap V})) = Rj''_*(E|_{U\cap V})$$

where the second equality is part (1) of Cohomology, Lemma 20.33.6. This implies that $E = R(U \to X)_* E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. $\square$

36.16. An example generator

0BQQ In this section we prove that the derived category of projective space over a ring is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf.

The following lemma says that $\bigoplus_{n \geq 0} \mathcal{L}^{\otimes -n}$ is a generator if $\mathcal{L}$ is ample.

0BQR **Lemma 36.16.1.** *Let X be a scheme and $\mathcal{L}$ an ample invertible $\mathcal{O}_X$-module. If K is a nonzero object of $D_{QCoh}(\mathcal{O}_X)$, then for some $n \geq 0$ and $p \in \mathbf{Z}$ the cohomology group $H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ is nonzero.*

Proof. Recall that as X has an ample invertible sheaf, it is quasi-compact and separated (Properties, Definition 28.26.1 and Lemma 28.26.7). Thus we may apply Proposition 36.7.5 and represent K by a complex $\mathcal{F}^\bullet$ of quasi-coherent modules. Pick any p such that $\mathcal{H}^p = \mathrm{Ker}(\mathcal{F}^p \to \mathcal{F}^{p+1}) / \mathrm{Im}(\mathcal{F}^{p-1} \to \mathcal{F}^p)$ is nonzero. Choose a point $x \in X$ such that the stalk $\mathcal{H}^p_x$ is nonzero. Choose an $n \geq 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is an affine open neighbourhood of x. Choose $\tau \in \mathcal{H}^p(X_s)$ which maps to a nonzero element of the stalk $\mathcal{H}^p_x$; this is possible as $\mathcal{H}^p$ is quasi-coherent and X_s is affine. Since taking sections over X_s is an exact functor on quasi-coherent modules, we can find a section $\tau' \in \mathcal{F}^p(X_s)$ mapping to zero in $\mathcal{F}^{p+1}(X_s)$ and mapping to τ in $\mathcal{H}^p(X_s)$. By Properties, Lemma 28.17.2 there exists an m such that $\tau' \otimes s^{\otimes m}$ is the image of a section $\tau'' \in \Gamma(X, \mathcal{F}^p \otimes \mathcal{L}^{\otimes mn})$. Applying the same lemma once more, we find $l \geq 0$ such that $\tau'' \otimes s^{\otimes l}$ maps to zero in $\mathcal{F}^{p+1} \otimes \mathcal{L}^{\otimes (m+l)n}$. Then τ'' gives a nonzero class in $H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{(m+l)n})$ as desired. $\square$

0BQS **Lemma 36.16.2.** *Let A be a ring. Let $X = \mathbf{P}^n_A$. For every $a \in \mathbf{Z}$ there exists an exact complex*

$$0 \to \mathcal{O}_X(a) \to \ldots \to \mathcal{O}_X(a+i)^{\oplus \binom{n+1}{i}} \to \ldots \to \mathcal{O}_X(a+n+1) \to 0$$

of vector bundles on X.

Proof. Recall that $\mathbf{P}^n_A$ is $\text{Proj}(A[X_0, \ldots, X_n])$, see Constructions, Definition 27.13.2. Consider the Koszul complex

$$K_\bullet = K_\bullet(A[X_0, \ldots, X_n], X_0, \ldots, X_n)$$

over $S = A[X_0, \ldots, X_n]$ on $X_0, \ldots, X_n$. Since $X_0, \ldots, X_n$ is clearly a regular sequence in the polynomial ring S, we see that (More on Algebra, Lemma 15.30.2) that the Koszul complex $K_\bullet$ is exact, except in degree 0 where the cohomology is $S/(X_0, \ldots, X_n)$. Note that $K_\bullet$ becomes a complex of graded modules if we put the generators of K_i in degree $+i$. In other words an exact complex

$$0 \to S(-n-1) \to \ldots \to S(-n-1+i)^{\oplus\binom{n}{i}} \to \ldots \to S \to S/(X_0, \ldots, X_n) \to 0$$

Applying the exact functor $\tilde{}$ functor of Constructions, Lemma 27.8.4 and using that the last term is in the kernel of this functor, we obtain the exact complex

$$0 \to \mathcal{O}_X(-n-1) \to \ldots \to \mathcal{O}_X(-n-1+i)^{\oplus\binom{n+1}{i}} \to \ldots \to \mathcal{O}_X \to 0$$

Twisting by the invertible sheaves $\mathcal{O}_X(n+a)$ we get the exact complexes of the lemma. □

0A9V Lemma 36.16.3. Let A be a ring. Let $X = \mathbf{P}^n_A$. Then

$$E = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n)$$

is a generator (Derived Categories, Definition 13.36.3) of $D_{QCoh}(X)$.

Proof. Let $K \in D_{QCoh}(\mathcal{O}_X)$. Assume $\text{Hom}(E, K[p]) = 0$ for all $p \in \mathbf{Z}$. We have to show that $K = 0$. By Derived Categories, Lemma 13.36.4 we see that $\text{Hom}(E', K[p])$ is zero for all $E' \in \langle E \rangle$ and $p \in \mathbf{Z}$. By Lemma 36.16.2 applied with $a = -n-1$ we see that $\mathcal{O}_X(-n-1) \in \langle E \rangle$ because it is quasi-isomorphic to a finite complex whose terms are finite direct sums of summands of E. Repeating the argument with $a = -n-2$ we see that $\mathcal{O}_X(-n-2) \in \langle E \rangle$. Arguing by induction we find that $\mathcal{O}_X(-m) \in \langle E \rangle$ for all $m \geq 0$. Since

$$\text{Hom}(\mathcal{O}_X(-m), K[p]) = H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X(m)) = H^p(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m})$$

we conclude that $K = 0$ by Lemma 36.16.1. (This also uses that $\mathcal{O}_X(1)$ is an ample invertible sheaf on X which follows from Properties, Lemma 28.26.12.) □

0BQT Remark 36.16.4. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Let $E \in D_{QCoh}(\mathcal{O}_Y)$ be a generator (see Theorem 36.15.3). Then the following are equivalent

(1) for $K \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf_*K = 0$ if and only if $K = 0$,
(2) $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms, and
(3) Lf^*E is a generator for $D_{QCoh}(\mathcal{O}_X)$.

The equivalence between (1) and (2) is a formal consequence of the fact that $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ is an exact functor of triangulated categories. Similarly, the equivalence between (1) and (3) follows formally from the fact that Lf^* is the left adjoint to Rf_*. These conditions hold if f is affine (Lemma 36.5.2) or if f is an open immersion, or if f is a composition of such. We conclude that

(1) if X is a quasi-affine scheme then $\mathcal{O}_X$ is a generator for $D_{QCoh}(\mathcal{O}_X)$,
(2) if $X \subset \mathbf{P}^n_A$ is a quasi-compact locally closed subscheme, then $\mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n)$ is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.16.3.

36.17. Compact and perfect objects

09M0 Let X be a Noetherian scheme of finite dimension. By Cohomology, Proposition 20.20.7 and Cohomology on Sites, Lemma 21.52.5 the sheaves of modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O}_X)$ for all opens $U \subset X$. These sheaves are typically not quasi-coherent, hence these do not give perfect objects of the derived category $D(\mathcal{O}_X)$. However, if we restrict ourselves to complexes with quasi-coherent cohomology sheaves, then this does not happen. Here is the precise statement.

09M1 **Proposition 36.17.1.** Let X be a quasi-compact and quasi-separated scheme. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.

Proof. If K is a perfect object of $D(\mathcal{O}_X)$ with dual $K^\vee$ (Cohomology, Lemma 20.50.5) we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M. Since $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums and since $H^0(X, -)$ commutes with direct sums on $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.4.5 we conclude that K is compact in $D_{QCoh}(\mathcal{O}_X)$.

Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine open $U \subset X$, see Cohomology, Lemma 20.49.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 36.4.5. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a compact object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine.

Assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 36.3.5 the problem is translated into the same problem for $D(A)$. For $D(A)$ the result is More on Algebra, Proposition 15.78.3. $\square$

0GEF **Remark 36.17.2.** Let X be a quasi-compact and quasi-separated scheme. Let G be a perfect object of $D(\mathcal{O}_X)$ which is a generator for $D_{QCoh}(\mathcal{O}_X)$. By Theorem 36.15.3 there is at least one of these. Combining Lemma 36.3.1 with Proposition 36.17.1 and with Derived Categories, Proposition 13.37.6 we see that G is a classical generator for $D_{perf}(\mathcal{O}_X)$.

The following result is a strengthening of Proposition 36.17.1. Let $T \subset X$ be a closed subset of a scheme X. As before $D_T(\mathcal{O}_X)$ denotes the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of objects supported on T (Definition 36.6.1). Since taking direct sums commutes with taking cohomology sheaves, it follows that $D_T(\mathcal{O}_X)$ has direct sums and that they are equal to direct sums in $D(\mathcal{O}_X)$.

0A9B **Lemma 36.17.3.** Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. An object of $D_{QCoh,T}(\mathcal{O}_X)$ is compact if and only if it is perfect as an object of $D(\mathcal{O}_X)$.

Proof. We observe that $D_{QCoh,T}(\mathcal{O}_X)$ is a triangulated category with direct sums by the remark preceding the lemma. By Proposition 36.17.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$ hence a fortiori of any subcategory preserved under taking direct sums. For the converse we will use there exists a generator $E \in D_{QCoh,T}(\mathcal{O}_X)$ which is a perfect complex of $\mathcal{O}_X$-modules, see Lemma 36.15.4.

Hence by the above, E is compact. Then it follows from Derived Categories, Proposition 13.37.6 that E is a classical generator of the full subcategory of compact objects of $D_{QCoh,T}(\mathcal{O}_X)$. Thus any compact object can be constructed out of E by a finite sequence of operations consisting of (a) taking shifts, (b) taking finite direct sums, (c) taking cones, and (d) taking direct summands. Each of these operations preserves the property of being perfect and the result follows. $\square$

0GEG Remark 36.17.4. Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let G be a perfect object of $D_{QCoh,T}(\mathcal{O}_X)$ which is a generator for $D_{QCoh,T}(\mathcal{O}_X)$. By Lemma 36.15.4 there is at least one of these. Combining the fact that $D_{QCoh,T}(\mathcal{O}_X)$ has direct sums with Lemma 36.17.3 and with Derived Categories, Proposition 13.37.6 we see that G is a classical generator for $D_{perf,T}(\mathcal{O}_X)$.

The following lemma is an application of the ideas that go into the proof of the preceding lemma.

0A9C Lemma 36.17.5. Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $U = X \setminus T$ is quasi-compact. Let $\alpha : P \to E$ be a morphism of $D_{QCoh}(\mathcal{O}_X)$ with either

(1) P is perfect and E supported on T, or
(2) P pseudo-coherent, E supported on T, and E bounded below.

Then there exists a perfect complex of $\mathcal{O}_X$-modules I and a map $I \to \mathcal{O}_X[0]$ such that $I \otimes^{\mathbf{L}} P \to E$ is zero and such that $I|_U \to \mathcal{O}_U[0]$ is an isomorphism.

Proof. Set $\mathcal{D} = D_{QCoh,T}(\mathcal{O}_X)$. In both cases the complex $K = R\mathcal{H}om(P, E)$ is an object of $\mathcal{D}$. See Lemma 36.10.8 for quasi-coherence. It is clear that K is supported on T as formation of $R\mathcal{H}om$ commutes with restriction to opens. The map α defines an element of $H^0(K) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{O}_X[0], K)$. Then it suffices to prove the result for the map $\alpha : \mathcal{O}_X[0] \to K$.

Let $E \in \mathcal{D}$ be a perfect generator, see Lemma 36.15.4. Write

$$K = \text{hocolim} K_n$$

as in Derived Categories, Lemma 13.37.3 using the generator E. Since the functor $\mathcal{D} \to D(\mathcal{O}_X)$ commutes with direct sums, we see that $K = \text{hocolim} K_n$ holds in $D(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is a compact object of $D(\mathcal{O}_X)$ we find an n and a morphism $\alpha_n : \mathcal{O}_X \to K_n$ which gives rise to α, see Derived Categories, Lemma 13.33.9. By Derived Categories, Lemma 13.37.4 applied to the morphism $\mathcal{O}_X[0] \to K_n$ in the ambient category $D(\mathcal{O}_X)$ we see that α_n factors as $\mathcal{O}_X[0] \to Q \to K_n$ where Q is an object of $\langle E \rangle$. We conclude that Q is a perfect complex supported on T.

Choose a distinguished triangle

$$I \to \mathcal{O}_X[0] \to Q \to I[1]$$

By construction I is perfect, the map $I \to \mathcal{O}_X[0]$ restricts to an isomorphism over U, and the composition $I \to K$ is zero as α factors through Q. This proves the lemma. $\square$

36.18. Derived categories as module categories

09M2 In this section we draw some conclusions of what has gone before. Before we do so we need a couple more lemmas.

09M3 **Lemma 36.18.1.** Let X be a scheme. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.35.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E-module with property (P) and let $F_\bullet$ be a filtration on P as in Differential Graded Algebra, Section 22.20. Then we have

$$P \otimes_E K^\bullet = \mathrm{hocolim}\ F_iP \otimes_E K^\bullet$$

Each of the F_iP has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. $\square$

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

09M4 **Lemma 36.18.2.** Let X be a quasi-compact and quasi-separated scheme. Let K be a perfect object of $D(\mathcal{O}_X)$. Then

1. there exist integers $a \leq b$ such that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = 0$ for $L \in D_{QCoh}(\mathcal{O}_X)$ with $H^i(L) = 0$ for $i \in [a, b]$, and
2. if L is bounded, then $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(K, L)$ is zero for all but finitely many n.

Proof. Part (2) follows from (1) as $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(K, L) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, L[n])$. We prove (1). Since K is perfect we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L) = H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

where $K^\vee$ is the "dual" perfect complex to K, see Cohomology, Lemma 20.50.5. Note that $K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(X)$ by Lemmas 36.3.9 and 36.10.1 (to see that a perfect complex has quasi-coherent cohomology sheaves). Say $K^\vee$ has tor amplitude in $[a, b]$. Then the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

shows that $H^j(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$ is zero if $H^q(L) = 0$ for $q \in [j-b, j-a]$. Let N be the integer d of Cohomology of Schemes, Lemma 30.4.4. Then $H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$ vanishes if the cohomology sheaves

$$H^{-N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L),\ H^{-N+1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L),\ \ldots,\ H^0(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

are zero. Namely, by the lemma cited and Lemma 36.3.4, we have

$$H^0(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^0(X, \tau_{\geq -N}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$$

and by the vanishing of cohomology sheaves, this is equal to $H^0(X, \tau_{\geq 1}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L))$ which is zero by Derived Categories, Lemma 13.16.1. It follows that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K, L)$ is zero if $H^i(L) = 0$ for $i \in [-b-N, -a]$. $\square$

The following result is taken from [BV03].

09M5 **Theorem 36.18.3.** Let X be a quasi-compact and quasi-separated scheme. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, \text{d})$.

Proof. Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 36.15.3 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\text{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 22.35. Since $K^\bullet$ is K-injective we have

09M6 (36.18.3.1) $$H^n(E) = \text{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 36.18.2. Consider the functor

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \text{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.35.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 36.17.1. Combined with (36.18.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 22.35.6. It has a right adjoint

$$R\,\text{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \text{d})$$

by Differential Graded Algebra, Lemmas 22.35.5 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 36.18.1 that we obtain

$$- \otimes_E^{\mathbf{L}} K^\bullet : D(E, \text{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 13.7.2. $\square$

0DJL **Remark 36.18.4** (Variant with support)**.** Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. The analogue of Theorem 36.18.3 holds for $D_{QCoh,T}(\mathcal{O}_X)$. This follows from the exact same argument as in the proof of the theorem, using Lemmas 36.15.4 and 36.17.3 and a variant of Lemma 36.18.1 with supports. If we ever need this, we will precisely state the result here and give a detailed proof.

09SU **Remark 36.18.5** (Uniqueness of dga)**.** Let X be a quasi-compact and quasi-separated scheme over a ring R. By the construction of the proof of Theorem 36.18.3 there exists a differential graded algebra (A, d) over R such that $D_{QCoh}(X)$ is R-linearly equivalent to $D(A, \text{d})$ as a triangulated category. One may ask: how unique is (A, d)? The answer is (only) slightly better than just saying that (A, d) is well defined up to derived equivalence. Namely, suppose that (B, d) is a second such pair. Then we have

$$(A, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, \text{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K-injective complexes $K^\bullet$ and $L^\bullet$ of $\mathcal{O}_X$-modules corresponding to perfect generators of $D_{QCoh}(\mathcal{O}_X)$. Set

$$\Omega = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then Ω is a differential graded $B^{opp} \otimes_R A$-module and Ω' is a differential graded $A^{opp} \otimes_R B$-module. Moreover, the equivalence

$$D(A, \mathrm{d}) \to D_{QCoh}(\mathcal{O}_X) \to D(B, \mathrm{d})$$

is given by the functor $- \otimes_A^{\mathbf{L}} \Omega'$ and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 22.37.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

36.19. Characterizing pseudo-coherent complexes, I

0DJM We can use the methods above to characterize pseudo-coherent objects as derived homotopy limits of approximations by perfect objects.

0DJN **Lemma 36.19.1.** Let X be a quasi-compact and quasi-separated scheme. Let $K \in D(\mathcal{O}_X)$. The following are equivalent

(1) K is pseudo-coherent, and
(2) $K = \mathrm{hocolim} K_n$ where K_n is perfect and $\tau_{\geq -n}K_n \to \tau_{\geq -n}K$ is an isomorphism for all n.

Proof. The implication (2) $\Rightarrow$ (1) is true on any ringed space. Namely, assume (2) holds. Recall that a perfect object of the derived category is pseudo-coherent, see Cohomology, Lemma 20.49.5. Then it follows from the definitions that $\tau_{\geq -n}K_n$ is $(-n+1)$-pseudo-coherent and hence $\tau_{\geq -n}K$ is $(-n+1)$-pseudo-coherent, hence K is $(-n+1)$-pseudo-coherent. This is true for all n, hence K is pseudo-coherent, see Cohomology, Definition 20.47.1.

Assume (1). We start by choosing an approximation $K_1 \to K$ of $(X, K, -2)$ by a perfect complex K_1, see Definitions 36.14.1 and 36.14.2 and Theorem 36.14.6. Suppose by induction we have

$$K_1 \to K_2 \to \ldots \to K_n \to K$$

with K_i perfect such that such that $\tau_{\geq -i}K_i \to \tau_{\geq -i}K$ is an isomorphism for all $1 \leq i \leq n$. Then we pick $a \leq b$ as in Lemma 36.18.2 for the perfect object K_n. Choose an approximation $K_{n+1} \to K$ of $(X, K, \min(a-1, -n-1))$. Choose a distinguished triangle

$$K_{n+1} \to K \to C \to K_{n+1}[1]$$

Then we see that $C \in D_{QCoh}(\mathcal{O}_X)$ has $H^i(C) = 0$ for $i \geq a$. Thus by our choice of a, b we see that $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_n, C) = 0$. Hence the composition $K_n \to K \to C$ is zero. Hence by Derived Categories, Lemma 13.4.2 we can factor $K_n \to K$ through K_{n+1} proving the induction step.

We still have to prove that $K = \mathrm{hocolim} K_n$. This follows by an application of Derived Categories, Lemma 13.33.8 to the functors $H^i(-) : D(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$ and our choice of K_n. $\square$

0DJP **Lemma 36.19.2.** Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $K \in D(\mathcal{O}_X)$ supported on T. The following are equivalent

(1) K is pseudo-coherent, and
(2) $K = \text{hocolim} K_n$ where K_n is perfect, supported on T, and $\tau_{\geq -n} K_n \to \tau_{\geq -n} K$ is an isomorphism for all n.

Proof. The proof of this lemma is exactly the same as the proof of Lemma 36.19.1 except that in the choice of the approximations we use the triples (T, K, m). $\square$

36.20. An example equivalence

0CS7 In Section 36.16 we proved that the derived category of projective space $\mathbf{P}^n_A$ over a ring A is generated by a vector bundle, in fact a direct sum of shifts of the structure sheaf. In this section we prove this determines an equivalence of $D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$ with the derived category of an A-algebra.

Before we can state the result we need some notation. Let A be a ring. Let $X = \mathbf{P}^n_A = \text{Proj}(S)$ where $S = A[X_0, \ldots, X_n]$. By Lemma 36.16.3 we know that

0CS8 (36.20.0.1)
$$P = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \ldots \oplus \mathcal{O}_X(-n)$$

is a perfect generator of $D_{QCoh}(\mathcal{O}_X)$. Consider the (noncommutative) A-algebra

0CS9 (36.20.0.2)
$$R = \text{Hom}_{\mathcal{O}_X}(P, P) = \begin{pmatrix} S_0 & S_1 & S_2 & \ldots & \ldots \\ 0 & S_0 & S_1 & \ldots & \ldots \\ 0 & 0 & S_0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & S_0 \end{pmatrix}$$

with obvious multiplication and addition. If we view P as a complex of $\mathcal{O}_X$-modules in the usual way (i.e., with P in degree 0 and zero in every other degree), then we have

$$R = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(P, P)$$

where on the right hand side we view R as a differential graded algebra over A with zero differential (i.e., with R in degree 0 and zero in every other degree). According to the discussion in Differential Graded Algebra, Section 22.35 we obtain a derived functor

$$- \otimes^{\mathbf{L}}_R P : D(R) \longrightarrow D(\mathcal{O}_X),$$

see especially Differential Graded Algebra, Lemma 22.35.3. By Lemma 36.18.1 we see that the essential image of this functor is contained in $D_{QCoh}(\mathcal{O}_X)$.

0BQU **Lemma 36.20.1.** Let A be a ring. Let $X = \mathbf{P}^n_A = \text{Proj}(S)$ where $S = A[X_0, \ldots, X_n]$. [Bei78]
With P as in (36.20.0.1) and R as in (36.20.0.2) the functor

$$- \otimes^{\mathbf{L}}_R P : D(R) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an A-linear equivalence of triangulated categories sending R to P.

In words: the derived category of quasi-coherent modules on projective space is equivalent to the derived category of modules over a (noncommutative) algebra. This property of projective space appears to be quite unusual among all projective schemes over A.

Proof. To prove that our functor is fully faithful it suffices to prove that $\text{Ext}^i_X(P, P)$ is zero for $i \neq 0$ and equal to R for $i = 0$, see Differential Graded Algebra, Lemma 22.35.6. As in the proof of Lemma 36.18.2 we see that

$$\text{Ext}^i_X(P, P) = H^i(X, P^\wedge \otimes P) = \bigoplus\nolimits_{0 \leq a, b \leq n} H^i(X, \mathcal{O}_X(a - b))$$

By the computation of cohomology of projective space (Cohomology of Schemes, Lemma 30.8.1) we find that these Ext-groups are zero unless $i = 0$. For $i = 0$ we recover R because this is how we defined R in (36.20.0.2). By Differential Graded Algebra, Lemma 22.35.5 our functor has a right adjoint, namely $R\operatorname{Hom}(P, -) : D_{QCoh}(\mathcal{O}_X) \to D(R)$. Since P is a generator for $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.16.3 we see that the kernel of $R\operatorname{Hom}(P, -)$ is zero. Hence our functor is an equivalence of triangulated categories by Derived Categories, Lemma 13.7.2. $\square$

36.21. The coherator revisited

0CQZ In Section 36.7 we constructed and studied the right adjoint RQ_X to the canonical functor $D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. It was constructed as the right derived extension of the coherator $Q_X : \operatorname{Mod}(\mathcal{O}_X) \to \mathit{QCoh}(\mathcal{O}_X)$. In this section, we study when the inclusion functor

$$D_{QCoh}(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

has a right adjoint. If this right adjoint exists, we will denote[3] it

$$DQ_X : D(\mathcal{O}_X) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

It turns out that quasi-compact and quasi-separated schemes have such a right adjoint.

0CR0 Lemma 36.21.1. Let X be a quasi-compact and quasi-separated scheme. The inclusion functor $D_{QCoh}(\mathcal{O}_X) \to D(\mathcal{O}_X)$ has a right adjoint DQ_X.

First proof. We will use the induction principle as in Cohomology of Schemes, Lemma 30.4.1 to prove this. If $D(\mathit{QCoh}(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ is an equivalence, then the lemma is true because the functor RQ_X of Section 36.7 is a right adjoint to the functor $D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$. In particular, our lemma is true for affine schemes, see Lemma 36.7.3. Thus we see that it suffices to show: if $X = U \cup V$ is a union of two quasi-compact opens and the lemma holds for U, V, and $U \cap V$, then the lemma holds for X.

The adjoint exists if and only if for every object K of $D(\mathcal{O}_X)$ we can find a distinguished triangle

$$E' \to E \to K \to E'[1]$$

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\operatorname{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. See Derived Categories, Lemma 13.40.7. Consider the distinguished triangle

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\cap V,*}E|_{U\cap V} \to E[1]$$

in $D(\mathcal{O}_X)$ of Cohomology, Lemma 20.33.2. By Derived Categories, Lemma 13.40.5 it suffices to construct the desired distinguished triangles for $Rj_{U,*}E|_U$, $Rj_{V,*}E|_V$, and $Rj_{U\cap V,*}E|_{U\cap V}$. This reduces us to the statement discussed in the next paragraph.

Let $j : U \to X$ be an open immersion corresponding with U a quasi-compact open for which the lemma is true. Let L be an object of $D(\mathcal{O}_U)$. Then there exists a distinguished triangle

$$E' \to Rj_*L \to K \to E'[1]$$

[3]This is probably nonstandard notation. However, we have already used Q_X for the coherator and RQ_X for its derived extension.

in $D(\mathcal{O}_X)$ such that E' is in $D_{QCoh}(\mathcal{O}_X)$ and such that $\mathrm{Hom}(M, K) = 0$ for all M in $D_{QCoh}(\mathcal{O}_X)$. To see this we choose a distinguished triangle

$$L' \to L \to Q \to L'[1]$$

in $D(\mathcal{O}_U)$ such that L' is in $D_{QCoh}(\mathcal{O}_U)$ and such that $\mathrm{Hom}(N, Q) = 0$ for all N in $D_{QCoh}(\mathcal{O}_U)$. This is possible because the statement in Derived Categories, Lemma 13.40.7 is an if and only if. We obtain a distinguished triangle

$$Rj_*L' \to Rj_*L \to Rj_*Q \to Rj_*L'[1]$$

in $D(\mathcal{O}_X)$. Observe that Rj_*L' is in $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.4.1. On the other hand, if M in $D_{QCoh}(\mathcal{O}_X)$, then

$$\mathrm{Hom}(M, Rj_*Q) = \mathrm{Hom}(Lj^*M, Q) = 0$$

because Lj^*M is in $D_{QCoh}(\mathcal{O}_U)$ by Lemma 36.3.8. This finishes the proof. □

Second proof. The adjoint exists by Derived Categories, Proposition 13.38.2. The hypotheses are satisfied: First, note that $D_{QCoh}(\mathcal{O}_X)$ has direct sums and direct sums commute with the inclusion functor (Lemma 36.3.1). On the other hand, $D_{QCoh}(\mathcal{O}_X)$ is compactly generated because it has a perfect generator Theorem 36.15.3 and because perfect objects are compact by Proposition 36.17.1. □

0CR1 Lemma 36.21.2. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. If the right adjoints DQ_X and DQ_Y of the inclusion functors $D_{QCoh} \to D$ exist for X and Y, then

$$Rf_* \circ DQ_X = DQ_Y \circ Rf_*$$

Proof. The statement makes sense because Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Lemma 36.4.1. The statement is true because Lf^* similarly maps $D_{QCoh}(\mathcal{O}_Y)$ into $D_{QCoh}(\mathcal{O}_X)$ (Lemma 36.3.8) and hence both $Rf_* \circ DQ_X$ and $DQ_Y \circ Rf_*$ are right adjoint to $Lf^* : D_{QCoh}(\mathcal{O}_Y) \to D(\mathcal{O}_X)$. □

0CR2 Remark 36.21.3. Let X be a quasi-compact and quasi-separated scheme. Let $X = U \cup V$ with U and V quasi-compact open. By Lemma 36.21.1 the functors DQ_X, DQ_U, DQ_V, $DQ_{U\cap V}$ exist. Moreover, there is a canonical distinguished triangle

$$DQ_X(K) \to Rj_{U,*}DQ_U(K|_U) \oplus Rj_{V,*}DQ_V(K|_V) \to Rj_{U\cap V,*}DQ_{U\cap V}(K|_{U\cap V}) \to$$

for any $K \in D(\mathcal{O}_X)$. This follows by applying the exact functor DQ_X to the distinguished triangle of Cohomology, Lemma 20.33.2 and using Lemma 36.21.2 three times.

0CSA Lemma 36.21.4. Let X be a quasi-compact and quasi-separated scheme. The functor DQ_X of Lemma 36.21.1 has the following boundedness property: there exists an integer $N = N(X)$ such that, if K in $D(\mathcal{O}_X)$ with $H^i(U, K) = 0$ for U affine open in X and $i \notin [a, b]$, then the cohomology sheaves $H^i(DQ_X(K))$ are zero for $i \notin [a, b + N]$.

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

If X is affine, then the lemma is true with $N = 0$ because then $RQ_X = DQ_X$ is given by taking the complex of quasi-coherent sheaves associated to $R\Gamma(X, K)$. See Lemmas 36.3.5 and 36.7.3.

Asssume U, V are quasi-compact open in X and the lemma holds for U, V, and $U \cap V$. Say with integers $N(U)$, $N(V)$, and $N(U \cap V)$. Now suppose K is in $D(\mathcal{O}_X)$ with $H^i(W, K) = 0$ for all affine open $W \subset X$ and all $i \not\in [a, b]$. Then $K|_U$, $K|_V$, $K|_{U \cap V}$ have the same property. Hence we see that $RQ_U(K|_U)$ and $RQ_V(K|_V)$ and $RQ_{U \cap V}(K|_{U \cap V})$ have vanishing cohomology sheaves outside the inverval $[a, b + \max(N(U), N(V), N(U \cap V))$. Since the functors $Rj_{U,*}$, $Rj_{V,*}$, $Rj_{U \cap V,*}$ have finite cohomological dimension on D_{QCoh} by Lemma 36.4.1 we see that there exists an N such that $Rj_{U,*}DQ_U(K|_U)$, $Rj_{V,*}DQ_V(K|_V)$, and $Rj_{U \cap V,*}DQ_{U \cap V}(K|_{U \cap V})$ have vanishing cohomology sheaves outside the interval $[a, b + N]$. Then finally we conclude by the distinguished triangle of Remark 36.21.3. □

0CSB Example 36.21.5. Let X be a quasi-compact and quasi-separated scheme. Let $(\mathcal{F}_n)$ be an inverse system of quasi-coherent sheaves. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence we see that

$$DQ_X(R\lim \mathcal{F}_n) = (R\lim \text{ in } D_{QCoh}(\mathcal{O}_X))(\mathcal{F}_n)$$

where the first $R\lim$ is taken in $D(\mathcal{O}_X)$. In fact, let's write $K = R\lim \mathcal{F}_n$ for this. For any affine open $U \subset X$ we have

$$H^i(U, K) = H^i(R\Gamma(U, R\lim \mathcal{F}_n)) = H^i(R\lim R\Gamma(U, \mathcal{F}_n)) = H^i(R\lim \Gamma(U, \mathcal{F}_n))$$

since cohomology commutes with derived limits and since the quasi-coherent sheaves $\mathcal{F}_n$ have no higher cohomology on affines. By the computation of $R\lim$ in the category of abelian groups, we see that $H^i(U, K) = 0$ unless $i \in [0, 1]$. Then finally we conclude that the $R\lim$ in $D_{QCoh}(\mathcal{O}_X)$, which is $DQ_X(K)$ by the above, is in $D^b_{QCoh}(\mathcal{O}_X)$ by Lemma 36.21.4.

36.22. Cohomology and base change, IV

08ET This section continues the discussion of Cohomology of Schemes, Section 30.22. First, we have a very general version of the projection formula for quasi-compact and quasi-separated morphisms of schemes and complexes with quasi-coherent cohomology sheaves.

08EU Lemma 36.22.1. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ the map

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

defined in Cohomology, Equation (20.54.2.1) is an isomorphism.

Proof. To check the map is an isomorphism we may work locally on Y. Hence we reduce to the case that Y is affine.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each K_i, then it holds for K. Namely, the functors Lf^* and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 36.4.5. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \mathrm{Spec}(A)$. The functor $\widetilde{\ } : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence (Lemma 36.3.5). Let T be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 15.59.11 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. $\square$

08IA Definition 36.22.2. Let S be a scheme. Let X, Y be schemes over S. We say X and Y are Tor independent over S if for every $x \in X$ and $y \in Y$ mapping to the same point $s \in S$ the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are Tor independent over $\mathcal{O}_{S,s}$ (see More on Algebra, Definition 15.61.1).

0FXV Lemma 36.22.3. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. The following are equivalent

(1) X and Y are tor independent over S, and
(2) for every affine opens $U \subset X$, $V \subset Y$, $W \subset S$ with $f(U) \subset W$ and $g(V) \subset W$ the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$ are tor independent over $\mathcal{O}_S(W)$.
(3) there exists an affine open overing $S = \bigcup W_i$ and for each i affine open coverings $f^{-1}(W_i) = \bigcup U_{ij}$ and $g^{-1}(W_i) = \bigcup V_{ik}$ such that the rings $\mathcal{O}_X(U_{ij})$ and $\mathcal{O}_Y(V_{ik})$ are tor independent over $\mathcal{O}_S(W_i)$ for all i, j, k.

Proof. Omitted. Hint: use More on Algebra, Lemma 15.61.6. $\square$

0FXW Lemma 36.22.4. Let $X \to S$ and $Y \to S$ be morphisms of schemes. Let $S' \to S$ be a morphism of schemes and denote $X' = X \times_S S'$ and $Y' = Y \times_S S'$. If X and Y are tor independent over S and $S' \to S$ is flat, then X' and Y' are tor independent over S'.

Proof. Omitted. Hint: use Lemma 36.22.3 and on affine opens use More on Algebra, Lemma 15.61.4. $\square$

08IB Lemma 36.22.5. Let $g : S' \to S$ be a morphism of schemes. Let $f : X \to S$ be quasi-compact and quasi-separated. Consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

If X and S' are Tor independent over S, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*L(g')^*E = Lg^*Rf_*E$.

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology, Remark 20.28.3 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*L(g')^*E$. To check this is an isomorphism we may work locally on S'. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*L(g')^*E = Rf_*(Rg'_*L(g')^*E)$$

is an isomorphism, see Lemma 36.5.2 (and use Lemmas 36.3.8, 36.3.9, and 36.4.1 to see that the objects $Rf'_*L(g')^*E$ and Lg^*Rf_*E have quasi-coherent cohomology sheaves). Note that g' is affine as well (Morphisms, Lemma 29.11.8). By Lemma 36.5.3 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_S} g_*\mathcal{O}_{S'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} g'_*\mathcal{O}_{X'})$$

Observe that $g'_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{S'}$. Thus by Lemma 36.22.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$. This follows from our assumption that X and S' are Tor independent over S. Namely, to check it we may work locally on X, hence we may also assume X is affine. Say $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (More on Algebra, Lemma 15.61.6), i.e., $\text{Tor}_i^R(A, R') = 0$ for $i > 0$. In other words $A \otimes_R^{\mathbf{L}} R' = A \otimes_R R'$ which exactly means that $Lf^* g_* \mathcal{O}_{S'} = f^* g_* \mathcal{O}_{S'}$ (use Lemma 36.3.8). □

The following lemma will be used in the chapter on dualizing complexes.

0AA7 Lemma 36.22.6. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of quasi-compact and quasi-separated schemes. Assume g and f Tor independent and $S = \text{Spec}(R)$, $S' = \text{Spec}(R')$ affine. For $M, K \in D(\mathcal{O}_X)$ the canonical map

$$R\,\text{Hom}_X(M, K) \otimes_R^{\mathbf{L}} R' \longrightarrow R\,\text{Hom}_{X'}(L(g')^*M, L(g')^*K)$$

in $D(R')$ is an isomorphism in the following two cases

(1) $M \in D(\mathcal{O}_X)$ is perfect and $K \in D_{QCoh}(X)$, or
(2) $M \in D(\mathcal{O}_X)$ is pseudo-coherent, $K \in D^+_{QCoh}(X)$, and R' has finite tor dimension over R.

Proof. There is a canonical map $R\,\text{Hom}_X(M, K) \to R\,\text{Hom}_{X'}(L(g')^*M, L(g')^*K)$ in $D(\Gamma(X, \mathcal{O}_X))$ of global hom complexes, see Cohomology, Section 20.44. Restricting scalars we can view this as a map in $D(R)$. Then we can use the adjointness of restriction and $- \otimes_R^{\mathbf{L}} R'$ to get the displayed map of the lemma. Having defined the map it suffices to prove it is an isomorphism in the derived category of abelian groups.

The right hand side is equal to

$$R\,\text{Hom}_X(M, R(g')_*L(g')^*K) = R\,\text{Hom}_X(M, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'})$$

by Lemma 36.5.3. In both cases the complex $R\,\mathcal{H}om(M, K)$ is an object of $D_{QCoh}(\mathcal{O}_X)$ by Lemma 36.10.8. There is a natural map

$$R\,\mathcal{H}om(M, K) \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'} \longrightarrow R\,\mathcal{H}om(M, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'})$$

which is an isomorphism in both cases by Lemma 36.10.9. To see that this lemma applies in case (2) we note that $g'_*\mathcal{O}_{X'} = Rg'_*\mathcal{O}_{X'} = Lf^*g_*\mathcal{O}_X$ the second equality by Lemma 36.22.5. Using Lemma 36.10.4 and Cohomology, Lemma 20.48.4 we conclude that $g'_*\mathcal{O}_{X'}$ has finite Tor dimension. Hence, in both cases by replacing K by $R\,\mathcal{H}om(M, K)$ we reduce to proving

$$R\Gamma(X, K) \otimes_A^{\mathbf{L}} A' \longrightarrow R\Gamma(X, K \otimes_{\mathcal{O}_X}^{\mathbf{L}} g'_*\mathcal{O}_{X'})$$

is an isomorphism. Note that the left hand side is equal to $R\Gamma(X', L(g')^*K)$ by Lemma 36.5.3. Hence the result follows from Lemma 36.22.5. □

0BZA Remark 36.22.7. With notation as in Lemma 36.22.6. The diagram

$$\begin{tikzcd} R\,\text{Hom}_X(M, Rg'_*L) \otimes_R^{\mathbf{L}} R' \arrow[r] \arrow[d, "\mu"'] & R\,\text{Hom}_{X'}(L(g')^*M, L(g')^*Rg'_*L) \arrow[d, "a"] \\ R\,\text{Hom}_X(M, R(g')_*L) \arrow[r, equal] & R\,\text{Hom}_{X'}(L(g')^*M, L) \end{tikzcd}$$

is commutative where the top horizontal arrow is the map from the lemma, μ is the multiplication map, and a comes from the adjunction map $L(g')^*Rg'_*L \to L$. The multiplication map is the adjunction map $K' \otimes_R^{\mathbf{L}} R' \to K'$ for any $K' \in D(R')$.

0C0V Lemma 36.22.8. Consider a cartesian square of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Assume g and f Tor independent.

(1) If $E \in D(\mathcal{O}_X)$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_S$-modules, then $L(g')^*E$ has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_{S'}$-modules.
(2) If $\mathcal{G}$ is an $\mathcal{O}_X$-module flat over S, then $L(g')^*\mathcal{G} = (g')^*\mathcal{G}$.

Proof. We can compute tor dimension at stalks, see Cohomology, Lemma 20.48.5. If $x' \in X'$ with image $x \in X$, then

$$(L(g')^*E)_{x'} = E_x \otimes_{\mathcal{O}_{X,x}}^{\mathbf{L}} \mathcal{O}_{X',x'}$$

Let $s' \in S'$ and $s \in S$ be the image of x' and x. Since X and S' are tor independent over S, we can apply More on Algebra, Lemma 15.61.2 to see that the right hand side of the displayed formula is equal to $E_x \otimes_{\mathcal{O}_{S,s}}^{\mathbf{L}} \mathcal{O}_{S',s'}$ in $D(\mathcal{O}_{S',s'})$. Thus (1) follows from More on Algebra, Lemma 15.66.13. To see (2) observe that flatness of $\mathcal{G}$ is equivalent to the condition that $\mathcal{G}[0]$ has tor amplitude in $[0, 0]$. Applying (1) we conclude. $\square$

0E23 Lemma 36.22.9. Consider a cartesian diagram of schemes

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where i is a closed immersion. If Z and X' are tor independent over X, then $Ri'_* \circ Lg^* = Lf^* \circ Ri_*$ as functors $D(\mathcal{O}_Z) \to D(\mathcal{O}_{X'})$.

Proof. Note that the lemma is supposed to hold for all $K \in D(\mathcal{O}_Z)$. Observe that i_* and i'_* are exact functors and hence Ri_* and Ri'_* are computed by applying i_* and i'_* to any representatives. Thus the base change map

$$Lf^*(Ri_*(K)) \longrightarrow Ri'_*(Lg^*(K))$$

on stalks at a point $z' \in Z'$ with image $z \in Z$ is given by

$$K_z \otimes_{\mathcal{O}_{X,z}}^{\mathbf{L}} \mathcal{O}_{X',z'} \longrightarrow K_z \otimes_{\mathcal{O}_{Z,z}}^{\mathbf{L}} \mathcal{O}_{Z',z'}$$

This map is an isomorphism by More on Algebra, Lemma 15.61.2 and the assumed tor independence. $\square$

36.23. Künneth formula, II

0FLN For the case where the base is a field, please see Varieties, Section 33.29. Consider a cartesian diagram of schemes

$$
\begin{tikzcd}
 & X \times_S Y \arrow[ld, "p"'] \arrow[d, "f"] \arrow[rd, "q"] & \\
X \arrow[rd, "a"'] & S & Y \arrow[ld, "b"] \\
 & S &
\end{tikzcd}
$$

Let $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$. There is a canonical map

0FLP (36.23.0.1)
$$Ra_*K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*M \longrightarrow Rf_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M)$$

Namely, we can use the maps $Ra_*K \to Ra_*Rp_*Lp^*K = Rf_*Lp^*K$ and $Rb_*M \to Rb_*Rq_*Lq^*M = Rf_*Lq^*M$ and then we can use the relative cup product (Cohomology, Remark 20.28.7).

Set $A = \Gamma(S, \mathcal{O}_S)$. There is a global Künneth map

0G7V (36.23.0.2)
$$R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma(X \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M)$$

in $D(A)$. This map is constructed using the pullback maps $R\Gamma(X, K) \to R\Gamma(X \times_S Y, Lp^*K)$ and $R\Gamma(Y, M) \to R\Gamma(X \times_S Y, Lq^*M)$ and the cup product constructed in Cohomology, Section 20.31.

0FLQ **Lemma 36.23.1.** *In the situation above, if a and b are quasi-compact and quasi-separated and X and Y are tor-independent over S, then (36.23.0.1) is an isomorphism for $K \in D_{QCoh}(\mathcal{O}_X)$ and $M \in D_{QCoh}(\mathcal{O}_Y)$. If in addition $S = \mathrm{Spec}(A)$ is affine, then the map (36.23.0.2) is an isomorphism.*

First proof. This follows from the following sequence of isomorphisms

$$
\begin{aligned}
Rf_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M) &= Ra_*Rp_*(Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times_S Y}} Lq^*M) \\
&= Ra_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Rp_*Lq^*M) \\
&= Ra_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} La^*Rb_*M) \\
&= Ra_*K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*M
\end{aligned}
$$

The first equality holds because $f = a \circ p$. The second equality by Lemma 36.22.1. The third equality by Lemma 36.22.5. The fourth equality by Lemma 36.22.1. We omit the verification that the composition of these isomorphisms is the same as the map (36.23.0.1). If S is affine, then the source and target of the arrow (36.23.0.2) are the result of applying $R\Gamma(S, -)$ to the source and target of (36.23.0.1) and we obtain the final statement; details omitted. □

Second proof. The construction of the arrow (36.23.0.1) is compatible with restricting to open subschemes of S as is immediate from the construction of the relative cup product. Thus it suffices to prove that (36.23.0.1) is an isomorphism when S is affine.

Assume $S = \mathrm{Spec}(A)$ is affine. By Leray we have $R\Gamma(S, Rf_*K) = R\Gamma(X, K)$ and similarly for the other cases. By Cohomology, Lemma 20.31.7 the map (36.23.0.1) induces the map (36.23.0.2) on taking $R\Gamma(S, -)$. On the other hand, by Lemmas 36.4.1 and 36.3.9 the source and target of the map (36.23.0.1) are in $D_{QCoh}(\mathcal{O}_S)$. Thus, by Lemma 36.3.5, it suffices to prove that (36.23.0.2) is an isomorphism.

Assume $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(C)$ are all affine. We will use Lemma 36.3.5 without further mention. In this case we can choose a K-flat complex $K^\bullet$ of B-modules whose terms are flat such that K is represented by $\widetilde{K}^\bullet$. Similarly, we can choose a K-flat complex $M^\bullet$ of C-modules whose terms are flat such that M is represented by $\widetilde{M}^\bullet$. See More on Algebra, Lemma 15.59.10. Then $\widetilde{K}^\bullet$ is a K-flat complex of $\mathcal{O}_X$-modules and similarly for $\widetilde{M}^\bullet$, see Lemma 36.3.6. Thus La^*K is represented by

$$a^*\widetilde{K}^\bullet = \widetilde{K^\bullet \otimes_A C}$$

and similarly for Lb^*M. This in turn is a K-flat complex of $\mathcal{O}_{X \times_S Y}$-modules by the lemma cited above and More on Algebra, Lemma 15.59.3. Thus we finally see that the complex of $\mathcal{O}_{X \times_S Y}$-modules associated to

$$\mathrm{Tot}((K^\bullet \otimes_A C) \otimes_{B \otimes_A C} (B \otimes_A M^\bullet)) = \mathrm{Tot}(K^\bullet \otimes_A M^\bullet)$$

represents $La^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lb^*M$ in the derived category of $X \times_S Y$. Taking global sections we obtain $\mathrm{Tot}(K^\bullet \otimes_A M^\bullet)$ which of course is also the complex representing $R\Gamma(X, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M)$. The fact that the isomorphism is given by cup product follows from the relationship between the genuine cup product and the naive one in Cohomology, Section 20.31 (and in particular Cohomology, Lemma 20.31.3 and the discussion following it).

Assume $S = \mathrm{Spec}(A)$ and Y are affine. We will use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove the statement. To do this we only have to show: if $X = U \cup V$ is an open covering with U and V quasi-compact and if the map (36.23.0.2)

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma(U \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for U and Y over S, the map (36.23.0.2)

$$R\Gamma(V, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma(V \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for V and Y over S, and the map (36.23.0.2)

$$R\Gamma(U \cap V, K) \otimes^{\mathbf{L}}_A R\Gamma(Y, M) \longrightarrow R\Gamma((U \cap V) \times_S Y, Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M)$$

for $U \cap V$ and Y over S are isomorphisms, then so is the map (36.23.0.2) for X and Y over S. However, by Cohomology, Lemma 20.33.7 these maps fit into a map of distinguished triangles with (36.23.0.2) the final leg and hence we conclude by Derived Categories, Lemma 13.4.3.

Assume $S = \mathrm{Spec}(A)$ is affine. To finish the proof we can use the induction principle of Cohomology of Schemes, Lemma 30.4.1 on Y. Namely, by the above we already know that our map is an isomorphism when Y is affine. The rest of the argument is exactly the same as in the previous paragraph but with the roles of X and Y switched. □

0FML Lemma 36.23.2. Let $a : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}^\bullet$ be a locally bounded complex of $a^{-1}\mathcal{O}_S$-modules. Assume for all $n \in \mathbf{Z}$ the sheaf $\mathcal{F}^n$ is a flat $a^{-1}\mathcal{O}_S$-module and $\mathcal{F}^n$ has the structure of a quasi-coherent $\mathcal{O}_X$-module compatible with the given $a^{-1}\mathcal{O}_S$-module structure (but the differentials in the complex $\mathcal{F}^\bullet$ need not be $\mathcal{O}_X$-linear). Then the following hold

(1) $Ra_*\mathcal{F}^\bullet$ is locally bounded,
(2) $Ra_*\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_S)$,
(3) $Ra_*\mathcal{F}^\bullet$ locally has finite tor dimension,
(4) $\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra_*\mathcal{F}^\bullet = Ra_*(a^{-1}\mathcal{G} \otimes_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ for $\mathcal{G} \in QCoh(\mathcal{O}_S)$, and
(5) $K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra_*\mathcal{F}^\bullet = Ra_*(a^{-1}K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ for $K \in D_{QCoh}(\mathcal{O}_S)$.

Proof. Parts (1), (2), (3) are local on S hence we may and do assume S is affine. Since a is quasi-compact we conclude that X is quasi-compact. Since $\mathcal{F}^\bullet$ is locally bounded, we conclude that $\mathcal{F}^\bullet$ is bounded.

For (1) and (2) we can use the first spectral sequence $R^pa_*\mathcal{F}^q \Rightarrow R^{p+q}a_*\mathcal{F}^\bullet$ of Derived Categories, Lemma 13.21.3. Combining Cohomology of Schemes, Lemma 30.4.5 and Homology, Lemma 12.24.11 we conclude.

Let us prove (3) by the induction principle of Cohomology of Schemes, Lemma 30.4.1. Namely, for a quasi-compact open of U of X consider the condition that $R(a|_U)_*(\mathcal{F}^\bullet|_U)$ has finite tor dimension. If U, V are quasi-compact open in X, then we have a relative Mayer-Vietoris distinguished triangle

$$R(a|_{U\cup V})_*\mathcal{F}^\bullet|_{U\cup V} \to R(a|_U)_*\mathcal{F}^\bullet|_U \oplus R(a|_V)_*\mathcal{F}^\bullet|_V \to R(a|_{U\cap V})_*\mathcal{F}^\bullet|_{U\cap V} \to$$

by Cohomology, Lemma 20.33.5. By the behaviour of tor amplitude in distinguished triangles (see Cohomology, Lemma 20.48.6) we see that if we know the result for U, V, $U \cap V$, then the result holds for $U \cup V$. This reduces us to the case where X is affine. In this case we have

$$Ra_*\mathcal{F}^\bullet = a_*\mathcal{F}^\bullet$$

by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and the vanishing of higher direct images of quasi-coherent modules under an affine morphism (Cohomology of Schemes, Lemma 30.2.3). Since $\mathcal{F}^n$ is S-flat by assumption and X affine, the modules $a_*\mathcal{F}^n$ are flat for all n. Hence $a_*\mathcal{F}^\bullet$ is a bounded complex of flat $\mathcal{O}_S$-modules and hence has finite tor dimension.

Proof of part (5). Denote $a' : (X, a^{-1}\mathcal{O}_S) \to (S, \mathcal{O}_S)$ the obvious flat morphism of ringed spaces. Part (5) says that

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} Ra'_*\mathcal{F}^\bullet = Ra'_*(L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$$

Thus Cohomology, Equation (20.54.2.1) gives a functorial map from the left to the right and we want to show this map is an isomorphism. This question is local on S hence we may and do assume S is affine. The rest of the proof is exactly the same as the proof of Lemma 36.22.1 except that we have to show that the functor $K \mapsto Ra'_*(L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet)$ commutes with direct sums. This is where we will use $\mathcal{F}^n$ has the structure of a quasi-coherent $\mathcal{O}_X$-module. Namely, observe that $K \mapsto L(a')^*K \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ commutes with arbitrary direct sums. Next, if $\mathcal{F}^\bullet$ consists of a single quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}^\bullet = \mathcal{F}^n[-n]$ then we have $L(a')^*G \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet = La^*K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^n[-n]$, see Cohomology, Lemma 20.27.4. Hence in this case the commutation with direct sums follows from Lemma 36.4.5. Now, in general, since S is affine (hence X quasi-compact) and $\mathcal{F}^\bullet$ is locally bounded, we see that

$$\mathcal{F}^\bullet = (\mathcal{F}^a \to \ldots \to \mathcal{F}^b)$$

is bounded. Arguing by induction on $b-a$ and considering the distinguished triangle

$$\mathcal{F}^b[-b] \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^b) \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}) \to \mathcal{F}^b[-b+1]$$

the proof of this part is finished. Some details omitted.

Proof of part (4). Let $a' : (X, a^{-1}\mathcal{O}_S) \to (S, \mathcal{O}_S)$ be as above. Since $\mathcal{F}^\bullet$ is a locally bounded complex of flat $a^{-1}\mathcal{O}_S$-modules we see the complex $a^{-1}\mathcal{G} \otimes_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ represents $L(a')^*\mathcal{G} \otimes^{\mathbf{L}}_{a^{-1}\mathcal{O}_S} \mathcal{F}^\bullet$ in $D(a^{-1}\mathcal{O}_S)$. Hence (4) follows from (5). $\square$

0FMQ Lemma 36.23.3. Let $f : X \to Y$ be a morphism of schemes with $Y = \text{Spec}(A)$ affine. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite affine open covering such that all the finite intersections $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ are affine. Let $\mathcal{F}^\bullet$ be a bounded complex of $f^{-1}\mathcal{O}_Y$-modules. Assume for all $n \in \mathbf{Z}$ the sheaf $\mathcal{F}^n$ is a flat $f^{-1}\mathcal{O}_Y$-module and $\mathcal{F}^n$ has the structure of a quasi-coherent $\mathcal{O}_X$-module compatible with the given $p^{-1}\mathcal{O}_Y$-module structure (but the differentials in the complex $\mathcal{F}^\bullet$ need not be $\mathcal{O}_X$-linear). Then the complex $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is K-flat as a complex of A-modules.

Proof. We may write

$$\mathcal{F}^\bullet = (\mathcal{F}^a \to \ldots \to \mathcal{F}^b)$$

Arguing by induction on $b - a$ and considering the distinguished triangle

$$\mathcal{F}^b[-b] \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^b) \to (\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}) \to \mathcal{F}^b[-b+1]$$

and using More on Algebra, Lemma 15.59.5 we reduce to the case where $\mathcal{F}^\bullet$ consists of a single quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ placed in degree 0. In this case the Čech complex for $\mathcal{F}$ and $\mathcal{U}$ is homotopy equivalent to the alternating Čech complex, see Cohomology, Lemma 20.23.6. Since $U_{i_0 \ldots i_p}$ is always affine, we see that $\mathcal{F}(U_{i_0 \ldots i_p})$ is A-flat. Hence $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is a bounded complex of flat A-modules and hence K-flat by More on Algebra, Lemma 15.59.7. $\square$

Let X, Y, S, a, b, p, q, f be as in the introduction to this section. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Set $A = \Gamma(S, \mathcal{O}_S)$. Consider the map

0G49 (36.23.3.1)
$$R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

in $D(A)$. This map is constructed using the pullback maps $R\Gamma(X, \mathcal{F}) \to R\Gamma(X \times_S Y, p^*\mathcal{F})$ and $R\Gamma(Y, \mathcal{G}) \to R\Gamma(X \times_S Y, q^*\mathcal{G})$, the cup product constructed in Cohomology, Section 20.31, and the canonical map $p^*\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G} \to p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}$.

0FU4 Lemma 36.23.4. In the situation above the map (36.23.3.1) is an isomorphism if S is affine, $\mathcal{F}$ and $\mathcal{G}$ are S-flat and quasi-coherent and X and Y are quasi-compact with affine diagonal.

Proof. We strongly urge the reader to read the proof of Varieties, Lemma 33.29.1 first. Choose finite affine open coverings $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$. This determines an affine open covering $\mathcal{W} : X \times_S Y = \bigcup_{(i,j) \in I \times J} U_i \times_S V_j$. Note that $\mathcal{W}$ is a refinement of $\text{pr}_1^{-1}\mathcal{U}$ and of $\text{pr}_2^{-1}\mathcal{V}$. Thus by the discussion in Cohomology, Section 20.25 we obtain maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}) \quad \text{and} \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, q^*\mathcal{G})$$

well defined up to homotopy and compatible with pullback maps on cohomology. In Cohomology, Equation (20.25.3.2) we have constructed a map of complexes

$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{W}, q^*\mathcal{G})) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

which is compatible with the cup product on cohomology by Cohomology, Lemma 20.31.4. Combining the above we obtain a map of complexes

0FLU (36.23.4.1)
$$\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

We claim this is the map in the statement of the lemma, i.e., the source and target of this arrow are the same as the source and target of (36.23.3.1). Namely, by Cohomology of Schemes, Lemma 30.2.2 and Cohomology, Lemma 20.25.2 the canonical maps

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to R\Gamma(X, \mathcal{F}), \quad \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}) \to R\Gamma(Y, \mathcal{G})$$

and

$$\check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G})$$

are isomorphisms. On the other hand, the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is K-flat by Lemma 36.23.3 and we conclude that $\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G}))$ represents the derived tensor product $R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G})$ as claimed.

We still have to show that (36.23.4.1) is a quasi-isomorphism. We will do this using dimension shifting. Set $d(\mathcal{F}) = \max\{d \mid H^d(X, \mathcal{F}) \neq 0\}$. Assume $d(\mathcal{F}) > 0$. Set $U = \coprod_{i \in I} U_i$. This is an affine scheme as I is finite. Denote $j : U \to X$ the morphism which is the inclusion $U_i \to X$ on each U_i. Since the diagonal of X is affine, the morphism j is affine, see Morphisms, Lemma 29.11.11. It follows that $\mathcal{F}' = j_*j^*\mathcal{F}$ is S-flat, see Morphisms, Lemma 29.25.4. It also follows that $d(\mathcal{F}') = 0$ by combining Cohomology of Schemes, Lemmas 30.2.4 and 30.2.2. For all $x \in X$ we have $\mathcal{F}_x \to \mathcal{F}'_x$ is the inclusion of a direct summand: if $x \in U_i$, then $\mathcal{F}' \to (U_i \to X)_*\mathcal{F}|_{U_i}$ gives a splitting. We conclude that $\mathcal{F} \to \mathcal{F}'$ is injective and $\mathcal{F}'' = \mathcal{F}'/\mathcal{F}$ is S-flat as well. The short exact sequence $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ of flat quasi-coherent $\mathcal{O}_X$-modules produces a short exact sequence of complexes

$$0 \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}') \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to \text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}'') \otimes_A \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})) \to 0$$

and a short exact sequence of complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}' \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to \check{\mathcal{C}}^\bullet(\mathcal{W}, p^*\mathcal{F}'' \otimes_{\mathcal{O}_{X \times_S Y}} q^*\mathcal{G}) \to 0$$

Moreover, the maps (36.23.4.1) between these are compatible with these short exact sequences. Hence it suffices to prove (36.23.4.1) is an isomorphism for $\mathcal{F}'$ and $\mathcal{F}''$. Finally, we have $d(\mathcal{F}'') < d(\mathcal{F})$. In this way we reduce to the case $d(\mathcal{F}) = 0$.

Arguing in the same fashion for $\mathcal{G}$ we find that we may assume that both $\mathcal{F}$ and $\mathcal{G}$ have nonzero cohomology only in degree 0. Observe that this means that $\Gamma(X, \mathcal{F})$ is quasi-isomorphic to the K-flat complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of A-modules sitting in degrees ≥ 0. It follows that $\Gamma(X, \mathcal{F})$ is a flat A-module (because we can compute higher Tor's against this module by tensoring with the Cech complex). Let $V \subset Y$ be an affine open. Consider the affine open covering $\mathcal{U}_V : X \times_S V = \bigcup_{i \in I} U_i \times_S V$. It is immediate that

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \mathcal{G}(V) = \check{\mathcal{C}}^\bullet(\mathcal{U}_V, p^*\mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} q^*\mathcal{G})$$

(equality of complexes). By the flatness of $\mathcal{G}(V)$ over A we see that $\Gamma(X, \mathcal{F}) \otimes_A \mathcal{G}(V) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A \mathcal{G}(V)$ is a quasi-isomorphism. Since the sheafification of

$V \mapsto \check{\mathcal{C}}^\bullet(\mathcal{U}_V, p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{G})$ represents $Rq_*(p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{G})$ by Cohomology of Schemes, Lemma 30.7.1 we conclude that

$$Rq_*(p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes_A \mathcal{G}$$

on Y where the notation on the right hand side indicates the module

$$b^*\widetilde{\Gamma(X, \mathcal{F})} \otimes_{\mathcal{O}_Y} \mathcal{G}$$

Using the Leray spectral sequence for q we find

$$H^n(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{G}) = H^n(Y, b^*\widetilde{\Gamma(X, \mathcal{F})} \otimes_{\mathcal{O}_Y} \mathcal{G})$$

Using Lemma 36.22.1 for the morphism $b : Y \to S = \text{Spec}(A)$ and using that $\Gamma(X, \mathcal{F})$ is A-flat we conclude that $H^n(X \times_S Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{G})$ is zero for $n > 0$ and isomorphic to $H^0(X, \mathcal{F}) \otimes_A H^0(Y, \mathcal{G})$ for $n = 0$. Of course, here we also use that $\mathcal{G}$ only has cohomology in degree 0. This finishes the proof (except that we should check that the isomorphism is indeed given by cup product in degree 0; we omit the verification). $\square$

0G7W Remark 36.23.5. Let $S = \text{Spec}(A)$ be an affine scheme. Let $a : X \to S$ and $b : Y \to S$ be morphisms of schemes. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules and let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $\xi \in H^i(X, \mathcal{G})$ with pullback $p^*\xi \in H^i(X \times_S Y, p^*\mathcal{G})$. Then the following diagram is commutative

$$\begin{tikzcd}
R\Gamma(X, \mathcal{F})[-i] \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{E}) \arrow[r, "\xi\otimes\mathrm{id}"'] \arrow[d] & R\Gamma(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{E}) \arrow[d] \\
R\Gamma(X \times_S Y, p^*\mathcal{F} \otimes q^*\mathcal{E})[-i] \arrow[r, "p^*\xi"] & R\Gamma(X \times_S Y, p^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes q^*\mathcal{E})
\end{tikzcd}$$

where the unadorned tensor products are over $\mathcal{O}_{X\times_S Y}$. The horizontal arrows are from Cohomology, Remark 20.31.2 and the vertical arrows are (36.23.0.2) hence given by pulling back followed by cup product on $X \times_S Y$. The diagram commutes because the global cup product (on $X \times_S Y$ with the sheaves $p^*\mathcal{G}$, $p^*\mathcal{F}$, and $q^*\mathcal{E}$) is associative, see Cohomology, Lemma 20.31.5.

36.24. Künneth formula, III

0G4A Let X, Y, S, a, b, p, q, f be as in the introduction to Section 36.23. In this section, given an $\mathcal{O}_X$-module $\mathcal{F}$ and a $\mathcal{O}_Y$-module $\mathcal{G}$ let us set

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_S Y}} q^*\mathcal{G}$$

Note that, contrary to what happens in a future section, we take the nonderived tensor product here.

On X let $\mathcal{F}^\bullet$ be a complex of sheaves of abelian groups whose terms are quasi-coherent $\mathcal{O}_X$-modules such that the differentials $d^i_\mathcal{F} : \mathcal{F}^i \to \mathcal{F}^{i+1}$ are differential operators on X/S of finite order, see Morphisms, Section 29.33. Simlarly, on Y let $\mathcal{G}^\bullet$ be a complex of sheaves of abelian groups whose terms are quasi-coherent $\mathcal{O}_Y$-modules such that the differentials $d^j_\mathcal{G} : \mathcal{G}^j \to \mathcal{G}^{j+1}$ are differential operators on Y/S of finite order. Applying the construction of Morphisms, Lemma 29.33.2 we obtain a double complex

$$\begin{tikzcd} \ldots & \ldots & \ldots & \ldots \\ \ldots \arrow[r] & \mathcal{F}^i \boxtimes \mathcal{G}^{j+1} \arrow[r, "d_1^{i,j+1}"] \arrow[u] & \mathcal{F}^{i+1} \boxtimes \mathcal{G}^{j+1} \arrow[r] \arrow[u] & \ldots \\ \ldots \arrow[r] & \mathcal{F}^i \boxtimes \mathcal{G}^j \arrow[r, "d_1^{i,j}"] \arrow[u, "d_2^{i,j}"] & \mathcal{F}^{i+1} \boxtimes \mathcal{G}^j \arrow[r] \arrow[u, "d_2^{i+1,j}"'] & \ldots \\ \ldots & \ldots \arrow[u] & \ldots \arrow[u] & \ldots \end{tikzcd}$$

of quasi-coherent modules whose maps are differential operators of finite order on $X \times_S Y/S$. Please see the discussion in Morphisms, Remark 29.33.3 and Homology, Example 12.18.2. To be explicit, we set

$$d_1^{i,j} = d_{\mathcal{F}}^i \boxtimes 1 \quad \text{and} \quad d_2^{i,j} = 1 \boxtimes d_{\mathcal{G}}^j$$

In the discussion below the notation

$$\text{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$$

refers to the total complex associated to this double complex. This complex has terms which are quasi-coherent $\mathcal{O}_{X \times_S Y}$-modules and whose differentials are differential operators of finite order on $X \times_S Y/S$.

In the situation above there exists a "relative cup product" map

0G4B (36.24.0.1)
$$Ra_*(\mathcal{F}^\bullet) \otimes_{\mathcal{O}_S}^{\mathbf{L}} Rb_*(\mathcal{G}^\bullet) \longrightarrow Rf_* \left(\text{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)\right)$$

Namely, we can construct this map by combining

(1) $Ra_*(\mathcal{F}^\bullet) \to Rf_*(p^{-1}\mathcal{F}^\bullet)$,
(2) $Rb_*(\mathcal{G}^\bullet) \to Rf_*(q^{-1}\mathcal{G}^\bullet)$,
(3) $Rf_*(p^{-1}\mathcal{F}^\bullet) \otimes_{\mathcal{O}_S}^{\mathbf{L}} Rf_*(q^{-1}\mathcal{G}^\bullet) \to Rf_*(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} q^{-1}\mathcal{G}^\bullet)$,
(4) $p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} q^{-1}\mathcal{G}^\bullet \to \text{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet)$
(5) $\text{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet) \to \text{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$.

Maps (1) and (2) are pullback maps, map (3) is the relative cup product, see Cohomology, Remark 20.28.7, map (4) compares the derived and nonderived tensor products, and map (5) is given by the obvious maps $p^{-1}\mathcal{F}^i \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^j \to \mathcal{F}^i \boxtimes \mathcal{G}^j$ on the underlying double complexes.

Set $A = \Gamma(S, \mathcal{O}_S)$. There exists a "global cup product" map

0FLR (36.24.0.2)
$$R\Gamma(X, \mathcal{F}^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \longrightarrow R\Gamma(X \times_S Y, \text{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet))$$

in $D(A)$. This is constructed similarly to the relative cup product above using

(1) $R\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet)$
(2) $R\Gamma(Y, \mathcal{G}^\bullet) \to R\Gamma(X \times_S Y, q^{-1}\mathcal{G}^\bullet)$,
(3) $R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(X \times_S Y, q^{-1}\mathcal{G}^\bullet) \to R\Gamma(X \times_S Y, p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} q^{-1}\mathcal{G}^\bullet)$,
(4) $p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} q^{-1}\mathcal{G}^\bullet \to \text{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet)$
(5) $\text{Tot}(p^{-1}\mathcal{F}^\bullet \otimes_{f^{-1}\mathcal{O}_S} q^{-1}\mathcal{G}^\bullet) \to \text{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$.

Here maps (1) and (2) are the pullback maps, map (3) is the cup product constructed in Cohomology, Section 20.31. Maps (4) and (5) are as indicated in the previous paragraph.

0FLT Lemma 36.24.1. In the situation above the cup product (36.24.0.2) is an isomorphism in $D(A)$ if the following assumptions hold

(1) $S = \mathrm{Spec}(A)$ is affine,
(2) X and Y are quasi-compact with affine diagonal,
(3) $\mathcal{F}^\bullet$ is bounded,
(4) $\mathcal{G}^\bullet$ is bounded below,
(5) $\mathcal{F}^n$ is S-flat, and
(6) $\mathcal{G}^m$ is S-flat.

Proof. We will use the notation $\mathcal{A}_{X/S}$ and $\mathcal{A}_{Y/S}$ introduced in Morphisms, Remark 29.33.3. Suppose that we have maps of complexes

$$\mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to \mathcal{F}_1^\bullet[1]$$

in the category $\mathcal{A}_{X/S}$. Then by the functoriality of the cup product we obtain a commutative diagram

$$\begin{tikzcd}
R\Gamma(X, \mathcal{F}_1^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \arrow[r] \arrow[d] & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_1^\bullet \boxtimes \mathcal{G}^\bullet)) \arrow[d] \\
R\Gamma(X, \mathcal{F}_2^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \arrow[r] \arrow[d] & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_2^\bullet \boxtimes \mathcal{G}^\bullet)) \arrow[d] \\
R\Gamma(X, \mathcal{F}_3^\bullet) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \arrow[r] \arrow[d] & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_3^\bullet \boxtimes \mathcal{G}^\bullet)) \arrow[d] \\
R\Gamma(X, \mathcal{F}_1^\bullet[1]) \otimes_A^{\mathbf{L}} R\Gamma(Y, \mathcal{G}^\bullet) \arrow[r] & R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F}_1^\bullet[1] \boxtimes \mathcal{G}^\bullet))
\end{tikzcd}$$

If the original maps form a distinguished triangle in the homotopy category of $\mathcal{A}_{X/S}$, then the columns of this diagram form distinguished triangles in $D(A)$.

In the situation of the lemma, suppose that $\mathcal{F}^n = 0$ for $n < i$. Then we may consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq i+1}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}^i[-i] \to 0$$

where the truncation is as in Homology, Section 12.15. This produces the distinguished triangle

$$\sigma_{\geq i+1}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}^i[-i] \to (\sigma_{\geq i+1}\mathcal{F}^\bullet)[1]$$

in the homotopy category of $\mathcal{A}_{X/S}$ where the final arrow is given by the boundary map $\mathcal{F}^i \to \mathcal{F}^{i+1}$. It follows from the discussion above that it suffices to prove the lemma for $\mathcal{F}^i[-i]$ and $\sigma_{\geq i+1}\mathcal{F}^\bullet$. Since $\sigma_{\geq i+1}\mathcal{F}^\bullet$ has fewer nonzero terms, by induction, if we can prove the lemma if $\mathcal{F}^\bullet$ is nonzero only in single degree, then the lemma follows. Thus we may assume $\mathcal{F}^\bullet$ is nonzero only in one degree.

Assume $\mathcal{F}^\bullet$ is the complex which has an S-flat quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ sitting in degree 0 and is zero in other degrees. Observe that $R\Gamma(X, \mathcal{F})$ has finite tor dimension by Lemma 36.23.2 for example. Say it has tor amplitude in $[i, j]$. Pick $N \gg 0$ and consider the distinguished triangle

$$\sigma_{\geq N+1}\mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \sigma_{\leq N}\mathcal{G}^\bullet \to (\sigma_{\geq N+1}\mathcal{G}^\bullet)[1]$$

in the homotopy category of $\mathcal{A}_{Y/S}$. Now observe that both

$$R\Gamma(X, \mathcal{F}) \otimes_A^{\mathbf{L}} R\Gamma(Y, \sigma_{\geq N+1}\mathcal{G}^\bullet) \quad \text{and} \quad R\Gamma(X \times_S Y, \mathrm{Tot}(\mathcal{F} \boxtimes \sigma_{\geq N+1}\mathcal{G}^\bullet))$$

have vanishing cohomology in degrees $\leq N + i$. Thus, using the arguments given above, if we want to prove our statement in a given degree, then we may assume $\mathcal{G}^\bullet$ is bounded. Repeating the arguments above one more time we may also assume $\mathcal{G}^\bullet$ is nonzero only in one degree. This case is handled by Lemma 36.23.4. □

36.25. Künneth formula for Ext

0FXX Consider a cartesian diagram of schemes

$$\begin{tikzcd} & X \times_S Y \arrow[ld, "p"'] \arrow[d, "f"] \arrow[rd, "q"] & \\ X \arrow[rd, "a"'] & \arrow[d] & Y \arrow[ld, "b"] \\ & S & \end{tikzcd}$$

For $K \in D(\mathcal{O}_X)$ and $M \in D(\mathcal{O}_Y)$ in this section let us define

$$K \boxtimes M = Lp^*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*M$$

We claim there is a canonical map

(36.25.0.1)

0FXY $$Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*R\mathcal{H}om(M, M') \longrightarrow Rf_*(R\mathcal{H}om(K \boxtimes M, K' \boxtimes M'))$$

for $K, K' \in D(\mathcal{O}_X)$ and $M, M' \in D(\mathcal{O}_Y)$. Namely, we can take the map adjoint to the map

$$\begin{gathered} Lf^*\left(Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rb_*R\mathcal{H}om(M, M')\right) = \\ Lf^*Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lf^*Rb_*R\mathcal{H}om(M, M') = \\ Lp^*La^*Ra_*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*Lb^*Rb_*R\mathcal{H}om(M, M') \to \\ Lp^*R\mathcal{H}om(K, K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} Lq^*R\mathcal{H}om(M, M') \to \\ R\mathcal{H}om(Lp^*K, Lp^*K') \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_S Y}} R\mathcal{H}om(Lq^*M, Lq^*M') \to \\ R\mathcal{H}om(K \boxtimes M, K' \boxtimes M') \end{gathered}$$

Here the first equality is compatibility of pullbacks with tensor products, Cohomology, Lemma 20.27.3. The second equality is $f = a \circ p = b \circ q$ and composition of pullbacks, Cohomology, Lemma 20.27.2. The first arrow is given by the adjunction maps $La^*Ra_* \to \mathrm{id}$ and $Lb^*Rb_* \to \mathrm{id}$ because pushforward and pullback are adjoint, Cohomology, Lemma 20.28.1. The second arrow is given by Cohomology, Remark 20.42.13. The third and final arrow is Cohomology, Remark 20.42.10. A simple special case of this is the following result.

0FXZ **Lemma 36.25.1.** *In the situation above, assume a and b are quasi-compact and quasi-separated and X and Y are tor independent over S. If K is perfect, $K' \in D_{QCoh}(\mathcal{O}_X)$, M is perfect, and $M' \in D_{QCoh}(\mathcal{O}_Y)$, then (36.25.0.1) is an isomorphism.*

Proof. In this case we have $R\mathcal{H}om(K, K') = K' \otimes^{\mathbf{L}} K^\vee$, $R\mathcal{H}om(M, M') = M' \otimes^{\mathbf{L}} M^\vee$, and

$$R\mathcal{H}om(K \boxtimes M, K' \boxtimes M') = (K' \otimes^{\mathbf{L}} K^\vee) \boxtimes (M' \otimes^{\mathbf{L}} M^\vee)$$

See Cohomology, Lemma 20.50.5 and we also use that being perfect is preserved by pullback and by tensor products. Hence this case follows from Lemma 36.23.1. (We omit the verification that with these identifications we obtain the same map.) $\square$

36.26. Cohomology and base change, V

0DJ6 In Section 36.22 we saw a base change theorem holds when the morphisms are tor independent. Even in the affine case there cannot be a base change theorem without such a condition, see More on Algebra, Section 15.61. In this section we analyze when one can get a base change result "one complex at a time".

To make this work, suppose we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes (usually we will assume it is cartesian). Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. For a point $x' \in X'$ set $x = g'(x') \in X$, $s' = f'(x') \in S'$ and $s = f(x) = g(s')$. Then we can consider the maps

$$K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'} \to K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'} \to K'_{x'}$$

where the first arrow is More on Algebra, Equation (15.61.0.1) and the second comes from $(L(g')^*K)_{x'} = K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ and the given map $L(g')^*K \to K'$. For each $i \in \mathbf{Z}$ we obtain a $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$-module structure on $H^i(K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'})$. Putting everything together we obtain canonical maps

0DJ7 (36.26.0.1)
$$H^i(K_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}) \otimes_{(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'})} \mathcal{O}_{X',x'} \longrightarrow H^i(K'_{x'})$$

of $\mathcal{O}_{X',x'}$-modules.

0DJ8 Lemma 36.26.1. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. The following are equivalent

1. for any $x' \in X'$ and $i \in \mathbf{Z}$ the map (36.26.0.1) is an isomorphism,
2. for $U \subset X$, $V' \subset S'$ affine open both mapping into the affine open $V \subset S$ with $U' = V' \times_V U$ the composition
$$R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_S(U)} \mathcal{O}_{S'}(V') \to R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(U)} \mathcal{O}_{X'}(U') \to R\Gamma(U', K')$$
is an isomorphism in $D(\mathcal{O}_{S'}(V'))$, and
3. there is a set I of quadruples U_i, V'_i, V_i, U'_i, $i \in I$ as in (2) with $X' = \bigcup U'_i$.

Proof. The second arrow in (2) comes from the equality

$$R\Gamma(U, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(U)} \mathcal{O}_{X'}(U') = R\Gamma(U', L(g')^*K)$$

of Lemma 36.3.8 and the given arrow $L(g')^*K \to K'$. The first arrow of (2) is More on Algebra, Equation (15.61.0.1). It is clear that (2) implies (3). Observe that (1) is local on X'. Therefore it suffices to show that if X, S, S', X' are affine, then (1) is equivalent to the condition that

$$R\Gamma(X, K) \otimes^{\mathbf{L}}_{\mathcal{O}_S(S)} \mathcal{O}_{S'}(S') \to R\Gamma(X, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X(X)} \mathcal{O}_{X'}(X') \to R\Gamma(X', K')$$

is an isomorphism in $D(\mathcal{O}_{S'}(S'))$. Say $S = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(R')$, $X' = \mathrm{Spec}(A')$, K corresponds to the complex $M^\bullet$ of A-modules, and K' corresponds to the complex $N^\bullet$ of A'-modules. Note that $A' = A \otimes_R R'$. The condition above is that the composition

$$M^\bullet \otimes^{\mathbf{L}}_R R' \to M^\bullet \otimes^{\mathbf{L}}_A A' \to N^\bullet$$

is an isomorphism in $D(R')$. Equivalently, it is that for all $i \in \mathbf{Z}$ the map

$$H^i(M^\bullet \otimes^{\mathbf{L}}_R R') \to H^i(M^\bullet \otimes^{\mathbf{L}}_A A') \to H^i(N^\bullet)$$

is an isomorphism. Observe that this is a map of $A \otimes_R R'$-modules, i.e., of A'-modules. On the other hand, (1) is the requirement that for compatible primes $\mathfrak{q}' \subset A'$, $\mathfrak{q} \subset A$, $\mathfrak{p}' \subset R'$, $\mathfrak{p} \subset R$ the composition

$$H^i(M^\bullet_\mathfrak{q} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} R'_{\mathfrak{p}'}) \otimes_{(A_\mathfrak{q} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'})} A'_{\mathfrak{q}'} \to H^i(M^\bullet_\mathfrak{q} \otimes^{\mathbf{L}}_{A_\mathfrak{q}} A'_{\mathfrak{q}'}) \to H^i(N^\bullet_{\mathfrak{q}'})$$

is an isomorphism. Since

$$H^i(M^\bullet_\mathfrak{q} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} R'_{\mathfrak{p}'}) \otimes_{(A_\mathfrak{q} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'})} A'_{\mathfrak{q}'} = H^i(M^\bullet \otimes^{\mathbf{L}}_R R') \otimes_{A'} A'_{\mathfrak{q}'}$$

is the localization at $\mathfrak{q}'$, we see that these two conditions are equivalent by Algebra, Lemma 10.23.1. □

0DJ9 Lemma 36.26.2. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If

(1) the equivalent conditions of Lemma 36.26.1 hold, and
(2) f is quasi-compact and quasi-separated,

then the composition $Lg^*Rf_*K \to Rf'_*L(g')^*K \to Rf'_*K'$ is an isomorphism.

Proof. We could prove this using the same method as in the proof of Lemma 36.22.5 but instead we will prove it using the induction principle and relative Mayer-Vietoris.

To check the map is an isomorphism we may work locally on S'. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular X is a quasi-compact and quasi-separated scheme. We will use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove that for any quasi-compact open $U \subset X$ the similarly constructed map $Lg^*R(U \to S)_*K|_U \to R(U' \to S')_*K'|_{U'}$ is an isomorphism. Here $U' = (g')^{-1}(U)$.

If $U \subset X$ is an affine open, then we find that the result is true by assumption, see Lemma 36.26.1 part (2) and the translation into algebra afforded to us by Lemmas 36.3.5 and 36.3.8.

The induction step. Suppose that $X = U \cup V$ is an open covering with U, V, $U \cap V$ quasi-compact such that the result holds for U, V, and $U \cap V$. Denote $a = f|_U$, $b = f|_V$ and $c = f|_{U\cap V}$. Let $a' : U' \to S'$, $b' : V' \to S'$ and $c' : U' \cap V' \to S'$ be the base changes of a, b, and c. Using the distinguished triangles from relative Mayer-Vietoris (Cohomology, Lemma 20.33.5) we obtain a commutative diagram

$$\begin{tikzcd}
Lg^*Rf_*K \arrow[r] \arrow[d] & Rf'_*K' \arrow[d] \\
Lg^*Ra_*K|_U \oplus Lg^*Rb_*K|_V \arrow[r] \arrow[d] & Ra'_*K'|_{U'} \oplus Rb'_*K'|_{V'} \arrow[d] \\
Lg^*Rc_*K|_{U\cap V} \arrow[r] \arrow[d] & Rc'_*K'|_{U'\cap V'} \arrow[d] \\
Lg^*Rf_*K[1] \arrow[r] & Rf'_*K'[1]
\end{tikzcd}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 13.4.3) and the proof of the lemma is finished. $\square$

0DJA **Lemma 36.26.3.** Let

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

be a cartesian diagram of schemes. Let $K \in D_{QCoh}(\mathcal{O}_X)$ and let $L(g')^*K \to K'$ be a map in $D_{QCoh}(\mathcal{O}_{X'})$. If the equivalent conditions of Lemma 36.26.1 hold, then

(1) for $E \in D_{QCoh}(\mathcal{O}_X)$ the equivalent conditions of Lemma 36.26.1 hold for $L(g')^*(E \otimes^{\mathbf{L}} K) \to L(g')^*E \otimes^{\mathbf{L}} K'$,
(2) if E in $D(\mathcal{O}_X)$ is perfect the equivalent conditions of Lemma 36.26.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$, and
(3) if K is bounded below and E in $D(\mathcal{O}_X)$ pseudo-coherent the equivalent conditions of Lemma 36.26.1 hold for $L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, K')$.

Proof. The statement makes sense as the complexes involved have quasi-coherent cohomology sheaves by Lemmas 36.3.8, 36.3.9, and 36.10.8 and Cohomology, Lemmas 20.47.3 and 20.49.6. Having said this, we can check the maps (36.26.0.1) are isomorphisms in case (1) by computing the source and target of (36.26.0.1) using the transitive property of tensor product, see More on Algebra, Lemma 15.59.15. The map in (2) and (3) is the composition

$$L(g')^*R\mathcal{H}om(E, K) \to R\mathcal{H}om(L(g')^*E, L(g')^*K) \to R\mathcal{H}om(L(g')^*E, K')$$

where the first arrow is Cohomology, Remark 20.42.13 and the second arrow comes from the given map $L(g')^*K \to K'$. To prove the maps (36.26.0.1) are isomorphisms one represents E_x by a bounded complex of finite projective $\mathcal{O}_{X,x}$-modules in case

(2) or by a bounded above complex of finite free modules in case (3) and computes the source and target of the arrow. Some details omitted. $\square$

0A1D Lemma 36.26.4. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a bounded above complex of quasi-coherent $\mathcal{O}_X$-modules flat over S. Then formation of

$$Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in other words $X' = S' \times_S X$. The lemma asserts that

$$Lg^*Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \longrightarrow Rf'_*\left(L(g')^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} (g')^*\mathcal{G}^\bullet\right)$$

is an isomorphism. Observe that on the right hand side we do not use the derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 36.26.2 and 36.26.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 36.26.1. This follows by checking the condition on stalks, where it immediately follows from the fact that $\mathcal{G}^\bullet_x \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ computes the derived tensor product by our assumptions on the complex $\mathcal{G}^\bullet$. $\square$

08IE Lemma 36.26.5. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{G}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. If

(1) E is perfect, $\mathcal{G}^\bullet$ is a bounded above, and $\mathcal{G}^n$ is flat over S, or
(2) E is pseudo-coherent, $\mathcal{G}^\bullet$ is bounded, and $\mathcal{G}^n$ is flat over S,

then formation of

$$Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

in other words $X' = S' \times_S X$. The lemma asserts that

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet) \longrightarrow R(f')_*R\mathcal{H}om(L(g')^*E, (g')^*\mathcal{G}^\bullet)$$

is an isomorphism. Observe that on the right hand side we do not use the derived pullback on $\mathcal{G}^\bullet$. To prove this, we apply Lemmas 36.26.2 and 36.26.3 to see that it suffices to prove the canonical map

$$L(g')^*\mathcal{G}^\bullet \to (g')^*\mathcal{G}^\bullet$$

satisfies the equivalent conditions of Lemma 36.26.1. This was shown in the proof of Lemma 36.26.4. $\square$

36.27. Producing perfect complexes

0A1E The following lemma is our main technical tool for producing perfect complexes. Later versions of this result will reduce to this by Noetherian approximation, see Section 36.30.

08EV **Lemma 36.27.1.** Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ such that

(1) $E \in D^b_{\text{Coh}}(\mathcal{O}_X)$,
(2) the support of $H^i(E)$ is proper over S for all i, and
(3) E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$.

Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.

Proof. By Lemma 36.11.3 we see that Rf_*E is an object of $D^b_{\text{Coh}}(\mathcal{O}_S)$. Hence Rf_*E is pseudo-coherent (Lemma 36.10.3). Hence it suffices to show that Rf_*E has finite tor dimension, see Cohomology, Lemma 20.49.5. By Lemma 36.10.6 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves $\mathcal{F}$ on S. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all i. Then T is proper over S by assumptions (1) and (2), see Cohomology of Schemes, Lemma 30.26.6. In particular there exists a quasi-compact open $X' \subset X$ containing T. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. Moreover, f is quasi-separated by Morphisms, Lemma 29.15.7. Now

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F} = Rf_*\left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}\right) = Rf_*\left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}\right)$$

by Lemma 36.22.1 and Cohomology, Lemma 20.27.4. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. $\square$

0DJQ **Lemma 36.27.2.** Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S. Then $K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_S)$.

Proof. The object K is perfect by Lemma 36.27.1. We check the lemma applies: Locally E is isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules. Hence locally $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ is isomorphic to a finite complex whose terms are of the form

$$\bigoplus\nolimits_{i=a,\ldots,b} (\mathcal{G}^i)^{\oplus r_i}$$

for some integers $a, b, r_a, \ldots, r_b$. This immediately implies the cohomology sheaves $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ are coherent. The hypothesis on the tor dimension also follows as $\mathcal{G}^i$ is flat over $f^{-1}\mathcal{O}_S$. $\square$

0DJR Lemma 36.27.3. Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ be perfect. Let $\mathcal{G}^\bullet$ be a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S. Then $K = Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$ is a perfect object of $D(\mathcal{O}_S)$.

Proof. Since E is a perfect complex there exists a dual perfect complex $E^\vee$, see Cohomology, Lemma 20.50.5. Observe that $R\mathcal{H}om(E, \mathcal{G}^\bullet) = E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$. Thus the perfectness of K follows from Lemma 36.27.2. $\square$

We will generalize the following lemma to flat and proper morphisms over general bases in Lemma 36.30.4 and to perfect proper morphisms in More on Morphisms, Lemma 37.61.13.

0B6F Lemma 36.27.4. Let S be a Noetherian scheme. Let $f : X \to S$ be a flat proper morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.

Proof. We claim that Lemma 36.27.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and S affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Since $\mathcal{O}_X$ is flat as a sheaf of $f^{-1}\mathcal{O}_S$-modules we find that condition (3) is satisfied. $\square$

36.28. A projection formula for Ext

08IC Lemma 36.28.3 (or similar results in the literature) is sometimes used to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems. Suppose that $f : X \to S$ is a proper, flat, finitely presented morphism of schemes and $E \in D(\mathcal{O}_X)$ is perfect. Here the lemma says

$$\mathrm{Ext}^i_X(E, f^*\mathcal{F}) = \mathrm{Ext}^i_S((Rf_*E^\vee)^\vee, \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S. Writing it this way makes it look like a projection formula for Ext and indeed the result follows rather easily from Lemma 36.22.1.

0A1F Lemma 36.28.1. Assumptions and notation as in Lemma 36.27.2. Then there are functorial isomorphisms

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof).

Proof. We have

$$\mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology, Lemma 20.27.4, the second as $\mathcal{G}^n$ is a flat $f^{-1}\mathcal{O}_S$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(S, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(S, Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \\ &= H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology, Lemma 20.13.1), and the third equality by Lemma 36.22.1. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of quasi-coherent $\mathcal{O}_S$-modules, the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & H^i(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3)) \arrow[d, "\delta"] \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) \arrow[r] & H^{i+1}(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1)) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes of $\mathcal{O}_X$-modules. This sequence is exact because $\mathcal{G}^n$ is flat over S. We omit the verification of the commutativity of the displayed diagram. □

08ID **Lemma 36.28.2.** Assumptions and notation as in Lemma 36.27.3. Then there are functorial isomorphisms

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow \operatorname{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof).

Proof. As in the proof of Lemma 36.27.3 let $E^\vee$ be the dual perfect complex and recall that $K = Rf_*(E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$. Since we also have

$$\operatorname{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of $E^\vee$, the existence of the isomorphisms follows from Lemma 36.28.1 applied to $E^\vee$ and $\mathcal{G}^\bullet$. The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{tikzcd} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) \arrow[r] \arrow[d, "\delta"'] & \operatorname{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \arrow[d, "\delta"] \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) \arrow[r] & \operatorname{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{tikzcd}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

of complexes. This sequence is exact because $\mathcal{G}$ is flat over S. We omit the verification of the commutativity of the displayed diagram. □

08IF **Lemma 36.28.3.** Let $f : X \to S$ be a morphism of schemes, $E \in D(\mathcal{O}_X)$ and $\mathcal{G}^\bullet$ a complex of $\mathcal{O}_X$-modules. Assume

(1) S is Noetherian,
(2) f is locally of finite type,
(3) $E \in D^-_{Coh}(\mathcal{O}_X)$,

(4) $\mathcal{G}^\bullet$ is a bounded complex of coherent $\mathcal{O}_X$-modules flat over S with support proper over S.

Then the following two statements are true

(A) for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_S)$ and functorial maps
$$\alpha^i_\mathcal{F} : \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F})$$
for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof) such that $\alpha^i_\mathcal{F}$ is an isomorphism for $i \leq m$

(B) there exists a pseudo-coherent $L \in D(\mathcal{O}_S)$ and functorial isomorphisms
$$\mathrm{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$
for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps.

Proof. Proof of (A). Suppose $\mathcal{G}^i$ is nonzero only for $i \in [a, b]$. We may replace X by a quasi-compact open neighbourhood of the union of the supports of $\mathcal{G}^i$. Hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, -m-1+a)$ (possible by Theorem 36.14.6). Then the induced map
$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$
is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of
$$\mathrm{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$
where C is the cone of $P \to E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1+a$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 13.27.3. This reduces us to the case that E is a perfect complex which is Lemma 36.28.2. The statement on boundaries is explained in the proof of Lemma 36.28.2.

Proof of (B). As in the proof of (A) we may assume X is Noetherian. Observe that E is pseudo-coherent by Lemma 36.10.3. By Lemma 36.19.1 we can write $E = \mathrm{hocolim} E_n$ with E_n perfect and $E_n \to E$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $E_n^\vee$ be the dual perfect complex (Cohomology, Lemma 20.50.5). We obtain an inverse system $\ldots \to E_3^\vee \to E_2^\vee \to E_1^\vee$ of perfect objects. This in turn gives rise to an inverse system
$$\ldots \to K_3 \to K_2 \to K_1 \quad \text{with} \quad K_n = Rf_*(E_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$
perfect on S, see Lemma 36.27.2. By Lemma 36.28.2 and its proof and by the arguments in the previous paragraph (with $P = E_n$) for any quasi-coherent $\mathcal{F}$ on S we have functorial canonical maps
$$\begin{tikzcd}[column sep=small] & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \arrow[dl] \arrow[dr] & \\ H^i(S, K_{n+1} \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \arrow[rr] & & H^i(S, K_n \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \end{tikzcd}$$
which are isomorphisms for $i \leq n + a$. Let $L_n = K_n^\vee$ be the dual perfect complex. Then we see that $L_1 \to L_2 \to L_3 \to \ldots$ is a system of perfect objects in $D(\mathcal{O}_S)$ such that for any quasi-coherent $\mathcal{F}$ on S the maps
$$\mathrm{Ext}^i_{\mathcal{O}_S}(L_{n+1}, \mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_S}(L_n, \mathcal{F})$$

are isomorphisms for $i \leq n + a - 1$. This implies that $L_n \to L_{n+1}$ induces an isomorphism on truncations $\tau_{\geq -n-a+2}$ (hint: take cone of $L_n \to L_{n+1}$ and look at its last nonvanishing cohomology sheaf). Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Lemma 36.19.1. The mapping property of homotopy colimits gives that $\text{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) = \text{Ext}^i_{\mathcal{O}_S}(L_n, \mathcal{F})$ for $i \leq n + a - 3$ which finishes the proof. $\square$

0DJS Remark 36.28.4. The pseudo-coherent complex L of part (B) of Lemma 36.28.3 is canonically associated to the situation. For example, formation of L as in (B) is compatible with base change. In other words, given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes we have canonical functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_{S'}}(Lg^*L, \mathcal{F}') \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(L(g')^*E, (g')^*\mathcal{G}^\bullet \otimes_{\mathcal{O}_{X'}} (f')^*\mathcal{F}')$$

for $\mathcal{F}'$ quasi-coherent on S'. Obsere that we do not use derived pullback on $\mathcal{G}^\bullet$ on the right hand side. If we ever need this, we will formulate a precise result here and give a detailed proof.

36.29. Limits and derived categories

09RC In this section we collect some results about the derived category of a scheme which is the limit of an inverse system of schemes. More precisely, we will work in the following setting.

09RD Situation 36.29.1. Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also fix an element $0 \in I$.

09RE Lemma 36.29.2. In Situation 36.29.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{S_0})$. Set $E_i = Lf_{i0}^*E_0$ and $K_i = Lf_{i0}^*K_0$ for $i \geq 0$ and set $E = Lf_0^*E_0$ and $K = Lf_0^*K_0$. Then the map

$$\text{colim}_{i \geq 0} \text{Hom}_{D(\mathcal{O}_{S_i})}(E_i, K_i) \longrightarrow \text{Hom}_{D(\mathcal{O}_S)}(E, K)$$

is an isomorphism if either

(1) E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$, or
(2) E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$ has finite tor dimension.

Proof. For every open $U_0 \subset S_0$ consider the condition P that the canonical map

$$\text{colim}_{i \geq 0} \text{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \text{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 30.4.1. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 20.33.3. Thus it suffices to prove the lemma when S_0 is affine.

Assume S_0 is affine. Say $S_0 = \text{Spec}(A_0)$, $S_i = \text{Spec}(A_i)$, and $S = \text{Spec}(A)$. We will use Lemma 36.3.5 without further mention.

In case (1) the object $E_0^\bullet$ corresponds to a finite complex of finite projective A_0-modules, see Lemma 36.10.7. We may represent the object K_0 by a K-flat complex $K_0^\bullet$ of A_0-modules. In this situation we are trying to prove

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(A_i)}(E_0^\bullet \otimes_{A_0} A_i, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{D(A)}(E_0^\bullet \otimes_{A_0} A, K_0^\bullet \otimes_{A_0} A)$$

Because $E_0^\bullet$ is a bounded above complex of projective modules we can rewrite this as

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

Since there are only a finite number of nonzero modules E_0^n and since these are all finitely presented modules, this map is an isomorphism.

In case (2) the object E_0 corresponds to a bounded above complex $E_0^\bullet$ of finite free A_0-modules, see Lemma 36.10.2. We may represent K_0 by a finite complex $K_0^\bullet$ of flat A_0-modules, see Lemma 36.10.4 and More on Algebra, Lemma 15.66.3. In particular $K_0^\bullet$ is K-flat and we can argue as before to arrive at the map

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \mathrm{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

It is clear that this map is an isomorphism (only a finite number of terms are involved since $K_0^\bullet$ is bounded). □

09RF Lemma 36.29.3. In Situation 36.29.1 the category of perfect objects of $D(\mathcal{O}_S)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{S_i})$.

Proof. For every open $U_0 \subset S_0$ consider the condition P that the functor

$$\mathrm{colim}_{i \geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 30.4.1. First, we observe that we already know the functor is fully faithful by Lemma 36.29.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have $S_0 = U_0 \cup V_0$ and that P holds for U_0, V_0, and $U_0 \cap V_0$. Let E be a perfect object of $D(\mathcal{O}_S)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \to E|_{U \cap V} \to Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 20.45.1 to get an object E_i on S_i and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when S_0 is affine. Say $S_0 = \mathrm{Spec}(A_0)$, $S_i = \mathrm{Spec}(A_i)$, and $S = \mathrm{Spec}(A)$. Using Lemmas 36.3.5 and 36.10.7 we see that we have to show that

$$D_{perf}(A) = \mathrm{colim}\, D_{perf}(A_i)$$

This is clear from the fact that perfect complexes over rings are given by finite complexes of finite projective (hence finitely presented) modules. See More on Algebra, Lemma 15.74.17 for details. □

36.30. Cohomology and base change, VI

0A1G A final section on cohomology and base change continuing the discussion of Sections 36.22, 36.26, and 36.27. An easy to grok special case is given in Remark 36.30.2.

0A1H Lemma 36.30.1. Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 36.26.4. Thus it suffices to show that K is a perfect object. If S is Noetherian, then this follows from Lemma 36.27.2. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on S, hence we may assume S is affine. Say $S = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 32.10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemma 32.10.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}^\bullet$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits, Lemma 32.10.4. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits, Lemma 32.13.5 and Cohomology of Schemes, Lemma 30.26.7. Finally, by Lemma 36.29.3 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 36.27.2 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i^\bullet$ and using the base change property already shown we obtain the result. □

0A1I Remark 36.30.2. Let R be a ring. Let X be a scheme of finite presentation over R. Let $\mathcal{G}$ be a finitely presented $\mathcal{O}_X$-module flat over R with support proper over R. By Lemma 36.30.1 there exists a finite complex of finite projective R-modules $M^\bullet$ such that we have

$$R\Gamma(X_{R'}, \mathcal{G}_{R'}) = M^\bullet \otimes_R R'$$

functorially in the R-algebra R'.

0CSC Lemma 36.30.3. Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a pseudo-coherent object. Let $\mathcal{G}^\bullet$ be a bounded above complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then

$$K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 36.26.4. Thus it suffices to show that K is a pseudo-coherent object. This will follow from Lemma 36.30.1 by approximation by perfect complexes. We encourage the reader to skip the rest of the proof.

The question is local on S, hence we may assume S is affine. Then X is quasi-compact and quasi-separated. Moreover, there exists an integer N such that total direct image $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ has cohomological dimension N as explained in Lemma 36.4.1. Choose an integer b such that $\mathcal{G}^i = 0$ for $i > b$. It suffices to show that K is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m-N-1-b)$. This is possible by Theorem 36.14.6. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ are zero in degrees $\geq m - N - 1$. Thus the cohomology sheaves of $Rf_*(C \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$ are zero in degrees $\geq m - 1$. Hence

$$Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet) \to Rf_*(E \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. Next, suppose that $H^i(P) = 0$ for $i > a$. Then $P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet \longrightarrow P \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ is an isomorphism on cohomology sheaves in degrees $\geq m - N - 1$. Thus again we find that

$$Rf_*(P \otimes^{\mathbf{L}} \sigma_{\geq m-N-1-a}\mathcal{G}^\bullet) \to Rf_*(P \otimes^{\mathbf{L}} \mathcal{G}^\bullet)$$

is an isomorphism on cohomology sheaves in degrees $\geq m$. By Lemma 36.30.1 the source is a perfect complex. We conclude that K is m-pseudo-coherent as desired. $\square$

0B91 Lemma 36.30.4. Let S be a scheme. Let $f : X \to S$ be a proper morphism of finite presentation.

(1) Let $E \in D(\mathcal{O}_X)$ be perfect and f flat. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.
(2) Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Then $Rf_*\mathcal{G}$ is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. Special cases of Lemma 36.30.1 applied with (1) $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0 and (2) $E = \mathcal{O}_X$ and $\mathcal{G}^\bullet$ consisting of $\mathcal{G}$ sitting in degree 0. $\square$

0CSD Lemma 36.30.5. Let S be a scheme. Let $f : X \to S$ be a flat proper morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Then Rf_*E is a pseudo-coherent object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

More generally, if $f : X \to S$ is proper and E on X is pseudo-coherent relative to S (More on Morphisms, Definition 37.59.2), then Rf_*E is pseudo-coherent (but formation does not commute with base change in this generality). See [Kie72].

Proof. Special case of Lemma 36.30.3 applied with $\mathcal{G}^\bullet$ equal to $\mathcal{O}_X$ in degree 0. $\square$

0D2Q Lemma 36.30.6. Let R be a ring. Let X be a scheme and let $f : X \to \mathrm{Spec}(R)$ be proper, flat, and of finite presentation. Let (M_n) be an inverse system of R-modules with surjective transition maps. Then the canonical map

$$\mathcal{O}_X \otimes_R (\lim M_n) \longrightarrow \lim \mathcal{O}_X \otimes_R M_n$$

induces an isomorphism from the source to DQ_X applied to the target.

Proof. The statement means that for any object E of $D_{QCoh}(\mathcal{O}_X)$ the induced map

$$\mathrm{Hom}(E, \mathcal{O}_X \otimes_R (\lim M_n)) \longrightarrow \mathrm{Hom}(E, \lim \mathcal{O}_X \otimes_R M_n)$$

is an isomorphism. Since $D_{QCoh}(\mathcal{O}_X)$ has a perfect generator (Theorem 36.15.3) it suffices to check this for perfect E. By Lemma 36.3.2 we have $\lim \mathcal{O}_X \otimes_R M_n = R\lim \mathcal{O}_X \otimes_R M_n$. The exact functor $R\mathrm{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(R)$ of Cohomology, Section 20.44 commutes with products and hence with derived limits, whence

$$R\mathrm{Hom}_X(E, \lim \mathcal{O}_X \otimes_R M_n) = R\lim R\mathrm{Hom}_X(E, \mathcal{O}_X \otimes_R M_n)$$

Let $E^\vee$ be the dual perfect complex, see Cohomology, Lemma 20.50.5. We have

$$R\mathrm{Hom}_X(E, \mathcal{O}_X \otimes_R M_n) = R\Gamma(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*M_n) = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n$$

by Lemma 36.22.1. From Lemma 36.30.4 we see $R\Gamma(X, E^\vee)$ is a perfect complex of R-modules. In particular it is a pseudo-coherent complex and by More on Algebra, Lemma 15.102.3 we obtain

$$R\lim R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R M_n = R\Gamma(X, E^\vee) \otimes^{\mathbf{L}}_R \lim M_n$$

as desired. □

0A1J Lemma 36.30.7. Let $f : X \to S$ be a morphism of finite presentation. Let $E \in D(\mathcal{O}_X)$ be a perfect object. Let $\mathcal{G}^\bullet$ be a bounded complex of finitely presented $\mathcal{O}_X$-modules, flat over S, with support proper over S. Then

$$K = Rf_*R\mathcal{H}om(E, \mathcal{G}^\bullet)$$

is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 36.26.5. Thus it suffices to show that K is a perfect object. If S is Noetherian, then this follows from Lemma 36.27.3. We will reduce to this case by Noetherian approximation. We encourage the reader to skip the rest of this proof.

The question is local on S, hence we may assume S is affine. Say $S = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 32.10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemma 32.10.2 we may assume after increasing i, that there exists a bounded complex of finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{G}_i^\bullet$ whose pullback to X is $\mathcal{G}^\bullet$. After increasing i we may assume $\mathcal{G}_i^n$ is flat over R_i, see Limits, Lemma 32.10.4. After increasing i we may assume the support of $\mathcal{G}_i^n$ is proper over R_i, see Limits, Lemma 32.13.5 and Cohomology of Schemes, Lemma 30.26.7. Finally, by Lemma 36.29.3 we may, after increasing i, assume there exists a perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 36.27.3 to $X_i \to \mathrm{Spec}(R_i)$, E_i, $\mathcal{G}_i^\bullet$ and using the base change property already shown we obtain the result. □

36.31. Perfect complexes

0BDH We first talk about jumping loci for betti numbers of perfect complexes. Given a complex E on a scheme X and a point x of X we often write $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x)$ instead of the more correct Li_x^*E, where $i_x : x \to X$ is the canonical morphism.

0BDI **Lemma 36.31.1.** Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent (for example perfect). For any $i \in \mathbf{Z}$ consider the function

$$\beta_i : X \longrightarrow \{0, 1, 2, \ldots\}, \quad x \longmapsto \dim_{\kappa(x)} H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x))$$

Then we have

(1) formation of β_i commutes with arbitrary base change,
(2) the functions β_i are upper semi-continuous, and
(3) the level sets of β_i are locally constructible in X.

Proof. Consider a morphism of schemes $f : Y \to X$ and a point $y \in Y$. Let x be the image of y and consider the commutative diagram

$$\begin{tikzcd} y \arrow[r, "j"'] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ x \arrow[r, "i"] & X \end{tikzcd}$$

Then we see that $Lg^* \circ Li^* = Lj^* \circ Lf^*$. This implies that the function β'_i associated to the pseudo-coherent complex Lf^*E is the pullback of the function β_i, in a formula: $\beta'_i = \beta_i \circ f$. This is the meaning of (1).

Fix i and let $x \in X$. It is enough to prove (2) and (3) holds in an open neighbourhood of x, hence we may assume X affine. Then we can represent E by a bounded above complex $\mathcal{F}^\bullet$ of finite free modules (Lemma 36.13.3). Then $P = \sigma_{\geq i-1}\mathcal{F}^\bullet$ is a perfect object and $P \to E$ induces an isomorphism

$$H^i(P \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x')) \to H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x'))$$

for all $x' \in X$. Thus we may assume E is perfect. In this case by More on Algebra, Lemma 15.75.6 there exists an affine open neighbourhood U of x and $a \leq b$ such that $E|_U$ is represented by a complex

$$\ldots \to 0 \to \mathcal{O}_U^{\oplus \beta_a(x)} \to \mathcal{O}_U^{\oplus \beta_{a+1}(x)} \to \ldots \to \mathcal{O}_U^{\oplus \beta_{b-1}(x)} \to \mathcal{O}_U^{\oplus \beta_b(x)} \to 0 \to \ldots$$

(This also uses earlier results to turn the problem into algebra, for example Lemmas 36.3.5 and 36.10.7.) It follows immediately that $\beta_i(x') \leq \beta_i(x)$ for all $x' \in U$. This proves that β_i is upper semi-continuous.

To prove (3) we may assume that X is affine and E is given by a complex of finite free $\mathcal{O}_X$-modules (for example by arguing as in the previous paragraph, or by using Cohomology, Lemma 20.49.3). Thus we have to show that given a complex

$$\mathcal{O}_X^{\oplus a} \to \mathcal{O}_X^{\oplus b} \to \mathcal{O}_X^{\oplus c}$$

the function associated to a point $x \in X$ the dimension of the cohomology of $\kappa_x^{\oplus a} \to \kappa_x^{\oplus b} \to \kappa_x^{\oplus c}$ in the middle has constructible level sets. Let $A \in \text{Mat}(a \times b, \Gamma(X, \mathcal{O}_X))$ be the matrix of the first arrow. The rank of the image of A in $\text{Mat}(a \times b, \kappa(x))$ is equal to r if all $(r+1) \times (r+1)$-minors of A vanish at x and there is some $r \times r$-minor of A which does not vanish at x. Thus the set of points where the rank is r is a constructible locally closed set. Arguing similarly for the second arrow and putting everything together we obtain the desired result. $\square$

0BDJ **Lemma 36.31.2.** Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. The function

$$\chi_E : X \longrightarrow \mathbf{Z}, \quad x \longmapsto \sum (-1)^i \dim_{\kappa(x)} H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa(x))$$

is locally constant on X.

Proof. By Cohomology, Lemma 20.49.3 we see that we can, locally on X, represent E by a finite complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules. On such an open the function χ_E is constant with value $\sum(-1)^i\text{rank}(\mathcal{E}^i)$. □

0BDK Lemma 36.31.3. Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect. Given $i, r \in \mathbf{Z}$, there exists an open subscheme $U \subset X$ characterized by the following

(1) $E|_U \cong H^i(E|_U)[-i]$ and $H^i(E|_U)$ is a locally free $\mathcal{O}_U$-module of rank r,
(2) a morphism $f : Y \to X$ factors through U if and only if Lf^*E is isomorphic to a locally free module of rank r placed in degree i.

Proof. Let $\beta_j : X \to \{0, 1, 2, \ldots\}$ for $j \in \mathbf{Z}$ be the functions of Lemma 36.31.1. Then the set

$$W = \{x \in X \mid \beta_j(x) \leq 0 \text{ for all } j \neq i\}$$

is open in X and its formation commutes with pullback to any Y over X. This follows from the lemma using that apriori in a neighbourhood of any point only a finite number of the β_j are nonzero. Thus we may replace X by W and assume that $\beta_j(x) = 0$ for all $x \in X$ and all $j \neq i$. In this case $H^i(E)$ is a finite locally free module and $E \cong H^i(E)[-i]$, see for example More on Algebra, Lemma 15.75.6. Thus X is the disjoint union of the open subschemes where the rank of $H^i(E)$ is fixed and we win. □

0BDL Lemma 36.31.4. Let X be a scheme. Let $E \in D(\mathcal{O}_X)$ be perfect of tor-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$. Let $r \geq 0$. Then there exists a locally closed subscheme $j : Z \to X$ characterized by the following

(1) $H^a(Lj^*E)$ is a locally free $\mathcal{O}_Z$-module of rank r, and
(2) a morphism $f : Y \to X$ factors through Z if and only if for all morphisms $g : Y' \to Y$ the $\mathcal{O}_{Y'}$-module $H^a(L(f \circ g)^*E)$ is locally free of rank r.

Moreover, $j : Z \to X$ is of finite presentation and we have

(3) if $f : Y \to X$ factors as $Y \xrightarrow{g} Z \to X$, then $H^a(Lf^*E) = g^*H^a(Lj^*E)$,
(4) if $\beta_a(x) \leq r$ for all $x \in X$, then j is a closed immersion and given $f : Y \to X$ the following are equivalent
    (a) $f : Y \to X$ factors through Z,
    (b) $H^0(Lf^*E)$ is a locally free $\mathcal{O}_Y$-module of rank r,
    and if $r = 1$ these are also equivalent to
    (c) $\mathcal{O}_Y \to \mathcal{H}om_{\mathcal{O}_Y}(H^0(Lf^*E), H^0(Lf^*E))$ is injective.

Proof. First, let $U \subset X$ be the locally constructible open subscheme where the function β_a of Lemma 36.31.1 has values $\leq r$. Let $f : Y \to X$ be as in (2). Then for any $y \in Y$ we have $\beta_a(Lf^*E) = r$ hence y maps into U by Lemma 36.31.1. Hence f as in (2) factors through U. Thus we may replace X by U and assume that $\beta_a(x) \in \{0, 1, \ldots, r\}$ for all $x \in X$. We will show that in this case there is a closed subscheme $Z \subset X$ cut out by a finite type quasi-coherent ideal characterized by the equivalence of (4) (a), (b) and (4)(c) if $r = 1$ and that (3) holds. This will finish the proof because it will a fortiori show that morphisms as in (2) factor through Z.

If $x \in X$ and $\beta_a(x) < r$, then there is an open neighbourhood of x where $\beta_a < r$ (Lemma 36.31.1). In this way we see that set theoretically at least Z is a closed subset.

To get a scheme theoretic structure, consider a point $x \in X$ with $\beta_a(x) = r$. Set $\beta = \beta_{a+1}(x)$. By More on Algebra, Lemma 15.75.6 there exists an affine open neighbourhood U of x such that $K|_U$ is represented by a complex

$$\ldots \to 0 \to \mathcal{O}_U^{\oplus r} \xrightarrow{(f_{ij})} \mathcal{O}_U^{\oplus \beta} \to \ldots \to \mathcal{O}_U^{\oplus \beta_{b-1}(x)} \to \mathcal{O}_U^{\oplus \beta_b(x)} \to 0 \to \ldots$$

(This also uses earlier results to turn the problem into algebra, for example Lemmas 36.3.5 and 36.10.7.) Now, if $g : Y \to U$ is any morphism of schemes such that $g^\sharp(f_{ij})$ is nonzero for some pair i, j, then $H^0(Lg^*E)$ is not a locally free $\mathcal{O}_Y$-module of rank r. See More on Algebra, Lemma 15.15.7. Trivially $H^0(Lg^*E)$ is a locally free $\mathcal{O}_Y$-module if $g^\sharp(f_{ij}) = 0$ for all i, j. Thus we see that over U the closed subscheme cut out by all f_{ij} satisfies (3) and we have the equivalence of (4)(a) and (b). The characterization of Z shows that the locally constructed patches glue (details omitted). Finally, if $r = 1$ then (4)(c) is equivalent to (4)(b) because in this case locally $H^0(Lg^*E) \subset \mathcal{O}_Y$ is the annihilator of the ideal generated by the elements $g^\sharp(f_{ij})$. □

36.32. Applications

0BDM Mostly applications of cohomology and base change. In the future we may generalize these results to the situation discussed in Lemma 36.30.1.

0BDN Lemma 36.32.1. Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. For fixed $i \in \mathbf{Z}$ consider the function

$$\beta_i : S \to \{0, 1, 2, \ldots\}, \quad s \longmapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$$

Then we have

(1) formation of β_i commutes with arbitrary base change,
(2) the functions β_i are upper semi-continuous, and
(3) the level sets of β_i are locally constructible in S.

Proof. By cohomology and base change (more precisely by Lemma 36.30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. In particular we have

$$H^i(X_s, \mathcal{F}_s) = H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s))$$

Thus the lemma follows from Lemma 36.31.1. □

0B9T Lemma 36.32.2. Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. The function

$$s \longmapsto \chi(X_s, \mathcal{F}_s)$$

is locally constant on S. Formation of this function commutes with base change.

Proof. By cohomology and base change (more precisely by Lemma 36.30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. Thus we have to show the map

$$s \longmapsto \sum (-1)^i \dim_{\kappa(s)} H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s))$$

is locally constant on S. This is Lemma 36.31.2. □

0B9S Lemma 36.32.3. Let $f : X \to S$ be a proper morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Fix $i, r \in \mathbf{Z}$. Then there exists an open subscheme $U \subset S$ with the following property: A morphism $T \to S$ factors through U if and only if $Rf_{T,*}\mathcal{F}_T$ is isomorphic to a finite locally free module of rank r placed in degree i.

Proof. By cohomology and base change (more precisely by Lemma 36.30.4) the object $K = Rf_*\mathcal{F}$ is a perfect object of the derived category of S whose formation commutes with arbitrary base change. Thus this lemma follows immediately from Lemma 36.31.3. $\square$

0D4E Lemma 36.32.4. Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S with support proper over S. If $R^if_*\mathcal{F} = 0$ for $i > 0$, then $f_*\mathcal{F}$ is locally free and its formation commutes with arbitrary base change (see proof for explanation).

Proof. By Lemma 36.30.1 the object $E = Rf_*\mathcal{F}$ of $D(\mathcal{O}_S)$ is perfect and its formation commutes with arbitrary base change, in the sense that $Rf'_*(g')^*\mathcal{F} = Lg^*E$ for any cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

of schemes. Since there is never any cohomology in degrees < 0, we see that E (locally) has tor-amplitude in $[0, b]$ for some b. If $H^i(E) = R^if_*\mathcal{F} = 0$ for $i > 0$, then E has tor amplitude in $[0, 0]$. Whence $E = H^0(E)[0]$. We conclude $H^0(E) = f_*\mathcal{F}$ is finite locally free by More on Algebra, Lemma 15.74.2 (and the characterization of finite projective modules in Algebra, Lemma 10.78.2). Commutation with base change means that $g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}$ for a diagram as above and it follows from the already established commutation of base change for E. $\square$

0E62 Lemma 36.32.5. Let $f : X \to S$ be a morphism of schemes. Assume

(1) f is proper, flat, and of finite presentation, and
(2) for all $s \in S$ we have $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$.

Then we have

(a) $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change,
(b) locally on S we have
$$Rf_*\mathcal{O}_X = \mathcal{O}_S \oplus P$$
in $D(\mathcal{O}_S)$ where P is perfect of tor amplitude in $[1, \infty)$.

Proof. By cohomology and base change (Lemma 36.30.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. This first implies that E has tor aplitude in $[0, \infty)$. Second, it implies that for $s \in S$ we have $H^0(E \otimes^{\mathbf{L}} \kappa(s)) = H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. It follows that the map $\mathcal{O}_S \to Rf_*\mathcal{O}_X = E$ induces an isomorphism $\mathcal{O}_S \otimes \kappa(s) \to H^0(E \otimes^{\mathbf{L}} \kappa(s))$. Hence $H^0(E) \otimes \kappa(s) \to H^0(E \otimes^{\mathbf{L}} \kappa(s))$ is surjective and we may apply More on Algebra, Lemma 15.76.2 to see that, after replacing S by an affine open neighbourhood of s, we have a decomposition $E = H^0(E) \oplus \tau_{\geq 1}E$ with $\tau_{\geq 1}E$ perfect of tor amplitude in $[1, \infty)$. Since E has tor amplitude in $[0, \infty)$ we find that $H^0(E)$ is a flat $\mathcal{O}_S$-module. It follows that $H^0(E)$ is a flat, perfect $\mathcal{O}_S$-module, hence finite locally free, see More on Algebra, Lemma 15.74.2 (and the fact that finite projective modules are finite locally free by Algebra, Lemma 10.78.2). It follows that the map $\mathcal{O}_S \to H^0(E)$ is an isomorphism as we can check this after tensoring with residue fields (Algebra, Lemma 10.79.4). $\square$

0E0L Lemma 36.32.6. Let $f : X \to S$ be a morphism of schemes. Assume

(1) f is proper, flat, and of finite presentation, and
(2) the geometric fibres of f are reduced and connected.

Then $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds after any base change.

Proof. By Lemma 36.32.5 it suffices to show that $\kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$ for all $s \in S$. This follows from Varieties, Lemma 33.9.3 and the fact that X_s is geometrically connected and geometrically reduced. $\square$

0G7X Lemma 36.32.7. Let $f : X \to S$ be a proper morphism of schemes. Let $s \in S$ and let $e \in H^0(X_s, \mathcal{O}_{X_s})$ be an idempotent. Then e is in the image of the map $(f_*\mathcal{O}_X)_s \to H^0(X_s, \mathcal{O}_{X_s})$.

Proof. Let $X_s = T_1 \amalg T_2$ be the disjoint union decomposition with T_1 and T_2 nonempty and open and closed in X_s corresponding to e, i.e., such that e is identically 1 on T_1 and identically 0 on T_2.

Assume S is Noetherian. We will use the theorem on formal functions in the form of Cohomology of Schemes, Lemma 30.20.7. It tells us that

$$(f_*\mathcal{O}_X)^\wedge_s = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where X_n is the nth infinitesimal neighbourhood of X_s. Since the underlying topological space of X_n is equal to that of X_s we obtain for all n a disjoint union decomposition of schemes $X_n = T_{1,n} \amalg T_{2,n}$ where the underlying topological space of $T_{i,n}$ is T_i for $i = 1, 2$. This means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent e_n, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that e_{n+1} restricts to e_n on X_n. Hence $e_\infty = \lim e_n$ is a nontrivial idempotent of the limit. Thus e_∞ is an element of the completion of $(f_*\mathcal{O}_X)_s$ mapping to e in $H^0(X_s, \mathcal{O}_{X_s})$. Since the map $(f_*\mathcal{O}_X)^\wedge_s \to H^0(X_s, \mathcal{O}_{X_s})$ factors through $(f_*\mathcal{O}_X)^\wedge_s/\mathfrak{m}_s(f_*\mathcal{O}_X)^\wedge_s = (f_*\mathcal{O}_X)_s/\mathfrak{m}_s(f_*\mathcal{O}_X)_s$ (Algebra, Lemma 10.96.3) we conclude that e is in the image of the map $(f_*\mathcal{O}_X)_s \to H^0(X_s, \mathcal{O}_{X_s})$ as desired.

General case: we reduce the general case to the Noetherian case by limit arguments. We urge the reader to skip the proof. We may replace S by an affine open neighbourhood of s. Thus we may and do assume that S is affine. By Limits, Lemma 32.13.3 we can write $(f : X \to S) = \lim(f_i : X_i \to S_i)$ with f_i proper and S_i Noetherian. Denote $s_i \in S_i$ the image of s. Then $s = \lim s_i$, see Limits, Lemma 32.4.4. Then $X_s = X \times_S s = \lim X_i \times_{S_i} s_i = \lim X_{i,s_i}$ because limits commute with limits (Categories, Lemma 4.14.10). Hence e is the image of some idempotent $e_i \in H^0(X_{i,s_i}, \mathcal{O}_{X_{i,s_i}})$ by Limits, Lemma 32.4.7. By the Noetherian case there is an element $\tilde{e}_i$ in the stalk $(f_{i,*}\mathcal{O}_{X_i})_{s_i}$ mapping to e_i. Taking the pullback of $\tilde{e}_i$ we get an element $\tilde{e}$ of $(f_*\mathcal{O}_X)_s$ mapping to e and the proof is complete. $\square$

0G7Y Lemma 36.32.8. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume

(1) f is proper, flat, and of finite presentation, and
(2) the fibre X_s is geometrically reduced.

Then, after replacing S by an open neighbourhood of s, there exists a direct sum decomposition $Rf_*\mathcal{O}_X = f_*\mathcal{O}_X \oplus P$ in $D(\mathcal{O}_S)$ where $f_*\mathcal{O}_X$ is a finite étale $\mathcal{O}_S$-algebra and P is a perfect of tor amplitude in $[1, \infty)$.

Proof. The proof of this lemma is similar to the proof of Lemma 36.32.5 which we suggest the reader read first. By cohomology and base change (Lemma 36.30.4) the complex $E = Rf_*\mathcal{O}_X$ is perfect and its formation commutes with arbitrary base change. This first implies that E has tor aplitude in $[0, \infty)$.

We claim that after replacing S by an open neighbourhood of s we can find a direct sum decomposition $E = H^0(E) \oplus \tau_{\geq 1}E$ in $D(\mathcal{O}_S)$ with $\tau_{\geq 1}E$ of tor amplitude in $[1, \infty)$. Assume the claim is true for now and assume we've made the replacement so we have the direct sum decomposition. Since E has tor amplitude in $[0, \infty)$ we find that $H^0(E)$ is a flat $\mathcal{O}_S$-module. Hence $H^0(E)$ is a flat, perfect $\mathcal{O}_S$-module, hence finite locally free, see More on Algebra, Lemma 15.74.2 (and the fact that finite projective modules are finite locally free by Algebra, Lemma 10.78.2). Of course $H^0(E) = f_*\mathcal{O}_X$ is an $\mathcal{O}_S$-algebra. By cohomology and base change we obtain $H^0(E) \otimes \kappa(s) = H^0(X_s, \mathcal{O}_{X_s})$. By Varieties, Lemma 33.9.3 and the assumption that X_s is geometrically reduced, we see that $\kappa(s) \to H^0(E) \otimes \kappa(s)$ is finite étale. By Morphisms, Lemma 29.36.17 applied to the finite locally free morphism $\underline{\mathrm{Spec}}_S(H^0(E)) \to S$, we conclude that after shrinking S the $\mathcal{O}_S$-algebra $H^0(E)$ is finite étale.

It remains to prove the claim. For this it suffices to prove that the map

$$(f_*\mathcal{O}_X)_s \longrightarrow H^0(X_s, \mathcal{O}_{X_s}) = H^0(E \otimes^{\mathbf{L}} \kappa(s))$$

is surjective, see More on Algebra, Lemma 15.76.2. Choose a flat local ring homomorphism $\mathcal{O}_{S,s} \to A$ such that the residue field k of A is algebraically closed, see Algebra, Lemma 10.159.1. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we get $H^0(X_A, \mathcal{O}_{X_A}) = (f_*\mathcal{O}_X)_s \otimes_{\mathcal{O}_{S,s}} A$ and $H^0(X_k, \mathcal{O}_{X_k}) = H^0(X_s, \mathcal{O}_{X_s}) \otimes_{\kappa(s)} k$. Hence it suffices to prove that $H^0(X_A, \mathcal{O}_{X_A}) \to H^0(X_k, \mathcal{O}_{X_k})$ is surjective. Since X_k is a reduced proper scheme over k and since k is algebraically closed, we see that $H^0(X_k, \mathcal{O}_{X_k})$ is a finite product of copies of k by the already used Varieties, Lemma 33.9.3. Since by Lemma 36.32.7 the idempotents of this k-algebra are in the image of $H^0(X_A, \mathcal{O}_{X_A}) \to H^0(X_k, \mathcal{O}_{X_k})$ we conclude. $\square$

36.33. Other applications

0CRN In this section we state and prove some results that can be deduced from the theory worked out above.

0EX6 Lemma 36.33.1. Let R be a coherent ring. Let X be a scheme of finite presentation over R. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation, flat over R, with support proper over R. Then $H^i(X, \mathcal{G})$ is a coherent R-module.

Proof. Combine Lemma 36.30.1 with More on Algebra, Lemmas 15.64.18 and 15.74.2. $\square$

0CRP Lemma 36.33.2. Let X be a quasi-compact and quasi-separated scheme. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$ such that the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens. Then for any quasi-compact open $U \subset X$ and any perfect object E in $D(\mathcal{O}_X)$ the sets

$$H^i(U, K \otimes^{\mathbf{L}} E), \quad \mathrm{Ext}^i(E|_U, K|_U)$$

are countable.

Proof. Using Cohomology, Lemma 20.50.5 we see that it suffices to prove the result for the groups $H^i(U, K \otimes^{\mathbf{L}} E)$. We will use the induction principle to prove the lemma, see Cohomology of Schemes, Lemma 30.4.1.

First we show that it holds when $U = \mathrm{Spec}(A)$ is affine. Namely, we can represent K by a complex of A-modules $K^\bullet$ and E by a finite complex of finite projective A-modules $P^\bullet$. See Lemmas 36.3.5 and 36.10.7 and our definition of perfect complexes of A-modules (More on Algebra, Definition 15.74.1). Then $(E \otimes^{\mathbf{L}} K)|_U$ is represented by the total complex associated to the double complex $P^\bullet \otimes_A K^\bullet$ (Lemma 36.3.9). Using induction on the length of the complex $P^\bullet$ (or using a suitable spectral sequence) we see that it suffices to show that $H^i(P^a \otimes_A K^\bullet)$ is countable for each a. Since P^a is a direct summand of $A^{\oplus n}$ for some n this follows from the assumption that the cohomology group $H^i(K^\bullet)$ is countable.

To finish the proof it suffices to show: if $U = V \cup W$ and the result holds for V, W, and $V \cap W$, then the result holds for U. This is an immediate consquence of the Mayer-Vietoris sequence, see Cohomology, Lemma 20.33.4. □

0CRQ Lemma 36.33.3. Let X be a quasi-compact and quasi-separated scheme such that the sets of sections of $\mathcal{O}_X$ over affine opens are countable. Let K be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

(1) $K = \mathrm{hocolim} E_n$ with E_n a perfect object of $D(\mathcal{O}_X)$, and
(2) the cohomology sheaves $H^i(K)$ have countable sets of sections over affine opens.

Proof. If (1) is true, then (2) is true because homotopy colimits commutes with taking cohomology sheaves (by Derived Categories, Lemma 13.33.8) and because a perfect complex is locally isomorphic to a finite complex of finite free $\mathcal{O}_X$-modules and therefore satisfies (2) by assumption on X.

Assume (2). Choose a K-injective complex $\mathcal{K}^\bullet$ representing K. Choose a perfect generator E of $D_{QCoh}(\mathcal{O}_X)$ and represent it by a K-injective complex $\mathcal{I}^\bullet$. According to Theorem 36.18.3 and its proof there is an equivalence of triangulated categories $F : D_{QCoh}(\mathcal{O}_X) \to D(A, \mathrm{d})$ where (A, d) is the differential graded algebra

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

which maps K to the differential graded module

$$M = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(\mathcal{I}^\bullet, \mathcal{K}^\bullet)$$

Note that $H^i(A) = \mathrm{Ext}^i(E, E)$ and $H^i(M) = \mathrm{Ext}^i(E, K)$. Moreover, since F is an equivalence it and its quasi-inverse commute with homotopy colimits. Therefore, it suffices to write M as a homotopy colimit of compact objects of $D(A, \mathrm{d})$. By Differential Graded Algebra, Lemma 22.38.3 it suffices show that $\mathrm{Ext}^i(E, E)$ and $\mathrm{Ext}^i(E, K)$ are countable for each i. This follows from Lemma 36.33.2. □

0CRR Lemma 36.33.4. Let A be a ring. Let X be a scheme of finite presentation over A. Let $f : U \to X$ be a flat morphism of finite presentation. Then

(1) there exists an inverse system of perfect objects L_n of $D(\mathcal{O}_X)$ such that
$$R\Gamma(U, Lf^*K) = \mathrm{hocolim}\ R\,\mathrm{Hom}_X(L_n, K)$$
in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$, and

(2) there exists a system of perfect objects E_n of $D(\mathcal{O}_X)$ such that
$$R\Gamma(U, Lf^*K) = \text{hocolim } R\Gamma(X, E_n \otimes^{\mathbf{L}} K)$$
in $D(A)$ functorially in K in $D_{QCoh}(\mathcal{O}_X)$.

Proof. By Lemma 36.22.1 we have
$$R\Gamma(U, Lf^*K) = R\Gamma(X, Rf_*\mathcal{O}_U \otimes^{\mathbf{L}} K)$$
functorially in K. Observe that $R\Gamma(X, -)$ commutes with homotopy colimits because it commutes with direct sums by Lemma 36.4.5. Similarly, $- \otimes^{\mathbf{L}} K$ commutes with derived colimits because $- \otimes^{\mathbf{L}} K$ commutes with direct sums (because direct sums in $D(\mathcal{O}_X)$ are given by direct sums of representing complexes). Hence to prove (2) it suffices to write $Rf_*\mathcal{O}_U = \text{hocolim} E_n$ for a system of perfect objects E_n of $D(\mathcal{O}_X)$. Once this is done we obtain (1) by setting $L_n = E_n^\vee$, see Cohomology, Lemma 20.50.5.

Write $A = \operatorname{colim} A_i$ with A_i of finite type over $\mathbf{Z}$. By Limits, Lemma 32.10.1 we can find an i and morphisms $U_i \to X_i \to \operatorname{Spec}(A_i)$ of finite presentation whose base change to $\operatorname{Spec}(A)$ recovers $U \to X \to \operatorname{Spec}(A)$. After increasing i we may assume that $f_i : U_i \to X_i$ is flat, see Limits, Lemma 32.8.7. By Lemma 36.22.5 the derived pullback of $Rf_{i,*}\mathcal{O}_{U_i}$ by $g : X \to X_i$ is equal to $Rf_*\mathcal{O}_U$. Since Lg^* commutes with derived colimits, it suffices to prove what we want for f_i. Hence we may assume that U and X are of finite type over $\mathbf{Z}$.

Assume $f : U \to X$ is a morphism of schemes of finite type over $\mathbf{Z}$. To finish the proof we will show that $Rf_*\mathcal{O}_U$ is a homotopy colimit of perfect complexes. To see this we apply Lemma 36.33.3. Thus it suffices to show that $R^if_*\mathcal{O}_U$ has countable sets of sections over affine opens. This follows from Lemma 36.33.2 applied to the structure sheaf. □

36.34. Characterizing pseudo-coherent complexes, II

0CSE This section is a continuation of Section 36.19. In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. More results of this nature can be found in More on Morphisms, Section 37.69.

0CSF Lemma 36.34.1. Let A be a ring. Let R be a (possibly noncommutative) A-algebra which is finite free as an A-module. Then any object M of $D(R)$ which is pseudo-coherent in $D(A)$ can be represented by a bounded above complex of finite free (right) R-modules.

Proof. Choose a complex $M^\bullet$ of right R-modules representing M. Since M is pseudo-coherent we have $H^i(M) = 0$ for large enough i. Let m be the smallest index such that $H^m(M)$ is nonzero. Then $H^m(M)$ is a finite A-module by More on Algebra, Lemma 15.64.3. Thus we can choose a finite free R-module F^m and a map $F^m \to M^m$ such that $F^m \to M^m \to M^{m+1}$ is zero and such that $F^m \to H^m(M)$ is surjective. Picture:

$$\begin{tikzcd}
 & F^m \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & \ldots \\
M^{m-1} \arrow[r] & M^m \arrow[r] & M^{m+1} \arrow[r] & \ldots
\end{tikzcd}$$

By descending induction on $n \leq m$ we are going to construct finite free R-modules F^i for $i \geq n$, differentials $d^i : F^i \to F^{i+1}$ for $i \geq n$, maps $\alpha : F^i \to K^i$ compatible with differentials, such that (1) $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$, and (2) $F^i = 0$ for $i > m$. Picture

$$\begin{tikzcd} & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \arrow[r] & F^i \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & \ldots \\ M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \arrow[r] & M^i \arrow[r] & M^{i+1} \arrow[r] & \ldots \end{tikzcd}$$

The base case is $n = m$ which we've done above. Induction step. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 13.9.1). The long exact sequence of cohomology shows that $H^i(C^\bullet) = 0$ for $i \geq n$. Observe that $F^\bullet$ is pseudo-coherent as a complex of A-modules because R is finite free as an A-module. Hence by More on Algebra, Lemma 15.64.2 we see that $C^\bullet$ is $(n-1)$-pseudo-coherent as a complex of A-modules. By More on Algebra, Lemma 15.64.3 we see that $H^{n-1}(C^\bullet)$ is a finite A-module. Choose a finite free R-module F^{n-1} and a map $\beta : F^{n-1} \to C^{n-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that F^{n-1} surjects onto $H^{n-1}(C^\bullet)$. Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{tikzcd} & F^{n-1} \arrow[r, "d^{n-1}"'] \arrow[d, "\alpha^{n-1}"] & F^n \arrow[r] \arrow[d, "\alpha"] & F^{n+1} \arrow[r] \arrow[d, "\alpha"] & \ldots \\ \ldots \arrow[r] & M^{n-1} \arrow[r] & M^n \arrow[r] & M^{n+1} \arrow[r] & \ldots \end{tikzcd}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{tikzcd} (F^n \to \ldots) \arrow[r] \arrow[d] & (F^{n-1} \to \ldots) \arrow[r] \arrow[d] & F^{n-1} \arrow[r] \arrow[d, "\beta"'] & (F^n \to \ldots)[1] \arrow[d] \\ (F^n \to \ldots) \arrow[r] & M^\bullet \arrow[r] & C^\bullet \arrow[r] & (F^n \to \ldots)[1] \end{tikzcd}$$

Hence our choice of β implies that the map of complexes $(F^{n-1} \to \ldots) \to M^\bullet$ induces an isomorphism on cohomology in degrees $\geq n$ and a surjection in degree $n-1$. This finishes the proof of the lemma. ☐

0CSG Lemma 36.34.2. Let A be a ring. Let $n \geq 0$. Let $K \in D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$. The following are equivalent

(1) K is pseudo-coherent,
(2) $R\Gamma(\mathbf{P}^n_A, E \otimes^{\mathbf{L}} K)$ is a pseudo-coherent object of $D(A)$ for each pseudo-coherent object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,
(3) $R\Gamma(\mathbf{P}^n_A, E \otimes^{\mathbf{L}} K)$ is a pseudo-coherent object of $D(A)$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,
(4) $R\operatorname{Hom}_{\mathbf{P}^n_A}(E, K)$ is a pseudo-coherent object of $D(A)$ for each perfect object E of $D(\mathcal{O}_{\mathbf{P}^n_A})$,
(5) $R\Gamma(\mathbf{P}^n_A, K \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(d))$ is pseudo-coherent object of $D(A)$ for $d = 0, 1, \ldots, n$.

Proof. Recall that

$$R\operatorname{Hom}_{\mathbf{P}^n_A}(E, K) = R\Gamma(\mathbf{P}^n_A, R\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n_A}}(E, K))$$

by definition, see Cohomology, Section 20.44. Thus parts (4) and (3) are equivalent by Cohomology, Lemma 20.50.5.

Since every perfect complex is pseudo-coherent, it is clear that (2) implies (3).

Assume (1) holds. Then $E \otimes^{\mathbf{L}} K$ is pseudo-coherent for every pseudo-coherent E, see Cohomology, Lemma 20.47.5. By Lemma 36.30.5 the direct image of such a pseudo-coherent complex is pseudo-coherent and we see that (2) is true.

Part (3) implies (5) because we can take $E = \mathcal{O}_{\mathbf{P}^n_A}(d)$ for $d = 0, 1, \ldots, n$.

To finish the proof we have to show that (5) implies (1). Let P be as in (36.20.0.1) and R as in (36.20.0.2). By Lemma 36.20.1 we have an equivalence

$$- \otimes_R^{\mathbf{L}} P : D(R) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathbf{P}^n_A})$$

Let $M \in D(R)$ be an object such that $M \otimes^{\mathbf{L}} P = K$. By Differential Graded Algebra, Lemma 22.35.4 there is an isomorphism

$$R\operatorname{Hom}(R, M) = R\operatorname{Hom}_{\mathbf{P}^n_A}(P, K)$$

in $D(A)$. Arguing as above we obtain

$$R\operatorname{Hom}_{\mathbf{P}^n_A}(P, K) = R\Gamma(\mathbf{P}^n_A, R\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n_A}}(E, K)) = R\Gamma(\mathbf{P}^n_A, P^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{P}^n_A}} K).$$

Using that $P^\vee$ is the direct sum of $\mathcal{O}_{\mathbf{P}^n_A}(d)$ for $d = 0, 1, \ldots, n$ and (5) we conclude $R\operatorname{Hom}(R, M)$ is pseudo-coherent as a complex of A-modules. Of course $M = R\operatorname{Hom}(R, M)$ in $D(A)$. Thus M is pseudo-coherent as a complex of A-modules. By Lemma 36.34.1 we may represent M by a bounded above complex $F^\bullet$ of finite free R-modules. Then $F^\bullet = \bigcup_{p \geq 0} \sigma_{\geq p} F^\bullet$ is a filtration which shows that $F^\bullet$ is a differential graded R-module with property (P), see Differential Graded Algebra, Section 22.20. Hence $K = M \otimes_R^{\mathbf{L}} P$ is represented by $F^\bullet \otimes_R P$ (follows from the construction of the derived tensor functor, see for example the proof of Differential Graded Algebra, Lemma 22.35.3). Since $F^\bullet \otimes_R P$ is a bounded above complex whose terms are direct sums of copies of P we conclude that the lemma is true. $\square$

0CSH Lemma 36.34.3. Let A be a ring. Let X be a scheme over A which is quasi-compact and quasi-separated. Let $K \in D^-_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect E in $D(\mathcal{O}_X)$, then $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$.

Proof. There exists an integer N such that $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ has cohomological dimension N as explained in Lemma 36.4.1. Let $b \in \mathbf{Z}$ be such that $H^i(K) = 0$ for $i > b$. Let E be pseudo-coherent on X. It suffices to show that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent for every m. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, m - N - 1 - b)$. This is possible by Theorem 36.14.6. Choose a distinguished triangle

$$P \to E \to C \to P[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. The cohomology sheaves of C are zero in degrees $\geq m - N - 1 - b$. Hence the cohomology sheaves of $C \otimes^{\mathbf{L}} K$ are zero in degrees $\geq m - N - 1$. Thus the cohomology of $R\Gamma(X, C \otimes^{\mathbf{L}} K)$ are zero in degrees $\geq m - 1$. Hence

$$R\Gamma(X, P \otimes^{\mathbf{L}} K) \to R\Gamma(X, E \otimes^{\mathbf{L}} K)$$

is an isomorphism on cohomology in degrees $\geq m$. By assumption the source is pseudo-coherent. We conclude that $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is m-pseudo-coherent as desired. $\square$

36.35. Relatively perfect objects

0DHZ In this section we introduce a notion from [Lie06a].

0DI0 Definition 36.35.1. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. An object E of $D(\mathcal{O}_X)$ is perfect relative to S or S-perfect if E is pseudo-coherent (Cohomology, Definition 20.47.1) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition 20.48.1).

Please see Remark 36.35.14 for a discussion.

0DI1 Example 36.35.2. Let k be a field. Let X be a scheme of finite presentation over k (in particular X is quasi-compact). Then an object E of $D(\mathcal{O}_X)$ is k-perfect if and only if it is bounded and pseudo-coherent (by definition), i.e., if and only if it is in $D^b_{\text{Coh}}(X)$ (by Lemma 36.10.3). Thus being relatively perfect does not mean "perfect on the fibres".

The corresponding algebra concept is studied in More on Algebra, Section 15.83. We can link the notion for schemes with the algebraic notion as follows.

0DI2 Lemma 36.35.3. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

(1) E is S-perfect,
(2) for any affine open $U \subset X$ mapping into an affine open $V \subset S$ the complex $R\Gamma(U, E)$ is $\mathcal{O}_S(V)$-perfect.
(3) there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex $R\Gamma(U_{ij}, E)$ is $\mathcal{O}_S(V_i)$-perfect.

Proof. Being pseudo-coherent is a local property and "locally having finite tor dimension" is a local property. Hence this lemma immediately reduces to the statement: if X and S are affine, then E is S-perfect if and only if $K = R\Gamma(X, E)$ is $\mathcal{O}_S(S)$-perfect. Say $X = \text{Spec}(A)$, $S = \text{Spec}(R)$ and E corresponds to $K \in D(A)$, i.e., $K = R\Gamma(X, E)$, see Lemma 36.3.5.

Observe that K is R-perfect if and only if K is pseudo-coherent and has finite tor dimension as a complex of R-modules (More on Algebra, Definition 15.83.1). By Lemma 36.10.2 we see that E is pseudo-coherent if and only if K is pseudo-coherent. By Lemma 36.10.5 we see that E has finite tor dimension over $f^{-1}\mathcal{O}_S$ if and only if K has finite tor dimension as a complex of R-modules. $\square$

0DI3 Lemma 36.35.4. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. The full subcategory of $D(\mathcal{O}_X)$ consisting of S-perfect objects is a saturated[4] triangulated subcategory.

Proof. This follows from Cohomology, Lemmas 20.47.4, 20.47.6, 20.48.6, and 20.48.8. $\square$

[4]Derived Categories, Definition 13.6.1.

0DI4 Lemma 36.35.5. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. A perfect object of $D(\mathcal{O}_X)$ is S-perfect. If $K, M \in D(\mathcal{O}_X)$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} M$ is S-perfect if K is perfect and M is S-perfect.

Proof. First proof: reduce to the affine case using Lemma 36.35.3 and then apply More on Algebra, Lemma 15.83.3. □

0DI5 Lemma 36.35.6. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $g : S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$ and denote $g' : X' \to X$ the projection. If $K \in D(\mathcal{O}_X)$ is S-perfect, then $L(g')^*K$ is S'-perfect.

Proof. First proof: reduce to the affine case using Lemma 36.35.3 and then apply More on Algebra, Lemma 15.83.5.

Second proof: $L(g')^*K$ is pseudo-coherent by Cohomology, Lemma 20.47.3 and the bounded tor dimension property follows from Lemma 36.22.8. □

0DI6 Situation 36.35.7. Let $S = \lim_{i\in I} S_i$ be a limit of a directed system of schemes with affine transition morphisms $g_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $g_i : S \to S_i$ the projection. We fix an element $0 \in I$ and a flat morphism of finite presentation $X_0 \to S_0$. We set $X_i = S_i \times_{S_0} X_0$ and $X = S \times_{S_0} X_0$ and we denote the transition morphisms $f_{i'i} : X_{i'} \to X_i$ and $f_i : X \to X_i$ the projections.

0DI7 Lemma 36.35.8. In Situation 36.35.7. Let K_0 and L_0 be objects of $D(\mathcal{O}_{X_0})$. Set $K_i = Lf_{i0}^*K_0$ and $L_i = Lf_{i0}^*L_0$ for $i \geq 0$ and set $K = Lf_0^*K_0$ and $L = Lf_0^*L_0$. Then the map

$$\operatorname{colim}_{i\geq 0} \operatorname{Hom}_{D(\mathcal{O}_{X_i})}(K_i, L_i) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_X)}(K, L)$$

is an isomorphism if K_0 is pseudo-coherent and $L_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has (locally) finite tor dimension as an object of $D((X_0 \to S_0)^{-1}\mathcal{O}_{S_0})$

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition P that

$$\operatorname{colim}_{i\geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(K_i|_{U_i}, L_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U)$$

is an isomorphism where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. If P holds for U_0, V_0 and $U_0 \cap V_0$, then it holds for $U_0 \cup V_0$ by Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 20.33.3.

Denote $\pi_0 : X_0 \to S_0$ the given morphism. Then we can first consider $U_0 = \pi_0^{-1}(W_0)$ with $W_0 \subset S_0$ quasi-compact open. By the induction principle of Cohomology of Schemes, Lemma 30.4.1 applied to quasi-compact opens of S_0 and the remark above, we find that it is enough to prove P for $U_0 = \pi_0^{-1}(W_0)$ with W_0 affine. In other words, we have reduced to the case where S_0 is affine. Next, we apply the induction principle again, this time to all quasi-compact and quasi-separated opens of X_0, to reduce to the case where X_0 is affine as well.

If X_0 and S_0 are affine, the result follows from More on Algebra, Lemma 15.83.7. Namely, by Lemmas 36.10.1 and 36.3.5 the statement is translated into computations of homs in the derived categories of modules. Then Lemma 36.10.2 shows that the complex of modules corresponding to K_0 is pseudo-coherent. And Lemma

36.10.5 shows that the complex of modules corresponding to L_0 has finite tor dimension over $\mathcal{O}_{S_0}(S_0)$. Thus the assumptions of More on Algebra, Lemma 15.83.7 are satisfied and we win. $\square$

0DI8 Lemma 36.35.9. In Situation 36.35.7 the category of S-perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of S_i-perfect objects of $D(\mathcal{O}_{X_i})$.

Proof. For every quasi-compact open $U_0 \subset X_0$ consider the condition P that the functor

$$\mathrm{colim}_{i \geq 0} D_{S_i\text{-perfect}}(\mathcal{O}_{U_i}) \longrightarrow D_{S\text{-perfect}}(\mathcal{O}_U)$$

is an equivalence where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We observe that we already know this functor is fully faithful by Lemma 36.35.8. Thus it suffices to prove essential surjectivity.

Suppose that P holds for quasi-compact opens U_0, V_0 of X_0. We claim that P holds for $U_0 \cup V_0$. We will use the notation $U_i = f_{i0}^{-1}U_0$, $U = f_0^{-1}U_0$, $V_i = f_{i0}^{-1}V_0$, and $V = f_0^{-1}V_0$ and we will abusively use the symbol f_i for all the morphisms $U \to U_i$, $V \to V_i$, $U \cap V \to U_i \cap V_i$, and $U \cup V \to U_i \cup V_i$. Suppose E is an S-perfect object of $D(\mathcal{O}_{U \cup V})$. Goal: show E is in the essential image of the functor. By assumption, we can find $i \geq 0$, an S_i-perfect object $E_{U,i}$ on U_i, an S_i-perfect object $E_{V,i}$ on V_i, and isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. Let

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \to E|_{U \cap V} \to Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 20.45.1 to get an object E_i on $U_i \cup V_i$ and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is S_i-perfect (because being relatively perfect is a local property) and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

By exactly the same argument as used in the proof of Lemma 36.35.8 using the induction principle (Cohomology of Schemes, Lemma 30.4.1) we reduce to the case where both X_0 and S_0 are affine. (First work with opens in S_0 to reduce to S_0 affine, then work with opens in X_0 to reduce to X_0 affine.) In the affine case the result follows from More on Algebra, Lemma 15.83.7. The translation into algebra is done by Lemma 36.35.3. $\square$

0DJT Lemma 36.35.10. Let $f : X \to S$ be a morphism of schemes which is flat, proper, and of finite presentation. Let $E \in D(\mathcal{O}_X)$ be S-perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$ and its formation commutes with arbitrary base change.

Proof. The statement on base change is Lemma 36.22.5. Thus it suffices to show that Rf_*E is a perfect object. We will reduce to the case where S is Noetherian affine by a limit argument.

The question is local on S, hence we may assume S is affine. Say $S = \mathrm{Spec}(R)$. We write $R = \mathrm{colim}\, R_i$ as a filtered colimit of Noetherian rings R_i. By Limits, Lemma 32.10.1 there exists an i and a scheme X_i of finite presentation over R_i whose base change to R is X. By Limits, Lemmas 32.13.1 and 32.8.7 we may assume X_i is proper and flat over R_i. By Lemma 36.35.9 we may assume there exists a R_i-perfect object E_i of $D(\mathcal{O}_{X_i})$ whose pullback to X is E. Applying Lemma 36.27.1 to $X_i \to \mathrm{Spec}(R_i)$ and E_i and using the base change property already shown we obtain the result. $\square$

0DJU Lemma 36.35.11. Let $f : X \to S$ be a morphism of schemes. Let $E, K \in D(\mathcal{O}_X)$. Assume

(1) S is quasi-compact and quasi-separated,
(2) f is proper, flat, and of finite presentation,
(3) E is S-perfect,
(4) K is pseudo-coherent.

Then there exists a pseudo-coherent $L \in D(\mathcal{O}_S)$ such that

$$Rf_*R\mathcal{H}om(K, E) = R\mathcal{H}om(L, \mathcal{O}_S)$$

and the same is true after arbitrary base change: given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd} \quad \begin{array}{c} \text{cartesian, then we have} \\ Rf'_*R\mathcal{H}om(L(g')^*K, L(g')^*E) \\ = R\mathcal{H}om(Lg^*L, \mathcal{O}_{S'}) \end{array}$$

Proof. Since S is quasi-compact and quasi-separated, the same is true for X. By Lemma 36.19.1 we can write $K = \mathrm{hocolim} K_n$ with K_n perfect and $K_n \to K$ inducing an isomorphism on truncations $\tau_{\geq -n}$. Let $K_n^\vee$ be the dual perfect complex (Cohomology, Lemma 20.50.5). We obtain an inverse system $\ldots \to K_3^\vee \to K_2^\vee \to K_1^\vee$ of perfect objects. By Lemma 36.35.5 we see that $K_n^\vee \otimes_{\mathcal{O}_X} E$ is S-perfect. Thus we may apply Lemma 36.35.10 to $K_n^\vee \otimes_{\mathcal{O}_X} E$ and we obtain an inverse system

$$\ldots \to M_3 \to M_2 \to M_1$$

of perfect complexes on S with

$$M_n = Rf_*(K_n^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = Rf_*R\mathcal{H}om(K_n, E)$$

Moreover, the formation of these complexes commutes with any base change, namely $Lg^*M_n = Rf'_*((L(g')^*K_n)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} L(g')^*E) = Rf'_*R\mathcal{H}om(L(g')^*K_n, L(g')^*E)$.

As $K_n \to K$ induces an isomorphism on $\tau_{\geq -n}$, we see that $K_n \to K_{n+1}$ induces an isomorphism on $\tau_{\geq -n}$. It follows that $K_{n+1}^\vee \to K_n^\vee$ induces an isomorphism on $\tau_{\leq n}$ as $K_n^\vee = R\mathcal{H}om(K_n, \mathcal{O}_X)$. Suppose that E has tor amplitude in $[a, b]$ as a complex of $f^{-1}\mathcal{O}_Y$-modules. Then the same is true after any base change, see Lemma 36.22.8. We find that $K_{n+1}^\vee \otimes_{\mathcal{O}_X} E \to K_n^\vee \otimes_{\mathcal{O}_X} E$ induces an isomorphism on $\tau_{\leq n+a}$ and the same is true after any base change. Applying the right derived functor Rf_* we conclude the maps $M_{n+1} \to M_n$ induce isomorphisms on $\tau_{\leq n+a}$ and the same is true after any base change. Choose a distinguished triangle

$$M_{n+1} \to M_n \to C_n \to M_{n+1}[1]$$

Take S' equal to the spectrum of the residue field at a point $s \in S$ and pull back to see that $C_n \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s)$ has nonzero cohomology only in degrees $\geq n + a$. By More on Algebra, Lemma 15.75.6 we see that the perfect complex C_n has tor amplitude in $[n + a, m_n]$ for some integer m_n. In particular, the dual perfect complex $C_n^\vee$ has tor amplitude in $[-m_n, -n - a]$.

Let $L_n = M_n^\vee$ be the dual perfect complex. The conclusion from the discussion in the previous paragraph is that $L_n \to L_{n+1}$ induces isomorphisms on $\tau_{\geq -n-a}$. Thus $L = \text{hocolim} L_n$ is pseudo-coherent, see Lemma 36.19.1. Since we have

$$R\mathcal{H}om(K, E) = R\mathcal{H}om(\text{hocolim} K_n, E) = R\lim R\mathcal{H}om(K_n, E) = R\lim K_n^\vee \otimes_{\mathcal{O}_X} E$$

(Cohomology, Lemma 20.51.1) and since $R\lim$ commutes with Rf_* we find that

$$Rf_* R\mathcal{H}om(K, E) = R\lim M_n = R\lim R\mathcal{H}om(L_n, \mathcal{O}_S) = R\mathcal{H}om(L, \mathcal{O}_S)$$

This proves the formula over S. Since the construction of M_n is compatible with base chance, the formula continues to hold after any base change. ☐

0DJV Remark 36.35.12. The reader may have noticed the similarity between Lemma 36.35.11 and Lemma 36.28.3. Indeed, the pseudo-coherent complex L of Lemma 36.35.11 may be characterized as the unique pseudo-coherent complex on S such that there are functorial isomorphisms

$$\text{Ext}^i_{\mathcal{O}_S}(L, \mathcal{F}) \longrightarrow \text{Ext}^i_{\mathcal{O}_X}(K, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})$$

compatible with boundary maps for $\mathcal{F}$ ranging over $QCoh(\mathcal{O}_S)$. If we ever need this we will formulate a precise result here and give a detailed proof.

0GEH Lemma 36.35.13. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let E be a pseudo-coherent object of $D(\mathcal{O}_X)$. The following are equivalent

(1) E is S-perfect, and
(2) E is locally bounded below and for every point $s \in S$ the object $L(X_s \to X)^*E$ of $D(\mathcal{O}_{X_s})$ is locally bounded below.

Proof. Since everything is local we immediately reduce to the case that X and S are affine, see Lemma 36.35.3. Say $X \to S$ corresponds to $\text{Spec}(A) \to \text{Spec}(R)$ and E corresponds to K in $D(A)$. If s corresponds to the prime $\mathfrak{p} \subset R$, then $L(X_s \to X)^*E$ corresponds to $K \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})$ as $R \to A$ is flat, see for example Lemma 36.22.5. Thus we see that our lemma follows from the corresponding algebra result, see More on Algebra, Lemma 15.83.10. ☐

0DI9 Remark 36.35.14. Our Definition 36.35.1 of a relatively perfect complex is equivalent to the one given in [Lie06a] whenever our definition applies[5]. Next, suppose that $f : X \to S$ is only assumed to be locally of finite type (not necessarily flat, nor locally of finite presentation). The definition in the paper cited above is that $E \in D(\mathcal{O}_X)$ is relatively perfect if

(A) locally on X the object E should be quasi-isomorphic to a finite complex of S-flat, finitely presented $\mathcal{O}_X$-modules.

On the other hand, the natural generalization of our Definition 36.35.1 is

(B) E is pseudo-coherent relative to S (More on Morphisms, Definition 37.59.2) and E locally has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$ (Cohomology, Definition 20.48.1).

The advantage of condition (B) is that it clearly defines a triangulated subcategory of $D(\mathcal{O}_X)$, whereas we suspect this is not the case for condition (A). The advantage of condition (A) is that it is easier to work with in particular in regards to limits.

[5] To see this, use Lemma 36.35.3 and More on Algebra, Lemma 15.83.4.

36.36. The resolution property

0F85 This notion is discussed in the paper [Tot04]; the discussion is continued in [Gro10], [Gro12], and [Gro17]. It is currently not known if a proper scheme over a field always has the resolution property or if this is false. If you know the answer to this question, please email stacks.project@gmail.com.

We can make the following definition although it scarcely makes sense to consider it for general schemes.

0F86 **Definition 36.36.1.** Let X be a scheme. We say X has the *resolution property* if every quasi-coherent $\mathcal{O}_X$-module of finite type is the quotient of a finite locally free $\mathcal{O}_X$-module.

If X is a quasi-compact and quasi-separated scheme, then it suffices to check every $\mathcal{O}_X$-module module of finite presentation (automatically quasi-coherent) is the quotient of a finite locally free $\mathcal{O}_X$-module, see Properties, Lemma 28.22.8. If X is a Noetherian scheme, then finite type quasi-coherent modules are exactly the coherent $\mathcal{O}_X$-modules, see Cohomology of Schemes, Lemma 30.9.1.

0F87 **Lemma 36.36.2.** *Let X be a scheme. If X has an ample invertible $\mathcal{O}_X$-module, then X has the resolution property.*

Proof. Immediate consquence of Properties, Proposition 28.26.13. $\square$

0FDD **Lemma 36.36.3.** *Let $f : X \to Y$ be a morphism of schemes. Assume*

1. *Y is quasi-compact and quasi-separated and has the resolution property,*
2. *there exists an f-ample invertible module on X.*

Then X has the resolution property.

Proof. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an f-ample invertible module. Choose an affine open covering $Y = V_1 \cup \ldots \cup V_m$. Set $U_j = f^{-1}(V_j)$. By Properties, Proposition 28.26.13 for each j we know there exists finitely many maps $s_{j,i} : \mathcal{L}^{\otimes n_{j,i}}|_{U_j} \to \mathcal{F}|_{U_j}$ which are jointly surjective. Consider the quasi-coherent $\mathcal{O}_Y$-modules

$$\mathcal{H}_n = f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

We may think of $s_{j,i}$ as a section over V_j of the sheaf $\mathcal{H}_{-n_{j,i}}$. Suppose we can find finite locally free $\mathcal{O}_Y$-modules $\mathcal{E}_{i,j}$ and maps $\mathcal{E}_{i,j} \to \mathcal{H}_{-n_{j,i}}$ such that $s_{j,i}$ is in the image. Then the corresponding maps

$$f^*\mathcal{E}_{i,j} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n_{i,j}} \longrightarrow \mathcal{F}$$

are going to be jointly surjective and the lemma is proved. By Properties, Lemma 28.22.3 for each i, j we can find a finite type quasi-coherent submodule $\mathcal{H}'_{i,j} \subset \mathcal{H}_{-n_{j,i}}$ which contains the section $s_{i,j}$ over V_j. Thus the resolution property of Y produces surjections $\mathcal{E}_{i,j} \to \mathcal{H}'_{j,i}$ and we conclude. $\square$

0F88 **Lemma 36.36.4.** *Let $f : X \to Y$ be an affine or quasi-affine morphism of schemes with Y quasi-compact and quasi-separated. If Y has the resolution property, so does X.*

Proof. By Morphisms, Lemma 29.37.6 this is a special case of Lemma 36.36.3. $\square$

Here is a case where one can prove the resolution property goes down.

0GTC Lemma 36.36.5. Let $f : X \to Y$ be a surjective finite locally free morphism of schemes. If X has the resolution property, so does Y.

Proof. The condition means that f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module of positive rank. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module of finite type. By assumption there exists a surjection $\mathcal{E} \to f^*\mathcal{G}$ for some finite locally free $\mathcal{O}_X$-module $\mathcal{E}$. Since f_* is exact on quasi-coherent modules (Cohomology of Schemes, Lemma 30.2.3) we get a surjection

$$f_*\mathcal{E} \longrightarrow f_*f^*\mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X$$

Taking duals we get a surjection

$$f_*\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathcal{G}$$

Since $f_*\mathcal{E}$ is finite locally free[6], we conclude. □

0F89 Lemma 36.36.6. Let X be a scheme. Suppose given

(1) a finite affine open covering $X = U_1 \cup \ldots \cup U_m$
(2) finite type quasi-coherent ideals $\mathcal{I}_j$ with $V(\mathcal{I}_j) = X \setminus U_j$

Then X has the resolution property if and only if $\mathcal{I}_j$ is the quotient of a finite locally free $\mathcal{O}_X$-module for $j = 1, \ldots, m$.

Proof. One direction of the lemma is trivial. For the other, say $\mathcal{E}_j \to \mathcal{I}_j$ is a surjection with $\mathcal{E}_j$ finite locally free. In the next paragraph, we reduce to the Noetherian case; we suggest the reader skip it.

The first observation is that $U_j \cap U_{j'}$ is quasi-compact as the complement of the zero scheme of the quasi-coherent finite type ideal $\mathcal{I}_{j'}|U_j$ on the affine scheme U_j, see Properties, Lemma 28.24.1. Hence X is quasi-compact and quasi-separated, see Schemes, Lemma 26.21.6. By Limits, Proposition 32.5.4 we can write $X = \lim X_i$ as the limit of a direct system of Noetherian schemes with affine transition morphisms. For each j we can find an i and a finite locally free $\mathcal{O}_{X_i}$-module $\mathcal{E}_{i,j}$ pulling back to $\mathcal{E}_j$, see Limits, Lemma 32.10.3. After increasing i we may assume that the composition $\mathcal{E}_j \to \mathcal{I}_j \to \mathcal{O}_X$ is the pullback of a map $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$, see Limits, Lemma 32.10.2. Denote $\mathcal{I}_{i,j} \subset \mathcal{O}_{X_i}$ the image of this map; this is a quasi-coherent ideal sheaf on the Noetherian scheme X_i whose pullback to X is $\mathcal{I}_j$. Denoting $U_{i,j} \subset X_i$ the complementary opens, we may assume these are affine for all i, j, see Limits, Lemma 32.4.13. If we can prove the lemma for the opens $U_{i,j}$ and the ideal sheaves $\mathcal{I}_{i,j}$ on X_i then X, being affine over X_i, will have the resolution property by Lemma 36.36.4. In this way we reduce to the case of a Noetherian scheme.

Assume X is Noetherian. For every coherent module $\mathcal{F}$ we can choose a finite list of sections $s_{jk} \in \mathcal{F}(U_j)$, $k = 1, \ldots, e_j$ which generate the restriction of $\mathcal{F}$ to U_j. By Cohomology of Schemes, Lemma 30.10.5 we can extend s_{jk} to a map $s'_{jk} : \mathcal{I}_i^{n_{jk}} \to \mathcal{F}$ for some $n_{jk} \geq 1$. Then we can consider the compositions

$$\mathcal{E}_j^{\otimes n_{jk}} \to \mathcal{I}_j^{n_{jk}} \to \mathcal{F}$$

to conclude. □

[6]Namely, if $A \to B$ is a finite locally free ring map and N is a finite locally free B-module, then N is a finite locally free A-module. To see this, first note that N finite locally free over B implies N is flat and finitely presented as a B-module, see Algebra, Lemma 10.78.2. Then N is an A-module of finite presentation by Algebra, Lemma 10.36.23 and a flat A-module by Algebra, Lemma 10.39.4. Then conclude by using Algebra, Lemma 10.78.2 over A.

0GMM Lemma 36.36.7. Let X be a scheme. If X has an ample family of invertible modules (Morphisms, Definition 29.12.1), then X has the resolution property.

Proof. Since X is quasi-compact, there exists n and pairs $(\mathcal{L}_i, s_i)$, $i = 1, \ldots, n$ where $\mathcal{L}_i$ is an invertible $\mathcal{O}_X$-module and $s_i \in \Gamma(X, \mathcal{L}_i)$ is a section such that the set of points $U_i \subset X$ where s_i is nonvanishing is affine and $X = U_1 \cup \ldots \cup U_n$. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the image of $s_i : \mathcal{L}_i^{\otimes -1} \to \mathcal{O}_X$. Applying Lemma 36.36.6 we find that X has the resolution property. □

0F8A Lemma 36.36.8. Let X be a quasi-compact, regular scheme with affine diagonal. Then X has the resolution property.

Proof. Combine Divisors, Lemma 31.16.8 and the above Lemma 36.36.7. □

0F8B Lemma 36.36.9. Let $X = \lim X_i$ be a limit of a direct system of quasi-compact and quasi-separated schemes with affine transition morphisms. Then X has the resolution property if and only if X_i has the resolution properties for some i.

Proof. If X_i has the resolution property, then X does by Lemma 36.36.4. Assume X has the resolution property. Choose $i \in I$. Choose a finite affine open covering $X_i = U_{i,1} \cup \ldots \cup U_{i,m}$. For each j choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}_{i,j} \subset \mathcal{O}_{X_i}$ such that $X_i \setminus V(\mathcal{I}_{i,j}) = U_{i,j}$, see Properties, Lemma 28.24.1. Denote $U_j \subset X$ the inverse image of $U_{i,j}$ and denote $\mathcal{I}_j \subset \mathcal{O}_X$ the pullback of $\mathcal{I}_{i,j}$. Since X has the resolution property, we may choose finite locally free $\mathcal{O}_X$-modules $\mathcal{E}_j$ and surjections $\mathcal{E}_j \to \mathcal{I}_j$. By Limits, Lemmas 32.10.3 and 32.10.2 after increasing i we can find finite locally free $\mathcal{O}_{X_i}$-modules $\mathcal{E}_{i,j}$ and maps $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$ whose base changes to X recover the compositions $\mathcal{E}_j \to \mathcal{I}_j \to \mathcal{O}_X$, $j = 1, \ldots, m$. The pullbacks of the finitely presented $\mathcal{O}_{X_i}$-modules $\text{Coker}(\mathcal{E}_{i,j} \to \mathcal{O}_{X_i})$ and $\mathcal{O}_{X_i}/\mathcal{I}_{i,j}$ to X agree as quotients of $\mathcal{O}_X$. Hence by Limits, Lemma 32.10.2 we may assume that these agree, in other words that the image of $\mathcal{E}_{i,j} \to \mathcal{O}_{X_i}$ is equal to $\mathcal{I}_{i,j}$. Then we conclude that X_i has the resolution property by Lemma 36.36.6. □

0F8C Lemma 36.36.10. Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then X has affine diagonal. Special case of [Tot04, Proposition 1.3].

Proof. Combining Limits, Proposition 32.5.4 and Lemma 36.36.9 this reduces to the case where X is Noetherian (small detail omitted). Assume X is Noetherian. Recall that $X \times X$ is covered by the affine opens $U \times V$ for affine opens U, V of X, see Schemes, Section 26.17. Hence to show that the diagonal $\Delta : X \to X \times X$ is affine, it suffices to show that $U \cap V = \Delta^{-1}(U \times V)$ is affine for all affine opens U, V of X, see Morphisms, Lemma 29.11.3. In particular, it suffices to show that the inclusion morphism $j : U \to X$ is affine if U is an affine open of X. By Cohomology of Schemes, Lemma 30.3.4 it suffices to show that $R^1 j_*\mathcal{G} = 0$ for any quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$. By Proposition 36.8.3 (this is where we use that we've reduced to the Noetherian case) we can represent $Rj_*\mathcal{G}$ by a complex $\mathcal{H}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules. Assume

$$H^1(\mathcal{H}^\bullet) = \text{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)/\operatorname{Im}(\mathcal{H}^0 \to \mathcal{H}^1)$$

is nonzero in order to get a contradiction. Then we can find a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a map

$$\mathcal{F} \longrightarrow \text{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)$$

such that the composition with the projection onto $H^1(\mathcal{H}^\bullet)$ is nonzero. Namely, we can write $\text{Ker}(\mathcal{H}^1 \to \mathcal{H}^2)$ as the filtered union of its coherent submodules by Properties, Lemma 28.22.3 and then one of these will do the job. Next, we choose a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and a surjection $\mathcal{E} \to \mathcal{F}$ using the resolution property of X. This produces a map in the derived category

$$\mathcal{E}[-1] \longrightarrow Rj_*\mathcal{G}$$

which is nonzero on cohomology sheaves and hence nonzero in $D(\mathcal{O}_X)$. By adjunction, this is the same thing as a map

$$j^*\mathcal{E}[-1] \to \mathcal{G}$$

nonzero in $D(\mathcal{O}_U)$. Since $\mathcal{E}$ is finite locally free this is the same thing as a nonzero element of

$$H^1(U, j^*\mathcal{E}^\vee \otimes_{\mathcal{O}_U} \mathcal{G})$$

where $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual finite locally free module. However, this group is zero by Cohomology of Schemes, Lemma 30.2.2 which is the desired contradiction. (If in doubt about the step using duals, please see the more general Cohomology, Lemma 20.50.5.) $\square$

36.37. The resolution property and perfect complexes

0F8D In this section we discuss the relationship between perfect complexes and strictly perfect complexes on schemes which have the resolution property.

0F8E **Lemma 36.37.1.** Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{F}^\bullet$ be a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules representing a perfect object of $D(\mathcal{O}_X)$. Then there exists a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$.

Proof. Let $a, b \in \mathbf{Z}$ be integers such that $\mathcal{F}^\bullet$ has tor amplitude in $[a, b]$ and such that $\mathcal{F}^n = 0$ for $n < a$. The existence of such a pair of integers follows from Cohomology, Lemma 20.49.5 and the fact that X is quasi-compact. If $b < a$, then $\mathcal{F}^\bullet$ is zero in the derived category and the lemma holds. We will prove by induction on $b - a \geq 0$ that there exists a complex $\mathcal{E}^a \to \ldots \to \mathcal{E}^b$ with $\mathcal{E}^i$ finite locally free and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$.

The base case is the case $b - a = 0$. In this case $H^b(\mathcal{F}^\bullet) = H^a(\mathcal{F}^\bullet) = \text{Ker}(\mathcal{F}^a \to \mathcal{F}^{a+1})$ is finite locally free. Namely, it is a finitely presented $\mathcal{O}_X$-module of tor dimension 0 and hence finite locally free. See Cohomology, Lemmas 20.49.5 and 20.47.9 and Properties, Lemma 28.20.2. Thus we can take $\mathcal{E}^\bullet$ to be $H^b(\mathcal{F}^\bullet)$ sitting in degree b. The rest of the proof is dedicated to the induction step.

Assume $b > a$. Observe that

$$H^b(\mathcal{F}^\bullet) = \text{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1}) / \text{Im}(\mathcal{F}^{b-1} \to \mathcal{F}^b)$$

is a finite type quasi-coherent $\mathcal{O}_X$-module, see Cohomology, Lemmas 20.49.5 and 20.47.9. Then we can find a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a map

$$\mathcal{F} \longrightarrow \text{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})$$

such that the composition with the projection onto $H^b(\mathcal{F}^\bullet)$ is surjective. Namely, we can write $\text{Ker}(\mathcal{F}^b \to \mathcal{F}^{b+1})$ as the filtered union of its finite type quasi-coherent submodules by Properties, Lemma 28.22.3 and then one of these will do the job.

Next, we choose a finite locally free $\mathcal{O}_X$-module $\mathcal{E}^b$ and a surjection $\mathcal{E}^b \to \mathcal{F}$ using the resolution property of X. Consider the map of complexes

$$\alpha : \mathcal{E}^b[-b] \to \mathcal{F}^\bullet$$

and its cone $C(\alpha)^\bullet$, see Derived Categories, Definition 13.9.1. We observe that $C(\alpha)^\bullet$ is nonzero only in degrees $\geq a$, has tor amplitude in $[a, b]$ by Cohomology, Lemma 20.48.6, and has $H^b(C(\alpha)^\bullet) = 0$ by construction. Thus we actually find that $C(\alpha)^\bullet$ has tor amplitude in $[a, b-1]$. Hence the induction hypothesis applies to $C(\alpha)^\bullet$ and we find a map of complexes

$$(\mathcal{E}^a \to \ldots \to \mathcal{E}^{b-1}) \longrightarrow C(\alpha)^\bullet$$

with properties as stated in the induction hypothesis. Unwinding the definition of the cone this gives a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathcal{E}^{b-2} \arrow[r] \arrow[d] & \mathcal{E}^{b-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & \mathcal{F}^{b-2} \arrow[r] & \mathcal{F}^{b-1} \oplus \mathcal{E}^b \arrow[r] & \mathcal{F}^b \arrow[r] & \ldots
\end{tikzcd}$$

It is clear that we obtain a map of complexes $(\mathcal{E}^a \to \ldots \to \mathcal{E}^b) \to \mathcal{F}^\bullet$. We omit the verification that this map is a quasi-isomorphism. $\square$

0F8F Lemma 36.37.2. Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then every perfect object of $D(\mathcal{O}_X)$ can be represented by a bounded complex of finite locally free $\mathcal{O}_X$-modules.

Proof. Let E be a perfect object of $D(\mathcal{O}_X)$. By Lemma 36.36.10 we see that X has affine diagonal. Hence by Proposition 36.7.5 we can represent E by a complex $\mathcal{F}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules. Observe that E is in $D^b(\mathcal{O}_X)$ because X is quasi-compact. Hence $\tau_{\geq n}\mathcal{F}^\bullet$ is a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules which represents E if $n \ll 0$. Thus we may apply Lemma 36.37.1 to conclude. $\square$

0F8G Lemma 36.37.3. Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be finite complexes of finite locally free $\mathcal{O}_X$-modules. Then any $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ can be represented by a diagram

$$\mathcal{E}^\bullet \leftarrow \mathcal{G}^\bullet \to \mathcal{F}^\bullet$$

where $\mathcal{G}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_X$-modules and where $\mathcal{G}^\bullet \to \mathcal{E}^\bullet$ is a quasi-isomorphism.

Proof. By Lemma 36.36.10 we see that X has affine diagonal. Hence by Proposition 36.7.5 we can represent α by a diagram

$$\mathcal{E}^\bullet \leftarrow \mathcal{H}^\bullet \to \mathcal{F}^\bullet$$

where $\mathcal{H}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules and where $\mathcal{H}^\bullet \to \mathcal{E}^\bullet$ is a quasi-isomorphism. For $n \ll 0$ the maps $\mathcal{H}^\bullet \to \mathcal{E}^\bullet$ and $\mathcal{H}^\bullet \to \mathcal{F}^\bullet$ factor through the quasi-isomorphism $\mathcal{H}^\bullet \to \tau_{\geq n}\mathcal{H}^\bullet$ simply because $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are bounded complexes. Thus we may replace $\mathcal{H}^\bullet$ by $\tau_{\geq n}\mathcal{H}^\bullet$ and assume that $\mathcal{H}^\bullet$ is bounded below. Then we may apply Lemma 36.37.1 to conclude. $\square$

0F8H Lemma 36.37.4. Let X be a quasi-compact and quasi-separated scheme with the resolution property. Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be finite complexes of finite locally free $\mathcal{O}_X$-modules. Let $\alpha^\bullet, \beta^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be two maps of complexes defining the same map in $D(\mathcal{O}_X)$. Then there exists a quasi-isomorphism $\gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ where $\mathcal{G}^\bullet$ is a bounded complex of finite locally free $\mathcal{O}_X$-modules such that $\alpha^\bullet \circ \gamma^\bullet$ and $\beta^\bullet \circ \gamma^\bullet$ are homotopic maps of complexes.

Proof. By Lemma 36.36.10 we see that X has affine diagonal. Hence by Proposition 36.7.5 (and the definition of the derived category) there exists a quasi-isomorphism $\gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ where $\mathcal{G}^\bullet$ is a complex of quasi-coherent $\mathcal{O}_X$-modules such that $\alpha^\bullet \circ \gamma^\bullet$ and $\beta^\bullet \circ \gamma^\bullet$ are homotopic maps of complexes. Choose a homotopy $h^i : \mathcal{G}^i \to \mathcal{F}^{i-1}$ witnessing this fact. Choose $n \ll 0$. Then the map $\gamma^\bullet$ factors canonically over the quotient map $\mathcal{G}^\bullet \to \tau_{\geq n}\mathcal{G}^\bullet$ as $\mathcal{E}^\bullet$ is bounded below. For the exact same reason the maps h^i will factor over the surjections $\mathcal{G}^i \to (\tau_{\geq n}\mathcal{G})^i$. Hence we see that we may replace $\mathcal{G}^\bullet$ by $\tau_{\geq n}\mathcal{G}^\bullet$. Then we may apply Lemma 36.37.1 to conclude. □

0F8I Proposition 36.37.5. Let X be a quasi-compact and quasi-separated scheme with the resolution property. Denote

(1) $\mathcal{A}$ the additive category of finite locally free $\mathcal{O}_X$-modules,
(2) $K^b(\mathcal{A})$ the homotopy category of bounded complexes in $\mathcal{A}$, see Derived Categories, Section 13.8, and
(3) $D_{perf}(\mathcal{O}_X)$ the strictly full, saturated, triangulated subcategory of $D(\mathcal{O}_X)$ consisting of perfect objects.

With this notation the obvious functor

$$K^b(\mathcal{A}) \longrightarrow D_{perf}(\mathcal{O}_X)$$

is an exact functor of trianglated categories which factors through an equivalence $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ of triangulated categories where S is the saturated multiplicative system of quasi-isomorphisms in $K^b(\mathcal{A})$.

Proof. If you can parse the statement of the proposition, then please skip this first paragraph. For some of the definitions used, please see Derived Categories, Definition 13.3.4 (triangulated subcategory), Derived Categories, Definition 13.6.1 (saturated triangulated subcategory), Derived Categories, Definition 13.5.1 (multiplicative system compatible with the triangulated structure), and Categories, Definition 4.27.20 (saturated multiplicative system). Observe that $D_{perf}(\mathcal{O}_X)$ is a saturated triangulated subcategory of $D(\mathcal{O}_X)$ by Cohomology, Lemmas 20.49.7 and 20.49.9. Also, note that $K^b(\mathcal{A})$ is a triangulated category, see Derived Categories, Lemma 13.10.5.

It is clear that the functor sends distinguished triangles to distinguished triangles, i.e., is exact. Then S is a saturated multiplicative system compatible with the triangulated structure on $K^b(\mathcal{A})$ by Derived Categories, Lemma 13.5.4. Hence the localization $S^{-1}K^b(\mathcal{A})$ exists and is a triangulated category by Derived Categories, Proposition 13.5.6. We get an exact factorization $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ by Derived Categories, Lemma 13.5.7. By Lemmas 36.37.2, 36.37.3, and 36.37.4 this functor is an equivalence. Then finally the functor $S^{-1}K^b(\mathcal{A}) \to D_{perf}(\mathcal{O}_X)$ is an equivalence of triangulated categories (in the sense that distinguished triangles correspond) by Derived Categories, Lemma 13.4.18. □

36.38. K-groups

0FDE A tiny bit about K_0 of various categories associated to schemes. Previous material can be found in Algebra, Section 10.55, Homology, Section 12.11, Derived Categories, Section 13.28, and More on Algebra, Lemma 15.119.2.

Analogous to Algebra, Section 10.55 we will define two K-groups $K'_0(X)$ and $K_0(X)$ for any Noetherian scheme X. The first will use coherent $\mathcal{O}_X$-modules and the second will use finite locally free $\mathcal{O}_X$-modules.

0FDF **Lemma 36.38.1.** *Let X be a Noetherian scheme. Then*

$$K_0(\text{Coh}(\mathcal{O}_X)) = K_0(D^b(\text{Coh}(\mathcal{O}_X)) = K_0(D^b_{\text{Coh}}(\mathcal{O}_X))$$

Proof. The first equality is Derived Categories, Lemma 13.28.2. We have $K_0(\text{Coh}(\mathcal{O}_X)) = K_0(D^b_{\text{Coh}}(\mathcal{O}_X))$ by Derived Categories, Lemma 13.28.5. This proves the lemma. (We can also use that $D^b(\text{Coh}(\mathcal{O}_X)) = D^b_{\text{Coh}}(\mathcal{O}_X)$ by Proposition 36.11.2 to see the second equality.) $\square$

Here is the definition.

0FDG **Definition 36.38.2.** Let X be a scheme.

(1) We denote $K_0(X)$ the *Grothendieck group of X*. It is the zeroth K-group of the strictly full, saturated, triangulated subcategory $D_{perf}(\mathcal{O}_X)$ of $D(\mathcal{O}_X)$ consisting of perfect objects. In a formula

$$K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$$

(2) If X is locally Noetherian, then we denote $K'_0(X)$ the *Grothendieck group of coherent sheaves on X*. It is the is the zeroth K-group of the abelian category of coherent $\mathcal{O}_X$-modules. In a formula

$$K'_0(X) = K_0(\text{Coh}(\mathcal{O}_X))$$

We will show that our definition of $K_0(X)$ agrees with the often used definition in terms of finite locally free modules if X has the resolution property (for example if X has an ample invertible module). See Lemma 36.38.5.

0FDH **Lemma 36.38.3.** *Let $X = \text{Spec}(R)$ be an affine scheme. Then $K_0(X) = K_0(R)$ and if R is Noetherian then $K'_0(X) = K'_0(R)$.*

Proof. Recall that $K'_0(R)$ and $K_0(R)$ have been defined in Algebra, Section 10.55.

By More on Algebra, Lemma 15.119.2 we have $K_0(R) = K_0(D_{perf}(R))$. By Lemmas 36.10.7 and 36.3.5 we have $D_{perf}(R) = D_{perf}(\mathcal{O}_X)$. This proves the equality $K_0(R) = K_0(X)$.

The equality $K'_0(R) = K'_0(X)$ holds because $\text{Coh}(\mathcal{O}_X)$ is equivalent to the category of finite R-modules by Cohomology of Schemes, Lemma 30.9.1. Moreover it is clear that $K'_0(R)$ is the zeroth K-group of the category of finite R-modules from the definitions. $\square$

Let X be a Noetherian scheme. Then both $K'_0(X)$ and $K_0(X)$ are defined. In this case there is a canonical map

$$K_0(X) = K_0(D_{perf}(\mathcal{O}_X)) \longrightarrow K_0(D^b_{\text{Coh}}(\mathcal{O}_X)) = K'_0(X)$$

Namely, perfect complexes are in $D^b_{\text{Coh}}(\mathcal{O}_X)$ (by Lemma 36.10.3), the inclusion functor $D_{perf}(\mathcal{O}_X) \to D^b_{\text{Coh}}(\mathcal{O}_X)$ induces a map on zeroth K-groups (Derived

Categories, Lemma 13.28.3), and we have the equality on the right by Lemma 36.38.1.

0FDI Lemma 36.38.4. Let X be a Noetherian regular scheme. Then the map $K_0(X) \to K'_0(X)$ is an isomorphism.

Proof. Follows immediately from Lemma 36.11.8 and our construction of the map $K_0(X) \to K'_0(X)$ above. $\square$

Let X be a scheme. Let us denote $\text{Vect}(X)$ the category of finite locally free $\mathcal{O}_X$-modules. Although $\text{Vect}(X)$ isn't an abelian category in general, it is clear what a short exact sequence of $\text{Vect}(X)$ is. Denote $K_0(\text{Vect}(X))$ the unique abelian group with the following properties[7]:

(1) For every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ there is given an element $[\mathcal{E}]$ in $K_0(\text{Vect}(X))$,
(2) for every short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules we have the relation $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ in $K_0(\text{Vect}(X))$,
(3) the group $K_0(\text{Vect}(X))$ is generated by the elements $[\mathcal{E}]$, and
(4) all relations in $K_0(\text{Vect}(X))$ among the generators $[\mathcal{E}]$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

We omit the detailed construction of $K_0(\text{Vect}(X))$. There is a natural map

$$K_0(\text{Vect}(X)) \longrightarrow K_0(X)$$

Namely, given a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ let us denote $\mathcal{E}[0]$ the perfect complex on X which has $\mathcal{E}$ sitting in degree 0 and zero in other degrees. Given a short exact sequence $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$ of finite locally free $\mathcal{O}_X$-modules we obtain a distinguished triangle $\mathcal{E}[0] \to \mathcal{E}'[0] \to \mathcal{E}''[0] \to \mathcal{E}[1]$, see Derived Categories, Section 13.12. This shows that we obtain a map $K_0(\text{Vect}(X)) \to K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$ by sending $[\mathcal{E}]$ to $[\mathcal{E}[0]]$ with apologies for the horrendous notation.

0FDJ Lemma 36.38.5. Let X be a quasi-compact and quasi-separated scheme with the resolution property. Then the map $K_0(\text{Vect}(X)) \to K_0(X)$ is an isomorphism.

Proof. This lemma will follow in a straightforward manner from Lemmas 36.37.2, 36.37.3, and 36.37.4 whose results we will use without further mention. Let us construct an inverse map

$$c : K_0(X) = K_0(D_{perf}(\mathcal{O}_X)) \longrightarrow K_0(\text{Vect}(X))$$

Namely, any object of $D_{perf}(\mathcal{O}_X)$ can be represented by a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules. Then we set

$$c([\mathcal{E}^\bullet]) = \sum (-1)^i [\mathcal{E}^i]$$

Of course we have to show that this is well defined. For the moment we view c as a map defined on bounded complexes of finite locally free $\mathcal{O}_X$-modules.

Suppose that $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is a surjective map of bounded complexes of finite locally free $\mathcal{O}_X$-modules. Let $\mathcal{K}^\bullet$ be the kernel. Then we obtain short exact sequences of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}^n \to \mathcal{E}^n \to \mathcal{F}^n \to 0$$

[7]The correct generality here would be to define K_0 for any exact category, see Injectives, Remark 19.9.6.

which are locally split because $\mathcal{F}^n$ is finite locally free. Hence $\mathcal{K}^\bullet$ is also a bounded complex of finite locally free $\mathcal{O}_X$-modules and we have $c(\mathcal{E}^\bullet) = c(\mathcal{K}^\bullet) + c(\mathcal{F}^\bullet)$ in $K_0(\text{Vect}(X))$.

Suppose given a bounded complex $\mathcal{E}^\bullet$ of finite locally free $\mathcal{O}_X$-modules which is acyclic. Say $\mathcal{E}^n = 0$ for $n \not\in [a, b]$. Then we can break $\mathcal{E}^\bullet$ into short exact sequences

$$\begin{gathered} 0 \to \mathcal{E}^a \to \mathcal{E}^{a+1} \to \mathcal{F}^{a+1} \to 0, \\ 0 \to \mathcal{F}^{a+1} \to \mathcal{E}^{a+2} \to \mathcal{F}^{a+3} \to 0, \\ \ldots \\ 0 \to \mathcal{F}^{b-3} \to \mathcal{E}^{b-2} \to \mathcal{F}^{b-2} \to 0, \\ 0 \to \mathcal{F}^{b-2} \to \mathcal{E}^{b-1} \to \mathcal{E}^b \to 0 \end{gathered}$$

Arguing by descending induction we see that $\mathcal{F}^{b-2}, \ldots, \mathcal{F}^{a+1}$ are finite locally free $\mathcal{O}_X$-modules, and

$$c(\mathcal{E}^\bullet) = \sum (-1)[\mathcal{E}^n] = \sum (-1)^n([\mathcal{F}^{n-1}] + [\mathcal{F}^n]) = 0$$

Thus our construction gives zero on acyclic complexes.

It follows from the results of the preceding two paragraphs that c is well defined. Namely, suppose the bounded complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ of finite locally free $\mathcal{O}_X$-modules represent the same object of $D(\mathcal{O}_X)$. Then we can find quasi-isomorphisms $a : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ and $b : \mathcal{G}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{G}^\bullet$ bounded complex of finite locally free $\mathcal{O}_X$-modules. We obtain a short exact sequence of complexes

$$0 \to \mathcal{E}^\bullet \to C(a)^\bullet \to \mathcal{G}^\bullet[1] \to 0$$

see Derived Categories, Definition 13.9.1. Since a is a quasi-isomorphism, the cone $C(a)^\bullet$ is acyclic (this follows for example from the discussion in Derived Categories, Section 13.12). Hence

$$0 = c(C(f)^\bullet) = c(\mathcal{E}^\bullet) + c(\mathcal{G}^\bullet[1]) = c(\mathcal{E}^\bullet) - c(\mathcal{G}^\bullet)$$

as desired. The same argument using b shows that $0 = c(\mathcal{F}^\bullet) - c(\mathcal{G}^\bullet)$. Hence we find that $c(\mathcal{E}^\bullet) = c(\mathcal{F}^\bullet)$ and c is well defined.

A similar argument using the cone on a map $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ of bounded complexes of finite locally free $\mathcal{O}_X$-modules shows that $c(Y) = c(X) + c(Z)$ if $X \to Y \to Z$ is a distinguished triangle in $D_{perf}(\mathcal{O}_X)$. Details omitted. Thus we get the desired homomorphism of abelian groups $c : K_0(X) \to K_0(\text{Vect}(X))$.

It is clear that the composition $K_0(\text{Vect}(X)) \to K_0(X) \to K_0(\text{Vect}(X))$ is the identity. On the other hand, let $\mathcal{E}^\bullet$ be a bounded complex of finite locally free $\mathcal{O}_X$-modules. Then the the existence of the distinguished triangles of "stupid truncations" (see Homology, Section 12.15)

$$\sigma_{\geq n}\mathcal{E}^\bullet \to \sigma_{\geq n-1}\mathcal{E}^\bullet \to \mathcal{E}^{n-1}[-n+1] \to (\sigma_{\geq n}\mathcal{E}^\bullet)[1]$$

and induction show that

$$[\mathcal{E}^\bullet] = \sum (-1)^i [\mathcal{E}^i[0]]$$

in $K_0(X) = K_0(D_{perf}(\mathcal{O}_X))$ with apologies for the notation. Hence the map $K_0(\text{Vect}(X)) \to K_0(D_{perf}(\mathcal{O}_X)) = K_0(X)$ is surjective which finishes the proof. $\square$

0FDK Remark 36.38.6. Let X be a scheme. The K-group $K_0(X)$ is canonically a commutative ring. Namely, using the derived tensor product

$$\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X} : D_{perf}(\mathcal{O}_X) \times D_{perf}(\mathcal{O}_X) \longrightarrow D_{perf}(\mathcal{O}_X)$$

and Derived Categories, Lemma 13.28.6 we obtain a bilinear multiplication. Since $K \otimes L \cong L \otimes K$ we see that this product is commutative. Since $(K \otimes L) \otimes M = K \otimes (L \otimes M)$ we see that this product is associative. Finally, the unit of $K_0(X)$ is the element $1 = [\mathcal{O}_X]$.

If $\text{Vect}(X)$ and $K_0(\text{Vect}(X))$ are as above, then it is clearly the case that $K_0(\text{Vect}(X))$ also has a ring structure: if $\mathcal{E}$ and $\mathcal{F}$ are finite locally free $\mathcal{O}_X$-modules, then we set

$$[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}]$$

The reader easily verifies that this indeed defines a bilinear commutative, associative product. Details omitted. The map

$$K_0(\text{Vect}(X)) \longrightarrow K_0(X)$$

constructed above is a ring map with these definitions.

Now assume X is Noetherian. The derived tensor product also produces a map

$$\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_X} : D_{perf}(\mathcal{O}_X) \times D^b_{\text{Coh}}(\mathcal{O}_X) \longrightarrow D^b_{\text{Coh}}(\mathcal{O}_X)$$

Again using Derived Categories, Lemma 13.28.6 we obtain a bilinear multiplication $K_0(X) \times K'_0(X) \to K'_0(X)$ since $K'_0(X) = K_0(D^b_{\text{Coh}}(\mathcal{O}_X))$ by Lemma 36.38.1. The reader easily shows that this gives $K'_0(X)$ the structure of a module over the ring $K_0(X)$.

0FDL Remark 36.38.7. Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. There is a map

$$f_* : K'_0(X) \longrightarrow K'_0(Y)$$

which sends $[\mathcal{F}]$ to

$$[\bigoplus\nolimits_{i \geq 0} R^{2i}f_*\mathcal{F}] - [\bigoplus\nolimits_{i \geq 0} R^{2i+1}f_*\mathcal{F}]$$

This is well defined because the sheaves $R^if_*\mathcal{F}$ are coherent (Cohomology of Schemes, Proposition 30.19.1), because locally only a finite number are nonzero, and because a short exact sequence of coherent sheaves on X produces a long exact sequence of R^if_* on Y. If Y is quasi-compact (the only case most often used in practice), then we can rewrite the above as

$$f_*[\mathcal{F}] = \sum (-1)^i [R^if_*\mathcal{F}] = [Rf_*\mathcal{F}]$$

where we have used the equality $K'_0(Y) = K_0(D^b_{\text{Coh}}(Y))$ from Lemma 36.38.1.

0FDM Lemma 36.38.8. Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. Then we have $f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta$ for $\alpha \in K'_0(X)$ and $\beta \in K_0(Y)$.

Proof. Follows from Lemma 36.22.1, the discussion in Remark 36.38.7, and the definition of the product $K'_0(X) \times K_0(X) \to K'_0(X)$ in Remark 36.38.6. $\square$

0FDN Remark 36.38.9. Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Consider the strictly full, saturated, triangulated subcategory

$$D_{Z,perf}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

consisting of perfect complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are set-theoretically supported on Z. The zeroth K-group $K_0(D_{Z,perf}(\mathcal{O}_X))$ of this triangulated category is sometimes denoted $K_Z(X)$ or $K_{0,Z}(X)$. Using derived tensor product exactly as in Remark 36.38.6 we see that $K_0(D_{Z,perf}(\mathcal{O}_X))$ has a multiplication which is associative and commutative, but in general $K_0(D_{Z,perf}(\mathcal{O}_X))$ doesn't have a unit.

36.39. Determinants of complexes

0FJW This section is the continuation of More on Algebra, Section 15.122. For any ringed space $(X, \mathcal{O}_X)$ there is a functor

$$\det : \left\{\begin{matrix} \text{category of perfect complexes} \\ \text{morphisms are isomorphisms} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{category of invertible modules} \\ \text{morphisms are isomorphisms} \end{matrix}\right\}$$

Moreover, given an object (L, F) of the filtered derived category $DF(\mathcal{O}_X)$ whose filtration is finite and whose graded parts are perfect complexes, there is a canonical isomorphism $\det(\text{gr}L) \to \det(L)$. See [KM76] for the original exposition. We will add this material later (insert future reference).

For the moment we will present an ad hoc construction in the case where X is a scheme and where we consider perfect objects L in $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$.

0FJX **Lemma 36.39.1.** Let X be a scheme. There is a functor

$$\det : \left\{\begin{matrix} \text{category of perfect complexes} \\ \text{with tor amplitude in } [-1,0] \\ \text{morphisms are isomorphisms} \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{category of invertible modules} \\ \text{morphisms are isomorphisms} \end{matrix}\right\}$$

In addition, given a rank 0 perfect object L of $D(\mathcal{O}_X)$ with tor-amplitude in $[-1, 0]$ there is a canonical element $\delta(L) \in \Gamma(X, \det(L))$ such that for any isomorphism $a : L \to K$ in $D(\mathcal{O}_X)$ we have $\det(a)(\delta(L)) = \delta(K)$. Moreover, the construction is affine locally given by the construction of More on Algebra, Section 15.122.

Proof. Let L be an object of the left hand side. If $\text{Spec}(A) = U \subset X$ is an affine open, then $L|_U$ corresponds to a perfect complex $L^\bullet$ of A-modules with tor-amplitude in $[-1, 0]$, see Lemmas 36.3.5, 36.10.4, and 36.10.7. Then we can consider the invertible A-module $\det(L^\bullet)$ constructed in More on Algebra, Lemma 15.122.4. If $\text{Spec}(B) = V \subset U$ is another affine open contained in U, then $\det(L^\bullet) \otimes_A B = \det(L^\bullet \otimes_A B)$ and hence this construction is compatible with restriction mappings (see Lemma 36.3.8 and note $A \to B$ is flat). Thus we can glue these invertible modules to obtain an invertible module $\det(L)$ on X. The functoriality and canonical sections are constructed in exactly the same manner. Details omitted. $\square$

0FJY **Remark 36.39.2.** The construction of Lemma 36.39.1 is compatible with pullbacks. More precisely, given a morphism $f : X \to Y$ of schemes and a perfect object K of $D(\mathcal{O}_Y)$ of tor-amplitude in $[-1, 0]$ then Lf^*K is a perfect object K of $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$ and we have a canonical identification

$$f^* \det(K) \longrightarrow \det(Lf^*K)$$

Moreover, if K has rank 0, then $\delta(K)$ pulls back to $\delta(Lf^*K)$ via this map. This is clear from the affine local construction of the determinant.

36.40. Detecting Boundedness

0GEI In this section, we show that compact generators of D_{QCoh} of a quasi-compact, quasi-separated scheme, as constructed in Section 36.15, have a special property. We recommend reading that section first as it is very similar to this one.

0GEJ Lemma 36.40.1. In Situation 36.9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Let $E \in D_{QCoh}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$. Consider the following conditions

(1) The canonical map $\tau_{\geq a}E \to \tau_{\geq a}Rj_*(E|_U)$ is an isomorphism.
(2) We have $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], E) = 0$ for all $n \geq a$.

Then (2) implies (1) and (1) implies (2) with a replaced by $a + 1$.

Proof. Choose a distinguished triangle $N \to E \to Rj_*(E|_U) \to N[1]$. Then (1) implies $\tau_{\geq a+1}N = 0$ and (1) is implied by $\tau_{\geq a}N = 0$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[-n], E) = 0$$

for all n as $K|_U = 0$. Thus (2) is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \geq a$. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i+e''_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e''_i}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 15.28.11. Hence $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \geq a$ is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_e[-n], N) = 0$ for all $n \geq a$ and all $e \geq 1$ with K_e as in Lemma 36.9.6. Since $N|_U = 0$, that lemma implies that this in turn is equivalent to $H^n(X, N) = 0$ for $n \geq a$. We conclude that (2) is equivalent to $\tau_{\geq a}N = 0$ since N is determined by the complex of A-modules $R\Gamma(X, N)$, see Lemma 36.3.5. Thus we find that our lemma is true. $\square$

0GEK Lemma 36.40.2. In Situation 36.9.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Let $E \in D_{QCoh}(\mathcal{O}_X)$ and $a \in \mathbf{Z}$. Consider the following conditions

(1) The canonical map $\tau_{\leq a}E \to \tau_{\leq a}Rj_*(E|_U)$ is an isomorphism, and
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], E) = 0$ for all $n \leq a$.

Then (2) implies (1) and (1) implies (2) with a replaced by $a - 1$.

Proof. Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. Then (1) implies $\tau_{\leq a-1}N = 0$ and (1) is implied by $\tau_{\leq a}N = 0$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[-n], E) = 0$$

for all n as $K|_U = 0$. Thus (2) is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \leq a$. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e'_i+e''_i}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e''_i}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 15.28.11. Hence $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[-n], N) = 0$ for all $n \leq a$ is equivalent to $\mathrm{Hom}_{D(\mathcal{O}_X)}(K_e[-n], N) = 0$ for all $n \leq a$ and all $e \geq 1$ with K_e as in Lemma 36.9.6. Since $N|_U = 0$, that lemma implies that this in turn is equivalent to $H^n(X, N) = 0$ for $n \leq a$. We conclude that (2) is equivalent to $\tau_{\leq a}N = 0$ since N is determined by the complex of A-modules $R\Gamma(X, N)$, see Lemma 36.3.5. Thus we find that our lemma is true. $\square$

0GEL Lemma 36.40.3. Let X be a quasi-compact and quasi-separated scheme. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent

(1) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ for $i \gg 0$, and
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\geq a}E) = 0$ for $i \gg 0$.

Proof. Using the triangle $\tau_{<a}E \to E \to \tau_{\geq a}E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{<a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{<a}E)[i]) = 0$$

for $i \gg 0$. As P is perfect this is true by Lemma 36.18.2. $\square$

0GEM Lemma 36.40.4. Let X be a quasi-compact and quasi-separated scheme. Let $P \in D_{perf}(\mathcal{O}_X)$ and $E \in D_{QCoh}(\mathcal{O}_X)$. Let $a \in \mathbf{Z}$. The following are equivalent

(1) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], E) = 0$ for $i \ll 0$, and
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{\leq a}E) = 0$ for $i \ll 0$.

Proof. Using the triangle $\tau_{\leq a}E \to E \to \tau_{>a}E \to$ we see that the equivalence follows if we can show

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[-i], \tau_{>a}E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(P, (\tau_{>a}E)[i]) = 0$$

for $i \ll 0$. As P is perfect this is true by Lemma 36.18.2. $\square$

0GEN Proposition 36.40.5. Let X be a quasi-compact and quasi-separated scheme. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

(1) $E \in D^-_{QCoh}(\mathcal{O}_X)$,
(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \gg 0$,
(3) $\mathrm{Ext}^i_X(G, E) = 0$ for $i \gg 0$,
(4) $R\,\mathrm{Hom}_X(G, E)$ is in $D^-(\mathbf{Z})$,
(5) $H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \gg 0$,
(6) $R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^-(\mathbf{Z})$,
(7) for every perfect object P of $D(\mathcal{O}_X)$
 (a) the assertions (2), (3), (4) hold with G replaced by P, and
 (b) $H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \gg 0$,
 (c) $R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^-(\mathbf{Z})$.

Proof. Assume (1). Since $\mathrm{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \gg 0$ by Lemma 36.18.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology, Section 20.44. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology, Lemma 20.50.5 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 36.17.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\,\mathrm{Hom}_X(P, E)$ is in $D^-(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is perserved by shifts. Hence by Derived Categories, Remark 13.36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. Set $P = \mathcal{O}_X[0]$. From (7) we obtain $H^i(X, E) = 0$ for $i \gg 0$. Hence (1) follows since E is determined by $R\Gamma(X, E)$, see Lemma 36.3.5.

Now assume $X = U \cup V$ with U a quasi-compact open of X and V an affine open, and assume the implication (2) – (7) $\Rightarrow$ (1) is known for the schemes U, V, and $U \cap V$. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 36.15.1 and Theorem 36.15.3 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$. Let $f_1, \ldots, f_r \in \Gamma(V, \mathcal{O}_V)$ be such that $V \setminus U = V(f_1, \ldots, f_r)$ as subsets of V. Let $K \in D_{perf}(\mathcal{O}_V)$ be the object corresponding to the Koszul complex on $f_1, \ldots, f_r$. Let $K' \in D_{perf}(\mathcal{O}_X)$ be

0GEP (36.40.5.1) $$K' = R(V \to X)_* K = R(V \to X)_! K,$$

see Cohomology, Lemmas 20.33.6 and 20.49.10. This is a perfect complex on X supported on the closed set $X \setminus U \subset V$ and isomorphic to K on V. By assumption, we know $R\operatorname{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\operatorname{Hom}_{\mathcal{O}_X}(K', E)$ are bounded above.

By the second description of K' in (36.40.5.1) we have

$$\operatorname{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = \operatorname{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = 0$$

for $i \gg 0$. Therefore, we may apply Lemma 36.40.1 to $E|_V$ to obtain an integer a such that $\tau_{\geq a}(E|_V) = \tau_{\geq a} R(U \cap V \to V)_*(E|_{U \cap V})$. Then $\tau_{\geq a} E = \tau_{\geq a} R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 36.40.3 twice we see that

$$\operatorname{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \operatorname{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \gg 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^-_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves D^-_{QCoh} (by Lemma 36.4.1), we get $\tau_{\geq a} E \in D^-_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^-_{QCoh}(\mathcal{O}_X)$. $\square$

0GEQ **Proposition 36.40.6.** Let X be a quasi-compact and quasi-separated scheme. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$. Let $E \in D_{QCoh}(\mathcal{O}_X)$. The following are equivalent

1. $E \in D^+_{QCoh}(\mathcal{O}_X)$,
2. $\operatorname{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = 0$ for $i \ll 0$,
3. $\operatorname{Ext}^i_X(G, E) = 0$ for $i \ll 0$,
4. $R\operatorname{Hom}_X(G, E)$ is in $D^+(\mathbf{Z})$,
5. $H^i(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,
6. $R\Gamma(X, G^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$,
7. for every perfect object P of $D(\mathcal{O}_X)$
   (a) the assertions (2), (3), (4) hold with G replaced by P, and
   (b) $H^i(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E) = 0$ for $i \ll 0$,
   (c) $R\Gamma(X, P \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$ is in $D^+(\mathbf{Z})$.

Proof. Assume (1). Since $\operatorname{Hom}_{D(\mathcal{O}_X)}(G[-i], E) = \operatorname{Hom}_{D(\mathcal{O}_X)}(G, E[i])$ we see that this is zero for $i \ll 0$ by Lemma 36.18.2. This proves that (1) implies (2).

Parts (2), (3), (4) are equivalent by the discussion in Cohomology, Section 20.44. Part (5) and (6) are equivalent as $H^i(X, -) = H^i(R\Gamma(X, -))$ by definition. The equivalent conditions (2), (3), (4) are equivalent to the equivalent conditions (5), (6) by Cohomology, Lemma 20.50.5 and the fact that $(G[-i])^\vee = G^\vee[i]$.

It is clear that (7) implies (2). Conversely, let us prove that the equivalent conditions (2) – (6) imply (7). Recall that G is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Remark 36.17.2. For $P \in D_{perf}(\mathcal{O}_X)$ let $T(P)$ be the assertion that $R\operatorname{Hom}_X(P, E)$ is in $D^+(\mathbf{Z})$. Clearly, T is inherited by direct sums, satisfies the 2-out-of-three property for distinguished triangles, is inherited by direct summands, and is perserved by shifts. Hence by Derived Categories, Remark 13.36.7 we see that (4) implies T holds on all of $D_{perf}(\mathcal{O}_X)$. The same argument works for all other properties, except that for property (7)(b) and (7)(c) we also use that $P \mapsto P^\vee$ is a self equivalence of $D_{perf}(\mathcal{O}_X)$. Small detail omitted.

We will prove the equivalent conditions (2) – (7) imply (1) using the induction principle of Cohomology of Schemes, Lemma 30.4.1.

First, we prove (2) – (7) $\Rightarrow$ (1) if X is affine. Let $P = \mathcal{O}_X[0]$. From (7) we obtain $H^i(X, E) = 0$ for $i \ll 0$. Hence (1) follows since E is determined by $R\Gamma(X, E)$, see Lemma 36.3.5.

Now assume $X = U \cup V$ with U a quasi-compact open of X and V an affine open, and assume the implication (2) – (7) $\Rightarrow$ (1) is known for the schemes U, V, and $U \cap V$. Suppose $E \in D_{QCoh}(\mathcal{O}_X)$ satisfies (2) – (7). By Lemma 36.15.1 and Theorem 36.15.3 there exists a perfect complex Q on X such that $Q|_U$ generates $D_{QCoh}(\mathcal{O}_U)$. Let $f_1, \ldots, f_r \in \Gamma(V, \mathcal{O}_V)$ be such that $V \setminus U = V(f_1, \ldots, f_r)$ as subsets of V. Let $K \in D_{perf}(\mathcal{O}_V)$ be the object corresponding to the Koszul complex on $f_1, \ldots, f_r$. Let $K' \in D_{perf}(\mathcal{O}_X)$ be

0GER (36.40.6.1) $$K' = R(V \to X)_* K = R(V \to X)_! K,$$

see Cohomology, Lemmas 20.33.6 and 20.49.10. This is a perfect complex on X supported on the closed set $X \setminus U \subset V$ and isomorphic to K on V. By assumption, we know $R\operatorname{Hom}_{\mathcal{O}_X}(Q, E)$ and $R\operatorname{Hom}_{\mathcal{O}_X}(K', E)$ are bounded below.

By the second description of K' in (36.40.6.1) we have

$$\operatorname{Hom}_{D(\mathcal{O}_V)}(K[-i], E|_V) = \operatorname{Hom}_{D(\mathcal{O}_X)}(K'[-i], E) = 0$$

for $i \ll 0$. Therefore, we may apply Lemma 36.40.2 to $E|_V$ to obtain an integer a such that $\tau_{\leq a}(E|_V) = \tau_{\leq a} R(U \cap V \to V)_*(E|_{U \cap V})$. Then $\tau_{\leq a} E = \tau_{\leq a} R(U \to X)_*(E|_U)$ (check that the canonical map is an isomorphism after restricting to U and to V). Hence using Lemma 36.40.4 twice we see that

$$\operatorname{Hom}_{D(\mathcal{O}_U)}(Q|_U[-i], E|_U) = \operatorname{Hom}_{D(\mathcal{O}_X)}(Q[-i], R(U \to X)_*(E|_U)) = 0$$

for $i \ll 0$. Since the Proposition holds for U and the generator $Q|_U$, we have $E|_U \in D^+_{QCoh}(\mathcal{O}_U)$. But then since the functor $R(U \to X)_*$ preserves bounded below objects (see Cohomology, Section 20.3) we get $\tau_{\leq a} E \in D^+_{QCoh}(\mathcal{O}_X)$. Thus $E \in D^+_{QCoh}(\mathcal{O}_X)$. $\square$

36.41. Quasi-coherent objects in the derived category

0GZY Let X be a scheme. Recall that $X_{affine,Zar}$ denotes the category of affine opens of X with topology given by standard Zariski coverings, see Topologies, Definition 34.3.7. We remind the reader that the topos of $X_{affine,Zar}$ is the small Zariski topos of X, see Topologies, Lemma 34.3.11. The site $X_{affine,Zar}$ comes with a structure sheaf $\mathcal{O}$ and there is an equivalence of ringed topoi

$$(Sh(X_{affine,Zar}), \mathcal{O}) \longrightarrow (Sh(X_{Zar}), \mathcal{O})$$

See Descent, Equation (35.11.1.1) and the discussion in Descent, Section 35.11 surrounding it where a slightly different notation is used.

In this section we denote X_{affine} the underlying category of $X_{affine,Zar}$ endowed with the chaotic topology, i.e., such that sheaves agree with presheaves. In particular, the structure sheaf $\mathcal{O}$ becomes a sheaf on X_{affine} as well. We obtain a morphisms of ringed sites

$$\epsilon : (X_{affine,Zar}, \mathcal{O}) \longrightarrow (X_{affine}, \mathcal{O})$$

as in Cohomology on Sites, Section 21.27. In this section we will identify $D_{QCoh}(\mathcal{O}_X)$ with the category $QC(X_{affine}, \mathcal{O})$ introduced in Cohomology on Sites, Section 21.43.

0GZZ Lemma 36.41.1. In the sitation above there are canonical exact equivalences between the following triangulated categories

(1) $D_{QCoh}(\mathcal{O}_X)$,
(2) $D_{QCoh}(X_{Zar}, \mathcal{O})$,
(3) $D_{QCoh}(X_{affine,Zar}, \mathcal{O})$,
(4) $D_{QCoh}(X_{affine}, \mathcal{O}_X)$, and
(5) $QC(X_{affine}, \mathcal{O})$.

Proof. If $U \subset V \subset X$ are affine open, then the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$ is flat. Hence the equivalence between (4) and (5) is a special case of Cohomology on Sites, Lemma 21.43.11 (the proof also clarifies the statement).

The ringed site $(X_{Zar}, \mathcal{O})$ and the ringed space $(X, \mathcal{O}_X)$ have the same categories of modules by Descent, Remark 35.8.3. Via this equivalence the quasi-coherent modules correspond by Descent, Proposition 35.8.9. Hence we get a canonical exact equivalence between the triangulated categories in (1) and (2).

The discussion preceding the lemma shows that we have an equivalence of ringed topoi $(Sh(X_{affine,Zar}), \mathcal{O}) \to (Sh(X_{Zar}), \mathcal{O})$ and hence an equivalence between abelian categories of modules. Since the notion of quasi-coherent modules is intrinsic (Modules on Sites, Lemma 18.23.2) we see that this equivalence preserves the subcategories of quasi-coherent modules. Thus we get a canonical exact equivalence between the triangulated categories in (2) and (3).

To get an exact equivalence between the triangulated categories in (3) and (4) we will apply Cohomology on Sites, Lemma 21.29.1 to the morphism $\epsilon : (X_{affine,Zar}, \mathcal{O}) \to (X_{affine}, \mathcal{O})$ above. We take $\mathcal{B} = \text{Ob}(X_{affine})$ and we take $\mathcal{A} \subset \text{PMod}(X_{affine}, \mathcal{O})$ to be the full subcategory of those presheaves $\mathcal{F}$ such that $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to \mathcal{F}(U)$ is an isomorphism. Observe that by Descent, Lemma 35.11.2 objects of $\mathcal{A}$ are exactly those sheaves in the Zariski topology which are quasi-coherent modules on $(X_{affine,Zar}, \mathcal{O})$. On the other hand, by Modules on Sites, Lemma 18.24.2, the objects of $\mathcal{A}$ are exactly the quasi-coherent modules on $(X_{affine}, \mathcal{O})$, i.e., in the chaotic topology. Thus if we show that Cohomology on Sites, Lemma 21.29.1 applies, then we do indeed get the canonical equivalence between the categories of (3) and (4) using ϵ^* and $R\epsilon_*$.

We have to verify 4 conditions:

(1) Every object of $\mathcal{A}$ is a sheaf for the Zariski topology. This we have seen above.
(2) $\mathcal{A}$ is a weak Serre subcategory of $\text{Mod}(X_{affine,Zar}, \mathcal{O})$. Above we have seen that $\mathcal{A} = QCoh(X_{affine,Zar}, \mathcal{O})$ and we have seen above that these, via the equivalence $\text{Mod}(X_{affine,Zar}, \mathcal{O}) = \text{Mod}(X, \mathcal{O}_X)$, correspond to the quasi-coherent modules on X. Thus the result by the discussion in Schemes, Section 26.24.
(3) Every object of X_{affine} has a covering in the chaotic topology whose members are elements of $\mathcal{B}$. This holds because $\mathcal{B}$ contains all objects.
(4) For every object U of X_{affine} and $\mathcal{F}$ in $\mathcal{A}$ we have $H^p_{Zar}(U, \mathcal{F}) = 0$ for $p > 0$. This holds by the vanishing of cohomology of quasi-coherent modules on affines, see Cohomology of Schemes, Lemma 30.2.2.

This finishes the proof. □

0H00 Remark 36.41.2. Let S be a scheme. We will later show that also $QC((\text{Aff}/S), \mathcal{O})$ is canonically equivalent to $D_{QCoh}(\mathcal{O}_S)$. See Sheaves on Stacks, Proposition 96.26.4.

36.42. Other chapters

CHAPTER 37

More on Morphisms

02GX

37.1. Introduction

02GY In this chapter we continue our study of properties of morphisms of schemes. A fundamental reference is [DG67].

37.2. Thickenings

04EW The following terminology may not be completely standard, but it is convenient.

04EX Definition 37.2.1. Thickenings.

(1) We say a scheme X' is a thickening of a scheme X if X is a closed subscheme of X' and the underlying topological spaces are equal.
(2) We say a scheme X' is a first order thickening of a scheme X if X is a closed subscheme of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a morphism of thickenings is a morphism $f' : X' \to Y'$ such that $f'(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subscheme Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(4) Let S be a scheme. We similarly define thickenings over S, and morphisms of thickenings over S. This means that the schemes X, X', Y, Y' above are schemes over S, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over S.

Finite order thickenings. Let $i_X : X \to X'$ be a thickening. Any local section of the kernel $\mathcal{I} = \text{Ker}(i_X^\sharp)$ is locally nilpotent. Let us say that $X \subset X'$ is a finite order thickening if the ideal sheaf $\mathcal{I}$ is "globally" nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 = \mathcal{I}^{n+1} \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over S. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

First order thickening are described as follows (see Modules, Lemma 17.28.11).

05YV **Lemma 37.2.2.** Let X be a scheme over a base S. Consider a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on X where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I}$ is its kernel. If

(1) $\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and
(2) $\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module

then $X' = (X, \mathcal{A})$ is a scheme and $X \to X'$ is a first order thickening over S. Moreover, any first order thickening over S is of this form.

Proof. It is clear that X' is a locally ringed space. Let $U = \mathrm{Spec}(B)$ be an affine open of X. Set $A = \Gamma(U, \mathcal{A})$. Note that since $H^1(U, \mathcal{I}) = 0$ (see Cohomology of Schemes, Lemma 30.2.2) the map $A \to B$ is surjective. By assumption the kernel $I = \mathcal{I}(U)$ is an ideal of square zero in the ring A. By Schemes, Lemma 26.6.4 there is a canonical morphism of locally ringed spaces

$$(U, \mathcal{A}|_U) \longrightarrow \mathrm{Spec}(A)$$

coming from the map $B \to \Gamma(U, \mathcal{A})$. Since this morphism fits into the commutative diagram

$$\begin{tikzcd} (U, \mathcal{O}_X|_U) \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\ (U, \mathcal{A}|_U) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

we see that it is a homeomorphism on underlying topological spaces. Thus to see that it is an isomorphism, it suffices to check it induces an isomorphism on the local rings. For $u \in U$ corresponding to the prime $\mathfrak{p} \subset A$ we obtain a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & I_{\mathfrak{p}} \arrow[r] \arrow[d] & A_{\mathfrak{p}} \arrow[r] \arrow[d] & B_{\mathfrak{p}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}_u \arrow[r] & \mathcal{A}_u \arrow[r] & \mathcal{O}_{X,u} \arrow[r] & 0. \end{tikzcd}$$

The left and right vertical arrows are isomorphisms because $\mathcal{I}$ and $\mathcal{O}_X$ are quasi-coherent sheaves. Hence also the middle map is an isomorphism. Hence every point of $X' = (X, \mathcal{A})$ has an affine neighbourhood and X' is a scheme as desired. □

06AD **Lemma 37.2.3.** Any thickening of an affine scheme is affine.

The case of a finite order thickening is [GD60, Proposition 5.1.9].

Proof. This is a special case of Limits, Proposition 32.11.2. □

Proof for a finite order thickening. Suppose that $X \subset X'$ is a finite order thickening with X affine. Then we may use Serre's criterion to prove X' is affine. More precisely, we will use Cohomology of Schemes, Lemma 30.3.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{X'}$-module. It suffices to show that $H^1(X', \mathcal{F}) = 0$. Denote $i : X \to X'$ the given closed immersion and denote $\mathcal{I} = \mathrm{Ker}(i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X)$. By our discussion of finite order thickenings (following Definition 37.2.1) there exists an $n \geq 0$ and a filtration

$$0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F}$$

by quasi-coherent submodules such that $\mathcal{F}_a/\mathcal{F}_{a+1}$ is annihilated by $\mathcal{I}$. Namely, we can take $\mathcal{F}_a = \mathcal{I}^a\mathcal{F}$. Then $\mathcal{F}_a/\mathcal{F}_{a+1} = i_*\mathcal{G}_a$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}_a$, see Morphisms, Lemma 29.4.1. We obtain

$$H^1(X', \mathcal{F}_a/\mathcal{F}_{a+1}) = H^1(X', i_*\mathcal{G}_a) = H^1(X, \mathcal{G}_a) = 0$$

The second equality comes from Cohomology of Schemes, Lemma 30.2.4 and the last equality from Cohomology of Schemes, Lemma 30.2.2. Thus $\mathcal{F}$ has a finite filtration whose successive quotients have vanishing first cohomology and it follows by a simple induction argument that $H^1(X', \mathcal{F}) = 0$. □

09ZU Lemma 37.2.4. Let $S \subset S'$ be a thickening of schemes. Let $X' \to S'$ be a morphism and set $X = S \times_{S'} X'$. Then $(X \subset X') \to (S \subset S')$ is a morphism of thickenings. If $S \subset S'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.

Proof. Omitted. □

0BPE Lemma 37.2.5. If $S \subset S'$ and $S' \subset S''$ are thickenings, then so is $S \subset S''$.

Proof. Omitted. □

0BPF Lemma 37.2.6. The property of being a thickening is fpqc local. Similarly for first order thickenings.

Proof. The statement means the following: Let $X \to X'$ be a morphism of schemes and let $\{g_i : X'_i \to X'\}$ be an fpqc covering such that the base change $X_i \to X'_i$ is a thickening for all i. Then $X \to X'$ is a thickening. Since the morphisms g_i are jointly surjective we conclude that $X \to X'$ is surjective. By Descent, Lemma 35.23.19 we conclude that $X \to X'$ is a closed immersion. Thus $X \to X'$ is a thickening. We omit the proof in the case of first order thickenings. □

37.3. Morphisms of thickenings

0CF2 If $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings of schemes, then often properties of the morphism f are inherited by f'. There are several variants.

09ZV Lemma 37.3.1. Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings. Then

(1) f is an affine morphism if and only if f' is an affine morphism,
(2) f is a surjective morphism if and only if f' is a surjective morphism,
(3) f is quasi-compact if and only if f' quasi-compact,
(4) f is universally closed if and only if f' is universally closed,
(5) f is integral if and only if f' is integral,
(6) f is (quasi-)separated if and only if f' is (quasi-)separated,
(7) f is universally injective if and only if f' is universally injective,
(8) f is universally open if and only if f' is universally open,
(9) f is quasi-affine if and only if f' is quasi-affine, and
(10) add more here.

Proof. Observe that $S \to S'$ and $X \to X'$ are universal homeomorphisms (see for example Morphisms, Lemma 29.45.6). This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Lemma 37.2.3 which tells us that there is a 1-to-1 correspondence between affine opens of S and S' and between affine opens of

X and X'. Part (9) follows from Limits, Lemma 32.11.5 and the remark just made about affine opens of S and S'. Part (5) follows from (1) and (4) by Morphisms, Lemma 29.44.7. Finally, note that

$$S \times_X S = S \times_{X'} S \to S \times_{X'} S' \to S' \times_{X'} S'$$

is a thickening (the two arrows are thickenings by Lemma 37.2.4). Hence applying (3) and (4) to the morphism $(S \subset S') \to (S \times_X S \to S' \times_{X'} S')$ we obtain (6). $\square$

0D2R Lemma 37.3.2. Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings. Let $\mathcal{L}'$ be an invertible sheaf on X' and denote $\mathcal{L}$ the restriction to X. Then $\mathcal{L}'$ is f'-ample if and only if $\mathcal{L}$ is f-ample.

Proof. Recall that being relatively ample is a condition for each affine open in the base, see Morphisms, Definition 29.37.1. By Lemma 37.2.3 there is a 1-to-1 correspondence between affine opens of S and S'. Thus we may assume S and S' are affine and we reduce to proving that $\mathcal{L}'$ is ample if and only if $\mathcal{L}$ is ample. This is Limits, Lemma 32.11.4. $\square$

09ZW Lemma 37.3.3. Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings such that $X = S \times_{S'} X'$. If $S \subset S'$ is a finite order thickening, then

(1) f is a closed immersion if and only if f' is a closed immersion,
(2) f is locally of finite type if and only if f' is locally of finite type,
(3) f is locally quasi-finite if and only if f' is locally quasi-finite,
(4) f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,
(5) $\Omega_{X/S} = 0$ if and only if $\Omega_{X'/S'} = 0$,
(6) f is unramified if and only if f' is unramified,
(7) f is proper if and only if f' is proper,
(8) f is finite if and only if f' is finite,
(9) f is a monomorphism if and only if f' is a monomorphism,
(10) f is an immersion if and only if f' is an immersion, and
(11) add more here.

Proof. The properties $\mathcal{P}$ listed in the lemma are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 26.18.2 and 26.23.5 and Morphisms, Lemmas 29.15.4, 29.20.13, 29.29.2, 29.32.10, 29.35.5, 29.41.5, and 29.44.6.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 37.2.1.

Most of the proofs will use a reduction to the affine case. Let $U' \subset S'$ be an affine open and let $V' \subset X'$ be an affine open lying over U'. Let $U' = \mathrm{Spec}(A')$ and denote $I \subset A'$ be the ideal defining the closed subscheme $U' \cap S$. Say $V' = \mathrm{Spec}(B')$. Then $V' \cap X = \mathrm{Spec}(B'/IB')$. Setting $A = A'/I$ and $B = B'/IB'$ we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & IB' \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & IA' \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows and $I^2 = 0$.

The translation of (1) into algebra: If $A \to B$ is surjective, then $A' \to B'$ is surjective. This follows from Nakayama's lemma (Algebra, Lemma 10.20.1).

The translation of (2) into algebra: If $A \to B$ is a finite type ring map, then $A' \to B'$ is a finite type ring map. This follows from Nakayama's lemma (Algebra, Lemma 10.20.1) applied to a map $A'[x_1, \ldots, x_n] \to B'$ such that $A[x_1, \ldots, x_n] \to B$ is surjective.

Proof of (3). Follows from (2) and that quasi-finiteness of a morphism which is locally of finite type can be checked on fibres, see Morphisms, Lemma 29.20.6.

Proof of (4). Follows from (2) and that the additional property of "being of relative dimension d" can be checked on fibres (by definition, see Morphisms, Definition 29.29.1.

The translation of (5) into algebra: If $\Omega_{B/A} = 0$, then $\Omega_{B'/A'} = 0$. By Algebra, Lemma 10.131.12 we have $0 = \Omega_{B/A} = \Omega_{B'/A'}/I\Omega_{B'/A'}$. Hence $\Omega_{B'/A'} = 0$ by Nakayama's lemma (Algebra, Lemma 10.20.1).

The translation of (6) into algebra: If $A \to B$ is unramified map, then $A' \to B'$ is unramified. Since $A \to B$ is of finite type we see that $A' \to B'$ is of finite type by (2) above. Since $A \to B$ is unramified we have $\Omega_{B/A} = 0$. By part (5) we have $\Omega_{B'/A'} = 0$. Thus $A' \to B'$ is unramified.

Proof of (7). Follows by combining (2) with results of Lemma 37.3.1 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (8). Follows by combining (2) with results of Lemma 37.3.1 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 29.44.4).

Proof of (9). As f is a monomorphism we have $X = X \times_S X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_S X \subset X' \times_{S'} X')$. We conclude $X' \to X' \times_{S'} X'$ is a closed immersion by (1). In fact, it is a first order thickening as the ideal defining the closed immersion $X' \to X' \times_{S'} X'$ is contained in the pullback of the ideal $\mathcal{I} \subset \mathcal{O}_{S'}$ cutting out S in S'. Indeed, $X = X \times_S X = (X' \times_{S'} X') \times_{S'} S$ is contained in X'. Hence by Morphisms, Lemma 29.32.7 it suffices to show that $\Omega_{X'/S'} = 0$ which follows from (5) and the corresponding statement for X/S.

Proof of (10). If $f : X \to S$ is an immersion, then it factors as $X \to U \to S$ where $U \to S$ is an open immersion and $X \to U$ is a closed immersion. Let $U' \subset S'$ be the open subscheme whose underlying topological space is the same as U. Then $X' \to S'$ factors through U' and we conclude that $X' \to U'$ is a closed immersion by part (1). This finishes the proof. □

The following lemma is a variant on the preceding one. Rather than assume that the thickenings involved are finite order (which allows us to transfer the property of being locally of finite type from f to f'), we instead take as given that each of f and f' is locally of finite type.

0BPG Lemma 37.3.4. Let $(f, f') : (X \subset X') \to (Y \to Y')$ be a morphism of thickenings. Assume f and f' are locally of finite type and $X = Y \times_{Y'} X'$. Then

(1) f is locally quasi-finite if and only if f' is locally quasi-finite,

(2) f is finite if and only if f' is finite,
(3) f is a closed immersion if and only if f' is a closed immersion,
(4) $\Omega_{X/Y} = 0$ if and only if $\Omega_{X'/Y'} = 0$,
(5) f is unramified if and only if f' is unramified,
(6) f is a monomorphism if and only if f' is a monomorphism,
(7) f is an immersion if and only if f' is an immersion,
(8) f is proper if and only if f' is proper, and
(9) add more here.

Proof. The properties $\mathcal{P}$ listed in the lemma are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 26.18.2 and 26.23.5 and Morphisms, Lemmas 29.20.13, 29.29.2, 29.32.10, 29.35.5, 29.41.5, and 29.44.6. Hence in each case we need only to prove that if f has the desired property, so does f'.

A morphism is locally quasi-finite if and only if it is locally of finite type and the scheme theoretic fibres are discrete spaces, see Morphisms, Lemma 29.20.8. Since the underlying topological space is unchanged by passing to a thickening, we see that f' is locally quasi-finite if (and only if) f is. This proves (1).

Case (2) follows from case (5) of Lemma 37.3.1 and the fact that the finite morphisms are precisely the integral morphisms that are locally of finite type (Morphisms, Lemma 29.44.4).

Case (3). This follows immediately from Morphisms, Lemma 29.45.7.

Case (4) follows from the following algebra statement: Let A be a ring and let $I \subset A$ be a locally nilpotent ideal. Let B be a finite type A-algebra. If $\Omega_{(B/IB)/(A/I)} = 0$, then $\Omega_{B/A} = 0$. Namely, the assumption means that $I\Omega_{B/A} = 0$, see Algebra, Lemma 10.131.12. On the other hand $\Omega_{B/A}$ is a finite B-module, see Algebra, Lemma 10.131.16. Hence the vanishing of $\Omega_{B/A}$ follows from Nakayama's lemma (Algebra, Lemma 10.20.1) and the fact that IB is contained in the Jacobson radical of B.

Case (5) follows immediately from (4) and Morphisms, Lemma 29.35.2.

Proof of (6). As f is a monomorphism we have $X = X \times_Y X$. We may apply the results proved so far to the morphism of thickenings $(X \subset X') \to (X \times_Y X \subset X' \times_{Y'} X')$. We conclude $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$ is a closed immersion by (3). In fact $\Delta_{X'/Y'}$ is a bijection on underlying sets, hence $\Delta_{X'/Y'}$ is a thickening. On the other hand $\Delta_{X'/Y'}$ is locally of finite presentation by Morphisms, Lemma 29.21.12. In other words, $\Delta_{X'/Y'}(X')$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X' \times_{Y'} X'}$ of finite type. Since $\Omega_{X'/Y'} = 0$ by (5) we see that the conormal sheaf of $X' \to X' \times_{Y'} X'$ is zero by Morphisms, Lemma 29.32.7. In other words, $\mathcal{J}/\mathcal{J}^2 = 0$. This implies $\Delta_{X'/Y'}$ is an isomorphism, for example by Algebra, Lemma 10.21.5.

Proof of (7). If $f : X \to Y$ is an immersion, then it factors as $X \to V \to Y$ where $V \to Y$ is an open immersion and $X \to V$ is a closed immersion. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is the same as V. Then $X' \to V'$ factors through V' and we conclude that $X' \to V'$ is a closed immersion by part (3).

Case (8) follows from Lemma 37.3.1 and the definition of proper morphisms as being the quasi-compact, universally closed, and separated morphisms that are locally of finite type. □

37.4. Picard groups of thickenings

0C6Q Some material on Picard groups of thickenings.

0C6R Lemma 37.4.1. Let $X \subset X'$ be a first order thickening with ideal sheaf $\mathcal{I}$. Then there is a canonical exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{I}) \arrow[r] & H^0(X',\mathcal{O}_{X'}^*) \arrow[r] & H^0(X,\mathcal{O}_X^*) \arrow[dll] \\ & H^1(X,\mathcal{I}) \arrow[r] & \mathrm{Pic}(X') \arrow[r] & \mathrm{Pic}(X) \arrow[dll] \\ & H^2(X,\mathcal{I}) \arrow[r] & \ldots \arrow[r] & \ldots \end{tikzcd}$$

of abelian groups.

Proof. This is the long exact cohomology sequence associated to the short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{I} \to \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

where the first map sends a local section f of $\mathcal{I}$ to the invertible section $1 + f$ of $\mathcal{O}_{X'}$. We also use the identification of the Picard group of a ringed space with the first cohomology group of the sheaf of invertible functions, see Cohomology, Lemma 20.6.1. □

0C6S Lemma 37.4.2. Let $X \subset X'$ be a thickening. Let n be an integer invertible in $\mathcal{O}_X$. Then the map $\mathrm{Pic}(X')[n] \to \mathrm{Pic}(X)[n]$ is bijective.

Proof for a finite order thickening. By the general principle explained following Definition 37.2.1 this reduces to the case of a first order thickening. Then may use Lemma 37.4.1 to see that it suffices to show that $H^1(X,\mathcal{I})[n]$, $H^1(X,\mathcal{I})/n$, and $H^2(X,\mathcal{I})[n]$ are zero. This follows as multiplication by n on $\mathcal{I}$ is an isomorphism as it is an $\mathcal{O}_X$-module. □

Proof in general. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent ideal sheaf cutting out X. Then we have a short exact sequence of abelian groups

$$0 \to (1+\mathcal{I})^* \to \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0$$

We obtain a long exact cohomology sequence as in the statement of Lemma 37.4.1 with $H^i(X,\mathcal{I})$ replaced by $H^i(X,(1+\mathcal{I})^*)$. Thus it suffices to show that raising to the nth power is an isomorphism $(1+\mathcal{I})^* \to (1+\mathcal{I})^*$. Taking sections over affine opens this follows from Algebra, Lemma 10.32.8. □

37.5. First order infinitesimal neighbourhood

05YW A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme $Z \subset U$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U. Then we can consider the closed subscheme $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

04EY Definition 37.5.1. Let $i : Z \to X$ be an immersion of schemes. The first order infinitesimal neighbourhood of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

04EZ Lemma 37.5.2. Let $i : Z \to X$ be an immersion of schemes. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram

$$\begin{tikzcd} Z \arrow[d, "i"'] & T \arrow[l, "a"'] \arrow[d] \\ X & T' \arrow[l, "b"] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open used in the construction of Z', i.e., an open such that Z is identified with a closed subscheme of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $b(T') \subset U$. Hence we can think of b as a morphism into U. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the ideal cutting out T. Since $b(T) \subset Z$ by the diagram above we see that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Schemes, Lemma 26.4.6 this implies that b factors through Z'. Denote $a' : T' \to Z'$ this factorization and everything is clear. $\square$

04F0 Lemma 37.5.3. Let $i : Z \to X$ be an immersion of schemes. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Morphisms, Lemma 29.31.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. $\square$

37.6. Formally unramified morphisms

02H7

Recall that a ring map $R \to A$ is called formally unramified (see Algebra, Definition 10.148.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, at most one dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02H8 Definition 37.6.1. Let $f : X \to S$ be a morphism of schemes. We say f is formally unramified if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists at most one dotted arrow making the diagram commute.

We first prove some formal lemmas, i.e., lemmas which can be proved by drawing the corresponding diagrams.

04F1 Lemma 37.6.2. If $f : X \to S$ is a formally unramified morphism, then given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 37.6.1 the condition that T be affine may be dropped.

Proof. This is true because a morphism is determined by its restrictions to affine opens. $\square$

02HA Lemma 37.6.3. A composition of formally unramified morphisms is formally unramified.

Proof. This is formal. $\square$

02HB Lemma 37.6.4. A base change of a formally unramified morphism is formally unramified.

Proof. This is formal. $\square$

02HC Lemma 37.6.5. Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open such that $f(U) \subset V$. If f is formally unramified, so is $f|_U : U \to V$.

Proof. Consider a solid diagram

$$
\begin{tikzcd}
U \arrow[d, "f|_U"'] & T \arrow[l] \arrow[d, "i"] \arrow[l, phantom, "a"' near start] \\
V & T' \arrow[l] \arrow[lu, dashed]
\end{tikzcd}
$$

as in Definition 37.6.1. If f is formally ramified, then there exists at most one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Hence clearly there exists at most one such morphism into U. $\square$

02HD Lemma 37.6.6. Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally unramified if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally unramified ring map.

Proof. This is immediate from the definitions (Definition 37.6.1 and Algebra, Definition 10.148.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 26.6.5. $\square$

Here is a characterization in terms of the sheaf of differentials.

02H9 Lemma 37.6.7. Let $f : X \to S$ be a morphism of schemes. Then f is formally unramified if and only if $\Omega_{X/S} = 0$.

Proof. We recall some of the arguments of the proof of Morphisms, Lemma 29.32.5. Let $W \subset X \times_S X$ be an open such that $\Delta : X \to X \times_S X$ induces a closed immersion into W. Let $\mathcal{J} \subset \mathcal{O}_W$ be the ideal sheaf of this closed immersion. Let $X' \subset W$ be the closed subscheme defined by the quasi-coherent sheaf of ideals $\mathcal{J}^2$. Consider the two morphisms $p_1, p_2 : X' \to X$ induced by the two projections $X \times_S X \to X$. Note that p_1 and p_2 agree when composed with $\Delta : X \to X'$ and that $X \to X'$ is a closed immersion defined by a an ideal whose square is zero. Moreover there is a short exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2$. Moreover, $\mathcal{J}/\mathcal{J}^2$ is generated by the local sections $p_1^\sharp(f) - p_2^\sharp(f)$ for f a local section of $\mathcal{O}_X$.

Suppose that $f : X \to S$ is formally unramified. By assumption this means that $p_1 = p_2$ when restricted to any affine open $T' \subset X'$. Hence $p_1 = p_2$. By what was said above we conclude that $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2 = 0$.

Conversely, suppose that $\Omega_{X/S} = 0$. Then $X' = X$. Take any pair of morphisms $f'_1, f'_2 : T' \to X$ fitting as dotted arrows in the diagram of Definition 37.6.1. This gives a morphism $(f'_1, f'_2) : T' \to X \times_S X$. Since $f'_1|_T = f'_2|_T$ and $|T| = |T'|$ we see that the image of T' under (f'_1, f'_2) is contained in the open W chosen above. Since $(f'_1, f'_2)(T) \subset \Delta(X)$ and since T is defined by an ideal of square zero in T' we see that (f'_1, f'_2) factors through X'. As $X' = X$ we conclude $f'_1 = f'_2$ as desired. $\square$

02HE Lemma 37.6.8. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is unramified (resp. G-unramified), and
(2) the morphism f is locally of finite type (resp. locally of finite presentation) and formally unramified.

Proof. Use Lemma 37.6.7 and Morphisms, Lemma 29.35.2. $\square$

37.7. Universal first order thickenings

04F2 Let $h : Z \to X$ be a morphism of schemes. A universal first order thickening of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

$$\begin{tikzcd} & Z \arrow[ld] & & T \arrow[ll, "a"'] \arrow[rd] & \\ Z' \arrow[rrd] & & & & T' \arrow[llll, dotted, "a'"'] \arrow[lld, "b"] \\ & & X & & \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does.

04F3 **Lemma 37.7.1.** Let $h : Z \to X$ be a formally unramified morphism of schemes. There exists a universal first order thickening $Z \subset Z'$ of Z over X.

Proof. During this proof we will say $Z \subset Z'$ is a universal first order thickening of Z over X if it satisfies the condition of the lemma. We will construct the universal first order thickening $Z \subset Z'$ over X by glueing, starting with the affine case which is Algebra, Lemma 10.149.1. We begin with some general remarks.

If a universal first order thickening of Z over X exists, then it is unique up to unique isomorphism. Moreover, suppose that $V \subset Z$ and $U \subset X$ are open subschemes such that $h(V) \subset U$. Let $Z \subset Z'$ be a universal first order thickening of Z over X. Let $V' \subset Z'$ be the open subscheme such that $V = Z \cap V'$. Then we claim that $V \subset V'$ is the universal first order thickening of V over U. Namely, suppose given any diagram

$$\begin{tikzcd} V \arrow[d, "h"'] & T \arrow[l, "a"'] \arrow[d] \\ U & T' \arrow[l, "b"'] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening over U. By the universal property of Z' we obtain $(a, a') : (T \subset T') \to (Z \subset Z')$. But since we have equality $|T| = |T'|$ of underlying topological spaces we see that $a'(T') \subset V'$. Hence we may think of (a, a') as a morphism of thickenings $(a, a') : (T \subset T') \to (V \subset V')$ over U. Uniqueness is clear also. In a completely similar manner one proves that if $h(Z) \subset U$ and $Z \subset Z'$ is a universal first order thickening over U, then $Z \subset Z'$ is a universal first order thickening over X.

Before we glue affine pieces let us show that the lemma holds if Z and X are affine. Say $X = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(S)$. By Algebra, Lemma 10.149.1 there exists a first order thickening $Z \subset Z'$ over X which has the universal property of the lemma for diagrams

$$\begin{tikzcd} Z & T \arrow[l, "a"'] \arrow[d] \\ X \arrow[from=u, "h"'] & T' \arrow[l, "b"] \end{tikzcd}$$

where T, T' are affine. Given a general diagram we can choose an affine open covering $T' = \bigcup T'_i$ and we obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. By uniqueness we see that a'_i and a'_j agree on any affine open of $T'_i \cap T'_j$. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. Thus the lemma holds if X and Z are affine.

Choose an affine open covering $Z = \bigcup Z_i$ such that each Z_i maps into an affine open U_i of X. By Lemma 37.6.5 the morphisms $Z_i \to U_i$ are formally unramified. Hence by the affine case we obtain universal first order thickenings $Z_i \subset Z'_i$ over U_i. By the general remarks above $Z_i \subset Z'_i$ is also a universal first order thickening of Z_i over X. Let $Z'_{i,j} \subset Z'_i$ be the open subscheme such that $Z_i \cap Z_j = Z'_{i,j} \cap Z_i$. By the general remarks we see that both $Z'_{i,j}$ and $Z'_{j,i}$ are universal first order thickenings of $Z_i \cap Z_j$ over X. Thus, by the first of our general remarks, we see that there is a canonical isomorphism $\varphi_{ij} : Z'_{i,j} \to Z'_{j,i}$ inducing the identity on $Z_i \cap Z_j$. We claim that these morphisms satisfy the cocycle condition of Schemes, Section 26.14. (Verification omitted. Hint: Use that $Z'_{i,j} \cap Z'_{i,k}$ is the universal first order thickening of $Z_i \cap Z_j \cap Z_k$ which determines it up to unique isomorphism by what was said above.) Hence we can use the results of Schemes, Section 26.14 to get a first order thickening $Z \subset Z'$ over X which the property that the open subscheme $Z'_i \subset Z'$ with $Z_i = Z'_i \cap Z$ is a universal first order thickening of Z_i over X.

It turns out that this implies formally that Z' is a universal first order thickening of Z over X. Namely, we have the universal property for any diagram

$$\begin{tikzcd} Z & T \arrow[l, "a"'] \arrow[d] \\ X \arrow[from=u, "h"'] & T' \arrow[l, "b"] \end{tikzcd}$$

where $a(T)$ is contained in some Z_i. Given a general diagram we can choose an open covering $T' = \bigcup T'_i$ such that $a(T_i) \subset Z_i$. We obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. We see that a'_i and a'_j necessarily agree on $T'_i \cap T'_j$ since both $a'_i|_{T'_i \cap T'_j}$ and $a'_j|_{T'_i \cap T'_j}$ are solutions of the problem of mapping into the universal first order thickening $Z'_i \cap Z'_j$ of $Z_i \cap Z_j$ over X. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. This finishes the proof. ☐

04F4 Definition 37.7.2. Let $h : Z \to X$ be a formally unramified morphism of schemes.

(1) The universal first order thickening of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 37.7.1.
(2) The conormal sheaf of Z over X is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on Z. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

04F5 **Lemma 37.7.3.** Let $i : Z \to X$ be an immersion of schemes. Then

(1) i is formally unramified,
(2) the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 37.5.1, and
(3) the conormal sheaf of i in the sense of Morphisms, Definition 29.31.1 agrees with the conormal sheaf of i in the sense of Definition 37.7.2.

Proof. By Morphisms, Lemmas 29.35.7 and 29.35.8 an immersion is unramified, hence formally unramified by Lemma 37.6.8. The other assertions follow by combining Lemmas 37.5.2 and 37.5.3 and the definitions. □

04F6 **Lemma 37.7.4.** Let $Z \to X$ be a formally unramified morphism of schemes. Then the universal first order thickening Z' is formally unramified over X.

Proof. There are two proofs. The first is to show that $\Omega_{Z'/X} = 0$ by working affine locally and applying Algebra, Lemma 10.149.5. Then Lemma 37.6.7 implies what we want. The second is a direct argument as follows.

Let $T \subset T'$ be a first order thickening. Let

$$\begin{tikzcd} Z' \arrow[d] & T \arrow[l, "c"] \arrow[d] \\ X & T' \arrow[l] \arrow[ul, "a,b"'] \end{tikzcd}$$

be a commutative diagram. Consider two morphisms $a, b : T' \to Z'$ fitting into the diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □

04F7 **Lemma 37.7.5.** Consider a commutative diagram of schemes

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "h'"] & Y \end{tikzcd}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y.

There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram

$$\begin{tikzcd}
 & & & Z' \arrow[dl] \arrow[d, "f'"] \\
Z \arrow[urrr] \arrow[rr] \arrow[d, "f"'] & & X \arrow[d] & W' \arrow[dl] \\
W \arrow[urrr] \arrow[rr] & & Y &
\end{tikzcd}$$

In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Morphisms, Lemma 29.31.3 applied to $(Z \subset Z') \to (W \subset W')$. $\square$

04F8 Lemma 37.7.6. Let

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
W \arrow[r, "h'"] & Y
\end{tikzcd}$$

be a fibre product diagram in the category of schemes with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 37.7.5 is surjective.

Proof. The morphism h is formally unramified by Lemma 37.6.4. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 29.31.4 for why this implies that the map of conormal sheaves is surjective. $\square$

04F9 Lemma 37.7.7. Let

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
W \arrow[r, "h'"] & Y
\end{tikzcd}$$

be a fibre product diagram in the category of schemes with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms, Lemma 29.25.8. Hence the first statement follows from the description of W' in Lemma 37.7.6. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 29.31.4 for why this implies that the map of conormal sheaves is an isomorphism. $\square$

04FA Lemma 37.7.8. Taking the universal first order thickenings commutes with taking opens. More precisely, let $h : Z \to X$ be a formally unramified morphism of schemes. Let $V \subset Z$, $U \subset X$ be opens such that $h(V) \subset U$. Let Z' be the universal first order thickening of Z over X. Then $h|_V : V \to U$ is formally unramified and the universal first order thickening of V over U is the open subscheme $V' \subset Z'$ such that $V = Z \cap V'$. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.

Proof. The first statement is Lemma 37.6.5. The compatibility of universal thickenings can be deduced from the proof of Lemma 37.7.1, or from Algebra, Lemma 10.149.4 or deduced from Lemma 37.7.7. $\square$

04FB Lemma 37.7.9. Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map

$$c_{h'} : (h')^*\Omega_{X/S} \longrightarrow \Omega_{Z'/S}$$

induces an isomorphism $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Morphisms, Lemma 29.32.8. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the affine case by localization, see Lemma 37.7.8 and Morphisms, Lemma 29.32.3. In this case the result is Algebra, Lemma 10.149.5. $\square$

04FC Lemma 37.7.10. Let $h : Z \to X$ be a formally unramified morphism of schemes over S. There is a canonical exact sequence

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0.$$

The first arrow is induced by $\mathrm{d}_{Z'/S}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Morphisms, Lemma 29.32.15. Hence the result follows on applying Lemma 37.7.9. $\square$

067V Lemma 37.7.11. Let

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[dr, "j"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 37.7.5 and the second from Lemma 37.7.10.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 37.7.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"'] \arrow[dr, "j'"'] & Z' \arrow[r] \arrow[d] & X \arrow[d] \\ & Z'' \arrow[r] & Y \end{tikzcd}$$

Apply Morphisms, Lemma 29.32.18 to the left triangle to get an exact sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

As Z'' is formally unramified over Y (see Lemma 37.7.4) we have $\Omega_{Z'/Z''} = \Omega_{Z/Y}$ (by combining Lemma 37.6.7 and Morphisms, Lemma 29.32.9). Then we have $(i')^*\Omega_{Z'/Y} = i^*\Omega_{X/Y}$ by Lemma 37.7.9. □

06AE Lemma 37.7.12. Let $Z \to Y \to X$ be formally unramified morphisms of schemes.

(1) If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.
(2) There is a canonical exact sequence

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 37.7.5 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 37.7.5. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Morphisms, Lemma 29.31.5 we have an exact sequence

$$(i')^*\mathcal{C}_{Y \times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y \times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Morphisms, Lemma 29.31.4 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y \times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y \times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. □

37.8. Formally étale morphisms

02HF Recall that a ring map $R \to A$ is called formally étale (see Algebra, Definition 10.150.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal of square zero, there exists exactly one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02HG Definition 37.8.1. Let $f : X \to S$ be a morphism of schemes. We say f is formally étale if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists exactly one dotted arrow making the diagram commute.

It is clear that a formally étale morphism is formally unramified. Hence if $f : X \to S$ is formally étale, then $\Omega_{X/S}$ is zero, see Lemma 37.6.7.

04FD Lemma 37.8.2. If $f : X \to S$ is a formally étale morphism, then given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 37.8.1 the condition that T be affine may be dropped.

Proof. Let $T' = \bigcup T'_i$ be an affine open covering, and let $T_i = T \cap T'_i$. Then we get morphisms $a'_i : T'_i \to X$ fitting into the diagram. By uniqueness we see that a'_i and a'_j agree on any affine open subscheme of $T'_i \cap T'_j$. Hence a'_i and a'_j agree on $T'_i \cap T'_j$. Thus we see that the morphisms a'_i glue to a global morphism $a' : T' \to X$. The uniqueness of a' we have seen in Lemma 37.6.2. □

02HI Lemma 37.8.3. A composition of formally étale morphisms is formally étale.

Proof. This is formal. □

02HJ Lemma 37.8.4. A base change of a formally étale morphism is formally étale.

Proof. This is formal. □

02HK Lemma 37.8.5. Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally étale, so is $f|_U : U \to V$.

Proof. Consider a solid diagram

$$\begin{tikzcd} U \arrow[d, "f|_U"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ V & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 37.8.1. If f is formally ramified, then there exists exactly one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since $|T'| = |T|$ we conclude that $a'(T') \subset U$ which gives our unique morphism from T' into U. □

04FE Lemma 37.8.6. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) f is formally étale,

(2) f is formally unramified and the universal first order thickening of X over S is equal to X,
(3) f is formally unramified and $\mathcal{C}_{X/S} = 0$, and
(4) $\Omega_{X/S} = 0$ and $\mathcal{C}_{X/S} = 0$.

Proof. Actually, the last assertion only make sense because $\Omega_{X/S} = 0$ implies that $\mathcal{C}_{X/S}$ is defined via Lemma 37.6.7 and Definition 37.7.2. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/S} = 0$ since after all by definition $\mathcal{C}_{X/S} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. $\square$

04FF Lemma 37.8.7. An unramified flat morphism is formally étale.

Proof. Say $X \to S$ is unramified and flat. Then $\Delta : X \to X \times_S X$ is an open immersion, see Morphisms, Lemma 29.35.13. We have to show that $\mathcal{C}_{X/S}$ is zero. Consider the two projections $p, q : X \times_S X \to X$. As f is formally unramified (see Lemma 37.6.8), q is formally unramified (see Lemma 37.6.4). As f is flat, p is flat, see Morphisms, Lemma 29.25.8. Hence $p^*\mathcal{C}_{X/S} = \mathcal{C}_q$ by Lemma 37.7.7 where $\mathcal{C}_q$ denotes the conormal sheaf of the formally unramified morphism $q : X \times_S X \to X$. But $\Delta(X) \subset X \times_S X$ is an open subscheme which maps isomorphically to X via q. Hence by Lemma 37.7.8 we see that $\mathcal{C}_q|_{\Delta(X)} = \mathcal{C}_{X/X} = 0$. In other words, the pullback of $\mathcal{C}_{X/S}$ to X via the identity morphism is zero, i.e., $\mathcal{C}_{X/S} = 0$. $\square$

02HL Lemma 37.8.8. Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally étale if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally étale ring map.

Proof. This is immediate from the definitions (Definition 37.8.1 and Algebra, Definition 10.150.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 26.6.5. $\square$

02HM Lemma 37.8.9. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is étale, and
(2) the morphism f is locally of finite presentation and formally étale.

Proof. Assume f is étale. An étale morphism is locally of finite presentation, flat and unramified, see Morphisms, Section 29.36. Hence f is locally of finite presentation and formally étale, see Lemma 37.8.7.

Conversely, suppose that f is locally of finite presentation and formally étale. Being étale is local in the Zariski topology on X and S, see Morphisms, Lemma 29.36.2. By Lemma 37.8.5 we can cover X by affine opens U which map into affine opens V such that $U \to V$ is formally étale (and of finite presentation, see Morphisms, Lemma 29.21.2). By Lemma 37.8.8 we see that the ring maps $\mathcal{O}(V) \to \mathcal{O}(U)$ are formally étale (and of finite presentation). We win by Algebra, Lemma 10.150.2. (We will give another proof of this implication when we discuss formally smooth morphisms.) $\square$

37.9. Infinitesimal deformations of maps

04BU In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X.

04FG **Lemma 37.9.1.** Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings over S. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over S. Assume that

(1) $a = b$, and
(2) the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Morphisms, Lemma 29.31.3) are equal.

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_S$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_{X/X'} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & a^{-1}\mathcal{C}_{Y/Y'} \arrow[r] \arrow[u] & a^{-1}\mathcal{O}_{Y'} \arrow[r] \arrow[u, shift left, "(a')^\sharp"'] \arrow[u, shift right, "(b')^\sharp"] & a^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_S$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_S$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_S$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. $\square$

Note that in the situation of the lemma above we may write D as

04BV (37.9.1.1) $$D = \mathrm{d}_{Y/S} \circ \theta$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/S} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$.

02H5 **Lemma 37.9.2.** Let S be a scheme. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over S. Let

$$\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 37.9.1 hold and the derivation D and θ are related by Equation (37.9.1.1).

Proof. We simply set $b = a$ and we define $(b')^\sharp$ to be the map

$$(a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (37.9.1.1). We omit the verification that $(b')^\sharp$ is a map of sheaves of $\mathcal{O}_S$-algebras and that (1) and (2) of Lemma 37.9.1 hold. Equation (37.9.1.1) holds by construction. $\square$

0CK1 **Remark 37.9.3.** Assumptions and notation as in Lemma 37.9.2. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (37.9.1.1).

04FH **Lemma 37.9.4.** Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over S. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an open subscheme $U' \subset X'$ consider morphisms $a' : U' \to Y'$ such that

(1) a' is a morphism over S,
(2) $a'|_U = a|_U$, and
(3) the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.

Here $U = X \cap U'$. Then the rule

04FI (37.9.4.1) $$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\}$$

defines a sheaf of sets on X'.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms are defined locally. $\square$

In the following lemma we identify sheaves on X and any thickening of X.

04FJ **Lemma 37.9.5.** Same notation and assumptions as in Lemma 37.9.4. There is an action of the sheaf

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$$

on the sheaf (37.9.4.1). Moreover, the action is simply transitive for any open $U' \subset X'$ over which the sheaf (37.9.4.1) has a section.

Proof. This is a combination of Lemmas 37.9.1, 37.9.2, and 37.9.4. $\square$

04FK **Remark 37.9.6.** A special case of Lemmas 37.9.1, 37.9.2, 37.9.4, and 37.9.5 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 37.9.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$.

0CK2 **Remark 37.9.7.** Another special case of Lemmas 37.9.1, 37.9.2, 37.9.4, and 37.9.5 is where S itself is a thickening $Z \subset Z' = S$ and $Y = Z \times_{Z'} Y'$. Picture

$$\begin{tikzcd}
(X \subset X') \arrow[rr, dotted, "{(a,?)}"'] \arrow[dr, "{(g,g')}"'] & & (Y \subset Y') \arrow[dl, "{(h,h')}"] \\
& (Z \subset Z') &
\end{tikzcd}$$

In this case the map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is determined by a: the map $h^*\mathcal{C}_{Z/Z'} \to \mathcal{C}_{Y/Y'}$ is surjective (because we assumed $Y = Z \times_{Z'} Y'$), hence the pullback $g^*\mathcal{C}_{Z/Z'} = a^*h^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'}$ is surjective, and the composition $g^*\mathcal{C}_{Z/Z'} \to a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ has to be the canonical map induced by g'. Thus the sheaf of Lemma 37.9.4 is just given by the rule

$$U' \mapsto \{a' : U' \to Y' \text{ over } Z' \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/Z}, \mathcal{C}_{X/X'})$.

04FL Lemma 37.9.8. Let S be a scheme. Let $X \subset X'$ be a first order thickening over S. Let Y be a scheme over S. Let $a', b' : X' \to Y$ be two morphisms over S with $a = a'|_X = b'|_X$. This gives rise to a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"'] & X' \arrow[d, "{(b',a')}"] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

Since the horizontal arrows are immersions with conormal sheaves $\mathcal{C}_{X/X'}$ and $\Omega_{Y/S}$, by Morphisms, Lemma 29.31.3, we obtain a map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$. Then this θ and the derivation D of Lemma 37.9.1 are related by Equation (37.9.1.1).

Proof. Omitted. Hint: The equality may be checked on affine opens where it comes from the following computation. If f is a local section of $\mathcal{O}_Y$, then $1 \otimes f - f \otimes 1$ is a local section of $\mathcal{C}_{Y/(Y \times_S Y)}$ corresponding to $\mathrm{d}_{Y/S}(f)$. It is mapped to the local section $(a')^\sharp(f) - (b')^\sharp(f) = D(f)$ of $\mathcal{C}_{X/X'}$. In other words, $\theta(\mathrm{d}_{Y/S}(f)) = D(f)$. $\square$

For later purposes we need a result that roughly states that the construction of Lemma 37.9.2 is compatible with étale localization.

04BX Lemma 37.9.9. Let

$$\begin{tikzcd} X_1 \arrow[d] & X_2 \arrow[l, "f"'] \arrow[d] \\ S_1 & S_2 \arrow[l] \end{tikzcd}$$

be a commutative diagram of schemes with $X_2 \to X_1$ and $S_2 \to S_1$ étale. Then the map $c_f : f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_2}$ of Morphisms, Lemma 29.32.8 is an isomorphism.

Proof. We recall that an étale morphism $U \to V$ is a smooth morphism with $\Omega_{U/V} = 0$. Using this we see that Morphisms, Lemma 29.32.9 implies $\Omega_{X_2/S_2} = \Omega_{X_2/S_1}$ and Morphisms, Lemma 29.34.16 implies that the map $f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_1}$ (for the morphism f seen as a morphism over S_1) is an isomorphism. Hence the lemma follows. $\square$

04BY Lemma 37.9.10. Consider a commutative diagram of first order thickenings

$$\begin{tikzcd} (T_2 \subset T_2') \arrow[r, "{(a_2,a_2')}"'] \arrow[d, "{(h,h')}"'] & (X_2 \subset X_2') \arrow[d, "{(f,f')}"] \\ (T_1 \subset T_1') \arrow[r, "{(a_1,a_1')}"] & (X_1 \subset X_1') \end{tikzcd}$$

and a commutative diagram of schemes

$$\begin{tikzcd} X_2' \arrow[r] \arrow[d] & S_2 \arrow[d] \\ X_1' \arrow[r] & S_1 \end{tikzcd}$$

with $X_2 \to X_1$ and $S_2 \to S_1$ étale. For any $\mathcal{O}_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{C}_{T_1/T_1'}$ let θ_2 be the composition

$$a_2^*\Omega_{X_2/S_2} = h^*a_1^*\Omega_{X_1/S_1} \xrightarrow{h^*\theta_1} h^*\mathcal{C}_{T_1/T_1'} \longrightarrow \mathcal{C}_{T_2/T_2'}$$

(equality sign is explained in the proof). Then the diagram

$$\begin{tikzcd} T_2' \arrow[r, "\theta_2 \cdot a_2'"'] \arrow[d] & X_2' \arrow[d] \\ T_1' \arrow[r, "\theta_1 \cdot a_1'"] & X_1' \end{tikzcd}$$

commutes where the actions $\theta_2 \cdot a'_2$ and $\theta_1 \cdot a'_1$ are as in Remark 37.9.3.

Proof. The equality sign comes from the identification $f^*\Omega_{X_1/S_1} = \Omega_{X_2/S_2}$ of Lemma 37.9.9. Namely, using this we have $a_2^*\Omega_{X_2/S_2} = a_2^* f^*\Omega_{X_1/S_1} = h^* a_1^*\Omega_{X_1/S_1}$ because $f \circ a_2 = a_1 \circ h$. Having said this, the commutativity of the diagram may be checked on affine opens. Hence we may assume the schemes in the initial big diagram are affine. Thus we obtain commutative diagrams

$$\begin{tikzcd} (B'_2, I_2) & (A'_2, J_2) \arrow[l, "a'_2"'] \\ (B'_1, I_1) \arrow[u, "h'"] & (A'_1, J_1) \arrow[l, "a'_1"] \arrow[u, "f'"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A'_2 & R_2 \arrow[l] \\ A'_1 \arrow[u] & R_1 \arrow[l] \arrow[u] \end{tikzcd}$$

The notation signifies that I_1, I_2, J_1, J_2 are ideals of square zero and maps of pairs are ring maps sending ideals into ideals. Set $A_1 = A'_1/J_1$, $A_2 = A'_2/J_2$, $B_1 = B'_1/I_1$, and $B_2 = B'_2/I_2$. We are given that

$$A_2 \otimes_{A_1} \Omega_{A_1/R_1} \longrightarrow \Omega_{A_2/R_2}$$

is an isomorphism. Then $\theta_1 : B_1 \otimes_{A_1} \Omega_{A_1/R_1} \to I_1$ is B_1-linear. This gives an R_1-derivation $D_1 = \theta_1 \circ \mathrm{d}_{A_1/R_1} : A_1 \to I_1$. In a similar way we see that $\theta_2 : B_2 \otimes_{A_2} \Omega_{A_2/R_2} \to I_2$ gives rise to a R_2-derivation $D_2 = \theta_2 \circ \mathrm{d}_{A_2/R_2} : A_2 \to I_2$. The construction of θ_2 implies the following compatibility between θ_1 and θ_2: for every $x \in A_1$ we have

$$h'(D_1(x)) = D_2(f'(x))$$

as elements of I_2. We may view D_1 as a map $A'_1 \to B'_1$ using $A'_1 \to A_1 \xrightarrow{D_1} I_1 \to B_1$ similarly we may view D_2 as a map $A'_2 \to B'_2$. Then the displayed equality holds for $x \in A'_1$. By the construction of the action in Lemma 37.9.2 and Remark 37.9.3 we know that $\theta_1 \cdot a'_1$ corresponds to the ring map $a'_1 + D_1 : A'_1 \to B'_1$ and $\theta_2 \cdot a'_2$ corresponds to the ring map $a'_2 + D_2 : A'_2 \to B'_2$. By the displayed equality we obtain that $h' \circ (a'_1 + D_1) = (a'_2 + D_2) \circ f'$ as desired. □

04BZ Remark 37.9.11. Lemma 37.9.10 can be improved in the following way. Suppose that we have commutative diagrams as in Lemma 37.9.10 but we do not assume that $X_2 \to X_1$ and $S_2 \to S_1$ are étale. Next, suppose we have $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{C}_{T_1/T'_1}$ and $\theta_2 : a_2^*\Omega_{X_2/S_2} \to \mathcal{C}_{T_2/T'_2}$ such that

$$\begin{tikzcd} f_*\mathcal{O}_{X_2} \arrow[r, "f_*D_2"'] & f_*a_{2,*}\mathcal{C}_{T_2/T'_2} \\ \mathcal{O}_{X_1} \arrow[u, "f^\sharp"] \arrow[r, "D_1"] & a_{1,*}\mathcal{C}_{T_1/T'_1} \arrow[u, "\text{induced by } (h')^\sharp"'] \end{tikzcd}$$

is commutative where D_i corresponds to θ_i as in Equation (37.9.1.1). Then we have the conclusion of Lemma 37.9.10. The importance of the condition that both $X_2 \to X_1$ and $S_2 \to S_1$ are étale is that it allows us to construct a θ_2 from θ_1.

37.10. Infinitesimal deformations of schemes

063X The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

063Y Lemma 37.10.1. Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of first order thickenings. Assume that f is flat. Then the following are equivalent

(1) f' is flat and $X = S \times_{S'} X'$, and
(2) the canonical map $f^*\mathcal{C}_{S/S'} \to \mathcal{C}_{X/X'}$ is an isomorphism.

Proof. As the problem is local on X' we may assume that X, X', S, S' are affine schemes. Say $S' = \mathrm{Spec}(A')$, $X' = \mathrm{Spec}(B')$, $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ with $A = A'/I$ and $B = B'/J$ for some square zero ideals. Then we obtain the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. The canonical map of the lemma is the map

$$I \otimes_A B = I \otimes_{A'} B' \longrightarrow J.$$

The assumption that f is flat signifies that $A \to B$ is flat.

Assume (1). Then $A' \to B'$ is flat and $J = IB'$. Flatness implies $\mathrm{Tor}_1^{A'}(B', A) = 0$ (see Algebra, Lemma 10.75.8). This means $I \otimes_{A'} B' \to B'$ is injective (see Algebra, Remark 10.75.9). Hence we see that $I \otimes_A B \to J$ is an isomorphism.

Assume (2). Then it follows that $J = IB'$, so that $X = S \times_{S'} X'$. Moreover, we get $\mathrm{Tor}_1^{A'}(B', A'/I) = 0$ by reversing the implications in the previous paragraph. Hence B' is flat over A' by Algebra, Lemma 10.99.8. $\square$

The following lemma is the "nilpotent" version of the "critère de platitude par fibres", see Section 37.16.

06AF Lemma 37.10.2. Consider a commutative diagram

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume

(1) X' is flat over S',
(2) f is flat,
(3) $S \subset S'$ is a finite order thickening, and
(4) $X = S \times_{S'} X'$ and $Y = S \times_{S'} Y'$.

Then f' is flat and Y' is flat over S' at all points in the image of f'.

Proof. Immediate consequence of Algebra, Lemma 10.101.8. $\square$

Many properties of morphisms of schemes are preserved under flat deformations.

06AG Lemma 37.10.3. Consider a commutative diagram

$$\begin{tikzcd} (X \subset X') \arrow[rr, "{(f,f')}"'] \arrow[dr] & & (Y \subset Y') \arrow[dl] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume $S \subset S'$ is a finite order thickening, X' flat over S', $X = S \times_{S'} X'$, and $Y = S \times_{S'} Y'$. Then

06AH (1) f is flat if and only if f' is flat,
06AI (2) f is an isomorphism if and only if f' is an isomorphism,
06AJ (3) f is an open immersion if and only if f' is an open immersion,
06AK (4) f is quasi-compact if and only if f' is quasi-compact,
06AL (5) f is universally closed if and only if f' is universally closed,
06AM (6) f is (quasi-)separated if and only if f' is (quasi-)separated,
06AN (7) f is a monomorphism if and only if f' is a monomorphism,
06AP (8) f is surjective if and only if f' is surjective,
06AQ (9) f is universally injective if and only if f' is universally injective,
06AR (10) f is affine if and only if f' is affine,
06AS (11) f is locally of finite type if and only if f' is locally of finite type,
06AT (12) f is locally quasi-finite if and only if f' is locally quasi-finite,
06AU (13) f is locally of finite presentation if and only if f' is locally of finite presentation,
06AV (14) f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,
06AW (15) f is universally open if and only if f' is universally open,
06AX (16) f is syntomic if and only if f' is syntomic,
06AY (17) f is smooth if and only if f' is smooth,
06AZ (18) f is unramified if and only if f' is unramified,
06B0 (19) f is étale if and only if f' is étale,
06B1 (20) f is proper if and only if f' is proper,
06B2 (21) f is integral if and only if f' is integral,
06B3 (22) f is finite if and only if f' is finite,
06B4 (23) f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and
(24) add more here.

Proof. The assumptions on X and Y mean that f is the base change of f' by $X \to X'$. The properties $\mathcal{P}$ listed in (1) – (23) above are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 26.18.2, 26.19.3, 26.21.12, and 26.23.5 and Morphisms, Lemmas 29.9.4, 29.10.4, 29.11.8, 29.15.4, 29.20.13, 29.21.4, 29.29.2, 29.30.4, 29.34.5, 29.35.5, 29.36.4, 29.41.5, 29.44.6, and 29.48.4.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 37.2.1. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over W' with $f'(U') \subset V'$. Let $W' = \mathrm{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \mathrm{Spec}(B')$ and $V' = \mathrm{Spec}(A')$. Then we get a commutative

diagram

$$\begin{tikzcd} 0 \arrow[r] & IB' \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & IA' \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. Moreover $IB' \cong I \otimes_R B$, see proof of Lemma 37.10.1. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps f and f' (after any base change) is identical. (III) If f is flat, then f' is flat and $Y' \to S'$ is flat at every point in the image of f', see Lemma 37.10.2.

Ad (1). This is general remark (III).

Ad (2). Assume f is an isomorphism. By (III) we see that $Y' \to S'$ is flat. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 37.2.3 we see that U' is affine. Thus we have a diagram as in the general remark (I) and moreover $IA \cong I \otimes_R A$ because $R' \to A'$ is flat. Then $IB' \cong I \otimes_R B \cong I \otimes_R A \cong IA'$ and $A \cong B$. By the exactness of the rows in the diagram above we see that $A' \cong B'$, i.e., $U' \cong V'$. Thus f' is an isomorphism.

Ad (3). Assume f is an open immersion. Then f is an isomorphism of X with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is V. Then f' is a map from X' to V' which is an isomorphism by (2). Hence f' is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (6). Note that $X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X')$ so that $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma 37.3.1 for a more general statement.

Ad (7). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. The base change of $\Delta_{X'/Y'}$ by $S \to S'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (2) we conclude that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma 37.3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (10). Assume f is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_{Y'} U'$ is affine. By Lemma 37.2.3 we see that U' is affine. Hence f' is affine. See also Lemma 37.3.1 for a more general statement.

Ad (11). Via remark (I) comes down to proving $A' \to B'$ is of finite type if $A \to B$ is of finite type. Suppose that $x_1, \ldots, x_n \in B'$ are elements whose images in B generate B as an A-algebra. Then $A'[x_1, \ldots, x_n] \to B$ is surjective as both $A'[x_1, \ldots, x_n] \to B$ is surjective and $I \otimes_R A[x_1, \ldots, x_n] \to I \otimes_R B$ is surjective. See also Lemma 37.3.3 for a more general statement.

Ad (12). Follows from (11) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 29.20.6. See also Lemma 37.3.3 for a more general statement.

Ad (13). Via remark (I) comes down to proving $A' \to B'$ is of finite presentation if $A \to B$ is of finite presentation. We may assume that $B' = A'[x_1, \ldots, x_n]/K'$ for some ideal K' by (11). We get a short exact sequence

$$0 \to K' \to A'[x_1, \ldots, x_n] \to B' \to 0$$

As B' is flat over R' we see that $K' \otimes_{R'} R$ is the kernel of the surjection $A[x_1, \ldots, x_n] \to B$. By assumption on $A \to B$ there exist finitely many $f'_1, \ldots, f'_m \in K'$ whose images in $A[x_1, \ldots, x_n]$ generate this kernel. Since I is nilpotent we see that $f'_1, \ldots, f'_m$ generate K' by Nakayama's lemma, see Algebra, Lemma 10.20.1.

Ad (14). Follows from (11) and general remark (II). See also Lemma 37.3.3 for a more general statement.

Ad (15). Immediate from general remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (16). Assume f is syntomic. By (13) f' is locally of finite presentation, by general remark (III) f' is flat and the fibres of f' are the fibres of f. Hence f' is syntomic by Morphisms, Lemma 29.30.11.

Ad (17). Assume f is smooth. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is smooth by Morphisms, Lemma 29.34.3.

Ad (18). Assume f unramified. By (11) f' is locally of finite type and the fibres of f' are the fibres of f. Hence f' is unramified by Morphisms, Lemma 29.35.12. See also Lemma 37.3.3 for a more general statement.

Ad (19). Assume f étale. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is étale by Morphisms, Lemma 29.36.8.

Ad (20). This follows from a combination of (6), (11), (4), and (5). See also Lemma 37.3.3 for a more general statement.

Ad (21). Combine (5) and (10) with Morphisms, Lemma 29.44.7. See also Lemma 37.3.1 for a more general statement.

Ad (22). Combine (21), and (11) with Morphisms, Lemma 29.44.4. See also Lemma 37.3.3 for a more general statement.

Ad (23). Assume f finite locally free. By (22) we see that f' is finite, by general remark (III) f' is flat, and by (13) f' is locally of finite presentation. Hence f' is finite locally free by Morphisms, Lemma 29.48.2. □

The following lemma is the "locally nilpotent" version of the "critère de platitude par fibres", see Section 37.16.

0CF3 **Lemma 37.10.4.** Consider a commutative diagram

$$\begin{tikzcd} (X \subset X') \arrow[rr, "(f,f')"'] \arrow[rd] & & (Y \subset Y') \arrow[ld] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume

(1) $Y' \to S'$ is locally of finite type,
(2) $X' \to S'$ is flat and locally of finite presentation,
(3) f is flat, and
(4) $X = S \times_{S'} X'$ and $Y = S \times_{S'} Y'$.

Then f' is flat and for all $y' \in Y'$ in the image of f' the local ring $\mathcal{O}_{Y',y'}$ is flat and essentially of finite presentation over $\mathcal{O}_{S',s'}$.

Proof. Immediate consequence of Algebra, Lemma 10.128.10. $\square$

Many properties of morphisms of schemes are preserved under flat deformations as in the lemma above.

0CF4 **Lemma 37.10.5.** Consider a commutative diagram

$$\begin{tikzcd} (X \subset X') \arrow[rr, "(f,f')"'] \arrow[rd] & & (Y \subset Y') \arrow[ld] \\ & (S \subset S') & \end{tikzcd}$$

of thickenings. Assume $Y' \to S'$ locally of finite type, $X' \to S'$ flat and locally of finite presentation, $X = S \times_{S'} X'$, and $Y = S \times_{S'} Y'$. Then

0CF5 (1) f is flat if and only if f' is flat,
0CF6 (2) f is an isomorphism if and only if f' is an isomorphism,
0CF7 (3) f is an open immersion if and only if f' is an open immersion,
0CF8 (4) f is quasi-compact if and only if f' is quasi-compact,
0CF9 (5) f is universally closed if and only if f' is universally closed,
0CFA (6) f is (quasi-)separated if and only if f' is (quasi-)separated,
0CFB (7) f is a monomorphism if and only if f' is a monomorphism,
0CFC (8) f is surjective if and only if f' is surjective,
0CFD (9) f is universally injective if and only if f' is universally injective,
0CFE (10) f is affine if and only if f' is affine,
0CFF (11) f is locally quasi-finite if and only if f' is locally quasi-finite,
0CFG (12) f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,
0CFH (13) f is universally open if and only if f' is universally open,
0CFI (14) f is syntomic if and only if f' is syntomic,
0CFJ (15) f is smooth if and only if f' is smooth,
0CFK (16) f is unramified if and only if f' is unramified,
0CFL (17) f is étale if and only if f' is étale,
0CFM (18) f is proper if and only if f' is proper,
0CFN (19) f is finite if and only if f' is finite,

0CFP (20) f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and
(21) add more here.

Proof. The assumptions on X and Y mean that f is the base change of f' by $X \to X'$. The properties $\mathcal{P}$ listed in (1) – (20) above are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 26.18.2, 26.19.3, 26.21.12, and 26.23.5 and Morphisms, Lemmas 29.9.4, 29.10.4, 29.11.8, 29.20.13, 29.29.2, 29.30.4, 29.34.5, 29.35.5, 29.36.4, 29.41.5, 29.44.6, and 29.48.4.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over W' with $f'(U') \subset V'$. Let $W' = \mathrm{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \mathrm{Spec}(B')$ and $V' = \mathrm{Spec}(A')$. Then we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & IB' \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & IA' \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps f and f' (after any base change) is identical. (III) If f is flat, then f' is flat and $Y' \to S'$ is flat at every point in the image of f', see Lemma 37.10.2.

Ad (1). This is general remark (III).

Ad (2). Assume f is an isomorphism. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 37.2.3 we see that U' is affine. Thus we have a diagram as in the general remark (I). By Algebra, Lemma 10.126.11 we see that $A' \to B'$ is an isomorphism, i.e., $U' \cong V'$. Thus f' is an isomorphism.

Ad (3). Assume f is an open immersion. Then f is an isomorphism of X with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is V. Then f' is a map from X' to V' which is an isomorphism by (2). Hence f' is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (6). Note that $X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X')$ so that $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma 37.3.1 for a more general statement.

Ad (7). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. Observe that $X' \times_{Y'} X' \to S'$ is locally of finite type. The base change of $\Delta_{X'/Y'}$ by $S \to S'$ is $\Delta_{X/Y}$ which is an isomorphism by assumption. By (2) we conclude that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma 37.3.1 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (10). Assume f is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_{Y'} U'$ is affine. By Lemma 37.2.3 we see that U' is affine. Hence f' is affine. See also Lemma 37.3.1 for a more general statement.

Ad (11). Follows from the fact that f' is locally of finite type (by Morphisms, Lemma 29.15.8) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 29.20.6.

Ad (12). Follows from general remark (II) and the fact that f' is locally of finite type (Morphisms, Lemma 29.15.8).

Ad (13). Immediate from general remark (II). See also Lemma 37.3.1 for a more general statement.

Ad (14). Assume f is syntomic. By Morphisms, Lemma 29.21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is syntomic by Morphisms, Lemma 29.30.11.

Ad (15). Assume f is smooth. By Morphisms, Lemma 29.21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is smooth by Morphisms, Lemma 29.34.3.

Ad (16). Assume f unramified. By Morphisms, Lemma 29.15.8 f' is locally of finite type. The fibres of f' are the fibres of f. Hence f' is unramified by Morphisms, Lemma 29.35.12.

Ad (17). Assume f étale. By Morphisms, Lemma 29.21.11 f' is locally of finite presentation. By general remark (III) f' is flat. The fibres of f' are the fibres of f. Hence f' is étale by Morphisms, Lemma 29.36.8.

Ad (18). This follows from a combination of (6), the fact that f is locally of finite type (Morphisms, Lemma 29.15.8), (4), and (5).

Ad (19). Combine (5), (10), Morphisms, Lemma 29.44.7, the fact that f is locally of finite type (Morphisms, Lemma 29.15.8), and Morphisms, Lemma 29.44.4.

Ad (20). Assume f finite locally free. By (19) we see that f' is finite. By general remark (III) f' is flat. By Morphisms, Lemma 29.21.11 f' is locally of finite presentation. Hence f' is finite locally free by Morphisms, Lemma 29.48.2. $\square$

0D4F Lemma 37.10.6 (Deformations of projective schemes). Let $f : X \to S$ be a morphism of schemes which is proper, flat, and of finite presentation. Let $\mathcal{L}$ be f-ample. Assume S is quasi-compact. There exists a $d_0 \geq 0$ such that for every cartesian diagram

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "i"] & S' \end{tikzcd} \quad \text{and} \quad \begin{matrix} \text{invertible } \mathcal{O}_{X'}\text{-module} \\ \mathcal{L}' \text{ with } \mathcal{L} \cong (i')^*\mathcal{L}' \end{matrix}$$

where $S \subset S'$ is a thickening and f' is proper, flat, of finite presentation we have

(1) $R^p(f')_*(\mathcal{L}')^{\otimes d} = 0$ for all $p > 0$ and $d \geq d_0$,

(2) $\mathcal{A}'_d = (f')_*(\mathcal{L}')^{\otimes d}$ is finite locally free for $d \geq d_0$,
(3) $\mathcal{A}' = \mathcal{O}_{S'} \oplus \bigoplus_{d \geq d_0} \mathcal{A}'_d$ is a quasi-coherent $\mathcal{O}_{S'}$-algebra of finite presentation,
(4) there is a canonical isomorphism $r' : X' \to \underline{\text{Proj}}_{S'}(\mathcal{A}')$, and
(5) there is a canonical isomorphism $\theta' : (r')^*\mathcal{O}_{\underline{\text{Proj}}_{S'}(\mathcal{A}')}(1) \to \mathcal{L}'$.

The construction of $\mathcal{A}'$, r', θ' is functorial in the data $(X', S', i, i', f', \mathcal{L}')$.

Proof. We first describe the maps r' and θ'. Observe that $\mathcal{L}'$ is f'-ample, see Lemma 37.3.2. There is a canonical map of quasi-coherent graded $\mathcal{O}_{S'}$-algebras $\mathcal{A}' \to \bigoplus_{d \geq 0}(f')_*(\mathcal{L}')^{\otimes d}$ which is an isomorphism in degrees $\geq d_0$. Hence this induces an isomorphism on relative Proj compatible with the Serre twists of the structure sheaf, see Constructions, Lemma 27.18.4. Hence we get the morphism r' by Morphisms, Lemma 29.37.4 (which in turn appeals to the construction given in Constructions, Lemma 27.19.1) and it is an isomorphism by Morphisms, Lemma 29.43.17. We get the map θ' from Constructions, Lemma 27.19.1. By Properties, Lemma 28.28.2 we find that θ' is an isomorphism (this also uses that the morphism r' over affine opens of S' is the same as the morphism from Properties, Lemma 28.26.9 as is explained in the proof of Morphisms, Lemma 29.43.17).

Assuming the vanishing and local freeness stated in parts (1) and (2), the functoriality of the construction can be seen as follows. Suppose that $h : T \to S'$ is a morphism of schemes, denote $f_T : X'_T \to T$ the base change of f' and $\mathcal{L}_T$ the pullback of $\mathcal{L}$ to X'_T. By cohomology and base change (as formulated in Derived Categories of Schemes, Lemma 36.22.5 for example) we have the corresponding vanishing over T and moreover $h^*\mathcal{A}'_d = f_{T,*}\mathcal{L}_T^{\otimes d}$ (and thus the local freeness of pushforwards as well as the finite generation of the corresponding graded $\mathcal{O}_T$-algebra $\mathcal{A}_T$). Hence the morphism $r_T : X_T \to \underline{\text{Proj}}_T(\bigoplus f_{T,*}\mathcal{L}_T^{\otimes d})$ is simply the base change of r' to T and the pullback of θ' is the map θ_T.

Having said all of the above, we see that it suffices to prove (1), (2), and (3). Pick d_0 such that $R^pf_*\mathcal{L}^{\otimes d} = 0$ for all $d \geq d_0$ and $p > 0$, see Cohomology of Schemes, Lemma 30.16.1. We claim that d_0 works.

By cohomology and base change (Derived Categories of Schemes, Lemma 36.30.4) we see that $E'_d = Rf'_*(\mathcal{L}')^{\otimes d}$ is a perfect object of $D(\mathcal{O}_{S'})$ and its formation commutes with arbitrary base change. In particular, $E_d = Li^*E'_d = Rf_*\mathcal{L}^{\otimes d}$. By Derived Categories of Schemes, Lemma 36.32.4 we see that for $d \geq d_0$ the complex E_d is isomorphic to the finite locally free $\mathcal{O}_S$-module $f_*\mathcal{L}^{\otimes d}$ placed in cohomological degree 0. Then by Derived Categories of Schemes, Lemma 36.31.3 we conclude that E'_d is isomorphic to a finite locally free module placed in cohomological degree 0. Of course this means that $E'_d = \mathcal{A}'_d[0]$, that $R^pf'_*(\mathcal{L}')^{\otimes d} = 0$ for $p > 0$, and that $\mathcal{A}'_d$ is finite locally free. This proves (1) and (2).

The last thing we have to show is finite presentation of $\mathcal{A}'$ as a sheaf of $\mathcal{O}_{S'}$-algebras (this notion was introduced in Properties, Section 28.22). Let $U' = \text{Spec}(R') \subset S'$ be an affine open. Then $A' = \mathcal{A}'(U')$ is a graded R'-algebra whose graded parts are finite projective R'-modules. We have to show that A' is a finitely presented R'-algebra. We will prove this by reduction to the Noetherian case. Namely, we can find a finite type $\mathbf{Z}$-subalgebra $R'_0 \subset R'$ and a pair[1]

[1]With the same properties as those enjoyed by $X' \to S'$ and $\mathcal{L}'$, i.e., $X'_0 \to \text{Spec}(R'_0)$ is flat and proper and $\mathcal{L}'_0$ is ample.

$(X'_0, \mathcal{L}'_0)$ over R'_0 whose base change is $(X'_{U'}, \mathcal{L}'|_{X'_{U'}})$, see Limits, Lemmas 32.10.2, 32.10.3, 32.13.1, 32.8.7, and 32.4.15. Cohomology of Schemes, Lemma 30.16.1 implies $A'_0 = \bigoplus_{d \geq 0} H^0(X'_0, (\mathcal{L}'_0)^{\otimes d})$ is a finitely generated graded R'_0-algebra and implies there exists a d'_0 such that $H^p(X'_0, (\mathcal{L}'_0)^{\otimes d}) = 0$, $p > 0$ for $d \geq d'_0$. By the arguments given above applied to $X'_0 \to \mathrm{Spec}(R'_0)$ and $\mathcal{L}'_0$ we see that $(A'_0)_d$ is a finite projective R'_0-module and that

$$A'_d = \mathcal{A}'_d(U') = H^0(X'_{U'}, (\mathcal{L}')^{\otimes d}|_{X'_{U'}}) = H^0(X'_0, (\mathcal{L}'_0)^{\otimes d}) \otimes_{R'_0} R' = (A'_0)_d \otimes_{R'_0} R'$$

for $d \geq d'_0$. Now a small twist in the argument is that we don't know that we can choose d'_0 equal to d_0[2]. To get around this we use the following sequence of arguments to finish the proof:

(a) The algebra $B = R'_0 \oplus \bigoplus_{d \geq \max(d_0, d'_0)} (A'_0)_d$ is an R'_0-algebra of finite type: apply the Artin-Tate lemma to $B \subset A'_0$, see Algebra, Lemma 10.51.7.
(b) As R'_0 is Noetherian we see that B is an R'_0-algebra of finite presentation.
(c) By right exactness of tensor product we see that $B \otimes_{R'_0} R'$ is an R'-algebra of finite presentation.
(d) By the displayed equalities this exactly says that $C = R' \oplus \bigoplus_{d \geq \max(d_0, d'_0)} A'_d$ is an R'-algebra of finite presentation.
(e) The quotient A'/C is the direct sum of the finite projective R'-modules A'_d, $d_0 \leq d \leq \max(d_0, d'_0)$, hence finitely presented as R'-module.
(f) The quotient A'/C is finitely presented as a C-module by Algebra, Lemma 10.6.4.
(g) Thus A' is finitely presented as a C-module by Algebra, Lemma 10.5.3.
(h) By Algebra, Lemma 10.7.4 this implies A' is finitely presented as a C-algebra.
(i) Finally, by Algebra, Lemma 10.6.2 applied to $R' \to C \to A'$ this implies A' is finitely presented as an R'-algebra.

This finishes the proof. □

37.11. Formally smooth morphisms

02GZ Michael Artin's position on differential criteria of smoothness (e.g., Morphisms, Lemma 29.34.14) is that they are basically useless (in practice). In this section we introduce the notion of a formally smooth morphism $X \to S$. Such a morphism is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 37.11.7. It turns out that this criterion is often easier to use than the differential criteria mentioned above.

Recall that a ring map $R \to A$ is called formally smooth (see Algebra, Definition 10.138.1) if for every commutative solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rd, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

[2] Actually, one can reduce to this case by doing more limit arguments.

where $I \subset B$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

02H0 Definition 37.11.1. Let $f : X \to S$ be a morphism of schemes. We say f is formally smooth if given any solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists a dotted arrow making the diagram commute.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 37.6.2 and 37.8.2. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow Zariski locally on T'. In fact, analyzing the proof of Lemma 37.11.10 shows that this would be equivalent to the definition as it currently stands. In particular, being formally smooth is Zariski local on the source (and in fact it is smooth local on the source, insert future reference here).

02H1 Lemma 37.11.2. A composition of formally smooth morphisms is formally smooth.

Proof. Omitted. $\square$

02H2 Lemma 37.11.3. A base change of a formally smooth morphism is formally smooth.

Proof. Omitted, but see Algebra, Lemma 10.138.2 for the algebraic version. $\square$

02HH Lemma 37.11.4. Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.

Proof. Omitted. $\square$

02H3 Lemma 37.11.5. Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally smooth, so is $f|_U : U \to V$.

Proof. Consider a solid diagram

$$\begin{tikzcd} U \arrow[d, "f|_U"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ V & T' \arrow[l] \arrow[lu, dashed] \end{tikzcd}$$

as in Definition 37.11.1. If f is formally smooth, then there exists an S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since the underlying sets of T and T' are the same we see that a' is a morphism into U (see Schemes, Section 26.3). And it clearly is a V-morphism as well. Hence the dotted arrow above as desired. $\square$

02H4 Lemma 37.11.6. Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally smooth if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally smooth ring map.

Proof. This is immediate from the definitions (Definition 37.11.1 and Algebra, Definition 10.138.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 26.6.5. □

The following lemma is the main result of this section. It is a victory of the functorial point of view in that it implies (combined with Limits, Proposition 32.6.1) that we can recognize whether a morphism $f : X \to S$ is smooth in terms of "simple" properties of the functor $h_X : Sch/S \to \text{Sets}$.

02H6 Lemma 37.11.7 (Infinitesimal lifting criterion). Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) The morphism f is smooth, and
(2) the morphism f is locally of finite presentation and formally smooth.

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a pair of affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 37.11.5 we see that $U \to V$ is formally smooth. By Lemma 37.11.6 we see that $R \to A$ is formally smooth. By Morphisms, Lemma 29.21.2 we see that $R \to A$ is of finite presentation. By Algebra, Proposition 10.138.13 we see that $R \to A$ is smooth. Hence by the definition of a smooth morphism we see that $X \to S$ is smooth.

Conversely, assume that $f : X \to S$ is smooth. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ S & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 37.11.1. We will show the dotted arrow exists thereby proving that f is formally smooth.

Let $\mathcal{F}$ be the sheaf of sets on T' of Lemma 37.9.4 in the special case discussed in Remark 37.9.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 37.9.5. Our goal is simply to show that $\mathcal{F}(T) \neq \emptyset$. In other words we are trying to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor on T (see Cohomology, Section 20.4). There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}_t \neq \emptyset$ for all $t \in T$ (see Cohomology, Definition 20.4.1). (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$ (see Cohomology, Lemma 20.4.3 – we may use either cohomology of $\mathcal{H}$ as an abelian sheaf or as an $\mathcal{O}_T$-module, see Cohomology, Lemma 20.13.3).

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $U \subset T$ neighbourhood of t such that $a(U)$ is contained in an affine open $\text{Spec}(A) = W \subset X$ which maps to an affine open $\text{Spec}(R) = V \subset S$. By Morphisms, Lemma 29.34.2 the ring map $R \to A$ is smooth. Hence by Algebra, Proposition 10.138.13 the ring map $R \to A$ is formally smooth. Lemma 37.11.6 in turn implies that $W \to V$ is formally smooth. Hence we can lift $a|_U : U \to W$ to a V-morphism $a' : U' \to W \subset X$ showing that $\mathcal{F}(U) \neq \emptyset$.

Finally we prove (II). By Morphisms, Lemma 29.32.13 we see that $\Omega_{X/S}$ is of finite presentation (it is even finite locally free by Morphisms, Lemma 29.34.12). Hence

$a^*\Omega_{X/S}$ is of finite presentation (see Modules, Lemma 17.11.4). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$ is quasi-coherent by the discussion in Schemes, Section 26.24. Thus by Cohomology of Schemes, Lemma 30.2.2 we have $H^1(T, \mathcal{H}) = 0$ as desired. □

Locally projective quasi-coherent modules are defined in Properties, Section 28.21.

06B5 Lemma 37.11.8. Let $f : X \to Y$ be a formally smooth morphism of schemes. Then $\Omega_{X/Y}$ is locally projective on X.

Proof. Choose $U \subset X$ and $V \subset Y$ affine open such that $f(U) \subset V$. By Lemma 37.11.5 $f|_U : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see Lemma 37.11.6. Hence by Algebra, Lemma 10.138.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 28.21. □

0D0E Lemma 37.11.9. Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules. Consider $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$. If $\mathcal{F}$ is locally projective, then $H^1(T, \mathcal{H}) = 0$.

Proof. By the definition of a locally projective sheaf on a scheme (see Properties, Definition 28.21.1) we see that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module. Hence we may assume that $\mathcal{F} = \bigoplus_{i\in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i\in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology, Lemma 20.11.12 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Cohomology of Schemes, Lemma 30.2.2. □

0D0F Lemma 37.11.10. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

(1) f is formally smooth,
(2) for every $x \in X$ there exist opens $x \in U \subset X$ and $f(x) \in V \subset Y$ with $f(U) \subset V$ such that $f|_U : U \to V$ is formally smooth,
(3) for every pair of affine opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is formally smooth, and
(4) there exists an affine open covering $Y = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is a formally smooth ring map for all j and i.

Proof. The implications (1) ⇒ (2), (1) ⇒ (3), and (2) ⇒ (4) follow from Lemma 37.11.5. The implication (3) ⇒ (4) is immediate.

Assume (4). The proof that f is formally smooth is the same as the second part of the proof of Lemma 37.11.7. Consider a solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l, "a"'] \arrow[d, "i"] \\ Y & T' \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

as in Definition 37.11.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on T' of Lemma 37.9.4 as in the special case discussed in Remark 37.9.6. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on T with action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ as in Lemma 37.9.5. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology, Definition 20.4.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ locally has a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$, see Cohomology, Lemma 20.4.3.

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $W \subset T$ neighbourhood of t such that $a(W)$ is contained in U_{ji} for some i, j. Let $W' \subset T'$ be the corresponding open subscheme. By assumption (4) we can lift $a|_W : W \to U_{ji}$ to a V_j-morphism $a' : W' \to U_{ji}$ showing that $\mathcal{F}(W')$ is nonempty.

Finally we prove (II). By Lemma 37.11.8 we see that Ω_{U_{ji}/V_j} locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Properties, Lemma 28.21.2. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties, Lemma 28.21.3. Hence

$$H^1(T, \mathcal{H}) = H^1(T, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$$

by Lemma 37.11.9 as desired. □

06B6 Lemma 37.11.11. Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is formally smooth. Then

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Morphisms, Lemma 29.32.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ formally smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 10.131.7 is exact. This is Algebra, Lemma 10.138.9. □

06B7 Lemma 37.11.12. Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Assume that Z is formally smooth over S. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 37.7.10 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 37.7.10 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

Since $Z \to S$ is formally smooth we can locally on Z' find a left inverse $Z' \to Z$ over S to the inclusion map $Z \to Z'$. Thus the sequence is locally split, see Morphisms, Lemma 29.32.16. □

067W Lemma 37.11.13. Let

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[rd, "j"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

be a commutative diagram of schemes where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 37.7.11 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 37.7.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"'] \arrow[dr, "j'"'] & Z' \arrow[r, "a"'] \arrow[d, "k"] & X \arrow[d, "f"] \\ & Z'' \arrow[r, "b"] & Y \end{tikzcd}$$

In the proof of Lemma 37.7.11 we identified the sequence above with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

Let $U'' \subset Z''$ be an affine open. Denote $U \subset Z$ and $U' \subset Z'$ the corresponding affine open subschemes. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Z' \arrow[d, "k"] \\ U'' \arrow[r] \arrow[ur, "g"] & Z'' \end{tikzcd}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. $\square$

37.12. Smoothness over a Noetherian base

02HW It turns out that if the base is Noetherian then we can get away with less in the formulation of formal smoothness. In some sense the following lemmas are the beginning of deformation theory.

02HX **Lemma 37.12.1.** *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

1. *f is smooth at x,*
2. *for every solid commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ S & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

   where $B' \to B$ is a surjection of local rings with $\mathrm{Ker}(B' \to B)$ of square zero, and α mapping the closed point of $\mathrm{Spec}(B)$ to x there exists a dotted arrow making the diagram commute,
3. *same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 10.141.1), and*
4. *same as in (2) but with $B' \to B$ ranging over small extensions such that α induces an isomorphism $\kappa(x) \to \kappa(\mathfrak{m})$ where $\mathfrak{m} \subset B$ is the maximal ideal.*

Proof. Choose an affine neighbourhood $V \subset S$ of $f(x)$ and choose an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. For any "test" diagram as in (2) the morphism α will map $\mathrm{Spec}(B)$ into U and the morphism β will map $\mathrm{Spec}(B')$ into V (see Schemes, Section 26.13). Hence the lemma reduces to the morphism $f|_U : U \to V$ of affines. (Indeed, V is Noetherian and $f|_U$ is of finite type, see Properties, Lemma 28.5.2 and Morphisms, Lemma 29.15.2.) In this affine case the lemma is identical to Algebra, Lemma 10.141.2. □

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions "centered" at points of finite type (see Morphisms, Section 29.16).

02HY **Lemma 37.12.2.** Let $f : X \to S$ be a morphism of schemes. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:

(1) f is smooth,
(2) for every solid commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathrm{Spec}(B) \arrow[l, "\alpha"'] \arrow[d, "i"] \\ S & \mathrm{Spec}(B') \arrow[l, "\beta"] \arrow[ul, dashed] \end{tikzcd}$$

where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 37.11.7) says f is formally smooth and (2) holds.

Assume (2). The set of points $x \in X$ where f is not smooth forms a closed subset T of X. By the discussion in Morphisms, Section 29.16, if $T \neq \emptyset$ there exists a point $x \in T \subset X$ such that the morphism

$$\mathrm{Spec}(\kappa(x)) \to X \to S$$

is of finite type (namely, pick any point x of T which is closed in an affine open of X). By Morphisms, Lemma 29.16.2 given any local Artinian ring B' with residue field $\kappa(x)$ then any morphism $\beta : \mathrm{Spec}(B') \to S$ is of finite type. Thus we see that all the diagrams used in Lemma 37.12.1 (4) correspond to diagrams as in the current lemma (2). Whence $X \to S$ is smooth a x a contradiction. □

Here is a useful application.

0A43 **Lemma 37.12.3.** Let $f : X \to S$ be a finite type morphism of locally Noetherian schemes. Let $Z \subset S$ be a closed subscheme with nth infinitesimal neighbourhood $Z_n \subset S$. Set $X_n = Z_n \times_S X$.

(1) If $X_n \to Z_n$ is smooth for all n, then f is smooth at every point of $f^{-1}(Z)$.
(2) If $X_n \to Z_n$ is étale for all n, then f is étale at every point of $f^{-1}(Z)$.

Proof. Assume $X_n \to Z_n$ is smooth for all n. Let $x \in X$ be a point lying over a point of Z. Given a small extension $B' \to B$ and morphisms α, β as in Lemma 37.12.1 part (3) the maximal ideal of B' is nilpotent (as B' is Artinian) and hence the morphism β factors through Z_n and α factors through X_n for a suitable n. Thus the lifting property for $X_n \to Z_n$ kicks in to get the desired dotted arrow in the diagram. This proves (1). Part (2) follows from (1) and the fact that a morphism is étale if and only if it is smooth of relative dimension 0. □

0D4G Lemma 37.12.4. Let $f : X \to S$ be a morphism of locally Noetherian schemes. Let $Z \subset S$ be a closed subscheme with nth infinitesimal neighbourhood $Z_n \subset S$. Set $X_n = Z_n \times_S X$. If $X_n \to Z_n$ is flat for all n, then f is flat at every point of $f^{-1}(Z)$.

Proof. This is a translation of Algebra, Lemma 10.99.11 into the language of schemes. $\square$

37.13. The naive cotangent complex

0D0G This section is the continuation of Modules, Section 17.31 which in turn continues the discussion in Algebra, Section 10.134.

0D0H Definition 37.13.1. Let $f : X \to Y$ be a morphism of schemes. The naive cotangent complex of f is the complex defined in Modules, Definition 17.31.6. Notation: NL_f or $NL_{X/Y}$.

0D0I Lemma 37.13.2. Let $f : X \to Y$ be a morphism of schemes. Let $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ be affine opens with $f(U) \subset V$. There is a canonical map

$$\widetilde{NL_{A/R}} \longrightarrow NL_{X/Y}|_U$$

of complexes which is an isomorphism in $D(\mathcal{O}_U)$.

Proof. From the construction of $NL_{X/Y}$ in Modules, Section 17.31 we see there is a canonical map of complexes $NL_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)} \to NL_{X/Y}(U)$ of $A = \mathcal{O}_X(U)$-modules, which is compatible with further restrictions. Using the canonical map $R \to f^{-1}\mathcal{O}_Y(U)$ we obtain a canonical map $NL_{A/R} \to NL_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)}$ of complexes of A-modules. Using the universal property of the $\widetilde{\ }$ functor (see Schemes, Lemma 26.7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows from Algebra, Lemma 10.134.11 and 10.134.13 and Modules, Lemma 17.31.4 (and the description of the stalks of $\mathcal{O}_X$ and $f^{-1}\mathcal{O}_Y$ at a point $\mathfrak{p} \in \text{Spec}(A)$ as $A_\mathfrak{p}$ and $R_\mathfrak{q}$ where $\mathfrak{q} = R \cap \mathfrak{p}$; references used are Schemes, Lemma 26.5.4 and Sheaves, Lemma 6.21.5). $\square$

0D0J Lemma 37.13.3. Let $f : X \to Y$ be a morphism of schemes. The cohomology sheaves of the complex $NL_{X/Y}$ are quasi-coherent, zero outside degrees -1, 0 and equal to $\Omega_{X/Y}$ in degree 0.

Proof. By construction of the naive cotangent complex in Modules, Section 17.31 we have that $NL_{X/Y}$ is a complex sitting in degrees -1, 0 and that its cohomology in degree 0 is $\Omega_{X/Y}$. The sheaf of differentials is quasi-coherent (by Morphisms, Lemma 29.32.7). To finish the proof it suffices to show that $H^{-1}(NL_{X/Y})$ is quasi-coherent. This follows by checking over affines using Lemma 37.13.2. $\square$

0D0K Lemma 37.13.4. Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite presentation, then $NL_{X/Y}$ is locally on X quasi-isomorphic to a complex

$$\ldots \to 0 \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to 0 \to \ldots$$

of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{F}^0$ of finite presentation and $\mathcal{F}^{-1}$ of finite type.

Proof. By Lemma 37.13.2 it suffices to show that $NL_{A/R}$ has this shape if $R \to A$ is a finitely presented ring map. Write $A = R[x_1, \ldots, x_n]/I$ with I finitely generated. Then I/I^2 is a finite A-module and $NL_{A/R}$ is quasi-isomorphic to

$$\ldots \to 0 \to I/I^2 \to \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i \to 0 \to \ldots$$

by Algebra, Section 10.134 and in particular Algebra, Lemma 10.134.2. □

0D0L Lemma 37.13.5. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) f is formally smooth,
(2) $H^{-1}(NL_{X/Y}) = 0$ and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is locally projective.

Proof. This follows from Algebra, Proposition 10.138.8 and Lemma 37.11.10. □

0D0M Lemma 37.13.6. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) f is formally étale,
(2) $H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.

Proof. A formally étale morphism is formally smooth and hence we have $H^{-1}(NL_{X/Y}) = 0$ by Lemma 37.13.5. On the other hand, we have $\Omega_{X/Y} = 0$ by Lemma 37.8.6. Conversely, if (2) holds, then f is formally smooth by Lemma 37.13.5 and formally unramified by Lemma 37.6.7 and hence formally étale by Lemmas 37.11.4. □

0D0N Lemma 37.13.7. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) f is smooth, and
(2) f is locally of finite presentation, $H^{-1}(NL_{X/Y}) = 0$, and $H^0(NL_{X/Y}) = \Omega_{X/Y}$ is finite locally free.

Proof. This follows from the definition of a smooth ring homomorphism (Algebra, Definition 10.137.1), Lemma 37.13.2, and the definition of a smooth morphism of schemes (Morphisms, Definition 29.34.1). We also use that finite locally free is the same as finite projective for modules over rings (Algebra, Lemma 10.78.2). □

0G7Z Lemma 37.13.8. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) f is étale, and
(2) f is locally of finite presentation and $H^{-1}(NL_{X/Y}) = H^0(NL_{X/Y}) = 0$.

Proof. This follows from the definition of an étale ring homomorphism (Algebra, Definition 10.143.1), Lemma 37.13.2, and the definition of an étale morphism of schemes (Morphisms, Definition 29.36.1). □

0FV2 Lemma 37.13.9. Let $i : Z \to X$ be an immersion of schemes. Then $NL_{Z/X}$ is isomorphic to $\mathcal{C}_{Z/X}[1]$ in $D(\mathcal{O}_Z)$ where $\mathcal{C}_{Z/X}$ is the conormal sheaf of Z in X.

Proof. This follows from Algebra, Lemma 10.134.6, Morphisms, Lemma 29.31.2, and Lemma 37.13.2. □

0E44 Lemma 37.13.10. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. There is a canonical six term exact sequence

$$H^{-1}(f^* NL_{Y/Z}) \to H^{-1}(NL_{X/Z}) \to H^{-1}(NL_{X/Y}) \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of cohomology sheaves.

Proof. Special case of Modules, Lemma 17.31.7. $\square$

0FV3 Lemma 37.13.11. Let $f : X \to Y$ and $Y \to Z$ be morphisms of schemes. Assume $X \to Y$ is a complete intersection morphism. Then there is a canonical distinguished triangle

$$f^* NL_{Y/Z} \to NL_{X/Z} \to NL_{X/Y} \to f^* NL_{Y/Z}[1]$$

in $D(\mathcal{O}_X)$ which recovers the 6-term exact sequence of Lemma 37.13.10.

Proof. It suffices to show the canonical map

$$f^* NL_{Y/Z} \to \text{Cone}(NL_{X/Y} \to NL_{X/Z})[-1]$$

of Modules, Lemma 17.31.7 is an isomorphism in $D(\mathcal{O}_X)$. In order to show this, it suffices to show that the 6-term sequence has a zero on the left, i.e., that $H^{-1}(f^* NL_{Y/Z}) \to H^{-1}(NL_{X/Z})$ is injective. Affine locally this follows from the corresponding algebra result in More on Algebra, Lemma 15.33.6. To translate into algebra use Lemma 37.13.2. $\square$

0G80 Lemma 37.13.12. Let $X \to Y \to Z$ be morphisms of schemes. Assume $X \to Z$ smooth and $Y \to Z$ étale. Then $X \to Y$ is smooth.

Proof. The morphism $X \to Y$ is locally of finite presentation by Morphisms, Lemma 29.21.11. By Lemma 37.13.7 we have $H^{-1}(NL_{X/Z}) = 0$ and the module $\Omega_{X/Z}$ is finite locally free. By Lemma 37.13.8 we have $H^{-1}(NL_{Y/Z}) = H^0(NL_{Y/Z}) = 0$. By Lemma 37.13.10 we get $H^{-1}(NL_{X/Y}) = 0$ and $\Omega_{X/Y} \cong \Omega_{X/Z}$ is finite locally free. By Lemma 37.13.7 the morphism $X \to Y$ is smooth. $\square$

0FV4 Lemma 37.13.13. Let $f : X \to Y$ be a morphism of schemes which factors as $f = g \circ i$ with i an immersion and $g : P \to Y$ formally smooth (for example smooth). Then there is a canonical isomorphism

$$NL_{X/Y} \cong \left(\mathcal{C}_{X/P} \to i^*\Omega_{P/Y}\right)$$

in $D(\mathcal{O}_X)$ where the conormal sheaf $\mathcal{C}_{X/P}$ is placed in degree -1.

Proof. (For the parenthetical statement see Lemma 37.11.7.) By Lemmas 37.13.9 and 37.13.5 we have $NL_{X/P} = \mathcal{C}_{X/P}[1]$ and $NL_{P/Y} = \Omega_{P/Y}$ with $\Omega_{P/Y}$ locally projective. This implies that $i^* NL_{P/Y} \to i^*\Omega_{P/Y}$ is a quasi-isomorphism too (small detail omitted; the reason is that $i^* NL_{P/Y}$ is the same thing as $\tau_{\geq -1} Li^* NL_{P/Y}$, see More on Algebra, Lemma 15.85.1). Thus the canonical map

$$i^* NL_{P/Y} \to \text{Cone}(NL_{X/Y} \to NL_{X/P})[-1]$$

of Modules, Lemma 17.31.7 is an isomorphism in $D(\mathcal{O}_X)$ because the cohomology group $H^{-1}(i^* NL_{P/Y})$ is zero by what we said above. In other words, we have a distinguished triangle

$$i^* NL_{P/Y} \to NL_{X/Y} \to NL_{X/P} \to i^* NL_{P/Y}[1]$$

Clearly, this means that $NL_{X/Y}$ is the cone on the map $NL_{X/P}[-1] \to i^* NL_{P/Y}$ which is equivalent to the statement of the lemma by our computation of the cohomology sheaves of these objects in the derived category given above. $\square$

0FV5 **Lemma 37.13.14.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

The canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ induces an isomorphism on H^0 and a surjection on H^{-1}.

Proof. Translated into algebra this is More on Algebra, Lemma 15.85.2. To do the translation use Lemma 37.13.2. $\square$

0FJZ **Lemma 37.13.15.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

If $Y' \to Y$ is flat, then the canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism.

Proof. By Lemma 37.13.2 this follows from Algebra, Lemma 10.134.8. $\square$

0FK0 **Lemma 37.13.16.** Consider a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

If $X \to Y$ is flat, then the canonical map $(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism. If in addition $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$ then $L(g')^* NL_{X/Y} \to NL_{X'/Y'}$ is a quasi-isomorphism too.

Proof. Translated into algebra this is More on Algebra, Lemma 15.85.3. To do the translation use Lemma 37.13.2 and Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.4. $\square$

37.14. Pushouts in the category of schemes, I

07RS In this section we construct pushouts of $Y \leftarrow X \to X'$ where $X \to Y$ is affine and $X \to X'$ is a thickening. This will actually be an important case for us, hence a detailed discussion is merited. In Section 37.67 we discuss a more interesting and more difficult case. See Categories, Section 4.9 for a general discussion of pushouts in any category.

0ET0 Lemma 37.14.1. Let $A' \to A$ be a surjection of rings and let $B \to A$ be a ring map. Let $B' = B \times_A A'$ be the fibre product of rings. Set $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, $T = \mathrm{Spec}(B)$, and $T' = \mathrm{Spec}(B')$. Then

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "f"'] & S' \arrow[d, "f'"] \\ T \arrow[r, "i'"] & T' \end{tikzcd} \quad \text{corresponding to} \quad \begin{tikzcd} A & A' \arrow[l] \\ B \arrow[u] & B' \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout of schemes.

Proof. By More on Algebra, Lemma 15.6.2 we have $T' = T \amalg_S S'$ as topological spaces, i.e., the diagram is a pushout in the category of topological spaces. Next, consider the map

$$((i')^\sharp, (f')^\sharp) : \mathcal{O}_{T'} \longrightarrow i'_*\mathcal{O}_T \times_{g_*\mathcal{O}_S} f'_*\mathcal{O}_{S'}$$

where $g = i' \circ f = f' \circ i$. We claim this map is an isomorphism of sheaves of rings. Namely, we can view both sides as quasi-coherent $\mathcal{O}_{T'}$-modules (use Schemes, Lemmas 26.24.1 for the right hand side) and the map is $\mathcal{O}_{T'}$-linear. Thus it suffices to show the map is an isomorphism on the level of global sections (Schemes, Lemma 26.7.5). On global sections we recover the identification $B' \to B \times_A A'$ from statement of the lemma (this is how we chose B').

Let X be a scheme. Suppose we are given morphisms of schemes $m' : S' \to X$ and $n : T \to X$ such that $m' \circ i = n \circ f$ (call this m). We get a unique map of topological spaces $n' : T' \to X$ compatible with m' and n as $T' = T \amalg_S S'$ (see above). By the description of $\mathcal{O}_{T'}$ in the previous paragraph we obtain a unique homomorphism of sheaves of rings

$$(n')^\sharp : \mathcal{O}_X \longrightarrow (n')_*\mathcal{O}_{T'} = m'_*\mathcal{O}_T \times_{m_*\mathcal{O}_T} n_*\mathcal{O}_S$$

given by $(m')^\sharp$ and $n^\sharp$. Thus $(n', (n')^\sharp)$ is the unque morphism of ringed spaces $T' \to X$ compatible with m' and n. To finish the proof it suffices to show that n' is a morphism of schemes, i.e., a morphism of locally ringed spaces.

Let $t' \in T'$ with image $x \in X$. We have to show that $\mathcal{O}_{X,x} \to \mathcal{O}_{T',t'}$ is local. If $t' \notin T$, then t' is the image of a unique point $s' \in S'$ and $\mathcal{O}_{T',t'} = \mathcal{O}_{S',s'}$. Namely, $S' \setminus S \to T' \setminus T$ is an isomorphism of schemes as $B' \to A'$ induces an isomorphism $\mathrm{Ker}(B' \to B) = \mathrm{Ker}(A' \to A)$. If t' is the image of $t \in T$, then we know that the composition $\mathcal{O}_{X,x} \to \mathcal{O}_{T',t'} \to \mathcal{O}_{T,t}$ is local and we conclude also. □

0BMP Lemma 37.14.2. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram of schemes. Let $(W, X_i \to W)$ be a cocone for the diagram in the category of schemes (Categories, Remark 4.14.5). If there exists a fpqc covering $\{W_a \to W\}_{a \in A}$ of schemes such that

(1) for all $a \in A$ we have $W_a = \mathrm{colim}\, X_i \times_W W_a$ in the category of schemes, and
(2) for all $a, b \in A$ we have $W_a \times_W W_b = \mathrm{colim}\, X_i \times_W W_a \times_W W_b$ in the category of schemes,

then $W = \mathrm{colim}\, X_i$ in the category of schemes.

Proof. Namely, for a scheme T a morphism $W \to T$ is the same thing as collection of morphism $W_a \to T$, $a \in A$ which agree on the overlaps $W_a \times_W W_b$, see Descent, Lemma 35.13.7. □

07RT Lemma 37.14.3. Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Then there exists a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

in the category of schemes. Moreover, $Y \subset Y'$ is a thickening, $X = Y \times_{Y'} X'$, and

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

as sheaves on $|Y| = |Y'|$.

Proof. We first construct Y' as a ringed space. Namely, as topological space we take $Y' = Y$. Denote $f' : X' \to Y'$ the map of topological spaces which equals f. As structure sheaf $\mathcal{O}_{Y'}$ we take the right hand side of the equation of the lemma. To see that Y' is a scheme, we have to show that any point has an affine neighbourhood. Since the formation of the fibre product of sheaves commutes with restricting to opens, we may assume Y is affine. Then X is affine (as f is affine) and X' is affine as well (see Lemma 37.2.3). Say $Y \leftarrow X \to X'$ corresponds to $B \to A \leftarrow A'$. Set $B' = B \times_A A'$; this is the global sections of $\mathcal{O}_{Y'}$. As $A' \to A$ is surjective with locally nilpotent kernel we see that $B' \to B$ is surjective with locally nilpotent kernel. Hence $\text{Spec}(B') = \text{Spec}(B)$ (as topological spaces). We claim that $Y' = \text{Spec}(B')$. To see this we will show for $g' \in B'$ with image $g \in B$ that $\mathcal{O}_{Y'}(D(g)) = B'_{g'}$. Namely, by More on Algebra, Lemma 15.5.3 we see that

$$(B')_{g'} = B_g \times_{A_h} A'_{h'}$$

where $h \in A$, $h' \in A'$ are the images of g'. Since B_g, resp. A_h, resp. $A'_{h'}$ is equal to $\mathcal{O}_Y(D(g))$, resp. $f_*\mathcal{O}_X(D(g))$, resp. $f'_*\mathcal{O}_{X'}(D(g))$ the claim follows.

It remains to show that Y' is the pushout. The discussion above shows the scheme Y' has an affine open covering $Y' = \bigcup W'_i$ such that the corresponding opens $U'_i \subset X'$, $W_i \subset Y$, and $U_i \subset X$ are affine open. Moreover, if A'_i, B_i, A_i are the rings corresponding to U'_i, W_i, U_i, then W'_i corresponds to $B_i \times_{A_i} A'_i$. Thus we can apply Lemmas 37.14.1 and 37.14.2 to conclude our construction is a pushout in the category of schemes. $\square$

In the following lemma we use the fibre product of categories as defined in Categories, Example 4.31.3.

07RV Lemma 37.14.4. Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 37.14.3). Base change gives a functor

$$F : (Sch/Y') \longrightarrow (Sch/Y) \times_{(Sch/Y')} (Sch/X')$$

given by $V' \longmapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which has a left adjoint

$$G : (Sch/Y) \times_{(Sch/Y')} (Sch/X') \longrightarrow (Sch/Y')$$

which sends the triple (V, U', φ) to the pushout $V \amalg_{(V \times_Y X)} U'$. Finally, $F \circ G$ is isomorphic to the identity functor.

Proof. Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies U with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms, Lemma 29.11.8. Hence we can apply Lemma 37.14.3 to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on U as morphisms into Y'. Setting $G(V, U', \varphi) = V'$ gives the functor G.

Let us prove that G is a left adjoint to F. Let Z be a scheme over Y'. We have to show that

$$\mathrm{Mor}(V', Z) = \mathrm{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of g' with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then (g, f') is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of g, resp. f by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from U to Z. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse.

To prove that $F \circ G$ is isomorphic to the identity we have to show that the adjunction mapping $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. To do this we may work affine locally. Say $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, and $Y = \mathrm{Spec}(B)$. Then $A' \to A$ and $B \to A$ are ring maps as in More on Algebra, Lemma 15.6.4 and $Y' = \mathrm{Spec}(B')$ with $B' = B \times_A A'$. Next, suppose that $V = \mathrm{Spec}(D)$, $U' = \mathrm{Spec}(C')$ and φ is given by an A-algebra isomorphism $D \otimes_B A \to C' \otimes_{A'} A = C'/IC'$. Set $D' = D \times_{C'/IC'} C'$. In this case the statement we have to prove is that $D' \otimes_{B'} B \cong D$ and $D' \otimes_{B'} A' \cong C'$. This is a special case of More on Algebra, Lemma 15.6.4. □

08KU Lemma 37.14.5. Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 37.14.3). Let $V' \to Y'$ be a morphism of schemes. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, and $U = X \times_{Y'} V'$. There is an equivalence of categories between

1. quasi-coherent $\mathcal{O}_{V'}$-modules flat over Y', and
2. the category of triples $(\mathcal{G}, \mathcal{F}', \varphi)$ where
   - (a) $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_V$-module flat over Y,
   - (b) $\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{U'}$-module flat over X', and
   - (c) $\varphi : (U \to V)^*\mathcal{G} \to (U \to U')^*\mathcal{F}'$ is an isomorphism of $\mathcal{O}_U$-modules.

The equivalence maps $\mathcal{G}'$ to $((V \to V')^*\mathcal{G}', (U' \to V')^*\mathcal{G}', can)$. Suppose $\mathcal{G}'$ corresponds to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$. Then

- (a) $\mathcal{G}'$ is a finite type $\mathcal{O}_{V'}$-module if and only if $\mathcal{G}$ and $\mathcal{F}'$ are finite type $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules.
- (b) if $V' \to Y'$ is locally of finite presentation, then $\mathcal{G}'$ is an $\mathcal{O}_{V'}$-module of finite presentation if and only if $\mathcal{G}$ and $\mathcal{F}'$ are $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules of finite presentation.

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \to V')_*\mathcal{G} \times_{(U \to V')_*\mathcal{F}} (U' \to V')_*\mathcal{F}'$$

where $\mathcal{F} = (U \to U')^*\mathcal{F}'$. This works, because on affines we recover the equivalence of More on Algebra, Lemma 15.7.5. Some details omitted.

Parts (a) and (b) follow from More on Algebra, Lemmas 15.7.4 and 15.7.6. □

07RX Lemma 37.14.6. In the situation of Lemma 37.14.4. If $V' = G(V, U', \varphi)$ for some triple (V, U', φ), then

(1) $V' \to Y'$ is locally of finite type if and only if $V \to Y$ and $U' \to X'$ are locally of finite type,
(2) $V' \to Y'$ is flat if and only if $V \to Y$ and $U' \to X'$ are flat,
(3) $V' \to Y'$ is flat and locally of finite presentation if and only if $V \to Y$ and $U' \to X'$ are flat and locally of finite presentation,
(4) $V' \to Y'$ is smooth if and only if $V \to Y$ and $U' \to X'$ are smooth,
(5) $V' \to Y'$ is étale if and only if $V \to Y$ and $U' \to X'$ are étale, and
(6) add more here as needed.

If W' is flat over Y', then the adjunction mapping $G(F(W')) \to W'$ is an isomorphism. Hence F and G define mutually quasi-inverse functors between the category of schemes flat over Y' and the category of triples (V, U', φ) with $V \to Y$ and $U' \to X'$ flat.

Proof. Looking over affine pieces the assertions of this lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 15.7.7. □

37.15. Openness of the flat locus

0398 This result takes some work to prove, and (perhaps) deserves its own section. Here it is.

0399 Theorem 37.15.1. Let S be a scheme. Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is locally of finite presentation. Then [DG67, IV Theorem 11.3.1]

$$U = \{x \in X \mid \mathcal{F} \text{ is flat over } S \text{ at } x\}$$

is open in X.

Proof. We may test for openness locally on X hence we may assume that f is a morphism of affine schemes. In this case the theorem is exactly Algebra, Theorem 10.129.4. □

047C Lemma 37.15.2. Let S be a scheme. Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x' \in X'$ with images $x = g'(x')$ and $s' = f'(x')$.

(1) If $\mathcal{F}$ is flat over S at x, then $(g')^*\mathcal{F}$ is flat over S' at x'.
(2) If g is flat at s' and $(g')^*\mathcal{F}$ is flat over S' at x', then $\mathcal{F}$ is flat over S at x.

In particular, if g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation, then formation of the open subset of Theorem 37.15.1 commutes with base change.

Proof. Consider the commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{S',s'} \arrow[u] & \mathcal{O}_{S,s} \arrow[l] \arrow[u] \end{tikzcd}$$

Note that $\mathcal{O}_{X',x'}$ is a localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$, and that $((g')^*\mathcal{F})_{x'}$ is equal to $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$. Hence the lemma follows from Algebra, Lemma 10.100.1. □

37.16. Critère de platitude par fibres

039A Consider a commutative diagram of schemes (left hand diagram)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} X_s \arrow[rr, "f_s"] \arrow[dr] & & Y_s \arrow[dl] \\ & \mathrm{Spec}(\kappa(s)) & \end{tikzcd}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in X$ lying over $s \in S$ with image $y = f(x)$ we consider the question: Is $\mathcal{F}$ flat over Y at x? If $\mathcal{F}$ is flat over S at x, then the theorem states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre

$$\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$$

is flat over Y_s at x. Below you will find a "Noetherian" version, a "finitely presented" version, and earlier we treated a "nilpotent" version, see Lemma 37.10.2.

039B Theorem 37.16.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$. Set $y = f(x)$ and $s \in S$ the image of x in S. Assume S, X, Y locally Noetherian, $\mathcal{F}$ coherent, and $\mathcal{F}_x \neq 0$. Then the following are equivalent:

(1) $\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and
(2) Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) ⇒ (2) is Algebra, Lemma 10.99.15. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 10.39.4 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 10.39.7. □

Here is the non-Noetherian version.

039C Theorem 37.16.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) X is locally of finite presentation over S,

(2) $\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and
(3) Y is locally of finite type over S.

Let $x \in X$. Set $y = f(x)$ and let $s \in S$ be the image of x in S. If $\mathcal{F}_x \neq 0$, then the following are equivalent:

(1) $\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and
(2) Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.

Moreover, the set of points x where (1) and (2) hold is open in $\text{Supp}(\mathcal{F})$.

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 10.128.9. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 10.39.4 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 10.39.7.

By Morphisms, Lemma 29.21.11 the morphism f is locally of finite presentation. Consider the set

05VI (37.16.2.1) $$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over both } Y \text{ and } S\}.$$

This set is open in X by Theorem 37.15.1. Note that if $x \in U$, then $\mathcal{F}_s$ is flat at x over Y_s as a base change of a flat module under the morphism $Y_s \to Y$, see Morphisms, Lemma 29.25.7. Hence at every point of $U \cap \text{Supp}(\mathcal{F})$ condition (1) is satisfied. On the other hand, it is clear that if $x \in \text{Supp}(\mathcal{F})$ satisfies (1) and (2), then $x \in U$. Thus the open set we are looking for is $U \cap \text{Supp}(\mathcal{F})$. □

These theorems are often used in the following simplified forms. We give only the global statements – of course there are also pointwise versions.

039D Lemma 37.16.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume

(1) S, X, Y are locally Noetherian,
(2) X is flat over S,
(3) for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat.

Then f is flat. If f is also surjective, then Y is flat over S.

Proof. This is a special case of Theorem 37.16.1. □

039E Lemma 37.16.4. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume

(1) X is locally of finite presentation over S,
(2) X is flat over S,
(3) for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat, and
(4) Y is locally of finite type over S.

Then f is flat. If f is also surjective, then Y is flat over S.

Proof. This is a special case of Theorem 37.16.2. □

05VJ Lemma 37.16.5. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) X is locally of finite presentation over S,
(2) $\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,
(3) $\mathcal{F}$ is flat over S, and
(4) Y is locally of finite type over S.

Then the set

$$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change: If $S' \to S$ is a morphism of schemes, and U' is the set of points of $X' = X \times_S S'$ where $\mathcal{F}' = \mathcal{F} \times_S S'$ is flat over $Y' = Y \times_S S'$, then $U' = U \times_S S'$.

Proof. By Morphisms, Lemma 29.21.11 the morphism f is locally of finite presentation. Hence U is open by Theorem 37.15.1. Because we have assumed that $\mathcal{F}$ is flat over S we see that Theorem 37.16.2 implies

$$U = \{x \in X \mid \mathcal{F}_s \text{ flat at } x \text{ over } Y_s\}.$$

where s always denotes the image of x in S. (This description also works trivially when $\mathcal{F}_x = 0$.) Moreover, the assumptions of the lemma remain in force for the morphism $f' : X' \to Y'$ and the sheaf $\mathcal{F}'$. Hence U' has a similar description. In other words, it suffices to prove that given $s' \in S'$ mapping to $s \in S$ we have

$$\{x' \in X'_{s'} \mid \mathcal{F}'_{s'} \text{ flat at } x' \text{ over } Y'_{s'}\}$$

is the inverse image of the corresponding locus in X_s. This is true by Lemma 37.15.2 because in the cartesian diagram

$$\begin{tikzcd} X'_{s'} \arrow[r] \arrow[d] & X_s \arrow[d] \\ Y'_{s'} \arrow[r] & Y_s \end{tikzcd}$$

the horizontal morphisms are flat as they are base changes by the flat morphism $\mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s))$. □

05VK Lemma 37.16.6. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume

(1) X is locally of finite presentation over S,
(2) X is flat over S, and
(3) Y is locally of finite type over S.

Then the set

$$U = \{x \in X \mid X \text{ flat at } x \text{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change.

Proof. This is a special case of Lemma 37.16.5. □

The following lemma is a variant of Algebra, Lemma 10.99.4. Note that the hypothesis that $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module means that $(\mathcal{F}_s)_x$ is a free $\mathcal{O}_{X_s,x}$-module which is always the case if $x \in X_s$ is a generic point of an irreducible component of X_s and X_s is reduced (namely, in this case $\mathcal{O}_{X_s,x}$ is a field, see Algebra, Lemma 10.25.1).

080Q Lemma 37.16.7. Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in X$ with image $s \in S$. If $\mathcal{F}$ is flat at x over S and $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module, then $\mathcal{F}$ is finite free in a neighbourhood of x.

Proof. If $\mathcal{F}_x \otimes \kappa(x)$ is zero, then $\mathcal{F}_x = 0$ by Nakayama's lemma (Algebra, Lemma 10.20.1) and hence $\mathcal{F}$ is zero in a neighbourhood of x (Modules, Lemma 17.9.5) and the lemma holds. Thus we may assume $\mathcal{F}_x \otimes \kappa(x)$ is not zero and we see that Theorem 37.16.2 applies with $f = \text{id} : X \to X$. We conclude that $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$. Hence $\mathcal{F}_x$ is free, see Algebra, Lemma 10.78.5 for example. Choose an open neighbourhood $x \in U \subset X$ and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ which map to a basis in $\mathcal{F}_x$. The corresponding map $\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$ is surjective after shrinking U (Modules, Lemma 17.9.5). Then $\text{Ker}(\psi)$ is of finite type (see Modules, Lemma 17.11.3) and $\text{Ker}(\psi)_x = 0$. Whence after shrinking U once more ψ is an isomorphism. □

0CZR Lemma 37.16.8. Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module flat over S. Then the set

$$\{x \in X : \mathcal{F} \text{ free in a neighbourhood of } x\}$$

is open in X and its formation commutes with arbitrary base change $S' \to S$.

Proof. Openness holds trivially. Let $x \in X$ mapping to $s \in S$. By Lemma 37.16.7 we see that x is in our set if and only if $\mathcal{F}|_{X_s}$ is flat at x over X_s. Clearly this is also equivalent to $\mathcal{F}$ being flat at x over X (because this statement is implied by freeness of $\mathcal{F}_x$ and implies flatness of $\mathcal{F}|_{X_s}$ at x over X_s). Thus the base change statement follows from Lemma 37.16.5 applied to $\text{id} : X \to X$ over S. □

37.17. Closed immersions between smooth schemes

0H1G Some results that do not fit elsewhere very well.

0FUE Lemma 37.17.1. Let S be a scheme. Let $Y \to X$ be a closed immersion of schemes smooth over S. For every $y \in Y$ there exist integers $0 \leq m, n$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & \mathbf{A}^m_S \arrow[d, "{(a_1,\ldots,a_m)\mapsto(a_1,\ldots,a_m,0\ldots,0)}"] \\ X & U \arrow[l] \arrow[r, "\pi"] & \mathbf{A}^{m+n}_S \end{tikzcd}$$

where $U \subset X$ is open, $V = Y \cap U$, π is étale, $V = \pi^{-1}(\mathbf{A}^m_S)$, and $y \in V$.

Proof. The question is local on X hence we may replace X by an open neighbourhood of y. Since $Y \to X$ is a regular immersion by Divisors, Lemma 31.22.11 we may assume $X = \text{Spec}(A)$ is affine and there exists a regular sequence $f_1, \ldots, f_n \in A$ such that $Y = V(f_1, \ldots, f_n)$. After shrinking X (and hence Y) further we may assume there exists an étale morphism $Y \to \mathbf{A}^m_S$, see Morphisms, Lemma 29.36.20. Let $\overline{g}_1, \ldots, \overline{g}_m$ in $\mathcal{O}_Y(Y)$ be the coordinate functions of this étale morphism. Choose lifts $g_1, \ldots, g_m \in A$ of these functions and consider the morphism

$$(g_1, \ldots, g_m, f_1, \ldots, f_n) : X \longrightarrow \mathbf{A}^{m+n}_S$$

over S. This is a morphism of schemes locally of finite presentation over S and hence is locally of finite presentation (Morphisms, Lemma 29.21.11). The restriction of this morphism to $\mathbf{A}^m_S \subset \mathbf{A}^{m+n}_S$ is étale by construction. Thus, in order to show that $X \to \mathbf{A}^{m+n}_S$ is étale at y it suffices to show that $X \to \mathbf{A}^{m+n}_S$ is flat at y, see Morphisms, Lemma 29.36.15. Let $s \in S$ be the image of y. It suffices to check that $X_s \to \mathbf{A}^{m+n}_s$ is flat at y, see Theorem 37.16.2. Let $z \in \mathbf{A}^{m+n}_s$ be the image of y. The local ring map

$$\mathcal{O}_{\mathbf{A}^{m+n}_s, z} \longrightarrow \mathcal{O}_{X_s, y}$$

is flat by Algebra, Lemma 10.128.1. Namely, schemes smooth over fields are regular and regular rings are Cohen-Macaulay, see Varieties, Lemma 33.25.3 and Algebra, Lemma 10.106.3. Thus both source and target are regular local rings (and hence CM). The source and target have the same dimension: namely, we have $\dim(\mathcal{O}_{Y_s,y}) = \dim(\mathcal{O}_{\mathbf{A}^m_s,z})$ by More on Algebra, Lemma 15.44.2, we have $\dim(\mathcal{O}_{\mathbf{A}^{m+n}_s,z}) = n + \dim(\mathcal{O}_{\mathbf{A}^m_s,z})$, and we have $\dim(\mathcal{O}_{X_s,y}) = n + \dim(\mathcal{O}_{Y_s,y})$ because $\mathcal{O}_{Y_s,y}$ is the quotient of $\mathcal{O}_{X_s,y}$ by the regular sequence $f_1, \ldots, f_n$ of length n (see Divisors, Remark 31.22.5). Finally, the fibre ring of the displayed arrow is finite over $\kappa(z)$ since $Y_s \to \mathbf{A}^m_s$ is étale at y. This finishes the proof. $\square$

0H1H Remark 37.17.2. We fix a ring R and we set $S = \mathrm{Spec}(R)$. Fix integers $0 \leq m$ and $1 \leq n$. Consider the closed immersion

$$Z = \mathbf{A}^m_S \longrightarrow \mathbf{A}^{m+n}_S = X, \quad (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_m, 0, \ldots 0).$$

We are going to consider the blowing up X' of X along the closed subscheme Z. Write

$$X = \mathrm{Spec}(A) \quad \text{with} \quad A = R[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

Then X' is the Proj of the Rees algebra of A with respect ot the ideal $(y_1, \ldots, y_n)$. This Rees algebra is equal to $B = A[T_1, \ldots, T_n]/(y_iT_j - y_jT_i)$; details omitted. Hence $X' = \mathrm{Proj}(B)$ is smooth over S as it is covered by the affine opens

$$\begin{aligned} D_+(T_i) &= \mathrm{Spec}(B_{(T_i)}) \\ &= \mathrm{Spec}(A[t_1, \ldots, \hat{t}_i, \ldots t_n]/(y_j - y_it_j)) \\ &= \mathrm{Spec}(R[x_1, \ldots, x_m, y_i, t_1, \ldots, \hat{t}_i, \ldots, t_n]) \end{aligned}$$

which are isomorphic to $\mathbf{A}^{n+m}_S$. In this chart the exceptional divisor is cut out by setting $y_i = 0$ hence the exceptional divisor is smooth over S as well.

0FUT Lemma 37.17.3. Let S be a scheme. Let $Z \to X$ be a closed immersion of schemes smooth over S. Let $b : X' \to X$ be the blowing up of Z with exceptional divisor $E \subset X'$. Then X' and E are smooth over S. The morphism $p : E \to Z$ is canonically isomorphic to the projective space bundle

$$\mathbf{P}(\mathcal{I}/\mathcal{I}^2) \longrightarrow Z$$

where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Z. The relative $\mathcal{O}_E(1)$ coming from the projective space bundle structure is isomorphic to the restriction of $\mathcal{O}_{X'}(-E)$ to E.

Proof. By Divisors, Lemma 31.22.11 the immersion $Z \to X$ is a regular immmersion, hence the ideal sheaf $\mathcal{I}$ is of finite type, hence b is a projective morphism with relatively ample invertible sheaf $\mathcal{O}_{X'}(1) = \mathcal{O}_{X'}(-E)$, see Divisors, Lemmas 31.32.4 and 31.32.13. The canonical map $\mathcal{I} \to b_*\mathcal{O}_{X'}(1)$ gives a closed immersion

$$X' \longrightarrow \mathbf{P}\left(\bigoplus\nolimits_{n \geq 0} \mathrm{Sym}^n_{\mathcal{O}_X}(\mathcal{I})\right)$$

by the very construction of the blowup. The restriction of this morphism to E gives a canonical map

$$E \longrightarrow \mathbf{P}\left(\bigoplus\nolimits_{n\geq 0} \mathrm{Sym}^n_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2)\right)$$

over Z. Since $\mathcal{I}/\mathcal{I}^2$ is finite locally free if this canonical map is an isomorphism, then the final part of the lemma holds. Having said all of this, now the question is étale local on X. Namely, blowing up commutes with flat base change by Divisors, Lemma 31.32.3 and we can check smoothness after precomposing with a surjective étale morphism. Thus by the étale local structure of a closed immersion of schemes over S given in Lemma 37.17.1 this reduces us to the case discussed in Remark 37.17.2. □

37.18. Flat modules and relative assassins

0GSF In this section we will prove that the support of a flat module is (in some sense) equidimensional over the base in geometric situations. For the Noetherian case we refer the reader to [DG67, IV Proposition 12.1.1.5]. First, we prove two helper lemmas.

0GSG Lemma 37.18.1. Let A be a valuation ring. Let $A \to B$ is a local homomorphism of local rings which is essentially of finite type. Let $u : N \to M$ be a map of finite B-modules. Assume M is flat over A and $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is injective and $M/u(N)$ is flat over A.

Proof. We will deduce this lemma from Algebra, Lemma 10.128.4 (please note that we exchanged the roles of M and N). To do the reduction we will use More on Algebra, Lemma 15.25.7 to reduce to the finitely presented case.

By assumption we can write B as a quotient of the localization of a polynomial algebra $P = A[x_1, \ldots, x_n]$ at a prime ideal $\mathfrak{q}$. Then we can think of $u : N \to M$ as a map of finite $P_\mathfrak{q}$-modules. Hence we may and do assume that B is essentially of finite presentation over A.

Next, the B-module N is finite but perhaps not of finite presentation. Write $N = \mathrm{colim}\, N_\lambda$ as a filtered colimit of finitely presented B-modules with surjective transition maps. For example choose a presentation $0 \to K \to B^{\oplus r} \to N \to 0$, write K as the union of its finite submodules K_λ, and set $N_\lambda = \mathrm{Coker}(K_\lambda \to B^{\oplus r})$. The module $N/\mathfrak{m}_A N$ is of finite presentation over the Noetherian ring $B/\mathfrak{m}_A B$. Hence for λ large enough we have $N_\lambda/\mathfrak{m}_A N_\lambda = N/\mathfrak{m}_A N$. Now, if we can show the lemma for the composition $u_\lambda : N_\lambda \to M$, then we conclude that $N_\lambda = N$ and the result holds for u. Hence we may and do assume N is of finite presentation over B.

By More on Algebra, Lemma 15.25.7 the module M is of finite presentation over B. Thus all the assumptions of Algebra, Lemma 10.128.4 hold and we conclude. □

0GSH Lemma 37.18.2. Let $f : X \to S$ be a morphism of schemes. Let $y \in X$ be a point with image $t \in S$. Denote $Y \subset X$ the closure of $\{y\}$ viewed as an integral closed subscheme of X. Let $s \in S$ and let $x \in Y_s$ be a generic point of an irreducible component of Y_s. There exists a cartesian diagram

This can be found in the proof of [DG67, IV Proposition 12.1.1.5]

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with the following properties:

(1) S' is the spectrum of a valuation ring with generic point t' and closed point s',
(2) $g(t') = t$ and $g(s') = s$,
(3) there exists a point $y' \in X'_{t'}$ which is a generic point of an irreducible component of $(S' \times_S Y)_{t'} = Y_t \times_t t'$ and satisfies $g'(y') = y$,
(4) denoting $Y' \subset X'$ the closure of $\{y'\}$ viewed as an integral closed subscheme of X' there exists a point $x' \in Y'_{s'}$ which is a generic point of an irreducible component of $Y'_{s'}$ with $g'(x') = x$.

Proof. We choose a valuation ring R, we set $S' = \mathrm{Spec}(R)$ with generic point t' and closed point s', and we choose a morphism $h : S' \to X$ with $h(t') = y$ and $h(s') = x$. See Schemes, Lemma 26.20.4. Set $g = f \circ h$ so that $g(t') = t$ and $g(s') = s$. Consider the base change

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "g"] \arrow[u, bend left, "\sigma"] & S \end{tikzcd}$$

We obtain a section σ of the base change such that $h = g' \circ \sigma$.

Of course σ factors through the base change $S' \times_S Y$ of Y as h factors through Y. Let $y' \in X'_{t'} \subset X'$ be the generic point of an irreducible component of the fibre

$$(S' \times_S Y)_{t'} = Y_t \times_t t'$$

containing the point $\sigma(t')$, i.e., such that $y' \rightsquigarrow \sigma(t')$. Since $g'(y') \in Y_t$ and $g(y') \rightsquigarrow g(\sigma(t')) = y$ we find that $g'(y') = y$ because y is the generic point of the fibre Y_t. Denote $Y' \subset X'$ the closure of $\{y'\}$ in X' viewed as an integral closed subscheme. Then σ factors through Y' as $\sigma(t') \in Y'$. Choose a generic point $x' \in Y'_{s'}$ of an irreducible component of $Y'_{s'}$ which contains $\sigma(s')$, i.e., we get $x' \rightsquigarrow \sigma(s')$ and hence $g'(x') \rightsquigarrow g'(\sigma(s')) = x$. Again as x is a generic point of an irreducible component of Y_s by assumption and as $g'(Y') \subset Y$ we conclude that $g'(x') = x$. □

0GSI Lemma 37.18.3. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent finite type $\mathcal{O}_X$-module. Let $y \in \mathrm{Ass}_{X/S}(\mathcal{F})$ with image $t \in S$. Denote $Y \subset X$ the closure of $\{y\}$ in X viewed as an integral closed subscheme. Let $s \in S$ and let $x \in Y_s$ be a generic point of an irreducible component of Y_s. If $\mathcal{F}$ is flat over S at x, then $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$ and $\dim_x(Y_s) = \dim(Y_t)$.

Proof. Choose a diagram as in Lemma 37.18.2. Set $\mathcal{F}' = (g')^*\mathcal{F}$. Divisors, Lemma 31.7.3 implies that $y' \in \mathrm{Ass}_{X'/S'}(\mathcal{F}')$. By our choice of y' we also see that $\dim(Y'_{t'}) = \dim(Y_t)$, see for example Algebra, Lemma 10.116.7. By Algebra, Lemma 10.125.9 we see that $Y'_{s'}$ is equidimensional of dimension equal to $\dim(Y_t)$. Since $\mathcal{F}$ is flat at x over S we see that $\mathcal{F}'$ is flat at x' over S', see Morphisms, Lemma 29.25.7.

Suppose that we can show $x' \in \mathrm{Ass}_{X'/S}(\mathcal{F}')$. Then Divisors, Lemma 31.7.3 implies that $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$ and that the irreducible component C' of $Y'_{s'}$ containing x' is an irreducible component of $C \times_s s'$ where $C \subset Y_s$ is the irreducible component containing x. Whence $\dim(C) = \dim(C') = \dim(Y_t)$ (see above) and the proof is complete. This reduces us to the case discussed in the next paragraph.

Assume $S = \mathrm{Spec}(A)$ where A is a valuation ring and t and s are the generic and closed points of S. We will assume $x \not\in \mathrm{Ass}_{X/S}(\mathcal{F})$ in order to get a contradiction. In other words, we assume $x \not\in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ where $\mathcal{F}_s$ is the pullback of $\mathcal{F}$ to X_s. Consider the ring map

$$A \longrightarrow \mathcal{O}_{X,x} = B$$

and the module $N = \mathcal{F}_x$ over $B = \mathcal{O}_{X,x}$. Then $B/\mathfrak{m}_A B = \mathcal{O}_{X_s,x}$ and $N/\mathfrak{m}_A N$ is the stalk of $\mathcal{F}_s$ at the point x. Denote $\mathfrak{q} \subset B$ the prime ideal corresponding to the point y, see Schemes, Lemma 26.13.2. Since x is a generic point of Y_s we see that the radical of $\mathfrak{q} + \mathfrak{m}_A B$ is $\mathfrak{m}_B$. Then $\mathrm{Ass}_{B/\mathfrak{m}_A B}(N/\mathfrak{m}_A N)$ is a finite set of prime ideals (Algebra, Lemma 10.63.5) which doesn't contain the maximal ideal of $B/\mathfrak{m}_A B$ since $x \not\in \mathrm{Ass}_{X/S}(\mathcal{F})$. Thus the image of of $\mathfrak{q}$ in $B/\mathfrak{m}_A B$ is not contained in any of those prime ideals. Hence by prime avoidance (Algebra, Lemma 10.15.2) we can find an element $g \in \mathfrak{q}$ whose image in $B/\mathfrak{m}_A B$ is a nonzerodivisor on $N/\mathfrak{m}_A N$ (this uses the description of zerodivisors in Algebra, Lemma 10.63.9). Since $N = \mathcal{F}_x$ is A-flat by Lemma 37.18.1 we see that

$$g : N \longrightarrow N$$

is injective. In particular, if $K = \mathrm{Frac}(A)$ is the fraction field of A, then we see that

$$g : N \otimes_A K \longrightarrow N \otimes_A K$$

is injective. Observe that $\mathfrak{q}$ corresponds to a prime ideal of $B \otimes_A K$. Denote $\mathcal{F}_t$ the restriction of $\mathcal{F}$ to the generic fibre X_t. We have $(B \otimes_A K)_\mathfrak{q} = \mathcal{O}_{X_t,y}$ and $(N \otimes_A K)_\mathfrak{q}$ is the stalk at y of $\mathcal{F}_t$. Hence we find that $g \in \mathfrak{m}_y \subset \mathcal{O}_{X_t,y}$ is a nonzerodivisor on the stalk $(\mathcal{F}_t)_y$ which contradicts our assumption that $y \in \mathrm{Ass}_{X/S}(\mathcal{F})$. $\square$

0H3X Lemma 37.18.4. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module flat over S. Assume S is irreducible with generic point η. If $\dim(\mathrm{Supp}(\mathcal{F}_\eta)) \leq r$ then for all $s \in S$ we have $\dim(\mathrm{Supp}(\mathcal{F}_s)) \leq r$.

Proof. Let $x \in \mathrm{Supp}(\mathcal{F}_s)$ be a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}_s)$. By Algebra, Lemma 10.41.12 we can find a specialization $y \rightsquigarrow x$ in $\mathrm{Supp}(\mathcal{F})$ with $f(y) = \eta$. Of course we may assume y is a generic point of an irreducible component of $\mathrm{Supp}(\mathcal{F}_\eta)$. We conclude from Lemma 37.18.3 that the dimension of $\overline{\{x\}}$ is at most r. $\square$

0GSJ Lemma 37.18.5. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $y \in \mathrm{Ass}_{X/S}(\mathcal{F})$. Denote $Y \subset X$ the closure of $\{y\}$ in X viewed as an integral closed subscheme. Denote $T \subset S$ the closure of $\{f(y)\}$ viewed as an integral closed subscheme. We obtain a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

where $Y \to T$ is dominant. Assume $\mathcal{F}$ is flat over S at all generic points of irreducible components of fibres of $Y \to T$ (for example if $\mathcal{F}$ is flat over S). Then

(1) if $s \in S$ and $x \in Y_s$ is the generic point of an irreducible component of Y_s, then $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$, and

(2) there is an integer $d \geq 0$ such that $Y \to T$ is of relative dimension d, see Morphisms, Definition 29.29.1.

Proof. This follows immediately from the pointwise version Lemma 37.18.3. Note that to compute the dimension of the locally algebraic schemes Y_s it suffices to look near the generic points, see Varieties, Section 33.20. $\square$

0GSK Remark 37.18.6. Here are some cases where the material above, especially Lemma 37.18.5, allows one to conclude that a morphism $f : X \to S$ of schemes has relative dimension d as defined in Morphisms, Definition 29.29.1. For example, this is true if

(1) X is integral with generic point ξ,
(2) the transcendence degree of $\kappa(\xi)$ over $\kappa(f(\xi))$ is d,
(3) f is locally of finite type, and
(4) there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type which is flat over S with $\text{Supp}(\mathcal{F}) = X$.

Another set of hypotheses that work are the following:

(1) S is irreducible with generic point η,
(2) X_η is dense in X,
(3) every irreducible component of X_η has dimension d,
(4) f is locally of finite type, and
(5) there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type which is flat over S with $\text{Supp}(\mathcal{F}) = X$.

Of course, we can relax the flatness condition on $\mathcal{F}$ and require only that $\mathcal{F}$ is flat over S in codimension 0, i.e., that $\mathcal{F}$ is flat over S at every generic point of every fibre. If we ever need these results, we will carefully state and prove them here.

37.19. Normalization revisited

081J Normalization commutes with smooth base change.

081K Lemma 37.19.1. Let $f : Y \to X$ be a smooth morphism of schemes. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The integral closure of $\mathcal{O}_Y$ in $f^*\mathcal{A}$ is equal to $f^*\mathcal{A}'$ where $\mathcal{A}' \subset \mathcal{A}$ is the integral closure of $\mathcal{O}_X$ in $\mathcal{A}$.

Proof. This is a translation of Algebra, Lemma 10.147.4 into the language of schemes. Details omitted. $\square$

03GV Lemma 37.19.2 (Normalization commutes with smooth base change). Let

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "f_2"'] & Y_1 \arrow[d, "f_1"] \\ X_2 \arrow[r, "\varphi"] & X_1 \end{tikzcd}$$

be a fibre square in the category of schemes. Assume f_1 is quasi-compact and quasi-separated, and φ is smooth. Let $Y_i \to X_i' \to X_i$ be the normalization of X_i in Y_i. Then $X_2' \cong X_2 \times_{X_1} X_1'$.

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_2$ and $X_2 \times_{X_1} X_1' \to X_2$ is integral (Morphisms, Lemma 29.44.6). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of

Morphisms, Lemma 29.53.4. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_1}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$, see Constructions, Lemma 27.4.6. By Cohomology of Schemes, Lemma 30.5.2 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 37.19.1. $\square$

07TD Lemma 37.19.3 (Normalization and smooth morphisms). Let $X \to Y$ be a smooth morphism of schemes. Assume every quasi-compact open of Y has finitely many irreducible components. Then the same is true for X and there is a unique isomorphism $X^\nu = X \times_Y Y^\nu$ over X where X^ν, Y^ν are the normalizations of X, Y.

Proof. By Descent, Lemma 35.16.3 every quasi-compact open of X has finitely many irreducible components. Note that $X_{red} = X \times_Y Y_{red}$ as a scheme smooth over a reduced scheme is reduced, see Descent, Lemma 35.18.1. Hence we may assume that X and Y are reduced (as the normalization of a scheme is equal to the normalization of its reduction by definition). Next, note that $X' = X \times_Y Y^\nu$ is a normal scheme by Descent, Lemma 35.18.2. The morphism $X' \to Y^\nu$ is smooth (hence flat) thus the generic points of irreducible components of X' lie over generic points of irreducible components of Y^ν. Since $Y^\nu \to Y$ is birational we conclude that $X' \to X$ is birational too (because $X' \to Y^\nu$ induces an isomorphism on fibres over generic points of Y). We conclude that there exists a factorization $X^\nu \to X' \to X$, see Morphisms, Lemma 29.54.5 which is an isomorphism as X' is normal and integral over X. $\square$

0CBM Lemma 37.19.4 (Normalization and henselization). Let X be a locally Noetherian scheme. Let $\nu : X^\nu \to X$ be the normalization morphism. Then for any point $x \in X$ the base change

$$X^\nu \times_X \mathrm{Spec}(\mathcal{O}^h_{X,x}) \to \mathrm{Spec}(\mathcal{O}^h_{X,x}), \quad \text{resp.} \quad X^\nu \times_X \mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \to \mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$$

is the normalization of $\mathrm{Spec}(\mathcal{O}^h_{X,x})$, resp. $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$.

Proof. Let $\eta_1, \ldots, \eta_r$ be the generic points of the irreducible components of X passing through x. The base change of the normalization to $\mathrm{Spec}(\mathcal{O}_{X,x})$ is the spectrum of the integral closure of $\mathcal{O}_{X,x}$ in $\prod \kappa(\eta_i)$. This follows from our construction of the normalization of X in Morphisms, Definition 29.54.1 and Morphisms, Lemma 29.53.1; you can also use the description of the normalization in Morphisms, Lemma 29.54.3. Thus we reduce to the following algebra problem. Let A be a Noetherian local ring; recall that this implies the henselization A^h and strict henselization A^{sh} are Noetherian too (More on Algebra, Lemma 15.45.3). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be its minimal primes. Let A' be the integral closure of A in $\prod \kappa(\mathfrak{p}_i)$. Problem: show that $A' \otimes_A A^h$, resp. $A' \otimes_A A^{sh}$ is constructed from the Noetherian local ring A^h, resp. A^{sh} in the same manner.

Since A^h, resp. A^{sh} are colimits of étale A-algebras, we see that the minimal primes of A and A^{sh} are exactly the primes of A^h, resp. A^{sh} lying over the minimal primes of A (by going down, see Algebra, Lemmas 10.39.19 and 10.30.7). Thus More on Algebra, Lemma 15.45.13 tells us that $A^h \otimes_A \prod \kappa(\mathfrak{p}_i)$, resp. $A^{sh} \otimes_A \prod \kappa(\mathfrak{p}_i)$ is the product of the residue fields at the minimal primes of A^h, resp. A^{sh}. We know that taking the integral closure in an overring commutes with étale base change, see Algebra, Lemma 10.147.2. Writing A^h and A^{sh} as a limit of étale A-algebras we see that the same thing is true for the base change to A^h and A^{sh} (you can also use the more general Algebra, Lemma 10.147.5). □

37.20. Normal morphisms

038Z In the article [DM69] of Deligne and Mumford the notion of a normal morphism is mentioned. This is just one in a series of types[3] of morphisms that can all be defined similarly. Over time we will add these in their own sections as needed.

0390 Definition 37.20.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is normal at x if f is flat at x, and the scheme X_y is geometrically normal at x over $\kappa(y)$ (see Varieties, Definition 33.10.1).
(2) We say f is a normal morphism if f is normal at every point of X.

So the condition that the morphism $X \to Y$ is normal is stronger than just requiring all the fibres to be normal locally Noetherian schemes.

0391 Lemma 37.20.2. Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent

(1) f is normal, and
(2) f is flat and its fibres are geometrically normal schemes.

Proof. This follows directly from the definitions. □

056W Lemma 37.20.3. A smooth morphism is normal.

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 29.34.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 29.34.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 29.34.5) and hence geometrically normal by Varieties, Lemma 33.25.4. Thus f is normal by Lemma 37.20.2. □

We want to show that this notion is local on the source and target for the smooth topology. First we deal with the property of having locally Noetherian fibres.

0392 Lemma 37.20.4. The property $\mathcal{P}(f)$ ="the fibres of f are locally Noetherian" is local in the fppf topology on the source and the target.

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Moreover, as φ_i is of finite presentation the field extension $\kappa(y_i)/\kappa(y)$ is finitely generated. Hence in this situation we have that X_y is locally Noetherian if and only if X_{i,y_i} is locally Noetherian, see Varieties, Lemma 33.11.1. This fact implies locality on the target.

[3]The other types are coprof $\leq k$, Cohen-Macaulay, (S_k), regular, (R_k), and reduced. See [DG67, IV Definition 6.8.1.]. Gorenstein morphisms will be defined in Duality for Schemes, Section 48.24.

Let $\{X_i \to X\}$ be an fppf covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is an fppf covering of the fibre. Hence the locality on the source follows from Descent, Lemma 35.16.1. □

0393 Lemma 37.20.5. The property $\mathcal{P}(f)$ =“the fibres of f are locally Noetherian and f is normal” is local in the fppf topology on the target and local in the smooth topology on the source.

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f)$ =“the fibres of f are locally Noetherian”, $\mathcal{P}_2(f)$ =“f is flat”, and $\mathcal{P}_3(f)$ =“the fibres of f are geometrically normal”. We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 37.20.4, and Descent, Lemmas 35.23.15 and 35.27.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i\in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically normal if and only if X_{i,y_i} is geometrically normal, see Varieties, Lemma 33.10.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 35.18.2. Combining the above the lemma follows. □

37.21. Regular morphisms

07R6 Compare with Section 37.20. The algebraic version of this notion is discussed in More on Algebra, Section 15.41.

07R7 Definition 37.21.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is regular at x if f is flat at x, and the scheme X_y is geometrically regular at x over $\kappa(y)$ (see Varieties, Definition 33.12.1).
(2) We say f is a regular morphism if f is regular at every point of X.

The condition that the morphism $X \to Y$ is regular is stronger than just requiring all the fibres to be regular locally Noetherian schemes.

07R8 Lemma 37.21.2. Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent

(1) f is regular,
(2) f is flat and its fibres are geometrically regular schemes,
(3) for every pair of affine opens $U \subset X$, $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$ is regular,
(4) there exists an open covering $Y = \bigcup_{j\in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that each of the morphisms $U_i \to V_j$ is regular, and
(5) there exists an affine open covering $Y = \bigcup_{j\in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i\in I_j} U_i$ such that the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(U_i)$ are regular.

Proof. The equivalence of (1) and (2) is immediate from the definitions. Let $x \in X$ with $y = f(x)$. By definition f is flat at x if and only if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a flat ring map, and X_y is geometrically regular at x over $\kappa(y)$ if and only if $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a geometrically regular algebra over $\kappa(y)$. Hence Whether or not f is regular at x depends only on the local homomorphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Thus the equivalence of (1) and (4) is clear.

Recall (More on Algebra, Definition 15.41.1) that a ring map $A \to B$ is regular if and only if it is flat and the fibre rings $B \otimes_A \kappa(\mathfrak{p})$ are Noetherian and geometrically regular for all primes $\mathfrak{p} \subset A$. By Varieties, Lemma 33.12.3 this is equivalent to $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ being a geometrically regular scheme over $\kappa(\mathfrak{p})$. Thus we see that (2) implies (3). It is clear that (3) implies (5). Finally, assume (5). This implies that f is flat (see Morphisms, Lemma 29.25.3). Moreover, if $y \in Y$, then $y \in V_j$ for some j and we see that $X_y = \bigcup_{i \in I_j} U_{i,y}$ with each $U_{i,y}$ geometrically regular over $\kappa(y)$ by Varieties, Lemma 33.12.3. Another application of Varieties, Lemma 33.12.3 shows that X_y is geometrically regular. Hence (2) holds and the proof of the lemma is finished. $\square$

07R9 Lemma 37.21.3. A smooth morphism is regular.

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 29.34.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 29.34.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 29.34.5) and hence geometrically regular by Varieties, Lemma 33.25.4. Thus f is regular by Lemma 37.21.2. $\square$

07RA Lemma 37.21.4. The property $\mathcal{P}(f)$ ="the fibres of f are locally Noetherian and f is regular" is local in the fppf topology on the target and local in the smooth topology on the source.

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f)$ ="the fibres of f are locally Noetherian", $\mathcal{P}_2(f)$ ="f is flat", and $\mathcal{P}_3(f)$ ="the fibres of f are geometrically regular". We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 37.20.4, and Descent, Lemmas 35.23.15 and 35.27.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically regular if and only if X_{i,y_i} is geometrically regular, see Varieties, Lemma 33.12.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 35.18.4. Combining the above the lemma follows. $\square$

37.22. Cohen-Macaulay morphisms

045Q Compare with Section 37.20. Note that, as pointed out in Algebra, Section 10.167 and Varieties, Section 33.13 "geometrically Cohen-Macaulay" is the same as plain Cohen-Macaulay.

045R **Definition 37.22.1.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

1. Let $x \in X$, and $y = f(x)$. We say that f is Cohen-Macaulay at x if f is flat at x, and the local ring of the scheme X_y at x is Cohen-Macaulay.
2. We say f is a Cohen-Macaulay morphism if f is Cohen-Macaulay at every point of X.

Here is a translation.

045S **Lemma 37.22.2.** Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent

1. f is Cohen-Macaulay, and
2. f is flat and its fibres are Cohen-Macaulay schemes.

Proof. This follows directly from the definitions. $\square$

0AFG **Lemma 37.22.3.** Let $f : X \to Y$ be a morphism of locally Noetherian schemes which is locally of finite type and Cohen-Macaulay. For every point x in X with image y in Y,

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where X_y denotes the fiber over y.

Proof. After replacing X by an open neighborhood of x, there is a natural number d such that all fibers of $X \to Y$ have dimension d at every point, see Morphisms, Lemma 29.29.4. Then f is flat, locally of finite type and of relative dimension d. Hence the result follows from Morphisms, Lemma 29.29.6. $\square$

0C0W **Lemma 37.22.4.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume that the fibres of f, g, and $g \circ f$ are locally Noetherian. Let $x \in X$ with images $y \in Y$ and $z \in Z$.

1. If f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$, then $g \circ f$ is Cohen-Macaulay at x.
2. If f and g are Cohen-Macaulay, then $g \circ f$ is Cohen-Macaulay.
3. If $g \circ f$ is Cohen-Macaulay at x and f is flat at x, then f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$.
4. If $g \circ f$ is Cohen-Macaulay and f is flat, then f is Cohen-Macaulay and g is Cohen-Macaulay at every point in the image of f.

Proof. Consider the map of Noetherian local rings

$$\mathcal{O}_{Y_z,y} \to \mathcal{O}_{X_z,x}$$

and observe that its fibre is

$$\mathcal{O}_{X_z,x}/\mathfrak{m}_{Y_z,y}\mathcal{O}_{X_z,x} = \mathcal{O}_{X_y,x}$$

Thus the lemma this follows from Algebra, Lemma 10.163.3. $\square$

0C0X **Lemma 37.22.5.** Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes. If X is Cohen-Macaulay, then f is Cohen-Macaulay and $\mathcal{O}_{Y,f(x)}$ is Cohen-Macaulay for all $x \in X$.

Proof. After translating into algebra this follows from Algebra, Lemma 10.163.3. $\square$

045T **Lemma 37.22.6.** Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' \to Y'$ be the base change of f. Let $x' \in X'$ be a point with image $x \in X$.

(1) If f is Cohen-Macaulay at x, then $f' : X' \to Y'$ is Cohen-Macaulay at x'.
(2) If f is flat at x and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.
(3) If $Y' \to Y$ is flat at $f'(x')$ and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.

Proof. Note that the assumption on $Y' \to Y$ implies that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y')/\kappa(y)$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 33.11.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{Y',y'} \arrow[u] & \mathcal{O}_{Y,y} \arrow[l] \arrow[u] \end{tikzcd}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Cohen-Macaulay at x. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma 10.100.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay, see Varieties, Lemma 33.13.1. Hence we see that f' is Cohen-Macaulay at x'.

Assume f is flat at x and f' is Cohen-Macaulay at x'. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay, see Varieties, Lemma 33.13.1. Hence we see that f is Cohen-Macaulay at x.

Assume $Y' \to Y$ is flat at y' and f' is Cohen-Macaulay at x'. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 10.100.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay, see Varieties, Lemma 33.13.1. Hence we see that f is Cohen-Macaulay at x. $\square$

045U **Lemma 37.22.7.** Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let [DG67, IV Corollary 12.1.7(iii)]

$$W = \{x \in X \mid f \text{ is Cohen-Macaulay at } x\}$$

Then

(1) $W = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is Cohen-Macaulay}\}$,
(2) W is open in X,
(3) W dense in every fibre of $X \to S$,

(4) the formation of W commutes with arbitrary base change of f: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.

Proof. As f is flat with locally Noetherian fibres the equality in (1) holds by definition. Parts (2) and (3) follow from Algebra, Lemma 10.130.5. Part (4) follows either from Algebra, Lemma 10.130.7 or Varieties, Lemma 33.13.1. □

0BUU Lemma 37.22.8. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $x \in X$ with image $s \in S$. Set $d = \dim_x(X_s)$. The following are equivalent

(1) f is Cohen-Macaulay at x,
(2) there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite morphism $U \to \mathbf{A}^d_S$ over S which is flat at x,
(3) there exists an open neighbourhood $U \subset X$ of x and a locally quasi-finite flat morphism $U \to \mathbf{A}^d_S$ over S,
(4) for any S-morphism $g : U \to \mathbf{A}^d_S$ of an open neighbourhood $U \subset X$ of x we have: g is quasi-finite at $x \Rightarrow g$ is flat at x.

Proof. Openness of flatness shows (2) and (3) are equivalent, see Theorem 37.15.1.

Choose affine open $U = \text{Spec}(A) \subset X$ with $x \in U$ and $V = \text{Spec}(R) \subset S$ with $f(U) \subset V$. Then $R \to A$ is a flat ring map of finite presentation. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. After replacing A by a principal localization we may assume there exists a quasi-finite map $R[x_1, \ldots, x_d] \to A$, see Algebra, Lemma 10.125.2. Thus there exists at least one pair (U, g) consisting of an open neighbourhood $U \subset X$ of x and a locally[4] quasi-finite morphism $g : U \to \mathbf{A}^d_S$.

Claim: Given $R \to A$ flat and of finite presentation, a prime $\mathfrak{p} \subset A$ and $\varphi : R[x_1, \ldots, x_d] \to A$ quasi-finite at $\mathfrak{p}$ we have: $\text{Spec}(\varphi)$ is flat at $\mathfrak{p}$ if and only if $\text{Spec}(A) \to \text{Spec}(R)$ is Cohen-Macaulay at $\mathfrak{p}$. Namely, by Theorem 37.16.2 flatness may be checked on fibres. The same is true for being Cohen-Macaulay (as A is already assumed flat over R). Thus the claim follows from Algebra, Lemma 10.130.1.

The claim shows that (1) is equivalent to (4) and combined with the fact that we have constructed a suitable (U, g) in the second paragraph, the claim also shows that (1) is equivalent to (2). □

054T Lemma 37.22.9. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. For $d \geq 0$ there exist opens $U_d \subset X$ with the following properties

(1) $W = \bigcup_{d \geq 0} U_d$ is dense in every fibre of f, and
(2) $U_d \to S$ is of relative dimension d (see Morphisms, Definition 29.29.1).

Proof. This follows by combining Lemma 37.22.7 with Morphisms, Lemma 29.29.4. □

054U Lemma 37.22.10. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Suppose $x' \rightsquigarrow x$ is a specialization of points of X

[4] If S is quasi-separated, then g will be quasi-finite.

with image $s' \rightsquigarrow s$ in S. If x is a generic point of an irreducible component of X_s then $\dim_{x'}(X_{s'}) = \dim_x(X_s)$.

Proof. The point x is contained in U_d for some d, where U_d as in Lemma 37.22.9. $\square$

045V Lemma 37.22.11. The property $\mathcal{P}(f) =$"the fibres of f are locally Noetherian and f is Cohen-Macaulay" is local in the fppf topology on the target and local in the syntomic topology on the source.

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f) =$"f is flat", and $\mathcal{P}_2(f) =$"the fibres of f are locally Noetherian and Cohen-Macaulay". We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 35.23.15 and 35.27.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

and that $\kappa(y_i)/\kappa(y)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i,y_i} is locally Noetherian, see Varieties, Lemma 33.11.1. And if in addition X_y is Cohen-Macaulay, then X_{i,y_i} is Cohen-Macaulay, see Varieties, Lemma 33.13.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Descent, Lemma 35.17.2. Combining the above the lemma follows. $\square$

37.23. Slicing Cohen-Macaulay morphisms

056X The results in this section eventually lead to the assertion that the fppf topology is the same as the "finitely presented, flat, quasi-finite" topology. The following lemma is very closely related to Divisors, Lemma 31.18.9.

056Y Lemma 37.23.1. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume

(1) f is locally of finite presentation,
(2) f is flat at x, and
(3) the image $\overline{h}$ of h in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a nonzerodivisor.

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and locally of finite presentation.

Proof. We are going to prove this by reducing to the Noetherian case. By openness of flatness (see Theorem 37.15.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that there exists an $h \in \Gamma(X, \mathcal{O}_X)$ which maps to our given h.

We may write $S = \mathrm{Spec}(A)$ and we may write $A = \mathrm{colim}_i A_i$ as a directed colimit of finite type $\mathbf{Z}$ algebras. Then by Algebra, Lemma 10.168.1 or Limits, Lemmas

32.10.1, 32.8.2, and 32.10.1 we can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_0 \arrow[d, "f_0"] \\ S \arrow[r] & S_0 \end{tikzcd}$$

with f_0 flat and of finite presentation, X_0 affine, and S_0 affine and Noetherian. Let $x_0 \in X_0$, resp. $s_0 \in S_0$ be the image of x, resp. s. We may also assume there exists an element $h_0 \in \Gamma(X_0, \mathcal{O}_{X_0})$ which restricts to h on X. (If you used the algebra reference above then this is clear; if you used the references to the chapter on limits then this follows from Limits, Lemma 32.10.1 by thinking of h as a morphism $X \to \mathbf{A}^1_S$.) Note that $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{(X_0)_{s_0},x_0} \otimes_{\kappa(s_0)} \kappa(s)$, so that $\mathcal{O}_{(X_0)_{s_0},x_0} \to \mathcal{O}_{X_s,x}$ is a flat local ring map, in particular faithfully flat. Hence the image $\overline{h}_0 \in \mathcal{O}_{(X_0)_{s_0},x_0}$ is contained in $\mathfrak{m}_{(X_0)_{s_0},x_0}$ and is a nonzerodivisor. We claim that after replacing X_0 by a principal open neighbourhood of x_0 the element h_0 is a nonzerodivisor in $B_0 = \Gamma(X_0, \mathcal{O}_{X_0})$ such that B_0/h_0B_0 is flat over $A_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. If so then

$$0 \to B_0 \xrightarrow{h_0} B_0 \to B_0/h_0B_0 \to 0$$

is a short exact sequence of flat A_0-modules. Hence this remains exact on tensoring with A (by Algebra, Lemma 10.39.12) and the lemma follows.

It remains to prove the claim above. The corresponding algebra statement is the following (we drop the subscript $_0$ here): Let $A \to B$ be a flat, finite type ring map of Noetherian rings. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. Assume $h \in \mathfrak{q}$ maps to a nonzerodivisor in $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$. Goal: show that after possible replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$ the element h becomes a nonzerodivisor and B/hB becomes flat over A. By Algebra, Lemma 10.99.2 we see that h is a nonzerodivisor in $B_\mathfrak{q}$ and that $B_\mathfrak{q}/hB_\mathfrak{q}$ is flat over A. By openness of flatness, see Algebra, Theorem 10.129.4 or Theorem 37.15.1 we see that B/hB is flat over A after replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$. Finally, let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $IB_\mathfrak{q} = 0$ as h is a nonzerodivisor in $B_\mathfrak{q}$. Also I is finitely generated as B is Noetherian. Hence there exists a $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = 0$. After replacing B by B_g we see that h is a nonzerodivisor. $\square$

06LI Lemma 37.23.2. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h_1, \ldots, h_r \in \mathcal{O}_{X,x}$. Assume

(1) f is locally of finite presentation,
(2) f is flat at x, and
(3) the images of $h_1, \ldots, h_r$ in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ form a regular sequence.

Then there exists an affine open neighbourhood $U \subset X$ of x such that $h_1, \ldots, h_r$ come from $h_1, \ldots, h_r \in \Gamma(U, \mathcal{O}_U)$ and such that $Z = V(h_1, \ldots, h_r) \to U$ is a regular immersion with $x \in Z$ and $Z \to S$ flat and locally of finite presentation. Moreover, the base change $Z_{S'} \to U_{S'}$ is a regular immersion for any scheme S' over S.

Proof. (Our conventions on regular sequences imply that $h_i \in \mathfrak{m}_x$ for each i.) The case $r = 1$ follows from Lemma 37.23.1 combined with Divisors, Lemma 31.18.1 to see that $V(h_1)$ remains an effective Cartier divisor after base change. The case

$r > 1$ follows from a straightforward induction on r (applying the result for $r = 1$ exactly r times; details omitted).

Another way to prove the lemma is using the material from Divisors, Section 31.22. Namely, first by openness of flatness (see Theorem 37.15.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that we have $h_1, \ldots, h_r \in \Gamma(X, \mathcal{O}_X)$. Set $Z = V(h_1, \ldots, h_r)$. Note that X_s is a Noetherian scheme (because it is an algebraic $\kappa(s)$-scheme, see Varieties, Section 33.20) and that the topology on X_s is induced from the topology on X (see Schemes, Lemma 26.18.5). Hence after shrinking X a bit more we may assume that $Z_s \subset X_s$ is a regular immersion cut out by the r elements $h_i|_{X_s}$, see Divisors, Lemma 31.20.8 and its proof. It is also clear that $r = \dim_x(X_s) - \dim_x(Z_s)$ because

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(Z_s) &= \dim(\mathcal{O}_{Z_s,x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{Z_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 10.116.3 and the second by r times applying Algebra, Lemma 10.60.13. Hence Divisors, Lemma 31.22.7 part (3) applies to show that (after Zariski shrinking X) the morphism $Z \to X$ is a regular immersion to which Divisors, Lemma 31.22.4 applies (which gives the flatness and the statement on base change). $\square$

056Z Lemma 37.23.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

(1) f is locally of finite presentation,
(2) f is flat at x, and
(3) $\mathcal{O}_{X_s,x}$ has depth ≥ 1.

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is flat and of finite presentation.

Proof. Pick any $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ which maps to a nonzerodivisor in $\mathcal{O}_{X_s,x}$ and apply Lemma 37.23.1. $\square$

0570 Lemma 37.23.4. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume [DG67, IV Proposition 17.16.1]

(1) f is locally of finite presentation,
(2) f is Cohen-Macaulay at x, and
(3) x is a closed point of X_s.

Then there exists a regular immersion $Z \to X$ containing x such that

(a) $Z \to S$ is flat and locally of finite presentation,
(b) $Z \to S$ is locally quasi-finite, and
(c) $Z_s = \{x\}$ set theoretically.

Proof. We may and do replace S by an affine open neighbourhood of s. We will prove the lemma for affine S by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. (Note that an open immersion is a regular immersion.) Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 29.29.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 29.56.2. Thus after replacing X by U we see that the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neighbourhood of X we see that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 26.18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(x)/\kappa(s)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 29.20.3). Thus we see

$$\text{depth}(\mathcal{O}_{X_s,x}) = \dim(\mathcal{O}_{X_s,x}) = d > 0$$

the first equality as $\mathcal{O}_{X_s,x}$ is Cohen-Macaulay and the second by Morphisms, Lemma 29.28.1. Thus we may apply Lemma 37.23.3 to find a diagram

$$\begin{tikzcd} D \arrow[r] \arrow[drr] & U \arrow[r] \arrow[dr] & X \arrow[d] \\ & & S \end{tikzcd}$$

with $x \in D$. Note that $\mathcal{O}_{D_s,x} = \mathcal{O}_{X_s,x}/(\overline{h})$ for some nonzerodivisor $\overline{h}$, see Divisors, Lemma 31.18.1. Hence $\mathcal{O}_{D_s,x}$ is Cohen-Macaulay of dimension one less than the dimension of $\mathcal{O}_{X_s,x}$, see Algebra, Lemma 10.104.2 for example. Thus the morphism $D \to S$ is flat, locally of finite presentation, and Cohen-Macaulay at x with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find a regular immersion $Z \to D$ having properties (a), (b), (c). As $Z \to D \to U$ are both regular immersions, we see that also $Z \to U$ is a regular immersion by Divisors, Lemma 31.21.7. This finishes the proof. □

0571 Lemma 37.23.5. Let $f : X \to S$ be a flat morphism of schemes which is locally of finite presentation. Let $s \in S$ be a point in the image of f. Then there exists a commutative diagram

$$\begin{tikzcd} S' \arrow[rr] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

where $g : S' \to S$ is flat, locally of finite presentation, locally quasi-finite, and $s \in g(S')$.

Proof. The fibre X_s is not empty by assumption. Hence there exists a closed point $x \in X_s$ where f is Cohen-Macaulay, see Lemma 37.22.7. Apply Lemma 37.23.4 and set $S' = S$. □

The following lemma shows that sheaves for the fppf topology are the same thing as sheaves for the "quasi-finite, flat, finite presentation" topology.

0572 Lemma 37.23.6. Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be an fppf covering of S, see Topologies, Definition 34.7.1. Then there exists an fppf covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ which refines (see Sites, Definition 7.8.1) $\mathcal{U}$ such that each $T_j \to S$ is locally quasi-finite.

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 37.23.5 we can find a morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ which is flat, locally of finite presentation and locally quasi-finite and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is the desired covering of S that refines $\mathcal{U}$. □

37.24. Generic fibres

054V Some results on the relationship between generic fibres and nearby fibres.

054W **Lemma 37.24.1.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta = \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X = \emptyset$.*

Proof. Follows immediately from the more general Morphisms, Lemma 29.8.5. □

05F5 **Lemma 37.24.2.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta \neq \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X \to V$ is surjective.*

Proof. This follows, upon taking affine opens, from Algebra, Lemma 10.30.2. (Of course it also follows from generic flatness.) □

054X **Lemma 37.24.3.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $Z \subset X$ is a closed subset with Z_η nowhere dense in X_η, then there exists a nonempty open $V \subset Y$ such that Z_y is nowhere dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Set $X' = Y' \times_Y X$ and $Z' = Y' \times_Y Z$. As $Y' \to Y$ is a universal homeomorphism by Morphisms, Lemma 29.45.6 we see that it suffices to prove the lemma for $Z' \subset X' \to Y'$. Thus we may assume that Y is integral, see Properties, Lemma 28.3.4. By Morphisms, Proposition 29.27.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ and $Z_V \to V$ are flat and of finite presentation. We claim that V works. Pick $y \in V$. If Z_y has a nonempty interior, then Z_y contains a generic point ξ of an irreducible component of X_y. Note that $\eta \rightsquigarrow f(\xi)$. Since $Z_V \to V$ is flat we can choose a specialization $\xi' \rightsquigarrow \xi$, $\xi' \in Z$ with $f(\xi') = \eta$, see Morphisms, Lemma 29.25.9. By Lemma 37.22.10 we see that

$$\dim_{\xi'}(Z_\eta) = \dim_\xi(Z_y) = \dim_\xi(X_y) = \dim_{\xi'}(X_\eta).$$

Hence some irreducible component of Z_η passing through ξ' has dimension $\dim_{\xi'}(X_\eta)$ which contradicts the assumption that Z_η is nowhere dense in X_η and we win. □

0573 **Lemma 37.24.4.** *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $U \subset X$ be an open subscheme such that U_η is scheme theoretically dense in X_η. Then there exists a nonempty open $V \subset Y$ such that U_y is scheme theoretically dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' = Y' \times_Y X$ and $U' = Y' \times_Y U$. As $Y' \to Y$ induces a bijection on points, and as $U' \to U$ and $X' \to X$ induce isomorphisms of scheme theoretic fibres, we may replace Y by Y' and X by X'. Thus we may assume that Y is integral, see Properties, Lemma 28.3.4. We may also replace Y by a nonempty affine open. In other words we may assume that $Y = \mathrm{Spec}(A)$ where A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i. By Morphisms, Definition 29.7.1 we see that $U_i = X_i \cap U$ is an open subscheme of X_i such that $U_{i,\eta}$ is scheme theoretically dense in $X_{i,\eta}$. Thus it suffices to prove the result for the pairs (X_i, U_i), in other words we may assume that X is affine.

Write $X = \mathrm{Spec}(B)$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. The fact that U_η is scheme theoretically dense in X_η means that $B_K \to \bigoplus_j (B_K)_{g_j}$ is injective, see Morphisms, Example 29.7.4. By Algebra, Lemma 10.24.4 this is equivalent to the injectivity of $B_K \to \bigoplus_{j=1,\ldots,m} B_K$, $b \mapsto (g_1 b, \ldots, g_m b)$. Let M be the cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra, and that M is flat over A, see Algebra, Lemma 10.118.3. The flatness of B over A implies that B is torsion free as an A-module, see More on Algebra, Lemma 15.22.9. Hence $B \subset B_K$. By assumption $I_K = 0$ which implies that $I = 0$ (as $I \subset B \subset B_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to B \otimes_A \kappa \xrightarrow{(g_1 \otimes 1,\ldots,g_m \otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} B \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 10.39.12. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ is scheme theoretically dense in $\mathrm{Spec}(B \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ is scheme theoretically dense in X_κ which is what we wanted to prove. $\square$

Suppose given a morphism of schemes $f : X \to Y$ and a point $y \in Y$. Recall that the fibre X_y is homeomorphic to the subset $f^{-1}(\{y\})$ of X with induced topology, see Schemes, Lemma 26.18.5. Suppose given a closed subset $T(y) \subset X_y$. Let T be the closure of $T(y)$ in X. Endow T with the induced reduced scheme structure. Then T is a closed subscheme of X with the property that $T_y = T(y)$ set-theoretically. In fact T is the smallest closed subscheme of X with this property. Thus it is "harmless" to denote a closed subset of X_y by T_y if we so desire. In the following lemma we apply this to the generic fibre of f.

054Y Lemma 37.24.5. Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $X_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$ be a covering of the generic fibre by closed subsets of X_η. Let Z_i be the closure of $Z_{i,\eta}$ in X (see discussion above). Then there exists a nonempty open $V \subset Y$ such that $X_y = Z_{1,y} \cup \ldots \cup Z_{n,y}$ for all $y \in V$.

Proof. If Y is Noetherian then $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ is of finite type over Y and we can directly apply Lemma 37.24.1 to get that $U_V = \emptyset$ for a nonempty open $V \subset Y$. In general we argue as follows. As the question is topological we may replace Y by its reduction. Thus Y is integral, see Properties, Lemma 28.3.4. After shrinking Y we may assume that $X \to Y$ is flat, see Morphisms, Proposition 29.27.1. In this case every point x in X_y is a specialization of a point $x' \in X_\eta$ by Morphisms, Lemma 29.25.9. As the Z_i are closed in X and cover the generic fibre this implies that $X_y = \bigcup Z_{i,y}$ for $y \in Y$ as desired. $\square$

The following lemma says that generic fibres of morphisms whose source is reduced are reduced.

054Z Lemma 37.24.6. Let $f : X \to Y$ be a morphism of schemes. Let $\eta \in Y$ be a generic point of an irreducible component of Y. Then $(X_\eta)_{red} = (X_{red})_\eta$.

Proof. Choose an affine neighbourhood $\text{Spec}(A) \subset Y$ of η. Choose an affine open $\text{Spec}(B) \subset X$ mapping into $\text{Spec}(A)$ via the morphism f. Let $\mathfrak p \subset A$ be the minimal prime corresponding to η. Let B_{red} be the quotient of B by the nilradical $\sqrt{(0)}$. The algebraic content of the lemma is that $C = B_{red} \otimes_A \kappa(\mathfrak p)$ is reduced. Denote $I \subset A$ the nilradical so that $A_{red} = A/I$. Denote $\mathfrak p_{red} = \mathfrak p/I$ which is a minimal prime of A_{red} with $\kappa(\mathfrak p) = \kappa(\mathfrak p_{red})$. Since $A \to B_{red}$ and $A \to \kappa(\mathfrak p)$ both factor through $A \to A_{red}$ we have $C = B_{red} \otimes_{A_{red}} \kappa(\mathfrak p_{red})$. Now $\kappa(\mathfrak p_{red}) = (A_{red})_{\mathfrak p_{red}}$ is a localization by Algebra, Lemma 10.25.1. Hence C is a localization of B_{red} (Algebra, Lemma 10.12.15) and hence reduced. $\square$

0550 Lemma 37.24.7. Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

where

(1) V is a nonempty open of Y,
(2) $X_V = V \times_Y X$,
(3) $g : Y' \to V$ is a finite universal homeomorphism,
(4) $X' = (Y' \times_Y X)_{red} = (Y' \times_V X_V)_{red}$,
(5) g' is a finite universal homeomorphism,
(6) Y' is an integral affine scheme,
(7) f' is flat and of finite presentation, and
(8) the generic fibre of f' is geometrically reduced.

Proof. Let $V = \text{Spec}(A)$ be a nonempty affine open of Y. By assumption the Jacobson radical of A is a prime ideal $\mathfrak p$. Let $K = \kappa(\mathfrak p)$. Let p be the characteristic of K if positive and 1 if the characteristic is zero. By Varieties, Lemma 33.6.11 there exists a finite purely inseparable field extension K'/K such that $X_{K'}$ is geometrically reduced over K'. Choose elements $x_1, \ldots, x_n \in K'$ which generate K' over K and such that some p-power of x_i is in $A/\mathfrak p$. Let $A' \subset K'$ be the finite A-subalgebra of K' generated by $x_1, \ldots, x_n$. Note that A' is a domain with fraction field K'. By Algebra, Lemma 10.46.7 we see that $A \to A'$ induces a universal homeomorphism on spectra. Set $Y' = \text{Spec}(A')$. Set $X' = (Y' \times_Y X)_{red}$. The generic fibre of $X' \to Y'$ is $(X_K)_{red}$ by Lemma 37.24.6 which is geometrically reduced by construction. Note that $X' \to X_V$ is a finite universal homeomorphism as the composition of the reduction morphism $X' \to Y' \times_Y X$ (see Morphisms, Lemma 29.45.6) and the base change of g. At this point all of the properties of the lemma hold except for possibly (7). This can be achieved by shrinking Y' and hence V, see Morphisms, Proposition 29.27.1. $\square$

0551 **Lemma 37.24.8.** Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

where

(1) V is a nonempty open of Y,
(2) $X_V = V \times_Y X$,
(3) $g : Y' \to V$ is surjective finite étale,
(4) $X' = Y' \times_Y X = Y' \times_V X_V$,
(5) g' is surjective finite étale,
(6) Y' is an irreducible affine scheme, and
(7) all irreducible components of the generic fibre of f' are geometrically irreducible.

Proof. Let $V = \mathrm{Spec}(A)$ be a nonempty affine open of Y. By assumption the Jacobson radical of A is a prime ideal $\mathfrak{p}$. Let $K = \kappa(\mathfrak{p})$. By Varieties, Lemma 33.8.15 there exists a finite separable field extension K'/K such that all irreducible components of $X_{K'}$ are geometrically irreducible over K'. Choose an element $\alpha \in K'$ which generates K' over K, see Fields, Lemma 9.19.1. Let $P(T) \in K[T]$ be the minimal polynomial for α over K. After replacing α by $f\alpha$ for some $f \in A$, $f \notin \mathfrak{p}$ we may assume that there exists a monic polynomial $T^d + a_1T^{d-1} + \ldots + a_d \in A[T]$ which maps to $P(T) \in K[T]$ under the map $A[T] \to K[T]$. Set $A' = A[T]/(P)$. Then $A \to A'$ is a finite free ring map such that there exists a unique prime $\mathfrak{q}$ lying over $\mathfrak{p}$, such that $K = \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) = K'$ is finite separable, and such that $\mathfrak{p}A'_{\mathfrak{q}}$ is the maximal ideal of $A'_{\mathfrak{q}}$. Hence $g : Y' = \mathrm{Spec}(A') \to V = \mathrm{Spec}(A)$ is étale at $\mathfrak{q}$, see Algebra, Lemma 10.143.7. This means that there exists an open $W \subset \mathrm{Spec}(A')$ such that $g|_W : W \to \mathrm{Spec}(A)$ is étale. Since g is finite and since $\mathfrak{q}$ is the only point lying over $\mathfrak{p}$ we see that $Z = g(Y' \setminus W)$ is a closed subset of V not containing $\mathfrak{p}$. Hence after replacing V by a principal affine open of V which does not meet Z we obtain that g is finite étale. $\square$

37.25. Relative assassins

05KM

05F1 **Lemma 37.25.1.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\xi \in \mathrm{Ass}_{X/S}(\mathcal{F})$ and set $Z = \overline{\{\xi\}} \subset X$. If f is locally of finite type and $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, then there exists a nonempty open $V \subset Z$ such that for every $s \in f(V)$ the generic points of V_s are elements of $\mathrm{Ass}_{X/S}(\mathcal{F})$.

Proof. We may replace S by an affine open neighbourhood of $f(\xi)$ and X by an affine open neighbourhood of ξ. Hence we may assume $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and that f is given by the finite type ring map $A \to B$, see Morphisms, Lemma 29.15.2. Moreover, we may write $\mathcal{F} = \widetilde{M}$ for some finite B-module M, see Properties, Lemma 28.16.1. Let $\mathfrak{q} \subset B$ be the prime corresponding to ξ and let $\mathfrak{p} \subset A$ be the corresponding prime of A. By assumption $\mathfrak{q} \in \mathrm{Ass}_B(M \otimes_A \kappa(\mathfrak{p}))$, see Algebra, Remark 10.65.6 and Divisors, Lemma 31.2.2. With this notation

$Z = V(\mathfrak{q}) \subset \mathrm{Spec}(B)$. In particular $f(Z) \subset V(\mathfrak{p})$. Hence clearly it suffices to prove the lemma after replacing A, B, and M by $A/\mathfrak{p}A$, $B/\mathfrak{p}B$, and $M/\mathfrak{p}M$. In other words we may assume that A is a domain with fraction field K and $\mathfrak{q} \subset B$ is an associated prime of $M \otimes_A K$.

At this point we can use generic flatness. Namely, by Algebra, Lemma 10.118.3 there exists a nonzero $g \in A$ such that M_g is flat as an A_g-module. After replacing A by A_g we may assume that M is flat as an A-module.

In this case, by Algebra, Lemma 10.65.4 we see that $\mathfrak{q}$ is also an associated prime of M. Hence we obtain an injective B-module map $B/\mathfrak{q} \to M$. Let Q be the cokernel so that we obtain a short exact sequence

$$0 \to B/\mathfrak{q} \to M \to Q \to 0$$

of finite B-modules. After applying generic flatness Algebra, Lemma 10.118.3 once more, this time to the B-module Q, we may assume that Q is a flat A-module. In particular we may assume the short exact sequence above is universally injective, see Algebra, Lemma 10.39.12. In this situation $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}') \subset M \otimes_A \kappa(\mathfrak{p}')$ for any prime $\mathfrak{p}'$ of A. The lemma follows as a minimal prime $\mathfrak{q}'$ of the support of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ is an associated prime of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ by Divisors, Lemma 31.2.9. $\square$

05KN Lemma 37.25.2. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume

(1) f is of finite type,
(2) $\mathcal{F}$ is of finite type,
(3) Y is irreducible with generic point η, and
(4) $\mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta)$ is not contained in U_η.

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ the set $\mathrm{Ass}_{X_y}(\mathcal{F}_y)$ is not contained in U_y.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 29.5.5. Then Z_η is the scheme theoretic support of $\mathcal{F}_\eta$ (Morphisms, Lemma 29.25.14). Hence the generic points of irreducible components of Z_η are contained in $\mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta)$ by Divisors, Lemma 31.2.9. Hence we see that $Z_\eta \cap U_\eta = \emptyset$. Thus $T = Z \setminus U$ is a closed subset of Z with $T_\eta = \emptyset$. If we endow T with the induced reduced scheme structure then $T \to Y$ is a morphism of finite type. By Lemma 37.24.1 there is a nonempty open $V \subset Y$ with $T_V = \emptyset$. Then V works. $\square$

05KP Lemma 37.25.3. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume

(1) f is of finite type,
(2) $\mathcal{F}$ is of finite type,
(3) Y is irreducible with generic point η, and
(4) $\mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$.

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ we have $\mathrm{Ass}_{X_y}(\mathcal{F}_y) \subset U_y$.

Proof. (This proof is the same as the proof of Lemma 37.24.4. We urge the reader to read that proof first.) Since the statement is about fibres it is clear that we may replace Y by its reduction. Hence we may assume that Y is integral, see

Properties, Lemma 28.3.4. We may also assume that $Y = \mathrm{Spec}(A)$ is affine. Then A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i and set $\mathcal{F}_i = \mathcal{F}|_{X_i}$. By assumption the generic fibre of $U_i = X_i \cap U$ contains $\mathrm{Ass}_{X_{i,\eta}}(\mathcal{F}_{i,\eta})$. Thus it suffices to prove the result for the triples $(X_i, \mathcal{F}_i, U_i)$, in other words we may assume that X is affine.

Write $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F} = \widetilde{N}$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. Since $\mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$ we see that $N_K \to \bigoplus_j (N_K)_{g_j}$ is injective. By Algebra, Lemma 10.24.4 this is equivalent to the injectivity of $N_K \to \bigoplus_{j=1,\ldots,m} N_K$, $n \mapsto (g_1 n, \ldots, g_m n)$. Let I and M be the kernel and cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra and that both M and N are flat over A, see Algebra, Lemma 10.118.3. The flatness of N over A implies that N is torsion free as an A-module, see More on Algebra, Lemma 15.22.9. Hence $N \subset N_K$. By construction $I_K = 0$ which implies that $I = 0$ (as $I \subset N \subset N_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to N \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} N \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 10.39.12. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ contains $\mathrm{Ass}_{B\otimes_A\kappa}(N \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ contains $\mathrm{Ass}_{X\otimes\kappa}(\mathcal{F} \otimes \kappa)$ which is what we wanted to prove. □

05KQ Lemma 37.25.4. Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $U \subset X$ be an open subscheme. Let $g : S' \to S$ be a morphism of schemes, let $f' : X' = X_{S'} \to S'$ be the base change of f, let $g' : X' \to X$ be the projection, set $\mathcal{F}' = (g')^*\mathcal{F}$, and set $U' = (g')^{-1}(U)$. Finally, let $s' \in S'$ with image $s = g(s')$. In this case

$$\mathrm{Ass}_{X_s}(\mathcal{F}_s) \subset U_s \Leftrightarrow \mathrm{Ass}_{X'_{s'}}(\mathcal{F}'_{s'}) \subset U'_{s'}.$$

Proof. This follows immediately from Divisors, Lemma 31.7.3. See also Divisors, Remark 31.7.4. □

05KR Lemma 37.25.5. Let $f : X \to Y$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Let $U \subset X$ be an open subscheme such that $U \to Y$ is quasi-compact. Then the set

$$E = \{y \in Y \mid \mathrm{Ass}_{X_y}(\mathcal{F}_y) \subset U_y\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. After possibly increasing i we may assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose pullback to X is isomorphic to $\mathcal{F}$, see Limits, Lemma 32.10.2. After possibly increasing i one more time we may assume there exists an open subscheme $U_i \subset X_i$ whose inverse image in X is U, see Limits, Lemma 32.4.11. By Lemma 37.25.4 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. This follows from Lemmas 37.25.2 and 37.25.3 applied to the base change $(X, \mathcal{F}, U) \times_Y Z$ over Z. □

37.26. Reduced fibres

0574

0575 Lemma 37.26.1. Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is nonreduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is nonreduced.

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 28.3.4. We may also replace Y by an affine open. Hence we may assume that $Y = \text{Spec}(A)$ with A a domain. Denote K the fraction field of A. Pick an affine open $\text{Spec}(B) = U \subset X$ and a section $h_\eta \in \Gamma(U_\eta, \mathcal{O}_{U_\eta}) = B_K$ which is nonzero and nilpotent. After shrinking Y we may assume that h comes from $h \in \Gamma(U, \mathcal{O}_U) = B$. After shrinking Y a bit more we may assume that h is nilpotent. Let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $C = B/I$ is a finite type A-algebra whose generic fiber $(B/I)_K$ is nonzero (as $h_\eta \neq 0$). We apply generic flatness to $A \to C$ and $A \to B/hB$, see Algebra, Lemma 10.118.3, and we obtain a $g \in A$, $g \neq 0$ such that C_g is free as an A_g-module and $(B/hB)_g$ is flat as an A_g-module. Replace Y by $D(g) \subset Y$. Now we have the short exact sequence

$$0 \to C \to B \to B/hB \to 0.$$

with B/hB flat over A and with C nonzero free as an A-module. It follows that for any homomorphism $A \to \kappa$ to a field the ring $C \otimes_A \kappa$ is nonzero and the sequence

$$0 \to C \otimes_A \kappa \to B \otimes_A \kappa \to B/hB \otimes_A \kappa \to 0$$

is exact, see Algebra, Lemma 10.39.12. Note that $B/hB \otimes_A \kappa = (B \otimes_A \kappa)/h(B \otimes_A \kappa)$ by right exactness of tensor product. Thus we conclude that multiplication by h is not zero on $B \otimes_A \kappa$. This clearly means that for any point $y \in Y$ the element h restricts to a nonzero element of U_y, whence X_y is nonreduced. □

0576 Lemma 37.26.2. Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then

$$\begin{aligned} &\{y' \in Y' \mid X'_{y'} \text{ is geometrically reduced}\} \\ &= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically reduced}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically reduced over $\kappa(y')$ if and only if X_y is geometrically reduced over $\kappa(y)$. This follows from Varieties, Lemma 33.6.6. □

0577 Lemma 37.26.3. Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is not geometrically reduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is not geometrically reduced.

Proof. Apply Lemma 37.24.7 to get

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

with all the properties mentioned in that lemma. Let η' be the generic point of Y'. Consider the morphism $X' \to X_{Y'}$ (which is the reduction morphism) and the resulting morphism of generic fibres $X'_{\eta'} \to X_{\eta'}$. Since $X'_{\eta'}$ is geometrically reduced, and X_η is not this cannot be an isomorphism, see Varieties, Lemma 33.6.6. Hence $X_{\eta'}$ is nonreduced. Hence by Lemma 37.26.1 the fibres of $X_{Y'} \to Y'$ are nonreduced at all points $y' \in V'$ of a nonempty open $V' \subset Y'$. Since $g : Y' \to V$ is a homeomorphism Lemma 37.26.2 proves that $g(V')$ is the open we are looking for. □

0578 Lemma 37.26.4. Let $f : X \to Y$ be a morphism of schemes. Assume

(1) Y is irreducible with generic point η,
(2) X_η is geometrically reduced, and
(3) f is of finite type.

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically reduced fibres.

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 28.3.4. We may also replace Y by an affine open. Hence we may assume that $Y = \text{Spec}(A)$ with A a domain. Denote K the fraction field of A. After shrinking Y a bit we may also assume that $X \to Y$ is flat and of finite presentation, see Morphisms, Proposition 29.27.1.

As X_η is geometrically reduced there exists an open dense subset $V \subset X_\eta$ such that $V \to \text{Spec}(K)$ is smooth, see Varieties, Lemma 33.25.7. Let $U \subset X$ be the set of points where f is smooth. By Morphisms, Lemma 29.34.15 we see that $V \subset U_\eta$. Thus the generic fibre of U is dense in the generic fibre of X. Since X_η is reduced, it follows that U_η is scheme theoretically dense in X_η, see Morphisms, Lemma 29.7.8. We note that as $U \to Y$ is smooth all the fibres of $U \to Y$ are geometrically reduced. Thus it suffices to show that, after shrinking Y, for all $y \in Y$ the scheme U_y is scheme theoretically dense in X_y, see Morphisms, Lemma 29.7.9. This follows from Lemma 37.24.4. □

0579 Lemma 37.26.5. Let $f : X \to Y$ be a morphism which is quasi-compact and locally of finite presentation. Then the set

$$E = \{y \in Y \mid X_y \text{ is geometrically reduced}\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Then X is quasi-compact. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. Then the fibres of $U_i \to Y$ at y form an affine open covering of the fibre of $X \to Y$ at y. Hence we may assume X is affine as well. Write $Y = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 37.26.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. If X_ξ is geometrically reduced, then Lemma 37.26.4 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. If X_ξ is not geometrically reduced, then Lemma 37.26.3 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. Thus we win. □

0C0D Lemma 37.26.6. Let $X \to \mathrm{Spec}(R)$ be a proper flat morphism where R is a discrete valuation ring. If the special fibre is reduced, then both X and the generic fibre X_η are reduced.

Proof. Assume the special fibre X_s is reduced. Let $x \in X$ be any point, and let us show that $\mathcal{O}_{X,x}$ is reduced; this will prove that X and X_η are reduced. Let $x \rightsquigarrow x'$ be a specialization with x' in the special fibre; such a specialization exists as a proper morphism is closed. Consider the local ring $A = \mathcal{O}_{X,x'}$. Then $\mathcal{O}_{X,x}$ is a localization of A, so it suffices to show that A is reduced. Let $\pi \in R$ be a uniformizer. If $a \in A$ then there exists an $n \geq 0$ and an element $a' \in A$ such that $a = \pi^n a'$ and $a' \notin \pi A$. This follows from Krull intersection theorem (Algebra, Lemma 10.51.4). If a is nilpotent, so is a', because π is a nonzerodivisor by flatness of A over R. But a' maps to a nonzero element of the reduced ring $A/\pi A = \mathcal{O}_{X_s,x'}$. This is a contradiction unless A is reduced, which is what we wanted to show. □

0C0E Lemma 37.26.7. Let $f : X \to Y$ be a flat proper morphism of finite presentation. Then the set $\{y \in Y \mid X_y \text{ is geometrically reduced}\}$ is open in Y.

Proof. We may assume Y is affine. Then Y is a cofiltered limit of affine schemes of finite type over $\mathbf{Z}$. Hence we can assume $X \to Y$ is the base change of $X_0 \to Y_0$ where Y_0 is the spectrum of a finite type $\mathbf{Z}$-algebra and $X_0 \to Y_0$ is flat and proper. See Limits, Lemma 32.10.1, 32.8.7, and 32.13.1. Since the formation of the set of points where the fibres are geometrically reduced commutes with base change (Lemma 37.26.2), we may assume the base is Noetherian.

Assume Y is Noetherian. The set is constructible by Lemma 37.26.5. Hence it suffices to show the set is stable under generalization (Topology, Lemma 5.19.10). By Properties, Lemma 28.5.10 we reduce to the case where $Y = \mathrm{Spec}(R)$, R is a discrete valuation ring, and the closed fibre X_y is geometrically reduced. To show: the generic fibre X_η is geometrically reduced.

If not then there exists a finite extension L of the fraction field of R such that X_L is not reduced, see Varieties, Lemma 33.6.4. There exists a discrete valuation ring $R' \subset L$ with fraction field L dominating R, see Algebra, Lemma 10.120.18. After replacing R by R' we reduce to Lemma 37.26.6. □

37.27. Irreducible components of fibres

0553

0554 Lemma 37.27.1. Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n irreducible components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n irreducible components.

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 28.3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into irreducible components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X_i whose generic fibre $Z_{i,j,\eta}$ is nowhere dense in $X_{i,\eta}$. Hence after shrinking Y we may assume that $Z_{i,j,y}$ is nowhere dense in $X_{i,y}$ for every $y \in Y$, see Lemma 37.24.3. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 37.24.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 29.27.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 29.25.10. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are not dense in $X_{i,y}$. Clearly this implies that X_y has at least n irreducible components. □

0555 Lemma 37.27.2. Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then

$$\begin{aligned} & \{y' \in Y' \mid X'_{y'} \text{ is geometrically irreducible}\} \\ & = g^{-1}(\{y \in Y \mid X_y \text{ is geometrically irreducible}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically irreducible over $\kappa(y')$ if and only if X_y is geometrically irreducible over $\kappa(y)$. This follows from Varieties, Lemma 33.8.2. □

0556 Lemma 37.27.3. Let $f : X \to Y$ be a morphism of schemes. Let

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of irreducible components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{tikzcd} K \arrow[r] & K'' & K' \arrow[l] \\ \kappa(y) \arrow[u] \arrow[rr] & & \kappa(y') \arrow[u] \end{tikzcd}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between irreducible components, see Varieties, Lemma 33.8.7. □

0557 Lemma 37.27.4. Let A be a domain with fraction field K. Let $P \in A[x_1, \ldots, x_n]$. Denote $\overline{K}$ the algebraic closure of K. Assume P is irreducible in $\overline{K}[x_1, \ldots, x_n]$. Then there exists a $f \in A$ such that $P^\varphi \in \kappa[x_1, \ldots, x_n]$ is irreducible for all homomorphisms $\varphi : A_f \to \kappa$ into fields.

Proof. There exists an automorphism Ψ of $A[x_1, \ldots, x_n]$ over A such that $\Psi(P) = ax_n^d +$ lower order terms in x_n with $a \neq 0$, see Algebra, Lemma 10.115.2. We may replace P by $\Psi(P)$ and we may replace A by A_a. Thus we may assume that P is monic in x_n of degree $d > 0$. For $i = 1, \ldots, n-1$ let d_i be the degree of P in x_i. Note that this implies that P^φ is monic of degree d in x_n and has degree $\leq d_i$ in x_i for every homomorphism $\varphi : A \to \kappa$ where κ is a field. Thus if P^φ is reducible, then we can write

$$P^\varphi = Q_1 Q_2$$

with Q_1, Q_2 monic of degree $e_1, e_2 \geq 0$ in x_n with $e_1 + e_2 = d$ and having degree $\leq d_i$ in x_i for $i = 1, \ldots, n-1$. In other words we can write

0558 (37.27.4.1) $$Q_j = x_n^{e_j} + \sum\nolimits_{0 \leq l < e_j} \left(\sum\nolimits_{L \in \mathcal{L}} a_{j,l,L} x^L \right) x_n^l$$

where the sum is over the set $\mathcal{L}$ of multi-indices L of the form $L = (l_1, \ldots, l_{n-1})$ with $0 \leq l_i \leq d_i$. For any $e_1, e_2 \geq 0$ with $e_1 + e_2 = d$ we consider the A-algebra

$$B_{e_1,e_2} = A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}]/(\text{relations})$$

where the (relations) is the ideal generated by the coefficients of the polynomial

$$P - Q_1 Q_2 \in A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}][x_1, \ldots, x_n]$$

with Q_1 and Q_2 defined as in (37.27.4.1). OK, and the assumption that P is irreducible over $\overline{K}$ implies that there does not exist any A-algebra homomorphism $B_{e_1,e_2} \to \overline{K}$. By the Hilbert Nullstellensatz, see Algebra, Theorem 10.34.1 this means that $B_{e_1,e_2} \otimes_A K = 0$. As B_{e_1,e_2} is a finitely generated A-algebra this signifies that we can find an $f_{e_1,e_2} \in A$ such that $(B_{e_1,e_2})_{f_{e_1,e_2}} = 0$. By construction this means that if $\varphi : A_{f_{e_1,e_2}} \to \kappa$ is a homomorphism to a field, then P^φ does not have a factorization $P^\varphi = Q_1 Q_2$ with Q_1 of degree e_1 in x_n and Q_2 of degree e_2 in x_n. Thus taking $f = \prod_{e1,e_2 \geq 0, e_1+e_2=d} f_{e_1,e_2}$ we win. □

0559 Lemma 37.27.5. Let $f : X \to Y$ be a morphism of schemes. Assume

(1) Y is irreducible with generic point η,
(2) X_η is geometrically irreducible, and
(3) f is of finite type.

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibres.

First proof of Lemma 37.27.5. We give two proofs of the lemma. These are essentially equivalent; the second is more self contained but a bit longer. Choose a diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

as in Lemma 37.24.7. Note that the generic fibre of f' is the reduction of the generic fibre of f (see Lemma 37.24.6) and hence is geometrically irreducible. Suppose that the lemma holds for the morphism f'. Then after shrinking V all the fibres of f' are geometrically irreducible. As $X' = (Y' \times_V X_V)_{red}$ this implies that all the fibres of $Y' \times_V X_V$ are geometrically irreducible. Hence by Lemma 37.27.2 all the fibres of $X_V \to V$ are geometrically irreducible and we win. In this way we see that we may assume that the generic fibre is geometrically reduced as well as geometrically irreducible and we may assume $Y = \mathrm{Spec}(A)$ with A a domain.

Let $x \in X_\eta$ be the generic point. As X_η is geometrically irreducible and reduced we see that $L = \kappa(x)$ is a finitely generated extension of $K = \kappa(\eta)$ which is geometrically reduced and geometrically irreducible, see Varieties, Lemmas 33.6.2 and 33.8.6. In particular the field extension L/K is separable, see Algebra, Lemma 10.44.1. Hence we can find $x_1, \ldots, x_{r+1} \in L$ which generate L over K and such that $x_1, \ldots, x_r$ is a transcendence basis for L over K, see Algebra, Lemma 10.42.3. Let $P \in K(x_1, \ldots, x_r)[T]$ be the minimal polynomial for x_{r+1}. Clearing denominators we may assume that P has coefficients in $A[x_1, \ldots, x_r]$. Note that as L is geometrically reduced and geometrically irreducible over K, the polynomial P is irreducible in $\overline{K}[x_1, \ldots, x_r, T]$ where $\overline{K}$ is the algebraic closure of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1}))$$

and set $X' = \mathrm{Spec}(B')$. By construction the fraction field of B' is isomorphic to $L = \kappa(x)$ as K-extensions. Hence there exists an open $U \subset X$, and open $U' \subset X'$ and a Y-isomorphism $U \to U'$, see Morphisms, Lemma 29.50.7. Here is a diagram:

$$\begin{tikzcd} X \arrow[rd] & U \arrow[l] \arrow[r, equal] \arrow[d] & U' \arrow[r] \arrow[d] & X' \arrow[r, equal] \arrow[ld] & \mathrm{Spec}(B') \\ & Y \arrow[r, equal] & Y & & \end{tikzcd}$$

Note that $U_\eta \subset X_\eta$ and $U'_\eta \subset X'_\eta$ are dense opens. Thus after shrinking Y by applying Lemma 37.24.3 we obtain that U_y is dense in X_y and U'_y is dense in X'_y

for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 37.27.4. $\square$

Second proof of Lemma 37.27.5. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to X$ be the reduction of X. Note that $X' \to X \to Y$ factors through Y', see Schemes, Lemma 26.12.7. As $Y' \to Y$ and $X' \to X$ are universal homeomorphisms by Morphisms, Lemma 29.45.6 we see that it suffices to prove the lemma for $X' \to Y'$. Thus we may assume that X and Y are reduced. In particular Y is integral, see Properties, Lemma 28.3.4. Thus by Morphisms, Proposition 29.27.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ is flat and of finite presentation. After replacing Y by V we may assume, in addition to (1), (2), (3) that Y is integral affine, X is reduced, and f is flat and of finite presentation. In particular f is universally open, see Morphisms, Lemma 29.25.10.

Pick a nonempty affine open $U \subset X$. Then $U \to Y$ is flat and of finite presentation with geometrically irreducible generic fibre. The complement $X_\eta \setminus U_\eta$ is nowhere dense. Thus after shrinking Y we may assume $U_y \subset X_y$ is open dense for all $y \in Y$, see Lemma 37.24.3. Thus we may replace X by U and we reduce to the case where Y is integral affine and X is reduced affine, flat and of finite presentation over Y with geometrically irreducible generic fibre X_η.

Write $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$. Then A is a domain, B is reduced, $A \to B$ is flat of finite presentation, and B_K is geometrically irreducible over the fraction field K of A. In particular we see that B_K is a domain. Let L be the fraction field of B_K. Note that L is a finitely generated field extension of K as B is an A-algebra of finite presentation. Let K'/K be a finite purely inseparable extension such that $(L \otimes_K K')_{red}$ is a separably generated field extension, see Algebra, Lemma 10.45.3. Choose $x_1, \ldots, x_n \in K'$ which generate the field extension K' over K, and such that $x_i^{q_i} \in A$ for some prime power q_i (proof existence x_i omitted). Let A' be the A-subalgebra of K' generated by $x_1, \ldots, x_n$. Then A' is a finite A-subalgebra $A' \subset K'$ whose fraction field is K'. Note that $\text{Spec}(A') \to \text{Spec}(A)$ is a universal homeomorphism, see Algebra, Lemma 10.46.7. Hence it suffices to prove the result after base changing to $\text{Spec}(A')$. We are going to replace A by A' and B by $(B \otimes_A A')_{red}$ to arrive at the situation where L is a separably generated field extension of K. Of course it may happen that $(B \otimes_A A')_{red}$ is no longer flat, or of finite presentation over A', but this can be remedied by replacing A' by A'_f for a suitable $f \in A'$, see Algebra, Lemma 10.118.3.

At this point we know that A is a domain, B is reduced, $A \to B$ is flat and of finite presentation, B_K is a domain whose fraction field L is a separably generated field extension of the fraction field K of A. By Algebra, Lemma 10.42.3 we may write $L = K(x_1, \ldots, x_{r+1})$ where $x_1, \ldots, x_r$ are algebraically independent over K, and x_{r+1} is separable over $K(x_1, \ldots, x_r)$. After clearing denominators we may assume that the minimal polynomial $P \in K(x_1, \ldots, x_r)[T]$ of x_{r+1} over $K(x_1, \ldots, x_r)$ has coefficients in $A[x_1, \ldots, x_r]$. Note that since L/K is separable and since L is geometrically irreducible over K, the polynomial P is irreducible over the algebraic closure $\overline{K}$ of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1})).$$

By construction the fraction fields of B and B' are isomorphic as K-extensions. Hence there exists an isomorphism of A-algebras $B_h \cong B'_{h'}$ for suitable $h \in B$ and

$h' \in B'$, see Morphisms, Lemma 29.50.7. In other words X and $X' = \text{Spec}(B')$ have a common affine open U. Here is a diagram:

$$\begin{tikzcd} X = \text{Spec}(B) \arrow[rd] & U \arrow[l] \arrow[r] \arrow[d] & \text{Spec}(B') = X' \arrow[ld] \\ & Y = \text{Spec}(A) & \end{tikzcd}$$

After shrinking Y once more (by applying Lemma 37.24.3 to $Z = X \setminus U$ in X and $Z' = X' \setminus U$ in X') we see that U_y is dense in X_y and U_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 37.27.4. □

055A Lemma 37.27.6. Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 37.27.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 28.3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 37.24.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 29.36.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 37.27.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 37.24.5. After shrinking Y some more we see that each fibre of f has at least n irreducible components, see Lemma 37.27.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see Lemma 37.27.5. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

055B Lemma 37.27.7. Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 37.27.3. Assume f of finite presentation. Then the level sets

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By

Lemma 37.27.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 37.27.6 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

37.28. Connected components of fibres

055C

055D Lemma 37.28.1. Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n connected components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n connected components.

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 28.3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into connected components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X whose generic fibre $Z_{i,j,\eta}$ is empty. Hence after shrinking Y we may assume that $Z_{i,j} = \emptyset$, see Lemma 37.24.1. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 37.24.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 29.27.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 29.25.10. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are empty! Clearly this implies that X_y has at least n connected components. □

055E Lemma 37.28.2. Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then

$$\begin{aligned} &\{y' \in Y' \mid X'_{y'} \text{ is geometrically connected}\} \\ &= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically connected}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically connected over $\kappa(y')$ if and only if X_y is geometrically connected over $\kappa(y)$. This follows from Varieties, Lemma 33.7.3. □

055F Lemma 37.28.3. Let $f : X \to Y$ be a morphism of schemes. Let

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of connected components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{tikzcd} K \arrow[r] & K'' & K' \arrow[l] \\ \kappa(y) \arrow[u] \arrow[rr] & & \kappa(y') \arrow[u] \end{tikzcd}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between connected components, see Varieties, Lemma 33.7.6. □

055G **Lemma 37.28.4.** Let $f : X \to Y$ be a morphism of schemes. Assume

(1) Y is irreducible with generic point η,
(2) X_η is geometrically connected, and
(3) f is of finite type.

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically connected fibres.

Proof. Choose a diagram

$$\begin{tikzcd} X' \arrow[r, "g'"'] \arrow[d, "f'"'] & X_V \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & V \arrow[r] & Y \end{tikzcd}$$

as in Lemma 37.24.8. Note that the generic fibre of f' is geometrically connected (for example by Lemma 37.28.3). Suppose that the lemma holds for the morphism f'. This means that there exists a nonempty open $W \subset Y'$ such that every fibre of $X' \to Y'$ over W is geometrically connected. Then, as g is an open morphism by Morphisms, Lemma 29.36.13 all the fibres of f at points of the nonempty open $V = g(W)$ are geometrically connected, see Lemma 37.28.3. In this way we see that we may assume that the irreducible components of the generic fibre X_η are geometrically irreducible.

Let Y' be the reduction of Y, and set $X' = Y' \times_Y X$. Then it suffices to prove the lemma for the morphism $X' \to Y'$ (for example by Lemma 37.28.3 once again). Since the generic fibre of $X' \to Y'$ is the same as the generic fibre of $X \to Y$ we see that we may assume that Y is irreducible and reduced (i.e., integral, see Properties, Lemma 28.3.4) and that the irreducible components of the generic fibre X_η are geometrically irreducible.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 37.24.5. Let $Z_{i,j} = X_i \cap X_j$. Let

$$\{1, \ldots, n\} \times \{1, \ldots, n\} = I \amalg J$$

where $(i, j) \in I$ if $Z_{i,j,\eta} = \emptyset$ and $(i, j) \in J$ if $Z_{i,j,\eta} \neq \emptyset$. After shrinking Y we may assume that $Z_{i,j} = \emptyset$ for all $(i, j) \in I$, see Lemma 37.24.1. After shrinking Y we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see

Lemma 37.27.5. After shrinking Y some more we achieve the situation where each $Z_{i,j} \to Y$ is flat and of finite presentation for all $(i,j) \in J$, see Morphisms, Proposition 29.27.1. This means that $f(Z_{i,j}) \subset Y$ is open, see Morphisms, Lemma 29.25.10. We claim that

$$V = \bigcap\nolimits_{(i,j) \in J} f(Z_{i,j})$$

works, i.e., that X_y is geometrically connected for each $y \in V$. Namely, the fact that X_η is connected implies that the equivalence relation generated by the pairs in J has only one equivalence class. Now if $y \in V$ and $K \supset \kappa(y)$ is a separably closed extension, then the irreducible components of $(X_y)_K$ are the fibres $(X_{i,y})_K$. Moreover, we see by construction and $y \in V$ that $(X_{i,y})_K$ meets $(X_{j,y})_K$ if and only if $(i,j) \in J$. Hence the remark on equivalence classes shows that $(X_y)_K$ is connected and we win. □

055H Lemma 37.28.5. Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically connected components of fibres of f introduced in Lemma 37.28.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 28.3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 37.24.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 29.36.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 37.27.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$. By Varieties, Lemma 33.8.16 this implies that also the connected components of X_η are geometrically connected.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) connected components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 37.24.5. After shrinking Y some more we see that each fibre of f has at least n connected components, see Lemma 37.28.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically connected for each i and all $y \in Y$, see Lemma 37.28.4. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

055I Lemma 37.28.6. Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometric connected components of fibres of f introduced in Lemma 37.28.3. Assume f of finite presentation. Then the level sets

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 37.28.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 37.28.5 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

055J Lemma 37.28.7. Let $f : X \to S$ be a morphism of schemes. Assume that

(1) S is the spectrum of a discrete valuation ring,
(2) f is flat,
(3) X is connected,
(4) the closed fibre X_s is reduced.

Then the generic fibre X_η is connected.

Proof. Write $S = \mathrm{Spec}(R)$ and let $\pi \in R$ be a uniformizer. To get a contradiction assume that X_η is disconnected. This means there exists a nontrivial idempotent $e \in \Gamma(X_\eta, \mathcal{O}_{X_\eta})$. Let $U = \mathrm{Spec}(A)$ be any affine open in X. Note that π is a nonzerodivisor on A as A is flat over R, see More on Algebra, Lemma 15.22.9 for example. Then $e|_{U_\eta}$ corresponds to an element $e \in A[1/\pi]$. Let $z \in A$ be an element such that $e = z/\pi^n$ with $n \geq 0$ minimal. Note that $z^2 = \pi^n z$. This means that $z \bmod \pi A$ is nilpotent if $n > 0$. By assumption $A/\pi A$ is reduced, and hence minimality of n implies $n = 0$. Thus we conclude that $e \in A$! In other words $e \in \Gamma(X, \mathcal{O}_X)$. As X is connected it follows that e is a trivial idempotent which is a contradiction. □

37.29. Connected components meeting a section

055K The results in this section are in particular applicable to a group scheme $G \to S$ and its neutral section $e : S \to G$.

055L Situation 37.29.1. Here $f : X \to Y$ be a morphism of schemes, and $s : Y \to X$ is a section of f. For every $y \in Y$ we denote X_y^0 the connected component of X_y containing $s(y)$. Finally, we set $X^0 = \bigcup_{y \in Y} X_y^0$.

055M Lemma 37.29.2. Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. If $g : Y' \to Y$ is any morphism, consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"'] \\ Y' \arrow[r, "g"] \arrow[u, bend left, "s'"] & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

so that we obtain $(X')^0 \subset X'$. Then $(X')^0 = (g')^{-1}(X^0)$.

Proof. Let $y' \in Y'$ with image $y \in Y$. We may think of X_y^0 as a closed subscheme of X_y, see for example Morphisms, Definition 29.26.3. As $s(y) \in X_y^0$ we conclude from Varieties, Lemma 33.7.14 that X_y^0 is a geometrically connected scheme over $\kappa(y)$. Hence $X_y^0 \times_y y' \to X'_{y'}$ is a connected closed subscheme which contains $s'(y')$.

Thus $X^0_y \times_y y' \subset (X'_{y'})^0$. The other inclusion $X^0_y \times_y y' \supset (X'_{y'})^0$ is clear as the image of $(X'_{y'})^0$ in X_y is a connected subset of X_y which contains $s(y)$. □

055N Lemma 37.29.3. Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that the inverse image of X^0 in the base change X_V is open and closed in X_V.

Proof. Let $Z \subset Y$ be the induced reduced closed subscheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ and $s_Z : Z \to X_Z$ be the base changes of f and s. By Lemma 37.29.2 we have $(X_Z)^0 = (X^0)_Z$. Hence it suffices to prove the lemma for the morphism $X_Z \to Z$ and the point $x \in X_Z$ which maps to the generic point of Z. In other words we have reduced the problem to the case where Y is an integral scheme (see Properties, Lemma 28.3.4) with generic point η. Our goal is to show that after shrinking Y the subset X^0 becomes an open and closed subset of X.

Note that the scheme X_η is of finite type over a field, hence Noetherian. Thus its connected components are open as well as closed. Hence we may write $X_\eta = X^0_\eta \amalg T_\eta$ for some open and closed subset T_η of X_η. Next, let $T \subset X$ be the closure of T_η and let $X^{00} \subset X$ be the closure of X^0_η. Note that T_η, resp. X^0_η is the generic fibre of T, resp. X^{00}, see discussion preceding Lemma 37.24.5. Moreover, that lemma implies that after shrinking Y we may assume that $X = X^{00} \cup T$ (set theoretically). Note that $(T \cap X^{00})_\eta = T_\eta \cap X^0_\eta = \emptyset$. Hence after shrinking Y we may assume that $T \cap X^{00} = \emptyset$, see Lemma 37.24.1. In particular X^{00} is open in X. Note that X^0_η is connected and has a rational point, namely $s(\eta)$, hence it is geometrically connected, see Varieties, Lemma 33.7.14. Thus after shrinking Y we may assume that all fibres of $X^{00} \to Y$ are geometrically connected, see Lemma 37.28.4. At this point it follows that the fibres X^{00}_y are open, closed, and connected subsets of X_y containing $\sigma(y)$. It follows that $X^0 = X^{00}$ and we win. □

055P Lemma 37.29.4. Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. If f is of finite presentation then X^0 is locally constructible in X.

Proof. Let $x \in X$. We have to show that there exists an open neighbourhood U of x such that $X^0 \cap U$ is constructible in U. This reduces us to the case where Y is affine. Write $Y = \text{Spec}(A)$ and $A = \text{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \text{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \text{Spec}(A_i) \to X_i$ whose base change to Y recovers f and the section s. By Lemma 37.29.2 it suffices to prove the lemma for f_i, s_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is a Noetherian affine scheme. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 29.15.6. In order to prove the lemma in this case it suffices to show that for every irreducible closed subset $Z \subset X$ the intersection $Z \cap X^0$ either contains a nonempty open of Z or is not dense in Z, see Topology, Lemma 5.16.3. Let $x \in Z$ be the generic point, and let $y = f(x)$. By Lemma 37.29.3 there exists a nonempty open subset $V \subset \overline{\{y\}}$ such that $X^0 \cap X_V$ is open and closed in X_V. Since $f(Z) \subset \overline{\{y\}}$ and $f(x) = y \in V$ we see that $W = f^{-1}(V) \cap Z$ is a nonempty open subset of Z. It follows that $X^0 \cap W$ is open and closed in W. Since W is irreducible we see that $X^0 \cap W$ is either empty or equal to W. This proves the lemma. □

055Q Lemma 37.29.5. Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. Let $y \in Y$ be a point. Assume

(1) f is of finite presentation and flat, and
(2) the fibre X_y is geometrically reduced.

Then X^0 is a neighbourhood of X^0_y in X.

Proof. We may replace Y with an affine open neighbourhood of y. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \mathrm{Spec}(A_i) \to X_i$ whose base change to Y recovers f and the section s. After possibly increasing i we may also assume that f_i is flat, see Limits, Lemma 32.8.7. Let y_i be the image of y in Y_i. Note that $X_y = (X_{i,y_i}) \times_{y_i} y$. Hence X_{i,y_i} is geometrically reduced, see Varieties, Lemma 33.6.6. By Lemma 37.29.2 it suffices to prove the lemma for the system $f_i, s_i, y_i \in Y_i$. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is the spectrum of a Noetherian ring. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 29.15.6. Let $x \in X^0$ be a point lying over y. By Topology, Lemma 5.16.4 it suffices to prove that for any irreducible closed $Z \subset X$ passing through x the intersection $X^0 \cap Z$ is dense in Z. In particular it suffices to prove that the generic point $x' \in Z$ is in X^0. By Properties, Lemma 28.5.10 we can find a discrete valuation ring R and a morphism $\mathrm{Spec}(R) \to X$ which maps the special point to x and the generic point to x'. We are going to think of $\mathrm{Spec}(R)$ as a scheme over Y via the composition $\mathrm{Spec}(R) \to X \to Y$. By Lemma 37.29.2 we have that $(X_R)^0$ is the inverse image of X^0. By construction we have a second section $t : \mathrm{Spec}(R) \to X_R$ (besides the base change s_R of s) of the structure morphism $X_R \to \mathrm{Spec}(R)$ such that $t(\eta_R)$ is a point of X_R which maps to x' and $t(0_R)$ is a point of X_R which maps to x. Note that $t(0_R)$ is in $(X_R)^0$ and that $t(\eta_R) \rightsquigarrow t(0_R)$. Thus it suffices to prove that this implies that $t(\eta_R) \in (X_R)^0$. Hence it suffices to prove the lemma in the case where Y is the spectrum of a discrete valuation ring and y its closed point.

Assume Y is the spectrum of a discrete valuation ring and y is its closed point. Our goal is to prove that X^0 is a neighbourhood of X^0_y. Note that X^0_y is open and closed in X_y as X_y has finitely many irreducible components. Hence the complement $C = X_y \setminus X^0_y$ is closed in X. Thus $U = X \setminus C$ is an open neighbourhood of X^0_y and $U^0 = X^0$. Hence it suffices to prove the result for the morphism $U \to Y$. In other words, we may assume that X_y is connected. Suppose that X is disconnected, say $X = X_1 \amalg \ldots \amalg X_n$ is a decomposition into connected components. Then $s(Y)$ is completely contained in one of the X_i. Say $s(Y) \subset X_1$. Then $X^0 \subset X_1$. Hence we may replace X by X_1 and assume that X is connected. At this point Lemma 37.28.7 implies that X_η is connected, i.e., $X^0 = X$ and we win. □

055R Lemma 37.29.6. Let $f : X \to Y$, $s : Y \to X$ be as in Situation 37.29.1. Assume

(1) f is of finite presentation and flat, and
(2) all fibres of f are geometrically reduced.

Then X^0 is open in X.

Proof. This is an immediate consequence of Lemma 37.29.5. □

37.30. Dimension of fibres

05F6

05F7 **Lemma 37.30.1.** Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has dimension n, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has dimension n.

Proof. Let $Z = \{x \in X \mid \dim_x(X_{f(x)}) > n\}$. By Morphisms, Lemma 29.28.4 this is a closed subset of X. By assumption $Z_\eta = \emptyset$. Hence by Lemma 37.24.1 we may shrink Y and assume that $Z = \emptyset$. Let $Z' = \{x \in X \mid \dim_x(X_{f(x)}) > n-1\} = \{x \in X \mid \dim_x(X_{f(x)}) = n\}$. As before this is a closed subset of X. By assumption we have $Z'_\eta \neq \emptyset$. Hence after shrinking Y we may assume that $Z' \to Y$ is surjective, see Lemma 37.24.2. Hence we win. □

05F8 **Lemma 37.30.2.** Let $f : X \to Y$ be a morphism of finite type. Let

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the dimension of X_y. If $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. This follows from Morphisms, Lemma 29.28.3. □

05F9 **Lemma 37.30.3.** Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 37.30.2. Assume f of finite presentation. Then the level sets

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 32.10.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 37.30.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.16.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 37.30.1 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This implies what we want. □

0D4H **Lemma 37.30.4.** Let $f : X \to Y$ be a flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 37.30.2. Then $n_{X/Y}$ is lower semi-continuous.

Proof. Let $W \subset X$, $W = \coprod_{d \geq 0} U_d$ be the open constructed in Lemmas 37.22.7 and 37.22.9. Let $y \in Y$ be a point. If $n_{X/Y}(y) = \dim(X_y) = n$, then y is in the image of $U_n \to Y$. By Morphisms, Lemma 29.25.10 we see that $f(U_n)$ is open in Y. Hence there is an open neighbourhoof of y where $n_{X/Y}$ is $\geq n$. □

0D4I Lemma 37.30.5. Let $f : X \to Y$ be a proper morphism of schemes. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 37.30.2. Then $n_{X/Y}$ is upper semi-continuous.

Proof. Let $Z_d = \{x \in X \mid \dim_x(X_{f(x)}) > d\}$. Then Z_d is a closed subset of X by Morphisms, Lemma 29.28.4. Since f is proper $f(Z_d)$ is closed. Since $y \in f(Z_d) \Leftrightarrow n_{X/Y}(y) > d$ we see that the lemma is true. □

0D4J Lemma 37.30.6. Let $f : X \to Y$ be a proper, flat morphism of schemes of finite presentation. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 37.30.2. Then $n_{X/Y}$ is locally constant.

Proof. Immediate consequence of Lemmas 37.30.4 and 37.30.5. □

37.31. Weak relative Noether normalization

0GTD The goal of this section is to prove Lemma 37.31.3.

0GTE Lemma 37.31.1. Let R be a ring. Let $\mathfrak p_1, \ldots, \mathfrak p_r$ be prime ideals of R with $\mathfrak p_i \not\subset \mathfrak p_j$ if $i \neq j$. Let $k_i \subset \kappa(\mathfrak p_i)$ be subfields such that the extensions $\kappa(\mathfrak p_i)/k_i$ are not algebraic. Let $J \subset R$ be an ideal not contained in any of the $\mathfrak p_i$. Then there exists an element $x \in J$ such that the image of x in $\kappa(\mathfrak p_i)$ is transcendental over k_i for $i = 1, \ldots, r$.

Proof. The ideal $J_i = J\mathfrak p_1 \ldots \hat{\mathfrak p}_i \ldots \mathfrak p_r$ is not contained in $\mathfrak p_i$, see Algebra, Lemma 10.15.1. It follows that every element ξ of $\kappa(\mathfrak p_i) = \text{Frac}(B/\mathfrak p_i)$ is of the form $\xi = a/b$ with $a, b \in J_i$ and $b \notin \mathfrak p_i$. Choosing ξ transcendental over k_i we see that either a or b maps to an element of $\kappa(\mathfrak p_i)$ transcendental over k_i. We conclude that for every $i = 1, \ldots, r$ we can find an element $x_i \in J_i = J\mathfrak p_1 \ldots \hat{\mathfrak p}_i \ldots \mathfrak p_r$ which maps to an element of $\kappa(\mathfrak p_i)$ transcendental over k_i. Then $x = x_1 + \ldots + x_r$ works. □

0GTF Lemma 37.31.2. Let $R \to S$ be a finite type ring map. Let $d \geq 0$. Let $a, b \in S$. Assume that the fibres of

$$f_a : \text{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

given by the R-algebra map $R[x] \to S$ sending x to a have dimension $\leq d$. Then there exists an n_0 such that for $n \geq n_0$ the fibres of

$$f_{a^n+b} : \text{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

given by the R-algebra map $R[x] \to S$ sending x to $a^n + b$ have dimension $\leq d$.

Proof. In this paragraph we reduce to the case where $R \to S$ is of finite presentation. Namely, write $S = R[A, B, x_1, \ldots, x_n]/J$ for some ideal $J \subset R[x_1, \ldots, x_n]$ where A and B map to a and b in S. Then J is the union of its finitely generated ideals $J_\lambda \subset J$. Set $S_\lambda = R[A, B, x_1, \ldots, x_n]/J_\lambda$ and denote $a_\lambda, b_\lambda \in S_\lambda$ the images of A and B. Then for some λ the fibres of

$$f_{a_\lambda} : \text{Spec}(S_\lambda) \longrightarrow \mathbf{A}^1_R$$

have dimension $\leq d$, see Limits, Lemma 32.18.1. Fix such a λ. If we can find n_0 which works for $R \to S_\lambda$, a_λ, b_λ, then n_0 works for $R \to S$. Namely, the fibres of $f_{a_\lambda^n+b_\lambda} : \text{Spec}(S_\lambda) \to \mathbf{A}^1_R$ contain the fibres of $f_{a^n+b} : \text{Spec}(S) \to \mathbf{A}^1_R$. This reduces us to the case discussed in the next paragraph.

Assume $R \to S$ is of finite presentation. In this paragraph we reduce to the case where R is of finite type over $\mathbf{Z}$. By Algebra, Lemma 10.127.18 we can find a directed set Λ and a system of ring maps $R_\lambda \to S_\lambda$ over Λ whose colimit is $R \to S$ such that $S_\mu = S_\lambda \otimes_{R_\lambda} R_\mu$ for $\mu \geq \lambda$ and such that each R_λ and S_λ is of finite type over $\mathbf{Z}$. Choose $\lambda_0 \in \Lambda$ and elements $a_{\lambda_0}, b_{\lambda_0} \in S_{\lambda_0}$ mapping to $a, b \in S$. For $\lambda \geq \lambda_0$ denote $a_\lambda, b_\lambda \in S_\lambda$ the image of $a_{\lambda_0}, b_{\lambda_0}$. Then for $\lambda \geq \lambda_0$ large enough the fibres of

$$f_{a_\lambda} : \mathrm{Spec}(S_\lambda) \longrightarrow \mathbf{A}^1_{R_\lambda}$$

have dimension $\leq d$, see Limits, Lemma 32.18.4. Fix such a λ. If we can find n_0 which works for $R_\lambda \to S_\lambda$, a_λ, b_λ, then n_0 works for $R \to S$. Namely, any fibre of $f_{a^n+b} : \mathrm{Spec}(S) \to \mathbf{A}^1_R$ has the same dimension as a fibre of $f_{a_\lambda^n+b_\lambda} : \mathrm{Spec}(S_\lambda) \to \mathbf{A}^1_{R_\lambda}$ by Morphisms, Lemma 29.28.3. This reduces us the the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$. In particular the dimension of R is finite, and we may use induction on $\dim(R)$. Thus we may assume the result holds for all situations with $R' \to S'$, a, b as in the lemma with R' and S' of finite type over $\mathbf{Z}$ but with $\dim(R') < \dim(R)$.

Since the statement is about the topology of the spectrum of S we may assume S is reduced. Let S^ν be the normalization of S. Then $S \subset S^\nu$ is a finite extension as S is excellent, see Algebra, Proposition 10.162.16 and Morphisms, Lemma 29.54.10. Thus $\mathrm{Spec}(S^\nu) \to \mathrm{Spec}(S)$ is surjective and finite (Algebra, Lemma 10.36.17). It follows that if the result holds for $R \to S^\nu$ and the images of a, b in S^ν, then the result holds for $R \to S$, a, b. (Small detail omitted.) This reduces us to the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$ and S normal. Then $S = S_1 \times \ldots \times S_r$ for some normal domains S_i. If the result holds for each $R \to S_i$ and the images of a, b in S_i, then the result holds for $R \to S$, a, b. (Small detail omitted.) This reduces us to the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$ and S a normal domain. We may replace R by the image of R in S (this does not increase the dimension of R). This reduces us to the case discussed in the next paragraph.

Assume $R \subset S$ are of finite type over $\mathbf{Z}$ and S a normal domain. Consider the morphism

$$f_a : \mathrm{Spec}(S) \to \mathbf{A}^1_R$$

The assumption tells us that f_a has fibres of dimension $\leq d$. Hence the fibres of $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ have dimension $\leq d + 1$ (Morphisms, Lemma 29.28.2). Consider the morphism of integral schemes

$$\phi : \mathrm{Spec}(S) \to \mathbf{A}^2_R = \mathrm{Spec}(R[x, y])$$

corresponding to the R-algebra map $R[x, y] \to S$ sending x to a and y to b. There are two cases to consider

(1) ϕ is dominant, and
(2) ϕ is not dominant.

We claim that in both cases there exists an integer n_0 and a nonempty open $V \subset \mathrm{Spec}(R)$ such that for $n \geq n_0$ the fibres of f_{a^n+b} at points $q \in \mathbf{A}^1_V$ have dimension $\leq d$.

Proof of the claim in case (1). We have $f_{a^n+b} = \pi_n \circ \phi$ where

$$\pi_n : \mathbf{A}^2_R \to \mathbf{A}^1_R$$

is the flat morphism corresponding to the R-algebra map $R[x] \to R[x, y]$ sending x to $x^n + y$. Since ϕ is dominant there is a dense open $U \subset \text{Spec}(S)$ such that $\phi|_U : U \to \mathbf{A}^2_R$ is flat (this follows for example from generic flatness, see Morphisms, Proposition 29.27.1). Then the composition

$$f_{a^n+b}|_U : U \xrightarrow{\phi|_U} \mathbf{A}^2_R \xrightarrow{\pi_n} \mathbf{A}^1_R$$

is flat as well. Hence the fibres of this morphism have at least codimension 1 in the fibres of $f|_U : U \to \text{Spec}(R)$ by Morphisms, Lemma 29.28.2. In other words, the fibres of $f_{a^n+b}|_U$ have dimension $\leq d$. On the other hand, since U is dense in $\text{Spec}(S)$, we can find a nonempty open $V \subset \text{Spec}(R)$ such that $U \cap f^{-1}(p) \subset f^{-1}(p)$ is dense for all $p \in V$ (see for example Lemma 37.24.3). Thus $\dim(f^{-1}(p) \setminus U \cap f^{-1}(p)) \leq d$ and we conclude that our claim is true (as any fibres of $f_{a^n+b} : \text{Spec}(S) \to \mathbf{A}^1_R$ is contained in a fibre of f).

Case (2). In this case we can find a nonzero $g = \sum c_{ij} x^i y^j$ in $R[x, y]$ such that $\text{Im}(\phi) \subset V(g)$. In fact, we may assume g is irreducible over $\text{Frac}(R)$. If $g \in R[x]$, say with leading coefficient c, then over $V = D(c) \subset \text{Spec}(R)$ the fibres of f already have dimension $\leq d$ (because the image of f_a is contained in $V(g) \subset \mathbf{A}^1_R$ which has finite fibres over V). Hence we may assume g is not contained in $R[x]$. Let $s \geq 1$ be the degree of g as a polynomial in y and let t be the degree of $\sum c_{is} x^i$ as a polynomial in x. Then c_{ts} is nonzero and

$$g(x, -x^n) = (-1)^s c_{ts} x^{t+sn} + l.o.t.$$

provided that n is bigger than the degree of g as a polynomial in x (small detail omitted). For such n the polynomial $g(x, -x^n)$ is a nonzero polynomial in x and maps to a nonzero polynomial in $\kappa(\mathfrak{p})[x]$ for all $\mathfrak{p} \subset R$, $c_{st} \notin \mathfrak{p}$. We conclude that our claim is true for V equal to the principal open $D(c_{ts})$ of $\text{Spec}(R)$.

OK, and now we can use induction on $\dim(R)$. Namely, let $I \subset R$ be an ideal such that $V(I) = \text{Spec}(R) \setminus V$. Observe that $\dim(R/I) < \dim(R)$ as R is a domain. Let n'_0 be the integer we have by induction on $\dim(R)$ for $R/I \to S/IS$ and the images of a and b in S/IS. Then $\max(n_0, n'_0)$ works. $\square$

0GTG Lemma 37.31.3. Let $R \to S$ be a finite type ring map. Let d be the maximum of the dimensions of fibres of $\text{Spec}(S) \to \text{Spec}(R)$. Then there exists a quasi-finite ring map $R[t_1, \ldots, t_d] \to S$.

Proof. In this paragraph we reduce to the case where $R \to S$ is of finite presentation. Namely, write $S = R[x_1, \ldots, x_n]/J$ for some ideal $J \subset R[x_1, \ldots, x_n]$. Then J is the union of its finitely generated ideals $J_\lambda \subset J$. Set $S_\lambda = R[x_1, \ldots, x_n]/J_\lambda$. Then for some λ the fibres of $\text{Spec}(S_\lambda) \to \text{Spec}(R)$ have dimension $\leq d$, see Limits, Lemma 32.18.1. Fix such a λ. If we can find a quasi-finite $R[t_1, \ldots, t_d] \to S_\lambda$, then of course the composition $R[t_1, \ldots, t_d] \to S$ is quasi-finite. This reduces us to the case discussed in the next paragraph.

Assume $R \to S$ is of finite presentation. In this paragraph we reduce to the case where R is of finite type over $\mathbf{Z}$. By Algebra, Lemma 10.127.18 we can find a directed set Λ and a system of ring maps $R_\lambda \to S_\lambda$ over Λ whose colimit is $R \to S$ such that $S_\mu = S_\lambda \otimes_{R_\lambda} R_\mu$ for $\mu \geq \lambda$ and such that each R_λ and S_λ is of finite type over $\mathbf{Z}$. Then for λ large enough the fibres of $\text{Spec}(S_\lambda) \to \text{Spec}(R_\lambda)$ have dimension $\leq d$, see Limits, Lemma 32.18.4. Fix such a λ. If we can find a quasi-finite ring map $R_\lambda[t_1, \ldots, t_d] \to S_\lambda$, then the base change $R[t_1, \ldots, t_d] \to S$ is quasi-finite too (Algebra, Lemma 10.122.8). This reduces us the the case discussed in the next paragraph.

Assume R and S are of finite type over $\mathbf{Z}$. If $d = 0$, then the ring map is quasi-finite and we are done. Assume $d > 0$. We will find an element $a \in S$ such that the fibres of the R-algebra map $R[x] \to S$, $x \mapsto a$ have dimension $< d$. This will finish the proof by induction.

We will prove the existence of a by induction on $\dim(R)$.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r \subset S$ be those among the minimal primes of S such that $\dim_{\mathfrak{q}_i}(S/R) = d$. For notation, see Algebra, Definition 10.125.1. Say $\mathfrak{q}_i$ lies over the prime $\mathfrak{p}_i \subset R$. We have $\text{trdeg}_{\kappa(\mathfrak{p}_i)}(\kappa(\mathfrak{q}_i)) = d$ as $\mathfrak{q}_i$ is a generic point of its fibre; for example apply Algebra, Lemma 10.116.3 to $S \otimes_R \kappa(\mathfrak{p}_i)$. Hence by Lemma 37.31.1 we can find an element $a \in S$ such that the image of a in $\kappa(\mathfrak{q}_i)$ is transcendental over $\kappa(\mathfrak{p}_i)$ for $i = 1, \ldots, r$. Consider the morphism

$$f_a : \text{Spec}(S) \longrightarrow \mathbf{A}^1_R$$

corresponding the R-algebra homomorphism $R[x] \to S$ to mapping x to a. Let $U \subset \text{Spec}(S)$ be the open subset where the fibres have dimension $\leq d - 1$, see Morphisms, Lemma 29.28.4. By construction U contains all the generic points of $\text{Spec}(S)$. In particular we see that U contains all generic points of all the generic fibres of $\text{Spec}(S) \to \text{Spec}(R)$ as such points are necessarily generic points of $\text{Spec}(S)$. Set $T = \text{Spec}(S) \setminus U$ viewed as a reduced closed subscheme of $\text{Spec}(S)$. It follows from what we just said and the assumption that $\dim(S/R) \leq d$ that the generic fibres of $T \to \text{Spec}(R)$ have dimension $\leq d - 1$. Hence by Lemma 37.30.1, applied several times to produce open neighbourhoods of the generic points of $\text{Spec}(R)$, we can find a dense open $V \subset \text{Spec}(R)$ such that $T_V \to V$ has fibres of dimension $\leq d - 1$. We conclude that for $q \in \mathbf{A}^1_V$ the fibre of f_a over q has dimension $\leq d - 1$ (as we have bounded the dimension of the fibre of $f_a|_U$ and of the fibre of $f_a|_T$).

By prime avoidance, we may assume that $V = D(f)$ for some $f \in R$. Then we see that the ring map $R_f[x] \to S_f$, $x \mapsto a$ has fibres of dimension $\leq d - 1$. We may replace a by fa and assume $a \in (f)$. By induction on $\dim(R)$ we can find an element $\overline{b} \in S/fS$ such that the fibres of $\text{Spec}(S/fS) \to \text{Spec}(R/fR[x])$, $x \mapsto \overline{b}$ have dimension $\leq d - 1$. Let $b \in S$ be a lift of $\overline{b}$. By Lemma 37.31.2 there exists an $n > 0$ such that $a^n + b$ still works for $R_f \to S_f$. On the other hand, the image of $a^n + b$ in S/fS is $\overline{b}$ and the proof is complete. $\square$

37.32. Bertini theorems

0G4C We continue the discussion started in Varieties, Section 33.47. In this section we prove that general hyperplane sections of geometrically irreducible varieties are geometrically irreducible following the remarkable argument given in [Jou83].

0G4D Lemma 37.32.1. Let K/k be a geometrically irreducible and finitely generated field extension. Let $n \geq 1$. Let $g_1, \ldots, g_n \in K$ be elements such that there exist $c_1, \ldots, c_n \in k$ such that the elements See pages 71 and 72 of [Jou83]

$$x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i \in K(x_1, \ldots, x_n)$$

are algebraically independent over k. Then $K(x_1, \ldots, x_n)$ is geometrically irreducible over $k(x_1, \ldots, x_n, \sum g_i x_i)$.

Proof. Let $c_1, \ldots, c_n \in k$ be as in the statement of the lemma. Write $\xi = \sum g_i x_i$ and $\delta = \sum c_i g_i$. For $a \in k$ consider the automorphism σ_a of $K(x_1, \ldots, x_n)$ given by the identity on K and the rules

$$\sigma_a(x_i) = x_i + ac_i$$

Observe that $\sigma_a(\xi) = \xi + a\delta$ and $\sigma_a(\delta) = \delta$. Consider the tower of fields

$$K_0 = k(x_1, \ldots, x_n) \subset K_1 = K_0(\xi) \subset K_2 = K_0(\xi, \delta) \subset K(x_1, \ldots, x_n) = \Omega$$

Observe that $\sigma_a(K_0) = K_0$ and $\sigma_a(K_2) = K_2$. Let $\theta \in \Omega$ be separable algebraic over K_1. We have to show $\theta \in K_1$, see Algebra, Lemma 10.47.12.

Denote K_2' the separable algebraic closure of K_2 in Ω. Since K_2'/K_2 is finite (Algebra, Lemma 10.47.13) and separable there are only a finite number of fields in between K_2' and K_2 (Fields, Lemma 9.19.1). If k is infinite[5], then we can find distinct elements a_1, a_2 of k such that

$$K_2(\sigma_{a_1}(\theta)) = K_2(\sigma_{a_2}(\theta))$$

as subfields of Ω. Write $\theta_i = \sigma_{a_i}(\theta)$ and $\xi_i = \sigma_{a_i}(\xi) = \xi + a_i\delta$. Observe that

$$K_2 = K_0(\xi_1, \xi_2)$$

as we have $\xi_i = \xi + a_i\delta$, $\xi = (a_2\xi_1 - a_1\xi_2)/(a_2 - a_1)$, and $\delta = (\xi_1 - \xi_2)/(a_1 - a_2)$. Since K_2/K_0 is purely transcendental of degree 2 we conclude that ξ_1 and ξ_2 are algebraically indepedent over K_0. Since θ_1 is algebraic over $K_0(\xi_1)$ we conclude that ξ_2 is transcendental over $K_0(\xi_1, \theta_1)$.

By assumption K/k is geometrically irreducible. This implies that $K(x_1, \ldots, x_n)/K_0$ is geometrically irreducible (Algebra, Lemma 10.47.10). This in turn implies that $K_0(\xi_1, \theta_1)/K_0$ is geometrically irreducible as a subextension (Algebra, Lemma 10.47.6). Since ξ_2 is transcendental over $K_0(\xi_1, \theta_1)$ we conclude that $K_0(\xi_1, \xi_2, \theta_1)/K_0(\xi_2)$ is geometrically irreducible (Algebra, Lemma 10.47.11). By our choice of a_1, a_2 above we have

$$K_0(\xi_1, \xi_2, \theta_1) = K_2(\sigma_{a_1}(\theta)) = K_2(\sigma_{a_2}(\theta)) = K_0(\xi_1, \xi_2, \theta_2)$$

Since θ_2 is separably algebraic over $K_0(\xi_2)$ we conclude by Algebra, Lemma 10.47.12 again that $\theta_2 \in K_0(\xi_2)$. Taking $\sigma_{a_2}^{-1}$ of this relation givens $\theta \in K_0(\xi) = K_1$ as desired.

This finishes the proof in case k is infinite. If k is finite, then we can choose a variable t and consider the extension $K(t)/k(t)$ which is geometrically irreducible by Algebra, Lemma 10.47.10. Since it is still be true that $x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i$ in $K(t, x_1, \ldots, x_n)$ are algebraically independent over $k(t)$ we conclude that $K(t, x_1, \ldots, x_n)$ is geometrically irreducible over $k(t, x_1, \ldots, x_n, \sum g_i x_i)$ by the argument already given. Then using Algebra, Lemma 10.47.10 once more finishes the job. □

0G4E Lemma 37.32.2. Let A be a domain of finite type over a field k. Let $n \geq 2$. Let $g_1, \ldots, g_n \in A$ be elements such that $V(g_1, g_2)$ has an irreducible component of dimension $\dim(A) - 2$. Then there exist $c_1, \ldots, c_n \in k$ such that the elements

$$x_1, \ldots, x_n, \sum g_i x_i, \sum c_i g_i \in \mathrm{Frac}(A)(x_1, \ldots, x_n)$$

[5]We will deal with the finite field case in the last paragraph of the proof.

are algebraically independent over k.

Proof. The algebraic independence over k means that the morphism

$$T = \mathrm{Spec}(A[x_1, \ldots, x_n]) \longrightarrow \mathrm{Spec}(k[x_1, \ldots, x_n, y, z]) = S$$

given by $y = \sum g_i x_i$ and $z = \sum c_i g_i$ is dominant. Set $d = \dim(A)$. If $T \to S$ is not dominant, then the image has dimension $< n + 2$ and hence every irreducible component of every fibre has dimension $> d + n - (n + 2) = d - 2$, see Varieties, Lemma 33.20.4. Choose a closed point $u \in V(g_1, g_2)$ contained in an irreducible component of dimension $d-2$ and in no other component of $V(g_1, g_2)$. Consider the closed point $t = (u, 1, 0, \ldots 0)$ of T lying over u. Set $(c_1, \ldots, c_n) = (0, 1, 0, \ldots, 0)$. Then t maps to the point $s = (1, 0, \ldots, 0)$ of S. The fibre of $T \to S$ over s is cut out by

$$x_1 - 1, x_2, \ldots, x_n, \sum x_i g_i, g_2$$

and hence equivalently is cut out by

$$x_1 - 1, x_2, \ldots, x_n, g_1, g_2$$

By our condition on g_1, g_2 this subscheme has an irreducible component of dimension $d - 2$. □

0G4F Lemma 37.32.3. In Varieties, Situation 33.47.2 assume [Jou83, Theorem 6.3 part 4)]

1. X is of finite type over k,
2. X is geometrically irreducible over k,
3. there exist $v_1, v_2, v_3 \in V$ and an irreducible component Z of $H_{v_2} \cap H_{v_3}$ such that $Z \not\subset H_{v_1}$ and $\mathrm{codim}(Z, X) = 2$, and
4. every irreducible component Y of $\bigcap_{v \in V} H_v$ has $\mathrm{codim}(Y, X) \geq 2$.

Then for general $v \in V \otimes_k k'$ the scheme H_v is geometrically irreducible over k'.

Proof. In order for assumption (3) to hold, the elements v_1, v_2, v_3 must be k-linearly independent in V (small detail omitted). Thus we may choose a basis $v_1, \ldots, v_r$ of V incorporating these elements as the first 3. Recall that $H_{univ} \subset \mathbf{A}^r_k \times_k X$ is the "universal divisor". Consider the projection $q : H_{univ} \to \mathbf{A}^r_k$ whose scheme theoretic fibres are the divisors H_v. By Lemma 37.27.5 it suffices to show that the generic fibre of q is geometrically irreducible. To prove this we may replace X by its reduction, hence we may assume X is an integral scheme of finite type over k.

Let $U \subset X$ be a nonempty affine open such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Write $U = \mathrm{Spec}(A)$. Denote $f_i \in A$ the element corresponding to section $\psi(v_i)|_U$ via the isomorphism $\mathcal{L}|_U \cong \mathcal{O}_U$. Then $H_{univ} \cap (\mathbf{A}^r_k \times_k U)$ is given by

$$H_U = \mathrm{Spec}(A[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r))$$

By our choice of basis we see that f_1 cannot be zero because this would mean $v_1 = 0$ and hence $H_{v_1} = X$ which contradicts assumption (3). Hence $\sum x_i f_i$ is a nonzerodivisor in $A[x_1, \ldots, x_r]$. It follows that every irreducible component of H_U has dimension $d + r - 1$ where $d = \dim(X) = \dim(A)$. If $U' = U \cap D(f_1)$ then we see that

$$H_{U'} = \mathrm{Spec}(A_{f_1}[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r)) \cong \mathrm{Spec}(A_{f_1}[x_2, \ldots x_r]) = \mathbf{A}^{r-1}_k \times_k U'$$

is irreducible. On the other hand, we have

$$H_U \setminus H_{U'} = \mathrm{Spec}(A/(f_1)[x_1, \ldots, x_r]/(x_2 f_2 + \ldots + x_r f_r))$$

which has dimension at most $d+r-2$. Namely, for $i \neq 1$ the scheme $(H_U \setminus H_{U'}) \times_U D(f_i)$ is either empty (if $f_i = 0$) or by the same argument as above isomorphic to an $r-1$ dimensional affine space over an open of $\text{Spec}(A/(f_1))$ and hence has dimension at most $d+r-2$. On the other hand, $(H_U \setminus H_{U'}) \times_U V(f_2, \ldots, f_r)$ is an r dimensional affine space over $\text{Spec}(A/(f_1, \ldots, f_r))$ and hence assumption (4) tells us this has dimension at most $d+r-2$. We conclude that H_U is irreducible for every U as above. It follows that H_{univ} is irreducible.

Thus it suffices to show that the generic point of H_{univ} is geometrically irreducible over the generic point of $\mathbf{A}^r_k$, see Varieties, Lemma 33.8.6. Choose a nonempty affine open $U = \text{Spec}(A)$ of X contained in $X \setminus H_{v_1}$ which meets the irreducible component Z of $H_{v_2} \cap H_{v_3}$ whose existence is asserted in assumption (3). With notation as above we have to prove that the field extension

$$\text{Frac}(A[x_1, \ldots, x_r]/(x_1 f_1 + \ldots + x_r f_r))/k(x_1, \ldots, x_r)$$

is geometrically irreducible. Observe that f_1 is invertible in A by our choice of U. Set $K = \text{Frac}(A)$ equal to the fraction field of A. Eliminating the variable x_1 as above, we find that we have to show that the field extension

$$K(x_2, \ldots, x_r)/k(x_2, \ldots, x_r, -\sum\nolimits_{i=2,\ldots,r} f_1^{-1} f_i x_i)$$

is geometrically irreducible. By Lemma 37.32.1 it suffices to show that for some $c_2, \ldots, c_r \in k$ the elements

$$x_2, \ldots, x_r, \sum\nolimits_{i=2,\ldots,r} f_1^{-1} f_i x_i, \sum\nolimits_{i=2,\ldots,r} c_i f_1^{-1} f_i$$

are algebraically independent over k in the fraction field of $A[x_2, \ldots, x_r]$. This follows from Lemma 37.32.2 and the fact that $Z \cap U$ is an irreducible component of $V(f_1^{-1} f_2, f_1^{-1} f_3) \subset U$. $\square$

0G4G Remark 37.32.4. Let us sketch a "geometric" proof of a special case of Lemma 37.32.3. Namely, say k is an algebraically closed field and $X \subset \mathbf{P}^n_k$ is smooth and irreducible of dimension ≥ 2. Then we claim there is a hyperplane $H \subset \mathbf{P}^n_k$ such that $X \cap H$ is smooth and irreducible. Namely, by Varieties, Lemma 33.47.3 for a general $v \in V = kT_0 \oplus \ldots \oplus kT_n$ the corresponding hyperplane section $X \cap H_v$ is smooth. On the other hand, by Enriques-Severi-Zariski the scheme $X \cap H_v$ is connected, see Varieties, Lemma 33.48.3. Hence $X \cap H_v$ is smooth and irreducible.

37.33. Theorem of the cube

0BEZ The following lemma tells us that the diagonal of the Picard functor is representable by locally closed immersions under the assumptions made in the lemma.

0BDP Lemma 37.33.1. Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. For a morphism $g : T \to S$ consider the base change diagram

$$\begin{tikzcd} X_T \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

Assume $\mathcal{O}_T \to p_*\mathcal{O}_{X_T}$ is an isomorphism for all $g : T \to S$. Then there exists an immersion $j : Z \to S$ of finite presentation such that a morphism $g : T \to S$

factors through Z if and only if there exists a finite locally free $\mathcal{O}_T$-module $\mathcal{N}$ with $p^*\mathcal{N} \cong q^*\mathcal{E}$.

Proof. Observe that the fibres X_s of f are connected by our assumption that $H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. Thus the rank of $\mathcal{E}$ is constant on the fibres. Since f is open (Morphisms, Lemma 29.25.10) and closed we conclude that there is a decomposition $S = \coprod S_r$ of S into open and closed subschemes such that $\mathcal{E}$ has constant rank r on the inverse image of S_r. Thus we may assume $\mathcal{E}$ has constant rank r. We will denote $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ the dual rank r module.

By cohomology and base change (more precisely by Derived Categories of Schemes, Lemma 36.30.4) we see that $E = Rf_*\mathcal{E}$ is a perfect object of the derived category of S and that its formation commutes with arbitrary change of base. Similarly for $E' = Rf_*\mathcal{E}^\vee$. Since there is never any cohomology in degrees < 0, we see that E and E' have (locally) tor-amplitude in $[0, b]$ for some b. Observe that for any $g : T \to S$ we have $p_*(q^*\mathcal{E}) = H^0(Lg^*E)$ and $p_*(q^*\mathcal{E}^\vee) = H^0(Lg^*E')$. Let $j : Z \to S$ and $j' : Z' \to S$ be immersions of finite presentation constructed in Derived Categories of Schemes, Lemma 36.31.4 for E and E' with $a = 0$ and $r = r$; these are roughly speaking characterized by the property that $H^0(Lj^*E)$ and $H^0((j')^*E')$ are finite locally free modules compatible with pullback.

Let $g : T \to S$ be a morphism. If there exists an $\mathcal{N}$ as in the lemma, then, using the projection formula Cohomology, Lemma 20.54.2, we see that the modules

$$p_*(q^*\mathcal{L}) \cong p_*(p^*\mathcal{N}) \cong \mathcal{N} \otimes_{\mathcal{O}_T} p_*\mathcal{O}_{X_T} \cong \mathcal{N} \quad \text{and similarly} \quad p_*(q^*\mathcal{E}^\vee) \cong \mathcal{N}^\vee$$

are finite locally free modules of rank r and remain so after any further base change $T' \to T$. Hence in this case $T \to S$ factors through j and through j'. Thus we may replace S by $Z \times_S Z'$ and assume that $f_*\mathcal{E}$ and $f_*\mathcal{E}^\vee$ are finite locally free $\mathcal{O}_S$-modules of rank r whose formation commutes with arbitrary change of base (small detail omitted).

In this sitation if $g : T \to S$ be a morphism and there exists an $\mathcal{N}$ as in the lemma, then the map (cup product in degree 0)

$$p_*(q^*\mathcal{E}) \otimes_{\mathcal{O}_T} p_*(q^*\mathcal{E}^\vee) \longrightarrow \mathcal{O}_T$$

is a perfect pairing. Conversely, if this cup product map is a perfect pairing, then we see that locally on T we may choose a basis of sections $\sigma_1, \ldots, \sigma_r$ in $p_*(q^*\mathcal{E})$ and $\tau_1, \ldots, \tau_r$ in $p_*(q^*\mathcal{E}^\vee)$ whose products satisfy $\sigma_i\tau_j = \delta_{ij}$. Thinking of σ_i as a section of $q^*\mathcal{E}$ on X_T and τ_j as a section of $q^*\mathcal{E}^\vee$ on X_T, we conclude that

$$\sigma_1, \ldots, \sigma_r : \mathcal{O}_{X_T}^{\oplus r} \longrightarrow q^*\mathcal{E}$$

is an isomorphism with inverse given by

$$\tau_1, \ldots, \tau_r : q^*\mathcal{E} \longrightarrow \mathcal{O}_{X_T}^{\oplus r}$$

In other words, we see that $p^*p_*q^*\mathcal{E} \cong q^*\mathcal{E}$. But the condition that the cupproduct is nondegenerate picks out a retrocompact open subscheme (namely, the locus where a suitable determinant is nonzero) and the proof is complete. $\square$

The lemma above in particular tells us, that if a vector bundle is trivial on fibres for a proper flat family of proper spaces, then it is the pull back of a vector bundle. Let's spell this out a bit.

0EX7 Lemma 37.33.2. Let $f : X \to S$ be a flat, proper morphism of finite presentation such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and this remains true after arbitrary base change. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Assume

(1) $\mathcal{E}|_{X_s}$ is isomorphic to $\mathcal{O}_{X_s}^{\oplus r_s}$ for all $s \in S$, and
(2) S is reduced.

Then $\mathcal{E} = f^*\mathcal{N}$ for some finite locally free $\mathcal{O}_S$-module $\mathcal{N}$.

Proof. Namely, in this case the locally closed immersion $j : Z \to S$ of Lemma 37.33.1 is bijective and hence a closed immersion. But since S is reduced, j is an isomorphism. □

0EX8 Lemma 37.33.3. Let $f : X \to S$ be a proper flat morphism of finite presentation. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume

(1) S is the spectrum of a valuation ring,
(2) $\mathcal{L}$ is trivial on the generic fibre X_η of f,
(3) the closed fibre X_0 of f is integral,
(4) $H^0(X_\eta, \mathcal{O}_{X_\eta})$ is equal to the function field of S.

Then $\mathcal{L}$ is trivial.

Proof. Write $S = \mathrm{Spec}(A)$. We will first prove the lemma when A is a discrete valuation ring (as this is the case most often used in practice). Let $\pi \in A$ be a uniformizer. Take a trivializing section $s \in \Gamma(X_\eta, \mathcal{L}_\eta)$. After replacing s by $\pi^n s$ if necessary we can assume that $s \in \Gamma(X, \mathcal{L})$. If $s|_{X_0} = 0$, then we see that s is divisible by π (because X_0 is the scheme theoretic fibre and X is flat over A). Thus we may assume that $s|_{X_0}$ is nonzero. Then the zero locus $Z(s)$ of s is contained in X_0 but does not contain the generic point of X_0 (because X_0 is integral). This means that the $Z(s)$ has codimension ≥ 2 in X which contradicts Divisors, Lemma 31.15.3 unless $Z(s) = \emptyset$ as desired.

Proof in the general case. Since the valuation ring A is coherent (Algebra, Example 10.90.2) we see that $H^0(X, \mathcal{L})$ is a coherent A-module, see Derived Categories of Schemes, Lemma 36.33.1. Equivalently, $H^0(X, \mathcal{L})$ is a finitely presented A-module (Algebra, Lemma 10.90.4). Since $H^0(X, \mathcal{L})$ is torsion free (by flatness of X over A), we see from More on Algebra, Lemma 15.124.3 that $H^0(X, \mathcal{L}) = A^{\oplus n}$ for some n. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we have

$$K = H^0(X_\eta, \mathcal{O}_{X_\eta}) \cong H^0(X_\eta, \mathcal{L}_\eta) = H^0(X, \mathcal{L}) \otimes_A K$$

where K is the fraction field of A. Thus $n = 1$. Pick a generator $s \in H^0(X, \mathcal{L})$. Let $\mathfrak{m} \subset A$ be the maximal ideal. Then $\kappa = A/\mathfrak{m} = \mathrm{colim}\, A/\pi$ where this is a filtered colimit over nonzero $\pi \in \mathfrak{m}$ (here we use that A is a valuation ring). Thus $X_0 = \lim X \times_S \mathrm{Spec}(A/\pi)$. If $s|_{X_0}$ is zero, then for some π we see that s restricts to zero on $X \times_S \mathrm{Spec}(A/\pi)$, see Limits, Lemma 32.4.7. But if this happens, then $\pi^{-1}s$ is a global section of $\mathcal{L}$ which contradicts the fact that s is a generator of $H^0(X, \mathcal{L})$. Thus $s|_{X_0}$ is not zero. Let $Z(s) \subset X$ be the zero scheme of s. Since $s|_{X_0}$ is not zero and since X_0 is integral, we see that $Z(s)_0 \subset X_0$ is an effective Cartier divisor. Since f is proper and S is local, every point of $Z(s)$ specializes to a point of $Z(s)_0$. Thus by Divisors, Lemma 31.18.9 part (3) we see that $Z(s)$ is a relative effective Cartier divisor, in particular $Z(s) \to S$ is flat. Hence if $Z(s)$ were nonemtpy, then $Z(s)_\eta$ would be nonempty which contradicts the fact that $s|_{X_\eta}$ is a trivialization of $\mathcal{L}_\eta$. Thus $Z(s) = \emptyset$ as desired. □

0BF0 **Lemma 37.33.4.** Let $f : X \to S$ and $\mathcal{E}$ be as in Lemma 37.33.1 and in addition assume $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module. If moreover the geometric fibres of f are integral, then Z is closed in S.

Proof. Since $j : Z \to S$ is of finite presentation, it suffices to show: for any morphism $g : \mathrm{Spec}(A) \to S$ where A is a valuation ring with fraction field K such that $g(\mathrm{Spec}(K)) \in j(Z)$ we have $g(\mathrm{Spec}(A)) \subset j(Z)$. See Morphisms, Lemma 29.6.5. This follows from Lemma 37.33.3 and the characterization of $j : Z \to S$ in Lemma 37.33.1. $\square$

0BF1 **Lemma 37.33.5.** Consider a commutative diagram of schemes

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

with $f' : X' \to S$ and $f : X \to S$ satisfying the hypotheses of Lemma 37.33.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathcal{L}'$ be the pullback to X'. Let $Z \subset S$, resp. $Z' \subset S$ be the locally closed subscheme constructed in Lemma 37.33.1 for $(f, \mathcal{L})$, resp. $(f', \mathcal{L}')$ so that $Z \subset Z'$. If $s \in Z$ and

$$H^1(X_s, \mathcal{O}) \longrightarrow H^1(X'_s, \mathcal{O})$$

is injective, then $Z \cap U = Z' \cap U$ for some open neighbourhood U of s.

Proof. We may replace S by Z'. After shrinking S to an affine open neighbourhood of s we may assume that $\mathcal{L}' = \mathcal{O}_{X'}$. Let $E = Rf_*\mathcal{L}$ and $E' = Rf'_*\mathcal{L}' = Rf'_*\mathcal{O}_{X'}$. These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 36.30.4). In particular we see that

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})$$

The second equality because $s \in Z$. Set $h_i = \dim_{\kappa(s)} H^i(X_s, \mathcal{O}_{X_s})$. After shrinking S we can represent E by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_1} \to \mathcal{O}_S^{\oplus h_2} \to \ldots$$

see More on Algebra, Lemma 15.75.6 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.7). Similarly, we may assume E' is represented by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h'_1} \to \mathcal{O}_S^{\oplus h'_2} \to \ldots$$

where $h'_i = \dim_{\kappa(s)} H^i(X'_s, \mathcal{O}_{X'_s})$. By functoriality of cohomology we have a map

$$E \longrightarrow E'$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing E is a finite complex of finite free modules and since S is affine, we can choose a map of complexes

$$\begin{tikzcd} \mathcal{O}_S \arrow[r, "d"'] \arrow[d, "a"'] & \mathcal{O}_S^{\oplus h_1} \arrow[r] \arrow[d, "b"'] & \mathcal{O}_S^{\oplus h_2} \arrow[r] \arrow[d, "c"'] & \ldots \\ \mathcal{O}_S \arrow[r, "d'"] & \mathcal{O}_S^{\oplus h'_1} \arrow[r] & \mathcal{O}_S^{\oplus h'_2} \arrow[r] & \ldots \end{tikzcd}$$

representing the given map $E \to E'$. Since $s \in Z$ we see that the trivializing section of $\mathcal{L}_s$ pulls back to a trivializing section of $\mathcal{L}'_s = \mathcal{O}_{X'_s}$. Thus $a \otimes \kappa(s)$ is an isomorphism, hence after shrinking S we see that a is an isomorphism. Finally, we use the hypothesis that $H^1(X_s, \mathcal{O}) \to H^1(X'_s, \mathcal{O})$ is injective, to see that there exists a $h_1 \times h_1$ minor of the matrix defining b which maps to a nonzero element in $\kappa(s)$. Hence after shrinking S we may assume that b is injective. However, since $\mathcal{L}' = \mathcal{O}_{X'}$ we see that $d' = 0$. It follows that $d = 0$. In this way we see that the trivializing section of $\mathcal{L}_s$ lifts to a section of $\mathcal{L}$ over X. A straightforward topological argument (omitted) shows that this means that $\mathcal{L}$ is trivial after possibly shrinking S a bit further. □

0BF2 Lemma 37.33.6. Consider n commutative diagrams of schemes

$$\begin{tikzcd} X_i \arrow[rr] \arrow[dr, "f_i"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

with $f_i : X_i \to S$ and $f : X \to S$ satisfying the hypotheses of Lemma 37.33.1. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathcal{L}_i$ be the pullback to X_i. Let $Z \subset S$, resp. $Z_i \subset S$ be the locally closed subscheme constructed in Lemma 37.33.1 for $(f, \mathcal{L})$, resp. $(f_i, \mathcal{L}_i)$ so that $Z \subset \bigcap_{i=1,\ldots,n} Z_i$. If $s \in Z$ and

$$H^1(X_s, \mathcal{O}) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} H^1(X_{i,s}, \mathcal{O})$$

is injective, then $Z \cap U = (\bigcap_{i=1,\ldots,n} Z_i) \cap U$ (scheme theoretic intersection) for some open neighbourhood U of s.

Proof. This lemma is a variant of Lemma 37.33.5 and we strongly urge the reader to read that proof first; this proof is basically a copy of that proof with minor modifications. It follows from the description of (scheme valued) points of Z and the Z_i that $Z \subset \bigcap_{i=1,\ldots,n} Z_i$ where we take the scheme theoretic intersection. Thus we may replace S by the scheme theoretic intersection $\bigcap_{i=1,\ldots,n} Z_i$. After shrinking S to an affine open neighbourhood of s we may assume that $\mathcal{L}_i = \mathcal{O}_{X_i}$ for $i = 1, \ldots, n$. Let $E = Rf_*\mathcal{L}$ and $E_i = Rf_{i,*}\mathcal{L}_i = Rf_{i,*}\mathcal{O}_{X_i}$. These are perfect complexes whose formation commutes with arbitrary change of base (Derived Categories of Schemes, Lemma 36.30.4). In particular we see that

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_S} \kappa(s) = R\Gamma(X_s, \mathcal{L}_s) = R\Gamma(X_s, \mathcal{O}_{X_s})$$

The second equality because $s \in Z$. Set $h_j = \dim_{\kappa(s)} H^j(X_s, \mathcal{O}_{X_s})$. After shrinking S we can represent E by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_1} \to \mathcal{O}_S^{\oplus h_2} \to \ldots$$

see More on Algebra, Lemma 15.75.6 (strictly speaking this also uses Derived Categories of Schemes, Lemmas 36.3.5 and 36.10.7). Similarly, we may assume E_i is represented by a complex

$$\mathcal{O}_S \to \mathcal{O}_S^{\oplus h_{i,1}} \to \mathcal{O}_S^{\oplus h_{i,2}} \to \ldots$$

where $h_{i,j} = \dim_{\kappa(s)} H^j(X_{i,s}, \mathcal{O}_{X_{i,s}})$. By functoriality of cohomology we have a map

$$E \longrightarrow E_i$$

in $D(\mathcal{O}_S)$ whose formation commutes with change of base. Since the complex representing E is a finite complex of finite free modules and since S is affine, we can choose a map of complexes

$$\begin{tikzcd} \mathcal{O}_S \arrow[r, "d"'] \arrow[d, "a_i"'] & \mathcal{O}_S^{\oplus h_1} \arrow[r] \arrow[d, "b_i"'] & \mathcal{O}_S^{\oplus h_2} \arrow[r] \arrow[d, "c_i"'] & \ldots \\ \mathcal{O}_S \arrow[r, "d_i"] & \mathcal{O}_S^{\oplus h_{i,1}} \arrow[r] & \mathcal{O}_S^{\oplus h_{i,2}} \arrow[r] & \ldots \end{tikzcd}$$

representing the given map $E \to E_i$. Since $s \in Z$ we see that the trivializing section of $\mathcal{L}_s$ pulls back to a trivializing section of $\mathcal{L}_{i,s} = \mathcal{O}_{X_{i,s}}$. Thus $a_i \otimes \kappa(s)$ is an isomorphism, hence after shrinking S we see that a_i is an isomorphism. Finally, we use the hypothesis that $H^1(X_s, \mathcal{O}) \to \bigoplus_{i=1,\ldots,n} H^1(X_{i,s}, \mathcal{O})$ is injective, to see that there exists a $h_1 \times h_1$ minor of the matrix defining $\oplus b_i$ which maps to a nonzero element in $\kappa(s)$. Hence after shrinking S we may assume that $(b_1, \ldots, b_n) : \mathcal{O}_S^{h_1} \to \bigoplus_{i=1,\ldots,n} \mathcal{O}_S^{h_{i,1}}$ is injective. However, since $\mathcal{L}_i = \mathcal{O}_{X_i}$ we see that $d_i = 0$ for $i = 1, \ldots n$. It follows that $d = 0$ because $(b_1, \ldots, b_n) \circ d = (\oplus d_i) \circ (a_1, \ldots, a_n)$. In this way we see that the trivializing section of $\mathcal{L}_s$ lifts to a section of $\mathcal{L}$ over X. A straightforward topological argument (omitted) shows that this means that $\mathcal{L}$ is trivial after possibly shrinking S a bit further. □

0BF3 Lemma 37.33.7. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes satisfying the hypotheses of Lemma 37.33.1. Let $\sigma : S \to X$ and $\tau : S \to Y$ be sections of f and g. Let $s \in S$. Let $\mathcal{L}$ be an invertible sheaf on $X \times_S Y$. If $(1 \times \tau)^*\mathcal{L}$ on X, $(\sigma \times 1)^*\mathcal{L}$ on Y, and $\mathcal{L}|_{(X \times_S Y)_s}$ are trivial, then there is an open neighbourhood U of s such that $\mathcal{L}$ is trivial over $(X \times_S Y)_U$.

Proof. By Künneth (Varieties, Lemma 33.29.1) the map

$$H^1(X_s \times_{\text{Spec}(\kappa(s))} Y_s, \mathcal{O}) \to H^1(X_s, \mathcal{O}) \oplus H^1(Y_s, \mathcal{O})$$

is injective. Thus we may apply Lemma 37.33.6 to the two morphisms

$$1 \times \tau : X \to X \times_S Y \quad \text{and} \quad \sigma \times 1 : Y \to X \times_S Y$$

to conclude. □

0BF4 Theorem 37.33.8 (Theorem of the cube). Let S be a scheme. Let X, Y, and Z be schemes over S. Let $x : S \to X$ and $y : S \to Y$ be sections of the structure morphisms. Let $\mathcal{L}$ be an invertible module on $X \times_S Y \times_S Z$. If

(1) $X \to S$ and $Y \to S$ are flat, proper morphisms of finite presentation with geometrically integral fibres,
(2) the pullbacks of $\mathcal{L}$ by $x \times \text{id}_Y \times \text{id}_Z$ and $\text{id}_X \times y \times \text{id}_Z$ are trivial over $Y \times_S Z$ and $X \times_S Z$,
(3) there is a point $z \in Z$ such that $\mathcal{L}$ restricted to $X \times_S Y \times_S z$ is trivial, and
(4) Z is connected,

then $\mathcal{L}$ is trivial.

An often used special case is the following. Let k be a field. Let X, Y, Z be varieties with k-rational points x, y, z. Let $\mathcal{L}$ be an invertible module on $X \times Y \times Z$. If

(1) $\mathcal{L}$ is trivial over $x \times Y \times Z$, $X \times y \times Z$, and $X \times Y \times z$, and
(2) X and Y are geometrically integral and proper over k,

then $\mathcal{L}$ is trivial.

Proof. Observe that the morphism $X \times_S Y \to S$ is a flat, proper morphism of finite presentation whose geometrically integral fibres (see Varieties, Lemmas 33.9.2, 33.8.4, and 33.6.7 for the statement about the fibres). By Derived Categories of Schemes, Lemma 36.32.6 we see that the pushforward of the structure sheaf by $X \to S$, $Y \to S$, or $X \times_S Y \to S$ is the structure sheaf of S and the same remains true after any base change. Thus we may apply Lemma 37.33.1 to the morphism

$$p : X \times_S Y \times_S Z \longrightarrow Z$$

and the invertible module $\mathcal{L}$ to get a "universal" locally closed subscheme $Z' \subset Z$ such that $\mathcal{L}|_{X \times_S Y \times_S Z'}$ is the pullback of an invertible module $\mathcal{N}$ on Z'. The existence of z shows that Z' is nonempty. By Lemma 37.33.4 we see that $Z' \subset Z$ is a closed subscheme. Let $z' \in Z'$ be a point. Observe that we may write p as the product morphism

$$(X \times_S Z) \times_Z (Y \times_S Z) \longrightarrow Z$$

Hence we may apply Lemma 37.33.7 to the morphism p, the point z', and the sections $\sigma : Z \to X \times_S Z$ and $\tau : Z \to Y \times_S Z$ given by x and y. We conclude that Z' is open. Hence $Z' = Z$ and $\mathcal{L} = p^*\mathcal{N}$ for some invertible module $\mathcal{N}$ on Z. Pulling back via $x \times y \times \mathrm{id}_Z : Z \to X \times_S Y \times_S Z$ we obtain on the one hand $\mathcal{N}$ and on the other hand we obtain the trivial invertible module by assumption (2). Thus $\mathcal{N} = \mathcal{O}_Z$ and the proof is complete. □

37.34. Limit arguments

05FA Some lemmas involving limits of schemes, and Noetherian approximation. We stick mostly to the affine case. Some of these lemmas are special cases of lemmas in the chapter on limits.

05FB Lemma 37.34.1. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Then there exists a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \longleftarrow & S \end{array}$$

such that

(1) X_0, S_0 are affine schemes,
(2) S_0 of finite type over $\mathbf{Z}$,
(3) f_0 is of finite type.

Proof. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 29.21.2. Thus the lemma follows from Algebra, Lemma 10.127.18. □

05FC Lemma 37.34.2. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then there exists a diagram as in Lemma 37.34.1 such that there exists a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ with $g^*\mathcal{F}_0 = \mathcal{F}$.

Proof. Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 29.21.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 28.16.2. Thus the lemma follows from Algebra, Lemma 10.127.18. $\square$

05FD Lemma 37.34.3. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation and flat over S. Then we may choose a diagram as in Lemma 37.34.2 and sheaf $\mathcal{F}_0$ such that in addition $\mathcal{F}_0$ is flat over S_0.

Proof. Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 29.21.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 28.16.2. As $\mathcal{F}$ is flat over S we see that M is flat over A, see Morphisms, Lemma 29.25.2. Thus the lemma follows from Algebra, Lemma 10.168.1. $\square$

05FE Lemma 37.34.4. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation and flat. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 is flat.

Proof. This is a special case of Lemma 37.34.3. $\square$

05FF Lemma 37.34.5. Let $f : X \to S$ be a morphism of affine schemes, which is smooth. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 is smooth.

Proof. Write $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and as f is smooth we see that B is smooth as an A-algebra, see Morphisms, Lemma 29.34.2. Hence the lemma follows from Algebra, Lemma 10.138.14. $\square$

05FG Lemma 37.34.6. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically reduced fibres. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 has geometrically reduced fibres.

Proof. Apply Lemma 37.34.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 37.26.5 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically reduced is a constructible subset. By Lemma 37.26.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 32.4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\text{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of f_0 are geometrically reduced. $\square$

05FH Lemma 37.34.7. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically irreducible fibres. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 has geometrically irreducible fibres.

Proof. Apply Lemma 37.34.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 37.27.7 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically irreducible is a constructible subset. By Lemma 37.27.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 32.4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\text{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of f_0 are geometrically irreducible. $\square$

05FI **Lemma 37.34.8.** *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically connected fibres. Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 has geometrically connected fibres.*

Proof. Apply Lemma 37.34.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 37.28.6 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically connected is a constructible subset. By Lemma 37.28.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 32.4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\text{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of f_0 are geometrically connected. $\square$

05FJ **Lemma 37.34.9.** *Let $d \geq 0$ be an integer. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation all of whose fibres have dimension d. Then there exists a diagram as in Lemma 37.34.1 such that in addition all fibres of f_0 have dimension d.*

Proof. Apply Lemma 37.34.1 to get a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l, "g"'] \arrow[d, "f"] \\ S_0 & S \arrow[l, "h"] \end{tikzcd}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 37.30.3 the set $E \subset S_0$ of points where the fibre of f_0 has dimension d is a constructible subset. By Lemma 37.30.2 we have $h(S) \subset E$. Write $S_0 = \text{Spec}(A_0)$ and $S = \text{Spec}(A)$. Write $A = \text{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 32.4.10 we see that $\text{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\text{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \text{Spec}(A_i)$ we see that all fibres of f_0 have dimension d. $\square$

05FK **Lemma 37.34.10.** Let $f : X \to S$ be a morphism of affine schemes, which is standard syntomic (see Morphisms, Definition 29.30.1). Then there exists a diagram as in Lemma 37.34.1 such that in addition f_0 is standard syntomic.

Proof. This lemma is a copy of Algebra, Lemma 10.136.11. $\square$

05FL **Lemma 37.34.11.** (Noetherian approximation and combining properties.) Let P, Q be properties of morphisms of schemes which are stable under base change. Let $f : X \to S$ be a morphism of finite presentation of affine schemes. Assume we can find cartesian diagrams

$$\begin{tikzcd} X_1 \arrow[d, "f_1"'] & X \arrow[l] \arrow[d, "f"] \\ S_1 & S \arrow[l] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2 \arrow[d, "f_2"'] & X \arrow[l] \arrow[d, "f"] \\ S_2 & S \arrow[l] \end{tikzcd}$$

of affine schemes, with S_1, S_2 of finite type over $\mathbf{Z}$ and f_1, f_2 of finite type such that f_1 has property P and f_2 has property Q. Then we can find a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X \arrow[l] \arrow[d, "f"] \\ S_0 & S \arrow[l] \end{tikzcd}$$

of affine schemes with S_0 of finite type over $\mathbf{Z}$ and f_0 of finite type such that f_0 has both property P and property Q.

Proof. The given pair of diagrams correspond to cocartesian diagrams of rings

$$\begin{tikzcd} B_1 \arrow[r] & B \\ A_1 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_2 \arrow[r] & B \\ A_2 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

Let $A_0 \subset A$ be a finite type $\mathbf{Z}$-subalgebra of A containing the image of both $A_1 \to A$ and $A_2 \to A$. Such a subalgebra exists because by assumption both A_1 and A_2 are of finite type over $\mathbf{Z}$. Note that the rings $B_{0,1} = B_1 \otimes_{A_1} A_0$ and $B_{0,2} = B_2 \otimes_{A_2} A_0$ are finite type A_0-algebras with the property that $B_{0,1} \otimes_{A_0} A \cong B \cong B_{0,2} \otimes_{A_0} A$ as A-algebras. As A is the directed colimit of its finite type A_0-subalgebras, by Limits, Lemma 32.10.1 we may assume after enlarging A_0 that there exists an isomorphism $B_{0,1} \cong B_{0,2}$ as A_0-algebras. Since properties P and Q are assumed stable under base change we conclude that setting $S_0 = \mathrm{Spec}(A_0)$ and

$$X_0 = X_1 \times_{S_1} S_0 = \mathrm{Spec}(B_{0,1}) \cong \mathrm{Spec}(B_{0,2}) = X_2 \times_{S_2} S_0$$

works. $\square$

37.35. Étale neighbourhoods

02LD It turns out that some properties of morphisms are easier to study after doing an étale base change. It is convenient to introduce the following terminology.

02LE **Definition 37.35.1.** Let S be a scheme. Let $s \in S$ be a point.

(1) An *étale neighbourhood* of (S, s) is a pair (U, u) together with an étale morphism of schemes $\varphi : U \to S$ such that $\varphi(u) = s$.

(2) A morphism of étale neighbourhoods $f : (V, v) \to (U, u)$ of (S, s) is simply a morphism of S-schemes $f : V \to U$ such that $f(v) = u$.
(3) An elementary étale neighbourhood is an étale neighbourhood $\varphi : (U, u) \to (S, s)$ such that $\kappa(s) = \kappa(u)$.

The notion of an elementary étale neighbourhood has many different names in the literature, for example these are sometimes called "étale neighbourhoods" ([Mil80, Page 36] or "strongly étale" ([KPR75, Page 108]). Here we follow the convention of the paper [GR71] by calling them elementary étale neighbourhoods.

If $f : (V, v) \to (U, u)$ is a morphism of étale neighbourhoods, then f is automatically étale, see Morphisms, Lemma 29.36.18. Hence it turns (V, v) into an étale neighbourhood of (U, u). Of course, since the composition of étale morphisms is étale (Morphisms, Lemma 29.36.3) we see that conversely any étale neighbourhood (V, v) of (U, u) is an étale neighbourhood of (S, s) as well. We also remark that if $U \subset S$ is an open neighbourhood of s, then $(U, s) \to (S, s)$ is an étale neighbourhood. This follows from the fact that an open immersion is étale (Morphisms, Lemma 29.36.9). We will use these remarks without further mention throughout this section.

Note that $\kappa(u)/\kappa(s)$ is a finite separable extension if $(U, u) \to (S, s)$ is an étale neighbourhood, see Morphisms, Lemma 29.36.15.

02LF Lemma 37.35.2. Let S be a scheme. Let $s \in S$. Let $k/\kappa(s)$ be a finite separable field extension. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ such that the field extension $\kappa(u)/\kappa(s)$ is isomorphic to $k/\kappa(s)$.

Proof. We may assume S is affine. In this case the lemma follows from Algebra, Lemma 10.144.3. $\square$

057A Lemma 37.35.3. Let S be a scheme, and let s be a point of S. The category of étale neighborhoods has the following properties:

(1) Let $(U_i, u_i)_{i=1,2}$ be two étale neighborhoods of s in S. Then there exists a third étale neighborhood (U, u) and morphisms $(U, u) \to (U_i, u_i)$, $i = 1, 2$.
(2) Let $h_1, h_2 : (U, u) \to (U', u')$ be two morphisms between étale neighborhoods of s. Assume h_1, h_2 induce the same map $\kappa(u') \to \kappa(u)$ of residue fields. Then there exist an étale neighborhood (U'', u'') and a morphism $h : (U'', u'') \to (U, u)$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Morphisms, Lemma 29.36.4. There is a point of U mapping to both u_1 and u_2 for example by the description of points of a fibre product in Schemes, Lemma 26.17.5. For part (2), define U'' as the fibre product

$$\begin{tikzcd} U'' \arrow[r] \arrow[d] & U \arrow[d, "{(h_1,h_2)}"] \\ U' \arrow[r, "\Delta"] & U' \times_S U'. \end{tikzcd}$$

Since h_1 and h_2 induce the same map of residue fields $\kappa(u') \to \kappa(u)$ there exists a point $u'' \in U''$ lying over u' with $\kappa(u'') = \kappa(u')$. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Morphisms, Lemmas 29.36.4 and 29.36.3). Hence the vertical arrow (h_1, h_2) is étale by Morphisms, Lemma 29.36.18. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus (U'', u'') is a solution to the problem. □

057B Lemma 37.35.4. Let S be a scheme, and let s be a point of S. The category of elementary étale neighborhoods of (S, s) is cofiltered (see Categories, Definition 4.20.1).

Proof. This is immediate from the definitions and Lemma 37.35.3. □

05KS Lemma 37.35.5. Let S be a scheme. Let $s \in S$. Then we have

$$\mathcal{O}_{S,s}^h = \operatorname{colim}_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category which is opposite to the category of elementary étale neighbourhoods (U, u) of (S, s).

Proof. Let $\operatorname{Spec}(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_\mathfrak{p}$ and $\kappa(s) = \kappa(\mathfrak{p})$. A cofinal system of elementary étale neighbourhoods is given by those elementary étale neighbourhoods (U, u) such that U is affine and $U \to S$ factors through $\operatorname{Spec}(A)$. In other words, we see that the right hand side is equal to $\operatorname{colim}_{(B,\mathfrak{q})} B$ where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ with $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. Thus the lemma follows from Algebra, Lemma 10.155.7. □

We can lift étale neighbourhoods of points on fibres to the total space.

0CAS Lemma 37.35.6. Let $X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $(V, v) \to (X_s, x)$ be an étale neighbourhood. Then there exists an étale neighbourhood $(U, u) \to (X, x)$ such that there exists a morphism $(U_s, u) \to (V, v)$ of étale neighbourhoods of (X_s, x) which is an open immersion.

Proof. We may assume X, V, and S affine. Say the morphism $X \to S$ is given by $A \to B$ the point x by a prime $\mathfrak{q} \subset B$, the point s by $\mathfrak{p} = A \cap \mathfrak{q}$, and the morphism $V \to X_s$ by $B \otimes_A \kappa(\mathfrak{p}) \to C$. Since $\kappa(\mathfrak{p})$ is a localization of $A/\mathfrak{p}$ there exists an $f \in A$, $f \notin \mathfrak{p}$ and an étale ring map $B \otimes_A (A/\mathfrak{p})_f \to D$ such that

$$C = (B \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A (A/\mathfrak{p})_f} D$$

See Algebra, Lemma 10.143.3 part (9). After replacing A by A_f and B by B_f we may assume D is étale over $B \otimes_A A/\mathfrak{p} = B/\mathfrak{p}B$. Then we can apply Algebra, Lemma 10.143.10. This proves the lemma. □

37.36. Étale neighbourhoods and branches

0CB2 The number of (geometric) branches of a scheme at a point was defined in Properties, Section 28.15. In Varieties, Section 33.40 we related this to fibres of the normalization morphism. In this section we discuss a characterization of this number in terms of étale neighbourhoods.

0CB3 **Lemma 37.36.1.** Let $R = \operatorname{colim} R_i$ be colimit of a directed system of rings whose transition maps are faithfully flat. Then the number of minimal primes of R taken as an element of $\{0, 1, 2, \ldots, \infty\}$ is the supremum of the numbers of minimal primes of the R_i.

Proof. If $A \to B$ is a flat ring map, then $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ maps minimal primes to minimal primes by going down (Algebra, Lemma 10.39.19). If $A \to B$ is faithfully flat, then every minimal prime is the image of a minimal prime (by Algebra, Lemma 10.39.16 and 10.30.7). Hence the number of minimal primes of R_i is $\geq$ the number of minimal primes of $R_{i'}$ if $i \leq i'$. By Algebra, Lemma 10.39.20 each of the maps $R_i \to R$ is faithfully flat and we also see that the number of minimal primes of R is $\geq$ the number of minimal primes of R_i. Finally, suppose that $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are pairwise distinct minimal primes of R. Then we can find an i such that $R_i \cap \mathfrak{q}_1, \ldots, R_i \cap \mathfrak{q}_n$ are pairwise distinct (as sets and hence as prime ideals). This implies the lemma. □

0CB4 **Lemma 37.36.2.** Let X be a scheme and $x \in X$ a point. Then

(1) the number of branches of X at x is equal to the supremum of the number of irreducible components of U passing through u taken over elementary étale neighbourhoods $(U, u) \to (X, x)$,
(2) the number of geometric branches of X at x is equal to the supremum of the number of irreducible components of U passing through u taken over étale neighbourhoods $(U, u) \to (X, x)$,
(3) X is unibranch at x if and only if for every elementary étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U passing through u, and
(4) X is geometrically unibranch at x if and only if for every étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U passing through u.

Proof. Parts (3) and (4) follow from parts (1) and (2) via Properties, Lemma 28.15.6.

Proof of (1). Let $\operatorname{Spec}(A)$ be an affine open neighbourhood of x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. We may replace X by $\operatorname{Spec}(A)$ and it suffices to consider affine elementary étale neighbourhoods (U, u) in the supremum as they form a cofinal subsystem. Recall that the henselization $A^h_{\mathfrak{p}}$ is the colimit of the rings $B_{\mathfrak{q}}$ over the category of pairs $(B, \mathfrak{q})$ where B is an étale A-algebra and $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$ with $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$, see Algebra, Lemma 10.155.7. These pairs $(B, \mathfrak{q})$ correspond exactly to the affine elementary étale neighbourhoods (U, u) by the correspondence between rings and affine schemes. Observe that irreducible components of $\operatorname{Spec}(B)$ passing through $\mathfrak{q}$ are exactly the minimal prime ideals of $B_{\mathfrak{q}}$. The number of minimal primes of $A^h_{\mathfrak{p}}$ is the number of branches of X at x by Properties, Definition 28.15.4. Observe that the transition maps $B_{\mathfrak{q}} \to B'_{\mathfrak{q}'}$ in the system are all flat. Since a flat local ring map is faithfully flat (Algebra, Lemma 10.39.17) we see that the lemma follows from Lemma 37.36.1.

Proof of (2). The proof is the same as the proof of (1), except that we use Algebra, Lemma 10.155.11. There is a tiny difference: given a separable algebraic closure κ^{sep} of $\kappa(x)$ for every étale neighbourhood (U, u) we can choose a $\kappa(x)$-embedding $\phi : \kappa(u) \to \kappa^{sep}$ because $\kappa(u)/\kappa(x)$ is finite separable (Morphisms, Lemma 29.36.15).

Hence we can look at the supremum over all triples (U, u, ϕ) where $(U, u) \to (X, x)$ is an affine étale neighbourhood and $\phi : \kappa(u) \to \kappa^{sep}$ is a $\kappa(x)$-embedding. These triples correspond exactly to the triples in Algebra, Lemma 10.155.11 and the rest of the proof is exactly the same. □

We will need a relative variant of the lemma above.

0CB5 Lemma 37.36.3. Let $X \to S$ be a morphism of schemes and $x \in X$ a point with image s. Then

(1) the number of branches of the fibre X_s at x is equal to the supremum of the number of irreducible components of the fibre U_s passing through u taken over elementary étale neighbourhoods $(U, u) \to (X, x)$,
(2) the number of geometric branches of the fibre X_s at x is equal to the supremum of the number of irreducible components of the fibre U_s passing through u taken over étale neighbourhoods $(U, u) \to (X, x)$,
(3) the fibre X_s is unibranch at x if and only if for every elementary étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of the fibre U_s passing through u, and
(4) X is geometrically unibranch at x if and only if for every étale neighbourhood $(U, u) \to (X, x)$ there is exactly one irreducible component of U_s passing through u.

Proof. Combine Lemmas 37.36.2 and 37.35.6. □

0DQ2 Lemma 37.36.4. Let $X \to S$ be a smooth morphism of schemes. Let $x \in X$ with image $s \in S$. Then

(1) The number of geometric branches of X at x is equal to the number of geometric branches of S at s.
(2) If $\kappa(x)/\kappa(s)$ is a purely inseparable[6] extension of fields, then number of branches of X at x is equal to the number of branches of S at s.

Proof. Follows immediately from More on Algebra, Lemma 15.106.8 and the definitions. □

37.37. Unramified and étale morphisms

0GS7 Sometimes unramified morphisms are automatically étale.

0GS8 Lemma 37.37.1. Let $f : X \to Y$ be a morphism of schemes. Let $x \in X$ with image $y \in Y$. Assume

(1) Y is integral and geometrically unibranch at y,
(2) f is locally of finite type,
(3) there is a specialization $x' \rightsquigarrow x$ such that $f(x')$ is the generic point of Y,
(4) f is unramified at x.

Then f is étale at x.

Proof. We may replace X and Y by suitable affine open neighbourhoods of x and y. Then Y is the spectrum of a domain A and X is the spectrum of a finite type A-algebra B. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and $\mathfrak{p} \subset A$ the prime ideal corresponding to y. The local ring $A_{\mathfrak{p}} = \mathcal{O}_{Y,y}$ is geometrically unibranch.

[6]In fact, it would suffice if $\kappa(x)$ is geometrically irreducible over $\kappa(s)$. If we ever need this we will add a detailed proof.

The ring map $A \to B$ is unramified at $\mathfrak{q}$. Also, the point x' in (3) corresponds to a prime ideal $\mathfrak{q}' \subset \mathfrak{q}$ such that $A \cap \mathfrak{q}' = (0)$. It follows that $A_\mathfrak{p} \to B_\mathfrak{q}$ is injective. We conclude by More on Algebra, Lemma 15.107.2. □

0GS9 Lemma 37.37.2. Let $f : X \to Y$ be a morphism of schemes. Assume [Gro71, Expose I, Corollary 9.11]

(1) Y is integral and geometrically unibranch,
(2) at least one irreducible component of X dominates Y,
(3) f is unramified, and
(4) X is connected.

Then f is étale and X is irreducible.

Proof. Let $X' \subset X$ be the irreducible component which dominates Y. This means that the generic point of X' maps to the generic point of Y (see for example Morphisms, Lemma 29.8.6). By Lemma 37.37.1 we see that f is étale at every point of X'. In particular, the open subscheme $U \subset X$ where f is étale contains X'. Note that every quasi-compact open of U has finitely many irreducible components, see Descent, Lemma 35.16.3. On the other hand since Y is geometrically unibranch and U is étale over Y, the scheme U is geometrically unibranch. In particular, through every point of U there passes at most one irreducible component. A simple topological argument now shows that $X' \subset U$ is both open and closed. Then of course X' is open and closed in X and by connectedness we find $X = U = X'$ as desired. □

0GSA Lemma 37.37.3. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Let $x \in X$ with image $y \in Y$. Assume

(1) Y is integral and geometrically unibranch at y,
(2) f is locally of finite type,
(3) $g \circ f$ is étale at x,
(4) there is a specialization $x' \rightsquigarrow x$ such that $f(x')$ is the generic point of Y.

Then f is étale at x and g is étale at y.

Proof. The morphism f is unramified at x by Morphisms, Lemmas 29.35.16 and 29.36.5. Hence f is étale at x by Lemma 37.37.1. Then by étale descent we see that g is étale at y, see for example Descent, Lemma 35.14.4. □

0GSB Lemma 37.37.4. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume

(1) Y is integral and geometrically unibranch,
(2) f is locally of finite type,
(3) $g \circ f$ is étale,
(4) every irreducible component of X dominates Y.

Then f is étale and g is étale at every point in the image of f.

Proof. Immediate from the pointwise version Lemma 37.37.3. □

37.38. Slicing smooth morphisms

055S In this section we explain a result that roughly states that smooth coverings of a scheme S can be refined by étale coverings. The technique to prove this relies on a slicing argument.

057C Lemma 37.38.1. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume

(1) f is smooth at x, and
(2) the image $\mathrm{d}\overline{h}$ of $\mathrm{d}h$ in
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is nonzero.

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ smooth.

Proof. As f is smooth at x we may assume, after replacing X by an open neighbourhood of x that f is smooth. In particular we see that f is flat and locally of finite presentation. By Lemma 37.23.1 we already know there exists an open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and of finite presentation. By Morphisms, Lemma 29.32.15 we have a short exact sequence
$$\mathcal{C}_{D/U} \to i^*\Omega_{U/S} \to \Omega_{D/S} \to 0$$
where $i : D \to U$ is the closed immersion and $\mathcal{C}_{D/U}$ is the conormal sheaf of D in U. As D is an effective Cartier divisor cut out by $h \in \Gamma(U, \mathcal{O}_U)$ we see that $\mathcal{C}_{D/U} = h \cdot \mathcal{O}_S$. Since $U \to S$ is smooth the sheaf $\Omega_{U/S}$ is finite locally free, hence its pullback $i^*\Omega_{U/S}$ is finite locally free also. The first arrow of the sequence maps the free generator h to the section $\mathrm{d}h|_D$ of $i^*\Omega_{U/S}$ which has nonzero value in the fibre $\Omega_{U/S,x} \otimes \kappa(x)$ by assumption. By right exactness of $\otimes\kappa(x)$ we conclude that
$$\dim_{\kappa(x)} \left(\Omega_{D/S,x} \otimes \kappa(x)\right) = \dim_{\kappa(x)} \left(\Omega_{U/S,x} \otimes \kappa(x)\right) - 1.$$
By Morphisms, Lemma 29.34.14 we see that $\Omega_{U/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s)$ elements. By the displayed formula we see that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s) - 1$ elements. Note that $\dim_x(D_s) = \dim_x(U_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{U_s,x}) - 1$ by Algebra, Lemma 10.60.13 (also $D_s \subset U_s$ is effective Cartier, see Divisors, Lemma 31.18.1) and then using Morphisms, Lemma 29.28.1. Thus we conclude that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(D_s)$ elements and we conclude that $D \to S$ is smooth at x by Morphisms, Lemma 29.34.14 again. After shrinking U we get that $D \to S$ is smooth and we win. $\square$

057D Lemma 37.38.2. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

(1) f is smooth at x, and
(2) the map
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) \longrightarrow \Omega_{\kappa(x)/\kappa(s)}$$
has a nonzero kernel.

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 10.131.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By Algebra, Lemma 10.131.9 there is an exact sequence
$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\mathrm{d}} \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0.$$

Hence if (2) holds, there exists an element $\overline{h} \in \mathfrak{m}$ such that $d\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 37.38.1. $\square$

057E Remark 37.38.3. The second condition in Lemma 37.38.2 is necessary even if x is a closed point of a positive dimensional fibre. An example is the following: Let k be a field of characteristic $p > 0$ which is imperfect. Let $a \in k$ be an element which is not a pth power. Let $\mathfrak{m} = (x, y^p - a) \subset k[x, y]$. This corresponds to a closed point w of $X = \mathbf{A}^2_k$. Set $S = \mathbf{A}^1_k$ and let $f : X \to S$ be the morphism corresponding to $k[x] \to k[x, y]$. Then there does not exist any commutative diagram

$$\begin{tikzcd} S' \arrow[rr, "h"'] \arrow[rd, "g"'] & & X \arrow[ld, "f"] \\ & S & \end{tikzcd}$$

with g étale and w in the image of h. This is clear as the residue field extension $\kappa(w)/\kappa(f(w))$ is purely inseparable, but for any $s' \in S'$ with $g(s') = f(w)$ the extension $\kappa(s')/\kappa(f(w))$ would be separable.

If you assume the residue field extension is separable then the phenomenon of Remark 37.38.3 does not happen. Here is the precise result.

057F Lemma 37.38.4. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

(1) f is smooth at x,
(2) the residue field extension $\kappa(x)/\kappa(s)$ is separable, and
(3) x is not a generic point of X_s.

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 10.131.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By assumption (2) and Algebra, Lemma 10.140.4 the map

$$d : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$

is injective. Assumption (3) implies that $\mathfrak{m}/\mathfrak{m}^2 \neq 0$. Thus there exists an element $\overline{h} \in \mathfrak{m}$ such that $d\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 37.38.1. $\square$

The subscheme Z constructed in the following lemma is really a complete intersection in an affine open neighbourhood of x. If we ever need this we will explicitly formulate a separate lemma stating this fact.

057G Lemma 37.38.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume

(1) f is smooth at x, and
(2) x is a closed point of X_s and $\kappa(s) \subset \kappa(x)$ is separable.

Then there exists an immersion $Z \to X$ containing x such that

(1) $Z \to S$ is étale, and
(2) $Z_s = \{x\}$ set theoretically.

Proof. We may and do replace S by an affine open neighbourhood of s. We may and do replace X by an affine open neighbourhood of x such that $X \to S$ is smooth. We will prove the lemma for smooth morphisms of affines by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 29.29.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 29.56.2. Thus after replacing X by U we see that X is quasi-finite and smooth over S, hence smooth of relative dimension 0 over S, hence étale over S. Moreover, the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neighbourhood of X we see that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 26.18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(x)/\kappa(s)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 29.20.3). Thus we see $\Omega_{\kappa(x)/\kappa(s)} = 0$ as this holds for algebraic separable field extensions. Thus we may apply Lemma 37.38.2 to find a diagram

$$\begin{tikzcd} D \arrow[r] \arrow[rrd] & U \arrow[r] \arrow[rd] & X \arrow[d] \\ & & S \end{tikzcd}$$

with $x \in D$. Note that $\dim_x(D_s) = \dim_x(X_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{X_s,x}) - 1$ by Algebra, Lemma 10.60.13 (also $D_s \subset X_s$ is effective Cartier, see Divisors, Lemma 31.18.1) and then using Morphisms, Lemma 29.28.1. Thus the morphism $D \to S$ is smooth with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find an immersion $Z \to D$ as desired, which finishes the proof. $\square$

055U Lemma 37.38.6. Let $f : X \to S$ be a smooth morphism of schemes. Let $s \in S$ be a point in the image of f. Then there exists an étale neighbourhood $(S', s') \to (S, s)$ and a S-morphism $S' \to X$.

First proof of Lemma 37.38.6. By assumption $X_s \neq \emptyset$. By Varieties, Lemma 33.25.6 there exists a closed point $x \in X_s$ such that $\kappa(x)$ is a finite separable field extension of $\kappa(s)$. Hence by Lemma 37.38.5 there exists an immersion $Z \to X$ such that $Z \to S$ is étale and such that $x \in Z$. Take $(S', s') = (Z, x)$. $\square$

Second proof of Lemma 37.38.6. Pick a point $x \in X$ with $f(x) = s$. Choose a diagram

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[r, "\pi"'] \arrow[d] & \mathbf{A}^d_V \arrow[ld] \\ S & V \arrow[l] \end{tikzcd}$$

with π étale, $x \in U$ and $V = \mathrm{Spec}(R)$ affine, see Morphisms, Lemma 29.36.20. In particular $s \in V$. The morphism $\pi : U \to \mathbf{A}^d_V$ is open, see Morphisms, Lemma 29.36.13. Thus $W = \pi(U) \cap \mathbf{A}^d_s$ is a nonempty open subset of $\mathbf{A}^d_s$. Let $w \in W$ be a point with $\kappa(s) \subset \kappa(w)$ finite separable, see Varieties, Lemma 33.25.5. By Algebra, Lemma 10.114.1 there exist d elements $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$ which generate the maximal ideal corresponding to w in $\kappa(s)[x_1, \ldots, x_d]$. After replacing R by a principal localization we may assume there are $f_1, \ldots, f_d \in R[x_1, \ldots, x_d]$ which map to $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$. Consider the R-algebra

$$R' = R[x_1, \ldots, x_d]/(f_1, \ldots, f_d)$$

and set $S' = \mathrm{Spec}(R')$. By construction we have a closed immersion $j : S' \to \mathbf{A}^d_V$ over V. By construction the fibre of $S' \to V$ over s is a single point s' whose residue field is finite separable over $\kappa(s)$. Let $\mathfrak{q}' \subset R'$ be the corresponding prime. By Algebra, Lemma 10.136.10 we see that $(R')_g$ is a relative global complete intersection over R for some $g \in R'$, $g \notin \mathfrak{q}$. Thus $S' \to V$ is flat and of finite presentation in a neighbourhood of s', see Algebra, Lemma 10.136.13. By construction the scheme theoretic fibre of $S' \to V$ over s is $\mathrm{Spec}(\kappa(s'))$. Hence it follows from Morphisms, Lemma 29.36.15 that $S' \to S$ is étale at s'. Set

$$S'' = U \times_{\pi, \mathbf{A}^d_V, j} S'.$$

By construction there exists a point $s'' \in S''$ which maps to s' via the projection $p : S'' \to S'$. Note that p is étale as the base change of the étale morphism π, see Morphisms, Lemma 29.36.4. Choose a small affine neighbourhood $S''' \subset S''$ of s'' which maps into the nonempty open neighbourhood of $s' \in S'$ where the morphism $S' \to S$ is étale. Then the étale neighbourhood $(S''', s'') \to (S, s)$ is a solution to the problem posed by the lemma. $\square$

The following lemma shows that sheaves for the smooth topology are the same thing as sheaves for the étale topology.

055V Lemma 37.38.7. Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be a smooth covering of S, see Topologies, Definition 34.5.1. Then there exists an étale covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ (see Topologies, Definition 34.4.1) which refines (see Sites, Definition 7.8.1) $\mathcal{U}$.

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 37.38.6 we can find an étale morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is an étale covering of S that refines $\mathcal{U}$. $\square$

0EY4 Lemma 37.38.8. Let $f : X \to S$ be a smooth morphism of schemes. Then there exists an étale covering $\{U_i \to X\}_{i \in I}$ such that $U_i \to S$ factors as $U_i \to V_i \to S$ where $V_i \to S$ is étale and $U_i \to V_i$ is a smooth morphism of affine schemes, which has a section, and has geometrically connected fibres.

Proof. Let $s \in S$. By Varieties, Lemma 33.25.6 the set of closed points $x \in X_s$ such that $\kappa(x)/\kappa(s)$ is separable is dense in X_s. Thus it suffices to construct an étale morphism $U \to X$ with x in the image such that $U \to S$ factors in the manner described in the lemma. To do this, choose an immersion $Z \to X$ passing through x such that $Z \to S$ is étale (Lemma 37.38.5). After replacing S by Z and X by $Z \times_S X$ we see that we may assume $X \to S$ has a section $\sigma : S \to X$ with $\sigma(s) = x$. Then we can first replace S by an affine open neighbourhood of s and next replace X by an affine open neighbourhood of x. Then finally, we consider the subset $X^0 \subset X$ of Section 37.29. By Lemmas 37.29.6 and 37.29.4 this is a retrocompact open subscheme containing σ such that the fibres $X^0 \to S$ are geometrically connected. If X^0 is not affine, then we choose an affine open $U \subset X^0$

containing x. Since $X^0 \to S$ is smooth, the image of U is open. Choose an affine open neighbourhood $V \subset S$ of s contained in $\sigma^{-1}(U)$ and in the image of $U \to S$. Finally, the reader sees that $U \cap f^{-1}(V) \to V$ has all the desired properties. For example $U \cap f^{-1}(V)$ is equal to $U \times_S V$ is affine as a fibre product of affine schemes. Also, the geometric fibres of $U \cap f^{-1}(V) \to V$ are nonempty open subschemes of the irreducible fibres of $X^0 \to S$ and hence connected. Some details omitted. $\square$

37.39. Étale neighbourhoods and Artin approximation

0CAT In this section we prove results of the form: if two pointed schemes have isomorphic complete local rings, then they have isomorphic étale neighbourhoods. We will rely on Popescu's theorem, see Smoothing Ring Maps, Theorem 16.12.1.

0CAU Lemma 37.39.1. Let S be a locally Noetherian scheme. Let X, Y be schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume further we are given a local $\mathcal{O}_{S,s}$-algebra map

$$\varphi : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}^\wedge$$

For every $N \geq 1$ there exists an elementary étale neighbourhood $(U, u) \to (X, x)$ and an S-morphism $f : U \to Y$ mapping u to y such that the diagram

$$\begin{tikzcd} \mathcal{O}_{X,x}^\wedge \arrow[r] & \mathcal{O}_{U,u}^\wedge \\ \mathcal{O}_{Y,y} \arrow[u, "\varphi"] \arrow[r, "f_u^\sharp"] & \mathcal{O}_{U,u} \arrow[u] \end{tikzcd}$$

commutes modulo $\mathfrak{m}_u^N$.

Proof. The question is local on X hence we may assume X, Y, S are affine. Say $S = \text{Spec}(R)$, $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$. Write $B = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to x. The local ring $\mathcal{O}_{X,x} = A_\mathfrak{p}$ is a G-ring by More on Algebra, Proposition 15.50.10. The map φ is a map

$$B_\mathfrak{q}^\wedge \longrightarrow A_\mathfrak{p}^\wedge$$

where $\mathfrak{q} \subset B$ is the prime corresponding to y. Let $a_1, \ldots, a_n \in A_\mathfrak{p}^\wedge$ be the images of $x_1, \ldots, x_n$ via $R[x_1, \ldots, x_n] \to B \to B_\mathfrak{q}^\wedge \to A_\mathfrak{p}^\wedge$. Then we can apply Smoothing Ring Maps, Lemma 16.13.4 to get an étale ring map $A \to A'$ and a prime ideal $\mathfrak{p}' \subset A'$ and $b_1, \ldots, b_n \in A'$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$, $a_i - b_i \in (\mathfrak{p}')^N (A'_{\mathfrak{p}'})^\wedge$, and $f_j(b_1, \ldots, b_n) = 0$ for $j = 1, \ldots, n$. This determines an R-algebra map $B \to A'$ by sending the class of x_i to $b_i \in A'$. This finishes the proof by taking $U = \text{Spec}(A') \to \text{Spec}(B)$ as the morphism f and $u = \mathfrak{p}'$. $\square$

0CAV Lemma 37.39.2. Let S be a locally Noetherian scheme. Let X, Y be schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be points lying over the same point $s \in S$. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume we have an $\mathcal{O}_{S,s}$-algebra isomorphism

$$\varphi : \mathcal{O}_{Y,y}^\wedge \longrightarrow \mathcal{O}_{X,x}^\wedge$$

between the complete local rings. Then for every $N \geq 1$ there exists morphisms

$$(X, x) \leftarrow (U, u) \to (Y, y)$$

of pointed schemes over S such that both arrows define elementary étale neighbourhoods and such that the diagram

$$\begin{tikzcd} & \mathcal{O}^\wedge_{U,u} & \\ \mathcal{O}^\wedge_{Y,y} \arrow[ru] \arrow[rr, "\varphi"] & & \mathcal{O}^\wedge_{X,x} \arrow[lu] \end{tikzcd}$$

commutes modulo $\mathfrak{m}_u^N$.

Proof. We may assume $N \geq 2$. Apply Lemma 37.39.1 to get $(U, u) \to (X, x)$ and $f : (U, u) \to (Y, y)$. We claim that f is étale at u which will finish the proof. In fact, we will show that the induced map $\mathcal{O}^\wedge_{Y,y} \to \mathcal{O}^\wedge_{U,u}$ is an isomorphism. Having proved this, Lemma 37.12.1 will show that f is smooth at u and of course f is unramified at u as well, so Morphisms, Lemma 29.36.5 tells us f is étale at u. For a local ring $(R, \mathfrak{m})$ we set $\text{Gr}_\mathfrak{m}(R) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. To prove the claim we look at the induced diagram of graded rings

$$\begin{tikzcd} & \text{Gr}_{\mathfrak{m}_u}(\mathcal{O}_{U,u}) & \\ \text{Gr}_{\mathfrak{m}_y}(\mathcal{O}_{Y,y}) \arrow[ru] \arrow[rr, "\varphi"] & & \text{Gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \arrow[lu] \end{tikzcd}$$

Since $N \geq 2$ this diagram is actually commutative as the displayed graded algebras are generated in degree 1! By assumption the lower arrow is an isomorphism. By More on Algebra, Lemma 15.43.9 (for example) the map $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{U,u}$ is an isomorphism and hence the north-west arrow in the diagram is an isomorphism. We conclude that f induces an isomorphism $\text{Gr}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \to \text{Gr}_{\mathfrak{m}_y}(\mathcal{O}_{U,u})$. Using induction and the short exact sequences

$$0 \to \text{Gr}^n_\mathfrak{m}(R) \to R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n \to 0$$

for both local rings we conclude (from the snake lemma) that f induces isomorphisms $\mathcal{O}_{Y,y}/\mathfrak{m}_y^n \to \mathcal{O}_{U,u}/\mathfrak{m}_u^n$ for all n which is what we wanted to show. $\square$

0GDX Lemma 37.39.3. Let $X \to S$, $Y \to T$, x, s, y, t, σ, y_σ, and φ be given as follows: we have morphisms of schemes

$$\begin{matrix} X & & Y & & & x & & y \\ \downarrow & & \downarrow & & \text{with points} & \downarrow & & \downarrow \\ S & & T & & & s & & t \end{matrix}$$

Here S is locally Noetherian and T is of finite type over $\mathbf{Z}$. The morphisms $X \to S$ and $Y \to T$ are locally of finite type. The local ring $\mathcal{O}_{S,s}$ is a G-ring. The map

$$\sigma : \mathcal{O}_{T,t} \longrightarrow \mathcal{O}^\wedge_{S,s}$$

is a local homomorphism. Set $Y_\sigma = Y \times_{T,\sigma} \text{Spec}(\mathcal{O}^\wedge_{S,s})$. Next, y_σ is a point of Y_σ mapping to y and the closed point of $\text{Spec}(\mathcal{O}^\wedge_{S,s})$. Finally

$$\varphi : \mathcal{O}^\wedge_{X,x} \longrightarrow \mathcal{O}^\wedge_{Y_\sigma, y_\sigma}$$

is an isomorphism of $\mathcal{O}^\wedge_{S,s}$-algebras. In this situation there exists a commutative diagram

$$\begin{tikzcd} X & W \arrow[l] \arrow[r] \arrow[dr] & Y \times_{T,\tau} V \arrow[r] \arrow[d] & Y \arrow[d] \\ S & & V \arrow[ll] \arrow[r, "\tau"] & T \end{tikzcd}$$

(with $X \to S$ the left vertical arrow)

of schemes and points $w \in W$, $v \in V$ such that

(1) $(V, v) \to (S, s)$ is an elementary étale neighbourhood,
(2) $(W, w) \to (X, x)$ is an elementary étale neighbourhood, and
(3) $\tau(v) = t$.

Let $y_\tau \in Y \times_T V$ correspond to y_σ via the identification $(Y_\sigma)_s = (Y \times_T V)_v$. Then

(4) $(W, w) \to (Y \times_{T,\tau} V, y_\tau)$ is an elementary étale neighbourhood.

Proof. Denote $X_\sigma = X \times_S \mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$ and $x_\sigma \in X_\sigma$ the unique point lying over x. Observe that $\mathcal{O}^\wedge_{S,s}$ is a G-ring by More on Algebra, Proposition 15.50.6. By Lemma 37.39.2 we can choose

$$(X_\sigma, x_\sigma) \leftarrow (U, u) \to (Y_\sigma, y_\sigma)$$

where both arrows are elementary étale neighbourhoods.

After replacing S by an open neighbourhood of s, we may assume $S = \mathrm{Spec}(R)$ is affine. Since $\mathcal{O}_{S,s}$ is a G-ring by Smoothing Ring Maps, Theorem 16.12.1 the ring $\mathcal{O}^\wedge_{S,s}$ is a filtered colimit of smooth R-algebras. Thus we can write

$$\mathrm{Spec}(\mathcal{O}^\wedge_{S,s}) = \lim S_i$$

as a directed limit of affine schemes S_i smooth over S. Denote $s_i \in S_i$ the image of the closed point of $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$. Observe that $\kappa(s) = \kappa(s_i)$. Set $X_i = X \times_S S_i$ and denote $x_i \in X_i$ the unique point mapping to x. Note that $\kappa(x) = \kappa(x_i)$. Since T is of finite type over $\mathbf{Z}$ by Limits, Proposition 32.6.1 we can choose an i and a morphism $\sigma_i : (S_i, s_i) \to (T, t)$ of pointed schemes whose composition with $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s}) \to S_i$ is equal to σ. Set $Y_i = Y \times_T S_i$ and denote y_i the image of y_σ. Note that $\kappa(y_i) = \kappa(y_\sigma)$. By Limits, Lemma 32.10.1 we can choose an i and a diagram

$$\begin{tikzcd} X_i \arrow[dr] & U_i \arrow[l] \arrow[r] \arrow[d] & Y_i \arrow[dl] \\ & S_i & \end{tikzcd}$$

whose base change to $\mathrm{Spec}(\mathcal{O}^\wedge_{S,s})$ recovers $X_\sigma \leftarrow U \to Y_\sigma$. By Limits, Lemma 32.8.10 after increasing i we may assume the morphisms $X_i \leftarrow U_i \to Y_i$ are étale. Let $u_i \in U_i$ be the image of u. Then $u_i \mapsto x_i$ hence $\kappa(x) = \kappa(x_\sigma) = \kappa(u) \supset \kappa(u_i) \supset \kappa(x_i) = \kappa(x)$ and we see that $\kappa(u_i) = \kappa(x_i)$. Hence $(X_i, x_i) \leftarrow (U_i, u_i)$ is an elementary étale neighbourhood. Since also $\kappa(y_i) = \kappa(y_\sigma) = \kappa(u)$ we see that also $(U_i, u_i) \to (Y_i, y_i)$ is an elementary étale neighbourhood.

At this point we have constructed a diagram

$$\begin{tikzcd} X \arrow[d] & X \times_S S_i \arrow[l] \arrow[dr] & U_i \arrow[l] \arrow[r] \arrow[d] & Y \times_T S_i \arrow[r] \arrow[dl] & Y \arrow[d] \\ S & & S_i \arrow[ll] \arrow[rr] & & T \end{tikzcd}$$

as in the statement of the lemma, except that $S_i \to S$ is smooth. By Lemma 37.38.5 and after shrinking S_i we can assume there exists a closed subscheme $V \subset S_i$ passing through s_i such that $V \to S$ is étale. Setting W equal to the scheme theoretic inverse image of V in U_i we conclude. $\square$

We strongly encourage the reader to skip the rest of this section.

0CAW **Lemma 37.39.4.** Consider a diagram

$$\begin{tikzcd} X \arrow[d] & Y \arrow[d] \\ S & T \arrow[l] \end{tikzcd} \quad \text{with points} \quad \begin{tikzcd} x \arrow[d] & y \arrow[d] \\ s & t \arrow[l] \end{tikzcd}$$

where S be a locally Noetherian scheme and the morphisms are locally of finite type. Assume $\mathcal{O}_{S,s}$ is a G-ring. Assume further we are given a local $\mathcal{O}_{S,s}$-algebra map

$$\sigma : \mathcal{O}_{T,t} \longrightarrow \mathcal{O}_{S,s}^\wedge$$

and a local $\mathcal{O}_{S,s}$-algebra map

$$\varphi : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y_\sigma, y_\sigma}^\wedge$$

where $Y_\sigma = Y \times_{T,\sigma} \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$ and y_σ is the unique point of Y_σ lying over y. For every $N \geq 1$ there exists a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X \times_S V \arrow[l] \arrow[rrd] & & W \arrow[ll, "f"'] \arrow[r] \arrow[d] & Y \times_{T,\tau} V \arrow[ld] \arrow[r] & Y \arrow[d] \\ S & & & V \arrow[lll] \arrow[rr, "\tau"] & & T \end{tikzcd}$$

of schemes over S and points $w \in W$, $v \in V$ such that

(1) $v \mapsto s$, $\tau(v) = t$, $f(w) = (x, v)$, and $w \mapsto (y, v)$,
(2) $(V, v) \to (S, s)$ is an elementary étale neighbourhood,
(3) the diagram

$$\begin{tikzcd} \mathcal{O}_{S,s}^\wedge \arrow[r] & \mathcal{O}_{V,v}^\wedge \\ \mathcal{O}_{T,t} \arrow[u, "\sigma"] \arrow[r, "\tau_v^\sharp"] & \mathcal{O}_{V,v} \arrow[u] \end{tikzcd}$$

commutes module $\mathfrak{m}_v^N$,
(4) $(W, w) \to (Y \times_{T,\tau} V, (y, v))$ is an elementary étale neighbourhood,
(5) the diagram

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "\varphi"'] \arrow[d, equal] & \mathcal{O}_{Y_\sigma, y_\sigma}^\wedge \arrow[r] & \mathcal{O}_{Y_\sigma, y_\sigma}/\mathfrak{m}_{y_\sigma}^N \arrow[r, equal] & \mathcal{O}_{Y \times_{T,\tau} V, (y,v)}/\mathfrak{m}_{(y,v)}^N \arrow[d, "\cong"'] \\ \mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{X \times_S V, (x,v)} \arrow[r, "f_w^\sharp"] & \mathcal{O}_{W,w} \arrow[r] & \mathcal{O}_{W,w}/\mathfrak{m}_w^N \end{tikzcd}$$

commutes. The equality comes from the fact that Y_σ and $Y \times_{T,\tau} V$ are canonically isomorphic over $\mathcal{O}_{V,v}/\mathfrak{m}_v^N = \mathcal{O}_{S,s}/\mathfrak{m}_s^N$ by parts (2) and (3).

Proof. After replacing X, S, T, Y by affine open subschemes we may assume the diagram in the statement of the lemma comes from applying Spec to a diagram

$$\begin{tikzcd} A & B \\ R \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd} \quad \text{with primes} \quad \begin{tikzcd} \mathfrak p_A \arrow[d, dash] & \mathfrak p_B \arrow[d, dash] \\ \mathfrak p_R \arrow[r, dash] & \mathfrak p_C \end{tikzcd}$$

of Noetherian rings and finite type ring maps. In this proof every ring E will be a Noetherian R-algebra endowed with a prime ideal $\mathfrak p_E$ lying over $\mathfrak p_R$ and all ring maps will be R-algebra maps compatible with the given primes. Moreover, if we write $E^\wedge$ we mean the completion of the localization of E at $\mathfrak p_E$. We will also use without further mention that an étale ring map $E_1 \to E_2$ such that $\kappa(\mathfrak p_{E_1}) = \kappa(\mathfrak p_{E_2})$ induces an isomorphism $E_1^\wedge = E_2^\wedge$ by More on Algebra, Lemma 15.43.9.

With this notation σ and φ correspond to ring maps

$$\sigma : C \to R^\wedge \quad \text{and} \quad \varphi : A \longrightarrow (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

Here is a picture

$$\begin{tikzcd} A \arrow[rrr, bend left, "\varphi"] & B \arrow[r] & B \otimes_{C,\sigma} R^\wedge \arrow[r] & (B \otimes_{C,\sigma} R^\wedge)^\wedge \\ R \arrow[u] \arrow[r] & C \arrow[u] \arrow[r, "\sigma"] & R^\wedge \arrow[u] \arrow[ur] & \end{tikzcd}$$

Observe that $R^\wedge$ is a G-ring by More on Algebra, Proposition 15.50.6. Thus $B \otimes_{C,\sigma} R^\wedge$ is a G-ring by More on Algebra, Proposition 15.50.10. By Lemma 37.39.1 (translated into algebra) there exists an étale ring map $B \otimes_{C,\sigma} R^\wedge \to B'$ inducing an isomorphism $\kappa(\mathfrak p_{B \otimes_{C,\sigma} R^\wedge}) \to \kappa(\mathfrak p_{B'})$ and an R-algebra map $A \to B'$ such that the composition

$$A \to B' \to (B')^\wedge = (B \otimes_{C,\sigma} R^\wedge)^\wedge$$

is the same as φ modulo $(\mathfrak p_{(B \otimes_{C,\sigma} R^\wedge)^\wedge})^N$. Thus we may replace φ by this composition because the only way φ enters the conclusion is via the commutativity requirement in part (5) of the statement of the lemma. Picture:

$$\begin{tikzcd} & & B' \arrow[r] & (B')^\wedge \arrow[d, equal] \\ A \arrow[urr] & B \arrow[r] & B \otimes_{C,\sigma} R^\wedge \arrow[u] \arrow[r] & (B \otimes_{C,\sigma} R^\wedge)^\wedge \\ R \arrow[u] \arrow[r] & C \arrow[u] \arrow[r, "\sigma"] & R^\wedge \arrow[u] \arrow[ur] & \end{tikzcd}$$

Next, we use that $R^\wedge$ is a filtered colimit of smooth R-algebras (Smoothing Ring Maps, Theorem 16.12.1) because $R_{\mathfrak p_R}$ is a G-ring by assumption. Since C is of finite presentation over R we get a factorization

$$C \to R' \to R^\wedge$$

for some $R \to R'$ smooth, see Algebra, Lemma 10.127.3. After increasing R' we may assume there exists an étale $B \otimes_C R'$-algebra B'' whose base change to $B \otimes_{C,\sigma} R^\wedge$ is B', see Algebra, Lemma 10.143.3. Then B' is the filtered colimit of these B''

and we conclude that after increasing R' we may assume there is an R-algebra map $A \to B''$ such that $A \to B'' \to B'$ is the previously constructed map (same reference as above). Picture

$$
\begin{tikzcd}
 & & B'' \arrow[r] & B' \arrow[r] & (B')^\wedge \arrow[d, equal] \\
A \arrow[urr] & B \arrow[r] & B \otimes_C R' \arrow[r] \arrow[u] & B \otimes_{C,\sigma} R^\wedge \arrow[r] \arrow[u] & (B \otimes_{C,\sigma} R^\wedge)^\wedge \\
R \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & R' \arrow[r] \arrow[u] & R^\wedge \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

and

$$B' = B'' \otimes_{(B \otimes_C R')} (B \otimes_{C,\sigma} R^\wedge)$$

This means that we may replace C by R', $\sigma : C \to R^\wedge$ by $R' \to R^\wedge$, and B by B'' so that we simplify to the diagram

$$
\begin{tikzcd}
A \arrow[r] & B \arrow[r] & B \otimes_{C,\sigma} R^\wedge \\
R \arrow[r] \arrow[u] & C \arrow[r, "\sigma"] \arrow[u] & R^\wedge \arrow[u]
\end{tikzcd}
$$

with φ equal to the composition of the horizontal arrows followed by the canonical map from $B \otimes_{C,\sigma} R^\wedge$ to its completion. The final step in the proof is to apply Lemma 37.39.1 (or its proof) one more time to $\text{Spec}(C)$ and $\text{Spec}(R)$ over $\text{Spec}(R)$ and the map $C \to R^\wedge$. The lemma produces a ring map $C \to D$ such that $R \to D$ is étale, such that $\kappa(\mathfrak{p}_R) = \kappa(\mathfrak{p}_D)$, and such that

$$C \to D \to D^\wedge = R^\wedge$$

is equal to $\sigma : C \to R^\wedge$ modulo $(\mathfrak{p}_{R^\wedge})^N$. Then we can take

$$V = \text{Spec}(D) \quad \text{and} \quad W = \text{Spec}(B \otimes_C D)$$

as our solution to the problem posed by the lemma. Namely the diagram

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & B \otimes_{C,\sigma} R^\wedge \arrow[r] & B \otimes_{C,\sigma} R^\wedge/(\mathfrak{p}_{R^\wedge})^N \arrow[r, equal] & B \otimes_C D/(\mathfrak{p}_D)^N \arrow[d, equal] \\
A \arrow[r] & A \otimes_R D \arrow[r] & B \otimes_R D \arrow[r] & B \otimes_C D/(\mathfrak{p}_D)^N
\end{tikzcd}
$$

commutes because $C \to D \to D^\wedge = R^\wedge$ is equal to σ modulo $(\mathfrak{p}_{R^\wedge})^N$. This proves part (5) and the other properties are immediate from the construction. □

0CAX Lemma 37.39.5. Let $T \to S$ be finite type morphisms of Noetherian schemes. Let $t \in T$ map to $s \in S$ and let $\sigma : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^\wedge$ be a local $\mathcal{O}_{S,s}$-algebra map. For every $N \geq 1$ there exists a finite type morphism $(T', t') \to (T, t)$ such that σ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ and such that for every local $\mathcal{O}_{S,s}$-algebra map $\sigma' : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^\wedge$ which factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ the maps σ and σ' agree modulo $\mathfrak{m}_s^N$.

Proof. We may assume S and T are affine. Say $S = \mathrm{Spec}(R)$ and $T = \mathrm{Spec}(C)$. Let $c_1, \ldots, c_n \in C$ be generators of C as an R-algebra. Let $\mathfrak p \subset R$ be the prime ideal corresponding to s. Say $\mathfrak p = (f_1, \ldots, f_m)$. After replacing R by a principal localization (to clear denominators in $R_\mathfrak p$) we may assume there exist $r_1, \ldots, r_n \in R$ and $a_{i,I} \in \mathcal{O}_{S,s}^\wedge$ where $I = (i_1, \ldots, i_m)$ with $\sum i_j = N$ such that

$$\sigma(c_i) = r_i + \sum\nolimits_I a_{i,I} f_1^{i_1} \ldots f_m^{i_m}$$

in $\mathcal{O}_{S,s}^\wedge$. Then we consider

$$C' = C[t_{i,I}]/\left(c_i - r_i - \sum\nolimits_I t_{i,I} f_1^{i_1} \ldots f_m^{i_m}\right)$$

with $\mathfrak p' = \mathfrak p C' + (t_{i,I})$ and factorization of $\sigma : C \to \mathcal{O}_{S,s}^\wedge$ through C' given by sending $t_{i,I}$ to $a_{i,I}$. Taking $T' = \mathrm{Spec}(C')$ works because any σ' as in the statement of the lemma will send c_i to r_i modulo the maximal ideal to the power N. $\square$

0CAY Lemma 37.39.6. Let $Y \to T \to S$ be finite type morphisms of Noetherian schemes. Let $t \in T$ map to $s \in S$ and let $\sigma : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^\wedge$ be a local $\mathcal{O}_{S,s}$-algebra map. There exists a finite type morphism $(T', t') \to (T, t)$ such that σ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ and such that for every local $\mathcal{O}_{S,s}$-algebra map $\sigma' : \mathcal{O}_{T,t} \to \mathcal{O}_{S,s}^\wedge$ which factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$ the closed immersions

$$Y \times_{T,\sigma} \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge) = Y_\sigma \longleftarrow Y_t \longrightarrow Y_{\sigma'} = Y \times_{T,\sigma'} \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$$

have isomorphic conormal algebras.

Proof. A useful observation is that $\kappa(s) = \kappa(t)$ by the existence of σ. Observe that the statement makes sense as the fibres of Y_σ and $Y_{\sigma'}$ over $s \in \mathrm{Spec}(\mathcal{O}_{S,s}^\wedge)$ are both canonically isomorphic to Y_t. We will think of the property "σ' factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$" as a constraint on σ'. If we have several such constraints, say given by $(T'_i, t'_i) \to (T, t)$, $i = 1, \ldots, n$ then we can combined them by considering $(T'_1 \times_T \ldots \times_T T'_n, (t'_1, \ldots, t'_n)) \to (T, t)$. We will use this without further mention in the following.

By Lemma 37.39.5 we can assume that any σ' as in the statement of the lemma is the same as σ modulo $\mathfrak m_s^2$. Note that the conormal algebra of Y_t in Y_σ is just the quasi-coherent graded $\mathcal{O}_{Y_t}$-algebra

$$\bigoplus\nolimits_{n \geq 0} \mathfrak m_s^n \mathcal{O}_{Y_\sigma} / \mathfrak m_s^{n+1} \mathcal{O}_{Y_\sigma}$$

and similarly for $Y_{\sigma'}$. Since σ and σ' agree modulo $\mathfrak m_s^2$ we see that these two algebras are the same in degrees 0 and 1. On the other hand, these conormal algebras are generated in degree 1 over degree 0. Hence if there is an isomorphism extending the isomorphism just constructed in degrees 0 and 1, then it is unique.

We may assume S and T are affine. Let $Y = Y_1 \cup \ldots \cup Y_n$ be an affine open covering. If we can construct $(T'_i, t'_i) \to (T, t)$ as in the lemma such that the desired isomorphism (see previous paragraph) exists for $Y_i \to T \to S$ and σ, then these glue by uniqueness to prove the result for $Y \to T$. Thus we may assume Y is affine.

Write $S = \mathrm{Spec}(R)$, $T = \mathrm{Spec}(C)$, and $Y = \mathrm{Spec}(B)$. Choose a presentation $B = C[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Denote $R^\wedge = \mathcal{O}_{S,s}^\wedge$. Let $a_{kj} \in R^\wedge[x_1, \ldots, x_n]$ be polynomials such that

$$\sum\nolimits_{j=1,\ldots,m} a_{kj} \sigma(f_j) = 0, \quad \text{for } k = 1, \ldots, K$$

is a set of generators for the module of relations among the $\sigma(f_j) \in R^\wedge[x_1, \ldots, x_n]$. Thus we have an exact sequence

(37.39.6.1)

0CAZ
$$R^\wedge[x_1, \ldots, x_n]^{\oplus K} \to R^\wedge[x_1, \ldots, x_n]^{\oplus m} \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C,\sigma} R^\wedge \to 0$$

Let c be an integer which works in the Artin-Rees lemma for both the first and the second map in this sequence and the ideal $\mathfrak{m}_{R^\wedge} R^\wedge[x_1, \ldots, x_n]$ as defined in More on Algebra, Section 15.4. Write

$$a_{kj} = \sum\nolimits_{I \in \Omega} a_{kj,I} x^I \quad \text{and} \quad f_j = \sum\nolimits_{I \in \Omega} f_{j,I} x^I$$

in multiindex notation where $a_{kj,I} \in R^\wedge$, $f_{j,I} \in C$, and Ω a finite set of multiindices. Then we see that

$$\sum\nolimits_{j=1,\ldots,m,\ I,I' \in \Omega,\ I+I'=I''} a_{kj,I}\sigma(f_{j,I'}) = 0, \quad I'' \text{ a multiindex}$$

in $R^\wedge$. Thus we take

$$C' = C[t_{jk,I}]/\left(\sum\nolimits_{j=1,\ldots,m,\ I,I' \in \Omega,\ I+I'=I''} t_{kj,I} f_{j,I'},\ I'' \text{ a multiindex}\right)$$

Then σ factors through a map $\tilde{\sigma} : C' \to R^\wedge$ sending $t_{kj,I}$ to $a_{jk,I}$. Thus $T' = \text{Spec}(C')$ comes with a point $t' \in T'$ such that σ factors through $\mathcal{O}_{T,t} \to \mathcal{O}_{T',t'}$. Let $t_{kj} = \sum t_{kj,I} x^I$ in $C'[x_1, \ldots, x_n]$. Then we see that we have a complex

(37.39.6.2)

0CB0
$$C'[x_1, \ldots, x_n]^{\oplus K} \to C'[x_1, \ldots, x_n]^{\oplus m} \to C'[x_1, \ldots, x_n] \to B \otimes_C C' \to 0$$

which is exact at $C'[x_1, \ldots, x_n]$ and whose base change by $\tilde{\sigma}$ gives (37.39.6.1).

By Lemma 37.39.5 we can find a further morphism $(T'', t'') \to (T', t')$ such that $\tilde{\sigma}$ factors through $\mathcal{O}_{T',t'} \to \mathcal{O}_{T'',t''}$ and such that if $\sigma' : C \to R^\wedge$ factors through $\mathcal{O}_{T'',t''}$, then the induced map $\tilde{\sigma}' : C' \to R^\wedge$ agrees modulo $\mathfrak{m}_s^{c+1}$ with $\tilde{\sigma}$. Thus if σ' is such a map, then we obtain a complex

$$R^\wedge[x_1, \ldots, x_n]^{\oplus K} \to R^\wedge[x_1, \ldots, x_n]^{\oplus m} \to R^\wedge[x_1, \ldots, x_n] \to B \otimes_{C,\sigma'} R^\wedge \to 0$$

over $R^\wedge[x_1, \ldots, x_n]$ by applying $\tilde{\sigma}'$ to the polynomials t_{kj} and f_j. In other words, this is the base change of the complex (37.39.6.2) by $\tilde{\sigma}'$. The matrices defining this complex are congruent modulo $\mathfrak{m}_s^{c+1}$ to the matrices defining the complex (37.39.6.1) because $\tilde{\sigma}$ and $\tilde{\sigma}'$ are congruent modulo $\mathfrak{m}_s^{c+1}$. Since (37.39.6.1) is exact, we can apply More on Algebra, Lemma 15.4.2 to conclude that

$$\text{Gr}_{\mathfrak{m}_s}(B \otimes_{C,\sigma'} R^\wedge) \cong \text{Gr}_{\mathfrak{m}_s}(B \otimes_{C,\sigma} R^\wedge)$$

as desired. $\square$

0CB1 Lemma 37.39.7. With notation an assumptions as in Lemma 37.39.4 assume that φ induces an isomorphism on completions. Then we can choose our diagram such that f is étale.

Proof. We may assume $N \geq 2$ and we may replace (T, t) with (T', t') as in Lemma 37.39.6. Since $(V, v) \to (S, s)$ is an elementary étale neighbourhood, so is $(X \times_S V, (x, v)) \to (X, x)$. Thus $\mathcal{O}_{X,x} \to \mathcal{O}_{X \times_S V,(x,v)}$ induces an isomorphism on completions by More on Algebra, Lemma 15.43.9. We claim $\mathcal{O}_{X,x} \to \mathcal{O}_{W,w}$ induces an isomorphism on completions. Having proved this, Lemma 37.12.1 will show that f is smooth at w and of course f is unramified at u as well, so Morphisms, Lemma 29.36.5 tells us f is étale at w.

First we use the commutativity in part (5) of Lemma 37.39.4 to see that for $i \leq N$ there is a commutative diagram

$$\begin{tikzcd} \mathrm{Gr}^i_{\mathfrak m_x}(\mathcal{O}_{X,x}) \arrow[r, "\varphi"'] \arrow[d, equal] & \mathrm{Gr}^i_{\mathfrak m_{y_\sigma}}(\mathcal{O}^\wedge_{Y_\sigma, y_\sigma}) \arrow[r, equal] & \mathrm{Gr}^i_{\mathfrak m_{(y,v)}}(\mathcal{O}_{Y \times_{T,\tau} V,(y,v)}) \arrow[d, "\cong"'] \\ \mathrm{Gr}^i_{\mathfrak m_x}(\mathcal{O}_{X,x}) \arrow[r, "\cong"] & \mathrm{Gr}^i_{\mathfrak m_{(x,v)}}(\mathcal{O}_{X \times_S V,(x,v)}) \arrow[r, "f^\sharp_w"] & \mathrm{Gr}^i_{\mathfrak m_w}(\mathcal{O}_{W,w}) \end{tikzcd}$$

This implies that $f^\sharp_w$ defines an isomorphism $\kappa(x) \to \kappa(w)$ on residue fields and an isomorphism $\mathfrak m_x/\mathfrak m_x^2 \to \mathfrak m_w/\mathfrak m_w^2$ on cotangent spaces. Hence $f^\sharp_w$ defines a surjection $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{W,w}$ on complete local rings.

By Lemma 37.39.6 there is an isomorphism of $\mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}_{(Y \times_{T,\tau} V,(y,v)})$ with $\mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}_{Y_\sigma, y_\sigma})$. This follows by taking stalks of the isomorphism of conormal sheaves at the point y. Since our local rings are Noetherian taking associated graded with respect to $\mathfrak m_s$ commutes with completion because completion with respect to an ideal is an exact functor on finite modules over Noetherian rings. This produces the right vertical isomorphism in the diagram of graded rings

$$\begin{tikzcd} \mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}^\wedge_{W,w}) & \mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}^\wedge_{(Y \times_{T,\tau} V,(y,v)}) \arrow[l] \\ \mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}^\wedge_{X,x}) \arrow[u] \arrow[r, "\varphi"] & \mathrm{Gr}_{\mathfrak m_s}(\mathcal{O}^\wedge_{Y_\sigma, y_\sigma}) \arrow[u, "\cong"'] \end{tikzcd}$$

We do not claim the diagram commutes. By the result of the previous paragraph the left arrow is surjective. The other three arrows are isomorphisms. It follows that the left arrow is a surjective map between isomorphic Noetherian rings. Hence it is an isomorphism by Algebra, Lemma 10.31.10 (you can argue this directly using Hilbert functions as well). In particular $\mathcal{O}^\wedge_{X,x} \to \mathcal{O}^\wedge_{W,w}$ must be injective as well as surjective which finishes the proof. $\square$

37.40. Finite free locally dominates étale

04HE In this section we explain a result that roughly states that étale coverings of a scheme S can be refined by Zariski coverings of finite locally free covers of S.

02LG **Lemma 37.40.1.** Let S be a scheme. Let $s \in S$. Let $f : (U, u) \to (S, s)$ be an étale neighbourhood. There exists an affine open neighbourhood $s \in V \subset S$ and a surjective, finite locally free morphism $\pi : T \to V$ such that for every $t \in \pi^{-1}(s)$ there exists an open neighbourhood $t \in W_t \subset T$ and a commutative diagram

$$\begin{tikzcd} T \arrow[d, "\pi"] & W_t \arrow[l] \arrow[r, "h_t"'] \arrow[dr] & U \arrow[d] \\ V \arrow[rr] & & S \end{tikzcd}$$

with $h_t(t) = u$.

Proof. The problem is local on S hence we may replace S by any open neighbourhood of s. We may also replace U by an open neighbourhood of u. Hence, by Morphisms, Lemma 29.36.14 we may assume that $U \to S$ is a standard étale morphism of affine schemes. In this case the lemma (with $V = S$) follows from Algebra, Lemma 10.144.5. $\square$

02LH Lemma 37.40.2. Let $f : U \to S$ be a surjective étale morphism of affine schemes. There exists a surjective, finite locally free morphism $\pi : T \to S$ and a finite open covering $T = T_1 \cup \ldots \cup T_n$ such that each $T_i \to S$ factors through $U \to S$. Diagram:

$$\begin{tikzcd} & \coprod T_i \arrow[ld] \arrow[rd] & \\ T \arrow[rd, "\pi"] & & U \arrow[ld, "f"'] \\ & S & \end{tikzcd}$$

where the south-west arrow is a Zariski-covering.

Proof. This is a restatement of Algebra, Lemma 10.144.6. □

02LI Remark 37.40.3. In terms of topologies Lemmas 37.40.1 and 37.40.2 mean the following. Let S be any scheme. Let $\{f_i : U_i \to S\}$ be an étale covering of S. There exists a Zariski open covering $S = \bigcup V_j$, for each j a finite locally free, surjective morphism $W_j \to V_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the given étale covering $\{f_i : U_i \to S\}$. What does this mean in practice? Well, for example, suppose we have a descent problem which we know how to solve for Zariski coverings and for fppf coverings of the form $\{\pi : T \to S\}$ with π finite locally free and surjective. Then this descent problem has an affirmative answer for étale coverings as well. This trick was used by Gabber in his proof that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for an affine scheme X, see [Hoo82].

37.41. Étale localization of quasi-finite morphisms

04HF Now we come to a series of lemmas around the theme "quasi-finite morphisms become finite after étale localization". The general idea is the following. Suppose given a morphism of schemes $f : X \to S$ and a point $s \in S$. Let $\varphi : (U, u) \to (S, s)$ be an étale neighbourhood of s in S. Consider the fibre product $X_U = U \times_S X$ and the basic diagram

02LJ (37.41.0.1)
$$\begin{tikzcd} V \arrow[r] \arrow[rd] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & U \arrow[r, "\varphi"] & S \end{tikzcd}$$

where $V \subset X_U$ is open. Is there some standard model for the morphism $f_U : X_U \to U$, or for the morphism $V \to U$ for suitable opens V? Of course the answer is no in general. But for quasi-finite morphisms we can say something.

02LK Lemma 37.41.1. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume that

(1) f is locally of finite type, and
(2) $x \in X_s$ is isolated[7].

Then there exist

(a) an elementary étale neighbourhood $(U, u) \to (S, s)$,
(b) an open subscheme $V \subset X_U$ (see 37.41.0.1)

[7] In the presence of (1) this means that f is quasi-finite at x, see Morphisms, Lemma 29.20.6.

such that

(i) $V \to U$ is a finite morphism,
(ii) there is a unique point v of V mapping to u in U, and
(iii) the point v maps to x under the morphism $X_U \to X$, inducing $\kappa(x) = \kappa(v)$.

Moreover, for any elementary étale neighbourhood $(U', u') \to (U, u)$ setting $V' = U' \times_U V \subset X_{U'}$ the triple (U', u', V') satisfies the properties (i), (ii), and (iii) as well.

Proof. Let $Y \subset X$, $W \subset S$ be affine opens such that $f(Y) \subset W$ and such that $x \in Y$. Note that x is also an isolated point of the fibre of the morphism $f|_Y : Y \to W$. If we can prove the theorem for $f|_Y : Y \to W$, then the theorem follows for f. Hence we reduce to the case where f is a morphism of affine schemes. This case is Algebra, Lemma 10.145.2. □

In the preceding and following lemma we do not assume that the morphism f is separated. This means that the opens V, V_i created in them are not necessarily closed in X_U. Moreover, if we choose the neighbourhood U to be affine, then each V_i is affine, but the intersections $V_i \cap V_j$ need not be affine (in the nonseparated case).

02LL Lemma 37.41.2. Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that

(1) f is locally of finite type, and
(2) $x_i \in X_s$ is isolated for $i = 1, \ldots, n$.

Then there exist

(a) an elementary étale neighbourhood $(U, u) \to (S, s)$,
(b) for each i an open subscheme $V_i \subset X_U$,

such that for each i we have

(i) $V_i \to U$ is a finite morphism,
(ii) there is a unique point v_i of V_i mapping to u in U, and
(iii) the point v_i maps to x_i in X and $\kappa(x_i) = \kappa(v_i)$.

Proof. We will use induction on n. Namely, suppose $(U, u) \to (S, s)$ and $V_i \subset X_U$, $i = 1, \ldots, n-1$ work for $x_1, \ldots, x_{n-1}$. Since $\kappa(s) = \kappa(u)$ the fibre $(X_U)_u = X_s$. Hence there exists a unique point $x'_n \in X_u \subset X_U$ corresponding to $x_n \in X_s$. Also x'_n is isolated in X_u. Hence by Lemma 37.41.1 there exists an elementary étale neighbourhood $(U', u') \to (U, u)$ and an open $V_n \subset X_{U'}$ which works for x'_n and hence for x_n. By the final assertion of Lemma 37.41.1 the open subschemes $V'_i = U' \times_U V_i$ for $i = 1, \ldots, n-1$ still work with respect to $x_1, \ldots, x_{n-1}$. Hence we win. □

If we allow a nontrivial field extension $\kappa(u)/\kappa(s)$, i.e., general étale neighbourhoods, then we can split the points as follows.

02LM Lemma 37.41.3. Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that

(1) f is locally of finite type, and
(2) $x_i \in X_s$ is isolated for $i = 1, \ldots, n$.

Then there exist

(a) an étale neighbourhood $(U, u) \to (S, s)$,
(b) for each i an integer m_i and open subschemes $V_{i,j} \subset X_U$, $j = 1, \ldots, m_i$

such that we have

(i) each $V_{i,j} \to U$ is a finite morphism,
(ii) there is a unique point $v_{i,j}$ of $V_{i,j}$ mapping to u in U with $\kappa(u) \subset \kappa(v_{i,j})$ finite purely inseparable,
(iv) if $v_{i,j} = v_{i',j'}$, then $i = i'$ and $j = j'$, and
(iii) the points $v_{i,j}$ map to x_i in X and no other points of $(X_U)_u$ map to x_i.

Proof. This proof is a variant of the proof of Algebra, Lemma 10.145.4 in the language of schemes. By Morphisms, Lemma 29.20.6 the morphism f is quasi-finite at each of the points x_i. Hence $\kappa(s) \subset \kappa(x_i)$ is finite for each i (Morphisms, Lemma 29.20.5). For each i, let $\kappa(s) \subset L_i \subset \kappa(x_i)$ be the subfield such that $L_i/\kappa(s)$ is separable, and $\kappa(x_i)/L_i$ is purely inseparable. Choose a finite Galois extension $L/\kappa(s)$ such that there exist $\kappa(s)$-embeddings $L_i \to L$ for $i = 1, \ldots, n$. Choose an étale neighbourhood $(U, u) \to (S, s)$ such that $L \cong \kappa(u)$ as $\kappa(s)$-extensions (Lemma 37.35.2).

Let $y_{i,j}$, $j = 1, \ldots, m_i$ be the points of X_U lying over $x_i \in X$ and $u \in U$. By Schemes, Lemma 26.17.5 these points $y_{i,j}$ correspond exactly to the primes in the rings $\kappa(u) \otimes_{\kappa(s)} \kappa(x_i)$. This also explains why there are finitely many; in fact $m_i = [L_i : \kappa(s)]$ but we do not need this. By our choice of L (and elementary field theory) we see that $\kappa(u) \subset \kappa(y_{i,j})$ is finite purely inseparable for each pair i, j. Also, by Morphisms, Lemma 29.20.13 for example, the morphism $X_U \to U$ is quasi-finite at the points $y_{i,j}$ for all i, j.

Apply Lemma 37.41.2 to the morphism $X_U \to U$, the point $u \in U$ and the points $y_{i,j} \in (X_U)_u$. This gives an étale neighbourhood $(U', u') \to (U, u)$ with $\kappa(u) = \kappa(u')$ and opens $V_{i,j} \subset X_{U'}$ with the properties (i), (ii), and (iii) of that lemma. We claim that the étale neighbourhood $(U', u') \to (S, s)$ and the opens $V_{i,j} \subset X_{U'}$ are a solution to the problem posed by the lemma. We omit the verifications. □

02LN Lemma 37.41.4. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that

(1) f is locally of finite type,
(2) f is separated, and
(3) $x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V_1 \amalg \ldots \amalg V_n$$

into open and closed subschemes such that the morphisms $V_i \to U$ are finite, the fibres of $V_i \to U$ over u are singletons $\{v_i\}$, each v_i maps to x_i with $\kappa(x_i) = \kappa(v_i)$, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. Choose $(U, u) \to (S, s)$ and $V_i \subset X_U$ as in Lemma 37.41.2. Since $X_U \to U$ is separated (Schemes, Lemma 26.21.12) and $V_i \to U$ is finite hence proper (Morphisms, Lemma 29.44.11) we see that $V_i \subset X_U$ is closed by Morphisms, Lemma 29.41.7. Hence $V_i \cap V_j$ is a closed subset of V_i which does not contain v_i. Hence the image of $V_i \cap V_j$ in U is a closed set (because $V_i \to U$ proper) not containing u. After shrinking U we may therefore assume that $V_i \cap V_j = \emptyset$ for all i, j. This gives the decomposition as in the lemma. □

Here is the variant where we reduce to purely inseparable field extensions.

02LO Lemma 37.41.5. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that

(1) f is locally of finite type,
(2) f is separated, and
(3) $x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.

Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

into open and closed subschemes such that the morphisms $V_{i,j} \to U$ are finite, the fibres of $V_{i,j} \to U$ over u are singletons $\{v_{i,j}\}$, each $v_{i,j}$ maps to x_i, $\kappa(u) \subset \kappa(v_{i,j})$ is purely inseparable, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. This is proved in exactly the same way as the proof of Lemma 37.41.4 except that it uses Lemma 37.41.3 instead of Lemma 37.41.2. □

The following version may be a little easier to parse.

02LP Lemma 37.41.6. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that

(1) f is locally of finite type,
(2) f is separated, and
(3) X_s has at most finitely many isolated points.

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \amalg V$$

into open and closed subschemes such that the morphism $V \to U$ is finite, and the fibre W_u of the morphism $W \to U$ contains no isolated points. In particular, if $f^{-1}(s)$ is a finite set, then $W_u = \emptyset$.

Proof. This is clear from Lemma 37.41.4 by choosing $x_1, \ldots, x_n$ the complete set of isolated points of X_s and setting $V = \bigcup V_i$. □

37.42. Étale localization of integral morphisms

0BUH Some variants of the results of Section 37.41 for the case of integral morphisms.

0BSR Lemma 37.42.1. Let $R \to S$ be an integral ring map. Let $\mathfrak{p} \subset R$ be a prime ideal. Assume

(1) there are finitely many primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ lying over $\mathfrak{p}$, and
(2) for each i the maximal separable subextension $\kappa(\mathfrak{q})/\kappa(\mathfrak{q}_i)_{sep}/\kappa(\mathfrak{p})$ (Fields, Lemma 9.14.6) is finite over $\kappa(\mathfrak{p})$.

Then there exists an étale ring map $R \to R'$ and a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that

$$S \otimes_R R' = A_1 \times \ldots \times A_m$$

with $R' \to A_j$ integral having a unique prime $\mathfrak{r}_j$ over $\mathfrak{p}'$ such that $\kappa(\mathfrak{r}_j)/\kappa(\mathfrak{p}')$ is purely inseparable.

First proof. This proof uses Algebra, Lemma 10.145.4. Namely, choose a generator $\theta_i \in \kappa(\mathfrak q_i)_{sep}$ of this field over $\kappa(\mathfrak p)$ (Fields, Lemma 9.19.1). The spectrum of the fibre ring $S \otimes_R \kappa(\mathfrak p)$ is finite discrete with points corresponding to $\mathfrak q_1, \ldots, \mathfrak q_n$. By the Chinese remainder theorem (Algebra, Lemma 10.15.4) we see that $S \otimes_R \kappa(\mathfrak p) \to \prod \kappa(\mathfrak q_i)$ is surjective. Hence after replacing R by R_g for some $g \in R$, $g \not\in \mathfrak p$ we may assume that $(0, \ldots, 0, \theta_i, 0, \ldots, 0) \in \prod \kappa(\mathfrak q_i)$ is the image of some $x_i \in S$. Let $S' \subset S$ be the R-subalgebra generated by our x_i. Since $\text{Spec}(S) \to \text{Spec}(S')$ is surjective (Algebra, Lemma 10.36.17) we conclude that $\mathfrak q'_i = S' \cap \mathfrak q_i$ are the primes of S' over $\mathfrak p$. By our choice of x_i we conclude these primes are distinct that and $\kappa(\mathfrak q'_i)_{sep} = \kappa(\mathfrak q_i)_{sep}$. In particular the field extensions $\kappa(\mathfrak q_i)/\kappa(\mathfrak q'_i)$ are purely inseparable. Since $R \to S'$ is finite we may apply Algebra, Lemma 10.145.4. and we get $R \to R'$ and $\mathfrak p'$ and a decomposition

$$S' \otimes_R R' = A'_1 \times \ldots \times A'_m \times B'$$

with $R' \to A'_j$ integral having a unique prime $\mathfrak r'_j$ over $\mathfrak p'$ such that $\kappa(\mathfrak r'_j)/\kappa(\mathfrak p')$ is purely inseparable and such that B' does not have a prime lying over $\mathfrak p'$. Since $R' \to B'$ is finite (as $R \to S'$ is finite) we can after localizing R' at some $g' \in R'$, $g' \not\in \mathfrak p'$ assume that $B' = 0$. Via the map $S' \otimes_R R' \to S \otimes_R R'$ we get the corresponding decomposition for S. $\square$

Second proof. This proof uses strict henselization. First, assume R is strictly henselization with maximal ideal $\mathfrak p$. Then $S/\mathfrak pS$ has finitely many primes corresponding to $\mathfrak q_1, \ldots, \mathfrak q_n$, each maximal, each with purely inseparable residue field over $\kappa(\mathfrak p)$. Hence $S/\mathfrak pS$ is equal to $\prod (S/\mathfrak pS)_{\mathfrak p_i}$. By More on Algebra, Lemma 15.11.6 we can lift this product decomposition to a product composition of S as in the statement.

In the general case, let R^{sh} be the strict henselization of $R_{\mathfrak p}$. Then we can apply the result of the first paragraph to $R^{sh} \to S \otimes_R R^{sh}$. Consider the m mutually orthogonal idempotents in $S \otimes_R R^{sh}$ corresponding to the product decomposition. Since R^{sh} is a filtered colimit of étale ring maps $(R, \mathfrak p) \to (R', \mathfrak p')$ by Algebra, Lemma 10.155.11 we see that these idempotents descend to some R' as desired. $\square$

37.43. Zariski's Main Theorem

02LQ In this section we prove Zariski's main theorem as reformulated by Grothendieck. Often when we say "Zariski's main theorem" in this content we mean either of Lemma 37.43.1, Lemma 37.43.2, or Lemma 37.43.3. In most texts people refer to the last of these as Zariski's main theorem.

We have already proved the algebraic version in Algebra, Theorem 10.123.12 and we have already restated this algebraic version in the language of schemes, see Morphisms, Theorem 29.56.1. The version in this section is more subtle; to get the full result we use the étale localization techniques of Section 37.41 to reduce to the algebraic case.

03GW Lemma 37.43.1. Let $f : X \to S$ be a morphism of schemes. Assume f is of finite type and separated. Let S' be the normalization of S in X, see Morphisms, Definition 29.53.3. Picture:

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\nu"] \\ & S & \end{tikzcd}$$

Then there exists an open subscheme $U' \subset S'$ such that

(1) $(f')^{-1}(U') \to U'$ is an isomorphism, and
(2) $(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.

Proof. By Morphisms, Lemma 29.56.2 the subset $U \subset X$ of points where f is quasi-finite is open. The lemma is equivalent to

(a) $U' = f'(U) \subset S'$ is open,
(b) $U = (f')^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Let $x \in U$ be arbitrary. We claim there exists an open neighbourhood $f'(x) \in V \subset S'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is both locally of finite type over S (as an open subscheme of X) and for $v \in V$ the residue field extension $\kappa(v)/\kappa(\nu(v))$ is algebraic (as $V \subset S'$ and S' is integral over S). Hence the fibres of $V \to S$ are discrete (Morphisms, Lemma 29.20.2) and $(f')^{-1}V \to S$ is locally quasi-finite (Morphisms, Lemma 29.20.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $s = f(x)$. Let $(T, t) \to (S, s)$ be an elementary étale neighbourhood. Denote by a subscript $_T$ the base change to T. Let $y = (x, t) \in X_T$ be the unique point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms, Lemma 29.20.13. Note that

$$X_T \xrightarrow{f'_T} S'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 37.19.2. Suppose that the claim holds for $y \in U_T \subset X_T \to S'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(y) \in V' \subset S'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $S'_T \to S'$ is étale hence the image $V \subset S'$ of V' is open. Observe that $f'(x) \in V$ as $f'_T(y) \in V'$. Observe that

$$\begin{tikzcd} (f'_T)^{-1}V' \arrow[r] \arrow[d] & (f')^{-1}(V) \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

is a fibre square (as $S'_T \times_{S'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent, Lemma 35.23.17. In other words, the claim holds for $x \in U \subset X \to S' \to S$.

By the result of the previous paragraph we may replace S by an elementary étale neighbourhood of $s = f(x)$ in order to prove the claim. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subschemes where $V \to S$ is finite and $x \in V$, see Lemma 37.41.4. Since X is a disjoint union of V and W over S and since $V \to S$ is finite we see that the normalization of S in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of S in W, see Morphisms, Lemmas 29.53.10, 29.44.4, and 29.53.12. The claim follows and we win. $\square$

02LR Lemma 37.43.2. Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated. Let S' be the normalization of S in X, see Morphisms, Definition 29.53.3. Picture:

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\nu"] \\ & S & \end{tikzcd}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 37.43.1. Namely, by that lemma there exists an open subscheme $U' \subset S'$ such that $(f')^{-1}(U') = X$ and $X \to U'$ is an isomorphism. In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : S' \to S$ is separated (Schemes, Lemma 26.21.14). It follows that f is quasi-affine by Morphisms, Lemma 29.13.3. $\square$

05K0 Lemma 37.43.3 (Zariski's Main Theorem). Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated and assume that S is quasi-compact and quasi-separated. Then there exists a factorization [DG67, IV Corollary 18.12.13]

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to S' \to S$ be as in the conclusion of Lemma 37.43.2. By Properties, Lemma 28.22.13 we can write $\nu_*\mathcal{O}_{S'} = \text{colim}_{i \in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{S'}$. Then $\pi_i : T_i = \underline{\text{Spec}}_S(\mathcal{A}_i) \to S$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $S' = \lim T_i$.

By Limits, Lemma 32.4.11 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in S' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits, Lemma 32.2.2. By Limits, Lemma 32.4.16 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms, Lemma 29.3.2 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. $\square$

0F2N Lemma 37.43.4. With notation and hypotheses as in Lemma 37.43.3. Assume moreover that f is locally of finite presentation. Then we can choose the factorization such that T is finite and of finite presentation over S.

Proof. By Limits, Lemma 32.9.8 we can write $T = \lim T_i$ where all T_i are finite and of finite presentation over Y and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits, Lemma 32.4.11 there exists an i and an open subscheme $U_i \subset T_i$ whose inverse image in T is X. By Limits, Lemma 32.4.16 we see that $X \cong U_i$ for large enough i. Replacing T by T_i finishes the proof. □

37.44. Applications of Zariski's Main Theorem, I

0F2P A first application is the characterization of finite morphisms as proper morphisms with finite fibres.

02LS Lemma 37.44.1. Let $f : X \to S$ be a morphism of schemes. The following are equivalent:

(1) f is finite,
(2) f is proper with finite fibres,
(3) f is proper and locally quasi-finite,
(4) f is universally closed, separated, locally of finite type and has finite fibres.

Proof. We have (1) implies (2) by Morphisms, Lemmas 29.44.11, 29.20.10, and 29.44.10. We have (2) implies (3) by Morphisms, Lemma 29.20.7. We have (3) implies (4) by the definition of proper morphisms and Morphisms, Lemmas 29.20.9 and 29.20.10.

Assume (4). Pick $s \in S$. By Morphisms, Lemma 29.20.7 we see that all the finitely many points of X_s are isolated in X_s. Choose an elementary étale neighbourhood $(U, u) \to (S, s)$ and decomposition $X_U = V \amalg W$ as in Lemma 37.41.6. Note that $W_u = \emptyset$ because all points of X_s are isolated. Since f is universally closed we see that the image of W in U is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $s \in S$ was arbitrary this means there exists a family $\{U_i \to S\}$ of étale morphisms whose images cover S such that the base changes $X_{U_i} \to U_i$ are finite. Note that $\{U_i \to S\}$ is an étale covering, see Topologies, Definition 34.4.1. Hence it is an fpqc covering, see Topologies, Lemma 34.9.6. Hence we conclude f is finite by Descent, Lemma 35.23.23. □

As a consequence we have the following useful results.

02UP Lemma 37.44.2. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that f is proper and $f^{-1}(\{s\})$ is a finite set. Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 29.20.7. By Morphisms, Lemma 29.56.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $f(Z) \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 29.20.9), hence has finite fibres (Morphisms, Lemma 29.20.10), hence is finite by Lemma 37.44.1. □

0AH8 Lemma 37.44.3. Consider a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Let $s \in S$. Assume

(1) $X \to S$ is a proper morphism,
(2) $Y \to S$ is separated and locally of finite type, and
(3) the image of $X_s \to Y_s$ is finite.

Then there is an open subspace $U \subset S$ containing s such that $X_U \to Y_U$ factors through a closed subscheme $Z \subset Y_U$ finite over U.

Proof. Let $Z \subset Y$ be the scheme theoretic image of h, see Morphisms, Section 29.6. By Morphisms, Lemma 29.41.10 the morphism $X \to Z$ is surjective and $Z \to S$ is proper. Thus $X_s \to Z_s$ is surjective. We see that either (3) implies Z_s is finite. Hence $Z \to S$ is finite in an open neighbourhood of s by Lemma 37.44.2. □

37.45. Applications of Zariski's Main Theorem, II

0F2Q In this section we give a few more consequences of Zariski's main theorem to the structure of quasi-finite morphisms.

07S0 Lemma 37.45.1. Let $f : X \to Y$ be a separated, locally quasi-finite morphism with Y affine. Then every finite set of points of X is contained in an open affine of X.

Proof. Let $x_1, \ldots, x_n \in X$. Choose a quasi-compact open $U \subset X$ with $x_i \in U$. Then $U \to Y$ is quasi-affine by Lemma 37.43.2. Hence there exists an affine open $V \subset U$ containing $x_1, \ldots, x_n$ by Properties, Lemma 28.29.5. □

03I1 Lemma 37.45.2. Let $f : Y \to X$ be a quasi-finite morphism. There exists a dense open $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.

Proof. If $U_i \subset X$, $i \in I$ is a collection of opens such that the restrictions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are finite, then with $U = \bigcup U_i$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite, see Morphisms, Lemma 29.44.3. Thus the problem is local on X and we may assume that X is affine.

Assume X is affine. Write $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine. This is possible since f is quasi-finite and hence in particular quasi-compact. Each $V_j \to X$ is quasi-finite and separated. Let $\eta \in X$ be a generic point of an irreducible component of X. We see from Morphisms, Lemmas 29.20.10 and 29.51.1 that there exists an open neighbourhood $\eta \in U_\eta$ such that $f^{-1}(U_\eta) \cap V_j \to U_\eta$ is finite. We may choose U_η such that it works for each $j = 1, \ldots, m$. Note that the collection of generic points of X is dense in X. Thus we see there exists a dense open $W = \bigcup_\eta U_\eta$ such that each $f^{-1}(W) \cap V_j \to W$ is finite. It suffices to show that there exists a dense open $U \subset W$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. Thus we may replace X by an affine open subscheme of W and assume that each $V_j \to X$ is finite.

Assume X is affine, $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine, and the restrictions $f|_{V_j} : V_j \to X$ are finite. Set

$$\Delta_{ij} = \left(\overline{V_i \cap V_j} \setminus V_i \cap V_j\right) \cap V_j.$$

This is a nowhere dense closed subset of V_j because it is the boundary of the open subset $V_i \cap V_j$ in V_j. By Morphisms, Lemma 29.48.7 the image $f(\Delta_{ij})$ is a nowhere dense closed subset of X. By Topology, Lemma 5.21.2 the union $T = \bigcup f(\Delta_{ij})$ is a nowhere dense closed subset of X. Thus $U = X \setminus T$ is a dense open subset of X. We claim that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. To see this let $U' \subset U$ be an affine open. Set $Y' = f^{-1}(U') = U' \times_X Y$, $V_j' = Y' \cap V_j = U' \times_X V_j$. Consider the restriction

$$f' = f|_{Y'} : Y' \longrightarrow U'$$

of f. This morphism now has the property that $Y' = \bigcup_{j=1,\ldots,m} V_j'$ is an affine open covering, each $V_j' \to U'$ is finite, and $V_i' \cap V_j'$ is (open and) closed both in V_i' and V_j'. Hence $V_i' \cap V_j'$ is affine, and the map

$$\mathcal{O}(V_i') \otimes_{\mathbf{Z}} \mathcal{O}(V_j') \longrightarrow \mathcal{O}(V_i' \cap V_j')$$

is surjective. This implies that Y' is separated, see Schemes, Lemma 26.21.7. Finally, consider the commutative diagram

$$\begin{tikzcd} \coprod_{j=1,\ldots,m} V_j' \arrow[rr] \arrow[rd] & & Y' \arrow[ld] \\ & U' & \end{tikzcd}$$

The south-east arrow is finite, hence proper, the horizontal arrow is surjective, and the south-west arrow is separated. Hence by Morphisms, Lemma 29.41.9 we conclude that $Y' \to U'$ is proper. Since it is also quasi-finite, we see that it is finite by Lemma 37.44.1, and we win. $\square$

07RY Lemma 37.45.3. Let $f : X \to S$ be flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. Then there exist closed subsets

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$$

such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \coprod_{r=0,\ldots,n} S_r$ is characterized by the following universal property: Given $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree r if and only if $g(T) \subset S_r$ (set theoretically).

Proof. Let n be an integer bounding the degree of the fibres of $X \to S$. By Morphisms, Lemma 29.57.5 we see that any base change has degrees of fibres bounded by n also. In particular, all the integers r that occur in the statement of the lemma will be $\leq n$. We will prove the lemma by induction on n. The base case is $n = 0$ which is obvious.

We claim the set of points $s \in S$ with $\deg_{\kappa(s)}(X_s) = n$ is an open subset $S_n \subset S$ and that $X \times_S S_n \to S_n$ is finite locally free of degree n. Namely, suppose that $s \in S$ is such a point. Choose an elementary étale morphism $(U, u) \to (S, s)$ and a decomposition $U \times_S X = W \amalg V$ as in Lemma 37.41.6. Since $V \to U$ is finite, flat, and locally of finite presentation, we see that $V \to U$ is finite locally free, see Morphisms, Lemma 29.48.2. After shrinking U to a smaller neighbourhood of u we may assume $V \to U$ is finite locally free of some degree d, see Morphisms, Lemma 29.48.5. As $u \mapsto s$ and $W_u = \emptyset$ we see that $d = n$. Since n is the maximum degree of a fibre we see that $W = \emptyset$! Thus $U \times_S X \to U$ is finite locally free of degree n. By Descent, Lemma 35.23.30 we conclude that $X \to S$ is finite locally free of degree n over $\mathrm{Im}(U \to S)$ which is an open neighbourhood of s (Morphisms, Lemma 29.36.13). This proves the claim.

Let $S' = S \setminus S_n$ endowed with the reduced induced scheme structure and set $X' = X \times_S S'$. Note that the degrees of fibres of $X' \to S'$ are universally bounded by $n - 1$. By induction we find a stratification $S' = S_0 \amalg \ldots \amalg S_{n-1}$ adapted to the morphism $X' \to S'$. We claim that $S = \coprod_{r=0,\ldots,n} S_r$ works for the morphism $X \to S$. Let $g : T \to S$ be a morphism of schemes and assume that $X \times_S T \to T$ is finite locally free of degree r. As remarked above this implies that $r \leq n$. If $r = n$, then it is clear that $T \to S$ factors through S_n. If $r < n$, then $g(T) \subset S' = S \setminus S_d$ (set theoretically) hence $T_{red} \to S$ factors through S', see Schemes, Lemma 26.12.7. Note that $X \times_S T_{red} \to T_{red}$ is also finite locally free of degree r as a base change. By the universal property of the stratification $S' = \coprod_{r=0,\ldots,n-1} S_r$ we see that $g(T) = g(T_{red})$ is contained in S_r. Conversely, suppose that we have $g : T \to S$ such that $g(T) \subset S_r$ (set theoretically). If $r = n$, then g factors through S_n and it is clear that $X \times_S T \to T$ is finite locally free of degree n as a base change. If $r < n$, then $X \times_S T \to T$ is a morphism which is separated, flat, and locally of finite presentation, such that the restriction to T_{red} is finite locally free of degree r. Since $T_{red} \to T$ is a universal homeomorphism, we conclude that $X \times_S T_{red} \to X \times_S T$ is a universal homeomorphism too and hence $X \times_S T \to T$ is universally closed (as this is true for the finite morphism $X \times_S T_{red} \to T_{red}$). It follows that $X \times_S T \to T$ is finite, for example by Lemma 37.44.1. Then we can use Morphisms, Lemma 29.48.2 to see that $X \times_S T \to T$ is finite locally free. Finally, the degree is r as all the fibres have degree r. □

07RZ Lemma 37.45.4. Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, separated, and quasi-finite. Then there exist closed subsets

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset S$$

such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \coprod S_r$ is characterized by the following universal property: Given a morphism $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree r if and only if $g(T) \subset S_r$ (set theoretically). Moreover, the inclusion maps $S_r \to S$ are quasi-compact.

Proof. The question is local on S, hence we may assume that S is affine. By Morphisms, Lemma 29.57.9 the fibres of f are universally bounded in this case. Hence the existence of the stratification follows from Lemma 37.45.3.

We will show that $U_r = S \setminus Z_r \to S$ is quasi-compact for each $r \geq 0$. This will prove the final statement by elementary topology. Since a composition of quasi-compact maps is quasi-compact it suffices to prove that $U_r \to U_{r-1}$ is quasi-compact. Choose an affine open $W \subset U_{r-1}$. Write $W = \mathrm{Spec}(A)$. Then $Z_r \cap W = V(I)$ for some ideal $I \subset A$ and $X \times_S \mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I)$ is finite locally free of degree r. Note that $A/I = \mathrm{colim}\, A/I_i$ where $I_i \subset I$ runs through the finitely generated ideals. By Limits, Lemma 32.8.8 we see that $X \times_S \mathrm{Spec}(A/I_i) \to \mathrm{Spec}(A/I_i)$ is finite locally free of degree r for some i. (This uses that $X \to S$ is of finite presentation, as it is locally of finite presentation, separated, and quasi-compact.) Hence $\mathrm{Spec}(A/I_i) \to \mathrm{Spec}(A) = W$ factors (set theoretically) through $Z_r \cap W$. It follows that $Z_r \cap W = V(I_i)$ is the zero set of a finite subset of elements of A. This means that $W \setminus Z_r$ is a finite union of standard opens, hence quasi-compact, as desired. □

086R Lemma 37.45.5. Let $f : X \to S$ be a flat, locally of finite presentation, separated, and locally quasi-finite morphism of schemes. Then there exist open subschemes

$$S = U_0 \supset U_1 \supset U_2 \supset \ldots$$

such that a morphism $\mathrm{Spec}(k) \to S$ where k is a field factors through U_d if and only if $X \times_S \mathrm{Spec}(k)$ has degree $\geq d$ over k.

Proof. The statement simply means that the collection of points where the degree of the fibre is $\geq d$ is open. Thus we can work locally on S and assume S is affine. In this case, for every $W \subset X$ quasi-compact open, the set of points $U_d(W)$ where the fibres of $W \to S$ have degree $\geq d$ is open by Lemma 37.45.4. Since $U_d = \bigcup_W U_d(W)$ the result follows. $\square$

082V Lemma 37.45.6. Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Let $g \in \Gamma(X, \mathcal{O}_X)$ nonzero. Then there exist an open $V \subset X$ such that $g|_V \neq 0$, an open $U \subset S$ fitting into a commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\ U \arrow[r] & S, \end{tikzcd}$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ whose image contains $g|_V$.

Proof. We may assume X and S affine. We obtain a filtration $\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$ as in Lemmas 37.45.3 and 37.45.4. Let $T \subset X$ be the scheme theoretic support of the finite $\mathcal{O}_X$-module $\mathrm{Im}(g : \mathcal{O}_X \to \mathcal{O}_X)$. Note that T is the support of g as a section of $\mathcal{O}_X$ (Modules, Definition 17.5.1) and for any open $V \subset X$ we have $g|_V \neq 0$ if and only if $V \cap T \neq \emptyset$. Let r be the smallest integer such that $f(T) \subset Z_r$ set theoretically. Let $\xi \in T$ be a generic point of an irreducible component of T such that $f(\xi) \notin Z_{r-1}$ (and hence $f(\xi) \in Z_r$). We may replace S by an affine neighbourhood of $f(\xi)$ contained in $S \setminus Z_{r-1}$. Write $S = \mathrm{Spec}(A)$ and let $I = (a_1, \ldots, a_m) \subset A$ be a finitely generated ideal such that $V(I) = Z_r$ (set theoretically, see Algebra, Lemma 10.29.1). Since the support of g is contained in $f^{-1}V(I)$ by our choice of r we see that there exists an integer N such that $a_j^N g = 0$ for $j = 1, \ldots, m$. Replacing a_j by a_j^r we may assume that $Ig = 0$. For any A-module M write $M[I]$ for the I-torsion of M, i.e., $M[I] = \{m \in M \mid Im = 0\}$. Write $X = \mathrm{Spec}(B)$, so $g \in B[I]$. Since $A \to B$ is flat we see that

$$B[I] = A[I] \otimes_A B \cong A[I] \otimes_{A/I} B/IB$$

By our choice of Z_r, the A/I-module B/IB is finite locally free of rank r. Hence after replacing S by a smaller affine open neighbourhood of $f(\xi)$ we may assume that $B/IB \cong (A/IA)^{\oplus r}$ as A/I-modules. Choose a map $\psi : A^{\oplus r} \to B$ which reduces modulo I to the isomorphism of the previous sentence. Then we see that the induced map

$$A[I]^{\oplus r} \longrightarrow B[I]$$

is an isomorphism. The lemma follows by taking $\mathcal{F}$ the quasi-coherent sheaf associated to the A-module $A[I]$ and the map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ the one corresponding to $A[I]^{\oplus r} \subset A^{\oplus r} \to B$. $\square$

09Z0 Lemma 37.45.7. Let $U \to X$ be a surjective étale morphism of schemes. Assume X is quasi-compact and quasi-separated. Then there exists a surjective integral morphism $Y \to X$, such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U. In fact, we may assume $Y \to X$ is finite and of finite presentation.

Proof. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing U by U', we see that we may assume U is affine. In particular $U \to X$ is separated (Schemes, Lemma 26.21.15). Then there exists an integer d bounding the degree of the geometric fibres of $U \to X$ (see Morphisms, Lemma 29.57.9). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Lemma 37.43.2 and we obtain a factorization

$$\begin{tikzcd} U \arrow[rr, "j"'] \arrow[rd] & & Y \arrow[ld, "\pi"] \\ & X & \end{tikzcd}$$

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y. Note that

$$U \times_X Y = U \amalg W$$

where the first summand is the image of $U \to U \times_X Y$ (which is closed by Schemes, Lemma 26.21.10 and open because it is étale as a morphism between schemes étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subscheme containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $U \subset Y$ is dense, it holds for all geometric points of Y for example by Lemma 37.45.3 (the degree of the fibres of a quasi-compact separated étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with Z a scheme, which Zariski locally factors through W. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Lemma 37.43.2). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z'. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced reduced closed subscheme structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through U. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U. If $z \in Z$, then we have a neighbourhood $\Omega \subset Z$ which factors through $W \subset U \times_X Y$ and hence through U. This proves existence.

Assume we have found $Y \to X$ integral and surjective which Zariski locally factors through U. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Limits, Lemma 32.7.3. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Limits, Lemma 32.4.11. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Limits, Proposition 32.6.1. This finishes the proof. □

37.46. Application to morphisms with connected fibres

057H In this section we prove some lemmas that produce morphisms all of whose fibres are geometrically connected or geometrically integral. This will be useful in our study of the local structure of morphisms of finite type later.

057I **Lemma 37.46.1.** Consider a diagram of morphisms of schemes

$$\begin{tikzcd} Z \arrow[r, "\sigma"'] \arrow[dr] & X \arrow[d] \\ & Y \end{tikzcd}$$

an a point $y \in Y$. Assume

(1) $X \to Y$ is of finite presentation and flat,
(2) $Z \to Y$ is finite locally free,
(3) $Z_y \neq \emptyset$,
(4) all fibres of $X \to Y$ are geometrically reduced, and
(5) X_y is geometrically connected over $\kappa(y)$.

Then there exists a quasi-compact open $X^0 \subset X$ such that $X^0_y = X_y$ and such that all nonempty fibres of $X^0 \to Y$ are geometrically connected.

Proof. In this proof we will use that flat, finite presentation, finite locally free are properties that are preserved under base change and composition. We will also use that a finite locally free morphism is both open and closed. You can find these facts as Morphisms, Lemmas 29.25.8, 29.21.4, 29.48.4, 29.25.6, 29.21.3, 29.48.3, 29.25.10, and 29.44.11.

Note that $X_Z \to Z$ is flat morphism of finite presentation which has a section s coming from σ. Let X^0_Z denote the subset of X_Z defined in Situation 37.29.1. By Lemma 37.29.6 it is an open subset of X_Z.

The pullback $X_{Z \times_Y Z}$ of X to $Z \times_Y Z$ comes equipped with two sections s_0, s_1, namely the base changes of s by $\text{pr}_0, \text{pr}_1 : Z \times_Y Z \to Z$. The construction of Situation 37.29.1 gives two subsets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$. By Lemma 37.29.2 these are the inverse images of X^0_Z under the morphisms $1_X \times \text{pr}_0, 1_X \times \text{pr}_1 : X_{Z \times_Y Z} \to X_Z$. In particular these subsets are open.

Let $(Z \times_Y Z)_y = \{z_1, \ldots, z_n\}$. As X_y is geometrically connected, we see that the fibres of $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$ over each z_i agree (being equal to the whole fibre). Another way to say this is that

$$s_0(z_i) \in (X_{Z \times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(z_i) \in (X_{Z \times_Y Z})^0_{s_0}.$$

Since the sets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$ are open in $X_{Z \times_Y Z}$ there exists an open neighbourhood $W \subset Z \times_Y Z$ of $(Z \times_Y Z)_y$ such that

$$s_0(W) \subset (X_{Z \times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(W) \subset (X_{Z \times_Y Z})^0_{s_0}.$$

Then it follows directly from the construction in Situation 37.29.1 that

$$p^{-1}(W) \cap (X_{Z\times_Y Z})^0_{s_0} = p^{-1}(W) \cap (X_{Z\times_Y Z})^0_{s_1}$$

where $p : X_{Z\times_Y Z} \to Z \times_W Z$ is the projection. Because $Z \times_Y Z \to Y$ is finite locally free, hence open and closed, there exists an affine open neighbourhood $V \subset Y$ of y such that $q^{-1}(V) \subset W$, where $q : Z \times_Y Z \to Y$ is the structure morphism. To prove the lemma we may replace Y by V. After we do this we see that $X^0_Z \subset Y_Z$ is an open such that

$$(1_X \times \mathrm{pr}_0)^{-1}(X^0_Z) = (1_X \times \mathrm{pr}_1)^{-1}(X^0_Z).$$

This means that the image $X^0 \subset X$ of X^0_Z is an open such that $(X_Z \to X)^{-1}(X^0) = X^0_Z$, see Descent, Lemma 35.13.6. Finally, X^0 is quasi-compact because X^0_Z is quasi-compact by Lemma 37.29.4 (use that at this point Y is affine, hence X is quasi-compact and quasi-separated, hence locally constructible is the same as constructible and in particular quasi-compact; details omitted). In this way we see that X^0 has all the desired properties. □

055W Lemma 37.46.2. Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume

(1) h is flat and of finite presentation,
(2) all fibres of h are geometrically reduced, and
(3) T is geometrically connected over $\kappa(s)$.

Then we can find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that

(a) all fibres of $V \to S'$ are geometrically connected,
(b) $V_{s'} = T \times_s s'$.

Proof. The problem is clearly local on S, hence we may replace S by an affine open neighbourhood of s. The topology on Y_s is induced from the topology on Y, see Schemes, Lemma 26.18.5. Hence we can find a quasi-compact open $V \subset Y$ such that $V_s = T$. The restriction of h to V is quasi-compact (as S affine and V quasi-compact), quasi-separated, locally of finite presentation, and flat hence flat of finite presentation. Thus after replacing Y by V we may assume, in addition to (1) and (2) that $Y_s = T$ and S affine.

Pick a closed point $y \in Y_s$ such that h is Cohen-Macaulay at y, see Lemma 37.22.7. By Lemma 37.23.4 there exists a diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[rd] & Y \arrow[d] \\ & S \end{tikzcd}$$

such that $Z \to S$ is flat, locally of finite presentation, locally quasi-finite with $Z_s = \{y\}$. Apply Lemma 37.41.1 to find an elementary neighbourhood $(S', s') \to (S, s)$ and an open $Z' \subset Z_{S'} = S' \times_S Z$ with $Z' \to S'$ finite with a unique point $z' \in Z'$ lying over s. Note that $Z' \to S'$ is also locally of finite presentation and flat (as an open of the base change of $Z \to S$), hence $Z' \to S'$ is finite locally free, see Morphisms, Lemma 29.48.2. Note that $Y_{S'} \to S'$ is flat and of finite presentation with geometrically reduced fibres as a base change of h. Also $Y_{s'} = Y_s$ is geometrically connected. Apply Lemma 37.46.1 to $Z' \to Y_{S'}$ over S' to get

$V \subset Y_{S'}$ quasi-compact open satisfying (2) whose fibres over S' are either empty or geometrically connected. As $V \to S'$ is open (Morphisms, Lemma 29.25.10), after replacing S' by an affine open neighbourhood of s' we may assume $V \to S'$ is surjective, whence (1) holds. $\square$

0EY5 Lemma 37.46.3. Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation and flat with geometrically reduced fibres. Then there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that $X_i \to S$ factors as $X_i \to S_i \to S$ where $S_i \to S$ is étale and $X_i \to S_i$ is flat of finite presentation with geometrically connected and geometrically reduced fibres.

Proof. Pick a point $x \in X$ with image $s \in S$. We will produce a diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[dr] & S' \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ & S' \arrow[r] & S \end{tikzcd}$$

and points $s' \in S'$, $x' \in X'$, $y \in S' \times_S X$ such that x' maps to x, $(S', s') \to (S, s)$ is an étale neighbourhood, $(X', x') \to (S' \times_S X, y)$ is an étale neighbourhood[8], and $X' \to S'$ has geometrically connected fibres. If we can do this for every $x \in X$, then the lemma follows (with members of the covering given by the collection of étale morphisms $X' \to X$ so produced). The first step is the replace X and S by affine open neighbourhoods of x and s which reduces us to the case that X and S are affine (and hence f of finite presentation).

Choose a separable algebraic extension $\overline{k}$ of $\kappa(s)$. Denote $X_{\overline{k}}$ the base change of X_s. Choose a point $\overline{x}$ in $X_{\overline{k}}$ mapping to $x \in X_s$. Choose a connected quasi-compact open neighbourhood $\overline{V} \subset X_{\overline{k}}$ of $\overline{x}$. (This is possible because any scheme locally of finite type over a field is locally connected as a locally Noetherian topological space.) By Varieties, Lemma 33.7.9 we can find a finite separable extension $k'/\kappa(s)$ and a quasi-compact open $V' \subset X_{k'}$ whose base change is $\overline{V}$. In particular V' is geometrically connected over k', see Varieties, Lemma 33.7.7. By Lemma 37.35.2 we can find an étale neighbourhood $(S', s') \to (S, s)$ such that $\kappa(s')$ is isomorphic to k' as an extension of $\kappa(s)$. Denote $x' \in (S' \times_S X)_{s'} = X_{k'}$ the image of $\overline{x}$. Thus after replacing (S, s) by (S', s') and (X, x) by $(S' \times_S X, x')$ we reduce to the case handled in the next paragrah.

Assume there is a quasi-compact open $V \subset X_s$ which contains x and is geometrically irreducible. Then we can apply Lemma 37.46.2 to find an affine étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $X' \subset S' \times_S X$ such that $X' \to S'$ has geometrically connected fibres and such that X' contains a point mapping to x. This finishes the proof. $\square$

057J Lemma 37.46.4. Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume

(1) h is of finite presentation,
(2) h is normal, and
(3) T is geometrically irreducible over $\kappa(s)$.

[8] The proof actually gives an open $X' \subset S' \times_S X$.

Then we can find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that

(a) all fibres of $V \to S'$ are geometrically integral,
(b) $V_{s'} = T \times_s s'$.

Proof. Apply Lemma 37.46.2 to find an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and a quasi-compact open $V \subset Y_{S'}$ such that all fibres of $V \to S'$ are geometrically connected and $V_{s'} = T \times_s s'$. As V is an open of the base change of h all fibres of $V \to S'$ are geometrically normal, see Lemma 37.20.2. In particular, they are geometrically reduced. To finish the proof we have to show they are geometrically irreducible. But, if $t \in S'$ then V_t is of finite type over $\kappa(t)$ and hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is of finite type over $\overline{\kappa(t)}$ hence Noetherian. By choice of $S' \to S$ the scheme $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is connected. Hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is irreducible by Properties, Lemma 28.7.6 and we win. $\square$

37.47. Application to the structure of finite type morphisms

052D The result in this section can be found in [GR71]. Loosely stated it says that a finite type morphism is étale locally on the source and target the composition of a finite morphism by a smooth morphism with geometrically connected fibres of relative dimension equal to the fibre dimension of the original morphism.

052E Lemma 37.47.1. Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y \arrow[d, "h"] \\ S \arrow[r, equal] & S \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & y \arrow[d, mapsto] \\ s \arrow[r, equal] & s \end{tikzcd}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y = \pi(x')$ we have

(1) $h : Y \to S$ is smooth of relative dimension n,
(2) $g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,
(3) π is finite, and $\pi^{-1}(\{y\}) = \{x'\}$, and
(4) $\kappa(y)$ is a purely transcendental extension of $\kappa(s)$.

Moreover, if f is locally of finite presentation then π is of finite presentation.

Proof. The problem is local on X and S, hence we may assume that X and S are affine. By Algebra, Lemma 10.125.3 after replacing X by a standard open neighbourhood of x in X we may assume there is a factorization

$$X \xrightarrow{\pi} \mathbf{A}^n_S \longrightarrow S$$

such that π is quasi-finite and such that $\kappa(\pi(x))$ is purely transcendental over $\kappa(s)$. By Lemma 37.41.1 there exists an elementary étale neighbourhood

$$(Y, y) \to (\mathbf{A}^n_S, \pi(x))$$

and an open $X' \subset X \times_{\mathbf{A}^n_S} Y$ which contains a unique point x' lying over y such that $X' \to Y$ is finite. This proves (1) – (4) hold. For the final assertion, use Morphisms, Lemma 29.21.11. $\square$

057K Lemma 37.47.2. Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & y' \arrow[d, mapsto] \\ s & s' \arrow[l, mapsto] \end{tikzcd}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y' = \pi(x')$, $s' = h(y')$ we have

(1) $h : Y' \to S'$ is smooth of relative dimension n,
(2) all fibres of $Y' \to S'$ are geometrically integral,
(3) $g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,
(4) π is finite, and $\pi^{-1}(\{y'\}) = \{x'\}$,
(5) $\kappa(y')$ is a purely transcendental extension of $\kappa(s')$, and
(6) $e : (S', s') \to (S, s)$ is an elementary étale neighbourhood.

Moreover, if f is locally of finite presentation, then π is of finite presentation.

Proof. The question is local on S, hence we may replace S by an affine open neighbourhood of s. Next, we apply Lemma 37.47.1 to get a commutative diagram

$$\begin{tikzcd} X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\ & Y \arrow[d, "h"] \\ S \arrow[r, equal] & S \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] \\ & y \arrow[d, mapsto] \\ s \arrow[r, equal] & s \end{tikzcd}$$

where h is smooth of relative dimension n and $\kappa(y)$ is a purely transcendental extension of $\kappa(s)$. Since the question is local on X also, we may replace Y by an affine neighbourhood of y (and X' by the inverse image of this under π). As S is affine this guarantees that $Y \to S$ is quasi-compact, separated and smooth, in particular of finite presentation. Let T be the connected component of Y_s containing y. As Y_s is Noetherian we see that T is open. We also see that T is geometrically connected over $\kappa(s)$ by Varieties, Lemma 33.7.14. Since T is also smooth over $\kappa(s)$ it is geometrically normal, see Varieties, Lemma 33.25.4. We conclude that T is geometrically irreducible over $\kappa(s)$ (as a connected Noetherian normal scheme is irreducible, see Properties, Lemma 28.7.6). Finally, note that the smooth morphism h is normal by Lemma 37.20.3. At this point we have verified all assumption of Lemma 37.46.4 hold for the morphism $h : Y \to S$ and open $T \subset Y_s$. As a result of applying Lemma 37.46.4 we obtain $e : S' \to S$, $s' \in S'$, Y' as in the commutative

diagram

$$
\begin{tikzcd}
X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] & X' \times_Y Y' \arrow[l] \arrow[d] \\
 & Y \arrow[d, "h"] & Y' \arrow[l] \arrow[d] \\
S \arrow[r, equal] & S & S' \arrow[l, "e"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[dd, mapsto] & x' \arrow[l, mapsto] \arrow[d, mapsto] & (x', s') \arrow[l, mapsto] \arrow[d, mapsto] \\
 & y \arrow[d, mapsto] & (y, s') \arrow[l, mapsto] \arrow[d, mapsto] \\
s \arrow[r, equal] & s & s' \arrow[l, mapsto]
\end{tikzcd}
$$

where $e : (S', s') \to (S, s)$ is an elementary étale neighbourhood, and where $Y' \subset Y_{S'}$ is an open neighbourhood all of whose fibres over S' are geometrically irreducible, such that $Y'_{s'} = T$ via the identification $Y_s = Y_{S', s'}$. Let $(y, s') \in Y'$ be the point corresponding to $y \in T$; this is also the unique point of $Y \times_S S'$ lying over y with residue field equal to $\kappa(y)$ which maps to s' in S'. Similarly, let $(x', s') \in X' \times_Y Y' \subset X' \times_S S'$ be the unique point over x' with residue field equal to $\kappa(x')$ lying over s'. Then the outer part of this diagram is a solution to the problem posed in the lemma. Some minor details omitted. $\square$

057L Lemma 37.47.3. Assumption and notation as in Lemma 37.47.2. In addition to properties (1) – (6) we may also arrange it so that

(7) S', Y', X' are affine.

Proof. Note that if Y' is affine, then X' is affine as π is finite. Choose an affine open neighbourhood $U' \subset S'$ of s'. Choose an affine open neighbourhood $V' \subset h^{-1}(U')$ of y'. Let $W' = h(V')$. This is an open neighbourhood of s' in S', see Morphisms, Lemma 29.34.10, contained in U'. Choose an affine open neighbourhood $U'' \subset W'$ of s'. Then $h^{-1}(U'') \cap V'$ is affine because it is equal to $U'' \times_{U'} V'$. By construction $h^{-1}(U'') \cap V' \to U''$ is a surjective smooth morphism whose fibres are (nonempty) open subschemes of geometrically integral fibres of $Y' \to S'$, and hence geometrically integral. Thus we may replace S' by U'' and Y' by $h^{-1}(U'') \cap V'$. $\square$

The significance of the property $\pi^{-1}(\{y'\}) = \{x'\}$ is partially explained by the following lemma.

05B8 Lemma 37.47.4. Let $\pi : X \to Y$ be a finite morphism. Let $x \in X$ with $y = \pi(x)$ such that $\pi^{-1}(\{y\}) = \{x\}$. Then

(1) For every neighbourhood $U \subset X$ of x in X, there exists a neighbourhood $V \subset Y$ of y such that $\pi^{-1}(V) \subset U$.
(2) The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite.
(3) If π is of finite presentation, then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is of finite presentation.
(4) For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $\mathcal{F}_x = \pi_*\mathcal{F}_y$ as $\mathcal{O}_{Y,y}$-modules.

Proof. The first assertion is purely topological; use that π is a continuous and closed map such that $\pi^{-1}(\{y\}) = \{x\}$. To prove the second and third parts we may assume $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. Then $A \to B$ is a finite ring map and y corresponds to a prime $\mathfrak{p}$ of A such that there exists a unique prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$. Then $B_\mathfrak{q} = B_\mathfrak{p}$, see Algebra, Lemma 10.41.11. In other words, the map $A_\mathfrak{p} \to B_\mathfrak{q}$ is equal to the map $A_\mathfrak{p} \to B_\mathfrak{p}$ you get from localizing $A \to B$ at $\mathfrak{p}$. Thus (2) and (3) follow from simple properties of localization (some details omitted). For the final statement, suppose that $\mathcal{F} = \widetilde{M}$ for some B-module M. Then $\mathcal{F} = M_\mathfrak{q}$ and $\pi_*\mathcal{F}_y = M_\mathfrak{p}$. By the above these localizations agree. Alternatively you can use part (1) and the definition of stalks to see that $\mathcal{F}_x = \pi_*\mathcal{F}_y$ directly. □

37.48. Application to the fppf topology

05WM We can use the above étale localization techniques to prove the following result describing the fppf topology as being equal to the topology "generated by" Zariski coverings and by coverings of the form $\{f : T \to S\}$ where f is surjective finite locally free.

0DET Lemma 37.48.1. Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exist

(1) an étale covering $\{S'_a \to S\}$,
(2) surjective finite locally free morphisms $V_a \to S'_a$,

such that the fppf covering $\{V_a \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. We may assume that each $S_i \to S$ is locally quasi-finite, see Lemma 37.23.6.

Fix a point $s \in S$. Pick an $i \in I$ and a point $s_i \in S_i$ mapping to s. Choose an elementary étale neighbourhood $(S', s) \to (S, s)$ such that there exists an open

$$S_i \times_S S' \supset V$$

which contains a unique point $v \in V$ mapping to $s \in S'$ and such that $V \to S'$ is finite, see Lemma 37.41.1. Then $V \to S'$ is finite locally free, because it is finite and because $S_i \times_S S' \to S'$ is flat and locally of finite presentation as a base change of the morphism $S_i \to S$, see Morphisms, Lemmas 29.21.4, 29.25.8, and 29.48.2. Hence $V \to S'$ is open, and after shrinking S' we may assume that $V \to S'$ is surjective finite locally free. Since we can do this for every point of S we conclude that $\{S_i \to S\}$ can be refined by a covering of the form $\{V_a \to S\}_{a \in A}$ where each $V_a \to S$ factors as $V_a \to S'_a \to S$ with $S'_a \to S$ étale and $V_a \to S'_a$ surjective finite locally free. □

05WN Lemma 37.48.2. Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exist

(1) a Zariski open covering $S = \bigcup U_j$,
(2) surjective finite locally free morphisms $W_j \to U_j$,
(3) Zariski open coverings $W_j = \bigcup_k W_{j,k}$,
(4) surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fppf covering $\{T_{j,k} \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. Let $\{V_a \to S\}_{a \in A}$ be the fppf covering found in Lemma 37.48.1. In other words, this covering refines $\{S_i \to S\}$ and each $V_a \to S$ factors as $V_a \to S'_a \to S$ with $S'_a \to S$ étale and $V_a \to S'_a$ surjective finite locally free.

By Remark 37.40.3 there exists a Zariski open covering $S = \bigcup U_j$, for each j a finite locally free, surjective morphism $W_j \to U_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the étale covering $\{S'_a \to S\}$, i.e., for each pair j, k there exists an $a(j, k)$ and a factorization $W_{j,k} \to S'_a \to S$ of the morphism $W_{j,k} \to S$. Set $T_{j,k} = W_{j,k} \times_{S'_a} V_a$ and everything is clear. □

0CNX Lemma 37.48.3. Let S be a scheme. If $U \subset S$ is open and $V \to U$ is a surjective integral morphism, then there exists a surjective integral morphism $\overline{V} \to S$ with $\overline{V} \times_S U$ isomorphic to V as schemes over U.

Proof. Let $V' \to S$ be the normalization of S in U, see Morphisms, Section 29.53. By construction $V' \to S$ is integral. By Morphisms, Lemmas 29.53.6 and 29.53.12 we see that the inverse image of U in V' is V. Let Z be the reduced induced scheme structure on $S \setminus U$. Then $\overline{V} = V' \amalg Z$ works. $\square$

0CNY Lemma 37.48.4. Let S be a quasi-compact and quasi-separated scheme. If $U \subset S$ is a quasi-compact open and $V \to U$ is a surjective finite morphism, then there exists a surjective finite morphism $\overline{V} \to S$ with $\overline{V} \times_S U$ isomorphic to V as schemes over U.

Proof. By Zariski's Main Theorem (Lemma 37.43.3) we can assume V is a quasi-compact open in a scheme V' finite over S. After replacing V' by the scheme theoretic image of V we may assume that V is dense in V'. It follows that $V' \times_S U = V$ because $V \to V' \times_S U$ is closed as V is finite over U. Let Z be the reduced induced scheme structure on $S \setminus U$. Then $\overline{V} = V' \amalg Z$ works. $\square$

0CNZ Lemma 37.48.5. Let S be a scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exists a surjective integral morphism $S' \to S$ and an open covering $S' = \bigcup U'_\alpha$ such that for each α the morphism $U'_\alpha \to S$ factors through $S_i \to S$ for some i.

Proof. Choose $S = \bigcup U_j$, $W_j \to U_j$, $W_j = \bigcup W_{j,k}$, and $T_{j,k} \to W_{j,k}$ as in Lemma 37.48.2. By Lemma 37.48.3 we can extend $W_j \to U_j$ to a surjective integral morphism $\overline{W}_j \to S$. After this we can extend $T_{j,k} \to W_{j,k}$ to a surjective integral morphism $\overline{T}_{j,k} \to \overline{W}_j$. We set $\overline{T}_j$ equal to the product of all the schemes $\overline{T}_{j,k}$ over $\overline{W}_j$ (Limits, Lemma 32.3.1). Then we set S' equal to the product of all the schemes $\overline{T}_j$ over S. If $x \in S'$, then there is a j such that the image of x in S lies in U_j. Hence there is a k such that the image of x under the projection $S' \to \overline{W}_j$ lies in $W_{j,k}$. Hence under the projection $S' \to \overline{T}_j \to \overline{T}_{j,k}$ the point x ends up in $T_{j,k}$. And $T_{j,k} \to S$ factors through S_i for some i. Finally, the morphism $S' \to S$ is integral and surjective by Limits, Lemmas 32.3.3 and 32.3.2. $\square$

0CP0 Lemma 37.48.6. Let S be a quasi-compact and quasi-separated scheme. Let $\{S_i \to S\}_{i \in I}$ be an fppf covering. Then there exists a surjective finite morphism $S' \to S$ of finite presentation and an open covering $S' = \bigcup U'_\alpha$ such that for each α the morphism $U'_\alpha \to S$ factors through $S_i \to S$ for some i.

Proof. Let $Y \to X$ be the integral surjective morphism found in Lemma 37.48.5. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through $S_{i(j)}$. We can write $Y = \lim Y_\lambda$ with $Y_\lambda \to X$ finite and of finite presentation, see Limits, Lemma 32.7.3. For large enough λ we can find affine opens $V_{\lambda,j} \subset Y_\lambda$ whose inverse image in Y recovers V_j, see Limits, Lemma 32.4.11. For even larger λ the morphisms $V_j \to S_{i(j)}$ over X come from morphisms $V_{\lambda,j} \to S_{i(j)}$ over X, see Limits, Proposition 32.6.1. Setting $S' = Y_\lambda$ for this λ finishes the proof. $\square$

0DBT Lemma 37.48.7. An fppf covering of schemes is a ph covering.

Proof. Let $\{T_i \to T\}$ be an fppf covering of schemes, see Topologies, Definition 34.7.1. Observe that $T_i \to T$ is locally of finite type. Let $U \subset T$ be an affine open. It suffices to show that $\{T_i \times_T U \to U\}$ can be refined by a standard ph covering, see Topologies, Definition 34.8.4. This follows immediately from Lemma 37.48.6 and the fact that a finite morphism is proper (Morphisms, Lemma 29.44.11). □

0DBU Remark 37.48.8. As a consequence of Lemma 37.48.7 we obtain a comparison morphism

$$\epsilon : (Sch/S)_{ph} \longrightarrow (Sch/S)_{fppf}$$

This is the morphism of sites given by the identity functor on underlying categories (with suitable choices of sites as in Topologies, Remark 34.11.1). The functor ϵ_* is the identity on underlying presheaves and the functor ϵ^{-1} associated to an fppf sheaf its ph sheafification. By composition we can in addition compare the ph topology with the syntomic, smooth, étale, and Zariski topologies.

37.49. Quasi-projective schemes

0B41 The term "quasi-projective scheme" has not yet been defined. A possible definition could be a scheme which has an ample invertible sheaf. However, if X is a scheme over a base scheme S, then we say that X is quasi-projective over S if the morphism $X \to S$ is quasi-projective (Morphisms, Definition 29.40.1). Since the identity morphism of any scheme is quasi-projective, we see that a scheme quasi-projective over S doesn't necessarily have an ample invertible sheaf. For this reason it seems better to leave the term "quasi-projective scheme" undefined.

0B42 Lemma 37.49.1. Let S be a scheme which has an ample invertible sheaf. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) $X \to S$ is quasi-projective,
(2) $X \to S$ is H-quasi-projective,
(3) there exists a quasi-compact open immersion $X \to X'$ of schemes over S with $X' \to S$ projective,
(4) $X \to S$ is of finite type and X has an ample invertible sheaf, and
(5) $X \to S$ is of finite type and there exists an f-very ample invertible sheaf.

Proof. The implication (2) ⇒ (1) is Morphisms, Lemma 29.40.5. The implication (1) ⇒ (2) is Morphisms, Lemma 29.43.16. The implication (2) ⇒ (3) is Morphisms, Lemma 29.43.11

Assume $X \subset X'$ is as in (3). In particular $X \to S$ is of finite type. By Morphisms, Lemma 29.43.11 the morphism $X \to S$ is H-projective. Thus there exists a quasi-compact immersion $i : X \to \mathbf{P}^n_S$. Hence $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$ is f-very ample. As $X \to S$ is quasi-compact we conclude from Morphisms, Lemma 29.38.2 that $\mathcal{L}$ is f-ample. Thus $X \to S$ is quasi-projective by definition.

The implication (4) ⇒ (2) is Morphisms, Lemma 29.39.3.

Assume the equivalent conditions (1), (2), (3) hold. Choose an immersion $i : X \to \mathbf{P}^n_S$ over S. Let $\mathcal{L}$ be an ample invertible sheaf on S. To finish the proof we will show that $\mathcal{N} = f^*\mathcal{L} \otimes_{\mathcal{O}_X} i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$ is ample on X. By Properties, Lemma 28.26.14 we reduce to the case $X = \mathbf{P}^n_S$. Let $s \in \Gamma(S, \mathcal{L}^{\otimes d})$ be a section such that the corresponding open S_s is affine. Say $S_s = \mathrm{Spec}(A)$. Recall that $\mathbf{P}^n_S$ is the projective bundle associated to $\mathcal{O}_S T_0 \oplus \ldots \oplus \mathcal{O}_S T_n$, see Constructions, Lemma 27.21.5 and its proof. Let $s_i \in \Gamma(\mathbf{P}^n_S, \mathcal{O}(1))$ be the global section corresponding to the section T_i of $\mathcal{O}_S T_0 \oplus \ldots \oplus \mathcal{O}_S T_n$. Then we see that $X_{f^*s \otimes s_i^{\otimes n}}$ is affine because it is equal to $\mathrm{Spec}(A[T_0/T_i, \ldots, T_n/T_i])$. This proves that $\mathcal{N}$ is ample by definition.

The equivalence of (1) and (5) follows from Morphisms, Lemmas 29.38.2 and 29.39.5. ☐

0B43 Lemma 37.49.2. Let S be a scheme which has an ample invertible sheaf. Let QP_S be the full subcategory of the category of schemes over S satisfying the equivalent conditions of Lemma 37.49.1.

(1) if $S' \to S$ is a morphism of schemes and S' has an ample invertible sheaf, then base change determines a functor $\text{QP}_S \to \text{QP}_{S'}$,
(2) if $X \in \text{QP}_S$ and $Y \in \text{QP}_X$, then $Y \in \text{QP}_S$,
(3) the category QP_S is closed under fibre products,
(4) the category QP_S is closed under finite disjoint unions,
(5) if $X \to S$ is projective, then $X \in \text{QP}_S$,
(6) if $X \to S$ is quasi-affine of finite type, then X is in QP_S,
(7) if $X \to S$ is quasi-finite and separated, then $X \in \text{QP}_S$,
(8) if $X \to S$ is a quasi-compact immersion, then $X \in \text{QP}_S$,
(9) add more here.

Proof. Part (1) follows from Morphisms, Lemma 29.40.2.

Part (2) follows from the fourth characterization of Lemma 37.49.1.

If $X \to S$ and $Y \to S$ are quasi-projective, then $X \times_S Y \to Y$ is quasi-projective by Morphisms, Lemma 29.40.2. Hence (3) follows from (2).

If $X = Y \amalg Z$ is a disjoint union of schemes and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module such that $\mathcal{L}|_Y$ and $\mathcal{L}|_Z$ are ample, then $\mathcal{L}$ is ample (details omitted). Thus part (4) follows from the fourth characterization of Lemma 37.49.1.

Part (5) follows from Morphisms, Lemma 29.43.10.

Part (6) follows from Morphisms, Lemma 29.40.7.

Part (7) follows from part (6) and Lemma 37.43.2.

Part (8) follows from part (7) and Morphisms, Lemma 29.20.16. ☐

The following lemma doesn't really belong in this section, but there does not seem to be a good spot for it anywhere else.

0EJY Lemma 37.49.3. Let X be a quasi-affine scheme. Let $f : U \to X$ be an integral morphism. Then U is quasi-affine and the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \text{Spec}(\Gamma(U, \mathcal{O}_U)) \arrow[d] \\ X \arrow[r] & \text{Spec}(\Gamma(X, \mathcal{O}_X)) \end{tikzcd}$$

is cartesian.

Proof. The scheme U is quasi-affine because integral morphisms are affine, affine morphisms are quasi-affine, a scheme is quasi-affine if and only if the structure morphism to $\text{Spec}(\mathbf{Z})$ is quasi-affine, and compositions of quasi-affine morphisms are quasi-affine. The first two statements follow immediately from the definition and the third is Morphisms, Lemma 29.13.4. Set $U' = X \times_{\mathrm{Spec}(\Gamma(X,\mathcal{O}_X))} \mathrm{Spec}(\Gamma(U,\mathcal{O}_U))$ and consider the extended diagram

$$\begin{tikzcd} U \arrow[r, "j"'] \arrow[rd] & U' \arrow[r] \arrow[d] & \mathrm{Spec}(\Gamma(U,\mathcal{O}_U)) \arrow[d] \\ & X \arrow[r] & \mathrm{Spec}(\Gamma(X,\mathcal{O}_X)) \end{tikzcd}$$

The morphism j is closed by Morphisms, Lemma 29.41.7 combined with the fact that an integral morphism is universally closed (Morphisms, Lemma 29.44.7) and the fact that the vertical arrows are in the diagram are separated. On the other hand, j is open because the horizontal arrows in the diagram of the lemma are open by Properties, Lemma 28.18.4. Thus j identifies U with an open and closed subscheme of U'. If $U \neq U'$ then U isn't dense in U' and a fortiori not dense in the spectrum of $\Gamma(U,\mathcal{O}_U)$. However, the scheme theoretic image of U in $\mathrm{Spec}(\Gamma(U,\mathcal{O}_U))$ is $\mathrm{Spec}(\Gamma(U,\mathcal{O}_U))$ because any ideal in $\Gamma(U,\mathcal{O}_U)$ cutting out a closed subscheme through which U factors would have to be zero. Hence U is dense in $\mathrm{Spec}(\Gamma(U,\mathcal{O}_U))$ for example by Morphisms, Lemma 29.6.3. Thus $U = U'$ and we win. $\square$

37.50. Projective schemes

0B44 This section is the analogue of Section 37.49 for projective morphisms.

0B45 Lemma 37.50.1. Let S be a scheme which has an ample invertible sheaf. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) $X \to S$ is projective,
(2) $X \to S$ is H-projective,
(3) $X \to S$ is quasi-projective and proper,
(4) $X \to S$ is H-quasi-projective and proper,
(5) $X \to S$ is proper and X has an ample invertible sheaf,
(6) $X \to S$ is proper and there exists an f-ample invertible sheaf,
(7) $X \to S$ is proper and there exists an f-very ample invertible sheaf,
(8) there is a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ generated by $\mathcal{A}_1$ over $\mathcal{A}_0$ with $\mathcal{A}_1$ a finite type $\mathcal{O}_S$-module such that $X = \underline{\mathrm{Proj}}_S(\mathcal{A})$.

Proof. Observe first that in each case the morphism f is proper, see Morphisms, Lemmas 29.43.3 and 29.43.5. Hence it suffices to prove the equivalence of the notions in case f is a proper morphism. We will use this without further mention in the following.

The equivalences (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) are Morphisms, Lemma 29.43.13.

The implication (2) $\Rightarrow$ (1) is Morphisms, Lemma 29.43.3.

The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are Morphisms, Lemma 29.43.16.

The implication (1) $\Rightarrow$ (7) is immediate from Morphisms, Definitions 29.43.1 and 29.38.1.

The conditions (3) and (6) are equivalent by Morphisms, Definition 29.40.1.

Thus (1) – (4), (6) are equivalent and imply (7). By Lemma 37.49.1 conditions (3), (5), and (7) are equivalent. Thus we see that (1) – (7) are equivalent.

By Divisors, Lemma 31.30.5 we see that (8) implies (1). Conversely, if (2) holds, then we can choose a closed immersion

$$i : X \longrightarrow \mathbf{P}^n_S = \underline{\text{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$

See Constructions, Lemma 27.21.5 for the equality. By Divisors, Lemma 31.31.1 we see that X is the relative Proj of a quasi-coherent graded quotient algebra $\mathcal{A}$ of $\mathcal{O}_S[T_0, \ldots, T_n]$. Then $\mathcal{A}$ satisfies the conditions of (8). $\square$

0B46 Lemma 37.50.2. Let S be a scheme which has an ample invertible sheaf. Let P_S be the full subcategory of the category of schemes over S satisfying the equivalent conditions of Lemma 37.50.1.

(1) if $S' \to S$ is a morphism of schemes and S' has an ample invertible sheaf, then base change determines a functor $\mathrm{P}_S \to \mathrm{P}_{S'}$,
(2) if $X \in \mathrm{P}_S$ and $Y \in \mathrm{P}_X$, then $Y \in \mathrm{P}_S$,
(3) the category P_S is closed under fibre products,
(4) the category P_S is closed under finite disjoint unions,
(5) if $X \to S$ is finite, then X is in P_S,
(6) add more here.

Proof. Part (1) follows from Morphisms, Lemma 29.43.9.

Part (2) follows from the fifth characterization of Lemma 37.50.1 and the fact that compositions of proper morphisms are proper (Morphisms, Lemma 29.41.4).

If $X \to S$ and $Y \to S$ are projective, then $X \times_S Y \to Y$ is projective by Morphisms, Lemma 29.43.9. Hence (3) follows from (2).

If $X = Y \amalg Z$ is a disjoint union of schemes and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module such that $\mathcal{L}|_Y$ and $\mathcal{L}|_Z$ are ample, then $\mathcal{L}$ is ample (details omitted). Thus part (4) follows from the fifth characterization of Lemma 37.50.1.

Part (5) follows from Morphisms, Lemma 29.44.16. $\square$

Here is a slightly different type of result.

0D2S Lemma 37.50.3. Let $f : X \to Y$ be a proper morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that $\mathcal{L}_y$ is ample on X_y. Then there is an open neighbourhood $V \subset Y$ of y such that $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)/V$. [DG67, IV Corollary 9.6.4]

Proof. We may assume Y is affine. Then we find a directed set I and an inverse system of morphisms $X_i \to Y_i$ of schemes with Y_i of finite type over $\mathbf{Z}$, with affine transition morphisms $X_i \to X_{i'}$ and $Y_i \to Y_{i'}$, with $X_i \to Y_i$ proper, such that $X \to Y = \lim(X_i \to Y_i)$. See Limits, Lemma 32.13.3. After shrinking I we can assume we have a compatible system of invertible $\mathcal{O}_{X_i}$-modules $\mathcal{L}_i$ pulling back to $\mathcal{L}$, see Limits, Lemma 32.10.3. Let $y_i \in Y_i$ be the image of y. Then $\kappa(y) = \text{colim}\, \kappa(y_i)$. Hence for some i we have $\mathcal{L}_{i,y_i}$ is ample on X_{i,y_i} by Limits, Lemma 32.4.15. By Cohomology of Schemes, Lemma 30.21.4 we find an open neigbourhood $V_i \subset Y_i$ of y_i such that $\mathcal{L}_i$ restricted to $f_i^{-1}(V_i)$ is ample relative to V_i. Letting $V \subset Y$ be the inverse image of V_i finishes the proof (hints: use Morphisms, Lemma 29.37.9 and the fact that $X \to Y \times_{Y_i} X_i$ is affine and the fact that the pullback of an ample invertible sheaf by an affine morphism is ample by Morphisms, Lemma 29.37.7). $\square$

37.51. Proj and Spec

0EKF In this section we clarify the relationship between the Proj and the spectrum of a graded ring.

Let R be a ring. Let A be a graded R-algebra, see Algebra, Section 10.56. For $m \geq 0$ we denote $A_{\geq m} = \bigoplus_{d \geq m} A_d$. Consider the graded ring

$$B = \bigoplus\nolimits_{d \geq 0} A_{\geq d}$$

For $d' \geq d$ and $a \in A_{d'}$ let us denote $a^{(d)} \in B$ the element in B_d corresponding to a. Let us denote $\sigma : A \to B$ and $\psi : A \to B$ the two obvious ring maps: if $a \in A_d$, then $\sigma(a) = a^{(0)}$ and $\psi(a) = a^{(d)}$. Then ψ is a graded ring map and σ turns B into a graded algebra over A. There is also a surjective graded ring map $\tau : B \to A$ which for $d' \geq d$ and $a \in A_{d'}$ sends $a^{(d)}$ to 0 if $d' > d$ and to a if $d' = d$.

Affine schemes and spectra. We set $X = \mathrm{Spec}(A)$. The irrelevant ideal A_+ cuts out a closed subscheme $Z = V(A_+) = \mathrm{Spec}(A/A_+) = \mathrm{Spec}(A_0)$. Set $U = X \setminus Z$.

$$U \longrightarrow X \longrightarrow Z$$

Projective schemes and Proj. Set $P = \mathrm{Proj}(A)$. We may and do view P as a scheme over $\mathrm{Spec}(A_0) = Z$. Set $L = \mathrm{Proj}(B)$. We may and do view L as a scheme over $\mathrm{Spec}(B_0) = \mathrm{Spec}(A) = X$; observe that the identification of B_0 with A is given by σ. The surjection τ defines a closed immersion $0 : P \to L$. Since $A \xrightarrow{\sigma} B \to A$ is equal to the map $A \to A_0 \to A$ we conclude that

$$\begin{tikzcd} P \arrow[r, "0"'] \arrow[d] & L \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

is commutative.

We claim that ψ defines a morphism $L \to P$. To see this, by Constructions, Lemma 27.11.1, it suffices to check $\psi(A_+) \not\subset \mathfrak{p}$ for every homogeneous prime ideal $\mathfrak{p} \subset B$ with $B_+ \not\subset \mathfrak{p}$. First, pick $g \in B_+$ homogeneous $g \notin \mathfrak{p}$. Then we can write g as a finite sum $g = \sum a_i^{(d)}$ with $a_i \in A_{d_i}$ for some $d_i \geq d$. We conclude that there exist $d' \geq d$ and $a \in A_{d'}$ such that $a^{(d)} \notin \mathfrak{p}$. Then

$$(a^{(d)})^{d'} = (a^{d'})^{(d'd)} = a^{(d)}(a^{d'-1})^{(d(d'-1))} = \psi(a)(a^{d'-1})^{(d(d'-1))}$$

(the notation leaves something to be desired) is not in $\mathfrak{p}$. Hence $\psi(a) \notin \mathfrak{p}$, proving the claim. Thus we can extend our diagram above to a commutative diagram

$$\begin{tikzcd} P \arrow[r, "0"'] \arrow[d] & L \arrow[r, "\pi"'] \arrow[d] & P \arrow[d] \\ Z \arrow[r] & X \arrow[r] & Z \end{tikzcd}$$

where $X \to Z$ is given by $A_0 \to A$. Since $\tau \circ \psi = \mathrm{id}_A$ we see $\pi \circ 0 = \mathrm{id}_P$.

Observe that π is an affine morphism. This is clear from the construction in Constructions, Lemma 27.11.1. In fact, if $f \in A_d$ for some $d > 0$, then setting $g = \psi(f)$

we have $\pi^{-1}(D_+(f)) = D_+(g)$. In this case we have the following equality of homogeneous parts

$$(B[1/g])_{m'} = \bigoplus\nolimits_{m \geq m'} (A[1/f])_m$$

This isomorphism is compatible with further localization. Taking $m' = 0$ we see that $\pi_*\mathcal{O}_L$ is the direct sum of $\mathcal{O}_P(m)$ for $m \geq 0$[9]. We conclude L is idendified with the relative spectrum:

$$L = \underline{\mathrm{Spec}}_P\left(\bigoplus\nolimits_{m \geq 0} \mathcal{O}_P(m)\right)$$

In particular $L \to P$ is a cone[10], see Constructions, Section 27.7. Moreover, it is clear that $0 : P \to L$ is the vertex of the cone.

Let $f \in A_d$ for some $d > 0$ and $g = \psi(f) \in B_d$ as in the previous paragraph. Looking at the structure of the ring maps

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\sigma"] & A_0 \arrow[d] \\
(A[1/f])_0 \arrow[r, "\psi"] & (B[1/g])_0 = \bigoplus_{m \geq 0} (A[1/f])_m \arrow[r, "\tau"] & (A[1/f])_0
\end{tikzcd}$$

some compuations[11] in graded rings will show that

(1) $\sigma(A_+)(B[1/g])_0 \subset \mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$,
(2) $\sigma(f) \in (B[1/g])_0$ is a nonzerodivisor,
(3) $\sigma(f)(B[1/g])_0 = \sigma(A_d)(B[1/g])_0$ as ideals,
(4) $\sigma(f)(B[1/g])_0$ and $\mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$ have the same radical,
(5) if $d = 1$, then $\sigma(f)(B[1/g])_0 = \mathrm{Ker}(\tau : (B[1/g])_0 \to (A[1/f])_0)$.

We see in particular that

$$0(D_+(f)) = V(\sigma(f)) \subset D_+(g) = \mathrm{Spec}((B[1/g])_0)$$

set theoretically. In other words, the ideal generated by $\sigma(A_d)$ cuts out an effective Cartier divisor on $D_+(g)$ which is set theoretically equal to the image of the closed immersion $0 : P \to L$.

We claim that $L \to X$ is an isomorphism over U. Namely, if $f \in A_d$ for some $d > 0$, then

$$\mathrm{Spec}(A_f) \times_X L = \mathrm{Proj}(A_f \otimes_A B) = \mathrm{Proj}(B_{\sigma(f)})$$

For each e we have $(B_{\sigma(f)})_e = A_f \otimes_B B_e = A_f \otimes_A A_{\geq e} = A_f$, the final equality induced by the injection $A_{\geq e} \subset A$. Hence $B_{\sigma(f)} \cong A_f[T]$ with T in degree 1. This proves the claim as $\mathrm{Proj}(A_f[T]) \to \mathrm{Spec}(A_f)$ is an isomorphism. From now on we identify U with the corresponding open of L.

---

[9] It similarly follows that $\pi_*\mathcal{O}_L(i) = \bigoplus_{m \geq -i} \mathcal{O}_P(m)$.

[10] Often L is a line bundle over P, see below.

[11] Parts (1) and (2) are clear. To see (3), note that if $a \in A_d$, then $\sigma(a) = \sigma(f)\psi(a/f)$. For (4) note that b/g^m is in the kernel of τ if and only if $b \in A_{\geq md}$ maps to zero in A_{md}. Thus it suffices to show if $m' > md$ and $a \in A_{m'}$, then some power of $a^{(md)}/g^m$ is in the ideal generated by $\sigma(f)$. Take e such that $em' - emd \geq d$. Then

$$(a^{(md)}/g^m)^e = (a^e)^{(emd)}/g^{em} = (fa^e)^{(emd+d)}/g^{em+1} = \sigma(f) \cdot (a^e)^{(emd+d)}/g^{em+1}$$

as desired (apologies for the terrible notation). To see (5) argue as before and note that $a^{(md)}/g^m = \sigma(f) \cdot a^{(md+1)}/g^{m+1}$ if $d = 1$.

The identification made in the previous paragraph lets us consider the restriction $\pi|_U : U \to P$. Pick $f \in A_d$ for some $d > 0$ and $g = \psi(f) \in B_d$ as we have done above several times. Then

$$U \cap \pi^{-1}(D_+(f)) = U \cap D_+(g)$$

is the complement of the zero locus of $\sigma(f) \in (B[1/g])_0$ via the identification of $D_+(g)$ with the spectrum of $(B[1/g])_0$. This is assertion (4) above. Therefore $U \cap D_+(g)$ is affine and

$$\mathcal{O}_L(U \cap D_+(g)) = (B[1/g])_0[1/\sigma(f)] = \bigoplus\nolimits_{m \in \mathbf{Z}} (A[1/f])_m$$

where the last equal sign is the natural extension of the identification $(B[1/g])_0 = \bigoplus_{m \geq 0}(A[1/f])_m$ made above. Exactly as we did before with $\pi : L \to P$ we conclude that $\pi|_U : U \to P$ is affine and

$$U = \underline{\mathrm{Spec}}_P\left(\bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{O}_P(m)\right)$$

as schemes over P.

Summarising the above, our constructions produce a commutative diagram

0EKG (37.51.0.1)
$$\begin{tikzcd}
\underline{\mathrm{Spec}}_P\left(\bigoplus_{m \in \mathbf{Z}} \mathcal{O}_P(m)\right) \arrow[r] \arrow[d, equal] & L = \underline{\mathrm{Spec}}_P\left(\bigoplus_{m \geq 0} \mathcal{O}_P(m)\right) \arrow[r, "\pi"'] \arrow[d, "\sigma"] & P \arrow[d] \\
U \arrow[r] & X \arrow[r] & Z
\end{tikzcd}$$

of schemes where π is a cone whose zero section $0 : P \to L$ maps set theoretically onto the inverse image of Z in L.

Let $W \subset P$ be the largest open such that $\mathcal{O}_P(1)|_W$ is invertible and the natural maps induce isomorphisms $\mathcal{O}_P(m)|_W \cong \mathcal{O}_P(1)^{\otimes m}|_W$ for all $m \in \mathbf{Z}$, i.e., the open of Constructions, Lemma 27.10.4 for $d = 1$. Then we see that $L|_W = \pi^{-1}(W) \to W$ is a vector bundle (Constructions, Section 27.6) of rank 1, namely,

$$L|_W = \mathbf{V}(\mathcal{O}_P(1)|_W)$$

in Grothendieckian notation. This is immediate from the above showing that $L|_W$ is equal to the relative spectrum of the symmetric algebra over $\mathcal{O}_W$ on $\mathcal{O}_P(1)|_W$. Then clearly the morphism $0|_W : W \to L|_W$ is the zero section of this vector bundle. In particular $0(W)$ is an effective Cartier divisor on $L|_W$. Moreover, the open $U|_W = (\pi|_U)^{-1}(W)$ is the complement of the zero section.

If A is generated by $f_1, \ldots, f_r \in A_1$ over A_0, then $(f_1, \ldots, f_r)^m = A_{\geq m}$ for all $m \geq 0$ and hence our B above is the Rees algebra for $A_+ = (f_1, \ldots, f_r)$. Thus in this case $L \to X$ is the blowup of Z and $W = P$ where W is as in the preceding paragraph.

If P is quasi-compact, then for d sufficiently divisible, the closed subscheme $D \subset L$ cut out by $\sigma(A_d)\mathcal{O}_L$ is an effective Cartier divisor, $0 : P \to L$ factors through D, and $0(P) = D$ set theoretically. This follows from Constructions, Lemma 27.8.9 and (1), (2), (3), and (4) proved above. (Take any d divisible by the lcm of the degrees of the elements found in the lemma.)

We continue to assume P is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_P$-module. Let us set $\mathcal{F}_U = \pi^*\mathcal{F}|_U$. Then we have

0EKH (37.51.0.2)
$$R\Gamma(U, \mathcal{F}_U) = \bigoplus\nolimits_{m \in \mathbf{Z}} R\Gamma(P, \mathcal{F} \otimes_{\mathcal{O}_P} \mathcal{O}_P(m))$$

Moreover, this direct sum decomposition is functorial in $\mathcal{F}$ and the induced A-module structure on the right is the same as the A-module structure on the left coming from $U \subset X$. To prove the formula, since $\pi|_U$ is affine and $(\pi|_U)_*\mathcal{O}_U = \bigoplus_{m \in \mathbf{Z}} \mathcal{O}_P(m)$ we get

$$\begin{aligned} R(\pi|_U)_*\mathcal{F}_U &= (\pi|_U)_*\mathcal{F}_U \\ &= (\pi|_U)_*(\pi|_U)^*\mathcal{F} \\ &= \mathcal{F} \otimes_{\mathcal{O}_P} \bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{O}_P(m) \\ &= \bigoplus\nolimits_{m \in \mathbf{Z}} \mathcal{F} \otimes_{\mathcal{O}_P} \mathcal{O}_P(m) \end{aligned}$$

By Leray we find that $R\Gamma(U, \mathcal{F}_U) = R\Gamma(P, R(\pi|_U)_*\mathcal{F}_U)$, see Cohomology, Lemma 20.13.6. The proof is finished because taking cohomology commutes with direct sums in this case, see Derived Categories of Schemes, Lemma 36.4.5. This is where we use that P is quasi-compact; P is separated by Constructions, Lemma 27.8.8.

0EKI Lemma 37.51.1. Let R be a ring. Let P be a proper scheme over R and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_P$-module. Set $A = \bigoplus_{m \geq 0} \Gamma(P, \mathcal{L}^{\otimes m})$. Then $P = \text{Proj}(A)$ and diagram (37.51.0.1) becomes the diagram

$$\begin{tikzcd} \underline{\text{Spec}}_P\left(\bigoplus_{m \in \mathbf{Z}} \mathcal{L}^{\otimes m}\right) \arrow[r] \arrow[d, equal] & L = \underline{\text{Spec}}_P\left(\bigoplus_{m \geq 0} \mathcal{L}^{\otimes m}\right) \arrow[r, "\pi"'] \arrow[d, "\sigma"] & P \arrow[d] \\ U \arrow[r] & X \arrow[r] & Z \end{tikzcd}$$

having the properties explained above.

Proof. We have $P = \text{Proj}(A)$ by Morphisms, Lemma 29.43.17. Moreover, by Properties, Lemma 28.28.2 via this identification we have $\mathcal{O}_P(m) = \mathcal{L}^{\otimes m}$ for all $m \in \mathbf{Z}$. $\square$

37.52. Closed points in fibres

053Q Some of the material in this section is taken from the preprint [OP10].

053R Lemma 37.52.1. Let $f : X \to S$ be a morphism of schemes. Let $Z \subset X$ be a closed subscheme. Let $s \in S$. Assume

(1) S is irreducible with generic point η,
(2) X is irreducible,
(3) f is dominant,
(4) f is locally of finite type,
(5) $\dim(X_s) \leq \dim(X_\eta)$,
(6) Z is locally principal in X, and
(7) $Z_\eta = \emptyset$.

Then the fibre Z_s is (set theoretically) a union of irreducible components of X_s.

Proof. Let X_{red} denote the reduction of X. Then $Z \cap X_{red}$ is a locally principal closed subscheme of X_{red}, see Divisors, Lemma 31.13.11. Hence we may assume that X is reduced. In other words X is integral, see Properties, Lemma 28.3.4. In this case the morphism $X \to S$ factors through S_{red}, see Schemes, Lemma 26.12.7. Thus we may replace S by S_{red} and assume that S is integral too.

The assertion that f is dominant signifies that the generic point of X is mapped to η, see Morphisms, Lemma 29.8.6. Moreover, the scheme X_η is an integral scheme which is locally of finite type over the field $\kappa(\eta)$. Hence $d = \dim(X_\eta) \geq 0$ is equal to $\dim_\xi(X_\eta)$ for every point ξ of X_η, see Algebra, Lemmas 10.114.4 and 10.114.5. In view of Morphisms, Lemma 29.28.4 and condition (5) we conclude that $\dim_x(X_s) = d$ for every $x \in X_s$.

In the Noetherian case the assertion can be proved as follows. If the lemma does not holds there exists $x \in Z_s$ which is a generic point of an irreducible component of Z_s but not a generic point of any irreducible component of X_s. Then we see that $\dim_x(Z_s) \leq d - 1$, because $\dim_x(X_s) = d$ and in a neighbourhood of x in X_s the closed subscheme Z_s does not contain any of the irreducible components of X_s. Hence after replacing X by an open neighbourhood of x we may assume that $\dim_z(Z_{f(z)}) \leq d - 1$ for all $z \in Z$, see Morphisms, Lemma 29.28.4. Let $\xi' \in Z$ be a generic point of an irreducible component of Z and set $s' = f(\xi)$. As $Z \neq X$ is locally principal we see that $\dim(\mathcal{O}_{X,\xi}) = 1$, see Algebra, Lemma 10.60.11 (this is where we use X is Noetherian). Let $\xi \in X$ be the generic point of X and let ξ_1 be a generic point of any irreducible component of $X_{s'}$ which contains ξ'. Then we see that we have the specializations

$$\xi \rightsquigarrow \xi_1 \rightsquigarrow \xi'.$$

As $\dim(\mathcal{O}_{X,\xi}) = 1$ one of the two specializations has to be an equality. By assumption $s' \neq \eta$, hence the first specialization is not an equality. Hence $\xi' = \xi_1$ is a generic point of an irreducible component of $X_{s'}$. Applying Morphisms, Lemma 29.28.4 one more time this implies $\dim_{\xi'}(Z_{s'}) = \dim_{\xi'}(X_{s'}) \geq \dim(X_\eta) = d$ which gives the desired contradiction.

In the general case we reduce to the Noetherian case as follows. If the lemma is false then there exists a point $x \in X$ lying over s such that x is a generic point of an irreducible component of Z_s, but not a generic point of any of the irreducible components of X_s. Let $U \subset S$ be an affine neighbourhood of s and let $V \subset X$ be an affine neighbourhood of x with $f(V) \subset U$. Write $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ so that $f|_V$ is given by a ring map $A \to B$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime corresponding to x, resp. s. After possibly shrinking V we may assume $Z \cap V$ is cut out by some element $g \in B$. Denote K the fraction field of A. What we know at this point is the following:

(1) $A \subset B$ is a finitely generated extension of domains,
(2) the element $g \otimes 1$ is invertible in $B \otimes_A K$,
(3) $d = \dim(B \otimes_A K) = \dim(B \otimes_A \kappa(\mathfrak{p}))$,
(4) $g \otimes 1$ is not a unit of $B \otimes_A \kappa(\mathfrak{p})$, and
(5) $g \otimes 1$ is not in any of the minimal primes of $B \otimes_A \kappa(\mathfrak{p})$.

We are seeking a contradiction.

Pick elements $x_1, \ldots, x_n \in B$ which generate B over A. For a finitely generated $\mathbf{Z}$-algebra $A_0 \subset A$ let $B_0 \subset B$ be the A_0-subalgebra generated by $x_1, \ldots, x_n$, denote

K_0 the fraction field of A_0, and set $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$. We claim that when A_0 is large enough then (1) – (5) also hold for the system $(A_0 \subset B_0, g, \mathfrak{p}_0)$.

We prove each of the conditions in turn. Part (1) holds by construction. For part (2) write $(g \otimes 1)h = 1$ for some $h \otimes 1/a \in B \otimes_A K$. Write $g = \sum a_I x^I$, $h = \sum a'_I x^I$ (multi-index notation) for some coefficients $a_I, a'_I \in A$. As soon as A_0 contains a and the a_I, a'_I then (2) holds because $B_0 \otimes_{A_0} K_0 \subset B \otimes_A K$ (as localizations of the injective map $B_0 \to B$). To achieve (3) consider the exact sequence

$$0 \to I \to A[X_1, \ldots, X_n] \to B \to 0$$

which defines I where the second map sends X_i to x_i. Since $\otimes$ is right exact we see that $I \otimes_A K$, respectively $I \otimes_A \kappa(\mathfrak{p})$ is the kernel of the surjection $K[X_1, \ldots, X_n] \to B \otimes_A K$, respectively $\kappa(\mathfrak{p})[X_1, \ldots, X_n] \to B \otimes_A \kappa(\mathfrak{p})$. As a polynomial ring over a field is Noetherian there exist finitely many elements $h_j \in I$, $j = 1, \ldots, m$ which generate $I \otimes_A K$ and $I \otimes_A \kappa(\mathfrak{p})$. Write $h_j = \sum a_{j,I} X^I$. As soon as A_0 contains all $a_{j,I}$ we get to the situation where

$$B_0 \otimes_{A_0} K_0 \otimes_{K_0} K = B \otimes_A K \quad \text{and} \quad B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \otimes_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{p}) = B \otimes_A \kappa(\mathfrak{p}).$$

By either Morphisms, Lemma 29.28.3 or Algebra, Lemma 10.116.5 we see that the dimension equalities of (3) are satisfied. Part (4) is immediate. As $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \subset B \otimes_A \kappa(\mathfrak{p})$ each minimal prime of $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$ lies under a minimal prime of $B \otimes_A \kappa(\mathfrak{p})$ by Algebra, Lemma 10.30.6. This implies that (5) holds. In this way we reduce the problem to the Noetherian case which we have dealt with above. □

Here is an algebraic application of the lemma above. The fourth assumption of the lemma holds if $A \to B$ is flat, see Lemma 37.52.3.

053S Lemma 37.52.2. Let $A \to B$ be a local homomorphism of local rings, and $g \in \mathfrak{m}_B$. Assume

(1) A and B are domains and $A \subset B$,
(2) B is essentially of finite type over A,
(3) g is not contained in any minimal prime over $\mathfrak{m}_A B$, and
(4) $\dim(B/\mathfrak{m}_A B) + \text{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = \text{trdeg}_A(B)$.

Then $A \subset B/gB$, i.e., the generic point of $\text{Spec}(A)$ is in the image of the morphism $\text{Spec}(B/gB) \to \text{Spec}(A)$.

Proof. Note that the two assertions are equivalent by Algebra, Lemma 10.30.6. To start the proof let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_\mathfrak{q}$. Of course we may assume that C is a domain and that $g \in C$. After replacing C by a localization we see that $\dim(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \text{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 29.28.1. Setting K equal to the fraction field of A we see by the same reference that $\dim(C \otimes_A K) = \text{trdeg}_A(B)$. Hence assumption (4) means that the generic and closed fibres of the morphism $\text{Spec}(C) \to \text{Spec}(A)$ have the same dimension.

Suppose that the lemma is false. Then $(B/gB) \otimes_A K = 0$. This means that $g \otimes 1$ is invertible in $B \otimes_A K = C_\mathfrak{q} \otimes_A K$. As $C_\mathfrak{q}$ is a limit of principal localizations we conclude that $g \otimes 1$ is invertible in $C_h \otimes_A K$ for some $h \in C$, $h \notin \mathfrak{q}$. Thus after replacing C by C_h we may assume that $(C/gC) \otimes_A K = 0$. We do one more replacement of C to make sure that the minimal primes of $C/\mathfrak{m}_A C$ correspond one-to-one with the minimal primes of $B/\mathfrak{m}_A B$. At this point we apply Lemma

37.52.1 to $X = \mathrm{Spec}(C) \to \mathrm{Spec}(A) = S$ and the locally closed subscheme $Z = \mathrm{Spec}(C/gC)$. Since $Z_K = \emptyset$ we see that $Z \otimes \kappa(\mathfrak m_A)$ has to contain an irreducible component of $X \otimes \kappa(\mathfrak m_A) = \mathrm{Spec}(C/\mathfrak m_A C)$. But this contradicts the assumption that g is not contained in any prime minimal over $\mathfrak m_A B$. The lemma follows. □

053T Lemma 37.52.3. Let $A \to B$ be a local homomorphism of local rings. Assume

(1) A and B are domains and $A \subset B$,
(2) B is essentially of finite type over A, and
(3) B is flat over A.

Then we have

$$\dim(B/\mathfrak m_A B) + \mathrm{trdeg}_{\kappa(\mathfrak m_A)}(\kappa(\mathfrak m_B)) = \mathrm{trdeg}_A(B).$$

Proof. Let C be an A-algebra of finite type and $\mathfrak q$ a prime of C such that $B = C_{\mathfrak q}$. We may assume C is a domain. We have $\dim_{\mathfrak q}(C/\mathfrak m_A C) = \dim(B/\mathfrak m_A B) + \mathrm{trdeg}_{\kappa(\mathfrak m_A)}(\kappa(\mathfrak m_B))$, see Morphisms, Lemma 29.28.1. Setting K equal to the fraction field of A we see by the same reference that $\dim(C \otimes_A K) = \mathrm{trdeg}_A(B)$. Thus we are really trying to prove that $\dim_{\mathfrak q}(C/\mathfrak m_A C) = \dim(C \otimes_A K)$. Choose a valuation ring A' in K dominating A, see Algebra, Lemma 10.50.2. Set $C' = C \otimes_A A'$. Choose a prime $\mathfrak q'$ of C' lying over $\mathfrak q$; such a prime exists because

$$C'/\mathfrak m_{A'} C' = C/\mathfrak m_A C \otimes_{\kappa(\mathfrak m_A)} \kappa(\mathfrak m_{A'})$$

which proves that $C/\mathfrak m_A C \to C'/\mathfrak m_{A'} C'$ is faithfully flat. This also proves that $\dim_{\mathfrak q}(C/\mathfrak m_A C) = \dim_{\mathfrak q'}(C'/\mathfrak m_{A'} C')$, see Algebra, Lemma 10.116.6. Note that $B' = C'_{\mathfrak q'}$ is a localization of $B \otimes_A A'$. Hence B' is flat over A'. The generic fibre $B' \otimes_{A'} K$ is a localization of $B \otimes_A K$. Hence B' is a domain. If we prove the lemma for $A' \subset B'$, then we get the equality $\dim_{\mathfrak q'}(C'/\mathfrak m_{A'} C') = \dim(C' \otimes_{A'} K)$ which implies the desired equality $\dim_{\mathfrak q}(C/\mathfrak m_A C) = \dim(C \otimes_A K)$ by what was said above. This reduces the lemma to the case where A is a valuation ring.

Let $A \subset B$ be as in the lemma with A a valuation ring. As before write $B = C_{\mathfrak q}$ for some domain C of finite type over A. By Algebra, Lemma 10.125.9 we obtain $\dim(C/\mathfrak m_A C) = \dim(C \otimes_A K)$ and we win. □

053U Lemma 37.52.4. Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a specialization of points in X. Set $s = f(x)$ and $s' = f(x')$. Assume

(1) x' is a closed point of $X_{s'}$, and
(2) f is locally of finite type.

Then the set

$$\{x_1 \in X \text{ such that } f(x_1) = s \text{ and } x_1 \text{ is closed in } X_s \text{ and } x \rightsquigarrow x_1 \rightsquigarrow x'\}$$

is dense in the closure of x in X_s.

Proof. We apply Schemes, Lemma 26.20.4 to the specialization $x \rightsquigarrow x'$. This produces a morphism $\varphi : \mathrm{Spec}(B) \to X$ where B is a valuation ring such that φ maps the generic point to x and the closed point to x'. We may also assume that $\kappa(x)$ is the fraction field of B. Let $A = B \cap \kappa(s)$. Note that this is a valuation ring (see Algebra, Lemma 10.50.7) which dominates the image of $\mathcal{O}_{S,s'} \to \kappa(s)$.

Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[dr] & X_A \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(A) \arrow[r] & S \end{tikzcd}$$

The generic (resp. closed) point of B maps to a point x_A (resp. x'_A) of X_A lying over the generic (resp. closed) point of $\mathrm{Spec}(A)$. Note that x'_A is a closed point of the special fibre of X_A by Morphisms, Lemma 29.20.4. Note that the generic fibre of $X_A \to \mathrm{Spec}(A)$ is isomorphic to X_s. Thus we have reduced the lemma to the case where S is the spectrum of a valuation ring, $s = \eta \in S$ is the generic point, and $s' \in S$ is the closed point.

We will prove the lemma by induction on $\dim_x(X_\eta)$. If $\dim_x(X_\eta) = 0$, then there are no other points of X_η specializing to x and x is closed in its fibre, see Morphisms, Lemma 29.20.6, and the result holds. Assume $\dim_x(X_\eta) > 0$.

Let $X' \subset X$ be the reduced induced scheme structure on the irreducible closed subscheme $\overline{\{x\}}$ of X, see Schemes, Definition 26.12.5. To prove the lemma we may replace X by X' as this only decreases $\dim_x(X_\eta)$. Hence we may also assume that X is an integral scheme and that x is its generic point. In addition, we may replace X by an affine neighbourhood of x'. Thus we have $X = \mathrm{Spec}(B)$ where $A \subset B$ is a finite type extension of domains. Note that in this case $\dim_x(X_\eta) = \dim(X_\eta) = \dim(X_{s'})$, and that in fact $X_{s'}$ is equidimensional, see Algebra, Lemma 10.125.9.

Let $W \subset X_\eta$ be a proper closed subset (this is the subset we want to "avoid"). As X_s is of finite type over a field we see that W has finitely many irreducible components $W = W_1 \cup \ldots \cup W_n$. Let $\mathfrak{q}_j \subset B$, $j = 1, \ldots, r$ be the corresponding prime ideals. Let $\mathfrak{q} \subset B$ be the maximal ideal corresponding to the point x'. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \subset B$ be the minimal primes lying over $\mathfrak{m}_A B$. There are finitely many as these correspond to the irreducible components of the Noetherian scheme $X_{s'}$. Moreover, each of these irreducible components has dimension > 0 (see above) hence we see that $\mathfrak{p}_i \neq \mathfrak{q}$ for all i. Now, pick an element $g \in \mathfrak{q}$ such that $g \notin \mathfrak{q}_j$ for all j and $g \notin \mathfrak{p}_i$ for all i, see Algebra, Lemma 10.15.2. Denote $Z \subset X$ the locally principal closed subscheme defined by g. Let $Z_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$, $n \geq 0$ be the decomposition of the generic fibre of Z into irreducible components (finitely many as the generic fibre is Noetherian). Denote $Z_i \subset X$ the closure of $Z_{i,\eta}$. After replacing X by a smaller affine neighbourhood we may assume that $x' \in Z_i$ for each $i = 1, \ldots, n$. By construction $Z \cap X_{s'}$ does not contain any irreducible component of $X_{s'}$. Hence by Lemma 37.52.1 we conclude that $Z_\eta \neq \emptyset$! In other words $n \geq 1$. Letting $x_1 \in Z_1$ be the generic point we see that $x_1 \rightsquigarrow x'$ and $f(x_1) = \eta$. Also, by construction $Z_{1,\eta} \cap W_j \subset W_j$ is a proper closed subset. Hence every irreducible component of $Z_{1,\eta} \cap W_j$ has codimension ≥ 2 in X_η whereas $\mathrm{codim}(Z_{1,\eta}, X_\eta) = 1$ by Algebra, Lemma 10.60.11. Thus $W \cap Z_{1,\eta}$ is a proper closed subset. At this point we see that the induction hypothesis applies to $Z_1 \to S$ and the specialization $x_1 \rightsquigarrow x'$. This produces a closed point x_2 of $Z_{1,\eta}$ not contained in W which specializes to x'. Thus we obtain $x \rightsquigarrow x_2 \rightsquigarrow x'$, the point x_2 is closed in X_η, and $x_2 \notin W$ as desired. $\square$

053V Remark 37.52.5. The proof of Lemma 37.52.4 actually shows that there exists a sequence of specializations

$$x \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \ldots \rightsquigarrow x_d \rightsquigarrow x'$$

where all x_i are in the fibre X_s, each specialization is immediate, and x_d is a closed point of X_s. The integer $d = \text{trdeg}_{\kappa(s)}(\kappa(x)) = \dim(\overline{\{x\}})$ where the closure is taken in X_s. Moreover, the points x_i can be chosen to avoid any closed subset of X_s which does not contain the point x.

Examples, Section 110.38 shows that the following lemma is false if A is not assumed Noetherian.

05GT Lemma 37.52.6. Let $\varphi : A \to B$ be a local ring map of local rings. Let $V \subset \text{Spec}(B)$ be an open subscheme which contains at least one prime not lying over $\mathfrak{m}_A$. Assume A is Noetherian, φ essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite. Then there exists a $\mathfrak{q} \in V$, $\mathfrak{m}_A \neq \mathfrak{q} \cap A$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map.

Proof. Since A is Noetherian and $A \to B$ is essentially of finite type, we know that B is Noetherian too. By Properties, Lemma 28.6.4 the topological space $\text{Spec}(B)\setminus\{\mathfrak{m}_B\}$ is Jacobson. Hence we can choose a closed point $\mathfrak{q}$ which is contained in the nonempty open

$$V \setminus \{\mathfrak{q} \subset B \mid \mathfrak{m}_A = \mathfrak{q} \cap A\}.$$

(Nonempty by assumption, open because $\{\mathfrak{m}_A\}$ is a closed subset of $\text{Spec}(A)$.) Then $\text{Spec}(B/\mathfrak{q})$ has two points, namely $\mathfrak{m}_B$ and $\mathfrak{q}$ and $\mathfrak{q}$ does not lie over $\mathfrak{m}_A$. Write $B/\mathfrak{q} = C_\mathfrak{m}$ for some finite type A-algebra C and prime ideal $\mathfrak{m}$. Then $A \to C$ is quasi-finite at $\mathfrak{m}$ by Algebra, Lemma 10.122.2 (2). Hence by Algebra, Lemma 10.123.13 we see that after replacing C by a principal localization the ring map $A \to C$ is quasi-finite. $\square$

05GU Lemma 37.52.7. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $U \subset X$ be an open subscheme. Assume f locally of finite type, S locally Noetherian, x a closed point of X_s, and assume there exists a point $x' \in U$ with $x' \rightsquigarrow x$ and $f(x') \neq s$. Then there exists a closed subscheme $Z \subset X$ such that (a) $x \in Z$, (b) $f|_Z : Z \to S$ is quasi-finite at x, and (c) there exists a $z \in Z$, $z \in U$, $z \rightsquigarrow x$ and $f(z) \neq s$.

Proof. This is a reformulation of Lemma 37.52.6. Namely, set $A = \mathcal{O}_{S,s}$ and $B = \mathcal{O}_{X,x}$. Denote $V \subset \text{Spec}(B)$ the inverse image of U. The ring map $f^\sharp : A \to B$ is essentially of finite type. By assumption there exists at least one point of V which does not map to the closed point of $\text{Spec}(A)$. Hence all the assumptions of Lemma 37.52.6 hold and we obtain a prime $\mathfrak{q} \subset B$ which does not lie over $\mathfrak{m}_A$ and such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. Let $z \in X$ be the image of the point $\mathfrak{q}$ under the canonical morphism $\text{Spec}(B) \to X$. Set $Z = \overline{\{z\}}$ with the induced reduced scheme structure. As $z \rightsquigarrow x$ we see that $x \in Z$ and $\mathcal{O}_{Z,x} = B/\mathfrak{q}$. By construction $Z \to S$ is quasi-finite at x. $\square$

05GV Remark 37.52.8. We can use Lemma 37.52.6 or its variant Lemma 37.52.7 to give an alternative proof of Lemma 37.52.4 in case S is locally Noetherian. Here is a rough sketch. Namely, first replace S by the spectrum of the local ring at s'. Then we may use induction on $\dim(S)$. The case $\dim(S) = 0$ is trivial because then $s' = s$. Replace X by the reduced induced scheme structure on $\overline{\{x\}}$. Apply Lemma 37.52.7 to $X \to S$ and $x' \mapsto s'$ and any nonempty open $U \subset X$ containing x. This gives us a closed subscheme $x' \in Z \subset X$ a point $z \in Z$ such that $Z \to S$ is quasi-finite at x' and such that $f(z) \neq s'$. Then z is a closed point of $X_{f(z)}$, and $z \rightsquigarrow x'$. As $f(z) \neq s'$ we see $\dim(\mathcal{O}_{S,f(z)}) < \dim(S)$. Since x is the generic point of X we see $x \rightsquigarrow z$, hence $s = f(x) \rightsquigarrow f(z)$. Apply the induction hypothesis to $s \rightsquigarrow f(z)$ and $z \mapsto f(z)$ to win.

05GW Lemma 37.52.9. Suppose that $f : X \to S$ is locally of finite type, S locally Noetherian, $x \in X$ a closed point of its fibre X_s, and $U \subset X$ an open subscheme such that $U \cap X_s = \emptyset$ and $x \in \overline{U}$, then the conclusions of Lemma 37.52.7 hold.

Proof. Namely, we can reduce this to the cited lemma as follows: First we replace X and S by affine neighbourhoods of x and s. Then X is Noetherian, in particular U is quasi-compact (see Morphisms, Lemma 29.15.6 and Topology, Lemmas 5.9.2 and 5.12.13). Hence there exists a specialization $x' \rightsquigarrow x$ with $x' \in U$ (see Morphisms, Lemma 29.6.5). Note that $f(x') \neq s$. Thus we see all hypotheses of the lemma are satisfied and we win. □

37.53. Stein factorization

03GX Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

03GY Lemma 37.53.1. Let S be a scheme. Let $f : X \to S$ be a universally closed and quasi-separated morphism. There exists a factorization

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

(1) the morphism f' is universally closed, quasi-compact, quasi-separated, and surjective,
(2) the morphism $\pi : S' \to S$ is integral,
(3) we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,
(4) we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and
(5) S' is the normalization of S in X, see Morphisms, Definition 29.53.3.

Formation of the factorization $f = \pi \circ f'$ commutes with flat base change.

Proof. By Morphisms, Lemma 29.41.8 the morphism f is quasi-compact. Hence the normalization S' of S in X is defined (Morphisms, Definition 29.53.3) and we have the factorization $X \to S' \to S$. By Morphisms, Lemma 29.53.11 we have (2), (4), and (5). The morphism f' is universally closed by Morphisms, Lemma 29.41.7. It is quasi-compact by Schemes, Lemma 26.21.14 and quasi-separated by Schemes, Lemma 26.21.13.

To show the remaining statements we may assume the base scheme S is affine, say $S = \mathrm{Spec}(R)$. Then $S' = \mathrm{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{S'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Schemes, Lemma 26.24.1, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset S' = \mathrm{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words I is contained in the Jacobson radical ideal of A and we conclude that $V(I) = S'$ as desired. ☐

0E0M Lemma 37.53.2. In Lemma 37.53.1 assume in addition that f is locally of finite type. Then for $s \in S$ the fibre $\pi^{-1}(\{s\}) = \{s_1, \ldots, s_n\}$ is finite and the field extensions $\kappa(s_i)/\kappa(s)$ are finite.

Proof. Recall that there are no specializations among the points of $\pi^{-1}(\{s\})$, see Algebra, Lemma 10.36.20. As f' is surjective, we find that $|X_s| \to \pi^{-1}(\{s\})$ is surjective. Observe that X_s is a quasi-separated scheme of finite type over a field (quasi-compactness was shown in the proof of the referenced lemma). Thus X_s is Noetherian (Morphisms, Lemma 29.15.6). A topological argument (omitted) now shows that $\pi^{-1}(\{s\})$ is finite. For each i we can pick a finite type point $x_i \in X_s$ mapping to s_i (Morphisms, Lemma 29.16.7). We conclude that $\kappa(s_i)/\kappa(s)$ is finite: x_i can be represented by a morphism $\mathrm{Spec}(k_i) \to X_s$ of finite type (by our definition of finite type points) and hence $\mathrm{Spec}(k_i) \to s = \mathrm{Spec}(\kappa(s))$ is of finite type (as a composition of finite type morphisms), hence $k_i/\kappa(s)$ is finite (Morphisms, Lemma 29.16.1). ☐

03GZ Lemma 37.53.3. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Then X_s is geometrically connected, if and only if for every étale neighbourhood $(U, u) \to (S, s)$ the base change $X_U \to U$ has connected fibre X_u.

Proof. If X_s is geometrically connected, then any base change of it is connected. On the other hand, suppose that X_s is not geometrically connected. Then by Varieties, Lemma 33.7.11 we see that $X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(k)$ is disconnected for some finite separable field extension $k/\kappa(s)$. By Lemma 37.35.2 there exists an affine étale neighbourhood $(U, u) \to (S, s)$ such that $\kappa(u)/\kappa(s)$ is identified with $k/\kappa(s)$. In this case X_u is disconnected. ☐

03H0 Theorem 37.53.4 (Stein factorization; Noetherian case). Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[rd, "f"'] & & S' \arrow[ld, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

(1) the morphism f' is proper with geometrically connected fibres,
(2) the morphism $\pi : S' \to S$ is finite,
(3) we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,
(4) we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and
(5) S' is the normalization of S in X, see Morphisms, Definition 29.53.3.

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 37.53.1. Note that besides the conclusions of Lemma 37.53.1 we also have that f' is separated (Schemes, Lemma 26.21.13) and finite type (Morphisms, Lemma 29.15.8). Hence f' is proper. By

Cohomology of Schemes, Proposition 30.19.1 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_S$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that all the fibres of f' are geometrically connected. It is clear from the discussion above that we may replace S by S', and we may therefore assume that S is Noetherian, affine, $f : X \to S$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_S$. Let $s \in S$ be a point of S. We have to show that X_s is geometrically connected. By Lemma 37.53.3 we see that it suffices to show X_u is connected for every étale neighbourhood $(U, u) \to (S, s)$. We may assume U is affine. Thus U is Noetherian (Morphisms, Lemma 29.15.6), the base change $f_U : X_U \to U$ is proper (Morphisms, Lemma 29.41.5), and that also $(f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U$ (Cohomology of Schemes, Lemma 30.5.2). Hence after replacing $(f : X \to S, s)$ by the base change $(f_U : X_U \to U, u)$ it suffices to prove that the fibre X_s is connected when $f_*\mathcal{O}_X = \mathcal{O}_S$. We can deduce this from Derived Categories of Schemes, Lemma 36.32.7 (by looking at idempotents in the structure sheaf of X_s) but we will also give a direct argument below.

Namely, we apply the theorem on formal functions, more precisely Cohomology of Schemes, Lemma 30.20.7. It tells us that

$$\mathcal{O}_{S,s}^\wedge = (f_*\mathcal{O}_X)_s^\wedge = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where X_n is the nth infinitesimal neighbourhood of X_s. Since the underlying topological space of X_n is equal to that of X_s we see that if $X_s = T_1 \amalg T_2$ is a disjoint union of nonempty open and closed subschemes, then similarly $X_n = T_{1,n} \amalg T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}_{S,s}^\wedge$ is a local ring. Thus the assumption was wrong, i.e., X_s is connected, and we win. □

03H2 Theorem 37.53.5 (Stein factorization; general case). Let S be a scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & S' \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with the following properties:

(1) the morphism f' is proper with geometrically connected fibres,
(2) the morphism $\pi : S' \to S$ is integral,
(3) we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,
(4) we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and
(5) S' is the normalization of S in X, see Morphisms, Definition 29.53.3.

Proof. We may apply Lemma 37.53.1 to get the morphism $f' : X \to S'$. Note that besides the conclusions of Lemma 37.53.1 we also have that f' is separated (Schemes, Lemma 26.21.13) and finite type (Morphisms, Lemma 29.15.8). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has geometrically connected fibres.

We may assume that $S = \mathrm{Spec}(R)$ is affine. Set $R' = \Gamma(X, \mathcal{O}_X)$. Then $S' = \mathrm{Spec}(R')$. Thus we may replace S by S' and assume that $S = \mathrm{Spec}(R)$ is affine

$R = \Gamma(X, \mathcal{O}_X)$. Next, let $s \in S$ be a point. Let $U \to S$ be an étale morphism of affine schemes and let $u \in U$ be a point mapping to s. Let $X_U \to U$ be the base change of X. By Lemma 37.53.3 it suffices to show that the fibre of $X_U \to U$ over u is connected. By Cohomology of Schemes, Lemma 30.5.2 we see that $\Gamma(X_U, \mathcal{O}_{X_U}) = \Gamma(U, \mathcal{O}_U)$. Hence we have to show: Given $S = \mathrm{Spec}(R)$ affine, $X \to S$ proper with $\Gamma(X, \mathcal{O}_X) = R$ and $s \in S$ is a point, the fibre X_s is connected.

To do this it suffices to show that the only idempotents $e \in H^0(X_s, \mathcal{O}_{X_s})$ are 0 and 1 (we already know that X_s is nonempty by Lemma 37.53.1). By Derived Categories of Schemes, Lemma 36.32.7 after replacing R by a principal localization we may assume e is the image of an element of R. Since $R \to H^0(X_s, \mathcal{O}_{X_s})$ factors through $\kappa(s)$ we conclude. $\square$

Here is an application.

0AY8 Lemma 37.53.6. Let $f : X \to S$ be a morphism of schemes. Assume

(1) f is proper,
(2) S is integral with generic point ξ,
(3) S is normal,
(4) X is reduced,
(5) every generic point of an irreducible component of X maps to ξ,
(6) we have $H^0(X_\xi, \mathcal{O}) = \kappa(\xi)$.

Then $f_*\mathcal{O}_X = \mathcal{O}_S$ and f has geometrically connected fibres.

Proof. Apply Theorem 37.53.5 to get a factorization $X \to S' \to S$. It is enough to show that $S' = S$. This will follow from Morphisms, Lemma 29.54.8. Namely, S' is reduced because X is reduced (Morphisms, Lemma 29.53.8). The morphism $S' \to S$ is integral by the theorem cited above. Every generic point of S' lies over ξ by Morphisms, Lemma 29.53.9 and assumption (5). On the other hand, since S' is the relative spectrum of $f_*\mathcal{O}_X$ we see that the scheme theoretic fibre S'_ξ is the spectrum of $H^0(X_\xi, \mathcal{O})$ which is equal to $\kappa(\xi)$ by assumption. Hence S' is an integral scheme with function field equal to the function field of S. This finishes the proof. $\square$

Here is another application.

0BUI Lemma 37.53.7. Let $X \to S$ be a flat proper morphism of finite presentation. Let $n_{X/S}$ be the function on S counting the numbers of geometric connected components of fibres of f introduced in Lemma 37.28.3. Then $n_{X/S}$ is lower semi-continuous.

Proof. Let $s \in S$. Set $n = n_{X/S}(s)$. Note that $n < \infty$ as the geometric fibre of $X \to S$ at s is a proper scheme over a field, hence Noetherian, hence has a finite number of connected components. We have to find an open neighbourhood V of s such that $n_{X/S}|_V \geq n$. Let $X \to S' \to S$ be the Stein factorization as in Theorem 37.53.5. By Lemma 37.53.2 there are finitely many points $s'_1, \ldots, s'_m \in S'$ lying over s and the extensions $\kappa(s'_i)/\kappa(s)$ are finite. Then Lemma 37.42.1 tells us that after replacing S by an étale neighbourhood of s we may assume $S' = V_1 \amalg \ldots \amalg V_m$ as a scheme with $s'_i \in V_i$ and $\kappa(s'_i)/\kappa(s)$ purely inseparable. Then the schemes $X_{s'_i}$ are geometrically connected over $\kappa(s)$, hence $m = n$. The schemes $X_i = (f')^{-1}(V_i)$, $i = 1, \ldots, n$ are flat and of finite presentation over S. Hence the image of $X_i \to S$

is open (Morphisms, Lemma 29.25.10). Thus in a neighbourhood of s we see that $n_{X/S}$ is at least n. $\square$

0E0N Lemma 37.53.8. Let $f : X \to S$ be a morphism of schemes. Assume

(1) f is proper, flat, and of finite presentation, and
(2) the geometric fibres of f are reduced.

Then the function $n_{X/S} : S \to \mathbf{Z}$ counting the numbers of geometric connected components of fibres of f is locally constant.

Proof. By Lemma 37.53.7 the function $n_{X/S}$ is lower semincontinuous. For $s \in S$ consider the $\kappa(s)$-algebra

$$A = H^0(X_s, \mathcal{O}_{X_s})$$

By Varieties, Lemma 33.9.3 and the fact that X_s is geometrically reduced A is finite product of finite separable extensions of $\kappa(s)$. Hence $A \otimes_{\kappa(s)} \kappa(\overline{s})$ is a product of $\beta_0(s) = \dim_{\kappa(s)} H^0(E \otimes^{\mathbf{L}} \kappa(s))$ copies of $\kappa(\overline{s})$. Thus $X_{\overline{s}}$ has $\beta_0(s) = \dim_{\kappa(s)} A$ connected components. In other words, we have $n_{X/S} = \beta_0$ as functions on S. Thus $n_{X/S}$ is upper semi-continuous by Derived Categories of Schemes, Lemma 36.32.1. This finishes the proof. $\square$

A final application.

0CT9 Lemma 37.53.9. Let (A, I) be a henselian pair. Let $X \to \mathrm{Spec}(A)$ be separated and of finite type. Set $X_0 = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. Let $Y \subset X_0$ be an open and closed subscheme such that $Y \to \mathrm{Spec}(A/I)$ is proper. Then there exists an open and closed subscheme $W \subset X$ which is proper over A with $W \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) = Y$.

A reference for the case of an adic Noetherian base is [DG67, III, Proposition 5.5.1]

Proof. We will denote $T \mapsto T_0$ the base change by $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$. By Chow's lemma (in the form of Limits, Lemma 32.12.1) there exists a surjective proper morphism $\varphi : X' \to X$ such that X' admits an immersion into $\mathbf{P}^n_A$. Set $Y' = \varphi^{-1}(Y)$. This is an open and closed subscheme of X'_0. Suppose the lemma holds for (X', Y'). Let $W' \subset X'$ be the open and closed subscheme proper over A such that $Y' = W'_0$. By Morphisms, Lemma 29.41.7 $W = \varphi(W') \subset X$ and $Q = \varphi(X' \setminus W') \subset X$ are closed subsets and by Morphisms, Lemma 29.41.9 W is proper over A. The image of $W \cap Q$ in $\mathrm{Spec}(A)$ is closed. Since (A, I) is henselian, if $W \cap Q$ is nonempty, then we find that $W \cap Q$ has a point lying over $\mathrm{Spec}(A/I)$. This is impossible as $W'_0 = Y' = \varphi^{-1}(Y)$. We conclude that W is an open and closed subscheme of X proper over A with $W_0 = Y$. Thus we reduce to the case described in the next paragraph.

Assume there exists an immersion $j : X \to \mathbf{P}^n_A$ over A. Let $\overline{X}$ be the scheme theoretic image of j. Since j is a quasi-compact morphism (Schemes, Lemma 26.21.14) we see that $j : X \to \overline{X}$ is an open immersion (Morphisms, Lemma 29.7.7). Hence the base change $j_0 : X_0 \to \overline{X}_0$ is an open immersion as well. Thus $j_0(Y) \subset \overline{X}_0$ is open. It is also closed by Morphisms, Lemma 29.41.7. Suppose that the lemma holds for $(\overline{X}, j_0(Y))$. Let $\overline{W} \subset \overline{X}$ be the corresponding open and closed subscheme proper over A such that $j_0(Y) = \overline{W}_0$. Then $T = \overline{W} \setminus j(X)$ is closed in $\overline{W}$, hence has closed image in $\mathrm{Spec}(A)$ by properness of $\overline{W}$ over A. Since (A, I) is henselian, we find that if T is nonempty, then there is a point of T mapping into $\mathrm{Spec}(A/I)$. This is impossible because $j_0(Y) = \overline{W}_0$ is contained in $j(X)$. Hence $\overline{W}$ is contained in $j(X)$ and we can set $W \subset X$ equal to the unique open and closed subscheme mapping isomorphically to $\overline{W}$ via j. Thus we reduce to the case described in the next paragraph.

Assume $X \subset \mathbf{P}^n_A$ is a closed subscheme. Then $X \to \mathrm{Spec}(A)$ is a proper morphism. Let $Z = X_0 \setminus Y$. This is an open and closed subscheme of X_0 and $X_0 = Y \amalg Z$. Let $X \to X' \to \mathrm{Spec}(A)$ be the Stein factorization as in Theorem 37.53.5. Let $Y' \subset X'_0$ and $Z' \subset X'_0$ be the images of Y and Z. Since the fibres of $X \to Z$ are geometrically connected, we see that $Y' \cap Z' = \emptyset$. Hence $X'_0 = Y' \amalg Z'$ as $X \to X'$ is surjective. Since $X' \to \mathrm{Spec}(A)$ is integral, we see that X' is the spectrum of an A-algebra integral over A. Recall that open and closed subsets of spectra correspond 1-to-1 with idempotents in the corresponding ring, see Algebra, Lemma 10.21.3. Hence by More on Algebra, Lemma 15.11.6 we see that we may write $X' = W' \amalg V'$ with W' and V' open and closed and with $Y' = W'_0$ and $Z' = V'_0$. Let W be the inverse image in X to finish the proof. □

37.54. Generic flatness stratification

0H3Y We can use generic flatness to construct a stratification of the base such that a given module becomes flat over the strata.

0ASY **Lemma 37.54.1** (Generic flatness stratification). Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Then there exists a $t \geq 0$ and closed subschemes

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that $S_i \to S$ is defined by a finite type ideal sheaf, $S_0 \subset S$ is a thickening, and $\mathcal{F}$ pulled back to $X \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$.

Proof. We can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0 \arrow[d] \\ S \arrow[r] & S_0 \end{tikzcd}$$

and a finitely presented $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ which pulls back to $\mathcal{F}$ such that X_0 and S_0 are of finite type over $\mathbf{Z}$. See Limits, Proposition 32.5.4 and Lemmas 32.10.1 and 32.10.2. Thus we may assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module.

Assume X and S are of finite type over $\mathbf{Z}$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. By generic flatness as stated in Morphisms, Proposition 29.27.2 there is a dense open $U_0 \subset S_0$ such that $\mathcal{F}$ pulled back to $X \times_S U_0$ is flat over U_0. Let $S_1 \subset S_0$ be the reduced closed subscheme whose underlying closed subset is $S \setminus U_0$. We continue in this way, provided $S_1 \neq \emptyset$, to find $S_0 \supset S_1 \supset \ldots$. Because S is Noetherian any descending chain of closed subsets stabilizes hence we see that $S_t = \emptyset$ for some $t \geq 0$. □

0H3Z Lemma 37.54.2. Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Then there exists a $t \geq 0$ and closed subschemes

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that $S_i \to S$ is defined by a finite type ideal sheaf, $S_0 \subset S$ is a thickening, and $X \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$.

Proof. Apply Lemma 37.54.1 with $\mathcal{F} = \mathcal{O}_X$. $\square$

0H40 Lemma 37.54.3. Let R be a Noetherian domain. Let $R \to A \to B$ be finite type ring maps. Let M be a finite A-module and let N a finite B-module. Let $M \to N$ be an A-linear map. There exists an nonzero $f \in R$ such that the cokernel of $M_f \to N_f$ is a flat R_f-module.

Proof. By replacing M by the image of $M \to N$, we may assume $M \subset N$. Choose a filtration $0 = N_0 \subset N_1 \subset \ldots \subset N_t = N$ such that $N_i/N_{i-1} = B/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i \subset B$, see Algebra, Lemma 10.62.1. Set $M_i = M \cap N_i$. Then $Q = N/M$ has a filtration by the submodules $Q_i = N_i/M_i$. It suffices to prove Q_i/Q_{i-1} becomes flat after localizing at a nonzero element of f (since extensions of flat modules are flat by Algebra, Lemma 10.39.13). Since Q_i/Q_{i-1} is isomorphic to the cokernel of the map $M_i/M_{i-1} \to N_i/N_{i-1}$, we reduce to the case discussed in the next paragraph.

Assume B is a domain and $M \subset N = B$. After replacing A by the image of A in B we may assume $A \subset B$. By generic flatness, we may assume A and B are flat over R (Algebra, Lemma 10.118.1). It now suffices to show $M \to B$ becomes R-universally injective after replacing R by a principal localization (Algebra, Lemma 10.82.7). By generic freeness, we can find a nonzero $g \in A$ such that B_g is a free A_g-module (Algebra, Lemma 10.118.1). Thus we may choose a direct summand $M' \subset B_g$ as an A_g-module, which is finite free as an A_g-module, and such that $M \to B \to B_g$ factors through M'. Clearly, it suffices to show that $M \to M'$ becomes R-universally injective after replacing R by a principal localization.

Say $M' = A_g^{\oplus n}$. Since $M \subset M'$ is a finite A-module, we see that M is contained in $(1/g^m)A^{\oplus n}$ for some $m \geq 0$. After changing our basis for M' we may assume $M \subset A^{\oplus n}$. Then it suffices to show that $A^{\oplus n}/M$ and A_g/A become R-flat after replacing R by a principal localization. Namely, then $M' \to A^{\oplus n}$ and $A^{\oplus n} \to A_g^{\oplus n}$ are universally injective by Algebra, Lemma 10.39.12 and consequently so is the composition $M \to M' = A_g^{\oplus n}$.

By generic flatness (see reference above), we may assume the module $A^{\oplus n}/M$ is R-flat. For the quotient A_g/A we use the fact that

$$A_g/A = \text{colim}(1/g^m)A/A \cong \text{colim}\, A/g^m A$$

and the module $A/g^m A$ has a filtration of length m whose succesive quotients are isomorphic to A/gA. Again by generic flatness we may assume A/gA is R-flat and hence each $A/g^m A$ is R-flat, and hence so is A_g/A. $\square$

Let $f : X \to Y$ be a morphism of schemes over a base scheme S. Let $Z \subset Y$ be the scheme theoretic image of f, see Morphisms, Section 29.6. Let $g : S' \to S$ be a morphism of schemes and let $f' : X \times_S S' \to Y \times_S S'$ be the base change of f by g. It is not always true that $Z \times_S S' \subset Y \times_S S'$ is the scheme theoretic image of f'. Let us say that formation of the scheme theoretic image of f/S commutes with arbitrary base change if for every g as above the scheme theoretic image of f' is equal to $Z \times_S S'$.

0H41 Lemma 37.54.4. Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes over S with both X and Y of finite presentation over S. Then there exists a $t \geq 0$ and closed subschemes

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

with the following properties:

(1) $S_i \to S$ is defined by a finite type ideal sheaf,
(2) $S_0 \subset S$ is a thickening, and
(3) with $T_i = S_i \setminus S_{i+1}$ and f_i the base change of f to T_i we have: formation of the scheme theoretic image of f_i/T_i commutes with arbitrary base change (see discussion above the lemma).

Proof. We can find a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & S \arrow[d] \\ U \arrow[r] & V \arrow[r] & W \end{tikzcd}$$

with cartesian squares such that U, V, W are of finite type over $\mathbf{Z}$. Namely, first write S as a cofiltered limit of finite type schemes over $\mathbf{Z}$ with affine transition morphisms using Limits, Proposition 32.5.4 and then descend the morphism $X \to Y$ using Limits, Lemma 32.10.1. This reduces us to the case discussed in the next paragraph.

Assume S is Noetherian. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. Let $\eta \in S_0$ be a generic point of an irreducible component of S_0. By Noetherian induction on the underlying topological space of S_0, we may assume the result holds for any closed subscheme of S_0 not containing η. Thus it suffices to show that there exists an open neighbourhood $U_0 \subset S_0$ such that the base change f_0 of f to U_0 has property (3).

Let R be a Noetherian domain. Let $f : X \to Y$ be a morphism of finite type schemes over R. By the discussion in the previous paragraph it suffices to show that after replacing R by R_g for some $g \in R$ nonzero and X, Y by their base changes to R_g, formation of the scheme theoretic image of f/R commutes with arbitrary base change.

Let $Y = V_1 \cup \ldots V_n$ be an affine open covering. Let $U_i = f^{-1}(V_i)$. If the statement is true for each of the morphisms $U_i \to V_i$ over R, then it holds for f. Namely, the scheme theoretic image of $U_i \to V_i$ is the intersection of V_i with the scheme theoretic image of $f : X \to Y$ by Morphisms, Lemma 29.6.3. Thus we may assume Y is affine.

Let $X = U_1 \cup \ldots U_n$ be an affine open covering. Then the scheme theoretic image of $X \to Y$ is the same as the scheme theoretic imge of $\coprod U_i \to Y$. Thus we may assume X is affine.

Say $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ and f corresponds to the R-algebra map $\varphi : A \to B$. Then the scheme theoretic image of f is $\text{Spec}(A/\text{Ker}(\varphi))$ and similarly after base change (by an affine morphism, but it is enough to check for those). Thus formation of the scheme theoretic image commutes with base change if $\text{Ker}(\varphi \otimes_R R') = \text{Ker}(\varphi) \otimes_R R'$ for all ring maps $R \to R'$.

After replacing R, A, B by R_g, A_g, B_g for a suitable nonzero g in R, we may assume A and B are flat over R. By Lemma 37.54.3 we may also assume B/A is a flat R-module. Then $0 \to \text{Ker}(\varphi) \to A \to B \to B/A \to 0$ is an exact sequence of flat R-modules, which implies the desired base change statement. □

37.55. Stratifying a morphism

0H42 Let $f : X \to S$ be a finitely presented morphism of quasi-compact and quasi-separated schemes. In Section 37.54 we have seen that we can stratify S such that X is flat over the strata. In this section look for stratifications of both S and X such that we obtain smooth strata; this won't quite work and we'll need a base change by finite locally free morphisms as well.

0H43 Lemma 37.55.1. Let $f : X \to S$ be a morphism of schemes of finite presentation. Let $\eta \in S$ be a generic point of an irreducible component of S. Assume S is reduced. Then there exist

(1) an open subscheme $U \subset S$ containing η,
(2) a surjective, universally injective, finite locally free morphism $V \to U$,
(3) a $t \geq 0$ and closed subschemes

$$X \times_S V \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X \times_S V$ is defined by a finite type ideal sheaf, $Z_0 \subset X \times_S V$ is a thickening, and such that the morphism $Z_i \setminus Z_{i+1} \to V$ is smooth.

Proof. It is clear that we may replace S by an open neighbourhood of η and X by the restriction to this open. Thus we may assume $S = \text{Spec}(A)$ where A is a reduced ring and η corresponds to a minimal prime ideal $\mathfrak{p}$. Recall that the local ring $\mathcal{O}_{S,\eta} = A_\mathfrak{p}$ is equal to $\kappa(\mathfrak{p})$ in this case, see Algebra, Lemma 10.25.1.

Apply Varieties, Lemma 33.25.11 to the scheme X_η over $k = \kappa(\eta)$. Denote k'/k the purely inseparable field extension this produces. In the next paragraph we reduce to the case $k' = k$. (This step corresponds to finding the morphism $V \to U$ in the statement of the lemma; in particular we can take $V = U$ if the characteristic of $\kappa(\mathfrak{p})$ is zero.)

If the characteristic of $k = \kappa(\mathfrak{p})$ is zero, then $k' = k$. If the characteristic of $k = \kappa(\mathfrak{p})$ is $p > 0$, then p maps to zero in $A_\mathfrak{p} = \kappa(\mathfrak{p})$. Hence after replacing A by a principal localization (i.e., shrinking S) we may assume $p = 0$ in A. If $k' \neq k$, then there exists an $\beta \in k'$, $\beta \notin k$ such that $\beta^p \in k$. After replacing A by a principal localization we may assume there exists an $a \in A$ such that $\beta^p = a$. Set $A' = A[x]/(x^p - a)$. Then $S' = \text{Spec}(A') \to \text{Spec}(A) = S$ is finite locally free, surjective, and universally injective. Moreover, if $\mathfrak{p}' \subset A'$ denotes the unique prime ideal lying over $\mathfrak{p}$, then $A'_{\mathfrak{p}'} = k(\beta)$ and $k'/k(\beta)$ has smaller degree. Thus after replacing S by S' and η by the point η' corresponding to $\mathfrak{p}'$ we see that the degree of k' over the residue field of η has decreased. Continuing like this, by induction we reduce to the case $k' = \kappa(\mathfrak{p}) = \kappa(\eta)$.

Thus we may assume S is affine, reduced, and that we have a $t \geq 0$ and closed subschemes

$$X_\eta \supset Z_{\eta,0} \supset Z_{\eta,1} \supset \ldots \supset Z_{\eta,t} = \emptyset$$

such that $Z_{\eta,0} = (X_\eta)_{red}$ and $Z_{\eta,i} \setminus Z_{\eta,i+1}$ is smooth over η for all i. Recall that $\kappa(\eta) = \kappa(\mathfrak p) = A_\mathfrak{p}$ is the filtered colimit of A_a for $a \in A$, $a \notin \mathfrak p$. See Algebra, Lemma 10.9.9. Thus we can descend the diagram above to a corresponding diagram over $\text{Spec}(A_a)$ for some $a \in A$, $a \notin \mathfrak p$. More precisely, after replacing S by $\text{Spec}(A_a)$ we may assume we have a $t \geq 0$ and closed subschemes

$$X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_t = \emptyset$$

such that $Z_i \to X$ is a closed immersion of finite presentation, such that $Z_0 \to X$ is a thickening, and such that $Z_i \setminus Z_{i+1}$ is smooth over S. In other words, the lemma holds. More precisely, we first use Limits, Lemma 32.10.1 to obtain morphisms

$$Z_t \to Z_{t-1} \to \ldots \to Z_0 \to X$$

over S, each of finite presentation, and whose base change to η produces the inclusions between the given closed subschemes above. After shrinking S further we may assume each of the morphisms is a closed immersion, see Limits, Lemma 32.8.5. After shrinking S we may assume $Z_0 \to X$ is surjective and hence a thickening, see Limits, Lemma 32.8.15. After shrinking S once more we may assume $Z_i \setminus Z_{i+1} \to S$ is smooth, see Limits, Lemma 32.8.9. This finishes the proof. $\square$

0H44 Lemma 37.55.2. Let $f : X \to S$ be a morphism of finite presentation between quasi-compact and quasi-separated schemes. Then there exists a $t \geq 0$ and closed subschemes

$$S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

such that

1. (1) $S_i \to S$ is defined by a finite type ideal sheaf,
2. (2) $S_0 \subset S$ is a thickening,
3. (3) for each i there exists a surjective finite locally free morphism $T_i \to S_i \setminus S_{i+1}$,
4. (4) for each i there exists a $t_i \geq 0$ and closed subschemes

$$X_i = X \times_S T_i \supset Z_{i,0} \supset Z_{i,1} \supset \ldots \supset Z_{i,t_i} = \emptyset$$

such that $Z_{i,j} \to X_i$ is defined by a finite type ideal sheaf, $Z_{i,0} \subset X_i$ is a thickening, and such that the morphism $Z_{i,j} \setminus Z_{i,j+1} \to T_i$ is smooth.

Proof. We can find a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_0 \arrow[d] \\ S \arrow[r] & S_0 \end{tikzcd}$$

such that X_0 and S_0 are of finite type over $\mathbf{Z}$. See Limits, Proposition 32.5.4 and Lemma 32.10.1. Thus we may assume X and S are of finite type over $\mathbf{Z}$. Namely, a solution of the problem posed by the lemma for $X_0 \to S_0$ will base change to a solution over S; details omitted.

Assume X and S are of finite type over $\mathbf{Z}$. In this case every quasi-coherent ideal is of finite type, hence we do not have to check the condition that S_i is cut out by a finite type ideal. Set $S_0 = S_{red}$ equal to the reduction of S. Let $\eta \in S_0$ be a generic point of an irreducible component. By Lemma 37.55.1 we can find an open subscheme $U \subset S_0$, a surjective, universally injective, finite locally free morphism $V \to U$, a $t_0 \geq 0$ and closed subschemes

$$X \times_S V \supset Z_{0,0} \supset Z_{0,1} \supset \ldots \supset Z_{0,t_0} = \emptyset$$

such that $Z_{0,i} \to X \times_S V$ is defined by a finite type ideal sheaf, $Z_{0,0} \subset X \times_S V$ is a thickening, and such that the morphism $Z_{0,i} \setminus Z_{0,i+1} \to V$ is smooth. Then we let $S_1 \subset S_0$ be the reduced induced subscheme structure on $S_0 \setminus U$. By Noetherian induction on the underlying topological space of S, we may assume that the lemma holds for $X \times_S S_1 \to S_1$. This produces $t \geq 1$ and

$$S_1 = S_1 \supset S_2 \supset \ldots \supset S_t = \emptyset$$

and t_i and $Z_{i,j}$ as in the statement of the lemma. This proves the lemma. $\square$

37.56. Improving morphisms of relative dimension one

0GK6 We can make any curve be smooth and projective after extending the ground field, compactifying, and normalizing. This also implies results about finite type morphisms whose generic fibres have dimension 1.

0GK7 **Lemma 37.56.1.** Let $f : X \to S$ be a morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S. Assume f is separated, of finite presentation, and $\dim(X_\eta) \leq 1$. Then there exists a commutative diagram

$$\begin{tikzcd}
\overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n \arrow[rd] & Y_1 \amalg \ldots \amalg Y_n \arrow[l, "j"'] \arrow[r, "\nu"'] \arrow[d] & X_V \arrow[r] \arrow[d] & X_U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
 & T_1 \amalg \ldots \amalg T_n \arrow[r] & V \arrow[r] & U \arrow[r] & S
\end{tikzcd}$$

of schemes with the following properties:

(1) $U \subset X$ is an open neighbourhood of η,
(2) $V \to U$ is a finite, surjective, universally injective morphism,
(3) $X_U = U \times_S X$ and $X_V = V \times_S X$ are the base changes,
(4) ν is finite, surjective, and there is an open $W \subset X_V$ such that
   (a) W is dense in all fibres of $X_V \to V$,
   (b) $\nu^{-1}(W) \cap Y_i$ is dense in all fibres of $Y_i \to T_i$, and
   (c) $\nu^{-1}(W) \to W$ is a thickening,
(5) j is an open immersion,
(6) $T_i \to V$ is finite étale,
(7) $Y_i \to T_i$ is surjective and smooth,
(8) $\overline{Y}_i \to T_i$ is smooth, proper, with geometrically connected fibres of dimension ≤ 1.

Proof. It is clear that we may replace S by an open neighbourhood of η and X by the restriction to this open. Moreover, we may replace S by its reduction and X by the base change to this reduction. Thus we may assume $S = \text{Spec}(A)$ where A is a reduced ring and η corresponds to a minimal prime ideal $\mathfrak{p}$. Recall that the local ring $\mathcal{O}_{S,\eta} = A_\mathfrak{p}$ is equal to $\kappa(\mathfrak{p})$ in this case, see Algebra, Lemma 10.25.1.

Apply Varieties, Lemma 33.43.7 to the scheme X_η over $k = \kappa(\eta)$. Denote k'/k the purely inseparable field extension this produces. In the next paragraph we reduce to the case $k' = k$. (This step corresponds to finding the morphism $V \to U$ in the statement of the lemma; in particular we can take $V = U$ if the characteristic of $\kappa(\mathfrak p)$ is zero.)

If the characteristic of $k = \kappa(\mathfrak p)$ is zero, then $k' = k$. If the characteristic of $k = \kappa(\mathfrak p)$ is $p > 0$, then p maps to zero in $A_\mathfrak p = \kappa(\mathfrak p)$. Hence after replacing A by a principal localization (i.e., shrinking S) we may assume $p = 0$ in A. If $k' \neq k$, then there exists an $\beta \in k'$, $\beta \notin k$ such that $\beta^p \in k$. After replacing A by a principal localization we may assume there exists an $a \in A$ such that $\beta^p = a$. Set $A' = A[x]/(x^p - a)$. Then $S' = \text{Spec}(A') \to \text{Spec}(A) = S$ is finite, surjective, and universally injective. Moreover, if $\mathfrak p' \subset A'$ denotes the unique prime ideal lying over $\mathfrak p$, then $A'_{\mathfrak p'} = k(\beta)$ and $k'/k(\beta)$ has smaller degree. Thus after replacing S by S' and η by the point η' corresponding to $\mathfrak p'$ we see that the degree of k' over the residue field of η has decreased. Continuing like this, by induction we reduce to the case $k' = \kappa(\mathfrak p) = \kappa(\eta)$.

Thus we may assume S is affine, reduced, and that we have a diagram

$$\begin{tikzcd} \overline{Y}_{1,\eta} \amalg \ldots \amalg \overline{Y}_{n,\eta} \arrow[drr] & & Y_{1,\eta} \amalg \ldots \amalg Y_{n,\eta} \arrow[ll, "j"'] \arrow[r, "\nu"'] \arrow[d] & X_\eta \arrow[d] \\ & & \text{Spec}(k_1) \amalg \ldots \amalg \text{Spec}(k_n) \arrow[r] & \eta \end{tikzcd}$$

of schemes with the following properties:

(1) ν is the normalization of X_η,
(2) j is an open immersion with dense image,
(3) $k_i/\kappa(\eta)$ is a finite separable extension for $i = 1, \ldots, n$,
(4) $\overline{Y}_{i,\eta}$ is smooth, projective, and geometrically irreducible of dimension ≤ 1 over k_i.

Recall that $\kappa(\eta) = \kappa(\mathfrak p) = A_\mathfrak p$ is the filtered colimit of A_a for $a \in A$, $a \notin \mathfrak p$. See Algebra, Lemma 10.9.9. Thus we can descend the diagram above to a corresponding diagram over $\text{Spec}(A_a)$ for some $a \in A$, $a \notin \mathfrak p$. More precisely, after replacing S by $\text{Spec}(A_a)$ we may assume we have a commutative diagram

$$\begin{tikzcd} \overline{Y}_1 \amalg \ldots \amalg \overline{Y}_n \arrow[drr] & & Y_1 \amalg \ldots \amalg Y_n \arrow[ll, "j"'] \arrow[r, "\nu"'] \arrow[d] & X \arrow[d] \\ & & T_1 \amalg \ldots \amalg T_n \arrow[r] & S \end{tikzcd}$$

of schemes whose base change to η is the diagram above with the following properties

(1) ν is a finite, surjective morphism,
(2) j is an open immersion,
(3) $T_i \to S$ is finite étale for $i = 1, \ldots, n$,
(4) $Y_i \to T_i$ is smooth and surjective,
(5) $\overline{Y}_i \to T_i$ is smooth and proper and has geometrically connected fibres of dimension ≤ 1.

For this we first use Limits, Lemma 32.10.1 to obtain the diagram base changing to the previous diagram. Then we use Limits, Lemmas 32.8.10, 32.8.9, 32.8.3, 32.4.13, 32.8.12, 32.13.1, and 32.8.15 to obtain ν finite, surjective, j open immersion, $T_i \to S$

finite étale, $Y_i \to T$ smooth, $\overline{Y}_i \to T_i$ proper and smooth. Since Y_i cannot be empty, since smooth morphisms are open, and since $T_i \to S$ is finite étale, after shrinking S we may assume $Y_i \to T_i$ is surjective. Finally, the fibre of $\overline{Y}_i \to T_i$ over the unique point $\eta_i = \mathrm{Spec}(k_i)$ of T_i lying over η is geometrically connected. Hence by another shrinking we may assume the same thing is true for all fibres, see Lemma 37.53.8.

It remains to prove the existence of an open $W \subset X$ satisfying (a), (b), and (c). Since $\nu_\eta : \coprod Y_{i,\eta} \to X_\eta$ is the normalization morphism, we know by Varieties, Lemma 33.27.1 there exists a dense open $W_\eta \subset X_\eta$ such that $\nu^{-1}(W_\eta) \to W_\eta$ is equal to the inclusion of the reduction of W_η into W_η. Let $W \subset X$ be a quasi-compact open whose fibre over η is the open W_η we just found. After replacing $A = \Gamma(S, \mathcal{O}_S)$ by another localization we may assume $\nu^{-1}(W) \to W$ is a closed immersion, see Limits, Lemma 32.8.5. Since ν is also surjective we conclude $\nu^{-1}(W) \to W$ is a thickening. Set $W_i = \nu^{-1}(W) \cap Y_i$. Shrinking S once more we can assume $W_i \to T_i$ is surjective for all i (same argument as above). Then we find that $W_i \subset Y_i$ is dense in all fibres of $Y_i \to T_i$ as $Y_i \to T_i$ has geometrically irreducible fibres. Since ν is finite and surjective, it then follows that $W = \nu(\nu^{-1}(W))$ is dense in all fibres of $X \to S$ too. □

37.57. Descending separated locally quasi-finite morphisms

02W7 In this section we show that "separated locally quasi-finite morphisms satisfy descent for fppf-coverings". See Descent, Definition 35.36.1 for terminology. This is in the marvellous (for many reasons) paper by Raynaud and Gruson hidden in the proof of [GR71, Lemma 5.7.1]. It can also be found in [Mur95], and [ABD+66, Exposé X, Lemma 5.4] under the additional hypothesis that the morphism is locally of finite presentation. Here is the formal statement.

02W8 **Lemma 37.57.1.** Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fppf covering, see Topologies, Definition 34.7.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is separated and locally quasi-finite, then the descent datum is effective.

Proof. Being separated and being locally quasi-finite are properties of morphisms of schemes which are preserved under any base change, see Schemes, Lemma 26.21.12 and Morphisms, Lemma 29.20.13. Hence Descent, Lemma 35.36.2 applies and it suffices to prove the statement of the lemma in case the fppf-covering is given by a single $\{X \to S\}$ flat surjective morphism of finite presentation of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is separated and locally quasi-finite.

Let $W^1 \subset V$ be any affine open. Consider $W = \text{pr}_1(\varphi(W^1 \times_S X)) \subset V$. Here is a picture

$$\begin{tikzcd}
W^1 \times_S X \arrow[rrrrr] \arrow[ddd] \arrow[dr] & & & & & \varphi(W^1 \times_S X) \arrow[dl] \arrow[ddd] \\
& V \times_S X \arrow[rrr, "\varphi"] \arrow[dd] \arrow[dr] & & & X \times_S V \arrow[dd] \arrow[dl] & \\
& & X \times_S X \arrow[r, "1"] \arrow[d, "\text{pr}_0"'] & X \times_S X \arrow[d, "\text{pr}_1"] & & \\
W^1 \arrow[r] & V \arrow[r] & X & X & V \arrow[l] & W \arrow[l]
\end{tikzcd}$$

Ok, and now since $X \to S$ is flat and of finite presentation it is universally open (Morphisms, Lemma 29.25.10). Hence we conclude that W is open. Moreover, it is also clearly the case that W is quasi-compact, and $W^1 \subset W$. Moreover, we note that $\varphi(W \times_S X) = X \times_S W$ by the cocycle condition for φ. Hence we obtain a new descent datum (W, φ') by restricting φ to $W \times_S X$. Note that the morphism $W \to X$ is quasi-compact, separated and locally quasi-finite. This implies that it is separated and quasi-finite by definition. Hence it is quasi-affine by Lemma 37.43.2. Thus by Descent, Lemma 35.38.1 we see that the descent datum (W, φ') is effective.

In other words, we find that there exists an open covering $V = \bigcup W_i$ by quasi-compact opens W_i which are stable for the descent morphism φ. Moreover, for each such quasi-compact open $W \subset V$ the corresponding descent data (W, φ') is effective. This means the original descent datum is effective by glueing the schemes obtained from descending the opens W_i, see Descent, Lemma 35.35.13. $\square$

37.58. Relative finite presentation

05GX Let $R \to A$ be a finite type ring map. Let M be an A-module. In More on Algebra, Section 15.80 we defined what it means for M to be finitely presented relative to R. We also proved this notion has good localization properties and glues. Hence we can define the corresponding global notion as follows.

05H1 **Definition 37.58.1.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *finitely presented relative to S* or *of finite presentation relative to S* if there exists an affine open covering $S = \bigcup V_i$ and for every i an affine open covering $f^{-1}(V_i) = \bigcup_j U_{ij}$ such that $\mathcal{F}(U_{ij})$ is a $\mathcal{O}_X(U_{ij})$-module of finite presentation relative to $\mathcal{O}_S(V_i)$.

Note that this implies that $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module. If $X \to S$ is just locally of finite type, then $\mathcal{F}$ may be of finite presentation relative to S, without $X \to S$ being locally of finite presentation. We will see that $X \to S$ is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to S.

09T7 **Lemma 37.58.2.** *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is of finite presentation relative to S,*
(2) *for every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is finitely presented relative to $\mathcal{O}_S(V)$.*

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is of finite presentation relative to V.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 37.58.1. Let $U \subset X$ and $V \subset S$ be as in (2). By More on Algebra, Lemma 15.80.8 it suffices to find a standard open covering $U = \bigcup U_k$ of U such that $\mathcal{F}(U_k)$ is finitely presented relative to $\mathcal{O}_S(V)$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. Pick i such that $f(u) \in V_i$ and then pick j such that $u \in U_{ij}$. By Schemes, Lemma 26.11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in V' and V_i. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of U, resp. U_{ij}. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus U' is also a standard affine open of U and U_{ij}. By More on Algebra, Lemma 15.80.4 the assumption that $\mathcal{F}(U_{ij})$ is finitely presented relative to $\mathcal{O}_S(V_i)$ implies that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V_i)$. Since $\mathcal{O}_X(U') = \mathcal{O}_X(U') \otimes_{\mathcal{O}_S(V_i)} \mathcal{O}_S(V')$ we see from More on Algebra, Lemma 15.80.5 that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V')$. Applying More on Algebra, Lemma 15.80.4 again we conclude that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. This finishes the proof. □

09T8 Lemma 37.58.3. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) If f is locally of finite presentation, then $\mathcal{F}$ is of finite presentation relative to S if and only if $\mathcal{F}$ is of finite presentation.
(2) The morphism f is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to S.

Proof. Follows immediately from the definitions, see discussion following More on Algebra, Definition 15.80.2. □

09T9 Lemma 37.58.4. Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is of finite presentation relative to S if and only if $\pi_*\mathcal{F}$ is of finite presentation relative to S.

Proof. Translation of the result of More on Algebra, Lemma 15.80.3 into the language of schemes. □

09TA Lemma 37.58.5. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes, set $X' = X \times_S S'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X'. If $\mathcal{F}$ is of finite presentation relative to S, then $\mathcal{F}'$ is of finite presentation relative to S'.

Proof. Translation of the result of More on Algebra, Lemma 15.80.5 into the language of schemes. □

09TB Lemma 37.58.6. Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $f : X \to Y$ is locally of finite presentation and $\mathcal{G}$ of finite presentation relative to S, then $f^*\mathcal{G}$ is of finite presentation relative to S.

Proof. Translation of the result of More on Algebra, Lemma 15.80.6 into the language of schemes. □

09TC Lemma 37.58.7. Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $Y \to S$ is locally of finite presentation and $\mathcal{F}$ is of finite presentation relative to Y, then $\mathcal{F}$ is of finite presentation relative to S.

Proof. Translation of the result of More on Algebra, Lemma 15.80.7 into the language of schemes. □

09TD Lemma 37.58.8. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules.

(1) If $\mathcal{F}', \mathcal{F}''$ are finitely presented relative to S, then so is $\mathcal{F}$.
(2) If $\mathcal{F}'$ is a finite type $\mathcal{O}_X$-module and $\mathcal{F}$ is finitely presented relative to S, then $\mathcal{F}''$ is finitely presented relative to S.

Proof. Translation of the result of More on Algebra, Lemma 15.80.9 into the language of schemes. □

09TE Lemma 37.58.9. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}, \mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{F} \oplus \mathcal{F}'$ is finitely presented relative to S, then so are $\mathcal{F}$ and $\mathcal{F}'$.

Proof. Translation of the result of More on Algebra, Lemma 15.80.10 into the language of schemes. □

37.59. Relative pseudo-coherence

09UH This section is the analogue of More on Algebra, Section 15.81 for schemes. We strongly urge the reader to take a look at that section first. Although we have developed the material in this section and the material on pseudo-coherent complexes in Cohomology, Sections 20.46, 20.47, 20.48, and 20.49 for arbitrary complexes of $\mathcal{O}_X$-modules, if X is a scheme then working exclusively with objects in $D_{QCoh}(\mathcal{O}_X)$ greatly simplifies many of the lemmmas and arguments, often reducing the problem at hand immediately to the algebraic counterpart. Moreover, one of the first thing we do is to show that being relatively pseudo-coherent implies the cohomology sheaves are quasi-coherent, see Lemma 37.59.3. Hence, on a first reading we suggest the reader work exclusively with objects in $D_{QCoh}(\mathcal{O}_X)$.

09VC Lemma 37.59.1. Let $X \to S$ be a finite type morphism of affine schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. The following are equivalent

(1) for some closed immersion $i : X \to \mathbf{A}^n_S$ the object Ri_*E of $D(\mathcal{O}_{\mathbf{A}^n_S})$ is m-pseudo-coherent, and
(2) for all closed immersions $i : X \to \mathbf{A}^n_S$ the object Ri_*E of $D(\mathcal{O}_{\mathbf{A}^n_S})$ is m-pseudo-coherent.

Proof. Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. Let i correspond to the surjection $\alpha : R[x_1, \ldots, x_n] \to A$ and let $X \to \mathbf{A}^m_S$ correspond to $\beta : R[y_1, \ldots, y_m] \to A$. Choose

$f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{tikzcd} R[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[r, "y_j \mapsto f_j"'] \arrow[d, "x_i \mapsto g_i"] & R[x_1, \ldots, x_n] \arrow[d] \\ R[y_1, \ldots, y_m] \arrow[r] & A \end{tikzcd}$$

corresponding to the commutative diagram of closed immersions

$$\begin{tikzcd} \mathbf{A}^{n+m}_S & \mathbf{A}^n_S \arrow[l] \\ \mathbf{A}^m_S \arrow[u] & X \arrow[l] \arrow[u] \end{tikzcd}$$

Thus it suffices to show that under a closed immersion

$$f : \mathbf{A}^m_S \to \mathbf{A}^{n+m}_S$$

an object E of $D(\mathcal{O}_{\mathbf{A}^m_S})$ is m-pseudo-coherent if and only if Rf_*E is m-pseudo-coherent. This follows from Derived Categories of Schemes, Lemma 36.12.5 and the fact that $f_*\mathcal{O}_{\mathbf{A}^m_S}$ is a pseudo-coherent $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-module. The pseudo-coherence of $f_*\mathcal{O}_{\mathbf{A}^m_S}$ is straightforward to prove directly, but it also follows from Derived Categories of Schemes, Lemma 36.10.2 and More on Algebra, Lemma 15.81.3. □

Recall that if $f : X \to S$ is a morphism of scheme which is locally of finite type, then for every pair of affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$, the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type (Morphisms, Lemma 29.15.2). Hence there always exist closed immersions $U \to \mathbf{A}^n_V$ and the following definition makes sense.

09UI Definition 37.59.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Fix $m \in \mathbf{Z}$.

(1) We say E is m-pseudo-coherent relative to S if there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 37.59.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$.
(2) We say E is pseudo-coherent relative to S if E is m-pseudo-coherent relative to S for all $m \in \mathbf{Z}$.
(3) We say $\mathcal{F}$ is m-pseudo-coherent relative to S if $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is m-pseudo-coherent relative to S.
(4) We say $\mathcal{F}$ is pseudo-coherent relative to S if $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is pseudo-coherent relative to S.

If X is quasi-compact and E is m-pseudo-coherent relative to S for some m, then E is bounded above. If E is pseudo-coherent relative to S, then E has quasi-coherent cohomology sheaves.

0CSU Lemma 37.59.3. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If E in $D(\mathcal{O}_X)$ is m-pseudo-coherent relative to S, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If E is pseudo-coherent relative to S, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.

Proof. Choose an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 37.59.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$. Since being quasi-coherent is local on X, we may assume that there exists an closed immersion $i : X \to \mathbf{A}^n_S$ such that Ri_*E is m-pseudo-coherent on $\mathbf{A}^n_S$. By Derived Categories of Schemes, Lemma 36.10.1 this means that $H^q(Ri_*E)$ is quasi-coherent for $q > m$. Since i_* is an exact functor, we have $i_*H^q(E) = H^q(Ri_*E)$ is quasi-coherent on $\mathbf{A}^n_S$. By Morphisms, Lemma 29.4.1 this implies that $H^q(E)$ is quasi-coherent as desired (strictly speaking it implies there exists some quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $i_*\mathcal{F} = i_*H^q(E)$ and then Modules, Lemma 17.13.4 tells us that $\mathcal{F} \cong H^q(E)$ hence the result). □

Next, we prove the condition of relative pseudo-coherence localizes well.

09VD Lemma 37.59.4. Let S be an affine scheme. Let $V \subset S$ be a standard open. Let $X \to V$ be a finite type morphism of affine schemes. Let $U \subset X$ be an affine open. Let E be an object of $D(\mathcal{O}_X)$. If the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(X \to V, E)$, then the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U \to S, E|_U)$.

Proof. Write $S = \text{Spec}(R)$, $V = D(f)$, $X = \text{Spec}(A)$, and $U = D(g)$. Assume the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(X \to V, E)$.

Choose $R_f[x_1, \ldots, x_n] \to A$ surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Thus we have

$$X \to \mathbf{A}^n_V \to \mathbf{A}^{n+1}_S$$

and we can apply Derived Categories of Schemes, Lemma 36.12.5 to conclude that the pushforward E' of E to $\mathbf{A}^{n+1}_S$ is m-pseudo-coherent.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. Consider the surjection $R[x_0, \ldots, x_{n+1}] \to R[x_0, \ldots, x_n, 1/g']$. We obtain

$$\begin{tikzcd} X \arrow[d] & U \arrow[l] \arrow[d] \arrow[dr] & \\ \mathbf{A}^{n+1}_S & D(g') \arrow[l] \arrow[r] & \mathbf{A}^{n+2}_S \end{tikzcd}$$

where the lower left arrow is an open immersion and the lower right arrow is a closed immersion. We conclude as before that the pushforward of $E'|_{D(g')}$ to $\mathbf{A}^{n+2}_S$ is m-pseudo-coherent. Since this is also the pushforward of $E|_U$ to $\mathbf{A}^{n+2}_S$ we conclude the lemma is true. □

09VE Lemma 37.59.5. Let $X \to S$ be a finite type morphism of affine schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Let $X = \bigcup U_i$ be a standard affine open covering. The following are equivalent

1. (1) the equivalent conditions of Lemma 37.59.1 hold for the pairs $(U_i \to S, E|_{U_i})$,
2. (2) the equivalent conditions of Lemma 37.59.1 hold for the pair $(X \to S, E)$.

Proof. The implication (2) ⇒ (1) is Lemma 37.59.4. Assume (1). Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ and $U_i = D(f_i)$. Write $1 = \sum f_i g_i$ in A. Consider the surjections

$$R[x_i, y_i, z_i] \to R[x_i, y_i, z_i]/(\sum y_i z_i - 1) \to A.$$

which sends y_i to f_i and z_i to g_i. Note that $R[x_i, y_i, z_i]/(\sum y_i z_i - 1)$ is pseudo-coherent as an $R[x_i, y_i, z_i]$-module. Thus it suffices to prove that the pushforward of E to $T = \mathrm{Spec}(R[x_i, y_i, z_i]/(\sum y_i z_i - 1))$ is m-pseudo-coherent, see Derived Categories of Schemes, Lemma 36.12.5. For each i_0 it suffices to prove the restriction of this pushforward to $W_{i_0} = \mathrm{Spec}(R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_i z_i - 1))$ is m-pseudo-coherent. Note that there is a commutative diagram

$$\begin{tikzcd} X \arrow[d] & U_{i_0} \arrow[l] \arrow[d] \\ T & W_{i_0} \arrow[l] \end{tikzcd}$$

which implies that the pushforward of E to T restricted to W_{i_0} is the pushforward of $E|_{U_{i_0}}$ to W_{i_0}. Since $R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_i z_i - 1)$ is isomorphic to a polynomial ring over R this proves what we want. □

09UJ Lemma 37.59.6. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D(\mathcal{O}_X)$. Fix $m \in \mathbf{Z}$. The following are equivalent

(1) E is m-pseudo-coherent relative to S,
(2) for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U \to V, E|_U)$.

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $E|_U$ is m-pseudo-coherent relative to V.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 37.59.2. Let $U \subset X$ and $V \subset S$ be as in (2). By Lemma 37.59.5 it suffices to find a standard open covering $U = \bigcup U_k$ of U such that the equivalent conditions of Lemma 37.59.1 are satisfied for the pairs $(U_k \to V, E|_{U_k})$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U' \to V, E|_{U'})$. Pick i such that $f(u) \in V_i$ and then pick j such that $u \in U_{ij}$. By Schemes, Lemma 26.11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in V' and V_i. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of U, resp. U_{ij}. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus U' is also a standard affine open of U and U_{ij}. By Lemma 37.59.4 the assumption that the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U_{ij} \to V_i, E|_{U_{ij}})$ implies that the equivalent conditions of Lemma 37.59.1 are satisfied for the pair $(U' \to V, E|_{U'})$. □

For objects of the derived category whose cohomology sheaves are quasi-coherent, we can relate relative m-pseudo-coherence to the notion defined in More on Algebra, Definition 15.81.4. We will use the fact that for an affine scheme $U = \mathrm{Spec}(A)$ the functor $R\Gamma(U, -)$ induces an equivalence between $D_{QCoh}(\mathcal{O}_U)$ and $D(A)$, see Derived Categories of Schemes, Lemma 36.3.5. This functor is compatible with pullbacks: if E is an object of $D_{QCoh}(\mathcal{O}_U)$ and $A \to B$ is a ring map corresponding to a morphism of affine schemes $g : V = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = U$, then $R\Gamma(V, Lg^*E) = R\Gamma(U, E) \otimes_A^{\mathbf{L}} B$. See Derived Categories of Schemes, Lemma 36.3.8.

09VF Lemma 37.59.7. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Fix $m \in \mathbf{Z}$. The following are equivalent

(1) E is m-pseudo-coherent relative to S,
(2) there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex of $\mathcal{O}_X(U_{ij})$-modules $R\Gamma(U_{ij}, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V_i)$, and
(3) for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the complex of $\mathcal{O}_X(U)$-modules $R\Gamma(U, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V)$.

Proof. Let U and V be as in (2) and choose a closed immersion $i : U \to \mathbf{A}^n_V$. A formal argument, using Lemma 37.59.6, shows it suffices to prove that $Ri_*(E|_U)$ is m-pseudo-coherent if and only if $R\Gamma(U, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V)$. Say $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$, and $\mathbf{A}^n_V = \mathrm{Spec}(R[x_1, \ldots, x_n]$. By the remarks preceding the lemma, $E|_U$ is quasi-isomorphic to the complex of quasi-coherent sheaves on U associated to the object $R\Gamma(U, E)$ of $D(A)$. Note that $R\Gamma(U, E) = R\Gamma(\mathbf{A}^n_V, Ri_*(E|_U))$ as i is a closed immersion (and hence i_* is exact). Thus Ri_*E is associated to $R\Gamma(U, E)$ viewed as an object of $D(R[x_1, \ldots, x_n])$. We conclude as m-pseudo-coherence of $Ri_*(E|_U)$ is equivalent to m-pseudo-coherence of $R\Gamma(U, E)$ in $D(R[x_1, \ldots, x_n])$ by Derived Categories of Schemes, Lemma 36.10.2 which is equivalent to $R\Gamma(U, E)$ is m-pseudo-coherent relative to $R = \mathcal{O}_S(V)$ by definition. $\square$

09VG Lemma 37.59.8. Let $i : X \to Y$ morphism of schemes locally of finite type over a base scheme S. Assume that i induces a homeomorphism of X with a closed subset of Y. Let E be an object of $D(\mathcal{O}_X)$. Then E is m-pseudo-coherent relative to S if and only if Ri_*E is m-pseudo-coherent relative to S.

Proof. By Morphisms, Lemma 29.45.4 the morphism i is affine. Thus we may assume S, Y, and X are affine. Say $S = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(A)$, and $X = \mathrm{Spec}(B)$. The condition means that $A/\mathrm{rad}(A) \to B/\mathrm{rad}(B)$ is surjective; here $\mathrm{rad}(A)$ and $\mathrm{rad}(B)$ denote the Jacobson radical of A and B. As B is of finite type over A, we can find $b_1, \ldots, b_m \in \mathrm{rad}(B)$ which generate B as an A-algebra. Say $b_j^N = 0$ for all j. Consider the diagram of rings

$$\begin{tikzcd} B & R[x_i, y_j]/(y_j^N) \arrow[l] & R[x_i, y_j] \arrow[l] \\ A \arrow[u] & R[x_i] \arrow[l] \arrow[u] \arrow[ur] & \end{tikzcd}$$

which translates into a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & \mathbf{A}^{n+m}_S \arrow[dl] \\ Y \arrow[r] & \mathbf{A}^n_S & \end{tikzcd}$$

of affine schemes. By Lemma 37.59.6 we see that E is m-pseudo-coherent relative to S if and only if its pushforward to $\mathbf{A}^{n+m}_S$ is m-pseudo-coherent. By Derived

Categories of Schemes, Lemma 36.12.5 we see that this is true if and only if its pushforward to T is m-pseudo-coherent. The same lemma shows that this holds if and only if the pushforward to $\mathbf{A}^n_S$ is m-pseudo-coherent. Again by Lemma 37.59.6 this holds if and only if Ri_*E is m-pseudo-coherent relative to S. □

09UK Lemma 37.59.9. Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then E is m-pseudo-coherent relative to S if and only if $R\pi_*E$ is m-pseudo-coherent relative to S.

Proof. Translation of the result of More on Algebra, Lemma 15.81.5 into the language of schemes. Observe that $R\pi_*$ indeed maps $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Schemes, Lemma 36.4.1. To do the translation use Lemma 37.59.6. □

09UL Lemma 37.59.10. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let (E, E', E'') be a distinguished triangle of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.

(1) If E is $(m+1)$-pseudo-coherent relative to S and E' is m-pseudo-coherent relative to S then E'' is m-pseudo-coherent relative to S.
(2) If E, E'' are m-pseudo-coherent relative to S, then E' is m-pseudo-coherent relative to S.
(3) If E' is $(m+1)$-pseudo-coherent relative to S and E'' is m-pseudo-coherent relative to S, then E is $(m+1)$-pseudo-coherent relative to S.

Moreover, if two out of three of E, E', E'' are pseudo-coherent relative to S, the so is the third.

Proof. Immediate from Lemma 37.59.6 and Cohomology, Lemma 20.47.4. □

09UM Lemma 37.59.11. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then

(1) $\mathcal{F}$ is m-pseudo-coherent relative to S for all $m > 0$,
(2) $\mathcal{F}$ is 0-pseudo-coherent relative to S if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,
(3) $\mathcal{F}$ is (-1)-pseudo-coherent relative to S if and only if $\mathcal{F}$ is quasi-coherent and finitely presented relative to S.

Proof. Part (1) is immediate from the definition. To see part (3) we may work locally on X (both properties are local). Thus we may assume X and S are affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$. Then we see that $\mathcal{F}$ is (-1)-pseudo-coherent relative to S if and only if $i_*\mathcal{F}$ is (-1)-pseudo-coherent, which is true if and only if $i_*\mathcal{F}$ is an $\mathcal{O}_{\mathbf{A}^n_S}$-module of finite presentation, see Cohomology, Lemma 20.47.9. A module of finite presentation is quasi-coherent, see Modules, Lemma 17.11.2. By Morphisms, Lemma 29.4.1 we see that $\mathcal{F}$ is quasi-coherent if and only if $i_*\mathcal{F}$ is quasi-coherent. Having said this part (3) follows. The proof of (2) is similar but less involved. □

09UN Lemma 37.59.12. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E, K be objects of $D(\mathcal{O}_X)$. If $E \oplus K$ is m-pseudo-coherent relative to S so are E and K.

Proof. Follows from Cohomology, Lemma 20.47.6 and the definitions. □

09UP Lemma 37.59.13. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent relative to S for all i. Then $\mathcal{F}^\bullet$ is m-pseudo-coherent relative to S.

Proof. Follows from Cohomology, Lemma 20.47.7 and the definitions. $\square$

09UQ Lemma 37.59.14. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. If E is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent relative to S for all i, then E is m-pseudo-coherent relative to S.

Proof. Follows from Cohomology, Lemma 20.47.8 and the definitions. $\square$

09UR Lemma 37.59.15. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$ which is m-pseudo-coherent relative to S. Let $S' \to S$ be a morphism of schemes. Set $X' = X \times_S S'$ and denote E' the derived pullback of E to X'. If S' and X are Tor independent over S, then E' is m-pseudo-coherent relative to S'.

Proof. The problem is local on X and X' hence we may assume X, S, S', and X' are affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and denote $i' : X' \to \mathbf{A}^n_{S'}$ the base change to S'. Denote $g : X' \to X$ and $g' : \mathbf{A}^n_{S'} \to \mathbf{A}^n_S$ the projections, so $E' = Lg^*E$. Since X and S' are tor-independent over S, the base change map (Cohomology, Remark 20.28.3) induces an isomorphism

$$Ri'_*(Lg^*E) = L(g')^*Ri_*E$$

Namely, for a point $x' \in X'$ lying over $x \in X$ the base change map on stalks at x' is the map

$$E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{A}^n_S,x}} \mathcal{O}_{\mathbf{A}^n_{S'},x'} \longrightarrow E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

coming from the closed immersions i and i'. Note that the source is quasi-isomorphic to a localization of $E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ which is isomorphic to the target as $\mathcal{O}_{X',x'}$ is isomorphic to (the same) localization of $\mathcal{O}_{X,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ by assumption. We conclude the lemma holds by an application of Cohomology, Lemma 20.47.3. $\square$

09US Lemma 37.59.16. Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base S. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_Y)$. Assume

(1) $\mathcal{O}_X$ is pseudo-coherent relative to Y[12], and
(2) E is m-pseudo-coherent relative to S.

Then Lf^*E is m-pseudo-coherent relative to S.

Proof. The problem is local on X. Thus we may assume X, Y, and S are affine. Arguing as in the proof of More on Algebra, Lemma 15.81.13 we can find a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] & \mathbf{A}^d_Y \arrow[r, "j"'] \arrow[dl, "p"] & \mathbf{A}^{n+d}_S \arrow[dl] \\ Y \arrow[r] & \mathbf{A}^n_S & \end{tikzcd}$$

[12]This means f is pseudo-coherent, see Definition 37.60.2.

Observe that

$$Ri_*Lf^*E = Ri_*Li^*Lp^*E = Lp^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{A}^n_Y}} Ri_*\mathcal{O}_X$$

by Cohomology, Lemma 20.54.4. By assumption and the fact that Y is affine, we can represent $Ri_*\mathcal{O}_X = i_*\mathcal{O}_X$ by a complexes of finite free $\mathcal{O}_{\mathbf{A}^n_Y}$-modules $\mathcal{F}^\bullet$, with $\mathcal{F}^q = 0$ for $q > 0$ (details omitted; use Derived Categories of Schemes, Lemma 36.10.2 and More on Algebra, Lemma 15.81.7). By assumption E is bounded above, say $H^q(E) = 0$ for $q > a$. Represent E by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules with $\mathcal{E}^q = 0$ for $q > a$. Then the derived tensor product above is represented by $\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$.

Since j is a closed immersion, the functor j_* is exact and Rj_* is computed by applying j_* to any representating complex of sheaves. Thus we have to show that $j_*\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$ is m-pseudo-coherent as a complex of $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-modules. Note that $\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$. Note that for $q \ll 0$ the complexes $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent. Hence, applying Lemma 37.59.10 and induction, it suffices to show that $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^q[-q]$ is pseudo-coherent relative to S for all q. Note that $\mathcal{F}^q = 0$ for $q > 0$. Since also $\mathcal{F}^q$ is finite free this reduces to proving that $p^*\mathcal{E}^\bullet$ is m-pseudo-coherent relative to S which follows from Lemma 37.59.15 for instance. □

09UT Lemma 37.59.17. Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base S. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. Assume $\mathcal{O}_Y$ is pseudo-coherent relative to S[13]. Then the following are equivalent

(1) E is m-pseudo-coherent relative to Y, and
(2) E is m-pseudo-coherent relative to S.

Proof. The question is local on X, hence we may assume X, Y, and S are affine. Arguing as in the proof of More on Algebra, Lemma 15.81.13 we can find a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] & \mathbf{A}^m_Y \arrow[r, "j"'] \arrow[dl, "p"] & \mathbf{A}^{n+m}_S \arrow[dl] \\ Y \arrow[r] & \mathbf{A}^n_S & \end{tikzcd}$$

The assumption that $\mathcal{O}_Y$ is pseudo-coherent relative to S implies that $\mathcal{O}_{\mathbf{A}^m_Y}$ is pseudo-coherent relative to $\mathbf{A}^m_S$ (by flat base change; this can be seen by using for example Lemma 37.59.15). This in turn implies that $j_*\mathcal{O}_{\mathbf{A}^n_Y}$ is pseudo-coherent as an $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-module. Then the equivalence of the lemma follows from Derived Categories of Schemes, Lemma 36.12.5. □

09UU Lemma 37.59.18. Let

$$\begin{tikzcd} X \arrow[rr, "i"'] \arrow[dr] & & P \arrow[dl] \\ & S & \end{tikzcd}$$

[13] This means $Y \to S$ is pseudo-coherent, see Definition 37.60.2.

be a commutative diagram of schemes. Assume i is a closed immersion and $P \to S$ flat and locally of finite presentation. Let E be an object of $D(\mathcal{O}_X)$. Then the following are equivalent

(1) E is m-pseudo-coherent relative to S,
(2) Ri_*E is m-pseudo-coherent relative to S, and
(3) Ri_*E is m-pseudo-coherent on P.

Proof. The equivalence of (1) and (2) is Lemma 37.59.9. The equivalence of (2) and (3) follows from Lemma 37.59.17 applied to $\mathrm{id} : P \to P$ provided we can show that $\mathcal{O}_P$ is pseudo-coherent relative to S. This follows from More on Algebra, Lemma 15.82.4 and the definitions. $\square$

37.60. Pseudo-coherent morphisms

067X Avoid reading this section at all cost. If you need some of this material, first take a look at the corresponding algebra sections, see More on Algebra, Sections 15.64, 15.81, and 15.82. For now the only thing you need to know is that a ring map $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules.

067Y Lemma 37.60.1. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a pseudo-coherent ring map,
(2) for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent, and
(3) f is locally of finite type and $\mathcal{O}_X$ is pseudo-coherent relative to S.

Proof. To see the equivalence of (1) and (2) it suffices to check conditions (1)(a), (b), (c) of Morphisms, Definition 29.14.1 for the property of being a pseudo-coherent ring map. These properties follow (using localization is flat) from More on Algebra, Lemmas 15.81.12, 15.81.11, and 15.81.16.

If (1) holds, then f is locally of finite type as a pseudo-coherent ring map is of finite type by definition. Moreover, (1) implies via Lemma 37.59.7 and the definitions that $\mathcal{O}_X$ is pseudo-coherent relative to S. Conversely, if (3) holds, then we see that for every U and V as in (2) the ring $\mathcal{O}_X(U)$ is of finite type over $\mathcal{O}_S(V)$ and $\mathcal{O}_X(U)$ is as a module pseudo-coherent relative to $\mathcal{O}_S(V)$, see Lemmas 37.59.6 and 37.59.7. This is the definition of a pseudo-coherent ring map, hence (2) and (1) hold. $\square$

067Z Definition 37.60.2. A morphism of schemes $f : X \to S$ is called pseudo-coherent if the equivalent conditions of Lemma 37.60.1 are satisfied. In this case we also say that X is pseudo-coherent over S.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

0680 Lemma 37.60.3. A flat base change of a pseudo-coherent morphism is pseudo-coherent.

Proof. This translates into the following algebra result: Let $A \to B$ be a pseudo-coherent ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is pseudo-coherent. This follows from the more general More on Algebra, Lemma 15.81.12. $\square$

0681 Lemma 37.60.4. A composition of pseudo-coherent morphisms of schemes is pseudo-coherent.

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable pseudo-coherent ring maps then $A \to C$ is pseudo-coherent. This follows from either More on Algebra, Lemma 15.81.13 or More on Algebra, Lemma 15.81.15. □

0682 Lemma 37.60.5. A pseudo-coherent morphism is locally of finite presentation.

Proof. Immediate from the definitions. □

0695 Lemma 37.60.6. A flat morphism which is locally of finite presentation is pseudo-coherent.

Proof. This follows from the fact that a flat ring map of finite presentation is pseudo-coherent (and even perfect), see More on Algebra, Lemma 15.82.4. □

0683 Lemma 37.60.7. Let $f : X \to Y$ be a morphism of schemes pseudo-coherent over a base scheme S. Then f is pseudo-coherent.

Proof. This translates into the following algebra result: If $R \to A \to B$ are composable ring maps and $R \to A$, $R \to B$ pseudo-coherent, then $R \to B$ is pseudo-coherent. This follows from More on Algebra, Lemma 15.81.15. □

0AVX Lemma 37.60.8. Let $f : X \to S$ be a finite morphism of schemes. Then f is pseudo-coherent if and only if $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_S$-module.

Proof. Translated into algebra this lemma says the following: If $R \to A$ is a finite ring map, then $R \to A$ is pseudo-coherent as a ring map (which means by definition that A as an A-module is pseudo-coherent relative to R) if and only if A is pseudo-coherent as an R-module. This follows from the more general More on Algebra, Lemma 15.81.5. □

0684 Lemma 37.60.9. Let $f : X \to S$ be a morphism of schemes. If S is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.

Proof. This translates into the following algebra result: If $R \to A$ is a finite type ring map with R Noetherian, then $R \to A$ is pseudo-coherent if and only if $R \to A$ is of finite type. To see this, note that a pseudo-coherent ring map is of finite type by definition. Conversely, if $R \to A$ is of finite type, then we can write $A = R[x_1, \ldots, x_n]/I$ and it follows from More on Algebra, Lemma 15.64.17 that A is pseudo-coherent as an $R[x_1, \ldots, x_n]$-module, i.e., $R \to A$ is a pseudo-coherent ring map. □

0696 Lemma 37.60.10. The property $\mathcal{P}(f) =$"f is pseudo-coherent" is fpqc local on the base.

Proof. We will use the criterion of Descent, Lemma 35.22.4 to prove this. By Definition 37.60.2 being pseudo-coherent is Zariski local on the base. By Lemma 37.60.3 being pseudo-coherent is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 35.22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is pseudo-coherent as an $B[x_1, \ldots, x_n]$-module, then C is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 15.64.15. □

0697 Lemma 37.60.11. Let $A \to B$ be a flat ring map of finite presentation. Let $I \subset B$ be an ideal. Then $A \to B/I$ is pseudo-coherent if and only if I is pseudo-coherent as a B-module.

Proof. Choose a presentation $B = A[x_1, \ldots, x_n]/J$. Note that B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module because $A \to B$ is a pseudo-coherent ring map by Lemma 37.60.6. Note that $A \to B/I$ is pseudo-coherent if and only if B/I is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 15.64.11 we see this is equivalent to the condition that B/I is pseudo-coherent as an B-module. This proves the lemma as the short exact sequence $0 \to I \to B \to B/I \to 0$ shows that I is pseudo-coherent if and only if B/I is (see More on Algebra, Lemma 15.64.6). □

The following lemma will be obsoleted by the stronger Lemma 37.60.13.

0698 Lemma 37.60.12. The property $\mathcal{P}(f) =$"f is pseudo-coherent" is syntomic local on the source.

Proof. We will use the criterion of Descent, Lemma 35.26.4 to prove this. It follows from Lemmas 37.60.6 and 37.60.4 that being pseudo-coherent is preserved under precomposing with flat morphisms locally of finite presentation, in particular under precomposing with syntomic morphisms (see Morphisms, Lemmas 29.30.7 and 29.30.6). It is clear from Definition 37.60.2 that being pseudo-coherent is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 35.26.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ pseudo-coherent. Then $X \to Y$ is pseudo-coherent. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 35.14.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 10.136.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{tikzcd} X'_i \arrow[r] \arrow[d] & W_i \arrow[d] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[d, "h"] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

with $W \to \mathbf{A}^n_Y$ flat and of finite presentation and $X' \to Y$ still pseudo-coherent. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 29.25.10) and $X' \to X$ is surjective we can find $f \in \Gamma(\mathbf{A}^n_Y, \mathcal{O})$ such that $X \subset D(f) \subset \mathrm{Im}(h)$. Write $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$ and $W = \mathrm{Spec}(B)$, $A = R[x_1, \ldots, x_n]/I$ and $A' = B/IB$.

Then $R \to A'$ is pseudo-coherent. Picture

$$\begin{tikzcd} A' = B/IB & B \arrow[l] \\ A = R[x_1, \ldots, x_n]/I \arrow[u] & R[x_1, \ldots, x_n] \arrow[l] \arrow[u] \end{tikzcd}$$

By Lemma 37.60.11 we see that IB is pseudo-coherent as a B-module. The ring map $R[x_1, \ldots, x_n]_f \to B_f$ is faithfully flat by our choice of f above. This implies that $I_f \subset R[x_1, \ldots, x_n]_f$ is pseudo-coherent, see More on Algebra, Lemma 15.64.15. Applying Lemma 37.60.11 one more time we see that $R \to A$ is pseudo-coherent. $\square$

0699 Lemma 37.60.13. The property $\mathcal{P}(f) =$"f is pseudo-coherent" is fppf local on the source.

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{g_i : X_i \to X\}$ be an fppf covering such that each composition $f \circ g_i$ is pseudo-coherent. According to Lemma 37.48.2 there exist

(1) a Zariski open covering $X = \bigcup U_j$,
(2) surjective finite locally free morphisms $W_j \to U_j$,
(3) Zariski open coverings $W_j = \bigcup_k W_{j,k}$,
(4) surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fppf covering $\{h_{j,k} : T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Denote $\psi_{j,k} : T_{j,k} \to X_{\alpha(j,k)}$ the morphisms that witness the fact that $\{T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Note that $T_{j,k} \to X$ is a flat, locally finitely presented morphism, so both X_i and $T_{j,k}$ are pseudo-coherent over X by Lemma 37.60.6. Hence $\psi_{j,k} : T_{j,k} \to X_i$ is pseudo-coherent, see Lemma 37.60.7. Hence $T_{j,k} \to S$ is pseudo coherent as the composition of $\psi_{j,k}$ and $f \circ g_{\alpha(j,k)}$, see Lemma 37.60.4. Thus we see we have reduced the lemma to the case of a Zariski open covering (which is OK) and the case of a covering given by a single surjective finite locally free morphism which we deal with in the following paragraph.

Assume that $X' \to X \to S$ is a sequence of morphisms of schemes with $X' \to X$ surjective finite locally free and $X' \to Y$ pseudo-coherent. Our goal is to show that $X \to S$ is pseudo-coherent. Note that by Descent, Lemma 35.14.3 the morphism $X \to S$ is locally of finite presentation. It is clear that the problem reduces to the case that X', X and S are affine and $X' \to X$ is free of some rank $r > 0$. The corresponding algebra problem is the following: Suppose $R \to A \to A'$ are ring maps such that $R \to A'$ is pseudo-coherent, $R \to A$ is of finite presentation, and $A' \cong A^{\oplus r}$ as an A-module. Goal: Show $R \to A$ is pseudo-coherent. The assumption that $R \to A'$ is pseudo-coherent means that A' as an A'-module is pseudo-coherent relative to R. By More on Algebra, Lemma 15.81.5 this implies that A' as an A-module is pseudo-coherent relative to R. Since $A' \cong A^{\oplus r}$ as an A-module we see that A as an A-module is pseudo-coherent relative to R, see More on Algebra, Lemma 15.81.8. This by definition means that $R \to A$ is pseudo-coherent and we win. $\square$

37.61. Perfect morphisms

0685 In order to understand the material in this section you have to understand the material of the section on pseudo-coherent morphisms just a little bit. For now the only thing you need to know is that a ring map $A \to B$ is perfect if and only if it is pseudo-coherent and B has finite tor dimension as an A-module.

0686 Lemma 37.61.1. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The following are equivalent

(1) there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a perfect ring map, and
(2) for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is perfect.

Proof. Assume (1) and let U, V be as in (2). It follows from Lemma 37.60.1 that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent. Hence it suffices to prove that the property of a ring map being "of finite tor dimension" satisfies conditions (1)(a), (b), (c) of Morphisms, Definition 29.14.1. These properties follow from More on Algebra, Lemmas 15.66.11, 15.66.14, and 15.66.16. Some details omitted. $\square$

0687 Definition 37.61.2. A morphism of schemes $f : X \to S$ is called perfect if the equivalent conditions of Lemma 37.61.1 are satisfied. In this case we also say that X is perfect over S.

Note that a perfect morphism is in particular pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

0688 Lemma 37.61.3. A flat base change of a perfect morphism is perfect.

Proof. This translates into the following algebra result: Let $A \to B$ be a perfect ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is perfect. This result for pseudo-coherent ring maps we have seen in Lemma 37.60.3. The corresponding fact for finite tor dimension follows from More on Algebra, Lemma 15.66.14. $\square$

0689 Lemma 37.61.4. A composition of perfect morphisms of schemes is perfect.

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable perfect ring maps then $A \to C$ is perfect. We have seen this is the case for pseudo-coherent in Lemma 37.60.4 and its proof. By assumption there exist integers n, m such that B has tor dimension $\leq n$ over A and C has tor dimension $\leq m$ over B. Then for any A-module M we have

$$M \otimes_A^{\mathbf{L}} C = (M \otimes_A^{\mathbf{L}} B) \otimes_B^{\mathbf{L}} C$$

and the spectral sequence of More on Algebra, Example 15.62.4 shows that $\text{Tor}_p^A(M, C) = 0$ for $p > n + m$ as desired. $\square$

068A Lemma 37.61.5. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) f is flat and perfect, and
(2) f is flat and locally of finite presentation.

Proof. The implication (2) $\Rightarrow$ (1) is More on Algebra, Lemma 15.82.4. The converse follows from the fact that a pseudo-coherent morphism is locally of finite presentation, see Lemma 37.60.5. $\square$

068B Lemma 37.61.6. Let $f : X \to S$ be a morphism of schemes. Assume S is regular and f is locally of finite type. Then f is perfect.

Proof. See More on Algebra, Lemma 15.82.5. □

068C Lemma 37.61.7. A regular immersion of schemes is perfect. A Koszul-regular immersion of schemes is perfect.

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 31.21.2, it suffices to prove the second statement. This translates into the following algebraic statement: Suppose that $I \subset A$ is an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_r$ of A. Then $A \to A/I$ is a perfect ring map. Since $A \to A/I$ is surjective this is a presentation of A/I by a polynomial algebra over A. Hence it suffices to see that A/I is pseudo-coherent as an A-module and has finite tor dimension. By definition of a Koszul sequence the Koszul complex $K(A, f_1, \ldots, f_r)$ is a finite free resolution of A/I. Hence A/I is a perfect complex of A-modules and we win. □

068D Lemma 37.61.8. Let

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ perfect. Then $f : X \to Y$ is perfect.

Proof. We can factor f as the composition

$$X \longrightarrow X \times_S Y \longrightarrow Y$$

where the first morphism is the map $i = (1, f)$ and the second morphism is the projection. Since $Y \to S$ is flat, see Morphisms, Lemma 29.34.9, we see that $X \times_S Y \to Y$ is perfect by Lemma 37.61.3. As $Y \to S$ is smooth, also $X \times_S Y \to X$ is smooth, see Morphisms, Lemma 29.34.5. Hence i is a section of a smooth morphism, therefore i is a regular immersion, see Divisors, Lemma 31.22.8. This implies that i is perfect, see Lemma 37.61.7. We conclude that f is perfect because the composition of perfect morphisms is perfect, see Lemma 37.61.4. □

069A Remark 37.61.9. It is not true that a morphism between schemes X, Y perfect over a base S is perfect. An example is $S = \text{Spec}(k)$, $X = \text{Spec}(k)$, $Y = \text{Spec}(k[x]/(x^2)$ and $X \to Y$ the unique S-morphism.

069B Lemma 37.61.10. The property $\mathcal{P}(f)$ ="f is perfect" is fpqc local on the base.

Proof. We will use the criterion of Descent, Lemma 35.22.4 to prove this. By Definition 37.61.2 being perfect is Zariski local on the base. By Lemma 37.61.3 being perfect is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 35.22.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is perfect as an $B[x_1, \ldots, x_n]$-module, then C is perfect as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 15.74.13. □

069C Lemma 37.61.11. Let $f : X \to S$ be a pseudo-coherent morphism of schemes. The following are equivalent

(1) f is perfect,
(2) $\mathcal{O}_X$ locally has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_S$-modules, and
(3) for all $x \in X$ the ring $\mathcal{O}_{X,x}$ has finite tor dimension as an $\mathcal{O}_{S,f(x)}$-module.

Proof. The problem is local on X and S. Hence we may assume that $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and f corresponds to a pseudo-coherent ring map $A \to B$.

If (1) holds, then B has finite tor dimension d as A-module. Then $B_\mathfrak{q}$ has tor dimension d as an $A_\mathfrak{p}$-module for all primes $\mathfrak{q} \subset B$ with $\mathfrak{p} = A \cap \mathfrak{q}$, see More on Algebra, Lemma 15.66.15. Then $\mathcal{O}_X$ has tor dimension d as a sheaf of $f^{-1}\mathcal{O}_S$-modules by Cohomology, Lemma 20.48.5. Thus (1) implies (2).

By Cohomology, Lemma 20.48.5 (2) implies (3).

Assume (3). We cannot use More on Algebra, Lemma 15.66.15 to conclude as we are not given that the tor dimension of $B_\mathfrak{q}$ over $A_\mathfrak{p}$ is bounded independent of $\mathfrak{q}$. Choose a presentation $A[x_1, \ldots, x_n] \to B$. Then B is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. Let $\mathfrak{q} \subset A[x_1, \ldots, x_n]$ be a prime ideal lying over $\mathfrak{p} \subset A$. Then either $B_\mathfrak{q}$ is zero or by assumption it has finite tor dimension as an $A_\mathfrak{p}$-module. Since the fibres of $A \to A[x_1, \ldots, x_n]$ have finite global dimension, we can apply More on Algebra, Lemma 15.77.5 to $A_\mathfrak{p} \to A[x_1, \ldots, x_n]_\mathfrak{q}$ to see that $B_\mathfrak{q}$ is a perfect $A[x_1, \ldots, x_n]_\mathfrak{q}$-module. Hence B is a perfect $A[x_1, \ldots, x_n]$-module by More on Algebra, Lemma 15.77.3. Thus $A \to B$ is a perfect ring map by definition. $\square$

0G2E Lemma 37.61.12. Let $i : Z \to X$ be a perfect closed immersion of schemes. Then $i_*\mathcal{O}_Z$ is a perfect $\mathcal{O}_X$-module, i.e., it is a perfect object of $D(\mathcal{O}_X)$.

Proof. This is more or less immediate from the definition. Namely, let $U = \mathrm{Spec}(A)$ be an affine open of X. Then $i^{-1}(U) = \mathrm{Spec}(A/I)$ for some ideal $I \subset A$ and A/I has a finite resolution by finite projective A-modules by More on Algebra, Lemma 15.82.2. Hence $i_*\mathcal{O}_Z|_U$ can be represented by a finite length complex of finite locally free $\mathcal{O}_U$-modules. This is what we had to show, see Cohomology, Section 20.49. $\square$

0B6G Lemma 37.61.13. Let S be a Noetherian scheme. Let $f : X \to S$ be a perfect proper morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be perfect. Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.

Proof. We claim that Derived Categories of Schemes, Lemma 36.27.1 applies. Conditions (1) and (2) are immediate. Condition (3) is local on X. Thus we may assume X and S affine and E represented by a strictly perfect complex of $\mathcal{O}_X$-modules. Thus it suffices to show that $\mathcal{O}_X$ has finite tor dimension as a sheaf of $f^{-1}\mathcal{O}_S$-modules. This is equivalent to being perfect by Lemma 37.61.11. $\square$

069D Lemma 37.61.14. The property $\mathcal{P}(f) =$"f is perfect" is fppf local on the source.

Proof. Let $\{g_i : X_i \to X\}_{i \in I}$ be an fppf covering of schemes and let $f : X \to S$ be a morphism such that each $f \circ g_i$ is perfect. By Lemma 37.60.13 we conclude that f is pseudo-coherent. Hence by Lemma 37.61.11 it suffices to check that $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{S,f(x)}$-module of finite tor dimension for all $x \in X$. Pick $i \in I$ and $x_i \in X_i$ mapping to x. Then we see that $\mathcal{O}_{X_i,x_i}$ has finite tor dimension over $\mathcal{O}_{S,f(x)}$ and that $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is faithfully flat. The desired conclusion follows from More on Algebra, Lemma 15.66.17. $\square$

09RK Lemma 37.61.15. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume

(1) X is locally Noetherian,
(2) $j \circ i$ is a regular immersion, and
(3) i is perfect.

Then i and j are regular immersions.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Divisors, Lemma 31.20.8. Thus the result follows from Divided Power Algebra, Lemma 23.7.5. $\square$

37.62. Local complete intersection morphisms

068E In Divisors, Section 31.21 we have defined 4 different types of regular immersions: regular, Koszul-regular, H_1-regular, and quasi-regular. In this section we consider morphisms $f : X \to S$ which locally on X factor as

$$\begin{tikzcd} X \arrow[rr, "i"'] \arrow[dr] & & \mathbf{A}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where i is a $*$-regular immersion for $* \in \{\emptyset, Koszul, H_1, quasi\}$. However, we don't know how to prove that this condition is independent of the factorization if $* = \emptyset$, i.e., when we require i to be a regular immersion. On the other hand, we want a local complete intersection morphism to be perfect, which is only going to be true if $* = Koszul$ or $* = \emptyset$. Hence we will define a local complete intersection morphism or Koszul morphism to be a morphism of schemes $f : X \to S$ that locally on X has a factorization as above with i a Koszul-regular immersion. To see that this works we first prove this is independent of the chosen factorizations.

069E **Lemma 37.62.1.** Let S be a scheme. Let U, P, P' be schemes over S. Let $u \in U$. Let $i : U \to P$, $i' : U \to P'$ be immersions over S. Assume P and P' smooth over S. Then the following are equivalent

(1) i is a Koszul-regular immersion in a neighbourhood of x, and
(2) i' is a Koszul-regular immersion in a neighbourhood of x.

Proof. Assume i is a Koszul-regular immersion in a neighbourhood of x. Consider the morphism $j = (i, i') : U \to P \times_S P' = P''$. Since $P'' = P \times_S P' \to P$ is smooth, it follows from Divisors, Lemma 31.22.9 that j is a Koszul-regular immersion, whereupon it follows from Divisors, Lemma 31.22.12 that i' is a Koszul-regular immersion. $\square$

Before we state the definition, let us make the following simple remark. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ and a factorization of $f|_U$ as the composition of an immersion $i : U \to \mathbf{A}^n_S$ followed by the projection $\mathbf{A}^n_S \to S$ which is smooth. Picture

$$\begin{tikzcd} X \arrow[dr] & U \arrow[l] \arrow[r, "i"'] \arrow[d] & \mathbf{A}^n_S = P \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

In fact you can do this with any affine open neighbourhood U of x in X, see Morphisms, Lemma 29.39.2.

069F Definition 37.62.2. Let $f : X \to S$ be a morphism of schemes.

(1) Let $x \in X$. We say that f is Koszul at x if f is of finite type at x and there exists an open neighbourhood and a factorization of $f|_U$ as $\pi \circ i$ where $i : U \to P$ is a Koszul-regular immersion and $\pi : P \to S$ is smooth.
(2) We say f is a Koszul morphism, or that f is a local complete intersection morphism if f is Koszul at every point.

We have seen above that the choice of the factorization $f|_U = \pi \circ i$ is irrelevant, i.e., given a factorization of $f|_U$ as an immersion i followed by a smooth morphism π, whether or not i is Koszul regular in a neighbourhood of x is an intrinsic property of f at x. Let us record this here explicitly as a lemma so that we can refer to it

069G Lemma 37.62.3. Let $f : X \to S$ be a local complete intersection morphism. Let P be a scheme smooth over S. Let $U \subset X$ be an open subscheme and $i : U \to P$ an immersion of schemes over S. Then i is a Koszul-regular immersion.

Proof. This is the defining property of a local complete intersection morphism. See discussion above. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

069H Lemma 37.62.4. Let $f : X \to S$ be a local complete intersection morphism. Then

(1) f is locally of finite presentation,
(2) f is pseudo-coherent, and
(3) f is perfect.

Proof. Since a perfect morphism is pseudo-coherent (because a perfect ring map is pseudo-coherent) and a pseudo-coherent morphism is locally of finite presentation (because a pseudo-coherent ring map is of finite presentation) it suffices to prove the last statement. Being perfect is a local property, hence we may assume that f factors as $\pi \circ i$ where π is smooth and i is a Koszul-regular immersion. A Koszul-regular immersion is perfect, see Lemma 37.61.7. A smooth morphism is perfect as it is flat and locally of finite presentation, see Lemma 37.61.5. Finally a composition of perfect morphisms is perfect, see Lemma 37.61.4. □

07DB Lemma 37.62.5. Let $f : X = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$ be a morphism of affine schemes. Then f is a local complete intersection morphism if and only if $A \to B$ is a local complete intersection homomorphism, see More on Algebra, Definition 15.33.2.

Proof. Follows immediately from the definitions. □

Beware that a base change of a Koszul morphism is not Koszul in general.

069I Lemma 37.62.6. A flat base change of a local complete intersection morphism is a local complete intersection morphism.

Proof. Omitted. Hint: This is true because a base change of a smooth morphism is smooth and a flat base change of a Koszul-regular immersion is a Koszul-regular immersion, see Divisors, Lemma 31.21.3. □

069J **Lemma 37.62.7.** A composition of local complete intersection morphisms is a local complete intersection morphism.

Proof. Let $g : Y \to S$ and $f : X \to Y$ be local complete intersection morphisms. Let $x \in X$ and set $y = f(x)$. Choose an open neighbourhood $V \subset Y$ of y and a factorization $g|_V = \pi \circ i$ for some Koszul-regular immersion $i : V \to P$ and smooth morphism $\pi : P \to S$. Next choose an open neighbourhood U of $x \in X$ and a factorization $f|_U = \pi' \circ i'$ for some Koszul-regular immersion $i' : U \to P'$ and smooth morphism $\pi' : P' \to Y$. In fact, we may assume that $P' = \mathbf{A}^n_V$, see discussion preceding and following Definition 37.62.2. Picture:

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[rr, "i'"'] & & P' = \mathbf{A}^n_V \arrow[d] & \\
Y \arrow[d] & & & V \arrow[lll] \arrow[r, "i"'] & P \arrow[d] \\
S & & & & S \arrow[llll]
\end{tikzcd}$$

Set $P'' = \mathbf{A}^n_P$. Then $U \to P' \to P''$ is a Koszul-regular immersion as a composition of Koszul-regular immersions, namely i' and the flat base change of i via $P'' \to P$, see Divisors, Lemma 31.21.3 and Divisors, Lemma 31.21.7. Also $P'' \to P \to S$ is smooth as a composition of smooth morphisms, see Morphisms, Lemma 29.34.4. Hence we conclude that $X \to S$ is Koszul at x as desired. $\square$

069K **Lemma 37.62.8.** Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) f is flat and a local complete intersection morphism, and
(2) f is syntomic.

Proof. Working affine locally this is More on Algebra, Lemma 15.33.5. We also give a more geometric proof.

Assume (2). By Morphisms, Lemma 29.30.10 for every point x of X there exist affine open neighbourhoods U of x and V of $f(x)$ such that $f|_U : U \to V$ is standard syntomic. This means that $U = \mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to V = \mathrm{Spec}(R)$ where $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. By Algebra, Lemma 10.136.12 the sequence $f_1, \ldots, f_c$ is a regular sequence in each local ring $R[x_1, \ldots, x_n]_\mathfrak{q}$ for every prime $\mathfrak{q} \supset (f_1, \ldots, f_c)$. Consider the Koszul complex $K_\bullet = K_\bullet(R[x_1, \ldots, x_n], f_1, \ldots, f_c)$ with homology groups $H_i = H_i(K_\bullet)$. By More on Algebra, Lemma 15.30.2 we see that $(H_i)_\mathfrak{q} = 0$, $i > 0$ for every $\mathfrak{q}$ as above. On the other hand, by More on Algebra, Lemma 15.28.6 we see that H_i is annihilated by $(f_1, \ldots, f_c)$. Hence we see that $H_i = 0$, $i > 0$ and $f_1, \ldots, f_c$ is a Koszul-regular sequence. This proves that $U \to V$ factors as a Koszul-regular immersion $U \to \mathbf{A}^n_V$ followed by a smooth morphism as desired.

Assume (1). Then f is a flat and locally of finite presentation (Lemma 37.62.4). Hence, according to Morphisms, Lemma 29.30.10 it suffices to show that the local rings $\mathcal{O}_{X_s,x}$ are local complete intersection rings. Choose, locally on X, a factorization $f = \pi \circ i$ for some Koszul-regular immersion $i : X \to P$ and smooth morphism $\pi : P \to S$. Note that $X \to P$ is a relative quasi-regular immersion over S, see Divisors, Definition 31.22.2. Hence according to Divisors, Lemma 31.22.4 we see that

$X \to P$ is a regular immersion and the same remains true after any base change. Thus each fibre is a regular immersion, whence all the local rings of all the fibres of X are local complete intersections. □

069L Lemma 37.62.9. A regular immersion of schemes is a local complete intersection morphism. A Koszul-regular immersion of schemes is a local complete intersection morphism.

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 31.21.2, it suffices to prove the second statement. The second statement follows immediately from the definition. □

069M Lemma 37.62.10. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Immediate from the definitions. □

0E9K Lemma 37.62.11. Let $f : X \to Y$ be a morphism of schemes. If f is locally of finite type and X and Y are regular, then f is a local complete intersection morphism.

Proof. We may assume there is a factorization $X \to \mathbf{A}^n_Y \to Y$ where the first arrow is an immersion. As Y is regular also $\mathbf{A}^n_Y$ is regular by Algebra, Lemma 10.163.10. Hence $X \to \mathbf{A}^n_Y$ is a regular immersion by Divisors, Lemma 31.21.12. □

The following lemma is of a different nature.

09RL Lemma 37.62.12. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume

(1) S is locally Noetherian,
(2) $Y \to S$ is locally of finite type,
(3) $f : X \to Y$ is perfect,
(4) $X \to S$ is a local complete intersection morphism.

Then $X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

Proof. In the course of this proof all schemes will be locally Noetherian and all rings will be Noetherian. We will use without further mention that regular sequences and Koszul regular sequences agree in this setting, see More on Algebra, Lemma 15.30.7. Moreover, whether an ideal (resp. ideal sheaf) is regular may be checked on local rings (resp. stalks), see Algebra, Lemma 10.68.6 (resp. Divisors, Lemma 31.20.8)

The question is local. Hence we may assume S, X, Y are affine. In this situation we may choose a commutative diagram

$$\begin{tikzcd} \mathbf{A}^{n+m}_S \arrow[d] & X \arrow[l] \arrow[d] \\ \mathbf{A}^n_S \arrow[d] & Y \arrow[l] \arrow[dl] \\ S & \end{tikzcd}$$

whose horizontal arrows are closed immersions. Let $x \in X$ be a point and consider the corresponding commutative diagram of local rings

$$\begin{tikzcd} J \arrow[r] & \mathcal{O}_{\mathbf{A}^{n+m}_S, x} \arrow[r] & \mathcal{O}_{X,x} \\ I \arrow[r] \arrow[u] & \mathcal{O}_{\mathbf{A}^n_S, f(x)} \arrow[r] \arrow[u] & \mathcal{O}_{Y,f(x)} \arrow[u] \end{tikzcd}$$

where J and I are the kernels of the horizontal arrows. Since $X \to S$ is a local complete intersection morphism, the ideal J is generated by a regular sequence. Since $X \to Y$ is perfect the ring $\mathcal{O}_{X,x}$ has finite tor dimension over $\mathcal{O}_{Y,f(x)}$. Hence we may apply Divided Power Algebra, Lemma 23.7.6 to conclude that I and J/I are generated by regular sequences. By our initial remarks, this finishes the proof. $\square$

0FJ2 Lemma 37.62.13. Let

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume S is locally Noetherian, $Y \to S$ is locally of finite type, Y is regular, and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

Proof. This is a special case of Lemma 37.62.12 in view of Lemma 37.61.6 (and Morphisms, Lemma 29.15.8). $\square$

0FK1 Lemma 37.62.14. Let $i : X \to Y$ be an immersion. If

(1) i is perfect,
(2) Y is locally Noetherian, and
(3) the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free,

then i is a regular immersion.

Proof. Translated into algebra, this is Divided Power Algebra, Proposition 23.11.3. $\square$

0FV6 Lemma 37.62.15. Let $f : X \to Y$ be a local complete intersection homomorphism. Then the naive cotangent complex $NL_{X/Y}$ is a perfect object of $D(\mathcal{O}_X)$ of tor-amplitude in $[-1, 0]$.

Proof. Translated into algebra this is More on Algebra, Lemma 15.85.4. To do the translation use Lemmas 37.62.5 and 37.13.2 as well as Derived Categories of Schemes, Lemmas 36.3.5, 36.10.4 and 36.10.7. □

0FK2 Lemma 37.62.16. Let $f : X \to Y$ be a perfect morphism of locally Noetherian schemes. The following are equivalent

(1) f is a local complete intersection morphism,
(2) $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and
(3) $NL_{X/Y}$ is perfect with tor-amplitude in $[-1, 0]$.

Proof. Translated into algebra this is Divided Power Algebra, Lemma 23.11.4. To do the translation use Lemmas 37.62.5 and 37.13.2 as well as Derived Categories of Schemes, Lemmas 36.3.5, 36.10.4 and 36.10.7. □

0FK3 Lemma 37.62.17. Let $f : X \to Y$ be a flat morphism of finite presentation. The following are equivalent

(1) f is a local complete intersection morphism,
(2) f is syntomic,
(3) $NL_{X/Y}$ has tor-amplitude in $[-1, 0]$, and
(4) $NL_{X/Y}$ is perfect with tor-amplitude in $[-1, 0]$.

Proof. Translated into algebra this is Divided Power Algebra, Lemma 23.11.5. To do the translation use Lemmas 37.62.5 and 37.13.2 as well as Derived Categories of Schemes, Lemmas 36.3.5, 36.10.4 and 36.10.7. □

The following lemma gives a characterization of smooth morphisms as flat morphisms whose diagonal is perfect.

0FDP Lemma 37.62.18. Let $f : X \to Y$ be a finite type morphism of locally Noetherian schemes. Denote $\Delta : X \to X \times_Y X$ the diagonal morphism. The following are equivalent

(1) f is smooth,
(2) f is flat and $\Delta : X \to X \times_Y X$ is a regular immersion,
(3) f is flat and $\Delta : X \to X \times_Y X$ is a local complete intersection morphism,
(4) f is flat and $\Delta : X \to X \times_Y X$ is perfect.

Proof. Assume (1). Then f is flat by Morphisms, Lemma 29.34.9. The projections $X \times_Y X \to X$ are smooth by Morphisms, Lemma 29.34.5. Hence the diagonal is a section to a smooth morphism and hence a regular immersion, see Divisors, Lemma 31.22.8. Hence (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is Lemma 37.62.9. The implication (3) $\Rightarrow$ (4) is Lemma 37.62.4. The interesting implication (4) $\Rightarrow$ (1) follows immediately from Divided Power Algebra, Lemma 23.10.2. □

069N Lemma 37.62.19. The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is fpqc local on the base.

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering of S. Assume that each base change $f_i : X_i \to S_i$ of f is a local complete intersection morphism. Note that this implies in particular that f is locally of finite type, see Lemma 37.62.4 and Descent, Lemma 35.23.10. Let $x \in X$. Choose an open neighbourhood U of x and an immersion $j : U \to \mathbf{A}^n_S$ over S (see discussion preceding Definition 37.62.2). We have to show that j is a Koszul-regular immersion. Since f_i is a local complete intersection morphism, we see that the base change

$j_i : U \times_S S_i \to \mathbf{A}^n_{S_i}$ is a Koszul-regular immersion, see Lemma 37.62.3. Because $\{\mathbf{A}^n_{S_i} \to \mathbf{A}^n_S\}$ is a fpqc covering we see from Descent, Lemma 35.23.32 that j is a Koszul-regular immersion as desired. □

069P Lemma 37.62.20. The property $\mathcal{P}(f) =$"f is a local complete intersection morphism" is syntomic local on the source.

Proof. We will use the criterion of Descent, Lemma 35.26.4 to prove this. It follows from Lemmas 37.62.8 and 37.62.7 that being a local complete intersection morphism is preserved under precomposing with syntomic morphisms. It is clear from Definition 37.62.2 that being a local complete intersection morphism is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 35.26.4 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ a local complete intersection morphism. Then $X \to Y$ is a local complete intersection morphism. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 35.14.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 10.136.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{tikzcd} X'_i \arrow[r] \arrow[d] & W_i \arrow[d] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram of affine schemes

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[d, "h"] \\ X \arrow[r] & \mathbf{A}^n_Y \end{tikzcd}$$

with $h : W \to \mathbf{A}^n_Y$ syntomic and $X' \to Y$ still a local complete intersection morphism. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 29.25.10) and $X' \to X$ is surjective we see that X is contained in the image of $W \to \mathbf{A}^n_Y$. Choose a closed immersion $W \to \mathbf{A}^{n+m}_Y$ over $\mathbf{A}^n_Y$. Now the diagram looks like

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "h"] & \mathbf{A}^{n+m}_Y \arrow[dl] \\ X \arrow[r] & \mathbf{A}^n_Y & \end{tikzcd}$$

Because h is syntomic and hence a local complete intersection morphism (see above) the morphism $W \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. Because $X' \to Y$ is a local complete intersection morphism the morphism $X' \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. We conclude from Divisors, Lemma 31.21.8 that $X' \to W$ is a Koszul-regular immersion. Hence, since being a Koszul-regular immersion is fpqc local on the target (see Descent, Lemma 35.23.32) we conclude that $X \to \mathbf{A}^n_Y$ is a Koszul-regular immersion which is what we had to show. □

06B8 Lemma 37.62.21. Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume both X and Y are flat and locally of finite presentation over S. Then the set

$$\{x \in X \mid f \text{ Koszul at } x\}.$$

is open in X and its formation commutes with arbitrary base change $S' \to S$.

Proof. The set is open by definition (see Definition 37.62.2). Let $S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$, $Y' = S' \times_S Y$, and denote $f' : X' \to Y'$ the base change of f. Let $x' \in X'$ be a point such that f' is Koszul at x'. Denote $s' \in S'$, $x \in X$, $y' \in Y'$, $y \in Y$, $s \in S$ the image of x'. Note that f is locally of finite presentation, see Morphisms, Lemma 29.21.11. Hence we may choose an affine neighbourhood $U \subset X$ of x and an immersion $i : U \to \mathbf{A}^n_Y$. Denote $U' = S' \times_S U$ and $i' : U' \to \mathbf{A}^n_{Y'}$ the base change of i. The assumption that f' is Koszul at x' implies that i' is a Koszul-regular immersion in a neighbourhood of x', see Lemma 37.62.3. The scheme X' is flat and locally of finite presentation over S' as a base change of X (see Morphisms, Lemmas 29.25.8 and 29.21.4). Hence i' is a relative H_1-regular immersion over S' in a neighbourhood of x' (see Divisors, Definition 31.22.2). Thus the base change $i'_{s'} : U'_{s'} \to \mathbf{A}^n_{Y'_{s'}}$ is a H_1-regular immersion in an open neighbourhood of x', see Divisors, Lemma 31.22.1 and the discussion following Divisors, Definition 31.22.2. Since $s' = \mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s)) = s$ is a surjective flat universally open morphism (see Morphisms, Lemma 29.23.4) we conclude that the base change $i_s : U_s \to \mathbf{A}^n_{Y_s}$ is an H_1-regular immersion in a neighbourhood of x, see Descent, Lemma 35.23.32. Finally, note that $\mathbf{A}^n_Y$ is flat and locally of finite presentation over S, hence Divisors, Lemma 31.22.7 implies that i is a (Koszul-)regular immersion in a neighbourhood of x as desired. □

06B9 Lemma 37.62.22. Let $f : X \to Y$ be a local complete intersection morphism of schemes. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.

Proof. The first assertion follows immediately from Lemma 37.6.8 and the fact that a local complete intersection morphism is locally of finite type. To compute the conormal sheaf of f we choose, locally on X, a factorization of f as $f = p \circ i$ where $i : X \to V$ is a Koszul-regular immersion and $V \to Y$ is smooth. By Lemma 37.11.13 we see that $\mathcal{C}_{X/Y}$ is a locally direct summand of $\mathcal{C}_{X/V}$ which is finite locally free as i is a Koszul-regular (hence quasi-regular) immersion, see Divisors, Lemma 31.21.5. □

06BA Lemma 37.62.23. Let $Z \to Y \to X$ be formally unramified morphisms of schemes. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 37.7.12 is short exact.

Proof. The question is local on Z hence we may assume there exists a factorization $Z \to \mathbf{A}^n_Y \to Y$ of the morphism $Z \to Y$. Then we get a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i'"'] \arrow[d, equal] & \mathbf{A}^n_Y \arrow[r] \arrow[d] & \mathbf{A}^n_X \arrow[d] \\ Z \arrow[r, "i"] & Y \arrow[r] & X \end{tikzcd}$$

As $Z \to Y$ is a local complete intersection morphism, we see that $Z \to \mathbf{A}^n_Y$ is a Koszul-regular immersion. Hence by Divisors, Lemma 31.21.6 the sequence

$$0 \to (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_Y} \to 0$$

is exact and locally split. Note that $i^*\mathcal{C}_{Y/X} = (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X}$ by Lemma 37.7.7 and note that the diagram

$$\begin{tikzcd} (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} \arrow[r] & \mathcal{C}_{Z/\mathbf{A}^n_X} \\ i^*\mathcal{C}_{Y/X} \arrow[r] \arrow[u, "\cong"] & \mathcal{C}_{Z/X} \arrow[u] \end{tikzcd}$$

is commutative. Hence the lower horizontal arrow is a locally split injection. This proves the lemma. ☐

37.63. Exact sequences of differentials and conormal sheaves

06BB In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to a pair of composable morphisms of schemes.

Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of schemes.

(1) There is a canonical exact sequence
$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$
see Morphisms, Lemma 29.32.9. If $g : Z \to Y$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 29.34.16 or see Lemma 37.11.11.

(2) If g is an immersion or more generally formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$
see Morphisms, Lemma 29.32.15 or see Lemma 37.7.10. If $f \circ g : Z \to X$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 29.34.17 or see Lemma 37.11.12.

(3) If g and $f \circ g$ are immersions or more generally formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$
see Morphisms, Lemma 29.32.18 or see Lemma 37.7.11. If $f : Y \to X$ is smooth or more generally formally smooth, then this sequence is a short exact sequence, see Morphisms, Lemma 29.34.18 or see Lemma 37.11.13.

(4) If g and f are immersions or more generally formally unramified, then there is a canonical exact sequence
$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$
see Morphisms, Lemma 29.31.5 or see Lemma 37.7.12. If $g : Z \to Y$ is a regular immersion[14] or more generally a local complete intersection morphism, then this sequence is a short exact sequence, see Divisors, Lemma 31.21.6 or see Lemma 37.62.23.

[14]It suffices for g to be a H_1-regular immersion. Observe that an immersion which is a local complete intersection morphism is Koszul regular.

37.64. Weakly étale morphisms

094N A ring homomorphism $A \to B$ is weakly étale if both $A \to B$ and $B \otimes_A B \to B$ are flat, see More on Algebra, Definition 15.104.1. The analogous notion for morphisms of schemes is the following.

094P Definition 37.64.1. A morphism of schemes $X \to Y$ is weakly étale or absolutely flat if both $X \to Y$ and the diagonal morphism $X \to X \times_Y X$ are flat.

An étale morphism is weakly étale and conversely it turns out that a weakly étale morphism is indeed somewhat like an étale morphism. For example, if $X \to Y$ is weakly étale, then $L_{X/Y} = 0$, as follows from Cotangent, Lemma 92.8.4. We will prove a very precise result relating weakly étale morphisms to étale morphisms later (see Pro-étale Cohomology, Section 61.9). In this section we stick with the basics.

094Q Lemma 37.64.2. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) $X \to Y$ is weakly étale, and
(2) for every $x \in X$ the ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is weakly étale.

Proof. Observe that under both assumptions (1) and (2) the morphism f is flat. Thus we may assume f is flat. Let $x \in X$ with image $y = f(x)$ in Y. There are canonical maps of rings

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_Y X, \Delta_{X/Y}(x)} \longrightarrow \mathcal{O}_{X,x}$$

where the first map is a localization (hence flat) and the second map is a surjection (hence an epimorphism of rings). Condition (1) means that for all x the second arrow is flat. Condition (2) is that for all x the composition is flat. These conditions are equivalent by Algebra, Lemma 10.39.4 and More on Algebra, Lemma 15.104.2. □

094R Lemma 37.64.3. Let $X \to Y$ be a morphism of schemes such that $X \to X \times_Y X$ is flat. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over Y, then $\mathcal{F}$ is flat over X.

Proof. Let $x \in X$ with image $y = f(x)$ in Y. Since $X \to X \times_Y X$ is flat, we see that $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is flat. Hence the result follows from More on Algebra, Lemma 15.104.2 and the definitions. □

094S Lemma 37.64.4. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) The morphism f is weakly étale.
(2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is weakly étale.
(3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is weakly étale.
(4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of weakly étale, for all $j \in J, i \in I_j$.

Moreover, if f is weakly étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is weakly-étale.

Proof. Suppose given open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$. Then $U \times_V U \subset X \times_Y X$ is open (Schemes, Lemma 26.17.3) and the diagonal $\Delta_{U/V}$ of $f|_U : U \to V$ is the restriction $\Delta_{X/Y}|_U : U \to U \times_V U$. Since flatness is a local property of morphisms of schemes (Morphisms, Lemma 29.25.3) the final statement of the lemma is follows as well as the equivalence of (1) and (3). If X and Y are affine, then $X \to Y$ is weakly étale if and only if $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is weakly étale (use again Morphisms, Lemma 29.25.3). Thus (1) and (3) are also equivalent to (2) and (4). □

094T Lemma 37.64.5. Let $X \to Y \to Z$ be morphisms of schemes.

(1) If $X \to X \times_Y X$ and $Y \to Y \times_Z Y$ are flat, then $X \to X \times_Z X$ is flat.
(2) If $X \to Y$ and $Y \to Z$ are weakly étale, then $X \to Z$ is weakly étale.

Proof. Part (1) follows from the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal of X over Z, the fact that

$$X \times_Y X = (X \times_Z X) \times_{(Y \times_Z Y)} Y,$$

the fact that a base change of a flat morphism is flat, and the fact that the composition of flat morphisms is flat (Morphisms, Lemmas 29.25.8 and 29.25.6). Part (2) follows from part (1) and the fact (just used) that the composition of flat morphisms is flat. □

094U Lemma 37.64.6. Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes and let $X' = Y' \times_Y X$ be the base change of X.

(1) If $X \to X \times_Y X$ is flat, then $X' \to X' \times_{Y'} X'$ is flat.
(2) If $X \to Y$ is weakly étale, then $X' \to Y'$ is weakly étale.

Proof. Assume $X \to X \times_Y X$ is flat. The morphism $X' \to X' \times_{Y'} X'$ is the base change of $X \to X \times_Y X$ by $Y' \to Y$. Hence it is flat by Morphisms, Lemmas 29.25.8. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat morphism is flat. □

094V Lemma 37.64.7. Let $X \to Y \to Z$ be morphisms of schemes. Assume that $X \to Y$ is flat and surjective and that $X \to X \times_Z X$ is flat. Then $Y \to Y \times_Z Y$ is flat.

Proof. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times_Z X \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

The top horizontal arrow is flat and the vertical arrows are flat. Hence X is flat over $Y \times_Z Y$. By Morphisms, Lemma 29.25.13 we see that Y is flat over $Y \times_Z Y$. □

094W Lemma 37.64.8. Let $f : X \to Y$ be a weakly étale morphism of schemes. Then f is formally unramified, i.e., $\Omega_{X/Y} = 0$.

Proof. Recall that f is formally unramified if and only if $\Omega_{X/Y} = 0$ by Lemma 37.6.7. Via Lemma 37.64.4 and Morphisms, Lemma 29.32.5 this follows from the case of rings which is More on Algebra, Lemma 15.104.12. □

094X Lemma 37.64.9. Let $f : X \to Y$ be a morphism of schemes. Then $X \to Y$ is weakly étale in each of the following cases

(1) $X \to Y$ is a flat monomorphism,
(2) $X \to Y$ is an open immersion,
(3) $X \to Y$ is flat and unramified,
(4) $X \to Y$ is étale.

Proof. If (1) holds, then $\Delta_{X/Y}$ is an isomorphism (Schemes, Lemma 26.23.2), hence certainly f is weakly étale. Case (2) is a special case of (1). The diagonal of an unramified morphism is an open immersion (Morphisms, Lemma 29.35.13), hence flat. Thus a flat unramified morphism is weakly étale. An étale morphism is flat and unramified (Morphisms, Lemma 29.36.5), hence (4) follows from (3). □

094Y Lemma 37.64.10. Let $f : X \to Y$ be a morphism of schemes. If Y is reduced and f weakly étale, then X is reduced.

Proof. Via Lemma 37.64.4 this follows from the case of rings which is More on Algebra, Lemma 15.104.8. □

The following lemma uses a nontrivial result about weakly étale ring maps.

094Z Lemma 37.64.11. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent

(1) f is weakly étale, and
(2) for $x \in X$ the local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism on strict henselizations.

Proof. Let $x \in X$ be a point with image $y = f(x)$ in Y. Choose a separable algebraic closure κ^{sep} of $\kappa(x)$. Let $\mathcal{O}_{X,x}^{sh}$ be the strict henselization corresponding to κ^{sep} and $\mathcal{O}_{Y,y}^{sh}$ the strict henselization relative to the separable algebraic closure of $\kappa(y)$ in κ^{sep}. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{X,x}^{sh} \\ \mathcal{O}_{Y,y} \arrow[r] \arrow[u] & \mathcal{O}_{Y,y}^{sh} \arrow[u] \end{tikzcd}$$

local homomorphisms of local rings, see Algebra, Lemma 10.155.10. Since the strict henselization is a filtered colimit of étale ring maps, More on Algebra, Lemma 15.104.14 shows the horizontal maps are weakly étale. Moreover, the horizontal maps are faithfully flat by More on Algebra, Lemma 15.45.1.

Assume f weakly étale. By Lemma 37.64.2 the left vertical arrow is weakly étale. By More on Algebra, Lemmas 15.104.9 and 15.104.11 the right vertical arrow is weakly étale. By More on Algebra, Theorem 15.104.24 we conclude the right vertical map is an isomorphism.

Assume $\mathcal{O}_{Y,y}^{sh} \to \mathcal{O}_{X,x}^{sh}$ is an isomorphism. Then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}^{sh}$ is weakly étale. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^{sh}$ is faithfully flat we conclude that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is weakly étale by More on Algebra, Lemma 15.104.10. Thus (2) implies (1) by Lemma 37.64.2. □

0950 Lemma 37.64.12. Let $f : X \to Y$ be a morphism of schemes. If Y is a normal scheme and f weakly étale, then X is a normal scheme.

Proof. By More on Algebra, Lemma 15.45.6 a scheme S is normal if and only if for all $s \in S$ the strict henselization of $\mathcal{O}_{S,s}$ is a normal domain. Hence the lemma follows from Lemma 37.64.11. $\square$

0951 **Lemma 37.64.13.** Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X, Y are weakly étale over S, then f is weakly étale.

Proof. We will use Morphisms, Lemmas 29.25.8 and 29.25.6 without further mention. Write $X \to Y$ as the composition $X \to X \times_S Y \to Y$. The second morphism is flat as the base change of the flat morphism $X \to S$. The first is the base change of the flat morphism $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$, hence flat. Thus $X \to Y$ is flat. The morphism $X \times_Y X \to X \times_S X$ is an immersion. Thus Lemma 37.64.3 implies, that since X is flat over $X \times_S X$ it follows that X is flat over $X \times_Y X$. $\square$

The following is a scheme theoretic generalization of the observation that a field extension that is simultaneously separable and purely inseparable must be an isomorphism.

0F6V **Lemma 37.64.14.** Let $f : X \to Y$ be a morphism of schemes. If f is weakly étale and a universal homeomorphism, it is an isomorphism.

Proof. Since f is a universal homeomorphism, the diagonal $\Delta : X \to X \times_Y X$ is a surjective closed immersion by Morphisms, Lemmas 29.45.4 and 29.10.2. Since Δ is also flat, we see that Δ must be an isomorphism by Morphisms, Lemma 29.26.1. In other words, f is a monomorphism (Schemes, Lemma 26.23.2). Since f is a universal homeomorphism it is certainly quasi-compact. Hence by Descent, Lemma 35.25.1 we find that f is an isomorphism. $\square$

The following is a weakly étale generalization of Étale Morphisms, Lemma 41.14.3.

0F6W **Lemma 37.64.15.** Let $U \to X$ be a weakly étale morphism of schemes where X is a scheme in characteristic p. Then the relative Frobenius $F_{U/X} : U \to U \times_{X,F_X} X$ is an isomorphism.

Proof. The morphism $F_{U/X}$ is a universal homeomorphism by Varieties, Lemma 33.36.6. The morphism $F_{U/X}$ is weakly étale as a morphism between schemes weakly étale over X by Lemma 37.64.13. Hence $F_{U/X}$ is an isomorphism by Lemma 37.64.14. $\square$

37.65. Reduced fibre theorem

09IJ In this section we discuss the simplest kind of theorem of the kind advertised by the title. Although the proof of the result is kind of laborious, in essence it follows in a straightforward manner from Epp's result on eliminating ramification, see More on Algebra, Theorem 15.115.18.

Let A be a Dedekind domain with fraction field K. Let X be a scheme flat and of finite type over A. Let L be a finite extension of K. Let B be the integral closure of A in L. Then B is a Dedekind domain (Algebra, Lemma 10.120.18). Let $X_B = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ be the base change. Then $X_B \to \mathrm{Spec}(B)$ is of finite type (Morphisms, Lemma 29.15.4). Hence X_B is Noetherian (Morphisms, Lemma

29.15.6). Thus the normalization $\nu : Y \to X_B$ exists (see Morphisms, Definition 29.54.1 and the discussion following). Picture

09IK (37.65.0.1)
$$\begin{tikzcd} Y \arrow[r, "\nu"'] \arrow[dr] & X_B \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

We sometimes call Y the normalized base change of X. In general the morphism ν may not be finite. But if A is a Nagata ring (a condition that is virtually always satisfied in practice) then ν is finite and Y is of finite type over B, see Morphisms, Lemmas 29.54.10 and 29.18.1.

Taking the normalized base change commutes with composition. More precisely, if $M/L/K$ are finite extensions of fields with integral closures $A \subset B \subset C$ then the normalized base change Z of $Y \to \mathrm{Spec}(B)$ relative to M/L is equal to the normalized base change of $X \to \mathrm{Spec}(A)$ relative to M/K.

09IL **Theorem 37.65.1.** Let A be a Dedekind ring with fraction field K. Let X be a scheme flat and of finite type over A. Assume A is a Nagata ring. There exists a finite extension L/K such that the normalized base change Y is smooth over $\mathrm{Spec}(B)$ at all generic points of all fibres.

Proof. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like $X \to \mathrm{Spec}(A)$ is smooth commutes with base change, see Morphisms, Lemma 29.34.15.

We first choose a finite extension L/K such that $(X_L)_{red}$ is geometrically reduced over L, see Varieties, Lemma 33.6.11. Since $Y \to (X_B)_{red}$ is birational we see applying Varieties, Lemma 33.6.8 that Y_L is geometrically reduced over L as well. Hence $Y_L \to \mathrm{Spec}(L)$ is smooth on a dense open $V \subset Y_L$ by Varieties, Lemma 33.25.7. Thus the smooth locus $U \subset Y$ of the morphism $Y \to \mathrm{Spec}(B)$ is open (by Morphisms, Definition 29.34.1) and is dense in the generic fibre. Replacing A by B and X by Y we reduce to the case treated in the next paragraph.

Assume X is normal and the smooth locus $U \subset X$ of $X \to \mathrm{Spec}(A)$ is dense in the generic fibre. This implies that U is dense in all but finitely many fibres, see Lemma 37.24.3. Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \mathrm{Spec}(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Let A_i be the localization of A at the maximal ideal corresponding to the image of x_i in $\mathrm{Spec}(A)$. By More on Algebra, Proposition 15.116.8 there exist finite extensions K_i/K which are solutions for the extension of discrete valuation rings $A_i \to \mathcal{O}_i$. Let L/K be a finite extension dominating all of the extensions K_i/K. Then L/K is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma 15.116.1.

Consider the diagram (37.65.0.1) with the extension L/K we just produced. Note that $U_B \subset X_B$ is smooth over B, hence normal (for example use Algebra, Lemma 10.163.9). Thus $Y \to X_B$ is an isomorphism over U_B. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \mathrm{Spec}(B)$ lying over the maximal ideal $\mathfrak{m} \subset B$. Assume that $y \notin U_B$. Then y maps to one of the points x_i. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because L/K is a solution for $A_i \to \mathcal{O}_i$ we see that $B_\mathfrak{m} \to \mathcal{O}_{Y,y}$ is formally smooth in the $\mathfrak{m}_y$-adic topology (this is the definition of being a "solution"). In other words, $\mathfrak{m}\mathcal{O}_{Y,y} = \mathfrak{m}_y$ and the residue field extension is separable, see More on Algebra, Lemma 15.111.5. Hence the local ring of the fibre at y is $\kappa(y)$. This implies the fibre is smooth over $\kappa(\mathfrak{m})$ at y for example by Algebra, Lemma 10.140.5. This finishes the proof. $\square$

0BRQ Lemma 37.65.2 (Variant over curves). Let $f : X \to S$ be a flat, finite type morphism of schemes. Assume S is Nagata, integral with function field K, and regular of dimension 1. Then there exists a finite extension L/K such that in the diagram

$$\begin{tikzcd} Y \arrow[r, "\nu"] \arrow[dr, "g"'] & X \times_S T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & T \arrow[r] & S \end{tikzcd}$$

the morphism g is smooth at all generic points of fibres. Here T is the normalization of S in $\mathrm{Spec}(L)$ and $\nu : Y \to X \times_S T$ is the normalization.

Proof. Choose a finite affine open covering $S = \bigcup \mathrm{Spec}(A_i)$. Then K is equal to the fraction field of A_i for all i. Let $X_i = X \times_S \mathrm{Spec}(A_i)$. Choose L_i/K as in Theorem 37.65.1 for the morphism $X_i \to \mathrm{Spec}(A_i)$. Let $B_i \subset L_i$ be the integral closure of A_i and let Y_i be the normalized base change of X to B_i. Let L/K be a finite extension dominating each L_i. Let $T_i \subset T$ be the inverse image of $\mathrm{Spec}(A_i)$. For each i we get a commutative diagram

$$\begin{tikzcd} g^{-1}(T_i) \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & X \times_S \mathrm{Spec}(A_i) \arrow[d] \\ T_i \arrow[r] & \mathrm{Spec}(B_i) \arrow[r] & \mathrm{Spec}(A_i) \end{tikzcd}$$

and in fact the left hand square is a normalized base change as discussed at the beginning of the section. In the proof of Theorem 37.65.1 we have seen that the smooth locus of $Y \to T$ contains the inverse image in $g^{-1}(T_i)$ of the set of points where Y_i is smooth over B_i. This proves the lemma. $\square$

0BRR Lemma 37.65.3 (Variant with separable extension). Let A be a Dedekind ring with fraction field K. Let X be a scheme flat and of finite type over A. Assume A is a Nagata ring and that for every generic point η of an irreducible component of X the field extension $\kappa(\eta)/K$ is separable. Then there exists a finite separable extension L/K such that the normalized base change Y is smooth over $\mathrm{Spec}(B)$ at all generic points of all fibres.

Proof. This is proved in exactly the same manner as Theorem 37.65.1 with a few minor modifications. The most important change is to use More on Algebra, Lemma 15.116.9 instead of More on Algebra, Proposition 15.116.8. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like $X \to \mathrm{Spec}(A)$ is smooth commutes with base change, see Morphisms, Lemma 29.34.15.

Since X is flat over A every generic point η of X maps to the generic point of $\mathrm{Spec}(A)$. After replacing X by its reduction we may assume X is reduced. In this case X_K is geometrically reduced over K by Varieties, Lemma 33.6.8. Hence $X_K \to \mathrm{Spec}(K)$ is smooth on a dense open by Varieties, Lemma 33.25.7. Thus the smooth locus $U \subset X$ of the morphism $X \to \text{Spec}(A)$ is open (by Morphisms, Definition 29.34.1) and is dense in the generic fibre. This reduces us to the situation of the following paragraph.

Assume X is normal and the smooth locus $U \subset X$ of $X \to \text{Spec}(A)$ is dense in the generic fibre. This implies that U is dense in all but finitely many fibres, see Lemma 37.24.3. Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \text{Spec}(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Observe that the fraction field of $\mathcal{O}_i$ is the residue field of a generic point of X. Let A_i be the localization of A at the maximal ideal corresponding to the image of x_i in $\text{Spec}(A)$. We may apply More on Algebra, Lemma 15.116.9 and we find finite separable extensions K_i/K which are solutions for $A_i \to \mathcal{O}_i$. Let L/K be a finite separable extension dominating all of the extensions K_i/K. Then L/K is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma 15.116.1.

Consider the diagram (37.65.0.1) with the extension L/K we just produced. Note that $U_B \subset X_B$ is smooth over B, hence normal (for example use Algebra, Lemma 10.163.9). Thus $Y \to X_B$ is an isomorphism over U_B. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \text{Spec}(B)$ lying over the maximal ideal $\mathfrak{m} \subset B$. Assume that $y \notin U_B$. Then y maps to one of the points x_i. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because L/K is a solution for $A_i \to \mathcal{O}_i$ we see that $B_\mathfrak{m} \to \mathcal{O}_{Y,y}$ is formally smooth (this is the definition of being a "solution"). In other words, $\mathfrak{m}\mathcal{O}_{Y,y} = \mathfrak{m}_y$ and the residue field extension is separable. Hence the local ring of the fibre at y is $\kappa(y)$. This implies the fibre is smooth over $\kappa(\mathfrak{m})$ at y for example by Algebra, Lemma 10.140.5. This finishes the proof. □

0BRS Lemma 37.65.4 (Variant with separable extensions over curves). Let $f : X \to S$ be a flat, finite type morphism of schemes. Assume S is Nagata, integral with function field K, and regular of dimension 1. Assume the field extensions $\kappa(\eta)/K$ are separable for every generic point η of an irreducible component of X. Then there exists a finite separable extension L/K such that in the diagram

$$\begin{tikzcd} Y \arrow[r, "\nu"'] \arrow[rd, "g"'] & X \times_S T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & T \arrow[r] & S \end{tikzcd}$$

the morphism g is smooth at all generic points of fibres. Here T is the normalization of S in $\text{Spec}(L)$ and $\nu : Y \to X \times_S T$ is the normalization.

Proof. This follows from Lemma 37.65.3 in exactly the same manner that Lemma 37.65.2 follows from Theorem 37.65.1. □

37.66. Ind-quasi-affine morphisms

0AP5 A bit of theory to be used later.

0AP6 Definition 37.66.1. A scheme X is ind-quasi-affine if every quasi-compact open of X is quasi-affine. Similarly, a morphism of schemes $X \to Y$ is ind-quasi-affine if $f^{-1}(V)$ is ind-quasi-affine for each affine open V in Y.

An example of an ind-quasi-affine scheme is an open of an affine scheme. If $X = \bigcup_{i \in I} U_i$ is a union of quasi-affine opens such that any two U_i are contained in a third, then X is ind-quasi-affine. An ind-quasi-affine scheme X is separated because any two affine opens U, V are contained in a separated open subscheme of X, namely $U \cup V$. Similarly an ind-quasi-affine morphism is separated.

0F1U Lemma 37.66.2. For a morphism of schemes $f : X \to Y$, the following are equivalent:

(1) f is ind-quasi-affine,
(2) for every affine open subscheme $V \subset Y$ and every quasi-compact open subscheme $U \subset f^{-1}(V)$, the induced morphism $U \to V$ is quasi-affine.
(3) for some cover $\{V_j\}_{j \in J}$ of Y by quasi-compact and quasi-separated open subschemes $V_j \subset Y$, every $j \in J$, and every quasi-compact open subscheme $U \subset f^{-1}(V_j)$, the induced morphism $U \to V_j$ is quasi-affine.
(4) for every quasi-compact and quasi-separated open subscheme $V \subset Y$ and every quasi-compact open subscheme $U \subset f^{-1}(V)$, the induced morphism $U \to V$ is quasi-affine.

In particular, the property of being an ind-quasi-affine morphism is Zariski local on the base.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from the definitions and Morphisms, Lemma 29.13.3. For (2) $\Rightarrow$ (4), let U and V be as in (4). By Schemes, Lemma 26.21.14, the induced morphism $U \to V$ is quasi-compact. Thus, for every affine open $V' \subset V$, the fiber product $V' \times_V U$ is quasi-compact, so, by (2), the induced map $V' \times_V U \to V'$ is quasi-affine. Thus, $U \to V$ is also quasi-affine by Morphisms, Lemma 29.13.3. This argument also gives (3) $\Rightarrow$ (4): indeed, keeping the same notation, those affine opens $V' \subset V$ that lie in one of the V_j cover V, so one needs to argue that the quasi-compact map $V' \times_V U \to V'$ is quasi-affine. However, by (3), the composition $V' \times_V U \to V' \to V_j$ is quasi-affine and, by Schemes, Lemma 26.21.13, the map $V' \to V_j$ is quasi-separated. Thus, $V' \times_V U \to V'$ is quasi-affine by Morphisms, Lemma 29.13.8. The final implications (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are evident. $\square$

0F1V Lemma 37.66.3. The property of being an ind-quasi-affine morphism is stable under composition.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be ind-quasi-affine morphisms. Let $V \subset Z$ and $U \subset f^{-1}(g^{-1}(V))$ be quasi-compact opens such that V is also quasi-separated. The image $f(U)$ is a quasi-compact subset of $g^{-1}(V)$, so it is contained in some quasi-compact open $W \subset g^{-1}(V)$ (a union of finitely many affines). We obtain a factorization $U \to W \to V$. The map $W \to V$ is quasi-affine by Lemma 37.66.2, so, in particular, W is quasi-separated. Then, by Lemma 37.66.2 again, $U \to W$ is quasi-affine as well. Consequently, by Morphisms, Lemma 29.13.4, the composition $U \to V$ is also quasi-affine, and it remains to apply Lemma 37.66.2 once more. $\square$

0F1W Lemma 37.66.4. Any quasi-affine morphism is ind-quasi-affine. Any immersion is ind-quasi-affine.

Proof. The first assertion is immediate from the definitions. In particular, affine morphisms, such as closed immersions, are ind-quasi-affine. Thus, by Lemma 37.66.3, it remains to show that an open immersion is ind-quasi-affine. This, however, is immediate from the definitions. $\square$

0F1X Lemma 37.66.5. If $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes such that $g \circ f$ is ind-quasi-affine, then f is ind-quasi-affine.

Proof. By Lemma 37.66.2, we may work Zariski locally on Z and then on Y, so we lose no generality by assuming that Z, and then also Y, is affine. Then any quasi-compact open of X is quasi-affine, so Lemma 37.66.2 gives the claim. □

0AP7 Lemma 37.66.6. The property of being ind-quasi-affine is stable under base change.

Proof. Let $f : X \to Y$ be an ind-quasi-affine morphism. For checking that every base change of f is ind-quasi-affine, by Lemma 37.66.2, we may work Zariski locally on Y, so we assume that Y is affine. Furthermore, we may also assume that in the base change morphism $Z \to Y$ the scheme Z is affine, too. The base change $X \times_Y Z \to X$ is an affine morphism, so, by Lemmas 37.66.3 and 37.66.4, the map $X \times_Y Z \to Y$ is ind-quasi-affine. Then, by Lemma 37.66.5, the base change $X \times_Y Z \to Z$ is ind-quasi-affine, as desired. □

0AP8 Lemma 37.66.7. The property of being ind-quasi-affine is fpqc local on the base.

Proof. The stability of ind-quasi-affineness under base change supplied by Lemma 37.66.6 gives one direction. For the other, let $f : X \to Y$ be a morphism of schemes and let $\{g_i : Y_i \to Y\}$ be an fpqc covering such that the base change $f_i : X_i \to Y_i$ is ind-quasi-affine for all i. We need to show f is ind-quasi-affine.

By Lemma 37.66.2, we may work Zariski locally on Y, so we assume that Y is affine. Then we use stability under base change ensured by Lemma 37.66.6 to refine the cover and assume that it is given by a single affine, faithfully flat morphism $g : Y' \to Y$. For any quasi-compact open $U \subset X$, its Y'-base change $U \times_Y Y' \subset X \times_Y Y'$ is also quasi-compact. It remains to observe that, by Descent, Lemma 35.23.20, the map $U \to Y$ is quasi-affine if and only if so is $U \times_Y Y' \to Y'$. □

0AP9 Lemma 37.66.8. A separated locally quasi-finite morphism of schemes is ind-quasi-affine.

Proof. Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Let $V \subset Y$ be affine and $U \subset f^{-1}(V)$ quasi-compact open. We have to show U is quasi-affine. Since $U \to V$ is a separated quasi-finite morphism of schemes, this follows from Zariski's Main Theorem. See Lemma 37.43.2. □

37.67. Pushouts in the category of schemes, II

0ECH This section is a continuation of Section 37.14. In this section we construct pushouts of $Y \leftarrow Z \to X$ where $Z \to X$ is a closed immersion and $Z \to Y$ is integral and an additional condition is satisfied. Please see the detailed discussion in [Fer03].

0ECI Situation 37.67.1. Here S is a scheme and $i : Z \to X$ and $j : Z \to Y$ are morphisms of schemes over S. We assume

(1) i is a closed immersion,
(2) j is an integral morphism of schemes,
(3) for $y \in Y$ there exists an affine open $U \subset X$ with $j^{-1}(\{y\}) \subset i^{-1}(U)$.

0ECJ Lemma 37.67.2. In Situation 37.67.1 then for $y \in Y$ there exist affine opens $U \subset X$ and $V \subset Y$ with $i^{-1}(U) = j^{-1}(V)$ and $y \in V$.

Proof. Let $y \in Y$. Choose an affine open $U \subset X$ such that $j^{-1}(\{y\}) \subset i^{-1}(U)$ (possible by assumption). Choose an affine open $V \subset Y$ neighbourhood of y such that $j^{-1}(V) \subset i^{-1}(U)$. This is possible because $j : Z \to Y$ is a closed morphism (Morphisms, Lemma 29.44.7) and $i^{-1}(U)$ contains the fibre over y. Since j is integral, the scheme theoretic fibre Z_y is the spectrum of an algebra integral over a field. By Limits, Lemma 32.11.6 we can find an $\overline{f} \in \Gamma(i^{-1}(U), \mathcal{O}_{i^{-1}(U)})$ such that $Z_y \subset D(\overline{f}) \subset j^{-1}(V)$. Since $i|_{i^{-1}(U)} : i^{-1}(U) \to U$ is a closed immersion of affines, we can choose an $f \in \Gamma(U, \mathcal{O}_U)$ whose restriction to $i^{-1}(U)$ is $\overline{f}$. After replacing U by the principal open $D(f) \subset U$ we find affine opens $y \in V \subset Y$ and $U \subset X$ with

$$j^{-1}(\{y\}) \subset i^{-1}(U) \subset j^{-1}(V)$$

Now we (in some sense) repeat the argument. Namely, we choose $g \in \Gamma(V, \mathcal{O}_V)$ such that $y \in D(g)$ and $j^{-1}(D(g)) \subset i^{-1}(U)$ (possible by the same argument as above). Then we can pick $f \in \Gamma(U, \mathcal{O}_U)$ whose restriction to $i^{-1}(U)$ is the pullback of g by $i^{-1}(U) \to V$ (again possible by the same reason as above). Then we finally have affine opens $y \in V' = D(g) \subset V \subset Y$ and $U' = D(f) \subset U \subset X$ with $j^{-1}(V') = i^{-1}(V')$. $\square$

0E25 **Proposition 37.67.3.** In Situation 37.67.1 the pushout $Y \amalg_Z X$ exists in the category of schemes. Picture [Fer03, Theorem 7.1 part iii]

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "j"] & X \arrow[d, "a"] \\ Y \arrow[r, "b"] & Y \amalg_Z X \end{tikzcd}$$

The diagram is a fibre square, the morphism a is integral, the morphism b is a closed immersion, and

$$\mathcal{O}_{Y \amalg_Z X} = b_*\mathcal{O}_Y \times_{c_*\mathcal{O}_Z} a_*\mathcal{O}_X$$

as sheaves of rings where $c = a \circ i = b \circ j$.

Proof. As a topological space we set $Y \amalg_Z X$ equal to the pushout of the diagram in the category of topological spaces (Topology, Section 5.29). This is just the pushout of the underlying sets (Topology, Lemma 5.29.1) endowed with the quotient topology. On $Y \amalg_Z X$ we have the maps of sheaves of rings

$$b_*\mathcal{O}_Y \longrightarrow c_*\mathcal{O}_Z \longleftarrow a_*\mathcal{O}_X$$

and we can define

$$\mathcal{O}_{Y \amalg_Z X} = b_*\mathcal{O}_Y \times_{c_*\mathcal{O}_Z} a_*\mathcal{O}_X$$

as the fibre product in the category of sheaves of rings. To prove that we obtain a scheme we have to show that every point has an affine open neighbourhood. This is clear for points not in the image of c as the image of c is a closed subset whose complement is isomorphic as a ringed space to $(Y \setminus j(Z)) \amalg (X \setminus i(Z))$.

A point in the image of c corresponds to a unique $y \in Y$ in the image of j. By Lemma 37.67.2 we find affine opens $U \subset X$ and $V \subset Y$ with $y \in V$ and $i^{-1}(U) = j^{-1}(V)$. Since the construction of the first paragraph is clearly compatible with restriction to compatible open subschemes, to prove that it produces a scheme we may assume X, Y, and Z are affine.

If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $Z = \mathrm{Spec}(C)$ are affine, then More on Algebra, Lemma 15.6.2 shows that $Y \amalg_Z X = \mathrm{Spec}(B \times_C A)$ as topological spaces. To finish the proof that $Y \times_Z X$ is a scheme, it suffices to show that on $\text{Spec}(B \times_C A)$ the structure sheaf is the fibre product of the pushforwards. This follows by applying More on Algebra, Lemma 15.5.3 to principal affine opens of $\text{Spec}(B \times_C A)$.

The discussion above shows the scheme $Y \amalg_X Z$ has an affine open covering $Y \amalg_X Z = \bigcup W_i$ such that $U_i = a^{-1}(W_i)$, $V_i = b^{-1}(W_i)$, and $\Omega_i = c^{-1}(W_i)$ are affine open in X, Y, and Z. Thus a and b are affine. Moreover, if A_i, B_i, C_i are the rings corresponding to U_i, V_i, Ω_i, then $A_i \to C_i$ is surjective and W_i corresponds to $A_i \times_{C_i} B_i$ which surjects onto B_i. Hence b is a closed immersion. The ring map $A_i \times_{C_i} B_i \to A_i$ is integral by More on Algebra, Lemma 15.6.3 hence a is integral. The diagram is cartesian because

$$C_i \cong B_i \otimes_{B_i \times_{C_i} A_i} A_i$$

This follows as $B_i \times_{C_i} A_i \to B_i$ and $A_i \to C_i$ are surjective maps whose kernels are the same.

Finally, we can apply Lemmas 37.14.1 and 37.14.2 to conclude our construction is a pushout in the category of schemes. □

0E26 Lemma 37.67.4. In Situation 37.67.1. If X and Y are separated, then the pushout $Y \amalg_Z X$ (Proposition 37.67.3) is separated. Same with "separated over S", "quasi-separated", and "quasi-separated over S".

Proof. The morphism $Y \amalg X \to Y \amalg_Z X$ is surjective and universall closed. Thus we may apply Morphisms, Lemma 29.41.11. □

0E27 Lemma 37.67.5. In Situation 37.67.1 assume S is a locally Noetherian scheme and X, Y, and Z are locally of finite type over S. Then the pushout $Y \amalg_Z X$ (Proposition 37.67.3) is locally of finite type over S.

Proof. Looking on affine opens we recover the result of More on Algebra, Lemma 15.5.1. □

0ECK Lemma 37.67.6. In Situation 37.67.1 suppose given a commutative diagram

$$\begin{tikzcd} Y' \arrow[d, "g"] & Z' \arrow[l, "j'"] \arrow[r, "i'"] \arrow[d, "h"] & X' \arrow[d, "f"] \\ Y & Z \arrow[l] \arrow[r] & X \end{tikzcd}$$

with cartesian squares and f, g, h separated and locally quasi-finite. Then

(1) the pushouts $Y \amalg_Z X$ and $Y' \amalg_{Z'} X'$ exist,
(2) $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is separated and locally quasi-finite, and
(3) the squares

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y' \amalg_{Z'} X' \arrow[d] & X' \arrow[l] \arrow[d] \\ Y \arrow[r] & Y \amalg_Z X & X \arrow[l] \end{tikzcd}$$

are cartesian.

Proof. The pushout $Y \amalg_Z X$ exists by Proposition 37.67.3. To see that the pushout $Y' \amalg_{Z'} X'$ exists, we check condition (3) of Situation 37.67.1 holds for (X', Y', Z', i', j'). Namely, let $y' \in Y'$ and denote $y \in Y$ the image. Choose $U \subset X$ affine open with

$i(j^{-1}(y)) \subset U$. Choose a quasi-compact open $U' \subset X'$ contained in $f^{-1}(U)$ containing the quasi-compact subset $i'((j')^{-1}(\{y'\}))$. By Lemma 37.66.8 we see that U' is quasi-affine. Since $Z'_{y'}$ is the spectrum of an algebra integral over a field, we can apply Limits, Lemma 32.11.6 and we find there exists an affine open subscheme of U' containing $i'((j')^{-1}(\{y'\}))$ as desired.

Having verified existence we check the other assertions. Affine locally we are exactly in the situation of More on Algebra, Lemma 15.7.7 with $B \to D$ and $A' \to C'$ locally quasi-finite[15]. In particular, the morphism $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is locally of finite type. The squares in of the diagram are cartesian by More on Algebra, Lemma 15.6.4. Since being locally quasi-finite can be checked on fibres (Morphisms, Lemma 29.20.6) we conclude that $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is locally quasi-finite.

We still have to check $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is separated. Observe that $Y' \amalg X' \to Y' \amalg_{Z'} X'$ is universally closed and surjective by Proposition 37.67.3. Since also the morphism $Y' \amalg X' \to Y \amalg_Z X$ is separated (as it factors as $Y' \amalg X' \to Y \amalg X \to Y \amalg_Z X$) we conclude by Morphisms, Lemma 29.41.11. □

0ECL Lemma 37.67.7. In Situation 37.67.1 the category of schemes flat, separated, and locally quasi-finite over the pushout $Y \amalg_Z X$ is equivalent to the category of $(X', Y', Z', i', j', f, g, h)$ as in Lemma 37.67.6 with f, g, h flat. Similarly with "flat" replaced with "étale".

Proof. If we start with $(X', Y', Z', i', j', f, g, h)$ as in Lemma 37.67.6 with f, g, h flat or étale, then $Y' \amalg_{Z'} X' \to Y \amalg_Z X$ is flat or étale by More on Algebra, Lemma 15.7.7.

For the converse, let $W \to Y \amalg_Z X$ be a separated and locally quasi-finite morphism. Set $X' = W \times_{Y \amalg_Z X} X$, $Y' = W \times_{Y \amalg_Z X} Y$, and $Z' = W \times_{Y \amalg_Z X} Z$ with obvious morphisms i', j', f, g, h. Form the pushout $Y' \amalg_{Z'} X'$. We obtain a morphism

$$Y' \amalg_{Z'} X' \longrightarrow W$$

of schemes over $Y \amalg_X Z$ by the universal property of the pushout. If we do not assume that $W \to Y \amalg_Z X$ is flat, then in general this morphism won't be an isomorphism. (In fact, More on Algebra, Lemma 15.6.5 shows the displayed arrow is a closed immersion but not an isomorphism in general.) However, if $W \to Y \times_Z X$ is flat, then it is an isomorphism by More on Algebra, Lemma 15.7.7. □

Next, we discuss existence in the case where both morphisms are closed immersions.

0B7M Lemma 37.67.8. Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Then the pushout $Y \amalg_Z X$ exists in the category of schemes. Picture

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[d, "j"'] & X \arrow[d, "a"] \\ Y \arrow[r, "b"] & Y \amalg_Z X \end{tikzcd}$$

The diagram is a fibre square, the morphisms a and b are closed immersions, and there is a short exact sequence

$$0 \to \mathcal{O}_{Y \amalg_Z X} \to a_*\mathcal{O}_X \oplus b_*\mathcal{O}_Y \to c_*\mathcal{O}_Z \to 0$$

where $c = a \circ i = b \circ j$.

[15]To be precise $X, Y, Z, Y \amalg_Z X, X', Y', Z', Y' \amalg_{Z'} X'$ correspond to $A', B, A, B', C', D, C, D'$.

Proof. This is a special case of Proposition 37.67.3. Observe that hypothesis (3) in Situation 37.67.1 is immediate because the fibres of j are singletons. Finally, reverse the roles of the arrows to conclude that both a and b are closed immersions. □

0CYY Lemma 37.67.9. Let $i : Z \to X$ and $j : Z \to Y$ be closed immersions of schemes. Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms of schemes and let $\varphi : X' \times_{X,i} Z \to Y' \times_{Y,j} Z$ be an isomorphism of schemes over Z. Consider the morphism

$$h : X' \amalg_{X' \times_{X,i} Z, \varphi} Y' \longrightarrow X \amalg_Z Y$$

Then we have

(1) h is locally of finite type if and only if f and g are locally of finite type,
(2) h is flat if and only if f and g are flat,
(3) h is flat and locally of finite presentation if and only if f and g are flat and locally of finite presentation,
(4) h is smooth if and only if f and g are smooth,
(5) h is étale if and only if f and g are étale, and
(6) add more here as needed.

Proof. We know that the pushouts exist by Lemma 37.67.8. In particular we get the morphism h. Hence we may replace all schemes in sight by affine schemes. In this case the assertions of the lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 15.7.7. □

37.68. Relative morphisms

0BL0 In this section we prove a representability result which we will use in Fundamental Groups, Section 58.5 to prove a result on the category of finite étale coverings of a scheme. The material in this section is discussed in the correct generality in Criteria for Representability, Section 97.10.

Let S be a scheme. Let Z and X be schemes over S. Given a scheme T over S we can consider morphisms $b : T \times_S Z \to T \times_S X$ over S. Picture

0BL1 (37.68.0.1)
$$\begin{tikzcd} T \times_S Z \arrow[rr, "b"'] \arrow[rd] & & T \times_S X \arrow[ld] & Z \arrow[rd] & & X \arrow[ld] \\ & T \arrow[rrrr] & & & S & \end{tikzcd}$$

Of course, we can also think of b as a morphism $b : T \times_S Z \to X$ such that

$$\begin{tikzcd} T \times_S Z \arrow[r] \arrow[d] \arrow[rr, bend left, "b"] & Z \arrow[rd] & X \arrow[d] \\ T \arrow[rr] & & S \end{tikzcd}$$

commutes. In this situation we can define a functor

0BL2 (37.68.0.2) $$Mor_S(Z, X) : (Sch/S)^{opp} \longrightarrow \text{Sets}, \quad T \longmapsto \{b \text{ as above}\}$$

Here is a basic representability result.

05Y6 Lemma 37.68.1. Let $Z \to S$ and $X \to S$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(S, \mathcal{O}_S)$-module. Then $Mor_S(Z, X)$ is representable by an affine scheme over S.

Proof. Write $S = \text{Spec}(R)$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$ over R. Choose a presentation

$$\Gamma(X, \mathcal{O}_X) = R[\{x_i\}_{i \in I}]/(\{f_k\}_{k \in K}).$$

We will denote $\overline{x}_i$ the image of x_i in this quotient. Write

$$P = R[\{a_{ij}\}_{i \in I, 1 \leq j \leq m}].$$

Consider the R-algebra map

$$\Psi : R[\{x_i\}_{i \in I}] \longrightarrow P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \longmapsto \sum\nolimits_j a_{ij} \otimes e_j.$$

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements c_{kj}, $k \in K$, $1 \leq j \leq m$. We claim that $W = \text{Spec}(P/J)$ represents the functor $Mor_S(Z, X)$.

First, note that by construction P/J is an R-algebra, hence a morphism $W \to S$. Second, by construction the map Ψ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an P/J-algebra homomorphism

$$P/J \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)$$

which determines a morphism $b_{univ} : W \times_S Z \to W \times_S X$. By the Yoneda lemma b_{univ} determines a transformation of functors $W \to Mor_S(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \to Mor_S(Z, X)(T)$ over any affine scheme T.

Suppose $T = \text{Spec}(R')$ is an affine scheme over S and $b \in Mor_S(Z, X)(T)$. The structure morphism $T \to S$ defines an R-algebra structure on R' and b defines an R'-algebra map

$$b^\sharp : R' \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow R' \otimes_R \Gamma(Z, \mathcal{O}_Z).$$

In particular we can write $b^\sharp(1 \otimes \overline{x}_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an R-algebra map $P \to R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$. This map factors through the quotient P/J by the construction of the ideal J to give a map $P/J \to R'$. This in turn corresponds to a morphism $T \to W$ such that b is the pullback of b_{univ}. Some details omitted. $\square$

0BL3 Lemma 37.68.2. Let $Z \to S$ and $X \to S$ be morphisms of schemes. If $Z \to S$ is finite locally free and $X \to S$ is affine, then $Mor_S(Z, X)$ is representable by a scheme affine over S.

Proof. Choose an affine open covering $S = \bigcup U_i$ such that $\Gamma(Z \times_S U_i, \mathcal{O}_{Z \times_S U_i})$ is finite free over $\mathcal{O}_S(U_i)$. Let $F_i \subset Mor_S(Z, X)$ be the subfunctor which assigns to T/S the empty set if $T \to S$ does not factor through U_i and $Mor_S(Z, X)(T)$ otherwise. Then the collection of these subfunctors satisfy the conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 26.15.4 which proves the lemma. Condition (2)(a) follows from Lemma 37.68.1 and the other two follow from straightforward arguments. $\square$

The condition on the morphism $f : X \to S$ in the lemma below is very useful to prove statements like it. It holds if one of the following is true: X is quasi-affine, f is quasi-affine, f is quasi-projective, f is locally projective, there exists an ample invertible sheaf on X, there exists an f-ample invertible sheaf on X, or there exists an f-very ample invertible sheaf on X.

0BL4 **Lemma 37.68.3.** Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume

(1) $Z \to S$ is finite locally free, and
(2) for all $(s, x_1, \ldots, x_d)$ where $s \in S$ and $x_1, \ldots, x_d \in X_s$ there exists an affine open $U \subset X$ with $x_1, \ldots, x_d \in U$.

Then $Mor_S(Z, X)$ is representable by a scheme.

Proof. Consider the set I of pairs (U, V) where $U \subset X$ and $V \subset S$ are affine open and $U \to S$ factors through V. For $i \in I$ denote (U_i, V_i) the corresponding pair. Set $F_i = Mor_{V_i}(Z_{V_i}, U_i)$. It is immediate that F_i is a subfunctor of $Mor_S(Z, X)$. Then we claim that conditions (2)(a), (2)(b), (2)(c) of Schemes, Lemma 26.15.4 which proves the lemma.

Condition (2)(a) follows from Lemma 37.68.2.

To check condition (2)(b) consider T/S and $b \in Mor_S(Z, X)$. Thinking of b as a morphism $T \times_S Z \to X$ we find an open $b^{-1}(U_i) \subset T \times_S Z$. Clearly, $b \in F_i(T)$ if and only if $b^{-1}(U_i) = T \times_S Z$. Since the projection $p : T \times_S Z \to T$ is finite hence closed, the set $U_{i,b} \subset T$ of points $t \in T$ with $p^{-1}(\{t\}) \subset b^{-1}(U_i)$ is open. Then $f : T' \to T$ factors through $U_{i,b}$ if and only if $b \circ f \in F_i(T')$ and we are done checking (2)(b).

Finally, we check condition (2)(c) and this is where our condition on $X \to S$ is used. Namely, consider T/S and $b \in Mor_S(Z, X)$. It suffices to prove that every $t \in T$ is contained in one of the opens $U_{i,b}$ defined in the previous paragraph. This is equivalent to the condition that $b(p^{-1}(\{t\})) \subset U_i$ for some i where $p : T \times_S Z \to T$ is the projection and $b : T \times_S Z \to X$ is the given morphism. Since p is finite, the set $b(p^{-1}(\{t\})) \subset X$ is finite and contained in the fibre of $X \to S$ over the image s of t in S. Thus our condition on $X \to S$ exactly shows a suitable pair exists. $\square$

0BL5 **Lemma 37.68.4.** Let $Z \to S$ and $X \to S$ be morphisms of schemes. Assume $Z \to S$ is finite locally free and $X \to S$ is separated and locally quasi-finite. Then $Mor_S(Z, X)$ is representable by a scheme.

Proof. This follows from Lemmas 37.68.3 and 37.45.1. $\square$

37.69. Characterizing pseudo-coherent complexes, III

0CSI In this section we discuss characterizations of pseudo-coherent complexes in terms of cohomology. This is a continuation of Derived Categories of Schemes, Section 36.34. A basic tool will be to reduce to the case of projective space using a derived version of Chow's lemma, see Lemma 37.69.2.

0CTA **Lemma 37.69.1.** Consider a commutative diagram of schemes

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X' \arrow[r] & S' \end{tikzcd}$$

Let $S \to S'$ be a morphism. Denote by X and Y the base changes of X' and Y' to S. Assume $Y' \to S'$ and $Z' \to X'$ are flat. Then $X \times_S Y$ and Z' are Tor independent over $X' \times_{S'} Y'$.

Proof. The question is local, hence we may assume all schemes are affine (some details omitted). Observe that

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X' \times_{S'} Y' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

is cartesian with flat vertical arrows. Write $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, $X' \times_{S'} Y' = \mathrm{Spec}(B')$. Then $X \times_S Y = \mathrm{Spec}(A \otimes_{A'} B')$. Write $Z' = \mathrm{Spec}(C')$. We have to show

$$\mathrm{Tor}_p^{B'}(A \otimes_{A'} B', C') = 0, \quad \text{for } p > 0$$

Since $A' \to B'$ is flat we have $A \otimes_{A'} B' = A \otimes_{A'}^{\mathbf{L}} B'$. Hence

$$(A \otimes_{A'} B') \otimes_{B'}^{\mathbf{L}} C' = (A \otimes_{A'}^{\mathbf{L}} B') \otimes_{B'}^{\mathbf{L}} C' = A \otimes_{A'}^{\mathbf{L}} C' = A \otimes_{A'} C'$$

The second equality by More on Algebra, Lemma 15.60.5. The last equality because $A' \to C'$ is flat. This proves the lemma. □

0CSJ Lemma 37.69.2 (Derived Chow's lemma). Let A be a ring. Let X be a separated scheme of finite presentation over A. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ of x, an $n \geq 0$, an open $V \subset \mathbf{P}^n_A$, a closed subscheme $Z \subset X \times_A \mathbf{P}^n_A$, a point $z \in Z$, and an object E in $D(\mathcal{O}_{X \times_A \mathbf{P}^n_A})$ such that

(1) $Z \to X \times_A \mathbf{P}^n_A$ is of finite presentation,
(2) $b : Z \to X$ is an isomorphism over U and $b(z) = x$,
(3) $c : Z \to \mathbf{P}^n_A$ is a closed immersion over V,
(4) $b^{-1}(U) = c^{-1}(V)$, in particular $c(z) \in V$,
(5) $E|_{X \times_A V} \cong (b, c)_* \mathcal{O}_Z|_{X \times_A V}$,
(6) E is pseudo-coherent and supported on Z.

Proof. We can find a finite type $\mathbf{Z}$-subalgebra $A' \subset A$ and a scheme X' separated and of finite presentation over A' whose base change to A is X. See Limits, Lemmas 32.10.1 and 32.8.6. Let $x' \in X'$ be the image of x. If we can prove the lemma for $x' \in X'/A'$, then the lemma follows for $x \in X/A$. Namely, if U', n', V', Z', z', E' provide the solution for $x' \in X'/A'$, then we can let $U \subset X$ be the inverse image of U', let $n = n'$, let $V \subset \mathbf{P}^n_A$ be the inverse image of V', let $Z \subset X \times \mathbf{P}^n$ be the scheme theoretic inverse image of Z', let $z \in Z$ be the unique point mapping to x, and let E be the derived pullback of E'. Observe that E is pseudo-coherent by Cohomology, Lemma 20.47.3. It only remains to check (5). To see this set $W = b^{-1}(U) = c^{-1}(V)$ and $W' = (b')^{-1}(U) = (c')^{-1}(V')$ and consider the cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "{(b,c)}"'] & W' \arrow[d, "{(b',c')}"] \\ X \times_A V \arrow[r] & X' \times_{A'} V' \end{tikzcd}$$

By Lemma 37.69.1 the schemes $X \times_A V$ and W' are Tor independent over $X' \times_{A'} V'$. Hence the derived pullback of $(b', c')_* \mathcal{O}_{W'}$ to $X \times_A V$ is $(b, c)_* \mathcal{O}_W$ by Derived Categories of Schemes, Lemma 36.22.5. This also uses that $R(b', c')_* \mathcal{O}_{Z'} = (b', c')_* \mathcal{O}_{Z'}$ because (b', c') is a closed immersion and simiarly for $(b, c)_* \mathcal{O}_Z$. Since $E'|_{U' \times_{A'} V'} = (b', c')_* \mathcal{O}_{W'}$ we obtain $E|_{U \times_A V} = (b, c)_* \mathcal{O}_W$ and (5) holds. This reduces us to the situation described in the next paragraph.

Assume A is of finite type over $\mathbf{Z}$. Choose an affine open neighbourhood $U \subset X$ of x. Then U is of finite type over A. Choose a closed immersion $U \to \mathbf{A}^n_A$ and denote $j : U \to \mathbf{P}^n_A$ the immersion we get by composing with the open immersion $\mathbf{A}^n_A \to \mathbf{P}^n_A$. Let Z be the scheme theoretic closure of

$$(\mathrm{id}_U, j) : U \longrightarrow X \times_A \mathbf{P}^n_A$$

Since the projection $X \times \mathbf{P}^n \to X$ is separated, we conclude from Morphisms, Lemma 29.6.8 that $b : Z \to X$ is an isomorphism over U. Let $z \in Z$ be the unique point lying over x.

Let $Y \subset \mathbf{P}^n_A$ be the scheme theoretic closure of j. Then it is clear that $Z \subset X \times_A Y$ is the scheme theoretic closure of $(\mathrm{id}_U, j) : U \to X \times_A Y$. As X is separated, the morphism $X \times_A Y \to Y$ is separated as well. Hence we see that $Z \to Y$ is an isomorphism over the open subscheme $j(U) \subset Y$ by the same lemma we used above. Choose $V \subset \mathbf{P}^n_A$ open with $V \cap Y = j(U)$. Then we see that (3) and (4) hold.

Because A is Noetherian we see that X and $X \times_A \mathbf{P}^n_A$ are Noetherian schemes. Hence we can take $E = (b, c)_*\mathcal{O}_Z$ in this case, see Derived Categories of Schemes, Lemma 36.10.3. This finishes the proof. $\square$

0CSK Lemma 37.69.3. Let A, $x \in X$, and U, n, V, Z, z, E be as in Lemma 37.69.2. For any $K \in D_{QCoh}(\mathcal{O}_X)$ we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = R(U \to V)_*K|_U$$

where $p : X \times_A \mathbf{P}^n_A \to X$ and $q : X \times_A \mathbf{P}^n_A \to \mathbf{P}^n_A$ are the projections and where the morphism $U \to V$ is the finitely presented closed immersion $c \circ (b|_U)^{-1}$.

Proof. Since $b^{-1}(U) = c^{-1}(V)$ and since c is a closed immersion over V, we see that $c \circ (b|_U)^{-1}$ is a closed immersion. It is of finite presentation because U and V are of finite presentation over A, see Morphisms, Lemma 29.21.11. First we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_V = Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right)$$

where $q' : X \times_A V \to V$ is the projection because formation of total direct image commutes with localization. Set $W = b^{-1}(U) = c^{-1}(V)$ and denote $i : W \to X \times_A V$ the closed immersion $i = (b, c)|_W$. Then

$$Rq'_*\left((Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V}\right) = Rq'_*(Lp^*K|_{X \times_A V} \otimes^{\mathbf{L}} i_*\mathcal{O}_W)$$

by property (5). Since i is a closed immersion we have $i_*\mathcal{O}_W = Ri_*\mathcal{O}_W$. Using Derived Categories of Schemes, Lemma 36.22.1 we can rewrite this as

$$Rq'_*Ri_*Li^*Lp^*K|_{X \times_A V} = R(q' \circ i)_*Lb^*K|_W = R(U \to V)_*K|_U$$

which is what we want. $\square$

0CSL Lemma 37.69.4. Let A be a ring. Let X be a scheme separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.

Proof. Assume $K \in D_{QCoh}(\mathcal{O}_X)$ and $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent E in $D(\mathcal{O}_X)$. Let $x \in X$. We will show that K is pseudo-coherent relative to A in a neighbourhood of x and this will prove the lemma.

Choose U, n, V, Z, z, E as in Lemma 37.69.2. Denote $p : X \times \mathbf{P}^n \to X$ and $q : X \times \mathbf{P}^n \to \mathbf{P}^n_A$ the projections. Then for any $i \in \mathbf{Z}$ we have

$$\begin{aligned} & R\Gamma(\mathbf{P}^n_A, Rq_*(Lp^*K \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X \times \mathbf{P}^n, Lp^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i)) \\ & = R\Gamma(X, K \otimes^{\mathbf{L}} Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))) \end{aligned}$$

by Derived Categories of Schemes, Lemma 36.22.1. By Derived Categories of Schemes, Lemma 36.30.5 the complex $Rq_*(E \otimes^{\mathbf{L}} Lq^*\mathcal{O}_{\mathbf{P}^n_A}(i))$ is pseudo-coherent on X. Hence the assumption tells us the expression in the displayed formula is a pseudo-coherent object of $D(A)$. By Derived Categories of Schemes, Lemma 36.34.2 we conclude that $Rq_*(Lp^*K \otimes^{\mathbf{L}} E)$ is pseudo-coherent on $\mathbf{P}^n_A$. By Lemma 37.69.3 we have

$$Rq_*(Lp^*K \otimes^{\mathbf{L}} E)|_{X \times_A V} = R(U \to V)_*K|_U$$

Since $U \to V$ is a closed immersion into an open subscheme of $\mathbf{P}^n_A$ this means $K|_U$ is pseudo-coherent relative to A by Lemma 37.59.18. □

0GES Lemma 37.69.5. Let A be a ring. Let X be a scheme separated and of finite presentation over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is pseudo-coherent relative to A.

Proof. In view of Lemma 37.69.4, it suffices to show $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is pseudo-coherent in $D(A)$ for every pseudo-coherent $E \in D(\mathcal{O}_X)$. By Derived Categories of Schemes, Proposition 36.40.5 it follows that $K \in D^-_{QCoh}(\mathcal{O}_X)$. Now the result follows by Derived Categories of Schemes, Lemma 36.34.3. □

0GET Lemma 37.69.6. Let A be a ring. Let X be a scheme separated, of finite presentation, and flat over A. Let $K \in D_{QCoh}(\mathcal{O}_X)$. If $R\Gamma(X, E \otimes^{\mathbf{L}} K)$ is perfect in $D(A)$ for every perfect $E \in D(\mathcal{O}_X)$, then K is $\mathrm{Spec}(A)$-perfect.

Proof. By Lemma 37.69.5, K is pseudo-coherent relative to A. By Lemma 37.59.18, K is pseudo-coherent in $D(\mathcal{O}_X)$. By Derived Categories of Schemes, Proposition 36.40.6 we see that K is in $D^-(\mathcal{O}_X)$. Let $\mathfrak{p}$ be a prime ideal of A and denote $i : Y \to X$ the inclusion of the scheme theoretic fibre over $\mathfrak{p}$, i.e., Y is a scheme over $\kappa(\mathfrak{p})$. By Derived Categories of Schemes, Lemma 36.35.13, we will be done if we can show $Li^*(K)$ is bounded below. Let $G \in D_{perf}(\mathcal{O}_X)$ be a perfect complex which generates $D_{QCoh}(\mathcal{O}_X)$, see Derived Categories of Schemes, Theorem 36.15.3. We have

$$\begin{aligned} R\,\mathrm{Hom}_{\mathcal{O}_Y}(Li^*(G), Li^*(K)) & = R\Gamma(Y, Li^*(G^\vee \otimes^{\mathbf{L}} K)) \\ & = R\Gamma(X, G^\vee \otimes^{\mathbf{L}} K) \otimes^{\mathbf{L}}_A \kappa(\mathfrak{p}) \end{aligned}$$

The first equality uses that Li^* preserves perfect objects and duals and Cohomology, Lemma 20.50.5; we omit some details. The second equality follows from Derived Categories of Schemes, Lemma 36.22.5 as X is flat over A. It follows from our hypothesis that this is a perfect object of $D(\kappa(\mathfrak{p}))$. The object $Li^*(G) \in D_{perf}(\mathcal{O}_Y)$ generates $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Schemes, Remark 36.16.4. Hence Derived Categories of Schemes, Proposition 36.40.6 now implies that $Li^*(K)$ is bounded below and we win. □

37.70. Descent finiteness properties of complexes

0CSM This section is the continuation of Derived Categories of Schemes, Section 36.12.

0CSN Lemma 37.70.1. Let $X \to S$ be locally of finite type. Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if each Lf_i^*E is m-pseudo-coherent relative to S.

Proof. Assume E is m-pseudo-coherent relative to S. The morphisms f_i are pseudo-coherent by Lemma 37.60.6. Hence Lf_i^*E is m-pseudo-coherent relative to S by Lemma 37.59.16.

Conversely, assume that Lf_i^*E is m-pseudo-coherent relative to S for each i. Pick $S = \bigcup U_j$, $W_j \to U_j$, $W_j = \bigcup W_{j,k}$, $T_{j,k} \to W_{j,k}$, and morphisms $\alpha_{j,k} : T_{j,k} \to X_{i(j,k)}$ over S as in Lemma 37.48.2. Since the morphism $T_{j,K} \to S$ is flat and of finite presentation, we see that $\alpha_{j,k}$ is pseudo-coherent by Lemma 37.60.7. Hence

$$L\alpha_{j,k}^* Lf_{i(j,k)}^* E = L(T_{i,k} \to S)^* E$$

is m-pseudo-coherent relative to S by Lemma 37.59.16. Now we want to descend this property through the coverings $\{T_{j,k} \to W_{j,k}\}$, $W_j = \bigcup W_{j,k}$, $\{W_j \to U_j\}$, and $S = \bigcup U_j$. Since for Zariski coverings the result is true (by the definition of m-pseudo-coherence relative to S), this means we may assume we have a single surjective finite locally free morphism $\pi : Y \to X$ such that $L\pi^*E$ is pseudo-coherent relative to S. In this case $R\pi_*L\pi^*E$ is pseudo-coherent relative to S by Lemma 37.59.9 (this is the first time we use that E has quasi-coherent cohomology sheaves). We have $R\pi_*L\pi^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \pi_*\mathcal{O}_Y$ for example by Derived Categories of Schemes, Lemma 36.22.1 and locally on X the map $\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is the inclusion of a direct summand. Hence we conclude by Lemma 37.59.12. □

0CSP Lemma 37.70.2. Let $X \to T \to S$ be morphisms of schemes. Assume $T \to S$ is flat and locally of finite presentation and $X \to T$ locally of finite type. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if E is m-pseudo-coherent relative to T.

Proof. Locally on X we can choose a closed immersion $i : X \to \mathbf{A}^n_T$. Then $\mathbf{A}^n_T \to S$ is flat and locally of finite presentation. Thus we may apply Lemma 37.59.17 to see the equivalence holds. □

0CSQ Lemma 37.70.3. Let $f : X \to S$ be locally of finite type. Let $\{S_i \to S\}$ be an fppf covering of schemes. Denote $f_i : X_i \to S_i$ the base change of f and $g_i : X_i \to X$ the projection. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent relative to S if and only if each Lg_i^*E is m-pseudo-coherent relative to S_i.

Proof. This follows formally from Lemmas 37.70.1 and 37.70.2. Namely, if E is m-pseudo-coherent relative to S, then Lg_i^*E is m-pseudo-coherent relative to S (by the first lemma), hence Lg_i^*E is m-pseudo-coherent relative to S_i (by the second). Conversely, if Lg_i^*E is m-pseudo-coherent relative to S_i, then Lg_i^*E is m-pseudo-coherent relative to S (by the second lemma), hence E is m-pseudo-coherent relative to S (by the first lemma). □

37.71. Relatively perfect objects

0DJW This section is a continuation of the discussion in Derived Categories of Schemes, Section 36.35.

0DJX **Lemma 37.71.1.** *Let $i : X \to X'$ be a finite order thickening of schemes. Let $K' \in D(\mathcal{O}_{X'})$ be an object such that $K = Li^*K'$ is pseudo-coherent. Then K' is pseudo-coherent.*

Proof. We first prove K' has quasi-coherent cohomology sheaves. To do this, we may reduce to the case of a first order thickening, see Section 37.2. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals cutting out X. Tensoring the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to i_*\mathcal{O}_X \to 0$$

with K' we obtain a distinguished triangle

$$K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I} \to K' \to K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} i_*\mathcal{O}_X \to (K' \otimes^{\mathbf{L}}_{\mathcal{O}_{X'}} \mathcal{I})[1]$$

Since $i_* = Ri_*$ and since we may view $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_X$-module (as we have a first order thickening) we may rewrite this as

$$i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I}) \to K' \to i_*K \to i_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{I})[1]$$

Please use Cohomology, Lemma 20.54.4 to identify the terms. Since K is in $D_{QCoh}(\mathcal{O}_X)$ we conclude that K' is in $D_{QCoh}(\mathcal{O}_{X'})$; this uses Derived Categories of Schemes, Lemmas 36.10.1, 36.3.9, and 36.4.1.

Assume K' is in $D_{QCoh}(\mathcal{O}_{X'})$. The question is local on X' hence we may assume X' is affine. Say $X' = \mathrm{Spec}(A')$ and $X = \mathrm{Spec}(A)$ with $A = A'/I$ and I nilpotent. Then K' comes from an object $M' \in D(A')$, see Derived Categories of Schemes, Lemma 36.3.5. Thus $M = M' \otimes^{\mathbf{L}}_{A'} A$ is a pseudo-coherent object of $D(A)$ by Derived Categories of Schemes, Lemma 36.10.2 and our assumption on K. Hence we can represent M by a bounded above complex of finite free A-modules $E^\bullet$, see More on Algebra, Lemma 15.64.5. By More on Algebra, Lemma 15.75.3 we conclude that M' is pseudo-coherent as desired. □

0DJY **Lemma 37.71.2.** *Consider a cartesian diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of schemes. Assume $X' \to Y'$ is flat and locally of finite presentation and $Y \to Y'$ is a finite order thickening. Let $E' \in D(\mathcal{O}_{X'})$. If $E = Li^*(E')$ is Y-perfect, then E' is Y'-perfect.

Proof. Recall that being Y-perfect for E means E is pseudo-coherent and locally has finite tor dimension as a complex of $f^{-1}\mathcal{O}_Y$-modules (Derived Categories of Schemes, Definition 36.35.1). By Lemma 37.71.1 we find that E' is pseudo-coherent. In particular, E' is in $D_{QCoh}(\mathcal{O}_{X'})$, see Derived Categories of Schemes, Lemma 36.10.1. To prove that E' locally has finite tor dimension we may work locally on X'. Hence we may assume X', S', X, S are affine, say given by rings A', R', A, R. Then we reduce to the commutative algebra version by Derived Categories of

Schemes, Lemma 36.35.3. The commutative algebra version in More on Algebra, Lemma 15.83.8. □

0DJZ Lemma 37.71.3. Let (R, I) be a pair consisting of a ring and an ideal I contained in the Jacobson radical. Set $S = \text{Spec}(R)$ and $S_0 = \text{Spec}(R/I)$. Let $f : X \to S$ be proper, flat, and of finite presentation. Denote $X_0 = S_0 \times_S X$. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. If the derived restriction E_0 of E to X_0 is S_0-perfect, then E is S-perfect.

Proof. Choose a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. For each i we can choose a closed immersion $U_i \to \mathbf{A}^{d_i}_S$. Set $U_{i,0} = S_0 \times_S U_i$. For each i the complex $E_0|_{U_{i,0}}$ has tor amplitude in $[a_i, b_i]$ for some $a_i, b_i \in \mathbf{Z}$. Let $x \in X$ be a point. We will show that the tor amplitude of E_x over R is in $[a_i - d_i, b_i]$ for some i. This will finish the proof as the tor amplitude can be read off from the stalks by Cohomology, Lemma 20.48.5.

Since f is proper $f(\overline{\{x\}})$ is a closed subset of S. Since I is contained in the Jacobson radical, we see that $f(\overline{\{x\}})$ meeting the closed subset $S_0 \subset S$. Hence there is a specialization $x \rightsquigarrow x_0$ with $x_0 \in X_0$. Pick an i with $x_0 \in U_i$, so $x_0 \in U_{i,0}$. We will fix i for the rest of the proof. Write $U_i = \text{Spec}(A)$. Then A is a flat, finitely presented R-algebra which is a quotient of a polynomial R-algebra in d_i-variables. The restriction $E|_{U_i}$ corresponds (by Derived Categories of Schemes, Lemma 36.3.5 and 36.10.2) to a pseudo-coherent object K of $D(A)$. Observe that E_0 corresponds to $K \otimes_A^{\mathbf{L}} A/IA$. Let $\mathfrak{q} \subset \mathfrak{q}_0 \subset A$ be the prime ideals corresponding to $x \rightsquigarrow x_0$. Then $E_x = K_\mathfrak{q}$ and $K_\mathfrak{q}$ is a localization of $K_{\mathfrak{q}_0}$. Hence it suffices to show that $K_{\mathfrak{q}_0}$ has tor amplitude in $[a_i - d_i, b_i]$ as a complex of R-modules. Let $I \subset \mathfrak{p}_0 \subset R$ be the prime ideal corresponding to $f(x_0)$. Then we have

$$\begin{aligned} K \otimes_R^{\mathbf{L}} \kappa(\mathfrak{p}_0) &= (K \otimes_R^{\mathbf{L}} R/I) \otimes_{R/I}^{\mathbf{L}} \kappa(\mathfrak{p}_0) \\ &= (K \otimes_A^{\mathbf{L}} A/IA) \otimes_{R/I}^{\mathbf{L}} \kappa(\mathfrak{p}_0) \end{aligned}$$

the second equality because $R \to A$ is flat. By our choice of a_i, b_i this complex has cohomology only in degrees in the interval $[a_i, b_i]$. Thus we may finally apply More on Algebra, Lemma 15.83.9 to $R \to A$, $\mathfrak{q}_0$, $\mathfrak{p}_0$ and K to conclude. □

37.72. Contracting rational curves

0E7E In this section we study proper morphisms $f : X \to Y$ whose fibres have dimension ≤ 1 having $R^1 f_* \mathcal{O}_X = 0$. To understand the title of this section, please take a look at Algebraic Curves, Sections 53.22, 53.23, and 53.24.

0E7F Lemma 37.72.1. Let $f : X \to Y$ be a proper morphism of schemes. Let $y \in Y$ be a point with $\dim(X_y) \leq 1$. If

(1) $R^1 f_* \mathcal{O}_X = 0$, or more generally
(2) there is a morphism $g : Y' \to Y$ such that y is in the image of g and such that $R' f'_* \mathcal{O}_{X'} = 0$ where $f' : X' \to Y'$ is the base change of f by g.

Then $H^1(X_y, \mathcal{O}_{X_y}) = 0$.

Proof. To prove the lemma we may replace Y by an open neighbourhood of y. Thus we may assume Y is affine and that all fibres of f have dimension ≤ 1, see Morphisms, Lemma 29.28.4. In this case $R^1 f_* \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type and its formation commutes with arbitrary base change, see Limits, Lemmas 32.19.3 and 32.19.2. The lemma follows immediately. □

0E7G **Lemma 37.72.2.** Let $f : X \to Y$ be a proper morphism of schemes. Let $y \in Y$ be a point with $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$. Then there is an open neighbourhood $V \subset Y$ of y such that $R^1 f_*\mathcal{O}_X|_V = 0$ and the same is true after base change by any $Y' \to V$.

Proof. To prove the lemma we may replace Y by an open neighbourhood of y. Thus we may assume Y is affine and that all fibres of f have dimension ≤ 1, see Morphisms, Lemma 29.28.4. In this case $R^1 f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type and its formation commutes with arbitrary base change, see Limits, Lemmas 32.19.3 and 32.19.2. Say $Y = \mathrm{Spec}(A)$, y corresponds to the prime $\mathfrak{p} \subset A$, and $R^1 f_*\mathcal{O}_X$ corresponds to the finite A-module M. Then $H^1(X_y, \mathcal{O}_{X_y}) = 0$ means that $\mathfrak{p}M_\mathfrak{p} = M_\mathfrak{p}$ by the statement on base change. By Nakayama's lemma we conclude $M_\mathfrak{p} = 0$. Since M is finite, we find an $f \in A$, $f \notin \mathfrak{p}$ such that $M_f = 0$. Thus taking V the principal open $D(f)$ we obtain the desired result. $\square$

0E7H **Lemma 37.72.3.** Let $f : X \to Y$ be a proper morphism of schemes such that $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for all $y \in Y$. Let $\mathcal{F}$ be quasi-coherent on X. Then

(1) $R^p f_*\mathcal{F} = 0$ for $p > 1$, and
(2) $R^1 f_*\mathcal{F} = 0$ if there is a surjection $f^*\mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ quasi-coherent on Y.

If Y is affine, then we also have

(3) $H^p(X, \mathcal{F}) = 0$ for $p \notin \{0, 1\}$, and
(4) $H^1(X, \mathcal{F}) = 0$ if $\mathcal{F}$ is globally generated.

Proof. The vanishing in (1) is Limits, Lemma 32.19.2. To prove (2) we may work locally on Y and assume Y is affine. Then $R^1 f_*\mathcal{F}$ is the quasi-coherent module on Y associated to the module $H^1(X, \mathcal{F})$. Here we use that Y is affine, quasi-coherence of higher direct images (Cohomology of Schemes, Lemma 30.4.5), and Cohomology of Schemes, Lemma 30.4.6. Since Y is affine, the quasi-coherent module $\mathcal{G}$ is globally generated, and hence so is $f^*\mathcal{G}$ and $\mathcal{F}$. In this way we see that (4) implies (2). Part (3) follows from (1) as well as the remarks on quasi-coherence of direct images just made. Thus all that remains is the prove (4). If $\mathcal{F}$ is globally generated, then there is a surjection $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. By part (1) and the long exact sequence of cohomology this induces a surjection on H^1. Since $H^1(X, \mathcal{O}_X) = 0$ because $R^1 f_*\mathcal{O}_X = 0$ by Lemma 37.72.2, and since $H^1(X, -)$ commutes with direct sums (Cohomology, Lemma 20.19.1) we conclude. $\square$

0E7I **Lemma 37.72.4.** Let $f : X \to Y$ be a proper morphism of schemes. Assume

(1) for all $y \in Y$ we have $\dim(X_y) \leq 1$ and $H^1(X_y, \mathcal{O}_{X_y}) = 0$, and
(2) $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

Then $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}$ is surjective for any base change $f' : X' \to Y'$ of f.

Proof. We may assume Y and Y' affine. Then we can choose a closed immersion $Y' \to Y''$ with $Y'' \to Y$ a flat morphism of affines. By flat base change (Cohomology of Schemes, Lemma 30.5.2) we see that the result holds for $X'' \to Y''$. Thus we may assume Y' is a closed subscheme of Y. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal cutting out Y'. Then there is a short exact sequence

$$0 \to \mathcal{I}\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

where we view $\mathcal{O}_{X'}$ as a quasi-coherent module on X. By Lemma 37.72.3 we have $H^1(X, \mathcal{I}\mathcal{O}_X) = 0$. It follows that

$$H^0(Y, \mathcal{O}_Y) \to H^0(Y, f_*\mathcal{O}_X) = H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{X'})$$

is surjective as desired. The first arrow is surjective as Y is affine and since we assumed $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective and the second by the long exact sequence of cohomology associated to the short exact sequence above and the vanishing just proved. ☐

0E7J Lemma 37.72.5. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of morphisms of schemes. Let $s \in S$ be a point. Assume

1. $X \to S$ is locally of finite presentation and flat at points of X_s,
2. f is proper,
3. the fibres of $f_s : X_s \to Y_s$ have dimension ≤ 1 and $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$,
4. $\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is surjective.

Then there is an open $Y_s \subset V \subset Y$ such that (a) $f^{-1}(V)$ is flat over S, (b) $\dim(X_y) \leq 1$ for $y \in V$, (c) $R^1 f_*\mathcal{O}_X|_V = 0$, (d) $\mathcal{O}_V \to f_*\mathcal{O}_X|_V$ is surjective, and (b), (c), and (d) remain true after base change by any $Y' \to V$.

Proof. Let $y \in Y$ be a point over s. It suffices to find an open neighbourhood of y with the desired properties. As a first step, we replace Y by the open V found in Lemma 37.72.2 so that $R^1 f_*\mathcal{O}_X$ is zero universally (the hypothesis of the lemma holds by Lemma 37.72.1). We also shrink Y so that all fibres of f have dimension ≤ 1 (use Morphisms, Lemma 29.28.4 and properness of f). Thus we may assume we have (b) and (c) with $V = Y$ and after any base change $Y' \to Y$. Thus by Lemma 37.72.4 it now suffices to show (d) over Y. We may still shrink Y further; for example, we may and do assume Y and S are affine.

By Theorem 37.15.1 there is an open subset $U \subset X$ where $X \to S$ is flat which contains X_s by hypothesis. Then $f(X \setminus U)$ is a closed subset not containing y. Thus after shrinking Y we may assume X is flat over S.

Say $S = \mathrm{Spec}(R)$. Choose a closed immersion $Y \to Y'$ where Y' is the spectrum of a polynomial ring $R[x_e; e \in E]$ on a set E. Denote $f' : X \to Y'$ the composition of f with $Y \to Y'$. Then the hypotheses (1) – (4) as well as (b) and (c) hold for f' and s. If we we show $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_X$ is surjective in an open neighbourhood of y, then the same is true for $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Thus we may assume Y is the spectrum of $R[x_e; e \in E]$.

At this point X and Y are flat over S. Then Y_s and X are tor independent over Y. We urge the reader to find their own proof, but it also follows from Lemma 37.69.1 applied to the square with corners X, Y, S, S and its base change by $s \to S$. Hence

$$Rf_{s,*}\mathcal{O}_{X_s} = L(Y_s \to Y)^* Rf_*\mathcal{O}_X$$

by Derived Categories of Schemes, Lemma 36.22.5. Because of the vanishing already established this implies $f_{s,*}\mathcal{O}_{X_s} = (Y_s \to Y)^* f_*\mathcal{O}_X$. We conclude that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of quasi-coherent $\mathcal{O}_Y$-modules whose pullback to Y_s is surjective. We claim

$f_*\mathcal{O}_X$ is a finite type $\mathcal{O}_Y$-module. If true, then the cokernel $\mathcal{F}$ of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a finite type quasi-coherent $\mathcal{O}_Y$-module such that $\mathcal{F}_y \otimes \kappa(y) = 0$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we have $\mathcal{F}_y = 0$. Thus $\mathcal{F}$ is zero in an open neighbourhood of y (Modules, Lemma 17.9.5) and the proof is complete.

Proof of the claim. For a finite subset $E' \subset E$ set $Y' = \mathrm{Spec}(R[x_e; e \in E'])$. For large enough E' the morphism $f' : X \to Y \to Y'$ is proper, see Limits, Lemma 32.13.4. We fix E' and Y' in the following. Write $R = \mathrm{colim}\, R_i$ as the colimit of its finite type $\mathbf{Z}$-subalgebras. Set $S_i = \mathrm{Spec}(R_i)$ and $Y'_i = \mathrm{Spec}(R_i[x_e; e \in E'])$. For i large enough we can find a diagram

$$\begin{tikzcd} X \arrow[r, "f'"'] \arrow[d] & Y' \arrow[r] \arrow[d] & S \arrow[d] \\ X_i \arrow[r, "f'_i"] & Y'_i \arrow[r] & S_i \end{tikzcd}$$

with cartesian squares such that X_i is flat over S_i and $X_i \to Y'_i$ is proper. See Limits, Lemmas 32.10.1, 32.8.7, and 32.13.1. The same argument as above shows Y' and X_i are tor independent over Y'_i and hence

$$R\Gamma(X, \mathcal{O}_X) = R\Gamma(X_i, \mathcal{O}_{X_i}) \otimes^{\mathbf{L}}_{R_i[x_e; e \in E']} R[x_e; e \in E']$$

by the same reference as above. By Cohomology of Schemes, Lemma 30.19.2 the complex $R\Gamma(X_i, \mathcal{O}_{X_i})$ is pseudo-coherent in the derived category of the Noetherian ring $R_i[x_e; e \in E']$ (see More on Algebra, Lemma 15.64.17). Hence $R\Gamma(X, \mathcal{O}_X)$ is pseudo-coherent in the derived category of $R[x_e; e \in E']$, see More on Algebra, Lemma 15.64.12. Since the only nonvanishing cohomology module is $H^0(X, \mathcal{O}_X)$ we conclude it is a finite $R[x_e; e \in E']$-module, see More on Algebra, Lemma 15.64.4. This concludes the proof. □

0E7K Lemma 37.72.6. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of morphisms of schemes. Assume $X \to S$ is flat, f is proper, $\dim(X_y) \leq 1$ for $y \in Y$, and $R^1 f_*\mathcal{O}_X = 0$. Then $f_*\mathcal{O}_X$ is S-flat and formation of $f_*\mathcal{O}_X$ commutes with arbitrary base change $S' \to S$.

Proof. We may assume Y and S are affine, say $S = \mathrm{Spec}(A)$. To show the quasi-coherent $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$ is flat relative to S it suffices to show that $H^0(X, \mathcal{O}_X)$ is flat over A (some details omitted). By Lemma 37.72.3 we have $H^1(X, \mathcal{O}_X \otimes_A M) = 0$ for every A-module M. Since also $\mathcal{O}_X$ is flat over A we deduce the functor $M \mapsto H^0(X, \mathcal{O}_X \otimes_A M)$ is exact. Moreover, this functor commutes with direct sums by Cohomology, Lemma 20.19.1. Then it is an exercise to see that $H^0(X, \mathcal{O}_X \otimes_A M) = M \otimes_A H^0(X, \mathcal{O}_X)$ functorially in M and this gives the desired flatness. Finally, if $S' \to S$ is a morphism of affines given by the ring map $A \to A'$, then in the affine case just discussed we see that

$$H^0(X \times_S S', \mathcal{O}_{X \times_S S'}) = H^0(X, \mathcal{O}_X \otimes_A A') = H^0(X, \mathcal{O}_X) \otimes_A A'$$

This shows that formation of $f_*\mathcal{O}_X$ commutes with any base change $S' \to S$. Some details omitted. □

0E7L **Lemma 37.72.7.** Consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[rd] & & Y \arrow[ld] \\ & S & \end{tikzcd}$$

of morphisms of schemes. Let $s \in S$ be a point. Assume

1. $X \to S$ is locally of finite presentation and flat at points of X_s,
2. $Y \to S$ is locally of finite presentation,
3. f is proper,
4. the fibres of $f_s : X_s \to Y_s$ have dimension ≤ 1 and $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$,
5. $\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is an isomorphism.

Then there is an open $Y_s \subset V \subset Y$ such that (a) V is flat over S, (b) $f^{-1}(V)$ is flat over S, (c) $\dim(X_y) \leq 1$ for $y \in V$, (d) $R^1 f_*\mathcal{O}_X|_V = 0$, (e) $\mathcal{O}_V \to f_*\mathcal{O}_X|_V$ is an isomorphism, and (a) – (e) remain true after base change of $f^{-1}(V) \to V$ by any $S' \to S$.

Proof. Let $y \in Y_s$. We may always replace Y by an open neighbourhood of y. Thus we may assume Y and S affine. We may also assume that X is flat over S, $\dim(X_y) \leq 1$ for $y \in Y$, $R^1 f_*\mathcal{O}_X = 0$ universally, and that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, see Lemma 37.72.5. (We won't use all of this.)

Assume S and Y affine. Write $S = \lim S_i$ as a cofiltered of affine Noetherian schemes S_i. By Limits, Lemma 32.10.1 there exists an element $0 \in I$ and a diagram

$$\begin{tikzcd} X_0 \arrow[rr, "f_0"] \arrow[rd] & & Y_0 \arrow[ld] \\ & S_0 & \end{tikzcd}$$

of finite type morphisms of schemes whose base change to S is the diagram of the lemma. After increasing 0 we may assume Y_0 is affine and $X_0 \to S_0$ proper, see Limits, Lemmas 32.13.1 and 32.4.13. Let $s_0 \in S_0$ be the image of s. As Y_s is affine, we see that $R^1 f_{s,*}\mathcal{O}_{X_s} = 0$ is equivalent to $H^1(X_s, \mathcal{O}_{X_s}) = 0$. Since X_s is the base change of X_{0,s_0} by the faithfully flat map $\kappa(s_0) \to \kappa(s)$ we see that $H^1(X_{0,s_0}, \mathcal{O}_{X_{0,s_0}}) = 0$ and hence $R^1 f_{0,*}\mathcal{O}_{X_{0,s_0}} = 0$. Similarly, as $\mathcal{O}_{Y_s} \to f_{s,*}\mathcal{O}_{X_s}$ is an isomorphism, so is $\mathcal{O}_{Y_{0,s_0}} \to f_{0,*}\mathcal{O}_{X_{0,s_0}}$. Since the dimensions of the fibres of $X_s \to Y_s$ are at most 1, the same is true for the morphism $X_{0,s_0} \to Y_{0,s_0}$. Finally, since $X \to S$ is flat, after increasing 0 we may assume X_0 is flat over S_0, see Limits, Lemma 32.8.7. Thus it suffices to prove the lemma for $X_0 \to Y_0 \to S_0$ and the point s_0.

Combining the reduction arguments above we reduce to the case where S and Y affine, S Noetherian, the fibres of f have dimension ≤ 1, and $R^1 f_*\mathcal{O}_X = 0$ universally. Let $y \in Y_s$ be a point. Claim:

$$\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$$

is an isomorphism. The claim implies the lemma. Namely, since $f_*\mathcal{O}_X$ is coherent (Cohomology of Schemes, Proposition 30.19.1) the claim means we can replace Y by an open neighbourhood of y and obtain an isomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Then we conclude that Y is flat over S by Lemma 37.72.6. Finally, the isomorphism

$\mathcal{O}_Y \to f_*\mathcal{O}_X$ remains an isomorphism after any base change $S' \to S$ by the final statement of Lemma 37.72.6.

Proof of the claim. We already know that $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ is surjective (Lemma 37.72.5) and that $(f_*\mathcal{O}_X)_y$ is $\mathcal{O}_{S,s}$-flat (Lemma 37.72.6) and that the induced map

$$\mathcal{O}_{Y_s,y} = \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y/\mathfrak{m}_s(f_*\mathcal{O}_X)_y \to (f_{s,*}\mathcal{O}_{X_s})_y$$

is injective by the assumption in the lemma. Then it follows from Algebra, Lemma 10.99.1 that $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ is injective as desired. □

0E24 Lemma 37.72.8. Let $f : X \to Y$ be a proper morphism of Noetherian schemes such that $f_*\mathcal{O}_X = \mathcal{O}_Y$, such that the fibres of f have dimension ≤ 1, and such that $H^1(X_y, \mathcal{O}_{X_y}) = 0$ for $y \in Y$. Then $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is a bijection onto the subgroup of $\mathcal{L} \in \mathrm{Pic}(X)$ with $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$.

Proof. By the projection formula (Cohomology, Lemma 20.54.2) we see that $f_*f^*\mathcal{N} \cong \mathcal{N}$ for $\mathcal{N} \in \mathrm{Pic}(Y)$. We claim that for $\mathcal{L} \in \mathrm{Pic}(X)$ with $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$ we have $\mathcal{N} = f_*\mathcal{L}$ is invertible and $\mathcal{L} \cong f^*\mathcal{N}$. This will finish the proof.

The $\mathcal{O}_Y$-module $\mathcal{N} = f_*\mathcal{L}$ is coherent by Cohomology of Schemes, Proposition 30.19.1. Thus to see that it is an invertible $\mathcal{O}_Y$-module, it suffices to check on stalks (Algebra, Lemma 10.78.2). Since the map from a Noetherian local ring to its completion is faithfully flat, it suffices to check the completion $(f_*\mathcal{L})^\wedge_y$ is free (see Algebra, Section 10.97 and Lemma 10.78.6). For this we will use the theorem of formal functions as formulated in Cohomology of Schemes, Lemma 30.20.7. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$ and hence $(f_*\mathcal{O}_X)^\wedge_y \cong \mathcal{O}^\wedge_{Y,y}$, it suffices to show that $\mathcal{L}|_{X_n} \cong \mathcal{O}_{X_n}$ for each n (compatibly for varying n. By Lemma 37.4.1 we have an exact sequence

$$H^1(X_y, \mathfrak{m}_y^n\mathcal{O}_X/\mathfrak{m}_y^{n+1}\mathcal{O}_X) \to \mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$$

with notation as in the theorem on formal functions. Observe that we have a surjection

$$\mathcal{O}_{X_y}^{\oplus r_n} \cong \mathfrak{m}_y^n/\mathfrak{m}_y^{n+1} \otimes_{\kappa(y)} \mathcal{O}_{X_y} \longrightarrow \mathfrak{m}_y^n\mathcal{O}_X/\mathfrak{m}_y^{n+1}\mathcal{O}_X$$

for some integers $r_n \geq 0$. Since $\dim(X_y) \leq 1$ this surjection induces a surjection on first cohomology groups (by the vanishing of cohomology in degrees ≥ 2 coming from Cohomology, Proposition 20.20.7). Hence the H^1 in the sequence is zero and the transition maps $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ are injective as desired.

We still have to show that $f^*\mathcal{N} \cong \mathcal{L}$. This is proved by the same method and we omit the details. □

37.73. Affine stratifications

0F2R This material is taken from [RV04]. Please read a little bit about stratifications in Topology, Section 5.28 before reading this section.

If X is a scheme, then a stratification of X usually means a stratification of the underlying topological space of X. The strata are locally closed subsets. We will view these strata as reduced locally closed subschemes of X using Schemes, Remark 26.12.6.

0F2S Definition 37.73.1. Let X be a scheme. An affine stratification is a locally finite stratification $X = \coprod_{i \in I} X_i$ whose strata X_i are affine and such that the inclusion morphisms $X_i \to X$ are affine.

The condition that a stratification $X = \coprod X_i$ is locally finite is, in the presence of the condition that the inclusion morphisms $X_i \to X$ are quasi-compact, equivalent to the condition that the strata are locally constructible subsets of X, see Properties, Lemma 28.2.7.

The condition that $X_i \to X$ is an affine morphism is independent on the scheme structure we put on the locally closed subset X_i, see Lemma 37.3.1. Moreover, if X is separated (or more generally has affine diagonal) and $X = \coprod X_i$ is a locally finite stratification with affine strata, then the morphisms $X_i \to X$ are affine. See Morphisms, Lemma 29.11.11. This allows us to disregard the condition of affineness of the inclusion morphisms $X_i \to X$ in most cases of interest.

We are often interested in the case where the partially ordered index set I of the stratification is finite. Recall that the length of a partially ordered set I is the supremum of the lengths p of chains $i_0 < i_1 < \ldots < i_p$ of elements of I.

0F2T Lemma 37.73.2. Let X be a scheme. Let $X = \coprod_{i \in I} X_i$ be a finite affine stratification. There exists an affine stratification with index set $\{0, \ldots, n\}$ where n is the length of I.

Proof. Recall that we have a partial ordering on I such that the closure of X_i is contained in $\bigcup_{j \leq i} X_j$ for all $i \in I$. Let $I' \subset I$ be the set of maximal indices of I. If $i \in I'$, then X_i is open in X because the union of the closures of the other strata is the complement of X_i. Let $U = \bigcup_{i \in I'} X_i$ viewed as an open subscheme of X so that $U_{red} = \coprod_{i \in I'} X_i$ as schemes. Then U is an affine scheme by Schemes, Lemma 26.6.8 and Lemma 37.2.3. The morphism $U \to X$ is affine as each $X_i \to X$, $i \in I'$ is affine by the same reasoning using Lemma 37.3.1. The complement $Z = X \setminus U$ endowed with the reduced induced scheme structure has the affine stratification $Z = \bigcup_{i \in I \setminus I'} X_i$. Here we use that a morphism of schemes $T \to Z$ is affine if and only if the composition $T \to X$ is affine; this follows from Morphisms, Lemmas 29.11.9, 29.11.7, and 29.11.11. Observe that the partially ordered set $I \setminus I'$ has length exactly one less than the length of I. Hence by induction we find that Z has an affine stratification $Z = Z_0 \amalg \ldots \amalg Z_{n-1}$ with index set $\{1, \ldots, n\}$. Setting $Z_n = U$ we obtain the desired stratification of X. $\square$

If a scheme X has a finite affine stratification, then of course X is quasi-compact. A bit less obvious is the fact that it forces X to be quasi-separated as well.

0F2U Lemma 37.73.3. Let X be a scheme. The following are equivalent

(1) X has a finite affine stratification, and
(2) X is quasi-compact and quasi-separated.

Proof. Let $X = \bigcup X_i$ be a finite affine stratification. Since each X_i is affine hence quasi-compact, we conclude that X is quasi-compact. Let $U, V \subset X$ be affine open. Then $U \cap X_i$ and $V \cap X_i$ are affine open in X_i since $X_i \to X$ is an affine morphism. Hence $U \cap V \cap X_i$ is an affine open of the affine scheme X_i (see Schemes, Lemma 26.21.7 for example). Therefore $U \cap V = \coprod U \cap V \cap X_i$ is quasi-compact as a finite union of affine strata. We conclude that X is quasi-separated by Schemes, Lemma 26.21.6.

Assume X is quasi-compact and quasi-separated. We may use the induction principle of Cohomology of Schemes, Lemma 30.4.1 to prove the assertion that X has a finite affine stratification. If X is empty, then it has an empty affine stratification.

If X is nonempty affine then it has an affine stratification with one stratum. Next, asssume $X = U \cup V$ where U is quasi-compact open, V is affine open, and we have a finite affine stratifications $U = \bigcup_{i \in I} U_i$ and $U \cap V = \coprod_{j \in J} W_j$. Denote $Z = X \setminus V$ and $Z' = X \setminus U$. Note that Z is closed in U and Z' is closed in V. Observe that $U_i \cap Z$ and $U_i \cap W_j = U_i \times_U W_j$ are affine schemes affine over U. (Hints: use that $U_i \times_U W_j \to W_j$ is affine as a base change of $U_i \to U$, hence $U_i \cap W_j$ is affine, hence $U_i \cap W_j \to U_i$ is affine, hence $U_i \cap W_j \to U$ is affine.) It follows that

$$U = \coprod\nolimits_{i \in I} (U_i \cap Z) \amalg \coprod\nolimits_{(i,j) \in I \times J} (U_i \cap W_j)$$

is a finite affine stratification with partial ordering on $I \amalg I \times J$ given by $i' \leq (i,j) \Leftrightarrow i' \leq i$ and $(i',j') \leq (i,j) \Leftrightarrow i' \leq i$ and $j' \leq j$. Observe that $(U_i \cap Z) \times_X V = \emptyset$ and $(U_i \cap W_j) \times_X V = U_i \cap W_j$ are affine. Hence the morphisms $U_i \cap Z \to X$ and $U_i \cap W_j \to X$ are affine because we can check affineness of a morphism locally on the target (Morphisms, Lemma 29.11.3) and we have affineness over both U and V. To finish the proof we take the stratification above and we add one additional stratum, namely Z', whose index we add as a minimal element to the partially ordered set. $\square$

0F2V Definition 37.73.4. Let X be a nonempty quasi-compact and quasi-separated scheme. The affine stratification number is the smallest integer $n \geq 0$ such that the following equivalent conditions are satisfied

(1) there exists a finite affine stratification $X = \coprod_{i \in I} X_i$ where I has length n,
(2) there exists an affine stratification $X = X_0 \amalg X_1 \amalg \ldots \amalg X_n$ with index set $\{0, \ldots, n\}$.

The equivalence of the conditions holds by Lemma 37.73.2. The existence of a finite affine stratification is proven in Lemma 37.73.3.

0F2W Lemma 37.73.5. Let X be a separated scheme which has an open covering by $n+1$ affines. Then the affine stratification number of X is at most n.

Proof. Say $X = U_0 \cup \ldots \cup U_n$ is an affine open covering. Set

$$X_i = (U_i \cup \ldots \cup U_n) \setminus (U_{i+1} \cup \ldots \cup U_n)$$

Then X_i is affine as a closed subscheme of U_i. The morphism $X_i \to X$ is affine by Morphisms, Lemma 29.11.11. Finally, we have $\overline{X_i} \subset X_i \cup X_{i-1} \cup \ldots X_0$. $\square$

0F2X Lemma 37.73.6. Let X be a Noetherian scheme of dimension $\infty > d \geq 0$. Then the affine stratification number of X is at most d.

Proof. By induction on d. If $d = 0$, then X is affine, see Properties, Lemma 28.10.5. Assume $d > 0$. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of X (Properties, Lemma 28.5.7). We can cover X by affine opens containing $\eta_1, \ldots, \eta_n$, see Properties, Lemma 28.29.4. Since X is quasi-compact we can find a finite affine open covering $X = \bigcup_{j=1,\ldots,m} U_j$ with $\eta_1, \ldots, \eta_n \in U_j$ for all $j = 1, \ldots, m$. Choose an affine open $U \subset U_1 \cap \ldots \cap U_m$ containing $\eta_1, \ldots, \eta_n$ (possible by the lemma already quoted). Then the morphism $U \to X$ is affine because $U \to U_j$ is affine for all j, see Morphisms, Lemma 29.11.3. Let $Z = X \setminus U$. By construction $\dim(Z) < \dim(X)$. By induction hypothesis we can find an affine stratification $Z = \bigcup_{i \in \{0,\ldots,n\}} Z_i$ of Z with $n \leq \dim(Z)$. Setting $U = X_{n+1}$ and $X_i = Z_i$ for $i \leq n$ we conclude. $\square$

0F2Y Proposition 37.73.7. Let X be a nonempty quasi-compact and quasi-separated scheme with affine stratification number n. Then $H^p(X, \mathcal{F}) = 0$, $p > n$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.

Proof. We will prove this by induction on the affine stratification number n. If $n = 0$, then X is affine and the result is Cohomology of Schemes, Lemma 30.2.2. Assume $n > 0$. By Definition 37.73.4 there is an affine scheme U and an affine open immersion $j : U \to X$ such that the complement Z has affine stratification number $n - 1$. As U and j are affine we have $H^p(X, j_*(\mathcal{F}|_U)) = 0$ for $p > 0$, see Cohomology of Schemes, Lemmas 30.2.4 and 30.2.3. Denote $\mathcal{K}$ and $\mathcal{Q}$ the kernel and cokernel of the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$. Thus we obtain an exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to \mathcal{Q} \to 0$$

of quasi-coherent $\mathcal{O}_X$-modules (see Schemes, Section 26.24). A standard argument, breaking our exact sequence into short exact sequences and using the long exact cohomology sequence, shows it suffices to prove $H^p(X, \mathcal{K}) = 0$ and $H^p(X, \mathcal{Q}) = 0$ for $p \geq n$. Since $\mathcal{F} \to j_*(\mathcal{F}|_U)$ restricts to an isomorphism over U, we see that $\mathcal{K}$ and $\mathcal{Q}$ are supported on Z. By Properties, Lemma 28.22.3 we can write these modules as the filtered colimits of their finite type quasi-coherent submodules. Using the fact that cohomology of sheaves on X commutes with filtered colimits, see Cohomology, Lemma 20.19.1, we conclude it suffices to show that if $\mathcal{G}$ is a finite type quasi-coherent module whose support is contained in Z, then $H^p(X, \mathcal{G}) = 0$ for $p \geq n$. Let $Z' \subset X$ be the scheme theoretic support of $\mathcal{G} \oplus \mathcal{O}_Z$; we may and do think of $\mathcal{G}$ as a quasi-coherent module on Z', see Morphisms, Section 29.5. Then Z' and Z have the same underlying topological space and hence the same affine stratification number, namely $n - 1$. Hence $H^p(X, \mathcal{G}) = H^p(Z', \mathcal{G})$ (equality by Cohomology of Schemes, Lemma 30.2.4) vanishes for $p \geq n$ by induction hypothesis. $\square$

0F2Z Example 37.73.8. Let k be a field and let $X = \mathbf{P}^n_k$ be n-dimensional projective space over k. Lemma 37.73.5 applies to this by Constructions, Lemma 27.13.3. Hence the affine stratification number of $\mathbf{P}^n_k$ is at most n. On the other hand, we have nonzero cohomology in degree n for some quasi-coherent modules on $\mathbf{P}^n_k$, see Cohomology of Schemes, Lemma 30.8.1. Using Proposition 37.73.7 we conclude that the affine stratification number of $\mathbf{P}^n_k$ is equal to n.

37.74. Universally open morphisms

0F30 Some material on universally open morphisms.

0F31 Lemma 37.74.1. Let $f : X \to S$ be a morphism of schemes. The following are equivalent

(1) f is universally open,
(2) for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is open, and
(3) for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is open.

Proof. It is clear that (1) implies (2) and (2) implies (3). Let us prove that (3) implies (1). Suppose that the base change $X_T \to T$ is not open for some morphism of schemes $g : T \to S$. Then we can find some affine opens $V \subset S$, $U \subset X$, $W \subset T$ with $f(U) \subset V$ and $g(W) \subset V$ such that $U \times_V W \to W$ is not open. If we can show that this implies $\mathbf{A}^n \times U \to \mathbf{A}^n \times V$ is not open, then $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$

is not open and the proof is complete. This reduces us to the result proved in the next paragraph.

Let $A \to B$ be a ring map such that $A' \to B' = A' \otimes_A B$ does not induce an open map of spectra for some A-algebra A'. As the principal opens give a basis for the topology of $\text{Spec}(B')$ we conclude that the image of $D(g)$ in $\text{Spec}(A')$ is not open for some $g \in B'$. Write $g = \sum_{i=1,\ldots,n} a'_i \otimes b_i$ for some n, $a'_i \in A'$, and $b_i \in B$. Consider the element $h = \sum_{i=1,\ldots,n} x_i b_i$ in $B[x_1, \ldots, x_n]$. Assume that $D(h)$ maps to an open subset under the morphism

$$\text{Spec}(B[x_1, \ldots, x_n]) \longrightarrow \text{Spec}(A[x_1, \ldots, x_n])$$

in order to get a contradiction. Then $D(h)$ would map surjectively onto a quasi-compact open $U \subset \text{Spec}(A[x_1, \ldots, x_n])$. Let $A[x_1, \ldots, x_n] \to A'$ be the A-algebra homomorphism sending x_i to a'_i. This also induces a B-algebra homomorphism $B[x_1, \ldots, x_n] \to B'$ sending h to g. Since

$$\begin{tikzcd} \text{Spec}(B[x_1, \ldots, x_n]) \arrow[d] & \text{Spec}(B') \arrow[l] \arrow[d] \\ \text{Spec}(A[x_1, \ldots, x_n]) & \text{Spec}(A') \arrow[l] \end{tikzcd}$$

is cartesian the image of $D(g)$ in $\text{Spec}(A')$ is equal to the inverse image of U in $\text{Spec}(A')$ and hence open which is the desired contradiction. $\square$

0F32 **Lemma 37.74.2.** Let $f : X \to Y$ be a morphism of schemes. If

(1) f is locally quasi-finite,
(2) Y is geometrically unibranch and locally Noetherian, and
(3) every irreducible component of X dominates an irreducible component of Y,

then f is universally open.

Proof. For any n the scheme $\mathbf{A}^n \times Y$ is geometrically unibranch by Lemma 37.36.4 and Properties, Lemma 28.15.6. Hence the hypotheses of the lemma hold for the morphisms $\mathbf{A}^n \times X \to \mathbf{A}^n \times Y$ for all n. By Lemma 37.74.1 it suffices to prove f is open. By Morphisms, Lemma 29.23.2 it suffices to show that generalizations lift along f. Suppose that $y' \rightsquigarrow y$ is a specialization of points in Y and $x \in X$ is a point mapping to y. As in Lemma 37.41.1 choose a diagram

$$\begin{tikzcd} u \arrow[d] & U \arrow[r] \arrow[d] & X \arrow[d] \\ v & V \arrow[r] & Y \end{tikzcd}$$

where $(V, v) \to (Y, y)$ is an elementary étale neighbourhood, $U \to V$ is finite, u is the unique point of U mapping to v, $U \subset V \times_Y X$ is open, and $v \mapsto y$ and $u \mapsto x$. Let E be an irreducible component of U passing through u (there is at least one of these). Since $U \to X$ is étale, E maps to an irreducible component of X, which in turn dominates an irreducible component of Y (by assumption). Since $U \to V$ is finite hence closed, we conclude that the image $E' \subset V$ of E is an irreducible closed subset passing through v which dominates an irreducible component of Y. Since $V \to Y$ is étale E' must be an irreducible component of V passing through v. Since Y is geometrically unibranch we see that E' is the unique irreducible component of

V passing through v (Lemma 37.36.2). Since V is locally Noetherian we may after shrinking V assume that $E' = V$ (equality of sets).

Since $V \to Y$ is étale we can find a specialization $v' \rightsquigarrow v$ whose image is $y' \rightsquigarrow y$. By the above we can find $u' \in U$ mapping to v'. Then $u' \rightsquigarrow u$ because u is the only point of U mapping to v and $U \to V$ is closed. Then finally the image $x' \in X$ of u' is a point specializing to x and mapping to y' and the proof is complete. □

0F33 Lemma 37.74.3. Let $A \to B$ be a ring map. Say B is generated as an A-module by $b_1, \ldots, b_d \in B$. Set $h = \sum x_i b_i \in B[x_1, \ldots, x_d]$. Then $\text{Spec}(B) \to \text{Spec}(A)$ is universally open if and only if the image of $D(h)$ in $\text{Spec}(A[x_1, \ldots, x_d])$ is open.

Proof. If $\text{Spec}(B) \to \text{Spec}(A)$ is universally open, then of course the image of $D(h)$ is open. Conversely, assume the image U of $D(h)$ is open. Let $A \to A'$ be a ring map. It suffices to show that the image of any principal open $D(g) \subset \text{Spec}(A' \otimes_A B)$ in $\text{Spec}(A')$ is open. We may write $g = \sum_{i=1,\ldots,d} a'_i \otimes b_i$ for some $a'_i \in A'$. Let $A[x_1, \ldots, x_n] \to A'$ be the A-algebra homomorphism sending x_i to a'_i. This also induces a B-algebra homomorphism $B[x_1, \ldots, x_n] \to A' \otimes_A B$ sending h to g. Since

$$\begin{tikzcd}
\text{Spec}(B[x_1, \ldots, x_n]) \arrow[d] & \text{Spec}(B') \arrow[l] \arrow[d] \\
\text{Spec}(A[x_1, \ldots, x_n]) & \text{Spec}(A') \arrow[l]
\end{tikzcd}$$

is cartesian the image of $D(g)$ in $\text{Spec}(A')$ is equal to the inverse image of U in $\text{Spec}(A')$ and hence open. □

0F34 Lemma 37.74.4. Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S. If

(1) f is locally quasi-finite and universally open, and
(2) f_0 is locally of finite presentation,

then there exists an $i \geq 0$ such that f_i is locally quasi-finite and universally open.

Proof. By Limits, Lemma 32.18.2 after increasing 0 we may assume f_0 is locally quasi-finite. Let $x \in X$. By étale localization of quasi-finite morphisms we can find a diagram

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l] \arrow[d] \\
Y & V \arrow[l]
\end{tikzcd}$$

where $V \to Y$ is étale, $U \subset X_V$ is open, $U \to V$ is finite, and x is in the image of $U \to X$, see Lemma 37.41.1. After shrinking V we may assume V and U are affine. Since X is quasi-compact, it follows, by taking a finite disjoint union of such V and U, that we can make a diagram as above such that $U \to X$ is surjective. By Limits, Lemmas 32.10.1, 32.4.11, 32.8.15, 32.8.3, 32.8.10, and 32.4.13 after possibly increasing 0 we may assume we have a diagram

$$\begin{tikzcd} X_0 \arrow[d] & U_0 \arrow[l] \arrow[d] \\ Y_0 & V_0 \arrow[l] \end{tikzcd}$$

where V_0 is affine, $V_0 \to Y_0$ is étale, $U_0 \subset (X_0)_{V_0}$ is open, $U_0 \to V_0$ is finite, and $U_0 \to X_0$ is surjective. Since $V_i \to Y_i$ is étale and hence universally open, follows that it suffices to prove that $U_i \to V_i$ is universally open for large enough i. This reduces us to the case discussed in the next paragraph.

Let $A = \text{colim}\, A_i$ be a filtered colimit of rings. Let $A_0 \to B_0$ be a ring map. Set $B = A \otimes_{A_0} B_0$ and $B_i = A_i \otimes_{A_0} B_0$. Assume $A_0 \to B_0$ is finite, of finite presentation, and $A \to B$ is universally open. We have to show that $A_i \to B_i$ is universally open for i large enough. Pick $b_{0,1}, \ldots, b_{0,d} \in B_0$ which generate B_0 as an A_0-module. Set $h_0 = \sum_{j=1,\ldots,d} x_j b_{0,j}$ in $B_0[x_1, \ldots, x_d]$. Denote h, resp. h_i the image of h_0 in $B[x_1, \ldots, x_d]$, resp. $B_i[x_1, \ldots, x_d]$. The image U of $D(h)$ in $\text{Spec}(A[x_1, \ldots, x_d])$ is open as $A \to B$ is universally open. Of course U is quasi-compact as the image of an affine scheme. For i large enough there is a quasi-compact open $U_i \subset \text{Spec}(A_i[x_1, \ldots, x_d])$ whose inverse image in $\text{Spec}(A[x_1, \ldots, x_d])$ is U, see Limits, Lemma 32.4.11. After increasing i we may assume that $D(h_i)$ maps into U_i; this follows from the same lemma by considering the pullback of U_i in $D(h_i)$. Finally, for i even larger the morphism of schemes $D(h_i) \to U_i$ will be surjective by an application of the already used Limits, Lemma 32.8.15. We conclude $A_i \to B_i$ is universally open by Lemma 37.74.3. $\square$

0F35 Lemma 37.74.5. Let $f : X \to Y$ be a locally quasi-finite morphism. Then

(1) the functions $n_{X/Y}$ of Lemmas 37.27.3 and 37.28.3 agree,
(2) if X is quasi-compact, then $n_{X/Y}$ attains a maximum $d < \infty$.

Proof. Agreement of the functions is immediate from the fact that the (geometric) fibres of a locally quasi-finite morphism are discrete, see Morphisms, Lemma 29.20.8. Boundedness follows from Morphisms, Lemmas 29.57.2 and 29.57.9. $\square$

0F36 Lemma 37.74.6. Let $f : X \to Y$ be a separated, locally quasi-finite, and universally open morphism of schemes. Let $n_{X/Y}$ be as in Lemma 37.74.5. If $n_{X/Y}(y) \geq d$ for some $y \in Y$ and $d \geq 0$, then $n_{X/Y} \geq d$ in an open neighbourhood of y.

Proof. The question is local on Y hence we may assume Y affine. Let K be an algebraic closure of the residue field $\kappa(y)$. Our assumption is that $(X_y)_K$ has $\geq d$ connected components. Then for a suitable quasi-compact open $X' \subset X$ the scheme $(X'_y)_K$ has $\geq d$ connected components; details omitted. After replacing X by X' we may assume X is quasi-compact. Then f is quasi-finite. Let $x_1, \ldots, x_n$ be the points of X lying over y. Apply Lemma 37.41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $n_{X/Y}(y) = \sum_i m_i$ in this situation; some details omitted. Since f is universally open, we see that $V_{i,j} \to U$ is open for all i, j. Hence after shrinking U we may assume $V_{i,j} \to U$ is surjective for all i, j. This proves that

$n_{U \times_Y X/U} \geq \sum_i m_i = n_{X/Y}(y) \geq d$. Since the construction of $n_{X/Y}$ is compatible with base change the proof is complete. $\square$

0F37 Lemma 37.74.7. Let $f : X \to Y$ be a separated, locally quasi-finite, and universally open morphism of schemes. Let $n_{X/Y}$ be as in Lemma 37.74.5. If $n_{X/Y}$ attains a maximum $d < \infty$, then the set

$$Y_d = \{y \in Y \mid n_{X/Y}(y) = d\}$$

is open in Y and the morphism $f^{-1}(Y_d) \to Y_d$ is finite.

Proof. The openness of Y_d is immediate from Lemma 37.74.6. To prove finiteness over Y_d we redo the argument of the proof of that lemma. Namely, let $y \in Y_d$. Then there are at most d points of X lying over y. Say $x_1, \ldots, x_n$ are the points of X lying over y. Apply Lemma 37.41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $d = n_{X/Y}(y) = \sum_i m_i$ in this situation; some details omitted. Since f is universally open, we see that $V_{i,j} \to U$ is open for all i, j. Hence after shrinking U we may assume $V_{i,j} \to U$ is surjective for all i, j and we may assume U maps into W. This proves that $n_{U \times_Y X/U} \geq \sum_i m_i = d$. Since the construction of $n_{X/Y}$ is compatible with base change we know that $n_{U \times_Y X/U} = d$. This means that W has to be empty and we conclude that $U \times_Y X \to U$ is finite. By Descent, Lemma 35.23.23 this implies that $X \to Y$ is finite over the image of the open morphism $U \to Y$. In other words, we see that f is finite over an open neighbourhood of y as desired. $\square$

37.75. Weightings

0F38 The material in this section is taken from [AGV71, Exposee XVII, 6.2.4].

Let $\pi : U \to V$ be a locally quasi-finite morphism of schemes with finite fibres. Given a function $w : U \to \mathbf{Z}$ we define a function

$$\textstyle\int_\pi w : V \longrightarrow \mathbf{Z}, \quad v \longmapsto \sum_{u \in U,\ \pi(u) = v} w(u)[\kappa(u) : \kappa(v)]_s$$

Note that the field extensions are finite (Morphisms, Lemma 29.20.5), $[\kappa' : \kappa]_s$ is the separable degree (Fields, Definition 9.14.7), and the sum is finite as the fibres of π are assumed finite. Another way to compute the value of $\int_\pi w$ at a point $v \in V$ is as follows. Choose an algebraically closed field k and a morphism $\overline{v} : \mathrm{Spec}(k) \to V$ whose image is v. Then we have

$$(\textstyle\int_\pi w)(v) = \sum_{\overline{u} \in U_{\overline{v}}} w(\overline{u})$$

where of course $w(\overline{u})$ denotes the value of w at the image u of the point $\overline{u}$ under the morphism $U_{\overline{v}} \to U$. Note that we may view $\overline{u} \in U_{\overline{v}}$ as morphisms $\overline{u} : \mathrm{Spec}(k) \to U$ such that $\pi \circ \overline{u} = \overline{v}$. Namely, since $U \to V$ is locally quasi-finite with finite fibres, the scheme $U_{\overline{v}}$ is the spectrum of a finite dimension algebra over k and all of whose prime ideals are maximal ideals with residue field k. To see that the equality holds, note that the number of morphisms $\overline{u}$ lying over a given u is equal to $[\kappa(u) : \kappa(v)]_s$ by Fields, Lemma 9.14.8.

0F39 **Lemma 37.75.1.** Given a cartesian square

$$\begin{tikzcd} U \arrow[d, "\pi"'] & U' \arrow[l, "h"] \arrow[d, "\pi'"] \\ V & V' \arrow[l, "g"] \end{tikzcd}$$

with π locally quasi-finite with finite fibres and a function $w : U \to \mathbf{Z}$ we have $(\int_\pi w) \circ g = \int_{\pi'}(w \circ h)$.

Proof. This follows immediately from the second description of $\int_\pi w$ given above. To prove it from the definition, you use that if E/F is a finite extension of fields and F'/F is another field extension, then writing $(E \otimes_F F')_{red} = \prod E'_i$ as a product of fields finite over F', we have

$$[E : F]_s = \sum [E'_i : F']_s$$

To prove this equality pick an algebraically closed field extension Ω/F' and observe that

$$\begin{aligned} [E : F]_s &= |\operatorname{Mor}_F(E, \Omega)| \\ &= |\operatorname{Mor}_{F'}(E \otimes_F F', \Omega)| \\ &= |\operatorname{Mor}_{F'}((E \otimes_F F')_{red}, \Omega)| \\ &= \sum |\operatorname{Mor}_{F'}(E'_i, \Omega)| \\ &= \sum [E'_i : F']_s \end{aligned}$$

where we have used Fields, Lemma 9.14.8. □

0F3A **Definition 37.75.2.** Let $f : X \to Y$ be a locally quasi-finite morphism. A *weighting* or a *pondération* of f is a map $w : X \to \mathbf{Z}$ such that for any diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & U \arrow[l, "h"] \arrow[d, "\pi"] \\ Y & V \arrow[l, "g"] \end{tikzcd}$$

where $V \to Y$ is étale, $U \subset X_V$ is open, and $U \to V$ finite, the function $\int_\pi(w \circ h)$ is locally constant.

Of course taking $w = 0$ we obtain a weighting of any locally quasi-finite morphism f, albeit not a very interesting one. It will turn out that positive weightings, i.e., $w : X \to \mathbf{Z}_{>0}$ are the most interesting ones for various purposes.

0F3B **Lemma 37.75.3.** Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting. Let $f' : X' \to Y'$ be the base change of f by a morphism $Y' \to Y$. Then the composition $w' : X' \to \mathbf{Z}$ of w and the projection $X' \to X$ is a weighting of f'.

Proof. Consider a diagram

$$\begin{tikzcd} X' \arrow[d, "f'"'] & U' \arrow[l, "h'"] \arrow[d, "\pi'"] \\ Y' & V' \arrow[l, "g'"] \end{tikzcd}$$

as in Definition 37.75.2 for the morphism f'. For any $v' \in V'$ we have to show that $\int_{\pi'}(w' \circ h')$ is constant in an open neighbourhood of v'. By Lemma 37.75.1 (and the fact that étale morphisms are open) we may replace V' by any étale neighbourhood of v'. After replacing V' by an étale neighbourhood of v' we may assume that $U' = U'_1 \amalg \ldots \amalg U'_n$ where each U'_i has a unique point u'_i lying over v' such that $\kappa(u'_i)/\kappa(v')$ is purely inseparable, see Lemma 37.41.5. Clearly, it suffices to prove that $\int_{U'_i \to V'} w'|_{U'_i}$ is constant in a neighbourhood of v'. This reduces us to the case discussed in the next paragraph.

We have $v' \in V'$ and there is a unique point u' of U' lying over v' with $\kappa(u')/\kappa(v')$ purely inseparable. Denote $x \in X$ and $y \in Y$ the image of u' and v'. We can find an étale neighbourhood $(V, v) \to (Y, y)$ and an open $U \subset X_V$ such that $\pi : U \to V$ is finite and such that there is a unique point $u \in U$ lying over v which maps to $x \in X$ via the projection $h : U \to X$ such that moreover $\kappa(u)/\kappa(v)$ is purely inseparable. This is possible by the lemma used above. Consider the morphism

$$U'' = U \times_X U' \longrightarrow V \times_Y V' = V''$$

Since u and u' both map to $x \in X$ there is a point $u'' \in U''$ mapping to (u, u'). Denote $v'' \in V''$ the image of u''. After replacing V', v' by V'', v'' we may assume that the composition $V' \to Y' \to Y$ factors through a map of étale neighbourhoods $(V', v') \to (V, v)$ such that the induced morphism $X'_{V'} = X_{V'} \to X_V$ sends u' to u. Inside the base change $X'_{V'} = X_{V'}$ we have two open subschemes, namely U' and the inverse image $U_{V'}$ of $U \subset X_V$. By construction both contain a unique point lying over v', namely u' for both of them. Thus after shrinking V' we may assume these open subsets are the same; namely, $U' \setminus (U' \cap U_{V'})$ and $U_{V'} \setminus (U' \cap U_{V'})$ have a closed image in V' and these images do not contain v'. Thus $U' = U_{V'}$ and we find a cartesian diagram as in Lemma 37.75.1. Since $\int_\pi (w \circ h)$ is locally constant by assumption we conclude. $\square$

0GK8 **Lemma 37.75.4.** Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting of f. If $X' \subset X$ is open, then $w|_{X'}$ is a weighting of $f|_{X'} : X' \to Y$.

Proof. Immediate from the definition. $\square$

0GK9 **Lemma 37.75.5.** Let $f : X \to Y$ and $g : Y \to Z$ be locally quasi-finite morphisms. Let $w_f : X \to \mathbf{Z}$ be a weighting of f and let $w_g : Y \to \mathbf{Z}$ be a weighting of g. Then the function

$$X \longrightarrow \mathbf{Z}, \quad x \longmapsto w_f(x) w_g(f(x))$$

is a weighting of $g \circ f$.

Proof. Let us set $w_{g \circ f}(x) = w_f(x) w_g(f(x))$ for $x \in X$. Consider a diagram

$$\begin{tikzcd} X \arrow[d, "g \circ f"'] & U \arrow[l] \arrow[d, "\pi"] \\ Z & W \arrow[l] \end{tikzcd}$$

where $W \to Z$ is étale, $U \subset X_W$ is open, and $U \to W$ finite. We have to show that $\int_\pi w_{g \circ f}|_U$ is locally constant. Choose a point $w \in W$. By Lemma 37.75.1 (and the fact that étale morphisms are open) it suffices to show that $\int_\pi w_{g \circ f}|_U$ is constant after replacing (W, w) by an étale neighbourhood. After replacing (W, w) by an étale neighbourhood we may assume $U = U_1 \amalg \ldots \amalg U_n$ where each U_i

has a unique point u_i lying over w such that $\kappa(u_i)/\kappa(w)$ is purely inseparable, see Lemma 37.41.5. Clearly, it suffices to show that $\int_{U_i \to W} w_{g\circ f}|_{U_i}$ is constant in an étale neighbourhood of w. This reduces us to the case discussed in the next paragraph.

We have $w \in W$ and there is a unique point $u \in U$ lying over w with $\kappa(u)/\kappa(w)$ purely inseparable. Consider the point $v = f(u) \in Y$. After replacing (W, w) by an elementary étale neighbourhood we may assume there is an open neighbourhood $V \subset Y_W$ of v such that $V \to W$ is finite, see Lemma 37.41.1. Then $f_W^{-1}(V) \cap U$ is an open neighbourhood of u where $f_W : X_W \to Y_W$ is the base change of f to W. Hence after Zariski shrinking W, we may assume $f_W(U) \subset V$. Thus we obtain morphisms

$$U \xrightarrow{a} V \xrightarrow{b} W$$

and $U \to V$ is finite as $V \to W$ is separated (because finite). Since w_f and w_g are weightings of f and g we see that $\int_a w_f|_U$ is locally constant on V and $\int_b w_g|_V$ is locally constant on W. Thus after shrinking W one more time we may assume these functions are constant say with values n and m. It follows immediately that $\int_\pi w_{g\circ f}|_U = \int_{b\circ a} w_{g\circ f}|_U$ is constant with value nm as desired. □

0F3C Lemma 37.75.6. Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting. If $w(x) > 0$ for all $x \in X$, then f is universally open.

Proof. Since the property is preserved by base change, see Lemma 37.75.3, it suffices to prove that f is open. Since we may also replace X by any open of X, it suffices to prove that $f(X)$ is open. Let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. It suffices to prove that $f(X)$ contains an open neighbourhood of y and it suffices to do so after replacing Y by an étale neighbourhood of y. By étale localization of quasi-finite morphisms, see Section 37.41, we may assume there is an open neighbourhood $U \subset X$ of x such that $\pi = f|_U : U \to Y$ is finite. Then $\int_\pi w|_U$ is locally constant and has positive value at y. Hence $\pi(U)$ contains an open neighbourhood of y and the proof is complete. □

0F3D Lemma 37.75.7. Let $f : X \to Y$ be a morphism of schemes. Assume f is locally quasi-finite, locally of finite presentation, and flat. Then there is a positive weighting $w : X \to \mathbf{Z}_{>0}$ of f given by the rule that sends $x \in X$ lying over $y \in Y$ to

$$w(x) = \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x) : \kappa(y)]_i$$

where $[\kappa' : \kappa]_i$ is the inseparable degree (Fields, Definition 9.14.7).

Proof. Consider a diagram as in Definition 37.75.2. Let $u \in U$ with images x, y, v in X, Y, V. Then we claim that

$$\text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}) = \text{length}_{\mathcal{O}_{U,u}}(\mathcal{O}_{U,u}/\mathfrak{m}_v\mathcal{O}_{U,u})$$

and

$$[\kappa(x) : \kappa(y)]_i = [\kappa(u) : \kappa(v)]_i$$

The first equality follows as $\mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ is a flat local homomorphism such that $\mathfrak{m}_y\mathcal{O}_{U,u} = \mathfrak{m}_v\mathcal{O}_{U,u}$ and $\mathfrak{m}_x\mathcal{O}_{U,u} = \mathfrak{m}_u$ (because $\mathcal{O}_{Y,y} \to \mathcal{O}_{V,v}$ and $\mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ are unramified) and hence the equality by Algebra, Lemma 10.52.13. The second equality follows because $\kappa(v)/\kappa(y)$ is a finite separable extension and $\kappa(u)$ is a factor of $\kappa(x) \otimes_{\kappa(y)} \kappa(v)$ and hence the inseparable degree is unchanged. Having said this, we see that formation of the function in the lemma commutes with étale base change. This reduces the problem to the discussion of the next paragraph.

Assume that f is a finite, flat morphism of finite presentation. We have to show that $\int_f w$ is locally constant on Y. In fact, f is finite locally free (Morphisms, Lemma 29.48.2) and we will show that $\int_f w$ is equal to the degree of f (which is a locally constant function on Y). Namely, for $y \in Y$ we see that

$$\begin{aligned}(\textstyle\int_f w)(y) &= \sum\nolimits_{f(x)=y} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x):\kappa(y)]_i[\kappa(x):\kappa(y)]_s \\ &= \sum\nolimits_{f(x)=y} \text{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})[\kappa(x):\kappa(y)] \\ &= \text{length}_{\mathcal{O}_{Y,y}}((f_*\mathcal{O}_X)_y/\mathfrak{m}_y(f_*\mathcal{O}_X)_y)\end{aligned}$$

Last equality by Algebra, Lemma 10.52.12. The final number is the rank of $f_*\mathcal{O}_X$ at y as desired. $\square$

0F3E Lemma 37.75.8. Let $f : X \to Y$ be a morphism of schemes. Assume

(1) f is locally quasi-finite, and
(2) Y is geometrically unibranch and locally Noetherian.

Then there is a weighting $w : X \to \mathbf{Z}_{\geq 0}$ given by the rule that sends $x \in X$ lying over $y \in Y$ to the "generic separable degree" of $\mathcal{O}^{sh}_{X,x}$ over $\mathcal{O}^{sh}_{Y,y}$.

Proof. It follows from Algebra, Lemma 10.156.3 that $\mathcal{O}^{sh}_{Y,y} \to \mathcal{O}^{sh}_{X,x}$ is finite. Since Y is geometrically unibranch there is a unique minimal prime $\mathfrak{p}$ in $\mathcal{O}^{sh}_{Y,y}$, see More on Algebra, Lemma 15.106.5. Write

$$(\kappa(\mathfrak{p}) \otimes_{\mathcal{O}^{sh}_{Y,y}} \mathcal{O}^{sh}_{X,x})_{red} = \prod K_i$$

as a finite product of fields. We set $w(x) = \sum [K_i : \kappa(\mathfrak{p})]_s$.

Since this definition is clearly insensitive to étale localization, in order to show that w is a weighting we reduce to showing that if f is a finite morphism, then $\int_f w$ is locally constant. Observe that the value of $\int_f w$ in a generic point η of Y is just the number of points of the geometric fibre $X_{\overline{\eta}}$ of $X \to Y$ over η. Moreover, since Y is unibranch a point y of Y is the specialization of a unique generic point η. Hence it suffices to show that $(\int_f w)(y)$ is equal to the number of points of $X_{\overline{\eta}}$. After passing to an affine neighbourhood of y we may assume $X \to Y$ is given by a finite ring map $A \to B$. Suppose $\mathcal{O}^{sh}_{Y,y}$ is constructed using a map $\kappa(y) \to k$ into an algebraically closed field k. Then

$$\mathcal{O}^{sh}_{Y,y} \otimes_A B = \prod\nolimits_{f(x)=y} \prod\nolimits_{\varphi \in \text{Mor}_{\kappa(y)}(\kappa(x), k)} \mathcal{O}^{sh}_{X,x}$$

by Algebra, Lemma 10.153.4 and the lemma used above. Observe that the minimal prime $\mathfrak{p}$ of $\mathcal{O}^{sh}_{Y,y}$ maps to the prime of A corresponding to η. Hence we see that the desired equality holds because the number of points of a geometric fibre is unchanged by a field extension. $\square$

37.76. More on weightings

0F3F We prove a few more basic properties of weightings. Allthough at first it appears that weightings can be very wild, it actually turns out the condition imposed in Definition 37.75.2 is rather strong.

0F3G Lemma 37.76.1. Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then the level sets of the function w are locally constructible in X.

Proof. In the proof below we will use Lemmas 37.75.4 and 37.75.3 without further mention. We will also use elementary properties of constructible subsets of schemes and topological spaces, see Topology, Section 5.15 and Properties, Section 28.2. Using this the reader sees question is local on X and Y; details omitted. Hence we may assume X and Y are affine. If we can find a surjective morphism $Y' \to Y$ of finite presentation such that the level sets of w pull back to locally constructible subsets of $X' = Y' \times_Y X$, then we conclude by Morphisms, Theorem 29.22.3.

Assume X and Y affine. We may choose an immersion $X \to T$ where $T \to Y$ is finite, see Lemma 37.43.3. By Morphisms, Lemma 29.48.6 after replacing Y by Y' surjective finite locally free over Y, replacing X by $Y' \times_Y X$ and T by a scheme finite locally free over Y' containing $Y' \times_Y T$ as a closed subscheme, we may assume T is finite locally free over Y, contains closed subschemes T_i mapping isomorphically to Y such that $T = \bigcup_{i=1,\ldots,n} T_i$ (set theoretically). Since $T_i \subset T$ is a constructible closed subset (as the image of a finitely presented morphism $Y \to T$ of schemes), we see that for $I \subset \{1, \ldots, n\}$ the intersection $\bigcap_{i \in I} T_i$ is a constructible closed subset of T and hence maps to a constructible closed subset of Y.

For a disjoint union decomposition $\{1, \ldots, n\} = I_1 \amalg \ldots \amalg I_r$ with nonempty parts consider the subset $Y_{I_1,\ldots,I_r} \subset Y$ consisting of points $y \in Y$ such that $T_y = \{x_1, \ldots, x_r\}$ consists of exactly r points with $x_j \in T_i \Leftrightarrow i \in I_j$. By our remarks above this is a constructible partition of Y. There exists an affine scheme Y' of finite presentation over Y such that the image of $Y' \to Y$ is exactly $Y_{I_1,\ldots,I_r}$, see Algebra, Lemma 10.29.4. Hence we may assume that $Y = Y_{I_1,\ldots,I_r}$ for some disjoint union decomposition $\{1, \ldots, n\} = I_1 \amalg \ldots \amalg I_r$. In this case $T = T(1) \amalg \ldots \amalg T(r)$ with $T(j) = \bigcap_{i \in I_j} T_i$ is a decomposition of T into disjoint closed (and hence open) subsets. Intersecting with the locally closed subscheme X we obtain an analogous decomposition $X = X(1) \amalg \ldots \amalg X(r)$ into open and closed parts. The morphism $X(j) \to Y$ an immersion. Since w is a weighting, it follows that $w|_{X(j)}$ is locally constant[16] and we conclude. □

0F3H Lemma 37.76.2. Let $f : X \to Y$ be a locally quasi-finite morphism of finite presentation. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then the level sets of the function $\int_f w$ are locally constructible in Y.

Proof. By Lemma 37.75.1 formation of the function $\int_f w$ commutes with arbitrary base change and by Lemma 37.75.3 after base change we still have a weighthing. This means that if we can find $Y' \to Y$ surjective and of finite presentation, then it suffices to prove the result after base change to Y', see Morphisms, Theorem 29.22.3.

The question is local on Y hence we may assume Y is affine. Then X is quasi-compact and quasi-separated (as f is of finite presentation). Suppose that $X = U \cup V$ are quasi-compact open. Then we have

$$\int_f w = \int_{f|_U} w|_U + \int_{f|_V} w|_V - \int_{f|_{U \cap V}} w|_{U \cap V}$$

[16]In fact, if $f : X \to Y$ is an immersion and w is a weighting of f, then f restricts to an open map on the locus where w is nonzero.

Thus if we know the result for $w|_U$, $w|_V$, $w|_{U\cap V}$ then we know the result for w. By the induction principle (Cohomology of Schemes, Lemma 30.4.1) it suffices to prove the lemma when X is affine.

Assume X and Y are affine. We may choose an open immersion $X \to T$ where $T \to Y$ is finite, see Lemma 37.43.3. Because we may still base change with a suitable $Y' \to Y$ we can use Morphisms, Lemma 29.48.6 to reduce to the case where all residue field extensions induced by the morphism $T \to Y$ (and a foriori induced by $X \to Y$) are trivial. In this situation $\int_f w$ is just taking the sums of the values of w in fibres. The level sets of w are locally constructible in X (Lemma 37.76.1). The function w only takes a finite number of values by Properties, Lemma 28.2.7. Hence we conclude by Morphisms, Theorem 29.22.3 and some elementary arguments on sums of integers. □

0F3I Lemma 37.76.3. Let $f : X \to Y$ be a locally quasi-finite morphism. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Then w is upper semi-continuous.

Proof. Let $x \in X$ with image $y \in Y$. Choose an étale neighbourhood $(V, v) \to (Y, y)$ and an open $U \subset X_V$ such that $\pi : U \to V$ is finite and there is a unique point $u \in U$ mapping to v with $\kappa(u)/\kappa(v)$ purely inseparable. See Lemma 37.41.3. Then $(\int_\pi w|_U)(v) = w(u)$. It follows from Definition 37.75.2 that after replacing V by a neighbourhood of v we we have $w|_U(u') \leq w|_U(u) = w(x)$ for all $u' \in U$. Namely, $w|_U(u')$ occurs as a summand in the expression for $(\int_\pi w|_U)(\pi(u'))$. This proves the lemma because the étale morphism $U \to X$ is open. □

0F3J Lemma 37.76.4. Let $f : X \to Y$ be a separated, locally quasi-finite morphism with finite fibres. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Then $\int_f w$ is lower semi-continuous.

Proof. Let $y \in Y$. Let $x_1, \ldots, x_r \in X$ be the points lying over y. Apply Lemma 37.41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $(\int_f w)(y) = \sum w(v_{i,j})$ where $w(v_{i,j}) = w(x_i)$. Since $\int_{V_{i,j}\to U} w|_{V_{i,j}}$ is locally constant by definition, we may after shrinking U assume these functions are constant with value $w(v_{i,j})$. We conclude that

$$\int\nolimits_{U\times_Y X\to U} w|_{U\times_Y X} = \int\nolimits_{W\to U} w|_W + \sum \int\nolimits_{V_{i,j}\to U} w|_{V_{i,j}} = \int\nolimits_{W\to U} w|_W + (\int\nolimits_f w)(y)$$

This is $\geq (\int_f w)(y)$ and we conclude because $U \to Y$ is open and formation of the integral commutes with base change (Lemma 37.75.1). □

0F3K Lemma 37.76.5. Let $f : X \to Y$ be a locally quasi-finite morphism with X quasi-compact. Let $w : X \to \mathbf{Z}$ be a weighting of f. Then $\int_f w$ attains its maximum.

Proof. It follows from Lemma 37.76.1 and Properties, Lemma 28.2.7 that w only takes a finite number of values on X. It follows from Morphisms, Lemma 29.57.9 that $X \to Y$ has bounded geometric fibres. This shows that $\int_f w$ is bounded. □

0F3L Lemma 37.76.6. Let $f : X \to Y$ be a separated, locally quasi-finite morphism. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Assume $\int_w f$ attains its maximum d and let $Y_d \subset Y$ be the open set of points y with $(\int_f w)(y) = d$. Then the morphism $f^{-1}(Y_d) \to Y_d$ is finite.

Proof. Observe that Y_d is open by Lemma 37.76.4. Let $y \in Y_d$. Say $x_1, \ldots, x_n$ are the points of X lying over y. Apply Lemma 37.41.5 to get an étale neighbourhood $(U, u) \to (Y, y)$ and a decomposition

$$U \times_Y X = W \amalg \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_i} V_{i,j}$$

as in locus citatus. Observe that $d = \sum w(v_{i,j})$ where $w(v_{i,j}) = w(x_i)$. Since $\int_{V_{i,j} \to U} w|_{V_{i,j}}$ is locally constant by definition, we may after shrinking U assume these functions are constant with value $w(v_{i,j})$. We conclude that

$$\int_{U \times_Y X \to U} w|_{U \times_Y X} = \int_{W \to U} w|_W + \sum \int_{V_{i,j} \to U} w|_{V_{i,j}} = \int_{W \to U} w|_W + (\int_f w)(y)$$

This is $\geq (\int_f w)(y) = d$ and we conclude that W must be the emptyset. Thus $U \times_Y X \to U$ is finite. By Descent, Lemma 35.23.23 this implies that $X \to Y$ is finite over the image of the open morphism $U \to Y$. In other words, we see that f is finite over an open neighbourhood of y as desired. $\square$

0F3M Lemma 37.76.7. Let $A \to B$ be a ring map which is finite and of finite presentation. There exists a finitely presented ring map $A \to A_{univ}$ and an idempotent $e_{univ} \in B \otimes_A A_{univ}$ such that for any ring map $A \to A'$ and idempotent $e \in B \otimes_A A'$ there is a ring map $A_{univ} \to A'$ mapping e_{univ} to e.

Proof. Choose $b_1, \ldots, b_n \in B$ generating B as an A-module. For each i choose a monic $P_i \in A[x]$ such that $P_i(b_i) = 0$ in B, see Algebra, Lemma 10.36.3. Thus B is a quotient of the finite free A-algebra $B' = A[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n))$. Let $J \subset B'$ be the kernel of the surjection $B' \to B$. Then $J = (f_1, \ldots, f_m)$ is finitely generated as B is a finitely generated A-algebra, see Algebra, Lemma 10.6.2. Choose an A-basis $b'_1, \ldots, b'_N$ of B'. Consider the algebra

$$A_{univ} = A[z_1, \ldots, z_N, y_1, \ldots, y_m]/I$$

where I is the ideal generated by the coefficients in $A[z_1, \ldots, z_n, y_1, \ldots, y_m]$ of the basis elements $b'_1, \ldots, b'_N$ of the expresssion

$$(\sum z_j b'_j)^2 - \sum z_j b'_j + \sum y_k f_k$$

in $B'[z_1, \ldots, z_N, y_1, \ldots, y_m]$. By construction the element $\sum z_j b'_j$ maps to an idempotent e_{univ} in the algebra $B \otimes_A A_{univ}$. Moreover, if $e \in B \otimes_A A'$ is an idempotent, then we can lift e to an element of the form $\sum b'_j \otimes a'_j$ in $B' \otimes_A A'$ and we can find $a''_k \in A'$ such that

$$(\sum b'_j \otimes a'_j)^2 - \sum b'_j \otimes a'_j + \sum f_k \otimes a''_k$$

is zero in $B' \otimes_A A'$. Hence we get an A-algebra map $A_{univ} \to A$ sending z_j to a'_j and y_k to a''_k mapping e_{univ} to e. This finishes the proof. $\square$

0F3N Lemma 37.76.8. Let $X \to Y$ be a morphism of affine schemes which is quasi-finite and of finite presentation. There exists a morphism $Y_{univ} \to Y$ of finite presentation and an open subscheme $U_{univ} \subset Y_{univ} \times_Y X$ such that $U_{univ} \to Y_{univ}$ is finite with the following property: given any morphism $Y' \to Y$ of affine schemes and an open subscheme $U' \subset Y' \times_Y X$ such that $U' \to Y'$ is finite, there exists a morphism $Y' \to Y_{univ}$ such that the inverse image of U_{univ} is U'.

Proof. Recall that a finite type morphism is quasi-finite if and only if it has relative dimension 0, see Morphisms, Lemma 29.29.5. By Lemma 37.34.9 applied with $d = 0$ we reduce to the case where X and Y are Noetherian. We may choose an open immersion $X \to X'$ such that $X' \to Y$ is finite, see Algebra, Lemma 10.123.14. Note that if we have $Y' \to Y$ and U' as in (2), then

$$U' \to Y' \times_Y X \to Y' \times_Y X'$$

is open immersion between schemes finite over Y' and hence is closed as well. We conclude that U' corresponds to an idempotent in

$$\Gamma(Y', \mathcal{O}_{Y'}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(X', \mathcal{O}_{X'})$$

whose corresponding open and closed subset is contained in the open $Y' \times_Y X$. Let $Y'_{univ} \to Y$ and idempotent

$$e'_{univ} \in \Gamma(Y_{univ}, \mathcal{O}_{Y_{univ}}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(X', \mathcal{O}_{X'})$$

be the pair constructed in Lemma 37.76.7 for the ring map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X', \mathcal{O}_{X'})$ (here we use that Y is Noetherian to see that X' is of finite presentation over Y). Let $U'_{univ} \subset Y'_{univ} \times_Y X'$ be the corresponding open and closed subscheme. Then we see that

$$U'_{univ} \setminus Y'_{univ} \times_Y X$$

is a closed subset of U'_{univ} and hence has closed image $T \subset Y'_{univ}$. If we set $Y_{univ} = Y'_{univ} \setminus T$ and U_{univ} the restriction of U'_{univ} to $Y_{univ} \times_Y X$, then we see that the lemma is true. □

0F3P Lemma 37.76.9. Let $Y = \lim Y_i$ be a directed limit of affine schemes. Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of affine schemes which is quasi-finite and of finite presentation. Let $f : X \to Y$ and $f_i : X_i \to Y_i$ for $i \geq 0$ be the base changes of f_0. If $w : X \to \mathbf{Z}$ is a weighting of f, then for sufficiently large i there exists a weighting $w_i : X_i \to \mathbf{Z}$ of f_i whose pullback to X is w.

Proof. By Lemma 37.76.1 the level sets of w are constructible subsets E_k of X. This implies the function w only takes a finite number of values by Properties, Lemma 28.2.7. Thus there exists an i such that E_k descends to a construcible subset $E_{i,k}$ in X_i for all k; moreover, we may assume $X_i = \coprod E_{i,k}$. This follows as the topological space of X is the limit in the category of topological spaces of the spectral spaces X_i along a directed system with spectral transition maps. See Limits, Section 32.4 and Topology, Section 5.24. We define $w_i : X_i \to \mathbf{Z}$ such that its level sets are the constructible sets $E_{i,k}$.

Choose $Y_{i,univ} \to Y_i$ and $U_{i,univ} \subset Y_{i,univ} \times_{Y_i} X_i$ as in Lemma 37.76.8. By the universal property of the construction, in order to show that w_i is a weighting, it would suffice to show that

$$\tau_i = \int_{U_{i,univ} \to Y_{i,univ}} w_i|_{U_{i,univ}}$$

is locally constant on $Y_{i,univ}$. By Lemma 37.76.2 this function has constructible level sets but it may not (yet) be locally constant. Set $Y_{univ} = Y_{i,univ} \times_{Y_i} Y$ and let $U_{univ} \subset Y_{univ} \times_Y X$ be the inverse image of $U_{i,univ}$. Then, since the pullback of w to $Y_{univ} \times_Y X$ is a weighting for $Y_{univ} \times_Y X \to Y_{univ}$ (Lemma 37.75.3) we do have that

$$\tau = \int_{U_{univ} \to Y_{univ}} w_i|_{U_{univ}}$$

is locally constant on Y_{univ}. Thus the level sets of τ are open and closed. Finally, we have $Y_{univ} = \lim_{i' \geq i} Y_{i',univ}$ and the level sets of τ are the inverse limits of the level sets of $\tau_{i'}$ (similarly defined). Hence the references above imply that for sufficiently large i' the level sets of $\tau_{i'}$ are open as well. For such an index i' we conclude that $w_{i'}$ is a weighting of $f_{i'}$ as desired. $\square$

37.77. Weightings and affine stratification numbers

0F3Q In this section we give a bound for the affine stratification number of a scheme which has a certain kind of cover by an affine scheme.

0F3R **Lemma 37.77.1.** Let $f : X \to Y$ be a morphism of affine schemes which is quasi-finite and of finite presentation. Let $w : X \to \mathbf{Z}_{>0}$ be a postive weighting of f. Let $d < \infty$ be the maximum value of $\int_f w$. The open

$$Y_d = \{y \in Y \mid (\textstyle\int_f w)(y) = d\}$$

of Y is affine.

Proof. Observe that $\int_f w$ attains its maximum by Lemma 37.76.5. The set Y_d is open by Lemma 37.76.4. Thus the statement of the lemma makes sense.

Reduction to the Noetherian case; please skip this paragraph. Recall that a finite type morphism is quasi-finite if and only if it has relative dimension 0, see Morphisms, Lemma 29.29.5. By Lemma 37.34.9 applied with $d = 0$ we can find a quasi-finite morphism $f_0 : X_0 \to Y_0$ of affine Noetherian schemes and a morphism $Y \to Y_0$ such that f is the base change of f_0. Then we can write $Y = \lim Y_i$ as a directed limit of affine schemes of finite type over Y_0, see Algebra, Lemma 10.127.2. By Lemma 37.76.9 we can find an i such that our weighting w descends to a weighting w_i of the base change $f_i : X_i \to Y_i$ of f_0. Now if the lemma holds for f_i, w_i, then it implies the lemma for f as formation of $\int_f w$ commutes with base change, see Lemma 37.75.1.

Assume X and Y Noetherian. Let $X' \to Y'$ be the base change of f by a morphism $g : Y' \to Y$. The formation of $\int_f w$ and hence the open Y_d commute with base change. If g is finite and surjective, then $Y'_d \to Y_d$ is finite and surjective. In this case proving that Y_d is affine is equivalent to showing that Y'_d is affine, see Cohomology of Schemes, Lemma 30.13.3.

We may choose an immersion $X \to T$ with T finite over Y, see Lemma 37.43.3. We are going to apply Morphisms, Lemma 29.48.6 to the finite morphism $T \to Y$. This lemma tells us that there is a finite surjective morphism $Y' \to Y$ such that $Y' \times_Y T$ is a closed subscheme of a scheme T' finite over Y' which has a special form. By the discussion in the first paragraph, we may replace Y by Y', T by T', and X by $Y' \times_Y X$. Thus we may assume there is an immersion $X \to T$ (not necessarily open or closed) and closed subschemes $T_i \subset T$, $i = 1, \ldots, n$ where

1. $T \to Y$ is finite (and locally free),
2. $T_i \to Y$ is an isomorphism, and
3. $T = \bigcup_{i=1,\ldots,n} T_i$ set theoretically.

Let $Y' = \coprod Y_k$ be the disjoint union of the irreducible components of Y (viewed as integral closed subschemes of Y). Then we may base change once more by $Y' \to Y$; here we are using that Y is Noetherian. Thus we may in addition assume Y is integral and Noetherian.

We also may and do assume that $T_i \neq T_j$ if $i \neq j$ by removing repeats. Since Y and hence all T_i are integral, this means that if T_i and T_j intersect, then they intersect in a closed subset which maps to a proper closed subset of Y.

Observe that $V_i = X \cap T_i$ is a locally closed subset which is in addition a closed subscheme of X hence affine. Let $\eta \in Y$ and $\eta_i \in T_i$ be the generic points. If $\eta \notin Y_d$, then $Y_d = \emptyset$ and we're done. Assume $\eta \in Y_d$. Denote $I \in \{1, \ldots, n\}$ the subset of indices i such that $\eta_i \in V_i$. For $i \in I$ the locally closed subset $V_i \subset T_i$ contains the generic point of the irreducible space T_i and hence is open. On the other hand, since f is open (Lemma 37.75.6), for any $x \in X$ we can find an $i \in I$ and a specialization $\eta_i \rightsquigarrow x$. It follows that $x \in T_i$ and hence $x \in V_i$. In other words, we see that $X = \bigcup_{i \in I} V_i$ set theoretically. We claim that $Y_d = \bigcap_{i \in I} \mathrm{Im}(V_i \to Y)$; this will finish the proof as the intersection of affine opens $\mathrm{Im}(V_i \to Y)$ of Y is affine.

For $y \in Y$ let $f^{-1}(\{y\}) = \{x_1, \ldots, x_r\}$ in X. For each $i \in I$ there is at most one $j(i) \in \{1, \ldots, x_r\}$ such that $\eta_i \rightsquigarrow x_{j(i)}$. In fact, $j(i)$ exists and is equal to j if and only if $x_j \in V_i$. If $i \in I$ is such that $j = j(i)$ exists, then $V_i \to Y$ is an isomorphism in a neighbourhood of $x_j \mapsto y$. Hence $\bigcup_{i \in I,\ j(i) = j} V_i \to Y$ is finite after replacing source and target by neighbourhoods of $x_j \mapsto y$. Thus the definition of a weighting tells us that $w(x_j) = \sum_{i \in I,\ j(i)=j} w(\eta_i)$. Thus we see that

$$(\textstyle\int_f w)(\eta) = \sum_{i \in I} w(\eta_i) \geq \sum_{j(i) \text{ exists}} w(\eta_i) = \sum_j w(x_j) = (\int_f w)(y)$$

Thus equality holds if and only if y is contained in $\bigcap_{i \in I} \mathrm{Im}(V_i \to Y)$ which is what we wanted to show. □

0F3S Proposition 37.77.2. Let $f : X \to Y$ be a surjective quasi-finite morphism of schemes. Let $w : X \to \mathbf{Z}_{>0}$ be a positive weighting of f. Assume X affine and Y separated[17]. Then the affine stratification number of Y is at most the number of distinct values of $\int_f w$.

Proof. Note that since Y is separated, the morphism $X \to Y$ is affine (Morphisms, Lemma 29.11.11). The function $\int_f w$ attains its maximum d by Lemma 37.76.5. We will use induction on d. Consider the open subscheme $Y_d = \{y \in Y \mid (\int_f w)(y) = d\}$ of Y and recall that $f^{-1}(Y_d) \to Y_d$ is finite, see Lemma 37.76.6. By Lemma 37.77.1 for every affine open $W \subset Y$ we have that $Y_d \cap W$ is affine (this uses that $W \times_Y X$ is affine, being affine over X). Hence $Y_d \to Y$ is an affine morphism of schemes. We conclude that $f^{-1}(Y_d) = Y_d \times_Y X$ is an affine scheme being affine over X. Then $f^{-1}(Y_d) \to Y_d$ is surjective and hence Y_d is affine by Limits, Lemma 32.11.1. Set $X' = X \setminus f^{-1}(Y_d)$ and $Y' = Y \setminus Y_d$ viewed as closed subschemes of X and Y. Since X' is closed in X it is affine. Since Y' is closed in Y it is separated. The morphism $f' : X' \to Y'$ is surjective and w induces a weighting w' of f', see Lemma 37.75.3. By induction Y' has an affine stratification of length $\leq$ the number of distinct values of $\int_{f'} w'$ and the proof is complete. □

37.78. Completely decomposed morphisms

0GTH Nishnevich studied the notion of a completely decomposed family of étale morphisms, in order to define what is now called the Nishnevich topology, see for example [Nis89].

[17]It suffices if the diagonal of Y is affine.

0GTI **Definition 37.78.1.** A morphism $f : X \to Y$ of schemes is said to be *completely decomposed*[18] if for all points $y \in Y$ there is a point $x \in X$ with $f(x) = y$ such that the field extension $\kappa(x)/\kappa(y)$ is trivial. A family of morphisms $\{f_i : X_i \to Y\}_{i \in I}$ of schemes with fixed target is said to be *completely decomposed* if $\coprod f_i : \coprod Y_i \to X$ is completely decomposed.

We start with some basic lemmas.

0GTJ **Lemma 37.78.2.** The composition of two completely decomposed morphisms of schemes is completely decomposed. If $\{f_i : X_i \to Y\}_{i \in I}$ is completely decomposed and for each i we have a family $\{X_{ij} \to X_i\}_{j \in J_i}$ which is completely decomposed, then the family $\{X_{ij} \to Y\}_{i \in I, j \in J_i}$ is completely decomposed.

Proof. Omitted. $\square$

0GTK **Lemma 37.78.3.** The base change of a completely decomposed morphism of schemes is completely decomposed. If $\{f_i : X_i \to Y\}_{i \in I}$ is completely decomposed and $Y' \to Y$ is a morphism of schemes, then $\{X_i \times_Y Y' \to Y'\}_{i \in I}$ is completely decomposed.

Proof. Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes. Let $y' \in Y'$ be a point with image $y = g(y')$ in Y. If $x \in X$ is a point such that $f(x) = y$ and $\kappa(x) = \kappa(y)$, then there exists a unique point $x' \in X' = X \times_Y Y'$ which maps to y' in Y' and to x in X and moreover $\kappa(x') = \kappa(y')$, see Schemes, Lemma 26.17.5. From this fact the lemma follows easily; we omit the details. $\square$

0GTL **Lemma 37.78.4.** Let $f : X \to Y$ be a morphism of schemes. Assume f is completely decomposed, f is locally of finite presentation, and Y is quasi-compact and quasi-separated. Then there exist $n \geq 0$ and morphisms $Z_i \to Y$, $i = 1, \ldots, n$ with the following properties [EHIK21, Lemma 2.1.2]

(1) $\coprod Z_i \to Y$ is surjective,
(2) $Z_i \to Y$ is an immersion for all i,
(3) $Z_i \to Y$ is of finite presentation for all i, and
(4) the base change $X \times_Y Z_i \to Z_i$ has a section for all i.

Proof. Let $y \in Y$. By assumption there is a morphism $\sigma : \mathrm{Spec}(\kappa(y)) \to X$ over Y. We can write $\mathrm{Spec}(\kappa(y))$ as a directed limit of affine schemes Z over Y such that $Z \to Y$ is an immersion of finite presentation. Namely, choose an affine open $y \in \mathrm{Spec}(A) \subset Y$ and say y corresponds to the prime ideal $\mathfrak{p}$ of A. Then $\kappa(\mathfrak{p})$ is the filtered colimit of the rings $(A/I)_f$ where $I \subset \mathfrak{p}$ is a finitely generated ideal and $f \in A$, $f \notin \mathfrak{p}$. The morphisms $Z = \mathrm{Spec}((A/I)_f) \to Y$ are immersions of finite presentation; quasi-compactness of $Z \to Y$ follows as Y is quasi-separated, see Schemes, Lemma 26.21.14. By Limits, Proposition 32.6.1 for some such Z there is a morphism $\sigma' : Z \to X$ over Y agreeing with σ on the spectrum of $\kappa(\mathfrak{p})$. Since σ' is a morphism over Y, we obtain a section of the projection $X \times_Y Z \to Z$

We conclude that Y is the union of the images of immersions $Z \to Y$ of finite presentation such that $X \times_Y Z \to Z$ has a section. Since the image of $Z \to Y$ is constructible (Morphisms, Lemma 29.22.2) and since Y is compact in the constructible topology (Properties, Lemma 28.2.4 and Topology, Lemma 5.23.2), we see that a finite number of these suffice. $\square$

[18]This may be nonstandard terminology.

0GTM Lemma 37.78.5. Let $S = \lim_{\lambda \in \Lambda} S_\lambda$ be a limit of a directed system of schemes with affine transition morphisms. Let $0 \in \Lambda$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. For $\lambda \geq 0$ let $f_\lambda : X_\lambda \to Y_\lambda$ be the base change of f_0 to S_λ and let $f : X \to Y$ be the base change of f_0 to S. If

(1) f is completely decomposed,
(2) Y_0 is quasi-compact and quasi-separated, and
(3) f_0 is locally of finite presentation,

then there exists an $\lambda \geq 0$ such that f_λ is completely decomposed.

Proof. Since Y_0 is quasi-compact and quasi-separated, the scheme Y, which is affine over Y_0, is quasi-compact and quasi-separated. Choose $n \geq 0$ and $Z_i \to Y$, $i = 1, \ldots, n$ as in Lemma 37.78.4. Denote $\sigma_i : Z_i \to X$ morphisms over Y which exist by our choice of Z_i. After increasing $0 \in \Lambda$ we may assume there exist morphisms $Z_{i,0} \to Y_0$ of finite presentation whose base changes to S are the morphisms $Z_i \to Y$, see Limits, Lemma 32.10.1. By Limits, Lemma 32.8.13 we may assume, after possibly increasing 0, that $Z_{i,0} \to Y_0$ is an immersion. Since $\coprod Z_i \to Y$ is surjective, we may assume, after possibly increasing 0, that $\coprod Z_{i,0} \to Y_0$ is surjective, see Limits, Lemma 32.8.15. Observe that $Z_i = \lim_{\lambda \geq 0} Z_{i,\lambda}$ where $Z_{i,\lambda} = Y_\lambda \times_{Y_0} Z_{i,0}$. Let us view the compositions

$$Z_i \xrightarrow{\sigma_i} X \to X_0$$

as morphisms over Y_0. Since f_0 is locally of finite presentation by Limits, Proposition 32.6.1 we can find a $\lambda \geq 0$ such that there exist morphisms $\sigma'_{i,\lambda} : Z_{i,\lambda} \to X_0$ over Y_0 whose precomposition with $Z_i \to Z_{i,\lambda}$ are the displayed arrows. Of course, then $\sigma'_{i,\lambda}$ determines a morphism $\sigma_{i,\lambda} : Z_{i,\lambda} \to X_\lambda = X_0 \times_{Y_0} Y_\lambda$ over Y_λ. Since $\coprod Z_{i,\lambda} \to Y_\lambda$ is surjective we conclude that $X_\lambda \to Y_\lambda$ is completely decomposed. □

37.79. Families of ample invertible modules

0GTN We continue the discussion from Morphisms, Section 29.12.

0GTP Lemma 37.79.1. Let $f : X \to Y$ be a morphism of schemes. Assume

(1) Y has an ample family of invertible modules,
(2) there exists an f-ample invertible module on X.

Then X has an ample family of invertible modules.

Proof. Let $\mathcal{L}$ be an f-ample invertible module on X. This in particular implies that f is quasi-compact, see Morphisms, Definition 29.37.1. Since Y is quasi-compact by Morphisms, Definition 29.12.1 we see that X is quasi-compact (and hence X itself satisfies the first condition of Morphisms, Definition 29.12.1). Let $x \in X$ with image $y \in Y$. By assumption (2) we can find an invertible $\mathcal{O}_Y$-module $\mathcal{N}$ and a section $t \in \Gamma(Y, \mathcal{N})$ such that the locus Y_t where t does not vanish is affine. Then $\mathcal{L}$ is ample over $f^{-1}(Y_t) = X_{f^*t}$ and hence we can find a section $s \in \Gamma(X_{f^*t}, \mathcal{L})$ such that $(X_{f^*t})_s$ is affine and contains x. By Properties, Lemma 28.17.2 for some $n \geq 0$ the product $(f^*t)^n s$ extends to a section $s' \in \Gamma(X, f^*\mathcal{N}^{\otimes n} \otimes \mathcal{L})$. Then finally the section $s'' = f^*ts'$ of $f^*\mathcal{N}^{\otimes n+1} \otimes \mathcal{L}$ vanishes at every point of $X \setminus X_{f^*t}$ hence we see that $X_{s''} = (X_{f^*t})_s$ is affine as desired. □

0GTQ Lemma 37.79.2. Let $f : X \to Y$ be an affine or quasi-affine morphism of schemes. If Y has an ample family of invertible modules, so does X.

Proof. By Morphisms, Lemma 29.37.6 this is a special case of Lemma 37.79.1. □

37.80. Blowing up and ample families of invertible modules

0GTR We prove a result from [Gro10].

0GTS **Lemma 37.80.1.** Let X be a scheme. Suppose given effective Cartier divisors $D_1, \ldots, D_m$ on X and invertible modules $\mathcal{L}_1, \ldots, \mathcal{L}_m$ such that $\bigcap D_i = \emptyset$ and $\mathcal{L}_i|_{X \setminus D_i}$ is ample. Then X has an ample family of invertible modules.

Proof. Let $x \in X$. Choose an index $i \in \{1, \ldots, m\}$ such that $x \notin D_i$. Set $U_i = X \setminus D_i$. Since $\mathcal{L}_i|_{U_i}$ we can find an $n \geq 1$ and a section $s \in \Gamma(U_i, \mathcal{L}_i^{\otimes n})$ such that the locus $(U_i)_s$ where s doesn't vanish is affine (Properties, Definition 28.26.1). Since U_i is the locus where the canonical section $1 \in \mathcal{O}_X(D_i)$ doesn't vanish, we see from Properties, Lemma 28.17.2 there exists an $N \geq 0$ such that s extends to a section

$$s' \in \Gamma(X, \mathcal{L}_i^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{O}_X(ND_i))$$

After replacing N by $N + 1$ we see that s' vanishes at every point of D_i and hence that $X_{s'} = (U_i)_s$ is affine. This proves that X has an ample family of invertible modules, see Morphisms, Definition 29.12.1. □

0GTT **Lemma 37.80.2.** Let X be a quasi-compact and quasi-separated scheme with finitely many irreducible components. There exists a quasi-compact dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that the scheme X' has an ample family of invertible modules. [Gro10, Proposition 1.3.1]

Proof. Let $\eta_1, \ldots, \eta_n \in X$ be the generic points of the irreducible components of X. By Properties, Lemma 28.29.4 and the fact that X is quasi-compact we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_n$. In particular the quasi-compact open subset $U = U_1 \cap \ldots \cap U_m$ is dense in X. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be a finite type quasi-coherent ideal sheaf such that $U_i = X \setminus Z_i$ where $Z_i = V(\mathcal{I}_i)$, see Properties, Lemma 28.24.1. Let

$$f : X' \longrightarrow X$$

be the blowing up of X in the ideal sheaf $\mathcal{I} = \mathcal{I}_1 \cdots \mathcal{I}_m$. Note that f is a U-admissible blowing up as $V(\mathcal{I})$ is (set theoretically) the union of the Z_i which are disjoint from U. Also, f is a projective morphism and $\mathcal{O}_{X'}(1)$ is f-relatively ample, see Divisors, Lemma 31.32.13. By Divisors, Lemma 31.32.12 for each i the morphism f' factors as $X' \to X'_i \to X$ where $X'_i \to X$ is the blowing up in $\mathcal{I}_i$ and $X' \to X'_i$ is another blowing up (namely in the pullback of the products of the ideals $\mathcal{I}_j$ omitting $\mathcal{I}_i$). It follows from this that $D_i = f^{-1}(Z_i) \subset X'$ is an effective Cartier divisor, see Divisors, Lemmas 31.32.11 and 31.32.4. We have $X' \setminus D_i = f^{-1}(U_i)$. As $\mathcal{O}_{X'}(1)$ is f-ample, the restriction of $\mathcal{O}_{X'}(1)$ to $X' \setminus D_i$ is ample. It follows from Lemma 37.80.1 that X' has an ample family of invertible modules. □

0GTU **Proposition 37.80.3.** Let X be a quasi-compact and quasi-separated scheme. There exists a morphism $f : Y \to X$ which is of finite presentation, proper, and completely decomposed (Definition 37.78.1) such that the scheme Y has an ample family of invertible modules.

Proof. By Limits, Proposition 32.5.4 there exists an affine morphism $X \to X_0$ where X_0 is a scheme of finite type over $\mathbf{Z}$. Below we produce a morphism $Y_0 \to X_0$ with all the desired properties. Then setting $Y = X \times_{X_0} Y_0$ and f equal to the projection $f : Y \to X$ we conclude. To see this observe that f is of finite presentation (Morphisms, Lemma 29.21.4), f is proper (Morphisms, Lemma 29.41.5), f

is completely decomposed (Lemma 37.78.3). Finally, since $Y \to Y_0$ is affine (as the base change of $X \to X_0$) we see that Y has an ample family of invertible modules by Lemma 37.79.2. This reduces us to the case discussed in the next paragraph.

Assume X is of finite type over $\mathbf{Z}$. In particular $\dim(X) < \infty$. We will argue by induction on $\dim(X)$. If $\dim(X) = 0$, then X is affine and has the resolution property. In general, there exists a dense open $U \subset X$ and a U-admissible blowing up $X' \to X$ such that X' has an ample family of invertible modules, see Lemma 37.80.2. Since $f : X' \to X$ is an isomorphism over U we see that every point of U lifts to a point of X' with the same residue field. Let $Z = X \setminus U$ with the reduced induced scheme structure. Then $\dim(Z) < \dim(X)$ as U is dense in X (see above). By induction we find a proper, completely decomposed morphism $W \to Z$ such that W has an ample family of invertible modules. Then it follows that $Y = X' \amalg W \to X$ is the desired morphism. ☐

37.81. The extensive criterion for closed immersions

0H2N In this section, we give a criterion for a morphism of schemes to be a closed immersion.

0H2P **Lemma 37.81.1.** *A morphism $f : X \to Y$ of affine schemes is a closed immersion if and only if for every injective ring map $A \to B$ and commutative square*

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there exists a lift $\mathrm{Spec}(A) \to X$ making the two triangles commute.

Proof. Let the morphism f be given by the ring map $\phi : R \to S$. Then f is a closed immersion if and only if ϕ is surjective.

First, we assume that ϕ is surjective. Let $\psi : A \to B$ be an injective ring map, and suppose we are given a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\phi"] & A \arrow[d, "\psi"] \\ S \arrow[r, "\beta"] \arrow[ur, dotted] & B \end{tikzcd}$$

Then we define a lift $S \to A$ by $s \mapsto \alpha(r)$, where $r \in R$ is such that $\phi(r) = s$. This is well-defined because ψ is injective and the square commutes. Since taking the ring spectrum defines an anti-equivalence between commutative rings and affine schemes, the desired lifting property for f holds.

Next, we assume that ϕ has lifts against all injective ring maps $\psi : A \to B$. Note that $\phi(R)$ is a subring of S, so we obtain a commutative square

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\phi"] & \phi(R) \arrow[d] \\ S \arrow[r, equal] \arrow[ur, dotted] & S \end{tikzcd}$$

in which a lift $S \to \phi(R)$ exists. Hence, the inclusion $\phi(R) \to S$ must be an isomorphism, which shows that ϕ is surjective, and we win. $\square$

0H2Q Lemma 37.81.2. Let X be a scheme. If the canonical morphism $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ of Schemes, Lemma 26.6.4 has a retraction, then X is an affine scheme.

Proof. Write $S = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ and $f : X \to S$ the morphism given in the lemma. Let $s : S \to X$ be a retraction; so $\mathrm{id}_X = sf$. Then $fsf = \mathrm{id}_S f$. Since f induces an isomorphism $\Gamma(S, \mathcal{O}_S) \to \Gamma(X, \mathcal{O}_X)$ this means that fs and id_S induce the same map on $\Gamma(S, \mathcal{O}_S)$. Whence $fs = \mathrm{id}_S$ as S is affine. Hence f is an isomorphism and X is an affine scheme, as was to be shown. $\square$

0H2R Lemma 37.81.3. Let X be a scheme. Let $f : X \to S = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ be the canonical morphism of Schemes, Lemma 26.6.4. The largest quasi-coherent $\mathcal{O}_S$-module contained in the kernel of $f^\sharp : \mathcal{O}_S \to f_*\mathcal{O}_X$ is zero. If X is quasi-compact, then $f^\sharp$ is injective. In particular, if X is quasi-compact, then f is a dominant morphism.

Proof. Let $M \subset \Gamma(S, \mathcal{O}_S)$ be the submodule corresponding to the largest quasi-coherent $\mathcal{O}_S$-module contained in the kernel of $f^\sharp$. Then any element $a \in M$ is mapped to zero by $f^\sharp$. However, $f^\sharp(a)$ is the element of

$$\Gamma(S, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \Gamma(S, \mathcal{O}_S)$$

corresponding to a itself! Thus $a = 0$. Hence $M = 0$ which proves the first assertion. Note that this is equivalent to the morphism $f : X \to S$ being scheme-theoretically surjective.

If X is quasi-compact, then $\mathrm{Ker}(f^\sharp)$ is quasi-coherent by Morphisms, Lemma 29.6.3. Hence $\mathrm{Ker}(f^\sharp) = 0$ and $f^\sharp$ is injective. In this case, f is a dominant morphism by part (4) of Morphisms, Lemma 29.6.3. $\square$

0H2S Lemma 37.81.4. Let $f : X \to Y$ be a quasi-compact morphism of schemes. Then f is a closed immersion if and only if for every injective ring map $A \to B$ and commutative square

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there exists a lift $\mathrm{Spec}\, A \to X$ making the diagram commute.

Proof. Assume that f is a closed immersion. Let $A \to B$ be an injective ring map and consider a commutative square

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Then $\mathrm{Spec}(A) \times_Y X \to \mathrm{Spec}(A)$ is a closed immersion and hence we get an ideal $I \subset A$ and a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dotted] & \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

We obtain a lift by Lemma 37.81.1.

Assume that f has the lifting property stated in the lemma. To prove that f is a closed immersion is local on Y, hence we may and do assume Y is affine. In particular, Y is quasi-compact and therefore X is quasi-compact. Hence there exists a finite affine open covering $X = U_1 \cup \ldots \cup U_n$. The source of the morphism

$$\pi : U = \coprod U_i \longrightarrow X$$

is affine and the induced ring map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$ is injective. By assumption, there exists a lift in the diagram

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) \arrow[r, "f'"] \arrow[ru, dotted, "h"] & Y \end{tikzcd}$$

where f' is the morphism of affine schemes corresponding to the ring map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$. It follows from the fact that π is an epimorphism that the morphism h is a retraction of the canonical morphism $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$; details omitted. Hence X is affine by Lemma 37.81.2. By Lemma 37.81.1 we conclude that f is a closed immersion. $\square$

37.82. Other chapters

CHAPTER 38

More on Flatness

057M 38.1. Introduction

057N In this chapter, we discuss some advanced results on flat modules and flat morphisms of schemes and applications. Most of the results on flatness can be found in the paper [GR71] by Raynaud and Gruson.

Before reading this chapter we advise the reader to take a look at the following results (this list also serves as a pointer to previous results):

(1) General discussion on flat modules in Algebra, Section 10.39.
(2) The relationship between Tor-groups and flatness, see Algebra, Section 10.75.
(3) Criteria for flatness, see Algebra, Section 10.99 (Noetherian case), Algebra, Section 10.101 (Artinian case), Algebra, Section 10.128 (non-Noetherian case), and finally More on Morphisms, Section 37.16.
(4) Generic flatness, see Algebra, Section 10.118 and Morphisms, Section 29.27.
(5) Openness of the flat locus, see Algebra, Section 10.129 and More on Morphisms, Section 37.15.
(6) Flattening, see More on Algebra, Sections 15.16, 15.17, 15.18, 15.19, and 15.20.
(7) Additional results in More on Algebra, Sections 15.21, 15.22, 15.25, and 15.26.

As applications of the material on flatness we discuss the following topics: a non-Noetherian version of Grothendieck's existence theorem, blowing up and flatness, Nagata's theorem on compactifications, the h topology, blow up squares and descent, weak normalization, descent of vector bundles in positive characteristic, and the local structure of perfect complexes in the h topology.

38.2. Lemmas on étale localization

05FM In this section we list some lemmas on étale localization which will be useful later in this chapter. Please skip this section on a first reading.

057R Lemma 38.2.1. Let $i : Z \to X$ be a closed immersion of affine schemes. Let $Z' \to Z$ be an étale morphism with Z' affine. Then there exists an étale morphism $X' \to X$ with X' affine such that $Z' \cong Z \times_X X'$ as schemes over Z.

Proof. See Algebra, Lemma 10.143.10. $\square$

05H2 Lemma 38.2.2. Let

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d] \\ S \arrow[u, leftarrow] & S' \arrow[l] \end{tikzcd}$$

be a commutative diagram of schemes with $X' \to X$ and $S' \to S$ étale. Let $s' \in S'$ be a point. Then

$$X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

is étale.

Proof. This is true because $X' \to X_{S'}$ is étale as a morphism of schemes étale over X, see Morphisms, Lemma 29.36.18 and the base change of an étale morphism is étale, see Morphisms, Lemma 29.36.4. $\square$

05B9 Lemma 38.2.3. Let $X \to T \to S$ be morphisms of schemes with $T \to S$ étale. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then

$$\mathcal{F} \text{ flat over } S \text{ at } x \Leftrightarrow \mathcal{F} \text{ flat over } T \text{ at } x$$

In particular $\mathcal{F}$ is flat over S if and only if $\mathcal{F}$ is flat over T.

Proof. As an étale morphism is a flat morphism (see Morphisms, Lemma 29.36.12) the implication "$\Leftarrow$" follows from Algebra, Lemma 10.39.4. For the converse assume that $\mathcal{F}$ is flat at x over S. Denote $\tilde{x} \in X \times_S T$ the point lying over x in X and over the image of x in T in T. Then $(X \times_S T \to X)^*\mathcal{F}$ is flat at $\tilde{x}$ over T via $\mathrm{pr}_2 : X \times_S T \to T$, see Morphisms, Lemma 29.25.7. The diagonal $\Delta_{T/S} : T \to T \times_S T$ is an open immersion; combine Morphisms, Lemmas 29.35.13 and 29.36.5. So X is identified with open subscheme of $X \times_S T$, the restriction of pr_2 to this open is the given morphism $X \to T$, the point $\tilde{x}$ corresponds to the point x in this open, and $(X \times_S T \to X)^*\mathcal{F}$ restricted to this open is $\mathcal{F}$. Whence we see that $\mathcal{F}$ is flat at x over T. $\square$

05BA Lemma 38.2.4. Let $T \to S$ be an étale morphism. Let $t \in T$ with image $s \in S$. Let M be a $\mathcal{O}_{T,t}$-module. Then

$$M \text{ flat over } \mathcal{O}_{S,s} \Leftrightarrow M \text{ flat over } \mathcal{O}_{T,t}.$$

Proof. We may replace S by an affine neighbourhood of s and after that T by an affine neighbourhood of t. Set $\mathcal{F} = (\mathrm{Spec}(\mathcal{O}_{T,t}) \to T)_*\widetilde{M}$. This is a quasi-coherent sheaf (see Schemes, Lemma 26.24.1 or argue directly) on T whose stalk at t is M (details omitted). Apply Lemma 38.2.3. $\square$

05VL Lemma 38.2.5. Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 10.155.3. Let M^{sh} be a $\mathcal{O}^{sh}_{S,s}$-module. The following are equivalent

(1) M^{sh} is flat over $\mathcal{O}_{S,s}$,
(2) M^{sh} is flat over $\mathcal{O}^h_{S,s}$, and
(3) M^{sh} is flat over $\mathcal{O}^{sh}_{S,s}$.

If $M^{sh} = M^h \otimes_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ this is also equivalent to

(4) M^h is flat over $\mathcal{O}_{S,s}$, and
(5) M^h is flat over $\mathcal{O}^h_{S,s}$.

If $M^h = M \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$ this is also equivalent to

(6) M is flat over $\mathcal{O}_{S,s}$.

Proof. By More on Algebra, Lemma 15.45.1 the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}^h_{S,s} \to \mathcal{O}^{sh}_{S,s}$ are faithfully flat. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from Algebra, Lemma 10.39.4. By faithful flatness the equivalences (6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from Algebra, Lemma 10.39.8. Thus it suffices to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). To prove these we may assume S is an affine scheme.

Assume (1). By Lemma 38.2.4 we see that M^{sh} is flat over $\mathcal{O}_{T,t}$ for any étale neighbourhood $(T, t) \to (S, s)$. Since $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ are directed colimits of local rings of the form $\mathcal{O}_{T,t}$ (see Algebra, Lemmas 10.155.7 and 10.155.11) we conclude that M^{sh} is flat over $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ by Algebra, Lemma 10.39.6. Thus (1) implies (2) and (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}^h_{S,s}$. The same argument applies to prove (4) $\Rightarrow$ (5). $\square$

0DK0 Lemma 38.2.6. Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 10.155.3. Let M^{sh} be an object of $D(\mathcal{O}^{sh}_{S,s})$. Let $a, b \in \mathbf{Z}$. The following are equivalent

(1) M^{sh} has tor amplitude in $[a, b]$ over $\mathcal{O}_{S,s}$,
(2) M^{sh} has tor amplitude in $[a, b]$ over $\mathcal{O}^h_{S,s}$, and
(3) M^{sh} has tor amplitude in $[a, b]$ over $\mathcal{O}^{sh}_{S,s}$.

If $M^{sh} = M^h \otimes^{\mathbf{L}}_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ for $M^h \in D(\mathcal{O}^h_{S,s})$ this is also equivalent to

(4) M^h has tor amplitude in $[a, b]$ over $\mathcal{O}_{S,s}$, and
(5) M^h has tor amplitude in $[a, b]$ over $\mathcal{O}^h_{S,s}$.

If $M^h = M \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$ for $M \in D(\mathcal{O}_{S,s})$ this is also equivalent to

(6) M has tor amplitude in $[a, b]$ over $\mathcal{O}_{S,s}$.

Proof. By More on Algebra, Lemma 15.45.1 the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}^h_{S,s} \to \mathcal{O}^{sh}_{S,s}$ are faithfully flat. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from More on Algebra, Lemma 15.66.11. By faithful flatness the equivalences (6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from More on Algebra, Lemma 15.66.17. Thus it suffices to show that (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), and (4) $\Rightarrow$ (5).

Assume (1). In particular M^{sh} has vanishing cohomology in degrees $< a$ and $> b$. Hence we can represent M^{sh} by a complex $P^\bullet$ of free $\mathcal{O}^{sh}_{X,x}$-modules with $P^i = 0$ for $i > b$ (see for example the very general Derived Categories, Lemma 13.15.4). Note that P^n is flat over $\mathcal{O}_{S,s}$ for all n. Consider $\text{Coker}(d_P^{a-1})$. By More on Algebra, Lemma 15.66.2 this is a flat $\mathcal{O}_{S,s}$-module. Hence by Lemma 38.2.5 this is a flat $\mathcal{O}^{sh}_{S,s}$-module. Thus $\tau_{\geq a} P^\bullet$ is a complex of flat $\mathcal{O}^{sh}_{S,s}$-modules representing M^{sh} in $D(\mathcal{O}^{sh}_{S,s}$ and we find that M^{sh} has tor amplitude in $[a, b]$, see More on Algebra, Lemma 15.66.3. Thus (1) implies (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}^h_{S,s}$. The same argument applies to prove (4) $\Rightarrow$ (5). $\square$

05FN Lemma 38.2.7. Let $g : T \to S$ be a finite flat morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $t \in T$ be a point with image $s \in S$. Then

$$t \in \text{WeakAss}(g^*\mathcal{G}) \Leftrightarrow s \in \text{WeakAss}(\mathcal{G})$$

Proof. The implication "$\Leftarrow$" follows immediately from Divisors, Lemma 31.6.4. Assume $t \in \text{WeakAss}(g^*\mathcal{G})$. Let $\text{Spec}(A) \subset S$ be an affine open neighbourhood of s. Let $\mathcal{G}$ be the quasi-coherent sheaf associated to the A-module M. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. As g is finite flat we have $g^{-1}(\text{Spec}(A)) = \text{Spec}(B)$ for some finite flat A-algebra B. Note that $g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\text{Spec}(B)}$-module associated to the B-module $M \otimes_A B$ and $g_*g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\text{Spec}(A)}$-module associated to the A-module $M \otimes_A B$. By Algebra, Lemma 10.78.5 we have $B_\mathfrak{p} \cong A_\mathfrak{p}^{\oplus n}$ for some integer $n \geq 0$. Note that $n \geq 1$ as we assumed there exists at least one point of T lying over s. Hence we see by looking at stalks that

$$s \in \text{WeakAss}(\mathcal{G}) \Leftrightarrow s \in \text{WeakAss}(g_*g^*\mathcal{G})$$

Now the assumption that $t \in \text{WeakAss}(g^*\mathcal{G})$ implies that $s \in \text{WeakAss}(g_*g^*\mathcal{G})$ by Divisors, Lemma 31.6.3 and hence by the above $s \in \text{WeakAss}(\mathcal{G})$. $\square$

05FP Lemma 38.2.8. Let $h : U \to S$ be an étale morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $u \in U$ be a point with image $s \in S$. Then

$$u \in \text{WeakAss}(h^*\mathcal{G}) \Leftrightarrow s \in \text{WeakAss}(\mathcal{G})$$

Proof. After replacing S and U by affine neighbourhoods of s and u we may assume that g is a standard étale morphism of affines, see Morphisms, Lemma 29.36.14. Thus we may assume $S = \text{Spec}(A)$ and $X = \text{Spec}(A[x, 1/g]/(f))$, where f is monic and f' is invertible in $A[x, 1/g]$. Note that $A[x, 1/g]/(f) = (A[x]/(f))_g$ is also the localization of the finite free A-algebra $A[x]/(f)$. Hence we may think of U as an open subscheme of the scheme $T = \text{Spec}(A[x]/(f))$ which is finite locally free over S. This reduces us to Lemma 38.2.7 above. $\square$

0CTU Lemma 38.2.9. Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 10.155.3. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. The following are equivalent

(1) s is a weakly associated point of $\mathcal{F}$,
(2) $\mathfrak{m}_s$ is a weakly associated prime of $\mathcal{F}_s$,
(3) $\mathfrak{m}_s^h$ is a weakly associated prime of $\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$, and
(4) $\mathfrak{m}_s^{sh}$ is a weakly associated prime of $\mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^{sh}_{S,s}$.

Proof. The equivalence of (1) and (2) is the definition, see Divisors, Definition 31.5.1. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from Divisors, Lemma 31.6.4 applied to the flat (More on Algebra, Lemma 15.45.1) morphisms

$$\text{Spec}(\mathcal{O}^{sh}_{S,s}) \to \text{Spec}(\mathcal{O}^h_{S,s}) \to \text{Spec}(\mathcal{O}_{S,s})$$

and the closed points. To prove (4) $\Rightarrow$ (2) we may replace S by an affine neighbourhood. Suppose that $x \in \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^{sh}_{S,s}$ is an element whose annihilator has radical equal to $\mathfrak{m}_s^{sh}$. (See Algebra, Lemma 10.66.2.) Since $\mathcal{O}^{sh}_{S,s}$ is equal to the limit of $\mathcal{O}_{U,u}$ over étale neighbourhoods $f : (U, u) \to (S, s)$ by Algebra, Lemma 10.155.11 we may assume that x is the image of some $x' \in \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{U,u}$. The local ring map $\mathcal{O}_{U,u} \to \mathcal{O}^{sh}_{S,s}$ is faithfully flat (as it is the strict henselization), hence universally injective (Algebra, Lemma 10.82.11). It follows that the annihilator of x' is the inverse image of the annihilator of x. Hence the radical of this annihilator is equal to $\mathfrak{m}_u$. Thus u is a weakly associated point of $f^*\mathcal{F}$. By Lemma 38.2.8 we see that s is a weakly associated point of $\mathcal{F}$. $\square$

38.3. The local structure of a finite type module

057P The key technical lemma that makes a lot of the arguments in this chapter work is the geometric Lemma 38.3.2.

057Q **Lemma 38.3.1.** Let $f : X \to S$ be a finite type morphism of affine schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$. Then there exist a closed immersion $i : Z \to X$ of finite presentation, and a quasi-coherent finite type $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} = \mathcal{F}$ and $Z_s = \text{Supp}(\mathcal{F}_s)$.

Proof. Say the morphism $f : X \to S$ is given by the ring map $A \to B$ and that $\mathcal{F}$ is the quasi-coherent sheaf associated to the B-module M. By Morphisms, Lemma 29.15.2 we know that $A \to B$ is a finite type ring map, and by Properties, Lemma 28.16.1 we know that M is a finite B-module. In particular the support of $\mathcal{F}$ is the closed subscheme of $\text{Spec}(B)$ cut out by the annihilator $I = \{x \in B \mid xm = 0 \ \forall m \in M\}$ of M, see Algebra, Lemma 10.40.5. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Note that $X_s = \text{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ and that $\mathcal{F}_s$ is the quasi-coherent sheaf associated to the $B \otimes_A \kappa(\mathfrak{p})$ module $M \otimes_A \kappa(\mathfrak{p})$. By Morphisms, Lemma 29.5.3 the support of $\mathcal{F}_s$ is equal to $V(I(B \otimes_A \kappa(\mathfrak{p})))$. Since $B \otimes_A \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$ there exist finitely many elements $f_1, \ldots, f_m \in I$ such that

$$I(B \otimes_A \kappa(\mathfrak{p})) = (f_1, \ldots, f_n)(B \otimes_A \kappa(\mathfrak{p})).$$

Denote $i : Z \to X$ the closed subscheme cut out by $(f_1, \ldots, f_m)$, in a formula $Z = \text{Spec}(B/(f_1, \ldots, f_m))$. Since M is annihilated by I we can think of M as an $B/(f_1, \ldots, f_m)$-module. In other words, $\mathcal{F}$ is the pushforward of a finite type module on Z. As $Z_s = \text{Supp}(\mathcal{F}_s)$ by construction, this proves the lemma. $\square$

057S **Lemma 38.3.2.** Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$ and $n = \dim_x(\text{Supp}(\mathcal{F}_s))$. Then we can construct

(1) elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,
(2) a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "g"] \arrow[dd] & Z' \arrow[l, "i"] \arrow[d, "\pi"] \\ & & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"] \arrow[r, equal] & S' \end{tikzcd}$$

(3) a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,
(4) a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,

such that the following properties hold

(1) X', Z', Y', S' are affine schemes,
(2) i is a closed immersion of finite presentation,
(3) $i_*(\mathcal{G}) \cong g^*\mathcal{F}$,
(4) π is finite and $\pi^{-1}(\{y'\}) = \{z'\}$,

(5) the extension $\kappa(y')/\kappa(s')$ is purely transcendental,
(6) h is smooth of relative dimension n with geometrically integral fibres.

Proof. Let $V \subset S$ be an affine neighbourhood of s. Let $U \subset f^{-1}(V)$ be an affine neighbourhood of x. Then it suffices to prove the lemma for $f|_U : U \to V$ and $\mathcal{F}|_U$. Hence in the rest of the proof we assume that X and S are affine.

First, suppose that $X_s = \text{Supp}(\mathcal{F}_s)$, in particular $n = \dim_x(X_s)$. Apply More on Morphisms, Lemmas 37.47.2 and 37.47.3. This gives us a commutative diagram

$$\begin{tikzcd}
X \arrow[dd] & X' \arrow[l, "g"'] \arrow[d, "\pi"] \\
 & Y' \arrow[d, "h"] \\
S & S' \arrow[l, "e"']
\end{tikzcd}$$

and point $x' \in X'$. We set $Z' = X'$, $i = \text{id}$, and $\mathcal{G} = g^*\mathcal{F}$ to obtain a solution in this case.

In general choose a closed immersion $Z \to X$ and a sheaf $\mathcal{G}$ on Z as in Lemma 38.3.1. Applying the result of the previous paragraph to $Z \to S$ and $\mathcal{G}$ we obtain a diagram

$$\begin{tikzcd}
X \arrow[dd, "f"'] & Z \arrow[l] \arrow[dd, "f|_Z"'] & Z' \arrow[l, "g"'] \arrow[d, "\pi"] \\
 & & Y' \arrow[d, "h"] \\
S \arrow[r, equal] & S & S' \arrow[l, "e"']
\end{tikzcd}$$

and point $z' \in Z'$ satisfying all the required properties. We will use Lemma 38.2.1 to embed Z' into a scheme étale over X. We cannot apply the lemma directly as we want X' to be a scheme over S'. Instead we consider the morphisms

$$Z' \longrightarrow Z \times_S S' \longrightarrow X \times_S S'$$

The first morphism is étale by Morphisms, Lemma 29.36.18. The second is a closed immersion as a base change of a closed immersion. Finally, as X, S, S', Z, Z' are all affine we may apply Lemma 38.2.1 to get an étale morphism of affine schemes $X' \to X \times_S S'$ such that

$$Z' = (Z \times_S S') \times_{(X \times_S S')} X' = Z \times_X X'.$$

As $Z \to X$ is a closed immersion of finite presentation, so is $Z' \to X'$. Let $x' \in X'$ be the point corresponding to $z' \in Z'$. Then the completed diagram

$$\begin{tikzcd}
X \arrow[dd] & X' \arrow[l] \arrow[dd] & Z' \arrow[l, "i"'] \arrow[d, "\pi"] \\
 & & Y' \arrow[d, "h"] \\
S & S' \arrow[l, "e"'] \arrow[r, equal] & S'
\end{tikzcd}$$

is a solution of the original problem. ☐

057T Lemma 38.3.3. Assumptions and notation as in Lemma 38.3.2. If f is locally of finite presentation then π is of finite presentation. In this case the following are equivalent

(1) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation in a neighbourhood of x,
(2) $\mathcal{G}$ is an $\mathcal{O}_{Z'}$-module of finite presentation in a neighbourhood of z', and
(3) $\pi_*\mathcal{G}$ is an $\mathcal{O}_{Y'}$-module of finite presentation in a neighbourhood of y'.

Still assuming f locally of finite presentation the following are equivalent to each other

(a) $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation,
(b) $\mathcal{G}_{z'}$ is an $\mathcal{O}_{Z',z'}$-module of finite presentation, and
(c) $(\pi_*\mathcal{G})_{y'}$ is an $\mathcal{O}_{Y',y'}$-module of finite presentation.

Proof. Assume f locally of finite presentation. Then $Z' \to S$ is locally of finite presentation as a composition of such, see Morphisms, Lemma 29.21.3. Note that $Y' \to S$ is also locally of finite presentation as a composition of a smooth and an étale morphism. Hence Morphisms, Lemma 29.21.11 implies π is locally of finite presentation. Since π is finite we conclude that it is also separated and quasi-compact, hence π is actually of finite presentation.

To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is a $\mathcal{O}_{X'}$-module of finite presentation in a neighbourhood of x'. The pullback of a module of finite presentation is of finite presentation, see Modules, Lemma 17.11.4. Hence (1) $\Rightarrow$ (4). The étale morphism g is open, see Morphisms, Lemma 29.36.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $\{U' \to g(U')\}$ is an étale covering. Thus (4) $\Rightarrow$ (1) by Descent, Lemma 35.7.3. Using Descent, Lemma 35.7.10 and some easy topological arguments (see More on Morphisms, Lemma 37.47.4) we see that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the ring maps

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{Z',z'} \leftarrow \mathcal{O}_{Y',y'}$$

The first ring map is faithfully flat. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ if and only if $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$, see Algebra, Lemma 10.83.2. The second ring map is surjective (hence finite) and finitely presented by assumption, hence $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$ if and only if $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$, see Algebra, Lemma 10.36.23. Because π is finite, of finite presentation, and $\pi^{-1}(\{y'\}) = \{x'\}$ the ring homomorphism $\mathcal{O}_{Y',y'} \leftarrow \mathcal{O}_{Z',z'}$ is finite and of finite presentation, see More on Morphisms, Lemma 37.47.4. Hence $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$ if and only if $\pi_*\mathcal{G}_{y'}$ is of finite presentation over $\mathcal{O}_{Y',y'}$, see Algebra, Lemma 10.36.23. ☐

057U Lemma 38.3.4. Assumptions and notation as in Lemma 38.3.2. The following are equivalent

(1) $\mathcal{F}$ is flat over S in a neighbourhood of x,
(2) $\mathcal{G}$ is flat over S' in a neighbourhood of z', and
(3) $\pi_*\mathcal{G}$ is flat over S' in a neighbourhood of y'.

The following are equivalent also

(a) $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$,

(b) $\mathcal{G}_{z'}$ is flat over $\mathcal{O}_{S',s'}$, and
(c) $(\pi_*\mathcal{G})_{y'}$ is flat over $\mathcal{O}_{S',s'}$.

Proof. To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is flat over S in a neighbourhood of x'. We will use Lemma 38.2.3 to equate flatness over S and S' without further mention. The étale morphism g is flat and open, see Morphisms, Lemma 29.36.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $U' \to g(U')$ is surjective and flat. Thus (4) $\Leftrightarrow$ (1) by Morphisms, Lemma 29.25.13. Note that

$$\Gamma(X', g^*\mathcal{F}) = \Gamma(Z', \mathcal{G}) = \Gamma(Y', \pi_*\mathcal{G})$$

Hence the flatness of $g^*\mathcal{F}$, $\mathcal{G}$ and $\pi_*\mathcal{G}$ over S' are all equivalent (this uses that X', Z', Y', and S' are all affine). Some omitted topological arguments (compare More on Morphisms, Lemma 37.47.4) regarding affine neighbourhoods now show that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the commutative diagram of local ring maps

$$\begin{tikzcd}
\mathcal{O}_{X',x'} \arrow[r, "\iota"'] & \mathcal{O}_{Z',z'} & \mathcal{O}_{Y',y'} \arrow[l, "\alpha"'] & \mathcal{O}_{S',s'} \arrow[l, "\beta"'] \\
\mathcal{O}_{X,x} \arrow[u, "\gamma"] & & & \mathcal{O}_{S,s} \arrow[lll, "\varphi"'] \arrow[u, "\epsilon"']
\end{tikzcd}$$

We will use Lemma 38.2.4 to equate flatness over $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$ without further mention. The map γ is faithfully flat. Hence $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$ if and only if $g^*\mathcal{F}_{x'}$ is flat over $\mathcal{O}_{S',s'}$, see Algebra, Lemma 10.39.9. As $\mathcal{O}_{S',s'}$-modules the modules $g^*\mathcal{F}_{x'}$, $\mathcal{G}_{z'}$, and $\pi_*\mathcal{G}_{y'}$ are all isomorphic, see More on Morphisms, Lemma 37.47.4. This finishes the proof. $\square$

38.4. One step dévissage

05H3 In this section we explain what is a one step dévissage of a module. A one step dévissage exist étale locally on base and target. We discuss base change, Zariski shrinking and étale localization of a one step dévissage.

05H4 Definition 38.4.1. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A one step dévissage of $\mathcal{F}/X/S$ over s is given by morphisms of schemes over S

$$X \xleftarrow{i} Z \xrightarrow{\pi} Y$$

and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type such that

(1) X, S, Z and Y are affine,
(2) i is a closed immersion of finite presentation,
(3) $\mathcal{F} \cong i_*\mathcal{G}$,
(4) π is finite, and
(5) the structure morphism $Y \to S$ is smooth with geometrically irreducible fibres of dimension $\dim(\text{Supp}(\mathcal{F}_s))$.

In this case we say $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s.

Note that such a one step dévissage can only exist if X and S are affine. In the definition above we only require X to be (locally) of finite type over S and we continue working in this setting below. In [GR71] the authors use consistently the setup where $X \to S$ is locally of finite presentation and $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module of finite type. The advantage of this choice is that it "makes sense" to ask for $\mathcal{F}$ to be of finite presentation as an $\mathcal{O}_X$-module, whereas in our setting it "does not make sense". Please see More on Morphisms, Section 37.58 for a discussion; the observations made there show that in our setup we may consider the condition of $\mathcal{F}$ being "locally of finite presentation relative to S", and we could work consistently with this notion. Instead however, we will rely on the results of Lemma 38.3.3 and the observations in Remark 38.6.3 to deal with this issue in an ad hoc fashion whenever it comes up.

05H5 Definition 38.4.2. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image s in S. A one step dévissage of $\mathcal{F}/X/S$ at x is a system $(Z, Y, i, \pi, \mathcal{G}, z, y)$, where $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s and

(1) $\dim_x(\text{Supp}(\mathcal{F}_s)) = \dim(\text{Supp}(\mathcal{F}_s))$,
(2) $z \in Z$ is a point with $i(z) = x$ and $\pi(z) = y$,
(3) we have $\pi^{-1}(\{y\}) = \{z\}$,
(4) the extension $\kappa(y)/\kappa(s)$ is purely transcendental.

A one step dévissage of $\mathcal{F}/X/S$ at x can only exist if X and S are affine. Condition (1) assures us that $Y \to S$ has relative dimension equal to $\dim_x(\text{Supp}(\mathcal{F}_s))$ via condition (5) of Definition 38.4.1.

05H6 Lemma 38.4.3. Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Then there exists a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d, "f"'] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

such that $(S', s') \to (S, s)$ and $(X', x') \to (X, x)$ are elementary étale neighbourhoods, and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'.

Proof. This is immediate from Definition 38.4.2 and Lemma 38.3.2. □

05H7 Lemma 38.4.4. Let S, X, $\mathcal{F}$, s be as in Definition 38.4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Given this data let X', Z', Y', i', π' be the base changes of X, Z, Y, i, π via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'$ be the pullback of $\mathcal{G}$ to Z'. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'.

Proof. Fibre products of affines are affine, see Schemes, Lemma 26.17.2. Base change preserves closed immersions, morphisms of finite presentation, finite morphisms, smooth morphisms, morphisms with geometrically irreducible fibres, and morphisms of relative dimension n, see Morphisms, Lemmas 29.2.4, 29.21.4, 29.44.6, 29.34.5, 29.29.2, and More on Morphisms, Lemma 37.27.2. We have $i'_*\mathcal{G}' \cong \mathcal{F}'$ because pushforward along the finite morphism i commutes with base change, see Cohomology of Schemes, Lemma 30.5.1. We have $\dim(\text{Supp}(\mathcal{F}_s)) = \dim(\text{Supp}(\mathcal{F}'_{s'}))$ by Morphisms, Lemma 29.28.3 because

$$\text{Supp}(\mathcal{F}_s) \times_s s' = \text{Supp}(\mathcal{F}'_{s'}).$$

This proves the lemma. □

05H8 Lemma 38.4.5. Let S, X, $\mathcal{F}$, x, s be as in Definition 38.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z', Y', i', \pi', \mathcal{G}')$ be as constructed in Lemma 38.4.4 and let $x' \in X'$ (resp. $z' \in Z'$, $y' \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z \in Z$, $y \in Y$) and $s' \in S'$. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}', z', y')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ at x'.

Proof. By Lemma 38.4.4 $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'. Properties (1) – (4) of Definition 38.4.2 hold for $(Z', Y', i', \pi', \mathcal{G}', z', y')$ as the assumption that $\kappa(s) = \kappa(s')$ insures that the fibres $X'_{s'}$, $Z'_{s'}$, and $Y'_{s'}$ are isomorphic to X_s, Z_s, and Y_s. □

05H9 Definition 38.4.6. Let S, X, $\mathcal{F}$, x, s be as in Definition 38.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let us define a standard shrinking of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z' \subset Z$, and $Y' \subset Y$ such that $s \in S'$, $x \in X'$, $z \in Z'$, and $y \in Y'$ and such that

$$(Z', Y', i|_{Z'}, \pi|_{Z'}, \mathcal{G}|_{Z'}, z, y)$$

is a one step dévissage of $\mathcal{F}|_{X'}/X'/S'$ at x.

05HA Lemma 38.4.7. With assumption and notation as in Definition 38.4.6 we have:

05HB (1) If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z' = Z_{S'}$ and $Y' = Y_{S'}$ we obtain a standard shrinking.

05HC (2) Let $W \subset Y$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y' = W \times_S S'$.

05HD (3) Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.

Proof. Part (1) is immediate from Lemma 38.4.5 and the fact that the inverse image of a standard open under a morphism of affine schemes is a standard open, see Algebra, Lemma 10.17.4.

Let $W \subset Y$ as in (2). Because $Y \to S$ is smooth it is open, see Morphisms, Lemma 29.34.10. Hence we can find a standard open neighbourhood S' of s contained in the image of W. Then the fibres of $W_{S'} \to S'$ are nonempty open subschemes of the fibres of $Y \to S$ over S' and hence geometrically irreducible too. Setting $Y' = W_{S'}$ and $Z' = \pi^{-1}(Y')$ we see that $Z' \subset Z$ is a standard open neighbourhood of z. Let $\overline{h} \in \Gamma(Z, \mathcal{O}_Z)$ be a function such that $Z' = D(\overline{h})$. As $i : Z \to X$ is a closed immersion, we can find a function $h \in \Gamma(X, \mathcal{O}_X)$ such that $i^\sharp(h) = \overline{h}$. Take $X' = D(h) \subset X$. In this way we obtain a standard shrinking as in (2).

Let $U \subset X$ be as in (3). We may after shrinking U assume that U is a standard open. By More on Morphisms, Lemma 37.47.4 there exists a standard open $W \subset Y$ neighbourhood of y such that $\pi^{-1}(W) \subset i^{-1}(U)$. Apply (2) to get a standard shrinking X', S', Z', Y' with $Y' = W_{S'}$. Since $Z' \subset \pi^{-1}(W) \subset i^{-1}(U)$ we may replace X' by $X' \cap U$ (still a standard open as U is also standard open) without violating any of the conditions defining a standard shrinking. Hence we win. □

05HE Lemma 38.4.8. Let S, X, $\mathcal{F}$, x, s be as in Definition 38.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let

$$\begin{tikzcd} (Y, y) \arrow[d] & (Y', y') \arrow[l] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

be a commutative diagram of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods. Then there exists a commutative diagram

$$\begin{tikzcd} & & (X'', x'') \arrow[dll] \arrow[d] & (Z'', z'') \arrow[l] \arrow[dll] \arrow[d] \\ (X, x) \arrow[d] & (Z, z) \arrow[l] \arrow[d] & (S'', s'') \arrow[dll] & (Y'', y'') \arrow[l] \arrow[dll] \\ (S, s) & (Y, y) \arrow[l] & & \end{tikzcd}$$

of pointed schemes with the following properties:

(1) $(S'', s'') \to (S', s')$ is an elementary étale neighbourhood and the morphism $S'' \to S$ is the composition $S'' \to S' \to S$,
(2) Y'' is an open subscheme of $Y' \times_{S'} S''$,
(3) $Z'' = Z \times_Y Y''$,
(4) $(X'', x'') \to (X, x)$ is an elementary étale neighbourhood, and
(5) $(Z'', Y'', i'', \pi'', \mathcal{G}'', z'', y'')$ is a one step dévissage at x'' of the sheaf $\mathcal{F}''$.

Here $\mathcal{F}''$ (resp. $\mathcal{G}''$) is the pullback of $\mathcal{F}$ (resp. $\mathcal{G}$) via the morphism $X'' \to X$ (resp. $Z'' \to Z$) and $i'' : Z'' \to X''$ and $\pi'' : Z'' \to Y''$ are as in the diagram.

Proof. Let $(S'', s'') \to (S', s')$ be any elementary étale neighbourhood with S'' affine. Let $Y'' \subset Y' \times_{S'} S''$ be any affine open neighbourhood containing the point $y'' = (y', s'')$. Then we obtain an affine (Z'', z'') by (3). Moreover $Z_{S''} \to X_{S''}$ is a closed immersion and $Z'' \to Z_{S''}$ is an étale morphism. Hence Lemma 38.2.1 applies and we can find an étale morphism $X'' \to X_{S'}$ of affines such that $Z'' \cong X'' \times_{X_{S'}} Z_{S'}$. Denote $i'' : Z'' \to X''$ the corresponding closed immersion. Setting $x'' = i''(z'')$ we obtain a commutative diagram as in the lemma. Properties (1), (2), (3), and (4) hold by construction. Thus it suffices to show that (5) holds for a suitable choice of $(S'', s'') \to (S', s')$ and Y''.

We first list those properties which hold for any choice of $(S'', s'') \to (S', s')$ and Y'' as in the first paragraph. As we have $Z'' = X'' \times_X Z$ by construction we see that $i''_*\mathcal{G}'' = \mathcal{F}''$ (with notation as in the statement of the lemma), see Cohomology of Schemes, Lemma 30.5.1. Set $n = \dim(\text{Supp}(\mathcal{F}_s)) = \dim_x(\text{Supp}(\mathcal{F}_s))$. The morphism $Y'' \to S''$ is smooth of relative dimension n (because $Y' \to S'$ is smooth of relative dimension n as the composition $Y' \to Y_{S'} \to S'$ of an étale and smooth morphism of relative dimension n and because base change preserves smooth morphisms of relative dimension n). We have $\kappa(y'') = \kappa(y)$ and $\kappa(s) = \kappa(s'')$ hence $\kappa(y'')$ is a purely transcendental extension of $\kappa(s'')$. The morphism of fibres $X''_{s''} \to X_s$ is an étale morphism of affine schemes over $\kappa(s) = \kappa(s'')$ mapping the point x'' to the point x and pulling back $\mathcal{F}_s$ to $\mathcal{F}''_{s''}$. Hence

$$\dim(\text{Supp}(\mathcal{F}''_{s''})) = \dim(\text{Supp}(\mathcal{F}_s)) = n = \dim_x(\text{Supp}(\mathcal{F}_s)) = \dim_{x''}(\text{Supp}(\mathcal{F}''_{s''}))$$

because dimension is invariant under étale localization, see Descent, Lemma 35.21.2. As $\pi'' : Z'' \to Y''$ is the base change of π we see that π'' is finite and as $\kappa(y) = \kappa(y'')$ we see that $\pi^{-1}(\{y''\}) = \{z''\}$.

At this point we have verified all the conditions of Definition 38.4.1 except we have not verified that $Y'' \to S''$ has geometrically irreducible fibres. Of course in general this is not going to be true, and it is at this point that we will use that $\kappa(s) \subset \kappa(y)$ is purely transcendental. Namely, let $T \subset Y'_{s'}$ be the irreducible component of $Y'_{s'}$ containing $y' = (y, s')$. Note that T is an open subscheme of $Y'_{s'}$ as this is a smooth scheme over $\kappa(s')$. By Varieties, Lemma 33.7.14 we see that T is geometrically connected because $\kappa(s') = \kappa(s)$ is algebraically closed in $\kappa(y') = \kappa(y)$. As T is smooth we see that T is geometrically irreducible. Hence More on Morphisms, Lemma 37.46.4 applies and we can find an elementary étale morphism $(S'', s'') \to (S', s')$ and an affine open $Y'' \subset Y'_{S''}$ such that all fibres of $Y'' \to S''$ are geometrically irreducible and such that $T = Y''_{s''}$. After shrinking (first Y'' and then S'') we may assume that both Y'' and S'' are affine. This finishes the proof of the lemma. □

05HF Lemma 38.4.9. Let S, X, $\mathcal{F}$, s be as in Definition 38.4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $\xi \in Y_s$ be the (unique) generic point. Then there exists an integer $r > 0$ and an $\mathcal{O}_Y$-module map

$$\alpha : \mathcal{O}_Y^{\oplus r} \longrightarrow \pi_*\mathcal{G}$$

such that

$$\alpha : \kappa(\xi)^{\oplus r} \longrightarrow (\pi_*\mathcal{G})_\xi \otimes_{\mathcal{O}_{Y,\xi}} \kappa(\xi)$$

is an isomorphism. Moreover, in this case we have

$$\dim(\text{Supp}(\text{Coker}(\alpha)_s)) < \dim(\text{Supp}(\mathcal{F}_s)).$$

Proof. By assumption the schemes S and Y are affine. Write $S = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. As π is finite the $\mathcal{O}_Y$-module $\pi_*\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_Y$-module. Hence $\pi_*\mathcal{G} = \widetilde{N}$ for some finite B-module N. Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to ξ. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_B \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_B \kappa(\mathfrak{p})$. Take $\alpha : B^{\oplus r} \to N$ to be the map given by the formula $\alpha(b_1, \ldots, b_r) = \sum b_i x_i$. It is clear that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism as desired. Finally, suppose α is any map with this property. Then $N' = \text{Coker}(\alpha)$ is a finite B-module such that $N' \otimes \kappa(\mathfrak{p}) = 0$. By Nakayama's lemma (Algebra, Lemma 10.20.1) we see that $N'_{\mathfrak{p}} = 0$. Since the fibre Y_s is geometrically irreducible of dimension n with generic point ξ and since we have just seen that ξ is not in the support of $\text{Coker}(\alpha)$ the last assertion of the lemma holds. □

38.5. Complete dévissage

05HG In this section we explain what is a complete dévissage of a module and prove that such exist. The material in this section is mainly bookkeeping.

05HH Definition 38.5.1. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A complete dévissage of $\mathcal{F}/X/S$ over s is given by a diagram

$$\begin{tikzcd}
X & Z_1 \arrow[l, "i_1"'] \arrow[d, "\pi_1"] & & & \\
& Y_1 & Z_2 \arrow[l, "i_2"'] \arrow[d, "\pi_2"] & & \\
& & Y_2 & Z_3 \arrow[l] \arrow[d] & \\
& & & \ldots & \ldots \arrow[l] \arrow[d] \\
& & & & Y_n
\end{tikzcd}$$

of schemes over S, finite type quasi-coherent $\mathcal{O}_{Z_k}$-modules $\mathcal{G}_k$, and $\mathcal{O}_{Y_k}$-module maps

$$\alpha_k : \mathcal{O}_{Y_k}^{\oplus r_k} \longrightarrow \pi_{k,*}\mathcal{G}_k, \quad k = 1, \ldots, n$$

satisfying the following properties:

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ is a one step dévissage of $\mathcal{F}/X/S$ over s,
(2) the map α_k induces an isomorphism

$$\kappa(\xi_k)^{\oplus r_k} \longrightarrow (\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k)$$

where $\xi_k \in (Y_k)_s$ is the unique generic point,
(3) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ over s,
(4) $\mathrm{Coker}(\alpha_n) = 0$.

In this case we say that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}/X/S$ over s.

05HI Definition 38.5.2. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image $s \in S$. A complete dévissage of $\mathcal{F}/X/S$ at x is given by a system

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$$

such that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)$ is a complete dévissage of $\mathcal{F}/X/S$ over s, and such that

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1, z_1, y_1)$ is a one step dévissage of $\mathcal{F}/X/S$ at x,
(2) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, z_k, y_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ at y_{k-1}.

Again we remark that a complete dévissage can only exist if X and S are affine.

05HJ Lemma 38.5.3. Let S, X, $\mathcal{F}$, s be as in Definition 38.5.1. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s. Given this data let $X', Z'_k, Y'_k, i'_k, \pi'_k$ be the base changes of X, Z_k, Y_k, i_k, π_k via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'_k$ be the pullback of $\mathcal{G}_k$ to Z'_k. Let α'_k be the pullback of α_k to Y'_k. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ over s'.

Proof. By Lemma 38.4.4 we know that the base change of a one step dévissage is a one step dévissage. Hence it suffices to prove that formation of $\mathrm{Coker}(\alpha_k)$ commutes with base change and that condition (2) of Definition 38.5.1 is preserved by base change. The first is true as $\pi'_{k,*}\mathcal{G}'_k$ is the pullback of $\pi_{k,*}\mathcal{G}_k$ (by Cohomology of Schemes, Lemma 30.5.1) and because $\otimes$ is right exact. The second because by the same token we have

$$(\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k) \otimes_{\kappa(\xi_k)} \kappa(\xi'_k) \cong (\pi'_{k,*}\mathcal{G}'_k)_{\xi'_k} \otimes_{\mathcal{O}_{Y'_k,\xi'_k}} \kappa(\xi'_k)$$

with obvious notation. □

05HK Lemma 38.5.4. Let S, X, $\mathcal{F}$, x, s be as in Definition 38.5.2. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ at x. Let $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ be as constructed in Lemma 38.5.3 and let $x' \in X'$ (resp. $z'_k \in Z'$, $y'_k \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z_k \in Z_k$, $y_k \in Y_k$) and $s' \in S'$. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z'_k, y'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ at x'.

Proof. Combine Lemma 38.5.3 and Lemma 38.4.5. □

05HL Definition 38.5.5. Let S, X, $\mathcal{F}$, x, s be as in Definition 38.5.2. Consider a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ of $\mathcal{F}/X/S$ at x. Let us define a standard shrinking of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z'_k \subset Z_k$, and $Y'_k \subset Y_k$ such that $s_k \in S'$, $x_k \in X'$, $z_k \in Z'$, and $y_k \in Y'$ and such that

$$(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z_k, y_k)_{k=1,\ldots,n}$$

is a one step dévissage of $\mathcal{F}'/X'/S'$ at x where $\mathcal{G}'_k = \mathcal{G}_k|_{Z'_k}$ and $\mathcal{F}' = \mathcal{F}|_{X'}$.

05HM Lemma 38.5.6. With assumption and notation as in Definition 38.5.5 we have:

05HN (1) If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z'_k = Z_{S'}$ and $Y'_k = Y_{S'}$ we obtain a standard shrinking.

05HP (2) Let $W \subset Y_n$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y'_n = W \times_S S'$.

05HQ (3) Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.

Proof. Part (1) is immediate from Lemmas 38.5.4 and 38.4.7.

Proof of (2). For convenience denote $X = Y_0$. We apply Lemma 38.4.7 (2) to find a standard shrinking S', Y'_{n-1}, Z'_n, Y'_n of the one step dévissage of $\mathrm{Coker}(\alpha_{n-1})/Y_{n-1}/S$ at y_{n-1} with $Y'_n = W \times_S S'$. We may repeat this procedure and find a standard shrinking $S'', Y''_{n-2}, Z''_{n-1}, Y''_{n-1}$ of the one step dévissage of $\mathrm{Coker}(\alpha_{n-2})/Y_{n-2}/S$ at y_{n-2} with $Y''_{n-1} = Y'_{n-1} \times_S S''$. We may continue in this manner until we obtain $S^{(n)}, Y_0^{(n)}, Z_1^{(n)}, Y_1^{(n)}$. At this point it is clear that we obtain our desired standard shrinking by taking $S^{(n)}$, $X^{(n)}$, $Z_k^{(n-k)} \times_S S^{(n)}$, and $Y_k^{(n-k)} \times_S S^{(n)}$ with the desired property.

Proof of (3). We use induction on the length of the complete dévissage. First we apply Lemma 38.4.7 (3) to find a standard shrinking S', X', Z'_1, Y'_1 of the one step dévissage of $\mathcal{F}/X/S$ at x with $X' \subset U$. If $n = 1$, then we are done. If $n > 1$, then by induction we can find a standard shrinking S'', Y_1'', Z_k'', and Y_k'' of the complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathrm{Coker}(\alpha_1)/Y_1/S$ at x such that $Y_1'' \subset Y_1'$. Using Lemma 38.4.7 (2) we can find $S''' \subset S'$, $X''' \subset X'$, Z_1''' and $Y_1''' = Y_1'' \times_S S'''$ which is a standard shrinking. The solution to our problem is to take

$$S''', X''', Z_1''', Y_1''', Z_2'' \times_S S''', Y_2'' \times_S S''', \ldots, Z_n'' \times_S S''', Y_n'' \times_S S'''$$

This ends the proof of the lemma. □

05HR Proposition 38.5.7. Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ be a point with image $s \in S$. There exists a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x.

Proof. We prove this by induction on the integer $d = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$. By Lemma 38.4.3 there exists a diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'. The local nature of the problem implies that we may replace $(X, x) \to (S, s)$ by $(X', x') \to (S', s')$. Thus after doing so we may assume that there exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x.

We apply Lemma 38.4.9 to find a map

$$\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \longrightarrow \pi_{1,*}\mathcal{G}_1$$

which induces an isomorphism of vector spaces over $\kappa(\xi_1)$ where $\xi_1 \in Y_1$ is the unique generic point of the fibre of Y_1 over s. Moreover $\dim_{y_1}(\mathrm{Supp}(\mathrm{Coker}(\alpha_1)_s)) < d$. It may happen that the stalk of $\mathrm{Coker}(\alpha_1)_s$ at y_1 is zero. In this case we may shrink Y_1 by Lemma 38.4.7 (2) and assume that $\mathrm{Coker}(\alpha_1) = 0$ so we obtain a complete dévissage of length zero.

Assume now that the stalk of $\mathrm{Coker}(\alpha_1)_s$ at y_1 is not zero. In this case, by induction, there exists a commutative diagram

05HS (38.5.7.1)
$$\begin{tikzcd} (Y_1, y_1) \arrow[d] & (Y_1', y_1') \arrow[l, "h"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $h^* \operatorname{Coker}(\alpha_1)/Y_1'/S'$ has a complete dévissage

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$$

at y_1'. (In particular $i_2 : Z_2 \to Y_1'$ is a closed immersion into Y_2'.) At this point we apply Lemma 38.4.8 to $S, X, \mathcal{F}, x, s$, the system $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ and diagram (38.5.7.1). We obtain a diagram

$$\begin{tikzcd}
 & & (X'', x'') \arrow[dll] \arrow[d] & (Z_1'', z_1'') \arrow[l] \arrow[dll] \arrow[d] \\
(X, x) \arrow[d] & (Z_1, z_1) \arrow[l] \arrow[d] & (S'', s'') \arrow[dll] & (Y_1'', y_1'') \arrow[l] \arrow[dll] \\
(S, s) & (Y_1, y_1) \arrow[l] & &
\end{tikzcd}$$

with all the properties as listed in the referenced lemma. In particular $Y_1'' \subset Y_1' \times_{S'} S''$. Set $X_1 = Y_1' \times_{S'} S''$ and let $\mathcal{F}_1$ denote the pullback of $\operatorname{Coker}(\alpha_1)$. By Lemma 38.5.4 the system

05HT (38.5.7.2) $$(Z_k \times_{S'} S'', Y_k \times_{S'} S'', i_k'', \pi_k'', \mathcal{G}_k'', \alpha_k'', z_k'', y_k'')_{k=2,\ldots,n}$$

is a complete dévissage of $\mathcal{F}_1$ to X_1. Again, the nature of the problem allows us to replace $(X, x) \to (S, s)$ by $(X'', x'') \to (S'', s'')$. In this we see that we may assume:

(a) There exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x,
(b) there exists an $\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \to \pi_{1,*}\mathcal{G}_1$ such that $\alpha \otimes \kappa(\xi_1)$ is an isomorphism,
(c) $Y_1 \subset X_1$ is open, $y_1 = x_1$, and $\mathcal{F}_1|_{Y_1} \cong \operatorname{Coker}(\alpha_1)$, and
(d) there exists a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathcal{F}_1/X_1/S$ at x_1.

To finish the proof all we have to do is shrink the one step dévissage and the complete dévissage such that they fit together to a complete dévissage. (We suggest the reader do this on their own using Lemmas 38.4.7 and 38.5.6 instead of reading the proof that follows.) Since $Y_1 \subset X_1$ is an open neighbourhood of x_1 we may apply Lemma 38.5.6 (3) to find a standard shrinking $S', X_1', Z_2', Y_2', \ldots, Y_n'$ of the datum (d) so that $X_1' \subset Y_1$. Note that X_1' is also a standard open of the affine scheme Y_1. Next, we shrink the datum (a) as follows: first we shrink the base S to S', see Lemma 38.4.7 (1) and then we shrink the result to S'', X'', Z_1'', Y_1'' using Lemma 38.4.7 (2) such that eventually $Y_1'' = X_1' \times_S S''$ and $S'' \subset S'$. Then we see that

$$Z_1'', Y_1'', Z_2' \times_{S'} S'', Y_2' \times_{S'} S'', \ldots, Y_n' \times_{S'} S''$$

gives the complete dévissage we were looking for. □

Some more bookkeeping gives the following consequence.

05HU Lemma 38.5.8. Let $X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point. There exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and étale morphisms $h_i : Y_i \to X_{S'}$, $i = 1, \ldots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/Y_i/S'$ over s', where $\mathcal{F}_i$ is the pullback of $\mathcal{F}$ to Y_i and such that $X_s = (X_{S'})_{s'} \subset \bigcup h_i(Y_i)$.

Proof. For every point $x \in X_s$ we can find a diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x'. As $X \to S$ is of finite type the fibre X_s is quasi-compact, and since each $g : X' \to X$ as above is open we can cover X_s by a finite union of $g(X'_{s'})$. Thus we can find a finite family of such diagrams

$$\begin{tikzcd} (X, x) \arrow[d] & (X'_i, x'_i) \arrow[l, "g_i"'] \arrow[d] \\ (S, s) & (S'_i, s'_i) \arrow[l] \end{tikzcd} \quad i = 1, \ldots, n$$

such that $X_s = \bigcup g_i(X'_i)$. Set $S' = S'_1 \times_S \ldots \times_S S'_n$ and let $Y_i = X_i \times_{S'_i} S'$ be the base change of X'_i to S'. By Lemma 38.5.3 we see that the pullback of $\mathcal{F}$ to Y_i has a complete dévissage over s and we win. □

38.6. Translation into algebra

05HV It may be useful to spell out algebraically what it means to have a complete dévissage. We introduce the following notion (which is not that useful so we give it an impossibly long name).

05HW Definition 38.6.1. Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. A elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$ is given by a commutative diagram of rings and accompanying primes

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd}[arrows=dash] \mathfrak{q} \arrow[r] \arrow[d] & \mathfrak{q}' \arrow[d] \\ \mathfrak{p} \arrow[r] & \mathfrak{p}' \end{tikzcd}$$

such that $R \to R'$ and $S \to S'$ are étale ring maps and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$ and $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}')$.

05HX Definition 38.6.2. Let $R \to S$ be a finite type ring map. Let $\mathfrak{r}$ be a prime of R. Let N be a finite S-module. A complete dévissage of $N/S/R$ over $\mathfrak{r}$ is given by R-algebra maps

$$\begin{tikzcd}[column sep=small] & A_1 & & A_2 & & \ldots & & A_n & \\ S \arrow[ur] & & B_1 \arrow[ul] \arrow[ur] & & \ldots \arrow[ul] \arrow[ur] & & \ldots \arrow[ul] \arrow[ur] & & B_n \arrow[ul] \end{tikzcd}$$

finite A_i-modules M_i and B_i-module maps $\alpha_i : B_i^{\oplus r_i} \to M_i$ such that

(1) $S \to A_1$ is surjective and of finite presentation,
(2) $B_i \to A_{i+1}$ is surjective and of finite presentation,
(3) $B_i \to A_i$ is finite,
(4) $R \to B_i$ is smooth with geometrically irreducible fibres,

(5) $N \cong M_1$ as S-modules,
(6) $\text{Coker}(\alpha_i) \cong M_{i+1}$ as B_i-modules,
(7) $\alpha_i : \kappa(\mathfrak{p}_i)^{\oplus r_i} \to M_i \otimes_{B_i} \kappa(\mathfrak{p}_i)$ is an isomorphism where $\mathfrak{p}_i = \mathfrak{r}B_i$, and
(8) $\text{Coker}(\alpha_n) = 0$.

In this situation we say that $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ is a complete dévissage of $N/S/R$ over $\mathfrak{r}$.

05HY Remark 38.6.3. Note that the R-algebras B_i for all i and A_i for $i \geq 2$ are of finite presentation over R. If S is of finite presentation over R, then it is also the case that A_1 is of finite presentation over R. In this case all the ring maps in the complete dévissage are of finite presentation. See Algebra, Lemma 10.6.2. Still assuming S of finite presentation over R the following are equivalent

(1) M is of finite presentation over S,
(2) M_1 is of finite presentation over A_1,
(3) M_1 is of finite presentation over B_1,
(4) each M_i is of finite presentation both as an A_i-module and as a B_i-module.

The equivalences (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) follow from Algebra, Lemma 10.36.23. If M_1 is finitely presented, so is $\text{Coker}(\alpha_1)$ (see Algebra, Lemma 10.5.3) and hence M_2, etc.

05HZ Definition 38.6.4. Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{r}$ of R. Let N be a finite S-module. A complete dévissage of $N/S/R$ at $\mathfrak{q}$ is given by a complete dévissage $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ of $N/S/R$ over $\mathfrak{r}$ and prime ideals $\mathfrak{q}_i \subset B_i$ lying over $\mathfrak{r}$ such that

(1) $\kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q}_i)$ is purely transcendental,
(2) there is a unique prime $\mathfrak{q}'_i \subset A_i$ lying over $\mathfrak{q}_i \subset B_i$,
(3) $\mathfrak{q} = \mathfrak{q}'_1 \cap S$ and $\mathfrak{q}_i = \mathfrak{q}'_{i+1} \cap A_i$,
(4) $R \to B_i$ has relative dimension $\dim_{\mathfrak{q}_i}(\text{Supp}(M_i \otimes_R \kappa(\mathfrak{r})))$.

05I0 Remark 38.6.5. Let $A \to B$ be a finite type ring map and let N be a finite B-module. Let $\mathfrak{q}$ be a prime of B lying over the prime $\mathfrak{r}$ of A. Set $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and $\mathcal{F} = \widetilde{N}$ on X. Let x be the point corresponding to $\mathfrak{q}$ and let $s \in S$ be the point corresponding to $\mathfrak{p}$. Then

(1) if there exists a complete dévissage of $\mathcal{F}/X/S$ over s then there exists a complete dévissage of $N/B/A$ over $\mathfrak{p}$, and
(2) there exists a complete dévissage of $\mathcal{F}/X/S$ at x if and only if there exists a complete dévissage of $N/B/A$ at $\mathfrak{q}$.

There is just a small twist in that we omitted the condition on the relative dimension in the formulation of "a complete dévissage of $N/B/A$ over $\mathfrak{p}$" which is why the implication in (1) only goes in one direction. The notion of a complete dévissage at $\mathfrak{q}$ does have this condition built in. In any case we will only use that existence for $\mathcal{F}/X/S$ implies the existence for $N/B/A$.

05I1 Lemma 38.6.6. Let $R \to S$ be a finite type ring map. Let M be a finite S-module. Let $\mathfrak{q}$ be a prime ideal of S. There exists an elementary étale localization $R' \to S', \mathfrak{q}', \mathfrak{p}'$ of the ring map $R \to S$ at $\mathfrak{q}$ such that there exists a complete dévissage of $(M \otimes_S S')/S'/R'$ at $\mathfrak{q}'$.

Proof. This is a reformulation of Proposition 38.5.7 via Remark 38.6.5 $\square$

38.7. Localization and universally injective maps

05DD

05DE **Lemma 38.7.1.** Let $R \to S$ be a ring map. Let N be a S-module. Assume

1. R is a local ring with maximal ideal $\mathfrak{m}$,
2. $\overline{S} = S/\mathfrak{m}S$ is Noetherian, and
3. $\overline{N} = N/\mathfrak{m}_R N$ is a finite $\overline{S}$-module.

Let $\Sigma \subset S$ be the multiplicative subset of elements which are not a zerodivisor on $\overline{N}$. Then $\Sigma^{-1}S$ is a semi-local ring whose spectrum consists of primes $\mathfrak{q} \subset S$ contained in an element of $\text{Ass}_S(\overline{N})$. Moreover, any maximal ideal of $\Sigma^{-1}S$ corresponds to an associated prime of $\overline{N}$ over $\overline{S}$.

Proof. Note that $\text{Ass}_S(\overline{N}) = \text{Ass}_{\overline{S}}(\overline{N})$, see Algebra, Lemma 10.63.14. This is a finite set by Algebra, Lemma 10.63.5. Say $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\} = \text{Ass}_S(\overline{N})$. We have $\Sigma = S \setminus (\bigcup \mathfrak{q}_i)$ by Algebra, Lemma 10.63.9. By the description of $\text{Spec}(\Sigma^{-1}S)$ in Algebra, Lemma 10.17.5 and by Algebra, Lemma 10.15.2 we see that the primes of $\Sigma^{-1}S$ correspond to the primes of S contained in one of the $\mathfrak{q}_i$. Hence the maximal ideals of $\Sigma^{-1}S$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. This proves the lemma. $\square$

05DF **Lemma 38.7.2.** Assumption and notation as in Lemma 38.7.1. Assume moreover that

1. S is local and $R \to S$ is a local homomorphism,
2. S is essentially of finite presentation over R,
3. N is finitely presented over S, and
4. N is flat over R.

Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.

Proof. By Algebra, Lemma 10.128.4 the sequence $0 \to N \to N \to N/sN \to 0$ is exact and N/sN is flat over R. This implies that $s : N \to N$ is universally injective, see Algebra, Lemma 10.39.12. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. $\square$

05DG **Lemma 38.7.3.** Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume

1. $R \to S$ is a local homomorphism of local rings
2. S is essentially of finite presentation over R,
3. N is of finite presentation over S,
4. N is flat over R,
5. $S \to S'$ is flat, and
6. the image of $\text{Spec}(S') \to \text{Spec}(S)$ contains all primes $\mathfrak{q}$ of S lying over $\mathfrak{m}_R$ such that $\mathfrak{q}$ is an associated prime of $N/\mathfrak{m}_R N$.

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. Set $N' = N \otimes_R S'$. Consider the commutative diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N' \arrow[d] \\ \Sigma^{-1}N \arrow[r] & \Sigma^{-1}N' \end{tikzcd}$$

where $\Sigma \subset S$ is the set of elements which are not a zerodivisor on $N/\mathfrak{m}_R N$. If we can show that the map $N \to \Sigma^{-1}N'$ is universally injective, then $N \to N'$ is too (see Algebra, Lemma 10.82.10).

By Lemma 38.7.1 the ring $\Sigma^{-1}S$ is a semi-local ring whose maximal ideals correspond to associated primes of $N/\mathfrak{m}_R N$. Hence the image of $\text{Spec}(\Sigma^{-1}S') \to \text{Spec}(\Sigma^{-1}S)$ contains all these maximal ideals by assumption. By Algebra, Lemma 10.39.16 the ring map $\Sigma^{-1}S \to \Sigma^{-1}S'$ is faithfully flat. Hence $\Sigma^{-1}N \to \Sigma^{-1}N'$, which is the map

$$N \otimes_S \Sigma^{-1}S \longrightarrow N \otimes_S \Sigma^{-1}S'$$

is universally injective, see Algebra, Lemmas 10.82.11 and 10.82.8. Finally, we apply Lemma 38.7.2 to see that $N \to \Sigma^{-1}N$ is universally injective. As the composition of universally injective module maps is universally injective (see Algebra, Lemma 10.82.9) we conclude that $N \to \Sigma^{-1}N'$ is universally injective and we win. $\square$

05DH Lemma 38.7.4. Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume

(1) $R \to S$ is of finite presentation and N is of finite presentation over S,
(2) N is flat over R,
(3) $S \to S'$ is flat, and
(4) the image of $\text{Spec}(S') \to \text{Spec}(S)$ contains all primes $\mathfrak{q}$ such that $\mathfrak{q}$ is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R.

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. By Algebra, Lemma 10.82.12 it suffices to show that $N_\mathfrak{q} \to (N \otimes_R S')_\mathfrak{q}$ is a $R_\mathfrak{p}$-universally injective for any prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ in R. Thus we may apply Lemma 38.7.3 to the ring maps $R_\mathfrak{p} \to S_\mathfrak{q} \to S'_\mathfrak{q}$ and the module $N_\mathfrak{q}$. $\square$

The reader may want to compare the following lemma to Algebra, Lemmas 10.99.1 and 10.128.4 and the results of Section 38.25. In each case the conclusion is that the map $u : M \to N$ is universally injective with flat cokernel.

05FQ Lemma 38.7.5. Let $(R, \mathfrak{m})$ be a local ring. Let $u : M \to N$ be an R-module map. If M is a projective R-module, N is a flat R-module, and $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective then u is universally injective.

Proof. By Algebra, Theorem 10.85.4 the module M is free. If we show the result holds for every finitely generated direct summand of M, then the lemma follows. Hence we may assume that M is finite free. Write $N = \text{colim}_i N_i$ as a directed colimit of finite free modules, see Algebra, Theorem 10.81.4. Note that $u : M \to N$ factors through N_i for some i (as M is finite free). Denote $u_i : M \to N_i$ the corresponding R-module map. As $\overline{u}$ is injective we see that $\overline{u_i} : M/\mathfrak{m}M \to N_i/\mathfrak{m}N_i$ is injective and remains injective on composing with the maps $N_i/\mathfrak{m}N_i \to N_{i'}/\mathfrak{m}N_{i'}$ for all $i' \geq i$. As M and $N_{i'}$ are finite free over the local ring R this implies that $M \to N_{i'}$ is a split injection for all $i' \geq i$. Hence for any R-module Q we see that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective for all $i' \geq i$. As $- \otimes_R Q$ commutes with colimits we conclude that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective as desired. $\square$

05FR Lemma 38.7.6. Assumption and notation as in Lemma 38.7.1. Assume moreover that N is projective as an R-module. Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.

Proof. Pick $s \in \Sigma$. By Lemma 38.7.5 the map $s : N \to N$ is universally injective. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

38.8. Completion and Mittag-Leffler modules

05DI

05DJ Lemma 38.8.1. Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. The completion $(\bigoplus_{\alpha \in A} R)^\wedge$ is flat and Mittag-Leffler.

Proof. By More on Algebra, Lemma 15.27.1 the map $(\bigoplus_{\alpha \in A} R)^\wedge \to \prod_{\alpha \in A} R$ is universally injective. Thus, by Algebra, Lemmas 10.82.7 and 10.89.7 it suffices to show that $\prod_{\alpha \in A} R$ is flat and Mittag-Leffler. By Algebra, Proposition 10.90.6 (and Algebra, Lemma 10.90.5) we see that $\prod_{\alpha \in A} R$ is flat. Thus we conclude because a product of copies of R is Mittag-Leffler, see Algebra, Lemma 10.91.3. □

05DK Lemma 38.8.2. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume

(1) R is Noetherian and I-adically complete,
(2) M is flat over R, and
(3) M/IM is a projective R/I-module.

Then the I-adic completion $M^\wedge$ is a flat Mittag-Leffler R-module.

Proof. Choose a surjection $F \to M$ where F is a free R-module. By Algebra, Lemma 10.97.9 the module $M^\wedge$ is a direct summand of the module $F^\wedge$. Hence it suffices to prove the lemma for F. In this case the lemma follows from Lemma 38.8.1. □

In Lemmas 38.8.3 and 38.8.4 the assumption that S be Noetherian holds if $R \to S$ is of finite type, see Algebra, Lemma 10.31.1.

05DL Lemma 38.8.3. Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume

(1) R is a Noetherian ring,
(2) S is a Noetherian ring,
(3) N is a finite S-module, and
(4) for any finite R-module Q, any $\mathfrak{q} \in \text{Ass}_S(Q \otimes_R N)$ satisfies $IS + \mathfrak{q} \neq S$.

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. We have to show that for any finite R-module Q the map $Q \otimes_R N \to Q \otimes_R N^\wedge$ is injective, see Algebra, Theorem 10.82.3. As there is a canonical map $Q \otimes_R N^\wedge \to (Q \otimes_R N)^\wedge$ it suffices to prove that the canonical map $Q \otimes_R N \to (Q \otimes_R N)^\wedge$ is injective. Hence we may replace N by $Q \otimes_R N$ and it suffices to prove the injectivity for the map $N \to N^\wedge$.

Let $K = \text{Ker}(N \to N^\wedge)$. It suffices to show that $K_\mathfrak{q} = 0$ for $\mathfrak{q} \in \text{Ass}(N)$ as N is a submodule of $\prod_{\mathfrak{q} \in \text{Ass}(N)} N_\mathfrak{q}$, see Algebra, Lemma 10.63.19. Pick $\mathfrak{q} \in \text{Ass}(N)$. By the last assumption we see that there exists a prime $\mathfrak{q}' \supset IS + \mathfrak{q}$. Since $K_\mathfrak{q}$ is a localization of $K_{\mathfrak{q}'}$ it suffices to prove the vanishing of $K_{\mathfrak{q}'}$. Note that $K = \bigcap I^n N$, hence $K_{\mathfrak{q}'} \subset \bigcap I^n N_{\mathfrak{q}'}$. Hence $K_{\mathfrak{q}'} = 0$ by Algebra, Lemma 10.51.4. □

05DM **Lemma 38.8.4.** Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume

(1) R is a Noetherian ring,
(2) S is a Noetherian ring,
(3) N is a finite S-module,
(4) N is flat over R, and
(5) for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. This follows from Lemma 38.8.3 because Algebra, Lemma 10.65.5 and Remark 10.65.6 guarantee that the set of associated primes of tensor products $N \otimes_R Q$ are contained in the set of associated primes of the modules $N \otimes_R \kappa(\mathfrak{p})$. $\square$

38.9. Projective modules

05DN The following lemma can be used to prove projectivity by Noetherian induction on the base, see Lemma 38.9.2.

05DP **Lemma 38.9.1.** Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume

(1) R is Noetherian and I-adically complete,
(2) $R \to S$ is of finite type,
(3) N is a finite S-module,
(4) N is flat over R,
(5) N/IN is projective as a R/I-module, and
(6) for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.

Then N is projective as an R-module.

Proof. By Lemma 38.8.4 the map $N \to N^\wedge$ is universally injective. By Lemma 38.8.2 the module $N^\wedge$ is Mittag-Leffler. By Algebra, Lemma 10.89.7 we conclude that N is Mittag-Leffler. Hence N is countably generated, flat and Mittag-Leffler as an R-module, whence projective by Algebra, Lemma 10.93.1. $\square$

05FS **Lemma 38.9.2.** Let R be a ring. Let $R \to S$ be a ring map. Assume

(1) R is Noetherian,
(2) $R \to S$ is of finite type and flat, and
(3) every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.

Then S is projective as an R-module.

Proof. Consider the set

$$\{I \subset R \mid S/IS \text{ not projective as } R/I\text{-module}\}$$

We have to show this set is empty. To get a contradiction assume it is nonempty. Then it contains a maximal element I. Let $J = \sqrt{I}$ be its radical. If $I \neq J$, then S/JS is projective as a R/J-module, and S/IS is flat over R/I and J/I is a nilpotent ideal in R/I. Applying Algebra, Lemma 10.77.7 we see that S/IS is a projective R/I-module, which is a contradiction. Hence we may assume that I is a radical ideal. In other words we are reduced to proving the lemma in case R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R.

Assume R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R. By generic flatness, Algebra, Lemma 10.118.1 (applied to a localization R_g which is a domain) or the more general Algebra, Lemma 10.118.7 there exists a nonzero $f \in R$ such that S_f is free as an R_f-module. Denote $R^\wedge = \lim R/(f^n)$ the (f)-adic completion of R. Note that the ring map

$$R \longrightarrow R_f \times R^\wedge$$

is a faithfully flat ring map, see Algebra, Lemma 10.97.2. Hence by faithfully flat descent of projectivity, see Algebra, Theorem 10.95.6 it suffices to prove that $S \otimes_R R^\wedge$ is a projective $R^\wedge$-module. To see this we will use the criterion of Lemma 38.9.1. First of all, note that $S/fS = (S \otimes_R R^\wedge)/f(S \otimes_R R^\wedge)$ is a projective $R/(f)$-module and that $S \otimes_R R^\wedge$ is flat and of finite type over $R^\wedge$ as a base change of such. Next, suppose that $\mathfrak p^\wedge$ is a prime ideal of $R^\wedge$. Let $\mathfrak p \subset R$ be the corresponding prime of R. As $R \to S$ has geometrically integral fibre rings, the same is true for the fibre rings of any base change. Hence $\mathfrak q^\wedge = \mathfrak p^\wedge(S \otimes_R R^\wedge)$, is a prime ideals lying over $\mathfrak p^\wedge$ and it is the unique associated prime of $S \otimes_R \kappa(\mathfrak p^\wedge)$. Thus we win if $f(S \otimes_R R^\wedge) + \mathfrak q^\wedge \neq S \otimes_R R^\wedge$. This is true because $\mathfrak p^\wedge + fR^\wedge \neq R^\wedge$ as f lies in the Jacobson radical of the f-adically complete ring $R^\wedge$ and because $R^\wedge \to S \otimes_R R^\wedge$ is surjective on spectra as its fibres are nonempty (irreducible spaces are nonempty). □

05FT Lemma 38.9.3. Let R be a ring. Let $R \to S$ be a ring map. Assume

(1) $R \to S$ is of finite presentation and flat, and
(2) every fibre ring $S \otimes_R \kappa(\mathfrak p)$ is geometrically integral over $\kappa(\mathfrak p)$.

Then S is projective as an R-module.

Proof. We can find a cocartesian diagram of rings

$$\begin{tikzcd} S_0 \arrow[r] & S \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

such that R_0 is of finite type over $\mathbf{Z}$, the map $R_0 \to S_0$ is of finite type and flat with geometrically integral fibres, see More on Morphisms, Lemmas 37.34.4, 37.34.6, 37.34.7, and 37.34.11. By Lemma 38.9.2 we see that S_0 is a projective R_0-module. Hence $S = S_0 \otimes_{R_0} R$ is a projective R-module, see Algebra, Lemma 10.94.1. □

05FU Remark 38.9.4. Lemma 38.9.3 is a key step in the development of results in this chapter. The analogue of this lemma in [GR71] is [GR71, I Proposition 3.3.1]: If $R \to S$ is smooth with geometrically integral fibres, then S is projective as an R-module. This is a special case of Lemma 38.9.3, but as we will later improve on this lemma anyway, we do not gain much from having a stronger result at this point. We briefly sketch the proof of this as it is given in [GR71].

(1) First reduce to the case where R is Noetherian as above.

(2) Since projectivity descends through faithfully flat ring maps, see Algebra, Theorem 10.95.6 we may work locally in the fppf topology on R, hence we may assume that $R \to S$ has a section $\sigma : S \to R$. (Just by the usual trick of base changing to S.) Set $I = \mathrm{Ker}(S \to R)$.
(3) Localizing a bit more on R we may assume that I/I^2 is a free R-module and that the completion $S^\wedge$ of S with respect to I is isomorphic to $R[[t_1, \ldots, t_n]]$, see Morphisms, Lemma 29.34.20. Here we are using that $R \to S$ is smooth.
(4) To prove that S is projective as an R-module, it suffices to prove that S is flat, countably generated and Mittag-Leffler as an R-module, see Algebra, Lemma 10.93.1. The first two properties are evident. Thus it suffices to prove that S is Mittag-Leffler as an R-module. By Algebra, Lemma 10.91.4 the module $R[[t_1, \ldots, t_n]]$ is Mittag-Leffler over R. Hence Algebra, Lemma 10.89.7 shows that it suffices to show that the $S \to S^\wedge$ is universally injective as a map of R-modules.
(5) Apply Lemma 38.7.4 to see that $S \to S^\wedge$ is R-universally injective. Namely, as $R \to S$ has geometrically integral fibres, any associated point of any fibre ring is just the generic point of the fibre ring which is in the image of $\mathrm{Spec}(S^\wedge) \to \mathrm{Spec}(S)$.

There is an analogy between the proof as sketched just now, and the development of the arguments leading to the proof of Lemma 38.9.3. In both a completion plays an essential role, and both times the assumption of having geometrically integral fibres assures one that the map from S to the completion of S is R-universally injective.

38.10. Flat finite type modules, Part I

05I2 In some cases given a ring map $R \to S$ of finite presentation and a finite S-module N the flatness of N over R implies that N is of finite presentation. In this section we prove this is true "pointwise". We remark that the first proof of Proposition 38.10.3 uses the geometric results of Section 38.3 but not the existence of a complete dévissage.

05I3 **Lemma 38.10.1.** Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Write $\mathfrak{p} = \mathfrak{m}S$. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$. Let N be a finite S-module. There exist $r \geq 0$ and an S-module map

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

(1) $N_\mathfrak{q}$ is R-flat,
(2) α is R-universally injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,
(3) α is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,
(4) $\alpha_\mathfrak{p}$ is an isomorphism and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat, and
(5) $\alpha_\mathfrak{q}$ is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat.

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix an α. The most interesting implication is (1) $\Rightarrow$ (2) which we prove first. Assume (1). Because $S/\mathfrak{m}S$ is a domain with fraction field $\kappa(\mathfrak{p})$ we see that

$(S/\mathfrak{m}S)^{\oplus r} \to N_\mathfrak{p}/\mathfrak{m}N_\mathfrak{p} = N \otimes_S \kappa(\mathfrak{p})$ is injective. Hence by Lemmas 38.7.5 and 38.9.3. the map $S^{\oplus r} \to N_\mathfrak{p}$ is R-universally injective. It follows that $S^{\oplus r} \to N$ is R-universally injective, see Algebra, Lemma 10.82.10. Then also the localization $\alpha_\mathfrak{q}$ is R-universally injective, see Algebra, Lemma 10.82.13. We conclude that $\text{Coker}(\alpha)_\mathfrak{q}$ is R-flat by Algebra, Lemma 10.82.7.

The implication (2) $\Rightarrow$ (3) is immediate. If (3) holds, then $\alpha_\mathfrak{p}$ is injective as a localization of an injective module map. By Nakayama's lemma (Algebra, Lemma 10.20.1) $\alpha_\mathfrak{p}$ is surjective too. Hence (3) $\Rightarrow$ (4). If (4) holds, then $\alpha_\mathfrak{p}$ is an isomorphism, so α is injective as $S_\mathfrak{q} \to S_\mathfrak{p}$ is injective. Namely, elements of $S \setminus \mathfrak{p}$ are nonzerodivisors on S by a combination of Lemmas 38.7.6 and 38.9.3. Hence (4) $\Rightarrow$ (5). Finally, if (5) holds, then $N_\mathfrak{q}$ is R-flat as an extension of flat modules, see Algebra, Lemma 10.39.13. Hence (5) $\Rightarrow$ (1) and the proof is finished. □

05I4 Lemma 38.10.2. Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a ring map of finite presentation. Let N be a finite S-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Assume that $N_\mathfrak{q}$ is flat over R, and assume there exists a complete dévissage of $N/S/R$ at $\mathfrak{q}$. Then N is a finitely presented S-module, free as an R-module, and there exists an isomorphism

$$N \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R-algebra with geometrically irreducible fibres.

Proof. Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that N is finitely presented as an S-module if and only if M_1 is finitely presented as an B_1-module, see Remark 38.6.3. Note that $N_\mathfrak{q} \cong (M_1)_{\mathfrak{q}_1}$ as R-modules because (a) $N_\mathfrak{q} \cong (M_1)_{\mathfrak{q}'_1}$ where $\mathfrak{q}'_1$ is the unique prime in A_1 lying over $\mathfrak{q}_1$ and (b) $(A_1)_{\mathfrak{q}'_1} = (A_1)_{\mathfrak{q}_1}$ by Algebra, Lemma 10.41.11, so (c) $(M_1)_{\mathfrak{q}'_1} \cong (M_1)_{\mathfrak{q}_1}$. Hence $(M_1)_{\mathfrak{q}_1}$ is a flat R-module. Thus we may replace (S, N) by (B_1, M_1) in order to prove the lemma. By Lemma 38.10.1 the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective and $\text{Coker}(\alpha_1)_\mathfrak{q}$ is R-flat. Note that $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=2,\ldots,n}$ is a complete dévissage of $\text{Coker}(\alpha_1)/B_1/R$ at $\mathfrak{q}_1$. Hence the induction hypothesis implies that $\text{Coker}(\alpha_1)$ is finitely presented as a B_1-module, free as an R-module, and has a decomposition as in the lemma. This implies that M_1 is finitely presented as a B_1-module, see Algebra, Lemma 10.5.3. It further implies that $M_1 \cong B_1^{\oplus r_1} \oplus \text{Coker}(\alpha_1)$ as R-modules, hence a decomposition as in the lemma. Finally, B_1 is projective as an R-module by Lemma 38.9.3 hence free as an R-module by Algebra, Theorem 10.85.4. This finishes the proof. □

05I5 Proposition 38.10.3. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that

(1) f is locally of finite presentation,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat at x over S.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \text{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \text{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

First proof. This proof is longer but does not use the existence of a complete dévissage. The problem is local around x and s, hence we may assume that X and S are affine. During the proof we will finitely many times replace S by an elementary étale neighbourhood of (S, s). The goal is then to find (after such a replacement) an open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ containing x such that $\mathcal{F}|_V$ is flat over S and finitely presented. Of course we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ at any point of the proof, i.e., we may assume S is a local scheme. We will prove the proposition by induction on the integer $n = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$.

We can choose

(1) elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,
(2) a commutative diagram

$$\begin{tikzcd} X \arrow[dd, "f"'] & X' \arrow[l, "g"'] \arrow[dd] & Z' \arrow[l, "i"'] \arrow[d, "\pi"] \\ & & Y' \arrow[d, "h"] \\ S & S' \arrow[l, "e"'] \arrow[r, equal] & S' \end{tikzcd}$$

(3) a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,
(4) a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,

as in Lemma 38.3.2. We are going to replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$, see remarks in first paragraph of the proof. Consider the diagram

$$\begin{tikzcd} X_{\mathcal{O}_{S',s'}} \arrow[ddr, "f"'] & X'_{\mathcal{O}_{S',s'}} \arrow[l, "g"'] \arrow[dd] & Z'_{\mathcal{O}_{S',s'}} \arrow[l, "i"'] \arrow[d, "\pi"] \\ & & Y'_{\mathcal{O}_{S',s'}} \arrow[dl, "h"] \\ & \mathrm{Spec}(\mathcal{O}_{S',s'}) & \end{tikzcd}$$

Here we have base changed the schemes X', Z', Y' over S' via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S'$ and the scheme X over S via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S$. It is still the case that g is étale, see Lemma 38.2.2. After replacing X by $X_{\mathcal{O}_{S',s'}}$, X' by $X'_{\mathcal{O}_{S',s'}}$, Z' by $Z'_{\mathcal{O}_{S',s'}}$, and Y' by $Y'_{\mathcal{O}_{S',s'}}$ we may assume we have a diagram as Lemma 38.3.2 where in addition $S = S'$ is a local scheme with closed point s. By Lemmas 38.3.3 and 38.3.4 the result for $Y' \to S$, the sheaf $\pi_*\mathcal{G}$, and the point y' implies the result for $X \to S$, $\mathcal{F}$ and x. Hence we may assume that S is local and $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension n.

The base case of the induction: $n = 0$. As $X \to S$ is smooth with geometrically irreducible fibres of dimension 0 we see that $X \to S$ is an open immersion, see Descent, Lemma 35.25.2. As S is local and the closed point is in the image of $X \to S$ we conclude that $X = S$. Thus we see that $\mathcal{F}$ corresponds to a finite flat $\mathcal{O}_{S,s}$ module. In this case the result follows from Algebra, Lemma 10.78.5 which tells us that $\mathcal{F}$ is in fact finite free.

The induction step. Assume the result holds whenever the dimension of the support in the closed fibre is $< n$. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $\mathcal{F} = \widetilde{N}$ for some B-module N. Note that A is a local ring; denote its maximal ideal $\mathfrak m$. Then $\mathfrak p = \mathfrak m B$ is the unique minimal prime lying over $\mathfrak m$ as $X \to S$ has geometrically irreducible fibres. Finally, let $\mathfrak q \subset B$ be the prime corresponding to x. By Lemma 38.10.1 we can choose a map

$$\alpha : B^{\oplus r} \to N$$

such that $\kappa(\mathfrak p)^{\oplus r} \to N \otimes_B \kappa(\mathfrak p)$ is an isomorphism. Moreover, as $N_{\mathfrak q}$ is A-flat the lemma also shows that α is injective and that $\mathrm{Coker}(\alpha)_{\mathfrak q}$ is A-flat. Set $Q = \mathrm{Coker}(\alpha)$. Note that the support of $Q/\mathfrak m Q$ does not contain $\mathfrak p$. Hence it is certainly the case that $\dim_{\mathfrak q}(\mathrm{Supp}(Q/\mathfrak m Q)) < n$. Combining everything we know about Q we see that the induction hypothesis applies to Q. It follows that there exists an elementary étale morphism $(S', s) \to (S, s)$ such that the conclusion holds for $Q \otimes_A A'$ over $B \otimes_A A'$ where $A' = \mathcal{O}_{S', s'}$. After replacing A by A' we have an exact sequence

$$0 \to B^{\oplus r} \to N \to Q \to 0$$

(here we use that α is injective as mentioned above) of finite B-modules and we also get an element $g \in B$, $g \notin \mathfrak q$ such that Q_g is finitely presented over B_g and flat over A. Since localization is exact we see that

$$0 \to B_g^{\oplus r} \to N_g \to Q_g \to 0$$

is still exact. As B_g and Q_g are flat over A we conclude that N_g is flat over A, see Algebra, Lemma 10.39.13, and as B_g and Q_g are finitely presented over B_g the same holds for N_g, see Algebra, Lemma 10.5.3. $\square$

Second proof. We apply Proposition 38.5.7 to find a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x. (In particular S' and X' are affine.) By Morphisms, Lemma 29.25.13 we see that $g^*\mathcal{F}$ is flat at x' over S and by Lemma 38.2.3 we see that it is flat at x' over S'. Via Remark 38.6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage at the prime of $\Gamma(X', \mathcal{O}_{X'})$ corresponding to x'. We may base change this complete dévissage to the local ring $\mathcal{O}_{S', s'}$ of $\Gamma(S', \mathcal{O}_{S'})$ at the prime corresponding to s'. Thus Lemma 38.10.2 implies that

$$\Gamma(X', \mathcal{F}') \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S', s'}$$

is flat over $\mathcal{O}_{S', s'}$ and of finite presentation over $\Gamma(X', \mathcal{O}_{X'}) \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S', s'}$. In other words, the restriction of $\mathcal{F}$ to $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S', s'})$ is of finite presentation and flat over $\mathcal{O}_{S', s'}$. Since the morphism $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S', s'}) \to X \times_S \mathrm{Spec}(\mathcal{O}_{S', s'})$ is étale (Lemma 38.2.2) its image $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S', s'})$ is an open subscheme, and by étale descent the restriction of $\mathcal{F}$ to V is of finite presentation and flat over $\mathcal{O}_{S', s'}$. (Results used: Morphisms, Lemma 29.36.13, Descent, Lemma 35.7.3, and Morphisms, Lemma 29.25.13.) $\square$

05M9 Lemma 38.10.4. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Then the set

$$\{x \in X_s \mid \mathcal{F} \text{ flat over } S \text{ at } x\}$$

is open in the fibre X_s.

Proof. Suppose $x \in U$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 38.10.3. Note that $X_{s'} = X_s$ as $\kappa(s) = \kappa(s')$. If $x' \in V \cap X_{s'}$, then the pullback of $\mathcal{F}$ to $X \times_S S'$ is flat over S' at x'. Hence $\mathcal{F}$ is flat at x' over S, see Morphisms, Lemma 29.25.13. In other words $X_s \cap V \subset U$ is an open neighbourhood of x in U. □

05KT Lemma 38.10.5. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that

(1) f is locally of finite type,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat at x over S.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Proposition 38.10.3 is that we do not assume f is of finite presentation.) The question is local on X and S, hence we may assume X and S are affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 38.10.3 to $\mathbf{A}^n_S \to S$, the sheaf $i_*\mathcal{F}$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{F}$ to W is flat over $\mathcal{O}_{S',s'}$. This means that $V := W \cap \big(X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

05KU Lemma 38.10.6. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that

(1) f is of finite presentation,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat over S at every point of the fibre X_s.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

Proof. For every point $x \in X_s$ we can use Proposition 38.10.3 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is an $\mathcal{O}_{V_x}$-module of finite presentation and flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$. □

05KV Lemma 38.10.7. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that

(1) f is of finite type,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat over S at every point of the fibre X_s.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Lemma 38.10.6 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 38.10.5 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$. □

05I6 Lemma 38.10.8. Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ with image $s \in S$. If X is flat at x over S, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that $V \to \operatorname{Spec}(\mathcal{O}_{S',s'})$ is flat and of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S are affine. Write $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply

Proposition 38.10.3 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{O}_X$ is flat and of finite presentation. This means that $V := W \cap \big(X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

05I7 Lemma 38.10.9. Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. If $x \in X$ and $\mathcal{F}$ is flat at x over S, then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation.

Proof. Let $s = f(x)$. By Proposition 38.10.3 there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the pullback of $\mathcal{F}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation in a neighbourhood of the point $x' \in X_{s'} = X_s$ corresponding to x. The ring map

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'}),x'} = \mathcal{O}_{X \times_S S',x'}$$

is flat and local as a localization of an étale ring map. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ by descent, see Algebra, Lemma 10.83.2 (and also that a flat local ring map is faithfully flat, see Algebra, Lemma 10.39.17). □

05I8 Lemma 38.10.10. Let $f : X \to S$ be a morphism which is locally of finite type. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then $\mathcal{O}_{X,x}$ is essentially of finite presentation over $\mathcal{O}_{S,s}$.

Proof. We may assume X and S affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Lemma 38.10.9 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We conclude that $\mathcal{O}_{X,x}$ is of finite presentation over $\mathcal{O}_{\mathbf{A}^n_S,t}$ which implies what we want. □

38.11. Extending properties from an open

0B47 In this section we collect a number of results of the form: If $f : X \to S$ is a flat morphism of schemes and f satisfies some property over a dense open of S, then f satisfies the same property over all of S.

081N Lemma 38.11.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset S$ be open. Assume

(1) f is locally of finite presentation,
(2) $\mathcal{F}$ is of finite type and flat over S,
(3) $U \subset S$ is retrocompact and scheme theoretically dense,
(4) $\mathcal{F}|_{f^{-1}U}$ is of finite presentation.

Then $\mathcal{F}$ is of finite presentation.

Proof. The problem is local on X and S, hence we may assume X and S affine. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F}$ is the quasi-coherent sheaf associated to N. We have $U = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 10.29.1. As U is schematically dense the map $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. Pick a prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ corresponding to $x \in X$ mapping to $s \in S$. By Lemma 38.10.9 the module $N_\mathfrak{q}$ is of finite presentation over $B_\mathfrak{q}$. Choose a surjection $\varphi : B^{\oplus m} \to N$ of B-modules.

Choose $k_1, \ldots, k_t \in \mathrm{Ker}(\varphi)$ and set $N' = B^{\oplus m}/\sum Bk_j$. There is a canonical surjection $N' \to N$ and N is the filtered colimit of the B-modules N' constructed in this manner. Thus we see that we can choose $k_1, \ldots, k_t$ such that (a) $N'_{f_i} \cong N_{f_i}$, $i = 1, \ldots, n$ and (b) $N'_{\mathfrak q} \cong N_{\mathfrak q}$. This in particular implies that $N'_{\mathfrak q}$ is flat over A. By openness of flatness, see Algebra, Theorem 10.129.4 we conclude that there exists a $g \in B$, $g \notin \mathfrak q$ such that N'_g is flat over A. Consider the commutative diagram

$$\begin{tikzcd} N'_g \arrow[r] \arrow[d] & N_g \arrow[d] \\ \prod N'_{gf_i} \arrow[r] & \prod N_{gf_i} \end{tikzcd}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $A \to \prod A_{f_i}$ is injective and N'_g is flat over A. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that N_g is of finite presentation over B_g. We conclude by applying Algebra, Lemma 10.23.2. □

081P Lemma 38.11.2. Let $f : X \to S$ be a morphism of schemes. Let $U \subset S$ be open. Assume

(1) f is locally of finite type and flat,
(2) $U \subset S$ is retrocompact and scheme theoretically dense,
(3) $f|_{f^{-1}U} : f^{-1}U \to U$ is locally of finite presentation.

Then f is of locally of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and apply Lemma 38.11.1 to $i_*\mathcal{O}_X$. Some details omitted. □

081L Lemma 38.11.3. Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite type. Let $U \subset S$ be a dense open such that $X_U \to U$ has relative dimension $\leq e$, see Morphisms, Definition 29.29.1. If also either

(1) f is locally of finite presentation, or
(2) $U \subset S$ is retrocompact,

then f has relative dimension $\leq e$.

Proof. Proof in case (1). Let $W \subset X$ be the open subscheme constructed and studied in More on Morphisms, Lemmas 37.22.7 and 37.22.9. Note that every generic point of every fibre is contained in W, hence it suffices to prove the result for W. Since $W = \bigcup_{d \geq 0} U_d$, it suffices to prove that $U_d = \emptyset$ for $d > e$. Since f is flat and locally of finite presentation it is open hence $f(U_d)$ is open (Morphisms, Lemma 29.25.10). Thus if U_d is not empty, then $f(U_d) \cap U \neq \emptyset$ as desired.

Proof in case (2). We may replace S by its reduction. Then U is scheme theoretically dense. Hence f is locally of finite presentation by Lemma 38.11.2. In this way we reduce to case (1). □

0B48 Lemma 38.11.4. Let $f : X \to S$ be a morphism of schemes which is flat and proper. Let $U \subset S$ be a dense open such that $X_U \to U$ is finite. If also either f is locally of finite presentation or $U \subset S$ is retrocompact, then f is finite.

Proof. By Lemma 38.11.3 the fibres of f have dimension zero. Hence f is quasi-finite (Morphisms, Lemma 29.29.5) whence has finite fibres (Morphisms, Lemma 29.20.10). Hence f is finite by More on Morphisms, Lemma 37.44.1. $\square$

081M **Lemma 38.11.5.** Let $f : X \to S$ be a morphism of schemes and $U \subset S$ an open. If

1. f is separated, locally of finite type, and flat,
2. $f^{-1}(U) \to U$ is an isomorphism, and
3. $U \subset S$ is retrocompact and scheme theoretically dense,

then f is an open immersion.

Proof. By Lemma 38.11.2 the morphism f is locally of finite presentation. The image $f(X) \subset S$ is open (Morphisms, Lemma 29.25.10) hence we may replace S by $f(X)$. Thus we have to prove that f is an isomorphism. We may assume S is affine. We can reduce to the case that X is quasi-compact because it suffices to show that any quasi-compact open $X' \subset X$ whose image is S maps isomorphically to S. Thus we may assume f is quasi-compact. All the fibers of f have dimension 0, see Lemma 38.11.3. Hence f is quasi-finite, see Morphisms, Lemma 29.29.5. Let $s \in S$. Choose an elementary étale neighbourhood $g : (T, t) \to (S, s)$ such that $X \times_S T = V \amalg W$ with $V \to T$ finite and $W_t = \emptyset$, see More on Morphisms, Lemma 37.41.6. Denote $\pi : V \amalg W \to T$ the given morphism. Since π is flat and locally of finite presentation, we see that $\pi(V)$ is open in T (Morphisms, Lemma 29.25.10). After shrinking T we may assume that $T = \pi(V)$. Since f is an isomorphism over U we see that π is an isomorphism over $g^{-1}U$. Since $\pi(V) = T$ this implies that $\pi^{-1}g^{-1}U$ is contained in V. By Morphisms, Lemma 29.25.15 we see that $\pi^{-1}g^{-1}U \subset V \amalg W$ is scheme theoretically dense. Hence we deduce that $W = \emptyset$. Thus $X \times_S T = V$ is finite over T. This implies that f is finite (after replacing S by an open neighbourhood of s), for example by Descent, Lemma 35.23.23. Then f is finite locally free (Morphisms, Lemma 29.48.2) and after shrinking S to a smaller open neighbourhood of s we see that f is finite locally free of some degree d (Morphisms, Lemma 29.48.5). But $d = 1$ as is clear from the fact that the degree is 1 over the dense open U. Hence f is an isomorphism. $\square$

38.12. Flat finitely presented modules

05I9 In some cases given a ring map $R \to S$ of finite presentation and a finitely presented S-module N the flatness of N over R implies that N is projective as an R-module, at least after replacing S by an étale extension. In this section we collect a some results of this nature.

05IA **Lemma 38.12.1.** Let R be a ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Let $\mathfrak q \subset S$ be a prime ideal lying over the prime $\mathfrak r \subset R$. Set $\mathfrak p = \mathfrak r S$. Let N be a finitely presented S-module. There exists $r \geq 0$ and an S-module map

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak p)^{\oplus r} \to N \otimes_S \kappa(\mathfrak p)$ is an isomorphism. For any such α the following are equivalent:

1. $N_{\mathfrak q}$ is R-flat,
2. there exists an $f \in R$, $f \notin \mathfrak r$ such that $\alpha_f : S_f^{\oplus r} \to N_f$ is R_f-universally injective and a $g \in S$, $g \notin \mathfrak q$ such that $\mathrm{Coker}(\alpha)_g$ is R-flat,
3. $\alpha_{\mathfrak r}$ is $R_{\mathfrak r}$-universally injective and $\mathrm{Coker}(\alpha)_{\mathfrak q}$ is R-flat

(4) $\alpha_\mathfrak{r}$ is injective and $\text{Coker}(\alpha)_\mathfrak{q}$ is R-flat,
(5) $\alpha_\mathfrak{p}$ is an isomorphism and $\text{Coker}(\alpha)_\mathfrak{q}$ is R-flat, and
(6) $\alpha_\mathfrak{q}$ is injective and $\text{Coker}(\alpha)_\mathfrak{q}$ is R-flat.

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix a choice of α. We may apply Lemma 38.10.1 to the map $\alpha_\mathfrak{r} : S_\mathfrak{r}^{\oplus r} \to N_\mathfrak{r}$. Hence we see that (1), (3), (4), (5), and (6) are all equivalent. Since it is also clear that (2) implies (3) we see that all we have to do is show that (1) implies (2).

Assume (1). By openness of flatness, see Algebra, Theorem 10.129.4, the set

$$U_1 = \{\mathfrak{q}' \subset S \mid N_{\mathfrak{q}'} \text{ is flat over } R\}$$

is open in $\text{Spec}(S)$. It contains $\mathfrak{q}$ by assumption and hence $\mathfrak{p}$. Because $S^{\oplus r}$ and N are finitely presented S-modules the set

$$U_2 = \{\mathfrak{q}' \subset S \mid \alpha_{\mathfrak{q}'} \text{ is an isomorphism}\}$$

is open in $\text{Spec}(S)$, see Algebra, Lemma 10.79.2. It contains $\mathfrak{p}$ by (5). As $R \to S$ is finitely presented and flat the map $\Phi : \text{Spec}(S) \to \text{Spec}(R)$ is open, see Algebra, Proposition 10.41.8. For any prime $\mathfrak{r}' \in \Phi(U_1 \cap U_2)$ we see that there exists a prime $\mathfrak{q}'$ lying over $\mathfrak{r}'$ such that $N_{\mathfrak{q}'}$ is flat and such that $\alpha_{\mathfrak{q}'}$ is an isomorphism, which implies that $\alpha \otimes \kappa(\mathfrak{p}')$ is an isomorphism where $\mathfrak{p}' = \mathfrak{r}'S$. Thus $\alpha_{\mathfrak{r}'}$ is $R_{\mathfrak{r}'}$-universally injective by the implication (1) $\Rightarrow$ (3). Hence if we pick $f \in R$, $f \notin \mathfrak{r}$ such that $D(f) \subset \Phi(U_1 \cap U_2)$ then we conclude that α_f is R_f-universally injective, see Algebra, Lemma 10.82.12. The same reasoning also shows that for any $\mathfrak{q}' \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ the module $\text{Coker}(\alpha)_{\mathfrak{q}'}$ is R-flat. Note that $\mathfrak{q} \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$. Hence we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g) \subset U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ and we win. □

05IB Lemma 38.12.2. Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module flat over R. Let $\mathfrak{r} \subset R$ be a prime ideal. Assume there exists a complete dévissage of $N/S/R$ over $\mathfrak{r}$. Then there exists an $f \in R$, $f \notin \mathfrak{r}$ such that

$$N_f \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R_f-algebra with geometrically irreducible fibres. Moreover, N_f is projective as an R_f-module.

Proof. Let $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that the assertions of the lemma are entirely about the structure of N as an R-module. Hence we may replace N by M_1, and we may think of M_1 as a B_1-module. See Remark 38.6.3 in order to see why M_1 is of finite presentation as a B_1-module. By Lemma 38.12.1 we may, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, assume the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective. Since M_1 and $B_1^{\oplus r_1}$ are R-flat and finitely presented as B_1-modules we see that $\text{Coker}(\alpha_1)$ is R-flat (Algebra, Lemma 10.82.7) and finitely presented as a B_1-module. Note that $(A_i, B_i, M_i, \alpha_i)_{i=2,\ldots,n}$ is a complete dévissage of $\text{Coker}(\alpha_1)$. Hence the induction hypothesis implies that, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, we may assume that $\text{Coker}(\alpha_1)$ has a decomposition as in the lemma and is projective. In particular $M_1 = B_1^{\oplus r_1} \oplus \text{Coker}(\alpha_1)$. This proves the statement regarding the decomposition. The statement on projectivity follows as B_1 is projective as an R-module by Lemma 38.9.3. □

05IC Remark 38.12.3. There is a variant of Lemma 38.12.2 where we weaken the flatness condition by assuming only that N is flat at some given prime $\mathfrak{q}$ lying over $\mathfrak{r}$ but where we strengthen the dévissage condition by assuming the existence of a complete dévissage at $\mathfrak{q}$. Compare with Lemma 38.10.2.

The following is the main result of this section.

05ID Proposition 38.12.4. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that

(1) f is locally of finite presentation,
(2) $\mathcal{F}$ is of finite presentation, and
(3) $\mathcal{F}$ is flat at x over S.

Then there exists a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. By openness of flatness, see More on Morphisms, Theorem 37.15.1 we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is flat over S. Next, we apply Proposition 38.5.7 to find a diagram as in the statement of the proposition such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage over s'. (In particular S' and X' are affine.) By Morphisms, Lemma 29.25.13 we see that $g^*\mathcal{F}$ is flat over S and by Lemma 38.2.3 we see that it is flat over S'. Via Remark 38.6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage over the prime of $\Gamma(S', \mathcal{O}_{S'})$ corresponding to s'. Thus Lemma 38.12.2 implies that the result of the proposition holds after replacing S' by a standard open neighbourhood of s'. $\square$

In the rest of this section we prove a number of variants on this result. The first is a “global” version.

05KW Lemma 38.12.5. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that

(1) f is of finite presentation,
(2) $\mathcal{F}$ is of finite presentation, and
(3) $\mathcal{F}$ is flat over S at every point of the fibre X_s.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. For every point $x \in X_s$ we can use Proposition 38.12.4 to find a commutative diagram

$$\begin{tikzcd} (X, x) & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\ (S, s) \arrow[u, leftarrow] & (S_x, s_x) \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that Y_x, S_x are affine and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a projective $\Gamma(S_x, \mathcal{O}_{S_x})$-module. In particular $g_x(Y_x) \cap X_s$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}(Y_{x_i}) \cap X_s$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. We may also assume that S' is affine. Set $X' = \coprod Y_{x_i} \times_{S_{x_i}} S'$ and endow it with the obvious morphism $g : X' \to X$. By construction $g(X')$ contains X_s and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\Gamma(S_{x_i}, \mathcal{O}_{S_{x_i}})} \Gamma(S', \mathcal{O}_{S'}).$$

This is a projective $\Gamma(S', \mathcal{O}_{S'})$-module, see Algebra, Lemma 10.94.1. $\square$

The following two lemmas are reformulations of the results above in case $\mathcal{F} = \mathcal{O}_X$.

05IE **Lemma 38.12.6.** Let $f : X \to S$ be locally of finite presentation. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then there exists a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (S', s') \arrow[l] \end{tikzcd}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Proposition 38.12.4. $\square$

05KX **Lemma 38.12.7.** Let $f : X \to S$ be of finite presentation. Let $s \in S$. If X is flat over S at all points of X_s, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

with g étale, $X_s \subset g(X')$, such that X', S' are affine, and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Lemma 38.12.5. $\square$

The following lemmas explain consequences of Proposition 38.12.4 in case we only assume the morphism and the sheaf are of finite type (and not necessarily of finite presentation).

05KY **Lemma 38.12.8.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that

(1) f is locally of finite presentation,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat at x over S.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. To prove the lemma we may replace (S, s) by any elementary étale neighbourhood, and we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$. Hence by Proposition 38.10.3 we may assume that $\mathcal{F}$ is finitely presented and flat over S in a neighbourhood of x. In this case the result follows from Proposition 38.12.4 because Algebra, Theorem 10.85.4 assures us that projective = free over a local ring. $\square$

05KZ Lemma 38.12.9. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that

(1) f is locally of finite type,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat at x over S.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd} (X, x) \arrow[d] & (X', x') \arrow[l, "g"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 38.12.8 is that we do not assume f is of finite presentation.) The problem is local on X and S. Hence we may assume X and S are affine, say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$. Since B is a finite type A-algebra we can find a surjection $A[x_1, \ldots, x_n] \to B$. In other words, we can choose a closed immersion $i : X \to \mathbf{A}^n_S$. Set $t = i(x)$ and $\mathcal{G} = i_*\mathcal{F}$. Note that $\mathcal{G}_t \cong \mathcal{F}_x$ are $\mathcal{O}_{S,s}$-modules. Hence $\mathcal{G}$ is flat over S at t. We apply Lemma 38.12.8 to the morphism $\mathbf{A}^n_S \to S$, the point t, and the sheaf $\mathcal{G}$. Thus we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{tikzcd} (\mathbf{A}^n_S, t) \arrow[d] & (Y, y) \arrow[l, "h"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \arrow[l] \end{tikzcd}$$

such that $Y \to \mathbf{A}^n_{\mathcal{O}_{S',s'}}$ is étale, $\kappa(t) = \kappa(y)$, the scheme Y is affine, and such that $\Gamma(Y, h^*\mathcal{G})$ is a projective $\mathcal{O}_{S',s'}$-module. Then a solution to the original problem is given by the closed subscheme $X' = Y \times_{\mathbf{A}^n_S} X$ of Y. $\square$

05L0 Lemma 38.12.10. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that

(1) f is of finite presentation,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat over S at all points of X_s.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & \mathrm{Spec}(\mathcal{O}_{S',s'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. For every point $x \in X_s$ we can use Lemma 38.12.8 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S_x,s_x}), s_x) \arrow[l] \end{tikzcd}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine of finite presentation over $\mathcal{O}_{S_x,s_x}$, the sheaf $g_x^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{Y_x}$, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S',s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i},s_{x_i}}} \mathcal{O}_{S',s'}.$$

This is a free $\mathcal{O}_{S',s'}$-module as a direct sum of base changes of free modules. Some minor details omitted. $\square$

05L1 Lemma 38.12.11. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that

(1) f is of finite type,
(2) $\mathcal{F}$ is of finite type, and
(3) $\mathcal{F}$ is flat over S at all points of X_s.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & \mathrm{Spec}(\mathcal{O}_{S',s'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 38.12.10 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 38.12.9 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} (X, x) \arrow[d] & (Y_x, y_x) \arrow[l, "g_x"'] \arrow[d] \\ (S, s) & (\mathrm{Spec}(\mathcal{O}_{S_x,s_x}), s_x) \arrow[l] \end{tikzcd}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 37.35.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S',s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i},s_{x_i}}} \mathcal{O}_{S',s'}.$$

This is a free $\mathcal{O}_{S',s'}$-module as a direct sum of base changes of free modules. $\square$

38.13. Flat finite type modules, Part II

05IF We will need the following lemma.

0CU6 Lemma 38.13.1. Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{p} \subset R$. Set $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N \otimes_R \kappa(\mathfrak{p})$. Then we can find a $g \in S$ with $g \notin \mathfrak{q}$ such that $\overline{g} \in \mathfrak{r}$ for all $\mathfrak{r} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$ such that $\mathfrak{r} \not\subset \overline{\mathfrak{q}}$.

Proof. Namely, if $\mathrm{Ass}_{\overline{S}}(\overline{N}) = \{\mathfrak{r}_1, \ldots, \mathfrak{r}_n\}$ (finiteness by Algebra, Lemma 10.63.5), then after renumbering we may assume that

$$\mathfrak{r}_1 \subset \overline{\mathfrak{q}}, \ldots, \mathfrak{r}_r \subset \overline{\mathfrak{q}}, \quad \mathfrak{r}_{r+1} \not\subset \overline{\mathfrak{q}}, \ldots, \mathfrak{r}_n \not\subset \overline{\mathfrak{q}}$$

Since $\overline{\mathfrak{q}}$ is a prime ideal we see that the product $\mathfrak{r}_{r+1} \ldots \mathfrak{r}_n$ is not contained in $\overline{\mathfrak{q}}$ and hence we can pick an element a of $\overline{S}$ contained in $\mathfrak{r}_{r+1}, \ldots, \mathfrak{r}_n$ but not in $\overline{\mathfrak{q}}$. If there exists $g \in S$ mapping to a, then g works. In general we can find a nonzero element $\lambda \in \kappa(\mathfrak{p})$ such that λa is the image of a $g \in S$. $\square$

The following lemma has a sligthly stronger variant Lemma 38.13.4 below.

05IG **Lemma 38.13.2.** Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module which is flat as an R-module. Let M be an R-module. Let $\mathfrak q$ be a prime of S lying over $\mathfrak p \subset R$. Then

$$\mathfrak q \in \text{WeakAss}_S(M \otimes_R N) \Leftrightarrow \Big(\mathfrak p \in \text{WeakAss}_R(M) \text{ and } \overline{\mathfrak q} \in \text{Ass}_{\overline{S}}(\overline{N})\Big)$$

Here $\overline{S} = S \otimes_R \kappa(\mathfrak p)$, $\overline{\mathfrak q} = \mathfrak q\overline{S}$, and $\overline{N} = N \otimes_R \kappa(\mathfrak p)$.

Proof. Pick $g \in S$ as in Lemma 38.13.1. Apply Proposition 38.12.4 to the morphism of schemes $\text{Spec}(S_g) \to \text{Spec}(R)$, the quasi-coherent module associated to N_g, and the points corresponding to the primes $\mathfrak qS_g$ and $\mathfrak p$. Translating into algebra we obtain a commutative diagram of rings

$$\begin{tikzcd} S \arrow[r] & S_g \arrow[r] & S' \\ & R \arrow[lu] \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak q \arrow[r, dash] & \mathfrak qS_g \arrow[r, dash] & \mathfrak q' \\ & \mathfrak p \arrow[lu, dash] \arrow[u, dash] \arrow[r, dash] & \mathfrak p' \arrow[u, dash] \end{tikzcd}$$

endowed with primes as shown, the horizontal arrows are étale, and $N \otimes_S S'$ is projective as an R'-module. Set $N' = N \otimes_S S'$, $M' = M \otimes_R R'$, $\overline{S}' = S' \otimes_{R'} \kappa(\mathfrak q')$, $\overline{\mathfrak q}' = \mathfrak q'\overline{S}'$, and

$$\overline{N}' = N' \otimes_{R'} \kappa(\mathfrak p') = \overline{N} \otimes_{\overline{S}} \overline{S}'$$

By Lemma 38.2.8 we have

$$\text{WeakAss}_{S'}(M' \otimes_{R'} N') = (\text{Spec}(S') \to \text{Spec}(S))^{-1}\text{WeakAss}_S(M \otimes_R N)$$
$$\text{WeakAss}_{R'}(M') = (\text{Spec}(R') \to \text{Spec}(R))^{-1}\text{WeakAss}_R(M)$$
$$\text{Ass}_{\overline{S}'}(\overline{N}') = (\text{Spec}(\overline{S}') \to \text{Spec}(\overline{S}))^{-1}\text{Ass}_{\overline{S}}(\overline{N})$$

Use Algebra, Lemma 10.66.9 for $\overline{N}$ and $\overline{N}'$. In particular we have

$$\mathfrak q \in \text{WeakAss}_S(M \otimes_R N) \Leftrightarrow \mathfrak q' \in \text{WeakAss}_{S'}(M' \otimes_{R'} N')$$
$$\mathfrak p \in \text{WeakAss}_R(M) \Leftrightarrow \mathfrak p' \in \text{WeakAss}_{R'}(M')$$
$$\overline{\mathfrak q} \in \text{Ass}_{\overline{S}}(\overline{N}) \Leftrightarrow \overline{\mathfrak q}' \in \text{WeakAss}_{\overline{S}'}(\overline{N}')$$

Our careful choice of g and the formula for $\text{Ass}_{\overline{S}'}(\overline{N}')$ above shows that

0CU7 (38.13.2.1) $$\text{if } \mathfrak r' \in \text{Ass}_{\overline{S}'}(\overline{N}') \text{ lies over } \mathfrak r \subset \overline{S} \text{ then } \mathfrak r \subset \overline{\mathfrak q}$$

This will be a key observation later in the proof. We will use the characterization of weakly associated primes given in Algebra, Lemma 10.66.2 without further mention.

Suppose that $\overline{\mathfrak q} \not\in \text{Ass}_{\overline{S}}(\overline{N})$. Then $\overline{\mathfrak q}' \not\in \text{Ass}_{\overline{S}'}(\overline{N}')$. By Algebra, Lemmas 10.63.9, 10.63.5, and 10.15.2 there exists an element $\overline{a}' \in \overline{\mathfrak q}'$ which is not a zerodivisor on $\overline{N}'$. After replacing $\overline{a}'$ by $\lambda\overline{a}'$ for some nonzero $\lambda \in \kappa(\mathfrak p)$ we can find $a' \in \mathfrak q'$ mapping to $\overline{a}'$. By Lemma 38.7.6 the map $a' : N'_{\mathfrak p'} \to N'_{\mathfrak p'}$ is $R'_{\mathfrak p'}$-universally injective. In particular we see that $a' : M' \otimes_{R'} N' \to M' \otimes_{R'} N'$ is injective after localizing at $\mathfrak p'$ and hence after localizing at $\mathfrak q'$. Clearly this implies that $\mathfrak q' \not\in \text{WeakAss}_{S'}(M' \otimes_{R'} N')$. We conclude that $\mathfrak q \in \text{WeakAss}_S(M \otimes_R N)$ implies $\overline{\mathfrak q} \in \text{Ass}_{\overline{S}}(\overline{N})$.

Assume $\mathfrak q \in \text{WeakAss}_S(M \otimes_R N)$. We want to show $\mathfrak p \in \text{WeakAss}_S(M)$. Let $z \in M \otimes_R N$ be an element such that $\mathfrak q$ is minimal over $J = \text{Ann}_S(z)$. Let $f_i \in \mathfrak p$,

$i \in I$ be a set of generators of the ideal $\mathfrak p$. Since $\mathfrak q$ lies over $\mathfrak p$, for every i we can choose an $n_i \geq 1$ and $g_i \in S$, $g_i \notin \mathfrak q$ with $g_i f_i^{n_i} \in J$, i.e., $g_i f_i^{n_i} z = 0$. Let $z' \in (M' \otimes_{R'} N')_{\mathfrak p'}$ be the image of z. Observe that z' is nonzero because z has nonzero image in $(M \otimes_R N)_{\mathfrak q}$ and because $S_{\mathfrak q} \to S'_{\mathfrak q'}$ is faithfully flat. We claim that $f_i^{n_i} z' = 0$.

Proof of the claim: Let $g'_i \in S'$ be the image of g_i. By the key observation (38.13.2.1) we find that the image $\overline{g}'_i \in \overline{S}'$ is not contained in $\mathfrak r'$ for any $\mathfrak r' \in \text{Ass}_{\overline{S}'}(\overline{N})$. Hence by Lemma 38.7.6 we see that $g'_i : N'_{\mathfrak p'} \to N'_{\mathfrak p'}$ is $R'_{\mathfrak p'}$-universally injective. In particular we see that $g'_i : M' \otimes_{R'} N' \to M' \otimes_{R'} N'$ is injective after localizating at $\mathfrak p'$. The claim follows because $g_i f_i^{n_i} z' = 0$.

Our claim shows that the annihilator of z' in $R'_{\mathfrak p'}$ contains the elements $f_i^{n_i}$. As $R \to R'$ is étale we have $\mathfrak p' R'_{\mathfrak p'} = \mathfrak p R'_{\mathfrak p'}$ by Algebra, Lemma 10.143.5. Hence the annihilator of z' in $R'_{\mathfrak p'}$ has radical equal to $\mathfrak p' R_{\mathfrak p'}$ (here we use z' is not zero). On the other hand

$$z' \in (M' \otimes_{R'} N')_{\mathfrak p'} = M'_{\mathfrak p'} \otimes_{R'_{\mathfrak p'}} N'_{\mathfrak p'}$$

The module $N'_{\mathfrak p'}$ is projective over the local ring $R'_{\mathfrak p'}$ and hence free (Algebra, Theorem 10.85.4). Thus we can find a finite free direct summand $F' \subset N'_{\mathfrak p'}$ such that $z' \in M'_{\mathfrak p'} \otimes_{R'_{\mathfrak p'}} F'$. If F' has rank n, then we deduce that $\mathfrak p' R'_{\mathfrak p'} \in \text{WeakAss}_{R'_{\mathfrak p'}}({M'_{\mathfrak p'}}^{\oplus n})$. This implies $\mathfrak p' R'_{\mathfrak p'} \in \text{WeakAss}(M'_{\mathfrak p'})$ for example by Algebra, Lemma 10.66.4. Then $\mathfrak p' \in \text{WeakAss}_{R'}(M')$ which in turn gives $\mathfrak p \in \text{WeakAss}_R(M)$. This finishes the proof of the implication "$\Rightarrow$" of the equivalence of the lemma.

Assume that $\mathfrak p \in \text{WeakAss}_R(M)$ and $\overline{\mathfrak q} \in \text{Ass}_{\overline{S}}(\overline{N})$. We want to show that $\mathfrak q$ is weakly associated to $M \otimes_R N$. Note that $\overline{\mathfrak q}'$ is a maximal element of $\text{Ass}_{\overline{S}'}(\overline{N}')$. This is a consequence of (38.13.2.1) and the fact that there are no inclusions among the primes of $\overline{S}'$ lying over $\overline{\mathfrak q}$ (as fibres of étale morphisms are discrete Morphisms, Lemma 29.36.7). Thus, after replacing $R, S, \mathfrak p, \mathfrak q, M, N$ by $R', S', \mathfrak p', \mathfrak q', M', N'$ we may assume, in addition to the assumptions of the lemma, that

1. $\mathfrak p \in \text{WeakAss}_R(M)$,
2. $\overline{\mathfrak q} \in \text{Ass}_{\overline{S}}(\overline{N})$,
3. N is projective as an R-module, and
4. $\overline{\mathfrak q}$ is maximal in $\text{Ass}_{\overline{S}}(\overline{N})$.

There is one more reduction, namely, we may replace R, S, M, N by their localizations at $\mathfrak p$. This leads to one more condition, namely,

5. R is a local ring with maximal ideal $\mathfrak p$.

We will finish by showing that (1) – (5) imply $\mathfrak q \in \text{WeakAss}(M \otimes_R N)$.

Since R is local and $\mathfrak p \in \text{WeakAss}_R(M)$ we can pick a $y \in M$ whose annihilator I has radical equal to $\mathfrak p$. Write $\overline{\mathfrak q} = (\overline{g}_1, \ldots, \overline{g}_n)$ for some $\overline{g}_i \in \overline{S}$. Choose $g_i \in S$ mapping to $\overline{g}_i$. Then $\mathfrak q = \mathfrak p S + g_1 S + \ldots + g_n S$. Consider the map

$$\Psi : N/IN \longrightarrow (N/IN)^{\oplus n}, \quad z \longmapsto (g_1 z, \ldots, g_n z).$$

This is a homomorphism of projective R/I-modules. The local ring R/I is auto-associated (More on Algebra, Definition 15.15.1) as $\mathfrak p/I$ is locally nilpotent. The map $\Psi \otimes \kappa(\mathfrak p)$ is not injective, because $\overline{\mathfrak q} \in \text{Ass}_{\overline{S}}(\overline{N})$. Hence More on Algebra, Lemma 15.15.4 implies Ψ is not injective. Pick $z \in N/IN$ nonzero in the kernel of Ψ. The annihilator $J = \text{Ann}_S(z)$ contains IS and g_i by construction. Thus

$\sqrt{J} \subset S$ contains $\mathfrak{q}$. Let $\mathfrak{s} \subset S$ be a prime minimal over J. Then $\mathfrak{q} \subset \mathfrak{s}$, $\mathfrak{s}$ lies over $\mathfrak{p}$, and $\mathfrak{s} \in \text{WeakAss}_S(N/IN)$. The last fact by definition of weakly associated primes. Apply the "$\Rightarrow$" part of the lemma (which we've already proven) to the ring map $R \to S$ and the modules R/I and N to conclude that $\overline{\mathfrak{s}} \in \text{Ass}_{\overline{S}}(\overline{N})$. Since $\overline{\mathfrak{q}} \subset \overline{\mathfrak{s}}$ the maximality of $\overline{\mathfrak{q}}$, see condition (4) above, implies that $\overline{\mathfrak{q}} = \overline{\mathfrak{s}}$. This shows that $\mathfrak{q} = \mathfrak{s}$ and we conlude what we want. □

05IH Lemma 38.13.3. Let S be a scheme. Let $f : X \to S$ be locally of finite type. Let $x \in X$ with image $s \in S$. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. If $\mathcal{F}$ is flat at x over S, then

$$x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow s \in \text{WeakAss}_S(\mathcal{G}) \text{ and } x \in \text{Ass}_{X_s}(\mathcal{F}_s).$$

Proof. In this paragraph we reduce to f being of finite presentation. The question is local on X and S, hence we may assume X and S are affine. Write $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$ over S. Denote $t = i(x) \in \mathbf{A}^n_S$. Note that $i_*\mathcal{F}$ is a finite type quasi-coherent sheaf on $\mathbf{A}^n_S$ which is flat at t over S and note that

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = i_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbf{A}^n_S}} p^*\mathcal{G}$$

where $p : \mathbf{A}^n_S \to S$ is the projection. Note that t is a weakly associated point of $i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if x is a weakly associated point of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$, see Divisors, Lemma 31.6.3. Similarly $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $t \in \text{Ass}_{\mathbf{A}^n_s}((i_*\mathcal{F})_s)$ (see Algebra, Lemma 10.63.14). Hence it suffices to prove the lemma in case $X = \mathbf{A}^n_S$. Thus we may assume that $X \to S$ is of finite presentation.

In this paragraph we reduce to $\mathcal{F}$ being of finite presentation and flat over S. Choose an elementary étale neighbourhood $e : (S', s') \to (S, s)$ and an open $V \subset X \times_S \text{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 38.10.3. Let $x' \in X' = X \times_S S'$ be the unique point mapping to x and s'. Then it suffices to prove the statement for $X' \to S'$, x', s', $(X' \to X)^*\mathcal{F}$, and $e^*\mathcal{G}$, see Lemma 38.2.8. Let $v \in V$ the unique point mapping to x' and let $s' \in \text{Spec}(\mathcal{O}_{S',s'})$ be the closed point. Then $\mathcal{O}_{V,v} = \mathcal{O}_{X',x'}$ and $\mathcal{O}_{\text{Spec}(\mathcal{O}_{S',s'}),s'} = \mathcal{O}_{S',s'}$ and similarly for the stalks of pullbacks of $\mathcal{F}$ and $\mathcal{G}$. Also $V_{s'} \subset X'_{s'}$ is an open subscheme. Since the condition of being a weakly associated point depend only on the stalk of the sheaf, we may replace $X' \to S'$, x', s', $(X' \to X)^*\mathcal{F}$, and $e^*\mathcal{G}$ by $V \to \text{Spec}(\mathcal{O}_{S',s'})$, v, s', $(V \to X)^*\mathcal{F}$, and $(\text{Spec}(\mathcal{O}_{S',s'}) \to S)^*\mathcal{G}$. Thus we may assume that f is of finite presentation and $\mathcal{F}$ of finite presentation and flat over S.

Assume f is of finite presentation and $\mathcal{F}$ of finite presentation and flat over S. After shrinking X and S to affine neighbourhoods of x and s, this case is handled by Lemma 38.13.2. □

05II Lemma 38.13.4. Let $R \to S$ be a ring map which is essentially of finite type. Let N be a localization of a finite S-module flat over R. Let M be an R-module. Then

$$\text{WeakAss}_S(M \otimes_R N) = \bigcup\nolimits_{\mathfrak{p} \in \text{WeakAss}_R(M)} \text{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

Proof. This lemma is a translation of Lemma 38.13.3 into algebra. Details of translation omitted. □

05IJ Lemma 38.13.5. Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have

$$\text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \bigcup\nolimits_{s \in \text{WeakAss}_S(\mathcal{G})} \text{Ass}_{X_s}(\mathcal{F}_s)$$

Proof. Immediate consequence of Lemma 38.13.3. □

05IK Theorem 38.13.6. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) $X \to S$ is locally of finite presentation,
(2) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and
(3) the set of weakly associated points of S is locally finite in S.

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module of finite presentation and flat over S.

Proof. Let $x \in X$ be such that $\mathcal{F}$ is flat at x over S. We have to find an open neighbourhood of x such that $\mathcal{F}$ restricts to a S-flat finitely presented module on this neighbourhood. The problem is local on X and S, hence we may assume that X and S are affine. As $\mathcal{F}_x$ is a finitely presented $\mathcal{O}_{X,x}$-module by Lemma 38.10.9 we conclude from Algebra, Lemma 10.126.5 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$ and a map $\varphi : \mathcal{F}' \to \mathcal{F}$ which induces an isomorphism $\varphi_x : \mathcal{F}'_x \to \mathcal{F}_x$. In particular we see that $\mathcal{F}'$ is flat over S at x, hence by openness of flatness More on Morphisms, Theorem 37.15.1 we see that after shrinking X we may assume that $\mathcal{F}'$ is flat over S. As $\mathcal{F}$ is of finite type after shrinking X we may assume that φ is surjective, see Modules, Lemma 17.9.4 or alternatively use Nakayama's lemma (Algebra, Lemma 10.20.1). By Lemma 38.13.5 we have

$$\text{WeakAss}_X(\mathcal{F}') \subset \bigcup\nolimits_{s \in \text{WeakAss}(S)} \text{Ass}_{X_s}(\mathcal{F}'_s)$$

As $\text{WeakAss}(S)$ is finite by assumption and since $\text{Ass}_{X_s}(\mathcal{F}'_s)$ is finite by Divisors, Lemma 31.2.5 we conclude that $\text{WeakAss}_X(\mathcal{F}')$ is finite. Using Algebra, Lemma 10.15.2 we may, after shrinking X once more, assume that $\text{WeakAss}_X(\mathcal{F}')$ is contained in the generalization of x. Now consider $\mathcal{K} = \text{Ker}(\varphi)$. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F}')$ (by Divisors, Lemma 31.5.4) but on the other hand, φ_x is an isomorphism, also $\varphi_{x'}$ is an isomorphism for all $x' \rightsquigarrow x$. We conclude that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$ whence $\mathcal{K} = 0$ by Divisors, Lemma 31.5.5. □

05IL Lemma 38.13.7. Let $R \to S$ be a ring map of finite presentation. Let M be a finite S-module. Assume $\text{WeakAss}_S(S)$ is finite. Then

$$U = \{\mathfrak{q} \subset S \mid M_\mathfrak{q} \text{ flat over } R\}$$

is open in $\text{Spec}(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is a finitely presented S_g-module flat over R.

Proof. Follows immediately from Theorem 38.13.6. □

05IM Lemma 38.13.8. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Assume the set of weakly associated points of S is locally finite in S. Then the set of points $x \in X$ where f is flat is an open subscheme $U \subset X$ and $U \to S$ is flat and locally of finite presentation.

Proof. The problem is local on X and S, hence we may assume that X and S are affine. Then $X \to S$ corresponds to a finite type ring map $A \to B$. Choose a surjection $A[x_1, \ldots, x_n] \to B$ and consider B as an $A[x_1, \ldots, x_n]$-module. An application of Lemma 38.13.7 finishes the proof. $\square$

05IN **Lemma 38.13.9.** Let $f : X \to S$ be a morphism of schemes which is locally of finite type and flat. If S is integral, then f is locally of finite presentation.

Proof. Special case of Lemma 38.13.8. $\square$

053G **Proposition 38.13.10.** Let R be a domain. Let $R \to S$ be a ring map of finite type. Let M be a finite S-module.

(1) If S is flat over R, then S is a finitely presented R-algebra.
(2) If M is flat as an R-module, then M is finitely presented as an S-module.

Proof. Part (1) is a special case of Lemma 38.13.9. For Part (2) choose a surjection $R[x_1, \ldots, x_n] \to S$. By Lemma 38.13.7 we find that M is finitely presented as an $R[x_1, \ldots, x_n]$-module. We conclude by Algebra, Lemma 10.6.4. $\square$

05IQ **Lemma 38.13.11** (Finite type version of Theorem 38.13.6)**.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) $X \to S$ is locally of finite type,
(2) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and
(3) the set of weakly associated points of S is locally finite in S.

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is flat over S and locally finitely presented relative to S (see More on Morphisms, Definition 37.58.1).

Proof. The question is local on X and S. Thus we may assume X and S are affine. Then we may choose a closed immersion $i : X \to \mathbf{A}^n_S$. We apply Theorem 38.13.6 to $X' = \mathbf{A}^n_S \to S$ and the quasi-coherent module $\mathcal{F}' = i_*\mathcal{F}$ of finite type and we find that

$$U' = \{x' \in X' \mid \mathcal{F}' \text{ flat at } x' \text{ over } S\}$$

is open in X' and that $\mathcal{F}'|_{U'}$ is of finite presentation. Since $\mathcal{F}'$ restricts to zero on $X' \setminus i(X)$ and since $\mathcal{F}'_{i(x)} \cong \mathcal{F}_x$ for all $x \in X$ we see that

$$U' = i(U) \amalg (X' \setminus i(X))$$

Hence $U = i^{-1}(U')$ is open. Moreover, it is clear that $\mathcal{F}'|_{U'} = (i|_U)_*(\mathcal{F}|_U)$. Hence we conclude that $\mathcal{F}|_U$ is finitely presented relative to S by More on Morphisms, Lemmas 37.58.3 and 37.58.4. $\square$

05IR **Lemma 38.13.12.** Let $R \to S$ be a ring map of finite type. Let M be a finite S-module. Assume $\text{WeakAss}_R(R)$ is finite. Then

$$U = \{\mathfrak{q} \subset S \mid M_\mathfrak{q} \text{ flat over } R\}$$

is open in $\text{Spec}(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is flat over R and an S_g-module finitely presented relative to R (see More on Algebra, Definition 15.80.2).

Proof. This is Lemma 38.13.11 translated into algebra. $\square$

38.14. Examples of relatively pure modules

05IS In the short section we discuss some examples of results that will serve as motivation for the notion of a relatively pure module and the concept of an impurity which we will introduce later. Each of the examples is stated as a lemma. Note the similarity with the condition on associated primes to the conditions appearing in Lemmas 38.7.4, 38.8.3, 38.8.4, and 38.9.1. See also Algebra, Lemma 10.65.1 for a discussion.

05FV **Lemma 38.14.1.** Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume

(1) N is projective as an R-module, and
(2) $S/\mathfrak{m}S$ is Noetherian and $N/\mathfrak{m}N$ is a finite $S/\mathfrak{m}S$-module.

Then for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. Note that the hypotheses of Lemmas 38.7.1 and 38.7.6 are satisfied. We will use the conclusions of these lemmas without further mention. Let $\Sigma \subset S$ be the multiplicative set of elements which are not zerodivisors on $N/\mathfrak{m}N$. The map $N \to \Sigma^{-1}N$ is R-universally injective. Hence we see that any $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ is also an associated prime of $\Sigma^{-1}N \otimes_R \kappa(\mathfrak{p})$. Clearly this implies that $\mathfrak{q}$ corresponds to a prime of $\Sigma^{-1}S$. Thus $\mathfrak{q} \subset \mathfrak{q}'$ where $\mathfrak{q}'$ corresponds to an associated prime of $N/\mathfrak{m}N$ and we win. $\square$

The following lemma gives another (slightly silly) example of this phenomenon.

05IT **Lemma 38.14.2.** Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map. Let N be an S-module. If N is I-adically complete, then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + IS \neq S$.

Proof. Let $S^\wedge$ denote the I-adic completion of S. Note that N is an $S^\wedge$-module, hence also $N \otimes_R M$ is an $S^\wedge$-module. Let $z \in N \otimes_R M$ be an element such that $\mathfrak{q} = \text{Ann}_S(z)$. Since $z \neq 0$ we see that $\text{Ann}_{S^\wedge}(z) \neq S^\wedge$. Hence $\mathfrak{q}S^\wedge \neq S^\wedge$. Hence there exists a maximal ideal $\mathfrak{m} \subset S^\wedge$ with $\mathfrak{q}S^\wedge \subset \mathfrak{m}$. Since $IS^\wedge \subset \mathfrak{m}$ by Algebra, Lemma 10.96.6 we win. $\square$

Note that the following lemma gives an alternative proof of Lemma 38.14.1 as a projective module over a local ring is free, see Algebra, Theorem 10.85.4.

05IU **Lemma 38.14.3.** Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is isomorphic as an R-module to a direct sum of finite R-modules. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. Write $N = \bigoplus_{i \in I} M_i$ with each M_i a finite R-module. Let M be an R-module and let $\mathfrak{q} \subset S$ be an associated prime of $N \otimes_R M$ such that $\mathfrak{q} + \mathfrak{m}S = S$. Let $z \in N \otimes_R M$ be an element with $\mathfrak{q} = \text{Ann}_S(z)$. After modifying the direct sum decomposition a little bit we may assume that $z \in M_1 \otimes_R M$ for some element $1 \in I$. Write $1 = f + \sum x_j g_j$ for some $f \in \mathfrak{q}$, $x_j \in \mathfrak{m}$, and $g_j \in S$. For any $g \in S$ denote g' the R-linear map

$$M_1 \to N \xrightarrow{g} N \to M_1$$

where the first arrow is the inclusion map, the second arrow is multiplication by g and the third arrow is the projection map. Because each $x_j \in R$ we obtain the equality

$$f' + \sum x_j g'_j = \mathrm{id}_{M_1} \in \mathrm{End}_R(M_1)$$

By Nakayama's lemma (Algebra, Lemma 10.20.1) we see that f' is surjective, hence by Algebra, Lemma 10.16.4 we see that f' is an isomorphism. In particular the map

$$M_1 \otimes_R M \to N \otimes_R M \xrightarrow{f} N \otimes_R M \to M_1 \otimes_R M$$

is an isomorphism. This contradicts the assumption that $fz = 0$. □

05IV Lemma 38.14.4. Let R be a henselian local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is countably generated and Mittag-Leffler as an R-module. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. This lemma reduces to Lemma 38.14.3 by Algebra, Lemma 10.153.13. □

Suppose $f : X \to S$ is a morphism of schemes and $\mathcal{F}$ is a quasi-coherent module on X. Let $\xi \in \mathrm{Ass}_{X/S}(\mathcal{F})$ and let $Z = \overline{\{\xi\}}$. Picture

$$\begin{tikzcd} \xi \arrow[d, mapsto] \\ f(\xi) \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[dr] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

Note that $f(Z) \subset \overline{\{f(\xi)\}}$ and that $f(Z)$ is closed if and only if equality holds, i.e., $f(Z) = \overline{\{f(\xi)\}}$. It follows from Lemma 38.14.1 that if S, X are affine, the fibres X_s are Noetherian, $\mathcal{F}$ is of finite type, and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module, then $f(Z) = \overline{\{f(\xi)\}}$ is a closed subset. Slightly different analogous statements holds for the cases described in Lemmas 38.14.2, 38.14.3, and 38.14.4.

38.15. Impurities

05IW We want to formalize the phenomenon of which we gave examples in Section 38.14 in terms of specializations of points of $\mathrm{Ass}_{X/S}(\mathcal{F})$. We also want to work locally around a point $s \in S$. In order to do so we make the following definitions.

05FW Situation 38.15.1. Here S, X are schemes and $f : X \to S$ is a finite type morphism. Also, $\mathcal{F}$ is a finite type quasi-coherent $\mathcal{O}_X$-module. Finally s is a point of S.

In this situation consider a morphism $g : T \to S$, a point $t \in T$ with $g(t) = s$, a specialization $t' \rightsquigarrow t$, and a point $\xi \in X_T$ in the base change of X lying over t'. Picture

05IX (38.15.1.1)
$$\begin{tikzcd} \xi \arrow[d, mapsto] & & \\ t' \arrow[r, squiggly] & t \arrow[r, mapsto] & s \end{tikzcd} \qquad \begin{tikzcd} X_T \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

Moreover, denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T.

05IY Definition 38.15.2. In Situation 38.15.1 we say a diagram (38.15.1.1) defines an impurity of $\mathcal{F}$ above s if $\xi \in \mathrm{Ass}_{X_T/T}(\mathcal{F}_T)$ and $\overline{\{\xi\}} \cap X_t = \emptyset$. We will indicate this by saying "let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s".

05FX Lemma 38.15.3. In Situation 38.15.1. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \leadsto t, \xi)$ of $\mathcal{F}$ above s such that g is locally of finite presentation and t a closed point of the fibre of g above s.

Proof. Let $(g : T \to S, t' \leadsto t, \xi)$ be any impurity of $\mathcal{F}$ above s. We apply Limits, Lemma 32.14.1 to $t \in T$ and $Z = \overline{\{\xi\}}$ to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

As t' specializes to t we may replace T by the open neighbourhood V of t. Thus we have a commutative diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X_{T'} \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "a"] & T' \arrow[r, "b"] & S \end{tikzcd}$$

where $b \circ a = g$. Let $\xi' \in X_{T'}$ denote the image of ξ. By Divisors, Lemma 31.7.3 we see that $\xi' \in \text{Ass}_{X_{T'}/T'}(\mathcal{F}_{T'})$. Moreover, by construction the closure of $\overline{\{\xi'\}}$ is contained in the closed subset Z' which avoids the fibre $X_{a(t)}$. In this way we see that $(T' \to S, a(t') \leadsto a(t), \xi')$ is an impurity of $\mathcal{F}$ above s.

Thus we may assume that $g : T \to S$ is locally of finite presentation. Let $Z = \overline{\{\xi\}}$. By assumption $Z_t = \emptyset$. By More on Morphisms, Lemma 37.24.1 this means that $Z_{t''} = \emptyset$ for t'' in an open subset of $\overline{\{t\}}$. Since the fibre of $T \to S$ over s is a Jacobson scheme, see Morphisms, Lemma 29.16.10 we find that there exist a closed point $t'' \in \overline{\{t\}}$ such that $Z_{t''} = \emptyset$. Then $(g : T \to S, t' \leadsto t'', \xi)$ is the desired impurity. $\square$

05IZ Lemma 38.15.4. In Situation 38.15.1. Let $(g : T \to S, t' \leadsto t, \xi)$ be an impurity of $\mathcal{F}$ above s. Assume $T = \lim_{i \in I} T_i$ is a directed limit of affine schemes over S. Then for some i the triple $(T_i \to S, t'_i \leadsto t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s.

Proof. The notation in the statement means this: Let $p_i : T \to T_i$ be the projection morphisms, let $t_i = p_i(t)$ and $t'_i = p_i(t')$. Finally $\xi_i \in X_{T_i}$ is the image of ξ. By Divisors, Lemma 31.7.3 it is true that ξ_i is a point of the relative assassin of $\mathcal{F}_{T_i}$ over T_i. Thus the only point is to show that $\overline{\{\xi_i\}} \cap X_{t_i} = \emptyset$ for some i.

First proof. Let $Z_i = \overline{\{\xi_i\}} \subset X_{T_i}$ and $Z = \overline{\{\xi\}} \subset X_T$ endowed with the reduced induced scheme structure. Then $Z = \lim Z_i$ by Limits, Lemma 32.4.4. Choose a field k and a morphism $\text{Spec}(k) \to T$ whose image is t. Then

$$\emptyset = Z \times_T \text{Spec}(k) = (\lim Z_i) \times_{(\lim T_i)} \text{Spec}(k) = \lim Z_i \times_{T_i} \text{Spec}(k)$$

because limits commute with fibred products (limits commute with limits). Each $Z_i \times_{T_i} \text{Spec}(k)$ is quasi-compact because $X_{T_i} \to T_i$ is of finite type and hence

$Z_i \to T_i$ is of finite type. Hence $Z_i \times_{T_i} \mathrm{Spec}(k)$ is empty for some i by Limits, Lemma 32.4.3. Since the image of the composition $\mathrm{Spec}(k) \to T \to T_i$ is t_i we obtain what we want.

Second proof. Set $Z = \overline{\{\xi\}}$. Apply Limits, Lemma 32.14.1 to this situation to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{tikzcd} V \arrow[r, "a"'] \arrow[d] & T' \arrow[d, "b"] \\ T \arrow[r, "g"] & S, \end{tikzcd}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

We may assume V is an affine open of T, hence by Limits, Lemmas 32.4.11 and 32.4.13 we can find an i and an affine open $V_i \subset T_i$ with $V = f_i^{-1}(V_i)$. By Limits, Proposition 32.6.1 after possibly increasing i a bit we can find a morphism $a_i : V_i \to T'$ such that $a = a_i \circ f_i|_V$. The induced morphism $X_{V_i} \to X_{T'}$ maps ξ_i into Z'. As $Z' \cap X_{a(t)} = \emptyset$ we conclude that $(T_i \to S, t_i' \leadsto t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s. $\square$

05J0 Lemma 38.15.5. In Situation 38.15.1. If there exists an impurity $(g : T \to S, t' \leadsto t, \xi)$ of $\mathcal{F}$ above s with g quasi-finite at t, then there exists an impurity $(g : T \to S, t' \leadsto t, \xi)$ such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood.

Proof. Let $(g : T \to S, t' \leadsto t, \xi)$ be an impurity of $\mathcal{F}$ above s such that g is quasi-finite at t. After shrinking T we may assume that g is locally of finite type. Apply More on Morphisms, Lemma 37.41.1 to $T \to S$ and $t \mapsto s$. This gives us a diagram

$$\begin{tikzcd} T \arrow[d] & T \times_S U \arrow[l] \arrow[d] & V \arrow[l] \arrow[dl] \\ S & U \arrow[l] & \end{tikzcd}$$

where $(U, u) \to (S, s)$ is an elementary étale neighbourhood and $V \subset T \times_S U$ is an open neighbourhood of $v = (t, u)$ such that $V \to U$ is finite and such that v is the unique point of V lying over u. Since the morphism $V \to T$ is étale hence flat we see that there exists a specialization $v' \leadsto v$ such that $v' \mapsto t'$. Note that $\kappa(t') \subset \kappa(v')$ is finite separable. Pick any point $\zeta \in X_{v'}$ mapping to $\xi \in X_{t'}$. By Divisors, Lemma 31.7.3 we see that $\zeta \in \mathrm{Ass}_{X_V/V}(\mathcal{F}_V)$. Moreover, the closure $\overline{\{\zeta\}}$ does not meet the fibre X_v as by assumption the closure $\overline{\{\xi\}}$ does not meet X_t. In other words $(V \to S, v' \leadsto v, \zeta)$ is an impurity of $\mathcal{F}$ above S.

Next, let $u' \in U'$ be the image of v' and let $\theta \in X_U$ be the image of ζ. Then $\theta \mapsto u'$ and $u' \leadsto u$. By Divisors, Lemma 31.7.3 we see that $\theta \in \mathrm{Ass}_{X_U/U}(\mathcal{F})$. Moreover, as $\pi : X_V \to X_U$ is finite we see that $\pi\big(\overline{\{\zeta\}}\big) = \overline{\{\pi(\zeta)\}}$. Since v is the unique point of V lying over u we see that $X_u \cap \overline{\{\pi(\zeta)\}} = \emptyset$ because $X_v \cap \overline{\{\zeta\}} = \emptyset$. In this way we conclude that $(U \to S, u' \leadsto u, \theta)$ is an impurity of $\mathcal{F}$ above s and we win. $\square$

05J1 Lemma 38.15.6. In Situation 38.15.1. Assume that S is locally Noetherian. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is quasi-finite at t.

Proof. We may replace S by an affine neighbourhood of s. By Lemma 38.15.3 we may assume that we have an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of such that g is locally of finite type and t a closed point of the fibre of g above s. We may replace T by the reduced induced scheme structure on $\overline{\{t'\}}$. Let $Z = \overline{\{\xi\}} \subset X_T$. By assumption $Z_t = \emptyset$ and the image of $Z \to T$ contains t'. By More on Morphisms, Lemma 37.25.1 there exists a nonempty open $V \subset Z$ such that for any $w \in f(V)$ any generic point ξ' of V_w is in $\text{Ass}_{X_T/T}(\mathcal{F}_T)$. By More on Morphisms, Lemma 37.24.2 there exists a nonempty open $W \subset T$ with $W \subset f(V)$. By More on Morphisms, Lemma 37.52.7 there exists a closed subscheme $T' \subset T$ such that $t \in T'$, $T' \to S$ is quasi-finite at t, and there exists a point $z \in T' \cap W$, $z \rightsquigarrow t$ which does not map to s. Choose any generic point ξ' of the nonempty scheme V_z. Then $(T' \to S, z \rightsquigarrow t, \xi')$ is the desired impurity. □

In the following we will use the henselization $S^h = \text{Spec}(\mathcal{O}_{S,s}^h)$ of S at s, see Étale Cohomology, Definition 59.33.2. Since $S^h \to S$ maps to closed point of S^h to s and induces an isomorphism of residue fields, we will indicate $s \in S^h$ this closed point also. Thus $(S^h, s) \to (S, s)$ is a morphism of pointed schemes.

05J2 Lemma 38.15.7. In Situation 38.15.1. If there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s then there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s where $(T, t) \to (S, s)$ is an elementary étale neighbourhood.

Proof. We may replace S by an affine neighbourhood of s. Say $S = \text{Spec}(A)$ and s corresponds to the prime $\mathfrak{p} \subset A$. Then $\mathcal{O}_{S,s}^h = \text{colim}_{(T,t)} \Gamma(T, \mathcal{O}_T)$ where the limit is over the opposite of the cofiltered category of affine elementary étale neighbourhoods (T, t) of (S, s), see More on Morphisms, Lemma 37.35.5 and its proof. Hence $S^h = \lim_i T_i$ and we win by Lemma 38.15.4. □

05J3 Lemma 38.15.8. In Situation 38.15.1 the following are equivalent

(1) there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s where S^h is the henselization of S at s,
(2) there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, and
(3) there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that $T \to S$ is quasi-finite at t.

Proof. As an étale morphism is locally quasi-finite it is clear that (2) implies (3). We have seen that (3) implies (2) in Lemma 38.15.5. We have seen that (1) implies (2) in Lemma 38.15.7. Finally, if $(T \to S, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then we can choose a factorization $S^h \to T \to S$ of the structure morphism $S^h \to S$. Choose any point $s' \in S^h$ mapping to t' and choose any $\xi' \in X_{s'}$ mapping to $\xi \in X_{t'}$. Then $(S^h \to S, s' \rightsquigarrow s, \xi')$ is an impurity of $\mathcal{F}$ above s. We omit the details. □

38.16. Relatively pure modules

05BB The notion of a module pure relative to a base was introduced in [GR71].

05J4 Definition 38.16.1. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.

(1) Let $s \in S$. We say $\mathcal{F}$ is pure along X_s if there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with $(T, t) \to (S, s)$ an elementary étale neighbourhood.
(2) We say $\mathcal{F}$ is universally pure along X_s if there does not exist any impurity of $\mathcal{F}$ above s.
(3) We say that X is pure along X_s if $\mathcal{O}_X$ is pure along X_s.
(4) We say $\mathcal{F}$ is universally S-pure, or universally pure relative to S if $\mathcal{F}$ is universally pure along X_s for every $s \in S$.
(5) We say $\mathcal{F}$ is S-pure, or pure relative to S if $\mathcal{F}$ is pure along X_s for every $s \in S$.
(6) We say that X is S-pure or pure relative to S if $\mathcal{O}_X$ is pure relative to S.

We intentionally restrict ourselves here to morphisms which are of finite type and not just morphisms which are locally of finite type, see Remark 38.16.2 for a discussion. In the situation of the definition Lemma 38.15.8 tells us that the following are equivalent

(1) $\mathcal{F}$ is pure along X_s,
(2) there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ with g quasi-finite at t,
(3) there does not exist any impurity of the form $(S^h \to S, s' \rightsquigarrow s, \xi)$, where S^h is the henselization of S at s.

If we denote $X^h = X \times_S S^h$ and $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h, then we can formulate the last condition in the following more positive way:

(4) All points of $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$ specialize to points of X_s.

In particular, it is clear that $\mathcal{F}$ is pure along X_s if and only if the pullback of $\mathcal{F}$ to $X \times_S \text{Spec}(\mathcal{O}_{S,s})$ is pure along X_s.

05J5 Remark 38.16.2. Let $f : X \to S$ be a morphism which is locally of finite type and $\mathcal{F}$ a quasi-coherent finite type $\mathcal{O}_X$-module. In this case it is still true that (1) and (2) above are equivalent because the proof of Lemma 38.15.5 does not use that f is quasi-compact. It is also clear that (3) and (4) are equivalent. However, we don't know if (1) and (3) are equivalent. In this case it may sometimes be more convenient to define purity using the equivalent conditions (3) and (4) as is done in [GR71]. On the other hand, for many applications it seems that the correct notion is really that of being universally pure.

A natural question to ask is if the property of being pure relative to the base is preserved by base change, i.e., if being pure is the same thing as being universally pure. It turns out that this is true over Noetherian base schemes (see Lemma 38.16.5), or if the sheaf is flat (see Lemmas 38.18.3 and 38.18.4). It is not true in general, even if the morphism and the sheaf are of finite presentation, see Examples, Section 110.39 for a counter example. First we match our usage of "universally" to the usual notion.

05J6 Lemma 38.16.3. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. The following are equivalent

(1) $\mathcal{F}$ is universally pure along X_s, and

(2) for every morphism of pointed schemes $(S', s') \to (S, s)$ the pullback $\mathcal{F}_{S'}$ is pure along $X_{s'}$.

In particular, $\mathcal{F}$ is universally pure relative to S if and only if every base change $\mathcal{F}_{S'}$ of $\mathcal{F}$ is pure relative to S'.

Proof. This is formal. $\square$

05J7 Lemma 38.16.4. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. If $S' \to S$ is quasi-finite at s' and $\mathcal{F}$ is pure along X_s, then $\mathcal{F}_{S'}$ is pure along $X_{s'}$.

Proof. It $(T \to S', t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}_{S'}$ above s' with $T \to S'$ quasi-finite at t, then $(T \to S, t' \to t, \xi)$ is an impurity of $\mathcal{F}$ above s with $T \to S$ quasi-finite at t, see Morphisms, Lemma 29.20.12. Hence the lemma follows immediately from the characterization (2) of purity given following Definition 38.16.1. $\square$

05J8 Lemma 38.16.5. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. If $\mathcal{O}_{S,s}$ is Noetherian then $\mathcal{F}$ is pure along X_s if and only if $\mathcal{F}$ is universally pure along X_s.

Proof. First we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$, i.e., we may assume that S is Noetherian. Next, use Lemma 38.15.6 and characterization (2) of purity given in discussion following Definition 38.16.1 to conclude. $\square$

Purity satisfies flat descent.

05J9 Lemma 38.16.6. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. Assume $S' \to S$ is flat at s'.

(1) If $\mathcal{F}_{S'}$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s.
(2) If $\mathcal{F}_{S'}$ is universally pure along $X_{s'}$, then $\mathcal{F}$ is universally pure along X_s.

Proof. Let $(T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Set $T_1 = T \times_S S'$, and let t_1 be the unique point of T_1 mapping to t and s'. Since $T_1 \to T$ is flat at t_1, see Morphisms, Lemma 29.25.8, there exists a specialization $t'_1 \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Algebra, Section 10.41. Choose a point $\xi_1 \in X_{t'_1}$ which corresponds to a generic point of $\mathrm{Spec}(\kappa(t'_1) \otimes_{\kappa(t')} \kappa(\xi))$, see Schemes, Lemma 26.17.5. By Divisors, Lemma 31.7.3 we see that $\xi_1 \in \mathrm{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. As the Zariski closure of $\{\xi_1\}$ in X_{T_1} maps into the Zariski closure of $\{\xi\}$ in X_T we conclude that this closure is disjoint from X_{t_1}. Hence $(T_1 \to S', t'_1 \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_{S'}$ above s'. In other words we have proved the contrapositive to part (2) of the lemma. Finally, if $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then $(T_1, t_1) \to (S', s')$ is an elementary étale neighbourhood too, and in this way we see that (1) holds. $\square$

05K1 Lemma 38.16.7. Let $i : Z \to X$ be a closed immersion of schemes of finite type over a scheme S. Let $s \in S$. Let $\mathcal{F}$ be a finite type, quasi-coherent sheaf on Z. Then $\mathcal{F}$ is (universally) pure along Z_s if and only if $i_*\mathcal{F}$ is (universally) pure along X_s.

Proof. This follows from Divisors, Lemma 31.8.3. $\square$

38.17. Examples of relatively pure sheaves

05K2 Here are some example cases where it is possible to see what purity means.

05K3 **Lemma 38.17.1.** Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.

(1) If the support of $\mathcal{F}$ is proper over S, then $\mathcal{F}$ is universally pure relative to S.
(2) If f is proper, then $\mathcal{F}$ is universally pure relative to S.
(3) If f is proper, then X is universally pure relative to S.

Proof. First we reduce (1) to (2). Namely, let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$. Let $i : Z \to X$ be the corresponding closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some finite type quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$, see Morphisms, Section 29.5. In case (1) $Z \to S$ is proper by assumption. Thus by Lemma 38.16.7 case (1) reduces to case (2).

Assume f is proper. Let $(g : T \to S, t' \leadsto t, \xi)$ be an impurity of $\mathcal{F}$ above $s \in S$. Since f is proper, it is universally closed. Hence $f_T : X_T \to T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. □

05K4 **Lemma 38.17.2.** Let $f : X \to S$ be a separated, finite type morphism of schemes. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Assume that $\text{Supp}(\mathcal{F}_s)$ is finite for every $s \in S$. Then the following are equivalent

(1) $\mathcal{F}$ is pure relative to S,
(2) the scheme theoretic support of $\mathcal{F}$ is finite over S, and
(3) $\mathcal{F}$ is universally pure relative to S.

In particular, given a quasi-finite separated morphism $X \to S$ we see that X is pure relative to S if and only if $X \to S$ is finite.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 29.5.5. Then $Z \to S$ is a separated, finite type morphism of schemes with finite fibres. Hence it is separated and quasi-finite, see Morphisms, Lemma 29.20.10. By Lemma 38.16.7 it suffices to prove the lemma for $Z \to S$ and the sheaf $\mathcal{F}$ viewed as a finite type quasi-coherent module on Z. Hence we may assume that $X \to S$ is separated and quasi-finite and that $\text{Supp}(\mathcal{F}) = X$.

It follows from Lemma 38.17.1 and Morphisms, Lemma 29.44.11 that (2) implies (3). Trivially (3) implies (1). Assume (1) holds. We will prove that (2) holds. It is clear that we may assume S is affine. By More on Morphisms, Lemma 37.43.3 we can find a diagram

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with π finite and j a quasi-compact open immersion. If we show that j is closed, then j is a closed immersion and we conclude that $f = \pi \circ j$ is finite. To show that j is closed it suffices to show that specializations lift along j, see Schemes, Lemma 26.19.8. Let $x \in X$, set $t' = j(x)$ and let $t' \leadsto t$ be a specialization. We have to show $t \in j(X)$. Set $s' = f(x)$ and $s = \pi(t)$ so $s' \leadsto s$. By More on Morphisms, Lemma 37.41.4 we can find an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$T_U = T \times_S U = V \amalg W$$

into open and closed subschemes, such that $V \to U$ is finite and there exists a unique point v of V mapping to u, and such that v maps to t in T. As $V \to T$ is étale, we can lift generalizations, see Morphisms, Lemmas 29.25.9 and 29.36.12. Hence there exists a specialization $v' \rightsquigarrow v$ such that v' maps to $t' \in T$. In particular we see that $v' \in X_U \subset T_U$. Denote $u' \in U$ the image of t'. Note that $v' \in \text{Ass}_{X_U/U}(\mathcal{F})$ because $X_{u'}$ is a finite discrete set and $X_{u'} = \text{Supp}(\mathcal{F}_{u'})$. As $\mathcal{F}$ is pure relative to S we see that v' must specialize to a point in X_u. Since v is the only point of V lying over u (and since no point of W can be a specialization of v') we see that $v \in X_u$. Hence $t \in X$. □

05K5 Lemma 38.17.3. Let $f : X \to S$ be a finite type, flat morphism of schemes with geometrically integral fibres. Then X is universally pure over S.

Proof. Let $\xi \in X$ with $s' = f(\xi)$ and $s' \rightsquigarrow s$ a specialization of S. If ξ is an associated point of $X_{s'}$, then ξ is the unique generic point because $X_{s'}$ is an integral scheme. Let ξ_0 be the unique generic point of X_s. As $X \to S$ is flat we can lift $s' \rightsquigarrow s$ to a specialization $\xi' \rightsquigarrow \xi_0$ in X, see Morphisms, Lemma 29.25.9. The $\xi \rightsquigarrow \xi'$ because ξ is the generic point of $X_{s'}$ hence $\xi \rightsquigarrow \xi_0$. This means that $(\text{id}_S, s' \to s, \xi)$ is not an impurity of $\mathcal{O}_X$ above s. Since the assumption that f is finite type, flat with geometrically integral fibres is preserved under base change, we see that there doesn't exist an impurity after any base change. In this way we see that X is universally S-pure. □

05K6 Lemma 38.17.4. Let $f : X \to S$ be a finite type, affine morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module such that $f_*\mathcal{F}$ is locally projective on S, see Properties, Definition 28.21.1. Then $\mathcal{F}$ is universally pure over S.

Proof. After reducing to the case where S is the spectrum of a henselian local ring this follows from Lemma 38.14.1. □

38.18. A criterion for purity

05L2 We first prove that given a flat family of finite type quasi-coherent sheaves the points in the relative assassin specialize to points in the relative assassins of nearby fibres (if they specialize at all).

05L3 Lemma 38.18.1. Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Assume that $\mathcal{F}$ is flat over S at all points of X_s. Let $x' \in \text{Ass}_{X/S}(\mathcal{F})$ with $f(x') = s'$ such that $s' \rightsquigarrow s$ is a specialization in S. If x' specializes to a point of X_s, then $x' \rightsquigarrow x$ with $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$.

Proof. Say $x' \rightsquigarrow t$ with $t \in X_s$. Then we can find specializations $x' \rightsquigarrow x \rightsquigarrow t$ with x corresponding to a generic point of an irreducible component of $\overline{\{x'\}} \cap f^{-1}(\{s\})$. By assumption $\mathcal{F}$ is flat over S at x. By More on Morphisms, Lemma 37.18.3 we see that $x \in \text{Ass}_{X/S}(\mathcal{F})$ as desired. □

05L4 Lemma 38.18.2. Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Let $(S', s') \to (S, s)$ be an elementary étale neighbourhood and let

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d] \\ S \arrow[from=u] & S' \arrow[l] \end{tikzcd}$$

be a commutative diagram of morphisms of schemes. Assume

(1) $\mathcal{F}$ is flat over S at all points of X_s,
(2) $X' \to S'$ is of finite type,
(3) $g^*\mathcal{F}$ is pure along $X'_{s'}$,
(4) $g : X' \to X$ is étale, and
(5) $g(X')$ contains $\text{Ass}_{X_s}(\mathcal{F}_s)$.

In this situation $\mathcal{F}$ is pure along X_s if and only if the image of $X' \to X \times_S S'$ contains the points of $\text{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S')$ lying over points in S' which specialize to s'.

Proof. Since the morphism $S' \to S$ is étale, we see that if $\mathcal{F}$ is pure along X_s, then $\mathcal{F} \times_S S'$ is pure along X_s, see Lemma 38.16.4. Since purity satisfies flat descent, see Lemma 38.16.6, we see that if $\mathcal{F} \times_S S'$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s. Hence we may replace S by S' and assume that $S = S'$ so that $g : X' \to X$ is an étale morphism between schemes of finite type over S. Moreover, we may replace S by $\text{Spec}(\mathcal{O}_{S,s})$ and assume that S is local.

First, assume that $\mathcal{F}$ is pure along X_s. In this case every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of X_s by purity. Hence by Lemma 38.18.1 we see that every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of $\text{Ass}_{X_s}(\mathcal{F}_s)$. Thus every point of $\text{Ass}_{X/S}(\mathcal{F})$ is in the image of g (as the image is open and contains $\text{Ass}_{X_s}(\mathcal{F}_s)$).

Conversely, assume that $g(X')$ contains $\text{Ass}_{X/S}(\mathcal{F})$. Let $S^h = \text{Spec}(\mathcal{O}^h_{S,s})$ be the henselization of S at s. Denote $g^h : (X')^h \to X^h$ the base change of g by $S^h \to S$, and denote $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h. By Divisors, Lemma 31.7.3 and Remark 31.7.4 the relative assassin $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$ is the inverse image of $\text{Ass}_{X/S}(\mathcal{F})$ via the projection $X^h \to X$. As we have assumed that $g(X')$ contains $\text{Ass}_{X/S}(\mathcal{F})$ we conclude that the base change $g^h((X')^h) = g(X') \times_S S^h$ contains $\text{Ass}_{X^h/S^h}(\mathcal{F}^h)$. In this way we reduce to the case where S is the spectrum of a henselian local ring. Let $x \in \text{Ass}_{X/S}(\mathcal{F})$. To finish the proof of the lemma we have to show that x specializes to a point of X_s, see criterion (4) for purity in discussion following Definition 38.16.1. By assumption there exists a $x' \in X'$ such that $g(x') = x$. As $g : X' \to X$ is étale, we see that $x' \in \text{Ass}_{X'/S}(g^*\mathcal{F})$, see Lemma 38.2.8 (applied to the morphism of fibres $X'_w \to X_w$ where $w \in S$ is the image of x'). Since $g^*\mathcal{F}$ is pure along X'_s we see that $x' \rightsquigarrow y$ for some $y \in X'_s$. Hence $x = g(x') \rightsquigarrow g(y)$ and $g(y) \in X_s$ as desired. $\square$

05L5 Lemma 38.18.3. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Assume

(1) f is of finite type,
(2) $\mathcal{F}$ is of finite type,
(3) $\mathcal{F}$ is flat over S at all points of X_s, and
(4) $\mathcal{F}$ is pure along X_s.

Then $\mathcal{F}$ is universally pure along X_s.

Proof. We first make a preliminary remark. Suppose that $(S', s') \to (S, s)$ is an elementary étale neighbourhood. Denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to $X' = X \times_S S'$. By the discussion following Definition 38.16.1 we see that $\mathcal{F}'$ is pure along $X'_{s'}$. Moreover, $\mathcal{F}'$ is flat over S' along $X'_{s'}$. Then it suffices to prove that $\mathcal{F}'$ is universally pure along $X'_{s'}$. Namely, given any morphism $(T, t) \to (S, s)$ of pointed schemes the fibre product $(T', t') = (T \times_S S', (t, s'))$ is flat over (T, t) and hence if $\mathcal{F}_{T'}$ is pure along $X_{t'}$ then $\mathcal{F}_T$ is pure along X_t by Lemma 38.16.6. Thus during the proof we may always replace (s, S) by an elementary étale neighbourhood. We may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ due to the local nature of the problem.

Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & \mathrm{Spec}(\mathcal{O}_{S',s'}) \arrow[l] \end{tikzcd}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module, see Lemma 38.12.11. Note that $X' \to \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite type (as a quasi-compact morphism which is the composition of an étale morphism and the base change of a finite type morphism). By our preliminary remarks in the first paragraph of the proof we may replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$. Hence we may assume there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes of finite type over S, where g is étale, $X_s \subset g(X')$, with S local with closed point s, with X' affine, and with $\Gamma(X', g^*\mathcal{F})$ a free $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 38.17.4.

In this situation we apply Lemma 38.18.2 to deduce that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ from our assumption that $\mathcal{F}$ is pure along X_s and flat over S along X_s. By Divisors, Lemma 31.7.3 and Remark 31.7.4 we see that for any morphism of pointed schemes $(T, t) \to (S, s)$ we have

$$\mathrm{Ass}_{X_T/T}(\mathcal{F}_T) \subset (X_T \to X)^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \subset g(X') \times_S T = g_T(X'_T).$$

Hence by Lemma 38.18.2 applied to the base change of our displayed diagram to (T, t) we conclude that $\mathcal{F}_T$ is pure along X_t as desired. $\square$

05L6 Lemma 38.18.4. Let $f : X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is flat over S. In this case $\mathcal{F}$ is pure relative to S if and only if $\mathcal{F}$ is universally pure relative to S.

Proof. Immediate consequence of Lemma 38.18.3 and the definitions. $\square$

05MA Lemma 38.18.5. Let I be a directed set. Let $(S_i, g_{ii'})$ be an inverse system of affine schemes over I. Set $S = \lim_i S_i$ and $s \in S$. Denote $g_i : S \to S_i$ the projections and set $s_i = g_i(s)$. Suppose that $f : X \to S$ is a morphism of finite presentation,

$\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation which is pure along X_s and flat over S at all points of X_s. Then there exists an $i \in I$, a morphism of finite presentation $X_i \to S_i$, a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation which is pure along $(X_i)_{s_i}$ and flat over S_i at all points of $(X_i)_{s_i}$ such that $X \cong X_i \times_{S_i} S$ and such that the pullback of $\mathcal{F}_i$ to X is isomorphic to $\mathcal{F}$.

Proof. Let $U \subset X$ be the set of points where $\mathcal{F}$ is flat over S. By More on Morphisms, Theorem 37.15.1 this is an open subscheme of X. By assumption $X_s \subset U$. As X_s is quasi-compact, we can find a quasi-compact open $U' \subset U$ with $X_s \subset U'$. By Limits, Lemma 32.10.1 we can find an $i \in I$ and a morphism of finite presentation $f_i : X_i \to S_i$ whose base change to S is isomorphic to f_i. Fix such a choice and set $X_{i'} = X_i \times_{S_i} S_{i'}$. Then $X = \lim_{i'} X_{i'}$ with affine transition morphisms. By Limits, Lemma 32.10.2 we can, after possible increasing i assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose base change to S is isomorphic to $\mathcal{F}$. By Limits, Lemma 32.4.11 after possibly increasing i we may assume there exists an open $U'_i \subset X_i$ whose inverse image in X is U'. Note that in particular $(X_i)_{s_i} \subset U'_i$. By Limits, Lemma 32.10.4 (after increasing i once more) we may assume that $\mathcal{F}_i$ is flat on U'_i. In particular we see that $\mathcal{F}_i$ is flat along $(X_i)_{s_i}$.

Next, we use Lemma 38.12.5 to choose an elementary étale neighbourhood $(S'_i, s'_i) \to (S_i, s_i)$ and a commutative diagram of schemes

$$\begin{tikzcd} X_i \arrow[d] & X'_i \arrow[l, "g_i"'] \arrow[d] \\ S_i & S'_i \arrow[l] \end{tikzcd}$$

such that g_i is étale, $(X_i)_{s_i} \subset g_i(X'_i)$, the schemes X'_i, S'_i are affine, and such that $\Gamma(X'_i, g_i^*\mathcal{F}_i)$ is a projective $\Gamma(S'_i, \mathcal{O}_{S'_i})$-module. Note that $g_i^*\mathcal{F}_i$ is universally pure over S'_i, see Lemma 38.17.4. We may base change the diagram above to a diagram with morphisms $(S'_{i'}, s'_{i'}) \to (S_{i'}, s_{i'})$ and $g_{i'} : X'_{i'} \to X_{i'}$ over $S_{i'}$ for any $i' \geq i$ and we may base change the diagram to a diagram with morphisms $(S', s') \to (S, s)$ and $g : X' \to X$ over S.

At this point we can use our criterion for purity. Set $W'_i \subset X_i \times_{S_i} S'_i$ equal to the image of the étale morphism $X'_i \to X_i \times_{S_i} S'_i$. For every $i' \geq i$ we have similarly the image $W'_{i'} \subset X_{i'} \times_{S_{i'}} S'_{i'}$ and we have the image $W' \subset X \times_S S'$. Taking images commutes with base change, hence $W'_{i'} = W'_i \times_{S'_i} S'_{i'}$ and $W' = W_i \times_{S'_i} S'$. Because $\mathcal{F}$ is pure along X_s the Lemma 38.18.2 implies that

05MB (38.18.5.1) $$f^{-1}(\mathrm{Spec}(\mathcal{O}_{S',s'})) \cap \mathrm{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S') \subset W'$$

By More on Morphisms, Lemma 37.25.5 we see that

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\} \quad \text{and} \quad E_{i'} = \{t \in S'_{i'} \mid \mathrm{Ass}_{X_t}(\mathcal{F}_{i',t}) \subset W'_{i'}\}$$

are locally constructible subsets of S' and $S'_{i'}$. By More on Morphisms, Lemma 37.25.4 we see that $E_{i'}$ is the inverse image of E_i under the morphism $S'_{i'} \to S'_i$ and that E is the inverse image of E_i under the morphism $S' \to S'_i$. Thus Equation (38.18.5.1) is equivalent to the assertion that $\mathrm{Spec}(\mathcal{O}_{S',s'})$ maps into E_i. As $\mathcal{O}_{S',s'} = \mathrm{colim}_{i' \geq i} \mathcal{O}_{S'_{i'},s'_{i'}}$ we see that $\mathrm{Spec}(\mathcal{O}_{S'_{i'},s'_{i'}})$ maps into E_i for some $i' \geq i$, see Limits, Lemma 32.4.10. Then, applying Lemma 38.18.2 to the situation over $S_{i'}$, we conclude that $\mathcal{F}_{i'}$ is pure along $(X_{i'})_{s_{i'}}$. □

05MC Lemma 38.18.6. Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation flat over S. Then the set

$$U = \{s \in S \mid \mathcal{F} \text{ is pure along } X_s\}$$

is open in S.

Proof. Let $s \in U$. Using Lemma 38.12.5 we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "g"'] \arrow[d] \\ S & S' \arrow[l] \end{tikzcd}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module. Note that $g^*\mathcal{F}$ is universally pure over S', see Lemma 38.17.4. Set $W' \subset X \times_S S'$ equal to the image of the étale morphism $X' \to X \times_S S'$. Note that W is open and quasi-compact over S'. Set

$$E = \{t \in S' \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset W'\}.$$

By More on Morphisms, Lemma 37.25.5 E is a constructible subset of S'. By Lemma 38.18.2 we see that $\text{Spec}(\mathcal{O}_{S',s'}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood V' of s'. Applying Lemma 38.18.2 once more we see that for any point s_1 in the image of V' in S the sheaf $\mathcal{F}$ is pure along X_{s_1}. Since $S' \to S$ is étale the image of V' in S is open and we win. □

38.19. How purity is used

05L7 Here are some examples of how purity can be used. The first lemma actually uses a slightly weaker form of purity.

05L8 Lemma 38.19.1. Let $f : X \to S$ be a morphism of finite type. Let $\mathcal{F}$ be a quasi-coherent sheaf of finite type on X. Assume S is local with closed point s. Assume $\mathcal{F}$ is pure along X_s and that $\mathcal{F}$ is flat over S. Let $\varphi : \mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules. Then the following are equivalent

(1) the map on stalks φ_x is injective for all $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$, and
(2) φ is injective.

Proof. Let $\mathcal{K} = \text{Ker}(\varphi)$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 31.5.5. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 31.5.4. As $\mathcal{F}$ is flat we see from Lemma 38.13.5 that $\text{WeakAss}_X(\mathcal{F}) \subset \text{Ass}_{X/S}(\mathcal{F})$. By purity any point x' of $\text{Ass}_{X/S}(\mathcal{F})$ is a generalization of a point of X_s, and hence is the specialization of a point $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$, by Lemma 38.18.1. Hence the injectivity of φ_x implies the injectivity of $\varphi_{x'}$, whence $\mathcal{K}_{x'} = 0$. □

05MD Proposition 38.19.2. Let $f : X \to S$ be an affine, finitely presented morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation, flat over S. Then the following are equivalent

(1) $f_*\mathcal{F}$ is locally projective on S, and

(2) $\mathcal{F}$ is pure relative to S.

In particular, given a ring map $A \to B$ of finite presentation and a finitely presented B-module N flat over A we have: N is projective as an A-module if and only if $\widetilde{N}$ on $\text{Spec}(B)$ is pure relative to $\text{Spec}(A)$.

Proof. The implication (1) ⇒ (2) is Lemma 38.17.4. Assume $\mathcal{F}$ is pure relative to S. Note that by Lemma 38.18.3 this implies $\mathcal{F}$ remains pure after any base change. By Descent, Lemma 35.7.7 it suffices to prove $f_*\mathcal{F}$ is fpqc locally projective on S. Pick $s \in S$. We will prove that the restriction of $f_*\mathcal{F}$ to an étale neighbourhood of s is locally projective. Namely, by Lemma 38.12.5, after replacing S by an affine elementary étale neighbourhood of s, we may assume there exists a diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 38.17.4. Hence by Lemma 38.18.2 we see that the open $g(X')$ contains the points of $\text{Ass}_{X/S}(\mathcal{F})$ lying over $\text{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 37.25.5 E is a constructible subset of S. We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of s. Hence after replacing S by an affine neighbourhood of s we may assume that $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$. By Lemma 38.7.4 this means that

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X', g^*\mathcal{F})$$

is $\Gamma(S, \mathcal{O}_S)$-universally injective. By Algebra, Lemma 10.89.7 we conclude that $\Gamma(X, \mathcal{F})$ is Mittag-Leffler as an $\Gamma(S, \mathcal{O}_S)$-module. Since $\Gamma(X, \mathcal{F})$ is countably generated and flat as a $\Gamma(S, \mathcal{O}_S)$-module, we conclude it is projective by Algebra, Lemma 10.93.1. □

We can use the proposition to improve some of our earlier results. The following lemma is an improvement of Proposition 38.12.4.

05ME Lemma 38.19.3. Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite presentation. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S S'$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. During the proof we may replace X by an open neighbourhood of x and we may replace S by an elementary étale neighbourhood of s. Hence, by openness of flatness (see More on Morphisms, Theorem 37.15.1) we may assume that $\mathcal{F}$ is flat over S. We may assume S and X are affine. After shrinking X some more we may assume that any point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a generalization of x. This property is preserved on replacing (S, s) by an elementary étale neighbourhood. Hence we may apply Lemma 38.12.5 to arrive at the situation where there exists a diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 38.17.4.

Let $U \subset g(X')$ be an affine open neighbourhood of x. We claim that $\mathcal{F}|_U$ is pure along U_s. If we prove this, then the lemma follows because $\mathcal{F}|_U$ will be pure relative to S after shrinking S, see Lemma 38.18.6, whereupon the projectivity follows from Proposition 38.19.2. To prove the claim we have to show, after replacing (S, s) by an arbitrary elementary étale neighbourhood, that any point ξ of $\text{Ass}_{U/S}(\mathcal{F}|_U)$ lying over some $s' \in S$, $s' \leadsto s$ specializes to a point of U_s. Since $U \subset g(X')$ we can find a $\xi' \in X'$ with $g(\xi') = \xi$. Because $g^*\mathcal{F}$ is pure over S, using Lemma 38.18.1, we see there exists a specialization $\xi' \leadsto x'$ with $x' \in \text{Ass}_{X'_s}(g^*\mathcal{F}_s)$. Then $g(x') \in \text{Ass}_{X_s}(\mathcal{F}_s)$ (see for example Lemma 38.2.8 applied to the étale morphism $X'_s \to X_s$ of Noetherian schemes) and hence $g(x') \leadsto x$ by our choice of X above! Since $x \in U$ we conclude that $g(x') \in U$. Thus $\xi = g(\xi') \leadsto g(x') \in U_s$ as desired. $\square$

The following lemma is an improvement of Lemma 38.12.9.

05MF Lemma 38.19.4. Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite type. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S \text{Spec}(\mathcal{O}_{S',s'})$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. The question is Zariski local on X and S. Hence we may assume that X and S are affine. Then we can find a closed immersion $i : X \to \mathbf{A}^n_S$ over S. It is clear that it suffices to prove the lemma for the sheaf $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x)$. In this way we reduce to the case where $X \to S$ is of finite presentation. After replacing S by $\text{Spec}(\mathcal{O}_{S',s'})$ and X by an open of $X \times_S \text{Spec}(\mathcal{O}_{S',s'})$ we may assume that $\mathcal{F}$ is of finite presentation, see Proposition 38.10.3. In this case we may appeal to Lemma 38.19.3 and Algebra, Theorem 10.85.4 to conclude. $\square$

05U7 Lemma 38.19.5. Let $A \to B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B-module which is flat as an A-module. If A is henselian, then N is a filtered colimit

$$N = \text{colim}_i F_i$$

of free A-modules F_i such that all transition maps $u_i : F_i \to F_{i'}$ of the system induce injective maps $\overline{u}_i : F_i/\mathfrak{m}_A F_i \to F_{i'}/\mathfrak{m}_A F_{i'}$. Also, N is a Mittag-Leffler A-module.

Proof. We can find a morphism of finite type $X \to S = \text{Spec}(A)$ and a point $x \in X$ lying over the closed point s of S and a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_x \cong N$ as an A-module. After shrinking X we may assume that each point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ specializes to x. By Lemma 38.19.4 we see that there exists a fundamental system of affine open neighbourhoods $U_i \subset X$ of x such that $\Gamma(U_i, \mathcal{F})$ is a free A-module F_i. Note that if $U_{i'} \subset U_i$, then

$$F_i/\mathfrak{m}_A F_i = \Gamma(U_{i,s}, \mathcal{F}_s) \longrightarrow \Gamma(U_{i',s}, \mathcal{F}_s) = F_{i'}/\mathfrak{m}_A F_{i'}$$

is injective because a section of the kernel would be supported at a closed subset of X_s not meeting x which is a contradiction to our choice of X above. Since the maps $F_i \to F_{i'}$ are A-universally injective (Lemma 38.7.5) it follows that N is Mittag-Leffler by Algebra, Lemma 10.89.9. □

The following lemma should be skipped if reading through for the first time.

0ASX Lemma 38.19.6. Let $A \to B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B-module which is flat as an A-module. If A is a valuation ring, then any element of N has a content ideal $I \subset A$ (More on Algebra, Definition 15.24.1). Also, I is a principal ideal.

Proof. The final statement follows from the fact that I is a finitely generated ideal by More on Algebra, Lemma 15.24.2 and Algebra, Lemma 10.50.15.

Proof of existence of I. Let $A \subset A^h$ be the henselization. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Then $B \to B'$ is flat, hence faithfully flat. Let $N' = N \otimes_B B'$. Let $x \in N$ and let $x' \in N'$ be the image. We claim that for an ideal $I \subset A$ we have $x \in IN \Leftrightarrow x' \in IN'$. Namely, $N/IN \to N'/IN'$ is the tensor product of $B \to B'$ with N/IN and $B \to B'$ is universally injective by Algebra, Lemma 10.82.11. By More on Algebra, Lemma 15.123.6 and Algebra, Lemma 10.50.17 the map $A \to A^h$ defines an inclusion preserving bijection $I \mapsto IA^h$ on sets of ideals. We conclude that x has a content ideal in A if and only if x' has a content ideal in A^h. The assertion for $x' \in N'$ follows from Lemma 38.19.5 and Algebra, Lemma 10.89.6. □

An application is the following.

0H2T Lemma 38.19.7. Let $X \to \mathrm{Spec}(R)$ be a proper flat morphism where R is a valuation ring. If the special fibre is reduced, then X and every fibre of $X \to \mathrm{Spec}(R)$ is reduced.

Proof. Assume the special fibre X_s is reduced. Let $x \in X$ be any point, and let us show that $\mathcal{O}_{X,x}$ is reduced; this will prove that X is reduced. Let $x \rightsquigarrow x'$ be a specialization with x' in the special fibre; such a specialization exists as a proper morphism is closed. Consider the local ring $A = \mathcal{O}_{X,x'}$. Then $\mathcal{O}_{X,x}$ is a localization of A, so it suffices to show that A is reduced. Let $a \in A$ and let $I = (\pi) \subset R$ be its content ideal, see Lemma 38.19.6. Then $a = \pi a'$ and a' maps to a nonzero element of $A/\mathfrak{m}A$ where $\mathfrak{m} \subset R$ is the maximal ideal. If a is nilpotent, so is a', because π is a nonzerodivisor by flatness of A over R. But a' maps to a nonzero element of the reduced ring $A/\mathfrak{m}A = \mathcal{O}_{X_s,x'}$. This is a contradiction unless A is reduced, which is what we wanted to show.

Of course, if X is reduced, so is the generic fibre of X over R. If $\mathfrak{p} \subset R$ is a prime ideal, then $R/\mathfrak{p}$ is a valuation ring by Algebra, Lemma 10.50.9. Hence redoing the argument with the base change of X to $R/\mathfrak{p}$ proves the fibre over $\mathfrak{p}$ is reduced. □

38.20. Flattening functors

05MG Let S be a scheme. Recall that a functor $F : (Sch/S)^{opp} \to \text{Sets}$ is called limit preserving if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \text{colim}_i F(T_i)$.

05MH Situation 38.20.1. Let $f : X \to S$ be a morphism of schemes. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over S we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_S T \to X$. In this situation we can consider the functor

(38.20.1.1)

05MI
$$F_{iso} : (Sch/S)^{opp} \longrightarrow \text{Sets}, \quad T \longrightarrow \begin{cases} \{*\} & \text{if} \quad u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

05MJ Lemma 38.20.2. In Situation 38.20.1.

(1) Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.
(2) If f is quasi-compact and $\mathcal{G}$ is of finite type, then F_{surj} is limit preserving.
(3) If f is quasi-compact and $\mathcal{F}$ of finite type, then F_{zero} is limit preserving.
(4) If f is quasi-compact, $\mathcal{F}$ is of finite type, and $\mathcal{G}$ is of finite presentation, then F_{iso} is limit preserving.

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies, Lemma 34.9.7. In particular, for every $x \in X_T$ there exists an $i \in I$ and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X_T,x} \to \mathcal{O}_{X_i,x_i}$ is flat, hence faithfully flat (see Algebra, Lemma 10.39.17) we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. Whence part (1) of the lemma.

Proof of (2). Assume f quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove part (2) we have to show that u_i is surjective for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since f is quasi-compact the scheme X_{i_0} is quasi-compact. Hence we may choose affine opens $W_1, \ldots, W_m \subset X$ and an affine open covering $X_{i_0} = U_{1,i_0} \cup \ldots \cup U_{m,i_0}$ such that U_{j,i_0} maps into W_j under the projection morphism $X_{i_0} \to X$. For any $i \in I$ let $U_{j,i}$ be the inverse image of U_{j,i_0}. Setting $U_j = \lim_i U_{j,i}$ we see that $X_T = U_1 \cup \ldots \cup U_m$ is an affine open covering of X_T. Now it suffices to show, for a given $j \in \{1, \ldots, m\}$ that $u_i|_{U_{j,i}}$ is surjective for some $i = i(j) \in I$. Using Properties, Lemma 28.16.1 this translates into the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \text{colim}_{i \in I} R_i$ is a directed colimit of A-algebras. If N is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective. This is Algebra, Lemma 10.127.5 part (2).

Proof of (3). Exactly the same arguments as given in the proof of (2) reduces this to the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \text{colim}_{i \in I} R_i$ is a directed colimit of A-algebras. If M is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is zero, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is zero. This is Algebra, Lemma 10.127.5 part (1).

Proof of (4). Assume f quasi-compact and $\mathcal{F}, \mathcal{G}$ of finite presentation. Arguing in exactly the same manner as in the previous paragraph (using in addition also Properties, Lemma 28.16.2) part (3) translates into the following algebra statement: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \text{colim}_{i \in I} R_i$ is a directed colimit of A-algebras. Assume M is a finite A-module, N is a finitely presented A-module, and $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism. Then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism. This is Algebra, Lemma 10.127.5 part (3). $\square$

05MK Situation 38.20.3. Let $(A, \mathfrak{m}_A)$ be a local ring. Denote $\mathcal{C}$ the category whose objects are A-algebras A' which are local rings such that the algebra structure $A \to A'$ is a local homomorphism of local rings. A morphism between objects A', A'' of $\mathcal{C}$ is a local homomorphism $A' \to A''$ of A-algebras. Let $A \to B$ be a local ring map of local rings and let M be a B-module. If A' is an object of $\mathcal{C}$ we set $B' = B \otimes_A A'$ and we set $M' = M \otimes_A A'$ as a B'-module. Given $A' \in \text{Ob}(\mathcal{C})$, consider the condition

05ML (38.20.3.1) $$\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \text{Spec}(B') : M'_{\mathfrak{q}} \text{ is flat over } A'.$$

Note the similarity with More on Algebra, Equation (15.19.1.1). In particular, if $A' \to A''$ is a morphism of $\mathcal{C}$ and (38.20.3.1) holds for A', then it holds for A'', see More on Algebra, Lemma 15.19.2. Hence we obtain a functor

05MM (38.20.3.2) $$F_{lf} : \mathcal{C} \longrightarrow \textit{Sets}, \quad A' \longrightarrow \begin{cases} \{*\} & \text{if (38.20.3.1) holds,} \\ \emptyset & \text{else.} \end{cases}$$

05MN Lemma 38.20.4. In Situation 38.20.3.

(1) If $A' \to A''$ is a flat morphism in $\mathcal{C}$ then $F_{lf}(A') = F_{lf}(A'')$.
(2) If $A \to B$ is essentially of finite presentation and M is a B-module of finite presentation, then F_{lf} is limit preserving: If $\{A_i\}_{i \in I}$ is a directed system of objects of $\mathcal{C}$, then $F_{lf}(\text{colim}_i A_i) = \text{colim}_i F_{lf}(A_i)$.

Proof. Part (1) is a special case of More on Algebra, Lemma 15.19.3. Part (2) is a special case of More on Algebra, Lemma 15.19.4. $\square$

05P4 Lemma 38.20.5. In Situation 38.20.3. Let $B \to C$ is a local map of local A-algebras and N a C-module. Denote $F'_{lf} : \mathcal{C} \to \textit{Sets}$ the functor associated to the pair (C, N). If $M \cong N$ as B-modules and $B \to C$ is finite, then $F_{lf} = F'_{lf}$.

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A'$ similarly to the definitions of B', M' in Situation 38.20.3. Note that $M' \cong N'$ as B'-modules. The assumption that $B \to C$ is finite has two consequences: (a) $\mathfrak{m}_C = \sqrt{\mathfrak{m}_B C}$ and (b) $B' \to C'$ is finite. Consequence (a) implies that

$$V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') = (\text{Spec}(C') \to \text{Spec}(B'))^{-1} V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B').$$

Suppose $\mathfrak{q} \subset V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$. Then $M'_{\mathfrak{q}}$ is flat over A' if and only if the $C'_{\mathfrak{q}}$-module $N'_{\mathfrak{q}}$ is flat over A' (because these are isomorphic as A'-modules) if and only if for every maximal ideal $\mathfrak{r}$ of $C'_{\mathfrak{q}}$ the module $N'_{\mathfrak{r}}$ is flat over A' (see Algebra, Lemma 10.39.18). As $B'_{\mathfrak{q}} \to C'_{\mathfrak{q}}$ is finite by (b), the maximal ideals of $C'_{\mathfrak{q}}$ correspond exactly to the primes of C' lying over $\mathfrak{q}$ (see Algebra, Lemma 10.36.22) and these primes are all contained in $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ by the displayed equation above. Thus the result of the lemma holds. $\square$

05P5 Lemma 38.20.6. In Situation 38.20.3 suppose that $B \to C$ is a flat local homomorphism of local rings. Set $N = M \otimes_B C$. Denote $F'_{lf} : \mathcal{C} \to Sets$ the functor associated to the pair (C, N). Then $F_{lf} = F'_{lf}$.

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A' = M' \otimes_{B'} C'$ similarly to the definitions of B', M' in Situation 38.20.3. Note that

$$V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') = \mathrm{Spec}(\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}))$$

and similarly for $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$. The ring map

$$\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}) \longrightarrow \kappa(\mathfrak{m}_C) \otimes_A \kappa(\mathfrak{m}_{A'})$$

is faithfully flat, hence $V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C') \to V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$ is surjective. Finally, if $\mathfrak{r} \in V(\mathfrak{m}_{A'}C' + \mathfrak{m}_C C')$ maps to $\mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B')$, then $M'_{\mathfrak{q}}$ is flat over A' if and only if $N'_{\mathfrak{r}}$ is flat over A' because $B' \to C'$ is flat, see Algebra, Lemma 10.39.9. The lemma follows formally from these remarks. $\square$

05MP Situation 38.20.7. Let $f : X \to S$ be a smooth morphism with geometrically irreducible fibres. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is smooth with geometrically irreducible fibres, see Morphisms, Lemma 29.34.5 and More on Morphisms, Lemma 37.27.2. Let $p \geq 0$ be an integer. Given a point $t \in T$ consider the condition

05MQ (38.20.7.1) $$\mathcal{F}_T \text{ is free of rank } p \text{ in a neighbourhood of } \xi_t$$

where ξ_t is the generic point of the fibre X_t. This condition for all $t \in T$ is stable under base change, and hence we obtain a functor

(38.20.7.2)

05MR $$H_p : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ satisfies (38.20.7.1) } \forall t \in T, \\ \emptyset & \text{else.} \end{cases}$$

05MS Lemma 38.20.8. In Situation 38.20.7.

(1) The functor H_p satisfies the sheaf property for the fpqc topology.
(2) If $\mathcal{F}$ is of finite presentation, then functor H_p is limit preserving.

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc[1] covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ satisfies (38.20.7.1) for all i. Pick $t \in T$ and let $\xi_t \in X_T$ denote the generic point of X_t. We have to show that $\mathcal{F}$ is free in a neighbourhood of ξ_t. For some $i \in I$ we can find a $t_i \in T_i$ mapping to t. Let $\xi_i \in X_i$ denote the generic point of X_{t_i}, so that ξ_i maps to ξ_t. The fact that $\mathcal{F}_i$ is free of rank p in a neighbourhood of ξ_i implies that $(\mathcal{F}_i)_{x_i} \cong \mathcal{O}_{X_i,x_i}^{\oplus p}$ which implies that $\mathcal{F}_{T,\xi_t} \cong \mathcal{O}_{X_T,\xi_t}^{\oplus p}$ as $\mathcal{O}_{X_T,\xi_t} \to \mathcal{O}_{X_i,x_i}$ is flat, see for example Algebra, Lemma 10.78.6. Thus there exists an affine neighbourhood U of ξ_t in X_T and a surjection $\mathcal{O}_U^{\oplus p} \to \mathcal{F}_U = \mathcal{F}_T|_U$, see Modules, Lemma 17.9.4. After shrinking T we may assume that $U \to T$ is surjective. Hence $U \to T$ is a smooth morphism

[1]It is quite easy to show that H_p is a sheaf for the fppf topology using that flat morphisms of finite presentation are open. This is all we really need later on. But it is kind of fun to prove directly that it also satisfies the sheaf condition for the fpqc topology.

of affines with geometrically irreducible fibres. Moreover, for every $t' \in T$ we see that the induced map

$$\alpha : \mathcal{O}_{U,\xi_{t'}}^{\oplus p} \longrightarrow \mathcal{F}_{U,\xi_{t'}}$$

is an isomorphism (since by the same argument as before the module on the right is free of rank p). It follows from Lemma 38.10.1 that

$$\Gamma(U, \mathcal{O}_U^{\oplus p}) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'} \longrightarrow \Gamma(U, \mathcal{F}_U) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'}$$

is injective for every $t' \in T$. Hence we see the surjection α is an isomorphism. This finishes the proof of (1).

Assume that $\mathcal{F}$ is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ satisfies (38.20.7.1). Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 37.15.1. By assumption every generic point of every fibre is a point of U, i.e., $U \to T$ is a smooth surjective morphism with geometrically irreducible fibres. We may shrink U a bit and assume that U is quasi-compact. Using Limits, Lemma 32.4.11 we can find an $i \in I$ and a quasi-compact open $U_i \subset X_i$ whose inverse image in X_T is U. After increasing i we may assume that $\mathcal{F}_i|_{U_i}$ is flat over T_i, see Limits, Lemma 32.10.4. In particular, $\mathcal{F}_i|_{U_i}$ is finite locally free hence defines a locally constant rank function $\rho : U_i \to \{0, 1, 2, \ldots\}$. Let $(U_i)_p \subset U_i$ denote the open and closed subset where ρ has value p. Let $V_i \subset T_i$ be the image of $(U_i)_p$; note that V_i is open and quasi-compact. By assumption the image of $T \to T_i$ is contained in V_i. Hence there exists an $i' \geq i$ such that $T_{i'} \to T_i$ factors through V_i by Limits, Lemma 32.4.11. Then $\mathcal{F}_{i'}$ satisfies (38.20.7.1) as desired. Some details omitted. □

0CWF Lemma 38.20.9. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. The following are equivalent

(1) for $s \in S$ the closed subset $Z \subset X_s$ of points where $\mathcal{F}$ is not flat over S (see Lemma 38.10.4) satisfies $\dim(Z) < n$, and
(2) for $x \in X$ such that $\mathcal{F}$ is not flat at x over S we have $\text{trdeg}_{\kappa(f(x))}(\kappa(x)) < n$.

If this is true, then it remains true after any base change.

Proof. Let $x \in X$ be a point over $s \in S$. Then the dimension of the closure of $\{x\}$ in X_s is $\text{trdeg}_{\kappa(s)}(\kappa(x))$ by Varieties, Lemma 33.20.3. Conversely, if $Z \subset X_s$ is a closed subset of dimension d, then there exists a point $x \in Z$ with $\text{trdeg}_{\kappa(s)}(\kappa(x)) = d$ (same reference). Therefore the equivalence of (1) and (2) holds (even fibre by fibre). The statement on base change follows from Morphisms, Lemmas 29.25.7 and 29.28.3. □

0CWG Definition 38.20.10. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $n \geq 0$. We say $\mathcal{F}$ is flat over S in dimensions $\geq n$ if the equivalent conditions of Lemma 38.20.9 are satisfied.

05MT Situation 38.20.11. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that

$X_T \to T$ is of finite type and that $\mathcal{F}_T$ is an $\mathcal{O}_{X_T}$-module of finite type (Morphisms, Lemma 29.15.4 and Modules, Lemma 17.9.2). Let $n \geq 0$. By Definition 38.20.10 and Lemma 38.20.9 we obtain a functor
(38.20.11.1)

05MU
$$F_n : (Sch/S)^{opp} \longrightarrow \text{Sets}, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

05MV **Lemma 38.20.12.** In Situation 38.20.11.

(1) The functor F_n satisfies the sheaf property for the fpqc topology.
(2) If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_n is limit preserving.

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ is flat over T_i in dimensions $\geq n$ for all i. Let $t \in T$. Choose an index i and a point $t_i \in T_i$ mapping to t. Consider the cartesian diagram

$$\begin{tikzcd} X_{\mathrm{Spec}(\mathcal{O}_{T,t})} \arrow[d] & X_{\mathrm{Spec}(\mathcal{O}_{T_i,t_i})} \arrow[l] \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{T,t}) & \mathrm{Spec}(\mathcal{O}_{T_i,t_i}) \arrow[l] \end{tikzcd}$$

As the lower horizontal morphism is flat we see from More on Morphisms, Lemma 37.15.2 that the set $Z_i \subset X_{t_i}$ where $\mathcal{F}_i$ is not flat over T_i and the set $Z \subset X_t$ where $\mathcal{F}_T$ is not flat over T are related by the rule $Z_i = Z_{\kappa(t_i)}$. Hence we see that $\mathcal{F}_T$ is flat over T in dimensions $\geq n$ by Morphisms, Lemma 29.28.3.

Assume that f is quasi-compact and locally of finite presentation and that $\mathcal{F}$ is of finite presentation. In this paragraph we first reduce the proof of (2) to the case where f is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ is flat in dimensions $\geq n$. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. We have to show that $\mathcal{F}_i$ is flat in dimensions $\geq n$ for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since T_{i_0} is affine (hence quasi-compact) there exist finitely many affine opens $W_j \subset S$, $j = 1, \ldots, m$ and an affine open overing $T_{i_0} = \bigcup_{j=1,\ldots,m} V_{j,i_0}$ such that $T_{i_0} \to S$ maps V_{j,i_0} into W_j. For $i \geq i_0$ denote $V_{j,i}$ the inverse image of V_{j,i_0} in T_i. If we can show, for each j, that there exists an i such that $\mathcal{F}_{V_{j,i_0}}$ is flat in dimensions $\geq n$, then we win. In this way we reduce to the case that S is affine. In this case X is quasi-compact and we can choose a finite affine open covering $X = W_1 \cup \ldots \cup W_m$. In this case the result for $(X \to S, \mathcal{F})$ is equivalent to the result for $(\coprod W_j, \coprod \mathcal{F}|_{W_j})$. Hence we may assume that f is of finite presentation.

Assume f is of finite presentation and $\mathcal{F}$ is of finite presentation. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 37.15.1. By assumption the dimension of every fibre of $Z = X_T \setminus U$ over T has dimension $< n$. By Limits, Lemma 32.18.5 we can find a closed subscheme $Z \subset Z' \subset X_T$ such that $\dim(Z'_t) < n$ for all $t \in T$ and such that $Z' \to X_T$ is of finite presentation. By Limits, Lemmas 32.10.1 and 32.8.5 there exists an $i \in I$ and a closed subscheme $Z'_i \subset X_i$ of finite presentation whose base change to T is Z'. By Limits, Lemma 32.18.1 we may assume all fibres of $Z'_i \to T_i$ have dimension $< n$. By Limits, Lemma 32.10.4 we may assume that $\mathcal{F}_i|_{X_i \setminus T'_i}$ is flat over T_i. This implies that $\mathcal{F}_i$ is flat in dimensions $\geq n$; here we use that $Z' \to X_T$ is of finite presentation, and hence the complement $X_T \setminus Z'$ is quasi-compact! Thus part (2) is proved and the proof of the lemma is complete. □

05MW Situation 38.20.13. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Since the base change of a flat module is flat we obtain a functor

05MX (38.20.13.1)
$$F_{flat} : (Sch/S)^{opp} \longrightarrow \text{Sets}, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases}$$

05MY Lemma 38.20.14. In Situation 38.20.13.

(1) The functor F_{flat} satisfies the sheaf property for the fpqc topology.
(2) If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_{flat} is limit preserving.

Proof. Part (1) follows from the following statement: If $T' \to T$ is a surjective flat morphism of schemes over S, then $\mathcal{F}_{T'}$ is flat over T' if and only if $\mathcal{F}_T$ is flat over T, see More on Morphisms, Lemma 37.15.2. Part (2) follows from Limits, Lemma 32.10.4 after reducing to the case where X and S are affine (compare with the proof of Lemma 38.20.12). □

38.21. Flattening stratifications

052F Just the definitions. The reader looking for a "generic flatness stratification", should consult More on Morphisms, Section 37.54.

05P6 Definition 38.21.1. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that the universal flattening of $\mathcal{F}$ exists if the functor F_{flat} defined in Situation 38.20.13 is representable by a scheme S' over S. We say that the universal flattening of X exists if the universal flattening of $\mathcal{O}_X$ exists.

Note that if the universal flattening S'[2] of $\mathcal{F}$ exists, then the morphism $S' \to S$ is a surjective monomorphism of schemes such that $\mathcal{F}_{S'}$ is flat over S' and such that a morphism $T \to S$ factors through S' if and only if $\mathcal{F}_T$ is flat over T.

0FJ1 Example 38.21.2. Let $X = S = \text{Spec}(k[x, y])$ where k is a field. Let $\mathcal{F} = \widetilde{M}$ where $M = k[x, x^{-1}, y]/(y)$. For a $k[x, y]$-algebra A set $F_{flat}(A) = F_{flat}(\text{Spec}(A))$. Then $F_{flat}(k[x, y]/(x, y)^n) = \{*\}$ for all n, while $F_{flat}(k[[x, y]]) = \emptyset$. This means that F_{flat} isn't representable (even by an algebraic space, see Formal Spaces, Lemma 87.33.3). Thus the universal flattening does not exist in this case.

We define (compare with Topology, Remark 5.28.5) a (locally finite, scheme theoretic) stratification of a scheme S to be given by closed subschemes $Z_i \subset S$ indexed by a partially ordered set I such that $S = \bigcup Z_i$ (set theoretically), such that every point of S has a neighbourhood meeting only a finite number of Z_i, and such that

$$Z_i \cap Z_j = \bigcup\nolimits_{k \leq i,j} Z_k.$$

[2] The scheme S' is sometimes called the universal flatificator. In [GR71] it is called the platificateur universel. Existence of the universal flattening should not be confused with the type of results discussed in More on Algebra, Section 15.26.

Setting $S_i = Z_i \setminus \bigcup_{j<i} Z_j$ the actual stratification is the decomposition $S = \coprod S_i$ into locally closed subschemes. We often only indicate the strata S_i and leave the construction of the closed subschemes Z_i to the reader. Given a stratification we obtain a monomorphism

$$S' = \coprod\nolimits_{i \in I} S_i \longrightarrow S.$$

We will call this the monomorphism associated to the stratification. With this terminology we can define what it means to have a flattening stratification.

05P7 Definition 38.21.3. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ has a flattening stratification if the functor F_{flat} defined in Situation 38.20.13 is representable by a monomorphism $S' \to S$ associated to a stratification of S by locally closed subschemes. We say that X has a flattening stratification if $\mathcal{O}_X$ has a flattening stratification.

When a flattening stratification exists, it is often important to understand the index set labeling the strata and its partial ordering. This often has to do with ranks of modules. For example if $X = S$ and $\mathcal{F}$ is a finitely presented $\mathcal{O}_S$-module, then the flattening stratification exists and is given by the Fitting ideals of $\mathcal{F}$, see Divisors, Lemma 31.9.7.

38.22. Flattening stratification over an Artinian ring

05PA A flatting stratification exists when the base scheme is the spectrum of an Artinian ring.

05PB Lemma 38.22.1. Let S be the spectrum of an Artinian ring. For any scheme X over S, and any quasi-coherent $\mathcal{O}_X$-module there exists a universal flattening. In fact the universal flattening is given by a closed immersion $S' \to S$, and hence is a flattening stratification for $\mathcal{F}$ as well.

Proof. Choose an affine open covering $X = \bigcup U_i$. Then F_{flat} is the product of the functors associated to each of the pairs $(U_i, \mathcal{F}|_{U_i})$. Hence it suffices to prove the result for each $(U_i, \mathcal{F}|_{U_i})$. In the affine case the lemma follows immediately from More on Algebra, Lemma 15.17.2. □

38.23. Flattening a map

05PC Theorem 38.23.3 is the key to further flattening statements.

05PD Lemma 38.23.1. Let S be a scheme. Let $g : X' \to X$ be a flat morphism of schemes over S with X locally of finite type over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module which is flat over S. If $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ then the canonical map

$$\mathcal{F} \longrightarrow g_* g^* \mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_T \to (g_T)_* g_T^* \mathcal{F}_T$ is injective for any morphism $T \to S$. The assumption $\text{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ is preserved by base change, see Divisors, Lemma 31.7.3 and Remark 31.7.4. The same holds for the assumption of flatness and finite type. Hence it suffices to prove the injectivity of the displayed arrow. Let $\mathcal{K} = \text{Ker}(\mathcal{F} \to g_* g^* \mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\text{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 31.5.5. We have $\text{WeakAss}_X(\mathcal{K}) \subset \text{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 31.5.4. As $\mathcal{F}$ is flat we see from Lemma 38.13.5 that $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/S}(\mathcal{F})$. By assumption any point x of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is the image of some $x' \in X'$. Since g is flat the local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat, hence the map

$$\mathcal{F}_x \longrightarrow g^*\mathcal{F}_{x'} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

is injective (see Algebra, Lemma 10.82.11). This implies that $\mathcal{K}_x = 0$ as desired. □

05PE Lemma 38.23.2. Let A be a ring. Let $u : M \to N$ be a surjective map of A-modules. If M is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent

(1) $u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is an isomorphism, and
(2) $\varphi(I) = 0$.

Proof. As M is projective we can find a projective A-module C such that $F = M \oplus C$ is a free A-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume M is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\mathrm{Ker}(u)$ with respect to some (fixed) basis of M. The reason this works is that, since u is surjective and $\otimes_A B$ is right exact, $\mathrm{Ker}(u \otimes 1)$ is the image of $\mathrm{Ker}(u) \otimes_A B$ in $M \otimes_A B$. □

05PF Theorem 38.23.3. In Situation 38.20.1 assume

(1) f is of finite presentation,
(2) $\mathcal{F}$ is of finite presentation, flat over S, and pure relative to S, and
(3) u is surjective.

Then F_{iso} is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $\mathcal{G}$ is of finite presentation.

Proof. We will use without further mention that $\mathcal{F}$ is universally pure over S, see Lemma 38.18.3. By Lemma 38.20.2 and Descent, Lemmas 35.37.2 and 35.39.1 the question is local for the étale topology on S. Hence it suffices to prove, given $s \in S$, that there exists an étale neighbourhood of (S, s) so that the theorem holds.

Using Lemma 38.12.5 and after replacing S by an elementary étale neighbourhood of s we may assume there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[ll, "g"'] \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes of finite presentation over S, where g is étale, $X_s \subset g(X')$, the schemes X' and S are affine, $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that $g^*\mathcal{F}$ is universally pure over S, see Lemma 38.17.4. Hence by Lemma 38.18.2 we see that the open $g(X')$ contains the points of $\mathrm{Ass}_{X/S}(\mathcal{F})$ lying over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 37.25.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of s. Hence after replacing S by a smaller affine neighbourhood of s we may assume that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$.

Since we have assumed that u is surjective we have $F_{iso} = F_{inj}$. From Lemma 38.23.1 it follows that $u : \mathcal{F} \to \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \to g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \to S$ is a morphism of affine schemes and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module. This case follows immediately from Lemma 38.23.2.

To see that Z is of finite presentation if $\mathcal{G}$ is of finite presentation, combine Lemma 38.20.2 part (4) with Limits, Remark 32.6.2. $\square$

07AI Lemma 38.23.4. Let $f : X \to S$ be a morphism of schemes which is of finite presentation, flat, and pure. Let Y be a closed subscheme of X. Let $F = f_*Y$ be the Weil restriction functor of Y along f, defined by

$$F : (Sch/S)^{opp} \to \text{Sets}, \quad T \mapsto \begin{cases} \{*\} & \text{if} \quad Y_T \to X_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

Then F is representable by a closed immersion $Z \to S$. Moreover $Z \to S$ is of finite presentation if $Y \to S$ is.

Proof. Let $\mathcal{I}$ be the ideal sheaf defining Y in X and let $u : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ be the surjection. Then for an S-scheme T, the closed immersion $Y_T \to X_T$ is an isomorphism if and only if u_T is an isomorphism. Hence the result follows from Theorem 38.23.3. $\square$

38.24. Flattening in the local case

05MZ In this section we start applying the earlier material to obtain a shadow of the flattening stratification.

05PG Theorem 38.24.1. In Situation 38.20.3 assume A is henselian, B is essentially of finite type over A, and M is a finite B-module. Then there exists an ideal $I \subset A$ such that A/I corepresents the functor F_{lf} on the category $\mathcal{C}$. In other words given a local homomorphism of local rings $\varphi : A \to A'$ with $B' = B \otimes_A A'$ and $M' = M \otimes_A A'$ the following are equivalent:

(1) $\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \text{Spec}(B') : M'_{\mathfrak{q}}$ is flat over A', and
(2) $\varphi(I) = 0$.

If B is essentially of finite presentation over A and M of finite presentation over B, then I is a finitely generated ideal.

Proof. Choose a finite type ring map $A \to C$ and a finite C-module N and a prime $\mathfrak{q}$ of C such that $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. In the following, when we say "the theorem holds for $(N/C/A, \mathfrak{q})$ we mean that it holds for $(A \to B, M)$ where $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. By Lemma 38.20.6 the functor F_{lf} is unchanged if we replace B by a local ring flat over B. Hence, since A is henselian, we may apply Lemma 38.6.6 and assume that there exists a complete dévissage of $N/C/A$ at $\mathfrak{q}$.

Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be such a complete dévissage of $N/C/A$ at $\mathfrak{q}$. Let $\mathfrak{q}'_i \subset A_i$ be the unique prime lying over $\mathfrak{q}_i \subset B_i$ as in Definition 38.6.4. Since $C \to A_1$ is surjective and $N \cong M_1$ as C-modules, we see by Lemma 38.20.5 it suffices to prove the theorem holds for $(M_1/A_1/A, \mathfrak{q}'_1)$. Since $B_1 \to A_1$ is finite and $\mathfrak{q}_1$ is the only prime of B_1 over $\mathfrak{q}'_1$ we see that $(A_1)_{\mathfrak{q}'_1} \to (B_1)_{\mathfrak{q}_1}$ is finite (see Algebra, Lemma 10.41.11 or More on Morphisms, Lemma 37.47.4). Hence by Lemma 38.20.5 it suffices to prove the theorem holds for $(M_1/B_1/A, \mathfrak{q}_1)$.

At this point we may assume, by induction on the length n of the dévissage, that the theorem holds for $(M_2/B_2/A, \mathfrak{q}_2)$. (If $n = 1$, then $M_2 = 0$ which is flat over A.) Reversing the last couple of steps of the previous paragraph, using that $M_2 \cong \text{Coker}(\alpha_2)$ as B_1-modules, we see that the theorem holds for $(\text{Coker}(\alpha_1)/B_1/A, \mathfrak{q}_1)$.

Let A' be an object of $\mathcal{C}$. At this point we use Lemma 38.10.1 to see that if $(M_1 \otimes_A A')_{\mathfrak{q}'}$ is flat over A' for a prime $\mathfrak{q}'$ of $B_1 \otimes_A A'$ lying over $\mathfrak{m}_{A'}$, then $(\text{Coker}(\alpha_1) \otimes_A A')_{\mathfrak{q}'}$ is flat over A'. Hence we conclude that F_{lf} is a subfunctor of the functor F'_{lf} associated to the module $\text{Coker}(\alpha_1)_{\mathfrak{q}_1}$ over $(B_1)_{\mathfrak{q}_1}$. By the previous paragraph we know F'_{lf} is corepresented by A/J for some ideal $J \subset A$. Hence we may replace A by A/J and assume that $\text{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A.

Since $\text{Coker}(\alpha_1)$ is a B_1-module for which there exist a complete dévissage of $N_1/B_1/A$ at $\mathfrak{q}_1$ and since $\text{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A by Lemma 38.10.2 we see that $\text{Coker}(\alpha_1)$ is free as an A-module, in particular flat as an A-module. Hence Lemma 38.10.1 implies $F_{lf}(A')$ is nonempty if and only if $\alpha \otimes 1_{A'}$ is injective. Let $N_1 = \text{Im}(\alpha_1) \subset M_1$ so that we have exact sequences

$$0 \to N_1 \to M_1 \to \text{Coker}(\alpha_1) \to 0 \quad \text{and} \quad B_1^{\oplus r_1} \to N_1 \to 0$$

The flatness of $\text{Coker}(\alpha_1)$ implies the first sequence is universally exact (see Algebra, Lemma 10.82.5). Hence $\alpha \otimes 1_{A'}$ is injective if and only if $B_1^{\oplus r_1} \otimes_A A' \to N_1 \otimes_A A'$ is an isomorphism. Finally, Theorem 38.23.3 applies to show this functor is corepresentable by A/I for some ideal I and we conclude F_{lf} is corepresentable by A/I also.

To prove the final statement, suppose that $A \to B$ is essentially of finite presentation and M of finite presentation over B. Let $I \subset A$ be the ideal such that F_{lf} is corepresented by A/I. Write $I = \bigcup I_\lambda$ where I_λ ranges over the finitely generated ideals contained in I. Then, since $F_{lf}(A/I) = \{*\}$ we see that $F_{lf}(A/I_\lambda) = \{*\}$ for some λ, see Lemma 38.20.4 part (2). Clearly this implies that $I = I_\lambda$. $\square$

05PH Remark 38.24.2. Here is a scheme theoretic reformulation of Theorem 38.24.1. Let $(X, x) \to (S, s)$ be a morphism of pointed schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume S henselian local with closed point s. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent

(1) $\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to $x \in X_s$, and
(2) $\text{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.

Moreover, if $X \to S$ is of finite presentation at x and $\mathcal{F}_x$ of finite presentation over $\mathcal{O}_{X,x}$, then $Z \to S$ is of finite presentation.

At this point we can obtain some very general results completely for free from the result above. Note that perhaps the most interesting case is when $E = X_s$!

05PI Lemma 38.24.3. Let S be the spectrum of a henselian local ring with closed point s. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $E \subset X_s$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent

(1) $\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to a point of $E \subset X_s$, and
(2) $\text{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.

Moreover, if $X \to S$ is locally of finite presentation, $\mathcal{F}$ is of finite presentation, and $E \subset X_s$ is closed and quasi-compact, then $Z \to S$ is of finite presentation.

Proof. For $x \in X_s$ denote $Z_x \subset S$ the closed subscheme we found in Remark 38.24.2. Then it is clear that $Z = \bigcap_{x \in E} Z_x$ works!

To prove the final statement assume X locally of finite presentation, $\mathcal{F}$ of finite presentation and Z closed and quasi-compact. First, choose finitely many affine opens $W_j \subset X$ such that $E \subset \bigcup W_j$. It clearly suffices to prove the result for each morphism $W_j \to S$ with sheaf $\mathcal{F}|_{X_j}$ and closed subset $E \cap W_j$. Hence we may assume X is affine. In this case, More on Algebra, Lemma 15.19.4 shows that the functor defined by (1) is "limit preserving". Hence we can show that $Z \to S$ is of finite presentation exactly as in the last part of the proof of Theorem 38.24.1. $\square$

052G Remark 38.24.4. Tracing the proof of Lemma 38.24.3 to its origins we find a long and winding road. But if we assume that

(1) f is of finite type,
(2) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,
(3) $E = X_s$, and
(4) S is the spectrum of a Noetherian complete local ring.

then there is a proof relying completely on more elementary algebra as follows: first we reduce to the case where X is affine by taking a finite affine open cover. In this case Z exists by More on Algebra, Lemma 15.20.3. The key step in this proof is constructing the closed subscheme Z step by step inside the truncations $\mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^n)$. This relies on the fact that flattening stratifications always exist when the base is Artinian, and the fact that $\mathcal{O}_{S,s} = \lim \mathcal{O}_{S,s}/\mathfrak{m}_s^n$.

38.25. Variants of a lemma

0ASZ In this section we discuss variants of Algebra, Lemmas 10.128.4 and 10.99.1. The most general version is Proposition 38.25.13; this was stated as [GR71, Lemma 4.2.2] but the proof in loc.cit. only gives the weaker result as stated in Lemma 38.25.5. The intricate proof of Proposition 38.25.13 is due to Ofer Gabber. As we currently have no application for the proposition we encourage the reader to skip to the next section after reading the proof of Lemma 38.25.5; this lemma will be used in the next section to prove Theorem 38.26.1.

0AT0 Situation 38.25.1. Let $\varphi : A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective.

In this situation it is our goal to show that u is A-universally injective, N is of finite presentation over B, and N is flat as an A-module. If this is true, we will say the lemma holds in the given situation.

0AT1 Lemma 38.25.2. If in Situation 38.25.1 the ring A is Noetherian then the lemma holds.

Proof. Applying Algebra, Lemma 10.99.1 we see that u is injective and that $N/u(M)$ is flat over A. Then u is A-universally injective (Algebra, Lemma 10.39.12) and N is A-flat (Algebra, Lemma 10.39.13). Since B is Noetherian in this case we see that N is of finite presentation. $\square$

0AT2 Lemma 38.25.3. Let A_0 be a local ring. If the lemma holds for every Situation 38.25.1 with $A = A_0$, with B a localization of a polynomial algebra over A, and N of finite presentation over B, then the lemma holds for every Situation 38.25.1 with $A = A_0$.

Proof. Let $A \to B$, $u : N \to M$ be as in Situation 38.25.1. Write $B = C/I$ where C is the localization of a polynomial algebra over A at a prime. If we can show that N is finitely presented as a C-module, then a fortiori this shows that N is finitely presented as a B-module (see Algebra, Lemma 10.6.4). Hence we may assume that B is the localization of a polynomial algebra. Next, write $N = B^{\oplus n}/K$ for some submodule $K \subset B^{\oplus n}$. Since $B/\mathfrak{m}_A B$ is Noetherian (as it is essentially of finite type over a field), there exist finitely many elements $k_1, \ldots, k_s \in K$ such that for $K' = \sum Bk_i$ and $N' = B^{\oplus n}/K'$ the canonical surjection $N' \to N$ induces an isomorphism $N'/\mathfrak{m}_A N' \cong N/\mathfrak{m}_A N$. Now, if the lemma holds for the composition $u' : N' \to M$, then u' is injective, hence $N' = N$ and $u' = u$. Thus the lemma holds for the original situation. □

0AT3 Lemma 38.25.4. If in Situation 38.25.1 the ring A is henselian then the lemma holds.

Proof. It suffices to prove this when B is essentially of finite presentation over A and N is of finite presentation over B, see Lemma 38.25.3. Let us temporarily make the additional assumption that N is flat over A. Then N is a filtered colimit $N = \mathrm{colim}_i F_i$ of free A-modules F_i such that the transition maps $u_{ii'} : F_i \to F_{i'}$ are injective modulo $\mathfrak{m}_A$, see Lemma 38.19.5. Each of the compositions $u_i : F_i \to M$ is A-universally injective by Lemma 38.7.5 wherefore $u = \mathrm{colim}\, u_i$ is A-universally injective as desired.

Assume A is a henselian local ring, B is essentially of finite presentation over A, N of finite presentation over B. By Theorem 38.24.1 there exists a finitely generated ideal $I \subset A$ such that N/IN is flat over A/I and such that N/I^2N is not flat over A/I^2 unless $I = 0$. The result of the previous paragraph shows that the lemma holds for $u \bmod I : N/IN \to M/IM$ over A/I. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \otimes_A I/I^2 \arrow[r] & M/I^2M \arrow[r] & M/IM \arrow[r] & 0 \\ & N \otimes_A I/I^2 \arrow[r] \arrow[u, "u"] & N/I^2N \arrow[r] \arrow[u, "u"] & N/IN \arrow[r] \arrow[u, "u"] & 0 \end{tikzcd}$$

whose rows are exact by right exactness of $\otimes$ and the fact that M is flat over A. Note that the left vertical arrow is the map $N/IN \otimes_{A/I} I/I^2 \to M/IM \otimes_{A/I} I/I^2$, hence is injective. A diagram chase shows that the lower left arrow is injective, i.e., $\mathrm{Tor}^1_{A/I^2}(I/I^2, M/I^2) = 0$ see Algebra, Remark 10.75.9. Hence N/I^2N is flat over A/I^2 by Algebra, Lemma 10.99.8 a contradiction unless $I = 0$. □

The following lemma discusses the special case of Situation 38.25.1 where M has a B-module structure and u is B-linear. This is the case most often used in practice and it is significantly easier to prove than the general case.

0AT4 Lemma 38.25.5. Let $A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let $u : N \to M$ be a B-module map. If N is a finite

B-module, M is flat over A, and $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective, then u is A-universally injective, N is of finite presentation over B, and N is flat over A.

Proof. Let $A \to A^h$ be the henselization of A. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Since $B \to B'$ is flat (hence faithfully flat, see Algebra, Lemma 10.39.17), we may replace $A \to B$ with $A^h \to B'$, the module M by $M \otimes_B B'$, the module N by $N \otimes_B B'$, and u by $u \otimes \mathrm{id}_{B'}$, see Algebra, Lemmas 10.83.2 and 10.39.9. Thus we may assume that A is a henselian local ring. In this case our lemma follows from the more general Lemma 38.25.4. $\square$

0AT5 Lemma 38.25.6. If in Situation 38.25.1 the ring A is a valuation ring then the lemma holds.

Proof. Recall that an A-module is flat if and only if it is torsion free, see More on Algebra, Lemma 15.22.10. Let $T \subset N$ be the A-torsion. Then $u(T) = 0$ and N/T is A-flat. Hence N/T is finitely presented over B, see More on Algebra, Lemma 15.25.6. Thus T is a finite B-module, see Algebra, Lemma 10.5.3. Since N/T is A-flat we see that $T/\mathfrak{m}_A T \subset N/\mathfrak{m}_A N$, see Algebra, Lemma 10.39.12. As $\overline{u}$ is injective but $u(T) = 0$, we conclude that $T/\mathfrak{m}_A T = 0$. Hence $T = 0$ by Nakayama's lemma, see Algebra, Lemma 10.20.1. At this point we have proved two out of the three assertions (N is A-flat and of finite presentation over B) and what is left is to show that u is universally injective.

By Algebra, Theorem 10.82.3 it suffices to show that $N \otimes_A Q \to M \otimes_A Q$ is injective for every finitely presented A-module Q. By More on Algebra, Lemma 15.124.3 we may assume $Q = A/(a)$ with $a \in \mathfrak{m}_A$ nonzero. Thus it suffices to show that $N/aN \to M/aM$ is injective. Let $x \in N$ with $u(x) \in aM$. By Lemma 38.19.6 we know that x has a content ideal $I \subset A$. Since I is finitely generated (More on Algebra, Lemma 15.24.2) and A is a valuation ring, we have $I = (b)$ for some b (by Algebra, Lemma 10.50.15). By More on Algebra, Lemma 15.24.3 the element $u(x)$ has content ideal I as well. Since $u(x) \in aM$ we see that $(b) \subset (a)$ by More on Algebra, Definition 15.24.1. Since $x \in bN$ we conclude $x \in aN$ as desired. $\square$

Consider the following situation

0AT6 (38.25.6.1)
$$\begin{gathered} A \to B \text{ of finite presentation, } S \subset B \text{ a multiplicative subset, and} \\ N \text{ a finitely presented } S^{-1}B\text{-module} \end{gathered}$$

In this situation a pure spreadout is an affine open $U \subset \mathrm{Spec}(B)$ with $\mathrm{Spec}(S^{-1}B) \subset U$ and a finitely presented $\mathcal{O}(U)$-module N' extending N such that N' is A-projective and $N' \to N = S^{-1}N'$ is A-universally injective.

In (38.25.6.1) if $A \to A_1$ is a ring map, then we can base change: take $B_1 = B \otimes_A A_1$, let $S_1 \subset B_1$ be the image of S, and let $N_1 = N \otimes_A A_1$. This works because $S_1^{-1}B_1 = S^{-1}B \otimes_A A_1$. We will use this without further mention in the following.

0AT7 Lemma 38.25.7. In (38.25.6.1) if there exists a pure spreadout, then

(1) elements of N have content ideals in A, and
(2) if $u : N \to M$ is a morphism to a flat A-module M such that $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is injective for all maximal ideals $\mathfrak{m}$ of A, then u is A-universally injective.

Proof. Choose U, N' as in the definition of a pure spreadout. Any element $x' \in N'$ has a content ideal in A because N' is A-projective (this can easily be seen directly, but it also follows from More on Algebra, Lemma 15.24.4 and Algebra, Example 10.91.1). Since $N' \to N$ is A-universally injective, we see that the image $x \in N$ of any $x' \in N'$ has a content ideal in A (it is the same as the content ideal of x'). For a general $x \in N$ we choose $s \in S$ such that sx is in the image of $N' \to N$ and we use that x and sx have the same content ideal.

Let $u : N \to M$ be as in (2). To show that u is A-universally injective, we may replace A by a localization at a maximal ideal (small detail omitted). Assume A is local with maximal ideal $\mathfrak{m}$. Pick $s \in S$ and consider the composition

$$N' \to N \xrightarrow{1/s} N \xrightarrow{u} M$$

Each of these maps is injective modulo $\mathfrak{m}$, hence the composition is A-universally injective by Lemma 38.7.5. Since $N = \mathrm{colim}_{s \in S}(1/s)N'$ we conclude that u is A-inversally injective as a colimit of universally injective maps. □

0AT8 Lemma 38.25.8. In (38.25.6.1) for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there is a finitely generated ideal $I \subset \mathfrak{p}A_{\mathfrak{p}}$ such that over $A_{\mathfrak{p}}/I$ we have a pure spreadout.

Proof. We may replace A by $A_{\mathfrak{p}}$. Thus we may assume A is local and $\mathfrak{p}$ is the maximal ideal $\mathfrak{m}$ of A. We may write $N = S^{-1}N'$ for some finitely presented B-module N' by clearing denominators in a presentation of N over $S^{-1}B$. Since $B/\mathfrak{m}B$ is Noetherian, the kernel K of $N'/\mathfrak{m}N' \to N/\mathfrak{m}N$ is finitely generated. Thus we can pick $s \in S$ such that K is annihilated by s. After replacing B by B_s which is allowed as it just means passing to an affine open subscheme of $\mathrm{Spec}(B)$, we find that the elements of S are injective on $N'/\mathfrak{m}N'$. At this point we choose a local subring $A_0 \subset A$ essentially of finite type over $\mathbf{Z}$, a finite type ring map $A_0 \to B_0$ such that $B = A \otimes_{A_0} B_0$, and a finite B_0-module N_0' such that $N' = B \otimes_{B_0} N_0' = A \otimes_{A_0} N_0'$. We claim that $I = \mathfrak{m}_{A_0}A$ works. Namely, we have

$$N'/IN' = N_0'/\mathfrak{m}_{A_0}N_0' \otimes_{\kappa_{A_0}} A/I$$

which is free over A/I. Multiplication by the elements of S is injective after dividing out by the maximal ideal, hence $N'/IN' \to N/IN$ is universally injective for example by Lemma 38.7.6. □

0AT9 Lemma 38.25.9. In (38.25.6.1) assume N is A-flat, M is a flat A-module, and $u : N \to M$ is an A-module map such that $u \otimes \mathrm{id}_{\kappa(\mathfrak{p})}$ is injective for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Then u is A-universally injective.

Proof. By Algebra, Lemma 10.82.14 it suffices to check that $N/IN \to M/IM$ is injective for every ideal $I \subset A$. After replacing A by A/I we see that it suffices to prove that u is injective.

Proof that u is injective. Let $x \in N$ be a nonzero element of the kernel of u. Then there exists a weakly associated prime $\mathfrak{p}$ of the module Ax, see Algebra, Lemma 10.66.5. Replacing A by $A_{\mathfrak{p}}$ we may assume A is local and we find a nonzero element $y \in Ax$ whose annihilator has radical equal to $\mathfrak{m}_A$, see Algebra, Lemma 10.66.2. Thus $\mathrm{Supp}(y) \subset \mathrm{Spec}(S^{-1}B)$ is nonempty and contained in the closed fibre of $\mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(A)$. Let $I \subset \mathfrak{m}_A$ be a finitely generated ideal so that we have a pure spreadout over A/I, see Lemma 38.25.8. Then $I^n y = 0$ for some n. Now $y \in \mathrm{Ann}_M(I^n) = \mathrm{Ann}_A(I^n) \otimes_R N$ by flatness. Thus, to get the desired contradiction, it suffices to show that

$$\mathrm{Ann}_A(I^n) \otimes_R N \longrightarrow \mathrm{Ann}_A(I^n) \otimes_R M$$

is injective. Since N and M are flat and since $\text{Ann}_A(I^n)$ is annihilated by I^n, it suffices to show that $Q \otimes_A N \to Q \otimes_A M$ is injective for every A-module Q annihilated by I. This holds by our choice of I and Lemma 38.25.7 part (2). $\square$

0ATA Lemma 38.25.10. Let A be a local domain which is not a field. Let S be a set of finitely generated ideals of A. Assume that S is closed under products and such that $\bigcup_{I \in S} V(I)$ is the complement of the generic point of $\text{Spec}(A)$. Then $\bigcap_{I \in S} I = (0)$.

Proof. Since $\mathfrak{m}_A \subset A$ is not the generic point of $\text{Spec}(A)$ we see that $I \subset \mathfrak{m}_A$ for at least one $I \in S$. Hence $\bigcap_{I \in S} I \subset \mathfrak{m}_A$. Let $f \in \mathfrak{m}_A$ be nonzero. Then $V(f) \subset \bigcup_{I \in S} V(I)$. Since the constructible topology on $V(f)$ is quasi-compact (Topology, Lemma 5.23.2 and Algebra, Lemma 10.26.2) we find that $V(f) \subset V(I_1) \cup \ldots \cup V(I_n)$ for some $I_j \in S$. Because $I_1 \ldots I_n \in S$ we see that $V(f) \subset V(I)$ for some I. As I is finitely generated this implies that $I^m \subset (f)$ for some m and since S is closed under products we see that $I \subset (f^2)$ for some $I \in S$. Then it is not possible to have $f \in I$. $\square$

0ATB Lemma 38.25.11. Let A be a local ring. Let $I, J \subset A$ be ideals. If J is finitely generated and $I \subset J^n$ for all $n \geq 1$, then $V(I)$ contains the closed points of $\text{Spec}(A) \setminus V(J)$.

Proof. Let $\mathfrak{p} \subset A$ be a closed point of $\text{Spec}(A) \setminus V(J)$. We want to show that $I \subset \mathfrak{p}$. If not, then some $f \in I$ maps to a nonzero element of $A/\mathfrak{p}$. Note that $V(J) \cap \text{Spec}(A/\mathfrak{p})$ is the set of non-generic points. Hence by Lemma 38.25.10 applied to the collection of ideals $J^n A/\mathfrak{p}$ we conclude that the image of f is zero in $A/\mathfrak{p}$. $\square$

0ATC Lemma 38.25.12. Let A be a local ring. Let $I \subset A$ be an ideal. Let $U \subset \text{Spec}(A)$ be quasi-compact open. Let M be an A-module. Assume that

(1) M/IM is flat over A/I,
(2) M is flat over U,

Then M/I_2M is flat over A/I_2 where $I_2 = \text{Ker}(I \to \Gamma(U, I/I^2))$.

Proof. It suffices to show that $M \otimes_A I/I_2 \to IM/I_2M$ is injective, see Algebra, Lemma 10.99.9. This is true over U by assumption (2). Thus it suffices to show that $M \otimes_A I/I_2$ injects into its sections over U. We have $M \otimes_A I/I_2 = M/IM \otimes_A I/I_2$ and M/IM is a filtered colimit of finite free A/I-modules (Algebra, Theorem 10.81.4). Hence it suffices to show that I/I_2 injects into its sections over U, which follows from the construction of I_2. $\square$

05U9 Proposition 38.25.13. Let $A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is A-universally injective, N is of finite presentation over B, and N is flat over A.

Proof. We may assume that B is the localization of a finitely presented A-algebra B_0 and that N is the localization of a finitely presented B_0-module M_0, see Lemma 38.25.3. By More on Morphisms, Lemma 37.54.1 there exists a "generic flatness stratification" for $\widetilde{M_0}$ on $\text{Spec}(B_0)$ over $\text{Spec}(A)$. Translating back to N we find a sequence of closed subschemes

$$S = \text{Spec}(A) \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

with $S_i \subset S$ cut out by a finitely generated ideal of A such that the pullback of $\widetilde{N}$ to $\mathrm{Spec}(B) \times_S (S_i \setminus S_{i+1})$ is flat over $S_i \setminus S_{i+1}$. We will prove the proposition by induction on t (the base case $t = 1$ will be proved in parallel with the other steps). Let $\mathrm{Spec}(A/J_i)$ be the scheme theoretic closure of $S_i \setminus S_{i+1}$.

Claim 1. N/J_iN is flat over A/J_i. This is immediate for $i = t-1$ and follows from the induction hypothesis for $i > 0$. Thus we may assume $t > 1$, $S_{t-1} \neq \emptyset$, and $J_0 = 0$ and we have to prove that N is flat. Let $J \subset A$ be the ideal defining S_1. By induction on t again, we also have flatness modulo powers of J. Let A^h be the henselization of A and let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak m_B \otimes A^h + B \otimes \mathfrak m_{A^h}$. Then $B \to B'$ is faithfully flat. Set $N' = N \otimes_B B'$. Note that N' is A^h-flat if and only if N is A-flat. By Theorem 38.24.1 there is a smallest ideal $I \subset A^h$ such that N'/IN' is flat over A^h/I, and I is finitely generated. By the above $I \subset J^nA^h$ for all $n \geq 1$. Let $S_i^h \subset \mathrm{Spec}(A^h)$ be the inverse image of $S_i \subset \mathrm{Spec}(A)$. By Lemma 38.25.11 we see that $V(I)$ contains the closed points of $U = \mathrm{Spec}(A^h) - S_1^h$. By construction N' is A^h-flat over U. By Lemma 38.25.12 we see that N'/I_2N' is flat over A/I_2, where $I_2 = \mathrm{Ker}(I \to \Gamma(U, I/I^2))$. Hence $I = I_2$ by minimality of I. This implies that $I = I^2$ locally on U, i.e., we have $I\mathcal{O}_{U,u} = (0)$ or $I\mathcal{O}_{U,u} = (1)$ for all $u \in U$. Since $V(I)$ contains the closed points of U we see that $I = 0$ on U. Since $U \subset \mathrm{Spec}(A^h)$ is scheme theoretically dense (because replaced A by A/J_0 in the beginning of this paragraph), we see that $I = 0$. Thus N' is A^h-flat and hence Claim 1 holds.

We return to the situation as laid out before Claim 1. With A^h the henselization of A, with B' the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak m_B \otimes A^h + B \otimes \mathfrak m_{A^h}$, and with $N' = N \otimes_B B'$ we now see that the flattening ideal $I \subset A^h$ of Theorem 38.24.1 is nilpotent. If $nil(A^h)$ denotes the ideal of nilpotent elements, then $nil(A^h) = nil(A)A^h$ (More on Algebra, Lemma 15.45.5). Hence there exists a finitely generated nilpotent ideal $I_0 \subset A$ such that N/I_0N is flat over A/I_0.

Claim 2. For every prime ideal $\mathfrak p \subset A$ the map $\kappa(\mathfrak p) \otimes_A N \to \kappa(\mathfrak p) \otimes_A M$ is injective. We say $\mathfrak p$ is bad it this is false. Suppose that C is a nonempty chain of bad primes and set $\mathfrak p^* = \bigcup_{\mathfrak p \in C} \mathfrak p$. By Lemma 38.25.8 there is a finitely generated ideal $\mathfrak a \subset \mathfrak p^*A_{\mathfrak p^*}$ such that there is a pure spreadout over $V(\mathfrak a)$. If $\mathfrak p^*$ were good, then it would follow from Lemma 38.25.7 that the points of $V(\mathfrak a)$ are good. However, since $\mathfrak a$ is finitely generated and since $\mathfrak p^*A_{\mathfrak p^*} = \bigcup_{\mathfrak p \in C} A_{\mathfrak p^*}$ we see that $V(\mathfrak a)$ contains a $\mathfrak p \in C$, contradiction. Hence $\mathfrak p^*$ is bad. By Zorn's lemma, if there exists a bad prime, there exists a maximal one, say $\mathfrak p$. In other words, we may assume every $\mathfrak p' \supset \mathfrak p$, $\mathfrak p' \neq \mathfrak p$ is good. In this case we see that for every $f \in A$, $f \not\in \mathfrak p$ the map $u \otimes \mathrm{id}_{A/(\mathfrak p + f)}$ is universally injective, see Lemma 38.25.9. Thus it suffices to show that $N/\mathfrak pN$ is separated for the topology defined by the submodules $f(N/\mathfrak pN)$. Since $B \to B'$ is faithfully flat, it is enough to prove the same for the module $N'/\mathfrak pN'$. By Lemma 38.19.5 and More on Algebra, Lemma 15.24.4 elements of $N'/\mathfrak pN'$ have content ideals in $A^h/\mathfrak pA^h$. Thus it suffices to show that $\bigcap_{f \in A, f \not\in \mathfrak p} f(A^h/\mathfrak pA^h) = 0$. Then it suffices to show the same for $A^h/\mathfrak qA^h$ for every prime $\mathfrak q \subset A^h$ minimal over $\mathfrak pA^h$. Because $A \to A^h$ is the henselization, every $\mathfrak q$ contracts to $\mathfrak p$ and every $\mathfrak q' \supset \mathfrak q$, $\mathfrak q' \neq \mathfrak q$ contracts to a prime $\mathfrak p'$ which strictly contains $\mathfrak p$. Thus we get the vanishing of the intersections from Lemma 38.25.10.

At this point we can put everything together. Namely, using Claim 1 and Claim 2 we see that $N/I_0N \to M/I_0M$ is A/I_0-universally injective by Lemma 38.25.9.

Then the diagrams

$$\begin{tikzcd} N \otimes_A (I_0^n/I_0^{n+1}) \arrow[r] \arrow[d] & M \otimes_A (I_0^n/I_0^{n+1}) \arrow[d, equal] \\ I_0^nN/I_0^{n+1}N \arrow[r] & I_0^nM/I_0^{n+1}M \end{tikzcd}$$

show that the left vertical arrows are injective. Hence by Algebra, Lemma 10.99.9 we see that N is flat. In a similar way the universal injectivity of u can be reduced (even without proving flatness of N first) to the one modulo I_0. This finishes the proof. □

38.26. Flat finite type modules, Part III

05U8 The following result is one of the main results of this chapter.

05UA **Theorem 38.26.1.** Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ with image $s \in S$. The following are equivalent

(1) $\mathcal{F}$ is flat at x over S, and
(2) for every $x' \in \text{Ass}_{X_s}(\mathcal{F}_s)$ which specializes to x we have that $\mathcal{F}$ is flat at x' over S.

Proof. It is clear that (1) implies (2) as $\mathcal{F}_{x'}$ is a localization of $\mathcal{F}_x$ for every point which specializes to x. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$ and $N = \mathcal{F}_x$. Let $\Sigma \subset B$ be the multiplicative subset of B of elements which act as nonzerodivisors on $N/\mathfrak{m}_AN$. Assumption (2) implies that $\Sigma^{-1}N$ is A-flat by the description of $\text{Spec}(\Sigma^{-1}N)$ in Lemma 38.7.1. On the other hand, the map $N \to \Sigma^{-1}N$ is injective modulo $\mathfrak{m}_A$ by construction. Hence applying Lemma 38.25.5 we win. □

Now we apply this directly to obtain the following useful results.

05UB **Lemma 38.26.2.** Let S be a local scheme with closed point s. Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume that

(1) every point of $\text{Ass}_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s[3],
(2) $\mathcal{F}$ is flat over S at every point of X_s.

Then $\mathcal{F}$ is flat over S.

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of $\mathcal{F}$ over S by Theorem 38.26.1. □

38.27. Universal flattening

05PS If $f : X \to S$ is a proper, finitely presented morphism of schemes then one can find a universal flattening of f. In this section we discuss this and some of its variants.

05UC **Lemma 38.27.1.** In Situation 38.20.7. For each $p \geq 0$ the functor H_p (38.20.7.2) is representable by a locally closed immersion $S_p \to S$. If $\mathcal{F}$ is of finite presentation, then $S_p \to S$ is of finite presentation.

[3]For example this holds if f is finite type and $\mathcal{F}$ is pure along X_s, or if f is proper.

Proof. For each S we will prove the statement for all $p \geq 0$ concurrently. The functor H_p is a sheaf for the fppf topology by Lemma 38.20.8. Hence combining Descent, Lemma 35.39.1, More on Morphisms, Lemma 37.57.1 , and Descent, Lemma 35.24.1 we see that the question is local for the étale topology on S. In particular, the question is Zariski local on S.

For $s \in S$ denote ξ_s the unique generic point of the fibre X_s. Note that for every $s \in S$ the restriction $\mathcal{F}_s$ of $\mathcal{F}$ is locally free of some rank $p(s) \geq 0$ in some neighbourhood of ξ_s. (As X_s is irreducible and smooth this follows from generic flatness for $\mathcal{F}_s$ over X_s, see Algebra, Lemma 10.118.1 although this is overkill.) For future reference we note that

$$p(s) = \dim_{\kappa(\xi_s)}(\mathcal{F}_{\xi_s} \otimes_{\mathcal{O}_{X,\xi_s}} \kappa(\xi_s)).$$

In particular $H_{p(s)}(s)$ is nonempty and $H_q(s)$ is empty if $q \neq p(s)$.

Let $U \subset X$ be an open subscheme. As $f : X \to S$ is smooth, it is open. It is immediate from (38.20.7.2) that the functor H_p for the pair $(f|_U : U \to f(U), \mathcal{F}|_U)$ and the functor H_p for the pair $(f|_{f^{-1}(f(U))}, \mathcal{F}|_{f^{-1}(f(U))})$ are the same. Hence to prove the existence of S_p over $f(U)$ we may always replace X by U.

Pick $s \in S$. There exists an affine open neighbourhood U of ξ_s such that $\mathcal{F}|_U$ can be generated by at most $p(s)$ elements. By the arguments above we see that in order to prove the statement for $H_{p(s)}$ in an neighbourhood of s we may assume that $\mathcal{F}$ is generated by $p(s)$ elements, i.e., that there exists a surjection

$$u : \mathcal{O}_X^{\oplus p(s)} \longrightarrow \mathcal{F}$$

In this case it is clear that $H_{p(s)}$ is equal to F_{iso} (38.20.1.1) for the map u (this follows immediately from Lemma 38.19.1 but also from Lemma 38.12.1 after shrinking a bit more so that both S and X are affine.) Thus we may apply Theorem 38.23.3 to see that $H_{p(s)}$ is representable by a closed immersion in a neighbourhood of s.

The result follows formally from the above. Namely, the arguments above show that locally on S the function $s \mapsto p(s)$ is bounded. Hence we may use induction on $p = \max_{s \in S} p(s)$. The functor H_p is representable by a closed immersion $S_p \to S$ by the above. Replace S by $S \setminus S_p$ which drops the maximum by at least one and we win by induction hypothesis.

Assume $\mathcal{F}$ is of finite presentation. Then $S_p \to S$ is locally of finite presentation by Lemma 38.20.8 part (2) combined with Limits, Remark 32.6.2. Then we redo the induction argument in the paragraph to see that each S_p is quasi-compact when S is affine: first if $p = \max_{s \in S} p(s)$, then $S_p \subset S$ is closed (see above) hence quasi-compact. Then $U = S \setminus S_p$ is quasi-compact open in S because $S_p \to S$ is a closed immersion of finite presentation (see discussion in Morphisms, Section 29.22 for example). Then $S_{p-1} \to U$ is a closed immersion of finite presentation, and so S_{p-1} is quasi-compact and $U' = S \setminus (S_p \cup S_{p-1})$ is quasi-compact. And so on. □

05UD Lemma 38.27.2. In Situation 38.20.11. Let $h : X' \to X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (38.20.11.1) associated to $(f' : X' \to S, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset \text{Ass}_{X/S}(\mathcal{F})$, then $F_n = F'_n$.

Proof. Let $T \to S$ be any morphism. Then $h_T : X'_T \to X_T$ is étale as a base change of the étale morphism g. For $t \in T$ denote $Z \subset X_t$ the set of points where $\mathcal{F}_T$ is not flat over T, and similarly denote $Z' \subset X'_t$ the set of points where $\mathcal{F}'_T$ is not flat over T. As $\mathcal{F}'_T = h_T^*\mathcal{F}_T$ we see that $Z' = h_t^{-1}(Z)$, see Morphisms, Lemma 29.25.13. Hence $Z' \to Z$ is an étale morphism, so $\dim(Z') \leq \dim(Z)$ (for example by Descent, Lemma 35.21.2 or just because an étale morphism is smooth of relative dimension 0). This implies that $F_n \subset F'_n$.

Finally, suppose that $h(X') \supset \text{Ass}_{X/S}(\mathcal{F})$ and that $T \to S$ is a morphism such that $F'_n(T)$ is nonempty, i.e., such that $\mathcal{F}'_T$ is flat in dimensions $\geq n$ over T. Pick a point $t \in T$ and let $Z \subset X_t$ and $Z' \subset X'_t$ be as above. To get a contradiction assume that $\dim(Z) \geq n$. Pick a generic point $\xi \in Z$ corresponding to a component of dimension $\geq n$. Let $x \in \text{Ass}_{X_t}(\mathcal{F}_t)$ be a generalization of ξ. Then x maps to a point of $\text{Ass}_{X/S}(\mathcal{F})$ by Divisors, Lemma 31.7.3 and Remark 31.7.4. Thus we see that x is in the image of h_T, say $x = h_T(x')$ for some $x' \in X'_T$. But $x' \notin Z'$ as $x \rightsquigarrow \xi$ and $\dim(Z') < n$. Hence $\mathcal{F}'_T$ is flat over T at x' which implies that $\mathcal{F}_T$ is flat at x over T (by Morphisms, Lemma 29.25.13). Since this holds for every such x we conclude that $\mathcal{F}_T$ is flat over T at ξ by Theorem 38.26.1 which is the desired contradiction. $\square$

05UE Lemma 38.27.3. Assume that $X \to S$ is a smooth morphism of affine schemes with geometrically irreducible fibres of dimension d and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then $F_d = \coprod_{p=0,\ldots,c} H_p$ for some $c \geq 0$ with F_d as in (38.20.11.1) and H_p as in (38.20.7.2).

Proof. As X is affine and $\mathcal{F}$ is quasi-coherent of finite presentation we know that $\mathcal{F}$ can be generated by $c \geq 0$ elements. Then $\dim_{\kappa(x)}(\mathcal{F}_x \otimes \kappa(x))$ in any point $x \in X$ never exceeds c. In particular $H_p = \emptyset$ for $p > c$. Moreover, note that there certainly is an inclusion $\coprod H_p \to F_d$. Having said this the content of the lemma is that, if a base change $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T and if $t \in T$, then $\mathcal{F}_T$ is free of some rank r in an open neighbourhood $U \subset X_T$ of the unique generic point ξ of X_t. Namely, then H_r contains the image of U which is an open neighbourhood of t. The existence of U follows from More on Morphisms, Lemma 37.16.7. $\square$

05UF Lemma 38.27.4. In Situation 38.20.11. Let $s \in S$ let $d \geq 0$. Assume

(1) there exists a complete dévissage of $\mathcal{F}/X/S$ over some point $s \in S$,
(2) X is of finite presentation over S,
(3) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and
(4) $\mathcal{F}$ is flat in dimensions $\geq d + 1$ over S.

Then after possibly replacing S by an open neighbourhood of s the functor F_d (38.20.11.1) is representable by a monomorphism $Z_d \to S$ of finite presentation.

Proof. A preliminary remark is that X, S are affine schemes and that it suffices to prove F_d is representable by a monomorphism of finite presentation $Z_d \to S$ on the category of affine schemes over S. (Of course we do not require Z_d to be affine.) Hence throughout the proof of the lemma we work in the category of affine schemes over S.

Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s, see Definition 38.5.1. We will use induction on the length n of the dévissage. Recall that $Y_k \to S$ is smooth with geometrically irreducible fibres, see Definition 38.4.1. Let d_k be the relative dimension of Y_k over S. Recall that $i_{k,*}\mathcal{G}_k = \text{Coker}(\alpha_k)$ and that i_k is a closed immersion. By the definitions referenced above we have

$d_1 = \dim(\text{Supp}(\mathcal{F}_s))$ and

$$d_k = \dim(\text{Supp}(\text{Coker}(\alpha_{k-1})_s)) = \dim(\text{Supp}(\mathcal{G}_{k,s}))$$

for $k = 2, \ldots, n$. It follows that $d_1 > d_2 > \ldots > d_n \geq 0$ because α_k is an isomorphism in the generic point of $(Y_k)_s$.

Note that i_1 is a closed immersion and $\mathcal{F} = i_{1,*}\mathcal{G}_1$. Hence for any morphism of schemes $T \to S$ with T affine, we have $\mathcal{F}_T = i_{1,T,*}\mathcal{G}_{1,T}$ and $i_{1,T}$ is still a closed immersion of schemes over T. Thus $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T if and only if $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. Because $\pi_1 : Z_1 \to Y_1$ is finite we see in the same manner that $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T if and only if $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. The same arguments work for "flat in dimensions $\geq d + 1$" and we conclude in particular that $\pi_{1,*}\mathcal{G}_1$ is flat over S in dimensions $\geq d + 1$ by our assumption on $\mathcal{F}$.

Suppose that $d_1 > d$. It follows from the discussion above that in particular $\pi_{1,*}\mathcal{G}_1$ is flat over S at the generic point of $(Y_1)_s$. By Lemma 38.12.1 we may replace S by an affine neighbourhood of s and assume that α_1 is S-universally injective. Because α_1 is S-universally injective, for any morphism $T \to S$ with T affine, we have a short exact sequence

$$0 \to \mathcal{O}_{Y_{1,T}}^{\oplus r_1} \to \pi_{1,T,*}\mathcal{G}_{1,T} \to \text{Coker}(\alpha_1)_T \to 0$$

and still the first arrow is T-universally injective. Hence the set of points of $(Y_1)_T$ where $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat over T is the same as the set of points of $(Y_1)_T$ where $\text{Coker}(\alpha_1)_T$ is flat over S. In this way the question reduces to the sheaf $\text{Coker}(\alpha_1)$ which has a complete dévissage of length $n - 1$ and we win by induction.

If $d_1 < d$ then F_d is represented by S and we win.

The last case is the case $d_1 = d$. This case follows from a combination of Lemma 38.27.3 and Lemma 38.27.1. □

05UG Theorem 38.27.5. In Situation 38.20.11. Assume moreover that f is of finite presentation, that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and that $\mathcal{F}$ is pure relative to S. Then F_n is representable by a monomorphism $Z_n \to S$ of finite presentation.

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 38.20.12. Observe that a monomorphism of finite presentation is separated and quasi-finite (Morphisms, Lemma 29.20.15). Hence combining Descent, Lemma 35.39.1, More on Morphisms, Lemma 37.57.1 , and Descent, Lemmas 35.23.31 and 35.23.13 we see that the question is local for the étale topology on S.

In particular the situation is local for the Zariski topology on S and we may assume that S is affine. In this case the dimension of the fibres of f is bounded above, hence we see that F_n is representable for n large enough. Thus we may use descending induction on n. Suppose that we know F_{n+1} is representable by a monomorphism $Z_{n+1} \to S$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_S X$ and the pullback $\mathcal{F}_{n+1}$ of $\mathcal{F}$ to X_{n+1}. The morphism $Z_{n+1} \to S$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 38.16.4 implies that $\mathcal{F}_{n+1}$ is pure relative to Z_{n+1}. Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the result for F_n in case $S_{n+1} = S$, i.e., we may assume that $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S.

Fix n and assume $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S. To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \to S$ of finite presentation. Since the question is local in the étale topology on S it suffices to show that for every $s \in S$ there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the result holds after base change to S'. Thus by Lemma 38.5.8 we may assume there exist étale morphisms $h_j : Y_j \to X$, $j = 1, \ldots, m$ such that for each j there exists a complete dévissage of $\mathcal{F}_j/Y_j/S$ over s, where $\mathcal{F}_j$ is the pullback of $\mathcal{F}$ to Y_j and such that $X_s \subset \bigcup h_j(Y_j)$. Note that by Lemma 38.27.2 the sheaves $\mathcal{F}_j$ are still flat over in dimensions $\geq n+1$ over S. Set $W = \bigcup h_j(Y_j)$, which is a quasi-compact open of X. As $\mathcal{F}$ is pure along X_s we see that

$$E = \{t \in S \mid \text{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of s. By More on Morphisms, Lemma 37.25.5 E is a constructible subset of S. We have seen that $\text{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 29.22.4 we see that E contains an open neighbourhood of s. Hence after shrinking S we may assume that $E = S$. It follows from Lemma 38.27.2 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod Y_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $Y_j \to S$ and the sheaf $\mathcal{F}_i$ we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \to S$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_S \ldots \times_S Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in Lemma 38.27.4. $\square$

We make explicit what the theorem means in terms of universal flattenings in the following lemma.

05UH Lemma 38.27.6. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) If f is of finite presentation, $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and $\mathcal{F}$ is pure relative to S, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.
(2) If f is of finite presentation and X is pure relative to S, then there exists a universal flattening $S' \to S$ of X. Moreover $S' \to S$ is a monomorphism of finite presentation.
(3) If f is proper and of finite presentation and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.
(4) If f is proper and of finite presentation then there exists a universal flattening $S' \to S$ of X.

Proof. These statements follow immediately from Theorem 38.27.5 applied to $F_0 = F_{flat}$ and the fact that if f is proper then $\mathcal{F}$ is automatically pure over the base, see Lemma 38.17.1. $\square$

38.28. Grothendieck's Existence Theorem, IV

0CTB This section continues the discussion in Cohomology of Schemes, Sections 30.24, 30.25, and 30.27. We will work in the following situation.

0CTC Situation 38.28.1. Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have a scheme X separated and of finite presentation over A. We set $X_n = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ and we view it as a closed subscheme of X. We assume further given a system $(\mathcal{F}_n, \varphi_n)$ where $\mathcal{F}_n$ is a finitely presented $\mathcal{O}_{X_n}$-module, flat over A_n, with support proper over A_n, and

$$\varphi_n : \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \longrightarrow \mathcal{F}_{n-1}$$

is an isomorphism (notation using the equivalence of Morphisms, Lemma 29.4.1).

Our goal is to see if we can find a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n.

0CTD Lemma 38.28.2. In Situation 38.28.1 consider

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(\mathcal{F}_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} \mathcal{F}_n)$$

Then K is in $D^b_{QCoh}(\mathcal{O}_X)$ and in fact K has nonzero cohomology sheaves only in degrees ≥ 0.

Proof. Special case of Derived Categories of Schemes, Example 36.21.5. $\square$

0CTE Lemma 38.28.3. In Situation 38.28.1 let K be as in Lemma 38.28.2. For any perfect object E of $D(\mathcal{O}_X)$ we have

(1) $M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes_A^{\mathbf{L}} A_n$ in $D(A_n)$,
(2) $N = R\operatorname{Hom}_X(E, K)$ is a perfect object of $D(A)$ and there is a canonical isomorphism $R\operatorname{Hom}_{X_n}(E|_{X_n}, \mathcal{F}_n) = N \otimes_A^{\mathbf{L}} A_n$ in $D(A_n)$.

In both statements $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Proof of (2). Write $E_n = E|_{X_n}$ and $N_n = R\operatorname{Hom}_{X_n}(E_n, \mathcal{F}_n)$. Recall that $R\operatorname{Hom}_{X_n}(-,-)$ is equal to $R\Gamma(X_n, R\mathcal{H}om(-,-))$, see Cohomology, Section 20.44. Hence by Derived Categories of Schemes, Lemma 36.30.7 we see that N_n is a perfect object of $D(A_n)$ whose formation commutes with base change. Thus the maps $N_n \otimes_{A_n}^{\mathbf{L}} A_{n-1} \to N_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 15.97.3 we find that $R\lim N_n$ is perfect and that its base change back to A_n recovers N_n. On the other hand, the exact functor $R\operatorname{Hom}_X(E, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\operatorname{Hom}_X(E, K) = R\lim R\operatorname{Hom}_X(E, \mathcal{F}_n) = R\lim R\operatorname{Hom}_X(E_n, \mathcal{F}_n) = R\lim N_n$$

This proves (2). To see that (1) holds, translate it into (2) using Cohomology, Lemma 20.50.5. $\square$

0CTF Lemma 38.28.4. In Situation 38.28.1 let K be as in Lemma 38.28.2. Then K is pseudo-coherent relative to A.

Proof. Combinging Lemma 38.28.3 and Derived Categories of Schemes, Lemma 36.34.3 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus the lemma follows from More on Morphisms, Lemma 37.69.4. $\square$

0CTG Lemma 38.28.5. In Situation 38.28.1 let K be as in Lemma 38.28.2. For any quasi-compact open $U \subset X$ we have

$$R\Gamma(U, K) \otimes_A^{\mathbf{L}} A_n = R\Gamma(U_n, \mathcal{F}_n)$$

in $D(A_n)$ where $U_n = U \cap X_n$.

Proof. Fix n. By Derived Categories of Schemes, Lemma 36.33.4 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to $\mathcal{F}_n$ we obtain

$$\begin{aligned} R\Gamma(U_n, \mathcal{F}_n) &= R\Gamma(U, \mathcal{F}_n) \\ &= \text{hocolim}_m R\Gamma(X, \mathcal{F}_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, \mathcal{F}_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 38.28.3 and the fact that $- \otimes_A^{\mathbf{L}} A_n$ commutes with homotopy colimits we obtain the result. $\square$

0CTH Lemma 38.28.6. In Situation 38.28.1 let K be as in Lemma 38.28.2. Denote $X_0 \subset X$ the closed subset consisting of points lying over the closed subset $\text{Spec}(A_1) = \text{Spec}(A_2) = \ldots$ of $\text{Spec}(A)$. There exists an open $W \subset X$ containing X_0 such that

(1) $H^i(K)|_W$ is zero unless $i = 0$,
(2) $\mathcal{F} = H^0(K)|_W$ is of finite presentation, and
(3) $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$.

Proof. Fix $n \geq 1$. By construction there is a canonical map $K \to \mathcal{F}_n$ in $D_{QCoh}(\mathcal{O}_X)$ and hence a canonical map $H^0(K) \to \mathcal{F}_n$ of quasi-coherent sheaves. This explains the meaning of part (3).

Let $x \in X_0$ be a point. We will find an open neighbourhood W of x such that (1), (2), and (3) are true. Since X_0 is quasi-compact this will prove the lemma. Let $U \subset X$ be an affine open neighbourhood of x. Say $U = \text{Spec}(B)$. Choose a surjection $P \to B$ with P smooth over A. By Lemma 38.28.4 and the definition of relative pseudo-coherence there exists a bounded above complex $F^\bullet$ of finite free P-modules representing Ri_*K where $i : U \to \text{Spec}(P)$ is the closed immersion induced by the presentation. Let M_n be the B-module corresponding to $\mathcal{F}_n|_U$. By Lemma 38.28.5

$$H^i(F^\bullet \otimes_A A_n) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ M_n & \text{if} \quad i = 0 \end{cases}$$

Let i be the maximal index such that F^i is nonzero. If $i \leq 0$, then (1), (2), and (3) are true. If not, then $i > 0$ and we see that the rank of the map

$$F^{i-1} \to F^i$$

in the point x is maximal. Hence in an open neighbourhood of x inside $\text{Spec}(P)$ the rank is maximal. Thus after replacing P by a principal localization we may assume that the displayed map is surjective. Since F^i is finite free we may choose a splitting $F^{i-1} = F' \oplus F^i$. Then we may replace $F^\bullet$ by the complex

$$\ldots \to F^{i-2} \to F' \to 0 \to \ldots$$

and we win by induction on i. $\square$

0CTI Lemma 38.28.7. In Situation 38.28.1 let K be as in Lemma 38.28.2. Let $W \subset X$ be as in Lemma 38.28.6. Set $\mathcal{F} = H^0(K)|_W$. Then, after possibly shrinking the open W, the support of $\mathcal{F}$ is proper over A.

Proof. Fix $n \geq 1$. Let $I_n = \mathrm{Ker}(A \to A_n)$. By More on Algebra, Lemma 15.11.3 the pair (A, I_n) is henselian. Let $Z \subset W$ be the support of $\mathcal{F}$. This is a closed subset as $\mathcal{F}$ is of finite presentation. By part (3) of Lemma 38.28.6 we see that $Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$ is equal to the support of $\mathcal{F}_n$ and hence proper over $\mathrm{Spec}(A/I)$. By More on Morphisms, Lemma 37.53.9 we can write $Z = Z_1 \amalg Z_2$ with Z_1, Z_2 open and closed in Z, with Z_1 proper over A, and with $Z_1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I_n)$ equal to the support of $\mathcal{F}_n$. In other words, Z_2 does not meet X_0. Hence after replacing W by $W \setminus Z_2$ we obtain the lemma. $\square$

0CTJ Lemma 38.28.8. Let $A = \lim A_n$ be a limit of a system of rings whose transition maps are surjective and with locally nilpotent kernels. Let $S = \mathrm{Spec}(A)$. Let $T \to S$ be a monomorphism which is locally of finite type. If $\mathrm{Spec}(A_n) \to S$ factors through T for all n, then $T = S$.

Proof. Set $S_n = \mathrm{Spec}(A_n)$. Let $T_0 \subset T$ be the common image of the factorizations $S_n \to T$. Then T_0 is quasi-compact. Let $T' \subset T$ be a quasi-compact open containing T_0. Then $S_n \to T$ factors through T'. If we can show that $T' = S$, then $T' = T = S$. Hence we may assume T is quasi-compact.

Assume T is quasi-compact. In this case $T \to S$ is separated and quasi-finite (Morphisms, Lemma 29.20.15). Using Zariski's Main Theorem (in the form of More on Morphisms, Lemma 37.43.3) we choose a factorization $T \to W \to S$ with $W \to S$ finite and $T \to W$ an open immersion. Write $W = \mathrm{Spec}(B)$. The (unique) factorizations $S_n \to T$ may be viewed as morphisms into W and we obtain

$$A \longrightarrow B \longrightarrow \lim A_n = A$$

Consider the morphism $h : S = \mathrm{Spec}(A) \to \mathrm{Spec}(B) = W$ coming from the arrow on the right. Then

$$T \times_{W,h} S$$

is an open subscheme of S containing the image of $S_n \to S$ for all n. To finish the proof it suffices to show that any open $U \subset S$ containing the image of $S_n \to S$ for some $n \geq 1$ is equal to S. This is true because $(A, \mathrm{Ker}(A \to A_n))$ is a henselian pair (More on Algebra, Lemma 15.11.3) and hence every closed point of S is contained in the image of $S_n \to S$. $\square$

0CTK Theorem 38.28.9 (Grothendieck Existence Theorem). In Situation 38.28.1 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$, flat over A, with support proper over A, such that $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.

Proof. Apply Lemmas 38.28.2, 38.28.3, 38.28.4, 38.28.5, 38.28.6, and 38.28.7 to get an open subscheme $W \subset X$ containing all points lying over $\mathrm{Spec}(A_n)$ and a finitely presented $\mathcal{O}_W$-module $\mathcal{F}$ whose support is proper over A with $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{O}_{X_n}$

for all $n \geq 1$. (This makes sense as $X_n \subset W$.) By Lemma 38.17.1 we see that $\mathcal{F}$ is universally pure relative to $\text{Spec}(A)$. By Theorem 38.27.5 (for explanation, see Lemma 38.27.6) there exists a universal flattening $S' \to \text{Spec}(A)$ of $\mathcal{F}$ and moreover the morphism $S' \to \text{Spec}(A)$ is a monomorphism of finite presentation. Since the base change of $\mathcal{F}$ to $\text{Spec}(A_n)$ is $\mathcal{F}_n$ we find that $\text{Spec}(A_n) \to \text{Spec}(A)$ factors (uniquely) through S' for each n. By Lemma 38.28.8 we see that $S' = \text{Spec}(A)$. This means that $\mathcal{F}$ is flat over A. Finally, since the scheme theoretic support Z of $\mathcal{F}$ is proper over $\text{Spec}(A)$, the morphism $Z \to X$ is closed. Hence the pushforward $(W \to X)_*\mathcal{F}$ is supported on W and has all the desired properties. □

38.29. Grothendieck's Existence Theorem, V

0DIA In this section we prove an analogue for Grothendieck's existence theorem in the derived category, following the method used in Section 38.28 for quasi-coherent modules. The classical case is discussed in Cohomology of Schemes, Sections 30.24, 30.25, and 30.27. We will work in the following situation.

0DIB Situation 38.29.1. Here we have an inverse system of rings (A_n) with surjective transition maps whose kernels are locally nilpotent. Set $A = \lim A_n$. We have a scheme X proper, flat, and of finite presentation over A. We set $X_n = X \times_{\text{Spec}(A)} \text{Spec}(A_n)$ and we view it as a closed subscheme of X. We assume further given a system (K_n, φ_n) where K_n is a pseudo-coherent object of $D(\mathcal{O}_{X_n})$ and

$$\varphi_n : K_n \longrightarrow K_{n-1}$$

is a map in $D(\mathcal{O}_{X_n})$ which induces an isomorphism $K_n \otimes^{\mathbf{L}}_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \to K_{n-1}$ in $D(\mathcal{O}_{X_{n-1}})$.

More precisely, we should write $\varphi_n : K_n \to Ri_{n-1,*}K_{n-1}$ where $i_{n-1} : X_{n-1} \to X_n$ is the inclusion morphism and in this notation the condition is that the adjoint map $Li^*_{n-1}K_n \to K_{n-1}$ is an isomorphism. Our goal is to find a pseudo-coherent $K \in D(\mathcal{O}_X)$ such that $K_n = K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n (with the same abuse of notation).

0DIC Lemma 38.29.2. In Situation 38.29.1 consider

$$K = R\lim\nolimits_{D_{QCoh}(\mathcal{O}_X)}(K_n) = DQ_X(R\lim\nolimits_{D(\mathcal{O}_X)} K_n)$$

Then K is in $D^-_{QCoh}(\mathcal{O}_X)$.

Proof. The functor DQ_X exists because X is quasi-compact and quasi-separated, see Derived Categories of Schemes, Lemma 36.21.1. Since DQ_X is a right adjoint it commutes with products and therefore with derived limits. Hence the equality in the statement of the lemma.

By Derived Categories of Schemes, Lemma 36.21.4 the functor DQ_X has bounded cohomological dimension. Hence it suffices to show that $R\lim K_n \in D^-(\mathcal{O}_X)$. To see this, let $U \subset X$ be an affine open. Then there is a canonical exact sequence

$$0 \to R^1\lim H^{m-1}(U, K_n) \to H^m(U, R\lim K_n) \to \lim H^m(U, K_n) \to 0$$

by Cohomology, Lemma 20.37.1. Since U is affine and K_n is pseudo-coherent (and hence has quasi-coherent cohomology sheaves by Derived Categories of Schemes, Lemma 36.10.1) we see that $H^m(U, K_n) = H^m(K_n)(U)$ by Derived Categories of Schemes, Lemma 36.3.5. Thus we conclude that it suffices to show that K_n is bounded above independent of n.

Since K_n is pseudo-coherent we have $K_n \in D^-(\mathcal{O}_{X_n})$. Suppose that a_n is maximal such that $H^{a_n}(K_n)$ is nonzero. Of course $a_1 \leq a_2 \leq a_3 \leq \ldots$. Note that $H^{a_n}(K_n)$ is an $\mathcal{O}_{X_n}$-module of finite presentation (Cohomology, Lemma 20.47.9). We have $H^{a_n}(K_{n-1}) = H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$. Since $X_{n-1} \to X_n$ is a thickening, it follows from Nakayama's lemma (Algebra, Lemma 10.20.1) that if $H^{a_n}(K_n) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$ is zero, then $H^{a_n}(K_n)$ is zero too. Thus $a_n = a_{n-1}$ for all n and we conclude. □

0DID Lemma 38.29.3. In Situation 38.29.1 let K be as in Lemma 38.29.2. For any perfect object E of $D(\mathcal{O}_X)$ the cohomology

$$M = R\Gamma(X, K \otimes^{\mathbf{L}} E)$$

is a pseudo-coherent object of $D(A)$ and there is a canonical isomorphism

$$R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n}) = M \otimes_A^{\mathbf{L}} A_n$$

in $D(A_n)$. Here $E|_{X_n}$ denotes the derived pullback of E to X_n.

Proof. Write $E_n = E|_{X_n}$ and $M_n = R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E|_{X_n})$. By Derived Categories of Schemes, Lemma 36.30.5 we see that M_n is a pseudo-coherent object of $D(A_n)$ whose formation commutes with base change. Thus the maps $M_n \otimes_{A_n}^{\mathbf{L}} A_{n-1} \to M_{n-1}$ coming from φ_n are isomorphisms. By More on Algebra, Lemma 15.97.1 we find that $R\lim M_n$ is pseudo-coherent and that its base change back to A_n recovers M_n. On the other hand, the exact functor $R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \to D(A)$ of triangulated categories commutes with products and hence with derived limits, whence

$$R\Gamma(X, E \otimes^{\mathbf{L}} K) = R\lim R\Gamma(X, E \otimes^{\mathbf{L}} K_n) = R\lim R\Gamma(X_n, E_n \otimes^{\mathbf{L}} K_n) = R\lim M_n$$

as desired. □

0DIE Lemma 38.29.4. In Situation 38.29.1 let K be as in Lemma 38.29.2. Then K is pseudo-coherent on X.

Proof. Combinging Lemma 38.29.3 and Derived Categories of Schemes, Lemma 36.34.3 we see that $R\Gamma(X, K \otimes^{\mathbf{L}} E)$ is pseudo-coherent in $D(A)$ for all pseudo-coherent E in $D(\mathcal{O}_X)$. Thus it follows from More on Morphisms, Lemma 37.69.4 that K is pseudo-coherent relative to A. Since X is of flat and of finite presentation over A, this is the same as being pseudo-coherent on X, see More on Morphisms, Lemma 37.59.18. □

0DIF Lemma 38.29.5. In Situation 38.29.1 let K be as in Lemma 38.29.2. For any quasi-compact open $U \subset X$ we have

$$R\Gamma(U, K) \otimes_A^{\mathbf{L}} A_n = R\Gamma(U_n, K_n)$$

in $D(A_n)$ where $U_n = U \cap X_n$.

Proof. Fix n. By Derived Categories of Schemes, Lemma 36.33.4 there exists a system of perfect complexes E_m on X such that $R\Gamma(U, K) = \text{hocolim} R\Gamma(X, K \otimes^{\mathbf{L}} E_m)$. In fact, this formula holds not just for K but for every object of $D_{QCoh}(\mathcal{O}_X)$. Applying this to K_n we obtain

$$\begin{aligned} R\Gamma(U_n, K_n) &= R\Gamma(U, K_n) \\ &= \text{hocolim}_m R\Gamma(X, K_n \otimes^{\mathbf{L}} E_m) \\ &= \text{hocolim}_m R\Gamma(X_n, K_n \otimes^{\mathbf{L}} E_m|_{X_n}) \end{aligned}$$

Using Lemma 38.29.3 and the fact that $-\otimes_A^{\mathbf{L}} A_n$ commutes with homotopy colimits we obtain the result. □

0DIG **Theorem 38.29.6** (Derived Grothendieck Existence Theorem). *In Situation 38.29.1 there exists a pseudo-coherent K in $D(\mathcal{O}_X)$ such that $K_n = K \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{X_n}$ for all n compatibly with the maps φ_n.*

Proof. Apply Lemmas 38.29.2, 38.29.3, 38.29.4 to get a pseudo-coherent object K of $D(\mathcal{O}_X)$. Choosing affine opens in Lemma 38.29.5 it follows immediately that K restricts to K_n over X_n. □

0DIH **Remark 38.29.7.** The result in this section can be generalized. It is probably correct if we only assume $X \to \text{Spec}(A)$ to be separated, of finite presentation, and K_n pseudo-coherent relative to A_n supported on a closed subset of X_n proper over A_n. The outcome will be a K which is pseudo-coherent relative to A supported on a closed subset proper over A. If we ever need this, we will formulate a precise statement and prove it here.

38.30. Blowing up and flatness

080X In this section we continue our discussion of results of the form: "After a blowup the strict transform becomes flat", see More on Algebra, Section 15.26 and Divisors, Section 31.35. We will use the following (more or less standard) notation in this section. If $X \to S$ is a morphism of schemes, $\mathcal{F}$ is a quasi-coherent module on X, and $T \to S$ is a morphism of schemes, then we denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to the base change $X_T = X \times_S T$.

080Y **Remark 38.30.1.** Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open subscheme. Given a U-admissible blowup $S' \to S$ we denote X' the strict transform of X and $\mathcal{F}'$ the strict transform of $\mathcal{F}$ which we think of as a quasi-coherent module on X' (via Divisors, Lemma 31.33.2). Let P be a property of $\mathcal{F}/X/S$ which is stable under strict transform (as above) for U-admissible blowups. The general problem in this section is: Show (under auxiliary conditions on $\mathcal{F}/X/S$) there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'/X'/S'$ has P.

The general strategy will be to use that a composition of U-admissible blowups is a U-admissible blowup, see Divisors, Lemma 31.34.2. In fact, we will make use of the more precise Divisors, Lemma 31.32.14 and combine it with Divisors, Lemma 31.33.6. The result is that it suffices to find a sequence of U-admissible blowups

$$S = S_0 \leftarrow S_1 \leftarrow \ldots \leftarrow S_n$$

such that, setting $\mathcal{F}_0 = \mathcal{F}$ and $X_0 = X$ and setting $\mathcal{F}_i/X_i$ equal to the strict transform of $\mathcal{F}_{i-1}/X_{i-1}$, we arrive at $\mathcal{F}_n/X_n/S_n$ with property P.

In particular, choose a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$ such that $V(\mathcal{I}) = S \setminus U$, see Properties, Lemma 28.24.1. Let $S' \to S$ be the blowup in $\mathcal{I}$ and let $E \subset S'$ be the exceptional divisor (Divisors, Lemma 31.32.4). Then we see that we've reduced the problem to the case where there exists an effective Cartier divisor $D \subset S$ whose support is $X \setminus U$. In particular we may assume U is scheme theoretically dense in S (Divisors, Lemma 31.13.4).

Suppose that P is local on S: If $S = \bigcup S_i$ is a finite open covering by quasi-compact opens and P holds for $\mathcal{F}_{S_i}/X_{S_i}/S_i$ then P holds for $\mathcal{F}/X/S$. In this case the general problem above is local on S as well, i.e., if given $s \in S$ we can find a quasi-compact open neighbourhood W of s such that the problem for $\mathcal{F}_W/X_W/W$ is solvable, then the problem is solvable for $\mathcal{F}/X/S$. This follows from Divisors, Lemmas 31.34.3 and 31.34.4.

0810 Lemma 38.30.2. Let R be a ring and let $f \in R$. Let $r \geq 0$ be an integer. Let $R \to S$ be a ring map and let M be an S-module. Assume

(1) $R \to S$ is of finite presentation and flat,
(2) every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over R,
(3) M is a finite S-module,
(4) M_f is a finitely presented S_f-module,
(5) for all $\mathfrak{p} \in R$, $f \notin \mathfrak{p}$ with $\mathfrak{q} = \mathfrak{p}S$ the module $M_\mathfrak{q}$ is free of rank r over $S_\mathfrak{q}$.

Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the quotient

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

over $S' = S \otimes_R R'$ satisfies the following: for every prime $\mathfrak{p}' \subset R'$ there exists a $g \in S'$, $g \notin \mathfrak{p}'S'$ such that M'_g is a free S'_g-module of rank r.

Proof. This lemma is a generalization of More on Algebra, Lemma 15.26.5; we urge the reader to read that proof first. Choose a surjection $S^{\oplus n} \to M$, which is possible by (1). Choose a finite submodule $K \subset \text{Ker}(S^{\oplus n} \to M)$ such that $S^{\oplus n}/K \to M$ becomes an isomorphism after inverting f. This is possible by (4). Set $M_1 = S^{\oplus n}/K$ and suppose we can prove the lemma for M_1. Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M'_1 = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 10.70.7. But a is a nonzerodivisor on M'_1, whence the displayed map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented S-module.

Assume M is a finitely presented S-module satisfying (3). Then $J = \text{Fit}_r(M) \subset S$ is a finitely generated ideal. By Lemma 38.9.3 we can write S as a direct summand of a free R-module: $\bigoplus_{\alpha \in A} R = S \oplus C$. For any element $h \in S$ writing $h = \sum a_\alpha$ in the decomposition above, we say that the a_α are the coefficients of h. Let $I' \subset R$ be the ideal of coefficients of elements of J. Multiplication by an element of S defines an R-linear map $S \to S$, hence I' is generated by the coefficients of the generators of J, i.e., I' is a finitely generated ideal. We claim that $I = fI'$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \notin \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}S$ is not an element of $V(J)$ by property (5) and More on Algebra, Lemma 15.8.6. Hence there is an element of J which does not map to zero in $S \otimes_R \kappa(\mathfrak{p})$. Thus there exists an element of I' which is not contained in $\mathfrak{p}$, so $\mathfrak{p} \notin V(fI') = V(I)$.

Let $a \in I$ and set $R' = R[\frac{I}{a}]$. We may write $a = fa'$ for some $a' \in I'$. By Algebra, Lemmas 10.70.2 and 10.70.8 we see that $I'R' = a'R'$ and a' is a nonzerodivisor in R'. Set $S' = S \otimes_S R'$. Every element g of $JS' = \text{Fit}_r(M \otimes_S S')$ can be written as $g = \sum_\alpha c_\alpha$ for some $c_\alpha \in I'R'$. Since $I'R' = a'R'$ we can write $c_\alpha = a'c'_\alpha$ for some $c'_\alpha \in R'$ and $g = (\sum c'_\alpha)a' = g'a'$ in S'. Moreover, there is an $g_0 \in J$ such that $a' = c_\alpha$ for some α. For this element we have $g_0 = g'_0 a'$ in S' where g'_0 is a unit in S'. Let $\mathfrak p' \subset R'$ be a prime ideal and $\mathfrak q' = \mathfrak p' S'$. By the above we see that $JS'_{\mathfrak q'}$ is the principal ideal generated by the nonzerodivisor a'. It follows from More on Algebra, Lemma 15.8.8 that $M'_{\mathfrak q'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \ldots, m_r \in M'$ and $g \in S'$, $g \notin \mathfrak q'$ such that the corresponding map $(S')^{\oplus r} \to M'$ becomes surjective after inverting g.

Finally, consider the ideal $J' = \text{Fit}_{k-1}(M')$. Note that $J'S'_g$ is generated by the coefficients of relations between $m_1, \ldots, m_r$ (compatibility of Fitting ideal with base change). Thus it suffices to show that $J' = 0$, see More on Algebra, Lemma 15.8.7. Since $R'_a = R_f$ (Algebra, Lemma 10.70.7) and $M'_a = M_f$ we see from (5) that J'_a maps to zero in $S_{\mathfrak q''}$ for any prime $\mathfrak q'' \subset S'$ of the form $\mathfrak q'' = \mathfrak p'' S'$ where $\mathfrak p'' \subset R'_a$. Since $S'_a \subset \prod_{\mathfrak q'' \text{ as above}} S'_{\mathfrak q''}$ (as $(S'_a)_{\mathfrak p''} \subset S'_{\mathfrak q''}$ by Lemma 38.7.4) we see that $J'R'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. $\square$

0811 Lemma 38.30.3. Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal F$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume

(1) $X \to S$ is affine, of finite presentation, flat, geometrically integral fibres,
(2) $\mathcal F$ is a module of finite type,
(3) $\mathcal F_U$ is of finite presentation,
(4) $\mathcal F$ is flat over S at all generic points of fibres lying over points of U.

Then there exists a U-admissible blowup $S' \to S$ and an open subscheme $V \subset X_{S'}$ such that (a) the strict transform $\mathcal F'$ of $\mathcal F$ restricts to a finitely locally free $\mathcal O_V$-module and (b) $V \to S'$ is surjective.

Proof. Given $\mathcal F/X/S$ and $U \subset S$ with hypotheses as in the lemma, denote P the property "$\mathcal F$ is flat over S at all generic points of fibres". It is clear that P is preserved under strict transform, see Divisors, Lemma 31.33.3 and Morphisms, Lemma 29.25.7. It is also clear that P is local on S. Hence any and all observations of Remark 38.30.1 apply to the problem posed by the lemma.

Consider the function $r : U \to \mathbf Z_{\geq 0}$ which assigns to $u \in U$ the integer

$$r(u) = \dim_{\kappa(\xi_u)}(\mathcal F_{\xi_u} \otimes \kappa(\xi_u))$$

where ξ_u is the generic point of the fibre X_u. By More on Morphisms, Lemma 37.16.7 and the fact that the image of an open in X_S in S is open, we see that $r(u)$ is locally constant. Accordingly $U = U_0 \amalg U_1 \amalg \ldots \amalg U_c$ is a finite disjoint union of open and closed subschemes where r is constant with value i on U_i. By Divisors, Lemma 31.34.5 we can find a U-admissible blowup to decompose S into the disjoint union of two schemes, the first containing U_0 and the second $U_1 \cup \ldots \cup U_c$. Repeating this $c-1$ more times we may assume that S is a disjoint union $S = S_0 \amalg S_1 \amalg \ldots \amalg S_c$ with $U_i \subset S_i$. Thus we may assume the function r defined above is constant, say with value r.

By Remark 38.30.1 we see that we may assume that we have an effective Cartier divisor $D \subset S$ whose support is $S \setminus U$. Another application of Remark 38.30.1 combined with Divisors, Lemma 31.13.2 tells us we may assume that $S = \text{Spec}(R)$

and $D = \mathrm{Spec}(R/(f))$ for some nonzerodivisor $f \in R$. This case is handled by Lemma 38.30.2. $\square$

0812 Lemma 38.30.4. Let $A \to C$ be a finite locally free ring map of rank d. Let $h \in C$ be an element such that C_h is étale over A. Let $J \subset C$ be an ideal. Set $I = \mathrm{Fit}_0(C/J)$ where we think of C/J as a finite A-module. Then $IC_h = JJ'$ for some ideal $J' \subset C_h$. If J is finitely generated so are I and J'.

Proof. We will use basic properties of Fitting ideals, see More on Algebra, Lemma 15.8.4. Then IC is the Fitting ideal of $C/J \otimes_A C$. Note that $C \to C \otimes_A C$, $c \mapsto 1 \otimes c$ has a section (the multiplication map). By assumption $C \to C \otimes_A C$ is étale at every prime in the image of $\mathrm{Spec}(C_h)$ under this section. Hence the multiplication map $C \otimes_A C_h \to C_h$ is étale in particular flat, see Algebra, Lemma 10.143.8. Hence there exists a C_h-algebra such that $C \otimes_A C_h \cong C_h \oplus C'$ as C_h-algebras, see Algebra, Lemma 10.143.9. Thus $(C/J) \otimes_A C_h \cong (C_h/J_h) \oplus C'/I'$ as C_h-modules for some ideal $I' \subset C'$. Hence $IC_h = JJ'$ with $J' = \mathrm{Fit}_0(C'/I')$ where we view C'/J' as a C_h-module. $\square$

0813 Lemma 38.30.5. Let $A \to B$ be an étale ring map. Let $a \in A$ be a nonzerodivisor. Let $J \subset B$ be a finite type ideal with $V(J) \subset V(aB)$. For every $\mathfrak{q} \subset B$ there exists a finite type ideal $I \subset A$ with $V(I) \subset V(a)$ and $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = JJ'$ for some finite type ideal $J' \subset B_g$.

Proof. We may replace B by a principal localization at an element $g \in B$, $g \notin \mathfrak{q}$. Thus we may assume that B is standard étale, see Algebra, Proposition 10.144.4. Thus we may assume B is a localization of $C = A[x]/(f)$ for some monic $f \in A[x]$ of some degree d. Say $B = C_h$ for some $h \in C$. Choose elements $h_1, \ldots, h_n \in C$ which generate J over B. The condition $V(J) \subset V(aB)$ signifies that $a^m = \sum b_i h_i$ in B for some large m. Set $h_{n+1} = a^m$. As in Lemma 38.30.4 we take $I = \mathrm{Fit}_0(C/(h_1, \ldots, h_{r+1}))$. Since the module $C/(h_1, \ldots, h_{r+1})$ is annihilated by a^m we see that $a^{dm} \in I$ which implies that $V(I) \subset V(a)$. $\square$

0814 Lemma 38.30.6. Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume there exist finitely many commutative diagrams

$$\begin{tikzcd} & X_i \arrow[r, "j_i"] \arrow[d] & X \arrow[d] \\ S_i^* \arrow[r] & S_i \arrow[r, "e_i"] & S \end{tikzcd}$$

where

(1) $e_i : S_i \to S$ are quasi-compact étale morphisms and $S = \bigcup e_i(S_i)$,
(2) $j_i : X_i \to X$ are étale morphisms and $X = \bigcup j_i(X_i)$,
(3) $S_i^* \to S_i$ is an $e_i^{-1}(U)$-admissible blowup such that the strict transform $\mathcal{F}_i^*$ of $j_i^*\mathcal{F}$ is flat over S_i^*.

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of $\mathcal{F}$ is flat over S'.

Proof. We claim that the hypotheses of the lemma are preserved under U-admissible blowups. Namely, suppose $b : S' \to S$ is a U-admissible blowup in the quasi-coherent sheaf of ideals $\mathcal{I}$. Moreover, let $S_i^* \to S_i$ be the blowup in the quasi-coherent sheaf of ideals $\mathcal{J}_i$. Then the collection of morphisms $e_i' : S_i' = S_i \times_S S' \to S'$ and $j_i' : X_i' = X_i \times_S S' \to X \times_S S'$ satisfy conditions (1), (2), (3) for the strict transform $\mathcal{F}'$ of $\mathcal{F}$ relative to the blowup $S' \to S$. First, observe that S_i' is the blowup of S_i in the pullback of $\mathcal{I}$, see Divisors, Lemma 31.32.3. Second, consider the blowup $S_i'^* \to S_i'$ of S_i' in the pullback of the ideal $\mathcal{J}_i$. By Divisors, Lemma 31.32.12 we get a commutative diagram

$$\begin{tikzcd} S_i'^* \arrow[r] \arrow[d] \arrow[dr] & S_i' \arrow[d] \\ S_i^* \arrow[r] & S_i \end{tikzcd}$$

and all the morphisms in the diagram above are blowups. Hence by Divisors, Lemmas 31.33.3 and 31.33.6 we see

$$\begin{aligned} & \text{the strict transform of } (j_i')^*\mathcal{F}' \text{ under } S_i'^* \to S_i' \\ = & \text{ the strict transform of } j_i^*\mathcal{F} \text{ under } S_i'^* \to S_i \\ = & \text{ the strict transform of } \mathcal{F}_i' \text{ under } S_i'^* \to S_i' \\ = & \text{ the pullback of } \mathcal{F}_i^* \text{ via } X_i \times_{S_i} S_i'^* \to X_i \end{aligned}$$

which is therefore flat over $S_i'^*$ (Morphisms, Lemma 29.25.7). Having said this, we see that all observations of Remark 38.30.1 apply to the problem of finding a U-admissible blowup such that the strict transform of $\mathcal{F}$ becomes flat over the base under assumptions as in the lemma. In particular, we may assume that $S \setminus U$ is the support of an effective Cartier divisor $D \subset S$. Another application of Remark 38.30.1 combined with Divisors, Lemma 31.13.2 shows we may assume that $S = \text{Spec}(A)$ and $D = \text{Spec}(A/(a))$ for some nonzerodivisor $a \in A$.

Pick an i and $s \in S_i$. Lemma 38.30.5 implies we can find an open neighbourhood $s \in W_i \subset S_i$ and a finite type quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I} \cdot \mathcal{O}_{W_i} = \mathcal{J}_i\mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{W_i}$ and such that $V(\mathcal{I}) \subset V(a) = S \setminus U$. Since S_i is quasi-compact we can replace S_i by a finite collection $W_1, \ldots, W_n$ of these opens and assume that for each i there exists a quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_S$ such that $\mathcal{I}_i \cdot \mathcal{O}_{S_i} = \mathcal{J}_i\mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{S_i}$. As in the discussion of the first paragraph of the proof, consider the blowup S' of S in the product $\mathcal{I}_1 \ldots \mathcal{I}_n$ (this blowup is U-admissible by construction). The base change of $S' \to S$ to S_i is the blowup in

$$\mathcal{J}_i \cdot \mathcal{J}_i'\mathcal{I}_1 \ldots \hat{\mathcal{I}}_i \ldots \mathcal{I}_n$$

which factors through the given blowup $S_i^* \to S_i$ (Divisors, Lemma 31.32.12). In the notation of the diagram above this means that $S_i'^* = S_i'$. Hence after replacing S by S' we arrive in the situation that $j_i^*\mathcal{F}$ is flat over S_i. Hence $j_i^*\mathcal{F}$ is flat over S, see Lemma 38.2.3. By Morphisms, Lemma 29.25.13 we see that $\mathcal{F}$ is flat over S. □

0815 Theorem 38.30.7. Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume

(1) X is quasi-compact,
(2) X is locally of finite presentation over S,
(3) $\mathcal{F}$ is a module of finite type,
(4) $\mathcal{F}_U$ is of finite presentation, and
(5) $\mathcal{F}_U$ is flat over U.

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_S S'}$-module of finite presentation and flat over S'.

Proof. We first prove that we can find a U-admissible blowup such that the strict transform is flat. The question is étale local on the source and the target, see Lemma 38.30.6 for a precise statement. In particular, we may assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ are affine. For $s \in S$ write $\mathcal{F}_s = \mathcal{F}|_{X_s}$ (pullback of $\mathcal{F}$ to the fibre). As $X \to S$ is of finite type $d = \max_{s \in S} \dim(\mathrm{Supp}(\mathcal{F}_s))$ is an integer. We will do induction on d.

Let $x \in X$ be a point of X lying over $s \in S$ with $\dim_x(\mathrm{Supp}(\mathcal{F}_s)) = d$. Apply Lemma 38.3.2 to get $g : X' \to X$, $e : S' \to S$, $i : Z' \to X'$, and $\pi : Z' \to Y'$. Observe that $Y' \to S'$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d. Because the problem is étale local it suffices to prove the theorem for $g^*\mathcal{F}/X'/S'$. Because $i : Z' \to X'$ is a closed immersion of finite presentation (and since strict transform commutes with affine pushforward, see Divisors, Lemma 31.33.4) it suffices to prove the flattening result for $\mathcal{G}$. Since π is finite (hence also affine) it suffices to prove the flattening result for $\pi_*\mathcal{G}/Y'/S'$. Thus we may assume that $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d.

Next, we apply a blowup as in Lemma 38.30.3. Doing so we reach the situation where there exists an open $V \subset X$ surjecting onto S such that $\mathcal{F}|_V$ is finite locally free. Let $\xi \in X$ be the generic point of X_s. Let $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi \otimes \kappa(\xi)$. Choose a map $\alpha : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ which induces an isomorphism $\kappa(\xi)^{\oplus r} \to \mathcal{F}_\xi \otimes \kappa(\xi)$. Because $\mathcal{F}$ is locally free over V we find an open neighbourhood W of ξ where α is an isomorphism. Shrink S to an affine open neighbourhood of s such that $W \to S$ is surjective. Say $\mathcal{F}$ is the quasi-coherent module associated to the A-module N. Since $\mathcal{F}$ is flat over S at all generic points of fibres (in fact at all points of W), we see that

$$\alpha_\mathfrak{p} : A_\mathfrak{p}^{\oplus r} \to N_\mathfrak{p}$$

is universally injective for all primes $\mathfrak{p}$ of R, see Lemma 38.10.1. Hence α is universally injective, see Algebra, Lemma 10.82.12. Set $\mathcal{H} = \mathrm{Coker}(\alpha)$. By Divisors, Lemma 31.33.7 we see that, given a U-admissible blowup $S' \to S$ the strict transforms of $\mathcal{F}'$ and $\mathcal{H}'$ fit into an exact sequence

$$0 \to \mathcal{O}_{X \times_S S'}^{\oplus r} \to \mathcal{F}' \to \mathcal{H}' \to 0$$

Hence Lemma 38.10.1 also shows that $\mathcal{F}'$ is flat at a point x' if and only if $\mathcal{H}'$ is flat at that point. In particular $\mathcal{H}_U$ is flat over U and $\mathcal{H}_U$ is a module of finite presentation. We may apply the induction hypothesis to $\mathcal{H}$ to see that there exists a U-admissible blowup such that the strict transform $\mathcal{H}'$ is flat as desired.

To finish the proof of the theorem we still have to show that $\mathcal{F}'$ is a module of finite presentation (after possibly another U-admissible blowup). This follows from Lemma 38.11.1 as we can assume $U \subset S$ is scheme theoretically dense (see third paragraph of Remark 38.30.1). This finishes the proof of the theorem. $\square$

38.31. Applications

081Q In this section we apply some of the results above.

081R **Lemma 38.31.1.** Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume

(1) $X \to S$ is of finite type and quasi-separated, and
(2) $X_U \to U$ is flat and locally of finite presentation.

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is flat and of finite presentation over S'.

Proof. Since $X \to S$ is quasi-compact and quasi-separated by assumption, the strict transform of X with respect to a blowing up $S' \to S$ is also quasi-compact and quasi-separated. Hence to prove the lemma it suffices to find a U-admissible blowup such that the strict transform is flat and locally of finite presentation. Let $X = W_1 \cup \ldots \cup W_n$ be a finite affine open covering. If we can find a U-admissible blowup $S_i \to S$ such that the strict transform of W_i is flat and locally of finite presentation, then there exists a U-admissible blowing up $S' \to S$ dominating all $S_i \to S$ which does the job (see Divisors, Lemma 31.34.4; see also Remark 38.30.1). Hence we may assume X is affine.

Assume X is affine. By Morphisms, Lemma 29.39.2 we can choose an immersion $j : X \to \mathbf{A}^n_S$ over S. Let $V \subset \mathbf{A}^n_S$ be a quasi-compact open subscheme such that j induces a closed immersion $i : X \to V$ over S. Apply Theorem 38.30.7 to $V \to S$ and the quasi-coherent module $i_*\mathcal{O}_X$ to obtain a U-admissible blowup $S' \to S$ such that the strict transform of $i_*\mathcal{O}_X$ is flat over S' and of finite presentation over $\mathcal{O}_{V \times_S S'}$. Let X' be the strict transform of X with respect to $S' \to S$. Let $i' : X' \to V \times_S S'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors, Lemma 31.33.4), we see that $i'_*\mathcal{O}_{X'}$ is flat over S and of finite presentation as a $\mathcal{O}_{V \times_S S'}$-module. This implies the lemma. □

0B49 **Lemma 38.31.2.** Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume

(1) $X \to S$ is proper, and
(2) $X_U \to U$ is finite locally free.

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is finite locally free over S'.

Proof. By Lemma 38.31.1 we may assume that $X \to S$ is flat and of finite presentation. After replacing S by a U-admissible blowup if necessary, we may assume that $U \subset S$ is scheme theoretically dense. Then f is finite by Lemma 38.11.4. Hence f is finite locally free by Morphisms, Lemma 29.48.2. □

081S **Lemma 38.31.3.** Let $\varphi : X \to S$ be a separated morphism of finite type with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ such that the strict transform X' of X is isomorphic to an open subscheme of S'.

Proof. The discussion in Remark 38.30.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Next, we do another U-admissible blowup to get to the situation where $X \to S$ is flat and of finite presentation, see Lemma 38.31.1. In this case the result follows from Lemma 38.11.5. □

The following lemma says that a proper modification can be dominated by a blowup.

081T Lemma 38.31.4. Let $\varphi : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ which dominates X, i.e., such that there exists a factorization $S' \to X \to S$ of the blowup morphism.

Proof. The discussion in Remark 38.30.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Choose another U-admissible blowup $S' \to S$ such that the strict transform X' of X is an open subscheme of S', see Lemma 38.31.3. Since $X' \to S'$ is proper, and $U \subset S'$ is dense, we see that $X' = S'$. Some details omitted. □

0CP1 Lemma 38.31.5. Let S be a scheme. Let $U \subset W \subset S$ be open subschemes. Let $f : X \to W$ be a morphism and let $s : U \to X$ be a morphism such that $f \circ s = \mathrm{id}_U$. Assume

(1) f is proper,
(2) S is quasi-compact and quasi-separated, and
(3) U and W are quasi-compact.

Then there exists a U-admissible blowup $b : S' \to S$ and a morphism $s' : b^{-1}(W) \to X$ extending s with $f \circ s' = b|_{b^{-1}(W)}$.

Proof. We may and do replace X by the scheme theoretic image of s. Then $X \to W$ is an isomorphism over U, see Morphisms, Lemma 29.6.8. By Lemma 38.31.4 there exists a U-admissible blowup $W' \to W$ and an extension $W' \to X$ of s. We finish the proof by applying Divisors, Lemma 31.34.3 to extend $W' \to W$ to a U-admissible blowup of S. □

38.32. Compactifications

0ATT Let S be a quasi-compact and quasi-separated scheme. We will say a scheme X over S has a compactification over S or is compactifyable over S if there exists a quasi-compact open immersion $X \to \overline{X}$ into a scheme $\overline{X}$ proper over S. If X has a compactification over S, then $X \to S$ is separated and of finite type. It is a theorem of Nagata, see [Lüt93], [Con07b], [Nag56], [Nag57a], [Nag62a], and [Nag63], that the converse is true as well. We will prove this theorem in the next section, see Theorem 38.33.8.

Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a separated finite type morphism of schemes. The category of compactifications of X over S is the category defined as follows:

(1) Objects are open immersions $j : X \to \overline{X}$ over S with $\overline{X} \to S$ proper.
(2) Morphisms $(j' : X \to \overline{X}') \to (j : X \to \overline{X})$ are morphisms $f : \overline{X}' \to \overline{X}$ of schemes over S such that $f \circ j' = j$.

If $j : X \to \overline{X}$ is a compactification, then j is a quasi-compact open immersion, see Schemes, Remark 26.21.18.

Warning. We do not assume compactifications $j : X \to \overline{X}$ to have dense image. Consequently, if $f : \overline{X}' \to \overline{X}$ is a morphism of compactifications, it may not be the case that $f^{-1}(j(X)) = j'(X)$.

0ATU Lemma 38.32.1. Let S be a quasi-compact and quasi-separated scheme. Let X be a compactifyable scheme over S.

(a) The category of compactifications of X over S is cofiltered.
(b) The full subcategory consisting of compactifications $j : X \to \overline{X}$ such that $j(X)$ is dense and scheme theoretically dense in $\overline{X}$ is initial (Categories, Definition 4.17.3).
(c) If $f : \overline{X}' \to \overline{X}$ is a morphism of compactifications of X such that $j'(X)$ is dense in $\overline{X}'$, then $f^{-1}(j(X)) = j'(X)$.

Proof. To prove part (a) we have to check conditions (1), (2), (3) of Categories, Definition 4.20.1. Condition (1) holds exactly because we assumed that X is compactifyable. Let $j_i : X \to \overline{X}_i$, $i = 1, 2$ be two compactifications. Then we can consider the scheme theoretic image $\overline{X}$ of $(j_1, j_2) : X \to \overline{X}_1 \times_S \overline{X}_2$. This determines a third compactification $j : X \to \overline{X}$ which dominates both j_i:

$$(X, \overline{X}_1) \longleftarrow (X, \overline{X}) \longrightarrow (X, \overline{X}_2)$$

Thus (2) holds. Let $f_1, f_2 : \overline{X}_1 \to \overline{X}_2$ be two morphisms between compactifications $j_i : X \to \overline{X}_i$, $i = 1, 2$. Let $\overline{X} \subset \overline{X}_1$ be the equalizer of f_1 and f_2. As $\overline{X}_2 \to S$ is separated, we see that $\overline{X}$ is a closed subscheme of $\overline{X}_1$ and hence proper over S. Moreover, we obtain an open immersion $X \to \overline{X}$ because $f_1|_X = f_2|_X = \mathrm{id}_X$. The morphism $(X \to \overline{X}) \to (j_1 : X \to \overline{X}_1)$ given by the closed immersion $\overline{X} \to \overline{X}_1$ equalizes f_1 and f_2 which proves condition (3).

Proof of (b). Let $j : X \to \overline{X}$ be a compactification. If $\overline{X}'$ denotes the scheme theoretic closure of X in $\overline{X}$, then X is dense and scheme theoretically dense in $\overline{X}'$ by Morphisms, Lemma 29.7.7. This proves the first condition of Categories, Definition 4.17.3. Since we have already shown the category of compactifications of X is cofiltered, the second condition of Categories, Definition 4.17.3 follows from the first (we omit the solution to this categorical exercise).

Proof of (c). After replacing $\overline{X}'$ with the scheme theoretic closure of $j'(X)$ (which doesn't change the underlying topological space) this follows from Morphisms, Lemma 29.6.8. □

We can also consider the category of all compactifications (for varying X). It turns out that this category, localized at the set of morphisms which induce an isomorphism on the interior is equivalent to the category of compactifyable schemes over S.

0A9Z Lemma 38.32.2. Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to Y$ be a morphism of schemes over S with Y separated and of finite type over S and X compactifyable over S. Then X has a compactification over Y.

Proof. Let $j : X \to \overline{X}$ be a compactification of X over S. Then we let $\overline{X}'$ be the scheme theoretic image of $(j, f) : X \to \overline{X} \times_S Y$. The morphism $\overline{X}' \to Y$ is proper because $\overline{X} \times_S Y \to Y$ is proper as a base change of $\overline{X} \to S$. On the other hand, since Y is separated over S, the morphism $(1, f) : X \to X \times_S Y$ is a closed immersion (Schemes, Lemma 26.21.10) and hence $X \to \overline{X}'$ is an open immersion by Morphisms, Lemma 29.6.8 applied to the "partial section" $s = (j, f)$ to the projection $\overline{X} \times_S Y \to \overline{X}$. $\square$

Let S be a quasi-compact and quasi-separated scheme. We define the category of compactifications to be the category whose objects are pairs $(X, \overline{X})$ where $\overline{X}$ is a scheme proper over S and $X \subset \overline{X}$ is a quasi-compact open and whose morphisms are commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & Y \arrow[d] \\ \overline{X} \arrow[r, "\overline{f}"] & \overline{Y} \end{tikzcd}$$

of morphisms of schemes over S.

0ATV Lemma 38.32.3. Let S be a quasi-compact and quasi-separated scheme. The collection of morphisms $(u, \overline{u}) : (X', \overline{X}') \to (X, \overline{X})$ such that u is an isomorphism forms a right multiplicative system (Categories, Definition 4.27.1) of arrows in the category of compactifications.

Proof. Axiom RMS1 is trivial to verify. Let us check RMS2 holds. Suppose given a diagram

$$\begin{tikzcd} & (X', \overline{X}') \arrow[d, "(u,\overline{u})"'] \\ (Y, \overline{Y}) \arrow[r, "(f,\overline{f})"] & (X, \overline{X}) \end{tikzcd}$$

with $u : X' \to X$ an isomorphism. Then we let $Y' = Y \times_X X'$ with the projection map $v : Y' \to Y$ (an isomorphism). We also set $\overline{Y}' = \overline{Y} \times_{\overline{X}} \overline{X}'$ with the projection map $\overline{v} : \overline{Y}' \to \overline{Y}$ It is clear that $Y' \to \overline{Y}'$ is an open immersion. The diagram

$$\begin{tikzcd} (Y', \overline{Y}') \arrow[r, "(g,\overline{g})"'] \arrow[d, "(v,\overline{v})"'] & (X', \overline{X}') \arrow[d, "(u,\overline{u})"'] \\ (Y, \overline{Y}) \arrow[r, "(f,\overline{f})"] & (X, \overline{X}) \end{tikzcd}$$

shows that axiom RMS2 holds.

Let us check RMS3 holds. Suppose given a pair of morphisms $(f, \overline{f}), (g, \overline{g}) : (X, \overline{X}) \to (Y, \overline{Y})$ of compactifications and a morphism $(v, \overline{v}) : (Y, \overline{Y}) \to (Y', \overline{Y}')$ such that v is an isomorphism and such that $(v, \overline{v}) \circ (f, \overline{f}) = (v, \overline{v}) \circ (g, \overline{g})$. Then $f = g$. Hence if we let $\overline{X}' \subset \overline{X}$ be the equalizer of $\overline{f}$ and $\overline{g}$, then $(u, \overline{u}) : (X, \overline{X}') \to (X, \overline{X})$ will be a morphism of the category of compactifications such that $(f, \overline{f}) \circ (u, \overline{u}) = (g, \overline{g}) \circ (u, \overline{u})$ as desired. $\square$

0ATW Lemma 38.32.4. Let S be a quasi-compact and quasi-separated scheme. The functor $(X, \overline{X}) \mapsto X$ defines an equivalence from the category of compactifications localized (Categories, Lemma 4.27.11) at the right multiplicative system of Lemma 38.32.3 to the category of compactifyable schemes over S.

Proof. Denote $\mathcal{C}$ the category of compactifications and denote $Q : \mathcal{C} \to \mathcal{C}'$ the localization functor of Categories, Lemma 4.27.16. Denote $\mathcal{D}$ the category of compactifyable schemes over S. It is clear from the lemma just cited and our choice of multiplicative system that we obtain a functor $\mathcal{C}' \to \mathcal{D}$. This functor is clearly essentially surjective. If $f : X \to Y$ is a morphism of compactifyable schemes, then we choose an open immersion $Y \to \overline{Y}$ into a scheme proper over S, and then we choose an embedding $X \to \overline{X}$ into a scheme $\overline{X}$ proper over $\overline{Y}$ (possible by Lemma 38.32.2 applied to $X \to \overline{Y}$). This gives a morphism $(X, \overline{X}) \to (Y, \overline{Y})$ of compactifications which produces our given morphism $X \to Y$. Finally, suppose given a pair of morphisms in the localized category with the same source and target: say

$$a = ((f, \overline{f}) : (X', \overline{X}') \to (Y, \overline{Y}), (u, \overline{u}) : (X', \overline{X}') \to (X, \overline{X}))$$

and

$$b = ((g, \overline{g}) : (X'', \overline{X}'') \to (Y, \overline{Y}), (v, \overline{v}) : (X'', \overline{X}'') \to (X, \overline{X}))$$

which produce the same morphism $X \to Y$ over S, in other words $f \circ u^{-1} = g \circ v^{-1}$. By Categories, Lemma 4.27.13 we may assume that $(X', \overline{X}') = (X'', \overline{X}'')$ and $(u, \overline{u}) = (v, \overline{v})$. In this case we can consider the equalizer $\overline{X}''' \subset \overline{X}'$ of $\overline{f}$ and $\overline{g}$. The morphism $(w, \overline{w}) : (X', \overline{X}''') \to (X', \overline{X}')$ is in the multiplicative subset and we see that $a = b$ in the localized category by precomposing with $(w, \overline{w})$. $\square$

38.33. Nagata compactification

0F3T In this section we prove the theorem announced in Section 38.32.

0F3U Lemma 38.33.1. Let $X \to S$ be a morphism of schemes. If $X = U \cup V$ is an open cover such that $U \to S$ and $V \to S$ are separated and $U \cap V \to U \times_S V$ is closed, then $X \to S$ is separated.

Proof. Omitted. Hint: check that $\Delta : X \to X \times_S X$ is closed by using the open covering of $X \times_S X$ given by $U \times_S U$, $U \times_S V$, $V \times_S U$, and $V \times_S V$. $\square$

0F3V Lemma 38.33.2. Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open.

(1) If $Z_1, Z_2 \subset X$ are closed subschemes of finite presentation such that $Z_1 \cap Z_2 \cap U = \emptyset$, then there exists a U-admissible blowing up $X' \to X$ such that the strict transforms of Z_1 and Z_2 are disjoint.
(2) If $T_1, T_2 \subset U$ are disjoint constructible closed subsets, then there is a U-admissible blowing up $X' \to X$ such that the closures of T_1 and T_2 are disjoint.

Proof. Proof of (1). The assumption that $Z_i \to X$ is of finite presentation signifies that the quasi-coherent ideal sheaf $\mathcal{I}_i$ of Z_i is of finite type, see Morphisms, Lemma 29.21.7. Denote $Z \subset X$ the closed subscheme cut out by the product $\mathcal{I}_1\mathcal{I}_2$. Observe that $Z \cap U$ is the disjoint union of $Z_1 \cap U$ and $Z_2 \cap U$. By Divisors, Lemma 31.34.5 there is a $U \cap Z$-admissible blowup $Z' \to Z$ such that the strict transforms of Z_1 and Z_2 are disjoint. Denote $Y \subset Z$ the center of this blowing up. Then $Y \to X$ is a closed immersion of finite presentation as the composition of $Y \to Z$ and $Z \to X$ (Divisors, Definition 31.34.1 and Morphisms, Lemma 29.21.3). Thus the blowing up $X' \to X$ of Y is a U-admissible blowing up. By general properties of strict transforms, the strict transform of Z_1, Z_2 with respect to $X' \to X$ is the same as the strict transform of Z_1, Z_2 with respect to $Z' \to Z$, see Divisors, Lemma 31.33.2. Thus (1) is proved.

Proof of (2). By Properties, Lemma 28.24.1 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J}_i \subset \mathcal{O}_U$ such that $T_i = V(\mathcal{J}_i)$ (set theoretically). By Properties, Lemma 28.22.2 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_X$ whose restriction to U is $\mathcal{J}_i$. Apply the result of part (1) to the closed subschemes $Z_i = V(\mathcal{I}_i)$ to conclude. $\square$

0F3W Lemma 38.33.3. Let $f : X \to Y$ be a proper morphism of quasi-compact and quasi-separated schemes. Let $V \subset Y$ be a quasi-compact open and $U = f^{-1}(V)$. Let $T \subset V$ be a closed subset such that $f|_U : U \to V$ is an isomorphism over an open neighbourhood of T in V. Then there exists a V-admissible blowing up $Y' \to Y$ such that the strict transform $f' : X' \to Y'$ of f is an isomorphism over an open neighbourhood of the closure of T in Y'.

Proof. Let $T' \subset V$ be the complement of the maximal open over which $f|_U$ is an isomorphism. Then T', T are closed in V and $T \cap T' = \emptyset$. Since V is a spectral topological space, we can find constructible closed subsets T_c, T'_c with $T \subset T_c$, $T' \subset T'_c$ such that $T_c \cap T'_c = \emptyset$ (choose a quasi-compact open W of V containing T' not meeting T and set $T_c = V \setminus W$, then choose a quasi-compact open W' of V containing T_c not meeting T' and set $T'_c = V \setminus W'$). By Lemma 38.33.2 we may, after replacing Y by a V-admissible blowing up, assume that T_c and T'_c have disjoint closures in Y. Set $Y_0 = Y \setminus \overline{T}'_c$, $V_0 = V \setminus T'_c$, $U_0 = U \times_V V_0$, and $X_0 = X \times_Y Y_0$. Since $U_0 \to V_0$ is an isomorphism, we can find a V_0-admissible blowing up $Y'_0 \to Y_0$ such that the strict transform X'_0 of X_0 maps isomorphically to Y'_0, see Lemma 38.31.3. By Divisors, Lemma 31.34.3 there exists a V-admissible blow up $Y' \to Y$ whose restriction to Y_0 is $Y'_0 \to Y_0$. If $f' : X' \to Y'$ denotes the strict transform of f, then we see what we want is true because f' restricts to an isomorphism over Y'_0. $\square$

0F3X Lemma 38.33.4. Let S be a quasi-compact and quasi-separated scheme. Let $U \to X_1$ and $U \to X_2$ be open immersions of schemes over S and assume U, X_1, X_2 of finite type and separated over S. Then there exists a commutative diagram

$$\begin{tikzcd} X'_1 \arrow[r] \arrow[d] & X & X'_2 \arrow[l] \arrow[d] \\ X_1 & U \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[ur] & X_2 \end{tikzcd}$$

of schemes over S where $X'_i \to X_i$ is a U-admissible blowup, $X'_i \to X$ is an open immersion, and X is separated and finite type over S.

Proof. Throughout the proof all schemes will be separated of finite type over S. This in particular implies these schemes are quasi-compact and quasi-separated and the morphisms between them are quasi-compact and separated. See Schemes, Sections 26.19 and 26.21. We will use that if $U \to W$ is an immersion of such schemes over S, then the scheme theoretic image Z of U in W is a closed subscheme of W and $U \to Z$ is an open immersion, $U \subset Z$ is scheme theoretically dense, and $U \subset Z$ is dense topologically. See Morphisms, Lemma 29.7.7.

Let $X_{12} \subset X_1 \times_S X_2$ be the scheme theoretic image of $U \to X_1 \times_S X_2$. The projections $p_i : X_{12} \to X_i$ induce isomorphisms $p_i^{-1}(U) \to U$ by Morphisms, Lemma

29.6.8. Choose a U-admissible blowup $X_i^i \to X_i$ such that the strict transform X_{12}^i of X_{12} is isomorphic to an open subscheme of X_i^i, see Lemma 38.31.3. Let $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ be the corresponding finite type quasi-coherent sheaf of ideals. Recall that $X_{12}^i \to X_{12}$ is the blowup in $p_i^{-1}\mathcal{I}_i\mathcal{O}_{X_{12}}$, see Divisors, Lemma 31.33.2. Let X_{12}' be the blowup of X_{12} in $p_1^{-1}\mathcal{I}_1 p_2^{-1}\mathcal{I}_2\mathcal{O}_{X_{12}}$, see Divisors, Lemma 31.32.12 for what this entails. We obtain in particular a commutative diagram

$$\begin{tikzcd} X'_{12} \arrow[r] \arrow[d] & X^2_{12} \arrow[d] \\ X^1_{12} \arrow[r] & X_{12} \end{tikzcd}$$

where all the morphisms are U-admissible blowing ups. Since $X_{12}^i \subset X_i^i$ is an open we may choose a U-admissible blowup $X_i' \to X_i^i$ restricting to $X_{12}' \to X_{12}^i$, see Divisors, Lemma 31.34.3. Then $X_{12}' \subset X_i'$ is an open subscheme and the diagram

$$\begin{tikzcd} X'_{12} \arrow[r] \arrow[d] & X'_i \arrow[d] \\ X^i_{12} \arrow[r] & X^i_i \end{tikzcd}$$

is commutative with vertical arrows blowing ups and horizontal arrows open immersions. Note that $X_{12}' \to X_1' \times_S X_2'$ is an immersion and proper (use that $X_{12}' \to X_{12}$ is proper and $X_{12} \to X_1 \times_S X_2$ is closed and $X_1' \times_S X_2' \to X_1 \times_S X_2$ is separated and apply Morphisms, Lemma 29.41.7). Thus $X_{12}' \to X_1' \times_S X_2'$ is a closed immersion. It follows that if we define X by glueing X_1' and X_2' along the common open subscheme X_{12}', then $X \to S$ is of finite type and separated (Lemma 38.33.1). As compositions of U-admissible blowups are U-admissible blowups (Divisors, Lemma 31.34.2) the lemma is proved. $\square$

0F3Y Lemma 38.33.5. Let $X \to S$ and $Y \to S$ be morphisms of schemes. Let $U \subset X$ be an open subscheme. Let $V \to X \times_S Y$ be a quasi-compact morphism whose composition with the first projection maps into U. Let $Z \subset X \times_S Y$ be the scheme theoretic image of $V \to X \times_S Y$. Let $X' \to X$ be a U-admissible blowup. Then the scheme theoretic image of $V \to X' \times_S Y$ is the strict transform of Z with respect to the blowing up.

Proof. Denote $Z' \to Z$ the strict transform. The morphism $Z' \to X'$ induces a morphism $Z' \to X' \times_S Y$ which is a closed immersion (as Z' is a closed subscheme of $X' \times_X Z$ by definition). Thus to finish the proof it suffices to show that the scheme theoretic image Z'' of $V \to Z'$ is Z'. Observe that $Z'' \subset Z'$ is a closed subscheme such that $V \to Z'$ factors through Z''. Since both $V \to X \times_S Y$ and $V \to X' \times_S Y$ are quasi-compact (for the latter this follows from Schemes, Lemma 26.21.14 and the fact that $X' \times_S Y \to X \times_S Y$ is separated as a base change of a proper morphism), by Morphisms, Lemma 29.6.3 we see that $Z \cap (U \times_S Y) = Z'' \cap (U \times_S Y)$. Thus the inclusion morphism $Z'' \to Z'$ is an isomorphism away from the exceptional divisor E of $Z' \to Z$. However, the structure sheaf of Z' does not have any nonzero sections supported on E (by definition of strict transforms) and we conclude that the surjection $\mathcal{O}_{Z'} \to \mathcal{O}_{Z''}$ must be an isomorphism. $\square$

0F3Z Lemma 38.33.6. Let S be a quasi-compact and quasi-separated scheme. Let U be a scheme of finite type and separated over S. Let $V \subset U$ be a quasi-compact open. If V has a compactification $V \subset Y$ over S, then there exists a V-admissible blowing up $Y' \to Y$ and an open $V \subset V' \subset Y'$ such that $V \to U$ extends to a proper morphism $V' \to U$.

Proof. Consider the scheme theoretic image $Z \subset Y \times_S U$ of the "diagonal" morphism $V \to Y \times_S U$. If we replace Y by a V-admissible blowing up, then Z is replaced by the strict transform with respect to this blowing up, see Lemma 38.33.5. Hence by Lemma 38.31.3 we may assume $Z \to Y$ is an open immersion. If $V' \subset Y$ denotes the image, then we see that the induced morphism $V' \to U$ is proper because the projection $Y \times_S U \to U$ is proper and $V' \cong Z$ is a closed subscheme of $Y \times_S U$. $\square$

The following lemma is formulated in the Noetherian case only. The version for quasi-compact and quasi-separated schemes is true as well, but will be trivially implied by the main theorem in this section.

0F40 Lemma 38.33.7. Let S be a Noetherian scheme. Let U be a scheme of finite type and separated over S. Let $U = U_1 \cup U_2$ be opens such that U_1 and U_2 have compactifications over S and such that $U_1 \cap U_2$ is dense in U. Then U has a compactification over S.

Proof. Choose a compactification $U_i \subset X_i$ for $i = 1, 2$. We may assume U_i is scheme theoretically dense in X_i. We may assume there is an open $V_i \subset X_i$ and a proper morphism $\psi_i : V_i \to U$ extending $\text{id} : U_i \to U_i$, see Lemma 38.33.6. Picture

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_i \arrow[r] \arrow[dl, "\psi_i"] & X_i \\ U & & \end{tikzcd}$$

If $\{i, j\} = \{1, 2\}$ denote $Z_i = U \setminus U_j = U_i \setminus (U_1 \cap U_2)$ and $Z_j = U \setminus U_i = U_j \setminus (U_1 \cap U_2)$. Thus we have

$$U = U_1 \amalg Z_2 = Z_1 \amalg U_2 = Z_1 \amalg (U_1 \cap U_2) \amalg Z_2$$

Denote $Z_{i,i} \subset V_i$ the inverse image of Z_i under ψ_i. Observe that ψ_i is an isomorphism over an open neighbourhood of Z_i. Denote $Z_{i,j} \subset V_i$ the inverse image of Z_j under ψ_i. Observe that $\psi_i : Z_{i,j} \to Z_j$ is a proper morphism. Since Z_i and Z_j are disjoint closed subsets of U, we see that $Z_{i,i}$ and $Z_{i,j}$ are disjoint closed subsets of V_i.

Denote $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ the closures of $Z_{i,i}$ and $Z_{i,j}$ in X_i. After replacing X_i by a V_i-admissible blowup we may assume that $\overline{Z}_{i,i}$ and $\overline{Z}_{i,j}$ are disjoint, see Lemma 38.33.2. We assume this holds for both X_1 and X_2. Observe that this property is preserved if we replace X_i by a further V_i-admissible blowup.

Set $V_{12} = V_1 \times_U V_2$. We have an immersion $V_{12} \to X_1 \times_S X_2$ which is the composition of the closed immersion $V_{12} = V_1 \times_U V_2 \to V_1 \times_S V_2$ (Schemes, Lemma 26.21.9) and the open immersion $V_1 \times_S V_2 \to X_1 \times_S X_2$. Let $X_{12} \subset X_1 \times_S X_2$ be the scheme theoretic image of $V_{12} \to X_1 \times_S X_2$. The projection morphisms

$$p_1 : X_{12} \to X_1 \quad \text{and} \quad p_2 : X_{12} \to X_2$$

are proper as X_1 and X_2 are proper over S. If we replace X_1 by a V_1-admissible blowing up, then X_{12} is replaced by the strict transform with respect to this blowing up, see Lemma 38.33.5.

Denote $\psi : V_{12} \to U$ the compositions $\psi = \psi_1 \circ p_1|_{V_{12}} = \psi_2 \circ p_2|_{V_{12}}$. Consider the closed subscheme

$$Z_{12,2} = (p_1|_{V_{12}})^{-1}(Z_{1,2}) = (p_2|_{V_{12}})^{-1}(Z_{2,2}) = \psi^{-1}(Z_2) \subset V_{12}$$

The morphism $p_1|_{V_{12}} : V_{12} \to V_1$ is an isomorphism over an open neighbourhood of $Z_{1,2}$ because $\psi_2 : V_2 \to U$ is an isomorphism over an open neighbourhood of Z_2 and $V_{12} = V_1 \times_U V_2$. By Lemma 38.33.3 there exists a V_1-admissible blowing up $X_1' \to X_1$ such that the strict tranform $p_1' : X_{12}' \to X_1'$ of p_1 is an isomorphism over an open neighbourhood of the closure of $Z_{1,2}$ in X_1'. After replacing X_1 by X_1' and X_{12} by X_{12}' we may assume that p_1 is an isomorphism over an open neighbourhood of $\overline{Z}_{1,2}$.

The reduction of the previous paragraph tells us that

$$X_{12} \cap (\overline{Z}_{1,2} \times_S \overline{Z}_{2,1}) = \emptyset$$

where the intersection taken in $X_1 \times_S X_2$. Namely, the inverse image $p_1^{-1}(\overline{Z}_{1,2})$ in X_{12} maps isomorphically to $\overline{Z}_{1,2}$. In particular, we see that $Z_{12,2}$ is dense in $p_1^{-1}(\overline{Z}_{1,2})$. Thus p_2 maps $p_1^{-1}(\overline{Z}_{1,2})$ into $\overline{Z}_{2,2}$. Since $\overline{Z}_{2,2} \cap \overline{Z}_{2,1} = \emptyset$ we conclude.

Consider the schemes

$$W_i = U \coprod\nolimits_{U_i} (X_i \setminus \overline{Z}_{i,j}), \quad i = 1, 2$$

obtained by glueing. Let us apply Lemma 38.33.1 to see that $W_i \to S$ is separated. First, $U \to S$ and $X_i \to S$ are separated. The immersion $U_i \to U \times_S (X_i \setminus \overline{Z}_{i,j})$ is closed because any specialization $u_i \rightsquigarrow u$ with $u_i \in U_i$ and $u \in U \setminus U_i$ can be lifted uniquely to a specialization $u_i \rightsquigarrow v_i$ in V_i along the proper morphism $\psi_i : V_i \to U$ and then v_i must be in $Z_{i,j}$. Thus the image of the immersion is closed, whence the immersion is a closed immersion.

On the other hand, for any valuation ring A over S with fraction field K and any morphism $\gamma : \mathrm{Spec}(K) \to (U_1 \cap U_2)$ over S, there is an i and an extension of γ to a morphism $h_i : \mathrm{Spec}(A) \to W_i$. Namely, for both $i = 1, 2$ there is a morphism $g_i : \mathrm{Spec}(A) \to X_i$ extending γ by the valuative criterion of properness for X_i over S, see Morphisms, Lemma 29.42.1. Thus we only are in trouble if $g_i(\mathfrak{m}_A) \in \overline{Z}_{i,j}$ for $i = 1, 2$. This is impossible by the emptyness of the intersection of X_{12} and $\overline{Z}_{1,2} \times_S \overline{Z}_{2,1}$ we proved above.

Consider a diagram

$$\begin{tikzcd}
W_1' \arrow[r] \arrow[d] & W & W_2' \arrow[l] \arrow[d] \\
W_1 & U \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[ur] & W_2
\end{tikzcd}$$

as in Lemma 38.33.4. By the previous paragraph for every solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "\gamma"'] \arrow[d] & W \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ru, dotted] & S \end{tikzcd}$$

where $\mathrm{Im}(\gamma) \subset U_1 \cap U_2$ there is an i and an extension $h_i : \mathrm{Spec}(A) \to W_i$ of γ. Using the valuative criterion of properness for $W_i' \to W_i$, we can then lift h_i to $h_i' : \mathrm{Spec}(A) \to W_i'$. Hence the dotted arrow in the diagram exists. Since W is separated over S, we see that the arrow is unique as well. This implies that $W \to S$ is universally closed by Morphisms, Lemma 29.42.2. As $W \to S$ is already of finite type and separated, we win. □

0F41 Theorem 38.33.8. Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a separated, finite type morphism. Then X has a compactification over S. See [Lüt93], [Con07b], [Nag56], [Nag57a], [Nag62a], and [Nag63]

Proof. We first reduce to the Noetherian case. We strongly urge the reader to skip this paragraph. There exists a closed immersion $X \to X'$ with $X' \to S$ of finite presentation and separated. See Limits, Proposition 32.9.6. If we find a compactification of X' over S, then taking the scheme theoretic image of X in this will give a compactification of X over S. Thus we may assume $X \to S$ is separated and of finite presentation. We may write $S = \lim S_i$ as a directed limit of a system of Noetherian schemes with affine transition morphisms. See Limits, Proposition 32.5.4. We can choose an i and a morphism $X_i \to S_i$ of finite presentation whose base change to S is $X \to S$, see Limits, Lemma 32.10.1. After increasing i we may assume $X_i \to S_i$ is separated, see Limits, Lemma 32.8.6. If we can find a compactification of X_i over S_i, then the base change of this to S will be a compactification of X over S. This reduces us to the case discussed in the next paragraph.

Assume S is Noetherian. We can choose a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$ such that $U_1 \cap \ldots \cap U_n$ is dense in X. This follows from Properties, Lemma 28.29.4 and the fact that X is quasi-compact with finitely many irreducible components. For each i we can choose an $n_i \geq 0$ and an immersion $U_i \to \mathbf{A}^{n_i}_S$ by Morphisms, Lemma 29.39.2. Hence U_i has a compactification over S for $i = 1, \ldots, n$ by taking the scheme theoretic image in $\mathbf{P}^{n_i}_S$. Applying Lemma 38.33.7 $(n-1)$ times we conclude that the theorem is true. □

38.34. The h topology

0ETQ For us, loosely speaking, an h sheaf is a sheaf for the Zariski topology which satisfies the sheaf property for surjective proper morphisms of finite presentation, see Lemma 38.34.17. However, it may be worth pointing out that the definition of the h topology on the category of schemes depends on the reference.

Voevodsky initially defined an h covering to be a finite collection of finite type morphisms which are jointly universally submersive (Morphisms, Definition 29.24.1). See [Voe96, Definition 3.1.2]. This definition works best if the underlying category of schemes is restricted to all schemes of finite type over a fixed Noetherian base scheme. In this setting, Voevodsky relates h coverings to ph coverings. The ph topology is generated by Zariski coverings and proper surjective morphisms. See Topologies, Section 34.8 for more information.

In Topologies, Section 34.10 we study the V topology. A quasi-compact morphism $X \to Y$ defines a V covering, if any specialization of points of Y is the image of a specialization of points in X and the same is true after any base change (Topologies, Lemma 34.10.13). In this case $X \to Y$ is universally submersive (Topologies, Lemma 34.10.14). It turns out the notion of a V covering is a good replacement for "families of morphisms with fixed target which are jointly universally submersive" when working with non-Noetherian schemes.

Our approach will be to first prove the equivalence between ph covers and V coverings for (possibly infinite) families of morphisms which are locally of finite presentation. We will then use these families as our notion of h coverings in the Stacks project. For Noetherian schemes and finite families these coverings match those in Voevodsky's definition, see Lemma 38.34.3. On the category of schemes of finite presentation over a fixed quasi-compact and quasi-separated scheme S these coverings determine the same topology as the one in [BS17, Definition 2.7].

0ETR Lemma 38.34.1. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of schemes with fixed target with f_i locally of finite presentation for all i. The following are equivalent

(1) $\{X_i \to X\}$ is a ph covering, and
(2) $\{X_i \to X\}$ is a V covering.

Proof. Let $U \subset X$ be affine open. Looking at Topologies, Definitions 34.8.4 and 34.10.7 it suffices to show that the base change $\{X_i \times_X U \to U\}$ can be refined by a standard ph covering if and only if it can be refined by a standard V covering. Thus we may assume X is affine and we have to show $\{X_i \to X\}$ can be refined by a standard ph covering if and only if it can be refined by a standard V covering. Since a standard ph covering is a standard V covering, see Topologies, Lemma 34.10.3 it suffices to prove the other implication.

Assume X is affine and assume $\{f_i : X_i \to X\}_{i \in I}$ can be refined by a standard V covering $\{g_j : Y_j \to X\}_{j=1,\ldots,m}$. For each j choose an i_j and a morphism $h_j : Y_j \to X_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since Y_j is affine hence quasi-compact, for each j we can find finitely many affine opens $U_{j,k} \subset X_{i_j}$ such that $\text{Im}(h_j) \subset \bigcup U_{j,k}$. Then $\{U_{j,k} \to X\}_{j,k}$ refines $\{X_i \to X\}$ and is a standard V covering (as it is a finite family of morphisms of affines and it inherits the lifting property for valuation rings from the corresponding property of $\{Y_j \to X\}$). Thus we reduce to the case discussed in the next paragraph.

Assume $\{f_i : X_i \to X\}_{i=1,\ldots,n}$ is a standard V covering with f_i of finite presentation. We have to show that $\{X_i \to X\}$ can be refined by a standard ph covering. Choose a generic flatness stratification

$$X = S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

as in More on Morphisms, Lemma 37.54.2 for the finitely presented morphism

$$\coprod\nolimits_{i=1,\ldots,n} f_i : \coprod\nolimits_{i=1,\ldots,n} X_i \longrightarrow X$$

of affines. We are going to use all the properties of the stratification without further mention. By construction the base change of each f_i to $U_k = S_k \setminus S_{k+1}$ is flat. Denote Y_k the scheme theoretic closure of U_k in S_k. Since $U_k \to S_k$ is a quasi-compact open immersion (see Properties, Lemma 28.24.1), we see that $U_k \subset Y_k$ is a quasi-compact dense (and scheme theoretically dense) open immersion, see Morphisms, Lemma 29.6.3. The morphism $\coprod_{k=0,\ldots,t-1} Y_k \to X$ is finite surjective, hence $\{Y_k \to X\}$ is a standard ph covering and hence a standard V covering (see above). By the transitivity property of standard V coverings (Topologies, Lemma 34.10.5) it suffices to show that the pullback of the covering $\{X_i \to X\}$ to each Y_k can be refined by a standard V covering. This reduces us to the case described in the next paragraph.

Assume $\{f_i : X_i \to X\}_{i=1,\ldots,n}$ is a standard V covering with f_i of finite presentation and there is a dense quasi-compact open $U \subset X$ such that $X_i \times_X U \to U$ is flat. By Theorem 38.30.7 there is a U-admissible blowup $X' \to X$ such that the strict transform $f'_i : X'_i \to X'$ of f_i is flat. Observe that the projective (hence closed) morphism $X' \to X$ is surjective as $U \subset X$ is dense and as U is identified with an open of X'. After replacing X' by a further U-admissible blowup if necessary, we may also assume $U \subset X'$ is scheme theoretically dense (see Remark 38.30.1). Hence for every point $x \in X'$ there is a valuation ring V and a morphism $g : \text{Spec}(V) \to X'$ such that the generic point of $\text{Spec}(V)$ maps into U and the closed point of $\text{Spec}(V)$ maps to x, see Morphisms, Lemma 29.6.5. Since $\{X_i \to X\}$ is a standard V covering, we can choose an extension of valuation rings $V \subset W$, an index i, and a morphism $\text{Spec}(W) \to X_i$ such that the diagram

$$\begin{tikzcd} \text{Spec}(W) \arrow[rr] \arrow[d] & & X_i \arrow[d] \\ \text{Spec}(V) \arrow[r] & X' \arrow[r] & X \end{tikzcd}$$

is commutative. Since $X'_i \subset X' \times_X X_i$ is a closed subscheme containing the open $U \times_X X_i$, since $\text{Spec}(W)$ is an integral scheme, and since the induced morphism $h : \text{Spec}(W) \to X' \times_X X_i$ maps the generic point of $\text{Spec}(W)$ into $U \times_X X_i$, we conclude that h factors through the closed subscheme $X'_i \subset X' \times_X X_i$. We conclude that $\{f'_i : X'_i \to X'\}$ is a V covering. In particular, $\coprod f'_i$ is surjective. In particular $\{X'_i \to X'\}$ is an fppf covering. Since an fppf covering is a ph covering (More on Morphisms, Lemma 37.48.7), we can find a standard ph covering $\{Y_j \to X'\}$ refining $\{X'_i \to X\}$. Say this covering is given by a proper surjective morphism $Y \to X'$ and a finite affine open covering $Y = \bigcup Y_j$. Then the composition $Y \to X$ is proper surjective and we conclude that $\{Y_j \to X\}$ is a standard ph covering. This finishes the proof. $\square$

Here is our definition.

0ETS Definition 38.34.2. Let T be a scheme. A h covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ such that each f_i is locally of finite presentation and one of the equivalent conditions of Lemma 38.34.1 is satisfied.

For Noetherian schemes this is the same thing as a ph covering (we record this in Lemma 38.34.4 below) and we recover Voevodsky's notion.

0ETT Lemma 38.34.3. Let X be a Noetherian scheme. Let $\{X_i \to X\}_{i \in I}$ be a finite family of finite type morphisms. The following are equivalent

(1) $\coprod_{i \in I} X_i \to X$ is universally submersive (Morphisms, Definition 29.24.1), and
(2) $\{X_i \to X\}_{i \in I}$ is an h covering.

Proof. The implication (2) $\Rightarrow$ (1) follows from the more general Topologies, Lemma 34.10.14 and our definition of h covers. Assume $\coprod X_i \to X$ is universally submersive. We will show that $\{X_i \to X\}$ can be refined by a ph covering; this will suffice by Topologies, Lemma 34.8.7 and our definition of h coverings. The argument will be the same as the one used in the proof of Lemma 38.34.1.

Choose a generic flatness stratification

$$X = S \supset S_0 \supset S_1 \supset \ldots \supset S_t = \emptyset$$

as in More on Morphisms, Lemma 37.54.2 for the finitely presented morphism

$$\coprod\nolimits_{i=1,\ldots,n} f_i : \coprod\nolimits_{i=1,\ldots,n} X_i \longrightarrow X$$

We are going to use all the properties of the stratification without further mention. By construction the base change of each f_i to $U_k = S_k \setminus S_{k+1}$ is flat. Denote Y_k the scheme theoretic closure of U_k in S_k. Since $U_k \to S_k$ is a quasi-compact open immersion (all schemes in this paragraph are Noetherian), we see that $U_k \subset Y_k$ is a quasi-compact dense (and scheme theoretically dense) open immersion, see Morphisms, Lemma 29.6.3. The morphism $\coprod_{k=0,\ldots,t-1} Y_k \to X$ is finite surjective, hence $\{Y_k \to X\}$ is a ph covering. By the transitivity property of ph coverings (Topologies, Lemma 34.8.8) it suffices to show that the pullback of the covering $\{X_i \to X\}$ to each Y_k can be refined by a ph covering. This reduces us to the case described in the next paragraph.

Assume $\coprod X_i \to X$ is universally submersive and there is a dense open $U \subset X$ such that $X_i \times_X U \to U$ is flat for all i. By Theorem 38.30.7 there is a U-admissible blowup $X' \to X$ such that the strict transform $f'_i : X'_i \to X'$ of f_i is flat for all i. Observe that the projective (hence closed) morphism $X' \to X$ is surjective as $U \subset X$ is dense and as U is identified with an open of X'. After replacing X' by a further U-admissible blowup if necessary, we may also assume $U \subset X'$ is dense (see Remark 38.30.1). Hence for every point $x \in X'$ there is a discrete valuation ring A and a morphism $g : \mathrm{Spec}(A) \to X'$ such that the generic point of $\mathrm{Spec}(A)$ maps into U and the closed point of $\mathrm{Spec}(A)$ maps to x, see Limits, Lemma 32.15.1. Set

$$W = \mathrm{Spec}(A) \times_X \coprod X_i = \coprod \mathrm{Spec}(A) \times_X X_i$$

Since $\coprod X_i \to X$ is universally submersive, there is a specialization $w' \leadsto w$ in W such that w' maps to the generic point of $\mathrm{Spec}(A)$ and w maps to the closed point of $\mathrm{Spec}(A)$. (If not, then the closed fibre of $W \to \mathrm{Spec}(A)$ is stable under generalizations, hence open, which contradicts the fact that $W \to \mathrm{Spec}(A)$ is submersive.) Say $w' \in \mathrm{Spec}(A) \times_X X_i$ so of course $w \in \mathrm{Spec}(A) \times_X X_i$ as well. Let $x'_i \leadsto x_i$ be the image of $w' \leadsto w$ in $X' \times_X X_i$. Since $x'_i \in X'_i$ and since $X'_i \subset X' \times_X X_i$ is a closed subscheme we see that $x_i \in X'_i$. Since x_i maps to $x \in X'$ we conclude that $\coprod X'_i \to X'$ is surjective! In particular $\{X'_i \to X'\}$ is an fppf covering. But an fppf covering is a ph covering (More on Morphisms, Lemma 37.48.7). Since $X' \to X$ is proper surjective, we conclude that $\{X'_i \to X\}$ is a ph covering and the proof is complete. $\square$

0H45 Lemma 38.34.4. Let X be a locally Noetherian scheme. A family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ with target X is an h covering if and only if it is a ph covering.

Proof. By Definition 38.34.2 a h covering is a ph covering. Conversely, if $\{f_i : X_i \to X\}$ is a ph covering, then the morphisms f_i are locally of finite type (Topologies, Definition 34.8.4). Since X is locally Noetherian, each f_i is locally of finite presentation and we see that we have a h covering by definition. □

The following lemma and [Ryd07b, Theorem 8.4] shows our definition agrees with (or at least is closely related to) the definition in the paper [Ryd07b] by David Rydh. We restrict to affine base for simplicity.

0ETU Lemma 38.34.5. Let X be an affine scheme. Let $\{X_i \to X\}_{i \in I}$ be an h covering. Then there exists a surjective proper morphism

$$Y \longrightarrow X$$

of finite presentation (!) and a finite affine open covering $Y = \bigcup_{j=1,\ldots,m} Y_j$ such that $\{Y_j \to X\}_{j=1,\ldots,m}$ refines $\{X_i \to X\}_{i \in I}$.

Proof. By assumption there exists a proper surjective morphism $Y \to X$ and a finite affine open covering $Y = \bigcup_{j=1,\ldots,m} Y_j$ such that $\{Y_j \to X\}_{j=1,\ldots,m}$ refines $\{X_i \to X\}_{i \in I}$. This means that for each j there is an index $i_j \in I$ and a morphism $h_j : Y_j \to X_{i_j}$ over X. See Definition 38.34.2 and Topologies, Definition 34.8.4. The problem is that we don't know that $Y \to X$ is of finite presentation. By Limits, Lemma 32.13.2 we can write

$$Y = \lim Y_\lambda$$

as a directed limit of schemes Y_λ proper and of finite presentation over X such that the morphisms $Y \to Y_\lambda$ and the the transition morphisms are closed immersions. Observe that each $Y_\lambda \to X$ is surjective. By Limits, Lemma 32.4.11 we can find a λ and quasi-compact opens $Y_{\lambda,j} \subset Y_\lambda$, $j = 1, \ldots, m$ covering Y_λ and restricting to Y_j in Y. Then $Y_j = \lim Y_{\lambda,j}$. After increasing λ we may assume $Y_{\lambda,j}$ is affine for all j, see Limits, Lemma 32.4.13. Finally, since $X_i \to X$ is locally of finite presentation we can use the functorial characterization of morphisms which are locally of finite presentation (Limits, Proposition 32.6.1) to find a λ such that for each j there is a morphism $h_{\lambda,j} : Y_{\lambda,j} \to X_{i_j}$ whose restriction to Y_j is the morphism h_j chosen above. Thus $\{Y_{\lambda,j} \to X\}$ refines $\{X_i \to X\}$ and the proof is complete. □

We return to the development of the general theory of h coverings.

0ETV Lemma 38.34.6. An fppf covering is a h covering. Hence syntomic, smooth, étale, and Zariski coverings are h coverings as well.

Proof. This is true because in an fppf covering the morphisms are required to be locally of finite presentation and because fppf coverings are ph covering, see More on Morphisms, Lemma 37.48.7. The second statement follows from the first and Topologies, Lemma 34.7.2. □

0ETW Lemma 38.34.7. Let $f : Y \to X$ be a surjective proper morphism of schemes which is of finite presentation. Then $\{Y \to X\}$ is an h covering.

Proof. Combine Topologies, Lemmas 34.10.10 and 34.8.6. □

0ETX Lemma 38.34.8. Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms such that f_i is locally of finite presentation for all i. The following are equivalent

(1) $\{T_i \to T\}_{i \in I}$ is an h covering,
(2) there is an h covering which refines $\{T_i \to T\}_{i \in I}$, and
(3) $\{\coprod_{i \in I} T_i \to T\}$ is an h covering.

Proof. This follows from the analogous statement for ph coverings (Topologies, Lemma 34.8.7) or from the analogous statement for V coverings (Topologies, Lemma 34.10.8). □

Next, we show that our notion of an h covering satisfies the conditions of Sites, Definition 7.6.2.

0ETY Lemma 38.34.9. Let T be a scheme.

(1) If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an h covering of T.
(2) If $\{T_i \to T\}_{i \in I}$ is an h covering and for each i we have an h covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an h covering.
(3) If $\{T_i \to T\}_{i \in I}$ is an h covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an h covering.

Proof. Follows immediately from the corresponding statement for either ph or V coverings (Topologies, Lemma 34.8.8 or 34.10.9) and the fact that the class of morphisms which are locally of finite presentation is preserved under base change and composition. □

Next, we define the big h sites we will work with in the Stacks project. It makes sense to read the general discussion in Topologies, Section 34.2 before proceeding.

0ETZ Definition 38.34.10. A big h site is any site Sch_h as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of h coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of h coverings, and the set Cov_0 chosen above.

See the remarks following Topologies, Definition 34.3.5 for motivation and explanation regarding the definition of big sites.

0EUY Definition 38.34.11. Let T be an affine scheme. A standard h covering of T is a family $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ with each T_i affine, with f_i of finite presentation satisfying either of the following equivalent conditions: (1) $\{U_i \to U\}$ can be refined by a standard ph covering or (2) $\{U_i \to U\}$ is a V covering.

The equivalence of the conditions follows from Lemma 38.34.1, Topologies, Definition 34.8.4, and Lemma 34.8.7.

Before we continue with the introduction of the big h site of a scheme S, let us point out that the topology on a big h site Sch_h is in some sense induced from the h topology on the category of all schemes.

0EU0 Lemma 38.34.12. Let Sch_h be a big h site as in Definition 38.34.10. Let $T \in \text{Ob}(Sch_h)$. Let $\{T_i \to T\}_{i \in I}$ be an arbitrary h covering of T.

(1) There exists a covering $\{U_j \to T\}_{j \in J}$ of T in the site Sch_h which refines $\{T_i \to T\}_{i \in I}$.

(2) If $\{T_i \to T\}_{i \in I}$ is a standard h covering, then it is tautologically equivalent to a covering of Sch_h.
(3) If $\{T_i \to T\}_{i \in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_h.

Proof. Omitted. Hint: this is exactly the same as the proof of Topologies, Lemma 34.8.10. $\square$

0EU1 Definition 38.34.13. Let S be a scheme. Let Sch_h be a big h site containing S.

(1) The big h site of S, denoted $(Sch/S)_h$, is the site Sch_h/S introduced in Sites, Section 7.25.
(2) The big affine h site of S, denoted $(\text{Aff}/S)_h$, is the full subcategory of $(Sch/S)_h$ whose objects are affine U/S. A covering of $(\text{Aff}/S)_h$ is any covering $\{U_i \to U\}$ of $(Sch/S)_h$ which is a standard h covering.

We explicitly state that the big affine h site is a site.

0EU2 Lemma 38.34.14. Let S be a scheme. Let Sch_h be a big h site containing S. Then $(\text{Aff}/S)_h$ is a site.

Proof. Reasoning as in the proof of Topologies, Lemma 34.4.9 it suffices to show that the collection of standard h coverings satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a standard h covering $\{T_i \to T\}_{i \in I}$ and for each i a standard h covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a h covering (Lemma 38.34.9), $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. Thus $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard h covering. $\square$

0EU3 Lemma 38.34.15. Let S be a scheme. Let Sch_h be a big h site containing S. The underlying categories of the sites Sch_h, $(Sch/S)_h$, and $(\text{Aff}/S)_h$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_h$ has a final object, namely S/S.

Proof. For Sch_h it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \text{Ob}(Sch_h)$. The fibre product $V \times_U W$ in Sch_h is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_h$. This proves the result for $(Sch/S)_h$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(\text{Aff}/S)_h$. $\square$

Next, we check that the big affine site defines the same topos as the big site.

0EU4 Lemma 38.34.16. Let S be a scheme. Let Sch_h be a big h site containing S. The functor $(\text{Aff}/S)_h \to (Sch/S)_h$ is cocontinuous and induces an equivalence of topoi from $Sh((\text{Aff}/S)_h)$ to $Sh((Sch/S)_h)$.

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.29.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.29.1. Denote the inclusion functor $u : (\text{Aff}/S)_h \to (Sch/S)_h$. Being cocontinuous follows because any h covering of T/S, T affine, can be refined by a standard h covering for example by Lemma 38.34.5. Hence (1) holds. We see u is continuous simply because a standard h covering is a h covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering (which is a h covering). $\square$

0EU5 **Lemma 38.34.17.** Let $\mathcal{F}$ be a presheaf on $(Sch/S)_h$. Then $\mathcal{F}$ is a sheaf if and only if

(1) $\mathcal{F}$ satisfies the sheaf condition for Zariski coverings, and
(2) if $f : V \to U$ is proper, surjective, and of finite presentation, then $\mathcal{F}(U)$ maps bijectively to the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$.

Moreover, in the presence of (1) property (2) is equivalent to property

(2') the sheaf property for $\{V \to U\}$ as in (2) with U affine.

Proof. We will show that if (1) and (2) hold, then $\mathcal{F}$ is sheaf. Let $\{T_i \to T\}$ be a covering in $(Sch/S)_h$. We will verify the sheaf condition for this covering. Let $s_i \in \mathcal{F}(T_i)$ be sections which restrict to the same section over $T_i \times_T T_{i'}$. We will show that there exists a unique section $s \in \mathcal{F}(T)$ restricting to s_i over T_i. Let $T = \bigcup U_j$ be an affine open covering. By property (1) it suffices to produce sections $s_j \in \mathcal{F}(U_j)$ which agree on $U_j \cap U_{j'}$ in order to produce s. Consider the coverings $\{T_i \times_T U_j \to U_j\}$. Then $s_{ji} = s_i|_{T_i \times_T U_j}$ are sections agreeing over $(T_i \times_T U_j) \times_{U_j} (T_{i'} \times_T U_j)$. Choose a proper surjective morphism $V_j \to U_j$ of finite presentation and a finite affine open covering $V_j = \bigcup V_{jk}$ such that $\{V_{jk} \to U_j\}$ refines $\{T_i \times_T U_j \to U_j\}$. See Lemma 38.34.5. If $s_{jk} \in \mathcal{F}(V_{jk})$ denotes the pullback of s_{ji} to V_{jk} by the implied morphisms, then we find that s_{jk} glue to a section $s'_j \in \mathcal{F}(V_j)$. Using the agreement on overlaps once more, we find that s'_j is in the equalizer of the two maps $\mathcal{F}(V_j) \to \mathcal{F}(V_j \times_{U_j} V_j)$. Hence by (2) we find that s'_j comes from a unique section $s_j \in \mathcal{F}(U_j)$. We omit the verification that these sections s_j have all the desired properties.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $V \to U$ is a morphism of $(Sch/S)_h$ which is proper, surjective, and of finite presentation. Choose an affine open covering $U = \bigcup U_i$ and set $V_i = V \times_U U_i$. Then we see that $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective because we know $\mathcal{F}(U_i) \to \mathcal{F}(V_i)$ is injective by (2') and we know $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective by (1). Finally, suppose that we are given an $t \in \mathcal{F}(V)$ in the equalizer of the two maps $\mathcal{F}(V) \to \mathcal{F}(V \times_U V)$. Then $t|_{V_i}$ is in the equalizer of the two maps $\mathcal{F}(V_i) \to \mathcal{F}(V_i \times_{U_i} V_i)$ for all i. Hence we obtain a unique section $s_i \in \mathcal{F}(U_i)$ mapping to $t|_{V_i}$ for all i by (2'). We omit the verification that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j; this uses the uniqueness property just shown. By the sheaf property for the covering $U = \bigcup U_i$ we obtain a section $s \in \mathcal{F}(U)$. We omit the proof that s maps to t in $\mathcal{F}(V)$. $\square$

Next, we establish some relationships between the topoi associated to these sites.

0EU6 **Lemma 38.34.18.** Let Sch_h be a big h site. Let $f : T \to S$ be a morphism in Sch_h. The functor

$$u : (Sch/T)_h \longrightarrow (Sch/S)_h, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_h \longrightarrow (Sch/T)_h, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_h) \longrightarrow Sh((Sch/S)_h)$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.21.5 and 7.21.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.22.1 and 7.22.2 to get the formula for $f_{big,*}$. $\square$

0EU7 Lemma 38.34.19. Given schemes X, Y, Y in $(Sch/S)_h$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 38.34.18. $\square$

38.35. More on the h topology

0EUZ In this section we prove a few more results on the h topology. First, some non-examples.

0EV0 Example 38.35.1. The "structure sheaf" $\mathcal{O}$ is not a sheaf in the h topology. For example, consider a surjective closed immersion of finite presentation $X \to Y$. Then $\{X \to Y\}$ is an h covering for example by Lemma 38.34.7. Moreover, note that $X \times_Y X = X$. Thus if $\mathcal{O}$ where a sheaf in the h topology, then $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ would be bijective. This is not the case as soon as X, Y are affine and the morphism $X \to Y$ is not an isomorphism.

0EV1 Example 38.35.2. On any of the sites $(Sch/S)_h$ the topology is not subcanonical, in other words, representable sheaves are not sheaves. Namely, the "structure sheaf" $\mathcal{O}$ is representable because $\mathcal{O}(X) = \mathrm{Mor}_S(X, \mathbf{A}^1_S)$ in $(Sch/S)_h$ and we saw in Example 38.35.1 that $\mathcal{O}$ is not a sheaf.

0EV2 Lemma 38.35.3. Let T be an affine scheme which is written as a limit $T = \lim_{i \in I} T_i$ of a directed inverse system of affine schemes.

(1) Let $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ be a standard h covering of T, see Definition 38.34.11. Then there exists an index i and a standard h covering $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ whose base change $T \times_{T_i} \mathcal{V}_i$ to T is isomorphic to $\mathcal{V}$.
(2) Let $\mathcal{V}_i$, $\mathcal{V}'_i$ be a pair of standard h coverings of T_i. If $f : T \times_{T_i} \mathcal{V}_i \to T \times_{T_i} \mathcal{V}'_i$ is a morphism of coverings of T, then there exists an index $i' \geq i$ and a morphism $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ whose base change to T is f.
(3) If $f, g : \mathcal{V} \to \mathcal{V}'_i$ are morphisms of standard h coverings of T_i whose base changes f_T, g_T to T are equal then there exists an index $i' \geq i$ such that $f_{T_{i'}} = g_{T_{i'}}$.

In other words, the category of standard h coverings of T is the colimit over I of the categories of standard h coverings of T_i.

Proof. By Limits, Lemma 32.10.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemma 32.8.2 the same is true for category of schemes which are affine and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of finitely presented morphisms with $V_{j,i}$ affine, and the base change family $\{T \times_{T_i} V_{j,i} \to T\}$ is an h covering, then for some $i' \geq i$ the family $\{T_{i'} \times_{T_i} V_{j,i} \to T_{i'}\}$ is an h covering. To see this we use Lemma 38.34.5 to choose a finitely presented, proper, surjective morphism $Y \to T$

and a finite affine open covering $Y = \bigcup_{k=1,\ldots,n} Y_k$ such that $\{Y_k \to T\}_{k=1,\ldots,n}$ refines $\{T \times_{T_i} V_{j,i} \to T\}$. Using the arguments above and Limits, Lemmas 32.13.1, 32.8.15, and 32.4.11 we can find an $i' \geq i$ and a finitely presented, surjective, proper morphism $Y_{i'} \to T_{i'}$ and an affine open covering $Y_{i'} = \bigcup_{k=1,\ldots,n} Y_{i',k}$ such that moreover $\{Y_{i',k} \to Y_{i'}\}$ refines $\{T_{i'} \times_{T_i} V_{j,i} \to T_{i'}\}$. It follows that this last mentioned family is a h covering and the proof is complete. □

0EV3 Lemma 38.35.4. Let S be a scheme contained in a big site Sch_h. Let $F : (Sch/S)_h^{opp} \to Sets$ be an h sheaf satisfying property (b) of Topologies, Lemma 34.13.1 with $\mathcal{C} = (Sch/S)_h$. Then the extension F' of F to the category of all schemes over S satisfies the sheaf condition for all h coverings and is limit preserving (Limits, Remark 32.6.2).

Proof. This is proven by the arguments given in the proofs of Topologies, Lemmas 34.13.3 and 34.13.4 using Lemmas 38.35.3 and 38.34.12. Details omitted. □

38.36. Blow up squares and the ph topology

0EW0 Let X be a scheme. Let $Z \subset X$ be a closed subscheme such that the inclusion morphism is of finite presentation, i.e., the quasi-coherent sheaf of ideals corresponding to Z is of finite type. Let $b : X' \to X$ be the blowup of X in Z and let $E = b^{-1}(Z)$ be the exceptional divisor. See Divisors, Section 31.32. In this situation and in this section, let us say

0EV5 (38.36.0.1)
$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d, "b"] \\ Z \arrow[r] & X \end{tikzcd}$$

is a blow up square.

0EW1 Lemma 38.36.1. Let $\mathcal{F}$ be a sheaf on a site $(Sch/S)_{ph}$, see Topologies, Definition 34.8.11. Then for any blow up square (38.36.0.1) in the category $(Sch/S)_{ph}$ the diagram

$$\begin{tikzcd} \mathcal{F}(E) & \mathcal{F}(X') \arrow[l] \\ \mathcal{F}(Z) \arrow[u] & \mathcal{F}(X) \arrow[l] \arrow[u] \end{tikzcd}$$

is cartesian in the category of sets.

Proof. Since $Z \amalg X' \to X$ is a surjective proper morphism we see that $\{Z \amalg X' \to X\}$ is a ph covering (Topologies, Lemma 34.8.6). We have

$$(Z \amalg X') \times_X (Z \amalg X') = Z \amalg E \amalg E \amalg X' \times_X X'$$

Since $\mathcal{F}$ is a Zariski sheaf we see that $\mathcal{F}$ sends disjoint unions to products. Thus the sheaf condition for the covering $\{Z \amalg X' \to X\}$ says that $\mathcal{F}(X) \to \mathcal{F}(Z) \times \mathcal{F}(X')$ is injective with image the set of pairs (t, s') such that (a) $t|_E = s'|_E$ and (b) s' is in the equalizer of the two maps $\mathcal{F}(X') \to \mathcal{F}(X' \times_X X')$. Next, observe that the obvious morphism

$$E \times_Z E \amalg X' \longrightarrow X' \times_X X'$$

is a surjective proper morphism as b induces an isomorphism $X' \setminus E \to X \setminus Z$. We conclude that $\mathcal{F}(X' \times_X X') \to \mathcal{F}(E \times_Z E) \times \mathcal{F}(X')$ is injective. It follows that (a) $\Rightarrow$ (b) which means that the lemma is true. $\square$

0EW2 Lemma 38.36.2. Let $\mathcal{F}$ be a sheaf on a site $(Sch/S)_{ph}$ as in Topologies, Definition 34.8.11. Let $X \to X'$ be a morphism of $(Sch/S)_{ph}$ which is a thickening. Then $\mathcal{F}(X') \to \mathcal{F}(X)$ is bijective.

Proof. Observe that $X \to X'$ is a proper surjective morphism of and $X \times_{X'} X = X$. By the sheaf property for the ph covering $\{X \to X'\}$ (Topologies, Lemma 34.8.6) we conclude. $\square$

38.37. Almost blow up squares and the h topology

0EV4 Consider a blow up square (38.36.0.1). Although the morphism $b : X' \to X$ is projective (Divisors, Lemma 31.32.13) in general there is no simple way to guarantee that b is of finite presentation. Since h coverings are constructed using morphisms of finite presentation, we need a variant. Namely, we will say a commutative diagram

0EV6 (38.37.0.1)
$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d, "b"] \\ Z \arrow[r] & X \end{tikzcd}$$

of schemes is an almost blow up square if the following conditions are satisfied

(1) $Z \to X$ is a closed immersion of finite presentation,
(2) $E = b^{-1}(Z)$ is a locally principal closed subscheme of X',
(3) b is proper and of finite presentation,
(4) the closed subscheme $X'' \subset X'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X'}$ supported on E (Properties, Lemma 28.24.5) is the blow up of X in Z.

It follows that the morphism b induces an isomorphism $X' \setminus E \to X \setminus Z$. For some very simple examples of almost blow up squares, see Examples 38.37.10 and 38.37.11.

The base change of a blow up usually isn't a blow up, but almost blow ups are compatible with base change.

0EV7 Lemma 38.37.1. Consider an almost blow up square (38.37.0.1). Let $Y \to X$ be any morphism. Then the base change

$$\begin{tikzcd} Y \times_X E \arrow[r] \arrow[d] & Y \times_X X' \arrow[d] \\ Y \times_X Z \arrow[r] & Y \end{tikzcd}$$

is an almost blow up square too.

Proof. The morphism $Y \times_X X' \to Y$ is proper and of finite presentation by Morphisms, Lemmas 29.41.5 and 29.21.4. The morphism $Y \times_X Z \to Y$ is a closed immersion (Morphisms, Lemma 29.2.4) of finite presentation. The inverse image of $Y \times_X Z$ in $Y \times_X X'$ is equal to the inverse image of E in $Y \times_X X'$ and hence is locally principal (Divisors, Lemma 31.13.11). Let $X'' \subset X'$, resp. $Y'' \subset Y \times_X X'$ be the closed subscheme corresponding to the quasi-coherent ideal of sections of

$\mathcal{O}_{X'}$, resp. $\mathcal{O}_{Y \times_Y X'}$ supported on E, resp. $Y \times_X E$. Clearly, $Y'' \subset Y \times_X X''$ is the closed subscheme corresponding to the quasi-coherent ideal of sections of $\mathcal{O}_{Y \times_Y X''}$ supported on $Y \times_X (E \cap X'')$. Thus Y'' is the strict transform of Y relative to the blowing up $X'' \to X$, see Divisors, Definition 31.33.1. Thus by Divisors, Lemma 31.33.2 we see that Y'' is the blow up of $Y \times_X Z$ on Y. □

One can shrink almost blow up squares.

0EV8 Lemma 38.37.2. Consider an almost blow up square (38.37.0.1). Let $W \to X'$ be a closed immersion of finite presentation. The following are equivalent

(1) $X' \setminus E$ is scheme theoretically contained in W,
(2) the blowup X'' of X in Z is scheme theoretically contained in W,
(3) the diagram

$$\begin{tikzcd} E \cap W \arrow[r] \arrow[d] & W \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

is an almost blow up square. Here $E \cap W$ is the scheme theoretic intersection.

Proof. Assume (1). Then the surjection $\mathcal{O}_{X'} \to \mathcal{O}_W$ is an isomorphism over the open $X' \subset E$. Since the ideal sheaf of $X'' \subset X'$ is the sections of $\mathcal{O}_{X'}$ supported on E (by our definition of almost blow up squares) we conclude (2) is true. If (2) is true, then (3) holds. If (3) holds, then (1) holds because $X'' \cap (X' \setminus E)$ is isomorphic to $X \setminus Z$ which in turn is isomorphic to $X' \setminus E$. □

The actual blowup is the limit of shrinkings of any given almost blowup.

0EV9 Lemma 38.37.3. Consider an almost blow up square (38.37.0.1) with X quasi-compact and quasi-separated. Then the blowup X'' of X in Z can be written as

$$X'' = \lim X'_i$$

where the limit is over the directed system of closed subschemes $X'_i \subset X'$ of finite presentation satisfying the equivalent conditions of Lemma 38.37.2.

Proof. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the quasi-coherent sheaf of ideals corresponding to X''. By Properties, Lemma 28.22.3 we can write $\mathcal{I}$ as the filtered colimit $\mathcal{I} = \operatorname{colim} \mathcal{I}_i$ of its quasi-coherent submodules of finite type. Since these modules correspond 1-to-1 to the closed subschemes X'_i the proof is complete. □

Almost blow up squares exist.

0EVA Lemma 38.37.4. Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Then there exists an almost blow up square as in (38.37.0.1).

Proof. We may write $X = \lim X_i$ as a directed limit of an inverse system of Noetherian schemes with affine transition morphisms, see Limits, Proposition 32.5.4. We can find an index i and a closed immersion $Z_i \to X_i$ whose base change to

X is the closed immersion $Z \to X$. See Limits, Lemmas 32.10.1 and 32.8.5. Let $b_i : X'_i \to X_i$ be the blowing up with center Z_i. This produces a blow up square

$$\begin{tikzcd} E_i \arrow[r] \arrow[d] & X'_i \arrow[d, "b_i"] \\ Z_i \arrow[r] & X_i \end{tikzcd}$$

where all the morphisms are finite type morphisms of Noetherian schemes and hence of finite presentation. Thus this is an almost blow up square. By Lemma 38.37.1 the base change of this diagram to X produces the desired almost blow up square. ☐

Almost blow up squares are unique up to shrinking as in Lemma 38.37.2.

0EVB Lemma 38.37.5. Let X be a quasi-compact and quasi-separated scheme and let $Z \subset X$ be a closed subscheme cut out by a finite type quasi-coherent sheaf of ideals. Suppose given almost blow up squares (38.37.0.1)

$$\begin{tikzcd} E_k \arrow[r] \arrow[d] & X'_k \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

for $k = 1, 2$, then there exists an almost blow up square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

and closed immersions $i_k : X' \to X'_k$ over X with $E = i_k^{-1}(E_k)$.

Proof. Denote $X'' \to X$ the blowing up of Z in X. We view X'' as a closed subscheme of both X'_1 and X'_2. Write $X'' = \lim X'_{1,i}$ as in Lemma 38.37.3. By Limits, Proposition 32.6.1 there exists an i and a morphism $h : X'_{1,i} \to X'_2$ agreeing with the inclusions $X'' \subset X'_{1,i}$ and $X'' \subset X'_2$. By Limits, Lemma 32.4.20 the restriction of h to $X'_{1,i'}$ is a closed immersion for some $i' \geq i$. This finishes the proof. ☐

Our flattening techniques for blowing up are inherited by almost blowups in favorable situations.

0EVC Lemma 38.37.6. Let Y be a quasi-compact and quasi-separated scheme. Let X be a scheme of finite presentation over Y. Let $V \subset Y$ be a quasi-compact open such that $X_V \to V$ is flat. Then there exist a commutative diagram

$$\begin{tikzcd}
E \arrow[dd] \arrow[dr] & & & D \arrow[lll] \arrow[dl] \arrow[dd] \\
& Y' \arrow[d] & X' \arrow[l] \arrow[d] & \\
& Y & X \arrow[l] & \\
Z \arrow[ur] & & & T \arrow[lll] \arrow[ul]
\end{tikzcd}$$

whose right and left hand squares are almost blow up squares, whose lower and top squares are cartesian, such that $Z \cap V = \emptyset$, and such that $X' \to Y'$ is flat (and of finite presentation).

Proof. If Y is a Noetherian scheme, then this lemma follows immediately from Lemma 38.31.1 because in this case blow up squares are almost blow up squares (we also use that strict transforms are blow ups). The general case is reduced to the Noetherian case by absolute Noetherian approximation.

We may write $Y = \lim Y_i$ as a directed limit of an inverse system of Noetherian schemes with affine transition morphisms, see Limits, Proposition 32.5.4. We can find an index i and a morphism $X_i \to Y_i$ of finite presentation whose base change to Y is $X \to Y$. See Limits, Lemmas 32.10.1. After increasing i we may assume V is the inverse image of an open subscheme $V_i \subset Y_i$, see Limits, Lemma 32.4.11. Finally, after increasing i we may assume that $X_{i,V_i} \to V_i$ is flat, see Limits, Lemma 32.8.7. By the Noetherian case, we may construct a diagram as in the lemma for $X_i \to Y_i \supset V_i$. The base change of this diagram by $Y \to Y_i$ provides the solution. Use that base change preserves properties of morphisms, see Morphisms, Lemmas 29.41.5, 29.21.4, 29.2.4, and 29.25.8 and that base change of an almost blow up square is an almost blow up square, see Lemma 38.37.1. □

0EVD Lemma 38.37.7. Let $\mathcal{F}$ be a sheaf on one of the sites $(Sch/S)_h$ constructed in Definition 38.34.13. Then for any almost blow up square (38.37.0.1) in the category $(Sch/S)_h$ the diagram

$$\begin{tikzcd}
\mathcal{F}(E) & \mathcal{F}(X') \arrow[l] \\
\mathcal{F}(Z) \arrow[u] & \mathcal{F}(X) \arrow[l] \arrow[u]
\end{tikzcd}$$

is cartesian in the category of sets.

Proof. Since $Z \amalg X' \to X$ is a surjective proper morphism of finite presentation we see that $\{Z \amalg X' \to X\}$ is an h covering (Lemma 38.34.7). We have

$$(Z \amalg X') \times_X (Z \amalg X') = Z \amalg E \amalg E \amalg X' \times_X X'$$

Since $\mathcal{F}$ is a Zariski sheaf we see that $\mathcal{F}$ sends disjoint unions to products. Thus the sheaf condition for the covering $\{Z \amalg X' \to X\}$ says that $\mathcal{F}(X) \to \mathcal{F}(Z) \times \mathcal{F}(X')$ is injective with image the set of pairs (t, s') such that (a) $t|_E = s'|_E$ and (b) s' is in the equalizer of the two maps $\mathcal{F}(X') \to \mathcal{F}(X' \times_X X')$. Next, observe that the obvious morphism

$$E \times_Z E \amalg X' \longrightarrow X' \times_X X'$$

is a surjective proper morphism of finite presentation as b induces an isomorphism $X' \setminus E \to X \setminus Z$. We conclude that $\mathcal{F}(X' \times_X X') \to \mathcal{F}(E \times_Z E) \times \mathcal{F}(X')$ is injective. It follows that (a) $\Rightarrow$ (b) which means that the lemma is true. $\square$

0EVE **Lemma 38.37.8.** Let $\mathcal{F}$ be a sheaf on one of the sites $(Sch/S)_h$ constructed in Definition 38.34.13. Let $X \to X'$ be a morphism of $(Sch/S)_h$ which is a thickening and of finite presentation. Then $\mathcal{F}(X') \to \mathcal{F}(X)$ is bijective.

Proof. First proof. Observe that $X \to X'$ is a proper surjective morphism of finite presentation and $X \times_{X'} X = X$. By the sheaf property for the h covering $\{X \to X'\}$ (Lemma 38.34.7) we conclude.

Second proof (silly). The blow up of X' in X is the empty scheme. The reason is that the affine blowup algebra $A[\frac{I}{a}]$ (Algebra, Section 10.70) is zero if a is a nilpotent element of A. Details omitted. Hence we get an almost blow up square of the form

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

Since $\mathcal{F}$ is a sheaf we have that $\mathcal{F}(\emptyset)$ is a singleton. Applying Lemma 38.37.7 we get the conclusion. $\square$

0EVF **Proposition 38.37.9.** Let $\mathcal{F}$ be a presheaf on one of the sites $(Sch/S)_h$ constructed in Definition 38.34.13. Then $\mathcal{F}$ is a sheaf if and only if the following conditions are satisfied

(1) $\mathcal{F}$ is a sheaf for the Zariski topology,
(2) given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$,
(3) given an almost blow up square (38.37.0.1) with X affine in the category $(Sch/S)_h$ the diagram

$$\begin{tikzcd} \mathcal{F}(E) & \mathcal{F}(X') \arrow[l] \\ \mathcal{F}(Z) \arrow[u] & \mathcal{F}(X) \arrow[l] \arrow[u] \end{tikzcd}$$

is cartesian in the category of sets.

Proof. Assume $\mathcal{F}$ is a sheaf. Condition (1) holds because a Zariski covering is a h covering, see Lemma 38.34.6. Condition (2) holds because for f as in (2) we have that $\{X \to Y\}$ is an fppf covering (this is clear) and hence an h covering, see Lemma 38.34.6. Condition (3) holds by Lemma 38.37.7.

Conversely, assume $\mathcal{F}$ satisfies (1), (2), and (3). We will prove $\mathcal{F}$ is a sheaf by applying Lemma 38.34.17. Consider a surjective, finitely presented, proper morphism $f : X \to Y$ in $(Sch/S)_h$ with Y affine. It suffices to show that $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$.

First, assume that $f : X \to Y$ is in addition a closed immersion (in other words, f is a thickening). Then the blow up of Y in X is the empty scheme and this produces an almost blow up square consisting with $\emptyset, \emptyset, X, Y$ at the vertices (compare with the second proof of Lemma 38.37.8). Hence we see that condition (3) tells us that

$$\begin{tikzcd} \mathcal{F}(\emptyset) & \mathcal{F}(\emptyset) \arrow[l] \\ \mathcal{F}(X) \arrow[u] & \mathcal{F}(Y) \arrow[l] \arrow[u] \end{tikzcd}$$

is cartesian in the category of sets. Since $\mathcal{F}$ is a sheaf for the Zariski topology, we see that $\mathcal{F}(\emptyset)$ is a singleton. Hence we see that $\mathcal{F}(X) = \mathcal{F}(Y)$.

Interlude A: let $T \to T'$ be a morphism of $(Sch/S)_h$ which is a thickening and of finite presentation. Then $\mathcal{F}(T') \to \mathcal{F}(T)$ is bijective. Namely, choose an affine open covering $T' = \bigcup T'_i$ and let $T_i = T \times_{T'} T'_i$ be the corresponding affine opens of T. Then we have $\mathcal{F}(T'_i) \to \mathcal{F}(T_i)$ is bijective for all i by the result of the previous paragraph. Using the Zariski sheaf property we see that $\mathcal{F}(T') \to \mathcal{F}(T)$ is injective. Repeating the argument we find that it is bijective. Minor details omitted.

Interlude B: consider an almost blow up square (38.37.0.1) in the category $(Sch/S)_h$. Then we claim the diagram

$$\begin{tikzcd} \mathcal{F}(E) & \mathcal{F}(X') \arrow[l] \\ \mathcal{F}(Z) \arrow[u] & \mathcal{F}(X) \arrow[l] \arrow[u] \end{tikzcd}$$

is cartesian in the category of sets. This is a consequence of condition (3) as follows by choosing an affine open covering of X and arguing as in Interlude A. We omit the details.

Next, let $f : X \to Y$ be a surjective, finitely presented, proper morphism in $(Sch/S)_h$ with Y affine. Choose a generic flatness stratification

$$Y \supset Y_0 \supset Y_1 \supset \ldots \supset Y_t = \emptyset$$

as in More on Morphisms, Lemma 37.54.2 for $f : X \to Y$. We are going to use all the properties of the stratification without further mention. Set $X_0 = X \times_Y Y_0$. By the Interlude B we have $\mathcal{F}(Y_0) = \mathcal{F}(Y)$, $\mathcal{F}(X_0) = \mathcal{F}(X)$, and $\mathcal{F}(X_0 \times_{Y_0} X_0) = \mathcal{F}(X \times_Y X)$.

We are going to prove the result by induction on t. If $t = 1$ then $X_0 \to Y_0$ is surjective, proper, flat, and of finite presentation and we see that the result holds by property (2). For $t > 1$ we may replace Y by Y_0 and X by X_0 (see above) and assume $Y = Y_0$.

Consider the quasi-compact open subscheme $V = Y \setminus Y_1 = Y_0 \setminus Y_1$. Choose a diagram

$$\begin{tikzcd} E \arrow[dd] \arrow[dr] & & & D \arrow[lll] \arrow[dl] \arrow[dd] \\ & Y' \arrow[d] & X' \arrow[l] \arrow[d] & \\ & Y & X \arrow[l] & \\ Z \arrow[ur] & & & T \arrow[lll] \arrow[ul] \end{tikzcd}$$

as in Lemma 38.37.6 for $f : X \to Y \supset V$. Then $f' : X' \to Y'$ is flat and of finite presentation. Also f' is proper (use Morphisms, Lemmas 29.41.4 and 29.41.7 to see this). Thus the image $W = f'(X') \subset Y'$ is an open (Morphisms, Lemma 29.25.10) and closed subscheme of Y'. Observe that $Y' \setminus E$ is contained in W. By Lemma 38.37.2 this means we may replace Y' by W in the above diagram. In other words, we may and do assume f' is surjective. At this point we know that

$$\begin{tikzcd} \mathcal{F}(E) & \mathcal{F}(Y') \arrow[l] \\ \mathcal{F}(Z) \arrow[u] & \mathcal{F}(Y) \arrow[l] \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}(D) & \mathcal{F}(X') \arrow[l] \\ \mathcal{F}(T) \arrow[u] & \mathcal{F}(X) \arrow[l] \arrow[u] \end{tikzcd}$$

are cartesian by Interlude B. Note that $Z \cap Y_1 \to Z$ is a thickening of finite presentation (as Z is set theoretically contained in Y_1 as a closed subscheme of Y disjoint from V). Thus we obtain a filtration

$$Z \supset Z \cap Y_1 \supset Z \cap Y_2 \subset \ldots \subset Z \cap Y_t = \emptyset$$

as above for the restriction $T = Z \times_Y X \to Z$ of f to T. Thus by induction hypothesis we find that $\mathcal{F}(Z) \to \mathcal{F}(T)$ is an injective map of sets whose image is the equalizer of the two maps $\mathcal{F}(T) \to \mathcal{F}(T \times_Z T)$.

Let $s \in \mathcal{F}(X)$ be in the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$. By the above we see that the restriction $s|_T$ comes from a unique element $t \in \mathcal{F}(Z)$ and similarly that the restriction $s|_{X'}$ comes from a unique element $t' \in \mathcal{F}(Y')$. Chasing sections using the restriction maps for $\mathcal{F}$ corresponding to the arrows in the huge commutative diagram above the reader finds that t and t' restrict to the same element of $\mathcal{F}(E)$ because they restrict to the same element of $\mathcal{F}(D)$ and we have (2); here we use that $D \to E$ is surjective, flat, proper, and of finite presentation as the restriction of $X' \to Y'$. Thus by the first of the two cartesian squares displayed above we get a unique section $u \in \mathcal{F}(Y)$ restricting to t and t' on Z and Y'. To see that u restrict to s on X use the second diagram. $\square$

0EVG Example 38.37.10. Let A be a ring. Let $f \in A$ be an element. Let $J \subset A$ be a finitely generated ideal annihilated by a power of f. Then

$$\begin{array}{ccc} E = \mathrm{Spec}(A/fA + J) & \longrightarrow & \mathrm{Spec}(A/J) = X' \\ \downarrow & & \downarrow \\ Z = \mathrm{Spec}(A/fA) & \longrightarrow & \mathrm{Spec}(A) = X \end{array}$$

is an almost blowup square.

0EVH Example 38.37.11. Let A be a ring. Let $f_1, f_2 \in A$ be elements.

$$\begin{array}{ccc} E = \mathrm{Proj}(A/(f_1, f_2)[T_0, T_1]) & \longrightarrow & \mathrm{Proj}(A[T_0, T_1]/(f_2T_0 - f_1T_1) = X' \\ \downarrow & & \downarrow \\ Z = \mathrm{Spec}(A/(f_1, f_2)) & \longrightarrow & \mathrm{Spec}(A) = X \end{array}$$

is an almost blowup square.

0EVI Lemma 38.37.12. Let $\mathcal{F}$ be a presheaf on one of the sites $(Sch/S)_h$ constructed in Definition 38.34.13. Then $\mathcal{F}$ is a sheaf if and only if the following conditions are satisfied

(1) $\mathcal{F}$ is a sheaf for the Zariski topology,
(2) given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then $\mathcal{F}(Y)$ is the equalizer of the two maps $\mathcal{F}(X) \to \mathcal{F}(X \times_Y X)$,
(3) $\mathcal{F}$ turns an almost blow up square as in Example 38.37.10 in the category $(Sch/S)_h$ into a cartesian diagram of sets, and
(4) $\mathcal{F}$ turns an almost blow up square as in Example 38.37.11 in the category $(Sch/S)_h$ into a cartesian diagram of sets.

Proof. By Proposition 38.37.9 it suffices to show that given an almost blow up square (38.37.0.1) with X affine in the category $(Sch/S)_h$ the diagram

$$\begin{array}{ccc} \mathcal{F}(E) & \longleftarrow & \mathcal{F}(X') \\ \uparrow & & \uparrow \\ \mathcal{F}(Z) & \longleftarrow & \mathcal{F}(X) \end{array}$$

is cartesian in the category of sets. The rough idea of the proof is to dominate the morphism by other almost blowup squares to which we can apply assumptions (3) and (4) locally.

Suppose we have an almost blow up square (38.37.0.1) in the category $(Sch/S)_h$, an open covering $X = \bigcup U_i$, and open coverings $U_i \cap U_j = \bigcup U_{ijk}$ such that the diagrams

$$\begin{array}{ccc} \mathcal{F}(E \cap b^{-1}(U_i)) & \longleftarrow & \mathcal{F}(b^{-1}(U_i)) \\ \uparrow & & \uparrow \\ \mathcal{F}(Z \cap U_i) & \longleftarrow & \mathcal{F}(U_i) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(E \cap b^{-1}(U_{ijk})) & \longleftarrow & \mathcal{F}(b^{-1}(U_{ijk})) \\ \uparrow & & \uparrow \\ \mathcal{F}(Z \cap U_{ijk}) & \longleftarrow & \mathcal{F}(U_{ijk}) \end{array}$$

are cartesian, then the same is true for

$$\begin{tikzcd} \mathcal{F}(E) & \mathcal{F}(X') \arrow[l] \\ \mathcal{F}(Z) \arrow[u] & \mathcal{F}(X) \arrow[l] \arrow[u] \end{tikzcd}$$

This follows as $\mathcal{F}$ is a sheaf in the Zariski topology.

In particular, if we have a blow up square (38.37.0.1) such that $b : X' \to X$ is a closed immersion and Z is a locally principal closed subscheme, then we see that $\mathcal{F}(X) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$. Namely, affine locally on X we obtain an almost blow up square as in (3).

Let $Z \subset X$, $E_k \subset X'_k \to X$, $E \subset X' \to X$, and $i_k : X' \to X'_k$ be as in the statement of Lemma 38.37.5. Then

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X' \arrow[d] \\ E_k \arrow[r] & X'_k \end{tikzcd}$$

is an almost blow up square of the kind discussed in the previous paragraph. Thus

$$\mathcal{F}(X'_k) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(E_k)$$

for $k = 1, 2$ by the result of the previous paragraph. It follows that

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(X'_k) \times_{\mathcal{F}(E_k)} \mathcal{F}(Z)$$

is bijective for $k = 1$ if and only if it is bijective for $k = 2$. Thus given a closed immersion $Z \to X$ of finite presentation with X quasi-compact and quasi-separated, whether or not $\mathcal{F}(X) = \mathcal{F}(X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$ is independent of the choice of the almost blow up square (38.37.0.1) one chooses. (Moreover, by Lemma 38.37.4 there does indeed exist an almost blow up square for $Z \subset X$.)

Finally, consider an affine object X of $(Sch/S)_h$ and a closed immersion $Z \to X$ of finite presentation. We will prove the desired property for the pair (X, Z) by induction on the number of generators r for the ideal defining Z in X. If the number of generators is ≤ 2, then we can choose our almost blow up square as in Example 38.37.11 and we conclude by assumption (4).

Induction step. Suppose $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/(f_1, \ldots, f_r))$ with $r > 2$. Choose a blow up square (38.37.0.1) for the pair (X, Z). Set $Z_1 = \text{Spec}(A/(f_1, f_2))$ and let

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & Y \arrow[d] \\ Z_1 \arrow[r] & X \end{tikzcd}$$

be the almost blow up square constructed in Example 38.37.11. By Lemma 38.37.1 the base changes

$$(I) \quad \begin{tikzcd} Y \times_X E \arrow[r] \arrow[d] & Y \times_X X' \arrow[d] \\ Y \times_X Z \arrow[r] & Y \end{tikzcd} \quad \text{and} \quad (II) \quad \begin{tikzcd} E \arrow[r] \arrow[d] & Z_1 \times_X X' \arrow[d] \\ Z \arrow[r] & Z_1 \end{tikzcd}$$

are almost blow up squares. The ideal of Z in Z_1 is generated by $r-2$ elements. The ideal of $Y \times_X Z$ is generated by the pullbacks of $f_1, \ldots, f_r$ to Y. Locally on Y the ideal generated by f_1, f_2 can be generated by one element, thus $Y \times_X Z$ is affine locally on Y cut out by at most $r-1$ elements. By induction hypotheses and the discussion above

$$\mathcal{F}(Y) = \mathcal{F}(Y \times_X X') \times_{\mathcal{F}(Y \times_X E)} \mathcal{F}(Y \times_X Z)$$

and

$$\mathcal{F}(Z_1) = \mathcal{F}(Z_1 \times_X X') \times_{\mathcal{F}(E)} \mathcal{F}(Z)$$

By assumption (4) we have

$$\mathcal{F}(X) = \mathcal{F}(Y) \times_{\mathcal{F}(E_1)} \mathcal{F}(Z_1)$$

Now suppose we have a pair (s', t) with $s' \in \mathcal{F}(X')$ and $t \in \mathcal{F}(Z)$ with same restriction in $\mathcal{F}(E)$. Then $(s'|Z_1 \times_X X', t)$ are the image of a unique element $t_1 \in \mathcal{F}(Z_1)$. Similarly, $(s'|_{Y \times_X X'}, t|_{Y \times_X Z})$ are the image of a unique element $s_Y \in \mathcal{F}(Y)$. We claim that s_Y and t_1 restrict to the same element of $\mathcal{F}(E_1)$. This is true because the almost blow up square

$$\begin{tikzcd} E_1 \times_X E \arrow[r] \arrow[d] & E_1 \times_X X' \arrow[d] \\ E_1 \times_X Z \arrow[r] & E_1 \end{tikzcd}$$

is the base change of almost blow up square (I) via $E_1 \to Y$ and the base change of almost blow up square (II) via $E_1 \to Z_1$ and because the pairs of sections used to construct s_Y and t_1 match. Thus by the third fibre product equality we see that there is a unique $s \in \mathcal{F}(X)$ mapping to s_Y in $\mathcal{F}(Y)$ and to t_1 in $\mathcal{F}(Z)$. We omit the verification that s maps to s' in $\mathcal{F}(X')$ and to t in $\mathcal{F}(Z)$; hint: use uniqueness of s just constructed and work affine locally. □

0EX9 Lemma 38.37.13. Let $p : \mathcal{S} \to (Sch/S)_h$ be a category fibred in groupoids. Then $\mathcal{S}$ is a stack in groupoids if and only if the following conditions are satisfied

(1) $\mathcal{S}$ is a stack in groupoids for the Zariski topology,
(2) given a morphism $f : X \to Y$ of $(Sch/S)_h$ with Y affine and f surjective, flat, proper, and of finite presentation, then
$$\mathcal{S}_Y \longrightarrow \mathcal{S}_X \times_{\mathcal{S}_{X \times_Y X}} \mathcal{S}_X$$
is an equivalence of categories,
(3) for an almost blow up square as in Example 38.37.10 or 38.37.11 in the category $(Sch/S)_h$ the functor
$$\mathcal{S}_X \longrightarrow \mathcal{S}_Z \times_{\mathcal{S}_E} \mathcal{S}_{X'}$$
is an equivalence of categories.

Proof. This lemma is a formal consequence of Lemma 38.37.12 and our defnition of stacks in groupoids. For example, assume (1), (2), (3). To show that $\mathcal{S}$ is a stack, we have to prove descent for morphisms and objects, see Stacks, Definition 8.5.1.

If x, y are objects of $\mathcal{S}$ over an object U of $(Sch/S)_h$, then our assumptions imply $Isom(x, y)$ is a presheaf on $(Sch/U)_h$ which satisfies (1), (2), (3), and (4) of Lemma 38.37.12 and therefore is a sheaf. Some details omitted.

Let $\{U_i \to U\}_{i \in I}$ be a covering of $(Sch/S)_h$. Let (x_i, φ_{ij}) be a descent datum in $\mathcal{S}$ relative to the family $\{U_i \to U\}_{i \in I}$, see Stacks, Definition 8.3.1. Consider the rule F which to V/U in $(Sch/U)_h$ associates the set of pairs (y, ψ_i) where y is an object of $\mathcal{S}_V$ and $\psi_i : y|_{U_i \times_U V} \to x_i|_{U_i \times_U V}$ is a morphism of $\mathcal{S}$ over $U_i \times_U V$ such that

$$\varphi_{ij}|_{U_i \times_U U_j \times_U V} \circ \psi_i|_{U_i \times_U U_j \times_U V} = \psi_j|_{U_i \times_U U_j \times_U V}$$

up to isomorphism. Since we already have descent for morphisms, it is clear that $F(V/U)$ is either empty or a singleton set. On the other hand, we have $F(U_{i_0}/U)$ is nonempty because it contains $(x_{i_0}, \varphi_{i_0 i})$. Since our goal is to prove that $F(U/U)$ is nonempty, it suffices to show that F is a sheaf on $(Sch/U)_h$. To do this we may use the criterion of Lemma 38.37.12. However, our assumptions (1), (2), (3) imply (by drawing some commutative diagrams which we omit), that properties (1), (2), (3), and (4) of Lemma 38.37.12 hold for F.

We omit the verification that if $\mathcal{S}$ is a stack in groupoids, then (1), (2), and (3) are satisfied. □

38.38. Absolute weak normalization and h coverings

0EVS In this section we use the criteria found in Section 38.37 to exhibit some h sheaves and we relate h sheafification of the structure sheaf to absolute weak normalization. We will need the following elementary lemma to do this.

0EVJ **Lemma 38.38.1.** Let Z, X, X', E be an almost blow up square as in Example 38.37.11. Then $H^p(X', \mathcal{O}_{X'}) = 0$ for $p > 0$ and $\Gamma(X, \mathcal{O}_X) \to \Gamma(X', \mathcal{O}_{X'})$ is a surjective map of rings whose kernel is an ideal of square zero.

Proof. First assume that $A = \mathbf{Z}[f_1, f_2]$ is the polynomial ring. In this case our almost blow up square is the blowing up of $X = \operatorname{Spec}(A)$ in the closed subscheme Z and in fact $X' \subset \mathbf{P}^1_X$ is an effective Cartier divisor cut out by the global section $f_2 T_0 - f_1 T_1$ of $\mathcal{O}_{\mathbf{P}^1_X}(1)$. Thus we have a resolution

$$0 \to \mathcal{O}_{\mathbf{P}^1_X}(-1) \to \mathcal{O}_{\mathbf{P}^1_X} \to \mathcal{O}_{X'} \to 0$$

Using the description of the cohomology given in Cohomology of Schemes, Section 30.8 it follows that in this case $\Gamma(X, \mathcal{O}_X) \to \Gamma(X', \mathcal{O}_{X'})$ is an isomorphism and $H^1(X', \mathcal{O}_{X'}) = 0$.

Next, we observe that any diagram as in Example 38.37.11 is the base change of the diagram in the previous paragraph by the ring map $\mathbf{Z}[f_1, f_2] \to A$. Hence by More on Morphisms, Lemmas 37.72.1, 37.72.2, and 37.72.4 we conclude that $H^1(X', \mathcal{O}_{X'})$ is zero in general and the surjectivity of the map $H^0(X, \mathcal{O}_X) \to H^0(X', \mathcal{O}_{X'})$ in general.

Next, in the general case, let us study the kernel. If $a \in A$ maps to zero, then looking on affine charts we see that

$$a = (f_1 x - f_2)(a_0 + a_1 x + \ldots + a_r x^r) \text{ in } A[x]$$

for some $r \geq 0$ and $a_0, \ldots, a_r \in A$ and similarly

$$a = (f_1 - f_2 y)(b_0 + b_1 y + \ldots + b_s y^s) \text{ in } A[y]$$

for some $s \geq 0$ and $b_0, \ldots, b_s \in A$. This means we have

$$a = f_2 a_0,\ f_1 a_0 = f_2 a_1,\ \ldots,\ f_1 a_r = 0,\ a = f_1 b_0,\ f_2 b_0 = f_1 b_1,\ \ldots,\ f_2 b_s = 0$$

If (a', r', a'_i, s', b'_j) is a second such system, then we have

$$aa' = f_1 f_2 a_0 b'_0 = f_1 f_2 a_1 b'_1 = f_1 f_2 a_2 b'_2 = \ldots = 0$$

as desired. □

For an $\mathbf{F}_p$-algebra A we set $\text{colim}_F A$ equal to the colimit of the system

$$A \xrightarrow{F} A \xrightarrow{F} A \xrightarrow{F} \ldots$$

where $F : A \to A$, $a \mapsto a^p$ is the Frobenius endomorphism.

0EVK Lemma 38.38.2. Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $(Sch/S)_h$ be a site as in Definition 38.34.13. There is a unique sheaf $\mathcal{F}$ on $(Sch/S)_h$ such that

$$\mathcal{F}(X) = \text{colim}_F \Gamma(X, \mathcal{O}_X)$$

for any quasi-compact and quasi-separated object X of $(Sch/S)_h$.

Proof. Denote $\mathcal{F}$ the Zariski sheafification of the functor

$$X \longrightarrow \text{colim}_F \Gamma(X, \mathcal{O}_X)$$

For quasi-compact and quasi-separated schemes X we have $\mathcal{F}(X) = \text{colim}_F \Gamma(X, \mathcal{O}_X)$. by Sheaves, Lemma 6.29.1 and the fact that $\mathcal{O}$ is a sheaf for the Zariski topology. Thus it suffices to show that $\mathcal{F}$ is a h sheaf. To prove this we check conditions (1), (2), (3), and (4) of Lemma 38.37.12. Condition (1) holds because we performed an (almost unnecessary) Zariski sheafification. Condition (2) holds because $\mathcal{O}$ is an fppf sheaf (Descent, Lemma 35.8.1) and if A is the equalizer of two maps $B \to C$ of $\mathbf{F}_p$-algebras, then $\text{colim}_F A$ is the equalizer of the two maps $\text{colim}_F B \to \text{colim}_F C$.

We check condition (3). Let A, f, J be as in Example 38.37.10. We have to show that

$$\text{colim}_F A = \text{colim}_F A/J \times_{\text{colim}_F A/fA+J} \text{colim}_F A/fA$$

This reduces to the following algebra question: suppose $a', a'' \in A$ are such that $F^n(a' - a'') \in fA + J$. Find $a \in A$ and $m \geq 0$ such that $a - F^m(a') \in J$ and $a - F^m(a'') \in fA$ and show that the pair (a, m) is uniquely determined up to a replacement of the form $(a, m) \mapsto (F(a), m+1)$. To do this just write $F^n(a'-a'') = fh + g$ with $h \in A$ and $g \in J$ and set $a = F^n(a') - g = F^n(a'') + fh$ and set $m = n$. To see uniqueness, suppose (a_1, m_1) is a second solution. By a replacement of the form given above we may assume $m = m_1$. Then we see that $a - a_1 \in J$ and $a - a_1 \in fA$. Since J is annihilated by a power of f we see that $a - a_1$ is a nilpotent element. Hence $F^k(a - a_1)$ is zero for some large k. Thus after doing more replacements we get $a = a_1$.

We check condition (4). Let X, X', Z, E be as in Example 38.37.11. By Lemma 38.38.1 we see that

$$\mathcal{F}(X) = \text{colim}_F \Gamma(X, \mathcal{O}_X) \longrightarrow \text{colim}_F \Gamma(X', \mathcal{O}_{X'}) = \mathcal{F}(X')$$

is bijective. Since $E = \mathbf{P}^1_Z$ in this case we also see that $\mathcal{F}(Z) \to \mathcal{F}(E)$ is bijective. Thus the conclusion holds in this case as well. □

Let p be a prime number. For an $\mathbf{F}_p$-algebra A we set $\lim_F A$ equal to the limit of the inverse system

$$\ldots \xrightarrow{F} A \xrightarrow{F} A \xrightarrow{F} A$$

where $F : A \to A$, $a \mapsto a^p$ is the Frobenius endomorphism.

0EVL Lemma 38.38.3. Let p be a prime number. Let S be a scheme over $\mathbf{F}_p$. Let $(Sch/S)_h$ be a site as in Definition 38.34.13. The rule

$$\mathcal{F}(X) = \lim_F \Gamma(X, \mathcal{O}_X)$$

defines a sheaf on $(Sch/S)_h$.

Proof. To prove $\mathcal{F}$ is a sheaf, let's check conditions (1), (2), (3), and (4) of Lemma 38.37.12. Condition (1) holds because limits of sheaves are sheaves and $\mathcal{O}$ is a Zariski sheaf. Condition (2) holds because $\mathcal{O}$ is an fppf sheaf (Descent, Lemma 35.8.1) and if A is the equalizer of two maps $B \to C$ of $\mathbf{F}_p$-algebras, then $\lim_F A$ is the equalizer of the two maps $\lim_F B \to \lim_F C$.

We check condition (3). Let A, f, J be as in Example 38.37.10. We have to show that

$$\begin{aligned} \lim_F A \to \lim_F A/J \times_{\lim_F A/fA+J} \lim_F A/fA \\ = \lim_F(A/J \times_{A/fA+J} A/fA) \\ = \lim_F A/(fA \cap J) \end{aligned}$$

is bijective. Since J is annihilated by a power of f we see that $\mathfrak{a} = fA \cap J$ is a nilpotent ideal, i.e., there exists an n such that $\mathfrak{a}^n = 0$. It is straightforward to verify that in this case $\lim_F A \to \lim_F A/\mathfrak{a}$ is bijective.

We check condition (4). Let X, X', Z, E be as in Example 38.37.11. By Lemma 38.38.1 and the same argument as above we see that

$$\mathcal{F}(X) = \lim_F \Gamma(X, \mathcal{O}_X) \longrightarrow \lim_F \Gamma(X', \mathcal{O}_{X'}) = \mathcal{F}(X')$$

is bijective. Since $E = \mathbf{P}^1_Z$ in this case we also see that $\mathcal{F}(Z) \to \mathcal{F}(E)$ is bijective. Thus the conclusion holds in this case as well. □

In the following lemma we use the absolute weak normalization X^{awn} of a scheme X, see Morphisms, Section 29.47.

0EVT Lemma 38.38.4. Let $(Sch/S)_{ph}$ be a site as in Topologies, Definition 34.8.11. The rule

$$X \longmapsto \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$$

is a sheaf on $(Sch/S)_{ph}$.

Proof. To prove $\mathcal{F}$ is a sheaf, let's check conditions (1) and (2) of Topologies, Lemma 34.8.15. Condition (1) holds because formation of X^{awn} commutes with open coverings, see Morphisms, Lemma 29.47.7 and its proof.

Let $\pi : Y \to X$ be a surjective proper morphism. We have to show that the equalizer of the two maps

$$\Gamma(Y^{awn}, \mathcal{O}_{Y^{awn}}) \to \Gamma((Y \times_X Y)^{awn}, \mathcal{O}_{(Y \times_X Y)^{awn}})$$

is equal to $\Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$. Let f be an element of this equalizer. Then we consider the morphism

$$f : Y^{awn} \longrightarrow \mathbf{A}^1_X$$

Since $Y^{awn} \to X$ is universally closed, the scheme theoretic image Z of f is a closed subscheme of $\mathbf{A}^1_X$ proper over X and $f : Y^{awn} \to Z$ is surjective. See Morphisms, Lemma 29.41.10. Thus $Z \to X$ is finite (Morphisms, Lemma 29.44.11) and surjective.

Let k be a field and let $z_1, z_2 : \mathrm{Spec}(k) \to Z$ be two morphisms equalized by $Z \to X$. We claim that $z_1 = z_2$. It suffices to show the images $\lambda_i = z_i^* f \in k$ agree (as the structure sheaf of Z is generated by f over the structure sheaf of X). To see this we choose a field extension K/k and morphisms $y_1, y_2 : \mathrm{Spec}(K) \to Y^{awn}$ such that $z_i \circ (\mathrm{Spec}(K) \to \mathrm{Spec}(k)) = f \circ y_i$. This is possible by the surjectivity of the map $Y^{awn} \to Z$. Choose an algebraically closed extension Ω/k of very large cardinality. For any k-algebra maps $\sigma_i : K \to \Omega$ we obtain

$$\mathrm{Spec}(\Omega) \xrightarrow{\sigma_1, \sigma_2} \mathrm{Spec}(K \otimes_k K) \xrightarrow{y_1, y_2} Y^{awn} \times_X Y^{awn}$$

Since the canonical morphism $(Y \times_X Y)^{awn} \to Y^{awn} \times_X Y^{awn}$ is a universal homeomorphism and since Ω is algebraically closed, we can lift the composition above uniquely to a morphism $\mathrm{Spec}(\Omega) \to (Y \times_X Y)^{awn}$. Since f is in the equalizer above, this proves that $\sigma_1(\lambda_1) = \sigma_2(\lambda_2)$. An easy lemma about field extensions shows that this implies $\lambda_1 = \lambda_2$; details omitted.

We conclude that $Z \to X$ is universally injective, i.e., $Z \to X$ is injective on points and induces purely inseparated residue field extensions (Morphisms, Lemma 29.10.2). All in all we conclude that $Z \to X$ is a universal homeomorphism, see Morphisms, Lemma 29.45.5.

Let $g : X^{awn} \to Z$ be the map obtained from the universal property of X^{awn}. Then $Y^{awn} \to X^{awn} \to Z$ and $f : Y^{awn} \to Z$ are two morphisms over X. By the universal property of $Y^{awn} \to Y$ the two corresponding morphisms $Y^{awn} \to Y \times_X Z$ over Y have to be equal. This implies that $g \circ \pi^{wan} = f$ as morphisms into $\mathbf{A}^1_X$ and we conclude that $g \in \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$ is the element we were looking for. $\square$

0EVU Lemma 38.38.5. Let S be a scheme. Choose a site $(Sch/S)_h$ as in Definition 38.34.13. The rule

$$X \longmapsto \Gamma(X^{awn}, \mathcal{O}_{X^{awn}})$$

is the sheafification of the "structure sheaf" $\mathcal{O}$ on $(Sch/S)_h$. Similarly for the ph topology.

Proof. In Lemma 38.38.4 we have seen that the rule $\mathcal{F}$ of the lemma defines a sheaf in the ph topology and hence a fortiori a sheaf for the h topology. Clearly, there is a canonical map of presheaves of rings $\mathcal{O} \to \mathcal{F}$. To finish the proof, it suffices to show

(1) if $f \in \mathcal{O}(X)$ maps to zero in $\mathcal{F}(X)$, then there is a h covering $\{X_i \to X\}$ such that $f|_{X_i} = 0$, and
(2) given $f \in \mathcal{F}(X)$ there is a h covering $\{X_i \to X\}$ such that $f|_{X_i}$ is the image of $f_i \in \mathcal{O}(X_i)$.

Let f be as in (1). Then $f|_{X^{awn}} = 0$. This means that f is locally nilpotent. Thus if $X' \subset X$ is the closed subscheme cut out by f, then $X' \to X$ is a surjective closed immersion of finite presentation. Hence $\{X' \to X\}$ is the desired h covering. Let f be as in (2). After replacing X by the members of an affine open covering we may assume $X = \mathrm{Spec}(A)$ is affine. Then $f \in A^{awn}$, see Morphisms, Lemma 29.47.6. By Morphisms, Lemma 29.46.11 we can find a ring map $A \to B$ of finite presentation such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism and such that f is the image of an element $b \in B$ under the canonical map $B \to A^{awn}$. Then $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an h covering and we conclude. The statement about the ph topology follows in the same manner (or it can be deduced from the statement for the h topology). ☐

Let p be a prime number. An $\mathbf{F}_p$-algebra A is called perfect if the map $F : A \to A$, $x \mapsto x^p$ is an automorphism of A.

0EVV Lemma 38.38.6. Let p be a prime number. An $\mathbf{F}_p$-algebra A is absolutely weakly normal if and only if it is perfect.

Proof. It is immediate from condition (2)(b) in Morphisms, Definition 29.47.1 that if A is absolutely weakly normal, then it is perfect.

Assume A is perfect. Suppose $x, y \in A$ with $x^3 = y^2$. If $p > 3$ then we can write $p = 2n + 3m$ for some $n, m > 0$. Choose $a, b \in A$ with $a^p = x$ and $b^p = y$. Setting $c = a^n b^m$ we have

$$c^{2p} = x^{2n} y^{2m} = x^{2n+3m} = x^p$$

and hence $c^2 = x$. Similarly $c^3 = y$. If $p = 2$, then write $x = a^2$ to get $a^6 = y^2$ which implies $a^3 = y$. If $p = 3$, then write $y = a^3$ to get $x^3 = a^6$ which implies $x = a^2$.

Suppose $x, y \in A$ with $\ell^\ell x = y^\ell$ for some prime number ℓ. If $\ell \neq p$, then $a = y/\ell$ satsifies $a^\ell = x$ and $\ell a = y$. If $\ell = p$, then $y = 0$ and $x = a^p$ for some a. ☐

0EVW Lemma 38.38.7. Let p be a prime number.

(1) If A is an $\mathbf{F}_p$-algebra, then $\mathrm{colim}_F\, A = A^{awn}$.
(2) If S is a scheme over $\mathbf{F}_p$, then the h sheafification of $\mathcal{O}$ sends a quasi-compact and quasi-separated X to $\mathrm{colim}_F\, \Gamma(X, \mathcal{O}_X)$.

Proof. Proof of (1). Observe that $A \to \mathrm{colim}_F\, A$ induces a universal homeomorphism on spectra by Algebra, Lemma 10.46.7. Thus it suffices to show that $B = \mathrm{colim}_F\, A$ is absolutely weakly normal, see Morphisms, Lemma 29.47.6. Note that the ring map $F : B \to B$ is an automorphism, in other words, B is a perfect ring. Hence Lemma 38.38.6 applies.

Proof of (2). This follows from (1) and Lemmas 38.38.2 and 38.38.5 by looking affine locally. ☐

38.39. Descent vector bundles in positive characteristic

0EXA A reference for this section is [BS17].

For a scheme S let us denote $\mathrm{Vect}(S)$ the category of finite locally free $\mathcal{O}_S$-modules. Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. In this section we will work with the category

$$\mathrm{colim}_F\, \mathrm{Vect}(S) = \mathrm{colim}\left(\mathrm{Vect}(S) \xrightarrow{F^*} \mathrm{Vect}(S) \xrightarrow{F^*} \mathrm{Vect}(S) \xrightarrow{F^*} \ldots\right)$$

where $F : S \to S$ is the absolute Frobenius morphism. In down to earth terms an object of this category is a pair $(\mathcal{E}, n)$ where $\mathcal{E}$ is a finite locally free $\mathcal{O}_S$-module and $n \geq 0$ is an integer. For morphisms we take

$$\mathrm{Hom}_{\mathrm{colim}_F \mathrm{Vect}(S)}((\mathcal{E}, n), (\mathcal{G}, m)) = \mathrm{colim}_N\, \mathrm{Hom}_S(F^{N-n,*}\mathcal{E}, F^{N-m,*}\mathcal{G})$$

where $F : S \to S$ is the absolute Frobenius morphism of S. Thus the object $(\mathcal{E}, n)$ is isomorphic to the object $(F^*\mathcal{E}, n + 1)$.

0EXB Lemma 38.39.1. Let p be a prime number. Let S be a quasi-compact and quasi-separated scheme over $\mathbf{F}_p$. The category $\mathrm{colim}_F \mathrm{Vect}(S)$ is equivalent to the category of finite locally free modules over the sheaf of rings $\mathrm{colim}_F \mathcal{O}_S$ on S.

Proof. Omitted. $\square$

0EXC Lemma 38.39.2. Let p be a prime number. Consider an almost blowup square X, X', Z, E in characteristic p as in Example 38.37.10. Then the functor

$$\mathrm{colim}_F \mathrm{Vect}(X) \longrightarrow \mathrm{colim}_F \mathrm{Vect}(Z) \times_{\mathrm{colim}_F \mathrm{Vect}(E)} \mathrm{colim}_F \mathrm{Vect}(X')$$

is an equivalence.

Proof. Let A, f, J be as in Example 38.37.10. Since all our schemes are affine and since we have internal Hom in the category of vector bundles, the fully faithfulness of the functor follows if we can show that

$$\mathrm{colim}\, P \otimes_{A,F^N} A = \mathrm{colim}\, P \otimes_{A,F^N} A/J \times_{\mathrm{colim}\, P \otimes_{A,F^N} A/fA+J} \mathrm{colim}\, P \otimes_{A,F^N} A/fA$$

for a finite projective A-module P. After writing P as a summand of a finite free module, this follows from the case where P is finite free. This case immediately reduces to the case $P = A$. The case $P = A$ follows from Lemma 38.38.2 (in fact we proved this case directly in the proof of this lemma).

Essential surjectivity. Here we obtain the following algebra problem. Suppose P_1 is a finite projective A/J-module, P_2 is a finite projective A/fA-module, and

$$\varphi : P_1 \otimes_{A/J} A/fA + J \longrightarrow P_2 \otimes_{A/fA} A/fA + J$$

is an isomorphism. Goal: show that there exists an N, a finite projective A-module P, an isomorphism $\varphi_1 : P \otimes_A A/J \to P_1 \otimes_{A/J,F^N} A/J$, and an isomorphism $\varphi_2 : P \otimes_A A/fA \to P_2 \otimes_{A/fA,F^N} A/fA$ compatible with φ in an obvious manner. This can be seen as follows. First, observe that

$$A/(J \cap fA) = A/J \times_{A/fA+J} A/fA$$

Hence by More on Algebra, Lemma 15.6.9 there is a finite projective module P' over $A/(J \cap fA)$ which comes with isomorphisms $\varphi'_1 : P' \otimes_A A/J \to P_1$ and $\varphi_2 : P' \otimes_A A/fA \to P_2$ compatible with φ. Since J is a finitely generated ideal and f-power torsion we see that $J \cap fA$ is a nilpotent ideal. Hence for some N there is a factorization

$$A \xrightarrow{\alpha} A/(J \cap fA) \xrightarrow{\beta} A$$

of F^N. Setting $P = P' \otimes_{A/(J\cap fA),\beta} A$ we conclude. $\square$

0EXD Lemma 38.39.3. Let p be a prime number. Consider an almost blowup square X, X', Z, E in characteristic p as in Example 38.37.11. Then the functor

$$G : \mathrm{colim}_F \mathrm{Vect}(X) \longrightarrow \mathrm{colim}_F \mathrm{Vect}(Z) \times_{\mathrm{colim}_F \mathrm{Vect}(E)} \mathrm{colim}_F \mathrm{Vect}(X')$$

is an equivalence.

Proof. Fully faithfulness. Suppose that $(\mathcal{E}, n)$ and $(\mathcal{F}, m)$ are objects of $\mathrm{colim}_F \mathrm{Vect}(X)$. Let $(a, b) : G(\mathcal{E}, n) \to G(\mathcal{F}, m)$ be a morphism in the RHS. We may choose $N \gg 0$ and think of a as a map $a : F^{N-n,*}\mathcal{E}|_Z \to F^{N-m,*}\mathcal{F}|_Z$ and b as a map

$b : F^{N-n,*}\mathcal{E}|_{X'} \to F^{N-m,*}\mathcal{F}|_{X'}$ agreeing over E. Choose a finite affine open covering $X = X_1 \cup \ldots \cup X_n$ such that $\mathcal{E}|_{X_i}$ and $\mathcal{F}|_{X_i}$ are finite free $\mathcal{O}_{X_i}$-modules. For each i the base change

$$\begin{tikzcd} E_i \arrow[r] \arrow[d] & X_i' \arrow[d] \\ Z_i \arrow[r] & X_i \end{tikzcd}$$

is another almost blow up square as in Example 38.37.11. For these squares we know that

$$\mathrm{colim}_F\, H^0(X_i, \mathcal{O}_{X_i}) = \mathrm{colim}_F\, H^0(Z_i, \mathcal{O}_{Z_i}) \times_{\mathrm{colim}_F\, H^0(E_i, \mathcal{O}_{E_i})} \mathrm{colim}_F\, H^0(X_i', \mathcal{O}_{X_i'})$$

by Lemma 38.38.2 (see proof of the lemma). Hence after increasing N we may assume the maps $a|_{Z_i}$ and $b|_{X_i'}$ come from maps $c_i : F^{N-n,*}\mathcal{E}|_{X_i} \to F^{N-m,*}\mathcal{F}|_{X_i}$. After possibly increasing N we may assume c_i and c_j agree on $X_i \cap X_j$. Thus these maps glue to give the desired morphism $(\mathcal{E}, n) \to (\mathcal{F}, m)$ in the LHS.

Essential surjectivity. Let $(\mathcal{F}, \mathcal{G}, \varphi)$ be a triple consisting of a finite locally free $\mathcal{O}_Z$-module $\mathcal{F}$, a finite locally free $\mathcal{O}_{X'}$-module $\mathcal{G}$, and an isomorphism $\varphi : \mathcal{F}|_E \to \mathcal{G}|_E$. We have to show that after replacing this triple by a Frobenius power pullback, it comes from a finite locally free $\mathcal{O}_X$-module.

Noetherian reduction; we urge the reader to skip this paragraph. Recall that $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/(f_1, f_2))$, $X' = \mathrm{Proj}(A[T_0, T_1]/(f_2T_0 - f_1T_1))$, and $E = \mathbf{P}^1_Z$. By Limits, Lemma 32.10.3 we can find a finitely generated $\mathbf{F}_p$-subalgebra $A_0 \subset A$ containing f_1 and f_2 such that the triple $(\mathcal{F}, \mathcal{G}, \varphi)$ descends to $X_0 = \mathrm{Spec}(A_0)$ and $Z_0 = \mathrm{Spec}(A_0/(f_1, f_2))$, $X_0' = \mathrm{Proj}(A_0[T_0, T_1]/(f_2T_0 - f_1T_1))$, and $E_0 = \mathbf{P}^1_{Z_0}$. Thus we may assume our schemes are Noetherian.

Assume X is Noetherian. We may choose a finite affine open covering $X = X_1 \cup \ldots \cup X_n$ such that $\mathcal{F}|_{Z \cap X_i}$ is free. Since we can glue objects of $\mathrm{colim}_F\, \mathrm{Vect}(X)$ in the Zariski topology (Lemma 38.39.1), and since we already know fully faithfulness over X_i and $X_i \cap X_j$ (see first paragraph of the proof), it suffices to prove the existence over each X_i. This reduces us to the case discussed in the next paragraph.

Assume X is Noetherian and $\mathcal{F} = \mathcal{O}_Z^{\oplus r}$. Using φ we get an isomorphism $\mathcal{O}_E^{\oplus r} \to \mathcal{G}|_E$. Let $I = (f_1, f_2) \subset A$. Let $\mathcal{I} \subset \mathcal{O}_{X'}$ be the ideal sheaf of E; it is globally generated by f_1 and f_2. For any n there is a surjection

$$(\mathcal{I}^n/\mathcal{I}^{n+1})^{\oplus r} = \mathcal{I}^n/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_E} \mathcal{G}|_E \longrightarrow \mathcal{I}^n\mathcal{G}/\mathcal{I}^{n+1}\mathcal{G}$$

Hence the first cohomology group of this module is zero. Here we use that $E = \mathbf{P}^1_Z$ and hence its structure sheaf and in fact any globally generated quasi-coherent module has vanishing H^1. Compare with More on Morphisms, Lemma 37.72.3. Then using the short exact sequences

$$0 \to \mathcal{I}^n\mathcal{G}/\mathcal{I}^{n+1}\mathcal{G} \to \mathcal{G}/\mathcal{I}^{n+1}\mathcal{G} \to \mathcal{G}/\mathcal{I}^n\mathcal{G} \to 0$$

and induction, we see that

$$\lim H^0(X', \mathcal{G}/\mathcal{I}^n\mathcal{G}) \to H^0(E, \mathcal{G}|_E) = H^0(E, \mathcal{O}_E^{\oplus r}) = A/I^{\oplus r}$$

is surjective. By the theorem on formal functions (Cohomology of Schemes, Theorem 30.20.5) this implies that

$$H^0(X', \mathcal{G}) \to H^0(E, \mathcal{G}|_E) = H^0(E, \mathcal{O}_E^{\oplus r}) = A/I^{\oplus r}$$